Constructing Arbitrage-free Binomial Models

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Abstract

In the past decades several versions of the binomial model for option pricing, originally introduced by COX, ROSS, AND RUBINSTEIN, have been discussed in the finance literature. Some of these approaches model an arbitrage-free market in the discrete setup whereas others attain this property only in the limit. We analyze the interrelation between the drift coefficient of price processes on arbitrage-free financial markets and the corresponding transition probabilities induced by a martingale measure. As a result, we obtain a flexible setting that encompasses most arbitrage-free binomial models and provides modifications for those that offer arbitrage opportunities. It is argued that the knowledge of the link between drift and transition probabilities may be useful for pricing derivatives such as barrier options. A simple example is presented to illustrate this idea.

Keywords: binomial model, martingale method, option pricing, barrier option
JEL Classification: G13, C60

1 Introduction

When COX, ROSS, AND RUBINSTEIN [3] developed the binomial model for pricing options in the late seventies, they tried to attain two major objectives. The first one was of didactic nature. They clarified the basic principles underlying the state-of-the-art valuation methods for derivatives at that time using a modelling framework with reduced mathematical requirements compared to techniques introduced e.g. by BLACK AND SCHOLES [2] or MERTON [11] some years before. The second thought was directed to implementing efficient numerical algorithms for pricing contracts with arrangements that can trigger payments before maturity. Indeed, their procedure is suitable to determine, or at least to approximate, the arbitrage-free price of an American option. Similar approaches have been devised by SHARPE [17] and RENDLE-MAN AND BARTTER [15] nearly at the same time. Subsequent works modified the stochastic processes of the basic securities to improve their properties in numerical applications. Those adjustments were suggested by JARROW AND RUDD [9], TRIGEORGIS [19], and TIAN [18].

Besides the contributions dealing with numerical issues, the pricing of contingent claims was enhanced by the theoretical ideas of HARRISON AND KREPS [4] and HARRISON AND PLISKA [5]. Based on the assumptions of an arbitrage-free and complete security market they reformulated the valuation principles transferring elements of the martingale theory to topics in finance. The so-called *martingale method* has extremely influenced the pricing of financial instruments, especially in continuous-time models. The *Girsanov theorem*, which they introduced into mathematical finance, is one of the most relevant tools in financial engineering today.

This paper is aimed at comprehensively embedding the binomial model into the martingale pricing methodology. It focuses solely on versions based on the classical market parameters μ (drift coefficient) and σ (diffusion coefficient). The framework contains the models of COX, ROSS, AND RUBINSTEIN [3] and AMIN [1] as special cases. Furthermore, the methodology is capable to adjust the processes of those numerical models that violate the assumption of an arbitrage-free security market. The emphasis of idealized market properties makes the framework a basis for discussing discrete models against a theoretical background. In addition, it has desirable convergence properties and can be easily calibrated to market price data.

The following representation is based on articles by AMIN [1], JARROW [8] as well as HEATH, JARROW, AND MORTON [6, 7]. AMIN emphasizes the algorithmic character of his model while the latter authors apply the martingale methodology exclusively to interest rate processes. All three aforementioned approaches deal with the special case of equal transition

probabilities under the martingale measure. In some cases one might wish to deviate from this specification. Therefore, in this paper the transition probabilities are not fixed in advance, and the interrelation between the drift parameter of the underlying processes and the martingale probabilities are exhaustedly explored. In some applications it is quite useful to have a thorough understanding of the link between the decisive quantities. The valuation of a barrier option is an example where it might be adequate to first fix the step size of the process and to then determine the corresponding transition probabilities. An illustration is given in the sixth section.

2 The Valuation Framework

It is assumed that the participants of a financial market have clear and homogenous ideas on the price evolution of some securities (basis securities). In accordance with the modelling of COX, ROSS, AND RUBINSTEIN [3], the future prices are expressed as the outcome of a binomial process. Each path can be associated with an element in the sample space Ω. Together with a σ-Algebra $\mathcal{A}$ and a probability measure $\mathbb{P}$ it forms a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The probability space is equipped with a filtration $\{\mathcal{F}_{t_n}\}_{n=0}^{N}$ having the characteristic property

$$\mathcal{F}_{t_0} \subseteq \mathcal{F}_{t_1} \subseteq \ldots \subseteq \mathcal{F}_{t_N} = \mathcal{F}$$

representing the evolution of information on the market, where no piece of information gets lost over time.

Trading only takes place at certain equidistant points in time contained in the set

$$\mathcal{T} = \{0 = t_0, t_1, \ldots, t_N = T\}$$
$$= \{t_0, t_0 + \Delta, \ldots, t_0 + N \cdot \Delta\}$$

with the overall time interval from 0 to T being fixed. Suppose that the price of the money market fund is determined by the non-stochastic one-period interest rate $r \geq 0$, such that its price evolution can be described by

$$B_{t_n} = \begin{cases} B_{t_0}, & \text{if } t_n = t_0; \\ B_{t_{n-1}} \exp(r\Delta), & \text{if } t_0 < t_n \leq t_N. \end{cases}$$

The stochastic process that governs the evolution of the stock price is given by

$$
S_{t_n} = \begin{cases} S_{t_0}, & \text{if } t_n = t_0; \\ S_{t_{n-1}} \exp(\mu \Delta + \sigma \sqrt{\Delta} X_{t_n}), & \text{if } t_0 < t_n \leq t_N; \end{cases}
$$

where X_{t_n} is a sequence of independently identically distributed (i.i.d.) Bernoulli random variables

$$
X_{t_n} : (\Omega_{t_n}, \mathcal{F}_{t_n}) \to (\mathcal{X}_{t_n}, \mathcal{B}_{t_n})
$$

with outcomes in the state space $\mathcal{X}_{t_n} = \{-1, 1\}$. Given the information at t_n, the probability that $X_{t_{n+1}} = 1$ is p and $X_{t_{n+1}} = -1$ is $(1-p)$. The parameter $\mu \in \mathbb{R}$ is referred to as the drift coefficient and the parameter $\sigma > 0$ as the diffusion coefficient of the process.

Under the specified assumptions the local expected value of the logarithmic return equals

$$
\mathbb{E}_{\mathbb{P}} \left[\ln \left(\frac{S_{t_n}}{S_{t_{n-1}}} \right) \Bigg| \mathcal{F}_{t_{n-1}} \right] = \mu \Delta - (1 - 2p) \sigma \sqrt{\Delta}
$$

and the local variance of the logarithmic return is

$$
\mathrm{Var}_{\mathbb{P}} \left[\ln \left(\frac{S_{t_n}}{S_{t_{n-1}}} \right) \Bigg| \mathcal{F}_{t_{n-1}} \right] = 4p(1 - p)\sigma^2 \Delta.
$$

Note that the diffusion parameter σ has no influence on the local expected value if and only if p equals $\frac{1}{2}$. In this case the local variance reduces to

$$
\mathrm{Var}_{\mathbb{P}} \left[\ln \left(\frac{S_{t_n}}{S_{t_{n-1}}} \right) \Bigg| \mathcal{F}_{t_{n-1}} \right] = \sigma^2 \Delta.
$$

Equal probabilities play a prominent role when interpreting μ and σ as distribution coefficients in a binomial model. Therefore, we frequently split a probability p into this reference probability and a resulting deviation according to

$$
p = \frac{1}{2} + \frac{1}{2}\eta_p \qquad , \eta_p \in (-1, 1),
$$

which leads to

$$\mathrm{E}_{\mathbb{P}}\left[\ln\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)\middle|\mathcal{F}_{t_{n-1}}\right]=\mu\Delta+\eta_p\sigma\sqrt{\Delta}$$

and

$$\mathrm{Var}_{\mathbb{P}}\left[\ln\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)\middle|\mathcal{F}_{t_{n-1}}\right]=\left(1-\eta_p^2\right)\sigma^2\Delta.$$

For the fixed period from 0 to T, one obtains an expected return of

$$\mathrm{E}_{\mathbb{P}}\left[\ln\left(\frac{S_0}{S_T}\right)\middle|\mathcal{F}_{t_0}\right]=N\left(\mu\Delta+\eta_p\sigma\sqrt{\Delta}\right)=\mu T+\eta_p\sigma\sqrt{N\cdot T}$$

and a variance of

$$\mathrm{Var}_{\mathbb{P}}\left[\ln\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)\middle|\mathcal{F}_{t_{n-1}}\right]=N\left(1-\eta_p^2\right)\sigma^2\Delta=\left(1-\eta_p^2\right)\sigma^2T.$$

The expected value does not depend on the number of trading days if and only if $\eta_p=0$, which again underlines the importance of this specification.

3   Martingales

Since the market is arbitrage-free and complete, there is a unique probability measure under which all price processes formulated in units of a suitable numeraire are martingales. Any security can be used as a numeraire as long as its prices are positive in every state that occurs with positive probability. Although both processes satisfy this property in our model, we follow HARRISON AND KREPS [4] and use the riskless security as the numeraire. Let

$$\hat{S}_{t_n}:=B_{t_n}^{-1}S_{t_n}$$

be the price of the stock in time t_n formulated in units of the money market fund, then

$$\hat{S}_{t_n}=\mathrm{E}_Q\left[\hat{S}_{t_{n+1}}\middle|\mathcal{F}_{t_n}\right]$$

is satiesfied for all $t_n \in \mathcal{T} \setminus \{t_N\}$. Applying the martingale condition to the binomial model, one obtains

$$
\begin{aligned}
& B_{t_n}^{-1} B_{t_{n+1}} = \mathrm{E}_Q \left[S_{t_n}^{-1} S_{t_{n+1}} \,\middle|\, \mathcal{F}_{t_n} \right] \\
\Leftrightarrow \quad & \exp(r\Delta) = q \exp\left(\alpha\Delta + \sigma \sqrt{\Delta} \right) + (1-q) \exp\left(\alpha\Delta - \sigma \sqrt{\Delta} \right) \\
\Leftrightarrow \quad & \exp(r\Delta) = \exp(\alpha\Delta) \left[q \exp\left(\sigma \sqrt{\Delta} \right) + (1-q) \exp\left(-\sigma \sqrt{\Delta} \right) \right]
\end{aligned}
\tag{1}
$$

and thus a constraint for the drift parameter α given by

$$
\alpha = r - \frac{1}{\Delta} \ln \left[q \exp\left(\sigma \sqrt{\Delta} \right) + (1-q) \exp\left(\sigma \sqrt{\Delta} \right) \right] .
\tag{2}
$$

If the transition probability q under the martingale measure is split into

$$
q = \frac{1}{2} + \frac{1}{2} \eta_q \qquad , \eta_q \in (-1, 1),
$$

condition (1) can be rewritten as

$$
\exp(r\Delta) = \exp(\alpha) \left[\cosh\left(\sigma \sqrt{\Delta} \right) + \eta_q \sinh\left(\sigma \sqrt{\Delta} \right) \right] ,
$$

where the definitions

$$
\cosh(x) := \frac{\exp(x) + \exp(-x)}{2}
$$

and

$$
\sinh(x) := \frac{\exp(x) - \exp(-x)}{2}
$$

have been used to simplify the notation. Finally, the restriction on the drift term is given by

$$
\alpha = r - \frac{1}{\Delta} \ln \left(\cosh\left(\sigma \sqrt{\Delta} \right) + \eta_q \sinh\left(\sigma \sqrt{\Delta} \right) \right) .
\tag{3}
$$

4 Change of Measure and Drift Transformation

Equation (3) reveals a strict relationship between the drift coefficient and the transition probabilities if assets are priced on a complete and arbitrage-free security market. If we choose a particular $\hat{\alpha}$, we get a unique $\hat{\eta}_q$ that in turn determines the transition probabilities. Solving for η_q and regarding this quantity as a function of α results in

$$\eta_q(\alpha) = \frac{\exp\left(-(\alpha - r)\Delta\right) - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sinh\left(\sigma\sqrt{\Delta}\right)}, \tag{4}$$

which we refer to as a probability spread.

Thus, the probabilities under the martingale measure $\mathbb{Q}$ are given by

$$\begin{aligned} q(\alpha) &= \frac{1}{2} + \frac{\exp\left(-(\alpha - r)\Delta\right) - \cosh\left(\sigma\sqrt{\Delta}\right)}{2\sinh\left(\sigma\sqrt{\Delta}\right)} \\ &= \frac{\exp\left(-(\alpha - r)\Delta\right) - \exp\left(-\sigma\sqrt{\Delta}\right)}{\exp\left(\sigma\sqrt{\Delta}\right) - \exp\left(-\sigma\sqrt{\Delta}\right)}. \end{aligned} \tag{5}$$

Of course, α cannot be chosen completely arbitrarily. It must be ensured that the resulting probabilities lie within the range of 0 to 1. The condition that forces $q(\alpha)$ and $1 - q(\alpha)$ to be positive probabilities is

$$-1 < \frac{\exp(-(\alpha - r)\Delta) - \cosh(\sigma\sqrt{\Delta})}{\sinh(\sigma\sqrt{\Delta})} < 1\,,$$

and implies that

$$\alpha\Delta - \sigma\sqrt{\Delta} < r\Delta < \alpha\Delta + \sigma\sqrt{\Delta}$$

must be satisfied. Note that there is a natural assignment of α that guarantees $q(\alpha)$ to have the required properties in any specification. Setting $\alpha = r$ will always lead to positive transition probabilities, though other drift coefficients will do in special constellations. On the other hand, one can always choose a probability $q \in (0,1)$ and calculate the corresponding arbitrage-free

drift coefficient by

$$\alpha(q) = r - \frac{1}{\Delta} \ln \left(\cosh \left(\sigma \sqrt{\Delta} \right) + (2q - 1) \sinh \left(\sigma \sqrt{\Delta} \right) \right).$$

4.1 Drift Shift Under a Measure Inducing Equal Probabilities

In section 2, the prominent role of a probability measure that generates equal transition probabilities has been emphasized. Although the specification was discussed in terms of the probability measure $\mathbb{P}$ and drift coefficient μ, a martingale measure $\mathbb{Q}_{\frac{1}{2}}$ with equal probabilities can also be constructed transferring the properties to the valuation world. We get a combination $\left(\mathbb{Q}_{\frac{1}{2}}, \alpha\left(\frac{1}{2}\right) \right)$ with

$$\alpha\left(\frac{1}{2}\right) = r - \frac{1}{\Delta} \ln \left(\cosh \left(\sigma \sqrt{\Delta} \right) \right). \tag{6}$$

This specification can be found quite frequently in the finance literature. It has been applied to stock processes in option pricing models by AMIN [1] and to interest rate processes in term structure models by Jarrow [8] and HEATH, JARROW, AND MORTON [6, 7]. Although the usage of this combination is not compelling its symmetry, shared with the limit distribution, might be advantageous.

4.2 Change of Measure Under the Drift of the Riskless Asset

In many continuous time models the stock process formulated in units of a money market fund has a drift coefficient of r. Moreover, we gave reasons for the usage of that specification in discrete time models above. So, setting

$$\alpha \stackrel{!}{=} r$$

yields, in combination with the definition

$$\tanh x := \frac{\sinh(x)}{\cosh(x)},$$

a probability spread of

$$\eta_q(r) = \frac{1 - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sinh\left(\sigma\sqrt{\Delta}\right)} = \frac{1 - \exp\left(\sigma\sqrt{\Delta}\right)}{1 + \exp\left(\sigma\sqrt{\Delta}\right)} = \tanh\left(-\frac{\sigma\sqrt{\Delta}}{2}\right)$$

and consequently a transition probability of

$$q(r) = \frac{1}{1 + \exp\left(\sigma\sqrt{\Delta}\right)}. \tag{7}$$

4.3   The Relation to the CRR Binomial Model

By setting $\alpha = 0$, there is another parameter constellation that is frequently used in the literature. However, this does not seem to be an obvious choice at first glance. As already mentioned, it is not the stock process itself that is driftless in continuous-time models but the stock process in units of a numeraire.

This setting is used if the state dependent payments can be derived more easily in a symmetric tree. Indeed, the (logarithmic) stock process has some symmetry properties if it is driftless. One major application is the valuation of barrier options where the reflection principle is used in a symmetric binomial tree. We will come to this point later in an example. Using equation (4), one obtains

$$\eta(0) = \frac{\exp(r\Delta) - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sinh\left(\sigma\sqrt{\Delta}\right)}$$

and correspondingly, the transition probability $q(0)$ is given by

$$\begin{aligned} q(0) &= \frac{1}{2} + \frac{\exp\left(r\Delta\right) - \cosh\left(\sigma\sqrt{\Delta}\right)}{2\sinh\left(\sigma\sqrt{\Delta}\right)} \\ &= \frac{\exp(r\Delta) - \exp\left(-\sigma\sqrt{\Delta}\right)}{\exp\left(\sigma\sqrt{\Delta}\right) - \exp\left(-\sigma\sqrt{\Delta}\right)}. \end{aligned}$$

It coincides with the parameter setting of the approach by COX, ROSS, AND RUBINSTEIN [3]. Thus, we just get the result presented in their seminal article, apart from the fact that we use a continuously compounded interest rate here instead of a one-period compounded rate.

5 Limit Results

So far we have only analyzed the behavior of binomial option pricing models in a discrete setting. We now address to the investigation of the convergence behavior if the number of trading days in the fixed interval $[0, T]$ increases without bounds. It turns out that the distribution originally developed by BLACK AND SCHOLES [2] is obtained. Considering the limit behavior of approximations to diffusion processes analyzed by NELSON AND RAMASWAMY [13], this is no surprise. The results presented here emphasize the role the probability spread plays in the convergence process.

For $t_n \in \mathcal{T} \setminus \{t_0\}$, let

$$Y_{t_n} := \ln\left(\frac{\hat{S}_{t_n}}{\hat{S}_{t_{n-1}}}\right),$$

then $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_N}$ is a sequence of i.i.d. random variables with a characteristic function

$$
\begin{aligned}
\chi_{Y_{t_n}}(\theta) &= E_{\mathbb{Q}}\left[e^{i\theta Y_{t_n}}\right] \\
&= q(\alpha)e^{i\theta\left(\alpha\Delta + \sigma\sqrt{\Delta}\right)} + (1 - q(\alpha))e^{i\theta\left(\alpha\Delta - \sigma\sqrt{\Delta}\right)}.
\end{aligned}
$$

Defining

$$Z_N := \sum_{n=1}^{N} Y_{t_n} = \ln\left(\frac{\hat{S}_T}{\hat{S}_0}\right)$$

yields the overall logarithmic stock return whose distribution under the martingale measure we are interested in. Since the random variables $(Y_{t_n})_{n=1}^{N}$ are independent, the characteristic function of the sum can be expressed as the product of the components' characteristic functions. We obtain

$$
\begin{aligned}
\chi_{Z_N}(\theta) &= E_{\mathbb{Q}}\left[e^{i\theta Z_N}\right] \\
&= \prod_{n=1}^{N} \chi_{Y_{t_n}}(\theta) = \left(q(\alpha)e^{i\theta\left(\alpha\Delta + \sigma\sqrt{\Delta}\right)} + (1 - q(\alpha))e^{i\theta\left(\alpha\Delta - \sigma\sqrt{\Delta}\right)}\right)^{N},
\end{aligned}
$$

the characteristic function of the overall logarithmic return. Its logarithm is given by

$$\ln\left(\chi_{Z_N}(\theta)\right) = \frac{1}{\Delta}\ln\left(q(\alpha)e^{i\theta\left(\alpha\Delta+\sigma\sqrt{\Delta}\right)} + (1-q(\alpha))e^{i\theta\left(\alpha\Delta-\sigma\sqrt{\Delta}\right)}\right)T$$

$$= \alpha\theta Ti + \frac{1}{\Delta}\ln\left(q(\alpha)e^{i\theta\sigma\sqrt{\Delta}} + (1-q(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}\right)T \qquad (8)$$

where we use this monotonic transformation for convenience.

Proposition 1 *Let* $\mathfrak{T} = \{0 = t_0,\ldots,t_N = T\}$ *be a discrete index set representing equidistant points of time. Let the price process of S in units of the numeraire B be given by*

$$\hat{S}_{t_n} = \begin{cases} \hat{S}_{t_0}, & \text{if } t_n = t_0; \\ \hat{S}_{t_{n-1}}\exp\left(\alpha\Delta + \sigma\sqrt{\Delta}X_{t_n}\right), & \text{if } t_0 < t_n \geq t_N; \end{cases}$$

where $X_{t_1},\ldots,X_{t_N}$ *is a sequence of i.i.d. random variables with*

$$X_{t_n} = \begin{cases} 1, & \text{with prob} \quad q(\alpha); \\ -1, & \text{with prob } 1-q(\alpha). \end{cases}$$

Then the distribution functions $F_N(z)$ *of a sequence of random variables*

$$Z_N := \sum_{n=1}^{N}\ln\left(\frac{\hat{S}_{t_n}}{\hat{S}_{t_{n-1}}}\right) = \ln\left(\frac{\hat{S}_T}{\hat{S}_0}\right)$$

converge for a fixed T and $N \to \infty$ *to the distribution function* $\Phi\left(z;\left(r-\frac{1}{2}\sigma^2\right)T,\sigma^2T\right)$ *of a normal distributed variable Z with mean* $\left(r-\frac{1}{2}\sigma^2\right)T$ *and variance* σ^2T, *symbolically*

$$F_N(z) \xrightarrow{d} \Phi\left(z;\left(r-\frac{1}{2}\sigma^2\right)T,\sigma^2T\right).$$

Equation (8) gives some further insights with respect to a given drift term and the necessary correction to obtain a martingale. For a given number of trading periods the first moment can

be evaluated (cf. MORAN [12], *p. 257*) as

$$\mathrm{E}_{\mathbb{Q}}\left[\ln\left(\frac{\hat{S}_T}{\hat{S}_{t_0}}\right)\right] = -i\left.\frac{\partial\chi_{Z_N}(\theta)}{\partial\theta}\right|_{\theta=0}$$

$$= \alpha T - \frac{1}{\sqrt{\Delta}}\left(2q(\alpha)-1\right)\sigma T$$

$$= \alpha T - \frac{\eta_q(\alpha)}{\sqrt{\Delta}}\sigma T.$$

Considering the limit,

$$\lim_{\Delta\to 0}\mathrm{E}_{\mathbb{Q}}\left[\ln\left(\frac{\hat{S}_T}{\hat{S}_{t_0}}\right)\right] = \alpha T + \lim_{\Delta\to 0}\frac{\eta_q(\alpha)}{\sqrt{\Delta}}\sigma T$$

results — after applying the *theorem of de l'Hospital* — in

$$\lim_{\Delta\to 0}\mathrm{E}_{\mathbb{Q}}\left[\ln\left(\frac{\hat{S}_T}{\hat{S}_{t_0}}\right)\right] = \alpha T - \lim_{\Delta\to 0}\frac{(\alpha-r)\exp(-(\alpha-r)\Delta)+\sinh\left(\sigma\sqrt{\Delta}\right)\frac{\sigma}{2\sqrt{\Delta}}}{\frac{1}{2\sqrt{\Delta}}\sinh\left(\sigma\sqrt{\Delta}\right)+\frac{\sigma}{2}\cosh\left(\sigma\sqrt{\Delta}\right)}\sigma T$$

$$= \alpha T - \frac{\left(\alpha-r+\frac{1}{2}\sigma^2\right)}{\sigma}\sigma T.$$

If the original drift coefficient μ is chosen (provided it is allowed to do so), then the correction term is given by

$$\gamma_T\sigma\sqrt{T} = \left(\frac{(\mu-r)T+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\sigma\sqrt{T}.$$

γ_T is called the market price of risk (of the logarithmic process) and is weighted with the corresponding risk measured by the standard deviation of the overall logarithmic stock return. This result is due to GIRSANOV and is one of the most important tools in the modern theory of derivative pricing.

6 The Valuation of Derivatives: An Example

It is assumed that a money market fund can be purchased promising a fixed interest rate quoted as an instantaneously compounded rate

$$r = 0.06 \,.$$

Moreover, a stock is traded whose price evolution is governed by the process

$$S_{t_n} = S_{t_{n-1}} \exp \left(0.12\Delta + 0.3 \sqrt{\Delta} X_{t_n} \right) \,.$$

In t_0 one share of the money market fund is normalized to a price of 1, the stock quotes at a price of 20. As already mentioned, the money market security serves as a numeraire. Suppose a European down-and-out barrier option is introduced to the market maturing in 3 months with a strike price $K = 18.40$. The hurdle, at which the writer is released from the obligation to purchase the stock, is at $H = 18.40$. Trading takes place once a month, i.e. $\Delta = \frac{1}{12}$, the resulting trading dates are contained in

$$\mathcal{T} = \{t_0, t_1, t_2, t_3\} \,.$$

Let us start with the specification that has been mentioned several times in the preceding sections. We choose a probability measure inducing constant transition probabilities, i.e. we assign

$$q = \frac{1}{2} \,.$$

Given the example data the drift coefficient results in

$$\alpha \left(\frac{1}{2} \right) = r - 12 \cdot \ln \left(\cosh \left(\sigma \sqrt{\frac{1}{12}} \right) \right)$$
$$\approx 0.015056$$

according to equation (6). It is interesting to compare this result with the specifications already developed in the literature. JARROW AND RUDD [9] have chosen equal transition probabilities

as well, but set the drift coefficient to the limit result given by

$$\nu^{JR}\left(\frac{1}{2}\right) = r - \frac{1}{2}\sigma^2$$

$$\approx 0.015.$$

Although the difference between ν^{JR} and α does not seem to be substantial, this model offers arbitrage opportunities in a time discrete setting. Since it is not difficult to avoid these arbitrage opportunities, there is no reason to refrain the correction.

Figure 1 shows the process of the stock price in monetary units (S_{t_n}) and in units of the numeraire $(\hat{S}_{t_n})$ as well as the evolution of the option value in units of the numeraire $(\hat{C}_{t_n})$ under the martingale measure. One obtains $\hat{C}_{t_0} = 1.7740$ which corresponds to the option value in monetary units since the numeraire is equal to 1 in t_0.
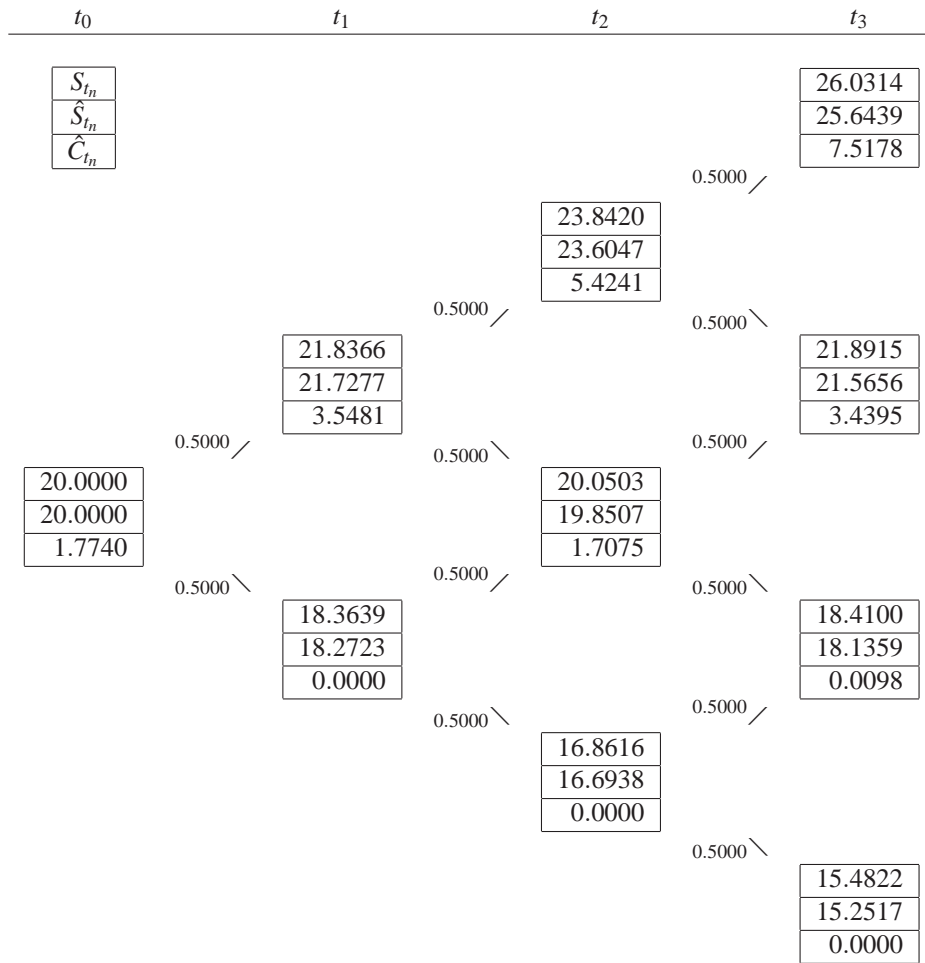

Figure 1: Binomial tree with equal transition probabilities

The next choice of parameter settings appears to be an obvious alternative. Defining the drift coefficient according to

$$\alpha = r = 0.06$$

seems to cover the characteristic properties of the continuous time process in the sense that the stock processes in units of the numeraire are driftless. Moreover, we have pointed out that a martingale measure exists in any case. The resulting transition probabilities can be calculated by (7) yielding

$$q(0.06) = \frac{1}{2} + \frac{1}{2} \tanh \left(-\frac{0.3 \sqrt{\frac{1}{12}}}{2} \right)$$
$$= \frac{1}{1 + \exp \left(0.3 \sqrt{\frac{1}{12}} \right)}$$
$$\approx 0.478362886 \,.$$

The second approach leads to an option value being 23.8% higher than before. The extent of this increase is due to the rough discretization of the time line. A closer look reveals the reason for the shortcoming of one of the solutions. Either figure 1 and 2 shows that the stock price in t_1 given a decline is close to the hurdle $H = 18.40$. However, the consequences are completely different. Whereas the stock price is above the hurdle in the binomial tree with a drift coefficient equal to the riskless interest rate maintaining the obligation of the writer to pay in the future, it has fallen below the hurdle in the binomial model with equal probabilities removing this obligation and leading to a value of zero in this knot. In other words, the quality of approximating the behavior of stock prices in the neighborhood of a hurdle depends largely and somewhat accidentally on the parameter specification and on the contract data.

The last case analyzed in this paper faces up to this problem and demonstrates the useful-ness of a thorough knowledge of the interrelationship between drift coefficients and transition probabilities. In contrast to the aforementioned models, the parameters are not kept constant over time but are fitted dynamically to satisfy some — possibly — advantageous properties. The dependency of the approximation quality on the exogenous data prompt us to control the behavior in the neighborhood of a hurdle, i.e. the drift in the first periods should be determined

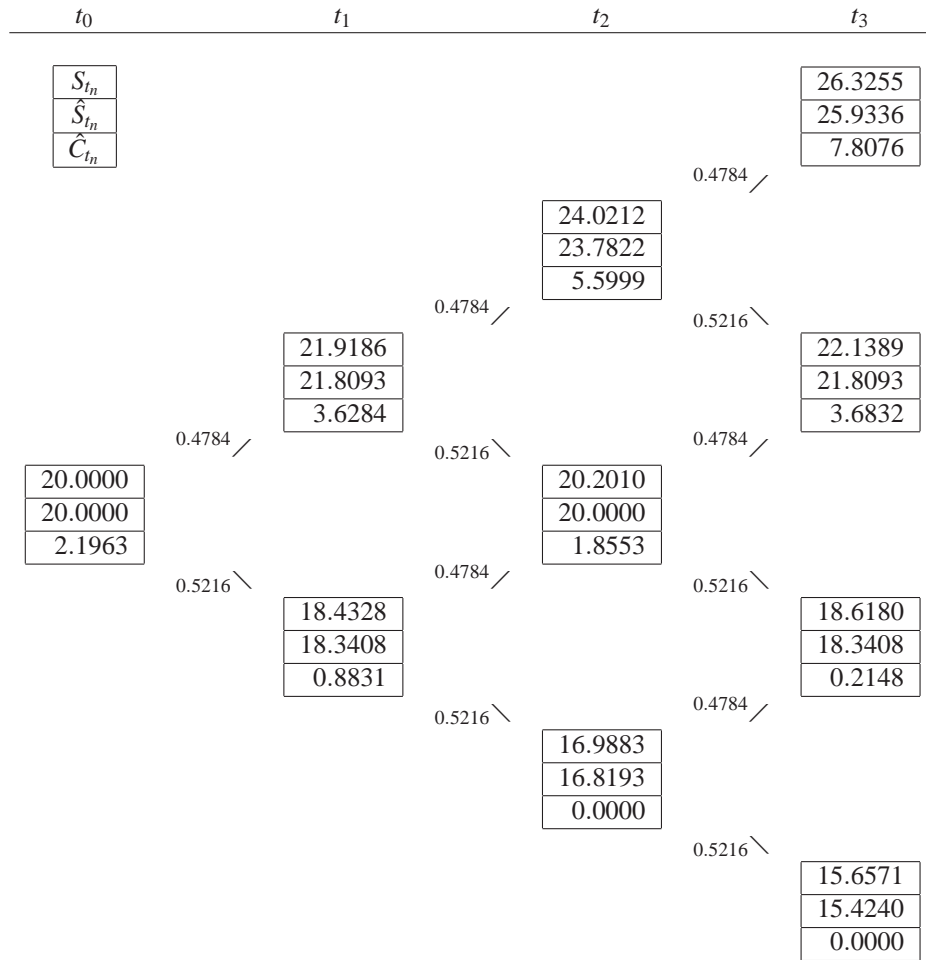Figure 2: Binomial tree with a drift coefficient $\alpha = r$

such that the hurdle is met exactly. If the drift coefficient is set to

$$\alpha = 0.03865\,,$$

then one gets a stock price in t_1 of

$$S_{t_1} = S_{t_0} \exp\left(2 \cdot 0.03865 \cdot \frac{1}{12} - 1 \cdot 0.3 \sqrt{\frac{1}{12}} \right) = 18.4\,,$$

which coincides with the hurdle H. From (5) the transition probability under the martingale

measure can be calculated

$$q(0.03865) = \frac{1}{2} + \frac{\exp\left(-(0.03865 - 0.06)\frac{1}{12}\right) - \cosh\left(0.3\sqrt{\frac{1}{12}}\right)}{2\sinh\left(0.3\sqrt{\frac{1}{12}}\right)}$$

$$\approx 0.494314.$$

First time the stock price meets the hurdle, the parameter are changed to attain a symmetric binomial tree. Thus, assigning

$$\alpha = 0$$

implies a symmetric logarithmic stock process and a transition probability of

$$q(0) \approx 0.507267$$

under the martingale measure. Figure 3 shows that one can imagine a line through the tree separating those knots where the stock price is 18.40 or below from those where the stock price is above 18.40.

These results can finally be compared with the arbitrage-free price of a barrier option obtained in a continuous-time model by MERTON [11] or RUBINSTEIN AND REINER [16]. One gets a price of

$$\begin{aligned}
C_{t_0} = {} & S_{t_0}\left(\Phi(d_1(K \vee H)) - \left(\frac{S_{t_0}}{H}\right)^{-1-\alpha}\Phi(g_1(K \vee H))\right) \\
& - Ke^{-r(T-t_0)}\left(\Phi(d_2(K \vee H)) - \left(\frac{S_{t_0}}{H}\right)^{1-\alpha}\Phi(g_2(K \vee H))\right) \\
\approx {} & 1.7723,
\end{aligned}$$

where $\Phi(z)$ is the p.d.f. of a standard normal random variable,

$$\alpha := \frac{2r}{\sigma^2},$$

Figure 3: Hurdle fitted binomial tree

$$d_{1/2}(x) := \frac{\ln\left(\frac{S_{t_0}}{xe^{-r(T-t_0)}}\right) \pm \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}$$

and

$$g_{1/2}(x) := \frac{\ln\left(\frac{H^2}{S_{t_0}xe^{-r(T-t_0)}}\right) \pm \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}.$$

The expression $K \vee H$ denotes the maximum of K and H, which is not relevant in this example since we have $K = H$.

The option values obtained from the binomial tree with equal probabilities and from the fitted model are both very close to the limit results. However, there is a significant difference. The first approach produces good results for the given data set, but it would perform poorly if the hurdle was lowered to $H = 18.30$. Our approach performs well even if the input data change

since the drift coefficient can always be fitted to meet the new requirements.

7 Concluding remarks

This paper is understood to be a link between the simple binomial model and the advanced methods of state-of-the-art derivative valuation. A representation was chosen that introduces the concept of the change of measure on a market being arbitrage-free for every way of discretization. The limit results lead to distributions known from the continuous-time financial mathematics literature. Furthermore, we showed the influence of the probability spread η at each step of the analysis.

Besides this didactic view, the results can be used to develop pricing models that benefit from a flexible setting of parameters. We illustrated the principles in a barrier option example, though other applications are conceivable. We changed the parameters after period t_1, which was easy to handle and appeared to be reasonable in this certain case. Sometimes there can be an ambiguous situation when a decision between an adjustment of the transition probabilities and the time of parameter change has to be made. This is of course a practical problem rather than a theoretical one.

The question which of the approaches result in the most stable numerical algorithm was not subject to this paper and needs to be analyzed in future research. We cannot even say that models of the class formulated in this paper produce better results than the discrete-time models using a drift coefficient obtained in the limit. From a theoretical point of view it seems to be favorable to use processes that are martingales in any specification.

If there are no restrictions on the choice of transition probabilities and drift coefficient, respectively, there remains the question which specification to chose. It might be possible that the choice of equal probabilities is the most stable approach, since it has for every discretization a symmetric distribution, a property it shares with its limit distribution. REIMER [14] has developed a comprehensive methodology for analyzing the convergence behavior of binomial models, especially the convergence speed. Though he analyzed just some special specification, the approach should be directly applicable for the generalized formulation presented in this paper.

Appendix

Proof of proposition 1 on page 10

Proof. Define the second term of expression (8) as

$$g_\Delta(\theta) = \frac{1}{\Delta} \ln\left(q(\alpha)e^{i\theta\sigma\sqrt{\Delta}} + (1-q(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}\right) T,$$

then g_Δ can be expressed as a Taylor series at $\theta = 0$ according to

$$g_\Delta(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} g_\Delta^{(n)}(0)\theta^n.$$

In the following, the limit of each term of the sum is analyzed as Δ tends to 0. Starting with

$$\lim_{\Delta \to 0} g_\Delta(0) = \lim_{\Delta \to 0} \frac{T}{\Delta} \ln(1) = 0$$

reveals that the first term can be neglected. For the second term one obtains

$$
\begin{aligned}
g_\Delta'(\theta) &= \frac{1}{\sqrt{\Delta}} \frac{q(\alpha)e^{i\theta\sigma\sqrt{\Delta}} - (1-q(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}}{q(\alpha)e^{i\theta\sigma\sqrt{\Delta}} + (1-q(\alpha))e^{-i\theta\sigma\sqrt{\Delta}}} i\sigma T \\
&= \frac{1}{\sqrt{\Delta}} \frac{\sinh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta_q(\alpha)\cosh\left(i\theta\sigma\sqrt{\Delta}\right)}{\cosh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta_q(\alpha)\sinh\left(i\theta\sigma\sqrt{\Delta}\right)} i\sigma T
\end{aligned}
$$

in general and

$$g_\Delta'(0) = \frac{1}{\sqrt{\Delta}} \eta_q(\alpha) i\sigma T \tag{9}$$

at $\theta = 0$. The limit of (9) as Δ tends to 0 is obtained by applying the *theorem of de l'Hospital* to

$$
\begin{aligned}
\lim_{\Delta \to 0} g_\Delta'(0) &= \lim_{\Delta \to 0} \frac{\exp\left(-(\alpha - r)\Delta\right) - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sqrt{\Delta}\sinh\left(\sigma\sqrt{\Delta}\right)} i\sigma T \\
&= \lim_{\Delta \to 0} -\frac{\exp\left(-(\alpha - r)\Delta\right)(\alpha - r) + \sinh\left(\sigma\sqrt{\Delta}\right)\frac{\sigma}{2\sqrt{\Delta}}}{\frac{1}{2\sqrt{\Delta}}\sinh\left(\sigma\sqrt{\Delta}\right) + \cosh\left(\sigma\sqrt{\Delta}\right)\frac{\sigma}{2}} i\sigma T \\
&= -(\alpha - r)iT - \frac{1}{2}\sigma^2 iT,
\end{aligned}
$$

where the result

$$\lim_{\Delta \to 0} \frac{1}{\sqrt{\Delta}} \sinh\left(\sigma \sqrt{\Delta}\right) = \sigma$$

has been used. Finally, the second derivative is given by

$$g_\Delta''(\theta) = -\left(1 - \frac{\left(\sinh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta_q(\alpha)\cosh\left(i\theta\sigma\sqrt{\Delta}\right)\right)^2}{\left(\cosh\left(i\theta\sigma\sqrt{\Delta}\right) + \eta_q(\alpha)\sinh\left(i\theta\sigma\sqrt{\Delta}\right)\right)^2}\right)\sigma^2 T\,.$$

Evaluated at $\theta = 0$, one gets

$$g_\Delta''(0) = -\left(1 - \left(\eta_q(\alpha)\right)^2\right)\sigma^2 T\,.$$

Taking the limit results in

$$\lim_{\Delta \to 0} g_\Delta''(0) = \lim_{\Delta \to 0} -\left(1 - \left(\eta_q(\alpha)\right)^2\right)\sigma^2 T$$

$$= -\sigma^2 T$$

since

$$\lim_{\Delta \to 0} \eta_q(\alpha) = \lim_{\Delta \to 0} \frac{\exp\left(-(\alpha - r)\Delta\right) - \cosh\left(\sigma\sqrt{\Delta}\right)}{\sinh\left(\sigma\sqrt{\Delta}\right)}$$

$$= \lim_{\Delta \to 0} \frac{-\frac{\alpha - r}{\sigma}\exp\left(-(\alpha - r)\Delta\right)2\sqrt{\Delta} - \sinh\left(\sigma\sqrt{\Delta}\right)}{\cosh\left(\sigma\sqrt{\Delta}\right)} = 0\,.$$

It is not difficult to see that all higher derivatives vanish as Δ tends to 0. Therefore the limit of the logarithmic characteristic function is

$$\ln\left(\chi_Z(\theta)\right) = \lim_{N \to \infty} \ln\left(\chi_{Z_N}(\theta)\right) = \left(rT - \frac{1}{2}\sigma^2 T\right)i\theta - \frac{1}{2}\sigma^2 T\theta^2$$

or equivalently

$$\chi_Z(\theta) = \exp\left(\left(rT - \frac{1}{2}\sigma^2 T\right)i\theta - \frac{1}{2}\sigma^2 T\theta^2\right)$$

which is the characteristic function of a normal distributed random variable Z with an expected value of $\left(r - \frac{1}{2}\sigma^2\right)T$ and a variance of $\sigma^2 T$. From the *uniqueness theorem* for characteristic functions (cf. Laha and Rohatgi [10], *144-149*) we know that a distribution function is determined uniquely by its characteristic function which completes the proof. ∎

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