Strategic Behavior on Financial Markets

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Abstract

We discuss the strategic behavior of agents on a financial market in the presence of a central bank which is borrowing and lending money.

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1 Introduction

The problem of modeling an exchange economy with money and credit as a non-cooperative game has been investigated for more than three decades (see e.g. [7] for an early contribution, and [3] for a survey and introduction to a recent issue of the Journal of Mathematical Economics devoted exclusively to strategic market games). While this substantial literature led to deep understanding of many issues, it is fair to observe that several key difficulties still need resolving.

Thus, it would be desirable to have a model for an exchange economy in which i) there exists money, serving both as a mean of exchange and as a store of value; ii) agents are price makers (and not just price takers); iii) there exists a central bank who issues money, accepts deposits, and lends; iv) bankruptcy is not ruled out, but is penalized. To the best of our knowledge, no model encompassing all these desiderata is available as yet. The early paper [7] does not discuss credit. The fairly recent study [2] deals with a continuum of agents. It turns out that agents behave like price takers and the system is in a (competitive) equilibrium.

A possible model including elements i) through iv) consists of a finite set of agents involved in trade of a Shapley-Shubik type ([5], [6]), along with a central bank able to issue money, distribute it as loans, and accept deposits. The central bank has the authority to determine the various interest rates. Agents would derive a negative utility from being bankrupt, whereas positive cash holdings at the end of the period have positive utility, the latter presumably deriving from subsequent use of money at a later period. We offer here such a one-shot model, and plan to construct a multi-period extension in subsequent work.

Our model is described in detail in SECTION 2. Each agent is endowed with positive amounts of a consumer nondurable commodity and money. Agents issue *bids* in terms of money towards purchasing a quantity of the consumption good. (Agents cannot consume directly their commodity endowment in whole or parts.) Agents may exceed their endowment (and thus take a loan from the bank), or else they may bid less than their endowment, their money surplus going to the bank as a deposit. There is a central bank in the market which controls the interest rates for deposits and loans and increases the total amount of money, if the books cannot be balanced otherwise. As soon as bids are announced, the price of the commodity is given by the equation (2.5) as the ratio of the aggregate bid to the aggregate supply of the good. Each agent then receives for consumption the good bought by his bid and the money proceeds of the selling of his commodity endowment. In addition, our agent receives returns from her bank deposit or has to pay the loan (with interest).

At the end of the day, each agent has 1) consumed an amount of the commodity (deriving from it a positive amount of utility), 2) is unable to repay his loan with the prescribed interest, so that he is bankrupt and derives a negative utility from this fact, or else 3) has a positive amount of cash left, from which she derives positive utility. These three components of the total utility are given in (2.3) of Definition 2.1. Besides the usual monotonicity and concavity assumptions we also require that the penalty for bankruptcy be sufficiently large so as to offset the positive utility of high consumption, and impose on the utility functions several technical assumptions in order to facilitate certain existence proofs. We do not try to optimize the bankruptcy rule. We trace the flow of cash in the economy and show that the bank never has to withdraw funds out of the economy.

The bank announces a policy concerning interest rates on deposits and loans. Formally, this policy is a (vector-valued) function of the agents' bids. The agents, in turn, may take into account the bank's policy. In this manner a well defined *game* (the *financial market game*) is specified. (As usual in this literature, bids play the role of strategies). The bank may try to achieve certain objectives through its policies. One such objective could be the wish to eliminate unnecessary bankruptcies. Another might be the desire to combat inflation. We exhibit a policy which leads to certain desirable outcomes.

In Section 3 we establish the existence of a Nash equilibrium for the financial market game. (An essential element of the proof is the construction of a compact set of strategies which is mapped into itself by the best response correspondence.) Under certain regularity conditions we demonstrate the existence of a (Nash) equilibrium in mixed strategies. For a specific policy we prove existence of an equilibrium in *pure* strategies.

Our original goal was to put forward a multi-period model where the utility for holding cash reserves at the end of the j-th period is derived from the utility of having this reserve as an endowment for the $j+1$-st period, and obtaining (using backward induction) a subgame-perfect equilibrium in pure strategies. We plan to achieve this goal in a sequel to this paper.

2 The Model

We consider an economy with a financial market. There is a central bank which is responsible for the supply of money and which has to support economic growth as well as to prevent inflationary development. The economic agents or players issue bids for consumption and are allowed to ask for loans from the bank. Following each period, the bank has to balance its books. Within this section we restrict the discussion to one period.

Definition 2.1. *A **financial market** (with one commodity and money) is a tripel of data*

$$\mathcal{M} := (\boldsymbol{I}, \boldsymbol{A}, U) \tag{2.1}$$

with the following specifications and interpretations.

1. $\boldsymbol{I} = \{1, \ldots, n\}$ *is a finite set, the set of **players** or **agents**.*
2. *The* $n \times 2$*-matrix*

$$\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2) > 0$$

*reflects the **initial assignments** of commodities and money to the agents. In particular, the* $i-th$ *coordinate of the vector* $\boldsymbol{a}_1 \in \mathbb{R}^n_+$*, written* $\boldsymbol{a}^i_1$*, denotes the initial endowment of player* $i \in \boldsymbol{I}$ *with the consumption good. Similarly, the vector* $\boldsymbol{a}_2 \in \mathbb{R}^n_+$ *indicates the initial allocation of money to the agents.*

3. *Finally,* $U = (U^i)_{i \in \boldsymbol{I}}$ *denotes the **utility functions** of the players; each one of them is a mapping*

$$U^i =: \mathbb{R}^3_+ \to \mathbb{R}. \tag{2.2}$$

The utilities are separable. That is, there are functions $u^i : \mathbb{R}_+ \to \mathbb{R}_+$*,* $w^i : \mathbb{R}_+ \to \mathbb{R}_+$*, and* $V^i : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the utilities can be written*

$$U^i(\alpha_1, \alpha_2, \alpha_3) = u^i(\alpha_1) - w^i(\alpha_2) + V^i(\alpha_3) \quad (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+. \tag{2.3}$$

The functions u^i*,*$-w^i$*, and* V^i *are continuous and strictly monotone (hence almost surely differentiable), moreover* u^i *and* $-w^i$ *are concave and second derivatives exist up to finitely many points.*

The a.s. derivative of V^i satisfies the following conditions: There is $t_0 > 0$ and $\varepsilon_0 > 0$ such that

$$
\begin{aligned}
V^{i\prime}(t) &\geq \varepsilon_0 \quad (t \leq t_0)\ , \\
V^{i\prime}(t) &\geq \frac{\varepsilon_0}{t} \quad (t \geq t_0)
\end{aligned}
\tag{2.4}
$$

holds true. Moreover, we require $u^i(0) = w^i(0) = 0,\ (u^i)' < (w^i)'\ \ (i \in \boldsymbol{I})$, so that $-w^i$ is negative for all arguments ("debts").

The agents derive utility from consuming the good as well as from having a cash balance. The consumption utility is represented by the functions u^i.

On the other hand, the agents derive utility from money in a two-fold fashion. On one hand, they may be punished for leaving the game being unable to pay their debts. This is is expressed by the functions w^i, which reflect the "bad reputation" or uneasiness of a player for being broke. Furthermore, the agents appreciate acquiring a large fortune. This component of the overall utility is taken care of by the functions V^i.

The monetarian part V^i reflects future possibilities resulting from holding money with the purpose of spending it in later periods.

Basically, the agents are allowed to act strategically in a very simple way. They issue ***bids*** in terms of money towards acquiring a quantity of the consumption good. Thus, at this stage, the action space of each agent is $\mathbb{R}_+$ and an action is denoted by $\boldsymbol{b}^i \in \mathbb{R}_+$ for agent $i \in \boldsymbol{I}$. (There is at present no difference between actions and strategies of the players or agents.) The bids are nonnegative. Agents may exceed their endowment (and thus take a loan from the bank) in which case they are called ***debtors***. Or else agents may also bid less than their endowment, their money surplus then goes to the bank as a deposit and we call the corresponding agent a ***depositor***.

There is a central bank in the market which is required to take two types of actions (not independently): it will control the interest rates and increase the total amount of money in order to balance its books.

The price generating mechanism is modeled in a rather straightforward fashion. Given that all agents made their bids, the price of the commodity is set to be

$$
\boldsymbol{p} := \frac{\sum_{i\in\boldsymbol{I}} \boldsymbol{b}^i}{\sum_{i\in\boldsymbol{I}} \boldsymbol{a}^i_1}. \tag{2.5}
$$

Introducing abbreviations

$$\bar{\boldsymbol{a}}_1 \; := \; \sum_{i \in \boldsymbol{I}} \boldsymbol{a}_1^i, \quad \bar{\boldsymbol{a}}_2 \; := \; \sum_{i \in \boldsymbol{I}} \boldsymbol{a}_2^i, \quad \bar{\boldsymbol{b}} \; := \; \sum_{i \in \boldsymbol{I}} \boldsymbol{b}^i, etc. \tag{2.6}$$

this reads also

$$\boldsymbol{p} \; = \; \frac{\bar{\boldsymbol{b}}}{\bar{\boldsymbol{a}}_1} \; . \tag{2.7}$$

Now, if the bank has determined the interest rates to be $r_1 = \rho_1 + 1 \geq 1$ for loans taken by the agents and $r_2 = \rho_2 + 1 \geq 1$ for deposits from the agents, then agent i's ***income*** is determined by

$$\boldsymbol{c}^i \; := \; \boldsymbol{p}\boldsymbol{a}_1^i - r_1(\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + r_2(\boldsymbol{a}_2^i - \boldsymbol{b}^i)^+ \; . \tag{2.8}$$

This should be interpreted as each agent selling his share of the consumption good, receiving the appropriate monetarian equivalent, while spending his bid entirely. He also receives the yield of his deposit with the bank or pays the returns of his debts.

Note that each agent's income may well be negative. He will feel these debts in his utility function but, at the end of the period, his debts will be cancelled.

On the other hand, each agent consumes his share of the consumption good determined by his bid and the prevailing price, that is, assuming a positive price, agent i receives

$$\frac{\boldsymbol{b}^i}{\boldsymbol{p}} \tag{2.9}$$

units for consumption.The total utility derived is now

$$U^i\left(\frac{\boldsymbol{b}^i}{\boldsymbol{p}}, -\frac{(\boldsymbol{c}^i)^-}{\boldsymbol{p}}, (\boldsymbol{c}^i)^+\right) \; = \; u^i\left(\frac{\boldsymbol{b}^i}{\boldsymbol{p}}\right) - w^i\left(-\frac{(\boldsymbol{c}^i)^-}{\boldsymbol{p}}\right) + V^i((\boldsymbol{c}^i)^+) \tag{2.10}$$

One of the main tasks of the bank is to ***balance the books*** in order to control the total amount of money available in the economy. Inevitably, this may require to add a certain amount of money. Thus the balancing equation is now

$$\sum_{i \in \boldsymbol{I}} \left(\boldsymbol{p}\boldsymbol{a}_1^i \bigwedge r_1(\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+\right) + \pi \; = \; \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \rho_2 \sum_{i \in \boldsymbol{I}} \left(\boldsymbol{a}_2^i - \boldsymbol{b}^i\right)^+ \tag{2.11}$$

This relation equates the money flow input and output as viewed by the bank. On the left hand side each term reflects the cash influx from a player who either returns his debts or, if he is unable to do so, leaves his total fortune acquired from turning in his commodities and receiving the corresponding payment given the prices.

On the right hand side we observe the money outflux: the bank issues loans to those agents with bids exceeding their initial money endowment. Agents who did not use up their initial money endowments receive the interest paid for the remainder.

It is now a simple exercise to verify that we have a law of conservation of money for our economy. This is expressed by the following lemma.

Lemma 2.2. *The balancing equation* (2.11) *is satisfied if and only if*

$$\sum_{i\in \boldsymbol{I}} (c^i)^+ = \sum_{i\in \boldsymbol{I}} a_2^i + \pi \tag{2.12}$$

is satisfied.

Proof: For any pair $(p, \boldsymbol{b}) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ and for any interest rate $r_1 \geq 1$ denote by

$$\mathcal{B}^{r_1} := \left\{ i \in \boldsymbol{I} \mid p\boldsymbol{a}_1^i < r_1(\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ \right\} \tag{2.13}$$

the set of agents who are bankrupt in the situation reflected by these data. Now, the total sum of surviving capital is

$$\sum_{i\in \boldsymbol{I}} (c^i)^+ = \sum_{i\notin \mathcal{B}^{r_1}} p\boldsymbol{a}_1^i - r_1 \sum_{i\notin \mathcal{B}^{r_1}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + r_2 \sum_{i\in \boldsymbol{I}} (\boldsymbol{a}_2^i - \boldsymbol{b}^i)^+ \quad . \tag{2.14}$$

The balancing equation (2.11) in turn can be written

$$\begin{aligned} \sum_{i\in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \rho_2 \sum_{i\in \boldsymbol{I}} (\boldsymbol{a}_2^i - \boldsymbol{b}^i)^+ &= \\ \sum_{i\in \mathcal{B}^{r_1}} p\boldsymbol{a}_1^i + r_1 \sum_{i\notin \mathcal{B}^{r_1}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \pi \ . & \end{aligned} \tag{2.15}$$

Adding up both equations and observing $r_2 = 1 + \rho_2$ we obtain

$$\sum_{i\in \boldsymbol{I}} (c^i)^+ = \sum_{i\in \boldsymbol{I}} p\boldsymbol{a}_1^i + \sum_{i\in \boldsymbol{I}} (\boldsymbol{a}_2^i - \boldsymbol{b}^i)^+ - \sum_{i\in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \pi \ , \tag{2.16}$$

or

$$\sum_{i\in \boldsymbol{I}} (c^i)^+ = p\bar{\boldsymbol{a}}_1 + \bar{\boldsymbol{a}}_2 - \bar{\boldsymbol{b}} + \pi \ . \tag{2.17}$$

Comparing the price setting equation (2.7) we obtain the desired result. **q.e.d.**

It is an essential feature of the model that the total amount of money within the economy has been increased by just the monetary corrections the bank issued in terms of booking money in order to balance its books.

At this stage it should be explained how the bank achieves the various goals that are assigned to it. The additional amount of booking money injected into the market for balancing will eventually have an inflationary effect. Interest rates should be determined so as to keep inflation low on one hand and to avoid unnecessary bankruptcy of agents on the other hand. It is not obvious *a priori* that all these goals are achievable simultaneously.

Observe that balancing the influx and outflux of money cannot always be achieved just by manipulating the interest rates, without an inflationary input of money. One might conjecture that, for $\pi = 0$, interest rates could be determined by the balancing equation (2.11). However, it is easy to see that this may fail, as the left side of (2.11) may be strictly smaller then the right side. We will come back to this problem below (Remark 2.4). Also, the interest rates may not be uniquely determined by (2.11), assuming that $\pi = 0$. Thus, it will be necessary to adjust the additional money input of the bank in view of the reactions of the players to the interest rates.

To this end the bank should take the strategically motivated bids of the agents into account. Hence, its behavior should be expressed as a reaction function defined on action $n-$tuples. Consequently, as prices depend on bids, there are some obvious coupling effects to be observed (and formulated) between the actions of the players and the banks policy.

Therefore, let us now present a precise description of the banks advanced planning, resulting in announcements concerning its reactive policy, ahead of the players bidding.

To this end, we consider $n-$tuples $\boldsymbol{b} \in \mathbb{R}^n_+$ of actions of the players and hypothetical price levels denoted by $p \in \mathbb{R}_+$.

Recall definition (2.13) of $\mathcal{B}^{r_1}$. For any $\rho_2 \geq 0$, it is obvious that

$$\sum_{i \in \boldsymbol{I}} \left(p\boldsymbol{a}^i_1 \bigwedge (\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ \right) \leq \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ + \rho_2 \sum_{i \in \boldsymbol{I}} \left(\boldsymbol{a}^i - \boldsymbol{b}^i \right)^+ \tag{2.18}$$

holds true. Hence, we may define the quantity

(2.19)
$$1 \le \bar{r}_1 = \bar{r}_1(p, \boldsymbol{b}, \rho_2) := \sup \left\{ r_1 \in \mathbb{R}_+ \,\middle|\, \mathcal{B}^{r_1} = \mathcal{B}^1, \sum_{i \in \boldsymbol{I}} \left(p\boldsymbol{a}_1^i \bigwedge r_1(\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ \right) \le \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \rho_2 \sum_{i \in \boldsymbol{I}} \left(\boldsymbol{a}_2^i - \boldsymbol{b}^i\right)^+ \right\},$$

where the value ∞ is permitted, e.g., if $\boldsymbol{b}^i \le a_2^i$ holds for all agents $i \in \boldsymbol{I}$, that is, if no one's bid exceeds his endowment.

Consider the lowest possible interest rate $r_1 = 1$. Then $\bar{r}_1$ is the largest rate the bank could choose without increasing the set of bankrupt agents, subject to balancing the books with a possibly positive amount of added booking money.

Let us collect a few simple relations concerning interest rates and the financial situation of agents.

Lemma 2.3. *Let* $(p, \boldsymbol{b}) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ *be such that*

$$p\bar{\boldsymbol{a}}_1 \ge \bar{\boldsymbol{b}} \tag{2.20}$$

and let $r_1 \ge 1$ *be an interest rate such that the balancing equation* (2.11) *is satisfied with some* $\pi \ge 0$. *Then, not all players are bankrupt, i.e.,*

$$\mathcal{B}^{r_1} \ne \boldsymbol{I} \quad . \tag{2.21}$$

Proof: Assume that, on the contrary,

$$0 \le p\boldsymbol{a}_1^i < r_1(b^i - \boldsymbol{a}_2^i) \tag{2.22}$$

is valid for *all* $i \in \boldsymbol{I}$. Then the balancing equation (2.11) reads

$$\sum_{i \in \boldsymbol{I}} p\boldsymbol{a}_1^i + \pi = \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i). \tag{2.23}$$

In view of (2.20) , it is clear that

$$\sum_{i \in \boldsymbol{I}} \boldsymbol{b}^i \le p \sum_{i \in \boldsymbol{I}} \boldsymbol{a}_1^i + \pi = \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}_2^i) < \sum_{i \in \boldsymbol{I}} \boldsymbol{b}^i \tag{2.24}$$

yields a contradiction. **q.e.d.**

Next, given $\boldsymbol{b} \in \mathbb{R}^n_+$, we denote by

$$\mathcal{E} \;:=\; \left\{ i \in \boldsymbol{I} \,\middle|\, \boldsymbol{b}^i > \boldsymbol{a}^i_2 \right\} \tag{2.25}$$

the set of those players whose bids exceed their endowments. Then we have

Remark 2.4. *Let $(p, \boldsymbol{b}, r_2) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times [1, \infty)$ be arbitrary. Then the following holds:*

1. $\mathcal{B}^1 \subseteq \mathcal{E}$.
2. *If* $\mathcal{E} = \boldsymbol{I}$ *and* $\mathcal{B}^1 = \emptyset$ *(i.e.,* $\mathcal{E} - \mathcal{B}^1 = \boldsymbol{I}$*), then* $\bar{r}_1 = 1$.
3. *If* $\mathcal{E} - \mathcal{B}^1 = \emptyset$*, then* $\bar{r}_1 = \infty$.
4. *If* $\emptyset \neq \mathcal{E} - \mathcal{B}^1 \neq \boldsymbol{I}$*, then* $\bar{r}_1$ *can be written*

$$\begin{aligned} \bar{r}_1 \;=\; \bar{r}_1(p, \boldsymbol{b}, \rho_2) \;:=\; \sup \Bigg\{ & r_1 \in \mathbb{R}_+ \;\Bigg|\; \sum_{i \in \mathcal{B}^1} p\boldsymbol{a}^i_1 + \sum_{i \in \mathcal{E} - \mathcal{B}^1} r_1(\boldsymbol{b}^i - \boldsymbol{a}^i_2) \\ & \leq \sum_{i \in \mathcal{E}} (\boldsymbol{b}^i - \boldsymbol{a}^i_2) + \rho_2 \sum_{i \in \boldsymbol{I} - \mathcal{E}} \left(\boldsymbol{a}^i_2 - \boldsymbol{b}^i\right) , \\ & r_1(\boldsymbol{b}^i - \boldsymbol{a}^i_2) \leq p\boldsymbol{a}^i_1 \quad (i \in \mathcal{E} - \mathcal{B}^1) \Bigg\} . \end{aligned} \tag{2.26}$$

From this it follows that $\bar{r}_1$ *can be expressed by a closed formula as follows:*

$$\begin{aligned} \bar{\rho}_1 \;=\; & \frac{\sum_{i \in \mathcal{B}^1} \left((\boldsymbol{b}^i - \boldsymbol{a}^i_2) - p\boldsymbol{a}^i_1 \right) + \rho_2 \sum_{i \in \boldsymbol{I} - \mathcal{E}} \left(\boldsymbol{a}^i_2 - \boldsymbol{b}^i \right)}{\sum_{i \in \mathcal{E} - \mathcal{B}^1} \left(\boldsymbol{b}^i - \boldsymbol{a}^i_2 \right)} \\ & \wedge \left\{ \bigwedge_{i \in \mathcal{E} - \mathcal{B}^1} \frac{p\boldsymbol{a}^i_1 - (\boldsymbol{b}^i - \boldsymbol{a}^i_2)}{\boldsymbol{b}^i - \boldsymbol{a}^i_2} \right\} . \end{aligned} \tag{2.27}$$

5. *In particular, if all agents have excess demand ($\mathcal{E} = \boldsymbol{I}$) and $n-1$ of them are bankrupt at interest rate 1 (i.e., $|\mathcal{B}^1| = n - 1$), then the remaining player, say $i_0 \in \mathcal{E} - \mathcal{B}^1$ cannot be broke at any r_1; this follows from Lemma 2.3. Therefore the $\bigwedge$-term in formula (2.27) vanishes and as a result we obtain a simpler formula for $\bar{r}_1$ which is*

$$\bar{\rho}_1 \;=\; \frac{\sum_{i \in \boldsymbol{I} - i_0} (\boldsymbol{b}^i - \boldsymbol{a}^i_2) - p\boldsymbol{a}^i_1}{\boldsymbol{b}^{i_0} - \boldsymbol{a}^{i_0}} . \tag{2.28}$$

The last formula reflects a situation in which the bank may be forced to set an arbitrarily large interest rate if it wishes to balance its books without adding inflationary money. In view of item 3 it is clear, however, that the situation may deteriorate to the case in which no balancing is possible at all without inflation.

Now, the bank's decision will be reached as follows. Given the data $(p, \boldsymbol{b}, \rho_2)$ the bank considers how much the interest rate r_1 should be raised and consequently how much additional booking money should be issued in order to balance the economy.

However, the interest rate $r_2 = 1 + \rho_2$ is also controlled by the bank. Thus we come up with a first definition concerning the bank's behavior:

Definition 2.5. *A **policy** of the bank is a pair of continuous functions (R_1, R_2) such that*

$$R_1 : \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times \mathbb{R}^1_+ \to [1, \infty], \quad R_2 : \mathbb{R}^1_+ \times \mathbb{R}^n_+ \to [1, \infty], \tag{2.29}$$

$$1 \leq R_1(p, \boldsymbol{b}, \rho) \leq \bar{r}_1(p, \boldsymbol{b}, \rho), \quad \left((p, \boldsymbol{b}, \rho) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times \mathbb{R}^1_+\right) \ . \tag{2.30}$$

Both functions admit of nonnegative partial derivatives outside a closed set which is the union of finitely many polyhedra of lower dimension.

With some abuse of notation we denote $R_2 - 1$ by ρ_2.

It follows from (2.19) that the two components of the banks policy, taken together, satisfy

$$\begin{aligned} &\sum_{i \in \boldsymbol{I}} \left(p\boldsymbol{a}^i_1 \bigwedge R_1(p, \boldsymbol{b}, \rho_2(p, \boldsymbol{b}))(\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ \right) \\ &\leq \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ + \rho_2(p, \boldsymbol{b}) \sum_{i \in \boldsymbol{I}} \left(\boldsymbol{a}^i - \boldsymbol{b}^i\right)^+ \ . \end{aligned} \tag{2.31}$$

The interest rate for depositors can be chosen arbitrarily. Thereafter the bank may formulate a policy concerning interest charged from borrowers. Then, finally, it may still be necessary to fill a gap in the balancing equation by adding a certain amount of money.

We now consider p not as a free variable but as a function of agents' bids as given by (2.5) or (2.7). Thus, we define a function

$$\boldsymbol{P} \ : \ \mathbb{R}^n_+ \to \mathbb{R}_+, \tag{2.32}$$

by

$$\boldsymbol{P}(\boldsymbol{b}) \ := \ \frac{\bar{\boldsymbol{b}}}{\bar{\boldsymbol{a}}_1} \ . \tag{2.33}$$

We may then define the relevant policy functions of the bank in terms of decisions of the players. The procedure permits the bank to determine, for any action $n-$tuple $\boldsymbol{b} \in \mathbb{R}^n_+$ of the agents, the necessary increase in money supply, provided its policy has been specified. Formally,

Definition 2.6. *Let (R_1, R_2) be a policy of the bank. Then the resulting banks' **monetarian strategy** is a triple $\boldsymbol{S} = (\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\Pi})$, given as follows: The domains and ranges are described by*

$$\begin{aligned} &\boldsymbol{r}_1 : \mathbb{R}^n_+ \to [1, \infty), \quad \boldsymbol{r}_2 : \mathbb{R}^n_+ \to [1, \infty), \\ &\boldsymbol{\Pi} : \mathbb{R}^n_+ \to \mathbb{R}_+ \ , \end{aligned} \tag{2.34}$$

and the first two functions are defined by the compositions

$$\begin{aligned} \boldsymbol{r}_2(\boldsymbol{b}) \ &:= \ R_2(\boldsymbol{P}(\boldsymbol{b}), \boldsymbol{b}), \\ \boldsymbol{r}_1(\boldsymbol{b}) \ &:= \ R_1(\boldsymbol{P}(\boldsymbol{b}), \boldsymbol{b}, \boldsymbol{r}_2(\boldsymbol{b})), \end{aligned} \tag{2.35}$$

where $\boldsymbol{P}$ is given by formula (2.33). The function $\boldsymbol{\Pi}$ is given for every $\boldsymbol{b} \in \mathbb{R}^n_+$ as the unique money supply

$$\pi \ = \ \boldsymbol{\Pi}(\boldsymbol{b}) \tag{2.36}$$

satisfying

$$\begin{aligned} &\sum_{i \in \boldsymbol{I}} \Big(\boldsymbol{P}(\boldsymbol{b}) \boldsymbol{a}^i_1 \bigwedge \boldsymbol{r}_1(\boldsymbol{b})(\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ \Big) + \pi \\ &= \ \sum_{i \in \boldsymbol{I}} (\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ + \boldsymbol{\rho}_2(\boldsymbol{b}) \sum_{i \in \boldsymbol{I}} \left(\boldsymbol{a}^i - \boldsymbol{b}^i \right)^+ \ . \end{aligned} \tag{2.37}$$

Thus, the bank internally fixes a policy and then computes the functions describing its behavior with respect to the setting of interest rates and the increase in money supply. These functions depend on the bids of the agents and will be announced publicly.

Hence, agents are made aware of consequences of their actions: they can compute the increase in money supply, the prices and the various interest rates, given everybody elses decision. In this manner, $n-$person game is defined and may reason strategically. So may the bank.

Remark 2.7. *Given a policy of the bank, the income of an agent is a function of the actions of all agents. We denote this function by*

$$\boldsymbol{C}^i \ : \ \mathbb{R}^n_+ \to \mathbb{R}^n \tag{2.38}$$

and compute it by inserting the action-dependent prices and interest rates, i.e., the functions $\Pi, \boldsymbol{r}_1$, and $\boldsymbol{r}_2$ into formula (2.8). More explicitely, we have

$$\boldsymbol{C}^i(\boldsymbol{b}) \ := \ \boldsymbol{P}(\boldsymbol{b})\boldsymbol{a}^i_1 - \boldsymbol{r}_1(\boldsymbol{b})(\boldsymbol{b}^i - \boldsymbol{a}^i_2)^+ + \boldsymbol{r}_2(\boldsymbol{b})(\boldsymbol{a}^i_2 - \boldsymbol{b}^i)^+ \quad (\boldsymbol{b} \in \mathbb{R}^n_+). \tag{2.39}$$

Similarly, we obtain the utility functions of the players depending on bids by composing the functions U^i introduced in Definition 2.1, 3 with the functions defined above, and get

(2.40)

$$\begin{aligned} \boldsymbol{U}^i \ : & \quad \mathbb{R}^n_+ \to \mathbb{R}_+ \\ \boldsymbol{U}^i(\boldsymbol{b}) \ := & \quad U^i\Big(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}, -\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}, (\boldsymbol{C}^i)^+\Big) \ = \ u^i\Big(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\Big) - w^i\Big(-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\Big) + V^i((\boldsymbol{C}^i)^+), \end{aligned}$$

(omitting the argument $\boldsymbol{b}$ in $\boldsymbol{P}$ and $\boldsymbol{C}^i$).

Definition 2.8. *Let $\mathcal{M}$ be a financial market and let $R = (R_1, R_2)$ be a policy of the bank. Then*

$$\Gamma \ = \ \Gamma_{\mathcal{M}} \ = \ \Gamma^R_{\mathcal{M}} \ := \ (\mathbb{R}^n_+; \boldsymbol{U}^1, \ldots, \boldsymbol{U}^n) \tag{2.41}$$

*is the **financial market game** (the one-shot game) generated by $\mathcal{M}$ and R.*

Remark 2.9. *We would like to specify policies of the bank that result in monetary strategies with certain desirable properties. For this purpose it may be useful to study properties of the function $\bar{r}_1$ which is defined by (2.19) and computed in some special cases in Remark 2.4.*

Consider first two disjoint subsets $J, L \subseteq \boldsymbol{I}$, none of which is the full set.

Then, regarding prices and bids as independent variables as in Remark 2.4, we consider the convex polyhedron

(2.42)

$$\begin{aligned} \big\{(p, \boldsymbol{b}) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+ \ \mid \quad & \boldsymbol{b}^i - \boldsymbol{a}^i_2 > p\boldsymbol{a}^i_1 \ (i \in J), \quad 0 < \boldsymbol{b}^i - \boldsymbol{a}^i_2 < p\boldsymbol{a}^i_1 \ (i \in L), \\ & \boldsymbol{b}^i - \boldsymbol{a}^i_2 \leq 0 \ (i \in \boldsymbol{I} - J - L)\big\} \end{aligned}$$

Within such a polyhedron, the sets $\mathcal{B}^1$ and $\mathcal{E}$ are obviously the same as J and $J \cup L$, So the function $\bar{r}_1$ has the shape indicated in formula (2.27). The polyhedron has a nonempty interior in which $\bar{r}_1$ is a fractionally linear function of $\boldsymbol{b}^i$ and p (in fact a linear function if $i \in \mathcal{B}^1$). If $L = I$, then $\bar{r}_1 = 1$, and if $J = I$ then $\bar{r}_1 = \infty$.

When varying the index sets J, L, we observe that there is a decomposition of $\mathbb{R}^1_+ \times \mathbb{R}^n_+$ into finitely many convex polyhedra with a nonempty interior, together *with lower dimensional polyhedra (common boundaries). In each of these polyhedra the function $\bar{r}_1$ is given by a formula of type (2.27) or else $\bar{r}_1$ equals 1 or ∞.*

Hence, the function $\bar{r}_1$ has easily computable partial derivatives outside a set of measure zero which consists of finitely many closed lower dimensional polyhedra.

Let us now exhibit two "nice" policies of the bank such that the interest rate $\boldsymbol{r}_1$ is monotone and has nonnegative partial derivatives with the exception of finitely many points.

Remark 2.10. *Let $R_2 = r_2$ be a constant and define $R_1 = R_1^M$ by*

$$R_1(p, \boldsymbol{b}, \rho) \ := \ \bar{r}_1(p, \boldsymbol{b}, \rho) \wedge M \quad \big((p, \boldsymbol{b}, \rho) \ \in \ \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times \mathbb{R}^1_+\big) \tag{2.43}$$

where M is large constant. This policy reflects the banks intent to avoid inflationary generation of money as long as possible, that is, as long as the demand for money does not exceed some huge amount. If this amount is exceeded, then the interest rate will be bounded by M and the necessary amount of booking money will be supplied.

Consider now the function $\boldsymbol{r}_1$ resulting from this policy via Definition 2.6. Let $\boldsymbol{b} \in \mathbb{R}^n_+$ be an $n-$tuple of bids such that some agent $i \in \boldsymbol{I}$ is bankrupt. We want to demonstrate that the partial derivative $\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{b}_i}$ exists, given the bids $\boldsymbol{b}^{-i}$ of the other players, with the exception of finitely many points.

Indeed, fix $p = \boldsymbol{P}(\boldsymbol{b})$ and $r_1 = \boldsymbol{r}_1(\boldsymbol{b})$ and consider the sets $\mathcal{B}^1$ and $\mathcal{E}$. If $\mathcal{E} = \mathcal{B}^1$, then the same equation holds true in a small neighborhood of $\boldsymbol{b}^i$ as well. Within this neighborhood, the function $\boldsymbol{r}_1$ equals the large constant M and hence the required partial derivative exists and equals zero.

Also, if $\mathcal{B}^1 \subseteq \mathcal{E}$, $\mathcal{B}^1 \neq \mathcal{E}$, then the same relations are valid in a neighborhood of $\boldsymbol{b}^i$ and hence the derivative can be computed with the aid of formula (2.27), substituting p by $\boldsymbol{P}(\boldsymbol{b})$ and applying the chain rule. This involves computing the partial derivative with respect to $\boldsymbol{b}^i$ of a function of the form

$$\frac{\boldsymbol{b}^i - \boldsymbol{P}(\boldsymbol{b})\boldsymbol{a}^i_1}{A} = \frac{\boldsymbol{b}^i - \frac{\bar{\boldsymbol{b}}}{\bar{\boldsymbol{a}}_1}\boldsymbol{a}^i_1}{A}, \tag{2.44}$$

where A is a positive constant independent of $\boldsymbol{b}^i$ as long as $\mathcal{B}$ and $\mathcal{E}$ do not vary. Hence

$$\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{b}_i} = \left(1 - \frac{\boldsymbol{a}^i_1}{\bar{\boldsymbol{a}}_1}\right)\frac{1}{A} > 0 \ . \tag{2.45}$$

The constancy of $\boldsymbol{b}^{-i}$ implies that the sets $\mathcal{B}$ and $\mathcal{E}$ change at most at finitely many values of $\boldsymbol{b}^i$. As long as $\mathcal{E} - \mathcal{B}$ is nonempty, formula (2.27) is valid, the constant A changes when passing a boundary. The only other case occurs when we pass from *the region where $\mathcal{E} = \mathcal{B}^1$ to the region where $\mathcal{B}^1 \subseteq \mathcal{E}$. However, at this point the right hand side partial derivative exists and equals 0, while the left hand side exists and is positive.*

Remark 2.11. *Let us assume that the bank varies the interest rate for depositors in proportion to the demand for loans demand. Hence, we set*

$$R_2(p, \boldsymbol{b}) \;:=\; 1 + C_0 \sum_{j \in \mathcal{E}} (\boldsymbol{b}^j - \boldsymbol{a}_2^j) \quad \big((p, \boldsymbol{b}) \;\in\; \mathbb{R}_+^1 \times \mathbb{R}_+^n\big)\,. \tag{2.46}$$

where C_0 is a positive constant.

The banks policy with respect to loan interest is the same as in (2.43). That is, the function R_1 is adapted until the interest rate exceeds a certain large and fixed amount M, after which this constant is the rate. The arguments concerning regularity of the functions involved are not affected, hence all statements concerning differentiability of the (composite) functions remain true.

3 Financial Market Equilibrium

Within this section we establish the existence of a Nash equilibrium of the financial market game $\Gamma^R_{\mathcal{M}}$ under rather mild conditions concerning the policy R of the bank. Let $\mathcal{M}$ be a financial market as defined in Definition 2.1 The following conditions are imposed upon the policy of the bank.

Definition 3.1.

1. *We call the bank's policy* ***regular*** *if for the resulting monetarian strategy, the function $\boldsymbol{r}_1$ is monotone and has nonnegative partial derivatives up to finitely many points.*
2. *The bank's policy is* ***strictly regular*** *if it is regular and:*
   (a) *there exists $b_0 > 0$ such that if $\boldsymbol{b}^i \geq b_0$ for some $i \in \boldsymbol{I}$, then*

   $$\frac{(u^i)'(0)\boldsymbol{a}^i_1}{\boldsymbol{r}_1(\boldsymbol{b}) - \frac{\boldsymbol{a}^i_1}{\bar{\boldsymbol{a}}_1}} < \varepsilon_0 \tag{3.1}$$

   where ε_0 is the constant appearing in formula (2.4);

   (b) *the function $\boldsymbol{r}_1$ is bounded by some constant $M > 0$;*

   (c) *there is a neighborhood of $0 \in \mathbb{R}^n_+$ such that for $\boldsymbol{b}$ within this neighborhood*

   $$\boldsymbol{\rho}_2(\boldsymbol{b}) < \frac{\boldsymbol{a}^i_1}{\bar{\boldsymbol{a}}_1} \quad (i \in \boldsymbol{I}) \tag{3.2}$$

   is valid.

Theorem 3.2. *Let $\mathcal{M}$ be a financial market and let R be a regular policy. Then, a bankrupt player has a negative marginal utility of bids and no player is bankrupt in a Nash equilibrium of $\Gamma^R_{\mathcal{M}}$.*

Proof:

The utility of a bankrupt player is computed by adding his reward for consumption and the punishment for being bankrupt; there is no capital left. Note that the price is positive. Therefore, the utility of a bankrupt player $i \in \boldsymbol{I}$ is given by

$$u^i\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right) - w^i\left(-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right), \tag{3.3}$$

where we assume, w.l.o.g., that $V^i(0) = 0$.

It suffices to show that

$$(u^i)' \left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right) \frac{\partial}{\partial \boldsymbol{b}^i} \left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right] < (w^i)' \left(-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right) \frac{\partial}{\partial \boldsymbol{b}^i} \left[-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right] \tag{3.4}$$

holds. It follows from Definition 2.1. *item* 3 that

$$(u^i)' \left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right) < (w^i)' \left(-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right) \tag{3.5}$$

is always valid. Hence, it is enough to verify that the relation

$$0 \leq \frac{\partial}{\partial \boldsymbol{b}^i} \left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right] < \frac{\partial}{\partial \boldsymbol{b}^i} \left[-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right] , \tag{3.6}$$

is true.

In the sequel we denote the partial derivative $\frac{\partial}{\partial \boldsymbol{b}^i}$ by $'$. Using this notation, the middle term of (3.6) is

$$\left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right]' = \frac{1}{\boldsymbol{P}} + \boldsymbol{b}^i \left(\frac{1}{\boldsymbol{P}}\right)' , \tag{3.7}$$

and the definition

$$\boldsymbol{P} = \frac{\bar{\boldsymbol{b}}}{\bar{\boldsymbol{a}}_1} = \frac{\sum_{j \in \boldsymbol{I}} \boldsymbol{b}^j}{\bar{\boldsymbol{a}}_1} \tag{3.8}$$

implies that

$$\left(\frac{1}{\boldsymbol{P}}\right)' = -\frac{1}{\boldsymbol{P}\bar{\boldsymbol{b}}} \tag{3.9}$$

is true. Hence, (3.6) is

$$\left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right]' = \frac{1}{\boldsymbol{P}} - \frac{1}{\boldsymbol{P}} \frac{\boldsymbol{b}^i}{\bar{\boldsymbol{b}}} = \frac{1}{\boldsymbol{P}} \left(1 - \frac{\boldsymbol{b}^i}{\bar{\boldsymbol{b}}}\right) \tag{3.10}$$

so that the middle term of (3.6) is nonnegative.

Next we consider the rightmost term of (3.6). Observe that

$$\left(-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right) = \frac{\boldsymbol{r}_1}{\boldsymbol{P}} \left(\boldsymbol{b}^i - \boldsymbol{a}_2^i\right) - \boldsymbol{a}_1^i \tag{3.11}$$

and a simple computation yields the formula

$$\left[-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right]' \;=\; \frac{\boldsymbol{r}_1'}{\boldsymbol{P}}\left(\boldsymbol{b}^i - \boldsymbol{a}_2^i\right) + \frac{\boldsymbol{r}_1}{\boldsymbol{P}} + \boldsymbol{r}_1\left(\frac{1}{\boldsymbol{P}}\right)'\left(\boldsymbol{b}^i - \boldsymbol{a}_2^i\right) \tag{3.12}$$

for the derivative. Inserting $\left(\frac{1}{\boldsymbol{P}}\right)'$ from (3.9), we obtain

$$\left[-\frac{(\boldsymbol{C}^i)^-}{\boldsymbol{P}}\right]' \;=\; \frac{\boldsymbol{r}_1'}{\boldsymbol{P}}\left(\boldsymbol{b}^i - \boldsymbol{a}_2^i\right) + \frac{\boldsymbol{r}_1}{\boldsymbol{P}}\left(1 - \frac{\boldsymbol{b}^i - \boldsymbol{a}_2^i}{\bar{\boldsymbol{b}}}\right) \;. \tag{3.13}$$

Now (3.6) is verified if the right hand side of (3.13) exceeds the rightmost term of (3.10). However, $\boldsymbol{r}_1' \geq 0$ and $\boldsymbol{r}_1 \geq 1$ follows from our assumptions and $\boldsymbol{b}^i > \boldsymbol{b}^i - \boldsymbol{a}_2^i$ is obvious.

q.e.d.

Theorem 3.3. *Let $\mathcal{M}$ be a financial market. Suppose the policy of the bank is strictly regular. Then there exists $b_1 > 0$ such that, for any $i \in \boldsymbol{I}$ satisfying $\boldsymbol{b}^i > b_1$, player i's marginal utility of bids is negative.*

Proof: If player i is not bankrupt (but demanding a loan), then the price is positive and this player's utility is given by

$$\boldsymbol{U}^i(\boldsymbol{b}) \;=\; u^i\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right) + V^i(\boldsymbol{C}^i) \tag{3.14}$$

with

$$\boldsymbol{C}^i(\boldsymbol{b}) \;=\; \boldsymbol{P}(\boldsymbol{b})\boldsymbol{a}_1^i - \boldsymbol{r}_1(\boldsymbol{b})(\boldsymbol{b}^i - \boldsymbol{a}_2^i) \quad (\boldsymbol{b} \in \mathbb{R}_+^n). \tag{3.15}$$

Using $'$ for $\frac{\partial}{\partial \boldsymbol{b}^i}$ we compute

$$(\boldsymbol{C}^i)' \;=\; \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_1 - \boldsymbol{r}_1'(\boldsymbol{b}^i - \boldsymbol{a}_2^i) \leq \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_1 \;. \tag{3.16}$$

Applying also (3.10) we can estimate player i's marginal utility, so as to obtain

$$\begin{aligned} (\boldsymbol{U}^i)' \;&=\; (u^i)'\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)' + (V^i)'\left(\boldsymbol{C}^i\right)(\boldsymbol{C}^i)' \\ &\leq\; (u^i)'\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\frac{1}{\boldsymbol{P}}\left(1 - \frac{\boldsymbol{b}^i}{\bar{\boldsymbol{b}}}\right) + (V^i)'\left(\boldsymbol{C}^i\right)\left(\frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_1\right). \end{aligned} \tag{3.17}$$

In order to show that $\left(\boldsymbol{U}^i\right)'$ is negative it suffices, therefore, to verify the inequality

$$(u^i)'(0)\,\frac{1}{\boldsymbol{P}} < (V^i)'\left(\boldsymbol{C}^i\right)\left(\boldsymbol{r}_1 - \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1}\right). \tag{3.18}$$

To this end we put $t := \boldsymbol{C}^i$. Now choose $b_1 \geq b_0$ such that for all $\boldsymbol{b}^i > b_1$ the inequality

$$\frac{(u^i)'(0)}{\boldsymbol{P}} < \varepsilon_0 \left(\boldsymbol{r}_1(\boldsymbol{b}) - \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1}\right) \tag{3.19}$$

is satisfied (strict regularity). Now, if we have $t \leq t_0$ (cf. (2.4)), then we have

$$V^{i\prime}(t) \geq \varepsilon_0 \tag{3.20}$$

and (3.18) follows from (3.19). If, on the other hand, we have $t > t_0$, then we use $\boldsymbol{C}^i \leq \boldsymbol{P}\boldsymbol{a}_1^i$ and by means of (3.1) we obtain the estimate

$$\begin{array}{lclcl} (V^i)'(\boldsymbol{C}^i) & = & (V^i)'(t) & > & \dfrac{\varepsilon_0}{t} \\ & \geq & \dfrac{\varepsilon_0}{\boldsymbol{P}\boldsymbol{a}_1^i} & > & \dfrac{1}{\boldsymbol{P}\boldsymbol{a}_1^i}\,\dfrac{(u^i)'(0)\boldsymbol{a}_1^i}{\boldsymbol{r}_1(\boldsymbol{b}) - \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1}} \end{array}, \tag{3.21}$$

which is (3.18).

q.e.d.

Theorem 3.4. *Let* $\mathcal{M}$ *be a financial market. Let* $(V^i)'$ *be bounded on compact subsets of* $\mathbb{R}_{++}$*. Then there exists a neighborhood of* $\boldsymbol{b} = \boldsymbol{0} \in \mathbb{R}_+^n$ *within which the marginal utility of at least* $n-1$ *players is positive.*

Proof: The utility of a player bidding a small amount is the same as in equation (3.14), however, the function $\boldsymbol{C}^i$ for small $\boldsymbol{b}$ is

$$\boldsymbol{C}^i(\boldsymbol{b}) \;=\; \boldsymbol{P}(\boldsymbol{b})\boldsymbol{a}_1^i + \boldsymbol{r}_2(\boldsymbol{a}_2^i - \boldsymbol{b}^i). \tag{3.22}$$

In analogy to (3.17) we get

$$(\boldsymbol{U}^i)' \;=\; (u^i)'\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\frac{1}{\boldsymbol{P}}\left(1 - \frac{\boldsymbol{b}^i}{\bar{\boldsymbol{b}}}\right) + (V^i)'\left(\boldsymbol{C}^i\right)\left(\boldsymbol{C}^i\right)' \tag{3.23}$$

It follows from (3.22) that $\boldsymbol{C}^i$ is bounded and is bounded away from 0 for small $\boldsymbol{b}$, hence the derivative of V^i is bounded. So is

$$(\boldsymbol{C}^i)' \;=\; \frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_2 + \boldsymbol{r}_2'(\boldsymbol{b}^i - \boldsymbol{a}_2^i).$$

Thus, it suffices to show that the first term in (3.23) is arbitrarily large for at least one player in a neighborhood of 0.

Because of $\sum_{i\in I}\boldsymbol{b}^i = \bar{\boldsymbol{b}}$ there is, for every $\boldsymbol{b} \in \mathbb{R}^n$, at most one $k \in I$ such that $\boldsymbol{b}^k \geq (1 - \frac{1}{n})\bar{\boldsymbol{b}}$ holds true. For all the others $i \neq k$ we have

$$1 - \frac{\boldsymbol{b}^i}{\bar{\boldsymbol{b}}} \geq \frac{1}{n}.$$

Now an inspection of (3.23) shows that indeed the marginal utility of these players is large whenever $\bar{\boldsymbol{b}}$ and $\boldsymbol{P}$ are small.

q.e.d. For convenience, let us summarize the results obtained by the previous theorems as follows.

Corollary 3.5. *Let $\mathcal{M}$ be a financial market and let the policy of the bank be strictly regular. Then there exists a compact convex set of strateg ies $\mathbb{B} \subseteq \mathbb{R}^n_+$ with the following properties:*

1. *There is $\varepsilon > 0$ and $\bar{\boldsymbol{\beta}} \in \mathbb{R}^n$ such that*

   $$\mathbb{B} \;=\; \left\{\boldsymbol{b} \in \mathbb{R}^n_+ \;\middle|\; \bar{\boldsymbol{b}} \geq \varepsilon,\; \boldsymbol{b} \leq \bar{\boldsymbol{\beta}}\right\}, \tag{3.24}$$

   (and hence prices are positive for $\boldsymbol{b} \in \mathbb{B}$.)

2. *For any $\boldsymbol{b} \in \mathbb{B}$, no player is bankrupt.*

3. *For any $\boldsymbol{b} \notin \mathbb{B}$ with $\boldsymbol{b}_i > \bar{\boldsymbol{b}}$ for some $i \in I$, the partial derivative of $\boldsymbol{U}^i$ is negative.*

4. *For any $\boldsymbol{b} \notin \mathbb{B}$ with $0 < \bar{\boldsymbol{b}} < \varepsilon$, the marginal utilities of at least $n-1$ players are positive.*

5. *For any $\boldsymbol{b} \notin \mathbb{B}$ with $0 < \bar{\boldsymbol{b}} < \varepsilon$, the best responses of at least $n-1$ players exceed ε.*

6. *The best response correspondence maps points on the lower boundary*

   $$\left\{b \in \mathbb{R}^n_+ \;\middle|\; \bar{\boldsymbol{b}} = \varepsilon\right\},$$

   into the interior of $\mathbb{B}$. The same holds true for points close to the lower boundary.

In view of the Corollary 3.5 it is clear that existence of Nash equilibria can be verified. We consider the compact set specified by the corollary and observe the behavior of the best response correspondence.

Theorem 3.6. *Let $\mathcal{M}$ be a financial market and let R be a strictly regular policy of the bank. Then the corresponding one shot game $\Gamma_{\mathcal{M}}^{R}$ has a Nash equilibrium in mixed strategies.*

Proof:

According to Corollary 3.5 there exists a compact convex domain $\mathbb{B}$ of strategies such that for any strategy n-tuple within $\mathbb{B}$, no player is bankrupt. The marginal utility outside this domain is positive for smaller bids and negative for larger ones. Also, zero bids are avoided so prices are well defined.

For every $\boldsymbol{b} \in \mathbb{B}$, consider the best response correspondence which necessarily yields vectors in $\mathbb{B}$. Introducing mixed strategies we find that the best response correspondence necessarily yields probabilities with carriers inside of $\mathbb{B}$. This correspondence is now convex valued as a mapping from probabilities on $\mathbb{B}$ into subsets of probabilities on $\mathbb{B}$. A fixed point obtained by the Kakutani Theorem yields the equilibrium. **q.e.d.**

For some policies of the bank we can establish an equilibrium in pure strategies.Consider the policy established in Remark 2.10. Assume that, accordingly, we have $\boldsymbol{r}_2 = 1$ while $R_1 = R_1^M$ is given by formula (2.43), i.e.,

$$R_1(p, \boldsymbol{b}, \rho) \ := \ \bar{r}_1(p, \boldsymbol{b}, \rho) \wedge M \quad \left((p, \boldsymbol{b}, \rho) \ \in \ \mathbb{R}_+^1 \times \mathbb{R}_+^n \times \mathbb{R}_+^1\right). \tag{3.25}$$

Here, $\bar{r}_1(p, \boldsymbol{b}, \rho)$ is given by (2.27).

Theorem 3.7. *Let $\mathcal{M}$ be a financial market. Assume that V^i is concave for all $i \in \boldsymbol{I}$. If the bank adopts a strictly regular policy (R_1, R_2) given by (3.25), then the game $\Gamma_{\mathcal{M}}^{R}$ admits of a Nash equilibrium in pure strategies. The same is true if the bank choses a strictly regular policy with constant interest rates $\boldsymbol{r}_2 \leq \boldsymbol{r}_1$.*

Proof:

We prove that the second derivatives of the players utilities within the compact set of strategies established by Corollary 3.5 are nonpositive. Within this set no one is bankrupt, therefore we have to consider the functions

$$u^i\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right) + V^i\left(\boldsymbol{C}^i\right) \ . \tag{3.26}$$

The second derivative of this expression is given by

$$u^{i\,\prime\prime}\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\left[\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)^{\prime}\right]^2 + V^{i\,\prime\prime}\left(\boldsymbol{C}^i\right)\left[\left(\boldsymbol{C}^i\right)^{\prime}\right]^2 + u^{i\,\prime}\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right]^{\prime\prime} + V^{i\,\prime}\left(\boldsymbol{C}^i\right)\left[\boldsymbol{C}^i\right]^{\prime\prime}\ . \tag{3.27}$$

We know that the second derivatives of u^i and V^i are negative, hence the first line in (3.27) is negative. Therefore, in order to obtain concave utilities, it is sufficient to show that

$$u^{i\,\prime}\left(\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right)\left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right]^{\prime\prime} + V^{i\,\prime}\left(\boldsymbol{C}^i\right)\right)\left[\boldsymbol{C}^i\right]^{\prime\prime} \tag{3.28}$$

is nonpositive..

Now, the first derivatives of both utilitiy functions that appear in (3.28) are nonnegative (positive for positive arguments). Also, the second derivative within the left term is

$$\left[\frac{\boldsymbol{b}^i}{\boldsymbol{P}}\right]^{\prime\prime} = -\frac{2\bar{\boldsymbol{a}}_1}{(\bar{\boldsymbol{b}})^3}(\bar{\boldsymbol{b}} - \boldsymbol{b}_i)\ <\ 0, \tag{3.29}$$

Hence, the first term in (3.28) is actually negative. In order to finish the proof it suffices, therefore, to show that $\boldsymbol{C}^{i\,\prime\prime}$ is nonpositive.

Recall that we have

$$\boldsymbol{C}^i = \boldsymbol{P}\boldsymbol{a}_1^i - \boldsymbol{r}_1(\boldsymbol{b}^i - \boldsymbol{a}_2^i)^+ + \boldsymbol{r}_2(\boldsymbol{a}_2^i - \boldsymbol{b}^i)^+. \tag{3.30}$$

Now consider formula (2.27) which defines $\bar{\rho}_1$. It turns out that $\bar{\rho}_1 = 0$ as no player is bankrupt and $\rho_2 = 0$. Hence, in the case described by (3.25), we have $\bar{r}_1 = \bar{r}_2 = 1$ within the critical domain described by Corollary 3.5. Therefore, we have to treat the second case of our theorem, i.e., the one in which both interest rates are constant and $\boldsymbol{r}_2$ does not exceed $\boldsymbol{r}_1$. In this case we se from (3.30) that $\boldsymbol{C}^i$ is a piecewise linear function of $\boldsymbol{b}^i$ with a possible kink at $\boldsymbol{a}_2^i$. The first derivative is $\frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_2$ for $\boldsymbol{b}^i \leq \boldsymbol{a}_2^i$ and $\frac{\boldsymbol{a}_1^i}{\bar{\boldsymbol{a}}_1} - \boldsymbol{r}_1$ for $\boldsymbol{b}^i > \boldsymbol{a}_2^i$. This function is concave exactly if $\boldsymbol{r}_1 \geq \boldsymbol{r}_2$ holds true.

Thus, we have completed the proof that players utilities are concave.

We now finish the existence proof for a Nash equilibrium in a standard way. According to Remark 3.5 there exists a compact convex domain $\mathbb{B}$ of strategies such that for any strategy n-tuple within $\mathbb{B}$, no player is bankrupt and the marginal utility outside this domain negative for large bids. Also, zero bids are avoided and for bids with a small total sum the marginal utility is positive for at least $n-1$ players. Hence, prices within $\mathbb{B}$ are well defined.

For every $\boldsymbol{b} \in \mathbb{B}$, consider the best response correspondence, say $\boldsymbol{D} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$. Setting $\boldsymbol{e} := (1, \ldots, 1) \in \mathbb{R}^n$, we define a function $F := \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(\boldsymbol{b}') := \boldsymbol{b}' + \frac{(\varepsilon - \overline{\boldsymbol{b}}')^+}{n}\boldsymbol{e} \quad (\boldsymbol{b}' \in \mathbb{R}^n_+). \tag{3.31}$$

This function projects all bids with total sum less than ε on the plane of all bids with total sum equal to ε. The composition, say $\boldsymbol{D}^\star := F \circ \boldsymbol{D}$, is an upper hemi continuous mapping from $\mathbb{B}$ to $\mathbb{B}$.

This correspondence is convex valued. A fixed point of $\boldsymbol{D}^\star$ is obtained by the Kakutani Theorem. This fixed point cannot be located on the lower boundary of $\mathbb{B}$, i.e., on $\{\boldsymbol{b} \in \mathbb{R}^n_+ \mid \boldsymbol{e}\boldsymbol{b} = \varepsilon\}$, as all of the points on this boundary are mapped into the interior of $\mathbb{B}$. Hence the fixed point is actually a fixed point of $\boldsymbol{D}$, hence it yields an equilibrium.

q.e.d.

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