# Thresholds in a Credit Market Model with Multiple Equilibria

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August, 2001

**Abstract**

The paper studies a credit market model with endogenous credit cost and debt constraints in which multiple candidates for steady state equilibria arise. We use dynamic programming (DP) with flexible grid size to locate thresholds that separate different domains of attraction. More specifically, we employ DP to (1) compute present value borrowing constraints and thus creditworthiness, (2) locate thresholds where the dynamics separate to different domains of attraction, (3) distinguish between optimal and non-optimal steady states and (5) demonstrate how the thresholds change with change of the credit cost function of the debtor and (6) explore the impact of debt ceilings and consumption paths on creditworthiness. The analytics is provided for a general model and some generic results are presented for a one state variable problem.

An earlier version of this paper has been prepared for the 1998 North American Winter Meeting of the Econometric Society, January 1998, Chicago. We want to thank Jess Benhabib, Gustav Feichtinger, Gerhard Sorger, Franz Wirl, Michael Woodford, Wolf-Jürgen Beyn and Thorsten Pampel for helpful discussions and comments. We also want to thank participants in a workshop at the University of Technology, Vienna, the Macroeconomic Workshop at Columbia University, and the SCE conference, at Yale University, June 2001.

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# 1 Introduction

Numerous examples of dynamic economic models with multiple equilibria exist in the economic literature. In earlier growth theory it has been shown that a convex-concave production function leads to multiple equilibria.[1] Multiple equilibria have further been studied in modern growth theory.[2] The literature on resource economics and ecological management problems show also numerous examples of models with multiple equilibria. Multiple equilibria are also important properties in trade models, in models of addiction and in labor market search and monetary policy models.[3] Yet, only recently it has been discovered that the study of the local dynamics needs to be complemented by the study of the global dynamics. In those models there is history dependence in the sense that there are thresholds where the dynamics separate to different domains of attraction. These thresholds may or may not coincide with the candidates for steady state equilibria. Since there do not seem to exist equations to locate those thresholds, the thresholds need to be detected by applying numerical methods. We propose a simple economic model with borrower's and lender's relationship which gives rise to multiple candidates of steady state equilibria and thresholds. Although our model lends itself to a multi-variable interpretation to which our methods can be applied to, for analytical purpose we restrict our study to a one control-one state variable model.

In this simple variant capital stock is the state variable and investment is the control variable. We take into account temporary and intertemporal budget constraints of the agent who is allowed to finance investment through credit market borrowing. We allow for adjustment cost of capital and state dependent credit cost which generate multiple candidates for

[1]Examples are given in the literature on development economics where a convex-concave production function arises which leads to a threshold that separates paths to low per capita income (poor) countries and high per capita income (rich) countries, see Skiba (1978) and Azariadis and Drazen (1990).

[2]In endogenous growth models of Lucas and Romer type multiple equilibria may arise by employing externalities or complementarities in the production function, for the Lucas model, see Benhabib, Perli and Xie (1994) and for the Romer model see Benhabib and Perli (1994) and Evans, Honkapohja and Romer (1997).

[3]On resources and the ecological management problem, see Brock and Starret (1999) and Sieveking and Semmler (1997); on trade theory, see Krugman (1991); on addiction, see Orphanides and Zervos (1998); on labor market search theory, see Mortensen (1989); and on monetary policy, see Benhabib, Schmitt-Grohe and Uribe (1998) and Greiner and Semmler (2000).

steady state equilibria. The model resembles the dynamic models with credit market borrowing such as employed in Blanchard (1983), Bhandari et al. (1990), Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999) and Miller and Stiglitz (1999). In these models the impact of credit market borrowing and debt dynamics on economic activity is studied.[4]

Most literature on dynamic credit market models, by assuming perfect credit markets, posits that, agents can borrow against future income as long as the discounted future income, the wealth of the agents, is no smaller than the debt that agents have incurred. There is no credit risk whenever the intertemporal budget constraint holds. Formally, often the so called transversality condition is invoked to provide a statement on the non-explosiveness of the debt of the economic agents. Models of this type have been discussed in the literature for households, firms, governments and countries (with access to international capital markets).[5] Here then, as long as the intertemporal budget constraint holds, debt dynamics do not impact economic activity.

There are other studies that assume credit market imperfections so that borrowing is constrained. Borrowing ceilings are assumed which are supposed to prevent agents from borrowing an unlimited amount. Presuming that agents' assets serve as collateral a convenient way to define the debt ceiling is then to assume the debt ceiling to be a fraction of the agents' wealth. The definition of debt ceilings have become standard, for example, in small scale macro dynamic models.[6]

In other research, also building on imperfect capital markets it is posited that borrowers face a risk dependent interest rate which is assumed to be composed by a market interest rate (for example, risk-free interest rate) and an idiosyncratic component determined by the individual degree of risk of the borrower. Employing the theory of asymmetric information and costly state verification in Bernanke et al. (1999), for example, credit cost is endogenous by making it dependent on net worth of the borrower, as collateral for borrowing.[7] This gives rise to an external finance premium that entrepreneurs

[4]The above models usually do not admit multiple steady states except the model by Miller and Stiglitz (1999). Yet, there is no study of thresholds. Note also that in contrast to some of the other models, for example, the one by Bernanke et al.(2000) we, in order to simplify matters, do not employ a stochastic version but rather employ a deterministic framework. A stochastic version is discussed in Sieveking and Semmler (1999).

[5]For a brief survey of such models for households, firms and governments or countries, see Blanchard and Fischer (1989, ch.2) and Turnovsky (1995).

[6]See, for example Barro, Mankiw and Sala-i-Martin (1995). It has also been pointed out that banks (like the World Bank), often define debt ceilings for their borrowers, see Bhandari, Haque and Turnovsky (1990).

[7]Recent work has been undertaken by nesting credit market imperfections and endogenous borrowing cost more formally in intertemporal models such as the standard stochastic

have to pay contingent on their net worth.[8]

We restrict our study to a simple credit market model and explore global dynamics of the model when agents, as in the latter case, face endogenous credit cost. We also study the impact of debt constraints and debt ceilings on the global dynamics. To study global dynamics we have to compute creditworthiness. We show that debt ceilings should not be arbitrarily defined but rather given by creditworthiness.[9] We use dynamic programming with flexible grid size to solve such type of models as well as to distinguish local from global dynamics. After having obtained candidates for steady state equilibria of the model we (1) compute the present value borrowing constraint and creditworthiness without and with endogenous credit cost, (2) compute thresholds of those types of models (in the sense of Skiba 1978) where the dynamics separate to different domains of attraction, (3) show that the policy function may be discontinuous and compute the jumps in the policy function, (4) distinguish between optimal and non-optimal steady states and (6) demonstrate how the thresholds change with change of the credit cost function and (7) compute creditworthiness curves and thresholds for model variants with debt ceilings and given consumption paths.

We want to note that since in this paper we are concentrating on methodological issues such as history dependence, thresholds, domains of attraction, suboptimal equilibria and jumps in the policy function we use a stylized model which is nested in an intertemporal model with utility maximization but can be studied independently of utility maximization. The relation of our model to a model with a utility functional is shown in appendix I.

Finally we want to remark, that such problems of dynamic models arising from multiple steady state equilibria have been studied also in earlier work, such as Skiba (1978).[10] Most researchers have employed the Hamiltonian equation derived from Pontryagin's maximum principle. As shown in Beyn, Pampel and Semmler (2000) Pontryagin's maximum principle and the associated Hamiltonian can also be applied to study the global dynamics of such a model with multiple steady states in restricted cases. In the current paper we propose the use of dynamic programming techniques on adaptively

growth model, see Carlstrom and Fuerst (1997), Cooley and Quadrini (1998) and Krieger (1999). We restrict our study to a simple investment model.

[8]Another development of the analysis of credit risk employs less the "ability to pay" but rather the "willingness to pay" approach to explain defaults. For the latter type of literature, in particular on the problem of incentive compatible contracts, see Eaton and Fernandez (1995).

[9]A more elaborate analysis of how credit ceilings affects welfare is given in Semmler and Sieveking (1996).

[10]An extensive discussion of the earlier work on studying thresholds using the Hamiltonian is given in Brock and Malliaris (1996, ch. 6))

refined grids which are well suited to study the above problems.

The remainder of the paper is organized as follows. Section 2 introduces the basic dynamic model. Section 3 describes numerical methods, in particular dynamic programming with flexible grid size that are used to study different variants of the dynamic model. Sections 4 reports the detailed results from our numerical study on the different variants of the model. In appendix I we briefly summarize the separation theorem and appendix II shows the relation of dynamic programming undertaken in discrete time[11] to the HJB-optimality equation.

# 2 The Dynamic Model

Next we want to specify the dynamic model that we study analytically and numerically. As above mentioned in the study of creditworthiness we can by-pass utility theory though the model is nested in a more fully developed model with utility theory. Economists have argued that analytical results in intertemporal models frequently depend on the form of the utility function employed. We show that we can study borrowing, lending and creditworthiness, without the direct use of utility theory. Although our model can be nested in utility theory, we use a separation theorem that permits us to separate the present value problem from the consumption problem. In Sieveking and Semmler (1998) an analytical treatment is given of why and under what conditions the subsequent credit market model can be separated from the consumption problem. A brief summary of the analytical result is given in appendix I. Note that by focusing on the entrepreneurs intertemporal optimal investment where debt can be continuously issued and retired we do not have to consider that in each period the agent is constrained by financial constraints but there will be intertemporal debt constraints where debt capacity will be defined by the agent's creditworthiness and credit constraints.

As to the more specifics of the credit market features of our model we presume credit market imperfections. Along the line of Bernanke, Gertler and Gilchrist (1999), henceforth BGG, we assume asymmetric information and agency costs in borrowing and lending relationships. BGG draw on the insight of the literature on costly state verification[12] in which lenders must pay a cost in order to observe the borrower's realized returns. This motivates the use of collaterals in credit market models. Uncollateralized borrowing is assumed to pay a larger finance premium than collateralized

[11] For the more extensive version of the generalized model, see Semmler and Sieveking (1999) which is available for the reader upon request.

[12] This literature originates in the seminal work by Townsend (1979).

borrowing or self-financing. The external finance premium is interpretable as the cost of bankruptcy (for example constituted by auditing, accounting, legal cost, as well as loss of assets arising from asset liquidation). Thus the external finance premium drives a wedge between the expected return of the borrower and the risk-free interest rate whereby the external finance premium is positively related to the default cost and inversely related to the borrowers net worth. Net worth is defined as the agents collateral value of the (illiquid) capital stock less the agent's outstanding obligations. Following BGG we can measure the inverse relationship between the external cost of finance and net worth in a function such as

$$H\left(k(t), B(t)\right)) = \frac{\alpha_1}{(\alpha_2 + \frac{N(t)}{k(t)})^{\mu}} \theta B(t) \tag{1}$$

with $H\left(k(t), B(t)\right))$ the credit cost depending on net worth, $N(t) = k(t) - B(t)$, with $k((t)$ as capital stock and $B(t)$ as debt. The parameters are $\alpha_1, \alpha_2, \mu > 0$ and $\theta$ is the risk-free interest rate. In the analytical and numerical study of the model below we presume that the external finance premium will be zero for $N(t) = k(t)$ and thus for $B(t) = 0$[13] so that in the limit the borrowing rate is the risk-free rate. Note, however, that even if the credit cost is endogenized we might want to define the agent's financial constraints which in our model will be given below by an upper bound of the debt-capital ratio.

Building on the separation theorem as briefly summarized in appendix I and employing the theory of asymmetric information and costly state verification which posits an inverse relation of external finance premium and net worth, we study the following intertemporal model with credit market borrowing

$$V(k) = \underset{j}{Max} \int_0^{\infty} e^{-\theta t} f\left(k(t), j(t)\right) dt \tag{2}$$

$$\dot{k}(t) = j(t) - \sigma k(t), \quad k(0) = k. \tag{3}$$

[13] Bernanke, Gertler and Gilchrist (1999) employ the same functional relationship as above. They state that ".. the external finance premium depends inversely on the share of the firm's capital investment that is financed by the entrepreneur's own net worth" (p.166).

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$$\dot{B}(t) = H\left(k(t), B(t)\right)\right) - \left(f\left(k(t), j(t)\right) - c(t)\right), \; B(0) = B_0 \tag{4}$$

The model represents an optimal investment problem with adjustment cost of capital and endogenous credit cost. In addition, debt constraints can be imposed. The agent‘s net income

$$f(k, j) = ak^{\alpha} - j - j^{\beta}k^{-\gamma} \tag{5}$$

is generated from capital stock, through a production function, $k^{\alpha}$, and investment, $j$, is undertaken so as to maximize the present value of net income in (5) given the adjustment cost of capital $j^{\beta}k^{-\gamma}$. Note that $\sigma > 0, \alpha > 0, \beta > 1$, $\gamma > 0$, are constants.[14] Equ. (3) represents capital accumulation and equ. (4) the evolution of debt of the economic agent. We allow for negative investment rates $j < 0$, i.e. reversible investment for simplicity. Note that in (4) $c(t)$ is a consumption stream that is, in the context of our model, treated as exogenous.[15]In the study of the next section the consumption stream will be specified further. Since net income in (5) less the consumption stream $c(t)$ can be negative the temporary budget constraint of the agent requires further borrowing from credit markets and if there is positive net income less consumption debt can be retired.

As above shown, along the line of the theory of imperfect capital markets we assume that the credit cost $H(k, B)$ may be state dependent, depending on the capital stock, $k$, and the level of debt $B$ with $H_k > 0$ and $H_B < 0$. Note, however, that if we assume that credit cost depends inversely on net worth as in equ. (1) we get a special case of our model when only the risk-free interest rate is accounted for in the credit cost. We then have a constant credit cost and a state equation for the evolution of debt such as

$$\dot{B}(t) = \theta B(t) - f(k, B), \qquad B(0) = B \tag{6}$$

In this case, we would only have to use the transversality condition $\lim_{t \to \infty} e^{-\theta t} B(t) = 0$ as the non-explosiveness condition for debt to close the model.

[14]Note that the production function may $k^{\alpha}$ may have to be multiplied by a scaling factor. For the analytics we leave it aside. Other authors have used the simplification $H(k, B) = \theta B, \beta = 2$, $\gamma = 1$ to study such a model, see Blanchard (1983).

[15]Note that in our model all variables are written in efficiency labor, therefore $\sigma$ represents sum of the capital depreciation rate, population growth and rate of exogenous technical change. Our model resembles the one by Blanchard (1983) but builds on imperfect capital markets and thus it endogenizes credit cost.

In our more general model (2)-(4) we define the limit of $B(t)$ equal to $B^*(t)$ which represents the present value borrowing constraint or the credit-worthiness of the agent. Note that $B^*(t)$ will be defined as curve and not as a point since it needs to be known for each point in the state space. As debt ceiling we take

$$\sup_{t\geq 0} B(t) < \infty \tag{7}$$

Let us call an initial indebtedness $B$ subcritical for an initial capital stock $k$ if there is an investment function $j(\cdot)$ such that the corresponding solution $t \rightarrow (B(t), k(t))$ of (2)- (4) satisfies (7). Let $B^*(k)$ be the supremum of all initial levels of debt which are subcritical for initial capital $k$. We call $B^*(k)$ the creditworthiness of the capital stock $k$, or the agent's present value borrowing constraint. The problem to be solved in this paper is how to compute $B^*(k)$. If the interest rate $\theta = \frac{H(k,B)}{B}$ is constant[16], then as is easy to see, $B^*(k)$ is the present value of $k$, exclusive of the initial value of debt

$$B^*(k) = \underset{j}{Max} \int_0^\infty e^{-\theta t} f\left(k(t), j(t)\right) dt - B(0) \tag{8}$$

$$s.t. \quad \dot{k}(t) = \theta B(t) - \left(f(k, B) - c(t)\right), \quad t \geq 0, \quad k = k(0). \tag{9}$$

with $B(0)$ is the initial value of debt. The more general case is, however, when the credit cost is endogenous. If we have $H(k, B)$, then, as argued above, not only the relation of the present value to creditworthiness but also the notion of present value itself becomes difficult to treat. Pontryagin's maximum principle is not suitable to solve the problem with endogenous credit cost and we thus prefer dynamic programming.

Below we will also presume a simplified credit cost function which has been proposed in literature on imperfect capital markets. Bhandari et al (1990), for example, propose a convex credit cost in the level debt. This helps to simplify the analytical treatment of the model. We simplify equ. (1) by assuming that $H(k, B) = h(B)$ and that the latter is twice differentiable $h'(B) > 0$, $h(0) = 0$, $\quad h' \geq \theta$ for some constant $\theta > 0$ and $h'' \geq 0$.

[16] As aforementioned in computing the present value of the future net income we do not have to assume a particular fixed interest rate, as in Kiyotaki and Moore (1997) and Miller and Stiglitz (1999), but the present value, defining debt capacity $B^*(t)$, will, for the optimal investment decision, enter as argument in the credit cost function $H(k(t), B(t)^*)$.

In the context of the model (2)-(4) we also explore the use of 'ceilings' in debt contracts if they differ from creditworthiness $B^*(k)$. Credit restrictions may affect welfare. Suppose the 'ceiling' is of the form $B(t) < C$, with $C$ a constant, for all $t$. Either $C > B^*(k)$, then the ceiling is too high because the debtor might be tempted to move close to the ceiling and then goes bankrupt if $B > B^*(k)$. Or $C < B^*(k)$, then the agent may not be able to develop his or her full potentials, and thus face a welfare loss[17], or it may be the case that the contract is not feasible whereas it would be feasible if $B(t) \leq C$ for all $t$ is replaced by

$$\lim_{t \to \infty} \sup B(t) \leq C. \tag{10}$$

On the other hand, the latter condition obviously is of no practical use if we can not say when $B(t) \leq C$. The major tasks of our methods will be to compute the creditworthiness-curve $B^*(k)$ even for the case of endogenous credit cost as in equ. (1). We also will compute thresholds that separate the optimal solution paths for $B^*(k)$ to different domains of attraction. We will presume different functional forms of the credit cost function and also consider the case of debt a constrained agent for whom holds that $B(t)/k(t) \leq c$ with $c$ a constant then study the creditworthiness curve.[18] Moreover, we also will presume in our study various paths for the consumption stream, $c(t)$, and their impact on the creditworthiness curve.

Before further analyzing the model we make a remark on the generalization of the model. A generalized version of model (2) -(4) is studied in appendix II where we state a theorem on $B^*(k)$ in the multi-variable case with capital stocks $k = (k_1, ...k_n)$ and investment vector $j = (j_1........j_n)$. We show that our model with state dependent credit cost lends itself to an iterative dynamic programming solution and that $B^*(k)$ satisfies the HJB-optimality equation. Since the latter is in continuous time and the former in discrete time we demonstrate the relation of our above stated continuous time model to a discrete time version used in dynamic programming.[19] We state the HJB-equation for $B^*(k)$ and show that $B^*(k)$ is the limit of a corresponding discrete time approximation.[20]

[17] In Semmler and Sieveking (1996) the welfare gains from borrowing are computed.

[18] Note that in the case of endogenous credit cost and/or debt ceilings the equity price of the firm – if we consider the agent to be a firm – will be affected.

[19] In the extended version we also discuss discretization errors, see Semmler and Sieveking (1999).

[20] It is worth noting that the model in this section has not quite the same mathematical properties as considered in the general case discussed in appendix II, because first, the

Next we want to give some economic intuition of the working of a univariate version of the model (2)-(4) in the context of the use of the HJB-equation. Presume the above suggested simplification of (1) with $H(k, B) = h(B)$, $h(0) = 0$, $h' \geq \theta$ for some constant $\theta > 0$ and $h'' \geq 0$, and take $h(B) = \theta B^{\kappa}$ for $\kappa \geq 1$. For this case we easily can study the number of candidates for equilibria of our model. The candidates for steady states can be computed from Proposition 1 below.

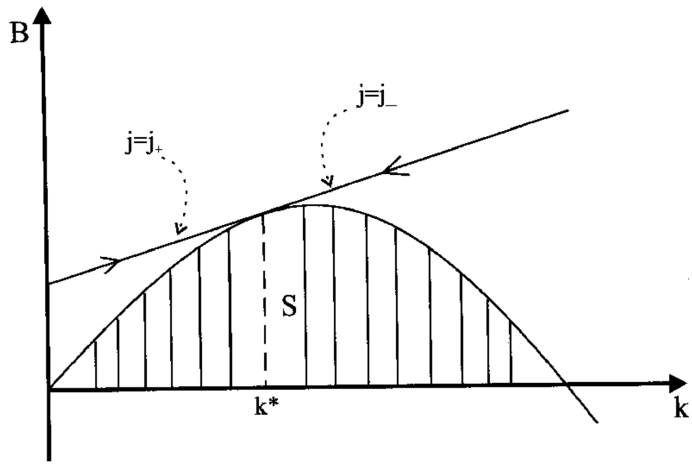


Figure 1: Debt control with a unique steady state equilibrium

There may be a uniquely determined steady state or multiple steady states. In the case of a unique steady state, as shown in Figure 1, there is a unique common stationary capital $k^*$ for the investment $j_-(k, B)$ with steepest descent as well as for the investment $j_+(k, B)$ with least steep ascent. [21] This solution path is shown in figure 1 where the set $S$ in the $(k, B)-$space is the set where investments $j = \sigma k$ decrease debt. As figure 1 shows if the debt is above $S$ we need to decrease debt most rapidly by decreasing capital $k$ while $k > k^*$ using $j_-(k, B)$ and increase debt least possible while $k < k^*$. The investment that achieves this can be called extremal investment. For this

range of the control variable $j$ is unbounded, second, the range of the state variable $(k, B)$ is unbounded, and third the net income function $f(k, j)$ has a singularity at $k = 0$. Yet, the present value for the above model can be approximated by one which belongs to the class of models considered in appendix II (to do so we can reduce the ranges and smooth $f$ out at $k = 0$). For details, see Semmler and Sieveking (1999).

[21]The detection of the solution path is based on the observation that the solution $k \rightarrow (k, B^*(k))$ of our problem consists of solutions to a differential equation which (i) use either steepest descent or the least steep ascent in the $(k, B)-$space and (ii) run into a stationary state. This method has been applied in Semmler and Sieveking (1999, 2000).

investment an unit of investment will give rise to the least increase of debt, if investment rises and the greatest decrease of debt if investment falls.[22]

A more complicated situation is shown in figure 2.

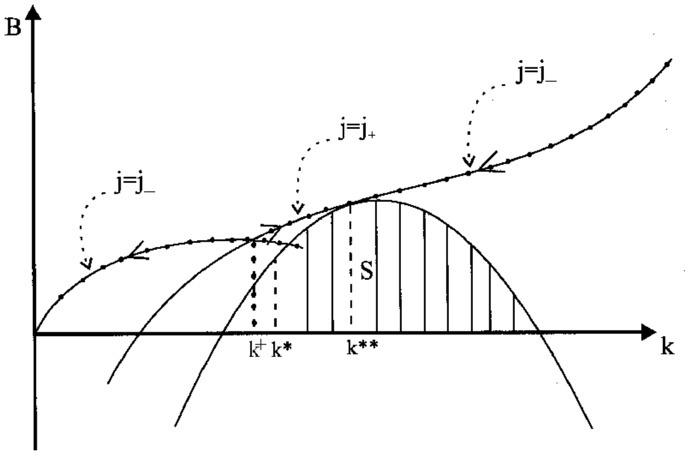


Figure 2: Debt control with multiple steady state equilibria

In addition to the attractor equilibrium $k^{**}$ there is a threshold $k^{+}$ where the solution path changes direction. As discussed in the introduction such thresholds turn up in many dynamic models. This generically happens if several attractor equilibria exist and there is, in between, a repeller equilibrium. In the current model, in the case of multiple equilibria, the threshold, the Skiba point, $k^{+}$, in figure 2, is economically significant: below $k^{+}$ the debtor has to reduce its capital to zero in order to keep debt bounded whereas above $k^{+}$ capital can be expanded.[23] Note, however, that such a threshold does not have to coincide with the candidate for the equilibrium $k^{*}$ which may be in the neighborhood of $k^{+}$. The threshold will be displaced from the neighboring unstable steady state if the latter is a focus.[24] Note also if $B(k^{*})$ is smaller than the trajectories $j_{-}$ leading to $k=0$ and $j_{+}$ leading to $k^{**}$ either of the latter two trajectories are better than staying at the steady state candidate $k^{*}, B(k^{*})$. Thus, $k^{*}$ is not optimal and the vicinity of the unstable steady

[22]Of course, in order to ensure that our creditworthiness curve will contain the equilibrium candidate $(k^{*}, B(k^{*}))$, we might invert the time and solve the initial value problem with $(k^{*}, B(k^{*}))$ as initial value (using $j_{-}$ for $k > k^{*}$ and $j_{+}$ for $k < k^{*}$).

[23]In the development literature such a threshold – which has, however, been identified with the middle unstable steady – has been called a development trap, see for example, Azariadis and Drazen (1990).

[24]For details, see Deissenberg et al. (2000).

state provides indeterminate solution.[25] A case like this will be numerically demonstrated below.[26]

The candidates for equilibria can be computed as follows. For our specification $h(B)$ where the critical curve $k \to B^*(k)$ is tangent to the point $B(k^*)$, we consider the set $S$ of states in the $(k, B)-$space where investments $j = \sigma k$ decrease debt.[27] Investment keeps capital constant precisely if

$$h(B) \leq f(k, \sigma k) \quad \text{or}$$

$$B \leq h^{-1}\left(f(k, \sigma k)\right) =: \varphi(k).$$

Let

$$S = \{(k, B) \mid 0 \leq B \leq \varphi(k), \quad k \geq 0\}$$

$S$ is bounded by the graph of the function $\varphi$, which we define only for $k \geq 0$ with $f(k, \sigma k) \geq 0$.

Note that if $h(B) = f(k, \sigma k)$ that is at the boundary of $S$ a trajectory starting from a candidate of an equilibrium is tangent to $S$. We thus can state a proposition for the candidates of steady state equilibria.

**Proposition 1** The candidates of equilibria satisfy $1 + 2\sigma k^{1-\gamma} = \varphi^{‘}(k)$.

For $h(B) = rB^{\kappa}$ this is equivalent to

$$1 + 2\sigma k^{1-\gamma} = \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2-\gamma)k^{1-\gamma}}{r^{1/\kappa}\kappa(k^{\alpha} - \sigma k - \sigma^2 k^{2-\gamma})^{(\kappa-1)/\kappa}} \tag{11}$$

For (11) there are likely to be multiple solutions depending on the parameters of the net income and credit cost functions. Thus, both the nonlinear adjustment cost of capital as well as the state dependent credit cost contribute to multiple candidates for steady state equilibria. Yet, one also may

[25]This type of indeterminacy in dynamic models is discussed in Benhabib and Gali (1995).

[26]Note that for a constant interest rate we might apply the Hamiltonian from Pontryagin's maximum principle to determine the optimal investment. In the situation of figure 2, we obtain two positive candidates for an optimal stationary capital stock such as $k^*$, $k^{**}$, but would, by local analysis, get no information that one of those (and which) is non-optimal.

[27]Note that here, for easons of simplicity, we have set consumption equal to zero. If it is positive it will move down the creditworthiness curve, see our results in section 4.

obtain only one (strictly positive) steady state equilibrium candidate inspite of both adjustment cost of capital and endogenous credit cost, see Semmler and Sieveking (1999). In general, however, if there is a state dependent credit cost and there is a change of the credit cost or net income[28] functions this will result in a change of the location of the thresholds. As above noted, for a constant credit cost, we can also use the maximum principle to obtain the candidates for equilibria. In fact the equilibria are the same as for $\kappa = 1$ in (11).[29] Since, however, we do not want to restrict our model (2)-( 4) we employ dynamic programming with flexible grid size to study the model and its extensions.

# 3 Numerical Dynamic Programming

In the literature it has been shown that dynamic optimization models giving rise to multiple steady state equilibria can be of concave[30] or non-concave type and yet generating multiple equilibria with thresholds. Although most of the historical models as discussed in the introduction build on non-concave models, yet recently examples have been given where such phenomena can also arise in concave models.[31] In the case of the existence of multiple steady state equilibria of system (2)-(4) a rigorous study of the dynamics of the model and the thresholds were the dynamics separate to different domains of attraction would require locating the thresholds analytically. This appears to be feasible only if the thresholds coincide with one of the steady state equilibria. As has been shown this occurs if the relevant (unstable) equilibria is a node. In the concave model the unstable equilibria is necessarily a node, but a node can also occur in a non-concave model.[32] Yet, it is impossible to locate the threshold analytically if the threshold does not coincide with the (unstable) equilibria. Thus, the thresholds that will exists in the vicinity of the unstable steady state – and will render the neighboring unstable steady

[28]In an open economy model with flexible exchange rates, see Miller and Siglitz (1999), net income can fall or endogenous credit cost rise with the devaluation of the currency due to an increase in the local currency value of total debt (if there is significant external borrowing in foreign currency). Thus the net income function– as well as the credit cost function – may change. This might give rise to a change of the thresholds.

[29]For details of the computation of the equlibria and creditworthiness curve for the case of a constant credit cost, $\theta$, using Pontryagin's maximum principle, see Beyn, Pample and Semmler (2000).

[30]The problem is concave if, following the Mangasarian definition, the Hamiltonian for the above problem P($\mathbf{a}$) is both concave in the state as well as control variables.

[31]See, for example, Hartl et al. (2000), and Deissenberg et al. (2000).

[32]See Hartl et al. (2000) and Deissenberg et al. (2000).

state to be non-optimal– have to be located by numerical methods.

In this section we describe two dynamic programming algorithms which enable us to compute the creditworthiness curve $B^*(k)$ and the thresholds. While the two algorithms presented here are of quite different nature, a common feature of both is the adaptive discretization of the state space which leads to high numerical accuracy with moderate use of memory.

## 3.1 The Discounted Infinite Horizon Problem

The first algorithm is applied to discounted infinite horizon optimal control problems of type (2)–(4) when, however, no restriction on the dynamics is present. In our model, this applies if the credit cost is constant, i.e., $H(k, B) = \theta B$ as in (6) and if in addition the constraint on $B$ is given by $\inf_j \sup_{t\geq 0} B(t) < \infty$, since in this case it follows from (8) that $B^*(k)$ is easily obtained from $V(k)$ in (2), namely from

$$V(k) = \underset{j}{Max} \int_0^\infty e^{-\theta t} f\left(k(t), j(t)\right) dt$$

We will briefly describe the algorithm which goes back to Capuzzo Dolcetta (1983), Falcone (1987) and Grüne (1997). For details and for a mathematically rigorous convergence analysis we refer to these papers as well as to Appendix A in the monograph by Bardi and Capuzzo Dolcetta (1997) and to Grüne, Metscher and Ohlberger (1999).

In the first step, the continuous time optimal control problem is replaced by a first order discrete time approximation given by

$$V_h(k) = \underset{j}{Max} J_h(k, j), \quad J_h(k, j) = h \sum_{i=0}^{\infty} (1 - \theta h)^i f(k_h(i), j_i) \tag{12}$$

where $k_h$ is defined by the discrete dynamics

$$k_h(0) = k, \quad k_h(i + 1) = k_h(i) + h(j_i - \sigma k_h(i)) \tag{13}$$

and $h > 0$ is the discretization time step. Note that $j = (j_i)_{i\in\mathbb{N}_0}$ here denotes a discrete control sequence.

The optimal value function is the unique solution of the discrete Hamilton–Jacobi–Bellman equation

$$V_h(k) = \underset{j}{Max} \left\{ hf(k, j_0) + (1 - \theta h)V_h(k_h(1)) \right\}, \tag{14}$$

where $k_h(1)$ denotes the discrete solution corresponding to the control $j$ and initial value $k$ after one time step $h$. Abbreviating

$$T_h(V_h)(k) = \underset{j}{Max} \{hf(k, j_0) + (1 - \theta h)V_h(k_h(1))\} \tag{15}$$

the second step of the algorithm now approximates the solution on a grid $\Gamma$ covering a compact subset of the state space, i.e., a compact interval $[0, K]$ in our setup. Denoting the nodes of $\Gamma$ by $k^i$, $i = 1, \dots, P$, we are now looking for an approximation $V_h^\Gamma$ satisfying

$$V_h^\Gamma(k^i) = T_h(V_h^\Gamma)(k^i) \tag{16}$$

for each node $k^i$ of the grid, where the value of $V_h^\Gamma$ for points $k$ which are not grid points (these are needed for the evaluation of $T_h$) is determined by linear interpolation. We refer to the papers cited above for the description of iterative methods for the solution of (16). Note that an approximately optimal control law (in feedback form for the discrete dynamics) can be obtained from this approximation by taking the value $j^*(k) = j$ for $j$ realizing the maximum in (14), where $V_h$ is replaced by $V_h^\Gamma$. This procedure in particular allows the numerical computation of approximately optimal trajectories.

In order to distribute the nodes of the grid efficiently, we make use of a posteriori error estimation. For each cell $C_l$ of the grid $\Gamma$ we compute

$$\eta_l := \underset{k \in C_l}{Max} |T_h(V_h^\Gamma)(k) - V_h^\Gamma(k)|$$

(more precisely we approximate this value by evaluating the right hand side in a number of test points). It can be shown that the error estimators $\eta_l$ give upper and lower bounds for the real error (i.e., the difference between $V_h$ and $V_h^\Gamma$) and hence serve as an indicator for a possible local refinement of the grid $\Gamma$. It should be noted that this adaptive refinement of the grid is very effective[33] for detecting thresholds, because the optimal value function typically fails to be differentiable in these points, resulting in large local errors and consequently in a fine grid, see Figure 4.

## 3.2 Domains of Attraction

For the general model, i.e., with endogenous credit cost $H(k, B)$ as defined in (1) and/or restrictions of the type $B/k \leq c$, this algorithm unfortunately

[33]Actually, for the one–dimensional problem at hand it is possible to compute rather accurate approximations $v_h^\Gamma$ also with equidistributed grid points. In higher dimensions the computational advantage of adaptive gridding is much more obvious, see, e.g., the examples in Grüne (1997) or Grüne et al. (1999).

is not applicable. Even though in certain cases a HJB equation for a discrete time version of the problem is available, see appendix II, it is not clear whether the full discretization procedure described above leads to a valid and convergent approximation of $B^*$

Hence we propose a different approach for the solution of this problem, based on a set oriented method for the computation of domains of attraction. The method relies on the following observation: For a given compact interval[34] $[0, K]$ for the capital stock $k$ one sees that there exists a constant $c^* > 0$ such that $B^*(k) \le c^*$ for all $k \in [0, K]$. Hence, for $k \in [0, K]$ the condition $\sup_{t\ge 0} B(t) < \infty$ can be replaced by

$$\sup_{t\ge 0} B(t) < c^*.$$

Hence both this constraint and the constraint $B(t) \le ck(t)$ can be expressed as

$$B(t) \le d(k(t)) \text{ for all } t \ge 0$$

for some suitable function $d$. In other words, the set of all initial values $(k_0, B_0)$ for which this constraint is violated is given by

$$D = \left\{ (k_0, B_0) \;\middle|\; \begin{array}{l} \text{there exists } T > 0 \text{ such that } B(t(j)) \ge d(k(t(j))) \\ \text{for all } j \text{ and some } t(j) \in [0, T] \end{array} \right\}$$

and the curve $B^*(k)$ is exactly the lower boundary of $D$.

The set $D$ is what is called a robust domain of attraction of the set $A = \{(k, B) \in \mathbb{R}^2 \,|\, B \ge d(k)\}$ and we will now give a brief description of an algorithm for the computation of such sets, for details we refer to Grüne (2001) and Chapter 7 of Grüne (2002).

Again we consider a first order discrete time approximate model, now both for $k$ and $B$ given by the Euler discretization[35]

$$\begin{aligned} k_h(i+1) &= k_h(i) + h(j_i - \sigma k_h(i)) \\ B_h(i+1) &= B_h(i) + hH(k_h(i), B_h(i)) - hf(k_h(i), j_i) \end{aligned}$$

and abbreviate the right hand side by $\Psi(k, B, j)$. Just as above, for the space discretization we use a grid $\Gamma$, now covering a two–dimensional rectangular

[34]In any numerical method we must restrict ourselves to a compact computational domain, hence this restriction is natural in this context.

[35]We use the simple first order Euler scheme here in order to avoid too much technicality in our presentation. For higher order discrete time approximations see, e.g., Chapter 5 of Grüne (2002).

domain $[0, K] \times [0, \overline{B}]$. For each cell $C_l$, $l = 1, \dots, Q$ of the grid we use a collection of test points $x_l^i = (k_l^i, B_l^i)$, $i = 1, \dots, N$ in order to compute the set image

$$\Phi(C_l, \bar{\jmath}) = \bigcup_m C_m \text{ for all } m \in \{1, \dots, Q\} \text{ with } \Psi(k_l^i, B_l^i, j^i) \in C_m$$

where $\bar{\jmath} = (j^1, \dots, j^N)$ is a vector of $N$ control values associated to the $N$ test points. For a sequence $(\bar{\jmath}_i)$, $i = 0, 1, 2, \dots$ of such control vectors we can iterate the map $\Phi$ and we denote the resulting iterated map by $\Phi_i(C_l, (\bar{\jmath}_i))$. Now we can define the following three sets

$$D_\Gamma = \bigcup_m C_m \ \text{ for all } m \text{ with } \Phi_i(C_m, (\bar{\jmath}_i)) \subseteq A \text{ for all } (\bar{\jmath}_i) \text{ and some } i$$

$$B_\Gamma = \bigcup_m C_m \ \text{ for all } m \text{ with } \Phi_i(C_m, (\bar{\jmath}_i)) \cap A = \emptyset \text{ for some } (\bar{\jmath}_i) \text{ and some } i$$

$$E_\Gamma = \bigcup_m C_m \ \text{ for all } m \text{ with } C_m \not\subseteq D_\Gamma \text{ and } C_m \not\subseteq E_\Gamma$$

These sets are easily computed by a dynamic programming type iteration and under appropriate conditions it can be shown that the set $D_\Gamma$ approximates $D$, the set $B_\Gamma$ approximates $D^c$ (the complement of $D$) and the set $E_\Gamma$ approximates $\partial D$ (the boundary of $D$), which in our case is exactly the curve $B^*(k)$. It turns out that for obtaining more and more accurate approximations (with respect to the space discretizations) it is sufficient to increase the accuracy on the set $E_\Gamma$, i.e., to refine the cells $C_m \subseteq E_\Gamma$.

While the convergence analysis in the general case is rather complicated and depends on certain properties of $D$, for our problem we can use the fact that the boundary $\partial D$ is given by the curve $B^*(k)$ which is monotone increasing in $k$. Hence, if we use a rectangular grid, and choose the test points in each cell to be the 4 corners of this rectangular cell, we obtain that if a cell $C_m$ intersects both $D$ and $D^c$, then there exist test points $x_m^{k_1}$ and $x_m^{k_2}$ in this set such that $x_m^{k_1} \in D$ and $x_m^{k_2} \notin D$. Consequently, the iterated cell image $\Phi_i$ cannot be contained in $A$ for all $(\bar{\jmath}_i)$ (implying that $C_m \not\subseteq D_\Gamma$) but it intersects $A$ for each $(\bar{\jmath}_i)$ (implying that $C_m \not\subseteq B_\Gamma$). Thus, if a cell $C_m$ intersects both $D$ and $D^c$ then we obtain that $C_m \subseteq E_\Gamma$ which finally yields that the set $E_\Gamma$ always covers the boundary $\partial D$ and hence gives an approximation of the curve $B^*(k)$ whose accuracy is equal to the width of the set $E_\Gamma$.[36]

[36] Of course, this discussion concerns the spatial discretization error only. For the analysis of the full error we refer to the cited references.

Since the problem which is solved by this algorithm is not a classical optimal control problem (though it can be interpreted as an optimal control problem for the set valued dynamics) it is not possible to obtain optimal trajectories with respect to some given functional. However, it is not too difficult to prove that the boundary of a domain of attraction $D$ is weakly invariant (i.e., for an initial value on the boundary $\partial D$ we can always find trajectories that remain on $\partial D$ for all future times), provided it is a "proper" domain of attraction, i.e., its boundary does not intersect with the boundary of $A$. Due to this fact, for each initial value $(k, B) \in B_\Gamma$ (recall that this set forms our numerical approximation of the set $\{(k, B) | B \leq B^*(k)\}$ of subcritical initial values) we can compute a control sequence $j_i$ realizing a (discrete time) trajectory for which $B_h(i)$ remains bounded for all times $i \geq 0$ and for initial values on the upper part of the boundary $\partial B_\Gamma$ we can even expect to find trajectories that stay on this upper part of $\partial B_\Gamma$ for all future times, i.e., they are (up to the numerical error) of the form $(k(t), B^*(k(t))$. The limiting behavior of these trajectories can then be used for the detection of the thresholds and it turns out that this procedure yields very good results.

# 4 The Numerical Study

In this section we present numerical results obtained for our model for different choices of $H(k, B)$. Throughout this section we specify the model parameters as $\sigma = 0.15$, $a = 0.29$, $\alpha = 1.1$, $\beta = 2$, $\gamma = 0.3$ and $\theta = 0.1$. The remaining parameters are specified below. Unless otherwise noted we use $c(t) \equiv 0$ in our experiments.

As for the numerical parameters, all examples were computed for $k$ in the compact interval $[0, 2]$ with control range $j \in [0, 0.25]$.[37] For the algorithm from Section 3.1 we have used the numerical time step $h = 0.05$ and an initial grid with 39 nodes. The final adapted grid consisted of 130 nodes. The range of control values was discretized using 101 equidistributed values. For the algorithm from Section 3.2 we used the time step $h = 0.5$, in order to generate the discrete time model $\Psi$ we used a highly accurate extrapolation method. For this algorithm the range of control values was discretized using 51 equidistributed values. The domain covered by the grid was chosen to be $[0, 2] \times [0, 3]$ where the upper value $\overline{B} = 3$ coincides with the value $c^* = 3$ used in order to implement the restriction $\sup_{t \geq 0} B(t) < \infty$, cf. the discussion at the beginning of Section 3.2. The initial grid was chosen with 1024 cells, while the final adapted grids consisted of about 100000 up to 500000 cells, depending on the example. For this algorithm the figures below always show

[37]In all our experiments larger control ranges did not yield different results.

the set $E_\Gamma$ which approximates the creditworthiness curve $B^*(k)$. Recall that the width of this set gives an estimate for the spatial discretization error.

### 4.1 $H(k, B) = \theta B$

In this case we can use the optimal control algorithm from Section 3.1 in order to solve the discounted infinite horizon problem (2)–(4). Figure 3 shows the corresponding optimal value function representing the creditworthiness curve (upper graph) and the related optimal control in feedback form (lower graph).

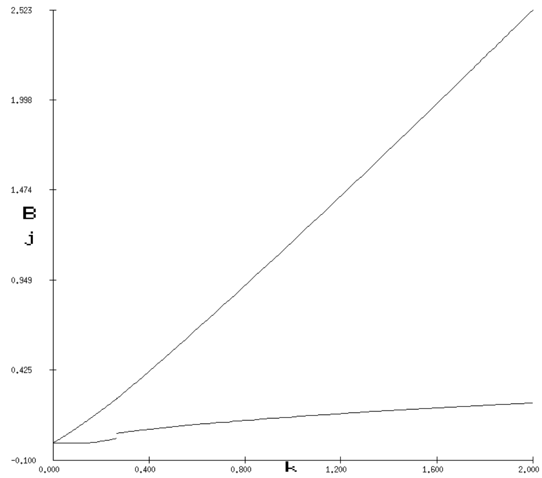


Figure 3: Optimal value function and optimal feedback law

The threshold $k^+ = 0.267$ is clearly visible in the optimal control law, which is discontinuous at this point. For initial values $k(0) < k^+$ the optimal trajectories tend to $k^* = 0$, for initial values $k(0) > k^+$ the optimal trajectories tend to the stable equilibrium $k^{**} = 0.996$.

Figure 4 shows the optimal feedback control in a neighborhood of the threshold. The discontinuity in the control variable is clearly observable. Investment to the left of $k^+$ is lower than $\sigma k$ and makes the capital stock shrinking whereas investment to the right of $k^+$ is larger than $\sigma k$ and increases the capital stock. At $k^+$ investment jumps.

In addition, in this figure the adaptively distributed grid points are shown. As mentioned in Section 3.1, the grid is in particular refined around the threshold, the reason for this is the (barely visible) kink in the optimal value function at this point, resulting in a non–differentiable value function and hence in large local errors.

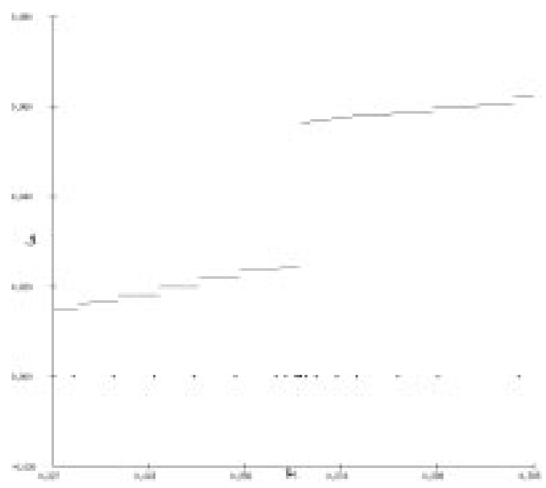

Figure 4: Optimal feedback law and distribution of grid points at threshold

## 4.2 $H(k, B)$ from (1)

For the more general model with

$$H\left(k(t), B(t)\right)) = \frac{\alpha_1}{(\alpha_2 + \frac{N(t)}{k(t)})^\mu} \theta B(t)$$

it is not possible to transform the problem into a standard infinite horizon optimal control problem, hence we will use the algorithm for the computation of domains of attractions from Section 3.2 and undertake experiments for different shapes of the credit cost function.

In this formula we specify $\mu = 2$. Taking into account that we want $\theta$ to be the risk–free interest rate, we obtain the condition $\alpha_1/(\alpha_2 + 1)^2 = 1$ and thus $\alpha_1 = (\alpha_2 + 1)^2$. Note that for $\alpha_2 \to \infty$ and $0 \leq B \leq k$ one obtains $H(k, B) = \theta B$, i.e., the model from the previous section. In order to compare these two models we use the formula $H(k, B) = \frac{\alpha_1}{\alpha_2^2} \theta B$ for $B > k$.[38]

Figure 5 shows the respective creditworthiness curves $B^*$ under the condition $\sup_{t \geq 0} B(t) < \infty$ for $\alpha_2 = 100,\ 10,\ 1,\ \sqrt{2} - 1$ (from top to bottom) and the corresponding $\alpha_1 = (\alpha_2 + 1)^2$.

For $\alpha_2 = 100$ the trajectories on the curve $B^*$ show almost the same behavior as the optimal trajectories in the previous section: There exists a threshold (now at $k^+ = 0.32$) and two stable equilibria at $k^* = 0$ and $k^{**} = 0.99$. For the smaller values of $\alpha_2$ there is no threshold observable

[38] For small values of $\alpha_2$ it turns out that the creditworthiness curve satisfies $B^*(k) < k$, hence this change of the formula has no effect on $B^*$.

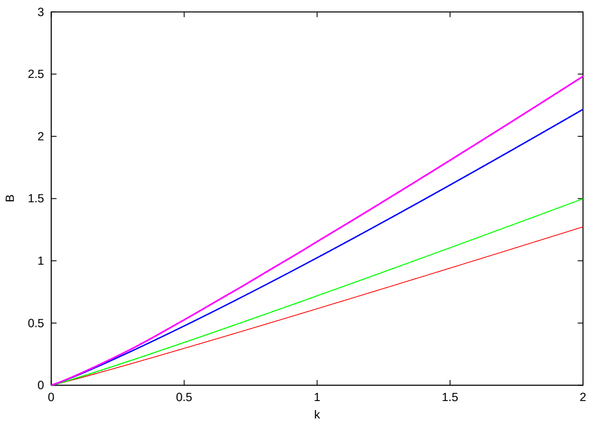


Figure 5: Creditworthiness curve $B^*$ for different $\alpha_2$

and there exists only one equilibrium at $k^* = 0$ which is stable. Further simulations have revealed that for decreasing values of $\alpha_2 \leq 100$ the threshold value $k^+$ increases (i.e., moves to the right) and the stable equilibrium $k^{**}$ decreases (i.e., moves to the left), until they meet at about $\alpha_2 = 31$. For all smaller values of $\alpha_2$ there exists just one equilibrium at $k^* = 0$ which is stable. The reason for this behavior lies in the fact that for decreasing $\alpha_2$ credit becomes more expensive, hence for small $\alpha_2$ it is no longer optimal to borrow large amounts and to increase the capital stock, instead it is optimal to shrink the capital stock and to reduce the stock of debt $B(t)$ to 0. Thus, with small $\alpha_2$ and thus large borrowing cost it is for any initial capital stock optimal to shrink the capital stock.

### 4.3 $H(k, B) = \theta B^\kappa$

In this section we use the algorithm from Section 3.2 and repeat the computations from the previous section for the credit cost function $H(k, B) = \theta B^\kappa$.[39] Figure 6 shows the respective curves for $\kappa = 1,\ 1.05,\ 1.25,\ 2$ (from top to bottom at the right boundary of the diagram).

For $\kappa = 1$ this is exactly the optimal value function from Figure 3, while for increasing $\kappa$ the values of $B^*(k)$ increase for small $k$ and decrease for larger $k$. This is due to the fact that for increasing $\kappa$ and $B > 1$ the credit cost increases whereas for increasing $\kappa$ and $B < 1$ the credit cost decreases,

[39]Note that this type of interest cost where the interest payment is convex in the agent's debt is frequently posited in the literature, see for example Bhandary, Haque and Turnovsky (1990)

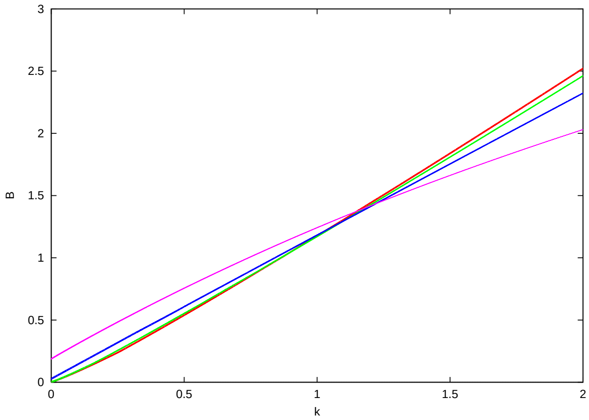


Figure 6: Creditworthiness curve $B^*$ for different $\kappa$

hence it is becomes possible to borrow larger amounts with small capital stock. It should also be noted that in all cases we have $B^*(0) = 0$, however, for larger $\kappa$ the creditworthiness curve becomes discontinuous at 0, i.e., $\lim_{k\to 0, k>0} B^*(k) > 0$. Again, this is due to the fact that for larger $\kappa$ the credit cost is small for small $B$.

This behavior of the creditworthiness is also reflected in the thresholds. For $\kappa = 1.05$ the qualitative behavior of the trajectories is just as in the case $\kappa = 1$: there exists a threshold $k^+ > 0$ in where the control is discontinuous and two stable equilibria $k^* = 0$ and $k^{**} > k^+$. For increasing values of $\kappa$ the threshold $k^+$ moves to the left until it hits 0 and vanishes; for $\kappa = 1.25$ and $\kappa = 2$ it has already vanished, implying that all trajectories with initial values on the $B^*(k)$ curve converge to a strictly positive stable equilibrium $k^{**}$. Note that this behavior is just the opposite to what happens for $H(k, B)$ from (1) in Section 4.2 for decreasing $\alpha_2$, which is due to the fact that the credit cost for small $B$ behaves the opposite way. We can thus observe that for both type of credit cost functions the asset price and thus the creditworthiness is affected, yet for the convex credit cost the asset price decreases (relative to the a credit cost with risk-free rate) only for large capital stock and borrowing. This rather unexpected behavior of the convex credit cost function – increasing creditworthiness with small capital stock and borrowing – makes the first formulation of endogenous credit cost, through equ. (1), a more reasonable approach to pursue.

## 4.4 Debt Ceilings

For $H(k, B)$ from (1) with $\alpha_2 = 100$ and for $H(k, B) = \theta B^\kappa$ with $\kappa = 2$ we now test a different criterion for the debt ceiling: instead of $\sup_{t \geq 0} B(t) < \infty$ we impose the restriction $B(t)/k(t) \leq c$ for some constant $c$. Again we use the algorithm from Section 3.2. Figure 7 shows the respective curves for the restriction $\sup_{t \geq 0} B(t) < \infty$ and for the ratio–restriction with $c = 1.2$ and $c = 0.6$ (from top to bottom). In addition, the restriction curves $B = ck$ are shown with dots for $c = 1.2$ and $c = 0.6$.

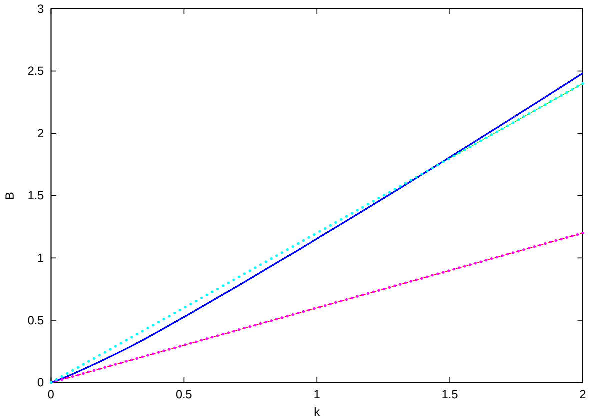


Figure 7: Creditworthiness curve $B^*$ for different ceilings, $H(k, B)$ from (1)

For $c = 0.6$ the creditworthiness curve $B^*(k)$ coincides with the "restriction curve" $B(k) = ck$; in this case the curve $(k, B^*(k))$ is no longer invariant for the dynamics,[40] i.e., each trajectory $B(t)$ with $B(t) \leq B^*(k(t))$ leaves the curve $(k, B^*(k))$ and eventually $B(t)$ tends to $-\infty$. For $c = 1.2$[41] the curves $B^*(k)$ and $B = ck$ coincide only for $k \geq 1.46$. Here one observes the same equilibria $k^*$ and $k^{**}$ and threshold $k^+$ as for the sup–restriction (cf. Section 4.2), however, in addition to these here a new threshold appears at $k^{++} = 1.54$. For initial values $(k, B^*(k))$ with $k^+ < k < k^{++}$ the trajectory tends to the stable equilibrium $k^{**}$, while for $k > k^{++}$ the behavior is the same as for $c = 0.6$, i.e., the corresponding trajectories leave the curve $B^*(k)$

[40] As mentioned in Section 3.2 the boundary of the domain of attraction $D = \{(k, B) \,|\, B > B^*(k)\}$ of the set $A = \{(k, B) \,|\, B \geq ck\}$ is invariant for the trajectories, provided $\partial D$ and $\partial A$ do not intersect. Here we have $B^*(k) = ck$, i.e., the boundaries $\partial D$ and $\partial A$ do intersect (they even coincide) and consequently we cannot expect invariance.

[41] This curve is difficult to see because it coincides with the curve for $\sup_{t \geq 0} B(t) < \infty$ for small $k$ and with the restriction curve $B = ck$ for large $k$.

and eventually $B(t)$ tends to zero.[42]

We have repeated these computations for $H(k, B) = \theta B^\kappa$ and $\kappa = 2$. Figure 8 shows the respective curves for the restriction $\sup_{t\geq 0} B(t) < \infty$ and for the ratio–restriction with $c = 1.2$ and $c = 0.6$ (from top to bottom). In addition, the restriction curves $B = ck$ are shown with dots for $c = 1.2$ and $c = 0.6$.

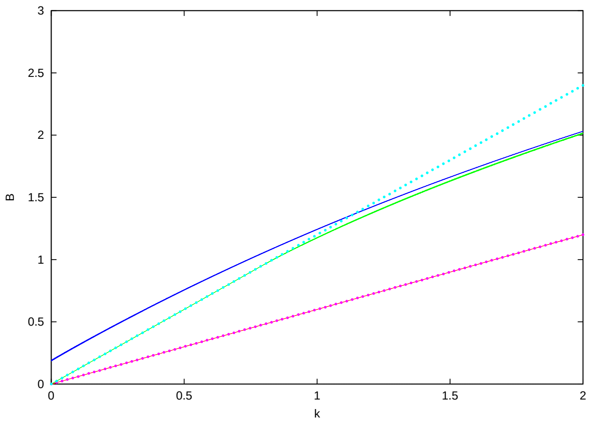


Figure 8: Creditworthiness curve $B^*$ for different ceilings, $H(k, B) = \theta B^2$

Again, for the sup–restriction the curve is discontinuous at $k = 0$. Just as for $H(k, B)$ from (1), for $c = 0.6$ the creditworthiness curve $B^*(k)$ coincides with the "restriction curve" $B(k) = ck$ and the curve $(k, B^*(k))$ is no longer invariant for the dynamics and for each trajectory with $B(t) \leq B^*(k(t))$ the second component $B(t)$ diverges to $-\infty$. For $c = 1.2$ the curves $B^*(k)$ and $B(k) = ck$ coincide for $k \in [0, k^{**}]$, where $k^{**} = 0.8$ is exactly the stable equilibrium for all trajectories starting on the curve $B^*(k)$.

## 4.5 Consumption

We finally investigate—again for $H(k, B)$ from (1) with $\alpha_2 = 100$ and for $H(k, B) = \theta B^\kappa$ with $\kappa = 2$—the case when the agent's net income $f$ is reduced by a constant consumption $c(t) \equiv \eta$. In this case the creditworthiness curve $B^*$ may become negative. This means that there is an initial level of capital stock required—the level of capital stock where the creditworthiness curve becomes positive—that supports the consumption path $c(t) = \eta$. All levels of capital stock below this point do not support the consumption path

[42]The simulation are halted at zero, but we would like to report if continued the $B(t)$ curve becomes negative and tends to $-\infty$.

$c(t) = \eta$. We have to specify the dynamics for $B(t) < 0$ which we choose to be $\dot{B}(t) = \theta B(t) - f$.

Note that for the linear model from Section 4.1 subtracting a constant $\eta$ from $f$ simply results in an optimal value function $V_\eta = V - \theta\eta$. Since for $\alpha_2 = 100$ the creditworthiness $B^*$ for $H(k, B)$ from (1) is very close to the model from Section 4.1 we would expect much the same behavior. Figure 9 shows that this is exactly what happens here.

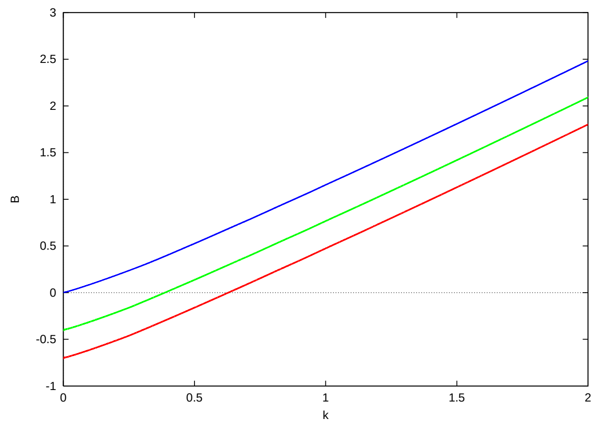


Figure 9: Creditworthiness curve $B^*$ for different $\eta$, $H(k, B)$ from (1)

The fact that the curves here are just shifted is also reflected in the stable equilibria and the threshold, which do not change their positions. In particular, the dynamical behavior does not depend on the consumption rate.[43]

Again, we have repeated our computations with $H(k, B) = \theta B^\kappa$ and $\kappa = 2$. Figure 10 shows the respective curves for $\eta = 0$, 0.04, 0.07, again from top to bottom.

In this nonlinear model here with $\kappa = 2$ the effect of $\eta$ is truly nonlinear, as it is easily seen from the figure, because the difference between the curves at the right boundary of the diagram is much smaller than on the left boundary. However, again dynamical behavior does not change: just as for $\eta = 0$, for both considered positive values of $\eta$ the resulting trajectories converge to a stable equilibrium $k^{**} > 0$ and no thresholds could be observed. The position of the equilibria $k^{**}$ depends on $\eta$, more precisely $k^{**}$ increases, i.e., moves to the right as $\eta$ increases. As concerns the behavior of the creditworthiness curve for our two different credit cost functions we here also might conclude that the equ. (1) represents a more reasonable approach.

[43]Note that this is an obvious case where our separation theorem of appendix I is valid.

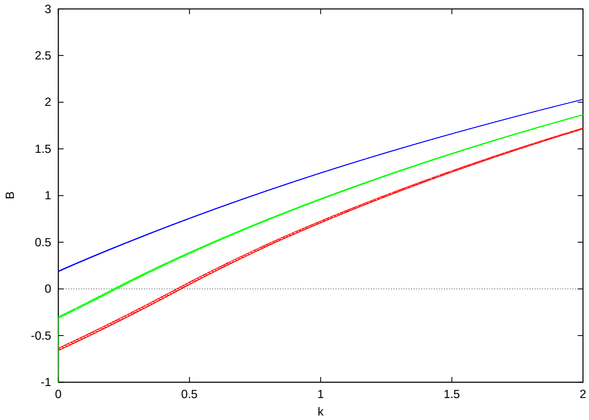


Figure 10: Creditworthiness curve $B^*$ for different $\eta$, $H(k, B) = \theta B^2$

# 5 Conclusions

We know that for a large number of dynamic models there may exist multiple equilibria and thresholds separating the global dynamics. These models give rise to history dependence and indeterminacy. Indeterminacy occurs if in the neighborhood of a steady state equilibrium it is preferable to move either to the higher or lower equilibria. In the literature mostly the thresholds separating the different domains of attraction have not been located. In this paper we apply dynamic programming with flexible grid size that proves to be useful to compute thresholds and global dynamics in models with multiple steady state equilibria. We study a credit market model where the agents can borrow from credit market for investment and where the credit cost may be state dependent and the agents may face debt ceilings. Using those methods we can compute the present value borrowing constraint and thus the region in which the borrower remains creditworthy. We apply dynamic programming to detect thresholds, domains of attraction, jumps in the policy function and suboptimal equilibria in a stylized model which is nested in utility maximization but can be studied independently. We also explore the impact of different shapes of the credit cost function, debt ceilings and given consumption paths on the equity price and creditworthiness, thresholds and domains of attraction. If the interest rate is a constant the Hamiltonian equation can be applied as well.

# 6 Appendices

## 6.1 Appendix I: The Optimal Consumption of Value.

In this appendix we briefly want to demonstrate of how our model of section 2 is nested in a more general model with utility functional. The essential feature of the more general model is that the study of the problem of creditworthiness and the present value borrowing constraints can be separated from the consumption problem. Here it is shown for a simple model.

We start with the more general problem where both $c$ and $j$ are control variables.[44] In order to optimize the utility functional $\hat{U}(c) := \int_0^\infty e^{-\theta t} U(c(t))dt$ we have to solve

$$\begin{cases} \max_{c,j} \int_0^\infty e^{-\theta t} U(c(t))dt, \\ \dot{k} = i(k,j); & k(0) = k_0 \\ \dot{B} = \theta B + c - f(k,j); & B(0) = B_0, \\ \lim_{t\to\infty} e^{-\theta t} B(t) = 0 \end{cases} \qquad (P_F(k_0, B_0))$$

where $j$ and $c$ are control variables and where $U$ is a strictly monotone increasing instantaneous utility function.

This problem can be separated into two optimization problems.

**1)** Solve the investment problem for $k_0 \in R_+$

$$\begin{cases} \max_j \int_0^\infty e^{-\theta s} f(k(s), j(s))ds, \\ \dot{k} = i(k,j); & k(0) = k_0. \end{cases} \qquad (P_I(k_0))$$

By using an optimal solution $(k^*(t), j^*(t))$ of $(P_I(k_0))$ we define the wealth of the economy at time $t = 0$ by $\omega^* := \int_0^\infty e^{\theta s} f(k^*(s), j^*(s))ds - B_0$.

**2)** Solve the problem of optimal consumption for given $(k, j, B_0)$, and $\omega \in R_+$

$$\begin{cases} \max_{c, \overline{c} \le \omega} \int_0^\infty U(c(s))e^{-\theta s} ds, \\ \dot{B} = \theta B + c - f(k,j), & B(0) = B_0, \\ \lim_{t\to\infty} e^{-\theta t} B(t) = 0 \end{cases} \qquad (P_C(\omega, k, j, B_0))$$

where $\overline{c} := \int_0^\infty e^{-\theta s} c(s)ds$. We denote a solution of $(P_C(\omega, k, j, B_0))$ by $(B^*(t), c^*(t))$.

[44] Note that this allows the use of external resources.

In Sieveking and Semmler (1998) it is shown that $(\widetilde{k}, \widetilde{B}, \widetilde{j}, \widetilde{c})$ is an optimal solution of $(P_F(k_0, B_0))$ if and only if $(\hat{k}, \overline{j})$ is an optimal solution of $(P_I(k_0))$ and $(\widetilde{B}, \overline{c})$ is an optimal solution of $(P_C((\widetilde{\omega}, k, j, B_0))$, where $\overline{\omega} := \int_0^\infty e^{-\theta s} f((\widetilde{k}(s), (\widetilde{j}(s))ds - b_0$.

A further analytical treatment why and under what conditions such problems can be separated as well as an example for the case of a utility function of CRRA type are given in Sieveking and Semmler (1998).

## 6.2 Appendix II: The Generalized Model and the HJB-Equation

Here we study a more general version of the credit market model discussed in section 2 in the sense that we refer to a multi-variable version of the model as well as to the relationship between discrete time and continuous time models.[45] We show in the general case that $B^*(k)$ fulfills the HJB-optimality equation.
Suppose capital $k(t+1)$ at time $t+1$ and debt $B(t+1)$ at time $t+1$ are determined by $k(t)$ and $B(t)$ and investment rate $j(t)$ through

$$k(t+1) = g\left(k(t), j(t)\right), \quad k(0) = k \tag{A7}$$

$$B(t+1) = H\left(k(t), B(t)\right) - f\left(k(t), j(t)\right), \quad B(0) = B \tag{A8}$$

$H(k, B)$ is the credit cost which we allow to depend on capital $k$, and $B$, $g(k, j)$ is the growth of capital due to investment $j$ and $f(k, j)$ the net income from capital stock $k$ and investment rate $j$.

More precise assumptions on $g, H, f$ and there domains of definition will be given below. We ask, if for a given pair $(k, B)$ is it possible to choose a sequence of investments $j(0), j(i)$, in such a way that the corresponding solution $t \to (k(t), B(t))$ of (A7) and (A8) satisfies

$$\sup_{t \geq 0} B(t) < \infty$$

[45] This relation is only sketched here. A detailed study of it can be found in Semmler and Sieveking (1999) where also an estimate of error bounds of the discretized version of the continuous time model is given.

If so we call $B$ subcritical for $k$. The supremum of all those $B$ which are subcritical for $k$ is denoted by $B^*(k)$. We propose to call $B^*(k)$ creditworthiness of $k$. The function $k \to v(k) = B^*(k)$ will be shown to satisfy the following HJB-equation or optimality equation.[46]

$$H\left(k, v(k)\right) = \sup_j \left[f(k,j) + v\left(g(k,j)\right)\right], \qquad k \in K \tag{A9}$$

Our assumptions below imply that the equation

$$H(k, C) = \sup_j \left[f(k,j) + v\left(g(k,j)\right)\right]$$

has a unique solution $C = C(v, k)$ for every capital stock $k$ and every continuous real valued function $v$ on capital stock $k$. Define the operator $T$ by

$$Tv(k) := C(v, k)$$

The assumptions stated below permit to demonstrate:

**Theorem 1**

(i) The HJB-equation (A9) admits a unique bounded continuous solution $B^*$, called creditworthiness.

(ii) If $v_0 = 0$ and $v_n$ is defined recursively by $v_{n+1} = Tv_n$ then for all $n$, $v_n \leq v_{n+1}$ and $\lim_n v_n = B^*$.

(iii) Suppose $\inf_{k \in K} \frac{H(k,B)}{B} > c$ for some $c > 1$. Then for every solution $(k(t), B(t), j(t))\,(t = 0, 1, 2, ...)$ if initially $B(0) > B^*(k)$ then for large $t$, $B(t) > c^t$; if, however, $B(0) < B^*(k)$ then $B(t) < -c^t$ for large $t$.

**Assumptions on $g, H, f$**

$A_1$: $K$ and $J$ are compact spaces and $g: K \times J \to K$ is continuous;

$A_2$: $f: K \times J \to \mathbb{R}$ is continuous, $\sup_j f(k,j) \geq 0$ for all $k \in K$;

$A_3$: $H: \mathbb{R} \times K \to \mathbb{R}$ is continuous, $B \to H(k,B)$ is differentiable and $\frac{\partial H(k,B)}{\partial B} > 1$, $H(k,0) = 0$ for all $k \in K$.

[46]Note that the subsequent equ. (A9) can be used as the basis for a dynamic programming algorithm to solve our debt control problem. If we take our simplifiction $H(k,B) = h(B)$ then we can invert the function $h(B)$ and obtain a dynamic programming algorithm in $B^*$.

**Remark** The state space $K$ of possible capital (resource) stocks, $k$, as well as the space $J$ of admissible investment rates is not bounded (or compact) in many models - such as the one treated in section 3. It makes sense however to assume that for a given initial state $(k, B)$ and with respect to a specific problem like debt control (HJB-equation) there is no loss in generality to restrict $k$ and $j$ respectively to some compact subspace. It is plausible that with a bounded investment rate:$\|j\| \leq c$ only a bounded set of stocks $k$ is reachable from some initial stock. In our continuous time model (in sections 3) we impose restriction $\|j\| \leq c =$ const.

**Proof of theorem 1 (i and ii)** Investment $j \in J$ applied in state $(k, B)$ produces a subsequent state

$$\{g(k, j), H(k, B) - f(k, j)\}.$$

The debt level $B$ is subcritical for $k$ iff for some $j \in J$

$$B^* (g(k, j)) + f(k, j) \geq H(k, B))$$

which implies

$$\sup_j [B^*(g(k, j)) + f(k, j)] \geq H(k, B^*(k))$$

If on the other hand in the above equation would hold, then for some $B > B^*(k)$ and some $j \in J$

$$B^* (g(k, j)) + f(k, j) \geq H(k, B)$$

This, however, implies $B < B^*(k)$, a contradiction. Therefore, $B^*$ satisfies the HJB-equation.

$B^*$ also is bounded. Let $F = \sup \{f(k, j) \,|k, j \in K \times J\}$. Since $K \times J$ is compact and $f$ is continuous, $F$ is finite. By assumption $A_3$

$$H(k, B) - f(k, j) \geq H(k, B) - F \geq cB$$

for sufficiently large B and some constant $c > 1$. Hence if $B(0)$ is large enough any solution $t \to (k(t), B(t))$ of (A7), (A8) satisfies

$$B(t) \geq c^t B(0)$$

which shows that $B^*$ is bounded. We now check that $T$ is a Lipschitz operator on the space of bounded functions $v : K \to \mathbb{R}$ with

$$\|v\| = \sup \{|v(k)| \mid k \in K\}$$

To do so let $v_1, v_2 : K \to R$ be bounded and

$$\sup_j [f(k, j) + v_1 (g(k, j))] \leq f (k, j(k)) + v_1 (g (k, j(k))) + \varepsilon$$

for some $\varepsilon > 0$. Then

$$H (Tv_1(k), k) - H (Tv_2(k), k) \leq v_1 (g(k, j(k))) - v_2 (g(k, j(k))) + \varepsilon \leq \|v_{1-}v_2\| + \varepsilon$$

Due to A$_3$ $|H(k, B_1) - H(k, B_2)| \geq \frac{1}{l} |B_1 - B_2|$ for some constant $l \in (0, 1)$ independently of $k$ and therefore

$$Tv_1(k) - Tv_2(k) \leq l \|v_{1-}v_2\| + l\varepsilon$$

Since $\varepsilon > 0$ and $k$ was arbitrary $\|Tv_1 - Tv_2\| \leq l \|v_1 - v_2\|$. This shows that $T$ is a Lipschitz transformation of the space of bounded functions $K \to \mathbb{R}$. Now if $v_0 = 0$, then since $\sup_j f(k, j) \geq 0$

$$v_0 = Tv_0 \geq v_0 \quad \text{and therefore}$$

$$v_{n+1} = Tv_n \geq v_n \quad \text{for all } n$$

Also, if $v$ is continuous, then so is $Tv$ since $K \times J$ is compact. Therefore $B^* = \lim_n v_n$ is continuous, this proves (ii).

**Proof of Theorem 1 (iii)** Suppose $B > B_1 > B^*(k)$ and let $t \to (k(t), B(t), j(t))$ solve (A7) and (A8) with $B = B(0), \quad k = k(0)$. Compare this to the solution $t \to (k_1(t), B_1(t), j(t))$ of (A7) and (A8) with $k_1(0) = k, \quad B_1(0) = B_1$ and the same investment.

$$B(t) - B_1(t) \geq c\,(B(t-1) - B_1(t-1)) \geq ... \geq c^t\,(B(0) - B_1(0))$$
$$B(t) \geq c^t\,(B(0) - B_1(0)) + B_1(t) \geq c^t\,(B(0) - B_1(0))$$

Similarly, if $B^*(k) > B_1 > B$ we find

$$B_1(t) - B(t) \geq c^t\,(B_1(0) - B(0))$$

and

$$-c^t\,(B_1(0) - B(0)) + B_1(t) \geq B(t)$$

As $B_1(\cdot)$ is bounded this proves 1 (iii), that is

$$B(t) \geq c_1^t \text{ for large } t, \quad c_1 < c$$

in the first case and

$$B(t) \leq -c_1^t \text{ for large } t, \quad c_1 < c$$

in the second case.

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