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Trading Bargaining Weights

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Abstract

We consider a model, in which two agents are engaged in two separate bargaining problems. We introduce a notion of bargaining power, which is basically given by asymmetric versions of the Perles-Maschler bargaining solution. Thereby, we view bargaining power as ordinary goods that can be traded in an exchange economy. With equal initial endowment of bargaining power there exists a Walrasian equilibrium in this exchange economy. The utility allocation in equilibrium coincides with the Perles-Maschler bargaining solution of the aggregate bargaining problem. Equilibrium prices are given by the standard traveling times of the two bargaining problems (see Perles-Maschler (1981)). As a version of the Second Fundamental Welfare Theorem, we show that any efficient allocation of bargaining power can be supported by this price system. Therefore, any asymmetric version of the PM solution can be achieved via suitably adjusted initial endowments.

Keywords: Bargaining Power, Perles-Maschler Solution, Equilibrium Model
JEL Classification: C78, C62, D51, D63

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1 Introduction

In this paper we consider a model for two agents who are involved in two separate bargaining problems A and B. The term "separate" reflects the fact that agents have to agree on two separate points. We will assume that utility scales that are used are compatible with each other in the sense that the total utility for agent i from an agreement is the sum of utilities he gets in A and B. It is well known that in case the bargaining problems are treated separately, the sum of agreements will not be Pareto optimal. Hence, agents should take both problems into account during negotiation. Clearly, this setup calls for superadditive solution concepts, which in particular assures that there is no dispute whether to treat the bargaining problems one after another or to connect them.

Perles & Maschler (1981b) introduced a superadditive bargaining solution for two person bargaining games. Thereby, they did not only discuss the symmetric version of this solution. Application of an asymmetric bargaining solution reflects that the conflicting parties may not have equal bargaining power. Such an unbalancedness of power may, for example, come from hierarchical structures, informational advantages or experience in bargaining situations. It is undisputed that there is a correlation between the outcome of an asymmetric bargaining solution and the effects of exerting bargaining power.

There are few references in the literature concerning superadditive solutions in the bargaining context. Definitely, one reason for this is that the superadditivity axiom is incompatible with the presence of more than two players. A counterexample is given in Perles (1982). However, Calvo & Gutherrez (1994) extend the construction of the PM solution to n-person bargaining games, but their solution of course loses the superadditivity property.

We exploit this relation to define a notion of bargaining power. Our idea now is to parameterize the set of (Pareto optimal) agreements in A and B, and let then agents agree on parameters. We identify such a parameter with a distribution of bargaining power. To reach an agreement, we treat parameters as "ordinary goods", initially endow agents with exactly half of each good and let them exchange these goods as in an ordinary exchange economy with two agents and two goods. We will show that an equilibrium allocation always exists. This specific pair of parameters in effect determines the final solution, which is in particular Pareto efficient (in the aggregate bargaining problem). It turns out that equilibrium prices are determined by standard traveling times of the underlying bargaining problems. In case that agents are equally endowed with "bargaining power in A and B", the utility allocation arising in equilibrium will be the symmetric version.

of the Perles-Maschler solution. As a version of the Second Welfare Theorem, we show that this equilibrium price system serves to support any efficient allocation of bargaining power. Moreover, asymmetric versions of the Perles-Maschler solution can be established by suitably adjusted initial endowments.

The organization of the paper is as follows: Section 2 provides the bargaining theoretic framework and reviews the definition of the Perles-Maschler solution. In Section 3 the basic model is introduced and discussed. Sections 4 and 5 discuss the main results of the paper on existence and uniqueness of equilibria in the exchange economy and the resulting utility allocations in the aggregate bargaining problem. Three examples are given in Section 6. Section 7 concludes.

2 Basic Definitions and Notation

An (axiomatic) **bargaining problem** for two persons is a pair $V := (U, \underline{x})$ consisting of a closed and convex set $U \subseteq \mathbb{R}^2$ describing feasible allocations of utilities and a vector $\underline{x} \in U$ that reflects the agents' utilities, when no agreement can be reached. Throughout the paper we will make the following assumption:

Assumption 1
For each bargaining problem $V = (U, \underline{x})$ the set U is comprehensive (i.e. $x \in U$ and $y \leq x$ implies $y \in U$). The set of individual rational allocations $U_{\underline{x}} := \{u \in U \,|\, u \geq \underline{x}\}$ is bounded (hence compact). Moreover, each U is generated by its individual rational utility allocations, i.e. $U = compH(U_{\underline{x}})$, where $compH(\cdot)$ denotes the comprehensive hull operator.

Let $\mathcal{U}^c$ denote the class of bargaining problems that satisfy Assumption 1 and denote by $\mathcal{U}_0^c$ the subclass in $\mathcal{U}^c$ that consists of bargaining problems having the common disagreement point $\underline{x} = 0$.

A mapping $\varphi : \mathcal{U}_0^c \longrightarrow \mathbb{R}^2$ is said to satisfy the symmetry axiom (SYM), if $\pi(\varphi(V)) = \varphi(\pi(V))$, where $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the function that "changes coordinates", i.e. $\pi(x_1, x_2) := (x_2, x_1)$[1]. Such a mapping φ is said to be covariant with (affine) linear transformations of utility (COV), if for each (affine) linear function $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the condition

[1]Here π applied to a bargaining problem in $\mathcal{U}^c$ yields a bargaining problem in $\mathcal{U}^c$ with exchanged roles of the agents.

$\varphi(L(V)) = L(\varphi(V))$ holds for each $V \in \mathcal{U}^c$. Since the class $\mathcal{U}_0^c$ is not invariant under affine transformations, we restrict transformations to be linear. But the analysis in the paper is valid on $\mathcal{U}^c$ and affine transformations.

A **bargaining solution (b.s.)** on $\mathcal{U}_0^c$ is a mapping $\varphi : \mathcal{U}_0^c \longrightarrow \mathbb{R}^2$ such that for each $V = (U, 0) \in \mathcal{U}_0^c$ the solution $\varphi(V)$ is feasible, individually rational (IR) and Pareto optimal (PO) in U. Moreover, φ has to satisfy SYM and COV.

We will use the following notation in the description of a bargaining problem. Fix $V = (U, 0) \in \mathcal{U}_0^c$. Let $\tau_i(V) = \tau_i := \max\{t \mid t e^i \in U\}$ denote the maximal possible utility for agent i among individual rational utility allocations.[2] By ∂U we denote the Pareto boundary of $U_{\underline{x}}$. W.l.o.g. we will also assume that ∂U does not contain line segments parallel to the axes. With such restrictions, we can describe ∂U as the graph of a function $C : [0, \tau_1] \longrightarrow \mathbb{R}$ with $C(t) := \max\{z \in \mathbb{R} \mid (t, z) \in U\}$. Due to the convexity assumption for bargaining problems the function C is continuous, strictly decreasing, concave and it is differentiable at all but at most countably many points $t \in [0, \tau_1]$. For this reason, we may use $C'(\cdot)$ to denote the first derivative of C, taking into account that this is almost everywhere well-defined.

A **parametrization** of V is a continuous mapping $x : [a, b] \longrightarrow \partial U$ with $a, b \in \mathbb{R}, a \leq b$ such that $x(a) = (0, \tau_2), x(b) = (\tau_1, 0)$ and $x_1(\cdot)$ is non-decreasing (which implies $x_2(\cdot)$ is non-increasing).

The mapping C itself generates a canonic parametrization $x^C : [0, \tau_1] \longrightarrow \mathbb{R}^2$ with $x^C(t) := (t, C(t))$.

For the canonic parametrization[3], we define a function $f : [0, \tau_1] \longrightarrow \mathbb{R}$ by

$$(1) \qquad f(t) := \int_0^t \sqrt{-\dot{x}_1^C(s) \cdot \dot{x}_2^C(s)}\, ds = \int_0^t \sqrt{-C'(s)}\, ds$$

(where $\dot{x}_i^C(\cdot)$ denotes the derivative of $x_i^C(\cdot)$).

The **Perles-Maschler bargaining solution** μ on $\mathcal{U}_0^c$ (hereafter PM solution) is now determined as follows: First, compute the real number $\bar{T} = \bar{T}(V)$ that satisfies

$$(2) \qquad \int_0^{\bar{T}} \sqrt{-C'(s)}\, ds = \int_{\bar{T}}^{\tau_1} \sqrt{-C'(s)}\, ds \qquad \text{i.e. } f(\bar{T}) = \frac{f(\tau_1)}{2}.$$

Second, the PM solution is defined by $\mu(V) := x^C(\bar{T}(V))$. In fact, μ is well defined as it does not depend on the parametrization used in (1) and (2) (for details see Rosenmüller

[2]By e^i we denote the i-th unit vector in $\mathbb{R}^2$.

[3]In fact, we can take any parametrization.

(2000) or Perles & Maschler (1981b)).

The function f is continuous and strictly increasing, because the integrand is strictly positive except for $s = 0$ (here $C'(0)$ might be zero). Denote by $\bar{b}(V) = \bar{b} := f(\tau_1)$ the largest possible value that f attains. One can show that $\bar{b}$ does not depend on the parametrization chosen in (1). Hence, f is a bijection from the interval $[0, \tau_1]$ onto the interval $[0, \bar{b}]$. By $h := f^{-1}$ we denote its inverse mapping. With the mapping h and the canonic parametrization x^C, we get a new parametrization $\xi : [0, \bar{b}] \longrightarrow \mathbb{R}^2$ with $\xi := x^C \circ h$. In fact, the mapping $h : [0, \bar{b}] \longrightarrow [0, \tau_1]$ describes the transformation of parameters when changing the parametrization from x^C to ξ.

A straightforward computation now yields

$$
\begin{aligned}
\dot{\xi}_1(s) \cdot \dot{\xi}_2(s) &= \dot{x_1}^C(h(s)) \cdot h'(s) \cdot \dot{x_2}^C(h(s)) \cdot h'(s) \\
&= C'(h(s)) \cdot (h'(s))^2 \qquad (3) \\
&= C'(h(s)) \cdot \left(\frac{1}{f'(h(s))} \right)^2 = -1
\end{aligned}
$$

Hence, computing the PM solution with the parametrization ξ, we get

$$
\bar{T} = \int_0^{\bar{T}} \sqrt{-\dot{\xi}_1(s) \cdot \dot{\xi}_2(s)}\; ds = \int_{\bar{T}}^{\bar{b}} \sqrt{-\dot{\xi}_1(s) \cdot \dot{\xi}_2(s)}\; ds = \bar{b} - \bar{T}
$$

and therefore obtain $\mu(V) = \xi(\bar{b}/2)$.

Let us pause for an interpretation. As Perles & Maschler (1981b) argue one can view the PM solution as follows. There are two particles moving along the Pareto frontier. We will associate each particle with one player. Player 1's particle starts at $(0, \tau_2)$ whereas player 2's particle starts at $(\tau_1, 0)$. The interval $[0, \bar{b}]$ reflects time. They "move" on the boundary according to the parametrization ξ, i.e. the product of coordinate velocities equals -1. In view of this, we detect $\bar{b}(V)$ as the time needed to traverse the whole boundary. We therefore call $\bar{b}$ the *standard traveling time*. Hence, after time $s \in [0, \bar{b}]$ player 1's particle is located at $\xi(s)$, whereas player 2's particle stands at $\xi(\bar{b} - s)$. At time $\bar{b}/2$ the two particles meet at the PM solution.

It is well known that the PM solution can be axiomatized using the superadditivity axiom. A bargaining solution φ on $\mathcal{U}_0^c$ is said to be **superadditive** (SUPA), if it satisfies $\varphi(V^1) + \varphi(V^2) \leq \varphi(V^1 + V^2)$ for any $V^1, V^2 \in \mathcal{U}_0^C$. Then the PM solution is the only continuous[4] bargaining solution that satisfies PO, IR, COV, SYM and SUPA (see Perles & Maschler (1981b), Peters (1992) or Rosenmüller (2000)).

[4]i.e. continuous with respect to the Hausdorff topology on $\mathcal{U}_0^c$

3 A Model for Bargaining Power Exchange

In this section we will discuss the basic model.
Suppose there are two agents being engaged in two (different) bargaining problems $V^A = (U^A, 0)$ and $V^B = (U^B, 0) \in \mathcal{U}_0^C$. An agreement consists of a pair $(u^A, u^B) \in U^A \times U^B$ specifying a utility allocation for each of the bargaining problems separately. We assume that utility scales are chosen in a way that does not only allow interpersonal utility comparison but also enables us to compute an agent's total utility by adding his utilities in V^A and V^B. The main problem, however, is that in general an agreement is not efficient w.r.t. the aggregate bargaining problem, which is given by the sum of V^A and V^B.

A first (naive) approach from bargaining theory would be the following. One computes the sum of the bargaining problems, and applies some bargaining solution φ to $V = V^A + V^B$ which automatically determines some agreement (u^A, u^B) that fulfills $u^A + u^B = \varphi(V^A + V^B)$. Of course, this final agreement should be compared with the utility allocations that $\varphi(V^A)$ and $\varphi(B)$, respectively. In case that φ is not superadditive, then in the final agreement one of the agents could be worse off in both bargaining problems compared to what φ dictates.

Yet, even with the PM solution which assures superadditivity, this "procedure" to achieve efficiency seems to be too "mechanical". It appears to be more realistic that agents start with an efficient focal point in each of the bargaining problems (e.g. they start with $\mu(V^A)$ and $\mu(V^B)$) and then deviate from this by favoring one agent in situation A and the other in situation B. The idea of our model is to engage a "Walrasian mechanism" to ensure efficiency. For this we construct an (artificial) exchange economy, in which, roughly speaking, bargaining power is traded and initial endowments are determined by the PM solution in A and B, respectively.

We keep the notation from the previous section and attach superscripts A and B to distinguish the quantities in the referring bargaining problems. With the standard parametrizations ξ^A, ξ^B we could interpret the quantities $\bar{b}^A$ and $\bar{b}^B$ as the time a particle needs to move from $(0, \tau_2^A)$ to $(\tau_1^A, 0)$ (or the other way round), when the law of motion is determined by (3). Starting with the PM solution corresponds to letting each agent's particle move half of the standard traveling time in each problem. We will now let agents trade fractions of these traveling times. For this, consider functions $w^A, w^B : [0,1] \longrightarrow \mathbb{R}^2$ given by

$$w_1^A(\alpha) := \xi_1^A(\alpha \cdot \bar{b}^A) = h^A(\alpha \cdot \bar{b}^A) \qquad w_2^A(\alpha) := \xi_2^A((1-\alpha) \cdot \bar{b}^A) = C^A(h^A((1-\alpha) \cdot \bar{b}^A))$$

$$w_1^B(\beta) := \xi_1^B(\beta \cdot \bar{b}^B) = h^B(\beta \cdot \bar{b}^B) \qquad w_2^B(\beta) := \xi_2^B((1-\beta) \cdot \bar{b}^B) = C^B(h^B((1-\beta) \cdot \bar{b}^B))$$

For example, the quantity $w_1^A(\alpha)$ denotes the utility that agent 1 obtains in A, if he were allowed to "move his particle" from $(0, \tau_2(V^A))$ according to ξ^A for $\alpha \cdot \bar{b}^A$ (units of time). Analogously, $w_2^A(\alpha)$ reflects agent 2's utility, if he were allowed to travel $\alpha \cdot \bar{b}^A$ units of time. By straightforward computations, the point on ∂U^A that agent 2's particle reaches is exactly the point, where agent 1's particle would be after $(1-\alpha) \cdot \bar{b}^A$ time units.

Lemma 1
The functions $w_i^K (K = A, B\,,\, i = 1, 2)$ are strictly increasing and concave. If $C^K (K = A, B)$ is strictly concave then so is $w_i^K (i = 1, 2)$.

Proof:
To start with, we assume that the functions C^A, C^B are twice continuously differentiable. Then we compute for the function f^A (analogously in problem B):

$$f^A(t) = \int_0^t \sqrt{-C^{A\prime}(s)}\, ds, \qquad f^{A\prime}(t) = \sqrt{-C^{A\prime}(t)} \geq 0$$

$$f^{A\prime\prime}(t) = \frac{-C^{A\prime\prime}(t)}{2\sqrt{-C^{A\prime}(t)}} \geq 0.$$

The first derivative of f^A is thereby strictly positive for $t > 0$. This shows that f^A is strictly decreasing and convex. The second derivative of f^A is strictly positive, if and only if $C^{A\prime\prime}$ is strictly negative, hence if C^A is strictly concave. For the derivatives of $h^A = (f^A)^{-1}$ we get:

$$(4) \qquad (h^A)'(s) = \frac{1}{(f^A)'(h^A(s))} = \frac{1}{\sqrt{-C^{A\prime}(h(s))}} > 0$$

$$(5) \qquad (h^A)''(t) = \frac{-f^{A\prime\prime}(h^A(s)) \cdot (h^A)'(s)}{(f^{A\prime}(h^A(s)))^2} = \frac{C^{A\prime\prime}(h(s))}{2\,(\sqrt{-C^{A\prime}(h(s))})^4} \leq 0.$$

This means that h^A is again strictly increasing (first derivative in (4) may not be defined for $h(s) = 0$) and concave. In view of (5) the function h^A is strictly concave, if and only if C^A is strictly concave. The same arguments show concavity of h^B.
Since the mappings w_1^A and w_1^B are linear transformations of h^A and h^B, respectively, (strict) concavity of w_1^A and w_1^B is also guaranteed. Let $r^A : [0, 1] \longrightarrow [0, \bar{b}^A]$ be the affine linear function defined by $r(\alpha) := (1-\alpha)\,\bar{b}^A$. Then $h^A \circ r^A$ is concave and strictly decreasing. Therefore $w_2^A = C^A \circ h^A \circ r^A$ is (strictly) concave as a composition of two

strictly decreasing (strictly) concave functions[5].
A re-inspection of (4) and (5) reveals that the differentiability assumptions are in fact not needed. The first derivative of h^A exists at all but at most countably many points in $[0, \bar{b}^A]$. With (strict) monotonicity of $C^{A'}$ we get (strict) monotonicity of $h^{A'}$, which implies the desired concavity property. □

Remark 1

1. It may appear slightly dubious to express the multi-faceted notion of *bargaining power* by a simple parameter $\alpha \in [0,1]$. However, we do not want to characterize *bargaining power* itself, but to describe the effects of "exerting bargaining power α". And these effects should be described by the utility allocation resulting in a specific bargaining problem. Hence, we can formally describe the effects of bargaining power by a mapping $P : \mathcal{U}_0^c \times [0,1] \longrightarrow \mathbb{R}^2$ that assigns to each bargaining problem V and each bargaining weight α (of agent 1) a utility allocation $P(V, \alpha)$. Generally, there are two kinds of plausible properties that P should satisfy. First, conditions for a fixed bargaining problem and varying weight, and second, conditions for fixed weight and varying bargaining problems. Thereby, we think of the following conditions: For fixed bargaining problem the mapping $P_1(V, \cdot)$ should be strictly increasing (i.e. a gain of power should always pay off), normalized (i.e. $P_1(V, 0) = 0$, "no power yields no utility") and concave (the additional gain of utility from an additional small unit of power should decrease with the amount of power the agent already possesses). For fixed weight α we want to require the "usual" regularity conditions, such as covariance with (affine) linear transformations. This means in effect we require $P(\cdot, \alpha)$ to be an (asymmetric) bargaining solution.
Lemma 1 shows that all these natural conditions are satisfied by our formal notion of bargaining weight. Set for example $P(V^A, \alpha) = \xi^A(\alpha\,\bar{b}^V) = (w_1^A(\alpha), w_2^A(1-\alpha))$. In this spirit, we view this as a justification to speak of a parameter α to represent (agent 1's) bargaining power in V^A.

2. Perles & Maschler (1981b) provide an economic interpretation of the "law of motion", according to which the two particles move along the Pareto boundary (see also Calvo & Gutherrez (1994)). Their idea can be described in the present context roughly as follows. We look at a fixed distribution of weights, say $(\alpha, 1-\alpha)$ and consider the ratio $\frac{w_1'(\alpha)}{w_1'(1-\alpha)}$. Linearizing first derivatives, this is roughly $\frac{w_1(\alpha+\varepsilon)-w_1(\alpha)}{w_1(1-\alpha-\varepsilon)-w_1(1-\alpha)}$ for small $\varepsilon > 0$. Thus, the numerator is agent 1's utility gain from an extra ε of

[5] see, e.g.,Rockafellar (1970)

power, whereas the denominator reflects his utility loss, when having weight $1-\alpha$ and losing ε. Hence the denominator describes his utility loss, when agent 2's bargaining weight were α and he gets an extra ε. Then the law of motion incorporated in eq. (3) requires such ratios of utility gain and loss to be equal, i.e.

$$\frac{\text{utility gain for 1}}{\text{utility loss for 1}} \simeq \frac{w_1'(\alpha)}{w_1'(1-\alpha)} = \frac{w_2'(\alpha)}{w_2'(1-\alpha)} \simeq \frac{\text{utility gain for 2}}{\text{utility loss for 2}}$$

has to be satisfied at each $\alpha \in [0,1]$.

With this interpretation in mind, we will now set up an exchange economy in which such bargaining weights can be traded. Formally, it is described by a tuple

$$\mathcal{E} = \mathcal{E}^{V^A,V^B} := ([0,1]\times[0,1], u_1, u_2, \omega_1, \omega_2)\,, \tag{6}$$

where $[0,1]^2$ reflects the commodity space for the two "commodities" *bargaining power in A and B*. Utilities are determined by adding utilities in the two bargaining problems, which means

$$u_i(\alpha,\beta) := w_i^A(\alpha) + w_i^B(\beta) \qquad (i=1,2).$$

Both agents are initially endowed with equal weights, i.e. $\omega_1 = \omega_2 = \left(\frac{1}{2},\frac{1}{2}\right)$.

Note that the initial utility allocation is

$$\begin{aligned}
(\ u_1(1/2,1/2)\ ,\ u_2(1/2,1/2)\ ) &= (\ \xi_1^A(\bar{b}^A/2) + \xi_1^B(\bar{b}^B/2)\ ,\ \xi_2^A(\bar{b}^A/2) + \xi_2^B(\bar{b}^B/2)\ ) \\
&= (\ \mu_1^A(V^A) + \mu_1^B(V^B)\ ,\ \mu_2^A(V^A) + \mu_2^B(V^B)\ ).
\end{aligned}$$

Thus, initial utilities are given by the sum of PM solutions in the two underlying bargaining problems.

Lemma 2
For each agent i the utility function u_i is concave and strictly increasing.
If both bargaining problems V^A, V^B are described by strictly concave functions C^A and C^B, then u_i is strictly concave.

Proof:
With the (strict) concavity of w_i^A and w_i^B $(i=1,2)$ one immediately obtains (strict) concavity of the utility functions u_1 and u_2, respectively. Use Lemma 1 to complete the proof. □

Lemma 2 guarantees existence of Walrasian equilibria in $\mathcal{E}$ and, as all assumptions of the First Welfare Theorem are satisfied, equilibrium allocations are Pareto efficient. Moreover, any Walrasian equilibrium is located in the Core of $\mathcal{E}$, so that neither agent will be worse off in equilibrium, compared to their initial endowments.

Note also, that the set of Pareto efficient allocations in $\mathcal{E}$ is mapped via (u_1, u_2) onto the set of Pareto efficient utility allocations in V.

4 Walrasian Equilibria and the PM Solution

Before we start equilibrium analysis in $\mathcal{E}$, we will have a closer look at the connection between standard traveling times and aggregation of bargaining problems.

The following lemma is discussed in Perles & Maschler (1981b).

Lemma 3
The function $\bar{b} : \mathcal{U}_0^c \longrightarrow \mathbb{R}$ that assigns to each bargaining problem its standard traveling time is additive on $\mathcal{U}_0^c$.

Moreover, we need a well known result on efficient points in aggregate bargaining problems.

Lemma 4
A utility allocation $z \in U$ is Pareto efficient ($z \in \partial U$), if and only if there exist points $z^A \in \partial U^A$ and $z^B \in \partial U^B$ satisfying

$$z = z^A + z^B, \qquad NC_U(z) \cap NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset,$$

where $NC_U(z)$ denotes the set of supporting normal vectors at $z \in \partial U$.

For $z = (z_1, z_2) \in \partial U$ define $T^l_{U,z} := compH((U - z_l \cdot e^l) \cap \mathbb{R}^2)$ $(l = 1, 2)$. We call $(T^1_{U,z}, 0) \in \mathcal{U}_0^c$ $((T^2_{U,z}, 0))$ the *truncated bargaining problem* of U in direction of the first (second) axis.

Lemmas 3 and 4 together yield a helpful connection between traveling times and efficient points.

Lemma 5

Suppose $z^A = (z_1^A, z_2^A) \in \partial U^A$ and $z^B = (z_1^B, z_2^B) \in \partial U^B$ are such that $NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset$ (which guarantees $z^A + z^B \in \partial U$).

1. For $l = 1, 2$ we have $T^l_{U^A,z^A} + T^l_{U^B,z^B} = T^l_{U,z^A+z^B}$.
2. Let s^A, s^B be determined by $\xi^A(s^A) = z^A, \xi^B(s^B) = z^B$. Then $\bar{b}(T^2_{U,z^A+z^B}) = s^A + s^B$ holds true.
3. Denote by s the corresponding traveling time for $z^A + z^B$, i.e. $\xi(s) = z^A + z^B = \xi^A(s^A) + \xi^B(s^B)$. Then we have $s = s^A + s^B$.

Proof: (Sketch)
To prove 1) use concavity for the functions C^A and C^B, which in particular means decreasing first derivatives. Then assertions 2) and 3) are a direct consequence of 1) and Lemma 3. □

Now, let $z = (z_1, z_2) \in \partial U$ be Pareto efficient in V and $s \in [0, \bar{b}]$ with $\xi(s) := z$. From the construction of aggregate bargaining problems we know that z_2 can be expressed as the value of the following maximization problem:

$$\begin{aligned} z_2 &= \max\left\{C^A(t^A) + C^B(t^B) \mid t^A \in [0, \tau_1^A], t^B \in [0, \tau_1^B], \quad t^A + t^B = z_1\right\} \\ &= \max\left\{C^A(h^A(s^A)) + C^B(h^B(s^B)) \mid s^A \in [0, \bar{b}^A],\right. \\ &\qquad\qquad \left. s^B \in [0, \bar{b}^B], \quad \underbrace{h^A(s^A) + h^B(s^B) = h(s)}_{\xi_1^A(s^A)+\xi_1^B(s^B)=\xi_1(s)}\right\}. \end{aligned} \tag{7}$$

First order conditions (in the differentiable case) require $C^{A\prime}(h^A(s^A)) = C^{B\prime}(h^B(s^B))$. This means that necessarily we are in the situation of Lemma 5 and can therefore rewrite (7) to

$$z_2 = \max\left\{\xi_2^A(s^A) + \xi_2^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = s\right\}. \tag{8}$$

Analogously for z_1 we have

$$z_1 = \max\left\{\xi_1^A(s^A) + \xi_1^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = s\right\}. \tag{9}$$

In particular, the coordinates of the PM solution $\mu(V)$ are obtained from (8) and (9) with $s = \bar{b}/2$, i.e.

(10)

$$\mu_i(V) = \max\left\{\xi_i^A(s^A) + \xi_i^B(s^B) \mid s^A \in [0, \bar{b}^A], s^B \in [0, \bar{b}^B], \quad s^A + s^B = \frac{\bar{b}}{2}\right\} \quad (i = 1, 2).$$

Roughly speaking, the PM solution is obtained by efficiently splitting a total traveling time of $\bar{b}/2$ in traveling times s^A and s^B in A and B.

Coming back to the exchange economy Lemma 4 has the following direct implication.

Lemma 6

Let $((\alpha,\beta);(1-\alpha,1-\beta))$ be an efficient allocation in $\mathcal{E}$. Then $\xi^A(\alpha\,\bar{b}^A) = \big(w_1^A(\alpha), w_2^A(1-\alpha)\big)$, which implies $h^A(\alpha\,\bar{b}^A) = w_1^A(\alpha)$ (analogously in situation B). Then the two derived utility allocations have a common normal vector, i.e. $NC_{U^A}(\xi^A(\alpha\,\bar{b}^A)) \cap NC_{U^B}(\xi^B(\beta\,\bar{b}^B)) \neq \emptyset$. Thus, in the differentiable case (and $0 < \alpha,\beta < 1$) the equation $C^{A'}(h^A(\alpha\,\bar{b}^A)) = C^{B'}(h^B(\beta\,\bar{b}^B))$ holds.

Proof:

Suppose to the contrary that α',β' are such that the referring utility allocations in V^A and V^B do not have a common normal vector, i.e. $NC_{U^A}(\xi^A(\alpha'\,\bar{b}^A)) \cap NC_{U^B}(\xi^B(\beta'\,\bar{b}^B)) = \emptyset$. By Lemma 4 this means that the sum $\xi^A(\alpha'\,\bar{b}^A)) + \xi^B(\beta'\,\bar{b})$ is not located in ∂U, i.e. it is not efficient. Hence, there exists $z \in \partial U$ that dominates this sum. Again, by use of Lemma 4 there exist $z^A \in \partial U^A$ and $z^B \in \partial U^B$ with $NC_{U^A}(z^A) \cap NC_{U^B}(z^B) \neq \emptyset$ and $z^A + z^B = z$. Let α,β now be defined to satisfy $\xi^A(\alpha\,\bar{b}^A) = z^A$ and $\xi^B(\beta\,\bar{b}^B) = z^B$. Then $(u_1(\alpha,\beta), u_2(1-\alpha,1-\beta)) = z^A + z^B \geq \xi^A(\alpha'\,\bar{b}^A)) + \xi^B(\beta'\,\bar{b}) = (u_1(\alpha',\beta'), u_2(1-\alpha',1-\beta'))$ shows that $((\alpha',\beta');(1-\alpha',1-\beta'))$ is not efficient and the lemma is proved. □

Next, we address the question how equilibrium prices in $\mathcal{E}$ look like? For this, we look at agent 1's utility maximization problem. Suppose u_1, u_2 are differentiable. Note that for $\alpha \in [0,1]$ we have

$$w_1^{A'}(\alpha) = h^{A'}(\alpha\cdot\bar{b}^A)\cdot\bar{b}^A = \frac{\bar{b}^A}{\sqrt{(-C^{A'}(h^A(\alpha\cdot\bar{b}^A)))}}.$$

Furthermore, we know that in an equilibrium $((\bar{\alpha},\bar{\beta};1-\bar{\alpha},1-\bar{\beta}),\bar{p}_1,\bar{p}_2)$ we have that the allocation is efficient and therefore $(u_1(\bar{\alpha},\bar{\beta}), u_2(1-\bar{\alpha},1-\bar{\beta}))$ is located in ∂U.

In the differentiable case[6] we can achieve a result on equilibrium prices.

Theorem 1

Let $\mathcal{E}$ be an exchange economy as in (6) with differentiable utility functions u_1, u_2. Then there exists a Walrasian equilibrium with equilibrium prices $(\bar{p}_1,\bar{p}_2)$ that satisfy $\bar{p}_1/\bar{p}_2 = \bar{b}^A/\bar{b}^B$.

[6] In the non-differentiable case the assertions have to be properly adjusted.

Proof:
We assume differentiability of C^A and C^B. For prices p_1 (for a unit of "power" in A) and p_2 we have the familiar first order conditions

$$\frac{\bar{p}_1}{\bar{p}_2} = \frac{\frac{\partial u_1}{\partial \alpha}(\bar{\alpha},\bar{\beta})}{\frac{\partial u_1}{\partial \beta}(\bar{\alpha},\bar{\beta})} = \frac{w^{A\prime}(\bar{\alpha})}{w^{B\prime}(\bar{\beta})} = \frac{\frac{\bar{b}^A}{\sqrt{(-C^{A\prime}(h^A(\alpha\cdot\bar{b}^A)))}}}{\frac{\bar{b}^B}{\sqrt{(-C^{B\prime}(h^B(\beta\cdot\bar{b}^B)))}}} = \frac{\bar{b}^A}{\bar{b}^B} \tag{11}$$

$$\bar{p}_1\cdot\alpha + \bar{p}_2\cdot\beta = \frac{1}{2}\cdot(\bar{p}_1+\bar{p}_2) \tag{12}$$

(the last equation in (11) holds due to Lemma 6). □

An inspection of agent 1's demand in an equilibrium for the two commodities with equilibrium prices $(\bar{p}_1,\bar{p}_2) = (\bar{b}^A,\bar{b}^B)$ now yields

$$\begin{aligned}
\max_{\alpha,\beta}\left\{u_1(\alpha,\beta)\mid \bar{p}_1\,\alpha+\bar{p}_2\,\beta = \frac{\bar{p}_1+\bar{p}_2}{2}\right\} &= \max_{\alpha,\beta}\left\{w_1^A(\alpha)+w_1^B(\beta)\mid \bar{b}^A\,\alpha+\bar{b}^B\,\beta = \frac{\bar{b}^A+\bar{b}^B}{2}\right\}\\
&= \max_{\alpha,\beta}\left\{h^A(\alpha\cdot\bar{b}^A)+h^B(\beta\cdot\bar{b}^B)\mid \bar{b}^A\,\alpha+\bar{b}^B\,\beta = \frac{\bar{b}}{2}\right\}\\
&= \max_{s^A,s^B}\left\{h^A(s^A)+h^B(s^B)\mid s^A+s^B = \frac{\bar{b}}{2}\right\}\\
&= \max_{s^A,s^B}\left\{\xi_1^A(s^A)+\xi_1^B(s^B)\mid s^A+s^B = \frac{\bar{b}}{2}\right\}
\end{aligned} \tag{13}$$

This means in view of (10) that given equilibrium prices as above agent 1 has to solve exactly the same maximization problem that also generates his coordinate of the PM solution. With similar considerations one obtains the same result for agent 2.

This establishes the following theorem.

Theorem 2
Let $\mathcal{E}$ be an exchange economy as in (6). Then there is an equilibrium $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}),\bar{p}_1,\bar{p}_2)$ with equilibrium prices $(\bar{p}_1,\bar{p}_2) = (\bar{b}^A,\bar{b}^B)$ and the utility allocation in equilibrium coincides with the Perles-Maschler solution of the aggregate bargaining problem, i.e.

$$u_1(\bar{\alpha},\bar{\beta}) = \mu_1(V), \qquad u_2(1-\bar{\alpha},1-\bar{\beta}) = \mu_2(V).$$

Theorem 2 guarantees that the PM solution is achieved in some equilibrium with equilibrium prices that reflect the different traveling times. But still, there may be a large set of equilibrium prices. This question will be addressed in the next section.

5 Uniqueness and the Second Welfare Theorem

In this section we will again have a closer look at Pareto efficient allocations and supporting prices. Due to concavity and monotonicity of utility functions the Second Fundamental Welfare Theorem applies to our exchange economy and therefore any efficient allocation can be described as an equilibrium with transfers (e.g. of initial endowments). Lemma 6 gives a necessary condition for efficient allocations in $\mathcal{E}$. In Theorem 1 this was exploited to show that the traveling times $\bar{b}^A$ and $\bar{b}^B$ in fact determine equilibrium prices. A re-inspection of the proof reveals that traveling times also determine supporting prices for arbitrary efficient allocations.

Lemma 7
Let $V = (U, 0) \in \mathcal{U}_0^c$ be a bargaining problem and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ be a normal vector at U in $(\bar{t}, C(\bar{t}))$, i.e. $\lambda \in NC_U(\bar{t}, C(\bar{t}))$. Let $\bar{s}$ be the corresponding traveling time, which means $\xi(\bar{s}) = (\bar{t}, C(\bar{t}))$. Set $\bar{\alpha} := \bar{s}/\bar{b}$ the corresponding weight (for agent 1). Then $\lambda' := \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)$ is a normal vector for $w_1(\cdot)$ at $\bar{\alpha}$. To be precise, we assert

$$\lambda\left(\bar{t}, C(\bar{t})\right) \geq \lambda\left(t, C(t)\right) \quad (t \in [0, \tau_1]) \quad \text{implies} \quad \lambda'\left(\bar{\alpha}, w_1(\bar{\alpha})\right) \geq \lambda'\left(\alpha, w_1(\alpha)\right) \quad (\alpha \in [0, 1]).$$

Proof:
Fix $\bar{t} \in [0, \tau_1]$. Since λ is a supporting normal vector at $(\bar{t}, C(\bar{t}))$ one immediately concludes that

$$\lambda_2 \cdot (-C'_{\searrow}(\bar{t})) \geq \lambda_1 \geq \lambda_2 \cdot (-C'_{\nearrow}(\bar{t})) \tag{14}$$

holds, where $C'_{\searrow}(\bar{t})$ ($C'_{\nearrow}(\bar{t})$) denotes the left-hand (right-hand) first derivative of C at $\bar{t}$. Due to concavity of the function C, the left inequality is valid for all $r \geq \bar{t}$ instead of $\bar{t}$, whereas the right inequality is valid for all $r \leq \bar{t}$. Taking appropriate integrals over square roots in (14) yields

$$\sqrt{\lambda_2} \cdot \int_{\bar{t}}^{t} \sqrt{-C'(r)}\, dr \geq \int_{\bar{t}}^{t} \sqrt{\lambda_1}\, dr \qquad (t \geq \bar{t})$$
$$\int_{t}^{\bar{t}} \sqrt{\lambda_1}\, dr \geq \sqrt{\lambda_2} \cdot \int_{t}^{\bar{t}} \sqrt{-C'(r)}\, dr \qquad (t \leq \bar{t}),$$

which is translated to

$$\sqrt{\lambda_1}(t - \bar{t}) \leq \sqrt{\lambda_2}(f(t) - f(\bar{t})) \qquad (t \in [0, \tau_1]). \tag{15}$$

Since the mapping h is a bijection from $[0, \bar{b}]$ onto $[0, \tau_1]$, inequality (15) can be rewritten as

$$\begin{aligned}
\sqrt{\lambda_1}(h(s) - h(\bar{s})) &\leq \sqrt{\lambda_2}\,(f(h(s)) - f(h(\bar{s}))) && (s \in [0, \bar{b}]) \\
\sqrt{\lambda_2}(s - \bar{s}) &\geq \sqrt{\lambda_1}(h(s) - h(\bar{s})) && (s \in [0, \bar{b}]) \\
\sqrt{\lambda_2}(\alpha\bar{b} - \bar{\alpha}\bar{b}) &\geq \sqrt{\lambda_1}(h(\alpha\bar{b}) - h(\bar{\alpha}\bar{b})) && (\alpha \in [0,1]) \\
\bar{b}\sqrt{\lambda_2}(\alpha - \bar{\alpha}) &\geq \sqrt{\lambda_1}(w_1(\alpha) - w_1(\bar{\alpha})) && (\alpha \in [0,1]) \\
\left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\bar{\alpha}, w_1(\bar{\alpha})) &\geq \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\alpha, w_1(\alpha)) && (\alpha \in [0,1])
\end{aligned}$$

The last inequality shows that $\lambda' = \left(-\bar{b}\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)$ is in fact a supporting normal vector for w_1 at $\bar{\alpha}$ and the lemma is proved. □

Lemma 7 now enables us to derive supporting prices at an efficient allocation $((\bar{\alpha}, \bar{\beta}); (1 - \bar{\alpha}, 1 - \bar{\beta})$ in the Edgeworth box. By Lemma 6 we know that the two corresponding utility allocations $\xi^A(\bar{\alpha}\,\bar{b}^A) \in U^A$ and $\xi^B(\bar{\beta}\,\bar{b}^B) \in U^B$ have a common normal vector, say λ. From Lemma 7 we obtain the inequalities

$$(16)\quad \left(-\bar{b}^A\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\bar{\alpha}, w_1^A(\bar{\alpha})) \geq \left(-\bar{b}^A\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\alpha, w_1^A(\alpha)) \qquad (\alpha \in [0,1])$$

$$(17)\quad \left(-\bar{b}^B\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\bar{\beta}, w_1^B(\bar{\beta})) \geq \left(-\bar{b}^B\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\beta, w_1^B(\beta)) \qquad (\beta \in [0,1]).$$

Adding up (16) and (17) we get

$$\left(-\bar{b}^A\sqrt{\lambda_2}, -\bar{b}^B\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\bar{\alpha}, \bar{\beta}, u_1(\bar{\alpha}, \bar{\beta})) \geq \left(-\bar{b}^A\sqrt{\lambda_2}, -\bar{b}^B\sqrt{\lambda_2}, \sqrt{\lambda_1}\right)(\alpha, \beta, u_1(\alpha, \beta))$$

$$(18)\quad \left(\bar{b}^A\sqrt{\lambda_2}, \bar{b}^B\sqrt{\lambda_2}\right)\left((\alpha, \beta) - (\bar{\alpha}, \bar{\beta})\right) \geq \sqrt{\lambda_1}\left(u_1(\alpha, \beta) - u_1(\bar{\alpha}, \bar{\beta})\right) \quad (\alpha, \beta \in [0,1]).$$

Inequality (18) now gives us the desired implication. Whenever agent 1 thinks the bundle (α, β) is at least as good as the "efficient bundle" $(\bar{\alpha}, \bar{\beta})$, then the right hand side in (18) is not negative. This implies that the left hand side has to be non-negative and therefore the value of $(\bar{\alpha}, \bar{\beta})$ under prices $(\bar{b}^A, \bar{b}^B)$ does not exceed the value of (α, β).

For agent 2 we get the analogous condition to (16) and (17) by interchanging λ_1 and λ_2. Thus the analogous inequality to (18) reads as

$$(19)\quad \left(\bar{b}^A\sqrt{\lambda_1}, \bar{b}^B\sqrt{\lambda_1}\right)\left((\alpha, \beta) - (\bar{\alpha}, \bar{\beta})\right) \geq \sqrt{\lambda_2}\left(u_2(\alpha, \beta) - u_2(\bar{\alpha}, \bar{\beta})\right) \quad (\alpha, \beta \in [0,1]).$$

This establishes the following theorem.

Theorem 3
Let $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}))$ be an efficient allocation in $\mathcal{E}$. Denote by K^A (K^B) the set of normal vectors supporting the respective utility allocations in U^A and U^B, i.e. $K^A := NC_{U^A}(\xi^A(\bar{\alpha}\,\bar{b}^A))$ ($K^B := NC_{U^B}(\xi^B(\bar{\beta}\,\bar{b}^B))$). Then the following statements hold:

1. The set of price vectors supporting u_1 at $(\bar{\alpha},\bar{\beta})$ is given by

   $$S_{u_1}(\bar{\alpha},\bar{\beta}) := \left\{p=(p_1,p_2)\in\mathbb{R}^2_+ \mid p=\left(\sqrt{\eta_2\,\rho_1}\,\bar{b}^A, \sqrt{\eta_1\,\rho_2}\,\bar{b}^B\right), \eta\in K^A, \rho\in K^B\right\}$$

   Analogously, the set of price vectors supporting u_2 at $(1-\bar{\alpha},1-\bar{\beta})$ is given by

   $$S_{u_2}(1-\bar{\alpha},1-\bar{\beta}) := \left\{p=(p_1,p_2)\in\mathbb{R}^2_+ \mid p=\left(\sqrt{\eta_1\,\rho_2}\,\bar{b}^A, \sqrt{\eta_2\,\rho_1}\,\bar{b}^B\right), \eta\in K^A, \rho\in K^B\right\}.$$

   Then the set of price vectors supporting the efficient allocation $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}))$ is given by the intersection $S_{u_1}(\bar{\alpha},\bar{\beta})\cap S_{u_2}(1-\bar{\alpha},1-\bar{\beta})$.

2. In particular, the price system $(\bar{b}^A,\bar{b}^B)$ is a supporting price system for any efficient allocation in $\mathcal{E}$.

Proof:
In order to determine subgradients of u_1, consider inequality (16) with $\eta\in K^A$ instead of λ and (17) with $\rho\in K^B$ instead of λ. Multiplying the first inequality with $\sqrt{\rho_1}$ and the second with $\sqrt{\eta_1}$ yields

$$\left(-\bar{b}^A\sqrt{\rho_2\,\eta_1}, \sqrt{\rho_1\,\eta_1}\right)\left(\bar{\alpha}, w_1^A(\bar{\alpha})\right) \geq \left(-\bar{b}^A\sqrt{\rho_2\,\eta_1}, \sqrt{\rho_1\,\eta_1}\right)\left(\alpha, w_1^A(\alpha)\right) \qquad (\alpha\in[0,1])$$
$$\left(-\bar{b}^B\sqrt{\eta_2\,\rho_1}, \sqrt{\eta_1\,\rho_1}\right)\left(\bar{\beta}, w_1^B(\bar{\beta})\right) \geq \left(-\bar{b}^B\sqrt{\eta_2\,\rho_1}, \sqrt{\eta_1\,\rho_1}\right)\left(\beta, w_1^B(\beta)\right) \qquad (\beta\in[0,1]).$$

Summation now yields

$$\left(\bar{b}^A\sqrt{\rho_2\,\eta_1}, \bar{b}^B\sqrt{\eta_2\,\rho_1}\right)\left((\alpha,\beta)-(\bar{\alpha},\bar{\beta})\right) \geq \sqrt{\eta_1\,\rho_1}\left(u_1(\alpha,\beta)-u_1(\bar{\alpha},\bar{\beta})\right) \quad (\alpha,\beta\in[0,1]).$$

This shows the support property for u_1 at $(\bar{\alpha},\bar{\beta})$. With analogous arguments and use of (19) we get the assertion for u_2.
The second part follows directly by taking $\rho=\eta\in K^A\cap K^B$. Then from part 1) the vector $\sqrt{\rho_1\,\rho_2}\left(\bar{b}^A,\bar{b}^B\right)$ (and hence the vector $\left(\bar{b}^A,\bar{b}^B\right)$) is located in $S_{u_1}(\bar{\alpha},\bar{\beta})\cap S_{u_2}(1-\bar{\alpha},1-\bar{\beta})$.
□

Corollary 1
Let $\mathcal{E}$ be an exchange economy as in (6). Assume that the functions C^A and C^B are differentiable. Let $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}))$ be an efficient allocation with $0<\bar{\alpha},\bar{\beta}<1$. Then $\left(\bar{b}^A,\bar{b}^B\right)$ is (up to normalization) the unique price vector supporting this allocation.

Proof:
Differentiability, efficiency and the non-boundary assumption together imply $NC_{U^A}(\xi^A(\bar{\alpha}\,\bar{b}^A)) = NC_{U^B}(\xi^B(\bar{\alpha}\,\bar{b}^B)) =: \{q\,\lambda \mid q \in \mathbb{R}_{++}\}$. From Theorem 3 it follows $S_{u_1}(\bar{\alpha},\bar{\beta}) = S_{u_2}(1-\bar{\alpha},1-\bar{\beta}) = \{r\,(\bar{b}^A,\bar{b}^B) \mid r \in \mathbb{R}_{++}\}$, which proves the corollary. □

Corollary 2
Let $\mathcal{E}$ be an exchange economy as in (6). Assume that the functions C^A and C^B are strictly concave, differentiable and satisfy

$$(20) \quad \lim_{t\searrow 0} C^{A'}(t) = 0 \qquad \lim_{t\nearrow\tau_1^A} C^{A'}(t) = \infty \qquad \lim_{t\searrow 0} C^{B'}(t) = 0 \qquad \lim_{t\nearrow\tau_1^B} C^{B'}(t) = \infty.$$

Then equilibrium prices are (up to normalization) uniquely determined by $\bar{p}_1/\bar{p}_2 = \bar{b}^A/\bar{b}^B$. Moreover, if C^A and C^B are strictly concave, then there exists exactly one equilibrium in $\mathcal{E}$.

Proof:
Condition (20) guarantees that the only efficient allocations, in which at least one of the weights is zero are those with either $(\bar{\alpha},\bar{\beta}) = (0,0)$ or $(\bar{\alpha},\bar{\beta}) = (1,1)$. But neither of these allocations can form an equilibrium. Thus, we are in the situation of Corollary 1 and therefore all efficient allocations are supported by a unique price vector. This establishes uniqueness of equilibrium prices. In case that C^A and C^B are strictly concave functions we know by Lemma 2 that utility functions u_1 and u_2 are strictly concave and therefore each agent's demand correspondence is single-valued, which implies that in this case there is exactly one equilibrium allocation. □

So far, we have treated the symmetric case, in the sense that initial endowments in $\mathcal{E}$ were determined by an equal endowment of bargaining weights. The second part of Theorem 3 in particular says that any efficient allocation in $\mathcal{E}$ can be achieved as a Walrasian equilibrium with equilibrium prices $\bar{p} = (\bar{b}^A,\bar{b}^B)$ after an appropriate redistribution of initial endowments. Such a redistribution can be performed by endowing agent 1 with a fraction $\eta \in [0,1]$ of bargaining power in each bargaining problem (agent 2's initial endowment is then $(1-\eta, 1-\eta)$). Given an efficient allocation, this fraction can be obtained as a specific convex combination.

Theorem 4
Let $\mathcal{E}_\eta$ be an exchange economy as in (6) but with initial endowments $\omega_1 = (\eta,\eta)$ and $\omega_2 = (1-\eta, 1-\eta)$.

1. Let $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta}))$ be an efficient allocation in $\mathcal{E}$. Then $((\bar{\alpha},\bar{\beta});(1-\bar{\alpha},1-\bar{\beta});(\bar{b}^A,\bar{b}^B))$ is an equilibrium in $\mathcal{E}_\eta$ for $\eta := \frac{\bar{b}^A}{\bar{b}}\,\bar{\alpha} + \frac{\bar{b}^B}{\bar{b}}\,\bar{\beta}$.
2. For each $\eta \in [0,1]$ there exists an equilibrium $((\hat{\alpha},\hat{\beta});(1-\hat{\alpha},1-\hat{\beta});(\bar{b}^A,\bar{b}^B))$ such that its derived utility allocation is the asymmetric PM solution with weights $(\eta, 1-\eta)$, i.e. $(u_1(\hat{\alpha},\hat{\beta}),u_2(1-\hat{\alpha},1-\hat{\beta})) = \xi(\eta\,\bar{b})$ holds true.

Proof:
Agent 1's budget constraint in $\mathcal{E}_\eta$ with prices $(\bar{b}^A,\bar{b}^B)$ is given by $\bar{b}^A\,\alpha+\bar{b}^B\,\beta = \eta\,(\bar{b}^A+\bar{b}^B)$. Rearranging and plugging in the efficient allocation yields the desired implication, since $(\bar{b}^A,\bar{b}^B)$ is by Theorem 3 a supporting price system.
The proof of the second part follows the proof of Theorems 1 and 2. Reconsider, e.g., equations (10) and (13) with $\eta\,\bar{b}$ instead of $\bar{b}/2$. With the same arguments as used above, this leads to the conclusion $(u_1(\hat{\alpha},\hat{\beta}),u_2(1-\hat{\alpha},1-\hat{\beta})) = \xi(\eta\,\bar{b})$, the asymmetric PM solution with weights $(\eta,1-\eta)$. □

Theorem 4 has an interesting interpretation. It tells us how to achieve the asymmetric PM solution with weights $(\eta, 1-\eta)$ of the *aggregate bargaining problem*. As with the symmetric version, agents may trade bargaining weights with prices $(\bar{b}^A,\bar{b}^B)$. All one has to do is to adjust initial endowments such that agent 1 initially receives a fraction of η of each commodity.
Alternatively, one can as well argue that the distribution of bargaining weights among the two agents is exogenously given. We should set initial endowments in the exchange economy according to these fixed weights and let agents trade with prices $(\bar{b}^A,\bar{b}^B)$. Then in equilibrium the pre-determined power distribution is preserved, because the resulting utility allocation in V exactly reflects these weights.

6 Examples

Example 1 (Non-differentiable case)
Consider the following setup:

$$C^A:[0,9]\longrightarrow\mathbb{R},\quad C^A(t):=\begin{cases}3-\frac{1}{8}t & , \; 0\le t\le 8\\ 18-2t & , \; 8<t\le 9\end{cases}\qquad C^B:[0,2]\longrightarrow\mathbb{R},\quad C^B(t):=2-t.$$

The bargaining problems are defined by

$$\begin{aligned} V^A &= (U^A, 0), \quad U^A := compH\left(\{z \in [0,9] \times [0,3] \,|\, z_2 \le C^A(z_1)\}\right) \\ V^B &= (U^B, 0), \quad U^B := compH\left(\{z \in [0,2] \times [0,2] \,|\, z_2 \le C^B(z_1)\}\right). \end{aligned}$$

Figure 1 illustrates the two bargaining problems and the aggregated one.

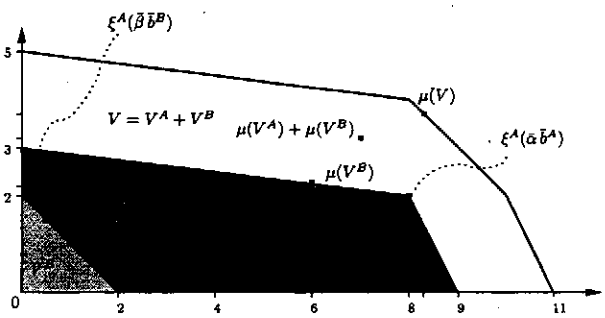

Figure 1: Bargaining Problems in Example 1

Standard traveling times are given by

$$\bar b^A = \int_0^9 \sqrt{-C^{A\prime}(s)}\, ds = 3\sqrt{2} \qquad\qquad \bar b^B = \int_0^2 \sqrt{-C^{B\prime}(s)}\, ds = 2.$$

Straightforward computations reveal that standard parametrizations are given by

$$h^A(s) = \begin{cases} \sqrt{8}\, s & , \quad 0 \le s \le \sqrt{8} \\ \frac{1}{\sqrt{2}}\, s + 6 & , \quad \sqrt{8} < s \le 3\sqrt{2} \end{cases} \qquad \xi^A(s) = \left(h^A(s), C^A(h^A(s))\right)$$

$$h^B(s) = s, \qquad \xi^B(s) = (s, 2 - s),$$

from which we can easily compute the PM solutions of V^A and V^B. We simply evaluate $\mu(V^A) = \xi^A(\frac{3}{2}\sqrt{2}) = (6, \frac{9}{4})$ and $\mu(V^B) = \xi^B(1) = (1, 1)$. As one immediately checks, the sum $\mu(V^A) + \mu(V^B)$ is not efficient in V. The PM solution of V is $\mu(V) = \left(9 - \frac{1}{2}\sqrt{2}, 3 + \frac{1}{2}\sqrt{2}\right)$ (see Figure 1).

From standard parametrizations we obtain the weight functions w^A, w^B

$$w_1^A(\alpha) = \begin{cases} \sqrt{8}\,(\alpha\,\bar b^A) = 12\,\alpha & , \quad 0 \le \alpha \le \frac{2}{3} \\ \frac{1}{\sqrt{2}}\,(\alpha\,\bar b^A) + 6 = 3\,\alpha + 6 & , \quad \frac{2}{3} < \alpha \le 1 \end{cases} \qquad w_2^A(\alpha) = \begin{cases} 6\,\alpha & , \quad 0 \le \alpha \le \frac{1}{3} \\ \frac{3}{2}\,\alpha + \frac{3}{2} & , \quad \frac{1}{3} < \alpha \le 1 \end{cases}$$

$$w_1^B(\beta) = 2\,\beta, \qquad w_2^B(\beta) = 2\,\beta.$$

which determines utilities as

$$u_1(\alpha,\beta)=\begin{cases}12\,\alpha+2\,\beta & ,\ 0\le\alpha\le\frac{2}{3}\\ 3\,\alpha+6+2\,\beta & ,\ \frac{2}{3}<\alpha\le 1\end{cases}\qquad u_2(\alpha,\beta)=\begin{cases}6\,\alpha+2\,\beta & ,\ 0\le\alpha\le\frac{1}{3}\\ \frac{3}{2}\,\alpha+\frac{3}{2}+2\,\beta & ,\ \frac{1}{3}<\alpha\le 1.\end{cases}$$

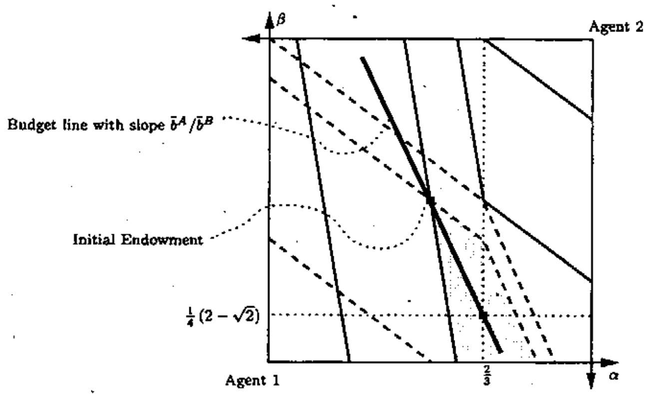

Figure 2: Edgeworth Box for Example 1

Thus, the exchange economy $\mathcal{E}$ (cf. (6)) is completely defined. Figure 2 illustrates utility functions in the corresponding Edgeworth box. The solid lines represent agent 1's indifference curves, whereas dashed lines describe agent 2's indifference curves. The shaded area represents all individually rational allocations. As one can immediately see, there are multiple equilibria in this exchange economy. If we computed the specific one with prices $(\bar{p}_1,\bar{p}_2)=(3\sqrt{2},2)$, we get a unique equilibrium allocation, which is $(\bar{\alpha},\bar{\beta};1-\bar{\alpha},1-\bar{\beta})$ with $\bar{\alpha}=\frac{2}{3}$ and $\bar{\beta}=\frac{1}{4}(2-\sqrt{2})$ (cf. Figure 2). The utility allocation in this equilibrium is $(u_1(\bar{\alpha},\bar{\beta}),u_2(1-\bar{\alpha},1-\bar{\beta}))=\left(9-\frac{1}{2}\sqrt{2},3+\frac{1}{2}\sqrt{2}\right)$, which is indeed the PM solution of the aggregate bargaining problem V. □

Example 2

We now consider an example with differentiable functions C^A and C^B, which are given by

$$C^A:[0,2]\longrightarrow\mathbb{R},\quad C^A(t):=4-t^2,\qquad C^B:[0,5]\longrightarrow\mathbb{R},\quad C^B(t):=5-t.$$

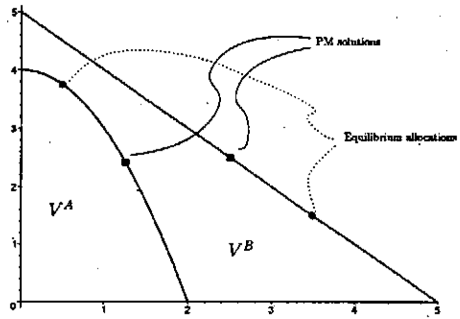

Figure 3: Bargaining Problems in Example 2

Straightforward computations reveal

$$\bar{b}^A = \frac{8}{3}, \qquad \bar{b}^B = 5,$$

$$\xi^A(s) = \left(\frac{1}{2}\,(3\,s)^{\frac{2}{3}}; 4 - \frac{1}{4}\,(3\,s)^{\frac{4}{3}}\right), \qquad \xi^B(s) = (s; 5 - s)\,,$$

$$w^A(\alpha) = \left(2\,\alpha^{\frac{2}{3}}; 4 - 4\,(1-\alpha)^{\frac{4}{3}}\right), \qquad w^B(\beta) = (5\,\beta; 5\,\beta)$$

$$u_1(\alpha,\beta) = 2\,\alpha^{\frac{2}{3}} + 5\,\beta \qquad u_2(\alpha,\beta) = 4 - 4\,(1-\alpha)^{\frac{4}{3}} + 5\,\beta.$$

Figure 4 displays utility functions in the Edgeworth box that corresponds to this problem. To compute efficient allocations of the two commodities, we have to determine solutions of the equation

$$\frac{\frac{\partial u_1}{\partial \alpha}(\alpha,\beta)}{\frac{\partial u_2}{\partial \alpha}(1-\alpha,1-\beta)} = \frac{\frac{\partial u_1}{\partial \beta}(\alpha,\beta)}{\frac{\partial u_2}{\partial \beta}(1-\alpha,1-\beta)} \iff \frac{\frac{4}{3}\,\alpha^{-\frac{1}{3}}}{\frac{16}{3}\,\alpha^{\frac{1}{3}}} = \frac{5}{5} \iff \alpha = \frac{1}{8}.$$

Thus the set of efficient allocations is

$$\left\{(\alpha,\beta;1-\alpha,1-\beta)\,|\,(\beta=0, 0\le\alpha\le\frac{1}{8}) \text{ or } (\alpha=\frac{1}{8}) \text{ or } (\beta=1,\frac{1}{8}\le\alpha\le 1)\right\}$$

Note that for a price (p_1,p_2) that supports an efficient allocation with $\alpha=\frac{1}{8}$ and $\beta\in(0,1)$ necessarily the equation

$$\frac{p_1}{p_2} = \frac{\frac{\partial u_1}{\partial \alpha}(\frac{1}{8},\beta)}{\frac{\partial u_1}{\partial \beta}(\frac{1}{8},\beta)} = \frac{8}{3\cdot 5} = \frac{\bar{b}^A}{\bar{b}^B}$$

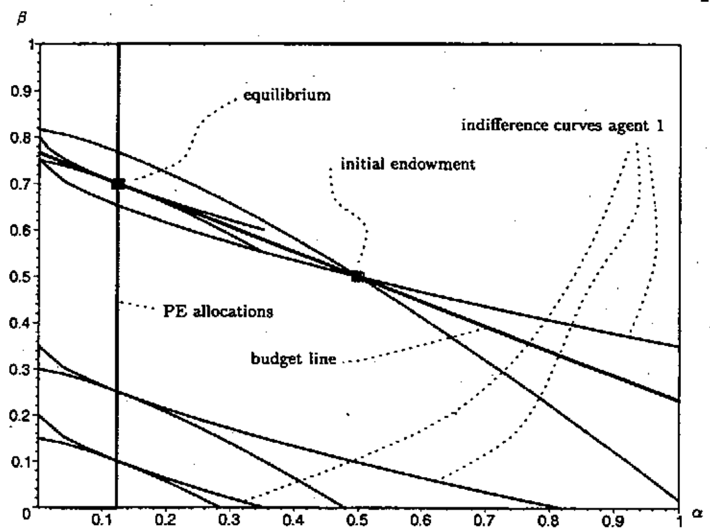

Figure 4: Edgeworth Box for Example 2

has to hold true. Therefore, taking the budget constraint, which amounts to $p_1\,\alpha + p_2\,\beta = \frac{1}{2}(p_1 + p_2)$ we get a unique equilibrium with

$$\bar{\alpha} = \frac{1}{8}, \qquad \bar{\beta} = \frac{7}{10}, \qquad \bar{p_1} = \frac{8}{3}, \qquad \bar{p_2} = 5.$$

Utilities in the equilibrium are $(4, \frac{21}{4})$, which is exactly the PM solution of the aggregate bargaining problem. □

Example 3

We now consider an example with strictly concave functions C^A and C^B. The are given by

$$C^A : [0, 2] \longrightarrow \mathbb{R}, \quad C^A(t) := 4 - t^2, \qquad C^B : [0, \ln 3] \longrightarrow \mathbb{R}, \quad C^B(t) := \frac{9}{2} - \frac{1}{2}e^{2t}.$$

Again, by the same computations as in the examples above, we get

$$\bar{b}^A = \frac{8}{3}, \qquad \bar{b}^B = 2,$$

$$\xi^A(s) = \left(\frac{1}{2}\,(3\,s)^{\frac{2}{3}}; 4 - \frac{1}{4}\,(3\,s)^{\frac{4}{3}}\right), \qquad \xi^B(s) = \left(\ln(s+1); \frac{9}{2}\,(s+1)^2\right),$$

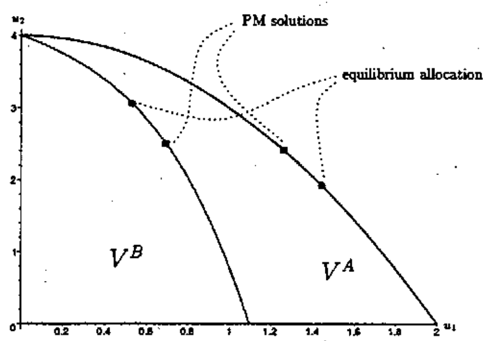

Figure 5: Bargaining Problems in Example 3

$$w^A(\alpha) = \left(2\,\alpha^{\frac{2}{3}}; 4 - 4\,(1-\alpha)^{\frac{4}{3}}\right), \qquad w^B(\beta) = \left(\ln(2\,\beta + 1); \frac{9}{2} - \frac{1}{2}(3 - 2\,\beta)^2\right)$$

$$u_1(\alpha, \beta) = 2\,\alpha^{\frac{2}{3}} + \ln(2\,\beta + 1) \qquad u_2(\alpha, \beta) = 4 - 4\,(1-\alpha)^{\frac{4}{3}} + \frac{9}{2} - \frac{1}{2}(3 - 2\,\beta)^2.$$

Figure 6 shows indifference curves for the two agents and the unique equilibrium alloca-

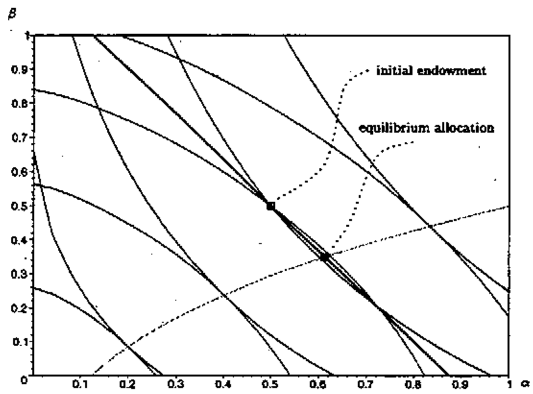

Figure 6: Edgeworth Box for Example 2

tion. Indeed, the equilibrium are according to Theorem 1 given by the standard traveling times. The equilibrium $((\bar{\alpha}, \bar{\beta}; 1 - \bar{\alpha}, 1 - \bar{\beta}), \bar{p_1}, \bar{p_2})$ is given by

$$\bar{\alpha} \approx 0.3494, \qquad \bar{\beta} \approx 0.6129, \qquad \bar{p_1} = \frac{8}{3}, \qquad \bar{p_2} = 2.$$

□

7 Concluding Remarks

One may as well think of other bargaining solutions and their asymmetric versions to get a similar construction for bargaining power. Yet, it turns out that it is the superadditivity that guarantees the desired properties that the weight functions $w^{\cdot}$ should have. For example, with the (asymmetric) Nash solution, such mappings may fail to be strictly increasing or to be concave. As a result, preferences in the Edgeworth box may no longer be convex and hence existence of an equilibrium is not guaranteed.

Some readers may feel uncomfortable with a seemingly conflicting mixture of cardinal and ordinal solution concepts. Indeed, as soon as we enter the Edgeworth box and apply the Walrasian equilibrium concept, we are no longer in a cardinal context. Yet, we view this way as a tool to come up with a certain allocation of bargaining power. And exactly this allocation is meant to "execute" the utilities, i.e. to determine the solution in the cardinal context. Note that agents' preferences in the Edgeworth box are not touched by the right transformations of the two bargaining problems. If we apply the same linear transformation to both bargaining problems, then the agents' utility functions will be linearly transformed and hence preferences will be preserved.[7]

The work in the paper can be extended in a couple of directions. First, the class of bargaining problems under consideration can be extended from $\mathcal{U}_0^c$ to $\mathcal{U}^c$ without substantial change of the results. This is as unproblematic as allowing boundaries of utility possibility sets to contain line segments that are parallel to some axis. Finally, there is nothing special with the fact that we consider two bargaining problems. With analogous arguments as used in the paper, one can consider the model with finitely many bargaining situations.

Since there is no superadditive bargaining solution for more than two persons (see Perles (1982)), we cannot hope for a straightforward extension of our model to the n-person case. Whether or not the extension of the PM solution to n-person bargaining problems can be used to define a notion of bargaining power is an open problem. But the lack of superadditivity may be an insurmountable obstacle for the process of finding an agreement.

[7]Application of different linear transformations to the bargaining problems should not be allowed, because this would violate our assumption that an agent's overall utility is the sum of utilities he gets in the two bargaining problems.

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