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An Algorithm for Incentive Compatible Mechanisms of Fee-Games

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Abstract

An algorithm for determining all vertices of the set of incentive compatible and individually rational mechanisms corresponding to a specific class of fee-games is presented. This method is based on the algorithm of Chernikova, which allows to find all extreme points of a convex polyhedron.

1 Introduction

This paper is devoted to the problem of finding all extreme points of the set of incentive compatible and individually rational mechanisms corresponding to a special version of an NTU-game with incomplete information.

Harsanyi and Selten [11] were the first who considered the cooperative games with incomplete information . The concept of the Bayesian incentive compatible mechanism was proposed by Myerson [17, 18, 19] for the framework of NTU - games with incomplete information. Rosenmüller [20, 21] dealt with the fee-game as a special form of the NTU-games with incomplete information and, particularly, considered the set of the 'feasible' - incentive compatible and individually rational mechanisms. It turned out that these mechanisms form convex compact polyhedral sets (see [20]) . Therefore, they can be described by their extreme points. The problem of finding the extreme mechanisms was also discussed by Rosenmüller [21]. Our approach is based on the idea that the extremal mechanisms can be viewed as vertices of some polyhedra.

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There are several algorithms for obtaining all vertices of convex polyhedral sets in the literature (e.g., see [22], p.p. 224 - 225, and [15]). A classification of published methods and computional comparisons have been made in [15]. A discussion of them has also appeared in [8]. All of the known algorithms fall into two main classes: pivoting methods and nonpivoting methods. According to some criteria the algorithms of Balinski [1], Manas- Nedoma [13] and Mattheiss [14] are considered in [15] as representative of the pivoting methods and the Chernikova's algorithm [5, 6, 7] as representative of the nonpivoting methods.

For a specific class of fee-games we provide here a procedure for finding the vertices of the set of feasible mechanisms which is based on the algorithm of Chernikova.

2 Fee-games: Definitions and Denotations

This Section provides the basic definitions concerning the fee-games. As already mentioned, as a special form of a NTU-game which displays some side-payment properties fee game was discussed by Rosenmüller in [20, 21].

A cooperative game with incomplete information (CII - Game) is said to be an object of the form

$$\Gamma = (\mathcal{I}, \mathcal{T}, p; \bar{\mathcal{X}}, \bar{x}, \mathcal{U})$$

with the following components:

$\mathcal{I} = \{1, \dots, n\}$ is the set of *players*;

$\mathcal{T} = \prod_{i \in \mathcal{I}} \mathcal{T}^i$, where for $i \in \mathcal{I}$ the finite set $\mathcal{T}^i$ represents player i's *types*;

p is a *probability* on T, the *distribution of types*, because we imagine that there is an abstract probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a random variable $\tau : \Omega \to \mathcal{T}$ with distribution $p = \mathcal{P} \circ \tau^{-1}$; τ 'chooses the types';

$\bar{\mathcal{X}} = \{x \in \mathbf{R}^n : ex \leq 1\}$, where $e = (1, \dots, 1) \in \mathbf{R}^n_+$, is the set of collective decisions, or *contracts* the players can agree upon;

$\partial\bar{\mathcal{X}} = \{x \in \mathbf{R}^n : ex = 1\}$ represents the Pareto efficient frontier of $\bar{\mathcal{X}}$;

$\bar{x} = 0$ - *status quo* parameter takes place if players fail to agree;

finally,the mapping

$$\mathcal{U} : \mathcal{I} \times \mathcal{T} \times \bar{\mathcal{X}} \to \mathbf{R}'$$

reflects the *utilities*, if chance chooses $t \in \mathcal{T}$ and the players agree upon $x \in \bar{\mathcal{X}}$, then the utility of player i is $\mathcal{U}_i^t(x)$. It is assumed that $\mathcal{U}_i^t$ satisfies certain conditions (see [21]).

A CII-Game Γ is said to be a *fee-game* if, for any $t \in \mathcal{T}$, there is a $b^t \in \mathbf{R}^n$, $o \leq eb^t \leq 1$, such that

$$\mathcal{U}^t(x) = x - (ex)b^t \qquad (x \in \bar{\mathcal{X}})$$

holds true; b^t is called *a fee.*

Γ is said to be *a game with incomplete information on one side*, if $n = 2$ and $|\mathcal{T}^2| = 1$, or, what is the same, $\mathcal{T} = \mathcal{T}^1 \times \{*\}$.

We use the denotation

$$\Sigma^1 := \{\Gamma : \Gamma \quad \text{is a fee-game with incomplete information on one side and} \quad |\mathcal{T}^1| = 2\}.$$

Further we will write

$$\mathcal{T} = \{\alpha, \beta\} \times \{*\}$$

for $\Gamma \in \Sigma^1$, we also may omit the index $*$, thus, in particular we will use the next denotations

$$\mathcal{U}^\alpha := \mathcal{U}^{(\alpha,*)}, \qquad \mathcal{U}^\beta := \mathcal{U}^{(\beta,*)};$$
$$b^\alpha := b^{(\alpha,*)}, \qquad b^\beta := b^{(\beta,*)}.$$

It is clear, that the two fee-vectors $b^\alpha, b^\beta \in \mathbf{R}^2$ essentially describe $\Gamma \in \Sigma^1$.

We will suppose that

$$b_1^\alpha < b_1^\beta \qquad (\Gamma \in \Sigma^1),$$

what means that state α is more preferable to state β for the informed player 1, because of the smaller fee.

The rules of the game are specified by Rosenmüller [20, 21] by the notation of a *mechanism.*

Let Γ be a *CII-Game*, then:

- A *mechanism* is a mapping

$$\mu : \mathcal{T} \to \bar{\mathcal{X}};$$

- A mechanism μ is called *Bayesian incentive compatible (BIC)* if

$$E(\mathcal{U}_i^\tau \circ \mu(\tau_1,\ldots,\tau_i,\ldots,\tau_n)|\tau_i = t_i) \geq$$
$$E(\mathcal{U}_i^\tau \circ \mu(\tau_1,\ldots,s_i,\ldots,\tau_n)|\tau_i = t_i)$$

holds true for every $i \in \mathcal{I}$ and $t_i, s_i \in \mathcal{T}^i$;

- μ is *individually rational (IR)* if

$$E(\mathcal{U}_i^\tau \circ \mu(\tau_1,\ldots,\tau_n)|\tau_i = t_i) \geq 0$$

holds true for every $i \in \mathcal{I}$ and $t_i \in \mathcal{T}^i$;

- The set of all mechanisms players can and will bargain about is denoted by

$$\Im := \Im(\Gamma) = \{\mu : \mu \quad \text{is BIC and IR}\}$$

Notice. Later on we will use the next notations:

$$\mu^\tau := \mu \circ \tau = \mu(\tau_1,\ldots,\tau_n), \quad (\tau_{-i}, s_i) = (\tau_1,\ldots,s_i,\ldots,\tau_n).$$

3 The Set of Incentive Compatible and Individually Rational Mechanisms

Now we are going to describe the set $\Im$ for the fee-game $\Gamma \in \Sigma^1$, given $b^\alpha, b^\beta \in \mathbf{R}^2$, such that $0 \leq eb^\alpha \leq 1$, $0 \leq eb^\beta \leq 1$, $b_1^\alpha < b_1^\beta$. We also have

$$\mathcal{U}^\alpha(x) = x - (ex)b^\alpha = (\mathcal{U}_1^\alpha(x), \quad \mathcal{U}_2^\alpha(x)) =$$
$$(x_1 - (ex)b_1^\alpha, \quad x_2 - (ex)b_2^\alpha) \qquad (x \in \bar{\mathcal{X}});$$
$$\mathcal{U}^\beta(x) = x - (ex)b^\beta = (\mathcal{U}_1^\beta(x), \quad \mathcal{U}_2^\beta(x)) =$$
$$(x_1 - (ex)b_1^\beta, \quad x_2 - (ex)b_2^\beta) \qquad (x \in \bar{\mathcal{X}}).$$

It is clear, that a mechanism $\mu : \{\alpha, \beta\} \times \{*\} \to \bar{\mathcal{X}}$ is BIC if

$$E(\mathcal{U}_i^\tau \circ \mu(\tau_1,\tau_2)|\tau_i = t_i) \geq$$
$$E(\mathcal{U}_i^\tau \circ \mu(\tau_{-i}, s_i)|\tau_i = t_i)$$

holds true for $i = 1, 2$ and $t_i, s_i \in \mathcal{T}^i \quad (i = 1, 2)$.
This means that the next inequalities

$$\begin{aligned}
&\textstyle\sum_{t_2 \in \mathcal{T}^2} \mathcal{U}_1^{(t_1,t_2)}(\mu^{(t_1,t_2)}) p(t|t_1) \geq \\
&\quad \textstyle\sum_{t_2 \in \mathcal{T}^2} \mathcal{U}_1^{(t_1,t_2)}(\mu^{(s_1,t_2)}) p(t|t_1); \\
&\textstyle\sum_{t_1 \in \mathcal{T}^1} \mathcal{U}_2^{(t_1,t_2)}(\mu^{(t_1,t_2)}) p(t|t_2) \geq \\
&\quad \textstyle\sum_{t_1 \in \mathcal{T}^1} \mathcal{U}_2^{(t_1,t_2)}(\mu^{(t_1,s_2)}) p(t|t_2),
\end{aligned}$$

where $p(\cdot) = \mathcal{P}(\tau = \cdot | \tau_i = t_i), \quad t_i, s_i \in \mathcal{T}^i \quad (i = 1, 2),$ define the BIC-mechanisms. As for the second of these inequalities, it is easy to see that this is an identity. From the first inequality it follows due to $\mathcal{T}^1 = \{\alpha, \beta\}, \quad |\mathcal{T}^2| = 1, \quad t_2 = \{*\},$ that

$$\begin{aligned}
\mathcal{U}_1^{\alpha}(\mu^{\alpha}) &\geq \mathcal{U}_1^{\alpha}(\mu^{\beta}), \\
\mathcal{U}_1^{\beta}(\mu^{\beta}) &\geq \mathcal{U}_1^{\beta}(\mu^{\alpha}).
\end{aligned}$$

Now we note that the IR-mechanisms satisfy the following system of inequalities:

$$\begin{aligned}
&\textstyle\sum_{t_2 \in \mathcal{T}^2} \mathcal{U}_1^{(t_1,t_2)}(\mu^{(t_1,t_2)}) p(t|t_1) \geq 0, \\
&\textstyle\sum_{t_1 \in \mathcal{T}^1} \mathcal{U}_2^{(t_1,t_2)}(\mu^{(t_1,t_2)}) p(t|t_2) \geq 0, \\
&\qquad \text{for all} \quad t_i \in \mathcal{T}^i \quad (i = 1, 2),
\end{aligned}$$

which in turn is tantamount to

$$\begin{aligned}
\mathcal{U}_1^{(t_1,*)}(\mu^{(t_1,*)}) p(t|t_1) &\geq 0, \\
\mathcal{U}_2^{(\alpha,*)}(\mu^{(\alpha,*)}) \mathcal{P}((\alpha,*)|t_2 = *) &+ \\
\mathcal{U}_2^{(\beta,*)}(\mu^{(\beta,*)}) \mathcal{P}((\alpha,*)|t_2 = *) &\geq 0.
\end{aligned}$$

And, finally, it follows that for the IR-mechanisms next inequalities

$$\begin{aligned}
\mathcal{U}_1^{\alpha}(\mu^{\alpha}) &\geq 0, \\
\mathcal{U}_1^{\beta}(\mu^{\beta}) &\geq 0, \\
\mathcal{U}_2^{\alpha}(\mu^{\alpha}) p_{\alpha} + \mathcal{U}_2^{\beta}(\mu^{\beta}) p_{\beta} &\geq 0,
\end{aligned}$$

where $p_\alpha := \mathcal{P}((\alpha,*)|t_2=*)$, $p_\beta := \mathcal{P}((\beta,*)|t_2=*)$, hold true. It is clear, that $p_\alpha \geq 0$, $p_\beta \geq 0$, $p_\alpha + p_\beta = 1$.

As already mentioned in [21] , a mechanism $\mu : \{\alpha,\beta\}\times\{*\} \to \bar{\mathcal{X}}$ is equivalent to a pair $\mu = (\mu^\alpha, \mu^\beta) \in \bar{\mathcal{X}} \times \bar{\mathcal{X}}$. This means, that

$$e\mu^\alpha \leq 1, \quad \mu_1^\alpha \geq 0, \quad \mu_2^\beta \geq 0,$$
$$\text{here} \quad e = (1,1) \in \mathbf{R}_+^2.$$

Thus, the conditions defining the set $\Im$ of BIC and IR mechanisms constitute the following system of linear inequalities:

$$\begin{aligned}
\mu_1^\alpha - (e\mu^\alpha)b_1^\alpha &\geq \mu_1^\beta - (e\mu^\beta)b_1^\alpha,\\
\mu_1^\beta - (e\mu^\beta)b_1^\beta &\geq \mu_1^\alpha - (e\mu^\alpha)b_1^\beta,\\
\mu_1^\alpha - (e\mu^\alpha)b_1^\alpha &\geq 0,\\
\mu_1^\beta - (e\mu^\beta)b_1^\beta &\geq 0,\\
(\mu_2^\alpha - (e\mu^\alpha)b_2^\alpha)p_\alpha + (\mu_2^\beta - (e\mu^\beta)b_2^\beta)p_\beta &\geq 0,\\
\mu_1^\alpha + \mu_2^\alpha &\leq 1,\\
\mu_1^\beta + \mu_2^\beta &\leq 1,\\
\mu_i^\alpha \geq 0 \qquad &(i=1,2),\\
\mu_i^\beta \geq 0 \qquad &(i=1,2).
\end{aligned}$$

Let us denote

$$\xi_1 := \mu_1^\alpha, \quad \xi_2 := \mu_2^\alpha, \quad \xi_3 := \mu_1^\beta, \quad \xi_4 := \mu_2^\beta.$$

Then the set $\Im$ can be represented as

$$\Im = \{\xi : \mathbf{A}\xi \geq \mathbf{B}, \quad \xi \geq 0\},$$

where by $\mathbf{A}$ is denoted the following 7×4 -matrix

$$\mathbf{A} = \begin{pmatrix}
1-b_1^\alpha & -b_1^\alpha & -(1-b_1^\alpha) & b_1^\alpha\\
-(1-b_1^\beta) & b_1^\beta & 1-b_1^\beta & -b_1^\beta\\
1-b_1^\alpha & -b_1^\alpha & 0 & 0\\
0 & 0 & 1-b_1^\beta & -b_1^\beta\\
-b_2^\alpha p_\alpha & (1-b_2^\alpha)p_\alpha & -b_2^\beta p_\beta & (1-b_2^\beta)p_\beta\\
-1 & -1 & 0 & 0\\
0 & 0 & -1 & -1
\end{pmatrix}$$

and **B** denotes a vector-column such that

$$\mathbf{B}^{\mathrm{T}} = (0, \quad 0, \quad 0, \quad 0, \quad 0, \quad -1, \quad -1).$$

To this end we note that $\Im$ is a convex compact polyhedron (see [20]).

4 Extreme Points of a Convex Polyhedron

The problem we are addressing is to find all vertices of the convex polyhedron $\Im(\Gamma)$, described as the intersection of a finite number of hyperplanes and closed half-spaces. More precisely, our $\Im(\Gamma)$ is described by a system of seven linear inequalities in four unknowns $\xi_i \quad (i = \overline{1,4})$:

$$\mathbf{A}\xi - \mathbf{B} \geq 0, \qquad \xi \geq 0.$$

In connection with this problem let us remind here that:

- A subsystem $\hat{\mathbf{A}}\xi - \hat{\mathbf{B}} = 0$ of equations, where $\hat{\mathbf{A}}$ is a nonsingular square submatrix consisting of n linearly independent rows of **A** , and $\hat{\mathbf{B}}$ is the column of the corresponding elements of **B**, is called a *fundamental subsystem* of the auxiliary system $\mathbf{A}\xi - \mathbf{B} = 0$;
- A fundamental subsystem has a unique solution ξ which is called a *point*;
- If the solution ξ satisfies $\mathbf{A}\xi - \mathbf{B} \geq 0$, then ξ is a *vertex*.

Hence the set of vertices of $\Im$ consists of the solutions ξ of all fundamental sudsystems of $\mathbf{A}\xi - \mathbf{B} \geq 0, \quad \xi \geq 0$, which also satisfy the inequalities $\mathbf{A}\xi - \mathbf{B} \geq 0$.

All methods for obtaining the vertices of convex polyhedral sets can be divided into two groups: pivoting algorithms and nonpivoting algorithms. As for the nonpivoting methods, they utilize the dual representation of a bounded polyhedron by convex combinations of its extreme points to describe the set of solutions to the inequalities. They can also be viewed as versions of the Double Description Method of Motzkin, Thompson, Raiffa, and Thrall [16] . The algebraic foundations of those methods are disscused by Burger [2] for cones and by Galperin [9] for polyhedra. They also are geometrically motivated. The geometrical interpretation essentially can be the following: Consider a convex bounded polyhedron **P**, whose vertices are already known. Assume that $\bar{\mathbf{P}}$ is the intersection of **P** and a hyperplane **H** or a closed half space $\mathbf{H}^+$, that means, $\bar{\mathbf{P}}$ is obtained from **P** by adding another constraint. Hence the vertices of $\bar{\mathbf{P}}$ are of two kinds: these of **P** which lie on **H** or in $\mathbf{H}^+$ and those which are convex combinations of vertices of **P** in $\mathbf{H}^+$ with other vertices of **P** in $\mathbf{H}^-$, where the weights of these combinations are choosen so that the all new vertices lie on **H**.

The method of Balinski [1] belongs to the group of pivoting algorithms, and it can be noted that this procedure is based on the basic iterative step of the simplex method - the 'pivotal operation'. We will not pursue here the details of the pivoting algorithms,we note only that the main difference between the two groups of methods is that the pivoting schemes have a 'hyperplane - oriented' bookkeeping system, while the nonpivoting schemes have a 'point - oriented' bookkeeping system. According to some criteria important from computional point of view, Mattheiss and Rubin [15] considered the methods of Balinski [1], Manas- Nedoma [13], and Mattheiss [14] as representative of the pivoting algorithms and the Chernikova's method [5, 6, 7]- as representative of the nonpivoting algorithms. The second group also contains methods of Burger [2], Chernikov [3], Galperin [9], Greenberg [10], Kuznetsov [12].

It should be mentioned, that many of nonpivoting methods are formulated by their authors in terms of finding all extreme rays of convex polyhedral cones. But it is easy to prove the next claim.

Proposition 1 *Let* **A** *be a* $m \times n$ *-matrix,* $\mathbf{B} \in \mathbf{R}^m$, *denote* $(x, 1) := \{(\lambda x, \lambda) : \lambda \geq 0\}$. *Then* $\bar{x}$ *is a vertex of the polyhedron*

$$\mathbf{P} = \{x \in \mathbf{R}^n : \mathbf{A}x \leq \mathbf{B}\}$$

if and only if $(\bar{x}, 1)$ *is an extreme ray of the cone*

$$\mathbf{C} = \{(x, \gamma) : -\mathbf{A}x + \mathbf{B}\gamma \geq 0, \gamma \geq 0\}$$

Proof: Assume $(\bar{x}, 1)$ is an extreme ray. Let us show that $\bar{x}$ is an extreme point of the polyhedron **P**. Let $\bar{x} = \nu x_1 + (1 - \nu)x_2, \quad \nu > 0$. Then we have

$(\bar{x}, 1) = (\nu x_1, \nu) + ((1-\nu)x_2, 1-\nu)$. Since $\mathbf{A}x \leq \mathbf{B}$ we have $\mathbf{A}(\nu x_1) \leq \nu \mathbf{B}$. Thus, $(\nu x_1, \nu) \in \mathbf{C}$. Analogously, $((1-\nu)x_2, 1-\nu) \in \mathbf{C}$. Since $(\bar{x}, 1)$ is an extreme ray of $\mathbf{C}$ and $(\nu x_1, \nu)$, $((1-\nu)x_2, 1-\nu) \in \mathbf{C}$ it follows that there exist $t_1 > 0$ and $t_2 > 0$ such that $(\nu x_1, \nu) = t_1(\bar{x}, 1)$ and $((1-\nu)x_2, 1-\nu) = t_2(\bar{x}, 1)$. Thus, $\nu x_1 = t_1 \bar{x}$, $\nu = t_1$ and $(1-\nu)x_2 = t_2 \bar{x}$, $1-\nu = t_2$, therefore $\nu x_1 = \nu \bar{x}$ and $(1-\nu)x_2 = (1-\nu)\bar{x}$. This means that $x_1 = \bar{x} = x_2$, or $\bar{x}$ is a vertex of $\mathbf{P}$.

Otherwise, let $\bar{x}$ be a vertex of $\mathbf{P}$. Suppose, that $(\bar{x}, 1)$ is a nonextremal ray of $\mathbf{C}$, it means there are $\lambda > 0, \mu > 0, \lambda + \mu = 1$ such that $(\bar{x}, 1) = \lambda(x_1, \nu_1) + \mu(x_2, \nu_2)$, where $(x_1, \nu_1) \neq (x_2, \nu_2)$. It follows that $\bar{x} = \lambda x_1 + \mu x_2$, $\lambda \nu_1 + \mu \nu_2 = 1$, what contradicts to the fact that $\bar{x}$ is a vertex of $\mathbf{P}$.

q.e.d.

5 The Algorithm of Chernikova

Chernikova [5, 6, 7] has given an algorithm for finding the extreme rays of a convex polyhedral cone. It can be adapted to determine the vertices of a convex polyhedron. The idea of this algorithm is similar to that of Motzkin et al. [16], but it is presented in a some different framework. The work of Burger [2] has served as algebraic foundation for this method. According to this algorithm, for determining all vertices and extreme rays (unbounded edges) of $\Im = \{\xi \in \mathbf{R}^4 : \mathbf{A}\xi \geq \mathbf{B}, \quad \xi \geq 0\}$ we have to find all extrme rays of the corresponding cone $\Re := \{(\xi, k) : \mathbf{A}\xi - \mathbf{B}k \geq 0, \quad \xi \geq 0, \quad k \geq 0\}$. Those with $k > 0$ correspond to vertices of $\Im$, those with $k = 0$ correspond to extreme rays of $\Im$.

Consider the matrix $(\mathbf{I} \mid \mathbf{A} - \mathbf{B})$, where $\mathbf{I}$ is a $(n+1) \times (n+1) = 5 \times 5$ - identity matrix. We make a series of transformations of this matrix which generates the solution. At any stage of the process we use the symbol $\mathbf{Z} := (\mathbf{M} \mid \mathbf{L})$ to denote the old matrix and $\bar{\mathbf{Z}}$ for the new matrix being generated. The matrices $\mathbf{M}$ and $\mathbf{L}$ will allways have $n+1 = 5$ and $m = 7$ columns, respectively; however, they will in generel not have $n+1 = 5$ rows. In most cases they will have more than $n+1 = 5$ rows, but if $\Re$ lies in some subspace of $\mathbf{R}^5$ they may have fewer than 5 rows. For the $(\xi, k) \in \mathbf{R}^5$ we use the symbol $((\xi, k))$ to denote the ray $\{(\delta\xi, \delta k) : \delta \geq 0\}$.

The computional scheme is as follows:

1. Consider the matrix $\mathbf{Z}$.

   (a) If any column of the matrix $\mathbf{L}$ has only negative elements, then $(\xi, k) = 0$ is the only solution.

   (b) If all elements of $\mathbf{L}$ are nonnegaive, then the process is finished, and to define the vertices of $\Im$ by the rows of $\mathbf{M}$ *go to step 6*.

   (c) Otherwise, go to step 2.

2. Take as leading column the first one of the matrix $\mathbf{Z}$, with at least one negative element.

3. Write down without change those rows of the matrix $\mathbf{Z}$ which intersect the leading column in nonnegative numbers.

   (a) If $\mathbf{Z}$ has only two rows and the elements of the leading column are of opposite signs, adjoin a linear combination of the rows of this matrix with positive coefficients, which are chosen by the same method as in general case. Go to step 5.

4. Select in turn those pairs of rows in which the elements of the leading column have opposite signs. For each such pair find all nonnegative columns of the matrix $\mathbf{Z}$ such that their intersections with the rows of the pair in question are zeros. The following cases may occur:

   (a) There are no such columns, or there is still at least one other row which intersects all such columns in zeros, then the pair in question does not contribute another row to the new matrix $\bar{\mathbf{Z}}$;

   (b) Otherwise, form a linear combination of this pair with the positive coefficients such that its element defined by the leading column is zero. Add this row to the new matrix $\bar{\mathbf{Z}}$.

5. When all such pair of rows have been examined and the additional rows (if any) have been adjoined then *the leading column is said to be processed.* Denote by $\mathbf{Z}$ the matrix $\bar{\mathbf{Z}}$ produced in processing the leading column, and return to the step 1 .

6. *Divide the matrix* **Z** *into two parts*

$$\mathbf{Z} = \left(\begin{array}{c|c} \hat{\mathbf{M}} & \hat{\mathbf{L}} \\ \check{\mathbf{M}} & \check{\mathbf{L}} \end{array} \right)$$

in accordance with the rule: to the upper part denoted by $\hat{\mathbf{Z}} = (\hat{\mathbf{M}}|\hat{\mathbf{L}})$ *belong those rows of the matrix* **Z** *which have on their right -hand side at least four zero columns, all other rows are in the lower part* $\check{\mathbf{Z}} = (\check{\mathbf{M}}|\check{\mathbf{L}})$.

7. The rows of $\hat{\mathbf{M}}$ represent the edges of the cone $\Re$, that means that the ray

$$(\hat{m}_j) = ((\xi^{(j)}, k^{(}j)) = \{\delta \hat{m}_j : \delta \geq 0\}$$

is an edge of $\Re$, where $\hat{m}_j$ denotes the jth row of $\hat{\mathbf{M}}$, and

$$\left(\frac{\xi_1^j}{k^{(j)}}, \frac{\xi_2^{(j)}}{k^{(j)}}, \frac{\xi_3^{(j)}}{k^{(j)}}, \frac{\xi_4^{(j)}}{k^j}\right), \quad (k^{(j)} > 0)$$

is a vetrex of the polyhedron $\Im$.

Note an important property of the matrices **Z** obtained by the application of the described above algorithm.

Proposition 2 *(see [5]) Let us denote*

$$\mathbf{Z} = (\mathbf{M}|\mathbf{L}) := ((m_{ir})|(l_{is})), (i = \overline{1,\nu}; r = \overline{1,5}, s = \overline{1,7}),$$

where ν *is the number of rows in* **Z**, *then*

$$l_{is} = m_i \cdot (a_s, -b_s) = \sum_{r=1}^{4} \xi_r^{(i)} a_{sr} - k^{(i)} b_s,$$

where a_{sr} $(s = \overline{1,7}, r = \overline{1,4})$ *are the elements of the matrix* **A**.

Proof: In other words, this means, that the scalar product of any vector-row of the left-hand side of the matix **Z** with the vector

$$(a_s, -b_s) = (a_{s1}, a_{s2}, a_{s3}, a_{s4}, -b_s),$$

where a_s $(s = \overline{1,7})$ are the rows of the matrix **A** and b_s $(s = \overline{1,7})$ are the elements of the vector-column **B**, is equal to the number at the intersection of the corresponding row of the matrix **Z** and the sth column of its right-hand side. This property is obvoius for the initial matrix, and for all subsequent matrices it follows from the well-known properties of the scalar product.

q.e.d.

Now let an $i \in \{\overline{1,\nu}\}$ be fixed and consider the i th row of the matrix $\mathbf{Z}$ denoted by $(m_i \mid l_i)$. Consider the right-hand part of this row, namely, l_i. The set of zero-elements of this row denote by $J_i := \{j : l_{is_j} = 0\}$.

Theorem 1 *If* $|J_i| < 4 \quad (i \in \{1,\ldots,\nu\})$ *then* $\left(\frac{\xi_1^{(i)}}{k^{(i)}}, \frac{\xi_2^{(i)}}{k^{(i)}}, \frac{\xi_3^{(i)}}{k^{(i)}}, \frac{\xi_4^{(i)}}{k^{(i)}}\right)$ *is not a vertex of the polyhedron* $\Im$.

Proof: First of all, assume $J_i = \emptyset$ for some $i \in \{1,\ldots,\nu\}$. It means that for all $s = \overline{1,7}$

$$l_{is} = \sum_{r=1}^{4} a_{sr}\xi_r^{(i)} - b_{sr}k^{(i)} > 0 \quad (k_i > 0).$$

Then, obviously, the point $\left(\frac{\xi_1^{(i)}}{k^{(i)}}, \frac{\xi_2^{(i)}}{k^{(i)}}, \frac{\xi_3^{(i)}}{k^{(i)}}, \frac{\xi_4^{(i)}}{k^{(i)}}\right)$ can not be a vertex of the polyhedron $\Im$.

Now assume , there is an $\bar{i} \in \{1,\ldots,\nu\}$ such that $0 < |J_{\bar{i}}| < 4$. It implies

$$\sum_{r=1}^{4} a_{s_j r}\frac{\xi_r^{(i)}}{k^{(i)}} = b_{s_j r} \quad (k^{(i)} > 0) \quad (j \in J_{\bar{i}}).$$

Clearly, the system constituted by those equalities is not a fundamental one. Hence its solution is not a vertex of the polyhedron $\Im$.

q.e.d.

Corollary 1 *If* $\mathbf{Z} = (\mathbf{M}|\mathbf{L})$ *is the last matrix produced by the application of the Chernikova's algorithm, then the vertices of the polyheder* $\Im$ *are delivered only by these rows* $m_i \quad (i \in \{1,\ldots,\nu\})$ *of the matrix* $\mathbf{M}$ *which corresponding rows* l_i *of the matrix* $\mathbf{L}$ *contain at least four zero-elements, i.e.* $|J_i| \geq 4$.

Remark 1 *It was due to this fact that the* step 6 *has been added to the computional scheme. Note that the Chernikova's description of the algorithm did not contain this step. Note also, that the computional programm of this algorithm, which was developed by Mattheis and Rubin [15], implements a similar test.*

Corollary 2 *Let the matrix* $\mathbf{Z}$ *be such that* $l_{is_k} \geq 0$ *for all* $i = 1,\ldots,\nu$ *and* $k = 1,\ldots,6$. *There is a pair* $(i,r) \quad (1 \leq i \leq \nu, \quad 1 \leq r \leq \nu)$, *such that* $l_{is_7} \cdot l_{rs_7} < 0$. *If* $|J_i \cap J_r| < 3$, *then the corresponding pair of rows* $(m_i|l_i)$ *and* $(m_r|l_r)$ *do not contribute a new row to the next matrix* $\bar{\mathbf{Z}}$.

6 The Vertices of the Polyhedron $\Im(\Gamma)$. The Extreme BIC and IR Mechanisms

Here we will consider a fee-game $\Gamma \in \Sigma^1$ with the fee-vectors b^α, b^β, such that $0 < eb^\alpha < 1, 0 < eb^\beta < 1$ and $b_1^\alpha < b_1^\beta$. We also will assume, that $p_\alpha > 0$, $p_\beta > 0$ $(p_\alpha + p_\beta = 1)$. The following claims will be proved under these assumptions.

Proposition 3 *Let $\Gamma \in \Sigma^1$ be a fee - game. Then the vertices of the polyhedron $\Im$ can be found by processing the matrix* $\mathbf{Z}^* =$

$$\left(\begin{array}{ccccc|cccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\
b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & (-b_2^\alpha p_\alpha - b_2^\beta p_\beta) & 0 & 0 \\
b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & (1-b_2^\alpha - b_1^\beta)p_\alpha & 0 & 1 \\
b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & (1-b_1^\beta - b_2^\alpha p_\alpha - b_2^\beta p_\beta) & 0 & 0
\end{array}\right).$$

Proof: It is not difficult to see that the initial matrix $\mathbf{Z}^0 = (\mathbf{I}|\mathbf{A}-\mathbf{B})$ corresponding to the Chernikova's computional scheme is the next one:

$$\left(\begin{array}{ccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 1-b_1^\alpha & b_1^\beta - 1 & 1-b_1^\alpha & 0 & -b_2^\alpha p_\alpha & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -b_1^\alpha & b_1^\beta & -b_1^\alpha & 0 & (1-b_2^\alpha)p_\alpha & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & b_1^\alpha - 1 & 1-b_1^\beta & 0 & 1-b_1^\beta & -b_2^\beta p_\beta & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & b_1^\alpha & -b_1^\beta & 0 & -b_1^\beta & (1-b_2^\beta)p_\beta & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right).$$

We transform this matrix according to the described algorithm taking as the leading column the first one. The new matrix is constituted by the first, the fourth and the fifth rows of the initial matrix, and also by the rows which are linear combinations of the first and the second, of the first and the third, of the second and the fourth, of the third and the fourth rows respectively. So we get the matrix $\mathbf{Z}^1 = (\mathbf{M}^1|\mathbf{L}^1)$ with $\mathbf{M}^1$ and $\mathbf{L}^1$ as follows:

$$\mathbf{M}^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & b_1^\alpha & 1-b_1^\alpha & 0 \end{pmatrix}$$

and

$$\mathbf{L}^1 = \begin{pmatrix} + & b_1^\beta - 1 & 1-b_1^\alpha & 0 & -b_2^\alpha p_\alpha & -1 & 0 \\ + & -b_1^\beta & 0 & -b_1^\beta & (1-b_2^\beta)p_\beta & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & b_1^\beta - b_1^\alpha & 0 & 0 & (1-eb^\alpha)p_\alpha & -1 & 0 \\ 0 & 0 & 1-b_1^\alpha & 1-b_1^\beta & (-b_2^\alpha p_\alpha - b_2^\beta p_\beta) & -1 & -1 \\ 0 & 0 & -b_1^\alpha & -b_1^\beta & (1-b_2^\alpha p_\alpha - b_2^\beta p_\beta) & -1 & -1 \\ 0 & b_1^\alpha - b_1^\beta & 0 & b_1^\alpha - b_1^\beta & (1-b_1^\alpha - b_2^\beta)p_\beta & 0 & -1 \end{pmatrix}.$$

Notice. The actual values of the positive elements of the processed columns play no part in the solution of our problem and therefore do not be evaluated, we denote them by the sign +.

Now we take the second column of the matrix $\mathbf{Z}^1$ as the leading one. The transformed martix $\mathbf{Z}^2$ has the next components:

$$\mathbf{M}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ b_1^\beta(1-b_1^\alpha) & (1-b_1^\alpha)(1-b_1^\beta) & 0 & 0 & 0 \end{pmatrix}.$$

and

$\mathbf{L}^2 =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & + & 0 & 0 & (1-eb^{\alpha})p_\alpha & -1 & 0 \\ 0 & 0 & 1-b_1^\alpha & 1-b_1^\beta & (-b_2^\alpha p_\alpha - b_2^\beta p_\beta) & -1 & -1 \\ 0 & 0 & -b_1^\alpha & b_1^\beta & (1-b_2^\alpha p_\alpha - b_2^\beta p_\beta) & -1 & -1 \\ + & 0 & (1-b_1^\alpha)(b_1^\beta - b_1^\alpha) & 0 & (1-b_1^\alpha)(1-b_2^\alpha - b_1^\beta)p_\alpha & (b_1^\alpha - 1) & 0 \end{pmatrix}.$$

As we can see, the matrix $\mathbf{Z}^2$ consists of four rows of the previous matrix and its last row is a linear combination of the first and the fourth row of $\mathbf{Z}^1$. Note that, for example, the second and the fourth roes of $\mathbf{Z}^1$ are not combined, according to the rule: their 'common ' zeros - in the third and the fifth columns - are also 'zeros' for the first row.

Let us denote

$$Eb_2^\tau := b_2^\alpha p_\alpha + b_2^\beta p_\beta, \quad \eta := 1 - b_2^\alpha - b_1^\beta, \quad \zeta := 1 - b_1^\beta - b_2^\alpha p_\alpha - b_2^\beta p_\beta.$$

Further we process the third column of the matrix $\mathbf{Z}^2$ and receive the components of the next matrix $\mathbf{Z}^3$:

$$\mathbf{M}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & (1-b_1^\alpha) & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ b_1^\beta & (1-b_1^\beta) & 0 & 0 & 0 \\ b_1^\alpha & (1-b_1^\alpha) & b_1^\alpha & (1-b_1^\alpha) & 0 \\ b_1^\alpha b_1^\beta & (1-b_1^\alpha)b_1^\beta & 0 & (b_1^\beta - b_1^\alpha) & 0 \end{pmatrix}$$

and

$$\mathbf{L}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & + & 0 & 0 & (1-eb^{\alpha})p_\alpha & -1 & 0 \\ 0 & 0 & + & (1-b_1^\beta) & -Eb_2^\tau & -1 & -1 \\ + & 0 & + & 0 & \eta p_\alpha & -1 & 0 \\ 0 & 0 & 0 & (b_1^\alpha - b_1^\beta) & \zeta & -1 & -1 \\ + & 0 & 0 & b_1^\beta(b_1^\alpha - b_1^\beta) & (1-Eb_2^\tau)(b_1^\beta - b_1^\alpha + \eta p_\alpha) & -b_1^\beta & (b_1^\alpha - b_1^\beta) \end{pmatrix}.$$

After processing the fourth column we have the following matrix
$\mathbf{Z}^4 =$

$$\left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ b_1^\alpha & (1-b_1^\alpha) & 0 & 0 & 0 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & + & + & -Eb_2^\tau & -1 & -1 \\ b_1^\beta & (1-b_1^\beta) & 0 & 0 & 0 & + & 0 & + & 0 & \eta p_\alpha & -1 & 0 \\ b_1^\beta & (1-b_1^\beta) & b_1^\beta & (1-b_1^\beta) & 0 & 0 & 0 & + & 0 & \zeta & -1 & -1 \end{array}\right).$$

Now we decide to take as the leading column the sixth one. As the resulting matrix we have
$\mathbf{Z}^5 =$

$$\left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\ b_1^\alpha & (1-b_1^\alpha) & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0 \\ b_1^\beta & (1-b_1^\beta) & 0 & 0 & 1 & + & 0 & + & 0 & \eta p_\alpha & 0 & 1 \\ b_1^\beta & (1-b_1^\beta) & b_1^\beta & (1-b_1^\beta) & 1 & 0 & 0 & + & 0 & \zeta & 0 & 1 \end{array}\right).$$

Note also that there is only one column,namely the fifth, which contains negative elements. Hence there is only one column to be processed. How many elements of this column are negative depends on the fee-vectors b^α and b^β. Finally, it is not difficult to see, that

$$\mathbf{Z}^5 = \mathbf{Z}^*.$$

q.e.d.

Lemma 1 *If*

$$\eta = 1 - b_1^\beta - b_2^\alpha \geq 0,$$

then

$$\zeta = 1 - b_1^\beta - Eb_2^\tau = 1 - b_1^\beta - b_2^\alpha p_\alpha - b_2^\beta p_\beta > 0.$$

Proof: First of all, we note that

$$\begin{aligned}\zeta = 1 - b_1^\beta - Eb_2^\tau = \\ 1 - b_1^\beta - b_2^\alpha p_\alpha - b_2^\beta p_\beta = \\ 1 - b_1^\beta - b_2^\alpha (1 - p_\beta) - b_2^\beta p_\beta = \\ 1 - b_1^\beta - b_2^\alpha + (b_2^\alpha - b_2^\beta) p_\beta = \eta + (b_2^\alpha - b_2^\beta) p_\beta.\end{aligned}$$

Let $\eta = 1 - b_2^\alpha - b_1^\beta = 0$, then $1 - b_1^\beta = b_2^\alpha$. From $1 - b_1^\beta - b_2^\beta > 0$ follows $1 - b_1^\beta > b_2^\beta$. Hence $b_2^\alpha = 1 - b_1^\beta > b_2^\beta$ and, finally, $b_2^\alpha > b_2^\beta$. So it is not difficult to see that

$$\zeta = 1 - b_1^\beta - Eb_2^\tau > 0.$$

Further take $\eta = 1 - b_2^\alpha - b_1^\beta > 0$. Since $1 - b_1^\beta - b_2^\beta > 0$, next inequalities hold true $-b_2^\beta > -1 + b_1^\beta$ and $b_2^\alpha - b_2^\beta > b_2^\alpha - 1 + b_1^\beta$. Then we have

$$\begin{aligned}\zeta = 1 - b_1^\beta - Eb_2^\tau = 1 - b_1^\beta - b_2^\alpha + (b_2^\alpha - b_2^\beta) p_\beta > \\ 1 - b_1^\beta - b_2^\alpha - (1 - b_1^\beta - b_2^\alpha) p_\beta = \\ (1 - b_1^\beta - b_2^\alpha)(1 - p_\beta) = \eta(1 - p_\beta) > 0.\end{aligned}$$

q.e.d.

Theorem 2 *Let $\Gamma \in \Sigma^1$ be a fee-game . If $\eta = 1 - b_2^\alpha - b_1^\beta > 0$, then the vertices of the corresponding polyhedron $\Im(\Gamma)$ are*

$$\begin{aligned}\xi^{(1)} = (0, 0, 0, 0), \\ \xi^{(2)} = (b_1^\alpha, 1 - b_1^\alpha, 0, 0), \\ \xi^{(3)} = (b_1^\beta, 1 - b_1^\beta, b_1^\beta, 1 - b_1^\beta), \\ \xi^{(4)} = (1 - Eb_2^\tau, Eb_2^\tau, 1 - Eb_2^\tau, Eb_2^\tau).\end{aligned}$$

Proof: As if follows from the **Proposition 3**, to find the vertices of the polyhedron $\Im$ we have to procees one column of the matrix
$\mathbf{Z}^* =$

$$\left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & + \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & + \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0 \\ b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & (1-b_2^\alpha-b_1^\beta)p_\alpha & 0 & + \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & (1-b_1^\beta-Eb_2^\tau) & 0 & 0 \end{array}\right).$$

In this case, due to **Lemma 1**, there is only one negative element, namely $-Eb_2^\tau$, in the leading column

$$(0,(1-eb^\alpha)p_\alpha,-Eb_2^\tau,(1-b_2^\alpha-b_1^\beta)p_\alpha,(1-b_1^\beta-Eb_2^\tau))^T.$$

Therefore all rows of $\mathbf{Z}^*$, except the third one, are included in the next matrix. To this matrix we should also add an additional row which is a linear combination of the third and the last rows. (Combinations of the third row with the second and the fourth rows do not contribute any additional rows because they have less then three 'common zeros'.) As a result, we have the following matrix $\tilde{\mathbf{Z}} = (\tilde{\mathbf{M}}|\tilde{\mathbf{L}})$, where
$\tilde{\mathbf{M}} =$

$$\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 \\ (1-b_1^\beta)(1-Eb_2^\tau) & (1-b_1^\beta)Eb_2^\tau & (1-b_1^\beta)(1-Eb_2^\tau) & (1-b_1^\beta)Eb_2^\tau & (1-b_1^\beta) \end{array}\right)$$

and

$$\tilde{\mathbf{L}} = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & + & + \\ 0 & + & 0 & 0 & + & 0 & + \\ + & 0 & + & 0 & + & 0 & + \\ 0 & 0 & + & 0 & + & 0 & 0 \\ 0 & 0 & + & + & 0 & 0 & 0 \end{array}\right).$$

The process is completed because there are only nonnegative elements on the right-hand side in the matrix $\tilde{\mathbf{L}}$. Now we choose rows $(\tilde{m}_i|\tilde{l}_i)$ such that $|J_i| \geq$

$0 \quad (i = \overline{1,4})$, or in other words, rows containing on the right at least four zeros . Their corresponding left parts are

$$\tilde{m}_1 = (0,0,0,0,1),$$
$$\tilde{m}_2 = (b_1^\alpha, 1-b_1^\alpha, 0, 0, 1),$$
$$\tilde{m}_3 = (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta, 1),$$
$$\tilde{m}_4 = ((1-b_1^\beta)(1-Eb_2^\tau), (1-b_1^\beta)Eb_2^\tau, (1-b_1^\beta)(1-Eb_2^\tau), (1-b_1^\beta)Eb_2^\tau, (1-b_1^\beta)).$$

They deliver the vertices of the polyhedron $\Im$:

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0),$$
$$\xi^{(3)} = (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta),$$
$$\xi^{(4)} = ((1-Eb_2^\tau), Eb_2^\tau, (1-Eb_2^\tau), Eb_2^\tau).$$

q.e.d.

Corollary 3 *Let $\Gamma \in \Sigma^1$ be a fee-game such as in the* **Theorem 2**. *Then the extreme BIC and IR mechanisms are as follows:*

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$
$$\mu^{(3)} = ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)),$$
$$\mu^{(4)} = ((1-Eb_2^\tau, Eb_2^\tau),(1-Eb_2^\tau, Eb_2^\tau)).$$

Proof : Going back to our previous notations, we can wrire:

$$\mu_1^{\alpha(i)} := \xi_1^{(i)}, \quad \mu_2^{\alpha(i)} := \xi_2^{(i)},$$
$$\mu_1^{\beta(i)} := \xi_1^{(i)}, \quad \mu_2^{\beta(i)} := \xi_2^{(i)}$$
$$(i = \overline{1,4}).$$

Then we also have

$$\mu^{(i)} = (\mu^{\alpha(i)}, \mu^{\beta(i)}) \quad (i = \overline{1,4}).$$

q.e.d.

Theorem 3 *Let $\Gamma \in \Sigma^1$ be a fee-game. If $\eta = 1 - b_2^\alpha - b_1^\beta = 0$, then the vertices of $\Im$ are as follows:*

$$\begin{aligned}
\xi^{(1)} &= (0,0,0,0),\\
\xi^{(2)} &= (b_1^\alpha, 1-b_1^\alpha, 0, 0),\\
\xi^{(3)} &= (b_1^\beta, 1-b_1^\beta, 0, 0),\\
\xi^{(4)} &= (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta),\\
\xi^{(5)} &= ((1-Eb_2^\tau), Eb_2^\tau, (1-Eb_2^\tau), Eb_2^\tau).
\end{aligned}$$

Proof : In view of the **Lemma 1** the matrix $\mathbf{Z}^*$ to be transformed is the next one:

$$\left(\begin{array}{ccccc|ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0\\
b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & 1\\
b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & 0 & 0 & 1\\
b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & \zeta & 0 & 0
\end{array}\right).$$

After processing the leading column we have the following matrix $\tilde{\mathbf{Z}} = (\tilde{\mathbf{M}}|\tilde{\mathbf{L}})$, where

$\tilde{\mathbf{M}} =$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1\\
b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1\\
b_1^\beta & 1-b_1^\beta & 0 & 0 & 1\\
b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1\\
(1-b_1^\beta)(1-Eb_2^\tau) & (1-b_1^\beta)Eb_2^\tau & (1-b_1^\beta)(1-Eb_2^\tau) & (1-b_1^\beta)Eb_2^\tau & (1-b_1^\beta)
\end{pmatrix}$$

and

$$\tilde{\mathbf{L}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & + & +\\
0 & + & 0 & 0 & + & 0 & +\\
+ & 0 & + & 0 & 0 & 0 & +\\
0 & 0 & + & 0 & + & 0 & 0\\
0 & 0 & + & + & 0 & 0 & 0
\end{pmatrix}.$$

It is obvious, that $|J_i| \geq 0$ for all $i = \overline{1,5}$. It means, that every row $(\tilde{m}_i)$ $(i = \overline{1,5})$ delivers a vertex of the polyhedron $\Im$. Then the vertices of $\Im$ are as follows:

$$\begin{aligned}
\xi^{(1)} &= (0,0,0,0),\\
\xi^{(2)} &= (b_1^\alpha, 1-b_1^\alpha, 0, 0),\\
\xi^{(3)} &= (b_1^\beta, 1-b_1^\beta, 0, 0),\\
\xi^{(4)} &= (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta),\\
\xi^{(5)} &= ((1-Eb_2^\tau), Eb_2^\tau, (1-Eb_2^\tau), Eb_2^\tau).
\end{aligned}$$

q.e.d.

Corollary 4 *Let $\Gamma \in \Sigma^1$ be a fee-game such as in the* **Theorem 3**. *Then the extreme BIC and IR mechanisms are as follows:*

$$\begin{aligned}
\mu^{(1)} &= ((0,0),(0,0)),\\
\mu^{(2)} &= ((b_1^\alpha, 1-b_1^\alpha),(0,0)),\\
\mu^{(3)} &= ((b_1^\beta, 1-b_1^\beta),(0,0)),\\
\mu^{(4)} &= ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)),\\
\mu^{(5)} &= ((1-Eb_2^\tau, Eb_2^\tau),(1-Eb_2^\tau, Eb_2^\tau)).
\end{aligned}$$

Proof is obvious.

q.e.d.

Theorem 4 *Let $\Gamma \in \Sigma^1$ be a fee-game . If $\eta = 1 - b_2^\alpha - b_1^\beta < 0$ and $\zeta = 1 - b_1^\beta - Eb_2^\tau > 0$ then the vertices of $\Im(\Gamma)$ are as follows:*

$$\begin{aligned}
\xi^{(1)} &= (0,0,0,0),\\
\xi^{(2)} &= (b_1^\alpha, 1-b_1^\alpha, 0, 0),\\
\xi^{(3)} &= (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta),\\
\xi^{(4)} &= ((1-Eb_2^\tau), Eb_2^\tau, (1-Eb_2^\tau), Eb_2^\tau),\\
\xi^{(5)} &= \left(b_1^\beta, 1-b_1^\beta, -\frac{b_1^\beta \eta p_\alpha}{(1-eb^\beta)p_\beta}, -\frac{(1-b_1^\beta)\eta p_\alpha}{(1-eb^\beta)p_\beta}\right).
\end{aligned}$$

Proof : Now we have to process the next matrix $\mathbf{Z}^* =$

$$\left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & \eta p_\alpha & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & \zeta & 0 & 0 \end{array}\right).$$

Taking into account that $\eta < 0$ and $\zeta > 0$, we have the following matrix $\tilde{\mathbf{Z}} = (\tilde{\mathbf{M}}|\tilde{\mathbf{L}})$, where
$\tilde{\mathbf{M}} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 \\ (1-Eb_2^\tau) & Eb_2^\tau & (1-Eb_2^\tau) & Eb_2^\tau & 1-b_1^\beta \\ b_1^\beta(1-eb^\beta)p_\beta & (1-b_1^\beta)(1-eb^\beta)p_\beta & b_1^\beta\eta p_\alpha & (1-b_1^\beta)\eta p_\alpha & (1-eb^\beta)p_\beta \end{pmatrix}$$

(note that here, for the sake of notation simplicity, all elements of the fourth row are devided by $1 - b_1^\beta > 0$)
and

$$\tilde{\mathbf{L}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & + & + \\ 0 & + & 0 & 0 & + & 0 & + \\ 0 & 0 & + & 0 & + & 0 & 0 \\ 0 & 0 & + & + & 0 & 0 & 0 \\ + & 0 & + & + & 0 & 0 & + \end{pmatrix}.$$

Since every row of the matrix $\tilde{\mathbf{L}}$ includes at least four zero-elements, all rows of the matrix $\tilde{\mathbf{M}}$ correspond to vertices of the polyhedron $\Im(\Gamma)$.

q.e.d.

Corollary 5 *Let* $\Gamma \in \Sigma^1$ *be a fee-game such as in the* **Theorem 4**. *Then the extreme BIC and IR mechanisms are as follows:*

$$\mu^{(1)} = ((0,0),(0,0)),$$

$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha), (0,0)),$$
$$\mu^{(3)} = ((b_1^\beta, 1-b_1^\beta), (b_1^\beta, 1-b_1^\beta)),$$
$$\mu^{(4)} = ((1-Eb_2^\tau, Eb_2^\tau), (1-Eb_2^\tau, Eb_2^\tau)),$$
$$\mu^{(5)} = \Big((b_1^\beta, 1-b_1^\beta), (-\frac{b_1^\beta \eta p_\alpha}{(1-eb^\beta)p_\beta}, -\frac{(1-b_1^\beta)\eta p_\alpha}{(1-eb^\beta)p_\beta})\Big).$$

Proof is obvious.

q.e.d.

Theorem 5 *Let* $\Gamma \in \Sigma^1$ *be a fee-game . If* $\eta = 1 - b_2^\alpha - b_1^\beta < 0$ *and* $\zeta = 1 - b_1^\beta - Eb_2^\tau = 0$ *then the vertices of* $\Im(\Gamma)$ *are as follows:*

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0),$$
$$\xi^{(3)} = (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta).$$

Proof : Now we have to transform the next matrix $\mathbf{Z}^* =$

$$\left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & (1-eb^\alpha)p_\alpha & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & \eta p_\alpha & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & 0 & 0 & 0 \end{array}\right).$$

As usual, we process the leading column

$$(0, -Eb_2^\tau, (1-eb^\alpha)p_\alpha, \eta p_\alpha, 0)^T.$$

So we receive the matrix

$$\tilde{\mathbf{Z}} = \left(\begin{array}{ccccc|ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & + & 0 & 1 \\ b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & 0 & 0 & 0 \end{array}\right).$$

Note, that no additional rows nave been adjoin to the matrix $\tilde{\mathbf{Z}}$. It is obvious, that there are three vertices of $\Im$ in this case:

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0),$$
$$\xi^{(3)} = (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta).$$

q.e.d.

Corollary 6 *Let $\Gamma \in \Sigma^1$ be a fee-game such as in the* **Theorem 5**. *Then the extreme BIC and IR mechanisms are as follows:*

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$
$$\mu^{(3)} = ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)).$$

Proof is obvious.

q.e.d.

Theorem 6 *Let $\Gamma \in \Sigma^1$ be a fee-game . If $\eta = 1 - b_2^\alpha - b_1^\beta < 0$ and $\zeta = 1 - b_1^\beta - Eb_2^\tau < 0$ then the vertices of $\Im(\Gamma)$ are as follows:*

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0),$$
$$\xi^{(3)} = \Big(\frac{b_1^\beta(1-eb^\alpha)p_\alpha - b_1^\alpha\zeta}{(1-eb^\alpha)p_\alpha - \zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha - (1-b_1^\alpha)\zeta}{(1-eb^\alpha)p_\alpha - \zeta},$$
$$\frac{b_1^\beta(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha - \zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha - \zeta}\Big).$$

Proof : We will use the next notation

$$\omega := (1-eb^\alpha)p_\alpha.$$

Now, processing the leading column

$$(0, -Eb_2^\tau, \omega, \eta p_\alpha, \zeta)^T$$

of the matrix $\mathbf{Z}^* =$

$$\left(\begin{array}{ccccc|ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & + & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & + & + & -Eb_2^\tau & 0 & 0 \\
b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 & 0 & + & 0 & 0 & \omega & 0 & 1 \\
b_1^\beta & 1-b_1^\beta & 0 & 0 & 1 & + & 0 & + & 0 & \eta p_\alpha & 0 & 1 \\
b_1^\beta & 1-b_1^\beta & b_1^\beta & 1-b_1^\beta & 1 & 0 & 0 & + & 0 & 0 & 0 & 0
\end{array}\right),$$

we receive the following matrix $\tilde{\mathbf{Z}} = (\tilde{\mathbf{M}}|\tilde{\mathbf{L}})$, where
$\tilde{\mathbf{M}} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ b_1^\alpha & 1-b_1^\alpha & 0 & 0 & 1 \\ (b_1^\beta\omega - b_1^\alpha\zeta) & ((1-b_1^\beta)\omega - (1-b_1^\alpha)\zeta) & b_1^\beta\omega & (1-b_1^\beta)\omega & (\omega-\zeta) \end{pmatrix}$$

and

$$\tilde{\mathbf{L}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & + & 1 \\ 0 & + & 0 & 0 & + & 0 & 1 \\ 0 & + & + & 0 & 0 & 0 & \omega-\zeta \end{pmatrix}.$$

Obviously, $\omega - \zeta > 0$. Hence the vertices of the polyhedron $\Im$ will be the next:

$$\xi^{(1)} = (0,0,0,0),$$

$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0),$$

$$\xi^{(3)} = \Big(\frac{b_1^\beta\omega - b_1^\alpha\zeta}{\omega-\zeta}, \frac{(1-b_1^\beta)\omega - (1-b_1^\alpha)\zeta}{\omega-\zeta}, \frac{b_1^\beta\omega}{\omega-\zeta}, \frac{(1-b_1^\beta)\omega}{\omega-\zeta}\Big).$$

q.e.d.

Corollary 7 *Let* $\Gamma \in \Sigma^1$ *be a fee-game such as in the* **Theorem 6**. *Then the extreme BIC and IR mechanisms are as follows:*

$$\mu^{(1)} = ((0,0),(0,0)),$$

$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$

$$\mu^{(3)} = \Big(\Big(\frac{b_1^\beta(1-eb^\alpha)p_\alpha - b_1^\alpha\zeta}{(1-eb^\alpha)p_\alpha - \zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha - (1-b_1^\alpha)\zeta}{(1-eb^\alpha)p_\alpha - \zeta}\Big), \Big(\frac{b_1^\beta(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha - \zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha - \zeta}\Big)\Big).$$

Proof is obvious.

q.e.d.

Remark 2 *It seems to be of interest to consider a fee-game* $\Gamma \in \Sigma^1$ *under more general assumtions and, for instance, to deal with a game* $\Gamma \in \Sigma^1$ *with fee-vectors* b^α, b^β, *such that* $eb^\alpha = 1, \quad 0 < eb^\beta < 1$.

7 The Algorithm for Finding the Extreme 'Feasible' Mechanisms

Consider a fee-game $\Gamma \in \Sigma^1$, given fee-vectors b^α, b^β, such that $0 < eb^\alpha < 1$, $0 < eb^\beta < 1$, $b_1^\alpha < b_1^\beta$, and $p_\alpha > 0, p_\beta > 0$ $(p_\alpha + p_\beta = 1)$. Then the results of the preceding Section 6 imply the following procedure for determining the extreme BIC and IR mechanisms:

1. Find the value of

$$Eb_2^\tau = b_2^\alpha p_\alpha + b_2^\beta p_\beta.$$

2. Find the value of

$$\eta = 1 - b_2^\alpha - b_1^\beta.$$

   Then

   (a) if $\eta > 0$ write the extreme mechanisms as follows:

$$\begin{aligned}
\mu^{(1)} &= ((0,0),(0,0)),\\
\mu^{(2)} &= ((b_1^\alpha, 1-b_1^\alpha),(0,0)),\\
\mu^{(3)} &= ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)),\\
\mu^{(4)} &= ((1-Eb_2^\tau, Eb_2^\tau),(1-Eb_2^\tau, Eb_2^\tau)).
\end{aligned}$$

   (b) if $\eta = 0$ write the extreme mechanisms as follows:

$$\begin{aligned}
\mu^{(1)} &= ((0,0),(0,0)),\\
\mu^{(2)} &= ((b_1^\alpha, 1-b_1^\alpha),(0,0)),\\
\mu^{(3)} &= ((b_1^\beta, 1-b_1^\beta),(0,0)),\\
\mu^{(4)} &= ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)),\\
\mu^{(5)} &= ((1-Eb_2^\tau, Eb_2^\tau),(1-Eb_2^\tau, Eb_2^\tau)).
\end{aligned}$$

   (c) if $\eta < 0$ go to step 3 .

3. Find the value of

$$\zeta = 1 - b_1^\beta - Eb_2^\tau.$$

   Then

(a) if $\zeta > 0$ write the extreme mechanisms as follows:

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$
$$\mu^{(3)} = ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta));$$
$$\mu^{(4)} = ((1-Eb_2^\tau, Eb_2^\tau),(1-Eb_2^\tau, Eb_2^\tau)),$$
$$\mu^{(5)} = \Big((b_1^\beta, 1-b_1^\beta),(-\frac{b_1^\beta \eta p_\alpha}{(1-eb^\beta)p_\beta}, -\frac{(1-b_1^\beta)\eta p_\alpha}{(1-eb^\beta)p_\beta})\Big).$$

(b) if $\zeta = 0$ write the extreme mechanisms as follows:

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$
$$\mu^{(3)} = ((b_1^\beta, 1-b_1^\beta),(b_1^\beta, 1-b_1^\beta)).$$

(c) if $\zeta > 0$ write the extreme mexanisms as follows:

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = ((b_1^\alpha, 1-b_1^\alpha),(0,0)),$$
$$\mu^{(3)} = \Big(\Big(\frac{b_1^\beta(1-eb^\alpha)p_\alpha - b_1^\alpha\zeta}{(1-eb^\alpha)p_\alpha-\zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha-(1-b_1^\alpha)\zeta}{(1-eb^\alpha)p_\alpha-\zeta}\Big), \Big(\frac{b_1^\beta(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha-\zeta}, \frac{(1-b_1^\beta)(1-eb^\alpha)p_\alpha}{(1-eb^\alpha)p_\alpha-\zeta}\Big)\Big).$$

Remark 3 *It should be useful to develop a computer program corresponding to this algorithm.*

Remark 4 *In the future research we aim to provide an algorithm similar to this one for finding all extreme BIC and IR mechanisms of a game* $\Gamma \in \Sigma^n$*, where*

$$\Sigma^n := \{\Gamma : \Gamma \quad \textit{is a fee-game with incomplete information on one side and} \quad |\mathcal{T}^1| = n\}.$$

8 Examples

Example 1 *(see [21]) Find all vertices of* $\Im = \Im(\Gamma)$*, given a* $\Gamma \in \Sigma^1$ *such that* $b^\alpha = (\frac{1}{10}, \frac{1}{10})$*,* $b^\beta = (\frac{7}{10}, \frac{1}{10})$*. Let* $p_\alpha = \frac{1}{10}$*,* $p_\beta = \frac{9}{10}$*.*

Since $\eta = 1 - b_1^\beta - b_2^\alpha = \frac{1}{5} > 0$ we make use of the **Theorem 2**. Then the vertices of the polyhedron $\Im$ are :

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1 - b_1^\alpha, 0, 0) = \Big(\frac{1}{10}, \frac{9}{10}, 0, 0\Big),$$
$$\xi^{(3)} = (b_1^\beta, 1 - b_1^\beta, b_1^\beta, 1 - b_1^\beta) = \Big(\frac{7}{10}, \frac{3}{10}, \frac{7}{10}, \frac{3}{10}\Big),$$
$$\xi^{(4)} = (1 - Eb_2^\tau, Eb_2^\tau, 1 - Eb_2^\tau, Eb_2^\tau) = \Big(\frac{9}{10}, \frac{1}{10}, \frac{9}{10}, \frac{1}{10}\Big).$$

According to the **Corollary 3** the extreme mechanisms are the following:

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = \Big(\Big(\frac{1}{10}, \frac{9}{10}\Big), \big(0,0\big)\Big),$$
$$\mu^{(3)} = \Big(\Big(\frac{7}{10}, \frac{3}{10}\Big), \Big(\frac{7}{10}, \frac{3}{10}\Big)\Big),$$
$$\mu^{(4)} = \Big(\Big(\frac{9}{10}, \frac{1}{10}\Big), \Big(\frac{9}{10}, \frac{1}{10}\Big)\Big).$$

Example 2 *Find all vertices of* $\Im = \Im(\Gamma)$, *given a* $\Gamma \in \Sigma^1$ *such that* $b^\alpha = (\frac{2}{3}, \frac{1}{4})$, $b^\beta = (\frac{3}{4}, \frac{1}{8})$. *Let* $p_\alpha = \frac{1}{3}$, $p_\beta = \frac{2}{3}$.

Here $\eta = 1 - b_1^\beta - b_2^\alpha = 0$, therefore in accordance with the **Theorem 3** we get the vertices of the polyhedron $\Im$:

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1 - b_1^\alpha, 0, 0) = \Big(\frac{2}{3}, \frac{1}{3}, 0, 0\Big),$$
$$\xi^{(3)} = (b_1^\beta, 1 - b_1^\beta, 0, 0) = \Big(\frac{3}{4}, \frac{1}{4}, 0, 0\Big),$$
$$\xi^{(4)} = (b_1^\beta, 1 - b_1^\beta, b_1^\beta, 1 - b_1^\beta) = \Big(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}\Big),$$
$$\xi^{(5)} = ((1 - Eb_2^\tau), Eb_2^\tau, (1 - Eb_2^\tau), Eb_2^\tau) = \Big(\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}\Big).$$

And from the **Corollary 4** it follows, that

$$\mu^{(1)} = (0,0,0,0),$$
$$\mu^{(2)} = \Big(\Big(\frac{2}{3}, \frac{1}{3}\Big), \big(0,0\big)\Big),$$

$$\mu^{(3)} = \Big(\Big(\frac{3}{4},\frac{1}{4}\Big),\Big(0,0\Big)\Big),$$
$$\mu^{(4)} = \Big(\Big(\frac{3}{4},\frac{1}{4}\Big),\Big(\frac{3}{4},\frac{1}{4}\Big)\Big),$$
$$\mu^{(5)} = \Big(\Big(\frac{5}{6},\frac{1}{6}\Big),\Big(\frac{5}{6},\frac{1}{6}\Big)\Big)$$

are the extreme BIC and IR mechanisms.

Example 3 *Find all vertices of* $\Im = \Im(\Gamma)$, *given a* $\Gamma \in \Sigma^1$ *such that* $b^\alpha = (\frac{1}{4},\frac{2}{7})$, $b^\beta = (\frac{4}{5},\frac{1}{7})$. *Let* $p_\alpha = \frac{1}{4}$, $p_\beta = \frac{3}{4}$.

Since $\eta = 1 - b_1^\beta - b_2^\alpha = -\frac{3}{35} < 0$ we have to compute the value of $\zeta = 1 - b_1^\beta - b_2^\alpha - b_2^\beta$. It turns out, that $\zeta = \frac{3}{140} > 0$. Hence, we use the results of the **Theorem 4** and of the **Corollary 5**. Therefore, the vertices of the polyhedron $\Im$ are as follows:

$$\xi^{(1)} = (0,0,0,0),$$
$$\xi^{(2)} = (b_1^\alpha, 1-b_1^\alpha, 0, 0) = \Big(\frac{1}{4},\frac{3}{4},0,0\Big),$$
$$\xi^{(3)} = (b_1^\beta, 1-b_1^\beta, b_1^\beta, 1-b_1^\beta) = \Big(\frac{4}{5},\frac{1}{5},\frac{4}{5},\frac{1}{5}\Big),$$
$$\xi^{(4)} = ((1-Eb_2^\tau), Eb_2^\tau, (1-Eb_2^\tau), Eb_2^\tau) = \Big(\frac{23}{28},\frac{5}{28},\frac{23}{28},\frac{5}{28}\Big),$$
$$\xi^{(5)} = \Big(b_1^\beta, 1-b_1^\beta, -\frac{b_1^\beta \eta p_\alpha}{(1-eb^\beta)p_\beta}, -\frac{(1-b_1^\beta)\eta p_\alpha}{(1-eb^\beta)p_\beta}\Big) = \Big(\frac{4}{5},\frac{1}{5},\frac{2}{5},\frac{1}{10}\Big).$$

The corresponding extreme mechanisms are the next:

$$\mu^{(1)} = ((0,0),(0,0)),$$
$$\mu^{(2)} = \Big(\Big(\frac{1}{4},\frac{3}{4}\Big),\Big(0,0\Big)\Big),$$
$$\mu^{(3)} = \Big(\Big(\frac{4}{5},\frac{1}{5}\Big),\Big(\frac{4}{5},\frac{1}{5}\Big)\Big),$$
$$\mu^{(4)} = \Big(\Big(\frac{23}{28},\frac{5}{28}\Big),\Big(\frac{23}{28},\frac{5}{28}\Big)\Big),$$
$$\mu^{(5)} = \Big(\Big(\frac{4}{5},\frac{1}{5}\Big),\Big(\frac{2}{5},\frac{1}{10}\Big)\Big).$$

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