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General aggregation problems and social structure: A model-theoretic generalisation of the Kirman-Sondermann correspondence*

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Abstract

This article proves a very general version of the Kirman-Sondermann [*Journal of Economic Theory*, 5(2):267–277, 1972] correspondence by extending the methodology of Lauwers and Van Liedekerke [*Journal of Mathematical Economics*, 24(3):217–237, 1995]. The paper first proposes a unified framework for the analysis of the relation between various aggregation problems and the social structure they induce, based on first-order predicate logic and model theory. Thereafter, aggregators satisfying Arrow-type rationality axioms are shown to be restricted reduced product constructions with respect to the filter of decisive coalitions; an oligarchic impossibility result follows. Under stronger assumptions, aggregators are restricted ultraproduct constructions, whence a generalised Kirman-Sondermann correspondence as well as a dictatorial impossibility result follow.

Key words: Arrow-type preference aggregation; judgment aggregation; systematicity; model theory; first-order predicate logic; filter; ultrafilter; reduced product; ultraproduct

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1 Introduction

The problem of collective decision making as formulated in Arrow's [1] famous impossibility theorem is a major puzzle in social philosophy, especially in the light of the difficult relation between power and rationality. According to Habermas [10], power neutrality is even a precondition of collective rationality. Recent extensions of the social choice literature from the aggregation of preferences to judgment aggregation however suggest that rationality even in

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the weakest possible sense of logical consistency bears a close relation to power: In fact, the literature on judgment aggregation (for a survey, cf. List and Puppe [17]) shows that the structure of a collective decision problem (essentially via the logical interconnections between the propositions in its agenda) determines the social structure (given by the structure of the set of decisive coalitions) and that this power structure can be as asymmetric as a dictatorship.

This relation between aggregation problems and the power structure they induce is well known since Kirman and Sondermann [13] established the correspondence between aggregation functions and ultrafilters on the set of individuals and has been systematically explored in the use of ultrafilters in the proof of Arrow's theorem (for a survey, cf. e.g. Monjardet [18]) and its analogues in judgment aggregation (cf. Dietrich and Mongin [5] as well as Eckert and Klamler [14]).[1] However, the natural foundation of this relation in model theory, namely via the ultraproduct of models, has only be explored in a seminal paper by Lauwers and Van Liedekerke [15] and some recent sequels (e.g. Herzberg [11]).

In this paper we show that first-order predicate logic and model theory provide a natural unified framework to study the relation between aggregation problems and social structure in great generality, including the aggregation of preferences, propositional judgments and modal propositional judgments. This framework can also be seen as the most natural one for the logical analysis of mathematical models like the ones used in social choice theory.

We shall then characterise aggregators satisfying Arrow-type rationality axioms: We prove that the collection of decisive coalitions forms a filter — and even an ultrafilter under stronger assumptions. Arrow-type aggregators map a given profile of models to a restriction of the reduced product. Under stronger assumptions, aggregators are in a one-to-one correspondence with ultrafilters. As corollaries for finite electorates, we obtain oligarchic and, under stronger assumptions, dictatorial impossibility results.

This paper can be seen as a natural continuation of the visionary article by Lauwers and Van Liedekerke [15] (with a recent correction by Herzberg *et al.* [12]), who were among the first to study aggregation problems using first-order predicate logic and model theory.[2]

2 A unified framework for general aggregation theory

Let A be a countable set. Let $\mathcal{L}$ be a language consisting of at most countably many predicate symbols $\dot{P}_n$, $n \in \mathbf{N}$, and constant symbols $\dot{a}$ for all elements a of A.[3] The arity of $\dot{P}_n$ will be denoted $\delta(n)$, for all $n \in \mathbf{N}$.

[1] Earlier approaches to judgment aggregation also based on the analysis of the structure of the sets of winning coalitions are due to Nehring and Puppe [20, 19].

[2] The work of Lauwers and Van Liedekerke has an early precursor in a paper by Rubinstein [21], who was presumably the first author to systematically explore the methodological value of mathematical logic for generalisations of results in social choice theory (in this case the extension to single profile analogues of classical results involving social welfare functions).

[3] The assumption that $\mathcal{L}$ is countable will not be used, but it is imposed for pedagogical reasons: For, we shall use the completeness of the $\mathcal{L}$-predicate calculus, and if $\mathcal{L}$ were uncountable, Henkin's completeness proof would have to invoke either Zorn's Lemma or the Löwenheim-Skolem theorem. The exposition of the required elements of predicate logic and

Let $\mathcal{S}$ be the set of atomic formulae in $\mathcal{L}$. Let $\mathcal{T}$ be the *Boolean closure* of $\mathcal{S}$, i.e. the closure of $\mathcal{S}$ under the logical connectives $\dot\neg$, $\dot\wedge$, $\dot\vee$. The elements of $\mathcal{T}$ are called *test sentences*, and the elements of $\mathcal{S}$ are called *basic test sentences*.

Let T be a consistent set of universal sentences in $\mathcal{L}$.[4] (As part of the aggregator axioms, an additional assumption on T and $\mathcal{T}$ will be imposed.)

The relational structure $\mathfrak{A} = \langle A, \{R_n : n \in \mathbf{N}\}\rangle$ with $R_n \subset A^n$ for each R_n is an interpretation of L if the arities of the relations R_n correspond to the arities of the predicate symbols $\dot P_n$. In this case, $\mathfrak{A}$ will be called an $\mathcal{L}$-structure. It is a model of the theory T if $\mathfrak{A} \models \varphi$ for all formulae $\varphi \in T$, i.e. if all formulae of the theory hold true in $\mathfrak{A}$.

Let Ω be the collection of models of T with domain A. The restriction of a model $\mathfrak{A} \in \Omega$ is the $\mathcal{L}$-structure that is obtained by restricting the interpretations of the relation symbol to the domain $B \subseteq A$; it is denoted $\mathrm{res}_B\mathfrak{A}$.

We assume that there are two sentences in $\mathcal{S}$, henceforth denoted $\mu, \nu \in \mathcal{S}$, such that each of $\mu\dot\wedge\nu$, $\mu\dot\wedge\dot\neg\nu$ and $\dot\neg\mu\dot\wedge\nu$ is consistent with T, in symbols,

$$T \cup \{\mu\dot\wedge\nu\} \not\vdash \bot, \quad T \cup \{\mu\dot\wedge\dot\neg\nu\} \not\vdash \bot, \quad T \cup \{\dot\neg\mu\dot\wedge\nu\} \not\vdash \bot \tag{1}$$

(wherein $\bot$ is shorthand for $\phi\dot\wedge\dot\neg\phi$ for some sentence ϕ). This assumption already appears in the paper of Lauwers and Van Liedekerke [15]. Its analogue in judgment aggregation can be found in the various assumptions about richness or logical connectedness of the agenda.

We assume that the following propositions hold for all $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$:

$$(\forall\lambda \in \mathcal{S} \qquad (\mathfrak{A}_1 \models \lambda \Leftrightarrow \mathfrak{A}_2 \models \lambda)) \Rightarrow \mathfrak{A}_1 = \mathfrak{A}_2. \tag{2}$$

$$\mathfrak{A} \models T \Rightarrow \mathrm{res}_A\mathfrak{A} \in \Omega \tag{3}$$

$$\forall\lambda \in \mathcal{T} \qquad (\mathfrak{A} \models \lambda \Leftrightarrow \mathrm{res}_A\mathfrak{A} \models \lambda)\,. \tag{4}$$

Remark 2.1. *If $\mathcal{S}$ is the set of all atomic $\mathcal{L}$-formulae, then the implication* (2) *holds for all $\mathfrak{A}_1, \mathfrak{A}_2$ with domain A.*

Proof by contraposition. If $\mathfrak{A}_1 \neq \mathfrak{A}_2$ for $\mathfrak{A}_1 = \left\langle A, \left\{R_n^1 : n \in \mathbf{N}\right\}\right\rangle$ and $\mathfrak{A}_2 = \left\langle A, \left\{R_n^2 : n \in \mathbf{N}\right\}\right\rangle$, then $R_n^1 \neq R_n^2$ for some $n \in \mathbf{N}$. Since $\mathfrak{A}_1, \mathfrak{A}_2 \in \Omega$, both R_n^1 and R_n^2 are (different) subsets of $A^{\delta(n)}$. Hence, there exists some $\langle a_1, \ldots, a_{\delta(n)}\rangle \in A^{\delta(n)}$ such that either $\langle a_1, \ldots, a_{\delta(n)}\rangle \in R_n^1$ and $\langle a_1, \ldots, a_{\delta(n)}\rangle \notin R_n^2$ or $\langle a_1, \ldots, a_{\delta(n)}\rangle \notin R_n^1$ and $\langle a_1, \ldots, a_{\delta(n)}\rangle \in R^2$. In both cases

$$\langle a_1, \ldots, a_{\delta(n)}\rangle \in R_n^1 \not\Leftrightarrow \langle a_1, \ldots, a_{\delta(n)}\rangle \notin R_n^2$$

hence

$$\mathfrak{A}_1 \models \dot P_n(\dot a_1, \ldots, \dot a_{\delta(n)}) \not\Leftrightarrow \mathfrak{A}_2 \models \dot P_n(\dot a_1, \ldots, \dot a_{\delta(n)}),$$

although $\dot P_n(\dot a_1, \ldots, \dot a_{\delta(n)}) \in \mathcal{S}$. □

Remark 2.2. *If T is universal, then the implication* (3) *holds for all $\mathcal{L}$-structures $\mathfrak{A}$.*

Proof. If T is universal and $\mathfrak{A} \models T$, then $\mathrm{res}_A\mathfrak{A} \models T$ and thus $\mathrm{res}_A\mathfrak{A} \in \Omega$. □

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model theory follows the textbook by Bell and Slomson [2].

[4] A sentence is *universal* if it (in its prenex normal form) has the form $(\dot\forall \dot v_{k_1}) \cdots (\dot\forall \dot v_{k_m})\phi$ for some formula ϕ that does not contain any quantifiers.

Remark 2.3. *If $\mathcal{S}$ is a set of atomic formulae, then the equivalence* (4) *holds for any $\mathcal{L}$-structure $\mathfrak{A}$.*

Proof. If $\mathcal{S}$ only consists of atomic formulae with constant symbols for elements of A, clearly

$$\forall \lambda \in \mathcal{S} \qquad (\mathfrak{A} \models \lambda \Leftrightarrow \mathrm{res}_A \mathfrak{A} \models \lambda)\,.$$

Since $\mathcal{T}$ is just the Boolean closure of $\mathcal{S}$, the equivalence (4) is established. $\square$

Elements of Ω^I will be called *profiles*.

An *aggregator* is a map f whose domain $\mathrm{dom}(f)$ is a subset of Ω^I and whose range is a subset of Ω.

For all $\lambda \in \mathcal{T}$ and all $\underline{\mathfrak{A}} \in \Omega^I$, we denote the *coalition supporting λ given profile $\underline{\mathfrak{A}}$*, by

$$C(\underline{\mathfrak{A}}, \lambda) := \{i \in I \ : \ \mathfrak{A}_i \models \lambda\}\,.$$

Let us fix an aggregator f, and let us also fix $\mathcal{S}_P, \mathcal{S}_S \subseteq \mathcal{T}$ (the significance of the subscripts will be explained later). Consider the following axioms:

(A1). $\mathrm{dom}(f) = \Omega^I$.

(A1'). There exist models $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \Omega$ such that

1. $\mathfrak{A}_1 \models \mu \dot\wedge \nu$, $\mathfrak{A}_2 \models \mu \dot\wedge \dot\neg \nu$, $\mathfrak{A}_3 \models \dot\neg \mu \dot\wedge \nu$, and
2. $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3\}^I \subseteq \mathrm{dom}(f)$.

(A2). For all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ and all $\lambda \in \mathcal{S}_P$, if $f(\underline{\mathfrak{A}}) \models \lambda$, then $C(\underline{\mathfrak{A}}, \lambda) \neq \emptyset$.

(A3). For all $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \mathrm{dom}(f)$ and all $\lambda, \lambda' \in \mathcal{S}_S$ such that $C(\underline{\mathfrak{A}}, \lambda) = C(\underline{\mathfrak{A}}', \lambda')$, one has $f(\underline{\mathfrak{A}}) \models \lambda$ if and only if $f(\underline{\mathfrak{A}}') \models \lambda'$.

(A1) is the axiom of *Universality*. Axiom (A2) is a generalised *Pareto Principle*. (A3) is a generalised form of the axiom of *Systematicity*, which itself is a strong variant of the axiom of *Independence of Irrelevant Alternatives*.[5]

If f satisfies (A2) and (A3), we shall call the sets $\mathcal{S}_P$ and $\mathcal{S}_S$, respectively, the *Pareto domain*, and the *systematicity domain* of f.

Axiom (A1') is simply a weak version of (A1): More precisely, (A1) implies (A1') under the assumption of (a), as the following remark shows.

Remark 2.4. *If there exist $\mu, \nu \in \mathcal{S}$ such that $\mu \dot\wedge \nu$, $\mu \dot\wedge \dot\neg \nu$, $\dot\neg \mu \dot\wedge \nu$ are each consistent with T, then there already exist three pairwise different models $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \Omega$ such that $\mathfrak{A}_1 \models \mu \dot\wedge \nu$, $\mathfrak{A}_2 \models \mu \dot\wedge \dot\neg \nu$, and $\mathfrak{A}_3 \models \dot\neg \mu \dot\wedge \nu$.*

[5] Systematicity vacuously implies Independence of Irrelevant Alternatives. The converse is true under additional hypotheses: In the preference aggregation framework, the combination of Independence of Irrelevant Alternatives and the Pareto Principle implies Systematicity if the individual preferences are complete and quasi-transitive (cf. Lauwers and Van Liedekerke [15, Section 6, p. 232]). In the judgment aggregation framework, the combination of Independence of Irrelevant Alternatives and the Pareto Principle implies Systematicity under an additional assumption on the logical interconnections of the propositions in the agenda known as total blockedness (cf. List and Pettit [16], Dietrich and Mongin [5, Lemma 5], Klamler and Eckert [14, Lemma 15]).

Proof. Since each of the three test sentences $\mu\dot{\wedge}\nu$, $\mu\dot{\wedge}\dot{\neg}\nu$, $\dot{\neg}\mu\dot{\wedge}\nu$ is consistent with T, the completeness of predicate logic yields models $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ such that $\mathfrak{B}_1 \models T \cup \{\mu\dot{\wedge}\nu\}$, $\mathfrak{B}_2 \models T \cup \{\mu\dot{\wedge}\dot{\neg}\nu\}$, and $\mathfrak{B}_3 \models T \cup \{\dot{\neg}\mu\dot{\wedge}\nu\}$. Define $\mathfrak{A}_1 := \mathrm{res}_A \mathfrak{B}_1$, $\mathfrak{A}_2 := \mathrm{res}_A \mathfrak{B}_2$, and $\mathfrak{A}_3 := \mathrm{res}_A \mathfrak{B}_3$. By implication (3), $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \Omega$, and by equivalence (4), we have $\mathfrak{A}_1 \models \mu\dot{\wedge}\nu$, $\mathfrak{A}_2 \models \mu\dot{\wedge}\dot{\neg}\nu$, and $\mathfrak{A}_3 \models \dot{\neg}\mu\dot{\wedge}\nu$. Finally, since the three test sentences are pairwise inconsistent, the three models $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ must be pairwise different. $\square$

An aggregator f will be called *Arrow-rational* if and only if the axioms (A1),(A2),(A3) are satisfied for $\mathcal{S}_P = \mathcal{S}_S = \mathcal{T}$; f will be called *weakly Arrow-rational* if and only if the axioms (A1'),(A2),(A3) are satisfied.

We denote by $\mathcal{AR}$ the set of all Arrow-rational aggregators, and by $\mathcal{AR}'$ the set of all weakly Arrow-rational aggregators. Since (A1) implies (A1') (see Remark 2.4), $\mathcal{AR} \subseteq \mathcal{AR}'$.

Given an aggregator f, we define the set of *decisive coalitions* by

$$\mathcal{D}_f := \{C(\underline{\mathfrak{A}}, \lambda) \ : \ \underline{\mathfrak{A}} \in \mathrm{dom}(f), \quad \lambda \in \mathcal{S}_P \cap \mathcal{S}_S, \quad f(\underline{\mathfrak{A}}) \models \lambda\}\,.$$

Remark 2.5. *If f satisfies (A3), then for all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ and $\lambda \in \mathcal{S}_P \cap \mathcal{S}_S$,*

$$C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}_f \Leftrightarrow f(\underline{\mathfrak{A}}) \models \lambda.$$

Proof. Suppose f satisfies (A3), let $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ and $\lambda \in \mathcal{S}_s$. By definition, $f(\underline{\mathfrak{A}}) \models \lambda$ implies $C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}_f$. Conversely, if $C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}_f$, then there exist $\underline{\mathfrak{A}}' \in \mathrm{dom}(f)$ and $\lambda' \in \mathcal{T}$ with $f(\underline{\mathfrak{A}}') \models \lambda'$ and $C(\underline{\mathfrak{A}}, \lambda) = C(\underline{\mathfrak{A}}', \lambda')$. As f satisfies (A3), this means $f(\underline{\mathfrak{A}}) \models \lambda$. $\square$

This framework is sufficiently general to cover the cases of preference aggregation, propositional judgment aggregation, and modal aggregation:

- For the case of preference aggregation, the centrality of binary relations makes it particularly natural to express preferences by a binary predicate in first order logic (cf. Rubinstein [21], Lauwers and Van Liedekerke [15]). A more elaborate formalisation and complete axiomatisation of Arrow's theorem in first order logic was recently given by Grandi and Endriss [7].

- For propositional judgment aggregation à la Dietrich and List [3], one lets $\mathcal{L}$ have a single unary predicate $\dot{B}$, modelling a belief operator. The set A will be the agenda. The interpretation of $\dot{B}\dot{a}$ is "a is accepted". (Thus, the interpretation of $\mathfrak{A}_i \models \dot{B}\dot{a}$ is "under profile $\underline{\mathfrak{A}}$, individual i accepts a", and the interpretation of $f(\underline{\mathfrak{A}}) \models \dot{B}\dot{a}$ is "under profile $\underline{\mathfrak{A}}$, a is socially accepted".) T can be any universal theory in that language.

- For modal propositional judgment aggregation, one simply uses the reduction of modal logic to first-order predicate logic, where the individuals correspond to possible states of the world. Thus, the set A will be the set of states of the world. Let there be in $\mathcal{L}$ one predicate M_p each for the elements p of the agenda, modelling a modal belief operator with world argument and proposition index. Let there also be a binary predicate $\dot{R}$ in $\mathcal{L}$, denoting the accessibility relation. The interpretation of $\dot{R}(\dot{a}, \dot{b})$ will thus be "b is accessible from world a". The interpretation of $\dot{M}_p\dot{a}$ will be "proposition p is accepted in world a". (The interpretation of $\mathfrak{A}_i \models \dot{M}_p \dot{a}$ is thus "under profile $\underline{\mathfrak{A}}$, individual i accepts p in world a", and the interpretation of $f(\underline{\mathfrak{A}}) \models \dot{M}_p \dot{a}$ is "under profile $\underline{\mathfrak{A}}$, p is socially accepted in world a". The modal operator $\Box$ will then not be an operator in the strict sense any longer, but in can be defined as a family of formulae, indexed by p:

$$\Box_p \dot{v}_0 :\equiv \dot{(}\dot{\forall}\dot{v}_1\dot{)}\dot{(}\dot{R}(\dot{v}_0, \dot{v}_1) \rightarrow \dot{M}_p \dot{v}_1\dot{)}$$

The interpretation of $\Box_p \dot{a}$ is "p is accepted in all worlds which are accessible from world a", or just "p is necessarily accepted in world a". T can be any universal theory in that language, which includes the axioms of the modal logical system employed (such as K, S4, S5, etc.).

3 Characterisation of aggregators. Impossibility results

In order to further analyse and characterise aggregators (in particular, through corresponding ultrafilters and related impossibility results), we need to make assumptions on the expressivity of the intersection of the Pareto domain and the systematicity domain of f. (In the judgment-aggregation terminology, these axioms would classify as assumptions about the richness and logical connectedness of the agenda, cf. e.g. Dietrich and Mongin [5] or Dietrich and List [4].)

(a). The Boolean closure of $\{\mu, \nu\}$ is a subset of $\mathcal{S}_P \cap \mathcal{S}_S$.

(a'). There exist sentences $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathcal{S}_P \cap \mathcal{S}_S$, such that

1. $T \vdash \dot{(}\lambda_1 \dot{\leftrightarrow} \mu \dot{\wedge} \nu\dot{)}$,
2. $T \vdash \dot{(}\lambda_2 \dot{\leftrightarrow} \mu \dot{\wedge} \dot{\neg} \nu\dot{)}$,
3. $T \vdash \dot{(}\lambda_3 \dot{\leftrightarrow} \dot{\neg} \mu \dot{\wedge} \nu\dot{)}$,
4. $T \vdash \dot{(}\lambda_4 \dot{\leftrightarrow} \mu\dot{)}$,
5. $T \vdash \dot{(}\lambda_5 \dot{\leftrightarrow} \nu\dot{)}$

(b). $\mathcal{S} \subseteq \mathcal{S}_P \cap \mathcal{S}_S$.

Clearly, (a') is a weak version of (a). The assumptions (a) and (b) are satisfied if $\mathcal{S}_P = \mathcal{S}_S = \mathcal{T}$. Thus, if $f \in \mathcal{AR}$, then (a),(a'),(b) hold.

Now we verify the (ultra)filter properties of D_f

Definition 3.1. *A collection $D \subseteq 2^I$ of coalitions is a* filter *on I if*

F1. $\varnothing \notin D$

F2. For all $C \in D$ and $C' \in 2^I$, if $C \subseteq C'$ then $C' \in D$ (closure under supersets)

F3. For all $C, C' \in D$ $C \cap C' \in D$ (closure unter intersections)

A filter $D \subseteq 2^I$ is an ultrafilter *if for all $C \in 2^I$ either $C \in D$ or $I \backslash C \in D$ but not both.*

Lemma 3.2. *Let $f \in \mathcal{AR}'$.*

1. *If f satisfies (a'), then $\mathcal{D}_f$ is a filter.*

2. *If f satisfies (a), then $\mathfrak{D}_f$ is an ultrafilter.*

Proof. We verify the (ultra)filter properties for $\mathfrak{D}_f$:

- Since f satisfies (A2), $\mathfrak{D}_f$ cannot contain $\emptyset$.
- Next we prove that $\mathfrak{D}_f$ is closed under supersets; we will need this result in order to show that $\mathfrak{D}_f$ is closed under intersections. Let $C \in \mathfrak{D}_f$ and $C' \subseteq I$ with $C \subseteq C'$. We shall show that $C' \in \mathfrak{D}_f$. Since f satisfies (A1'), the domain of f contains a profile $\underline{\mathfrak{A}}$ such that

$$\begin{array}{rcll} \forall i \in C & \mathfrak{A}_i & \models & \mu\dot{\wedge}\nu \\ \forall j \in C' \setminus C & \mathfrak{A}_j & \models & \mu\dot{\wedge}\dot{\neg}\nu \\ \forall k \in I \setminus C' & \mathfrak{A}_k & \models & \dot{\neg}\mu\dot{\wedge}\nu. \end{array}$$

  Note that $C\left(\underline{\mathfrak{A}}, \mu\dot{\wedge}\nu\right) = C \in \mathfrak{D}_f$, so $f(\underline{\mathfrak{A}}) \models \mu\dot{\wedge}\nu$ due to Remark 2.5. In particular, $f(\underline{\mathfrak{A}}) \models \mu$, whence readily $C\left(\underline{\mathfrak{A}}, \mu\right) \in \mathfrak{D}_f$. On the other hand, however, $C' = C\left(\underline{\mathfrak{A}}, \mu\right)$ by the choice of $\underline{\mathfrak{A}}$. Summarising this, we arrive at $C' \in \mathfrak{D}_f$.

- Now we prove that $\mathfrak{D}_f$ is closed under intersections. Let $C, C \subseteq I$. Again since f satisfies (A1'), there must be a profile $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ such that

$$\begin{array}{rcll} \forall i \in C \cap C' & \mathfrak{A}_i & \models & \mu\dot{\wedge}\nu \\ \forall j \in C \setminus (C \cap C') & \mathfrak{A}_j & \models & \mu\dot{\wedge}\dot{\neg}\nu \\ \forall k \in I \setminus C & \mathfrak{A}_k & \models & \dot{\neg}\mu\dot{\wedge}\nu. \end{array}$$

  Then $C = C(\underline{\mathfrak{A}}, \mu) \in \mathfrak{D}_f$, so $f(\underline{\mathfrak{A}}) \models \mu$ by Remark 2.5. Also $C' \subseteq C(\underline{\mathfrak{A}}, \nu)$ and $C' \in \mathfrak{D}_f$, therefore $C(\underline{\mathfrak{A}}, \nu) \in \mathfrak{D}_f$, as we have already shown that $\mathfrak{D}_f$ is closed under supersets. Again by Remark 2.5, we obtain $f(\underline{\mathfrak{A}}) \models \nu$, too. Thus, $f(\underline{\mathfrak{A}}) \models \mu\dot{\wedge}\nu$, therefore $C \cap C' = C(\underline{\mathfrak{A}}, \mu\dot{\wedge}\nu) \in \mathfrak{D}_f$.

- Let $C \subseteq I$, and let us show that $C \in \mathfrak{D}_f$ or $I \setminus C \in \mathfrak{D}_f$. Since f satisfies (A1'), the domain of f contains a profile $\underline{\mathfrak{A}}$ such that

$$\begin{array}{rcll} \forall i \in C & \mathfrak{A}_i & \models & \mu\dot{\wedge}\dot{\neg}\nu \\ \forall j \in I \setminus C & \mathfrak{A}_j & \models & \dot{\neg}\mu\dot{\wedge}\nu. \end{array}$$

  Then $\mathfrak{A}_i \models \dot{(}\mu\dot{\vee}\nu\dot{)}\dot{\wedge}\dot{\neg}\dot{(}\mu\dot{\wedge}\nu\dot{)}$ for all $i \in I$, therefore $\mathfrak{A}_i \models \dot{\neg}\dot{(}\dot{(}\mu\dot{\vee}\nu\dot{)}\dot{\wedge}\dot{\neg}\dot{(}\mu\dot{\wedge}\nu\dot{)}\dot{)}$ for no $i \in I$. In other words,

$$C\left(\underline{\mathfrak{A}}, \dot{\neg}\dot{(}\dot{(}\mu\dot{\vee}\nu\dot{)}\dot{\wedge}\dot{\neg}\dot{(}\mu\dot{\wedge}\nu\dot{)}\dot{)}\right) = \emptyset,$$

  whence

$$f(\underline{\mathfrak{A}}) \not\models \dot{\neg}\dot{(}\dot{(}\mu\dot{\vee}\nu\dot{)}\dot{\wedge}\dot{\neg}\dot{(}\mu\dot{\wedge}\nu\dot{)}\dot{)}$$

  as f satisfies (A2). Therefore,

$$f(\underline{\mathfrak{A}}) \models \dot{(}\mu\dot{\vee}\nu\dot{)}\dot{\wedge}\dot{\neg}\dot{(}\mu\dot{\wedge}\nu\dot{)}.$$

  This means that either $f(\underline{\mathfrak{A}}) \models \mu$ or $f(\underline{\mathfrak{A}}) \models \nu$, hence either $C(\underline{\mathfrak{A}}, \mu) \in \mathfrak{D}_f$ or $I \setminus C = C(\underline{\mathfrak{A}}, \nu) \in \mathfrak{D}_f$. However, $I \setminus C = C(\underline{\mathfrak{A}}, \nu)$ and $C = C(\underline{\mathfrak{A}}, \mu)$ by construction of $\underline{\mathfrak{A}}$. Hence, either $C \in \mathfrak{D}_f$ or $I \setminus C \in \mathfrak{D}_f$.

□

Remark 3.3. *If $\mathcal{D}$ is a filter, then*

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda \Leftrightarrow C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}$$

for all $\underline{\mathfrak{A}} \in \Omega^I$ and $\lambda \in \mathcal{S}$.

If $\mathcal{D}$ is an ultrafilter, then

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda \Leftrightarrow C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}$$

for all $\underline{\mathfrak{A}} \in \Omega^I$ and $\lambda \in \mathcal{T}$.

Proof. Let $\mathcal{D}$ be a filter, let $\underline{\mathfrak{A}} \in \Omega^I$ and let $\lambda \in \mathcal{T}$. By equivalence (4),

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda \Leftrightarrow \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda.$$

Now, if $\lambda \in \mathcal{S}$, then

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda \Leftrightarrow \{i \in I \ : \ \mathfrak{A}_i \models \lambda\} \in \mathcal{D}$$

by definition of the reduced product, and if $\mathcal{D}$ is an ultrafilter, then

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models \lambda \Leftrightarrow \{i \in I \ : \ \mathfrak{A}_i \models \lambda\} \in \mathcal{D}$$

holds for any $\lambda \in \mathcal{T}$ by Łoś's theorem. □

Lemma 3.4. *Let $f \in \mathcal{AR}'$. If (a') and (b) are satisfied, then $f(\underline{\mathfrak{A}}) = \mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f$ for all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$.*

Proof. By Lemma 3.2, $\mathcal{D}_f$ is a filter, whence $\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f$ is well-defined for all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$. Let us now fix arbitrary $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ and $\lambda \in \mathcal{S}$. By Remark 3.3,

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f \models \lambda \Leftrightarrow C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}_f.$$

Therefore, due to Remark 2.5,

$$\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f \models \lambda \Leftrightarrow f(\underline{\mathfrak{A}}) \models \lambda.$$

Since $\lambda \in \mathcal{S}$ was arbitrary, we deduce by means of implication (2) that $f(\underline{\mathfrak{A}}) = \mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f$. □

Lemma 3.5. *Suppose $\mathcal{D}$ is an ultrafilter, and consider the aggregator $\mathrm{res}_A \prod/\mathcal{D}$, defined by*

$$\mathrm{res}_A \prod/\mathcal{D} : \Omega^I \to \Omega, \qquad \underline{\mathfrak{A}} \mapsto \mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}.$$

Then $\mathrm{res}_A \prod/\mathcal{D} \in \mathcal{AR}$.

Proof. Let $\underline{\mathfrak{A}} \in \Omega^I$. Clearly, both the ultraproduct $\prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$ and its restriction to A are well-defined. By Łoś's theorem, $\prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \models T$, and by implication (3), $\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D} \in \Omega$. Therefore, $\mathrm{res}_A \prod /\mathcal{D} : \Omega^I \to \Omega$. Let us now verify the axioms (A1), (A2), and (A3) for $\mathrm{res}_A \prod /\mathcal{D}$.

(A1). As we have just seen, $\mathrm{res}_A \prod /\mathcal{D}$ is well-defined on Ω^I.

(A2). Let $\underline{\mathfrak{A}} \in \Omega^I$ and $\lambda \in \mathcal{T}$ such that $\mathrm{res}_A \prod /\mathcal{D}(\underline{\mathfrak{A}}) \models \lambda$, that is $\mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$. Then, by Remark 3.3, we have $C(\underline{\mathfrak{A}}, \lambda) \in \mathcal{D}$, hence $C(\underline{\mathfrak{A}}, \lambda) \neq \emptyset$ as $\mathcal{D}$ is an ultrafilter.

(A3). For all $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$ and all $\lambda, \lambda' \in \mathcal{T}$ such that $C(\underline{\mathfrak{A}}, \lambda) = C(\underline{\mathfrak{A}}', \lambda')$, Remark 3.3 entails that $\mathrm{res}_A \prod /\mathcal{D}(\underline{\mathfrak{A}}) \models \lambda$ if and only if $\mathrm{res}_A \prod /\mathcal{D}(\underline{\mathfrak{A}}') \models \lambda'$.

□

Let βI denote the set of all ultrafilters on the set I.

Theorem 3.6 (Kirman-Sondermann correspondence (generalised)). *There is a bijection $\Lambda : \mathcal{AR} \to \beta I$, given by*

$$\forall f \in \mathcal{AR} \qquad \Lambda(f) = \mathcal{D}_f.$$

Its inverse is given by

$$\forall \mathcal{D} \in \beta I \qquad \Lambda^{-1}(\mathcal{D}) = \mathrm{res}_A \prod /\mathcal{D},$$

wherein, as in Lemma 3.5, $\mathrm{res}_A \prod /\mathcal{D} : \underline{\mathfrak{A}} \mapsto \mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$.

Proof. 1. For all $f \in \mathcal{AR}$, $\mathcal{D}_f$ is an ultrafilter by Lemma 3.2, whence the range of Λ is a subset of βI.

2. $\mathcal{D} \mapsto \mathrm{res}_A \prod /\mathcal{D}$ is indeed the inverse of Λ as Lemma 3.4 teaches that $f = \mathrm{res}_A \prod /\mathcal{D}_f$, hence $\Lambda^{-1}(\Lambda(f)) = f$ for all $f \in \mathcal{AR}$.

3. Since Λ has an inverse, it must be injective. (Indeed, if $\Lambda(f) = \Lambda(g)$ for any $f, g \in \mathcal{AR}$, then $f = \Lambda^{-1}(\Lambda(f)) = \Lambda^{-1}(\Lambda(g)) = g$.)

4. By Lemma 3.5, the range of Λ^{-1} is contained in $\mathcal{AR}$. Hence, for any $\mathcal{D} \in \beta I$, the aggregator $\Lambda^{-1}(\mathcal{D})$ is in the domain of Λ, whence $\mathcal{D} = \Lambda\left(\Lambda^{-1}(\mathcal{D})\right)$ is in the range of Λ. Therefore, Λ is surjective.

□

As corollary of this result we immediately obtain the well-known impossibility results for a finite set of individuals.

We say that f is *oligarchic* if and only if there exists a finite subset $M_f \subseteq I$ such that $\mathcal{D}_f = \{J \subseteq I : M_f \subseteq J\}$. We say that f is dictatorial if and only if there exists some $i_f \in I$ such that $\mathcal{D}_f = \{J \subseteq I : i_f \in J\}$.

Remark 3.7. *Let f be an aggregator, and suppose I is finite.*

1. *f is oligarchic if and only if $\mathcal{D}_f$ is a filter.*

2. *f is dictatorial if and only if $\mathcal{D}_f$ is an ultrafilter.*

Proof. 1. A collection of subsets of a finite set is a filter if and only if it equals the set of all supersets of its intersection. Thus, $M_f = \bigcap \mathcal{D}_f$.

2. A filter on a finite set is an ultrafilter if and only if its intersection is a singleton. Thus, $\{i_f\} = \bigcap \mathcal{D}_f$.

$\square$

Corollary 3.8 (Impossibility theorem). *Suppose I is finite, and let $f \in \mathcal{AR}'$. If f satisfies (a'), then f is oligarchic. If f satisfies (a), then f is dictatorial. In particular, f is dictatorial if $f \in \mathcal{AR}$.*

4 Conclusion

This short note makes three contributions: At a conceptual level, we propose first order predicate logic and model theory as a unified framework which naturally extends the existing frameworks for the analyses of both preference aggregation and aggregation of logical propositions (judgment aggregation).

The value of having a unified framework for both preference and judgment aggregation is perhaps self-evident: We hope that it will allow for a systematic generalisation of existing techniques in social choice theory by fostering the exchange of methodologies between the areas of preference aggregation and judgment aggregation. (In addition, our proposal of a natural unified framework for a large class of aggregation problems might encourage the use of concepts and methods from mathematical logic in the mathematical modelling of socio-economic phenomena.)

In this general framework for aggregation theory, we have analyzed the relation between aggregation problems and the social structure they induce by generalising the Kirman-Sondermann correspondence between aggregators and ultrafilters on the set of individuals. This relation is of genuine interest besides its use for the proof of impossibility theorems. Under relatively mild assumptions, we have shown that aggregators are restricted reduced products with respect to the filter of decisive coalitions. Under stronger rationality axioms, a bijective correspondence between rational aggregators and ultrafilters has been established, generalising earlier results of Kirman and Sondermann [13] and Lauwers and Van Liedekerke [15] and allowing to derive the typical impossibility results known from preference and judgment aggregation.

Since this paper naturally extends an article by Lauwers and Van Liedekerke [15] written in 1992 or earlier, we have ultimately raised a hypothetical historical point: Already in the early 1990s a sufficiently general framework for analysing the aggregation of sets of logical propositions might perhaps have emerged, based on the work of Lauwers and Van Liedekerke — well before the first papers on judgment aggregation actually appeared.

The low level of attention which Lauwers and Van Liedekerke's [15] results received may be comparable to the oblivion in which the first use of the ultrafilter technique for a generalisation of Arrow's theorem to a "logical problem of aggregation" — by the French mathematician Georges Théodule Guilbaud [8] (translated into English by Monjardet [9]) — had fallen for half a century. (Cf. Eckert and Monjardet [6].) To be sure, no individual author may be held responsible for this development. Instead, the structural explanation for it lies in the separation of mathematical modelling from formal logical analysis that still — wittingly or unwittingly — prevails in the social sciences. Our hope is that the rich field of judgment aggregation will eventually help to (re)connect mathematical modelling and formal, i.e. mathematical, logic.

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