# On Core Stability and Apportionment Methods

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# Preface

In this thesis two distinct topics in cooperative game theory are investigated. The first problem analyzed in this dissertation is one of the oldest unsolved problems in cooperative game theory. The question asks, under what conditions does an $n$-person, cooperative, TU game have a stable core? This problem is fundamental for $n$-person, cooperative, TU game theory as the solution of this problem would provide vital insights into certain properties of the core as well as revealing certain aspects of von Neumann-Morgernstern stable sets.

The second topic examined in this PhD concerns what is known as the apportionment problem. The problem in question is how one can apportion seats, power, etc., in a parliament, committee, etc., corresponding to the size, power, etc., of certain states or parties within a country, company, etc. One is confronted with this problem as soon as one wishes to represent the interests of certain groups in some sort of committee. Hence, this problem is age old but has only recently received a proper mathematical treatment in the twentieth century. In this thesis, a new apportionment method based on game theoretical concepts is investigated for its suitability as an apportionment method to be applied in reality.

The realization of this thesis would not have been possible if it were not for the assistance of numerous people. The author must therefore thank, first and foremost, his supervisors Professor emeritus Joachim Rosenmüller and Associate Professor Peter Sudhölter. Both have contributed enormously not only to the theoretical results but also to the presentation, the questions which were investigated, to my financial, psychological and emotional support as well as contributing significantly to the enrichment of my stay in Germany in many other ways. I thank Claus-Jochen Haake for his assistance with many of my questions concerning all sorts of aspects of game theory. I must also thank Matthias Schleef for his unrelenting assistance with my Latex problems, Yaron Azrieli for reading and correcting some of the section on fuzzy games and the members of EBIM and BIGSEM for their numerous discussions and exchanges of opinions.

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# 1 Introduction

This thesis is devoted to the study of the characterization of core stability for $n$-person, cooperative, transferable utility (TU) games and to a question regarding apportionment methods. The methods employed to analyze these topics, and in the first case the topic itself, are based on the idea of a cooperative game introduced by the pioneers of Game Theory, John von Neumann and Oskar Morgernstern, in their book “The Theory of Games and Economic Behavior”, [47]. In this book, they introduced the aforementioned type of game and explored equilibrium concepts for these games. *Inter alia*, they introduced the von Neumann-Morgernstern (vNM) stable set. It is this solution concept and another solution concept introduced in [24], known as the core, which form the focus of the first part of this dissertation.

The core was one of the first solution concepts to be introduced in the field of $n$-person, cooperative, TU games. It represents the set of payoffs where no coalition receives less than what they can achieve on their own. Since its conception, the core has been extensively studied and characterized. However, demonstrations of some properties of games relating to the core have remained elusive and an open question is, when does a given game have a stable core?

The property of stability arises when the core coincides with the vNM stable set. This means that for every imputation not within the core there is an imputation within the core dominating it. Such a solution concept is very convincing, as there is always a “better” outcome in the core than an arbitrary, given imputation outside the core (i.e. the element(s) of the core that dominates the imputation outside the core). Hence, core stability is a desirable property of a game and it is the investigation of core stability and the tools of standard $n$-person, cooperative, TU game theory which form the basis of Chapter 2.

Numerous results have been developed in an attempt to characterize core stability in terms of some other more tractable properties, which are also simple to verify. Therefore, before the new results in this PhD are presented, it is worthwhile to have a look at the work already done on this problem. So, in the following paragraphs well-known results concerning core stability will be discussed. The reader is referred to either the original articles or the following chapters for any unfamiliar notation or concepts. In addition, because this dissertation focuses more on finding sufficient conditions for core stability, the following discussion will concentrate more on this endeavor.

The question of core stability is almost as old as the field of Game Theory itself. One of the earliest attempts to characterize when a game has a stable core was provided in [24] in 1959. In this article the idea of majorization was introduced and it was proven that a game has a stable core if and only if any imputation not belonging to the core is "majorizable" by a certain set of maximally majorizing elements. However, this solution only partially answered the question, as the conditions for core stability were not given via the coalition function (this is the most desirable form of solution for this problem as it allows one, for a given game, to immediately test, via the coalition function, if the game has a stable core).

Another result, proven in [20], stated general conditions under which a game has a stable core. The theorem is as follows.

**Theorem 1.0.1.** *Let $(N, v)$ be an $n$-person cooperative game in 0-1 normalized form. Let for any coalition $S$*

$$v(S) \leq \frac{1}{n - |S| + 1},$$

*where $|S|$ is the number of players in the coalition. Then the game $(N, v)$ has a stable core.*

In 1969 in [31], new conditions were provided under which a given game has a stable core. Under the assumptions of the theorem, one can easily show that the stated type of game has a large core and hence, a stable core. Just the statement of the theorem will be given here.

**Theorem 1.0.2.** *Let $(N, v)$ satisfy the following conditions. For each minimal covering $\mathcal{S}$ of $N$ let*

$$\sum_{Q \in \mathcal{S}} v(Q) \leq v(N).$$

*Then the game $(N, v)$ has a nonempty stable core.*

The main method of addressing the problem of core stability, which is utilized by game theorists today, can be traced back to a paper published in 1982, [42]. In this paper the concept of a large core was introduced and it was proven that if a game has a large core, then it has a stable core. Following the ideas of this result, in [30], a new concept was invented, that of extendability, and it was shown that if the game has a large core, then it is extendable and if the game is extendable, then it has a stable core. Researchers have also found examples of games which are extendable but do not have a large core and have a stable core but are not extendable, see [46]. Since then, game theorists have continued working in this direction in an attempt to generalize these results and to find a largeness concept that is both necessary and sufficient for core stability for all games in general.

That is not to say, however, that for specific classes of games core stability has not already been characterized. In [13] the authors provided necessary and sufficient conditions for core stability in the case of symmetric games based on previous results by L. Shapley (which were produced in an unpublished manuscript). Their theorem is as follows.

**Theorem 1.0.3.** *For a balanced symmetric game $(N, v)$, the statements $(N, v)$ has a large core, $(N, v)$ has a stable core and $(N, v)$ is exact are equivalent.*

As well, in the case of assignment games, minimum coloring games, a special class of flow games (see the corresponding sections for the relevant definitions) and other special classes of games (which won't be analyzed in this thesis), core stability has also been fully characterized.

These results may be satisfactory for the special classes of games considered, however, one still would like necessary and sufficient conditions for core stability for all games in general. So, game theorists are working in parallel by attempting to solve the problem for special classes of games and concomitantly hoping that these solutions will provide insights into the general case.

In this thesis, an attempt is made to contribute to this endeavor. The methods employed in Chapter 2 are mainly based on the aforementioned results in [30], namely extendability. The idea of extendability is extended and new, weaker conditions for core stability are given. These new concepts, however, turn out not to be necessary for core stability in general. Although the new types of extendability are not generally necessary for core stability, classes of games are considered for which at least one of the new concepts is both necessary and sufficient.

These methods, mentioned above, mainly employ the tools of standard cooperative game theory and results developed for the purposes of $n$-person, cooperative, TU games. Because half a century has past without anyone finding a solution to the problem of core stability along these lines of attack, this has led game theorists to utilize other mathematical tools to investigate the problem. One of the concepts recently brought to the study of core stability is that of fuzzy sets.

Using the idea of a fuzzy set one can define a fuzzy game. This approach is promising as it allows for a veritable augmentation of the available mathematical tools applicable to the problem of core stability. Within the field of fuzzy games, one is then able to use the tools of nonlinear analysis to analyze the question from a different point of view. Such a new perspective not only provides new insights but also stimulates other researchers not working directly on the problem of core stability to consider the problem of core stability.

It is this ansatz which is employed in Chapter 3 of this dissertation.

The fundamental definitions regarding fuzzy games were first introduced in [2]. The interpretation of a fuzzy game is that players are able to specify their level of participation within a certain coalition. This participation level is represented via a real number between zero and some positive real number. A similar concept considered by game theorists (in a countable context) is known as a multi-choice cooperative game (see, e.g., [28]). In this context players can choose an integer value to represent their level of participation. This latter style of game will not be considered in this thesis.

The idea of using fuzzy games to attack the problem concerning core stability has received surprisingly little attention. There has, however, been a recent revival of the usage of fuzzy games to study core stability, see, for example, [7] and [8]. Most articles using the tools of fuzzy games, however, tend to reformulate old, well known concepts from standard $n$-person, cooperative, TU games and have not really begun introducing new ideas and concepts engendered by the introduction of fuzzy sets.

By venturing into the realm of fuzzy games one is able to apply, with much more ease, the tools of nonlinear and convex analysis. In Chapter 3, an attempt is made to provide a basis for this analysis by stating the problem of core stability in a nonlinear/convex analysis context and, hence, providing one with the ability to analyze the problem of core stability with the full arsenal of aforementioned tools.

The above two ways of considering core stability (fuzzy games and standard cooperative games) form the focal point for the results presented in the first two chapters of this dissertation. The last chapter of this PhD addresses an age old question related to seat distributions in parliament. The investigation of this situation forms the basis of what is known as the apportionment problem.

The apportionment problem is a well known problem in political science concerned with determining how to divide a whole number of representatives or delegates among given states, territories, groups, etc., according to their respective sizes. The main goal of the theory is to assign a “fair” number of seats (representatives, etc.) to each state, territory, etc., according to their relative population (that is, relative to the total population). Difficulties in apportioning seats arise as the apportionment must provide integer values for each state, territory, etc., and the sum of the distributed seats must sum to some fixed number representing the number of seats in a parliament or committee.

Apportionment methods have been studied in detail by a number of researchers, in particular by M.L. Balinski and H.P. Young. The aforementioned authors developed a number of important results which culminated in the publishing of their book, [9]. This book summarizes their most significant results produced by their research up to 1982.

The investigation in the last chapter of this PhD is centered around the suitability of apportionment methods based on the idea of preserving the winning (or losing) coalitions of the simple game represented by the populations of the states. This is a rather salient idea that seems to have been neglected by the literature in the past. The idea however seems to be natural. One wishes to preserve the "power" of each represented state according to their "power" in the original game based on their populations.

This way of considering the problem of apportionments leads to a number of new schema, which one can use to select an apportionment method. Some of these methods are investigated in this last chapter and are analyzed regarding their suitability as an apportionment method. The judging criteria, employed here, are a number of properties expounded in the chapter and also in [9]. These criteria are natural conditions which could be required by any committee or parliament choosing a method to apportion seats.

The following thesis is structured as follows:
In Chapter 2 the necessary cooperative game theoretical concepts, pervading the entire dissertation, are given. In this chapter new weaker conditions for core stability are given as well as an analysis of these conditions. In Chapter 3 fuzzy games are introduced and the problem of core stability is analyzed from this perspective. In Chapter 4 the apportionment problem is analyzed and new apportionment methods are developed.

# 2 Cooperative Games

In this chapter the relevant game theoretical concepts employed in this thesis and new conditions guaranteeing core stability will be presented. The type of game analyzed in the following is that of a cooperative game as introduced by von Neumann and Morgenstern in [47]. The cooperative game was one of two of the game types introduced in their book, the second style of game falling under the rubric of a non-cooperative game. The major difference between a cooperative game and a non-cooperative game was elucidated in [25]. The basic idea was that a game is cooperative if one is allowed to make agreements, promises and other such contracts between players such that an accord between the participants of the game is enforceable. Non-cooperative games treat the situation where such enforceable contracts are not available. In this chapter, one of the central solution concepts in cooperative game theory, the core, is analyzed. In particular, the focus of this chapter is on providing sufficient conditions for core stability.

The chapter is organized as follows. In the next section, the necessary definitions pertaining to $n$-person, cooperative, TU games are given. In addition, new conditions guaranteeing core stability are also defined. An example showing that these conditions are not necessary for core stability is also given. In Section 2.2 consequences of the new sufficient conditions for core stability are discussed. Besides, a class of games introduced in [26] is also considered. In Section 2.3 assignment games are considered and it is shown that the new conditions are both necessary and sufficient for core stability in this case. In Section 2.4 "simple" flow games are considered and finally in Section 2.5 minimum coloring games are investigated. The results presented in this chapter are based on [43].

## 2.1 Notation and Results

The games treated in this section will encompass cooperative games with a finite number of players and transferable utility. This type of game will, from now on, simply be referred to as a game. In this setting one has pair, $(N, v)$, where $N$ represents the set of players and is usually taken to be a finite, nonempty subset of the natural numbers with numbers representing players (which is what will be adopted here), i.e. $N \subsetneqq \mathbb{N}$. For the sake of simplifying the notation it will be assumed that $|N| = n$ (here and in the following, if $D$ is a finite set, then $|D|$ denotes the cardinality of $D$). In addition, nonempty subsets of $N$ will be referred to as coalitions. Finally $v$ is the coalition function, $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, which intuitively describes the worth of a coalition. For $S \subseteq N$ denote by $\mathbb{R}^S$ the set of all real functions on $S$. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. A payoff to the players is generated by a vector $x$, $x \in \mathbb{R}^N$. To simplify the notation, one often introduces the following convention for a vector $x \in \mathbb{R}^N$ and a set $S \subseteq N$:

$$x(S) := \sum_{i \in S} x_i,$$

where each $x_i$ stands for the $i^{th}$ component of the vector $x$ ($x(\emptyset) = 0$). Let $x_S$ denote the restriction of $x$ to $S$, i.e. $x_S := (x_i)_{i \in S}$. In addition, if $x, y \in \mathbb{R}^S$, then write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$ and $x \gg y$ if $x_i > y_i$ for all $i \in S$. There are special classes of vectors in $\mathbb{R}^N$, for a given game $(N, v)$, which play an important role in defining a number of solution concepts, the set of preimputations and imputations. If $(N, v)$ is a game, then a payoff vector, $x \in \mathbb{R}^N$, is said to be *Pareto optimal* if $x(N) = v(N)$. If a payoff fulfills Pareto optimality, then this can be interpreted as the members of the grand coalition, $N$, dividing up the entire gain among themselves without any loss.

**Definition 2.1.1.** *The **set of preimputations**, $X(N, v)$, for a game $(N, v)$, is the set of Pareto optimal vectors, i.e. $X(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$.*

If $(N, v)$ is a game, then a payoff vector, $x \in \mathbb{R}^N$, is said to be *individually rational* if $x_i \geq v(\{i\})$ for all $i = 1, \ldots, n$. If a payoff fulfills individual rationality, then this can be interpreted as each player getting at least what he or she could achieve on his or her own.

**Definition 2.1.2.** *For a given game $(N, v)$, the **set of imputations**, $I(N, v)$, is*

$$I(N, v) := \{x \in X(N, v) \mid x_i \geq v(\{i\}) \ \forall \ i \in N\}.$$

It is a certain relationship between vectors in $\mathbb{R}^N$ that defines the requisite solution concept investigated in this thesis, that of domination. This idea is encompassed by the following definition.

**Definition 2.1.3.** *Let $(N, v)$ be a game. $\eta \in \mathbb{R}^N$ is said to **dominate** $\zeta \in \mathbb{R}^N$ via the coalition $D$ if $\eta$ satisfies $\eta(D) \leq v(D)$ as well as $\eta_D \gg \zeta_D$. In the case that $\eta$ dominates $\zeta$ via the coalition $D$ one writes $\eta \ \mathrm{dom}_D \ \zeta$ and one writes $\eta \ \mathrm{dom} \ \zeta$ in case there is a coalition $D$ such that $\eta \ \mathrm{dom}_D \ \zeta$.*

The idea of domination is used to define two important solution concepts for games. One solution concept is the vNM stable set. It was defined in [47] using the ideas of internal and external stability, which are as follows.

**Definition 2.1.4.** *A set $\mathcal{V} \subseteq \mathbb{R}^N$ is **internally stable** (with respect to (w.r.t.) $(N, v)$) if there do not exist $\eta, \zeta \in \mathcal{V}$ such that $\eta$ dominates $\zeta$.*

**Definition 2.1.5.** *A set* $\mathcal{V} \subseteq \mathbb{R}^N$ *is* ***externally stable*** *(w.r.t.* $(N,v)$*) if for all* $\zeta \in I(N,v)\backslash\mathcal{V}$ *it follows that there exists an* $\eta \in \mathcal{V}$ *such that* $\eta$ *dominates* $\zeta$*.*

**Definition 2.1.6.** *Let* $(N,v)$ *be a game. A set of imputations in* $I(N,v)$*,* $\mathcal{V}(N,v)$*, is said to be a* ***vNM stable set*** *if* $\mathcal{V}(N,v)$ *is both internally and externally stable. A set* $\mathcal{V}$ *is said to be* ***stable*** *if it is both internally and externally stable.*

Another solution concept, introduced in [24], is the core and is defined as follows.

**Definition 2.1.7.** *Let* $(N,v)$ *be a game. The* ***core****,* $\mathcal{C}(N,v)$*, is given by* $\mathcal{C}(N,v) := \{x \in X(N,v) \mid x(S) \geq v(S)\, \forall S \subseteq N\}$*.*

Note that $\mathcal{C}(N,v)$ is internally stable and that any externally stable set contains $\mathcal{C}(N,v)$.

**Definition 2.1.8.** *Let* $(N,v)$ *be a game.* $(N,v)$ ***has a stable core*** *if* $\mathcal{C}(N,v)$ *is stable, that is, externally stable, w.r.t.* $(N,v)$*.*

We also remark that, if $I(N,v) = \emptyset$, then $\emptyset = \mathcal{C}(N,v)$ is stable. Hence, the case $\sum_{i\in N} v(\{i\}) > v(N)$ shall no longer be considered.

Related to the concept of the core is that of the dominance core (also introduced in [24]). It is defined as follows.

**Definition 2.1.9.** *Let* $(N,v)$ *be a game. The* ***dominance core****,* $\mathcal{DC}(N,v)$*, is given by* $\mathcal{DC}(N,v) := \{x \in I(N,v) \mid \nexists\, y \in I(N,v), y \operatorname{dom} x\}$*.*

**Remark 2.1.10.** *In general,* $\mathcal{C}(N,v) \subseteq \mathcal{DC}(N,v)$ *however for the coming class of balanced games it follows that* $\mathcal{DC}(N,v) = \mathcal{C}(N,v)$ *(see [17]). Hence, for the class of balanced games, one can strategically interpret the core as the set of all undominated imputations. An example such that* $\mathcal{DC}(N,v) \neq \mathcal{C}(N,v)$ *can be constructed by letting* $N = \{1,2,3\}$*,* $v(N) = 1$*,* $v(\{1,2\}) = 2$ *and* $v(S) = 0$ *otherwise. Here* $\mathcal{C}(N,v) = \emptyset$ *and* $\mathcal{DC}(N,v) = \{(\alpha, 1-\alpha, 0) \mid \alpha \in [0,1]\}$*.*

It will also be necessary to consider subgames of the game $(N,v)$.

**Definition 2.1.11.** *Let* $(N,v)$ *be a game. A* ***subgame*** *of* $(N,v)$ *is a game* $(T, v_T)$ *where* $T$ *is a coalition and* $v_T(S) = v(S)$ *for all* $S \subseteq T$*. The subgame* $(T, v_T)$ *will also be denoted by* $(T,v)$*.*

We now recall some relevant results. The proof of the well-known Proposition 2.1.12 is presented, because its statement will be used several times.

**Proposition 2.1.12** ([24]). *Let $(N,v)$ be a game such that $I(N,v) \neq \emptyset$. If $(N,v)$ has a stable core, then, for each $i \in N$, there exists $x \in \mathcal{C}(N,v)$ such that $x_i = v(\{i\})$.*

**Proof:** As $(N,v)$ has a stable core and $I(N,v) \neq \emptyset$, $\mathcal{C}(N,v) \neq \emptyset$. Assume, on the contrary, that there exists $k \in N$ such that $x_k > v(\{k\})$ for all $x \in \mathcal{C}(N,v)$. As $\mathcal{C}(N,v)$ is a compact set, $t = \min\{x_k \mid x \in \mathcal{C}(N,v)\}$ exists so that $t > v(\{k\})$. Choose $x \in \mathcal{C}(N,v)$ with $x_k = t$, let $\varepsilon > 0$ satisfy $t - (|N|-1)\varepsilon \geq v(\{k\})$, and define $y \in \mathbb{R}^N$ by $y_i = x_i + \varepsilon$ for all $i \in N\backslash\{k\}$ and $y_k = x_k - (|N|-1)\varepsilon$. Then $y \in I(N,v)\backslash\mathcal{C}(N,v)$. Hence, there exists $z \in \mathcal{C}(N,v)$ and $\emptyset \neq T \subseteq N$ with $z \operatorname{dom}_T y$. For any $S \subseteq N\backslash\{k\}$, $y(S) = \varepsilon|S| + x(S) \geq x(S) \geq v(S)$. Hence, $k \in T$ so that $z_k \geq t = x_k$. As $y_{N\backslash\{k\}} \gg x_{N\backslash\{k\}}$, we conclude that $z(T) > x(T) \geq v(T)$ and the desired contradiction has been obtained. **q.e.d.**

The foregoing proposition has the following interesting consequence[1].

**Corollary 2.1.13.** *If the game $(N,v)$ has a stable core, then any preimputation of $(N,v)$ is dominated by some element of $\mathcal{C}(N,v)$, provided that $I(N,v) \neq \emptyset$.*

In order to recall the Bondareva-Shapley theorem (see [15] and [38]) which gives necessary and sufficient conditions for the non-emptiness of the core, the following notation is useful. For $T \subseteq N$, denote by $\chi^T \in \mathbb{R}^N$ the *characteristic vector* of $T$, defined by

$$\chi_i^T = \begin{cases} 1 & \text{, if } i \in T, \\ 0 & \text{, if } i \in N \setminus T. \end{cases}$$

**Definition 2.1.14.** *A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called* ***balanced*** *(over $N$) if positive numbers $\delta^S, S \in \mathcal{B}$, exist such that $\sum_{S\in\mathcal{B}} \delta^S \chi^S = \chi^N$. The collection $(\delta^S)_{S\in\mathcal{B}}$ is called a system of* ***balancing weights*** *for $\mathcal{B}$.*

**Theorem 2.1.15** (The Bondareva-Shapley Theorem)**.** *Let $(N,v)$ be a game. Then $\mathcal{C}(N,v) \neq \emptyset$ if and only if for each balanced collection $\mathcal{B}$ over $N$ and any system $(\delta^S)_{S\in\mathcal{B}}$ of balancing weights for $\mathcal{B}$, $\sum_{S\in\mathcal{B}} \delta^S v(S) \leq v(N)$.*

The foregoing theorem motivates calling a game $(N,v)$ a *balanced game* if $\mathcal{C}(N,v) \neq \emptyset$. Note that $(N,v)$ is *totally balanced* if, for any $\emptyset \neq S \subseteq N$, $(S,v)$ is balanced. The *totally balanced cover* of $(N,v)$, $(N,\bar{v})$, is given by

$$\bar{v}(S) = \max\left\{ \sum_{T\in\mathcal{B}} \delta^T v(T) \;\middle|\; \begin{array}{l} \mathcal{B} \text{ is a balanced collection over } S \text{ and} \\ (\delta^T)_{T\in\mathcal{B}} \text{ is system of balancing weights for } \mathcal{B} \end{array} \right\} \ \forall S \subseteq N. \tag{2.1}$$

[1] Corollary 2.1.13 may not hold for an arbitrary stable set. Indeed, if $(N,v)$ is the three-person majority game, defined by $N = \{1,2,3\}$, $v(N) = v(S) = 1$, if $|S| = 2$, and $v(T) = 0$, if $|T| \leq 1$, then $X = \left\{\left(c, \frac{3}{4} - c, \frac{1}{4}\right) \middle| \, 0 \leq c \leq \frac{3}{4}\right\}$ is a well-known stable set, but the preimputation $(1,1,-1)$ is not dominated by an element of $X$.

The following definition is also important for a number of results later on.

**Definition 2.1.16.** *Let $(N, v)$ be a game and let $S$ be a coalition. Call $y \in \mathcal{C}(S, v)$* ***extendable*** *(w.r.t $(N, v)$) if there exists an $x \in \mathcal{C}(N, v)$ such that $x_S = y$. A coalition $S$ is extendable if for all $y \in \mathcal{C}(S, v)$ there is $x \in \mathcal{C}(N, v)$ such that $x_S = y$. The game $(N, v)$ is extendable if every coalition is extendable.*

Although the extendability of every coalition is not a necessary condition for core stability (see [46]), one can show that the extendability of all coalitions with $n - 1$ players is necessary for core stability.

**Lemma 2.1.17.** *Let $(N, v)$ be a balanced game, $|N| \geq 2$, let $(N, v)$ have a stable core and let $i \in N$. Then $N \backslash \{i\}$ is extendable.*

**Proof:** Note first of all that for all $x \in \mathcal{C}(N, v)$ and $i \in N$, because $x(N \backslash \{i\}) \geq v(N \backslash \{i\})$ and $x(N) = v(N)$, it follows that $x_i \leq v(N) - v(N \backslash \{i\})$. Let $i \in N$ and let $x \in \mathcal{C}(N \backslash \{i\}, v)$. Let $\zeta \in \mathbb{R}^N$ be defined by $\zeta_i = v(N) - v(N \backslash \{i\}) \geq v(\{i\})$ and $\zeta_j = x_j$ for $j \in N \backslash \{i\}$. Then $\zeta \in I(N, v)$ and this element cannot be dominated by any element of the core and hence, must be an element of the core. As this is true for all $i \in N$ the result follows. **q.e.d.**

In addition, this lemma allows the classification of core stability for what are know as $[n, n-1]$ games. An $[n, n-1]$ game is a game $(N, v)$ such that $v(S) \leq \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$ such that $|S| < n - 1$. The proof of the following corollary is straightforward and hence omitted.

**Corollary 2.1.18.** *Let $(N, v)$ be a balanced, $[n, n-1]$ game with $|N| \geq 2$. Then the game $(N, v)$ has a stable core if and only if for all $i \in N$, $N \backslash \{i\}$ is extendable.*

The goal of the rest of the section is to extend the following result.

**Theorem 2.1.19** ([30])**.** *Any extendable game $(N, v)$ has a nonempty stable core.*

To continue the analysis the definition of exactness is requisite.

**Definition 2.1.20.** *Let $(N, v)$ be a game. A coalition $S$ is* ***exact*** *if there exists a $y \in \mathcal{C}(N, v)$ with $y(S) = v(S)$. In this case $S$ is* ***effective*** *for $y$. A game is said to be exact if every coalition is exact (see [37]).*

Call a balanced game $(N, v)$ *exact extendable* if all exact coalitions are extendable.

**Definition 2.1.21.** *A coalition $S$ is called* **vital** *(w.r.t. $(N, v)$) if there exists $x \in \mathcal{C}(S, v)$ such that $x(T) > v(T)$ for all $T \in 2^S \backslash \{\emptyset, S\}$*[2].

Call a balanced game $(N, v)$ *vital extendable* if all vital coalitions w.r.t. $(N, v)$ are extendable.

**Remark 2.1.22.** *Note that there is a simple characterization of a vital coalition (see [24]). Indeed, $S$ is vital if and only if for any balanced collection $\mathcal{B}$ over $S$, $S \notin \mathcal{B}$, and any system $(\delta^T)_{T\in\mathcal{B}}$ of balancing weights for $\mathcal{B}$, $\sum_{T\in\mathcal{B}} \delta^T v(T) < v(S)$.*

Denote by $\mathcal{E}(N, v)$ the set of all coalitions $S$ that are effective for $x$ for all $x \in \mathcal{C}(N, v)$ or $S = \emptyset$, that is,

$$\mathcal{E}(N, v) = \{S \subseteq N \mid x(S) = v(S) \; \forall x \in \mathcal{C}(N, v)\}. \tag{2.2}$$

**Definition 2.1.23.** *Let $(N, v)$ be a balanced game. A coalition $S$ is called* **strongly vital-exact** *(w.r.t. $(N, v)$) if $S$ is vital and if there exists $x \in \mathcal{C}(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus (\{S\} \cup \mathcal{E}(N, v))$. The game $(N, v)$ is* **vital-exact extendable** *if all strongly vital-exact coalitions are extendable.*

To prove the main result a number of lemmata are required. Let $(N, v)$ be a balanced game.

**Lemma 2.1.24.** *For any $x \in X(N, v) \backslash \mathcal{C}(N, v)$ there exists a strongly vital-exact coalition $P$ such that $x(P) < v(P)$.*

**Proof:** By the definition of $\mathcal{E}(N, v)$ and the convexity of the core, there exists $x^0 \in \mathcal{C}(N, v)$ such that $x^0(S) > v(S)$ for all $S \in 2^N \backslash \mathcal{E}(N, v)$. For $\lambda \in \mathbb{R}$ denote $x^\lambda = \lambda x + (1 - \lambda)x^0$. As $\mathcal{C}(N, v)$ is convex and closed, there exists $\widehat{\lambda}$, $0 \leq \widehat{\lambda} < 1$, such that

$$\lambda \geq 0 \text{ and } x^\lambda \in \mathcal{C}(N, v) \Longleftrightarrow 0 \leq \lambda \leq \widehat{\lambda}.$$

Then there exists $R \subseteq N$ such that $x(R) < v(R)$ and $x^{\widehat{\lambda}}(R) = v(R)$. Hence, $R$ is exact. Now, let $P \subseteq N$ be minimal (w.r.t. inclusion) such that $x(P) < v(P)$ and $x^{\widehat{\lambda}}(P) = v(P)$. By minimality of $P$,

$$Q \subsetneq P \text{ and } x(Q) < v(Q) \Longrightarrow x^{\widehat{\lambda}}(Q) > v(Q) \tag{2.3}$$

By (2.3), for all $Q \subsetneq P$ and all $\lambda, 0 < \lambda \leq 1$,

$$\left(Q \in \mathcal{E}(N, v) \Longrightarrow x^\lambda(Q) \geq v(Q)\right) \text{ and } \left(x(Q) > v(Q) \Longrightarrow x^\lambda(Q) > v(Q)\right).$$

[2][24] introduced vital coalitions of at least two elements, whereas according to our definition singletons are always vital.

Hence, $x_P^{\widehat{\lambda}} \in \mathcal{C}(P,v)$, $x^{\widehat{\lambda}}(Q) > v(Q)$ for all $Q \in 2^P \setminus \mathcal{E}(N,v)$, $Q \neq P$, and there exists $\varepsilon > 0$ such that $x^{\widehat{\lambda}+\varepsilon}(Q) \geq v(Q)$ for all $Q \subsetneqq P$. Then $d = v(P) - x^{\widehat{\lambda}+\varepsilon}(P) > 0$. Now, with $y = x^{\widehat{\lambda}+\varepsilon} + \frac{d}{|P|}\chi^P$ observe that $y(P) = v(P)$ and $y(Q) > v(Q)$ for all $Q \in 2^P \setminus \{\emptyset, P\}$. Hence, $P$ is strongly vital-exact. **q.e.d.**

It follows from the previous result that a preimputation $x$ satisfies $x \in \mathcal{C}(N,v)$ if and only if for all strongly vital-exact coalitions $S$ it follows that $x(S) \geq v(S)$.

**Lemma 2.1.25.** *If $(N,v)$ is vital-exact extendable and $x \in X(N,v) \setminus \mathcal{C}(N,v)$, then there exists a strongly vital-exact coalition $S$ such that $x(S) < v(S)$ and $x(T) \geq v(T)$ for all $T \subsetneqq S$.*

**Proof:** By Lemma 2.1.24 there exists a strongly vital-exact coalition $R$ such that $x(R) < v(R)$. Let $P$ be a minimal (w.r.t. inclusion) coalition that satisfies the foregoing condition. Assume, on the contrary, that there exists $Q \subsetneqq P$ such that $x(Q) < v(Q)$. Define

$$y = x + \frac{v(P) - x(P)}{|P \setminus Q|}\chi^{P \setminus Q}$$

and observe that $x \leq y$, $x(Q) = y(Q)$, and $y(P) = v(P)$. Hence $y_P \in X(P,v) \setminus \mathcal{C}(P,v)$. By Lemma 2.1.24 applied to $(P,v)$ and $y_P$, there exists a strongly vital-exact coalition $T$ w.r.t. $(P,v)$ such that $y(T) < v(T)$ and, hence, $x(T) < v(T)$. As $P$ is extendable, $T$ is strongly vital-exact w.r.t. $(N,v)$ so that the desired contradiction has been obtained. **q.e.d.**

**Theorem 2.1.26.** *Any balanced, vital-exact extendable game $(N,v)$ has a stable core.*

**Proof:** Let $z \in X(N,v) \setminus \mathcal{C}(N,v)$. By Lemma 2.1.25 there exists a strongly vital-exact $\emptyset \neq S \subseteq N$ such that $z(S) < v(S)$ and $z(T) \geq v(T)$ for all $T \subsetneqq S$. Let $y \in \mathbb{R}^S$ be given by $y_i = z_i + \frac{v(S)-z(S)}{|S|}$. Then $y(S) = v(S)$ and $y \gg z$, hence $y(T) > v(T)$ for all $\emptyset \neq T \subsetneqq S$. We conclude that $y \in \mathcal{C}(S,v)$. As $S$ is extendable, there exists $x \in \mathcal{C}(N,v)$ such that $x_S = y$. Thus $x \operatorname{dom}_S z$. **q.e.d.**

**Remark 2.1.27.** *If Definition 2.1.23 of a strongly vital-exact coalition $S$ is modified only inasmuch as "$T \in 2^S \setminus (\{S\} \cup \mathcal{E}(N,v))$" is replaced by "$T \in 2^S \setminus \{S, \emptyset\}$", then the arising relaxation of vital-exact extendability may not be sufficient for core stability as shown by the following example: Let $(N,v)$ be defined by $N = \{1,...,4\}$, $v(1,3,4) = v(2,3,4) = v(N) = 1, v(T) = 0$, otherwise. Then $\mathcal{C}(N,v) = \{(0,0,\alpha,1-\alpha)|0 \leq \alpha \leq 1\}$. Thus only the singletons are strongly vital-exact (using the modified definition) because the remaining vital coalitions, i.e., $\{1,3,4\}$ and $\{2,3,4\}$, violate the aforementioned modified definition of strong vital-exactness, because $x_1 = v(1) = x_2 = v(2) = 0$ for all $x \in \mathcal{C}(N,v)$. The* *singletons are also extendable, hence, the game is vital-exact extendable using the modified definition of strongly vital-exact. Clearly, $(N, v)$ does not have a stable core.*

Thus, Theorem 2.1.26 shows relations that may be summarized in the following diagram:

$$\text{extendability} \begin{array}{c} \nearrow\!\!\!\nearrow \\ \searrow\!\!\!\searrow \end{array} \begin{array}{c} \text{exact extendability} \\ \\ \text{vital extendability} \end{array} \begin{array}{c} \searrow\!\!\!\searrow \\ \nearrow\!\!\!\nearrow \end{array} \text{vital-exact extendability} \Rightarrow \text{core stability}. \tag{2.4}$$

By means of examples we will show that none of the opposite implications of (2.4) is valid and that exact extendability may not imply vital extendability and vice versa. Moreover, there are balanced games that are vital-exact extendable and have non-extendable coalitions that are vital and exact.

**Example 2.1.28.** Let $N = \{1, \ldots, 7\}$ and $(N, v_1)$ be defined as follows. Let $T = \{1, 2\}, T^i = \{2, i\}$ for $i = 3, 4, 5$, and $T^j = \{1, j\}$ for $j = 6, 7$, and $v_1(N) = 16, v_1(T^k) = 4$ for all $k = 3, \ldots, 7$, $v_1(T) = 1$, and for all other $S \subseteq N$, let $v_1(S) = 0$. Then $(3, 3, 2, 2, 2, 2, 2) \in \mathcal{C}(N, v_1)$ so that $\mathcal{E}(N, v_1) = \{\emptyset, N\}$. With

$$y^1 = (12, 4, 0, 0, 0, 0, 0), y^2 = (0, 2, 2, 2, 2, 4, 4), y^3 = (4, 0, 4, 4, 4, 0, 0)$$

note that $y^i \in \mathcal{C}(N, v_1)$ for $i = 1, 2, 3$. The coalition $T$ is vital, but not exact. Indeed, let $y \in \mathcal{C}(N, v_1)$. As $y(T^k) \geq 4$, $k = 3, \ldots, 7$,

$$y_i \geq 4 - y_2 \ \forall i \in \{3, 4, 5\} \text{ and } y_j \geq 4 - y_1 \ \forall j \in \{6, 7\} \tag{2.5}$$

so that $16 = y(N) \geq 20 - y(T) - y_2$, that is, $y(T) \geq 2$. We conclude that a coalition $S \subsetneqq N$ satisfying $v_1(S) > 0$ is exact if and only if it is one of the coalitions $T^j$, $j = 3, \ldots, 7$, and that these coalitions are extendable. An exact coalition $S$ with $v_1(S) = 0$ is also extendable, because $\mathcal{C}(S, v_1)$ is a singleton. Hence, $(N, v_1)$ is exact extendable, but not vital extendable. Let $(N, v_1')$ be the game that differs from $(N, v_1)$ only inasmuch as $v_1'(T) = 0$. Then $(N, v_1')$ is vital extendable (because $T$ is not vital w.r.t. $(N, v_1')$) and exact extendable, but $T$ is still not extendable.

**Example 2.1.29.** Now, let $(N, v_2)$ be the game that differs from $(N, v_1)$ defined in Example 2.1.28 only inasmuch as $v_2(N) = 18$. Any singleton and any of the coalitions $T^j, j = 3, \ldots, 7$, are still extendable which follows from the fact that $y^k + 2\chi^{\{i\}} \in \mathcal{C}(N, v_2)$ for any $k = 1, 2, 3$, and $i \in N$. Moreover, $z = (0, 1, 3, 3, 3, 4, 4)$ is the unique element in $\mathcal{C}(N, v_2)$ that satisfies $z(T) = v_2(T)$. Hence, $T$ is vital and exact, but not strongly vital-exact. We conclude that $(N, v_2)$ is vital-exact extendable, but neither exact extendable nor vital extendable. Now, if the worth of $N$ is further increased, that is, let

$0 < \varepsilon < 1$ and $(N, v_3)$ differ from $(N, v_2)$ only inasmuch as $v_3(N) = v_2(N) + \varepsilon$, then $(\varepsilon, 1-\varepsilon, 3+\varepsilon, 3+\varepsilon, 3+\varepsilon, 4-\varepsilon, 4-\varepsilon) \in \mathcal{C}(N, v)$ so that $T$ is strongly vital-exact. Now, $T$ is not extendable, because if $y \in \mathcal{C}(N, v_3)$ satisfies $y_2 = 0$, then $y_1 \geq 2 - \varepsilon > 1$, that is, $y(T) > v_3(T)$. Nevertheless, $(N, v_3)$ has a stable core. Indeed, if $x \in I(N, v_3) \setminus \mathcal{C}(N, v_3)$, then two cases may occur. If $x(T^j) \geq 4$ for all $j = 3, \ldots, 7$, then, by (2.5) applied to $x$, $x_2 + x(T) \geq 2 - \varepsilon$. As $x(T) < 1$, $x_2 > 1 - \varepsilon$ and $x_1 < \varepsilon$ so that $x$ is dominated by some core element via $T$. In the other case there exists $\ell \in \{3, \ldots, 7\}$ such that $x(T^\ell) < v(T^\ell)$ and extendability of $T^\ell$ guarantees that $x$ is dominated by some core element.

Together with Example 2.3.3 (the game $(N, v_4)$ discussed in Section 2.3) the foregoing examples show that the relations summarized in (2.4) are strict even if balancedness is assumed:

$$\text{core stability} \overset{v_3}{\not\Rightarrow} \text{vital-exact extendability} \begin{array}{c} \not\nearrow \\ v_2 \\ \not\searrow \end{array} \begin{array}{c} \text{vital extendability} \\ v_4 \not\Downarrow \qquad \not\Uparrow v_1 \\ \text{exact extendability} \end{array} \begin{array}{c} \not\searrow \\ v_1' \\ \not\nearrow \end{array} \text{extendability.} \quad (2.6)$$

The properties of the games $(N, v_1)$ and $(N, v_3)$ of Example 2.1.28 also show that neither "exact extendability" nor "vital-exact extendability" are strong *prosperity properties* in the sense of Definition 2.1 in [46], who showed that "extendability" is a strong prosperity property. Note that in a similar way it may be shown that "vital extendability" is a strong prosperity property (indeed, a nonempty, proper coalition in $N$ is or is not vital regardless of the "prosperity" of $N$).

An interesting invariance property shared by two of the new variants of "extendability" and by "core stability" is contained in the following statements. Let $(N, v)$ be a balanced game and $(N, \bar{v})$ its totally balanced cover (see (2.1)):

1. $(N, v)$ has a stable core $\Longleftrightarrow$ $(N, \bar{v})$ has a stable core.

2. $(N, v)$ is vital extendable $\Longleftrightarrow$ $(N, \bar{v})$ is vital extendable.

3. $(N, v)$ is vital-exact extendable $\Longleftrightarrow$ $(N, \bar{v})$ is vital-exact extendable.

For a proof of (1) see p. 220 in [46], who also show by means of Example 2 that there exists an extendable game whose totally balanced cover is not extendable. By (2.1), $\mathcal{C}(N, v) = \mathcal{C}(N, \bar{v})$. We conclude that a coalition is vital w.r.t. $(N, v)$ iff it is vital w.r.t. $(N, \bar{v})$. Moreover, if $S$ is vital, then $v(S) = \bar{v}(S)$. Therefore, $S$ is vital and exact w.r.t. $(N, v)$ iff $S$ is vital and exact w.r.t. $(N, \bar{v})$. Hence, if $S$ is vital and exact, then $\{x \in \mathcal{C}(N, v) \mid x(S) = v(S)\} = \{x \in \mathcal{C}(N, \bar{v}) \mid x(S) = \bar{v}(S)\}$. Again by (2.1), we may conclude that a coalition is strongly vital-exact w.r.t. $(N, v)$ iff it is strongly vital-exact w.r.t. $(N, \bar{v})$. Hence, (2) and (3) are valid. The totally balanced cover of $(N, v_1)$, $(N, \bar{v}_1)$, is not exact extendable. Indeed, it is straightforward to verify that $\bar{v}_1(\{1, 2, 3, 6\}) = 8$ and that $(1, 0, 4, 3) \in \mathcal{C}(\{1, 2, 3, 6\}, \bar{v}_1)$. However, this vector is not the restriction of any element of $\mathcal{C}(N, \bar{v}_1)$.

## 2.2 Consequences and monotone chains

This subsection serves to show that all strongly vital-exact coalition are extendable, if the set of strongly vital-exact coalitions exhibits a certain structure.

**Definition 2.2.1.** *A game $(N, v)$ has **disjoint antichains of strongly vital-exact coalitions** if, for all strongly vital-exact coalitions $S$ and $T$, $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$.*

The previous definition just states that the elements of any antichain of the partially ordered set of strongly vital-exact coalitions, ordered by inclusion, are pairwise disjoint.

**Theorem 2.2.2.** *If $(N, v)$ is a balanced game that has disjoint antichains of strongly vital-exact coalitions, then $(N, v)$ has a stable core.*

**Proof:** Let $S$ be a strongly vital-exact coalition. By Theorem 2.1.26 it suffices to show that $S$ is extendable. To this extent let $x \in \mathcal{C}(S, v)$. As $S$ is exact, there exists $y \in \mathcal{C}(N, v)$, $y(S) = v(S)$. Let $z \in \mathbb{R}^N$ be given by $z_S = x$ and $z_{N \setminus S} = y_{N \setminus S}$. We conclude that $z(N) = v(N)$, $z(T) = y(T) \geq v(T)$ for all $T \subseteq N \setminus S$ and all $S \subseteq T \subseteq N$, and $z(P) = x(P) \geq v(P)$ for all $P \subseteq S$. Hence, $z(Q) \geq v(Q)$ for all strongly vital-exact coalitions $Q$. By Lemma 2.1.24, $z \in \mathcal{C}(N, v)$ and the proof is complete. **q.e.d.**

Balanced games that have disjoint antichains of strongly vital-exact coalitions may be constructed as follows. Let $N$ be a finite nonempty set, $x \in \mathbb{R}^N$, and $(N, v)$ satisfy $v(S) = x(S)$ for all $S \subsetneq N$ and $v(N) \geq x(N)$. Then the strongly vital-exact coalitions are the singletons and $N$ provided that $v(N) > x(N)$. Hence $(N, v)$ has the desired property. Now let $(N^1, v^1), \ldots, (N^k, v^k)$ be $k$ balanced games that have disjoint antichains of strongly vital-exact coalitions such that the $N^\ell$ are pairwise disjoint. With $N = \bigcup_{\ell=1,\ldots,k} N^\ell$, let $(N, v)$ be a game that satisfies $v(S) = \sum_{\ell=1}^{k} v^\ell(S \cap N^\ell)$ for all $S \subsetneq N$ and $v(N) \geq \sum_{\ell=1}^{k} v^\ell(N^\ell)$. Then $(N, v)$ has the desired property.

The following theorem reveals some structure of the set of strongly vital-exact coalitions and will be used to show that vital-exact extendability is a necessary condition for core stability for the second class of games.

**Theorem 2.2.3.** *If $(N,v)$ is a balanced game, then there exist a balanced collection $\mathcal{P}$ of strongly vital-exact coalitions w.r.t. $(N,v)$ and a system $(\delta^P)_{P\in\mathcal{P}}$ of balancing weights for $\mathcal{P}$ such that*

$$\sum_{P\in\mathcal{P}} \delta^P v(P) = v(N).$$

**Proof:** Let $(N,v)$ be balanced. Claim:

$$\text{If } R \in \mathcal{E}(N,v), \emptyset \neq R, R \text{ is vital} \Rightarrow R \text{ is strongly vital-exact w.r.t. } (N,v). \tag{2.7}$$

In order to show the claim, note that by convexity of the core and the definition of $\mathcal{E}(N,v)$, there exists $x \in \mathcal{C}(N,v)$ such that $x(T) > v(T)$ for all $T \in 2^N \setminus \mathcal{E}(N,v)$. Hence, $x(R) = v(R)$ and $x(T) > v(T)$ for all $T \subsetneqq R$ with $T \notin \mathcal{E}(N,v)$ so that $R$ is strongly vital-exact.

We now proceed by induction on $n = |N|$. If $n = 1$, then $N$ is vital, hence strongly vital-exact, so that the proof is finished in this case. Let the theorem be true for $n \leq t$ and some $t \in \mathbb{N}$ and assume now that $n = t+1$. If $N$ is vital, then the theorem is true. Hence, we may assume that $N$ is not vital. Due to Remark 2.1.22 and Theorem 2.1.15, there exist a balanced collection $\widehat{\mathcal{R}}$ on $N$ and a system $(\widehat{\delta}^R)_{R\in\widehat{\mathcal{R}}}$ of balancing weights for $\widehat{\mathcal{R}}$ such that $N \notin \widehat{\mathcal{R}}$ and $\sum_{R\in\widehat{\mathcal{R}}} \widehat{\delta}^R v(R) = v(N)$. Moreover, for $x \in \mathcal{C}(N,v)$, $v(N) = x(N) = \sum_{R\in\widehat{\mathcal{R}}} \widehat{\delta}^R x(R) = \sum_{R\in\widehat{\mathcal{R}}} \widehat{\delta}^R v(R)$ so that $R \in \mathcal{E}(N,v)$ for all $R \in \widehat{\mathcal{R}}$. Moreover, $(R,v)$ is balanced so that, by the inductive hypothesis, there exist a balanced collection $\mathcal{P}_R$ of strongly vital-exact coalitions w.r.t. $(R,v)$ on $R$ and a system $(\delta_R^P)_{P\in\mathcal{P}_R}$ of balancing weights for $\mathcal{P}_R$ such that $v(R) = \sum_{P\in\mathcal{P}_R} \delta_R^P v(P)$. Define, for any $P \in \mathcal{P} = \bigcup_{R\in\widehat{\mathcal{R}}} \mathcal{P}_R$,

$$\delta^P = \sum_{R\in\{R\in\widehat{\mathcal{R}} \mid P\in\mathcal{P}_R\}} \widehat{\delta}^R \delta_R^P.$$

We conclude that $\sum_{P\in\mathcal{P}} \delta^P \chi^P = \chi^N$ and $\sum_{P\in\mathcal{P}} \delta^P v(P) = v(N)$. Thus, $\mathcal{P}$ is a balanced collection on $N$ and $\mathcal{P} \subseteq \mathcal{E}(N,v)$ so that the proof is finished by our (2.7). **q.e.d.**

Now, the second class of games is constructed as follows. Let $(N,v)$ be a game that satisfies the following property:

$$S \text{ is strongly vital-exact} \Longrightarrow |S| \leq 2. \tag{2.8}$$

For all $x, y \in X(N,v)$ and all $\alpha \geq 0$ define $z^{\alpha,x,y} \in \mathbb{R}^N$ by

$$z_i^{\alpha,x,y} = \begin{cases} x_i + \min\{y_i - x_i, \alpha\}, & \text{if } y_i \geq x_i, \\ x_i - \min\{x_i - y_i, \alpha\}, & \text{if } x_i \geq y_i \end{cases} \tag{2.9}$$

and note that $z$ is well-defined.

**Lemma 2.2.4.** *If $(N,v)$ satisfies (2.8), if $x, y \in \mathcal{C}(N,v)$, and if $\alpha \geq 0$, then $z^{\alpha,x,y} \in \mathcal{C}(N,v)$.*

**Proof:** If $\mathcal{C}(N,v) = \emptyset$, then the statement of the lemma is vacuously true. Hence, we assume now that $(N,v)$ is balanced. By Theorem 2.2.3 there exists a balanced collection $\mathcal{P}$ of strongly vital-exact coalitions on $N$ and a system $(\delta^P)_{P\in\mathcal{P}}$ of balancing weights for $\mathcal{P}$ such that $\sum_{P\in\mathcal{P}} \delta^P v(P) = v(N)$. Let $z = z^{\alpha,x,y}$ and $i \in N$. If $y_i \geq x_i$, then $z_i \geq x_i \geq v(\{i\})$. If $y_i < x_i$, then $z_i \geq y_i \geq v(\{i\})$. Hence, $z$ is individually rational. Let $P \in \mathcal{P}$. If $|P| = 1$, then $x(P) = y(P) = v(P)$ so that $z(P) = v(P)$. If $|P| = 2$, then $x(P) = y(P) = v(P)$ also implies $z(P) = v(P)$. By (2.8), $z(P) = v(P)$ for all $P \in \mathcal{P}$. We conclude that $z(N) = v(N)$. Now, let $S = \{i,j\}$, $i \neq j$, $i, j \in N$. By (2.8) and Lemma 2.1.24 it suffices to show that $z(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j \geq x_j$, then $z(S) \geq x(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j < x_j$, then the case $z(S) < y(S)$ may just occur, if $y_i - x_i > \alpha$. However, in this case $z(S) \geq x(S)$. The case $y_i < x_i$ and $y_j \geq x_j$ may be treated similarly. Finally, if $y_i < x_i$ and $y_j < x_j$, then $z(S) \geq y(S)$. Thus, $z \in \mathcal{C}(N,v)$. **q.e.d.**

**Proposition 2.2.5.** *If $(N,v)$ satisfies (2.8) and if each $\{i\}$, $i \in N$, is exact, then $(N,v)$ is vital-exact extendable.*

**Proof:** Let $S$ be a strongly vital-exact coalition and $x \in \mathcal{C}(N,v)$ be such that $x(S) = v(S)$. If $|S| = 1$, then the proof is already finished. Hence, we may assume that $S = \{k,\ell\}$ for some $k, \ell \in N$, $k \neq \ell$. Let $y \in \mathcal{C}(N,v)$ be such that $y_k = v(\{k\})$ and $\alpha := x_k - v(\{k\})$. By Lemma 2.2.4, $z = z^{\alpha,x,y} \in \mathcal{C}(N,v)$. Now, $z_k = y_k = v(\{k\})$ and $z_\ell = \alpha + x_\ell = v(\{k,\ell\}) - v(\{k\})$. By the convexity of $\mathcal{C}(N,v)$, $S$ is extendable. **q.e.d.**

Proposition 2.1.12 implies the following result.

**Corollary 2.2.6.** *If $(N,v)$ is a balanced game that satisfies (2.8), then the following conditions are equivalent:*

1. *$(N,v)$ has a stable core.*
2. *$(N,v)$ is vital-exact extendable.*
3. *For each $i \in N$, the singleton $\{i\}$ is an exact coalition.*

Note that (2.8) is sharp in the sense that if $|S| \leq 2$ is replaced by $|S| \leq 3$, then Corollary 2.2.6 is no longer valid. This statement may be shown by means of Example 2.4.1.

**Remark 2.2.7.** *Let $(N, v)$ be a balanced game. [37] presents a simple necessary and sufficient condition for the exactness of a coalition $T$: Define a $T$-covering, [37], as a set of non-negative numbers $\lambda^T = (\{\lambda_S^T\}_{S \subseteq N}, \gamma)$ such that*

$$\sum_{S \subseteq N} \lambda_S^T \chi^S - \gamma \chi^N = \chi^T.$$

*The coalition $T$ is exact if and only if for all $T$-coverings, $\lambda^T$, it follows that*

$$\max_{\lambda^T} \sum_{S \subseteq N} \lambda_S^T v(S) - \gamma v(N) = v(T). \tag{2.10}$$

*Note that by Corollary 2.2.6, (2.10) may be used to check whether a balanced game that satisfies (2.8) has a stable core. Moreover, a similar remark applies to the following theorem and minimum coloring games (see Theorem 2.5.3).*

Another class of games for which vital-exact extendability is both necessary and sufficient for core stability is the following. Let $(N, v)$ be a game satisfying the property that

$$S \text{ is strongly vital-exact} \Longrightarrow |S| \leq 2 \text{ or } S = N. \tag{2.11}$$

**Theorem 2.2.8.** *Let $(N, v)$ be a balanced game that satisfies (2.11) and such that for all $k \in N$ and all strongly vital-exact coalitions $S, T$, with $2 = |S| = |T|$ and $k \in S, T$, it follows that $v(S) - v(S \backslash k) = v(T) - v(T \backslash k)$. Then the following are equivalent:*

1. *$(N, v)$ has a stable core.*
2. *$(N, v)$ is vital-exact extendable.*
3. *For each $i \in N$, the singleton $\{i\}$ is an exact coalition.*

**Proof:** From Theorem 2.1.26 and Proposition 2.1.12 it just remains to be shown that (3) $\Rightarrow$ (2). First of all, it can be assumed that $|N| \geq 3$ as the case $|N| \leq 2$ follows directly. By (3) it follows that every singleton is extendable so, let $S$ be a strongly vital-exact coalition and $S = \{i, j\}$, $i, j \in N$ and $i \neq j$. By (3) there exists $x \in \mathcal{C}(N, v)$ such that $x_i = v(\{i\})$. Let $y \in \mathbb{R}^N$ be defined by $y_j = v(S) - v(\{i\})$, $y_i = v(\{i\})$ and $y_k = x_k + x_j - (v(S) - v(\{i\})) \geq x_k$ for some $k \in N \backslash \{i, j\}$ and otherwise $y_l = x_l$ for $l \in N \backslash \{i, j, k\}$. Then $y \in \mathcal{C}(N, v)$ by the assumptions of the theorem and by Lemma 2.1.24. By the convexity of $\mathcal{C}(N, v)$, $S$ is extendable. **q.e.d.**

The earlier example $(N, v_3)$ demonstrates that if one removes the assumption that for all $k \in N$, $v(S) - v(S \backslash k) = v(T) - v(T \backslash k)$ for all strongly vital-exact coalitions $S, T$, with $2 = |S| = |T|$ and $k \in S, T$, then the previous theorem is no longer valid.

Finally to finish this section a class of games introduced in [26] for which vital coalitions play an important role will be investigated. To introduce the class of games, a number of definitions must be given. The definitions here follow that of, e.g., [48].

**Definition 2.2.9.** *Let $(N, v)$ be a game. A **monotone chain** is a sequence of imputations and coalitions $(x_0, S_1, x_1, \dots, S_m, x_m)$ satisfying $x_{k-1} \operatorname{dom}_{S_k} x_k$ and $(x_0)_{S_k} \gg (x_k)_{S_k}$ for $1 \leq k \leq m$.*

**Definition 2.2.10.** *A game $(N, v)$ is **absolutely stable** if for every monotone chain $(x_0, S_1, x_1, \dots, S_m, x_m)$ it follows that $x_0$ dominates $x_m$.*

This is an interesting property as it can be shown that the class of such games always has a nonempty vNM stable set (see [49]). In addition, the class of absolutely stable games includes the classes of simple games and all $[n, n-1]$ games. This class of games was also characterized in [48]. A simpler characterization, without the assumptions of [48], will now be presented.

**Proposition 2.2.11.** *A game $(N, v)$ is absolutely stable iff for all vital coalitions $S, T$, with $|S|, |T| \geq 2$ and $v(S) + \sum_{i \in N \setminus S} v(\{i\}) < v(N)$, it follows that $S \cup T = N$.*

**Proof:** To prove the if direction let $(x_0, S_1, x_1, \dots, S_m, x_m)$ be a monotone chain. Note that it can be assumed that all $S_i$, $i = 1, \dots, m$, are vital[3]. It will be proven that $x_0(S_m) \leq v(S_m)$, whence $x_0 \operatorname{dom}_{S_m} x_m$. If $v(S_m) + \sum_{i \in N \setminus S_m} v(\{i\}) \geq v(N)$, then because $(x_0)_i \geq v(\{i\})$ for all $i \in N \setminus S_m$, it follows that $x_0(S_m) \leq v(S_m)$. So, let $v(S_m) + \sum_{i \in N \setminus S_m} v(\{i\}) < v(N)$. It will be proven by induction that $x_0(S_m) \leq v(S_m)$. So, as $x_{m-1}(S_m) \leq v(S_m)$, we now proceed to the induction step and assume that $x_k(S_m) \leq v(S_m)$. Assume, per absurdum, that $x_{k-1}(S_m) > v(S_m)$. Then because $(x_{k-1})_{S_k} \gg (x_k)_{S_k}$, and by the assumption $S_m \cup S_k = N$, one has

$$x_{k-1}(N) = x_{k-1}(S_m) + x_{k-1}(N \setminus S_m) > x_k(S_m) + x_k(N \setminus S_m) = x_k(N) = v(N).$$

The previous equation contradicts the fact that $x_{k-1}$ was an imputation. To demonstrate the other direction let $S, T$ be vital coalitions with $\sum_{i \in P} v(\{i\}) \geq v(P)$ for all coalitions $P \subsetneq S$, $|S|, |T| \geq 2$ and $v(S) + \sum_{i \in N \setminus S} v(\{i\}) < v(N)$ such that $S \cup T \neq N$. So, one has either $S \subsetneq T \neq N$ or $S \nsubseteq T$ and $T \nsubseteq S$. Define the following three vectors in $\mathbb{R}^N$ (the existence of the coming $\alpha$, $\sigma$, $\varepsilon$ and $\delta$ follows from the fact that $T$ and $S$ are vital,

[3] As demonstrated in [24], domination can always be achieved via a vital coalition

$v(S) + \sum_{i \in N \setminus S} v(\{i\}) < v(N)$ and also for all coalitions $P \subsetneqq S$, $\sum_{i \in P} v(\{i\}) \geq v(P)$),

$$x_i = \begin{cases} v(\{i\}) + \alpha, & \text{if } i \in S \setminus T, \\ v(\{i\}) + \sigma, & \text{if } i \in S \cap T, \\ v(\{i\}) + \varepsilon, & \text{if } i \in T \setminus S, \varepsilon > 0, \sigma > 0 \text{ and } \alpha > 0 : x(P) > v(P) \ \forall \ P \subseteq S, \\ & x(T) \leq v(T) \text{ and } v(N) - x(S \cup T) \geq \sum_{i \in N \setminus (S \cup T)} v(\{i\}), \\ v(\{i\}) + \kappa, & \text{if } i \in N \setminus (S \cup T) \text{ and } \kappa \geq 0 : x(N) = v(N). \end{cases}$$

$$y_i = \begin{cases} v(\{i\}) + \delta, & \text{if } i \in S \text{ and } 0 < \delta < \sigma : y(S) \leq v(S), \\ v(\{i\}), & \text{if } i \in T \setminus S, \\ v(\{i\}) + \tau, & \text{if } i \in N \setminus (S \cup T) \text{ and } \tau \geq 0 : y(N) = v(N). \end{cases}$$

$$z_i = \begin{cases} v(\{i\}), & \text{if } i \in S, \\ v(\{i\}) + \beta, & \text{if } i \in T \setminus S \text{ and } \beta > \varepsilon : z(T \cup S) = x(T \cup S), \\ v(\{i\}) + \kappa, & \text{if } i \in N \setminus (S \cup T). \end{cases}$$

It follows that $x \operatorname{dom}_T y \operatorname{dom}_S z$, however, $x$ does not dominate $z$ because for all $P \subseteq S$, $x(P) > v(P)$. Hence, the game is not absolutely stable contradicting the fact that $S \cup T \neq N$. The result then follows because if for all vital coalitions $S, T$ satisfying the conditions of the proposition and $\sum_{i \in P} v(\{i\}) \geq v(P)$, for all coalitions $P \subsetneqq S$, it follows that $N \subseteq S \cup T$, then there does not exist any vital coalition $P$ such that $S \subseteq P \subsetneqq N$.
**q.e.d.**

**Theorem 2.2.12.** *Let $(N, v)$ be an absolutely stable game. Then $(N, v)$ has a stable core if and only if $(N, v)$ is extendable.*

**Proof:** As the if direction is true in general, the other direction will now be proven. So, assume that $(N, v)$ has a stable core. Let $S$ be a vital coalition and $y \in \mathcal{C}(S, v)$. Let $\xi \in \mathbb{R}^{N \setminus S}$ be defined by $\xi_i \geq v(\{i\})$ for all $i \in N \setminus S$ and so that $\zeta \in \mathbb{R}^N$ defined by $\zeta_S = y_S$, $\zeta_{N \setminus S} = \xi_{N \setminus S}$ fulfills $\zeta(N) = v(N)$ (which is possible, otherwise the game is not even balanced). If $\zeta \notin \mathcal{C}(N, v)$, then there must exist $\eta \in \mathcal{C}(N, v)$ and a vital coalition $T$ such that $\eta \operatorname{dom}_T \zeta$. If $v(S) + \sum_{i \in N \setminus S} v(\{i\}) = v(N)$, then it follows that $T \subsetneqq S$, which is a contradiction. Hence, $v(S) + \sum_{i \in N \setminus S} v(\{i\}) < v(N)$. By the previous proposition, it follows that $N \setminus S \subseteq T$. Therefore, one has

$$\eta(N) = \eta(N \setminus S) + \eta(S) > \zeta(N \setminus S) + \zeta(S) = \zeta(N) = v(N).$$

This result contradicts the fact that $\eta \in \mathcal{C}(N, v)$, hence, $\zeta \in \mathcal{C}(N, v)$. The result then follows, as by Proposition 2.1.12, every singleton and, hence, every non-vital coalition with a nonempty core is extendable. **q.e.d.**

## 2.3 Assignment Games

[41] introduced assignment games. For finite sets $S$ and $T$, an *assignment* of $(S,T)$ is a bijection $b : S' \to T'$ such that $S' \subseteq S$, $T' \subseteq T$, and $|S'| = |T'| = \min\{|S|,|T|\}$. We shall identify $b$ with $\{(i,b(i)) \mid i \in S'\}$. Let $\mathcal{B}(S,T)$ denote the set of assignments.

**Definition 2.3.1.** *A game $(N,v)$ is an* ***assignment game*** *if there exist a partition $\{P,Q\}$ of $N$ and a nonnegative real matrix $A = (a_{ij})_{i\in P, j\in Q}$ such that*

$$v(S) = \max_{b\in\mathcal{B}(S\cap P, S\cap Q)} \sum_{(i,j)\in b} a_{ij}. \tag{2.12}$$

An assignment game is a model for an economic milieu in which there is a two-sided market composed of two types of agents, e.g., buyers and sellers, represented by two disjoint finite sets. These two types of agents are constrained to interact under exclusive, bilateral contracts. So, partnerships can be formed and the goods, money, etc. are then restricted only to be exchanged amongst partners. The number that is given to each possible matching represents the potential for profit if the particular partnership forms. The items being traded are also assumed to be indivisible and each transaction can only be carried out by a trade of all goods, money, etc. between the partners. The goal of the game is to suitably "assign" buyers to sellers.

Let $(N,v)$ be an assignment game defined by the matrix $A$ on $P \times Q$.

**Lemma 2.3.2.** *If $S \subseteq N$, $|S| \geq 2$, is a vital coalition, then $|S\cap P| = |S \cap Q| = 1$.*

**Proof:** Let $S \subseteq N$, $|S| \geq 2$. If $S \subseteq P$ or $S \subseteq Q$, then $0 = v(S) = \sum_{i\in S} v(\{i\})$ and, hence, $S$ is not vital. Assume now, that $|S\cap P| \geq 2$ or $|S \cap Q| \geq 2$, let us say $|S\cap P| \leq |S\cap Q|$, and let $b \in \mathcal{B}(S\cap P, S\cap Q)$ satisfy $v(S) = \sum_{i\in S\cap P} a_{ib(i)}$. Thus, for any $i \in S\cap P$, $v(S) = v(\{i,b(i)\}) + v(S \setminus \{i,b(i)\})$. We conclude that $S$ is not vital. **q.e.d.**

By Lemma 2.3.2 and Corollary 2.2.6, $(N,v)$ has a stable core if and only if $(N,v)$ is vital-exact extendable. The theory developed so far enables us to reprove Theorem 1 of [44]: Assume without loss of generality (w.l.o.g.) that $|P| \leq |Q|$ and let $b$ denote an optimal assignment. The assignment game $(N,v)$ has a stable core if and only if

$$a_{ib(i)} = \max_{r\in Q} a_{ir} = \max_{r\in P} a_{rb(i)} \text{ and } a_{ij} = 0 \ \forall i \in P, j \in Q \setminus \{b(r) \mid r \in P\}. \tag{2.13}$$

In order to verify the if direction, assume that (2.13) is valid and define $x, y \in \mathbb{R}^N$ by $x_i = a_{ib(i)} = y_{b(i)}$ and $x_{b(i)} = y_i = x_j = y_j = 0$ for all $i \in P$ and all $j \in Q \setminus b(P)$. Then

$x, y \in \mathcal{C}(N, v)$ and, hence, by Proposition 2.2.5 and Theorem 2.1.26, $(N, v)$ has a stable core.

In order to verify the only if direction, assume that $(N, v)$ has a stable core and let $i, j \in P$ and $r \in Q$. By Proposition 2.1.12 there exists $x \in \mathcal{C}(N, v)$ such that $x_r = 0$. Then $x \geq 0$, $x_i + x_{b(i)} = a_{ib(i)}$, and $x_i + x_r \geq a_{ir}$ so that $a_{ib(i)} \geq x_i \geq a_{ir}$. Similarly we may deduce that $a_{ib(i)} \geq a_{jb(i)}$ so that (2.13) is shown.

The following example shows that exact extendability and vital extendability are not necessary for core stability for assignment games.

**Example 2.3.3.** Let

$$A = \begin{pmatrix} 6 & 4 & 0 \\ 0 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix},$$

and $(N, v_4)$ be the assignment game defined by $A$, where $1, 2$, and $3$ are the "row" players and $4, 5$, and $6$ are the "column" players. The unique optimal assignment $b$ is given by $b(i) = 3 + i$ for $i = 1, 2, 3$. Hence, (2.13) is satisfied so that $(N, v_4)$ has a stable core. Moreover, $x = (3, 5, 1, 3, 1, 5) \in \mathcal{C}(N, v_4)$ and $x(S) = v_4(S)$, where $S = \{1, 3, 4, 5\}$. Now, $S$ is not extendable, because $(4, 0, 4, 0) \in \mathcal{C}(S, v_4)$ and any $y \in \mathcal{C}(N, v_4)$ must assign $a_{ib(i)}$ to any coalition $\{i, b(i)\}$ of optimally matched players, e.g., satisfies $y_1 + y_4 = a_{14} = 6$. We conclude that $(N, v_4)$ is not exact extendable. In order to show that $(N, v_4)$ is vital extendable, it suffices to show that $\{1, 5\}$ and $\{3, 4\}$ are extendable. A careful inspection of the core elements $(0, 2, 0, 6, 4, 6), (4, 6, 6, 2, 0, 0), (6, 6, 4, 0, 0, 2), (2, 0, 0, 4, 6, 6)$ shows that they are extendable. It should be noted that there are also assignment games with a stable core that are not vital extendable. Indeed, let $(N, v_5)$ be the assignment game defined by $B$. As each pair $(i, j)$, $i \in P, j \in Q$, belongs to an optimal matching except the pair $(3, 4)$, we conclude that $\mathcal{C}(N, v_5) = \{(\alpha, \alpha, \alpha, 2 - \alpha, 2 - \alpha, 2 - \alpha) \mid 0 \leq \alpha \leq 2\}$. Consequently, the vital coalition $\{3, 4\}$ is not exact and, hence, not extendable.

## 2.4 Simple Flow Games

[29] present two equivalent representations of totally balanced games: A game is totally balanced game if and only if (a) it is a *flow* game or (b) it is the minimum of finitely many additive games. The following example shows that even for the minimum of two additive games, the simplest nontrivial case in (b), vital-exact extendability may not be necessary for core stability. Moreover, for "simple" flow games we shall derive that vital extendability is necessary and sufficient for core stability.

**Example 2.4.1.** Let $N = \{1, \ldots, 6\}$, $\lambda = (2, 1, 1, 2, 1, 1)$, $N^1 = \{1, 2, 3\}$, $N^2 = \{4, 5, 6\}$ and $(N, v)$ be the game given by $v(S) = \min_{i=1,2} \lambda(S \cap N^i)$. The game $(N, v)$ is exact (see, e.g., [34]) and it has a stable core. Indeed, $\mathcal{C}(N, v)$ is the convex hull of $(2, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 2, 1, 1)$ (see, e.g., [36]) so that if $y \in I(N, v) \backslash \mathcal{C}(N, v)$ is not dominated by any core element via some 2-person coalition, then $y$ satisfies $y_2 = y_3$ and $y_5 = y_6$. Hence $y$ is dominated by some core element via $S = \{1, 5, 6\}$ or via $T = \{2, 3, 4\}$ and it follows that $S$ and $T$ are strongly vital-exact coalitions. These coalitions are not extendable, because for any $x \in \mathcal{C}(N, v)$, $x_2 = x_3$ and $x_5 = x_6$, but $\mathcal{C}(S, v)$ contains some $z$ with $z_5 \neq z_6$ (e.g., given by $z_1 = z_6 = 1/2$, $z_5 = 1$) and a similar statement holds for $T$.

Adopting the notation of [45], who characterized the simple flow games that have a stable core, $D = (V, E, s, t)$ is a *simple (directed) network* with source $s$ and sink $t$, if $V$ is the vertex set (a finite set), $E \neq \emptyset$ is the arc set (a subset of the set of ordered pairs of vertices, containing no loops or circles), and $s$ and $t$ are distinct vertices in $V$ such that there are no arcs leading into $s$ and no arcs coming out of $t$. The term "simple" refers to the fact that all arcs have the same capacity, let us say 1.

**Definition 2.4.2.** *The* ***flow game*** $(E, v^D)$ *associated with* $D = (V, E, s, t)$ *is the TU game defined by the requirement that, for any* $\emptyset \neq S \subseteq E$, $v^D(S)$ *is the maximal flow from* $s$ *to* $t$ *in the network* $(V, S, s, t)$*. A game* $(N, v)$ *is a* ***simple flow game*** *if it is the game associated with some simple directed network with a source and a sink.*

According to [29], a flow game is supposed to model "problems of profit sharing in an integrated production system with alternative production routes". Such production systems are modeled in terms of a "fluid" or "resource" which "flows" through the network. This "fluid" or "resource" accounts for the input at the $s$ vertex and is then converted to an output, which is available at the vertex $t$ (or it could also be simply transported to vertex $t$). The other vertices represent intermediate steps in the production process. The transition from one step of the production process to the next is achieved via the arcs, which are owned by the players. Each arc has a given capacity (in the current case 1) which represents the maximum amount of "fluid" or "resource" that can be directed through the arc. Sets of arcs are controlled by the players (in the current setup, each player only controls one arc) and players are then able to form coalitions by combining the arcs which they control in the network.

A *(simple) path* in a network $D = (V, E, s, t)$ is a sequence of arcs from $s$ to $t$ that visits each vertex at most once. It is well-known that

$$v^D(S) \text{ is the maximal number of arc-disjoint paths in } (V,S,s,t) \text{ for all } S \in 2^E \setminus \{\emptyset\}. \tag{2.14}$$

Let $D = (V, E, s, t)$ be a simple network with source and sink and denote $v = v^D$.

**Remark 2.4.3.** If a coalition $S$ is vital and $v(S) > 0$, then $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneqq S$. Indeed, by (2.14), the elements of $S$, suitably ordered, must form a path.

An arc $e \in E$ is called a *dummy arc*[4] if there exists a path containing $e$ and if $v(E \setminus \{e\}) = v(E)$. We recall that a *cut* of $D$ is a coalition $C \subseteq E$ such that each path contains an arc of $C$. For a proof of the following "max-flow min-cut" theorem see, e.g., [22]:

$$v^D(E) = \min\{|C| \mid C \text{ is a cut of } D\}. \tag{2.15}$$

We are now able to recall Theorem 3 of [45].

**Theorem 2.4.4.** *Let $D = (V, E, s, t)$ be a simple network with source and sink. Then $(E, v^D)$ has a stable core if and only if $E$ does not contain any dummy arc.*

We use the preceding theorem and the following lemma and remark to show that vital extendability is necessary for core stability in the case of simple flow games. Let $D = (V, E, s, t)$ be a simple network with source and sink.

**Lemma 2.4.5.** *If $E$ does not contain any dummy arc and if $e \in E$ satisfies $v^D(E \setminus \{e\}) < v^D(E)$, then there exists a minimum cut $C$ with $e \in C$.*

**Proof:** By (2.14) there are $v^D(E)$ arc-disjoint paths. We may assume that $v^D(E) > 1$. As $v^D(E) > v^D(E \setminus \{e\})$, the arc $e$ must be contained in one of the paths and $v^D(E \setminus \{e\}) = v^D(E) - 1$. Hence, if $C'$ is a minimum cut of $(V, E \setminus \{e\}, s, t)$, then $C' \cup \{e\}$ is a minimum cut of $D$ by (2.15). **q.e.d.**

**Remark 2.4.6.** In a constructive way, on p. 478 in [29] the authors show that the core of an arbitrary flow game is nonempty. Applied to a simple flow game $(N, v)$ associated with the simple network $D = (V, E, s, t)$ they prove that, for any minimum cut $C$ of $D$, $\chi^C \in \mathcal{C}(E, v^D)$.

**Proposition 2.4.7.** *A simple flow game $(N, v)$ has a stable core if and only if it is vital extendable.*

[4] [45] use this term although a dummy arc is not a *dummy player*. Indeed an arc is a dummy player if and only if it either connects $s$ and $t$ or it is not contained in any path

**Proof:** Let $D = (V, E, s, t)$ be a simple network with source and sink and $(E, v)$ be the associated simple flow game. As the if direction is valid by Theorem 2.1.26, we assume now that $(E, v)$ has a stable core. Let $S$ be a vital coalition. If $v(S) = 0$, then $|S| = 1$ and, by Proposition 2.1.12, $S$ is extendable. If $v(S) > 0$, then, by Remark 2.4.3, $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneqq S$ and the elements of $S$ form a path. By Lemma 2.4.5 and Remark 2.4.6, for any $e \in S$, there exists $x \in \mathcal{C}(E, v)$ such that $x_e = 1$ and $x_{e'} = 0$ for all $e' \in S \setminus \{e\}$. However, $\mathcal{C}(S, v)$ is the convex hull of those core elements when restricted to $S$. **q.e.d.**

It should be remarked that on p.444 in [21] the authors present an example of a simple flow game (associated with $G_3$) that has a stable core and is not extendable. (Indeed, the 4-person coalition corresponding to the arcs that are marked by $+$ is not exact, but the core of the corresponding subgame is nonempty.)

## 2.5 Minimum Coloring Games

[18] introduced minimum coloring games and we basically adopt the notation of [12]. A *graph* is a pair $G = (V, E)$, where $V$ is a finite nonempty set, called the set of *vertices*, and $E$ is a set of 2-element subsets of $V$, called the set of *edges*. For any $U \subseteq V, U \neq \emptyset$, let $G^U$ denote the subgraph of $G$ whose vertex set is $U$ and whose edges are those edges in $E$ that are subsets of $U$.

The graph $G$ is *complete* if $E$ is the set of all 2-element subsets. A nonempty set $U \subseteq V$ is a *clique* if $G^U$ is complete. A clique is maximum if it has the maximum size among all cliques. Let $\omega(G)$ denote the size of a maximum clique. A *coloring* of $G$ is a mapping $c : V \to \mathbb{R}$ satisfying $c(i) \neq c(j)$ for all $\{i, j\} \in E$. A *minimal* coloring is a coloring $c$ such that $|c(V)|$ is minimal. Let $\gamma(G)$ denote the *chromatic number* of $G$, i.e., $\gamma(G) = |c(V)|$ for any minimal coloring of $G$. A set $U \subseteq V$, $U \neq \emptyset$, is *independent* if $\gamma(G^U) = 1$. The graph $G$ is *perfect* if $\omega(G^U) = \gamma(G^U)$ for all $U \in 2^V \setminus \{\emptyset\}$.

**Definition 2.5.1.** *Let $G = (V, E)$ be a graph. The* ***minimum coloring game*** *on $G$ is the game $(N, v^G)$ defined by the following requirements: (1) $N = V$; (2) $v^G(S) = |S| - \gamma(G^S)$ for*[5] *all $S \in 2^V \setminus \{\emptyset\}$.*

According to [18], minimum coloring games can be used to model certain types of conflict situations amongst a group of people. In such applications, one is looking for the smallest number of conflict-free groups in a system where vertices represent members of

[5] [12] consider the "cost" game whose coalition function simply assigns $\gamma(G^S)$ to any coalition $S$. We consider the "cost sharing" game instead so that, e.g., the definition of the core remains unchanged.

the group and edges represent conflicts between the members of the group. This type of interpretation can be applied to resource sharing problems and similar situations (see [18]).

To prove that vital extendability is equivalent to core stability the following lemma is required.

**Lemma 2.5.2.** *Let $(N, v)$ be a minimum coloring game on the graph $G = (V, E)$. Then $\emptyset \neq S \subseteq N$ is vital if and only if $S$ is independent.*

**Proof:** If $S$ is independent, then $v(T) = |T| - 1$ for all $\emptyset \neq T \subseteq S$. Let $x \in \mathbb{R}^S$ be defined by $x_i = \frac{|S|-1}{|S|}$ for all $i \in S$. Then $x(S) = v(S)$ and $x(T) > v(T)$ for all $\emptyset \neq T \subsetneqq S$ so that $S$ is vital. Conversely, assume now that $S$ is a coalition with $v(S) < |S| - 1$. It remains to be shown that $S$ is not vital. Let $c : S \to \mathbb{R}$ be a minimal coloring of $G^S$ and $i \in S$. Then $T = \{j \in S \mid c(j) \neq c(i)\} \neq \emptyset$ and $c_T$ (the restriction of $c$ to $T$) is a minimal coloring of $G^T$. We conclude that $v(S) = v(T) + v(S \setminus T)$ and, hence, that $S$ is not vital. **q.e.d.**

**Theorem 2.5.3.** *Let $(N, v)$ be a balanced minimum coloring game. Then the following conditions are equivalent:*

1. *$(N, v)$ has a stable core.*
2. *$(N, v)$ is vital extendable.*
3. *Every singleton is exact w.r.t. $(N, v)$.*

**Proof Theorem 2.5.3:** By Proposition 2.1.12, Theorem 2.1.26 and (2.4) it remains to show that (3) implies (2). Let $S$ be a vital coalition and $y \in \mathcal{C}(N, v)$. For any $j \in N$, $v(N) - v(N \setminus \{j\}) + y(N \setminus \{j\}) \geq v(N) = y(N) = y_j + y(N \setminus \{j\})$. We conclude that $y_j \leq v(N) - v(N \setminus \{j\})$. As $v(N) - v(N \setminus \{j\}) \leq 1$ for any minimum coloring game, we conclude that $y_j \leq 1$. Now, let $i \in S$. By (3), there exists $x \in \mathcal{C}(N, v)$ with $x_i = v(\{i\}) = 0$. By Lemma 2.5.2, $v(S) = |S| - 1$. Therefore, $x_j = 1$ for all $j \in S \setminus \{i\}$ and convexity of the core completes the proof. **q.e.d.**

We now use Theorem 2.5.3 to characterize minimum coloring games that have stable cores.

**Theorem 2.5.4.** *Let $G = (N, E)$ be a graph, $c$ be a minimal coloring of $G$, and denote, for $k \in c(N)$, $T_k = \{i \in N \mid c(i) \neq k\}$. The minimum coloring game $(N, v^G)$ has a stable core if and only if for any $k \in c(N)$,*

$$\gamma\left(G^{T_k \cup \{i\}}\right) = \gamma(G) \text{ and } \left(T_k \cup \{i\}, v^{G^{T_k \cup \{i\}}}\right) \text{ is balanced } \forall i \in N \setminus T_k. \tag{2.16}$$

**Proof:** Let $v = v^G$. If $\gamma(G) = 1$, then $(N, v)$ has a stable core and the proof is immediate. Hence, we may assume that $\gamma(G) \geq 2$ so that for any $k \in c(N)$, $N \setminus T_k$ is independent and

$$v(T_k) + v(N \setminus T_k) = |T_k| - (\gamma(G) - 1) + |N \setminus T_k| - 1 = |N| - \gamma(G) = v(N). \qquad (2.17)$$

In order to verify the only if direction let $k \in c(N)$ and $i \in N \setminus T_k$. By Theorem 2.5.3 there exists $x \in \mathcal{C}(N, v)$ such that $x_i = 0$. By (2.17), $x(N \setminus T_k) = v(N \setminus T_k)$ and $x(T_k) = v(T_k)$. As $v(T_k) = x(T_k) = x(T_k \cup \{i\}) \geq v(T_k \cup \{i\}) \geq v(T_k)$, $\gamma\left(G^{T_k \cup \{i\}}\right) = \gamma(G)$ and $x_{T_k \cup \{i\}} \in \mathcal{C}(T_k \cup \{i\}, v)$.

In order to verify the if direction, let $i \in N$. By Theorem 2.5.3 it suffices to show that there exists $x \in \mathcal{C}(N, v)$, $x_i = 0$. Let $k = c(i)$. By (2.16), $v(T_k \cup \{i\}) = v(T_k)$ and there exists $y \in \mathcal{C}(T_k \cup \{i\}, v)$. As $v(T_k \cup \{i\}) = v(T_k) + 0 = v(T_k) + v(\{i\})$, we conclude that $y(T_k) = v(T_k)$ and $y_i = 0$. Let $x \in \mathbb{R}^N$ be given by $x_{T_k \cup \{i\}} = y$ and $x_j = 1$ for all $j \in N \setminus (T_k \cup \{i\})$. By (2.17), $x(N) = v(N)$ so that $x \in \mathcal{C}(N, v)$. **q.e.d.**

It should be noted that a minimum coloring game of a graph is balanced if there exists a coloring such that (2.16) is valid for some $k$ and some $i$. The foregoing theorem generalizes Theorem 4.1 of [12].

**Corollary 2.5.5.** *The minimum coloring game on a perfect graph $G$ has a stable core if and only if every vertex of $G$ belongs to a maximum clique of $G$.*

**Proof:** We may assume that $\gamma(G) > 1$. Let $G = (N, E)$ be a perfect graph, $c$ be a minimal coloring of $G$, $v = v^G$, let $i \in N$, $k = c(i)$, and $T = \{j \in N \mid c(j) \neq k\}$. If $(N, v)$ has a stable core, then, by (2.16), $\omega\left(G^T\right) = \gamma\left(G^T\right) = \gamma(G) - 1$ and $\omega\left(G^{T \cup \{i\}}\right) = \gamma\left(G^{T \cup \{i\}}\right) = \gamma(G) = \omega(G)$ so that $i$ is in a maximum clique. If, on the other hand, $i$ is in some maximum clique, then $\gamma(G) = \omega(G^{T \cup \{i\}})$ so that (2.16) is satisfied for $T = T_k$ and $i$. **q.e.d.**

Note that on p. 424 in [12], the authors present a perfect graph ($G_1$) with some vertex that does not belong to a maximal clique, that is, the first condition of (2.16) is violated for some $T_k$ and some $i$, so that the resulting minimum coloring game does not have a stable core.

**Example 2.5.6.** Let $G_2$ be the perfect graph that consists of two disjoint triangles that are connected via one edge and may be found on p. 424 in [12]. (For a characterization of extendable minimum coloring games on perfect graphs see their Theorem 4.2.) So, $G_2 = (N, E)$, where

$$N = \{1, \ldots, 6\} \text{ and } E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{4,5\}, \{4,6\}, \{5,6\}, \{1,4\}\}.$$

Let $v = v^{G_2}$ and $T = \{1,4\}$. Then $x(T) \geq 1$ for any $x \in \mathcal{C}(N,v)$. Note that $(0,0,0,1,1,1) \in \mathcal{C}(N,v)$ so that $S = \{1,2,4\}$ is exact. As $(0,1,0) \in \mathcal{C}(S,v)$, $S$ is exact and not extendable. By Corollary 2.5.5, $(N,v)$ has a stable core.

We present a graph that has a minimal coloring so that exclusively the second condition of (2.16) is violated for a unique $T_k$ and a unique $i$. Moreover, we present a minimum coloring game on a non-perfect graph that has a stable core.

**Example 2.5.7.** Let $G = (N,E)$ be defined by $N = \{1,\ldots,7\}$ and

$$E = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\} \cup \{\{1,6\},\{5,6\},\{3,7\},\{4,7\}\}.$$

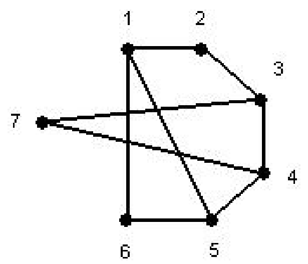


Figure 1. The graph from Example 2.5.7

Let $v = v^G$, $S = \{1,\ldots,5\}$, and note that $G^S$ is a pentagon so that $\gamma(G^S) = 3 > 2 = \omega(G^S)$. We conclude that $G$ is not perfect and it is well-known that $\mathcal{C}(S,v) = \emptyset$. Moreover, let $c : N \to \mathbb{R}$ be defined by $c(1) = c(4) = 1$, $c(3) = c(5) = 2$, and $c(2) = c(6) = c(7) = 3$. Thus, $c$ is a minimal coloring of $G$ and, with the notation of Theorem 2.5.4,

$$T_1 = N \setminus \{1,4\}, T_2 = N \setminus \{3,5\}, T_3 = N \setminus \{2,6,7\}.$$

As $S_1 = \{1,5,6\}$ and $S_2 = \{3,4,7\}$ are cliques, we conclude that $\omega\left(G^{T_k\cup\{i\}}\right) = \gamma\left(G^{T_k\cup\{i\}}\right)$ $= 3 = \gamma(G)$ and, hence, $\mathcal{C}(T_k \cup \{i\}, v) \neq \emptyset$ for $k = 1,2$ and all $i \in N \setminus T_k$. Similarly, if $i \in \{6,7\}$, then $\omega\left(G^{T_3\cup\{i\}}\right) = \gamma\left(G^{T_3\cup\{i\}}\right) = 3 = \gamma(G)$ and $\mathcal{C}(T_k \cup \{i\}, v) \neq \emptyset$. Finally, $T_3 \cup \{2\} = \{1,\ldots,5\}$ so that $\gamma\left(G^{T_3\cup\{2\}}\right) = 3 = \gamma(G)$ even in this case. Thus, $\mathcal{C}(T_3 \cup \{2\}, v) = \emptyset$ so that there is a unique balancedness condition that is violated in (2.16).

If $G' = (N,E')$ is the graph that differs from $G$ only inasmuch as $E'$ contains the additional edge $\{2,7\}$, then we may define a minimal coloring $c' : N \to \mathbb{R}$ by $c'(1) = c'(7) = 1, c'(3) = c'(5) = 2$, and $c'(2) = c'(4) = c'(6) = 3$. Now there is the additional maximal clique $S_3 = \{2,3,7\}$ so that (2.16) is valid. In fact, $\chi^{S_i} \in \mathcal{C}(N,v)$ for $i = 1,2,3$ so that any singleton is exact and Theorem 2.5.3 may be applied directly to show that the minimum coloring game on the non-perfect graph $G'$ has a stable core.

## 2.6 Conclusion

In the preceding chapter new conditions were given which were sufficient for core stability. For certain classes of games, it was demonstrated that these conditions were necessary and sufficient however they were shown not to be necessary in general. The main emphasis of the chapter was, however, on the importance of certain types of vital coalitions. It was shown that by studying certain properties of these types of vital coalitions, one is able to derive necessary and/or sufficient conditions for core stability for certain classes of games and games in general. The main conclusion, which can be drawn from the analysis here, is that a comprehensive study of vital coalitions (and variations thereof) could hold the key to core stability.

# 3 Fuzzy Games

In this chapter the question concerning core stability will be considered from a different perspective. Covers and extensions of $n$-person, cooperative, TU games that preserve core stability are considered here. The focus of this chapter is on the extension of the totally balanced cover of a game over the entire $n$-dimensional hypercube and its implications for core stability.

This chapter is organized as follows. As fuzzy games form the basis of this chapter the first section is devoted to the pertinent definitions and concepts. Section 3.2 then introduces the concavification of the coalition function and demonstrates a vital theorem providing the connection between core stability of the concave extension of a game and the original game. In Section 3.3 the superdifferential is introduced and important properties concerning it and the concave extension of a game are discussed. Section 3.4 then investigates exactness in this new setting. Finally the link between nonlinear analysis and core stability is given in Section 3.5.

## 3.1 Core Stability and Fuzzy Games

The idea of a fuzzy game was introduced in [2]. A fuzzy game, in some sense, represents a generalization of a normal, $n$-person, cooperative, TU game. In an $n$-person, cooperative, TU game the coalition function $v$ is a function from the corners of the $n$-dimensional cube to the reals. For fuzzy games one extends the domain of the function from the corner points of the $n$-dimensional cube over an entire given set (commonly over the entire $n$-dimensional cube). A fuzzy game represents the scenario where players do not have to fully commit themselves to being a member of a given coalition but are allowed to partially join a given coalition.

The idea of extending the coalition function does not always have to lead to fuzzy games. For example, in [5], the authors considered the problem from a measure theoretic point of view and [32] linearly extended the coalition function over the entire $n$-dimensional cube.

Numerous authors have already investigated the fuzzy equivalent of solution concepts from non fuzzy cooperative games. A recent paper exploring ideas germane to the following investigation is that of [8], in which they investigated, *inter alia*, the concepts of extendability, exactness and core stability. This section will build on the ideas presented in that paper and, unless stated otherwise, all definitions and results stated here can be found in [8].

Before the definitions are given, some notation needs to be introduced. Let $N \subsetneq \mathbb{N}$ be a nonempty, finite set with $|N| = n$. For a nonnegative vector $Q \in \mathbb{R}^N$, let $F(Q)$ be the box $F(Q) = \{c \in \mathbb{R}^N | 0 \leq c \leq Q\}$. The point $Q$ represents the grand coalition (normally $\chi^N$ for the fuzzy extension of a non-fuzzy, cooperative, TU game) and every $c \in F(Q)$ represents a fuzzy coalition. The support of a coalition $c = (c_1, \ldots, c_n) \in F(Q)$ is the set $supp(c) := \{i \in N \mid c_i > 0\}$. In addition, $|c|$ stands for the $l^1$ norm of $c$, that is $|c| = \sum_{i=1}^n |c_i|$ and for two vectors $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$, $x \cdot y = \sum_{i=1}^n x_i y_i$.

**Definition 3.1.1.** *A **fuzzy game** is a pair $(Q, v)$ so that*
*i) $Q \in \mathbb{R}_+^N$ and*
*ii) $v : F(Q) \to \mathbb{R}$ is bounded and satisfies $v(0) = 0$.*

The definition of the set of imputations and the core will now be given. To do so, for $Q \in \mathbb{R}^N$ define $Q^{\{i\}} = Q\chi^{\{i\}}$ for all $i \in N$.

**Definition 3.1.2.** *Let $(Q, v)$ be a fuzzy game. The **set of imputations**, $I(Q, v)$, is defined as*

$$I(Q, v) = \{x \in \mathbb{R}^N \mid x \cdot Q = v(Q),\ x_i \cdot Q_i \geq v(Q^{\{i\}}),\ \forall\, i \in N\}.$$

**Definition 3.1.3.** *The **core** of the fuzzy game $(Q, v)$, denoted by $\mathcal{C}(Q, v)$, is defined as*

$$\mathcal{C}(Q, v) = \{x \in \mathbb{R}^N \mid x \cdot Q = v(Q), x \cdot c \geq v(c)\ \forall\, c \in F(Q)\}.$$

In accordance with the terminology for $n$-person, cooperative, TU games, if the core of the fuzzy game is not empty, then the game will be called balanced. If the core of every subgame of $(Q, v)$ is not empty, then $(Q, v)$ is said to be totally balanced.

**Definition 3.1.4.** *Let $(Q, v)$ be a fuzzy game.*
*i) The **Strong Super-Additive** cover of $v$ is the function $\boldsymbol{SSa}v : F(Q) \to \mathbb{R}$ defined for each $d \in F(Q)$ by*

$$\boldsymbol{SSa}v(d) := \sup\{\sum_{j=1}^{L} \lambda_j v(c_j) \mid L \in \mathbb{N},\ \sum_{j=1}^{L} \lambda_j c_j = d,\ \lambda_j \geq 0, c_j \in F(d),\ j = 1, \ldots, L\}.$$

*ii) $v$ is called **Strongly Super-Additive** (SSA) if $v = \boldsymbol{SSa}v$ on $F(Q)$.*

The following theorem is proven in [6].

**Theorem 3.1.5.** *i) The fuzzy game $(Q, v)$ is balanced if and only if $\boldsymbol{SSa}v(Q) = v(Q)$*
*ii) The fuzzy game $(Q, v)$ is totally balanced if and only if $v$ is SSA*

The next most important and relevant definition is domination.

**Definition 3.1.6.** *Let $(Q, v)$ be a fuzzy game and $0 \neq c \in F(Q)$. Then $x \in \mathbb{R}^N$ is said to* ***dominate*** *$y \in \mathbb{R}^N$ via the fuzzy coalition $c$, $x \ \mathrm{dom}_c \ y$, if $x \cdot c \leq v(c)$ and $x_i > y_i$ for every $i \in supp(c)$. $x$ is said to dominate $y$, $x \ \mathrm{dom} \ y$, if there exists a non zero $c \in F(Q)$ such that $x \ \mathrm{dom}_c \ y$.*

The previous definition allows one to give the definition of a stable core.

**Definition 3.1.7.** *Let $(Q, v)$ be a fuzzy game. The fuzzy game $(Q, v)$ has a* ***stable core*** *if, for every imputation $y \notin \mathcal{C}(Q, v)$, there exists an $x \in \mathcal{C}(Q, v)$ such that $x \ \mathrm{dom} \ y$.*

To introduce the next few definitions, the definition of a subgame must be given.

**Definition 3.1.8.** *Let $(Q, v)$ be a fuzzy game and let $c \in F(Q)$. The* ***subgame of*** *$(Q, v)$ with respect to $c$ is the fuzzy game $(c, v_c)$, where for every $d \in F(c)$ it follows that $v_c(d) = v(d)$.*

**Definition 3.1.9.** *Let $(Q, v)$ be a fuzzy game. A fuzzy coalition $c \in F(Q)$ is* ***exact*** *if there exists $x \in \mathcal{C}(Q, v)$ such that $x \cdot c = v(c)$. $(Q, v)$ is exact if every $c \in F(Q)$ is exact.*

**Definition 3.1.10.** *Let $(Q, v)$ be a fuzzy game. $(Q, v)$ has a* ***large core*** *if for every $y \in \mathbb{R}^N$ that satisfies $y \cdot c \geq v(c)$ for every $c \in F(Q)$, there exists $x \in \mathcal{C}(Q, v)$ such that $x \leq y$.*

**Definition 3.1.11.** *Let $(Q, v)$ be a fuzzy game. $(Q, v)$ is* ***extendable*** *if for every $c \in F(Q)$ and $x \in \mathcal{C}(c, v_c)$, there exists a $y \in \mathcal{C}(Q, v)$ such that $y_{supp(c)} = x$.*

In [8] a number of relationships between the above concepts are demonstrated as shown in the following diagram for balanced fuzzy games. TB above an arrow means that the relationship only holds for the class of totally balanced (fuzzy) games or if there is a line through it, that the implication is also not valid for totally balanced games.

$$\begin{array}{ccccc}
\text{Extendability} & \begin{array}{c}\overset{TB}{\Rightarrow}\\ \Leftarrow\end{array} & \text{Exactness} & \begin{array}{c}\overset{TB}{\not\Leftarrow}\\ \Rightarrow\end{array} & \text{Largeness}\\
 & & \Downarrow & & \\
 & & \text{Stable core} & &
\end{array}$$

This diagram will be augmented, after displaying the relationship between the same properties for $n$-person, cooperative, TU games. It will be shown that core stability does not generally imply largeness of the core and that, unexpectedly, largeness of the core does not imply, in general, that the game has a stable core.

All these relationships are quite different from those of normal, cooperative, TU games in which the following relationships hold for balanced games. Examples exist to show that the arrows in the other directions, in the following diagram, do not hold in general (see [46] and [13]).

$$\begin{array}{ccc} \text{Largeness} & & \\ \Downarrow & & \\ \text{Extendability} & \overset{TB}{\Rightarrow} & \text{Exactness} \\ \Downarrow & & \\ \text{Stable core} & & \end{array}$$

An example showing that largeness of the core is not implied by core stability in fuzzy games is the following. The example is a variation of an example given in [46].

**Example 3.1.12.** Let $(Q, v)$ be a fuzzy game, $Q = (1, 1, 1, 1, 1, 1)$, defined as follows. $v(\chi^{\{1,2\}}) = v(\chi^{\{1,3\}}) = v(\chi^{\{4,5\}}) = v(\chi^{\{4,6\}}) = 1$, $v(Q) = 3.5$ and $v(c) = 0$ otherwise. The game $(Q, v)$ does not have a large core as there does not exist an element of the core, $x$, such that $x(\{1, 4\}) = 0$, whereas $y = (0, 1, 1, 0, 1, 1)$ satisfies $y \cdot c \geq v(c)$ for all $c \in F(Q)$. The game $(Q, v)$ has a stable core (the proof is identical to the non-fuzzy case, see [46]).

The following example demonstrates that largeness of the core does not imply core stability in general.

**Example 3.1.13.** Let $(Q, v)$ be a fuzzy game defined as follows. $Q = (1, 1)$ and for $x = (x_1, x_2) \in F(Q)$

$$v(x) = \begin{cases} 2(x_1 + x_2)^2, & \text{if } 0 \leq x_1 + x_2 < 1, \\ (x_1 + x_2)^2, & \text{if } 1 \leq x_1 + x_2 \leq 2. \end{cases}$$

Then it follows that $I(Q, v) = \{\alpha(1, 3) + (1 - \alpha)(3, 1) \mid 0 \leq \alpha \leq 1\}$. Also for $\zeta \in \mathbb{R}^2$,

$$\zeta \cdot x \geq v(x) \; \forall \, x \in F(Q) \iff \zeta_i \geq 2, \; i = 1, 2.$$

This will now be proven. To prove the if direction, let $\zeta \in \mathbb{R}^2$ and let $\zeta_i \geq 2, \; i = 1, 2$. Then for all $x \in F(Q)$ it follows that $\zeta \cdot x \geq 2(x_1 + x_2) \geq v(x)$. So, to prove the other direction let $x^1 = (x_1, 0)$ and $x^2 = (0, x_2)$ for $0 < x_i < 1$, $i = 1, 2$. Then $\zeta \cdot x^i = \zeta_i x_i \geq 2x_i^2$ for $i = 1, 2$ implies that $\zeta_i \geq 2x_i$ for all $0 < x_i < 1$ and $i = 1, 2$, hence $\zeta_i \geq 2$ for $i = 1, 2$. Therefore, $\mathcal{C}(Q, v) = (2, 2)$ and the game $(Q, v)$ has a large core. However, the game $(Q, v)$ does not have a stable core, as not a single imputation $y \neq (2, 2)$ can be dominated by $(2, 2)$. To demonstrate this, choose $\eta \in I(Q, v) \backslash \mathcal{C}(Q, v)$ and assume, w.l.o.g., that $\eta_1 < 2$. If $x \in F(Q)$ were to exist such that $(2, 2) \, \mathrm{dom}_x \, \eta$, then it would follow that $supp(x) = \{1\}$. However, if $x = (1, 0)$, then it follows that $2 = (2, 2)(1, 0) > v((1, 0)) = 1$ and if $x = (x_1, 0)$ with $0 < x_1 < 1$, then $(2, 2)(x_1, 0) = 2x_1 > 2x_1^2 = v(x)$.

## 3.2 The Concavification of the Coalition Function

One method, in cooperative Game Theory, of trying to find necessary and sufficient conditions for core stability is to study the stability of the core of the totally balanced cover of a game. The reason for studying the totally balanced cover was revealed in the first chapter and it was because a given game has a stable core if and only if the game arising from the totally balanced cover has a stable core. As was already stated in the previous chapter, a game is totally balanced if and only if it is a flow game. However, one can say more than that (for definitions of the coming different types of games, the reader is referred to the original articles, stated below, or to [1], where this proposition can be found).

**Proposition 3.2.1.** *Let $(N, v)$ be a game. The following are equivalent.*
*a) $(N, v)$ is a market game.*
*b) $(N, v)$ is a glove-market game.*
*c) $(N, v)$ is a linear production game.*
*d) $(N, v)$ is a flow game.*
*e) $(N, v)$ is totally balanced.*

In [40] it is proven that a cooperative game is totally balanced if and only if the game is a market game. [33] observed that a linear production game is a market game and in [29] the equivalence of *b*), *d*) and *e*) is proven. Finally, in [1] it is proven that a glove-market game is a linear production game.

The previous proposition implies that if one could find the necessary and sufficient conditions for core stability for one of the classes of games stated in the previous proposition, then one would have solved the problem concerning core stability. This is because one could calculate the totally balanced cover of a given game, then transform the game into one of the types of games listed above and check the necessary and sufficient conditions for the transformed game. The advantage with this method is that finding the necessary and sufficient conditions for core stability for the games listed above is possibly easier, as the games have a certain structure. The disadvantage is the circuitous path taken to test the conditions for core stability for a given game. Therefore, one would prefer a simple transformation of a given game (or also a transformation which ameliorated the analysis because, even for the games above, there still appears to be no breakthrough results coming close to finding the necessary and sufficient conditions for core stability), for which the necessary and sufficient conditions for core stability of the transformed game are known.

The previous discussion leads one quite naturally to the domain of fuzzy games via the concavification (or super additive extension) of a game (see below). Here one has much more structure and hence, perhaps a better chance of finding the necessary and sufficient conditions for core stability.

In [7] they introduced the concavification of the coalition function defined on the $(n-1)$-dimensional simplex. The notion which they developed turns out to be related to a well-known concept (albeit defined differently). This concept had already been studied in, e.g., [3].

When studying the concavification of the coalition function one usually extends the coalition function from a function defined on the vertices of the $n$-dimensional hypercube to a function defined over the entire hypercube. Hence this process naturally leads one to the realm of fuzzy games as defined and investigated in the previous section. The distinction with the previous section is that the coalition function of the fuzzy game has a definite form in this section.

The analysis presented in [7] will be given (for the $(n-1)$-dimensional simplex case) and then this idea will be extended over the entire $n$-dimensional hypercube,

$$\mathbb{H} := \{x \in \mathbb{R}^N \mid 0 \leq x_i \leq 1 \, \forall \, i \in N\}.$$

Let $(N, v)$ be a game and let $\Delta := \{x \in \mathbb{R}^N_+ \mid \sum_{i=1}^n x_i = 1\}$ stand for the $(n-1)$-dimensional simplex. In order to provide the definition given in [7], let $c_S = \frac{\chi^S}{|S|}$ and $\underline{v}(c_S) = \frac{v(S)}{|S|}$ for all nonempty $S \subseteq N$.

**Definition 3.2.2.** *Let $(N, v)$ be a game. The **concavification of the function** $\underline{v}$, $cav\underline{\boldsymbol{v}}$, is defined as the minimum of all concave functions $f : \Delta \to \mathbb{R}$ such that $f(c_S) \geq \underline{v}(c_S)$ for every coalition $S$.*

The following result is proven in [7].

**Proposition 3.2.3.** *Let $(N, v)$ be a game. For every $q \in \Delta$,*

$$cav\underline{\boldsymbol{v}}(q) = \max\{\sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S \underline{v}(S) \mid \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S c_S = q, \alpha_S \geq 0\}.$$

This result will now be extended over the entire $n$-dimensional hypercube. The corresponding definition of the concavification is as follows. Note that a function $f$ is positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$.

**Definition 3.2.4.** *Let $(N, v)$ be a game. The **concavification of the coalition function** $v$, $\hat{v}$, is defined as the minimum of all concave, positively homogeneous functions $f : \mathbb{H} \to \mathbb{R}$ such that $f(\chi^S) \geq v(S)$ for every coalition $S$.*

To prove the next result, the following lemma from [35] is required[6].

**Lemma 3.2.5.** *A function $f : \mathbb{R}^N \to \mathbb{R}$ is a positively homogeneous, concave function if and only if $f(x + y) \geq f(x) + f(y)$ for all $x, y \in \mathbb{R}^N$.*

The previous lemma clearly applies to the function $\hat{v}$ defined on $\mathbb{H}$.

**Remark 3.2.6.** *One should note that, since the minimum of a family of concave functions over $\mathbb{H}$ is concave it follows that the concavification of the coalition function is also concave (see [7]).*

**Proposition 3.2.7.** *Let $(N, v)$ be a game. For every $q \in \mathbb{H}$,*

$$\hat{v}(q) = \max\{\sum_{S\subseteq N} \alpha_S v(S) \mid \sum_{S\subseteq N} \alpha_S \chi^S = q, \alpha_S \geq 0\}.$$

**Proof:** For $q \in \mathbb{H}$ define $w(q) = \max\{\sum_{S\subseteq N} \alpha_S v(S) \mid \sum_{S\subseteq N} \alpha_S \chi^S = q, \alpha_S \geq 0\}$. Then $w$ is a concave and positively homogeneous function. Let $\hat{v}$ be as defined in Definition 3.2.4. As $v(S) \leq w(\chi^S)$ for $S \subseteq N$, it follows that $\hat{v}(q) \leq w(q)$ for all $q \in \mathbb{H}$. However, if $q = \sum_{S\subseteq N} \alpha_S \chi^S$, then it follows by the positive homogeneity and concavity of $\hat{v}$ and Lemma 3.2.5 that $\hat{v}(q) \geq \sum_{S\subseteq N} \alpha_S \hat{v}(\chi^S) \geq \sum_{S\subseteq N} \alpha_S v(S)$. Hence, $\hat{v}(q) \geq w(q)$ for all $q \in \mathbb{H}$, whence the result follows. **q.e.d.**

Let $(N, v)$ be a game. Note that the concavification of $v$, $\hat{v}$, coincides with the totally balanced cover of $v$, $\bar{v}$, on all $\chi^S$, that is for $S \subseteq N$, $\hat{v}(\chi^S) = \bar{v}(S)$. When referring to the fuzzy game $(\mathbb{H}, \hat{v})$ defined by the concavification of a game $(N, v)$, $(\mathbb{H}, \hat{v})$ will be referred to as the concave extension of $(N, v)$. The concavification of the coalition function $v$ is piecewise linear, positively homogeneous, and totally balanced (using the notation of the last section, it is **SSA**). Here it will also be proven to be continuous on $\mathbb{H}$. By noticing that the two functions $\hat{v}$ and $cav\underline{\mathbf{v}}$ coincide over $\Delta$ and that $\hat{v}$ is just the homogeneous extension of $cav\underline{\mathbf{v}}$, it suffices to prove the continuity of $cav\underline{\mathbf{v}}$. To this end, a number of definitions, which will also be relevant for later sections, will be given here. These definitions can be found in, e.g., [16]. Throughout, let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$.

**Definition 3.2.8.** *A **correspondence** is a map $\varphi : X \to 2^Y$. One writes $\varphi : X \rightsquigarrow Y$.*

**Definition 3.2.9.** *A correspondence $\varphi : X \rightsquigarrow Y$ is **lower hemi-continuous** (l.h.c.) at $x \in X$ if, for every $y \in \varphi(x)$ and for each neighborhood $V$ of $y$, there exists a neighborhood $U$ of $x$ so that $\varphi(z) \cap V \neq \emptyset$ for all $z \in U \cap X$. $\varphi$ is l.h.c. if it is l.h.c. at all $x \in X$.*

[6]The results from [35] cited in this chapter were all stated for convex functions. However, a function $f$ is concave if and only if $-f$ is convex and therefore the results apply to concave functions as well. Hence, the results in [35] can be applied equally well here, with changes to inequality signs when necessary.

**Definition 3.2.10.** *A correspondence $\varphi : X \rightsquigarrow Y$ is* ***upper hemi-continuous*** *(u.h.c.) at $x$ if, for every neighborhood $V \supseteq \varphi(x)$, there exists a neighborhood $U$ of $x$ such that $\varphi(y) \subseteq V$ for all $y \in U \cap X$. $\varphi$ is u.h.c. if it is u.h.c. at all $x \in X$.*

**Definition 3.2.11.** *A correspondence $\varphi : X \rightsquigarrow Y$ is* ***continuous*** *at $x$ if $\varphi$ is both u.h.c. and l.h.c. at $x$. $\varphi$ is continuous if it is continuous at all $x \in X$.*

It will now be proven that $cav\underline{\mathbf{v}}$ is continuous. To do so, consider the following correspondence, $\Phi : \Delta \to \mathbb{R}^{2^N \setminus \{\emptyset\}}$,

$$\Phi(x) := \{\eta = (\alpha_S)_{S \in 2^N \setminus \{\emptyset\}} \mid \alpha_S \geq 0, \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S c_S = x\}.$$

Note also that for a finite set $I$ and $\alpha_i \in \mathbb{R}_+$, $i \in I$, and $\gamma \in \mathbb{R}_+$, if $\gamma \geq \max_{i \in I} \alpha_i$, then $\sum_{i \in I}(\alpha_i - \gamma)_+ = 0$. If there exists $i^* \in I$ with $\alpha_{i^*} \geq \gamma$, then $\sum_{i \in I}(\alpha_i - \gamma)_+ \leq \sum_{i \in I \setminus i^*} \alpha_i + \alpha_{i^*} - \gamma = \sum_{i \in I} \alpha_i - \gamma$. In addition, the following simple fact will also be used in the proof, which can be seen by realizing that, for each coalition $S \subseteq N$, $|S| \geq 2$, the term $\frac{\alpha_S}{|S|}$ appears once for each $i \in S$,

$$\sum_{i \in N} \Big( \sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{\alpha_S}{|S|} \Big) = \sum_{\{S \in 2^N : |S| \geq 2\}} \alpha_S.$$

**Lemma 3.2.12.** $\Phi$ *is a continuous correspondence.*

**Proof:**

**1st STEP** :

It will first be proven that $\Phi$ is a l.h.c. correspondence. Take $\bar{x} \in \Delta$ and $\eta = (\alpha_S)_{S \in 2^N \setminus \{\emptyset\}} \in \Phi(\bar{x})$. Let $V$ be a neighborhood of $\eta$ and let $\varepsilon > 0$ such that if $\xi \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$ fulfills $|\xi - \eta| < \varepsilon$, then $\xi \in V$. For $x \in \Delta$, let $e^x = |N| \max_{i \in N}(\bar{x}_i - x_i)$ and define for all coalitions $S$, $|S| \geq 2$, $\alpha_S^x = (\alpha_S - e^x)_+$. Then, by the comments before the proposition, it follows, for all $i \in N$, that if $e^x \geq \alpha_S$ for all $S \subseteq N$, $|S| \geq 2$, then

$$\sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{\alpha_S^x}{|S|} = 0 \leq x_i,$$

otherwise

$$\sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{\alpha_S^x}{|S|} \leq \bar{x}_i - \frac{e^x}{|N|} \leq x_i.$$

It follows that one can choose $\alpha_{\{i\}}^x \geq 0$, $i \in N$, so that $\sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S^x c_S = x$. Choose now a neighborhood, $U \subseteq \Delta$, of $\bar{x}$ such that for all $x \in U$, it follows that $|x - \bar{x}| < \frac{\varepsilon}{2^{n+1}}$.

Whence it follows that $e^x < \frac{\varepsilon}{2^{n+1}}$. Let $x \in U$. Then, with $\xi := (\alpha_S^x)_{S \in 2^N \setminus \{\emptyset\}} \in \Phi(x)$, one has, first of all, that

$$\sum_{\{S \in 2^N : |S| \geq 2\}} |(\alpha_S - e^x)_+ - \alpha_S| \leq \sum_{\{S \in 2^N : |S| \geq 2\}} |(\alpha_S - \frac{\varepsilon}{2^{n+1}})_+ - \alpha_S| \leq \sum_{\{S \in 2^N : |S| \geq 2\}} \frac{\varepsilon}{2^{n+1}}.$$

In addition, because of the comments before the proposition and the fact that $\alpha_S \geq \alpha_S^x$ for all $S \subseteq N$, $|S| \geq 2$, one has

$$\sum_{i \in N} |\alpha_{\{i\}}^x - \alpha_{\{i\}}| = \sum_{i \in N} |(x_i - \sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{\alpha_S^x}{|S|}) - (\bar{x}_i - \sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{\alpha_S}{|S|})| \leq \sum_{i \in N} |x_i - \bar{x}_i|$$

$$+ \sum_{i \in N} | \sum_{\{S \in 2^N : |S| \geq 2, i \in S\}} \frac{(\alpha_S - \alpha_S^x)}{|S|}| = |x - \bar{x}| + \sum_{\{S \in 2^N : |S| \geq 2\}} |\alpha_S - \alpha_S^x| < \frac{\varepsilon}{2^{n+1}} + \sum_{\{S \in 2^N : |S| \geq 2\}} \frac{\varepsilon}{2^{n+1}}.$$

Hence, one can conclude

$$|\xi - \eta| = \sum_{S \in 2^N \setminus \{\emptyset\}} |\alpha_S^x - \alpha_S| = \sum_{\{S \in 2^N : |S| \geq 2\}} |(\alpha_S - e^x)_+ - \alpha_S| + \sum_{i \in N} |\alpha_{\{i\}}^x - \alpha_{\{i\}}|$$

$$< \sum_{\{S \in 2^N : |S| \geq 2\}} \frac{\varepsilon}{2^{n+1}} + \sum_{\{S \in 2^N : |S| \geq 2\}} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n+1}} < \varepsilon.$$

Whence one has that $\Phi(x) \cap V \neq \emptyset$.

**2$^{\text{nd}}$STEP** :

To prove that $\Phi$ is an u.h.c. correspondence, let $\bar{x} \in \Delta$ and let $V \subseteq \mathbb{R}^{2^N \setminus \{\emptyset\}}$ be a neighborhood of $\Phi(\bar{x})$ and let $\varepsilon > 0$ such that if for some $\xi \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, there exists $\eta \in \Phi(\bar{x})$ such that $|\eta - \xi| < \varepsilon$, then $\xi \in V$. For $x \in \Delta$, let $e^x = |N| \max_{i \in N}(x_i - \bar{x}_i)$. For $\xi = (\alpha_S)_{S \in 2^N \setminus \{\emptyset\}} \in \Phi(x)$ and $S \subseteq N$, with $|S| \geq 2$, define $\alpha_S^x = (\alpha_S - e^x)_+$. Then, as before, one has

$$\sum_{\{S \in 2^N : |S| \geq 2\}} \alpha_S^x c_S \leq \bar{x}.$$

Hence, one can choose $\alpha_{\{i\}}^x \geq 0$ so that $\sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S^x c_S = \bar{x}$. Choose now a neighborhood, $U \subseteq \Delta$, of $\bar{x}$ such that for all $x \in U$, it follows that $|x - \bar{x}| < \frac{\varepsilon}{2^{n+1}}$. Hence, for all $\xi \in \Phi(x)$, one can find a $\eta := (\alpha_S^x)_{S \in 2^N \setminus \{\emptyset\}} \in \Phi(\bar{x})$, as defined above such that (with identical steps as for the l.h.c. part of the proof) $|\eta - \xi| < \varepsilon$. From this it follows that $\xi \in V$. **q.e.d.**

**Definition 3.2.13.** *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ and let $\varphi : X \rightsquigarrow Y$ be a correspondence. $\varphi$ is **compact valued** if $\varphi(x)$ is compact for all $x \in X$.*

**Lemma 3.2.14.** *The correspondence $\Phi$ is compact valued.*

**Proof:** $\Phi$ is bounded because for all $x \in \Delta$ and $\eta = (\alpha_S)_{S \in 2^N \setminus \{\emptyset\}} \in \Phi(x)$, it follows that $\alpha_S \in [0, 1]$. To demonstrate that $\Phi$ is closed, let $\{\eta^t\}_{t \in \mathbb{N}}$ be a sequence such that $\eta^t \in \Phi(x)$, for all $t$, and $\eta^t \to \eta$. Let $\eta = (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}}$, $\eta^t = (\alpha_S^t)_{S\in 2^N\setminus\{\emptyset\}}$ and let $\bar{x} = \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S c_S$. Then for all $\varepsilon > 0$ it follows that there exists an $N \in \mathbb{N}$ such that for all $k > N$, $k \in \mathbb{N}$,

$$\varepsilon > |\eta^k - \eta| = \sum_{S\in 2^N\setminus\{\emptyset\}} |\alpha_S^k - \alpha_S| = \sum_{i\in N} \sum_{\{S\in 2^N : i\in S\}} \left|\frac{\alpha_S^k - \alpha_S}{|S|}\right|$$

$$\geq \sum_{i\in N} \left| \sum_{\{S\in 2^N : i\in S\}} \frac{\alpha_S^k}{|S|} - \sum_{\{S\in 2^N : i\in S\}} \frac{\alpha_S}{|S|} \right| = \sum_{i\in N} |x_i - \bar{x}_i| = |x - \bar{x}|.$$

Hence, one has that $\eta \in \Phi(x)$. **q.e.d.**

An application of the following theorem, first proven in [11], will then be used to show that $cav\underline{\mathbf{v}}$, and hence $\hat{v}$, is continuous.

**Theorem 3.2.15.** *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ and let $\varphi : X \rightsquigarrow Y$ be a compact valued correspondence. Let $f : Y \to \mathbb{R}$ be continuous. Define $\mu : X \rightsquigarrow Y$ by $\mu(x) = \{y \in \varphi(x) \mid y$ maximizes $f$ on $\varphi(x)\}$ and $F : X \to \mathbb{R}$ by $F(x) = f(y)$ for $y \in \mu(x)$. If $\varphi$ is continuous at $x$, then $F$ is continuous at $x$.*

Now take the correspondence $\Phi$ in the place of $\varphi$ in the statement of the previous theorem, as well as $cav\underline{\mathbf{v}}$ for $F$ and define, for $\eta \in \Phi(x)$ with $\eta := (\alpha_S)_{S\in 2^N\setminus\{\emptyset\}}$, the function $f : \mathbb{R}^{2^N\setminus\{\emptyset\}} \to \mathbb{R}$ by $f(\eta) = \sum_{S\in 2^N\setminus\{\emptyset\}} \alpha_S c_S$. As $\Phi$ is a compact valued, continuous correspondence and $f$ is clearly continuous, it follows that $cav\underline{\mathbf{v}}$ is continuous.

The main reason for studying the concave extension is because of a property that it shares with the totally balanced cover of a cooperative game. That is, the balanced game $(N, v)$ has a stable core if and only if the the concave extension $(\mathbb{H}, \hat{v})$ has a stable core. This result will now be proven. As the game $(\mathbb{H}, \hat{v})$ is a fuzzy game the definition of the core of $(\mathbb{H}, \hat{v})$ is that, which was given in the previous section. Note that $I(\mathbb{H}, \hat{v}) = I(N, v)$ for balanced games.

**Theorem 3.2.16.** *Let $(N, v)$ be a balanced game and let $(\mathbb{H}, \hat{v})$ be the concave extension of $(N, v)$. Then $(N, v)$ has a stable core if and only if $(\mathbb{H}, \hat{v})$ has a stable core.*

**Proof:**
**1$^{\text{st}}$STEP** :
Let $(N, v)$ have a stable core. It will be proven that for any imputation $x \notin \mathcal{C}(\mathbb{H}, \hat{v})$ there exists $y \in \mathcal{C}(\mathbb{H}, \hat{v})$ and $q \in \mathbb{H}$ such that $y \,\mathrm{dom}_q\, x$. If an imputation $x \notin \mathcal{C}(\mathbb{H}, \hat{v})$, then there exists a non zero $q \in \mathbb{H}$ such that $x \cdot q < \hat{v}(q)$. This implies, however, that there exists a coalition $\widehat{S}$ such that $x(\widehat{S}) < v(\widehat{S})$. If that were not the case, i.e., $x(S) \geq v(S)$ for all

$S \subseteq N$, then let $q = \sum_{S\subseteq N} \alpha_S \chi^S$ for the $\alpha_S \geq 0$ such that $\hat{v}(q) = \sum_{S\subseteq N} \alpha_S v(S)$. Then,

$$x \cdot q = x \cdot \sum_{S\subseteq N} \alpha_S \chi^S = \sum_{S\subseteq N} \alpha_S x(S) \geq \sum_{S\subseteq N} \alpha_S v(S) = \hat{v}(q).$$

Hence, there exists a coalition $\widehat{S}$ such that $x(\widehat{S}) < v(\widehat{S})$. As the game $(N, v)$ has a stable core and $x \in I(N, v)$, there exists a $y \in \mathcal{C}(N, v)$ and a vital coalition $R$ satisfying $v(R) = \hat{v}(\chi^R)$ and $y \operatorname{dom}_R x$. Now $y(P) \geq v(P)$ for all $P \subseteq N$ and hence, $y \cdot q \geq \hat{v}(q)$ for all $q \in \mathbb{H}$. Therefore, $y \in \mathcal{C}(\mathbb{H}, \hat{v})$. Hence, $(\mathbb{H}, \hat{v})$ has a stable core.
**2$^{\text{nd}}$STEP** :
It will now be proven that if the game $(\mathbb{H}, \hat{v})$ has a stable core, then $(N, v)$ has a stable core. So let $x \notin \mathcal{C}(N, v)$ be an imputation. Then there exists a coalition $S$ such that $x(S) < v(S)$. Hence, $x \cdot \chi^S < \hat{v}(\chi^S)$ and therefore $x \notin \mathcal{C}(\mathbb{H}, \hat{v})$ and $x \in I(\mathbb{H}, \hat{v})$. As the game $(\mathbb{H}, \hat{v})$ has a stable core, there exists $y \in \mathcal{C}(\mathbb{H}, \hat{v})$ and a non zero $q \in \mathbb{H}$ such that $y \operatorname{dom}_q x$ with $y \cdot q = \hat{v}(q)$. Now let $\alpha_S \geq 0$ such that $\hat{v}(q) = \sum_{S\subseteq N} \alpha_S v(S)$ and $q = \sum_{S\subseteq N} \alpha_S \chi^S$. Then there exists a coalition $\widehat{S}$ such that $y(\widehat{S}) = v(\widehat{S})$. To see this, assume the contrary, that is, $y(S) > v(S)$ for all $S \subseteq N$ such that $\alpha_S > 0$. As was done before, this would then imply that $y \cdot q > \hat{v}(q)$. Now $y_i > x_i$ for all $i \in \widehat{S}$ and therefore $y \operatorname{dom}_{\widehat{S}} x$ and one also has that $y \in \mathcal{C}(N, v)$. Hence, $(N, v)$ has a stable core. **q.e.d.**

The previous result provides one with a new approach to try and find the necessary and sufficient conditions under which a balanced, $n$-person, cooperative, TU game has a stable core. This analysis will be taken up again in Section 3.5, where it will be shown how the concave extension can be used to formulate the problem of core stability in a nonlinear analysis setting.

## 3.3 The Superdifferential

Let $(N, v)$ be a game and and let $(\mathbb{H}, \hat{v})$ be the concave extension of $(N, v)$. In this section properties of the superdifferential of $\hat{v}$, $\partial\hat{v}$, will be investigated. In the last section of this chapter, this analysis will be used to characterize when the game $(\mathbb{R}^N_+, \hat{v})$ (see below) has a stable core.

Let $(N, v)$ be a game. To simplify the following sections, the concavification $\hat{v}$ of $v$ will be extended over the entire $\mathbb{R}^N_+$ via its homogeneity. By a certain abuse of notation, the extension of $\hat{v}$ over $\mathbb{R}^N_+$ will also be denoted by $\hat{v}$. So, let $q \in \mathbb{R}^N_+$, $q \neq 0$. Then $\hat{v}(q) := q(N)\hat{v}(\frac{q}{q(N)})$. To make it clear that the extension over $\mathbb{R}^N_+$ is being considered the notation $(\mathbb{R}^N_+, \hat{v})$ will be used. One can then consider the pair $(\mathbb{R}^N_+, \hat{v})$ as an extended type of fuzzy game. So, $(\mathbb{R}^N_+, \hat{v})$ will also be called a game and for this game the core and the set of imputations will be defined as follows. That is, define

$$\mathcal{C}(\mathbb{R}^N_+, \hat{v}) = \{x \in \mathbb{R}^N \mid x \cdot q \geq \hat{v}(q)\ \forall q \in \mathbb{R}^N_+,\ x \cdot \chi^N = \hat{v}(\chi^N)\}$$

and

$$I(\mathbb{R}^N_+, \hat{v}) = \{x \in \mathbb{R}^N \mid x_i \geq \hat{v}(\chi^{\{i\}})\ \forall i \in N,\ x \cdot \chi^N = \hat{v}(\chi^N)\}.$$

Notice that for $x \in \mathcal{C}(\mathbb{H}, \hat{v})$, using this extension, it follows that, for all $q \in \mathbb{R}^N_+$, $q \neq 0$, one has $x \cdot q = q(N)x \cdot \frac{q}{q(N)} \geq q(N)\hat{v}(\frac{q}{q(N)}) = \hat{v}(q)$. It then clearly follows that $\mathcal{C}(\mathbb{R}^N_+, \hat{v}) = \mathcal{C}(\mathbb{H}, \hat{v})$. Hence, for balanced games $(N, v)$, and the respective extensions, $\mathcal{C}(N, v) = \mathcal{C}(\mathbb{R}^N_+, \hat{v}) = \mathcal{C}(\mathbb{H}, \hat{v})$. Again, by applying the definitions of domination and a stable core for fuzzy games to the game $(\mathbb{R}^N_+, \hat{v})$, one notices, because of the homogeneity of $\hat{v}$, that the game $(\mathbb{R}^N_+, \hat{v})$ has a stable core if and only if the game $(\mathbb{H}, \hat{v})$ has a stable core. In addition, for balanced games $(N, v)$, $I(\mathbb{R}^N_+, \hat{v}) = I(N, v)$.

To define the superdifferential of a concave function $f$, the definition of a supergradient must be given. The notation here follows that of, e.g., [35].

**Definition 3.3.1.** *A vector $x \in \mathbb{R}^N$ is a **supergradient** of a concave function $f : \mathbb{R}^N_+ \to \mathbb{R}$ at a point $q \in \mathbb{R}^N_+$ if*

$$f(z) \leq f(q) + x \cdot (z - q)\ \forall\ z \in \mathbb{R}^N_+.$$

*This condition will be referred to as the supergradient inequality. The set of all supergradients of $f$ at $q$ is called the **superdifferential** of $f$ at $q$ and is denoted by $\partial f(q)$.*

**Proposition 3.3.2.** *Let $(N, v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then*

$$\partial \hat{v}(q) = \{x \in \mathbb{R}^N \mid x \cdot p \geq \hat{v}(p)\ \forall p \in \mathbb{R}^N_+,\ x \cdot q = \hat{v}(q)\}.$$

**Proof:** Let $x \in \partial \hat{v}(q)$. To prove that $x \cdot p \geq \hat{v}(p)$ for all $p \in \mathbb{R}^N_+$, take $z = p + q$ in the supergradient inequality. Then it follows from

$$\hat{v}(p) + \hat{v}(q) \leq \hat{v}(p + q) \leq \hat{v}(q) + x \cdot (p + q - q)$$

that $x \cdot p \geq \hat{v}(p)$. To prove that $x \cdot q = \hat{v}(q)$, take $z = \frac{q}{2}$ in the supergradient inequality and one has

$$\hat{v}(\frac{q}{2}) \leq \hat{v}(q) + x \cdot (\frac{q}{2} - q).$$

From this, and the homogeneity of $\hat{v}$, it follows that $x \cdot q \leq \hat{v}(q)$, but as $x \cdot p \geq \hat{v}(p)$ for all $p \in \mathbb{R}^N_+$ one has $x \cdot q = \hat{v}(q)$. Now let $y \in \{x \in \mathbb{R}^N \mid x \cdot p \geq \hat{v}(p)\ \forall p \in \mathbb{R}^N_+,\ x \cdot q = \hat{v}(q)\}$. Then it follows from $y \cdot q = \hat{v}(q)$ and $y \cdot z \geq \hat{v}(z)$ for all $z \in \mathbb{R}^N_+$ that $y \cdot z - y \cdot q \geq \hat{v}(z) - \hat{v}(q)$. Hence, $y \cdot (z - q) + \hat{v}(q) \geq \hat{v}(z)$. **q.e.d.**

The following result, for super additive games, appears under a slightly different setting, e.g., in the first edition of [4], p. 213.

**Corollary 3.3.3.** *Let $(N, v)$ be a balanced game and let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. Then $\partial\hat{v}(\chi^N) = \mathcal{C}(N, v)$.*

Note also that for all $q \in \mathbb{R}^N_+$, $\partial\hat{v}(q) \neq \emptyset$. This can be proven by first noting that $\partial\hat{v}(r) \neq \emptyset$ for all $r \in int\mathbb{R}^N_+$, where *int* stands for the interior of a set (see, e.g., [35]). Let $q \in \mathbb{R}^N_+ \backslash int\mathbb{R}^N_+$. Because $\hat{v}$ is piece-wise linear, it follows that there exists a $\delta > 0$ so that for all $t \in \mathbb{R}^N_+$ with $|t - q| < \delta$ one has, for all $0 \leq \alpha \leq 1$,

$$\alpha\hat{v}(t) + (1 - \alpha)\hat{v}(q) = \hat{v}(\alpha t + (1 - \alpha)q).$$

Let $r \in int\mathbb{R}^N_+$, $|r - q| < \delta$ and define $p = \frac{1}{2}r + \frac{1}{2}q \in int\mathbb{R}^N_+$. Let $y \in \partial\hat{v}(p)$. Then one has, from the linearity of $y$ and $\hat{v}$, $\frac{1}{2}y \cdot r + \frac{1}{2}y \cdot q = \frac{1}{2}\hat{v}(r) + \frac{1}{2}\hat{v}(q)$, however, $y \cdot r \geq \hat{v}(r)$ and $y \cdot q \geq \hat{v}(q)$, hence, one can conclude that $y \in \partial\hat{v}(q)$.

Another well-known, simple result (see, e.g., [35]), which follows directly from the last proposition, is the following.

**Proposition 3.3.4.** *Let $(N, v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then $\partial\hat{v}(q)$ is a convex set.*

**Proof:** Let $q \in \mathbb{R}^N_+$ and let $\eta, \xi \in \partial\hat{v}(q)$. Then it follows for all $0 \leq \alpha \leq 1$ that

$$(\alpha\eta + (1 - \alpha)\xi) \cdot q = \alpha\eta \cdot q + (1 - \alpha)\xi \cdot q = \alpha\hat{v}(q) + (1 - \alpha)\hat{v}(q) = \hat{v}(q).$$

Also for all $p \in \mathbb{R}^N_+$ it follows that

$$(\alpha\eta + (1 - \alpha)\xi) \cdot p = \alpha\eta \cdot p + (1 - \alpha)\xi \cdot p \geq \alpha\hat{v}(p) + (1 - \alpha)\hat{v}(p) = \hat{v}(p).$$

**q.e.d.**

To prove some more results about $\partial\hat{v}$, which will be used in the last section of this chapter, a characterization of $\hat{v}$ will be given. In the following, let $(N, v)$ be a game and

$$U(N, v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S),\ \forall\ S \subseteq N\}.$$

Note also that

$$U(N, v) = U(\mathbb{H}, \hat{v}) := \{x \in \mathbb{R}^N \mid xq \geq \hat{v}(q),\ \forall\ 0 \leq q \leq \chi^N\}$$

$$= \{x \in \mathbb{R}^N \mid xq \geq \hat{v}(q),\ \forall\ q \in \mathbb{R}^N_+\} := U(\mathbb{R}^N_+, \hat{v}),$$

the last equality following from the comments at the beginning of this section. The new representation of $\hat{v}$ is as follows.

**Proposition 3.3.5.** *Let $(N, v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in \mathbb{R}^N_+$. Then*

$$\hat{v}(q) = \min_{x \in U(N,v)} \{x \cdot q\}. \tag{3.1}$$

**Proof:** Let $q \in \mathbb{R}^N_+$. As stated earlier, $\partial\hat{v}(q) \neq \emptyset$, however, all $y \in \partial\hat{v}(q)$ are such that $y \cdot q = \hat{v}(q)$ and $y \cdot p \geq \hat{v}(p)$ otherwise. Hence, all such $y$ are elements of $U(N, v)$ and satisfy $\hat{v}(q) = y \cdot q = \min_{x \in U(N,v)}\{x \cdot q\}$. **q.e.d.**

Let $extC$ stand for the extreme points of a set $C$. As is well-known, for each $q \in \mathbb{R}^N_+$, the minimum in Equation (3.1) is attained by an extreme point of the set $U(N, v)$. Note that the number of extreme points of $U(N, v)$ is finite (see [19]). Let $H \in extU(N, v)$ and let

$$def_{\hat{v}}H := \{x \in \mathbb{R}^N_+ \mid \hat{v}(x) = H \cdot x \text{ and } H \cdot x < G \cdot x \; \forall \; G \in extU(N, v)\backslash\{H\}\}.$$

Now define

$$\mathcal{H}(\mathbb{R}^N_+, \hat{v}) = \{H \in extU(N, v) \mid def_{\hat{v}}H \neq \emptyset\}.$$

Via the continuity of $\hat{v}$, this allows one, for $q \in \mathbb{R}^N_+$, to rewrite $\hat{v}$ in the following form.

$$\hat{v}(q) = \min_{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})} \{H \cdot q\}. \tag{3.2}$$

This representation will be used to characterize the regions $def_{\hat{v}}H$. In the following, a *convex cone* is a subset $C$ of $\mathbb{R}^N$ such that for $\alpha, \beta > 0$ and $x, y \in C$, it follows that $\alpha x + \beta y \in C$.

**Proposition 3.3.6.** *Let $(N, v)$ be a game and let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. If $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$, then $def_{\hat{v}}H$ is a convex cone.*

**Proof:** If $x, y \in def_{\hat{v}}H$, then for $\alpha, \beta > 0$,

$$H \cdot (\alpha x + \beta y) = \alpha H \cdot x + \beta H \cdot y = \alpha\hat{v}(x) + \beta\hat{v}(y) \leq \hat{v}(\alpha x + \beta y) \leq H \cdot (\alpha x + \beta y).$$

Hence, one has equalities throughout. Secondly, as for all $G \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})\backslash\{H\}$, it follows that $G \cdot x > H \cdot x$ and $G \cdot y > H \cdot y$, one has

$$H \cdot (\alpha x + \beta y) = \alpha H \cdot x + \beta H \cdot y < \alpha G \cdot x + \beta G \cdot y = G \cdot (\alpha x + \beta y).$$

Hence, $def_{\hat{v}}H$ is a convex cone. **q.e.d.**

Let $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$ (with $|\mathcal{H}(\mathbb{R}^N_+, \hat{v})| > 1$, as the following statements for the case $|\mathcal{H}(\mathbb{R}^N_+, \hat{v})| = 1$ follow immediately) and define $\overline{def_{\hat{v}}H} = \{x \in \mathbb{R}^N_+ \mid H \cdot x = \hat{v}(x)\}$. One notices that if the dimension of $\overline{def_{\hat{v}}H}$ is $n$, then the dimension of $def_{\hat{v}}H$ must also be $n$. This can be seen as follows. Let the dimension of $\overline{def_{\hat{v}}H}$ be $n$. Then if $Z := \overline{def_{\hat{v}}H} \backslash def_{\hat{v}}H$ is $n$-dimensional, it implies that there must exist a $G \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$, $G \neq H$, such that $G \cdot x = H \cdot x$ over an $n$-dimensional subset of $Z$ (as the set $\mathcal{H}(\mathbb{R}^N_+, \hat{v})$ is finite and the regions $\overline{def_{\hat{v}}H}$ are convex, the proof being analogous to the proof of the previous proposition). This would imply, however, that $G = H$ for all $x \in \mathbb{R}^N_+$, which is a contradiction. Hence, to prove that $def_{\hat{v}}H$ is $n$-dimensional, it suffices to prove that $\overline{def_{\hat{v}}H}$ is $n$-dimensional.

**Proposition 3.3.7.** *Let $(N, v)$ be a game and let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$. If $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$, then $def_{\hat{v}}H$ is $n$-dimensional.*

**Proof:**
As stated before the proposition, it suffices to prove that $\overline{def_{\hat{v}}H}$ is $n$-dimensional. Each $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$ is a linear function of each $z, y \in \mathbb{R}^N_+$ and so, it follows that there exists an $\alpha_H \in (0, \infty)$ such that $|H \cdot z - H \cdot y| < \alpha_H |z - y|$. Also let $\alpha < \infty$ be such that $\alpha > \max_{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})} \alpha_H$. Assume, per absurdum, that there exists a $G \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$ such that the set $\overline{def_{\hat{v}}G}$ is less than $n$-dimensional. Then, because $\overline{def_{\hat{v}}G}$ is less than $n$-dimensional, for $x \in \overline{def_{\hat{v}}G}$ there exists a $w \in \mathbb{R}^N_+$ such that $w \notin \overline{def_{\hat{v}}G}$ and also for all $\beta \in [0, 1)$, $\beta x + (1 - \beta) w \notin \overline{def_{\hat{v}}G}$ (one chooses a $w$ that cannot be represented as a linear combination of the elements of $\overline{def_{\hat{v}}G}$, which is not equal to $\mathbb{R}^N_+$, as $\overline{def_{\hat{v}}G}$ is less than $n$-dimensional). Let $x$ and $w$ be fixed as before and let

$$X := \{y \mid y = \beta x + (1 - \beta) w \text{ for } \beta \in [0, 1)\} \subseteq \mathbb{R}^N_+.$$

Also let $\sigma := \min_{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \backslash \{G\}} \{H \cdot x - G \cdot x\} > 0$. Choose $y \in X$ so that there exists $\varepsilon > 0$ with $|\hat{v}(x) - \hat{v}(y)| < \varepsilon$ and $\delta > 0$ with $|x - y| < \delta$ satisfying $\alpha\delta + \varepsilon < \sigma$ (such $\varepsilon$ and $\delta$ exist by the continuity of $\hat{v}$). Let $H' \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})$ be such that $H' \cdot y = \hat{v}(y)$ and note that $H' \neq G$ by definition of $\overline{def_{\hat{v}}G}$. Then

$$|H' \cdot x - \hat{v}(x)| = |H' \cdot x - H' \cdot y + H' \cdot y - \hat{v}(x)| \leq |H' \cdot x - H' \cdot y| + |\hat{v}(y) - \hat{v}(x)|$$

$$< \alpha |x - y| + \varepsilon < \alpha\delta + \varepsilon < \sigma.$$

This, however, contradicts the definition of $\sigma$. **q.e.d.**

Using the previous results another result concerning the structure of the set $\partial\hat{v}(q)$ for $q \in \mathbb{R}^N_+$ will be proven, which will be useful later on. Before that can be done, some well-known results, which will be relevant for later analysis, will be presented. The following two results can be found in [35].

**Proposition 3.3.8.** *Let $f : \mathbb{R}^N_+ \to \mathbb{R}$ be a closed*[7]*, proper*[8]*, convex (concave) function and let $q \in int\mathbb{R}^N_+$. Then $\partial f(q)$ is a nonempty, compact set.*

In the following, let $convH(C)$ stand for the convex hull of a set $C$.

**Proposition 3.3.9.** *Let $C \subseteq \mathbb{R}^N$ be a compact, convex set. Then $C = convH(ext\, C)$.*

To state the next result the following definition is required (see [35]).

**Definition 3.3.10.** *A hyperplane $H$ to a set $C$ is* ***supporting*** *if $C$ is contained in one of the closed half spaces defined by $H$ and $H \cap C \neq \emptyset$.*

Note that an extreme point, $x$, of a convex set, $C$, is such that there is a supporting hyperplane through $x$ which contains no other point of $C$ (see [35]). To simplify the statement of the next result, let $\mathcal{J}(q) := \{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \mid H \cdot q = \hat{v}(q)\}$.

**Proposition 3.3.11.** *Let $(N, v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in int\mathbb{R}^N_+$. Then $\partial\hat{v}(q) = convH(\{H \mid H \in \mathcal{J}(q)\})$.*

**Proof:** Let $q \in int\mathbb{R}^N_+$. First of all, it is clear from the definition of $\hat{v}$, in Equation (3.2), that $\partial\hat{v}(q) \supseteq \{H \mid H \in \mathcal{J}(q)\}$. By Proposition 3.3.4 and Proposition 3.3.8, it follows that $\partial\hat{v}(q)$ is a nonempty, compact, convex set. By Proposition 3.3.9, it follows that this set can be written as the convex hull of its extreme points. Assume now, per absurdum, that there exists an extreme point $\eta \in ext\,\partial\hat{v}(q)$ such that $\eta \neq H$ for all $H \in \mathcal{J}(q)$. Because $\eta$ is an extreme point there exists a vector $y \in \mathbb{R}^N$ (which defines a supporting hyperplane intersecting only $\eta$ in $\partial\hat{v}(q)$) so that $\eta \cdot y > H \cdot y$ for all $H \in \mathcal{J}(q)$. In addition, as $\hat{v}$ is continuous and all $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v})\backslash\mathcal{J}(q)$ are also continuous (considered as functions of $q \in \mathbb{R}^N_+$) and also $\hat{v}(q) < \min_{H\in\mathcal{H}(\mathbb{R}^N_+,\hat{v})\backslash\mathcal{J}(q)}\{H \cdot q\}$, it follows that there exists a $\delta > 0$ so that for $p \in N_\delta(q) := \{z \in \mathbb{R}^N_+ \mid |z - q| < \delta\}$ one has $\hat{v}(p) < \min_{H\in\mathcal{H}(\mathbb{R}^N_+,\hat{v})\backslash\mathcal{J}(q)}\{H \cdot p\}$. Let $\varepsilon > 0$ so that $q - \varepsilon y \in N_\delta(q)$. Then it follows, because $\eta \notin \mathcal{J}(q)$ and $\eta \cdot q = H \cdot q$ for all $H \in \mathcal{J}(q)$, that

$$\eta \cdot (q - \varepsilon y) < \min_{H\in\mathcal{J}(q)}\{H \cdot (q - \varepsilon y)\} = \hat{v}(q - \varepsilon y).$$

A contradiction because $\eta \in \partial\hat{v}(q)$. **q.e.d.**

For the general case, see [35]. Note that one can show more than the result in the previous proposition. Let $q \in int\mathbb{R}^N_+$. Then, because for each $H \in \mathcal{J}(q)$ there exists a $p \in \mathbb{R}^N_+$ such

[7] A concave function $f : \mathbb{R}^N_+ \to \mathbb{R}$ is closed if the set $\{(p, a) \in \mathbb{R}^N_+ \times \mathbb{R} \mid a \leq f(p)\}$ is closed.

[8] A function $f : \mathbb{R}^N_+ \to [-\infty, +\infty]$ is proper if $f(x) > -\infty$ for at least one $x$ and $f(x) < +\infty$ for every $x$.

that $H \cdot p < \min_{G \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \setminus \{H\}} \{G \cdot p\}$, it follows that $ext\partial\hat{v}(q) = \{H \mid H \in \mathcal{J}(q)\}$. Note, also, that for $q \in \mathbb{R}^N_+ \setminus int\mathbb{R}^N_+$, it follows that if $\eta \in \partial\hat{v}(q)$, then so is $\eta + \varepsilon\chi^{N\setminus supp(q)}$ for all $\varepsilon \geq 0$.

## 3.4 The Concavification and Exactness

Before the final section demonstrating the main result in this chapter is presented, some interesting results relating to the concavification $(\mathbb{H}, \hat{v})$ of a game $(N, v)$ and exactness will be presented. To avoid unnecessary complications, the analysis in this section will be restricted to the concave extension over the $n$-dimensional hypercube.

The property of exactness is neither necessary nor sufficient for core stability for the case of $n$-person, cooperative, TU games. This, however, is not the case for the concave extension, $(\mathbb{H}, \hat{v})$, of a game $(N, v)$. In this case, if the game $(\mathbb{H}, \hat{v})$ is exact, then it follows that the game $(\mathbb{H}, \hat{v})$ has a stable core (see Section 3.1). This is a general property shared by all fuzzy games. For the case of the concave extension, this is not that surprising due to the following result published in [7].

**Proposition 3.4.1.** *Let $(N, v)$ be a game and $(\mathbb{H}, \hat{v})$ its concave extension. The game $(N, v)$ has a large core if and only if $\hat{v}(q) = \min\{x \cdot q \mid x \in \mathcal{C}(N, v)\}$.*

The latter condition is equivalent to the exactness of the game $(\mathbb{H}, \hat{v})$ (see [8]) as $\mathcal{C}(N, v) = \mathcal{C}(\mathbb{H}, \hat{v})$. Hence, a game $(N, v)$ has a large core if and only if the concave extension of $(N, v)$, $(\mathbb{H}, \hat{v})$, is exact. Because of Theorem 3.2.16, it must follow that the exactness of $(\mathbb{H}, \hat{v})$ implies stability of the core (as largeness of the core of the game $(N, v)$ implies stability of the core). In addition, it is also shown in [7] that the exactness of certain points in $\mathbb{H}$ implies that the game $(N, v)$ is extendable and hence, that the game $(N, v)$ has a stable core. So, it would seem possible that a way of finding necessary and sufficient conditions for core stability would be to try and find which points of the concave extension of a game need to be exact to guarantee core stability. Work, on finding out which points need to be exact for core stability, has already been done in [24] (also independently proven in [14]) for normal, $n$-person, cooperative, TU games. The author showed that if the game $(N, v)$ has a stable core, then the coalitions consisting of single players must be exact (see the previous chapter).

The concave extension of a game not only presents a novel way to investigate the foregoing ideas but also allows one to prove interesting results concerning the exactness of the original game. To prove the coming proposition, the following lemma based on an incorrect statement in [50] is required (this result is also incorrectly reproduced in [7]).

**Lemma 3.4.2.** *Let $(N, v)$ be a balanced game. Then the coalition $S$ is exact if and only if the concave extension $(\mathbb{H}, \hat{v})$ satisfies $\hat{v}(\chi^S) = v(S)$ and for all $\alpha \in (0, 1)$*

$$\alpha\hat{v}(\chi^S) + (1-\alpha)\hat{v}(\chi^N) = \hat{v}(\alpha\chi^S + (1-\alpha)\chi^N). \tag{3.3}$$

**Proof:** To prove the only if direction let $x \in \mathcal{C}(N, v)$ be such that $x(S) = v(S) = \hat{v}(\chi^S)$. Then for all $\alpha \in (0, 1)$ it follows that

$$\hat{v}(\alpha\chi^S + (1-\alpha)\chi^N) \leq x \cdot (\alpha\chi^S + (1-\alpha)\chi^N) = \alpha x(S) + (1-\alpha)x(N)$$

$$= \alpha\hat{v}(\chi^S) + (1-\alpha)\hat{v}(\chi^N) \leq \hat{v}(\alpha\chi^S + (1-\alpha)\chi^N).$$

To prove the other direction let $\hat{v}(\chi^S) = v(S)$ and let $\hat{v}$ satisfy (3.3). Then it follows that there must exist a $\zeta \in \mathbb{R}^N$ defining $\hat{v}$ over the region $\alpha\chi^S + (1-\alpha)\chi^N \in \mathbb{H}$ for $\alpha \in [0, 1]$ as the function $\hat{v}$ is linear there. This $\zeta$ is also an extreme point of the set $U(N, v)$ and fulfills $\zeta(S) = \hat{v}(\chi^S) = v(S)$ and $\zeta(N) = \hat{v}(\chi^N) = v(N)$. Therefore, the coalition $S$ is exact. **q.e.d.**

The result proven in [37] is the following and to demonstrate the usefulness of the concave extension of a game, a proof of it, utilizing the concave extension, will now be given (the definition of a $T$-covering, as well as the following proposition, were already stated in Remark 2.2.7).

**Proposition 3.4.3.** *Let $(N, v)$ be a balanced game. The coalition $T$ is exact if and only if for all $T$-coverings, $\lambda^T$, it follows that*

$$\max_{\lambda^T} \sum_{S \subseteq N} \lambda_S^T v(S) - \gamma v(N) = v(T). \tag{3.4}$$

**Proof:**

**1$^{\text{st}}$STEP** :

First of all, it will be proven that if the coalition $T$ is exact, then Equation (3.4) is satisfied. As $T$ is exact, it follows that there exists an $x \in \mathcal{C}(N, v)$ such that $x(T) = v(T)$, $x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subseteq N$. By utilizing this, it follows that for all $T$-coverings, $\lambda^{\bar{T}}$,

$$\sum_{S \subseteq N} \lambda_S^{\bar{T}} v(S) - \gamma v(N) \leq v(T). \tag{3.5}$$

So, now assume to the contrary of Equation (3.4) that

$$\max_{\lambda^{\bar{T}}} \sum_{S \subseteq N} \lambda_S^{\bar{T}} v(S) - \gamma v(N) < v(T) \tag{3.6}$$

and let $\lambda^T$ be a $T$-covering maximizing the left hand side of (3.6). Then by rearranging Equation (3.6), it follows that $\sum_{S \subseteq N} \lambda_S^T v(S) < \gamma v(N) + v(T)$. Now choose $\beta \geq 1$ such that $\frac{\gamma}{\beta} + \frac{1}{\beta} = 1$ and let, for the current $T$-covering $\lambda^T$,

$$q := \frac{1}{\beta} \sum_{S \subseteq N} \lambda_S^T \chi^S = \frac{\gamma}{\beta} \chi^N + \frac{1}{\beta} \chi^T \in \mathbb{H},$$

by the choice of $\beta$. **Claim:**

$$\hat{v}(q) = \frac{1}{\beta} \sum_{S \subseteq N} \lambda_S^T v(S).$$

This claim will now be proven. Were this not the case, then it must follow that non-negative numbers, $\delta_S$, exist such that

$$\sum_{S \subseteq N} \delta_S \chi^S = \frac{1}{\beta} \sum_{S \subseteq N} \lambda_S^T \chi^S \text{ and} \sum_{S \subseteq N} \delta_S v(S) > \frac{1}{\beta} \sum_{S \subseteq N} \lambda_S^T v(S).$$

However, this contradicts the fact that the $T$-covering $\lambda^T$ was a maximum, as one could take the $T$-covering, $\lambda^{\mathcal{T}} = (\{\beta \delta_S\}_{S \subseteq N}, \gamma)$, and it follows that

$$\sum_{S \subseteq N} \beta \delta_S v(S) - \gamma v(N) > \sum_{S \subseteq N} \lambda_S^T v(S) - \gamma v(N).$$

Due to the claim, one then has that

$$\hat{v}(q) = \frac{1}{\beta} \sum_{S \subseteq N} \lambda_S^T v(S) < \frac{\gamma}{\beta} v(N) + \frac{1}{\beta} v(T).$$

This contradicts Lemma 3.4.2, as $v(T) = \hat{v}(\chi^T)$, $v(N) = \hat{v}(\chi^N)$ and the game was exact.

**2$^{\text{nd}}$STEP** :

To prove the other direction assume, per absurdum, that the coalition $T$ is not exact. It will be proven that the maximum over all $T$-coverings does not fulfill Equation (3.4). If $\hat{v}(\chi^T) > v(T)$, then the result follows straight away. So, assume that the coalition $T$ is not exact and $\hat{v}(\chi^T) = v(T)$. Then there exists an $\alpha \in (0, 1)$ such that

$$\alpha \hat{v}(\chi^T) + (1 - \alpha) \hat{v}(\chi^N) < \hat{v}(\alpha \chi^T + (1 - \alpha) \chi^N).$$

Hence, it follows that

$$\alpha \hat{v}(\chi^T) < \hat{v}(\alpha \chi^T + (1 - \alpha) \chi^N) - (1 - \alpha) \hat{v}(\chi^N).$$

Let, per definition, $\hat{v}(\alpha \chi^T + (1 - \alpha) \chi^N) = \sum_{S \subseteq N} \lambda_S v(S)$ for some $\lambda_S$ then

$$\chi^T = \frac{1}{\alpha} \sum_{S \subseteq N} \lambda_S \chi^S - \frac{(1 - \alpha)}{\alpha} \chi^N. \tag{3.7}$$

Hence it follows that $\{\frac{\lambda_S}{\alpha}, \frac{(1-\alpha)}{\alpha}\}$ is a $T$-covering. However $\hat{v}(\chi^T) = v(T)$ and $\hat{v}(\chi^N) = v(N)$ and so it follows that

$$v(T) < \sum_{S \subseteq N} \frac{\lambda_S}{\alpha} v(S) - \frac{(1-\alpha)}{\alpha} v(N) \leq \max_{\lambda^T} \sum_{S \subseteq N} \lambda_S^T v(S) - \gamma v(N).$$

and this contradicts Equation (3.4). **q.e.d.**

## 3.5 The Inverse Domination Correspondence

In this section the question of core stability will be considered from the perspective of nonlinear analysis. In addition, a closely related correspondence will be investigated and important properties of this correspondence will be proven. To begin the analysis, a number of correspondences shall be introduced.

**Definition 3.5.1.** *Let $(N, v)$ be a game. For $x \in \mathbb{R}^N$ let*

$$x \operatorname{dom}_S := \{y \in I(N, v) \mid x \operatorname{dom}_S y\}.$$

*Also*

$$x \operatorname{dom} := \{y \in I(N, v) \mid \exists\, S \subsetneqq N,\ x \operatorname{dom}_S y\}.$$

So, for each $x \in \mathbb{R}^N$, the set, $x \operatorname{dom}$, is the set of elements which $x$ dominates. These correspondences have been studied by numerous authors. For a discussion of them see, for example, [24]. Let $Q \operatorname{dom}$ be the corresponding set of all imputations dominated by an element of the set $Q$ and note that the game $(N, v)$ has a stable core if and only if $\mathcal{C}(N, v) = I(N, v) \backslash \mathcal{C}(N, v) \operatorname{dom}$. However, the focus of this section is not on the dom correspondence but on its inverse, defined as follows.

**Definition 3.5.2.** *Let $(N, v)$ be a game. For an imputation $x$, let*

$$\operatorname{dom}_S x := \{y \in I(N, v) \mid y \operatorname{dom}_S x\}.$$

*Also*

$$\operatorname{dom} x := \{y \in I(N, v) \mid \exists\, S \subsetneqq N,\ y \operatorname{dom}_S x\}.$$

In particular, one can say that the game $(N, v)$ has a stable core if and only if for all imputations $x \notin \mathcal{C}(N, v)$, it follows that $\operatorname{dom} x \cap \mathcal{C}(N, v) \neq \emptyset$. As it stands, this result is not very useful. In the following, however, it shall be rewritten in a form which is amenable to the techniques of nonlinear analysis. To do so, the extension of $\operatorname{dom} x$ over $\mathbb{R}_+^N$, for the concave extension $(\mathbb{R}_+^N, \hat{v})$ over $\mathbb{R}_+^N$ of a game $(N, v)$, will be considered. The extended definition of $\operatorname{dom} x$ is clear (one can now dominate via $q \in \mathbb{R}_+^N$ and not just via the $\chi^S$, $S \subseteq N$). To begin the analysis, however, some lemmata are required.

**Lemma 3.5.3.** *Let $(N, v)$ be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$ and let $q \in \mathbb{R}_+^N$ be an exact coalition. Then for all $\lambda > 0$ it follows that*

$$\hat{v}(q + \lambda\chi^N) = \hat{v}(q) + \lambda\hat{v}(\chi^N).$$

**Proof:** Let $q \in \mathbb{R}_+^N$ be an exact coalition. As $q$ is exact, there exists an $x \in \mathcal{C}(\mathbb{R}_+^N, \hat{v})$ such that $x \cdot q = \hat{v}(q)$. Let $\lambda > 0$. Then

$$\hat{v}(q + \lambda\chi^N) \leq x \cdot (q + \lambda\chi^N) = x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N) \leq \hat{v}(q + \lambda\chi^N)$$

and the result follows. **q.e.d.**

Using this, one can now prove the following result.

**Lemma 3.5.4.** *Let $(N, v)$ be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$, let $q \in \mathbb{R}_+^N$ be an exact coalition and let $\lambda > 0$. Then*

$$\partial\hat{v}(q + \lambda\chi^N) = \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N).$$

**Proof:** Let $q \in \mathbb{R}_+^N$ be an exact coalition and let $\lambda > 0$. If $x \in \partial\hat{v}(q + \lambda\chi^N)$, then $x \cdot (q + \lambda\chi^N) = \hat{v}(q + \lambda\chi^N)$, whence one has that $x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ from the exactness of $q$ and the previous result. However, $x \cdot q \geq \hat{v}(q)$ and $x \cdot \chi^N \geq \hat{v}(\chi^N)$, which imply that $x \cdot q = \hat{v}(q)$ and $x \cdot \chi^N = \hat{v}(\chi^N)$ (and $x \cdot p \geq \hat{v}(p)$ otherwise, as $x \in \partial\hat{v}(q + \lambda\chi^N)$). It then follows that $x \in \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N)$. The other direction is clear, as if $x \cdot q = \hat{v}(q)$ and $x \cdot \chi^N = \hat{v}(\chi^N)$ (and $x \cdot p \geq \hat{v}(p)$ otherwise), then $x \cdot q + \lambda x \cdot \chi^N = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ and, from the previous result, it follows that, because $q$ is an exact coalition, $\hat{v}(q + \lambda\chi^N) = \hat{v}(q) + \lambda\hat{v}(\chi^N)$ and hence, $x \cdot (q + \lambda\chi^N) = \hat{v}(q + \lambda\chi^N)$. Whence $x \in \partial\hat{v}(q + \lambda\chi^N)$. **q.e.d.**

Utilizing this result, one can now formulate necessary and sufficient conditions for core stability of the game $(\mathbb{R}_+^N, \hat{v})$. Before that is done, some more notation needs to be introduced. For two sets $C_1$ and $C_2$, let $C_1 - C_2 := \{x - y \mid x \in C_1, y \in C_2\}$. In addition, define the following set for a game $(N, v)$ and its concave extension $(\mathbb{R}_+^N, \hat{v})$ over $\mathbb{R}_+^N$.

**Definition 3.5.5.** *Let $(N, v)$ be a game, let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$, let $x \in \mathbb{R}^N$ and $q \in \mathbb{R}_+^N$.*

$$D(q, x) := \{y \in I(\mathbb{R}_+^N, \hat{v}) \mid y_i \geq x_i \; \forall \, i \in supp(q), y \cdot q = \hat{v}(q)\}.$$

The desired result is as follows (for a set $C$, $riC$ stands for the relative interior of $C$).

**Theorem 3.5.6.** *Let $(N, v)$ be a game and let $(\mathbb{R}_+^N, \hat{v})$ be its concave extension over $\mathbb{R}_+^N$. Then $(\mathbb{R}_+^N, \hat{v})$ has a stable core if and only if for all imputations (of $I(\mathbb{R}_+^N, \hat{v})$) $x \notin \mathcal{C}(\mathbb{R}_+^N, \hat{v})$ there exists a non-zero, exact $q \in \mathbb{R}_+^N$ and $\lambda > 0$ so that*

$$0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N).$$

**Proof:** First of all, if the game $(\mathbb{R}^N_+, \hat{v})$ has a stable core, then for all imputations $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ there exists a non zero $q \in \mathbb{R}^N_+$ and $y \in \mathcal{C}(\mathbb{R}^N_+, \hat{v}) \cap \mathrm{dom}_q\, x$. Hence, $q$ is exact, $y \in riD(q,x)$ and $y \in \partial\hat{v}(q)$. Therefore, for a $\lambda > 0$, $y \in \partial\hat{v}(q) \cap \partial\hat{v}(\chi^N) = \partial\hat{v}(q+\lambda\chi^N)$ by Lemma 3.5.4 and it follows that $0 \in riD(q,x) - \partial\hat{v}(q+\lambda\chi^N)$. To prove the other direction, assume that $x$ is an imputation and $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$. Then there exists a non zero, exact $q \in \mathbb{R}^N_+$ and $\lambda > 0$ such that $0 \in riD(q,x) - \partial\hat{v}(q+\lambda\chi^N)$. As $0 \in riD(q,x) - \partial\hat{v}(q+\lambda\chi^N)$, it follows that 0 can be written as a difference of a $y \in riD(q,x)$ and a $z \in \partial\hat{v}(q+\lambda\chi^N)$. Hence, $0 = y - z$. That is, there exists a $z \in \partial\hat{v}(q+\lambda\chi^N)$ such that $z\, \mathrm{dom}_q\, x$, whence the game $(\mathbb{R}^N_+, \hat{v})$ has a stable core, as by Lemma 3.5.4, $z \in \mathcal{C}(\mathbb{R}^N_+, \hat{v})$. **q.e.d.**

The reason for the interest in such a result is based on the properties of the following correspondence, called the inverse domination correspondence. Let an imputation $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ and $\lambda > 0$ be fixed and define for $q \in \mathbb{R}^N_+$ the correspondence $F_x : \mathbb{R}^N_+ \rightsquigarrow \mathbb{R}^N$ by

$$F_x(q) := D(q,x) - \partial\hat{v}(q+\lambda\chi^N). \tag{3.8}$$

Note that the choice of $\lambda$ in the definition is irrelevant.

**Definition 3.5.7.** *Let $X \subseteq \mathbb{R}^n_+$ and $Y \subseteq \mathbb{R}^m_+$. Let $\varphi : X \rightsquigarrow Y$ be a correspondence. $\varphi$ is **convex valued** if $\varphi(x)$ is convex for all $x \in X$.*

Let $x \in \mathbb{R}^N$. As is clear, $D(\cdot, x)$ is a correspondence taking compact and convex values. Note also that for $q \in int\mathbb{R}^N_+$, by Proposition 3.3.11, $\partial\hat{v}(q)$ is also a nonempty, compact and convex valued correspondence. Also, as is well-known, for two compact/convex sets $C_1$ and $C_2$ in $\mathbb{R}^N_+$, $C_1 - C_2$ is also compact/convex (see, e.g., Theorem 3.1 in [35] and for compactness, apply Tychonoff's Theorem then the continuous, affine transformation $A : C_1 \times C_2 \to \mathbb{R}^N, (x,y) \mapsto x - y$). Hence, $F_x$ is compact and convex valued and nonempty. Another important property shared by both correspondences, $\partial\hat{v}$ and $D(\cdot,x)$, is that they are u.h.c. correspondences.

**Proposition 3.5.8.** *Let $(N,v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $q \in int\mathbb{R}^N_+$. Then $\partial\hat{v}(q)$ is u.h.c. at $q$.*

**Proof:** Let $q \in int\mathbb{R}^N_+$ and let $\mathcal{J}(q) := \{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \mid H \cdot q = \hat{v}(q)\}$. As $\hat{v}$ is continuous and all $H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{J}(q)$ are also continuous and also $\hat{v}(q) < \min_{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{J}(q)} \{H \cdot q\}$, it follows that there exists a $\delta > 0$ so that for $p \in N_\delta(q) := \{z \in \mathbb{R}^N_+ \mid |z - q| < \delta\}$ one has $\hat{v}(p) < \min_{H \in \mathcal{H}(\mathbb{R}^N_+, \hat{v}) \backslash \mathcal{J}(q)} \{H \cdot p\}$. Hence, one has for all $p \in N_\delta(q) \cap int\mathbb{R}^N_+$ (by Proposition 3.3.11) that $\partial\hat{v}(p) \subseteq \partial\hat{v}(q)$. **q.e.d.**

For the general case, see [4]. To simplify the proof of the demonstration that $D(\cdot,x)$ is a u.h.c. correspondence, the following result, appearing in [16], will be used.

**Proposition 3.5.9.** *Let $X \subseteq \mathbb{R}^n_+$ and $Y \subseteq \mathbb{R}^m_+$ with $Y$ compact. A correspondence $\varphi : X \rightsquigarrow Y$ is u.h.c. at $\bar{x} \in X$, if for all sequences $\{x^t\}_{t\in\mathbb{N}}$ in $X$ with $x^t \to \bar{x}$ and all sequences $\{y^t\}_{t\in\mathbb{N}}$ with $y^t \to \bar{y}$ and $y^t \in \varphi(x^t)$ for all $t$, it follows that $\bar{y} \in \varphi(\bar{x})$.*

**Proposition 3.5.10.** *Let $(N, v)$ be a game, let $(\mathbb{R}^N_+, \hat{v})$ be its concave extension over $\mathbb{R}^N_+$ and let $x \in \mathbb{R}^N$. Then $D(\cdot, x)$ is a u.h.c. correspondence.*

**Proof:** Note that the range of $D(\cdot, x)$ is contained in $I(\mathbb{R}^N_+, \hat{v})$, a compact set. Let $q \in \mathbb{R}^N_+$. If $x \cdot q > \hat{v}(q)$, then $D(q, x) = \emptyset$ and the result clearly follows. Hence, let $x \cdot q \leq \hat{v}(q)$ and let $\{q^t\}_{t\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N_+$ such that $q^t \to q$ and $\{y^t\}_{t\in\mathbb{N}}$ be a sequence such that $y^t \to y$ and $y^t \in D(q^t, x)$ for all $t$. Hence, one clearly has $y_i \geq x_i$ for all $i \in supp(q)$ and also that $y \in I(\mathbb{R}^N_+, \hat{v})$ (because there exists $\bar{t} \in \mathbb{N}$ such that for all $t > \bar{t}$, $y^t_i \geq x^t_i$, for all $i \in supp(q)$, and $y^t \in I(\mathbb{R}^N_+, \hat{v})$). Because $\hat{v}$ is a continuous function, it follows that for $q^t \to q$ one has $\hat{v}(q^t) \to \hat{v}(q)$ and because $y^t \to y$ and $y^t q^t = \hat{v}(q^t)$ for all $t \in \mathbb{N}$, one can conclude that $yq = \hat{v}(q)$. **q.e.d.**

Finally, it also important to note that for two u.h.c. correspondences $\varphi_1$ and $\varphi_2$ that one also has the following result (see, e.g., [16]).

**Proposition 3.5.11.** *Let $\varphi_1 : X \rightsquigarrow Y$ and $\varphi_2 : X \rightsquigarrow Y$ be u.h.c. correspondences. Then $\varphi_1 - \varphi_2$ is also u.h.c.*

**Proof:** Let $O$ be a neighborhood of $\varphi_1(x) - \varphi_2(x)$ and let $O_1$ be a neighborhood of $\varphi_1(x)$ and $O_2$ a neighborhood of $\varphi_2(x)$ such that $O_1 - O_2 \subseteq O$. Then there exist neighborhoods $U_1$ and $U_2$ of $x$ so that $\varphi_1(x^1) \subseteq O_1$ for all $x^1 \in U_1$ and $\varphi_2(x^2) \subseteq O_2$ for all $x^2 \in U_2$. Choose a neighborhood, $U \subseteq U_1, U_2$, of $x$ and it follows that $\varphi_1(z) - \varphi_2(z) \subseteq O$ for all $z \in U$. **q.e.d.**

**Corollary 3.5.12.** *Let $x \notin \mathcal{C}(\mathbb{R}^N_+, \hat{v})$ be an imputation, let $\lambda > 0$ and let $F_x : \mathbb{R}^N_+ \rightsquigarrow \mathbb{R}^N$ be defined as in Equation (3.8). Then $F_x$ is a nonempty, u.h.c., compact and convex valued correspondence.*

**Proof:** As $\partial\hat{v}(q)$ is a compact and convex valued, nonempty, u.h.c. correspondence for all $q \in int\mathbb{R}^N_+$ the result follows from the previous results and the definition of $F_x$. **q.e.d.**

The properties of the correspondence $F_x$ now lend it to the results of nonlinear analysis. There exist numerous theorems (see, e.g., [4]) that state conditions under which 0 is contained in the range of a nonempty, u.h.c., compact and convex valued correspondence. Usually one does not deal with $riC - D$ for two sets $C$ and $D$ and hence, it may be difficult to find conditions such that $0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N)$. However, as is well-known (see, e.g., [35]), for two convex sets $C$ and $D$, $ri(C - D) = riC - riD$ and hence, if for a given $q \in \mathbb{R}_+^{\mathbb{N}}$, $0 \in riF_x(q)$, then $0 \in riD(q, x) - \partial\hat{v}(q + \lambda\chi^N)$. One may now be able to find sufficient conditions such that $0 \in riF_x(q)$, for some $q \in \mathbb{R}_+^{\mathbb{N}}$, using the properties of the correspondence $F_x$ established above.

## 3.6 Conclusion

In this chapter, fuzzy games and their pertinence to core stability were investigated. It was demonstrated that the stability of the core of a fuzzy extension of a game, the concave extension, reflects the core stability of the original game and vice versa. More importantly, it was demonstrated how the tools of nonlinear analysis can be applied to ordinary, $n$-person, cooperative, TU games to help analyze the question of core stability. This was achieved by first demonstrating the connection between the stability of the core of the concave extension of a game and then showing how the tools of nonlinear analysis can be applied to analyze the question of core stability. Finally, the direction of further analysis was given, in that it was elucidated how the results proven in the final section of this chapter can be used to give a characterization of core stability. This analysis will need to include, among others, an investigation of the equilibria of the correspondence $F_x$.

# 4 The Apportionment Problem

In this chapter, properties of new types of apportionment methods are investigated. The idea underlying the apportionment methods is that the populations of the states define a simple majority game and, for a given house size, this can be used to choose a seat distribution to define a new simple majority game, which preserves the winning (or losing) coalitions of the original game arising from the populations (when possible). If this is not possible, then the chosen seat distribution is the "nearest" of all simple majority weighted voting games, with the total weight of the players equaling the size of the house, to the original simple majority weighted voting game arising from the populations.

This chapter is organized as follows. Because the new methods are based on the comparison between the winning (or losing) coalitions of different simple games, in Section 4.1 all the necessary definitions concerning simple games are given. In Section 4.2 new apportionment methods are defined and these methods are then investigated using well-known properties of apportionment methods. In Section 4.3 variations of the methods introduced in Section 4.2 are considered. Finally, in Section 4.4 a new apportionment method based on the losing coalitions of a simple game is investigated.

## 4.1 Simple Games

Simple games form the basis upon which the ideas in the next sections are based. Hence, before the section on apportionment methods is introduced and the pertinent definitions concerning apportioning are expounded, definitions relating to winning coalitions and simple games will be presented in this section. As there will be a constant interchange between vectors and specific values related to these vectors, throughout this chapter bold font letters will mean that a vector is being referred to and not just a number. Unless mentioned otherwise, the definitions in this section can all be found in [39].

**Definition 4.1.1.** *A game $(N, v)$ is a **simple game** if $v(S) = 0$ or $1$ for all $S \subseteq N$, $v(N) = 1$ and if the function $v : 2^N \to \{0, 1\}$ satisfies $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.*

**Definition 4.1.2.** *Let $(N, v)$ be a simple game. A **winning coalition** is a coalition $S$ such that $v(S) = 1$. The set of winning coalitions will be denoted by $W$. A **minimal winning coalition** is a coalition $S$ such that $v(S) = 1$ but $v(T) = 0\ \forall\ T \subsetneqq S$. The set of minimal winning coalitions will be denoted by $W^{min}$. A coalition $S$ is a **losing coalition** if $S \notin W$. The set of losing coalitions will be denoted by $L$. A losing coalition $S$ is a **maximal losing coalition** if for all coalitions $T \supsetneqq S$, it follows that $T \in W$. The set of maximal losing coalitions will be denoted by $L^{max}$.*

Using this notation a simple game $(N, v)$ can also be unequivocally written as $(N, W)$ or $(N, L)$.

**Remark 4.1.3.** *As simple games are monotone, for two simple games $(N, W_1)$ and $(N, W_2)$, with corresponding losing coalitions $L_1$ and $L_2$, it follows that*

$$W_1^{min} = W_2^{min} \iff W_1 = W_2 \iff L_1 = L_2 \iff L_1^{max} = L_2^{max}.$$

The following definitions will be required later on.

**Definition 4.1.4.** *Let $(N, W)$ be a simple game. A game is called **dictatorial** if there exists $j \in N$ such that $S \in W \iff j \in S$. Player $j$ is referred to as a **dictator**.*

The next important definition is the following. As the term "voting game" is used in many different contexts with many different meanings, the following type of game will be called a simple majority weighted voting game to avoid confusion with other possible usages of a "voting game".

**Definition 4.1.5.** *Let $G = (N, W)$ be a simple game. $G$ is a **simple majority weighted voting game** if there exists weights $w_1 \geq 0, \ldots, w_n \geq 0$, $w_j \in \mathbb{N}_0$ for all $j \in \{1, \ldots, n\}$ and a quota*

$$\mu = \left\lceil \frac{w(N) + 1}{2} \right\rceil, \tag{4.1}$$

*where the brackets stand for the ceiling operator (that is, the smallest integer greater than or equal to $\frac{w(N)+1}{2}$) such that $S \in W \iff w(S) \geq \mu$. The $(n+1)$-tuple, $[\,\mu\,; w_1, \ldots, w_n]$, is called a representation of $G$ and one writes $G = [\,\mu\,; w_1, \ldots, w_n]$.*

In simple majority weighted voting games, the number $w_j$ $(j \in N)$ represents the votes that player $j$ possesses. The term "simple majority", in the previous definition, refers to the fact that for a coalition to be a winning coalition, it only needs strictly more than half the total number of votes. The requirement that a coalition possess strictly more than half the total number of votes to be winning is somewhat arbitrary. One could take any fraction of the total number of votes depending on one's model.

## 4.2 Apportionment Methods

In this section the concept of an apportionment method will be introduced. Afterwards, properties relating to two specific apportionment methods will be presented. As was mentioned in the last section, simple majority weighted voting games form the basis of the investigation to be presented here and hence, before the main definitions regarding apportionment methods are given, the notation used throughout this section will be fixed. To demonstrate that a simple majority weighted voting game $G = (N, W)$ is represented via a non-negative integer vector $\mathbf{m} \in \mathbb{N}_0^N$, i.e. $G = [\,\mu\,; m_1, \ldots, m_n]$, the notation $v^{\mathbf{m}}$ will be used for the coalition function and $W_{v^{\mathbf{m}}}$ will be used to represent the winning coalitions of such a game and $W_{v^{\mathbf{m}}}^{min}$ for the minimal winning coalitions. In addition, the vector $\mathbf{m} \in \mathbb{N}_0^N$ will be said to generate the representation $G = [\,\mu\,; m_1, \ldots, m_n]$.

The definition of an apportionment method is as follows.

**Definition 4.2.1.** *An* ***apportionment*** *for $h \in \mathbb{N}_0$ is an n-tuple of nonnegative integers $(a_1, \ldots, a_n)$ such that $a(N) = h$. An* ***apportionment method*** *is a correspondence*[9] *$H$ that, for a given $\boldsymbol{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$, assigns at least one apportionment for $h$, i.e.,*

$$H : \mathbb{N}_0^N \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0^N$$

*and $H$ is nonempty.*

An apportionment method represents a way of assigning seats within a parliament. The rule by which one apportions the seats is the correspondence $H$. The first component in the correspondence $H$ is a vector representing the population sizes of the states. The second component represents the size of parliament (house size), that is, the number of seats, $h$, to be distributed amongst the states.

The question that will be investigated in this section can now be outlined. Given a population vector $\mathbf{g} \in \mathbb{N}_0^N$, which apportionment method provides apportionments which generate representations of simple majority weighted voting games that preserve the (minimal) winning coalitions of the game $(N, v^{\mathbf{g}})$, when possible, and when not, are the "closest" simple majority weighted voting game(s) to the given game $(N, v^{\mathbf{g}})$ for a given $h \in \mathbb{N}_0$? To specify what was meant by closest one needs to be able to define the distance between simple majority weighted voting games. To do so, two metrics defined on simple games will be introduced. If there is no chance of confusion, to simplify the following, the game $(N, v)$ will simply be identified with the coalition function $v$.

**Definition 4.2.2.** *Let $(N_1, v_1)$ and $(N_2, v_2)$ be two simple games, with corresponding winning coalitions $W_1$ and $W_2$. Then define*

$$d_W(v_1, v_2) = |W_1 \backslash W_2| + |W_2 \backslash W_1|.$$

[9]Note that some authors define an apportionment method as a correspondence $H : \mathbb{Q}_0^N \times \mathbb{N}_0 \rightsquigarrow \mathbb{N}_0^N$, where $\mathbb{Q}_0$ stands for the rational numbers with zero.

By replacing $W_1$ and $W_2$ by $W_1^{min}$ and $W_2^{min}$, one can define the following metric.

**Definition 4.2.3.** *Let $(N_1, v_1)$ and $(N_2, v_2)$ be two simple games with corresponding minimal winning coalitions $W_1^{min}$ and $W_2^{min}$. Then define*

$$d_{min}(v_1, v_2) = |W_1^{min} \backslash W_2^{min}| + |W_2^{min} \backslash W_1^{min}|.$$

To show that the previous definitions define metrics, when $N_1 = N_2$ (the case that will be considered for the rest of this chapter), the $l^1$ metric between games needs to be introduced.

**Definition 4.2.4.** *The $l^1$ **distance** between two games $(N, v_1)$ and $(N, v_2)$ is defined as follows.*

$$l^1(v_1, v_2) = \sum_{S \subseteq N} |v_1(S) - v_2(S)|.$$

As is clear, for two simple games $(N, v_1)$ and $(N, v_2)$, with corresponding winning coalitions $W_1$ and $W_2$, it follows that $d_W(v_1, v_2) = l^1(v_1, v_2)$. In addition, define

$$\tilde{v}_1(S) = \begin{cases} 1, & \text{if } S \in W_1^{min} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{v}_2(S) = \begin{cases} 1, & \text{if } S \in W_2^{min} \\ 0, & \text{otherwise.} \end{cases}$$

One can now define two new games $(N, \tilde{v}_1)$ and $(N, \tilde{v}_2)$. Then one notices that $l^1(\tilde{v}_1, \tilde{v}_2) = d_{min}(v_1, v_2)$. The only property of a metric which $d_{min}(v_1, v_2)$ does not obviously fulfill is $d_{min}(v_1, v_2) = 0 \iff v_1 = v_2$. However, by Remark 4.1.3, it becomes clear that $\tilde{v}_1 = \tilde{v}_2 \iff v_1 = v_2$, from which the result follows. Hence, both $d_W$ and $d_{min}$ define metrics.

The goal of the rest of this section is to investigate properties of two apportionment methods that minimize the distance between the representations generated by the apportionment vectors and the original game $v^{\mathbf{g}}$, for a given population vector $\mathbf{g} \in \mathbb{N}_0^N$ and house size $h \in \mathbb{N}_0$, with respect to the metric $d_W$ or $d_{min}$. To define the coming apportionment methods, a number of definitions are required. Let $h \in \mathbb{N}_0$. In the following, let $\mathcal{V}$ be the set of simple majority weighted voting games with $n$ players and let

$$\mathbf{V}(h) := \{v \in \mathcal{V} \mid \exists\, \mathbf{m} \in \mathbb{N}_0^N,\ m(N) = h,\ v^{\mathbf{m}}(S) = v(S)\ \forall\, S \subseteq N\}.$$

Let $\mathbf{g} \in \mathbb{N}_0^N$ and define[10], for a function $f : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$,

$$m_f(v^{\mathbf{g}}, h) = \min_{v \in \mathbf{V}(h)} f(v, v^{\mathbf{g}})$$

and

$$M_f(v^{\mathbf{g}}, h) = \{v \in \mathbf{V}(h) \mid f(v, v^{\mathbf{g}}) = m_f(v^{\mathbf{g}}, h)\}.$$

[10] It suffices to consider the minimum and not the infimum because for all the functions considered in this chapter, to which this definition will be applied, the minimum exists.

**Definition 4.2.5.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$. The $\sigma_W$ **method** is the set of vectors $H_W(\boldsymbol{g}, h)$ defined as follows*

$$H_W(\boldsymbol{g}, h) = \{\boldsymbol{m} \in \mathbb{N}_0^N \mid v^{\boldsymbol{m}} \in M_{d_W}(v^{\boldsymbol{g}}, h)\}.$$

*The $\sigma_{min}$ **method** is defined as*

$$H_{min}(\boldsymbol{g}, h) = \{\boldsymbol{m} \in \mathbb{N}_0^N \mid v^{\boldsymbol{m}} \in M_{d_{min}}(v^{\boldsymbol{g}}, h)\}.$$

The reason for also considering minimal winning coalitions is because in a parliament it is normally the minimal winning coalitions that determine which party gains control over a parliament. Coalitions that are already winning would not normally invite new superfluous members to join the coalition because they would then have to share their power with the new otiose members. This would reduce the power and influence of all members in the original coalition.

As was mentioned in the Introduction, an apportionment method that preserves the (minimal) winning coalitions seems to have been neglected by the literature in the past. Publications in the area of apportionment methods have tended to focus on methods which fulfill other desirable criteria and for which uncomplicated algorithms can be given to calculate the methods (e.g. divisor methods, see [9]). The fact that an apportionment method, as defined in Definition 4.2.5, has been neglected seems inexplicable because it is the winning coalitions in a parliament which are able to provide majority rulings on decisions. This is not the only point that is surprising but the fact that some methods can sometimes dramatically change the coalitions which possess a simple majority also seems to have been neglected. For example, if one's goal is to apportion as close as possible to the proportions $\frac{g_j}{g(N)}h$ $(j \in N)$, then this can lead to fundamental differences in the structure of the winning coalitions as the following example demonstrates. To give the example, the definition of the Hare method (also known as the Hamilton method, see [9]) is required. To simplify the definition, let $h \in \mathbb{N}_0$ and let $\mathbb{N}_0^N(h) := \{\mathbf{m} \in \mathbb{N}_0^N \mid m(N) = h\}$.

**Definition 4.2.6.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$. The **Hare method**, $H_{Hare}(\boldsymbol{g}, h)$, is defined as follows (see [9]).*

$$H_{Hare}(\boldsymbol{g}, h) = \{\boldsymbol{m} \in \mathbb{N}_0^N(h) \mid \sum_{j \in N} |m_j - \frac{g_j}{g(N)}h| \leq \sum_{j \in N} |n_j - \frac{g_j}{g(N)}h| \ \forall\ \boldsymbol{n} \in \mathbb{N}_0^N(h)\}.$$

**Example 4.2.7.** Consider the following simple majority weighted voting game generated by the population vector $\mathbf{g} := (14, 14, 25)$, $v^{\mathbf{g}} = [\,27\,; 14, 14, 25]$. Consider the case when $h = 5$. Then $H_{Hare}(\mathbf{g}, 5) = (1, 1, 3)$. However, for $h = 5$, player 3 has now become a dictator. Note, however, that $H_W(\mathbf{g}, 5) = H_{min}(\mathbf{g}, 5) = \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$.

Consider now the following simple majority weighted voting game generated by the population vector $\mathbf{g} := (3,3,7)$, $v^{\mathbf{g}} = [\,7\,;3,3,7]$. Then for $h = 6$ one has that $H_{Hare}(\mathbf{g},6) = \{(1,2,3),(2,1,3)\}$. However, for $h = 6$, player 3 has now lost his or her dictator status. Note, however, that $H_W(\mathbf{g},6) = H_{min}(\mathbf{g},6) = \{\mathbf{m} \in \mathbb{N}_0^3(6) \mid m_3 \geq 4\}$.

Note also, that the $\sigma_W$ and $\sigma_{min}$ apportionment methods do not always generate the same set of apportionments (see Example 4.2.22). This result may seem unexpected, in light of Remark 4.1.3. Remark 4.1.3 merely implies, however, that for two simple majority weighted voting games $v, w$, $d_W(v,w) = 0$ if and only if $d_{min}(v,w) = 0$.

The following characterization of the $\sigma_W$ method provides one with a possible method of calculating the $\sigma_W$ method.

**Definition 4.2.8.** *Let $H$ be an apportionment method. Then $H$ fulfills* ***property 1*** *if when $\boldsymbol{f},\boldsymbol{g} \in \mathbb{N}_0^N$ and $W_{v^{\boldsymbol{f}}} = W_{v^{\boldsymbol{g}}}$, then for all $h \in \mathbb{N}_0$, $H(\boldsymbol{f},h) = H(\boldsymbol{g},h)$.*

**Definition 4.2.9.** *Let $v \in \mathcal{V}$. $\boldsymbol{f} \in \mathbb{N}_0^N$ is called a* ***minimal integer representation*** *for $v$, if $W_{v^{\boldsymbol{f}}} = W_v$ and for all $\boldsymbol{g} \in \mathbb{N}_0^N$ with $g(N) < f(N)$, it follows that $W_{v^{\boldsymbol{g}}} \neq W_v$.*

**Definition 4.2.10.** *An apportionment method $H$ fulfills* ***property 2*** *if for all $v \in \mathcal{V}$, all minimal integer representations $\boldsymbol{f} \in \mathbb{N}_0^N$ of $v$ and all $h \in \mathbb{N}_0$, $H(\boldsymbol{f},h) = H_W(\boldsymbol{f},h)$.*

**Theorem 4.2.11.** *An apportionment method $H$ fulfills property 1 and 2 if and only if $H = H_W$.*

**Proof:** The if direction is clear. So let $H$ be an apportionment method and let $H$ fulfill property 1 and 2. If $\mathbf{g} \in \mathbb{N}_0^N$ is a minimal integer representation of a simple majority weighted voting game $v$, then for all $h \in \mathbb{N}_0$ it follows that $H(\mathbf{g},h) = H_W(\mathbf{g},h)$. So, let $\mathbf{g} \in \mathbb{N}_0^N$ not be a minimal integer representation of the game $v^{\mathbf{g}}$. Then there exists a minimal integer representation for $v^{\mathbf{g}}$, $\mathbf{f} \in \mathbb{N}_0^N$ with $W_{v^{\mathbf{f}}} = W_{v^{\mathbf{g}}}$. As $H$ fulfills property 1 and 2, for all $h \in \mathbb{N}_0$, it follows that $H(\mathbf{g},h) = H(\mathbf{f},h) = H_W(\mathbf{f},h) = H_W(\mathbf{g},h)$, from which the result follows. **q.e.d.**

Note that it is clear that the properties 1 and 2 are independent of each other. It is also possible to give a characterization of the $\sigma_{min}$ method, similar to the above characterization of the $\sigma_W$ method.

In order to discuss the suitability of the $\sigma_W$ and $\sigma_{min}$ methods, general properties of apportionment methods need to be introduced. Firstly, four properties that all apportionment methods should satisfy will be considered and then other desirable properties will also be analyzed (except for *equal treatment*, the coming properties can all be found in [9]). The first property to be analyzed is anonymity ([9] call the property *symmetry*). It basically states that an apportionment method $H$ does not depend on the ordering of the players. To define it, let $\Pi$ stand for the set of all permutations $\pi : N \to N$. Let $\mathbf{a} \in \mathbb{N}_0^N$. Then for all permutations $\pi \in \Pi$, define $\pi(\mathbf{a})$ by $(\pi(\mathbf{a}))_i = a_{\pi^{-1}(i)}$.

**Definition 4.2.12.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$, $h \in \mathbb{N}_0$, let $H$ be an apportionment method and let $\boldsymbol{a} \in H(\boldsymbol{g}, h)$. An apportionment method $H$ satisfies* ***anonymity*** *if for all permutations $\pi \in \Pi$ it follows that $\pi(\boldsymbol{a}) \in H(\pi(\boldsymbol{g}), h)$*

Clearly the $\sigma_{min}$ and $\sigma_W$ methods fulfill anonymity. The second property is exactness.

**Definition 4.2.13.** *An apportionment method $H$ is* ***exact*** *if, for all $\boldsymbol{g} \in \mathbb{N}_0^N$, $H(\boldsymbol{g}, g(N)) = \boldsymbol{g}$.*

Note that neither the $\sigma_{min}$ method nor the $\sigma_W$ method necessarily fulfill exactness (e.g. $(5,6,7) \in H_W((6,6,6),18)$ and $(5,6,7) \in H_{min}((6,6,6),18)$). The third property is scale invariance ([9] call it homogeneity).

**Definition 4.2.14.** *An apportionment method $H$ fulfills* ***scale invariance*** *if, for all $\boldsymbol{g} \in \mathbb{N}_0^N$ and $\lambda > 0$ such that $\lambda\boldsymbol{g} \in \mathbb{N}_0^N$, then $H(\boldsymbol{g}, h) = H(\lambda\boldsymbol{g}, h)$ for all $h \in \mathbb{N}_0$.*

To prove that the $\sigma_W$ and $\sigma_{min}$ methods fulfill scale invariance the following lemma is required.

**Lemma 4.2.15.** *For all $\lambda > 1$,*

$$\left\lceil \frac{\lambda + 1}{2} \right\rceil < \lambda + 1.$$

**Proof:** Let $k \in \mathbb{N}$ so that $k < \lambda \leq k + 1$. Then

$$\left\lceil \frac{\lambda + 1}{2} \right\rceil \leq \left\lceil \frac{k + 2}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil + 1 \leq k + 1 < \lambda + 1.$$

**q.e.d.**

**Proposition 4.2.16.** *The $\sigma_W$ and $\sigma_{min}$ methods fulfill scale invariance.*

**Proof:** Let $\mathbf{g} \in \mathbb{N}_0^N$ and let $\lambda > 0$ such that $\lambda\mathbf{g} \in \mathbb{N}_0^N$. It will be shown that $L_{v^{\mathbf{g}}} = L_{v^{\lambda\mathbf{g}}}$, from which the result follows (see Remark 4.1.3). It can be assumed, w.l.o.g., that $\lambda > 1$ because if $1 > \lambda > 0$, then by letting $\mathbf{h} := \lambda\mathbf{g}$ and $\gamma = \frac{1}{\lambda} > 1$, it follows that $L_{v^{\mathbf{g}}} = L_{v^{\lambda\mathbf{g}}} \iff L_{v^{\mathbf{h}}} = L_{v^{\gamma\mathbf{h}}}$. So, let $S \in L_{v^{\mathbf{g}}}$. This implies that

$$\left\lceil \frac{g(N) + 1}{2} \right\rceil > g(S).$$

Let, first of all, $g(N) = 2k+1$ for some $k \in \mathbb{N}_0$. Then one has $k+1 > g(S)$, from which it follows that

$$\lambda g(S) \leq \lambda k < \lambda k + \left\lceil \frac{\lambda+1}{2} \right\rceil = \left\lceil \frac{\lambda g(N)+1}{2} \right\rceil.$$

Hence, $S \in L_{v^{\lambda \mathbf{g}}}$. Let now $g(N) = 2k$ for some $k \in \mathbb{N}_0$. Then, by similar steps as before, one has

$$\lambda g(S) \leq \lambda k < \lambda k + 1 = \left\lceil \frac{\lambda g(N)+1}{2} \right\rceil.$$

So, now let $S \in L_{v^{\lambda \mathbf{g}}}$. Let, first of all, $g(N) = 2k+1$ for some $k \in \mathbb{N}_0$. Then, by Lemma 4.2.15,

$$\lambda g(S) \leq \left\lceil \frac{\lambda g(N)+1}{2} \right\rceil - 1 = \lambda k + \left\lceil \frac{\lambda+1}{2} \right\rceil - 1 < \lambda k + \lambda = \lambda \left\lceil \frac{g(N)+1}{2} \right\rceil,$$

from which it follows that

$$g(S) < \left\lceil \frac{g(N)+1}{2} \right\rceil.$$

Hence, $S \in L_{v^{\mathbf{g}}}$. For the case $g(N) = 2k$ for some $k \in \mathbb{N}_0$ one has,

$$\lambda g(S) < \left\lceil \frac{\lambda g(N)+1}{2} \right\rceil = \lambda k + 1 < \lambda k + \lambda = \lambda \left\lceil \frac{g(N)+1}{2} \right\rceil,$$

from which the result follows. **q.e.d.**

Another desirable property, stronger than exactness, is weak proportionality. It is defined as follows.

**Definition 4.2.17.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$. An apportionment method $H$ is weakly proportional if for all $\lambda > 0$ and $\boldsymbol{a} \in \mathbb{N}_0^N$ such that $\lambda \boldsymbol{g} = \boldsymbol{a}$, it follows that $H(\boldsymbol{g}, a(N)) = \boldsymbol{a}$.*

The following relationship exists between exactness, scale invariance and weak proportionality.

**Proposition 4.2.18.** *Let $H$ be an apportionment method. If $H$ fulfills exactness and scale invariance, then $H$ is weakly proportional.*

**Proof:** Let $H$ be an apportionment method, let $H$ fulfill exactness and scale invariance, let $\mathbf{g} \in \mathbb{N}_0^N$. Let $\lambda > 0$ and let $\mathbf{a} \in \mathbb{N}_0^N$ such that $\mathbf{g} = \lambda \mathbf{a}$. Then it follows that

$$\mathbf{a} = H(\mathbf{a}, a(N)) = H(\lambda \mathbf{a}, a(N)) = H(\mathbf{g}, a(N)).$$

**q.e.d.**

Unfortunately, the $\sigma_{min}$ and $\sigma_W$ methods both do not fulfill weak proportionality because they do not fulfill exactness (which is the case $\lambda = 1$). The last fundamental property that all apportionment methods should fulfill is the following.

**Definition 4.2.19.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$, $h \in \mathbb{N}_0$ and let $S \subseteq N$. An apportionment method $H$ fulfills **equal treatment** if when $g_i = g_j$ for all $i, j \in S$, then for all $\boldsymbol{a} \in H(\boldsymbol{g}, h)$, $\pi(\boldsymbol{a}) \in H(\boldsymbol{g}, h)$ for all $\pi \in \Pi$ satisfying $\pi(k) = k$ for all $k \in N \backslash S$.*

**Proposition 4.2.20.** *The $\sigma_W$ and $\sigma_{min}$ methods fulfill equal treatment.*

**Proof:** It will be first shown that the $\sigma_W$ method fulfills equal treatment. Because every permutation can be written as the product of permutations which only permute two elements, it suffices to consider such permutations (see, e.g., [10]). So, let $S \subseteq N$, let $\mathbf{g} \in \mathbb{N}_0^N$, $h \in \mathbb{N}_0$, let $g_k = g_l$ for all $k, l \in S$ and let $\mathbf{a} \in H_W(\mathbf{g}, h)$. Fix $i, j \in S$, $i \neq j$. Let $\pi \in \Pi$ such that $\pi(i) = j$ and $\pi(j) = i$ and $\pi(k) = k$ otherwise. If $W_{v^{\mathbf{a}}} = W_{v^{\pi(\mathbf{a})}}$, then the result follows. So let $P \in W_{v^{\mathbf{a}}}$ such that $P \notin W_{v^{\pi(\mathbf{a})}}$. Then it follows that, w.l.o.g., $j \notin P$ and $i \in P$. Note then that $T := (P \backslash \{i\}) \cup \{j\}$ fulfills $T \in W_{v^{\pi(\mathbf{a})}}$ and $T \notin W_{v^{\mathbf{a}}}$. Because $g_i = g_j$, it follows that $g(P) = g(T)$ and either $P, T \notin W_{v^{\mathbf{g}}}$ or $P, T \in W_{v^{\mathbf{g}}}$, from which it follows that $d_W(v^{\mathbf{a}}, v^{\mathbf{g}}) = d_W(v^{\pi(\mathbf{a})}, v^{\mathbf{g}})$. Hence, $\pi(\mathbf{a}) \in H_W(\mathbf{g}, h)$. The proof for the $\sigma_{min}$ method follows analogously, the only additional step requiring to show that the above defined $T \in W^{min}_{v^{\pi(\mathbf{a})}}$. This can be shown by seeing that if $T \notin W^{min}_{v^{\pi(\mathbf{a})}}$, then $P \notin W^{min}_{v^{\mathbf{a}}}$, a contradiction. **q.e.d.**

Apart from these four fundamental properties, there are numerous other desirable properties that an apportionment method can satisfy. Two of the most important such properties are house monotonicity and quota. The first of these properties arose in connection with the so called "Alabama paradox" (see [9]). An apportionment method displays the "Alabama paradox" when it does not fulfill house monotonicity. The definition of house monotonicity is as follows.

**Definition 4.2.21.** *An apportionment method $H$ is called **house monotone** if for any $\boldsymbol{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$, there exist $\boldsymbol{b} \in H(\boldsymbol{g}, h+1)$ and $\boldsymbol{a} \in H(\boldsymbol{g}, h)$ such that $b_i \geq a_i$ for all $i \in N$.*

Were an apportionment method not to satisfy house monotonicity, then it would mean that when the size of parliament increases, then, counterintuitively, one or more states could lose seats. Such an occurrence would make no sense whatsoever and in the words of Balinski and Young, [9], on p.42 in their authoritative book on the subject of apportionments, *Fair Representation*,

"No apportionment method is reasonable that gives some state fewer seats when there are more seats to go around".

Hence, the requirement that an apportionment method fulfills house monotonicity is essential. A problem with the $\sigma_{min}$ and $\sigma_W$ methods is that they are not necessarily house monotone. An example demonstrating this is the following.

**Example 4.2.22.** Let $\mathbf{g} := (2, 2, 4, 4, 7)$. Then $\mathbf{g}$ provides the following representation for a simple majority weighted voting game $v^{\mathbf{g}} = [\,10\,; 2, 2, 4, 4, 7]$. If one considers the case $h = 5$, then $H_{min}(\mathbf{g}, 5) = \{(0, 1, 1, 1, 2), (1, 0, 1, 1, 2)\}$. Now consider the case when $h = 6$. Then $H_{min}(\mathbf{g}, 6) = \{(0, 0, 1, 2, 3), (0, 0, 2, 1, 3)\}$. Note that in this case, one has $H_W(\mathbf{g}, 6) = (0, 0, 2, 2, 2)$.

Consider now $\mathbf{g} := (1, 1, 2, 2, 4, 4, 7)$. Then $\mathbf{g}$ provides the following representation for a simple majority weighted voting game $v^{\mathbf{g}} = [\,11\,; 1, 1, 2, 2, 4, 4, 7]$. If one considers the case $h = 6$, then $H_W(\mathbf{g}, 6) = (0, 0, 0, 0, 2, 2, 2)$. Now consider the case when $h = 7$. Then

$$H_W(\mathbf{g}, 7) = \{(0, 0, 0, 1, 1, 2, 3),\ (0, 0, 0, 1, 2, 1, 3),\ (0, 0, 1, 0, 1, 2, 3),\ (0, 0, 1, 0, 2, 1, 3),$$
$$(0, 0, 2, 0, 1, 1, 3),\ (0, 0, 0, 2, 1, 1, 3)\}.$$

The requirement that an apportionment method fulfill house monotonicity is also important because, by results in [9], it follows that if an apportionment method does not fulfill house monotonicity, then it cannot be population monotone and it also does not fulfill uniformity (see [9] for definitions), both of which are desirable properties.

It is also of interest to see if the $\sigma_W$ and $\sigma_{min}$ methods satisfy what is known as quota.

**Definition 4.2.23.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$ $(\boldsymbol{g} \neq 0)$ and $h \in \mathbb{N}_0$. Define the vector $\boldsymbol{x}$ as follows*

$$x_i = \frac{g_i}{g(N)} h \;\forall\, i \in N.$$

*Then an apportionment method $H$ satisfies **quota** if for all $\boldsymbol{a} \in H(\boldsymbol{g}, h)$ and all $i \in N$,*

$$\lfloor x_i \rfloor \leq a_i \leq \lceil x_i \rceil,$$

*where $\lfloor x_i \rfloor$ stands for the largest integer smaller than or equal to $x_i$.*

Unfortunately the $\sigma_W$ and $\sigma_{min}$ methods do not satisfy quota, as the following example demonstrates.

**Example 4.2.24.** Consider the following simple majority weighted voting game generated by the population vector $\mathbf{g} := (2, 2, 2, 3)$, $v^{\mathbf{g}} = [\,5\,; 2, 2, 2, 3]$. Consider the case when $h = 6$. Then $H_W(\mathbf{g}, 6) = H_{min}(\mathbf{g}, 6) = (1, 1, 1, 3)$. However,

$$\frac{g_4}{g(N)} h = 2$$

and hence, the $\sigma_W$ and $\sigma_{min}$ methods do not satisfy quota.

In addition, not only do the $\sigma_W$ and $\sigma_{min}$ methods not fulfill quota they sometimes violate quota in extreme ways. For example, if one player is a dictator in the original game generated by the population vector, then for all house sizes there is a $\sigma_W$ (and $\sigma_{min}$) apportionment such that this player is assigned all seats (that is, the value $h$) regardless of the values of the other players (For example, $\mathbf{g} = (1, 2, 4)$. Then for $h = 6$ a $\sigma_W$ or $\sigma_{min}$ apportionment would be $(0, 0, 6)$, which does not seem "fair" at all). In addition, the $\sigma_W$ and $\sigma_{min}$ methods can also "misrepresent" dummy players. For example, if $i \in N$ is not in any minimal winning coalition (what is know as a dummy player), then the player $i$ can often receive less than what seems to be "fair" and in many examples it is possible to give this player 0 seats. For example, let $\mathbf{g} = (4, 6, 6, 6)$. Then for a house size of 15, it follows that a possible $\sigma_W$ or $\sigma_{min}$ apportionment is $(0, 5, 5, 5)$, which appears "unfair" for player 1.

The fact that the $\sigma_W$ and $\sigma_{min}$ methods do not satisfy quota is, however, not truly a major deficit in the apportionment methods. Methods applied today are not always chosen because they satisfy quota. However, as a trade off, they satisfy population monotonicity, house monotonicity or other important properties. As the $\sigma_{min}$ and $\sigma_W$ methods satisfy neither quota nor house monotonicity, this suggests that the $\sigma_{min}$ and $\sigma_W$ methods could not be taken seriously as apportionment methods to be applied in reality.

## 4.3 Varying the Method

In this section, variations of the $\sigma_W$ and $\sigma_{min}$ apportionment methods will be considered. This section is based on ideas presented in [23] of choosing an order to apply certain criteria. In particular, different orderings of house monotonicity, quota and preserving the (minimal) winning coalitions will be considered.

One needs to be prudent when selecting criteria that an apportionment method is required to satisfy. As is shown in [9], for certain desirable values of the house size, $h$, and the number of states, $n$, there does not exist an apportionment method that satisfies both quota and population monotonicity. In [9] it is also shown that there is one unique method that satisfies both quota and house monotonicity.

An adaptation on the $\sigma_W$ and $\sigma_{min}$ methods is to consider the apportionments of the $\sigma_W$ and $\sigma_{min}$ methods that are closest to the proportions $\frac{g_j}{g(N)}h$ $(j \in N)$. As mentioned in the previous section, the Hare method is the method that apportions closest to the proportions $\frac{g_j}{g(N)}h$. However, the Hare method also lacks desiderata of apportionment methods. For example, the Hare method does not fulfill house monotonicity.

**Definition 4.3.1.** *Let* $\boldsymbol{g} \in \mathbb{N}_0^N$ *and* $h \in \mathbb{N}_0$*. The* $\sigma_W$***-Hare method****,* $H_W^{Hare}(\boldsymbol{g}, h)$*, is defined as follows.*

$$H_W^{Hare}(\boldsymbol{g}, h) = \{\boldsymbol{m} \in H_W(\boldsymbol{g}, h) \mid \sum_{j \in N} |m_j - \frac{g_j}{g(N)}h| \leq \sum_{j \in N} |n_j - \frac{g_j}{g(N)}h| \; \forall \; \boldsymbol{n} \in H_W(\boldsymbol{g}, h)\}.$$

By replacing $H_W(\mathbf{g}, h)$ with $H_{min}(\mathbf{g}, h)$ in the previous definition one can similarly define a $\sigma_{min}$**-Hare method**. Both of these methods, however, do not fulfill desirable criteria. Example 4.2.22 demonstrates that the $\sigma_{min}$-Hare method and the $\sigma_W$-Hare method are not house monotone and Example 4.2.24 shows that both methods also do not satisfy quota. Hence, the $\sigma_W$-Hare and $\sigma_{min}$-Hare methods also cannot be considered as methods to be applied in real situations. One positive aspect of the $\sigma_W$-Hare and $\sigma_{min}$-Hare methods is that they fulfill exactness. Hence, because $\frac{\lambda g_i}{\lambda g(N)} = \frac{g_i}{g(N)}$, $i \in N$, for $\lambda > 0$, both methods are also scale invariant and, therefore, it follows that the $\sigma_W$-Hare and $\sigma_{min}$-Hare methods are weakly proportional.

Another possibility is to reverse the ordering of the criteria in the previous method. One way of reversing the order of the previous criteria is to consider the apportionment(s) that satisfy quota but minimize the distance to the original simple majority weighted voting game, hence a quota-$\sigma_{min}$ and quota-$\sigma_W$ method. However, by using Example 4.2.22 one can see that such a quota-$\sigma_{min}$ and quota-$\sigma_W$ method would not be house monotone and hence, they also must be rejected. Such a quota-$\sigma_{min}$ or quota-$\sigma_W$ method would, however, clearly fulfill weak proportionality.

Finally, one can also assure the nonemptiness of a house monotone-$\sigma_W$ or $\sigma_{min}$ apportionment method (i.e. choosing from the set of house monotone methods the method(s) which provides apportionments that generate games that minimize $d_W$ or $d_{min}$). As divisor methods are always house monotone, one can examine whether any of the traditional divisor methods correspond to such a house monotone-$\sigma_W$ or $\sigma_{min}$ method. However, by comparing the five traditional divisor methods, (see the appendix for definitions and examples) Jefferson, Webster, Huntington-Hill, Dean and Adam, one can find examples to show that not one of the five produces apportionments that generate representations of games that are consistently closer (viz. $d_W$ or $d_{min}$) to the original game $v^{\mathbf{g}}$ generated by the population vector $\mathbf{g} \in \mathbb{N}_0^N$.

## 4.4 Preserving the losing coalitions

The last possibility considered in this chapter will be to define an apportionment method that preserves the losing coalitions instead of the winning coalitions. An equally valid argument can be made to justify the importance of a method that preserves the losing coalitions as that for the $\sigma_W$ or $\sigma_{min}$ method. Analogous to the minimal winning coalitions, one can also consider maximal losing coalitions to define an apportionment method. To define the possible apportionment methods, two metrics shall be introduced.

**Definition 4.4.1.** *Let $(N_1, v_1)$ and $(N_2, v_2)$ be two simple games, with corresponding losing coalitions $L_1$ and $L_2$. Then define*

$$d_L(v_1, v_2) = |L_1 \backslash L_2| + |L_2 \backslash L_1|.$$

By noticing that $d_L(v_1, v_2) = l^1(v_1, v_2) = d_W(v_1, v_2)$, for two simple games $(N, v_1)$ and $(N, v_2)$, one can conclude that an apportionment method defined by the $d_L$ metric would be equivalent to the $\sigma_W$ method. Hence, just considering the losing coalitions does not grant anything new. By replacing $L_1$ and $L_2$ by $L_1^{max}$ and $L_2^{max}$, however, one can define the following metric and a new apportionment method (see below).

**Definition 4.4.2.** *Let $(N_1, v_1)$ and $(N_2, v_2)$ be two simple games with corresponding maximal losing coalitions $L_1^{max}$ and $L_2^{max}$. Then define*

$$d_{max}(v_1, v_2) = |L_1^{max} \backslash L_2^{max}| + |L_2^{max} \backslash L_1^{max}|.$$

To demonstrate that $d_{max}$ defines a metric for two simple games $(N, v_1)$ and $(N, v_2)$, define

$$\tilde{v}_1(S) = \begin{cases} 1, & \text{if } S \in L_1^{max} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{v}_2(S) = \begin{cases} 1, & \text{if } S \in L_2^{max} \\ 0, & \text{otherwise.} \end{cases}$$

One can now define two new games $(N, \tilde{v}_1)$ and $(N, \tilde{v}_2)$. Then one notices that $l^1(\tilde{v}_1, \tilde{v}_2) = d_{max}(v_1, v_2)$. The only property of a metric which $d_{max}(v_1, v_2)$ does not obviously fulfill is $d_{max}(v_1, v_2) = 0 \iff v_1 = v_2$. However, by Remark 4.1.3, it becomes clear that $\tilde{v}_1 = \tilde{v}_2 \iff v_1 = v_2$, from which the result follows. Hence, $d_{max}$ defines a metric.

**Definition 4.4.3.** *Let $\boldsymbol{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$. The $\sigma_{max}$ **method** is the set of vectors $H_{max}(\boldsymbol{g}, h)$ defined as follows.*

$$H_{max}(\boldsymbol{g}, h) = \{\boldsymbol{m} \in \mathbb{N}_0^N \mid v^{\boldsymbol{m}} \in M_{d_{max}}(v^{\boldsymbol{g}}, h)\}.$$

The following example demonstrates that the $\sigma_{max}$ method really defines a new apportionment method.

**Example 4.4.4.** Consider the following simple majority weighted voting game generated by the population vector $\mathbf{g} := (3, 4, 4, 5)$, $v^{\mathbf{g}} = [9\,; 3, 4, 4, 5]$. Consider the case when $h = 5$. Then it follows that $H_W(\mathbf{g}, 5) = \{(1, 1, 1, 2), (0, 1, 2, 2), (0, 2, 1, 2), (0, 2, 2, 1)\}$, $H_{min}(\mathbf{g}, 5) = (1, 1, 1, 2)$ and $H_{max}(\mathbf{g}, 5) = \{\pi(0, 1, 2, 2) \mid \pi \in \Pi\}$.

As for the $\sigma_W$ and $\sigma_{min}$ methods, the $\sigma_{max}$ method fulfills equal treatment, scale invariance, anonymity and also does not fulfill exactness. Unfortunately, as for the $\sigma_W$ and $\sigma_{min}$ methods, the $\sigma_{max}$ method fulfills neither house monotonicity nor quota. Two examples, to demonstrate this, are the following.

**Example 4.4.5.** Let $\mathbf{g} := (2, 2, 4, 4, 7)$. Then $\mathbf{g}$ provides the following representation for a simple majority weighted voting game $v^{\mathbf{g}} = [\,10\,; 2, 2, 4, 4, 7]$. If one considers the case $h = 5$, then $H_{max}(\mathbf{g}, 5) = \{(0, 1, 1, 1, 2), (1, 0, 1, 1, 2)\}$. Now consider the case when $h = 6$. Then $H_{max}(\mathbf{g}, 6) = (0, 0, 2, 2, 2)$. Hence, the $\sigma_{max}$ method is not house monotone.

**Example 4.4.6.** Consider the following simple majority weighted voting game generated by the population vector $\mathbf{g} := (3, 3, 3, 9)$, $v^{\mathbf{g}} = [\,10\,; 3, 3, 3, 9]$. Consider the case when $h = 10$. Then $(1, 1, 3, 5) \in H_{max}(\mathbf{g}, 6)$. However,

$$\frac{g_2}{g(N)} h < 2$$

and hence, the $\sigma_{max}$ method does not satisfy quota.

These examples demonstrate that the $\sigma_{max}$ method also cannot be considered as an apportionment method to be applied in the real world.

Another disadvantage of the three apportionment methods consider so far, $\sigma_W$, $\sigma_{min}$ and $\sigma_{max}$, is that they tend to allocate numerous possible apportionments for a given $\mathbf{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$. One possible way of reducing the possibilities (which is always desirable because only one apportionment is necessary for a given parliament) is to apply the different methods in a particular order (as in the previous section). For example, were one to minimize the distance with respect to the $d_W$ metric in Example 4.4.4 and then, for the representations of simple majority weighted voting games generated by the remaining apportionments, minimize the distance with respect to either the $d_{min}$ or $d_{max}$ metric, then the number of remaining apportionments for Example 4.4.4 would be reduced in both cases. To define one such apportionment method, let $\mathbf{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$ and define $H_W^{min}(\mathbf{g}, h)$ as follows.

$$H_W^{min}(\mathbf{g}, h) = \{\mathbf{m} \in H_W(\mathbf{g}, h) \mid d_{min}(v^{\mathbf{g}}, v^{\mathbf{m}}) \leq d_{min}(v^{\mathbf{g}}, v^{\mathbf{n}})\ \forall\, \mathbf{n} \in H_W(\mathbf{g}, h)\}.$$

Then it follows that $H_W^{min}(\mathbf{g}, 5) = (1, 1, 1, 2)$ for $\mathbf{g} = (3, 4, 4, 5)$. Also, define $H_W^{max}(\mathbf{g}, h)$ as follows.

$$H_W^{max}(\mathbf{g}, h) = \{\mathbf{m} \in H_W(\mathbf{g}, h) \mid d_{max}(v^{\mathbf{g}}, v^{\mathbf{m}}) \leq d_{max}(v^{\mathbf{g}}, v^{\mathbf{n}})\ \forall\, \mathbf{n} \in H_W(\mathbf{g}, h)\}.$$

Then it follows that $H_W^{max}(\mathbf{g}, 5) = \{(0, 1, 2, 2), (0, 2, 1, 2), (0, 2, 2, 1)\}$, again for $\mathbf{g} = (3, 4, 4, 5)$.

Hence, by applying different possible orderings of the three metrics, $d_W$, $d_{min}$ and $d_{max}$, one can narrow down the possible apportionment chosen for a given $\mathbf{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$. Note that, because of the earlier examples, no such method could satisfy house monotonicity or quota.

## 4.5 Conclusion

This chapter investigated the suitability of apportionment methods based on the idea of preserving the winning (or losing) coalitions of the simple game represented by the populations of the states. The results showed that the new apportionment methods (and variations thereof) do not satisfy desirable properties such as house monotonicity, quota, etc. Although the methods do not fulfill the aforementioned properties, an equally valid argument could be made for the preservation of the winning (or losing) coalitions and, hence, the methods investigated here could provide one with a basis for further investigations by combining methods as explained in Section 4.3. In particular, a house monotone-$\sigma_W$, -$\sigma_{min}$ or -$\sigma_{max}$ method deserves more attention. The methods investigated in this chapter could also be used, for example, to decide which player receives an additional seat when a different apportionment method does not provide a unique apportionment.

## 4.6 Appendix

The following definitions can be found in [9]. A *divisor function* $d$ is any monotone real valued function defined over the nonnegative integers that satisfies $d(k) \in [k, k+1]$ for all integers $k$, and for which there do not exist a pair of integers $a \geq 0$ and $b \geq 1$ with $d(a) = a + 1$ and $d(b) = b$. A *d-rounding* of a real number $z \geq 0$ is

$$[z]_d = \begin{cases} 0, & \text{if } z = 0; \\ a, & \text{if } d(a-1) \leq z \leq d(a). \end{cases}$$

Thus $[d(a)]_d = a$ or $a + 1$: at the threshold one can either round down or round up. For $\mathbf{g} \in \mathbb{N}_0^N$ and $h \in \mathbb{N}_0$, a *divisor method* $\varphi^d$ based on $d$ is

$$\varphi^d(\mathbf{g}, h) = \{\mathbf{x} = (x_j), j \in N \mid x_j = [\lambda g_j]_d, \lambda > 0 \text{ chosen so that } x(N) = h\}.$$

Jefferson's method is obtained by taking $d(a) = a + 1$. Adam's method is obtained by taking $d(a) = a$. Webster's method is obtained by taking $d(a) = a + \frac{1}{2}$. The Huntington-Hill method is obtained by taking $d(a) = \sqrt{a(a+1)}$. Dean's method is obtained by taking $d(a) = \frac{2a(a+1)}{2a+1}$. It is well-known that all divisor methods are house monotone, see [9].

The numbers in the first $\mathbf{g}$ vector are due to [27].

**Example 4.6.1.** Let $\mathbf{g} = (9988, 9084, 7182, 5260, 3321, 1185)$ and $h = 36$. Let $\mathbf{j} = (10, 10, 7, 5, 3, 1)$ be Jefferson's apportionment, $\mathbf{a} = (10, 10, 7, 5, 3, 1)$ be Adam's apportionment, $\mathbf{w} = (10, 9, 8, 5, 3, 1)$ be Webster's apportionment, $\mathbf{h} = (10, 9, 7, 6, 3, 1)$ be Huntington-Hill's apportionment and $\mathbf{d} = (10, 9, 7, 5, 4, 1)$ be Dean's apportionment. Then it follows that

$$d_{min}(v^{\mathbf{g}}, v^{\mathbf{d}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{h}}) < d_{min}(v^{\mathbf{g}}, v^{\mathbf{w}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{a}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{j}})$$

and

$$d_W(v^{\mathbf{g}}, v^{\mathbf{d}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{h}}) < d_W(v^{\mathbf{g}}, v^{\mathbf{w}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{a}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{j}}).$$

Now let $\mathbf{g} = (13536, 10545, 5216, 4518, 2772)$ and $h = 37$. Then it follows that $\mathbf{j} = (14, 11, 5, 4, 3)$, $\mathbf{a} = (14, 11, 5, 4, 3)$, $\mathbf{w} = (13, 11, 5, 5, 3)$, $\mathbf{h} = (13, 11, 5, 5, 3)$ and $\mathbf{d} = (13, 11, 5, 5, 3)$. These vectors satisfy

$$d_{min}(v^{\mathbf{g}}, v^{\mathbf{j}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{a}}) < d_{min}(v^{\mathbf{g}}, v^{\mathbf{h}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{d}}) = d_{min}(v^{\mathbf{g}}, v^{\mathbf{w}})$$

and

$$d_W(v^{\mathbf{g}}, v^{\mathbf{j}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{a}}) < d_W(v^{\mathbf{g}}, v^{\mathbf{h}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{d}}) = d_W(v^{\mathbf{g}}, v^{\mathbf{w}}).$$

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