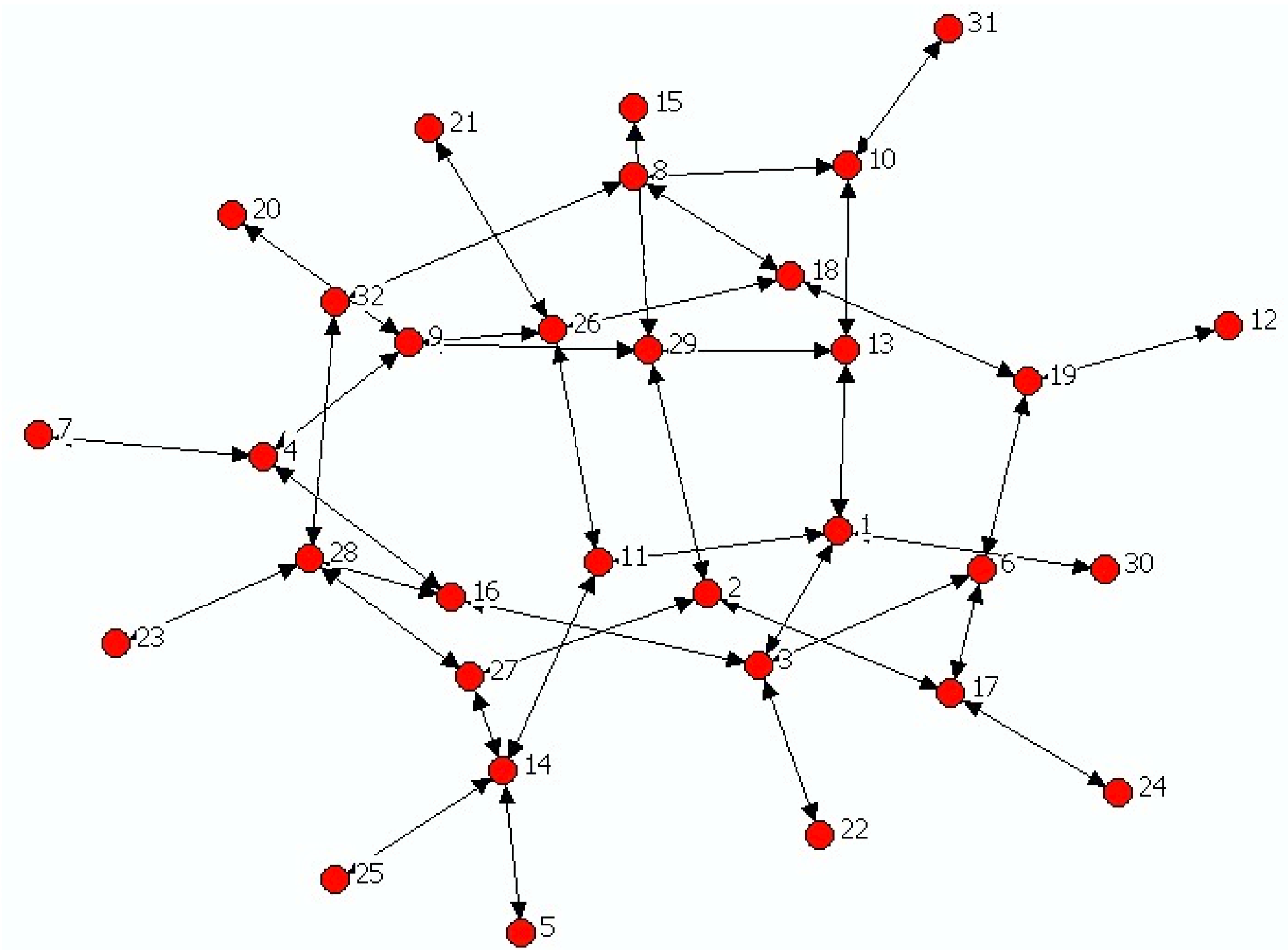

What would Happen if Everybody were Calculative in Building Relationships?

Advances in Strategic Network Formation: Preferences, Centrality, and Externalities

Dissertation zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.) an der Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

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Acknowledgments

“We constantly create [...] contacts to people who might become useful for us. Did NETWORKING by now erode our private lives and our idea of friendship?” This question – which was the headline in a German news magazine recently (see Dillig, 2008) – serves as a leitmotif of the work you are holding in your hands.
I would like to thank first and foremost Walter Trockel and Herbert Dawid. In their roles as my supervisors, as directors of our research training group, and as leaders of our institute they provided me with full support and the perfect environment to develop research skills. Walter Trockel's encouragement covered topics of mathematical economics but also exceeded work-related issues.

Important parts of this work are the product of fruitful collaborations with Tim Hellmann and Vincent Buskens. This work has also benefited heavily from countless inspiring discussions with colleagues and teachers at Bielefeld University, as well as from my stays at Paris I Panthéon-Sorbonne, Utrecht University, and the University of Konstanz.

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1 Introduction

This chapter motivates the topic and provides an outline of this work.

1.1 Problem and Research Questions

The importance of social and economic networks has meanwhile been widely acknowledged throughout social sciences. The textbook of Jackson (2008) synthesizes the main contributions of economists to that topic. Literature from Sociology and other social sciences is collected in Wasserman and Faust (1994), containing basic concepts, as well as many empirical examples. Examples for networks of bilateral relationships include R&D collaborations, strategic alliances, knowledge management within organizations, formal and informal leadership, personal (business) contacts, information about job openings, bargaining power, but are not restricted to these.

One famous example is given by Padgett and Ansell (1993) and illustrated in Figure 1. Depicted are the richest Italian families of the 15th century and relations among them. A link between two families indicates that there is a marriage relation. Note three typical aspects of that time: there were no trading companies, but family clans; trading was dangerous without trust in a long-term relationship; and marriages were arranged by the family leaders. Padgett and Ansell (1993) argue that besides the many explanations that historians already had for the rise of the Medici, it was their distinct position in this network of marriages that facilitated their prosperity. This is just one of various studies that emphasize the profitability of certain

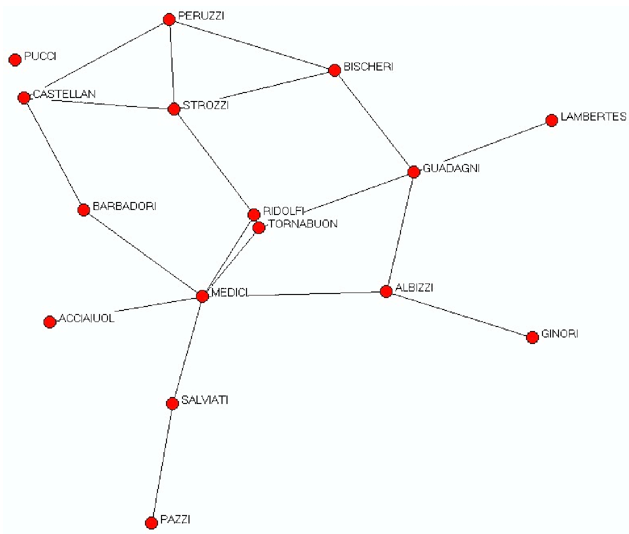

Figure 1: Marriage network of rich Italian families.

positions in a network. Other examples are applicants getting a job (Granovetter, 1974), networks facilitating companies' cooperation (Powell et al., 1996) and their success (Uzzi, 1996).

However, in these studies the networks are considered as fixed. While it might be true that a network (of bilateral relationships) is constraining the choices of an actor (person, company, organization), it is also true that the actor's choice on relationships influences the structure of the network. The dynamics of these two aspects are in question: *How do networks change when actors follow incentives for profitable network positions?* This question will be the recurrent theme throughout this work.

A special motivation to study this topic is the popularity of concepts from research on networks. Business consultants offer, e.g., to "study ways that Leaders can make better use of networks" (Leader Values, 2008) and also to "Find emergent leaders in fast growing companies [...] Determine influential journalists and analysts [...] Reveal key players [...] Reveal opinion leaders" (Krebs, 2008); other examples can be found in Weidner (2008). Web-based communities, such as Xing or Facebook, also draw attention to networks of bilateral relations. The fact that more and more people are aware of the importance of networks might not be without consequences for the way people decide about their relationships (be it in their personal network or as a decision maker for an organization, such as a company).

Approach (and Core Literature)

As suggested, one of the factors influencing the emergence of networks, is the purposeful actions of link formation and link dissolution in order to reach what we call "structural goals" (the term is borrowed from Doreian, 2006). Structural goals incorporate beneficial aspects of social networks that can be derived from the pure network structure, like the Medici's position in the marriage (resp. trade) network. This perspective is shared by Burt (1992), who stresses that some actors actively enter profitable network positions. Hummon (2000) also takes a rational choice perspective: Actors evaluate the consequences of network ties and make decision according to their goals. However an actor's position in a particular network does not only depend on his linking decisions, but also on the decisions of the other actors. Thus, decisions about profitable relations are not a situation of choice, but a situation of strategic interaction – an aspect that is best covered by game theory.

The field of 'Social network analysis' has a long tradition in studying the (positive) effects of network structure on individual well-being (see, e.g., Wasserman and Faust, 1994). However, the networks are usually treated as fixed. Models of dynamic networks, on the other hand, have long not considered aspects of agency. Random graph models (starting from Erdös and Renyi, 1960, through to Watts, 1999, Barabàsi and Albert, 1999) define probabilistic processes that are able to reconstruct different patterns of empirically observed social networks. Recently, there have been groundbreaking contributions in analyzing network dynamics theoretically, as well as empirically, in a more goal-oriented manner. The work of Snijders provides a statistical model relating changes in the network structure back to individual propensities to form and sever links (see Snijders, 1996, and Snijders, 2001). The works of Jackson and Wolinsky (1996) and Bala and Goyal (2000) provide concepts that allow us to study the formation of social networks as a game in the sense of cooperative, respectively non-cooperative, game theory. Thereafter, there has been a flourishing literature on specific situations of strategic network formation, of which two small surveys can be found in Jackson (2004) and Goyal and Joshi (2006a). The various network formation games provide micro-based models and analyze which networks are stable (and which are efficient).

Despite major advancements of this literature, there are still substantial open points. First, ideas from social network analysis are only integrated to a limited extent, (although this literature has a long tradition in studying the effects of network structure and it is those concepts, e.g. centrality, that are claimed to be important).[1] Secondly, while the number of specific models is constantly increasing, little is known about their interrelation. It can be expected that there are models with similar incentive structure leading to similar networks.

Considering the status of the research on strategic network formation, we can state more precisely the main question (How do networks change when actors follow incentives for profitable network positions?). The following research questions guided this work:

Q1 How much do we need to know about the structural goals of the actors (their preferences) in order to characterize endogenous network structure?

Q2 How do the stable network structures depend on the assumptions of increasing and decreasing marginal returns of linking?

Q3 Do typical structures of social networks persist if actors strive for central network posi-

[1] The work by Goyal and Vega-Redondo (2007), forms a beautiful exception.

tions?

Q4 Given two structural goals, to which extent do the induced networks differ, when actors follow one of these goals? How do different goals interact with each other?

Q5 How do stable network structures differ from efficient network structures?

Q6 Why does individual interest in forming networks not always lead to efficient outcomes?

1.2 Outline (of the Work) and Main Results

The work is organized into three parts. Chapter 2 discusses modeling aspects addressing *Q1* and *Q2*. Chapters 3 and 4 analyze a specific model targeted at *Q3* and *Q4*. Finally, Chapter 5 provides an answer for *Q5* and *Q6* in a general setting.

Chapter 2 first introduces the necessary formal framework. In the general model actors have preferences on all networks and form and sever links to reach better networks. Section 2.2 addresses how assumptions on the preferences affect the equilibrium outcomes, i.e. the stable networks. The first type of assumptions defines which aspects of the network are relevant for the actors. In two very simple models it is shown that such an assumption can already be sufficient to partially characterize the equilibria, i.e. assure existence of stable networks and describe some aspects of all stable networks. The second set of assumptions specifies the shape of the utility function. In Section 2.3 we study the role of increasing and decreasing marginal returns for a specific model and compare its results to a well-known model that is similar in spirit. It turns out that the results of the two models coincide for certain specifications. While specification details matter, they cannot arbitrarily determine the induced networks. A version of this section (Section 2.3) is published in Buechel (2008).

Chapter 3 introduces and analyzes a specific model (of strategic network formation), where actors strive for central network positions. We analyze this model not only by formal derivations, but also by numerical examples and simulations using a computer program written by Vincent Buskens from Utrecht University. The first set of results presents the basic findings with these three methods.

Then we contrast the dynamics of two structural goals, i.e. two types of centrality incentives, that are incorporated in the model. It turns out that each type of incentive leads to a special class of networks in most of the cases. For readers who are already familiar with terms that will be defined in the next chapter: Incentives for closeness centrality (capturing the gain of access to information and support by having many other actors in close reach) lead to tree networks, while incentives for betweenness centrality (capturing intermediation rents that are attained by being a broker for others) lead to complete bipartite networks.

Next, we turn to the interaction of both types of incentives and find non-trivial dynamics: a combination of the two structural goals frequently leads to networks that would have not been induced by just one of these goals. We relate this phenomenon to the observation that multiple structural goals allow two actors to form a link, each of them for very different reasons, depending on their current network position. In the last section of Chapter 3, we analyze one common effect of the two different structural goals. A typical feature of (friendship) networks is to contain small groups that are heavily interrelated (known as "closure" or "local density"). Section 3.4 shows that actors optimizing their centrality destroy such patterns.

Chapter 4 analyzes efficiency in the model introduced in Chapter 3. First, we show that utilitarian welfare in this model is determined by very basic network statistics. For instance, knowing the number of pairs at certain distances in the network is sufficient to compute its collective value ("welfare"). Section 4.2 shows which networks are efficient (welfare maximizing) for different parameters. Next, we combine these results with results of Chapter 3 to assess under which conditions the efficient networks are unstable, stable or uniquely stable.

In Section 4.3 we use simulations to estimate the welfare of an emerging network for some settings. Surprisingly, incentives for intermediation rents (i.e. betweenness) frequently lead to networks that have lower welfare than the starting networks. This fact is based on a divergence between individual and collective interest, since an individual link increases the (individual) utility of the two actors who form it but decreases (collective) welfare. For the second structural goal incorporated in the model, i.e. closeness centrality, we also find a systematic difference between stable and efficient networks: egoistic actors do not internalize positive spill-overs of link formation. The working paper Buechel and Buskens (2008) presents some of the main findings of Chapters 3 and 4. While it was the author, who initiated to study this model and carried out most of the analyses, Vincent Buskens provided invaluable assistance in interpreting and presenting the results.

Chapter 5 builds on the intuition gained in the example before: Negative externalities of link formation tend to induce networks that are too dense, while positive externalities of forming are assumed to lead to networks that are not dense enough. We introduce the notion of over-connected and under-connected networks to clarify this argument. A network that is not over-connected and not under-connected is locally efficient, which means that there is neither a subnetwork nor a supernetwork with higher welfare. We show that for situations of strategic network formation with positive externalities, no stable network can be over-connected; while for situations with negative externalities no stable network can be under-connected if some other conditions are met.

Those results contribute to a better understanding of the tension between stability and efficiency (in situations of strategic network formation) in two ways. First, they formalize an intuitive argument that provides a social planner a clear signal in which situations rather to impede and when to foster bilateral relationships. Secondly, the results can be used in specific network formation models to better characterize stable and efficient networks. We present this for a few examples (of positive and negative externalities), while there are many other models of strategic network formation that satisfy the required conditions. A similar version of Chapter 5 can be found in Buechel and Hellmann (2008), where we present some supporting results in addition. It seems fair to consider Chapter 5 as produced by Tim Hellmann and the author in equal parts.

2 Modeling Preference-based Network Formation

In the recent past, different approaches were developed to model the formation of networks as the result of strategic interactions. What they have in common is that agents can alter the network structure in certain ways and networks are considered as stable if there are no beneficial deviations left. It is not the focus of this work to describe and compare the different game theoretic approaches (i.e. stability notions); instead the emphasis is on discussing the assumptions on the preferences and analyzing their impact for stability.

For this purpose, we will distinguish network statistics and their evaluation when introducing the necessary framework in the next section. Section 2.2 addresses how network statistics drive the results and Section 2.3 addresses how the evaluation of network statistics can drive the results.

2.1 Framework

2.1.1 Definitions of Agents and Networks

Let $N = \{1, ..., n\}$ be a (finite, fixed) set of **agents/players**, with $n \geq 3$. We restrict attention to undirected and unweighted graphs. A **network/graph** g is a set of unordered pairs $\{i, j\}$ with $i, j \in N$. This set represents who is linked to whom, i.e. $\{i, j\} = ij \in g$ means that agent i and agent j are linked under g. Let g^N be the set of all subsets of N of size two and G be the set of all possible graphs, $G = \{g : g \subseteq g^N\}$. For conciseness we denote $g \cup \{i, j\} = g \cup ij$ and $g \setminus \{i, j\} = g \setminus ij$.

A network can be seen as a binary relation on the agent set. Sometimes it is also convenient to work with a matrix representation for a network called adjacency matrix.

$$A(g) = \begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}_{nxn} \text{, where } a_{ij} = \begin{cases} 1 \text{ , if } ij \in g \\ 0 \text{ , otherwise} \end{cases}$$

Note that we are only working with relations that are symmetric and irreflexive, which is also apparent in the matrices.

By $N_i(g)$ we denote the neighbors (or friends) of agent i in network g, $N_i(g) := \{j \in N : ij \in g\}$. Similarly, let $L_i(g)$ be the set of links that agent i is involved in, that is

$L_i(g) = \{ij \in g : j \in N\}$. We define $l_i(g) := |L_i(g)| = |N_i(g)|$ sometimes called agent i's degree. An isolate is an agent with degree zero and a pendant is an agent with degree one (the latter structure is called a **loose end**). Let $l(g) := \#\{ij \in g\}$ be the number of links of a network. The network density is canonically defined as $D(g) := \frac{l(g)}{l(g^N)} = \frac{l(g)}{\frac{1}{2}n(n-1)}$.

A **path** between two agents i and j is a sequence of distinct agents $(i_1, ..., i_K)$ such that $i_1 = i$, $i_K = j$, and $\{i_k, i_{k+1}\} \in g \ \forall k \in \{1, ..., K-1\}$. The (geodesic) **distance** between two agents is the length of their shortest path(s), where the length is the number of links in the sequence. Formally, we can define the distance ($d_{ij}(g)$) between two agents $i \neq j$ in a graph g by the corresponding adjacency matrix $A(g)$: $d_{ij}(g) := min\{k \in \mathbb{N} : A^k(g)_{ij} \geq 1\}$. If there does not exist such a k for two agents, there does not exist any path between them. Such a pair $\{i, j\}$ is not connected and their distance is defined as $d_{ij}(g) = M$, a fixed number that is bigger than the largest feasible distance $(n-1)$. As a convention we set $d_{ii}(g) = 0 \ \ \forall i \in N$.

A graph is called **connected** if there exists a path between any two agents in the graph. A set of connected agents is called a **component** if there is no path to agents outside of this set. The set of connected networks is formally defined as $\bar{G} := \{g \in G \mid \ \ \forall i, j \in N, \ \ d_{ij}(g) < M\}$. A link is called **critical** if its deletion increases the number of components in a graph. A graph is called **minimal** if all links are critical.

Let us define some classes of networks and special network architectures that will be used throughout the text. A **tree** is a connected network that is minimal. These networks are also characterized by having a unique path between any pair of agents. A **complete bipartite network** $g^{l:r}$ consists of two groups of nodes of sizes l and r (with $l + r = n$ and $l, r \geq 1$) such that there are no links within a group and all links are present across groups. If one group only consists of one agent this is a star network $g^{\star}$, which belongs to the class of trees, in addition. In a balanced complete bipartite network $g^{\frac{n}{2}:\frac{n}{2}}$ the groups are of equal size (for even n). A **circle** of size K is a sequence of K distinct agents $(i_1, ..., i_K)$ such that $\{i_k, i_{k+1}\} \ \in g \ \forall k \in \{1, ..., K\}$, where $i_{K+1} := i_1$. A circle network $g^{\bigcirc}$ is a graph with no links besides a circle of size $K = n$. Eliminating one link of a circle graph leads to a line graph $g^{|}$.

2.1.2 Network Statistics

A basic ingredient of all network formation models is the assumption that agents derive utility from the network structure. Let for each agent $i \in N$ a utility function $u_i : G \to \mathbb{R}$ represent his (complete and transitive) preferences on the set of networks. Let u denote a profile of n utility functions. A situation of strategic network formation can be defined as a society (N, G, u).

Before discussing the game theoretic modeling, let us have a closer look at the agents' preferences. To analyze and motivate the agents' preferences I would like to make an unusual but helpful distinction between relevant network statistics and their evaluation. This involves decomposing the utility of an agent derived from a network into (a) the measurement of certain criteria and (b) the evaluation of these criteria. Formally, we introduce a function $m : G \longrightarrow \mathbb{R}^{kn}$, which objectively measures k features of a graph for each agent. How these dimensions matter to a certain agent will be specified later by an evaluation function $U_i : X \to \mathbb{R}$ with $(X \subseteq \mathbb{R}^k)$.

Different network statistics summarize relevant aspects of the network structure:

- Number of contacts / degree, $l_i(g)$

  The number of links or neighbors that an agent has is an important network statistic in virtually any analysis of network positions. On one hand it can serve as a proxy for the strength of an agent's position (as proposed by Freeman, 1979). On the other hand the number of links certainly is also the main driver of costs – maintaining bilateral relationships is assumed to involve effort, time, or money. One example for degree is the number of contacts one has in an online community as Skype, MSN, Xing or Facebook.

- Number of connections, $\#\{j \neq i : d_{ij}(g) < M\}$

  The crucial feature of a network is sometimes the size of an agent's component – the number of people who are directly or indirectly connected to him. Empirical applications are networks that allow for an easy flow of resources (such as information). The number of connections also turns out to be the crucial feature of a more theoretical exercise: In an infinitely repeated game without discounting where only the actions of the neighbors can be observed, the set of equilibria depends on the number of connections, since this determines the potential punishment of deviating actions (see the "proximity game" by Renault and Tomala, 1998).

- Distance-based statistics

  An intermediate case between number of connections and number of contacts is to use an index that considers the distances of the different connections.[2] Jackson and Wolinsky

[2] The distances of one agent to all other agents is not a unidimensional feature of a network. But there are different indices computing one statistic for each agent based on distances.

(1996) introduce a model where the benefit of a connection decreases with distance ("connections model", discussed in the subsections 2.3.3 and 5.1.2). Freeman (1979) proposes a measure based on the average distance ($\frac{\sum_{j \in N} d_{ij}(g)}{n-1}$) to assess an agent's centrality: the smaller the sum of distances, the higher the "closeness". This has become one of the most applied network statistics.

- Intermediating position / brokerage
  If indirect connections are important, then it is reasonable that agents who connect others are in a special position. Lying on many paths between other agents can be harmful in the context of diseases or beneficial in a context of trading. Goyal and Vega-Redondo (2007) consider the agents' ability to block connections by being on every path for a certain other pair (what is called being "essential"). They use an indication function that counts for how many relationships a certain agent is essential. A more customary operationalization of intermediation rents is the measure "betweenness" (proposed by Freeman, 1979) that simply counts the number of shortest paths an agent lies on.

- Closure / closed triads $\#\{jk \in g : j, k \in N_i(g)\}$
  Another important aspect can be whether the neighbors of an agent are linked themselves, which is known as "network closure". As Burger and Buskens (2006) argue, there are cooperative contexts in the sense of Coleman (1988) where closing triads is beneficial (e.g. for building trust, see Buskens, 2002), while in "Burtian" contexts agents loose their advantageous (brokerage) position when two of their neighbors connect (see Burt, 1992).

- Number of a friend's friends $(l_j(g))_{j \in N_i(g)}$
  Goyal and Joshi (2006a) only consider this feature of networks in what they call "local spillover games". They present two examples where the number of friends of friends are considered beneficial ("provision of a pure public good" and "market sharing agreements") and two examples where these network statistics are considered as harmful ("free trade agreements among countries" and "friendship networks").

It is important to note that the different network statistics are not independent of each other. First, the k different network statistics of one agent may be highly correlated. Secondly, there are interrelations across the n agents, e.g. the sum of degrees cannot be odd.

Of course, there are many other aspects that can be important for specific applications. Solely to assess the centrality of an agent many other indices were developed besides closeness, betweenness and degree (e.g. Eigenvector centrality, see Bonacich, 1987, or the three centrality measures introduced by Friedkin, 1991). Moreover, the framework allows for general network statistics that are relevant for any agent, e.g. the number of links of the whole graph $l(g)$. Or it can be appropriate to define new network statistics for specific applications. For example, if there are two different characteristics of the agents, then one can split the degree into number of links to similar people and number of links to different people.

For this work, however, it suffices to consider the indices listed above. Degree and number of connections will be discussed in Section 2.2, distance-based statistics in Section 2.3 (both in this chapter). The model analyzed in the chapters 3 and 4 uses closeness, betweenness and degree (the three centrality measures proposed by Freeman, 1979). Finally, in the last chapter (Chapter 5), we will have a closer look at two examples where the number of a friend's friends is crucial.

2.1.3 Evaluation (of Network Statistics)

Once the relevant network statistics are specified, the question is how agents evaluate the (measured) qualities of certain graphs. Formally, for an agent i each graph can be represented by a vector $m_i(g) \in \mathbb{R}^k$, where all of these vectors form the feasible set, $F_i := \{y \in \mathbb{R}^k : \exists g : m_i(g) = y\}$. We assume for each agent i an evaluation function $U_i : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^k(: F_i \subseteq X)$ that expresses his preferences over the feasible set. Technically, this function's domain is larger than the feasible set, e.g. its convex hull, such that the function U_i can be assumed to be continuous and twice differentiable. So we write that an agent prefers a certain network if he prefers its statistics:[3]

$$u_i(g) \geq u_i(g') \iff U_i(m_i(g)) \geq U_i(m_i(g')).$$

When defining a certain model, one can describe an agent's evaluation of network statistics as assumptions on the evaluation function $U_i(\cdot)$. We will use the following categories of assumptions to specify different models. Consider an agent i, a set $X \subseteq \mathbb{R}^k$, and a vector $(x_1, x_2, ..., x_a, ..., x_k) \in X$.

A0 *Domain of U_i.*

The first – and most basic – assumption is the choice of the function $m_i(\cdot)$ that defines

[3]Without defining the network statistics, this is not restrictive.

the relevant network statistics for each agent.

A1 *Monotonicity* $U_i'(x_a) \geq (\leq) 0 \; \forall x_a$.

Interestingly, all of the network statistics listed in Subsection 2.1.2 can be beneficial in some contexts and harmful in others. So, a basic assumption is whether the evaluation function is increasing or decreasing in its arguments.

A2 *Additive separability* $U_i(x) = f_i^1(x_1) + ... + f_i^k(x_k)$.

It is very convenient to assume that the evaluation function is additively separable in its arguments: As the cross-derivatives are zero, this assumption uncouples the effects on utility coming from a change in one measure and the change (and level) in the other measures.

A3 *Curvature* $U_i''(x_a) \geq (\leq) 0$.

In many applications it is clear how the shape of the evaluation function should be modeled. For both beneficial and costly aspects, concave and convex functions might be appropriate. Concave benefits stand for decreasing marginal returns; convex benefits for increasing marginal returns. Concave costs may stem from the combination of fix costs and variable costs; convex costs represent the scarcity of resources (e.g. time).

A4 *Linearity* $U_i(x) = \alpha_0 + \alpha_1 x_1 + ... + \alpha_k x_k$, with $\alpha_a \in \mathbb{R}$.

Linear preferences satisfy additive separability and both convexity and concavity in each argument. In other words: the marginal rate of substitution is constant. The linearity assumption reduces the comparison of two networks $m_i(g) = x$ and $m_i(g') = y$ to the differences of their network statistics: $U(x) - U(y) = U(x - y)$. Although this assumption is standard in the literature, its only justification is computational ease.[4]

A5 *Homogeneity* $U_i(x) = f(x) \;\; \forall i \in N$.

It is an interesting question to ask how networks evolve when agents differ in their preferences (see, e.g., Galeotti et al., 2006). But it also complicates the analysis heavily. Homogeneity is a reasonable assumption if the emphasis is on the different contexts that influence everybody's choice, not on the difference between agents (as also argued in Burger and Buskens, 2006).

[4]We will also use this assumption in Chapter 3, but not without checking its importance in Section 2.3.

Throughout this work we will slowly increase the the number of assumptions on the evaluation function. Subsection 2.2.1 only uses A0, Subsection 2.2.2 uses A0-A1; Section 2.3 uses A0-A3 and A5; Chapter 3 uses A0-A5.

Conceptual Supplement: Microeconomic Theory of the Consumer

Since each graph is represented by a vector $m_i(g) \in \mathbb{R}^k$, the change of a network (from g to g') can be interpreted as a comparison of bundles of goods ($m_i(g)$ and $m_i(g')$). This perspective suggests to import concepts developed in the microeconomic theory of consumers. Among them are assumptions on preferences and the "trick" to treat indivisible goods (integer network statistics) as realizations of principally continuous dimensions.

For illustrative purposes, let $k = 2$, where there is one "good" X and one "bad" L and consider an agent i that has to decide between network g and network g'. He compares the two networks by their measured network statistics, $m_i(g) = (x, l)$ and $m_i(g) = (x', l')$ (with $x, x' \in X$ and $l, l' \in L$). Assuming additive separability, the evaluation of a point $(x, l) \in X \times L$ can be written as $U_i(x, l) = b(x) - c(l)$, where $b(\cdot)$ is considered the (differentiable) benefit function and $c(\cdot)$ the (differentiable) cost function. If additional assumptions on the curvature of the benefit function and cost function are reasonable, it can be handy to linearize the utility function in the following points $P := \{(x, l), (x', l'), (x, l'), (x', l)\}$. For instance, if agent i has a concave benefit function and a convex cost function, we can find a necessary and a sufficient condition on the marginal rate of substitution (MRS) for him to prefer g over g': $u_i(g) \geq u_i(g')$ if $MRS_i(x, l) \geq \frac{x'-x}{l'-l}$ and only if $MRS_i(x', l') \geq \frac{x'-x}{l'-l}$.

The MRS at the four points P allows such statements for any combination of concave and convex costs and benefits. To work with the marginal rate of substitution might be very handy, since $U_i(x, l) = b(x) - c(l)$ implies that the MRS for an agent i at a point (x, l) is simply $MRS_i(x, l) = \frac{c'(l)}{b'(x)}$. For the evaluation of these points we can also import well-known types of preferences. For example, let the agent's preferences be strictly convex and assume in addition that a Cobb-Douglas utility function can represent them: $U_i(x, l) = x^{\alpha}(-l)^{1-\alpha}$. Then his marginal rate of substitution is $MRS_i(x, l) = \frac{\alpha-1}{\alpha}\frac{x}{l}$. Thus, agent $i's$ evaluation is described by his elasticity of substitution.[5] More ideas follow this avenue (such as CES preferences, etc). However, there are also important differences between the choice of a consumption bundle from a budget set and the attempts to alter network structures, because the structure of interactions differs completely (as addressed in the next subsection). ♢

[5] Alternatively, one could also transform the utility function (by taking the logarithms) into an additive separated form.

2.1.4 Notions of Stability and Efficiency

Given a society (N, G, u), there are various ways to endogenize the network structure. We will restrict attention to one approach and mention some alternatives thereafter.

A strategic network game can be defined as a tupel $(N, \langle S_i \rangle, \langle \hat{u}_i \rangle)$ with the following specifications:

- The set of players is N as defined above.
- Following the idea of Myerson (1991), let an action be an announcement of requested links. A player's strategy space then can be defined as $S_i = \{0, 1\}^{n-1}$ such that a pure strategy s_i is a $(n-1)$-vector, $s_i = (s_{i1}, ..., s_{i(i-1)}, s_{i(i+1)}, ..., s_{in})$, where $s_{ij} = 1$ means that i requests a link to j and $s_{ij} = 0$ otherwise. The set of all strategy profiles is denoted by $S = S_1 \times ... \times S_n$, and an element of it is $s = (s_1, ..., s_n)$.
- The utility functions $\hat{u}_i : S \rightarrow \mathbb{R}$ stand for a player's preferences over the set of strategy profiles. These are induced by the original utility functions $u_i(g)$ and an outcome rule that maps strategy profiles into consequences $(r : S \rightarrow G)$, that is $\hat{u}_i(s) = u_i(r(s))$. We use the following outcome rule $g : S \rightarrow G$, where $g(s) = \{ij : s_{ij} = s_{ji} = 1\}$, that requires the request of both players to form a link.

Let $NE(u)$ denote the set of (pure) Nash equilibrium strategy profiles. A first approach is to consider the networks as stable if there is a corresponding Nash equilibrium. However, this turns out to be too weak a notion (e.g. the empty network is always stable in that sense). The issue with Nash stability is that networks are considered as stable even if there are two players who would both benefit from linking because it is also an equilibrium strategy not to request the link if the other player does not request it either. To account for this, a common refinement is pairwise Nash stability, sometimes also called pairwise equilibrium (see Goyal and Joshi, 2006a and Bloch and Jackson, 2006, among others).

Definition 2.1. *A network g is* ***pairwise Nash stable*** *(PNS) if there exists a Nash equilibrium in the corresponding link formation game that supports this network and no link will be added by two players, that is*

(i) $\exists s \in NE(u)$ *s.t.* $g(s) = g$ *and*

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

This notion is not free of criticism. First, there is an asymmetry of deviations, since it considers the addition of one link and the deletion of several links.[6] Secondly, a very basic deviation is not possible: replacing one link with another one. Buskens and Van de Rijt (2005) propose a refinement that accounts for these two problems. They introduce the definition of "initiative-proofness". Informally put, a strategy profile s is initiative-proof if there is no player who can make a proposition of rearranging his links (add some and/or delete some) without a negative response of one player who would have to form one link (see Buskens and Van de Rijt, 2005). Correspondingly, they define:

Definition 2.2. *A network g is* ***unilaterally stable*** *(US) if there exists a strategy profile s that is initiative proof and $g(s) = g$.*

In the context of cooperative games a simpler notion of stability was introduced by Jackson and Wolinsky (1996), that became the most used in strategic network formation.

Definition 2.3. *A network g is* ***pairwise stable*** *(PS) or just stable if no link will be added or cut (by two, respectively one player):*

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \backslash ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

Pairwise stability is a very basic notion that can be seen as minimal requirement for stability. In fact, for the notions introduced above, it holds that

$$[PS(u)] \overset{(1)}{\supseteq} [PNS(u)] \overset{(2)}{\supseteq} [US(u)], \tag{2.1}$$

where we denote by $[PS(u)]$, $[PNS(u)]$, $[US(u)]$ the sets of stable networks for a utility profile u. For (1) see Bloch and Jackson (2006); for (2) see Buskens and Van de Rijt (2005). Besides the two, there are other refinements, e.g. bilateral equilibrium (Goyal and Vega-Redondo, 2007), strict bilateral equilibrium (ibidem), or strong stability (Jackson and Van de Nouweland, 2005), allowing for coalitional deviations.

Despite its weakness, pairwise stability will often be sufficient to make a point. In this work, we will mostly work with this concept for three reasons. First, it is simple. Secondly, the results

[6] By a unilateral change of strategy a player can induce a network where some of his links are deleted. However, if no player is requesting a link to i, he cannot add any link by unilateral deviation. Condition (ii) allows for the addition of one such link but not more.

that exclude networks from being PS also hold for all stronger notions of stability. Thirdly, the refinements are not always able to exclude many networks, as we will see throughout the examples. When the set of stable networks is characterized for a specific setting, it is suggested to the reader not to interpret it as a prediction (of which networks will most likely emerge), but as a description of the candidates for reasonable predictions.

Working with the above notions of stability, there are some implicit assumptions attached, which shall not be hidden:

- We restrict attention to local actions: players can only influence the formation or severance of links they are involved in. Bloch and Jackson (2007) present an approach where this assumption is relaxed.
- We do not consider situations where players are incapable of denying links that somebody tries to form to them. Bala and Goyal (2000) introduce two such models of one-sided link formation.
- Decisions are made myopically. This means that agents consider the consequences of their actions on the current network structure, but do not anticipate the potential reactions of others. See Page (2004) for a concept of farsighted agents.
- The introduced game is static in principle. Aumann and Myerson (1988) propose a sequential game of link formation; Jackson and Watts (2001) present a dynamic model of myopic deviations.
- Since any player is assumed to have rational preferences on the set of networks, the change in utility by establishing a link equals the change by severing the link. In the framework of Snijders (2001) this assumption is relaxed, because it might also be reasonable that the cost to form a link is higher than the saving of severing an already established link (e.g. consider the presence of certain inertia).

While stability tries to answer which networks emerge based on individual preferences, efficiency addresses the evaluation of networks from a societal point of view. To formally capture efficiency, we use a welfare function $w : G \to \mathbb{R}$ that evaluates each graph. Welfare is typically, but not necessarily, only dependent on the $n-$vector of utility $u(g)$. If so, a basic property is monotonicity: a welfare function w satisfies monotonicity if $u_i(g) \geq u_i(g') \quad \forall i \in N \quad \Longrightarrow$

$w(g) \geq w(g')$. A special case of a monotonic welfare function is the utilitarian welfare function: $w^u(g) = \sum_{i \in N} u_i(g)$.

Definition 2.4. *A network g^* is called* **efficient** *with respect to the welfare function w if it is a welfare maximizing network, that is $w(g^*) \geq w(g) \ \ \forall g \in G$.*

Besides efficiency based on welfare, one can also consider Pareto efficiency: A network g is Pareto efficient if there is no network g' such that $u_i(g') \geq u_i(g) \ \ \forall i \in N$ and $u_i(g') > u_i(g)$ for some $i \in N$.

2.2 The Importance of Network Statistics

This section shall allow us to see how minimal assumptions on preferences can be sufficient to find characteristics of stable networks.

2.2.1 The Most Basic Case: Only Degree Matters

Consider a society (N, G, u) where agents only care about the number of relationships they maintain. In this model (M1) all relevant aspects of a network can be reduced to the degree: (A0) $m_i(g) = l_i(g)$ for all players.
The feasible set for one agent is $\{0, 1, ..., n-1\}$. For technical purposes let $X = [-1, n]$ and $U_i : [-1, n] \rightarrow \mathbb{R}$ be a differentiable function. U denotes a profile of n such functions.

The fact that only degree matters provides a lot of structure without strong (respectively any) assumptions on preferences. First of all, it allows us to establish existence.[7]

Proposition 2.1. *In a network formation game based on degree (M1), for any profile U there exists at least one stable network.*

All proofs of this chapter can be found in the Section 2.4.

In a situation of choice, each agent would choose the number of links ($l_i(g)$) that maximizes his utility ($g \in \operatorname{argmax} U_i(l_i(g))$). In a situation of strategic interaction, it is not clear that any player obtains the preferred number of links in equilibrium. The first reason is that not any distribution of degree is feasible in a network.[8] Secondly, the outcome of one player is still (partially) dependent on the action of others because it takes two to form a link.

[7] I am thankful to Nicolas Trotignon, who opened my eyes to the simple construction of a stable network.
[8] The feasible set was described by the theorem of Berge and Erdös (see, e.g., Diestel, 2005).

However, since foreign links are not part of the utility function, the equilibria can be partially characterized. For a graph g, let us partition the players into three (disjunct and exhaustive) groups A, B, and C:

- $A := \{i \in N : U_i(l_i(g) - 1) > U_i(l_i(g))\}$
- $B := \{i \in N : U_i(l_i(g) - 1) \leq U_i(l_i(g))$ and $U_i(l_i(g) + 1) > U_i(l_i(g))\}$
- $C := \{i \in N : U_i(l_i(g) - 1) \leq U_i(l_i(g))$ and $U_i(l_i(g) + 1) \leq U_i(l_i(g))\}$.

Players in A strictly prefer to have one link less, and players in B and C do not. The players in B strictly want to have (at least) one more link, the players in C do not. The players in C are necessarily close to a "local optimum" $U_i'(l^*) = 0$). Those players do not have an incentive to deviate. Players in A and B do, which leads to the following result.

Proposition 2.2. *In a network formation game based on degree (M1), if $g \in [PS]$ the following holds: $\forall i \in A$ it holds that $l_i(g^*) = 0$ and $\forall i \in B$ it holds that they are fully linked among themselves (and might be linked to some players in C). If there is a network where every agent belongs to group C, then this network is pairwise stable.*

The result follows straightforwardly from the notion of pairwise stability. This proposition helps characterize the equilibria a bit. If there are no players with optimal number of links ($U_i'(l_i(g^*) = 0)$), then only trivial networks are stable, which is called the dominant group architecture by Goyal and Joshi (2006a). The second part of the proposition states that if all players are close to a local optimum, the network is stable. This, however, does not imply that such a network is efficient – consider a stable network where some agent prefers to sever several of his links but not one. The refinement of pairwise Nash stability can rule out these equilibria.

Proposition 2.3. *In a network formation game based on degree (M1), the following holds: If $g \in [PNS]$, then $\forall i \in N$ it holds that $U_i(l_i(g)) \geq U_i(l')$ for any $l' \in \{0, 1, 2, ..., l_i(g)\}$.*

So the possibility to sever multiple links, guarantees that each player prefers this network over its subnetworks.[9]

The example of degree-based network formation shows that there are cases where the definition of a network statistic alone already sheds light into the set of stable and efficient networks.

[9] Unilateral Stability is not sufficient that all players are in the best of all subnetworks and supernetworks (a global optimum).

What makes this example so special is that the formation of a link between two agents has no effect on any other agent. Let us add another aspect to the model that incorporates indirect benefits.

2.2.2 Direct and Indirect Connections

Consider a society (N, G, u) where $k = 2$ and the most important network statistics are the number of links (degree) and the number of connections of a player: (A0) $m_i(g) = (\chi_i(g), l_i(g))$, where $\chi_i(g) := \#\{j \neq i : d_{ij}(g) < M\}$ for any player $i \in N$.

The feasible set for some player i is illustrated in Figure 2. Having degree of $l \geq 1$ corresponds to at least l and at most $n - 1$ connections.

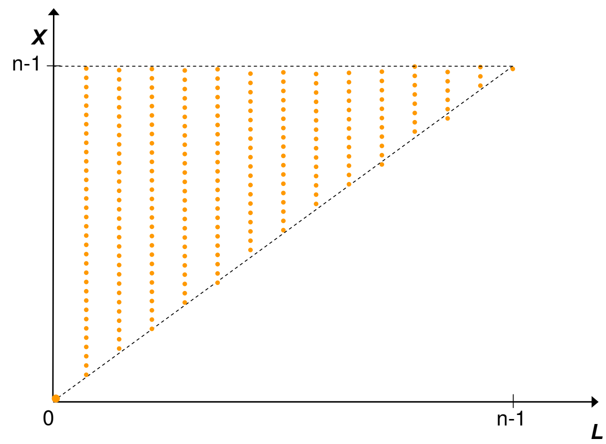

Figure 2: Set of feasible degree and number of connections for a player.

Let the domain of the evaluation function be $X = [0, n-1]^2$ and assume that any agent evaluates degree as costly, where connections are beneficial: (A1) the evaluation function of any agent $U_i(\cdot, \cdot)$ is strictly increasing in the first and strictly decreasing in the second argument. We refer to this model as M2. (A1) determines the preference relation on all networks that have the same number of links (vertically) or the same number of connections (horizontally). This leads to the following proposition.

Proposition 2.4. *In model M2, the following holds:*
(i) All stable networks are minimal (or empty).
(ii) Any Pareto efficient network is minimal or empty.

Both parts of the proposition follow the intuition that non-critical links are dispensable: causing costs (degree), but with a marginal benefit of zero since cutting a non-critical link does not decrease the number of connections (e.g. non-critical links lead to redundant information). In that sense the result can be interpreted as agents rationalizing their local network structure.

The models M1 and M2 are not chosen for their substance, but serve as examples showing that the definition of the relevant network statistics is able to drive results, without much structure on the evaluation function. The next section illustrates how (the shape of) the evaluation function affects results.

2.3 The Importance of Evaluation

Let us have a closer look at a specific example. We fix a beneficial network statistic – namely, closeness – and analyze how decreasing versus increasing marginal returns affect the stable networks. Finally, we compare the results to a slightly different network statistic.

2.3.1 A Closeness Model

In this section we consider a society (N, G, u) with $k = 2$, where the network statistics are closeness and number of links. Closeness incorporates the idea that an agent is considered as "central" in a social network if his distance to other agents is small (Sabidussi, 1966). In the literature on centrality it is standard to normalize an index between 0 and 1. We follow this convention by defining **closeness** of an agent i as the following affine transformation of his average distance $\frac{\sum_{j \in N} d_{ij}(g)}{n-1}$:

$$Close_i(g) = \frac{M}{M-1} - \frac{\sum_{j \in N} d_{ij}(g)}{(M-1)(n-1)}.$$

Then $Close_i(g) = 0$ for isolates, while $Close_i(g) = 1$ for an agent who is directly connected to all others in the network.

There is another operationalization which is more prominent in the literature: the closeness definition according to Freeman (1979),[10] $FrClose_i(g) := \frac{n-1}{\sum_{j \in N} d_{ij}(g)}$. While Freeman's version (inverse distances) is much more customary, our closeness definition (reverse distances) more naturally separates the measurement of a network statistic from its evaluation by keeping the

[10] In the original version, Freeman closeness is only defined for connected graphs. The extension to all networks works with the definition of the distance of unconnected players (as M).

units (as also argued in Valente and Foreman, 1998).[11]

This section uses three assumptions on the evaluation functions:

A0,A1 The agents make linking decisions in respect to their degree and their closeness, where closeness is beneficial and links are costly.

A2,A3 Each player's preferences are quasi-linear in degree. Although concave and convex cost functions are reasonable, we will focus on different shapes of the benefit function (which can be assumed to absorb the curvature of the cost function). For the benefit function we will distinguish three cases: concave shape, convex shape and linear shape; which stand for decreasing, increasing and constant marginal returns.

A5 The players are homogeneous in respect to preferences.

By introducing a (non-decreasing, twice differentiable) benefit function $b : [0, 1] \to \mathbb{R}$ one can put all assumptions together to what we denote as **closeness model**:

(M3) Each agent $i \in N$ decides about links according to preferences that can be represented by $u_i(g) = b(Close_i(g)) - \bar{c}l_i(g)$.

Remark 2.3.1 (Similar models)**.** *In the literature similar models to the closeness model are discussed. First, there is the connections model introduced in Jackson and Wolinsky (1996). We formally define this model in Subsection 5.1.2. A special case of it is the symmetric connections model:* $u_i^{SCO}(g) = \sum_{j \in N \setminus \{i\}} \delta^{d_{ij}(g)} - l_i(g)c$*, where* $\delta \in (0, 1)$ *and* $M = \infty$*. Another model stems from Fabrikant et al. (2003) and is adapted to a setting of bilateral link formation by Corbo and Parkes (2005):* $u_i^{Fabrikant}(g) = -\sum_{j \in N \setminus \{i\}} d_{ij}(g) - l_i(g)c$*, where* $M = \infty$*. Variations of the parameter* M *in this model are studied by Brandes et al. (2008) (but only in a setting of unilateral link formation).*

Jackson (2008) summarizes such models as "distance-based utility models" in the following way: $u_i^{distance-based}(g) = \tilde{b}(\sum_{j \in C_i(g) \setminus \{i\}} d_{ij}(g)) - l_i(g)c$*, where* $C_i(g)$ *is the set of players in* i*'s component and* $\tilde{b}$ *is some decreasing function. Set* $\tilde{b}(k) = -k$ *and* $\tilde{b}(k) = \delta^k$ *to see that the symmetric connections model and the model of Fabrikant are special cases of distance-based utility models (at least on the domain of connected networks).*

The closeness model (M3) introduced in this subsection does not fall into this class. For example,

[11] Without fixing an evaluation function, this choice does not restrict generality: By taking the following convex benefit function $f(x) = [M - x(M - 1)]^{-1}$, the benefits of closeness are equivalent to Freeman-closeness (with linear evaluation), because $f(Close_i(g)) = FrClose_i(g)$.

using a convex evaluation to get Freeman closeness, cannot be represented by a distance-based utility function. However, consider a linear version of the closeness model, i.e. $b(k) = k$, *and set* $M = \infty$: $u_i^{linear}(g) = Close_i(g) - \bar{c}l_i(g) = \frac{M}{M-1} - \frac{\sum_{j \in N} d_{ij}(g)}{(M-1)(n-1)} - \bar{c}l_i(g)$. *In this linear closeness model the benefits are just an affine linear transformation of the Fabrikant model, while for the costs* $\bar{c}$ *and* c *anyhow any possible value is considered. As a consequence the linear closeness model and the Fabrikant model are equivalent when analyzing which networks are stable and which networks are efficient. They are not equivalent in absolute values of utility, i.e. when computing ratios of the welfare of different networks, as Corbo and Parkes (2005) do. Virtually any result of this subsection can be found (in their corresponding formulation) in the literature on similar models: for the symmetric connections model see Jackson and Wolinsky (1996), for the Fabrikant model see Corbo and Parkes (2005), and for distance-based utility see Jackson (2008). The point here is not the substance of the results, but their robustness according to specification details.*

2.3.2 Increasing versus Decreasing Marginal Returns

To have a shorter notation, we substitute two often needed units of closeness:

1. $T1 := \frac{1}{(n-1)(M-1)}$. This is the smallest possible change in closeness, as it corresponds to a shift in distance of 1. It occurs when two players who were at distance two form a link and only the distance between these two changes, e.g. because they are already directly linked to everybody else.

2. $T2 := \frac{1}{(n-1)}$. This is the change in closeness of a player that links with an isolate. As his distance shifts from M to 1, his closeness increases by $\frac{M-1}{(n-1)(M-1)} = T2$.

The following results provide two characteristics of all stable networks.

Proposition 2.5. *In a closeness model (M3) with linear costs and* **concave** *benefits, the following holds:*
(i) If $\bar{c} < b(1) - b(1 - T2)$, *all stable graphs are connected.*
(ii) If $\bar{c} > b(T2) - b(0)$, *no stable graph exhibits loose ends.*

The intuition behind the result is that the thresholds of (i) and (ii) are just the minimal and the maximal marginal benefit that a link to an isolated node can mean.[12] If the benefit function

[12] That is, the threshold in (ii) is the marginal benefit of a new link in the empty graph $\beta_i^{ij}(g^{\emptyset})$; and the threshold in (i) is the marginal benefit that cutting a link means to the center of a star $\beta_c^{ci}(g^{\star})$.

is not concave but convex, these two thresholds just switch roles, as stated by the following proposition.

Proposition 2.6. *In a closeness model (M3) with linear costs and **convex** benefits, the following holds:*
(i) If $\bar{c} < b(T2) - b(0)$, all stable graphs are connected.
(ii) If $\bar{c} > b(1) - b(1 - T2)$, no stable graph exhibits loose ends.

To prove existence of a stable network the assumption A0 is not (obviously) sufficient. With the assumption of a convex benefit function, there is a very simple – admittedly not a very elegant – way of proving existence.

Proposition 2.7. *In a closeness model (M3) with linear costs and **convex** benefits, the following holds: for any marginal costs $\bar{c} \in (0, \infty)$ (parameter value) there exists at least one stable network.*

Figure 3 shows the idea of the proof: For any marginal cost, we can give a trivial example for a pairwise stable network. For low costs the complete graph, for high costs the empty network, and in the medium range the star. If the benefit function is not convex but concave, the thresholds shift such that these trivial graphs do not span the whole parameter space.

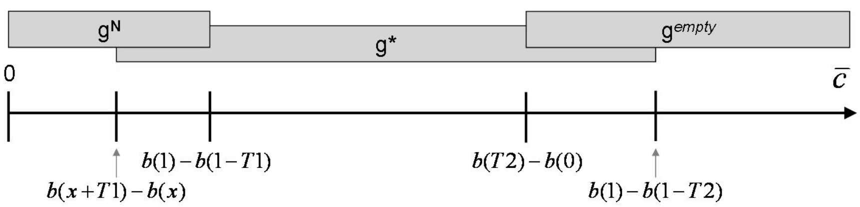

Figure 3: Existence of stable networks for convex benefits.

Remark 2.3.2. *Figure 3 also contains the thresholds for Prop. 2.6 (on the right). In the case of concave benefits these two thresholds not only switch positions, but also switch their roles as stated in Prop. 2.5.*

Pairwise Nash Stability

In the closeness model, (PNS) is not always a proper refinement of (PS):[13]

[13] The corresponding result in the Fabrikant model was already stated and proven in Corbo and Parkes (2005).

Proposition 2.8. *In a closeness model (M3) with linear costs and* **concave** *benefits, the set of pairwise stable networks $[PS]$ and the set of pairwise Nash stable networks $[PNS]$ coincide.*

One direction of the result follows directly from the definitions: $[PNS] \subseteq [PS]$. The other direction shall be briefly discussed. Calvó-Armengol and Ilkiliç (2007) show that $[PNS]$ and $[PS]$ coincide if the utility function $u(\cdot)$ satisfies a property called "α-convexity in own current links".[14] Moreover, if costs and benefits are additively separable and marginal costs are constant, it is enough to show that the benefit function satisfies $\forall i \in N, \forall g \in G, \forall l \subseteq L_i(g)$,

$$\beta_i^l(g) \geq \sum_{ij \in l} \beta_i^{ij}(g). \tag{2.2}$$

In essence, the condition says that the deletion of some of player i's links hurts him weakly more than the separate deletion of these links, one at the time.[15]

To show that condition 2.2 holds in a closeness model with concave benefits, we need two steps: one step shows that the shift in closeness on the lefthandside of 2.2 cannot be smaller than the shift in closeness on the righthandside. The other step exploits decreasing marginal returns (which guarantee, roughly, that multiple small reductions of closeness are not evaluated as severely as one big reduction).

The proof of Prop. 2.8 clarifies the role of network statistic and curvature of the evaluation function for the stability of networks: it is a genuine feature of the network statistic that cutting one link at a time shifts closeness (weakly) less than cutting them at once. The concave evaluation (of closeness benefits) is just used to preserve this feature.

2.3.3 Constant Marginal Returns

Let us compare constant marginal returns of closeness to a convex evaluation according to Freeman (1979) and to a similar distance-based network statistic. In addition to the assumptions before (M3), we assume preferences to be linear (A4). Without restriction of generality, we represent any player's preferences by $u_i^{linear}(g) = Close_i(g) - \bar{c}l_i(g)$, which is referred to as the linear closeness model (M3'). Note that by taking the id-function as benefit function,

[14] The label "convexity" can be misleading. We discuss this property in Section 5.1.1.

[15] Recall that (for $l \subset g$) $\beta_i^l(g) = b_i(Close_i(g)) - b_i(Close_i(g \backslash l))$ denotes the marginal benefit that the deletion of the links (in l) means to some player i. For constant marginal costs, it is intuitive that this is the condition requiring that deviations of cutting more than one link are only utility improving if deviations of cutting just one link are, which is sufficient for $[PS] = [PNS]$.

we mingle in this subsection what we distinguished before: the closeness of an agent and his benefit derived from closeness.

The first proposition is a corollary of Prop. 2.5 and 2.6, as the linear benefit function is a special case of both, concave and convex benefits functions.

Proposition 2.9. *Let again* $T2 := \frac{1}{(n-1)}$. *In the linear closeness model (M3'), the following holds:*
(i) For $\bar{c} < T2$, *all stable graphs are connected.*
(ii) For $\bar{c} > T2$, *no stable graph exhibits loose ends.*

Observe that in this result two thresholds (of the benefit function before) coincide: $b(1) - b(1-T2) = b(T2) - b(0) = T2$. This is also true for the next result.

Proposition 2.10. *Let again* $T1 := \frac{1}{(n-1)(M-1)}$. *In the linear closeness model (M3'), the following holds:*
(i) For $\bar{c} < T1$, *the unique stable network is the complete network.*
(ii) $T1 \leq \bar{c} \leq T2$, *a star network is stable, but not necessarily unique.*

Comparison between Linear Closeness Model and Connections Model

In the famous example of the symmetric connections model, basically the following benefit is used: $Connections_i(g) = \sum_{j \in N \setminus \{i\}} \delta^{d_{ij}(g)}$, where $\delta \in (0, 1)$.[16] So every reachable agent is of value but this diminishes with distance. Like in the closeness model, agents gain from short paths to other nodes. Consistently, Borgatti and Everett (2006) list the benefits of the connections model among the "closeness-like" centrality indices. But there is also a difference: In the connections model agents benefit from having many nodes close to them; while in the closeness model agents benefit from having a small average distance.

While the motivation of the two models is similar, the results turn out to be almost identical.

> Observe first that Prop. 2.9 and Prop. 2.10 correspond directly to the results of the connections model (see Jackson and Wolinsky, 1996), where $T1 \hat{=} \delta - \delta^2$ and $T2 \hat{=} \delta$.

For n not too big, a computer can enumerate all networks and check for stability.[17] We did this for $n = 8$ with the connections model (taking $\delta = 0.5$ and $\delta = 0.8$), and for the closeness

[16] By convention, here $M = \infty$ (see Jackson and Wolinsky, 1996).
[17] I thank Vincent Buskens for programming the routines to find all the stable networks for the various centrality measures. An extensive description of this method can be found in Subsection 3.2.2.

model once with the convex benefit function according to Freeman and once taking the linear closeness model (with $M = n$).

For $n = 8$ there are 12'346 non-isomorphic graphs (different network architectures). In the linear closeness model only 45 of them are stable for some parameter range (greater than 0).[18] As presented in Table 1, those 45 networks are not identical to the 63 stable networks with convex benefit function (Freeman) but overlap to some extent. The stable networks of the linear closeness model and the connections model overlap more heavily.

Table 1: Stable networks in the linear closeness model and related models for n=8.

Number of stable networks (for some cost range*)	Total	Also stable in linear closeness model
Freeman closeness	63	29
Connections $\delta = 0.5$	29	26
Connections $\delta = 0.8$	45	45

All of the above models are driven by similar linking behavior which we call "closeness-type" incentives: there is high incentive to link to agents who are at high distance (including those in other components) and there is low incentive to keep links that do not shorten some paths significantly. Interestingly, the differences within these models stem from specification details – be it increasing (instead of constant) marginal returns or level of decay – rather than from the choice of the network statistic (connections vs. closeness).

How the stable networks look like in the linear closeness model is addressed in Section 3.3.1. The lesson learned in this section is that assumptions on the curvature may strongly effect the resulting networks but not in an arbitrary way. The assumption of linearity (A4) allows us to state results in a easy and direct manner (since then the change of network statistics are proportional to the change in utility). When assuming linearity in the next sections, it should be kept in mind how relaxations of this assumption can affect the result.

2.4 Proofs of Chapter 2

Proof of Prop. 2.1. We construct a stable network. Start with the network $g^{\emptyset} \hat{=} g_0$. If $\exists i, j$ such that $u_i(g_0 \cup ij) > u_i(g_0)$ *and* $u_j(g_0 \cup ij) \geq u_j(g_0)$, then let $g_1 \hat{=} g_0 \cup ij$ for any of those ij's. Repeat the procedure for g_t to get g_{t+1}. Because there are finitely many links, at some point this

[18]*That is: we did not count the networks which are "stable" for only one point in the parameter space, e.g. the networks which are only stable if $\bar{c} = T1$.

process will stop: Let g_T be the network where $\nexists i, j$ such that $u_i(g_0 \cup ij) > u_i(g_0)$ *and* $u_j(g_0 \cup ij) \geq u_j(g_0)$. g_T clearly satisfies condition (ii) of pairwise stability. It remains to show that it also satisfies condition (i). Suppose not, then $\exists i, j$ such that $U_i(l_i(g_T) \setminus ij) > U_i(l_i(g_T))$. This implies that i strictly prefers $l_i(g_T) - 1$ links over $l_i(g_T)$ links which stands in contradiction to the procedure. □

Proof of Prop. 2.2. The result follows straightforwardly. (PS) requires the following, where d_i stands for $i's$ degree:
(A) $\forall i \in N : d_i > 0$, $U_i(d_i - 1) \leq U_i(d_i)$ and
(B) $\forall i \in N : U_i(d_i + 1) > U_i(d_i)$ it holds that they are fully linked to all $\forall i \in N : U_i(d_i + 1) \geq U_i(d_i)$. Moreover, for all players in group C, the two conditions of pairwise stability are satisfied. □

Proof of Prop. 2.3. This observation follows directly from condition (i) of pairwise Nash stability. Consider $l' \in \{0, 1, ..., l_i(g) - 1\}$. For any $s : g(s) = g$, it holds that $s \notin NE(u)$ because requiring l' links is a better response for i. □

Proof of Prop. 2.4. (i) Let g be a non-empty network that is not minimal. Then there exists a player who can cut a link without decreasing the size of his component. By (A1) his utility would increase, precluding PS.
(ii) Let g be a non-empty network that is not minimal. Then there exists a link ij that can be deleted without increasing the number of components. Severing that link ($g' = g \setminus ij$) implies that no agent's utility has decreased, some players' increased. Thus, g' Pareto dominates g. □

Some Remarks to the Proofs of Section 2.3

Formally, $\forall x, x', \Delta > 0$ a concave benefit function implies $b(x + \Delta) - b(x) \geq b(x' + \Delta) - b(x')$ whenever $x \leq x'$ (by the mean value theorem). Convexity implies increasing marginal returns: just let $x' \leq x$. **Marginal costs** $\bar{c}$ are constant and serve as the parameter for our model. Marginal benefits depend on the network g and on the shape of the benefit function. Let

$\beta_i^{ij}(g)$ denote the **marginal benefit** that link ij (either added or cut) means to player i in graph g. That is, $\beta_i^{ij}(g) := b(Close_i(g \cup ij)) - b(Close_i(g \setminus ij))$. When players make linking decisions, they compare marginal costs and marginal benefits: in graph g player i is eager to form a link to j $(ij \notin g)$ iff $\beta_i^{ij}(g) > \bar{c}$ and i wants to cut a link with k $(ik \in g)$ iff $\beta_i^{ik}(g) < \bar{c}$.[19]

Proof of Prop. 2.5. (i) Take any unconnected graph g. Take any player i and let $Close_i(g) =: x$. Linking with somebody of another component leads to a shift in closeness of at least *T2*. Because $x + T2 \leq 1$ and $b(\cdot)$ concave, it holds that $b(x + T2) - b(x) \geq b(1) - b(1 - T2)$. By assumption the marginal costs are lower, such that i wants to form this link. As in any unconnected graph, there exist two players who are not connected; they will alter the network structure, which makes g unstable.

(ii) Take any network g with at least one pendant and let i be his (only) neighbor. Denote $Close_i(g) =: x$. Cutting the link to the pendant means a shift in closeness of *T2*. Because $x \geq T2$ and $b(\cdot)$ concave, it holds that $b(x) - b(x - T2) \leq b(T2) - b(0)$. (By assumption the marginal costs are higher. Therefore, i will cut the link, which makes g unstable.) $\square$

The proof of Prop. 2.6 is analogue to the proof of 2.5.

Proof of Prop. 2.7. To show that for any marginal costs $\bar{c} \in (0, \infty)$ there exists a stable network, we take for low costs the complete graph, for high costs the empty network, and in the medium range the star. It is easy to verify that:

- The complete graph is stable if $\bar{c} \leq \beta_i^{ij}(g^N) = b(1) - b(1 - T1)$. Remember that *T1* is the shift in closeness when distance increases by 1.
- The empty network is stable if $\bar{c} \geq \beta_i^{ij}(g^{empty}) = b(T2) - b(0)$.
- A star is stable if $b(x + T1) - b(x) \leq \bar{c} \leq \min\{b(1) - b(1 - T2), b(x) - b(0)\}$, where $x := \frac{M}{M-1} - \frac{2n-3}{(M-1)(n-1)}$ is the closeness of a peripheral player (pendant). To verify the result, note that this condition precludes all possible deviations: (a) no peripheral players add a link $\bar{c} \geq b(x+T1) - b(x)$; and (b) the center does not cut a link $\bar{c} \leq b(1) - b(1-T2)$; and (c) no peripheral player cuts a link $\bar{c} \leq b(x) - b(0)$.

[19]When marginal benefits are equal to marginal costs, the player is indifferent. In this case he does not cut the link, respectively does not initiate the new link (but agrees when asked).

To prove existence for any marginal cost $\bar{c}$, it remains to show that 1. the lower bound of the star is below the upper bound of the complete network and 2. the upper bound of the star is above the lower bound of the empty network (see Figure 1).
1. $b(x+T1)-b(x) \leq b(1)-b(1-T1)$ follows from $x+T1 \leq 1$ and convexity of $b(\cdot)$. And 2. $b(1)-b(1-T2) \geq b(T2)-b(0)$ follows from convexity of $b(\cdot)$; and $b(x)-b(0) \geq b(T2)-b(0)$ follows from $b(\cdot)$ increasing and $x \geq T2$. □

Proof of Prop. 2.8. Calvó-Armengol and Ilkiliç (2007) show that [PNS] and [PS] coincide if the utility function $u(\cdot)$ is $\alpha - convex$ in its current links. Moreover, if costs and benefits are additively separable and marginal costs are constant, it is enough to show that the benefit function satisfies $\forall i \in N, \forall g \in G, \forall l \subseteq L_i(g)$,

$$\beta_i^l(g) \geq \sum_{ij \in l} \beta_i^{ij}(g),$$

where $\beta_i^l(g) = b_i(Close_i(g)) - b_i(Close_i(g \setminus l))$ denotes the marginal benefit that the deletion of the links (in l) means to some player i. Because of our homogeneity assumption, we can fix a player i without restricting the generality. So, we have to show that $\forall g \in G, \forall l \subseteq L_i(g)$ it holds that

$$b(Close_i(g)) - b(Close_i(g \setminus l)) \geq \sum_{ij \in l} [b(Close_i(g)) - b(Close_i(g \setminus ij))] \tag{2.3}$$

In words: the deletion of some of player i's links hurts him weakly more than the separate deletion of these links, one at the time.

Note the following property of concave functions: for any increasing concave function $f : \mathbb{R} \to \mathbb{R}$ it holds that $\forall x, \delta_1, ..., \delta_T, \Delta \in \mathbb{R}_{++},\ f(x) - f(x-\Delta) \geq \sum_{t=1}^{T} [f(x) - f(x-\delta_t)]$ if $\sum_{t=1}^{T} \delta_t \leq \Delta$.

Let's fix a graph g and a set of links $l \subset L_i(g)$. We substitute $x \equiv Close_i(g)$, $\Delta \equiv Close_i(g) - Close_i(g \setminus l)$, and for $t = 1, ..., T$, $\delta_t \equiv Close_i(g) - Close_i(g \setminus it)$, where every pair in l is renamed (in arbitrary order) as *i1, i2,. . . ,iT*. Using this substitution we learn from the result above that for the (concave and increasing) benefit function $b(\cdot)$, $b(Close_i(g)) - b(Close_i(g \setminus l)) \geq \sum_{ij \in l} [b(Close_i(g)) - b(Close_i(g \setminus ij))]$ is implied by

$$\sum_{ij \in l} (Close_i(g) - Close_i(g \setminus ij)) \leq Close_i(g) - Close_i(g \setminus l) \tag{2.4}$$

So, (2.4) is a sufficient condition for (2.3), for this particular combination of g and l. As we have to show that the statement (2.3) holds for any graph $g \in G$ and set of links $\forall l \subseteq L_i(g)$, we can use the substitution each time and check the sufficient condition 2.4. So to proof the result, it remains to show for $\forall g \in G, \forall l \subseteq L_i(g)$ statement (2.4) holds.

By using the definition of closeness and after straightforward simplifications, (2.4) can be rewritten as

$$\sum_{j \in N} d_{ij}(g \setminus l) - \sum_{j \in N} d_{ij}(g) \geq \sum_{ij \in l} (\sum_{j \in N} d_{ij}(g \setminus ij) - \sum_{j \in N} d_{ij}(g)). \tag{2.5}$$

We define $\kappa_i^l(g) := \{k \in N : d_{ik}(g) < d_{ik}(g \setminus l)\}$ for $l \subseteq g$ and for the ease of notation we write it afterwards as κ. We define $\bar{\kappa}_i^l(g) := \bigcup_{ij \in l} \kappa_i^{ij}(g)$ and write it for the ease of notation as $\bar{\kappa}$. κ is the set of players whose distance to i increases when the l links are cut from g. $\bar{\kappa}$ is the union of players whose distance to i increases when one of the links in l is cut.

Now we can transform condition (2.5) by using the kappa sets (as the summation over all j in N that are not in any kappa set, cancels out).

$$\begin{aligned} \sum_{k \in \kappa} d_{ik}(g \setminus l) - \sum_{k \in \kappa} d_{ik}(g) &\geq \sum_{ij \in l} \left(\sum_{k \in \kappa_i^{ij}(g)} d_{ik}(g \setminus ij) - \sum_{k \in \kappa_i^{ij}(g)} d_{ik}(g) \right) \\ \sum_{k \in \kappa} [d_{ik}(g \setminus l) - d_{ik}(g)] &\geq \sum_{ij \in l} \left(\sum_{k \in \kappa_i^{ij}(g)} [d_{ik}(g \setminus ij) - d_{ik}(g)] \right) \end{aligned} \tag{2.6}$$

Note three properties of the distances in graphs (which are also used in Calvó-Armengol and Ilkiliç, 2007).

- Note 1: $d_{ik}(g \setminus l) \geq d_{ik}(g \setminus ij)\ \forall ij \in l$.
- Note 2: $\bar{\kappa} \subseteq \kappa$.
- Note 3: $\kappa_i^{ij}(g) \bigcap \kappa_i^{ih}(g) = \emptyset\ \forall ij \neq ih \in l \subseteq L_i(g)$.

By note 2 we can split the sum of the LHS in (2.6), and we switch the summation on the RHS:

$$\begin{aligned} \sum_{k\in\bar{\kappa}}[d_{ik}(g\setminus l)-d_{ik}(g)]+\sum_{k\in\kappa\setminus\bar{\kappa}}[d_{ik}(g\setminus l)-d_{ik}(g)] &\geq \sum_{k\in\bar{\kappa}}\left(\sum_{ij\in l}[d_{ik}(g\setminus ij)-d_{ik}(g)]\right)\\ \sum_{k\in\kappa\setminus\bar{\kappa}}[d_{ik}(g\setminus l)-d_{ik}(g)] &\geq \sum_{k\in\bar{\kappa}}(-[d_{ik}(g\setminus l)-d_{ik}(g)]) \end{aligned} \quad (2.7)$$

By considering note 3 and note 1, the RHS is non-positive. So, (2.7) clearly holds.[20] □

Proof of Prop. 2.10. Remember that *T1* is the shift in closeness when distances shift by 1. (i) The minimal increase in benefit that a new link can lead to for both its owners is $T1$; because a new link reduces at least the distance to the other player from 2 to 1. So, for $\bar{c} < T1$, it follows immediately that nobody wants to cut a link in any graph (stability of complete graph) and any two players who are not directly linked will add a link (uniqueness).
(ii) Shown in proof of Prop. 2.7. □

[20] Hence, in this model deviations of cutting more than one link are only promising if deviations of cutting just one link are. Note that the result $[PS] = [PNS]$ together with the definition that $[PNS] = [NS] \cap [PS]$ imply that $[PS] \subseteq [NS]$.

3 The Centrality Model

This section extensively studies the stable and emerging networks in one specific model where agents try to optimize their centrality.

3.1 Introduction

3.1.1 Motivation and Research Questions

The examination of beneficial network positions is as old as social network analysis itself (see, e.g., Wasserman and Faust, 1994). But, until recently, the question has not been asked how incentive for central network positions affect the network structure. We introduce a model, in which agents strive for closeness and betweenness (centrality), while links are costly.

Three motivations justify such a model. First, it complements the theory of centrality that originally measured the effect of network positions on individual opportunities, but not the effect of individual behavior on network structure. Secondly, the centrality indices are based on network statistics that are relevant in many different applications – from ancient marriages (Padgett and Ansell, 1993) to R&D collaborations (Walker et al., 1997). In the same manner incentives for central positions are not restricted to single applications, but represent a general type of behavior in building networks. Third, there is empirical support for centrality being beneficial, e.g. Song et al. (2007) find that the centrality of a work unit has a positive impact on its creativity. But regardless of the empirical validity of centrality indices, there is justification to study network formation based on centrality incentives as long as there are researchers and businessmen who claim that central positions are desirable. In fact, this claim becomes more and more popular in the practice of business consulting, as argued in Section 1.1.

To cover the third aspect it is worthwhile to incorporate a centrality index that is well known. We chose the three indices based on Freeman (1979): degree centrality, closeness centrality and betweenness centrality. By this choice we cover the three most studied types of centrality measures according to the typology of Borgatti and Everett (2006).

This model is supposed to capture two types of linking incentives.[21] Closeness stands for all incentives to access resources (information and support) by having many other agents in close reach. Variants of closeness were defined in Subsection 2.3.1. Betweenness stands for

[21] We consider an environment where maintaining links is costly. Thus, degree centrality leading to costs (and benefits) of direct links is not considered as a goal in its own right.

the intermediation rents that stem from being a broker for others. Of course, there are other indices to measure the intermediation rents. Depending on the application it might be more appropriate to assume that only those players are getting intermediation rents who are essential for a pair of players.[22] A second shortcoming is that betweenness does not account for the length of a path (as we will see in the definition). For instance, if a player is the only one between a pair of agents his betweenness rent is the same as if there are 5 agents in a row that are on the only shortest path. These two variations were studied by Goyal and Vega-Redondo (2007). In their utility function they do not only incorporate incentives for intermediation rents, but also incentives to avoid being brokered as well as incentives to be connected. In contrast, the model studied here shall allow us to decouple effects of incentives for connections from incentives for intermediation rents.[23]

This is not the first work to analyze network formation based on centrality incentives. Rogers (2006) models the formation of weighted graphs using an index of social influence (Bonacich index). The beneficial aspects in the models of Buskens and Van de Rijt (2008), Goyal and Vega-Redondo (2007) and Jackson and Wolinsky (1996) can also be interpreted as measures of centrality. What has not been done in the literature is (a) to *contrast* and (b) to *combine* the dynamics of "closeness-type" incentives to the dynamics of "betweenness-type" incentives. Considering the latter point, there is hardly any research on the interplay between different types of incentives to predict network formation processes, although it is likely that multiple incentives are simultaneously important.[24]

3.1.2 (A Centrality) Model

Consider a society (N, G, u). Closeness is formally introduced in Section 2.3: $CLOSE_i(g) = \frac{M}{M-1} - \frac{\sum_{j \in N} d_{ij}(g)}{(M-1)(n-1)}$. Its idea reaches back to the origins of social network analysis (Sabidussi, 1966). Dekker et al. (2003) argue that closeness increases accuracy of information. Song et al. (2007) provide empirical evidence for the importance of closeness for the knowledge processing of organizational units. Moreover, in the study of Powell et al. (1996) experienced firms are

[22] Being essential here means that there is no path between the pair that the focal player cannot block.

[23] Although this analysis is sometimes rather a thought experiment than an empirical model. Especially, for λ close to 1.

[24] In fact, it seems difficult to find contexts where only one type of centrality is significant. For example, the Medici's position in the marriage network was important for their trading abilities (see Padgett and Ansell, 1993). Here betweenness centrality is stressed, but closeness should not be ignored. At least for agents with low betweenness, it is important to be close to others.

likely to occupy positions with high closeness.

Freeman (1979) clarifies that closeness measures one aspect of centrality, while it cannot sufficiently capture others. Some agents exhibit a mediating role between other agents, which can be beneficial for them. Burt (1992) emphasizes this idea by the term "tertius gaudens." To measure the brokerage role of a certain agent he not only proposes some new measures, but also employs betweenness centrality (see Burt, 2002). Betweenness was introduced by Freeman (1979) and was shown to be beneficial in many studies thereafter (e.g. Song et al., 2007).

The betweenness of an agent i is proportional to the number of pairs j and k for whom i lies on the shortest path. If there are more than one shortest paths between j and k, the fraction of shortest paths going through i is used. Formally,

$$BETW_i(g) = \frac{2}{(n-1)(n-2)} \sum_{j<k(j\neq i,k\neq i)} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}, \tag{3.1}$$

where $\tau_{jk}(g)$ is the number of shortest paths between j and k, and $\tau^i_{jk}(g)$ indicates the number of shortest paths between j and k that go through i; the fraction $\frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}$ is replaced by zero, when $\tau_{jk}(g) = 0$. The constant before the fraction normalizes betweenness to be between zero (an agent is on no shortest path between two other agents) and one (an agent is on all shortest paths).

Besides closeness and betweenness, we also incorporate a player's degree $l_i(g)$, as in the models (M1,M2,M3) before. Maintaining links is the source of costs in the network. On the other hand, degree can also be interpreted as a measure of centrality (see Freeman, 1979). We assume in our model that the costs of maintaining relationships exceed the benefits that are restricted to direct contacts such that the net benefit of degree is negative. Without this assumption every agent wants to be directly linked to every other agent, independently of the network structure.

Note that the network statistics closeness, betweenness and degree are interrelated. Trivially, an agent without any link, must have closeness and betweenness equal to zero. Such interdependencies already imply a trade-off between high closeness, high betweenness and low costs: while it is possible to reach two goals – having (a) high closeness and high betweenness (e.g. center of a star k has maximal closeness, maximal betweenness $CLOSE_k(g) = BETW_k(g) = 1$, but also maximal tie costs) or (b) having high closeness and low costs (e.g. a peripheral player in a star, his average distance is smaller than 2) or (c) having high betweenness and low costs (e.g. the center of a line) – it is not possible to satisfy all three goals at once. Thus, agents have to weigh the three aspects against each other.

In brief, the **centrality model** is based on the following assumptions (where the numbers refer to the categories of assumptions introduced in Subsection 2.1.3):

- A0 The relevant network statistics are closeness, betweenness and degree.
- A1 The evaluation function[25] is increasing in closeness, increasing in betweenness and decreasing in degree.
- A4 Each player's preferences are linear.
- A5 All players have homogeneous preferences.

Given these assumptions, we can represent the preferences of any agent $i \in N$ by a utility function $u_i : G \to \mathbb{R}$ with

$$u_i(g) = (1 - \lambda)CLOSE_i(g) + \lambda BETW_i(g) - cl_i(g).$$

The parameter $c > 0$ stands for the costs of one link (marginal costs). The parameter $\lambda \in [0, 1]$ stands for the weight of betweenness versus closeness in the benefits.[26] We will analyze the model for all points (λ, c) in the parameter space $[0, 1] \times \mathbb{R}_+$, as they represent different contexts.

3.2 Methods and Basic Results

To study which networks are likely to emerge in the centrality model, we employ three complementary methods: equilibrium analysis, enumeration, and simulation. In the following we introduce each method and show some of its basic results.

3.2.1 Equilibrium Analysis

As throughout the other chapters, one can use analytical means ("paper and pencil") to classify networks into stable and unstable according to the notion of pairwise stability.

[25] For the conciseness of the model, the function is not defined explicitly.

[26] Instead of setting the slopes (λ and $1 - \lambda$) in relation to each other, we could also have defined them independently. Both notations allow us to represent any linear preferences and there is no difference when examining stability and efficiency. The relative notation is advantageous for comparative statics, because c then measures the costs in comparison to one unit of benefit.

Analytical results on stability typically need the maximal incentive of any agent to sever a link and the maximal incentive of any two agents to add a link and compare them to linking costs c. Because benefits are based only on closeness and betweenness, the crucial aspects for a focal agent i are the change in distances ($\sum_{j \in N} d_{ij}(g) \hat{=}$ non-normalized closeness) and the change in the number of shortest paths he is on, which will be called "brokerage" ($\sum_{j<k(j \neq i, k \neq i)} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)} \hat{=}$ non-normalized betweenness). Specifically, *if a new link for some agent i in some network g means a decrease in distances of x and an increase in brokerage of y, then he is willing to form the link only if* $c \leq \frac{(1-\lambda)[x]}{(M-1)(n-1)} + \frac{\lambda 2[y]}{(n-1)(n-2)}$. As players compare marginal costs with marginal benefits. Although deriving the changes in distances and brokerage for a given situation might be tedious, it is a straightforward task.

Let us first have a look at some prominent networks. The following Prop. 3.1 presents the parameter combinations for which five prominent network structures are pairwise stable.

Proposition 3.1. *In the centrality model the following holds:*

(i) The complete network g^N is stable if and only if $c \leq \frac{1-\lambda}{(n-1)(M-1)}$.

(ii) The empty network $g^{\emptyset}$ is stable if and only if $c \geq \frac{1-\lambda}{n-1}$.

(iii) A star network $g^{\star}$ is stable if and only if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$.

(iv) Let n be a multiple of 4. Then a circle network $g^{\bigcirc}$ is stable if and only if $\frac{(1-\lambda)[\frac{1}{8}n^2-\frac{1}{2}n+1]}{(M-1)(n-1)} + \frac{2\lambda[\frac{1}{8}n^2-\frac{3}{4}n+1]}{(n-1)(n-2)} \leq c \leq \frac{(1-\lambda)[\frac{1}{4}n^2-\frac{1}{2}n]}{(M-1)(n-1)} + \frac{2\lambda[\frac{1}{8}n^2-\frac{1}{2}n+\frac{1}{2}]}{(n-1)(n-2)}$.

(v) A complete bipartite network $g^{l:r}$ with $2 \leq r \leq l$ (where l and r are the sizes of the two groups) is pairwise stable if and only if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq \frac{2(1-\lambda)}{(n-1)(M-1)} + \frac{2\lambda[\frac{r-1}{l}]}{(n-1)(n-2)}$.

All proofs of this chapter can be found in Section 3.5. The first implication of Prop. 3.1 is that non-existence of stable networks is not an issue in this model.

Proposition 3.2. *In the centrality model for any parameters $(\lambda, c) \in [0,1] \times \mathbb{R}_+$ there exists at least one stable network.*

As for the model in Section 2.3, at least one of the trivial networks (empty network, complete network or the star) is stable. Figure 4 depicts the parameter space with weight λ on the horizontal axis and marginal cost on the vertical axis. It illustrates (among other results) the "regions" of the parameter space where the complete network, the star network, the balanced complete bipartite network and the circle network are stable.[27] Not illustrated is the region where the empty network is stable.

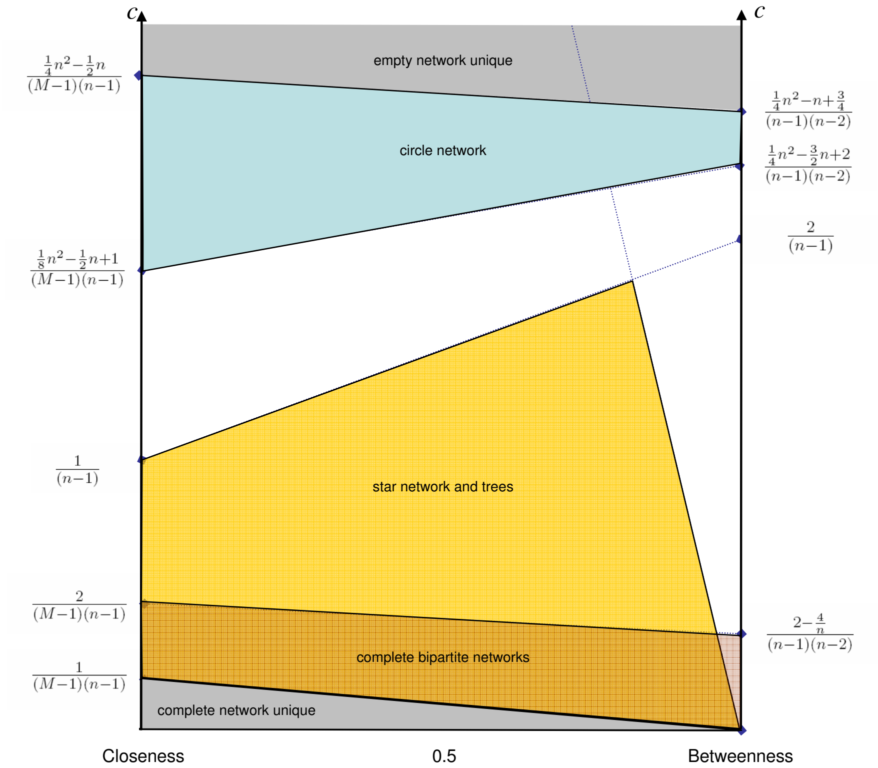

Figure 4: "Parameter map" with stability for some prominent networks.

It is intuitive that for sufficiently low c, the complete graph is stable, as long as there are some incentives for closeness. Above the upper bound for the stability of the complete network, complete bipartite networks can be stable. Among them is the star network that is stable for quite a range of the parameter space, but not for $\lambda = 1$. The figure indicates that the balanced complete bipartite network and the circle network, both can be stable for any weight λ. While the circle networks are high cost phenomena, the complete bipartite networks are low cost phenomena.

Some aspects deserve additional attention. The boundary between stability of the complete network and the complete bipartite networks marks a special border. At this border agents are indifferent between keeping and removing a link with the minimal possible benefits – that is a link that only serves to reducing the distance to one other agent by the amount of one,

[27]Results look different for small network size and slightly different for networks with an odd number of players. For example, Figure 6 and Figure 7 qualitatively present how these regions look like for $n = 8$ and for $n = 14$.

while it does not provide any brokerage. As a consequence, below this border the complete network is uniquely stable. A similar observation can be made for the upper boundary of the star network. This is the highest cost level, such that a network with loose ends can be stable. Finally, for high enough c, the empty network must be uniquely stable.

Proposition 3.3. *In the centrality model the following holds:*
(i) The complete network g^N *is uniquely stable if and only if* $c < \frac{1-\lambda}{(n-1)(M-1)}$.
(ii) If $c > \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$*, any network with pendants (players at loose ends) is unstable.*
(iii) For large enough c*, the empty network is uniquely stable.*

So, the first set of equilibrium analysis results structures the parameter space. Below the first frontier, the complete network is uniquely stable. As a second frontier one can consider the upper bound for the stability of the balanced complete bipartite network. It can be shown that any complete bipartite network with at least two agents in each group can only be stable below this frontier.[28] The intuition for this result is simple: If a complete bipartite network is not balanced, there is one group of agents with less beneficial ties than in the balanced network, such that high costs c lead to a deletion of ties. The third frontier has a "roof"-like shape and restricts stability of networks with loose ends. Among them are all minimal networks (that are non-empty). Consequently, trees can only be stable in the parameter region where the star network is stable.

The last frontier is the lower boundary for the uniqueness of the empty network. Prop. 3.3 shows existence of such a boundary without answering where it is. We depicted it at the upper boundary for stability of the circle network. Our conjecture that this is the maximal cost level where networks with non-critical links can be stable.[29] If this conjecture is true, it follows that above the third and the fourth frontier, neither a network with non-critical links nor a network with loose ends can be stable. Thus, only the empty network remains. Note that the empty

[28]This statement includes complete bipartite networks (with more than one player in both groups) with isolates, which will be discussed in Subsection 3.3.2.

[29]We argue that among all networks that contain non-critical links, the circle network has the highest marginal benefit for any of those (non-critical links). Define for $ij \in g$ the marginal benefit of agent i as $\beta_i^{ij}(g) := (1-\lambda)CLOSE_i(g)+\lambda BETW_i(g)-[(1-\lambda)CLOSE_i(g\backslash ij)+\lambda BETW_i(g\backslash ij)]$. And let T be the maximal marginal benefit that a non-critical link can mean to both its owners, that is $T := \max_{g\in\tilde{G}} \min\{\beta_i^{ij}(g), \beta_j^{ij}(g)\}$, where $\tilde{G} := \{g \in G : 1 < d_{ij}(g) < M\}$. The problem is to find the argmax of this expression.

Our conjecture is that $T = \beta_i^{ij}(g^{\bigcirc})$. For n odd this is $(1-\lambda)\frac{n-1}{4(M-1)} + \lambda\frac{n^2-4n+3}{4(n-1)(n-2)}$. For n even the threshold is slightly smaller. If the conjecture holds, then for $c > ub(g^{\bigcirc})$ in any network with circles (circles consist of non-critical links) at least one agent is willing to cut a link. Thus $ub(g^{\bigcirc})$ is the maximal cost level where any network with circles can be stable.

graph is stable "far below" this frontier,[30] and trivially stable if the importance of closeness is sufficiently low.

The equilibrium analysis is not only used to analyze for which parameter settings a particular network is stable, but also to characterize the stable networks by properties they must or must not satisfy. What is not possible by "paper and pencil", however, is to find all pairwise stable networks.

3.2.2 Enumeration

For n not too big (say $n < 12$) a computer can find all stable networks for a given weight of the incentives.[31] To be more precise, for a fixed weight λ, our algorithm takes a network and considers all possible deviations (in the sense of pairwise stability). This yields a lower bound ("addition proof") and an upper bound ("deletion proof") of costs c that are necessary and sufficient for the focal network to be pairwise stable.[32] For most of the networks the lower bound is greater than the upper bound implying they are not stable for any costs c (at the fixed weight λ). For some networks the lower bound coincides with the upper bound, implying that there is an infinitely small range of costs (c) such that this network is pairwise stable.[33] Those networks do not seem to be reasonable candidates for emerging networks, because the networks would lose their stability due to the smallest perturbations in the cost c. We will exclude such networks when counting the number of stable networks for some weight λ.

We computed all stable networks for $n = 3, ..., 8$ setting $M = n$ and fixing the weights at $\lambda = 0, 0.1, 0.2, ..., 1$. With this procedure we can find all stable networks, except those which are stable for some not used weight (say $\lambda = 0.724$) and not stable for all the used weights (i.e. $\lambda = 0.7, 0.8$).

[30]If $T = (1-\lambda)\frac{n-1}{4(M-1)} + \lambda\frac{n^2-4n+3}{4(n-1)(n-2)}$, we can assure that the empty network is stable above T ($\frac{1-\lambda}{n-1} < T$), by letting $M < \frac{1}{4}(n-1)^2 + 4$.

[31]Such a method was used by Corbo and Parkes (2005) to find all stable networks in the Fabrikant model for $n = 10$, which is equivalent to the centrality model if $\lambda = 0$ and $M = \infty$ (see Remark 2.3.1).

[32]To run the enumeration, we either have to fix λ and search for ranges of c for a given network, or fix c and search for a ranges of λ. We chose to fix λ, because there are some canonical candidates of λ to analyze, i.e. $\lambda = 0$ and $\lambda = 1$, while this is not true for c.

[33]As an example consider the network obtained when deleting one link from the complete network. This network is pairwise stable if and only if the parameters are such that both, the complete network and the star network are stable. In fact, this means an asymmetric treatment of the parameters c and λ. When we find a network to be stable for some fixed λ, we cannot generally exclude that the range of λ for which it is stable, is infinitely small.

Table 2 shows the number of all different stable networks found with this procedure.

Table 2: Number of stable networks in dependence of size.

Network size n	5	6	7	8
Non-isomorphic networks	34	156	1,044	12,346
Fraction of stable networks for $\lambda = 0.5$	26%	13%	4.3%	0.95%
For Closeness ($\lambda = 0$)	6	12	21	45
For $\lambda = 0.5$	9	20	45	117
For Betweenness ($\lambda = 1$)	4	9	18	37

While the number of possible networks explodes with the size n, the number of stable networks increases much more gradually. So our notion of stability – despite being a minimal requirement – can already exclude many networks from being part of a prediction.

Beyond the sheer number, the enumeration can identify the stable networks. For instance, Figure 5 shows the nine networks with five agents that are stable for $\lambda = 0.5$, as well as the six networks that are stable for $\lambda = 0$ and the four networks that are stable for $\lambda = 1$.

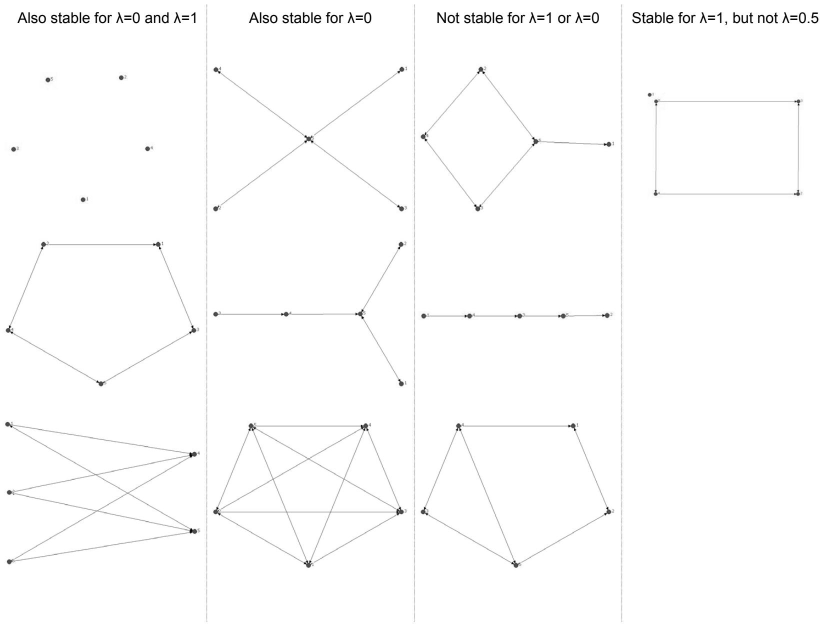

Figure 5: All stable networks for $\lambda = 0.5$ with indication of stability for $\lambda = 0$ and $\lambda = 1$.

While the enumeration provides a full picture of the candidates for emerging networks, it does not reveal which networks are most likely the endpoint of a dynamic process.[34]

[34]The equilibrium analysis and the enumeration are based on the notion of pairwise stability. Which is conceptually a weak requirement. However, we do not work with stronger notions of stability for three

3.2.3 Simulation

The third method is a simulation of myopic improvement dynamics. Consider a society where in any period two agents randomly meet. Each meeting is a possibility for those agents to change their relationship. To run a simulation one has to fix both behavioral parameters, weight λ and costs c (as well as the basic settings, n and M). Then, the simulation takes the following steps:

1. Start with some network g_0.

2. Pick a pair of players $\{i, j\}$ at random.

3. Allow (PS) deviations for the link ij. That is to form the link if both improve their utility (at least one strictly), $g_1 = g_0 \cup ij$. Cut the link if at least one improves strictly, $g_1 = g_0 \setminus ij$. Keep the graph in all the other cases, $g_1 = g_0$.

4. Take g_1 and go back to step 1. Repeat until a pairwise stable network is reached: $g_T \in [PS]$.

So the procedure starts with one given networks, follows the sequence of deviations and stops when no more changes occur. As starting networks we took a random sample (of all networks) stratified by density, with sample size around 2500. We ran the simulation starting with every network for $n = 3, ..., 8$, and a random sample for $n = 14$ and $n = 20$.

To give a specific example: For $n = 14$ we used a sample of 2'432 starting networks. Each of them was used for 42 runs (see next paragraph). All sequences converged to a stable network in less than 10'000 steps. The number of iterations is heavily skewed with a mean of 137 and a median of 56.

It lies in the nature of such a simulation that one has to choose a few parameter settings out of a continuum of possibilities. As in the enumeration, we fixed $M = n$ in any simulation. As parameter setting we chose the weights $\lambda = 0, 0.1, 0.5, 0.9, 1$ and four cost levels ($c = v\_lo, lo, med, hi$). The weights include the partial models – where only closeness incentives are present and only betweenness incentives are present – one balanced model $\lambda = 0.5$, and cases that check for jumps when going from a partial to a combined model.

reasons: a) As the enumeration shows, only a small subset of all networks are pairwise stable. b) We let the enumeration also check for unilateral stability – which is stronger than pairwise stability and stronger than pairwise Nash stability – but it turns out that this refinement does not heavily decrease the number of equilibrium networks in our model. c) The simulation partially serves as some kind of equilibrium selection device.

The cost levels are defined according to analytical considerations as follows (in increasing order): $v\_lo := \frac{1}{2^3 n} - \epsilon$, $low := \frac{1}{2^2 n} - \epsilon$, $med := \frac{1}{2^1 n} - \epsilon$, and $hi := \frac{1}{2^0 n} - \epsilon$, where $\epsilon = 0.001$.[35] We have not run the simulation for $c = v\_hi$, where typically the circle network is stable for any λ. The subtraction of $\epsilon = 0.001$ only serves to avoid prominent numbers. For $\lambda = 1$ there was an additional run for "epsilon costs" $c = \epsilon$, that is a cost level sufficiently small such that any increase in betweenness benefits would justify its costs (for not too high n). By starting twice (or three times) with each configuration there are $2(4 * 5 + 1) = 42$ (respectively 63) runs per starting network.

Figure 6 and Figure 7 represent the parameter settings of the simulations for $n = 8$ and $n = 14$. Each red dot stands for one setting of the parameters. While the proportions of the different areas do not represent their real size, the figures represent the qualitative properties. For example, for $n = 14$ and $(\lambda, c) = (0.5, med)$, it holds that the star network is stable, complete bipartite networks (with more than two players in each group) and the circle network are unstable, and the empty network is unstable (the dot is below the blue line). Since the qualitative position of a setting is important for the interpretation of the simulation result and due to the large number of results, we will restrict our argumentation to examinations of $n = 8$ and $n = 14$. For $n = 8$ we can compare the simulation results with the enumeration results to get a picture of how certain networks are selected from the set of stable networks. $n = 14$ seems sufficiently high to identify many of the network patterns (for small size, certain network patterns degenerate).

An alternative description of how these types of simulation work can be found in Buskens and Van de Rijt (2008). The purpose of the simulation is two-fold. Firstly, the simulation can select among the stable networks the ones that are more likely to emerge. This works well for n not too big. For instance, while the enumeration shows that for $n = 8$ there are around 10 to 20 stable networks (for each parameter combination), the simulation leads to two or three of them in at least 60 percent of the runs. Figure 8 exemplarily shows the most frequently emerging networks and their probabilities of occurrence, when starting with every non-isomorphic network three times.

The second purpose of the simulation is to provide a prediction of the patterns that the emerging network exhibit. Although for larger size, $n \geq 12$, we do not know all candidates

[35] Since we normalized betweenness and closeness to be within $[0, 1]$ and used one unit of benefit as a numeraire ($\lambda + (1 - \lambda) = 1$), costs, i.e. constant $\bar{c}$, across different settings of λ are interpretable. However, with some knowledge about aggregation of closeness and betweenness (see Section 4.1) there is a different perspective on this issue, as articulated in Remark 4.1.1.

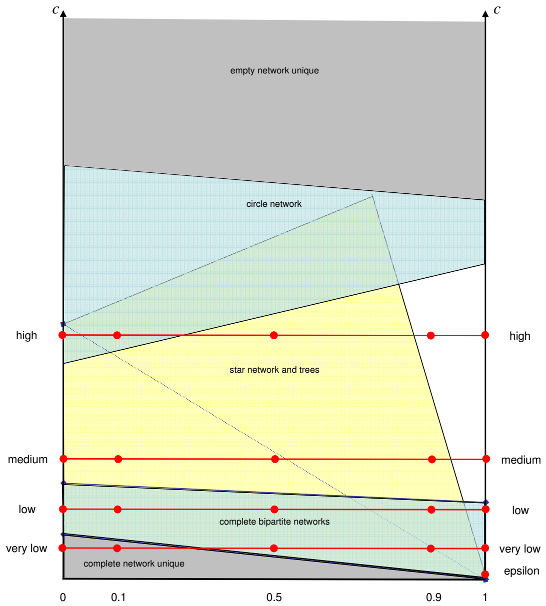

Figure 6: Setting of the parameters λ, c for simulation with $n = 8$.

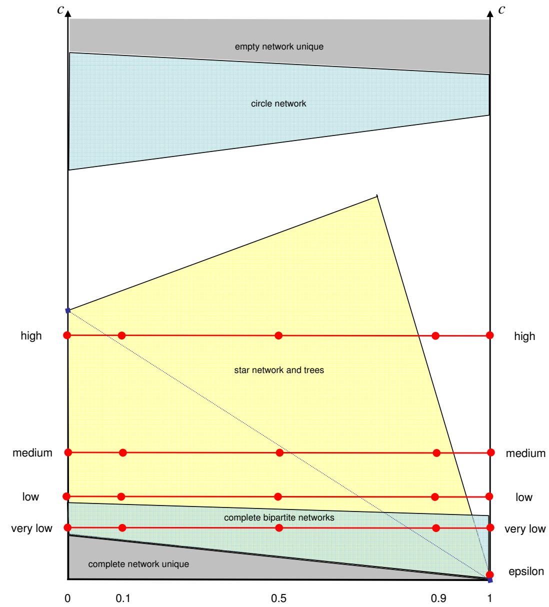

Figure 7: Setting of the parameters λ, c for simulation with $n = 14$.

for stable networks (as the enumeration is no longer feasible), the simulation is still a proper computational experiment. Starting with the same network structures, but using different behavioral parameters provides important evidence how changes in behavior affect the network structure. Table 3 shows the density of the emerging networks in the simulation with $n = 14$. There is a strong effect of the marginal costs on the density. The higher c, the lower the density. The weight λ does also have effects but those are not obvious.

Table 3: Density of emerging networks (sim. $n = 14$).

c λ	0 (Close.)	0.1	0.5	0.9	1 (Between.)
epsilon					78.2%
very low	34.7%	29.3%	35.5%	44.6%	25.7%
low	24.8%	24.7%	19.6%	29.9%	0.01%
medium	18.7%	18.3%	15.7%	16.1%	0.9%
high	15.1%	15.0%	14.7%	14.0%	0.1%

In the following we employ all three methods presented here (equilibrium analysis, enumeration, and simulation) to answer specific questions about the consequences of closeness and betweenness incentives on the network structure.

Remark 3.2.1 (Complementary methods)**.** *We argue that each of the three methods has its significant strengths and weaknesses such that omitting one of them would not lead to an appropriate examination of our model. Clearly, without equilibrium analysis, enumeration and*

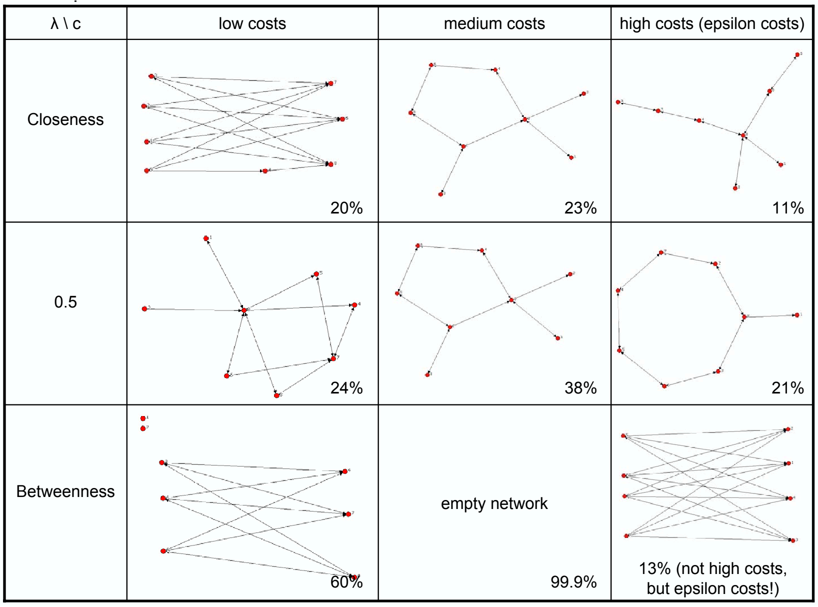

Figure 8: Most frequently emerging networks for different parameters (sim. $n = 8$).

simulation are black boxes leading to data which can be described but neither generalized nor explained.

Omitting the enumeration, we do not get a full picture of the candidates for stable networks. This is an issue, because the dynamics of the simulation are not only driven by the utility function – the point we are interested in – but also by the process of link formation.[36] *I.e., the rule that a pair of players is drawn to revise their relationship might induce different network structures than, e.g., the rule that one players is drawn, who can change the relationship that is most valuable for him.*

Finally, without the simulation we assess a dynamic question (which networks emerge when...) by only static methods. Moreover, we would not have had numerical examples for $n \geq 10$ *such that we might miss important features of emerging networks.*

3.3 Closeness versus Betweenness Incentives

This section first contrasts closeness dynamics with betweenness dynamics and then turns to the combination of both.

[36] I thank Ulrik Brandes for pointing out this issue.

3.3.1 Dynamics of Closeness

Although the centrality model in this form is new, there is sufficient literature to build expectations how the emerging networks look like for its extremes, full weight on closeness ($\lambda = 0$) or full weight on betweenness ($\lambda = 1$).

In the case of full weight on closeness, each agent strives for short distances to others, while optimizing costs. This is similar to the utility function of the connections model, discussed in Jackson and Wolinsky (1996), Hummon (2000), and Doreian (2006), where the value of each connected agent decreases with its distance. The most prominent result of the connections model is the star network, but besides different other stable networks were found (see Hummon, 2000).[37] In Section 2.3 we have shown that the analytical results of the (symmetric) connections model correspond one-to-one to the closeness model with linear evaluation (which is the centrality model for $\lambda = 0$). Furthermore, it was shown by enumeration that the sets of stable networks for both models are almost coinciding.

However, we have not characterized in that section how the stable networks look like. Corbo and Parkes (2005) identify some classes of stable networks and also mention the difficulty in finding all stable networks. So the question remains whether the star or star-like networks are a typical outcome for such a setting (closeness-type incentives) and which other networks can occur. The star belongs to the family of minimally connected networks, the tree networks. Among the trees the star is the network with the minimal sum of distances. Therefore, star-like networks can be described as **connected**, **sparse** with **short distances**. Let us analyze to which extent the stable and emerging networks for closeness incentives ($\lambda = 0$) satisfy these three properties.

Equilibrium Analysis

For $c < \frac{1}{(n-1)}$ all stable networks are connected, as shown in see Prop. 2.9. This threshold is slightly above $c = hi$.

Concerning distances, one can find an upper bound for the maximal distance in a stable network.[38] A tie that links two agents who were at a certain distance before means a certain amount of benefits in any case. Let the diameter of a network be the maximal distance between two connected agents in the network.

[37] The prominence of the star outcome is also due to a similar model by Bala and Goyal (2000).
[38] This result is also found by Fabrikant et al. (2003).

Proposition 3.4. *In the centrality model with $\lambda = 0$ and costs c, the following holds: The diameter of a stable network is smaller or equal to p,*
with $p = \max\{\sqrt{c4(n-1)(M-1)+1}, 1\}$.

Let us study the implications of this result in a numerical example: for $c = lo$ $(= \frac{1}{2^2 n} - \epsilon \approx \frac{1}{2^2 n})$, the boundary is $p = \sqrt{\frac{(n-1)(M-1)}{n} + 1}$. That means that the maximal real distance that can emerge in a simulation with $M = n = 14$ is three and in a simulation with $M = n = 8$ this is two. For $c = med$, the maximal possible distance is three for size 8 and five for size 14.

Also the sparsity of stable networks can be shown analytically. As for each level of c, we find an upper bound for the average degree. We cannot exclude a high degree player directly because a star-like position leads to high benefits which can compensate for the costs. But there is a link between the existence of small circles and the average degree $d(g) := \frac{l(g)}{n}$ that we can use to get the following result:

Proposition 3.5. *In the centrality model with $\lambda = 0$, the following holds: If $c > \frac{9n}{16(n-1)(M-1)}$, then $d(g) < \frac{1}{2}n + \frac{1}{2}$ for any stable network g and if $c > \frac{4n}{5(n-1)(M-1)}$ then $d(g) < \sqrt{n}$ for any stable network g.*

The first part does not drastically restrict the candidates for emerging networks. It restricts the density of the stable networks not to be higher than around 60 percent.[39] The second part applies for higher costs, e.g. $c = hi$. It restricts the stable networks of size 8 to have less than 11 links, networks of size 14 to have less than 25 links.[40]

Enumeration

While the equilibrium analysis provides upper bounds, the enumeration reveals to what extent the set of all stable networks for closeness incentives satisfies the three properties of interest (sparsity, connectedness and short distances). Table 4 shows the enumeration results. The first column describes the properties of all non-isomorphic networks and serves as a benchmark, the second column contains all stable networks for closeness incentives, the third column takes the same set excluding the complete and the empty network, which affect the result.

[39] From knowing the average degree, the density can be computed as $D(g) = \frac{\frac{1}{2}n * d(g)}{\frac{1}{2}n(n-1)} = \frac{d(g)}{n-1}$. We get: If $c > \frac{\frac{9}{16}n}{(n-1)(M-1)}$, then $D(g) < \frac{\frac{1}{2}n+\frac{1}{2}}{n-1}$ and if $c > \frac{\frac{4}{5}n}{(n-1)(M-1)}$, then $D(g) < \frac{\sqrt{n}}{n-1}$.

[40] The result on average degree and the result on the diameter get stronger for bigger sizes of the networks. For $n = M = 100$ and $c = lo$ the diameter is not higher than 9; and $c = hi$ restrict the density to be less than around 10 percent.

Table 4: Properties of stable networks for pure closeness incentives $\lambda = 0$ (enum. $n = 8$).

	all networks	stable networks	non-trivial stable n.
Number of networks	12'346	45	43
Number of trees	253	19	19
Number of connected networks	11'117	43	42
Mean number of links	14	9.09	8.86
Mean of average distance	1.779	2.149	2.040
Mean of average real distance	1.563	1.982	2.005

The first two rows of the table show that out of the $12'346$ non-isomorphic networks, only 253 (that is 0.2 percent) networks are trees. While for the set of stable networks for full weight on closeness $\lambda = 0$ there are 19 trees, which makes 42 percent.[41] To interpret the rows in the middle, note that a tree is connected with exactly 7 links. The table shows that, indeed, most stable networks are connected and sparse with an average of 9 links per network. The last two rows assess the distances. The average distance measures the distance between any pair of agents in a network; the average real distance only considers connected pairs. The set of stable networks exhibits relatively high distances. While a star has an average distance of 1.75, in the set of stable networks there are many with higher distances. In fact, only three of the 45 stable networks exhibit a lower average distance than an arbitrarily chosen network. This is the only aspect of star-like networks that is not consistent with the expectation that star-like networks are stable for pure closeness incentives ($\lambda = 0$). The stable networks are sparse and connected, but do not necessarily exhibit short distances.[42]

The results above do not differentiate outcome by the level of c. Analytically, it is easy to show that trees can only be stable for parameter values for which the star network is also stable (this is a consequence or Prop. 3.3 (ii); see also Figure 4). For pure closeness incentives $\lambda = 0$, this is the range $c \in [\frac{1}{(M-1)(n-1)}, \frac{1}{(n-1)}]$. Below that cost range the complete network is uniquely stable as shown in Prop. 3.3 (i). Above this range there are only few stable networks. For example, using the enumeration for $n = 8$ there are three networks that can be stable above this cost range. Those are the empty network, the circle network (see Prop. 3.1) and a network consisting of a circle of size 7 plus one isolate. Let us now analyze which networks emerge within this cost range.

[41] For other weights ($\lambda = 0.1, 0.2, ..., 1$), this fraction is not above 22 percent.

[42] We will return to this point, when discussing efficiency of the centrality model in Chapter 4.

Simulation

We ran a simulation with three settings of c where trees are stable, starting with any possible network. Table 5 shows the frequency that a tree and specifically the star emerges, as well as the density and average distance of the emerging networks. The average distance equals the average real distance, since all the emerging networks must be connected (by Prop. 2.9 (i)).

Table 5: Fraction of trees emerging for closeness incentives (sim. $n = 8$).

	low cost	medium cost	high cost
Stable Networks	12	10	20
Trees emerging	1%	11.4%	90.7%
Star emerging	1.0%	0.6%	0.1%
Number of links	12.29	8.58	7.09
Average distance	1.56	1.90	2.34

The table shows that the star network itself is not a good prediction for the dynamics of closeness.[43] Trees are the dominant structure for a certain level of c. The 4th row shows the average number of links for the emerging networks. The emerging networks are sparse, but become denser when reducing c.

By drawing all of the frequently emerging networks in this simulation, we made the following observations: For $c = med$ the dominant architecture consists of loose ends and some non-critical links forming a circle of size 4 or 5 (but not smaller). For $c = lo$ we find more of these circles in the dominant architecture, but there are typically no loose ends. The emerging networks for these cost levels do not belong to some class of bipartite networks (neither trees nor complete bipartite).

3.3.2 Dynamics of Betweenness

For full weight on betweenness ($\lambda = 1$), every agent is striving for brokerage opportunities. A similar model was studied by Buskens and Van de Rijt (2008) using Burt's network constraint measure as operationalization of the benefits of structural holes. By equilibrium analysis and simulation they find that complete bipartite networks emerge. More particularly, the balanced complete bipartite network is the most likely outcome. Moreover, Goyal and Vega-Redondo (2007) and Willer (2007) present models that (are supposed to) cover Burt's idea of structural holes. Willer (2007) finds the circle network as the most likely to emerge, but since he only

[43] This result is consistent with the argument of Watts (2001) analyzing the dynamics of the connections model. In particular, she shows that for a dynamic process like the one we consider here (in the simulation), the probability that the star network is reached, converges to zero for n going to infinity.

considers networks up to size $n = 4$, the circle network cannot be distinguished from a complete balanced bipartite network. Burger and Buskens (2006) use a simple utility function covering the disadvantage of closed triads (no brokering) on a local level. Computer simulations select the balanced complete bipartite network (for low capacity constraints), which is confirmed in laboratory experiments. Finally, in the model of Goyal and Vega-Redondo (2007) agents not only seek brokerage opportunity, but also derive benefits from the size of their component and try to avoid being mediated by others. With a strong notion of stability, they find the star network as most likely outcome. Since in our model for $\lambda = 1$ agents only optimize their brokerage benefits, we expect that the dynamics lead to complete bipartite networks.

Properties of Induced Networks

Bipartite networks are characterized by not containing any circle of odd length. Since this precludes complete triads, bipartite networks cannot be extremely dense. However, complete bipartite networks are quite dense and contain only distances of length 1 and 2. Table 6 shows to which extent the stable networks for $\lambda = 1$ satisfy those properties.

Table 6: Properties of stable networks for pure betweenness incentives $\lambda = 1$ (enum. $n = 8$).

	all networks	stable networks
Number of networks	12'346	37
Number of connected networks	11'117	19
Mean number of links (for connected subset)	14 (14.41)	12.7 (16.16)
Mean of average real distance	1.563	1.466
Fraction of networks with maximal real distance of 2	38.6%	86.5%
Mean of fraction of 3-circles	13.3%	9.0%
Number of networks without any 3-circle	3.3%	54%

First, it is notable that almost 50% of the stable networks are not connected. This fact has to be considered when interpreting the other statistics. The mean density for the stable networks is lower than in an arbitrary network, but this can be explained by the over-representation of unconnected networks. The last rows of the table show that a considerable number of stable networks satisfy the requirement of not containing a circle of length 3. The middle rows of the table show that the distances of the stable networks are short and, indeed, there are few networks with distances larger than 2. This observation is supported by the following analytical result.

Proposition 3.6. *In the centrality model with $\lambda = 1$, the following holds: (i) Any network with a diameter of size ($p \geq 4$) or larger is not stable if $c < \frac{(\lfloor \frac{p}{2} \rfloor - 1)\lfloor \frac{p}{2} \rfloor}{(n-1)(n-2)}$. Moreover, (ii) for* *sufficiently low c,*[44] *any network with a diameter of three or larger is not stable – in other words: the distance between any two players is either one, two or M.*

Those results give a first suggestion that the stable networks resemble complete bipartite networks. Let us now check how many of the emerging networks really are complete bipartite when starting with different values of c.

Emergence of Complete Bipartite Networks

As argued before, complete bipartite networks can only be stable for parameter values for which the balanced complete bipartite network is also stable (see Subsection 3.2.1). This part of the parameter space is depicted in Figure 4. So, we know that complete bipartite networks can only occur for relatively low c. Nonetheless, if only betweenness matters, as the enumeration shows, there are not many stable networks above this range, that is only 9 out of 37 stable networks for $n = 8$. Plotting the other 28 networks shows that many are or at least resemble complete bipartite networks. Some of them do not belong to this class in a strict sense, e.g., a network with two isolates and a (4:2)-complete-bipartite component.

We ran the simulation for three settings of c, where complete bipartite networks are stable. Table 7 presents the frequency of emergence for different sets of complete bipartite networks (with at least two agents in each group). As the table shows, the family of complete bipartite networks are, indeed, the dominant structure. Moreover, it can be observed that for small costs c, rather the connected ones emerge; for higher costs c, rather the balanced ones. The balanced complete bipartite network, belonging to both subclasses, is the most frequently emerging network. It is notable that for $c = lo$ the empty network emerges in 20.8% of the cases and for costs higher than depicted ($c = med$ and $c = hi$) the empty network emerges in 99.9% (resp. 94.0 %) of the simulation runs, while also the circle network is stable.

Table 7: Fraction of complete bipartite networks (CB) emerging (sim. $n = 8$).

	epsilon costs	very low costs	low cost
Stable Networks	19	9	4
All CBs with or without isolates	40.4%	78.3%	61.1%
CBs (2:6, 3:5, 4:4) without isolates	29.0%	38.7%	0.9%
Balanced CBs (4:4, 3:3, 2:2) with or without isolates	13.4%	37.6%	61.1%
Balanced CB (4:4) without isolates	12.5%	25.4%	0.9%

Summarizing the dynamics of closeness and betweenness, we find that the results of our model

[44]Epsilon costs ($c = 0.001$) satisfy this requirement for $n \leq 15$.

correspond with the literature on similar models. The results suggest as a rule of thumb that incentives for short paths (here closeness) lead to tree networks and incentives for intermediation rents (here betweenness) lead to complete bipartite networks – without implying that their most prominent representatives (the star and the balanced CB) emerge. Moreover, it must be stressed that the results depend on the costs c.

3.3.3 Interaction of Multiple Incentives

Having characterized the emerging networks for pure closeness incentives ($\lambda = 0$) and for pure betweenness incentives ($\lambda = 1$), the next question is how those results carry over to a scenario with a combination of the two, what we call "mixed incentives".

First, we check for trees and complete bipartite networks (the two dominant classes of emerging networks for pure incentives). Running simulations reveals that networks of both classes can also emerge quite frequently for mixed incentives. For example, in a simulation with $n = 8$ and $\lambda = 0.5$, there is a cost level (i.e. $c = v\_lo$) where complete bipartite networks emerge in 37.8 percent of the runs; and there is a cost level (i.e. $c = hi$) where trees emerge in 78 percent of the runs. This, however, is only part of the story.

By enumeration we compare all stable networks for different incentives. Figure 9 depicts the number of all stable networks found for different weights λ. The networks are organized by the range of weights λ for which they are stable.
Strikingly, there are more stable networks for mixed incentives ($\lambda = 0.1, ..., 0.9$) than for pure incentives ($\lambda \in \{0, 1\}$). All 45 networks that are stable for closeness incentives ($\lambda = 0$) are also stable for some other weight. Eight can be stable for any weight; fifteen are stable for any weight, except for pure betweenness ($\lambda = 1$). For pure betweenness incentives ($\lambda = 1$), there are 37 stable networks. Fifteen of them never occur for any other weight (we used). This is remarkable, as only three networks of the other categories are found stable for only one weight. The other stable networks for betweenness are typically also stable for any other weight, except for pure closeness ($\lambda = 0$). Thus, there is strong indication that the stable networks across certain weights do not differ heavily, except for the case of pure incentives. One network was even found that is stable for both pure incentives $\lambda = 0$ and $\lambda = 1$, but not for any mixed incentive $\lambda = 0.1, ..., 0.9$.

Already the sheer number of stable networks indicates that pure incentives are special cases. Measuring certain properties of the set of stable networks confirms that mixed incentives indeed lead to qualitatively different results from pure incentives. Figure 10 shows the cost ranges $[\underline{c}, \bar{c}]$ for all stable networks for different incentives. The two graphs depict the median of the lower bound $\underline{c}$ and the median of the upper bound $\bar{c}$ for stability.

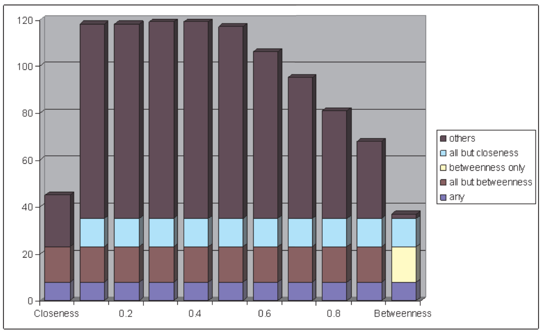

Figure 9: Number of stable networks by weights λ they are stable for (enumeration $n = 8$).

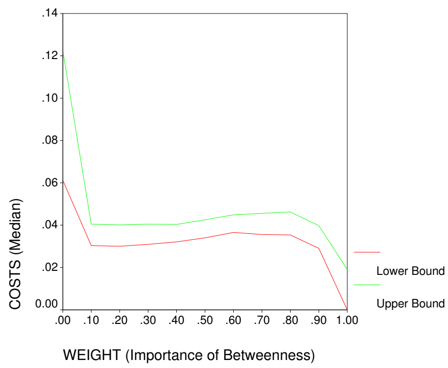

Figure 10: Cost range (median of all stable networks) across certain weights λ (enumeration $n = 8$).

For the mixed incentives ($\lambda = 0.1$ to 0.9), the typical costs for stability do not vary heavily, but the results differ drastically for the extremes $\lambda = 0$ and $\lambda = 1$. So we observe that, although the benefits of our model are a linear combination of closeness and betweenness, the results due to enumeration may exhibit jumps.[45] Thus, introducing a bit of closeness (betweenness) incentives into a pure betweenness (closeness) model may heavily affect the stable networks, changing and increasing the number of the candidates for emerging networks.

Integration of Isolates

To understand why such phenomena can occur, we analyze the interaction of closeness and betweenness incentives focusing on one structural feature: the integration of isolates. Consider a pendant i and his (only!) neighbor j, who is part of a larger component.

- When closeness only matters ($\lambda = 0$): i has a strong interest in the link ij, as this link is his only connection to the rest of the network (without ij, $CLOSE_i(g) = 0$). Player j's interest is restricted: cutting ij means not being connected to i, but does not have an impact on any other distance. So for high enough marginal costs c, i is willing to link with j but j rejects this offer.

[45]Not any characteristic of the stable networks is as discontinuous as the median costs, as can be seen in the next subsection.

- When betweenness only matters ($\lambda = 1$): j has a strong interest in the link ij because it provides a considerable amount of betweenness (increasing with the size of j's component). On the other hand i does not want to keep the link if he is only interested in betweenness (as his betweenness is zero with or without ij). So a network with loose ends cannot be stable when only betweenness matters.

- When both incentives are present ($\lambda \in [\epsilon, 1-\epsilon]$): both players do have high interest in keeping this link. But for different reasons: i wants to keep access to the community (closeness incentives); j enjoys mediating i with all his connections (betweenness incentives). While in the partial models networks with loose ends are not stable (for high enough c), they may well be stable in a mixed model.

The example here is a consequence of the players' incentives to form critical links, as stated in the next proposition. A bridge is a link that connects two otherwise unconnected components.

Proposition 3.7. *Consider two agents i and j in different components of size $l+1$ (respectively $r+1$). In the centrality model agent i is willing to form a critical link to j if $c < (1-\lambda)\frac{(r+1)(M-\frac{1}{2}r-1))}{(M-1)(n-1)} + \lambda\frac{2l(r+1)}{(n-1)(n-2)}$.*[46]

By setting one component to be zero, e.g. $r = 0$, we are in the situation discussed above. The threshold is sufficient but not necessary – it stems from considering a line structure in the non-critical component (that is the network structure with the least marginal closeness for the agent i). The necessary condition to be willing to form a link occurs when the non-trivial component forms a star-like network structure. This was used in Prop 3.3 (ii) showing a threshold for the stability of networks with loose ends. The result can be illustrated in the Figure 4 in Subsection 3.2.1 since the upper boundary for the trees (e.g., the star network) is driven by the fact that any tree contains loose ends. This area has a "roof"-like shape, achieving its maximum for a combination of closeness and betweenness incentives. Especially, for pure betweenness $\lambda = 1$, many networks fail to be stable because agents (i.e. pendants) do not have any incentive to keep a link.[47] Introducing a bit of closeness benefits can justify keeping these relationships.

[46]Interestingly, adding players (increasing the l and r) has an additive effect on the marginal closeness, but a multiplicative effect on the marginal betweenness of i and j.

[47]In 75 percent of all networks ($n = 8$) some agents are willing to sever a link even for the smallest costs $c = \epsilon$ for pure betweenness $\lambda = 1$. The reason is that in most networks there is an agent with no betweenness that already makes this network unstable.

The example of integrating an isolate only gives a partial explanation why mixed incentives lead to different network structures than pure incentives. Not only the number of stable networks with loose ends is bigger for intermediate weights, but also the number of stable networks without loose ends, as Figure 11 shows. The dotted line represents the number of stable networks with loose ends, e.g. for $\lambda \in [0.6, 0.9]$, the majority of the stable networks exhibits loose ends.

The example shows how in a mixed model different incentives can be at work, although all agents do have the same preferences.

3.3.4 Dynamics of Closeness and Betweenness

We discovered that stable networks for mixed incentives ($0 < \lambda < 1$) differ from stable networks for pure incentives ($\lambda \in \{0, 1\}$). The next question is *how* they differ. Our analysis of the dynamics for mixed incentives is rather exploratory and is limited to reporting basic results. The first figure shows three networks that are stable for mixed incentives only (that is: they are stable for some λ but not for $\lambda = 0$ or $\lambda = 1$).

The subsequent tables and figures report basic characteristics of the stable and emerging networks for different incentives. Figure 13 shows the distribution of density (number of links) in the stable networks. Figure 14 depicts the mean of this network statistic as well as of different other network statistics (stretched in a way that all shapes can be seen). MAXDEG and MINDEG stand for the maximal and minimal degree; DEGVAR is the variance of degree within a network; CONNECTED stands for the fraction of connected networks. While REAL DISTANCE measures the average distance between connected agents, MEAN DISTANCE

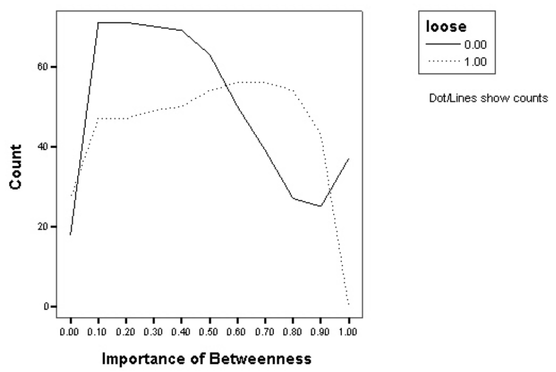

Figure 11: Number of stable networks with and without loose ends (enum. $n = 8$).

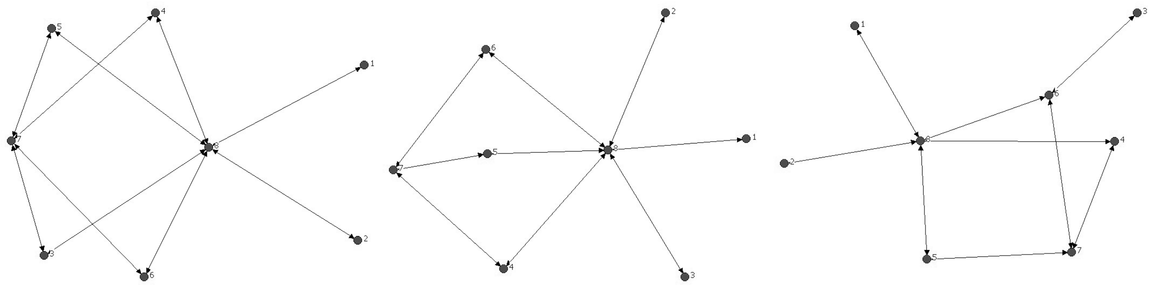

Figure 12: Examples of networks that are stable for mixed incentives only.

stands for the average distance between all pairs of players.[48]

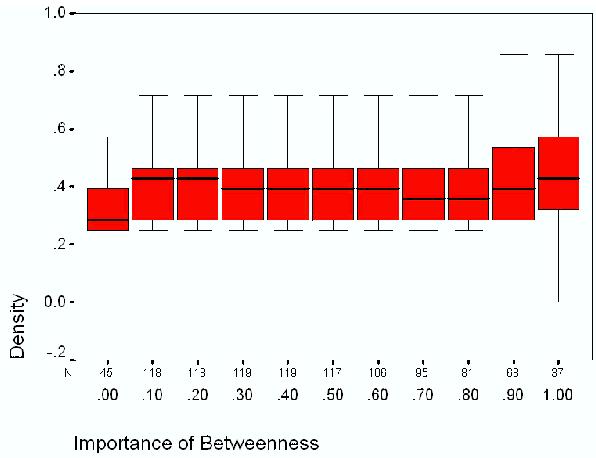

Figure 13: Distribution of density in stable networks (enum. $n = 8$).

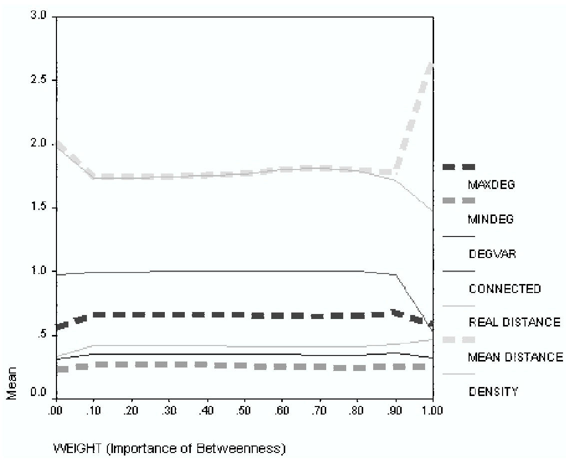

Figure 14: Properties of stable networks (enum. $n = 8$).

Simulation Results

It is important to keep in mind that the sets of stable networks above are not discriminated by costs c. Fixing different c, only some of them are stable. Table 8 presents simulation results for different settings of c. CONNECTED stands for the fraction of connected networks; LINKS for the number of links (which is proportional to the density); DEG'VAR stands for the variance of degree; AV'DIS measures the average distance (between all pairs) in a network; AV'RDIS stands for the average real distance. Since the simulation starts with any possible network, the first row stands for the properties of the starting networks and the mean values of the emerging networks can be interpreted as estimations for an arbitrary starting network.

[48] All these variables measure the mean of network statistics for different sets of stable networks, which has to be interpreted with caution.

Table 8: Properties of emerging networks (simulation with $n = 8$).

WEIGHT	COSTS	CONNECTED	LINKS	DEG'VAR	AV'DIS	AV'RDIS
All networks		90%	14.0	0.42	1.78	1.56
0 (Closeness)	very low	100%	28.0	0.00	1.00	1.00
	low	100%	12.3	0.75	1.56	1.56
	medium	100%	8.6	1.04	1.90	1.90
	high	100%	7.1	1.05	2.34	2.34
0.1	very low	100%	28.0	0.00	1.00	1.00
	low	100%	12.3	0.76	1.56	1.56
	medium	100%	8.7	1.04	1.88	1.88
	high	100%	7.2	1.01	2.30	2.30
0.5	very low	100%	17.4	0.66	1.38	1.38
	low	100%	11.5	1.43	1.61	1.61
	medium	100%	8.4	1.10	1.93	1.93
	high	100%	7.2	0.97	2.30	2.30
0.9	very low	100%	15.1	0.65	1.47	1.47
	low	100%	12.4	1.41	1.64	1.64
	medium	100%	8.1	1.52	2.04	2.02
	high	0%	0.4	0.03	7.76	2.00
1 (Betweenness)	epsilon	83%	17.2	1.03	1.69	1.36
	very low	45%	13.2	1.17	2.65	1.41
	low	1%	6.5	1.35	5.39	1.42
	medium	0%	0.0	0.00	7.99	1.71
	high	0%	0.4	0.03	7.73	2.00

Throughout any weight λ, there are some clear-cut relations between the costs of linking c and the properties of the emerging networks. The higher the costs, the lower the density and the higher the average distances. The effects of different incentives is not trivial. There is no property, except connectedness, that is influenced by weight (λ) in one clear direction.

While this section stresses the differences between closeness incentives, betweenness incentives and a mixture of them, in the next section we analyze the general effects that these different incentives have on social networks.

3.4 Lack of Clustering

The work of Watts and Strogatz (1998) has drawn attention to a structural feature of many social networks, that can be described as closure, transitivity or clustering. Small groups of agents are heavily linked among themselves such that "a friend of a friend is very likely to be also my friend". Formally, the clustering coefficient of an agent i is defined as the number of links among his neighbors $N_i(g)$ as a fraction of all possible links among them, $Clust_i(g) := \frac{2\zeta_i(g)}{(l_i(g)-1)l_i(g)}$, with $\zeta_i(g) := \#\{jk \in g | j, k \in N_i(g)\}$ (see Watts and Strogatz, 1998 or Watts, 1999). Let by convention the fraction be zero for $l_i(g) \leq 1$.

While many empirical studies measure closure – clustering indices of different networks can be found, e.g., in Newman et al. (2002) – theoretical works (mainly in sociology) stress the importance of closure for a society (e.g. Coleman, 1988).

The question arises whether such patterns persist when agents start to optimize their centrality. Granovetter (1973) and Burt (1992) argue that complete triads are not the source of the many network benefits, but the open triads providing connections to different areas of the network. Betweenness incorporates that idea by measuring the brokerage of a given node; but also closeness favors agents at distance two rather than one if links are sufficiently costly. So we would expect that agents optimizing their centrality replace links in complete triads by links that bridge higher distances.

Equilibrium Analysis: Cliques and Full Clustering

Consider a group of q fully connected agents (that is called a clique of size q). For pure closeness incentives $\lambda = 0$, we can show the following:

Proposition 3.8. *In the centrality model the following holds: For $\lambda = 0$ a network with a clique of size $q(\geq 3)$ or larger is not stable if $c > \frac{n}{q(M-1)(n-1)}$.*

Rather low costs (i.e. $c = lo$) are typically sufficient to rule out a clique of size 5. The result above is restricted to the case where only closeness incentives are present. The next result excludes clustering based on betweenness incentives.

Proposition 3.9. *In the centrality model for any network g, player i with $l_i(g) \geq 2$, and costs $c > \frac{1-\lambda}{l_i(g)(M-1)}$ the following holds:*
if i has full clustering ($Clust_i(g) = 1$), then the network is not stable ($g \notin [PS]$).

The idea of the proof is simple. First, there is a Lemma stating that full clustering of a player implies zero betweenness for him. As a consequence, betweenness incentives cannot justify clustering. Secondly, the closeness benefit of clustering is also very restricted: since any direct neighbor is also an indirect neighbor, cutting a link cannot decrease closeness heavily. Consequently, a player with full clustering will not keep all of his links, except if costs c are very low. In fact, the result leaves only a small "corridor" in the parameter map between the region, where the complete network is uniquely stable and no full clustering can occur. Figure 15 illustrates this and the last proposition. Above the thick (red) line there are no agents with full clustering and degree equal or higher than 3; and there are no cliques of size 4 or larger in stable networks above the thick (green) line on the left-hand side.

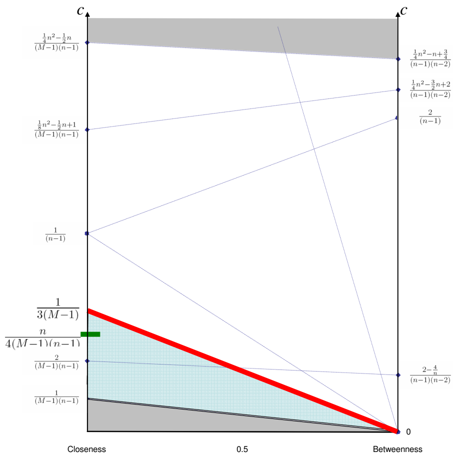

Figure 15: Illustration of closure results (Prop. 3.8 and Prop. 3.9).

Enumeration and Simulation: Clustering Coefficient

The two results above exclude networks with high clustering, but do not measure the extent of possible clustering. There are different ways to combine the individual clustering coefficients into one for the network.[49] We use a common transitivity index (see, e.g., Frank and Harary, 1982), that is called *the* clustering coefficient (of a network) by Newman et al. (2002) and overall clustering by Jackson (2008). This index is defined as the proportion of complete triads (three fully linked agents) among the triads with two or three links. The choice of operationalization, however, is not crucial at this point. Any index of network clustering is strongly related to the number of complete triads. In fact, if there are no complete triads in a network, then the clustering coefficient of every agent must be zero. We have already shown in Section 3.3 that many emerging networks are bipartite networks. Be it trees or complete bipartite networks, they all exhibit no complete triads (since a complete triad is a circle of odd length).

Table 9 shows the number of networks that exhibit no complete triad as a fraction of all networks. For example, 60 percent of all stable networks at $\lambda = 0.5$ do not have any complete triad. Thus, there is no player with positive clustering coefficient. Table 10 shows the clustering coefficient (transitivity index) of the emerging networks in a simulation with $n = 14$, where the starting networks have on average a clustering coefficient of 66 percent.

We observe that these networks with considerable clustering are transformed into networks with (almost) zero clustering by the dynamics of centrality incentives. Note that both types of incentives, closeness and betweenness, drive the closure results.

[49]One alternative is averaging the clustering coefficient of all individuals. However, by taking this average, the weight of an agent with few ties is the same as the weight of an agent with many ties and also the comparisons across networks of different size are problematic.

Table 9: Stable networks without any complete triad (enum. $n = 8$).

	fraction
All non-isomorphic networks	3.3%
Closeness	93%
$\lambda = 0.5$	60%
Betweenness	54%

Table 10: Clustering coefficient of emerging networks (simulation for $n = 14$).

	very low	low	medium
Closeness	0.19%	0.19%	0
$\lambda = 0.5$	4.86%	0.02%	0
Betweenness	0.49%	0	0

How can we explain the contradiction between empirical observations and the emerging networks in respect to clustering. As this model is about isolating the effect of centrality incentives on network structure, it does not consider many aspects that also drive the evolution of networks. First of all, there are external forces, like the opportunity to meet somebody, which increases the likelihood to become a friend of a friend. Secondly, by assuming homogeneous agents we exclude any difference between agents. For instance, the linking costs for geographically close agents are not lower than for other agents or there are no characteristics that lead to attractiveness (like homophily) etc. Finally, even if networks are determined by optimizing agents who are all the same, our benefit function does not incorporate the utility agents derive from closure (see Subsection 2.1.2) and strong ties (see Krackhardt, 1992, for example). So, it might not be a surprising result that agents striving for closeness and betweenness destroy the clustering patterns of a network, because it is in their best interest. The next chapter shows that the evolution of networks does not always obey the interests of optimizing agents.

3.5 Proofs of Chapter 3

In order to follow the results, it is important to keep in mind the utility function defined in Subsection 3.1.2. Moreover, it is helpful to read the first paragraph of Subsection 3.2.1. Let the marginal benefit be defined as $\beta_i^{ij}(g) := (1-\lambda)CLOSE_i(g \cup ij) + \lambda BETW_i(g \cup ij) - [(1-\lambda)CLOSE_i(g \setminus ij) + \lambda BETW_i(g \setminus ij)]$. The results are ordered according to their appearance in the main text.

Proof of Prop. 3.1. The results of Prop. 3.1 present lower and/or upper bounds of c where a network is claimed to be stable. For conciseness, we denote with $lb(g)$ the claimed lower bound of a network g and analogously the claimed upper bound with $ub(g)$.

(i) The complete network g^N can only be altered by deletion of a link. Any agent deleting any link increases his distances by 1 and does not change his brokerage. Therefore, no agent will sever a link for $c \leq ub(g^N)$ and any agent wants to sever a link for higher c.

(ii) The empty network $g^\emptyset$ can only be altered by the addition of links. Any agent adding any link decreases his distances by $M-1$, while his brokerage remains zero. Thus, no agent will do that for $c \geq lb(g^\emptyset)$ and any pair of agents is willing to add a link for $c < lb(g^\emptyset)$.

(iii) Only peripheral agents can add links. Any agent adding a link reduces his distances by 1 and does not change his brokerage. This leads to the $lb(g^\star)$. The central agent severing a link increases his distances by $M-1$ and decreases his brokerage by $n-2$. A peripheral agent cutting a link increases his distances by $M-1+(n-2)(M-2)$ and does not change his brokerage. Plugging into the utility function yields that no agent wants to sever a link for $c \leq \min\{ub1(g^\star); ub2(g^\star)\}$, while some agent is willing to sever a link for higher c.

(iv) Any agent severing any link increases his distances from the circle to the line network. For n even this is a change in distances of $\frac{1}{4}n^2 - \frac{1}{2}n$ and a change in brokerage from $\frac{1}{8}n^2 - \frac{1}{2}n + \frac{1}{2}$ to zero, yielding the upper bound. Two agents forming a link benefit the further away they are. For n a multiple of four, two agents on opposite sides (with two shortest paths) can form a link building a network with two circles of odd length. Their change in distances can be derived as $\frac{1}{8}n^2 - \frac{1}{2}n + 1$, while their brokerage changes by $\frac{1}{8}n^2 - \frac{3}{4}n + 1$. In the same way slightly different inequalities can be derived for other network sizes.

(v) In complete bipartite networks, additional links are only possible within a group. Since everybody is already indirectly linked, any agent adding a link reduces his distances by 1 without changing his brokerage. This yields the $lb(g^{l:r})$.
Since both groups consist of at least two agents, cutting one link only affects the distance between the focal agents. Their distance changes by 2. The brokerage for an agent in the group of size l changes by $\frac{1}{l}(r-1)$, because he is on one of l shortest paths between any pair of agents in the other group (he loses brokerage for any connection of j to its own group). Because $l \geq r$, the agents in the l-group benefit less from their links. They are indifferent about cutting if $c = \frac{2(1-\lambda)}{(n-1)(M-1)} + \frac{2\lambda[\frac{r-1}{l}]}{(n-1)(n-2)} = ub(g^{l:r})$. Therefore, for $c < lb(g^{l:r})$, two agents of the same group form a link; for $c > ub(g^{l:r})$ an agent of the larger group (of size l) will sever a link. No agent can improve by changing a link for $lb(g^{l:r}) \leq c \leq ub(g^{l:r})$.

Plugging in $l = r = \frac{n}{2}$ yields that a balanced complete bipartite network $g^{\frac{n}{2}:\frac{n}{2}}$ (for even n) is pairwise stable if and only if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq \frac{2(1-\lambda)}{(n-1)(M-1)} + \frac{2\lambda[1-\frac{2}{n}]}{(n-1)(n-2)}$. $\square$

Proof of Prop. 3.2. This theorem follows almost directly from Prop. 3.1 parts (i)-(iii). For $\lambda = 1$, the empty network $g^{\emptyset}$ is trivially stable for any c. For $\lambda < 1$, $g^{\emptyset}$ is stable if $c \geq \frac{1-\lambda}{n-1}$, g^N is stable if $c \leq \frac{1-\lambda}{(n-1)(M-1)}$ and $g^{\star}$ is stable if $\frac{1-\lambda}{(n-1)(M-1)} \leq c \leq \min\{ub1(g^{\star}) = \frac{1+\lambda}{n-1}, ub2(g^{\star}) = (1-\lambda)[\frac{M}{M-1} - \frac{2n-3}{(M-1)(n-1)}]\}$.

It remains to be shown that if $g^{\emptyset}$ and g^N are not stable, $g^{\star}$ is stable. This follows directly from $lb(g^{\star}) = ub(g^N)$, $ub1(g^{\star}) \geq lb(g^{\emptyset})$ as $\frac{1+\lambda}{n-1} \geq \frac{1-\lambda}{n-1}$, and $ub2(g^{\star}) \geq g^{\emptyset}$ (by definition $n \geq 3$ and $M \geq n-1$, which implies $\frac{M(n-1)-2n+3}{M-1} \geq 1$). $\square$

Proof of Prop. 3.3. We have to show three statements.

(i) The proof is analogue to the proof of Prop. 2.10 part (i). The complete network g^N is stable because $c \leq \frac{1-\lambda}{(n-1)(M-1)}$ (see Prop. 3.1). For any network $g \in \{G \setminus g^N\}$, $\exists\{i,j\} : d_{ij}(g) > 1$. By connecting, the distances of i and j decrease at least by 1, while their betweenness is not reduced. So for $c < \frac{1-\lambda}{(n-1)(M-1)}$ they want to connect and, therefore, the network is unstable. The complete network is not uniquely stable for other values of c because if $c \geq \frac{1-\lambda}{(n-1)(M-1)}$, the star or the empty network is stable (see Prop. 3.1).

(ii) Take any network g with a pendant i and his neighbor j. We show that the condition implies that one of the agents wants to sever this link.

a) Agent i does not reduce brokerage by severing this link. Removing the link increases his distances at least by $M-1$ (when agent j is also a pendant) and at most by $M-1+(n-2)(M-2)$ (when agent j is directly linked to all other agents). Therefore, agent i will not keep the link if $c > \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}$.

b) Similarly, for agent j: severing a link increases his distances by $M-1$ and hence decrease his closeness by $\frac{1}{n-1}$. Moreover, he was on the shortest path between i and any other agent in this component. The more agents in this component, the higher the incentive to keep this link. The maximum brokerage of $n-2$ is attained for a connected network. Therefore, agent j wants to sever the link for $c > \frac{1-\lambda}{n-1} + \frac{\lambda 2(n-2)}{(n-1)(n-2)}$ rendering the network unstable.

(iii) Prop. 3.1 (ii) shows that the empty network is stable for $c \geq \frac{1-\lambda}{n-1}$. In addition we have to show that any non-empty network is unstable for large enough c. This becomes obvious, when considering that the marginal benefits are bounded, while the marginal costs c can be arbitrarily large.[50] □

Proof of Prop. 3.4. We show that in a network with a diameter of $d > p$, there exists a pair of players who can increase their utility by forming a link.
Let Take any network g with a diameter of $d > p \geq 1$. Let i and j be two players at maximal real distance ($d_{ij}(g) = d$) and consider one shortest path between them. By forming the link ij, agent i does not only decrease his distance to j, but also to some players on this shortest path. The change in distances stemming from that path can easily be derived as:

$$\Delta d_i(d) = \begin{cases} 2+3+5+...+d-3+d-1 = \frac{1}{4}d^2 - \frac{1}{4} \text{ , for odd d} \\ 1+3+5+...+d-3+d-1 = \frac{1}{4}d^2 \text{ , for even d} \end{cases} \tag{3.2}$$

Therefore, $\beta_i^{ij}(g) \geq \frac{d^2-1}{4(M-1)(n-1)}$. This also holds for $\beta_j^{ij}(g)$. It remains to show that the marginal costs c are lower than this marginal benefit. $p = \sqrt{4c(n-1)(M-1)+1}$ implies that $c = \frac{p^2-1}{4(M-1)(n-1)}$. The marginal costs are smaller than the marginal benefit, because $p < d$. □

Lemma 3.5.1. *In the centrality model with $\lambda = 0$, the following holds: For any marginal costs $c \in (0, \infty)$ there exists a $q \in \mathbb{N}$ such that no network with a circle of size q or smaller is (PS), with q satisfying $q = max_{z \in \mathbb{N}} z$ s.t. $c > \frac{n[z-2+\frac{1}{4}(z-3)^2]}{z(M-1)(n-1)}$ and $z \geq 3$.*

Proof of Lemma 3.5.1. The idea of the proof is the following: We look – among all configurations with a circle of size q – for the one which maximizes the minimal marginal benefit of a link that belongs to the circle. For marginal costs c higher than this frontier, we conclude that all circles must break down.
Let $\bigcirc$ be the set of q nodes that belong to the circle and let $\circ$ be the set of the q links that form the circle, $ij \in \circ$ and $i, j \in \bigcirc$. A circle of length q can only be stable if no

[50]To verify the argument consider $\tilde{c} > 1$. Since for any g and any i, $CLOSE_i(g) \in [0, 1]$ and $BETW_i(g) \in [0, 1]$ while $\lambda \in [0, 1]$, it must hold that $\forall g, \forall i$, $0 \leq b_i(g) \leq 1$. Thus for any two networks g and g' it holds that $b_i(g) - b_i(g') \leq 1$. In any non-empty network, there is a player who can reduce his costs by $\tilde{c} > 1$, while he cannot loose more benefits than 1.

one wants to cut a link. This condition says that all players must face a marginal benefit of their links that is not lower than the marginal costs, that is $min_{i\in\bigcirc}\beta_i^{ij}(g) \geq c$, $ij \in \circ$, where $\beta_i^{ij}(g) = CLOSE_i(g) - CLOSE_i(g \setminus ij)$. If c is sufficiently high, no graph can satisfy this condition, that is when

$max_{g:\exists\circ\in g} min_{i,j\in\bigcirc}\beta_i^{ij}(g) < c$.

We argue that the maximizer $\hat{g}$ is one with all disposable nodes $k \notin \bigcirc$ (a) equally distributed among the nodes $i \in \bigcirc$ and (b) each is only attached to one node of the circle (meaning that all paths of a disposable node to other members of the circle go through one member of the circle) and (c) there are no links across the circle (this graph looks like a "sun": one circle and rays).

The reason for (c) is that a circle without crossing ties leads to higher damage if links are cut. The reason for (b) is that the marginal benefit of a link of the circle is maximal when the disposable nodes do not offer paths to avoid the circle. The reason for (a) is that the equal distribution leads to the maximin: By allocating the disposable players it is possible to construct higher marginal utility for certain players of the ring, but this will always mean that we reduce the marginal utility of other players of the ring. [51]

Derivation of the Maximin value:

Note first, the marginal benefit that the deletion of link $ij(\in \circ)$ means to i is the negative of the marginal increase in benefit that establishing the link ij in the graph $\hat{g} \setminus ij$ means. The marginal benefit for $\beta_i^{ij}(\hat{g}\setminus ij)$ always consists of the worths of j and the players on the geodesic. Besides them (they are exactly the members of the circle), there are disposable players that are not on the circle. The worth of the disposable players depends on the node on the circle they are attached to. In fact, they all have the same worth as their gatekeeper on the circle. So, the distribution of the disposable nodes can be seen as a weighting of the marginal benefits coming from the players on the circle.

Let now q be odd and a divisor of n such that we can distribute the disposable nodes equally among the circle members. Then, as the worths are summed up, we can just take the marginal benefit of a circle without disposable nodes and scale it by the number of disposable nodes plus

[51] Think of an extreme case where we maximize the marginal utility of one player i. We use the graph where all disposable players are attached to j (who is a neighbor of i in the circle). Whereas the deletion of ij has maximal consequences for i, they have only minimal consequences for j. So, this is "far" from being the maximin configuration.

one.

$$\beta_i^{ij}(\hat{g}) = \beta_i^{ij}(\circ)t$$
$$\beta_i^{ij}(\hat{g}) = \frac{(q-2)+\frac{1}{4}(q-3)^2}{(M-1)(n-1)}, \text{ where } t = \frac{n}{q}$$
$$\beta_i^{ij}(\hat{g}) = \frac{n[q-2+\frac{1}{4}(q-3)^2]}{q(M-1)(n-1)}$$

For c higher than that, no circle of size q can be stable. If we did not take q to be odd and if we did not take n to be a multiple of q the threshold would be lowered a bit. So, the restriction does also hold for these cases. □

Proof of Prop. 3.5. To proof the result, we employ the Lemma 3.5.1 and combine it with a theorem by Alon, Hoory, and Linial 02 (see Diestel, 2005, Th 1.3.4). Let $d(g) := \frac{1}{n}\sum_{i\in N} l_i(g)$ be the average degree and $q(g) :=$ the size of the smallest circle in g, which is defined to be large if there are no circles. Let $\delta \in \mathbb{R}$ and $\rho \in \mathbb{N}$. The theorem states:

If [A] $d(g) \geq \delta(\geq 2)$ and [B] $q(g) \geq \rho$, then [C] $n \geq n_0 = \begin{cases} 1+\delta\sum_{k=0,\ldots,\frac{\rho-3}{2}}(\delta-1)^k & for \quad \rho \quad odd \\ 2\sum_{k=0,\ldots,\frac{\rho}{2}-1}(\delta-1)^k & for \quad \rho \quad even. \end{cases}$

We transform the logical structure of [A] and [B] implies [C] into not[C] and [B] implies not[A]. To get [B], we fix a certain ρ, here $\rho = 4, 5$, and use the proposition that for $c > \frac{n[\rho-2+\frac{1}{4}(\rho-3)^2]}{\rho(M-1)(n-1)}$ there are no circles of size ρ or smaller in stable networks (see Lemma 3.5.1 above). To get not[C], we choose δ such that $n_0 = n+1$ (this is possible as n_0 is a function of δ and ρ). These conditions together imply not[A], that means that $d(g) < \delta$.

We now use this procedure for different values of ρ:
Let $\rho = 4$. Then [B] reduces to $c > \frac{9n}{16(M-1(n-1)}$. not[C] is achieved by choosing $\delta = \frac{1}{2}n + \frac{1}{2}$ because it implies that $n = n_0 - 1 = -1 + 2\sum_{k=0,1}(\delta-1)^k = 2\delta - 1$. not[A] tells that $d(g) < \delta$. So we get the result: If $c > \frac{9n}{16(M-1)(n-1)}$, then $d(g) < \frac{1}{2}n + \frac{1}{2}$.
Let $\rho = 5$. Then [B] reduces to $c > \frac{4n}{5(M-1)(n-1)}$. not[C] is achieved by choosing $\delta = \sqrt{n}$ because it implies that $n = n_0 - 1 = \delta\sum_{k=0,1}(\delta-1)^k = \delta^2$. not[A] tells that $d(g) < \delta$. So we get the result: If $c > \frac{4n}{5(M-1)(n-1)}$, then $d(g) < \sqrt{n}$. □

Proof of Prop. 3.6. For part (i) we show that in a network with a diameter of d, two players at maximal real distance increase their benefits at least by $\frac{(\lfloor\frac{d}{2}\rfloor-1)\lfloor\frac{d}{2}\rfloor}{(n-1)(n-2)}$ by establishing a link between them. This implies for c below that level, that the network is unstable. If a network has a larger diameter than d, then it also contains two players at distance d.

Take any network g with a diameter of $d \geq 4$. Let i and j be two players at maximal real distance $d_{ij} = d$ and consider one of their shortest paths. Consider two agents i' and j' on that geodesic such that $d_{ii'}(g) + d_{jj'}(g) < \frac{d-1}{2}$ ($i', j' \neq i$, but $j' = j$ is allowed). It holds that i is not on any shortest path between i' and j'. It must also hold that $d_{i'j'}(g) = d - d_{ii'}(g) - d_{jj'}(g)$.[52]

Establishing ij adds a new path from i' to j' (that uses i). This path is of length $d_{ii'}(g) + d_{jj'}(g) + 1 =: p_{i'j'}$. It is shorter than their former shortest path, as straightforward transformations show:

$$d_{ii'}(g) + d_{jj'}(g) < \frac{d-1}{2} \quad \Leftrightarrow \quad d_{ii'}(g) + d_{jj'}(g) < d - d_{ii'}(g) - d_{jj'}(g) \tag{3.3}$$

$$\Leftrightarrow \quad p_{i'j'} < d_{i'j'}(g). \tag{3.4}$$

Thus, $\frac{\tau^i_{i'j'}(g \cup ij)}{\tau_{i'j'}(g \cup ij)} - \frac{\tau^i_{i'j'}(g)}{\tau_{i'j'}(g)} = \frac{1}{1} - 0 = 1$. In words: player i increases his brokerage, since he is on the unique shortest path between i' and j' now, while he was not before. In order to compute the minimal change in brokerage, one can compute the number of pairs whose distance shortens. The straightforward derivation yields the following (where χ is the number of pairs whose distance shortens and $\lfloor x \rfloor$ stands for the next lower integer):

$$\chi(d) \geq 1 + 2 + 3 + 4 + ... + \lfloor \frac{d}{2} \rfloor \tag{3.5}$$

$$\Delta Brokerage \geq \frac{1}{2}(\lfloor \frac{d}{2} \rfloor - 1)\lfloor \frac{d}{2} \rfloor \tag{3.6}$$

$$\Delta BETW_i(g) \geq \frac{(\lfloor \frac{d}{2} \rfloor - 1)\lfloor \frac{d}{2} \rfloor}{(n-1)(n-2)}. \tag{3.7}$$

Since $\lambda = 1$, the marginal benefits are at least as high as the marginal costs c. The argument holds for both players i and j such that g is not pairwise stable.

Part (i) does not count the marginal benefits of pairs (i' and j') who are on such a distance that the establishment of ij builds an additional shortest path. Those pairs also increase the marginal benefit of i, but the amount depends on the number of shortest paths. For part (ii) we show that there exists such a pair.

Assume that for $g \in G$ $\exists i, j : 2 < d_{ij}(g) < M$. By the existence of a path longer than two we know that $N_i(g) \neq \emptyset$ and $N_j(g) \neq \emptyset$. As this path is a geodesic, we know that $\exists k : k \in N_i(g)$ and $k \notin N_j(g)$; and $\exists l : l \in N_j(g)$ and $l \notin N_i(g)$. In fact, $N_i(g) \cap N_j(g) = \emptyset$ which implies

[52] The distance cannot be shorter because this would imply that there exists a shorter path for i and j to connect. The distance cannot be longer since there is this path on the geodesic.

that $d_{kj}(g) \geq 2$. Let $g' := g \cup ij$, be the graph when we add the link ij. Then the path (k, i, j) is a geodesic between k and j in g'. This generates some betweenness value for i. Idem for j. As the marginal costs c are lower than any marginal benefit, we conclude $u_i(g) < u_i(g')$ and $u_j(g) < u_j(g')$ which contradicts (PS).

Proof of Prop. 3.7. Let g be a network as described in the Prop. 3.7 and let R be the set of agents in $j's$ component (excluding him). We have to show that the marginal benefit of a bridge is at least as high as the marginal costs, that is $\beta_i^{ij}(g) \geq (1-\lambda)\frac{(r+1)(M-\frac{1}{2}r-1))}{(M-1)(n-1)} + \lambda\frac{2l(r+1)}{(n-1)(n-2)}$. For this purpose we show that the change in distances for player i is at least $(r+1)(M-\frac{1}{2}r-1)$ and the change in brokerage is $l(r+1)$.

It holds that $d_{ik}(g) = M$ for all $k \in R \cup j$. In $g \cup ij$ a shortest path from i to $k \in R$ uses a shortest path of j to k. Thus, it holds that $d_{ik}(g \cup ij) = d_{ij}(g \cup ij) + d_{jk}(g \cup ij) = 1 + d_{jk}(g)$. Therefore, the change in distances is $M - 1 + \sum_{k \in R}(M - d_{jk}(g) - 1) = M - 1 + rM - r - \sum_{k \in R} d_{jk}(g)$. The change in distances is minimal if the sum of $j's$ distances is maximal. This is attained when the component of j forms a line network: $\max \sum_{k \in R} d_{jk}(g) = 1 + 2 + 3 + ... + r = \frac{1}{2}r(r+1)$. Therefore the change in distances is at least $M - 1 + rM - r - \frac{1}{2}r(r+1) = (r+1)[M - 1 - \frac{1}{2}r]$. By definition there are no paths between the two components. When forming ij, each path across the two components of g uses agent i. Thus, his brokerage (number of shortest paths) increases by $l(r+1)$. □

Proof of Prop. 3.8. Let $\kappa_i^{ij}(g) := \{k \in N : d_{ik}(g) < d_{ik}(g \setminus ij)\}$ be the number of players whose distance to player i increases, when cutting the link ij in network g (always $j \in \kappa_i^{ij}(g)$). Let $H \subset N$ be a completely linked group of size q and $h \in G$ its links. Note that cutting a link $ij \in h$ increases the distance to any agent $k \in \kappa_i^{ij}(g)$ by exactly 1, as i is linked to other agents, which are linked to j ($d_{ij}(g \setminus ij) = 2$). Therefore,

$$\sum_{k \in N} d_{ik}(g \setminus ij) - \sum_{k \in N} d_{ik}(g) = \mid \kappa_i^{ij}(g) \mid . \tag{3.8}$$

For all $j, h \in H$ it holds that $\kappa_i^{ij}(g) \cap \kappa_i^{ih}(g) = \emptyset$ and it also holds that $\kappa_i^{ij}(g) \cap \kappa_j^{ij}(g) = \emptyset$. Fixing a player $i \in H$ allows us to distinguish q distinct Kappa sets, one for each of his neighbors and one for some neighbor j ($N \supseteq \{(\kappa_i^{ij}(g))_{j \in N_i(g)}, \kappa_j^{ij}(g)\}$). As a consequence it holds that

$$\min_{i,j \in H} \mid \kappa_i^{ij}(g) \mid \leq \frac{n}{q}. \tag{3.9}$$

Implying that $\min_{i,j \in H} \beta_i^{ij}(g) \leq (1-\lambda)\frac{n}{q(M-1)(n-1)}$. □

Proof of Prop. 3.9. Consider $g \in G$, $i \in N$ with $l_i(g) \geq 2$ and $Clust_i(g) = 1$. We show that i is willing to sever a link. First, we observe that there is no betweenness incentive to keep the link.

Lemma 3.5.2. *For all graphs g' and players j with $l_j(g') \geq 2$, it holds that $Clust_j(g') = 1 \Leftrightarrow BETW_j(g') = 0$.*

For a proof of this Lemma see, e.g., Everett et al. (2004) or Gago Alvarez (2007). It follows that $BETW_i(g) = 0$ and cannot decrease by deletion of a link. So, the marginal benefit of a link only depends on the change in closeness.

Let $\kappa_i^{ij}(g) := \{k \in N : d_{ik}(g) < d_{ik}(g \setminus ij)\}$ be the set of players whose distance to player i increases, when cutting the link ij in network g (always $j \in \kappa_i^{ij}(g)$). Note that cutting a link to a neighbor (j) increases the distance to any agent $k \in \kappa_i^{ij}(g)$ by exactly 1, as i is linked to other agents, which are linked to j ($d_{ij}(g \setminus ij) = 2$). Therefore,

$$\sum_{k \in N} d_{ik}(g \setminus ij) - \sum_{k \in N} d_{ik}(g) = \mid \kappa_i^{ij}(g) \mid . \tag{3.10}$$

For all $j, h \in N_i(g)$ it holds that $\kappa_i^{ij}(g) \cap \kappa_i^{ih}(g) = \emptyset$ (this note is also used in Calvó-Armengol and Ilkiliç, 2007). So each player $k \in N \setminus \{i\}$ can only be in one of the kappa sets for player i. Therefore, it holds that

$$\min_{j \in N_i(g)} \mid \kappa_i^{ij}(g) \mid \leq \frac{n-1}{l_i(g)}. \tag{3.11}$$

Implying that $\min_{j \in N_i(g)} \beta_i^{ij}(g) \leq (1-\lambda)\frac{n-1}{l_i(g)(M-1)(n-1)} = \frac{1-\lambda}{l_i(g)(M-1)}$. □

4 Efficiency in the Centrality Model

In this chapter we want to assess the efficiency of the centrality model, which was introduced in the last section.

To motivate the discussion of efficiency let us have a look at the average closeness and average betweenness (among all players in a given network). One might expect that if agents strive for closeness (betweenness) centrality, the stable networks exhibit a high average closeness (betweenness). The following two figures depict the distribution of average betweenness and average closeness for different sets of stable networks. For example, the boxplot above .30 depicts the upper quartile and the lower quartile of average closeness (resp. betweenness) of all networks found stable for $\lambda = 0.3$ (the thick black line represents the mean, and the vertical black line reaches from the top ten to the bottom ten percent). As a reference point, the dashed lines mark the lower and upper quartile for the set of all non-isomorphic networks.

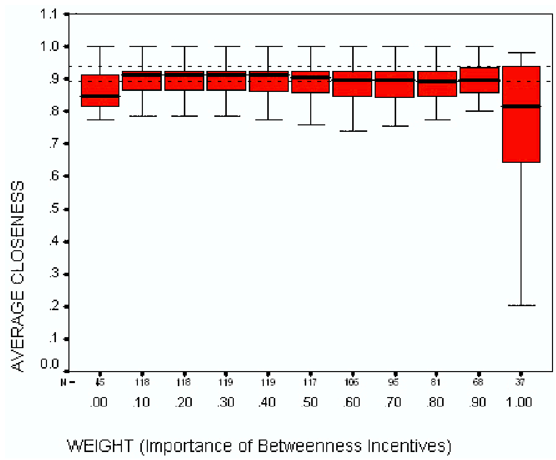

Figure 16: Average closeness of all stable networks (enum. $n = 8$).

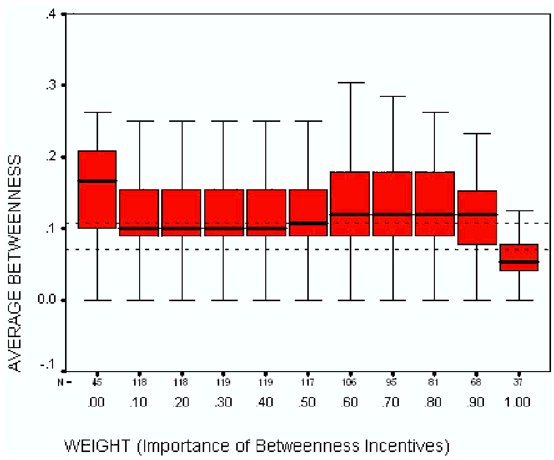

Figure 17: Average betweenness of all stable networks (enum. $n = 8$).

What we observe seems paradoxical: The stable networks for closeness incentives $\lambda = 0$ (betweenness incentives $\lambda = 1$) do not exhibit high average closeness (betweenness), in comparison to an arbitrary network. Moreover, average betweenness is specially high for the stable networks with full weight on closeness $\lambda = 0$. Note that this observation does not allow us to draw direct conclusions for the (in)efficiency of the stable networks – it serves as a motivation for further investigating this topic.

Throughout this chapter, we restrict attention to the utilitarian welfare function, $w := w^u = \sum_{i \in N} u_i(g)$. To be consistent with the other chapters, we speak of "welfare", while the adequacy of this term is questionable. It is not canonically defined (in a setting where centrality is important to the agents) which kinds of networks are socially desirable and which are not. The justification for utilitarian welfare is to measure the agents' actions by their own goals.[53]

To assess (in)efficiency (of the emerging networks) in this model it takes a systematic approach. Figure 18 illustrates some of the basic interdependencies. Starting point is the set of all stable networks depicted in the center. Those networks exhibit certain measurable properties, some of which are important determining the welfare. For each network one can evaluate its welfare according to the utilitarian welfare function. Note that the utilitarian welfare function is dependent on the parameters λ, c of the utility function. For each welfare function, that is for each setting of parameters (λ, c), there is a value of maximal welfare, which is depicted on the left bottom. Identifying the networks that maximize welfare means finding the efficient networks. On the right hand side of the scheme, the emerging networks are represented. They result of a dynamic process that starts with (a sample of) all networks and follows individual improvements determined by the utility function. The utility function, in turn, is also set by the parameters λ, c. Thus, a change in the parameters λ, c has three effects: (a) Since the utilitarian welfare function changes, a given network is evaluated differently (its welfare can change). (b) For the same reason different networks may be efficient (welfare maximizing); (c) changing the utility function affects the dynamics such that different networks might emerge and be stable.

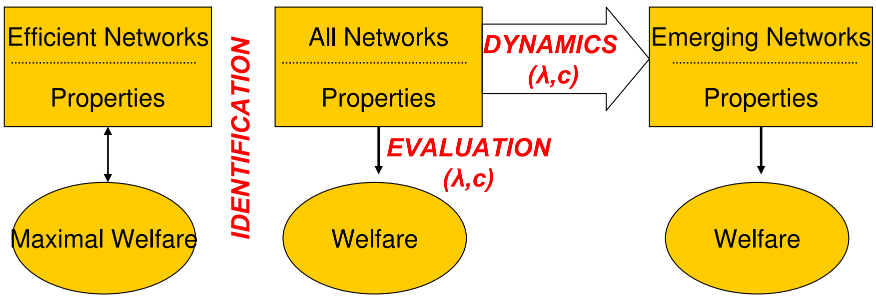

Figure 18: Scheme of basic interdependencies between starting networks, efficient networks and emerging networks.

This chapter is structured as follows. The first section analyzes how the utilitarian welfare of a network is determined by its properties. Section 4.2 identifies the efficient networks for

[53] It can be argued that benefits of central positions are not only based on the absolute centrality of an agent, but on his relative advantage (being more central than others). While this interpretation of the centrality model is consistent with chapter 3, where agents try to improve centrality, it is not consistent with this chapter, where we evaluate the efficiency of a network by the sum of utilities that consist of absolute centrality values.

different settings of λ, c. Section 4.3 compares the welfare of the emerging networks (in the simulation) to the welfare of the starting networks and the welfare of the efficient networks. Section 4.4 addresses why the emerging networks differ from the efficient networks in this model. As in any chapter, all proofs are collected in the last section.

4.1 Determinants of Welfare

In order to find the efficient networks it is necessary to understand the determining properties of welfare in our model. We will derive three expressions of utilitarian welfare in this section (W1, W2, W3), each based on a different set of network statistics. By linearity of the utility function and additivity of the welfare function we can separate welfare into contributions from aggregate closeness, aggregate betweenness and aggregate degree. While aggregate degree times c determines the total costs of a network, aggregate closeness and betweenness weighted by λ determine the total benefits.

$$w(g) = \sum_{i \in N} u_i(g) = (1-\lambda) \sum_{i \in N} CLOSE_i(g) + \lambda \sum_{i \in N} BETW_i(g) - c \sum_{i \in N} DEGREE_i(g) \tag{W1}$$

We will refer to this expression as W1, based on the first set of network statistics determining welfare: aggregate closeness, aggregate betweenness and aggregate degree. Those network statistics can be broken down into more basic network statistics.

Aggregate costs are clearly determined by the density ($\sim$ number of links) of a network: $\sum_{i \in N} DEGREE_i(g) = \sum_{i \in N} l_i(g) = 2l(g)$. When aggregating closeness, we sum up the sum of distances for each player. Denote by $SD(g) := \sum_{j<k} d_{jk}(g)$ the sum of all distances in a network (counting M for unconnected pairs). Then aggregate closeness can be rewritten as $\sum_{i \in N} CLOSE_i(g) = \sum_{i \in N} \left[\frac{M}{M-1} - \frac{\sum_{j \in N} d_{ij}(g)}{(M-1)(n-1)} \right] =$ $\frac{nM}{(M-1)} - \frac{\sum_{i \in N} \sum_{j \in N} d_{ij}(g)}{(n-1)(M-1)} = \frac{nM}{M-1} - \frac{2SD(g)}{(n-1)(M-1)}$. This is a linearly decreasing function in SD going from n for the complete network to 0 for the empty network.

The Determinants of Aggregate Betweenness

It is possible to find a similar result for aggregate betweenness, although the determinants of aggregate betweenness are not that obvious. The key idea is to switch the sequence of summation: While betweenness computation sums for each player i the derived benefit over all other pairs of agents $\{j, k : j \neq i \text{ and } k \neq i\}$; we can also fix a pair of agents $\{j, k\}$ and compute the betweenness benefit that they mean for other agents. This actually turns out to be a simple function of their distance (as stated in Lemma 4.5.1 in the appendix).[54] Applying the lemma, we get the following theorem:

Theorem 4.1. *Let* $\alpha = \frac{2}{(n-1)(n-2)}$ *(the normalization factor). Then aggregate betweenness of a network can be written as*

$$\sum_{i \in N} BETW_i(g) = \alpha \sum_{j<k:d_{jk}(g)<M} (d_{jk}(g) - 1). \tag{4.1}$$

The interpretation of this sum is straightforward: Any connected pair $\{j, k\}$ contributes to the aggregate betweenness proportionally to its distance. Two agents that are 4 steps away from each other, mean 3 units of betweenness for their brokers. (That might be three agents on the only geodesic or multiple agents sharing those units according to their position on the geodesics.) If two agents j, k are not connected or if they are directly linked, then they do not create any betweenness benefits for others. Gago Alvarez (2007) shows a similar result that links the diameter of a network (the longest distances between any two connected agents) with high aggregate betweenness.

So, we write the aggregate betweenness as a function of the "real distances", the distances between connected agents (not counting M for unconnected pairs). To rewrite the aggregate betweenness in dependence of the sum of distances $SD(g)$, we have to correct for the unconnected pairs. Let $\nu(g)$ be the number of unconnected pairs in g, that is $\nu(g) := \#\{i, j \in N \mid d_{ij}(g) = M\}$. Because $SD(g) = \sum_{j<k} d_{jk}(g) = \sum_{j<k:d_{jk}(g)<M}(d_{jk}(g)) + M * \nu(g)$, it holds that $\sum_{j<k:d_{jk}(g)<M} d_{jk}(g) = SD(g) - M\nu(g)$, while $\sum_{j<k:d_{jk}(g)<M} -1 = -\frac{1}{2}n(n-1) + \nu(g)$. Using this to rewrite (4.1) yields

$$\sum_{i \in N} BETW_i(g) = \frac{SD(g) - \frac{1}{2}n(n-1) - \nu(g)(M-1)}{\frac{1}{2}(n-1)(n-2)}. \tag{4.2}$$

This function is linearly increasing in the sum of all distances SD and goes from 0 to $\frac{1}{3}n$.

This implies for connected networks that both, aggregate closeness and aggregate betweenness are fully determined by SD, the sum of all distances.[55] Since the former is linearly increasing

[54] Arguably, the brokerage benefits that a pair offers are rather constant than increasing in their distances such that being on a long shortest path, is not as beneficial as being on a short one (see, e.g., Goyal and Vega-Redondo, 2007). But this is not true for the standard definition of betweenness (see, e.g., Wasserman and Faust, 1994) used here.

[55] For connected networks $g \in \bar{G}$ aggregate betweenness simplifies to $\sum_{i \in N} BETW_i(g) = \frac{2SD(g)}{(n-1)(n-2)} - \frac{n}{n-2}$.

and the latter is linearly decreasing in SD, we conclude that for connected networks aggregate closeness and aggregate betweenness are fully negatively correlated.[56]

How to Determine Welfare

Putting the considerations on aggregate degree, aggregate closeness, and aggregate betweenness together, we can write the welfare of a network in the following way:

$$w(g) = (1-\lambda)\left[\frac{nM}{M-1} - \frac{2SD(g)}{(n-1)(M-1)}\right] + \lambda\frac{2SD(g) - n(n-1) - 2\nu(g)(M-1)}{(n-1)(n-2)} - c2l(g) \quad \text{(W2)}$$

This very simple function is only dependent on three network statistics: the number of unconnected pairs $\nu(g)$, the number of links $l(g)$, and the sum of distances $SD(g)$. This is the second set of properties that suffices to compute the welfare of a given network. We will refer to this expression as W2.

Let $W : \mathbb{R}^3 \to \mathbb{R}$ be the following mapping: $W(l,\nu,SD) = (1-\lambda)\left[\frac{nM}{M-1} - \frac{2SD}{(n-1)(M-1)}\right] + \lambda\frac{2SD-n(n-1)-2\nu(M-1)}{(n-1)(n-2)} - c2l$ (representing the welfare in dependence of the second set of network statistics). Observe that this function is decreasing in l and ν. Its slope in respect to SD depends on the weight λ. Taking the first derivative we get $W'(\bar{l},\bar{\nu},SD) = -(1-\lambda)\frac{2}{(n-1)(M-1)} + \lambda\frac{2}{(n-1)(n-2)}$. Therefore, for small λ welfare is decreasing in the sum of distances and for large λ welfare is increasing. It is easy to compute that $W'(SD) = 0 \Leftrightarrow \lambda = \frac{n-2}{M+n-3} =: \hat{\lambda}$. Thus, for $\lambda = \hat{\lambda}$ the sum of distances cancels out from the welfare function. Recall that aggregate closeness and aggregate betweenness are fully negatively correlated for connected networks. For $\lambda = \hat{\lambda}$ aggregate betweenness and aggregate closeness are in balance. Some important conclusions will follow from this observation later on. For the moment, note that the disaggregation provides a handy way to compute the welfare of a given network.

For example, consider the circle network $g^{\bigcirc}$. If n is even its basic properties are $\nu(g) = 0$, $l(g) = n$ and $SD(g) = \frac{1}{8}n^3$.[57] Consequently, the first set of network statistics is $\sum_{i\in N} CLOSE_i(g) = \frac{Mn}{M-1} - \frac{n^3}{4(n-1)(M-1)}$, $\sum_{i\in N} BETW_i(g) = \frac{n^3-4n^2+4n}{4(n-1)(n-2)}$, and $\sum_{i\in N} DEGREE_i(g) = 2l(g)$. The welfare of this network is still dependent on the setting of the parameters: $w(g) = (1-\lambda)\frac{Mn}{M-1} - \frac{n^3}{4(n-1)(M-1)} + \lambda\frac{n^3-4n^2+4n}{4(n-1)(n-2)} - c2l(g)$. In the same way we can straightforwardly compute

[56] In this paper we work with closeness as the reverse distance, not the inverse distance according to Freeman (1979). Using Freeman's definition of closeness, then the correlation between aggregate (Freeman) closeness and aggregate betweenness is -0.978 (for all non-isomorphic networks of size 8), as its correlation to aggregate (inverse) closeness is +0.978.

[57] As always, the derivation of distances might be tedious, but is a straight forward task.

the welfare of the different prominent networks (empty, complete, star, and line network):

$$w(g^{\emptyset}) = (1-\lambda)0 + \lambda 0 - 2c0 = 0 \tag{4.3}$$

$$w(g^{N}) = (1-\lambda)n + \lambda 0 - cn(n-1) \tag{4.4}$$

$$w(g^{\star}) = (1-\lambda)\frac{nM-2n+2}{M-1} + \lambda 1 - 2c(n-1) \tag{4.5}$$

$$w(g^{|}) = (1-\lambda)[\frac{Mn}{M-1} - \frac{n^3-n}{3(M-1)(n-1)}] + \lambda\frac{1}{3} - 2c(n-1). \tag{4.6}$$

An Alternative Way to Determine Welfare

It is possible – and will be helpful – to further disaggregate the three network statistics used in W2 above. Let $\psi^x(g) := \#\{i < j : \quad d_{ij}(g) = x\}$ denote the number of pairs (of agents) who are in distance x to each other in network g. Clearly, $\psi^M(g) = \nu(g)$. Furthermore, it holds that $\psi^1(g) = l(g)$. Moreover, $SD(g) = \sum_{j<k} d_{jk}(g) = 1 * \psi^1(g) + 2 * \psi^2(g) + ... + (n-1)\psi^{n-1}(g) + M\psi^M(g)$. So, all network statistics needed to evaluate the welfare of a network can be disaggregated into the number of pairs at certain distances $\psi(g) \in \mathbb{N}^n$. Plugging in the "ψs" into W2 above and simplifying, yields the following expression (see "Derivation of W3" in Section 4.5):

$$w(g) = \alpha_0 + \alpha_1\psi^1(g) + \alpha_2\psi^2(g) + \alpha_3\psi^3(g) + ... + \alpha_{n-1}\psi^{n-1}(g) + \alpha_M\psi^M(g), \tag{W3}$$

$$\begin{aligned}
\alpha_0 &= \frac{(1-\lambda)Mn}{M-1} - \frac{\lambda n}{n-2} \\
\alpha_1 &= \frac{2\lambda}{(n-1)(n-2)} - \frac{(1-\lambda)2}{(M-1)(n-1)} - 2c \\
\alpha_x &= x\left[\frac{2\lambda}{(n-1)(n-2)} - \frac{(1-\lambda)2}{(M-1)(n-1)}\right] \qquad (x = 2,3,...,n-1) \\
\alpha_M &= \frac{2\lambda}{(n-1)(n-2)} - \frac{(1-\lambda)2M}{(M-1)(n-1)}.
\end{aligned}$$

Note that this function W3, such as W1 and W2, is linear in its network statistics ($\psi(g)$). The α-coefficients may be positive or negative. The relation of those coefficients (e.g. $\alpha_1 > \alpha_2$ or $\alpha_1 \leq \alpha_2$) will be crucial in determining the efficient networks.

Figure 19 summarizes the findings of this chapter. The utilitarian welfare of a network consists of total benefits and total costs. Total costs are determined by the aggregate degree and the marginal costs c. Total benefits are a linear combination of aggregate closeness and aggregate betweenness. Those properties can be further disaggregated: aggregate closeness is determined by the sum of distances. Aggregate betweenness depends on the sum of distances and the number of unconnected pairs. Aggregate degree is just twice the number of links in a network. A further disaggregation is possible by considering the number of pairs at each distance. This set of network statistics also contains all necessary information to compute the welfare of a network.

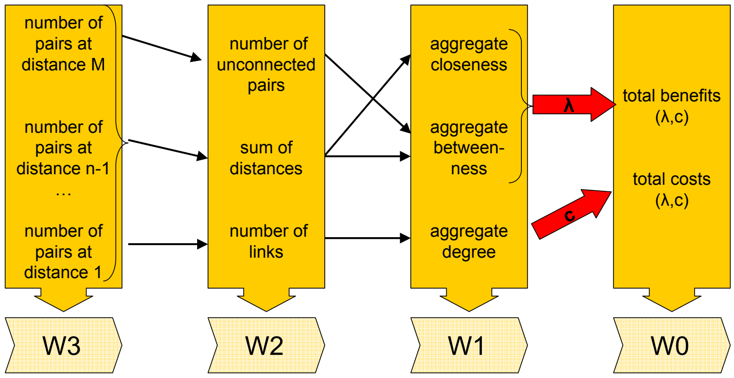

Figure 19: Scheme illustrating how utilitarian welfare can be disaggregated into different sets of network statistics.

Note that each expression of the welfare function W1, W2 and W3 is dependent on the parameters c and λ. Given a network, increasing the marginal costs c clearly decreases welfare (by increasing total costs). An analogue problem occurs for changes of the weight λ. Increasing λ (i.e. in W1) shifts more weight on aggregate betweenness and less on aggregate closeness. Since the dimensions of these numbers differ (see remark below), this results in an absolute decrease in welfare (without being a statement about the set of networks that is assessed).

In the following sections we will use the results of this section to find the efficient networks and discuss the tension between stability and efficiency.

Remark 4.1.1 (Relative costs)**.** *Although individual closeness and betweenness were normalized to take values between zero and one, aggregate closeness and aggregate betweenness differ in their range of possible realizations:* $\sum_{i\in N} CLOSE_i(g) \in [0, n]$, $\sum_{i\in N} BETW_i(g) \in [0, \frac{1}{3}n]$.[58] *Aggregate closeness can exhibit a higher value than aggregate betweenness, as betweenness is a more exclusive network statistic in the sense that there are networks where everybody has full closeness (g^N), but not so for betweenness.*

[58]For completeness, note that $\sum_{i\in N} DEGREE_i(g) \in [0, n^2 - n]$ because each agent can maximally have $n-1$ links.

This fact seems largely unproblematic, because we kept an individual unit of benefit as a numeraire throughout the model. However, one can also argue differently: Since the maximal total benefits are decreasing with λ, keeping c constant and increasing λ may also be interpreted as increase in some kind of "relative costs". In Chapter 3 and Chapter 4 we do not explicitly follow this line of interpretation; but we respect differing interpretations by avoiding direct comparisons of stable and emerging networks (across the weight λ) for fixed costs c.

4.2 Efficient Networks

In this section we identify the efficient networks depending on the parameter setting (λ, c) and check whether they are (uniquely) stable for those settings.

4.2.1 Finding the Efficient Network(s)

A special class of networks plays an important role in the following elaboration. A network g is called a local tree of size l if it holds that $l(g) = l$ and there is a component of size $l+1$ (that is: $l+1$ players are connected with l links). Clearly, any tree is a local tree, but also networks with isolates and one non-trivial component that is minimal is a local tree. According to this definition, also the empty network $g^{\emptyset}$ belongs to the class of local trees.

The proofs of this section operate with W3, expressing welfare as a linear function of the number of pairs at various distances. Clearly, each network has $\frac{1}{2}n(n-1)$ pairs of agents, any of which must be in exactly one distance: $\sum_{x=1,2,...,n-1,M} \psi^x(g) = \frac{1}{2}n(n-1)$. Thus, the welfare of a network depends on the distribution of those pairs and the relation of the α-coefficients. About the distribution we use two basic facts: first, $\psi^1(g) = l(g)$ for any network. Secondly, there is a relation between the number of links of a network and its minimal number of unconnected pairs (stated in Lemma 4.5.2 in the proof of Prop. 4.1).

Now, in order to find the efficient networks, the relation of the α-coefficients in W3 is crucial. Straightforward computations show the following:

$$\alpha_1 \geq \alpha_x \iff c \leq (x-1)\left[\frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)}\right] (x = 2, ..., n-1) \tag{4.7}$$

$$\alpha_1 \geq \alpha_M \iff c \leq \frac{1-\lambda}{n-1} \tag{4.8}$$

$$\alpha_2 > \alpha_3 > \alpha_4 > ... > \alpha_{n-1} \Longleftarrow \lambda < \hat{\lambda} \tag{4.9}$$

$$\alpha_{n-1} > \alpha_{n-2} > ... > \alpha_3 > \alpha_2 \Longleftarrow \lambda > \hat{\lambda} \tag{4.10}$$

$$\alpha_x \geq \alpha_M \quad (\forall x = 2, 3, ..., n-1). \tag{4.11}$$

where $\hat{\lambda} := \frac{n-2}{M+n-3}$, as before.[59] To see that expressions (4.9) and (4.10) hold, consider the definition of α_x for $x = 2, 3, ..., n-1$. This expression increases with x, if the term the brackets is positive. That is $\left[\frac{2\lambda}{(n-1)(n-2)} - \frac{(1-\lambda)2}{(M-1)(n-1)}\right] > 0 \iff \lambda > \frac{n-2}{M+n-3} = \hat{\lambda}$. From now on we distinguish the cases, where $\lambda < \hat{\lambda}$, $\lambda > \hat{\lambda}$ and $\lambda = \hat{\lambda}$.

Uniquely Efficient Networks

Proposition 4.1. *In the centrality model with $\lambda < \hat{\lambda}$, the following networks are uniquely efficient (according to the utilitarian welfare function):*
(i) the complete g^N network for $c < \frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)} =: T1$*,*
(ii) the star network $g^\star$ for $T1 < c < \frac{(1-\lambda)(Mn-2n+2)}{2(M-1)(n-1)} + \frac{\lambda}{2(n-1)} := T2$
(iii) the empty network $g^\emptyset$ for $c > T2$.

To get an intuition for this result consider W2 and the subcase that full weight is on closeness ($\lambda = 0$). In that case, welfare is determined by a trade-off between density ($l(g)$) and distances ($SD(g)$). Figure 20 shows this trade-off by depicting the maximal aggregate closeness for any level of density. For $l(g) \in \{0, 1, ..., n-1\}$, the networks maximizing aggregate closeness are local stars. For $l(g) \in \{n, n+1, ..., \frac{1}{2}n(n-1)\}$, among the networks maximizing aggregate closeness are supersets of the star $g \supseteq g^\star$.

Now, the trade-off between total costs and total benefits is evaluated by the welfare function. Figure 20 also represents one iso-welfare curve (iso-welfare curves have slope $2c$). For extremely low c, the network with the maximal closeness, that is the one with minimal SD, is efficient. This is clearly the complete network, where the distance between any two agents is one. For extremely high c, the network with minimal density, the empty network is efficient. For networks that are not connected, $l(g) \in \{0, 1, ..., n-2\}$, there are increasing marginal returns from density, while after the possibility of connected networks, the maximal closeness can only be linearly increased. Therefore, for moderate c an efficient network typically has $n-1$ links (i.e. seven links in Figure 20). An exception is the case where the slope of the depicted function coincides with the slope of the welfare function ($c = T1$), then there is a multitude of efficient networks as will be shown in Prop. 4.3 (i). Before, let us have a look at the case $\lambda > \hat{\lambda}$.

[59] Indeed, the threshold coincides with the threshold where distances cancel out in W2. That is why we denote it with the same letter $\hat{\lambda}$.

Proposition 4.2. *In the centrality model with $\lambda > \hat{\lambda}$, the following networks are uniquely efficient (according to the utilitarian welfare function):*
(i) the line network $g^{|}$ for $c < \frac{1-\lambda}{2(n-1)}\left[\frac{Mn}{4(M-1)} - \frac{n^3-n}{3(M-1)(n-1)}\right] + \lambda\frac{n}{6(n-1)} =: T3$,
(ii) the empty network $g^{\emptyset}$ for $c > T3$.

To grasp the intuition consider the subcase where $\lambda = 1$. In that case total benefits only depend on the aggregate betweenness. By Theorem 4.1, aggregate betweenness depends on the sum of real distances in a network (and total costs, as always, depend on the density). Thus, welfare of a network is determined by a trade-off between the sum of real distances and density ($\sim$ number of links). Figure 21 shows the maximal aggregate betweenness for different numbers of links. Clearly, the empty network has no links and zero aggregate betweenness. Additional links can first be used to increase aggregate betweenness until the point of a minimal connected network, $l(g) = n - 1$. Thereafter dense networks necessarily mean a low maximal aggregate betweenness – think of a very dense network: because distances are short, there are no intermediation rents (betweenness benefits). Since total costs are increasing in the number of links, a network with $l(g) > n - 1$ can never be efficient. This is shown as Lemma 4.5.3 in proof of Prop. 4.2 in the more general case $\lambda \geq \hat{\lambda}$. The second part of the proof is to establish that for $l \leq n - 1$ the efficient network must be a local line. Finally, we conclude that for high enough c, the empty network is efficient, while otherwise the line network is efficient.

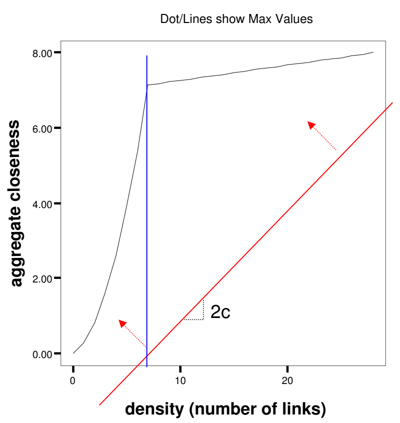

Figure 20: Maximal aggregate closeness by number of links ($n = 8$).

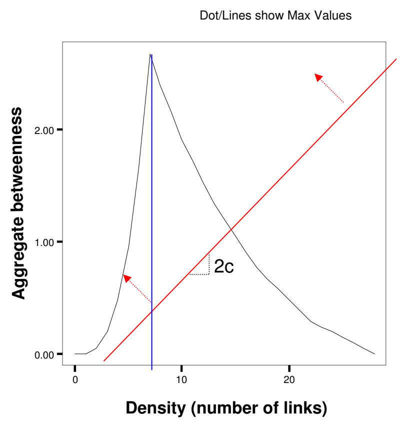

Figure 21: Maximal aggregate betweenness by number of links ($n = 8$).

Multiple Efficient Networks

The results of Prop. 4.1 and Prop. 4.2 almost cover the full parameter space $[0, 1] \times \mathbb{R}_+$.

Exceptions are situations where two of the prominent networks, e.g. line network and complete network, exhibit the same welfare. It can be checked that at the boundaries between the uniqueness of two such networks, both of them are efficient: i.e. the line network and the empty network are both efficient for $c = T3$ and $\lambda \geq \hat{\lambda}$ and (similarly for the complete network and the star, the star and the empty, and the star and the line). In special cases those networks are not the only efficient networks, as addressed by Prop. 4.3.

Proposition 4.3. *In the centrality model according to the utilitarian welfare function, the following holds:*
(i) For $\lambda = 0$ and $c = \frac{1}{(M-1)(n-1)}$, any network that does not have a distance larger than two (and is connected) is efficient.
(ii) For $\lambda = \hat{\lambda}$ and $c < \frac{n^2-n}{2n-3}$ $(= T2 = T3)$, a network is efficient if and only if it is a tree.
(iii) For $\lambda = \hat{\lambda}$ and $c > \frac{n^2-n}{2n-3}$, the empty network $g^{\emptyset}$ is uniquely efficient.

To get an intuition for the results (ii) and (iii) consider W2. Recall that for $\lambda = \hat{\lambda}$ $(= \frac{n-2}{M+n-3})$ the sum of distances SD cancels out of the welfare function (as noted in Section 4.1). Thus, welfare only depends on the number of unconnected pairs ν and the number of links l. For $l \leq n-1$, local trees handle this trade-off best, by connecting the maximal number of pairs with a given number of links (see Lemma 4.5.2). For $l > n-1$, it is possible to connect the network. Any connected network leads to the same aggregate benefits, $\hat{v} = \frac{n^2-n}{2n-3}$. Consequently, any network that is minimally connected (trees) is efficient for not too high c.

Figure 22 summarizes Prop. 4.1, Prop. 4.2, and Prop. 4.3. Roughly speaking, we have the empty network for extremely high c, the complete network for extremely low c (and some weight on closeness), the star network for high weight on closeness (low λ) and the line network for high weight on betweenness (high λ). Inside the regions a trivial network is unique. In the point $(\lambda, c) = (0, T1)$ there is a multitude of efficient networks. The second special case is the frontier where the star and the line network are efficient. Any tree is efficient for this setting.

It is not that surprising that for a large part of the parameter space the efficient networks are the star and the line. Both of them are minimal connected networks (trees). Moreover, the line is the tree with maximal sum of distances of $SD(g^{|}) = \frac{1}{6}n^3 - \frac{1}{6}n$ and the star network is the tree with minimal sum of distances $SD(g^{\star}) = (n-1)^2$.

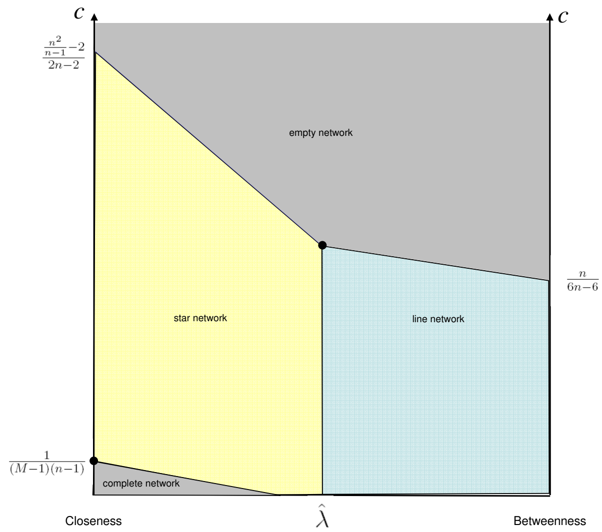

Figure 22: Efficient networks in the parameter space.

4.2.2 Stability of Efficient Networks

Having found the efficient networks for different parameter settings, we can now check whether those networks are also stable for the corresponding parameters. In fact, this is the usual way to assess the tension between stability and efficiency (see Jackson, 2004).

By combining the results of Section 3.2.1 (Prop. 3.1 and Prop. 3.3) with the results above (propositions 4.1, 4.2 and 4.3), we get the following partial characterization of the tension:

Proposition 4.4. *In the centrality model (according to the utilitarian welfare function), the following holds:*

(i) For $c < T1$, the complete network g^N is uniquely efficient and also uniquely stable.

(ii) For sufficiently large c, the empty network is uniquely stable and also uniquely efficient.

(iii) For $\frac{1-\lambda}{(n-1)(M-1)} < c < \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$ and $\lambda < \hat{\lambda}$, the star network is stable and uniquely efficient.

(iv) For $T1 < c < \frac{1-\lambda}{(n-1)(M-1)}$, the complete network is uniquely stable, while the star network is efficient.

(v) For $c > \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$, any stable network that is non-empty is inefficient.

(vi) For $c < T3$ and sufficiently large λ, the line network is uniquely efficient but not stable.

Figure 23 illustrates the results. Beginning with the bottom left, there is the area where the complete network is efficient and uniquely stable (i). In the region where the star network is efficient, there is an area such that the complete network is uniquely stable (iv). Then for some cost range, the star network is stable and efficient (iii). For higher c, the star network is still efficient, but not stable, because it is above the threshold of (v) which is depicted by the thick line with a kink. This result uses the fact that any (non-empty) efficient network contains loose ends, while Prop. 3.3 excludes such networks from being stable (this idea was already used in Jackson and Wolinsky, 1996). For λ arbitrarily close to 1, any c is above this threshold, which is used in statement (vi). For the area (*) Prop. 4.4 does not exclude the stability of the line network. However, for the stability of the line network it is required that two players, e.g., those at the ends, do not form a link. This condition is rarely satisfied for c below the threshold of (v), as discussed in Remark 4.5.1 in proof of Prop. 4.4.

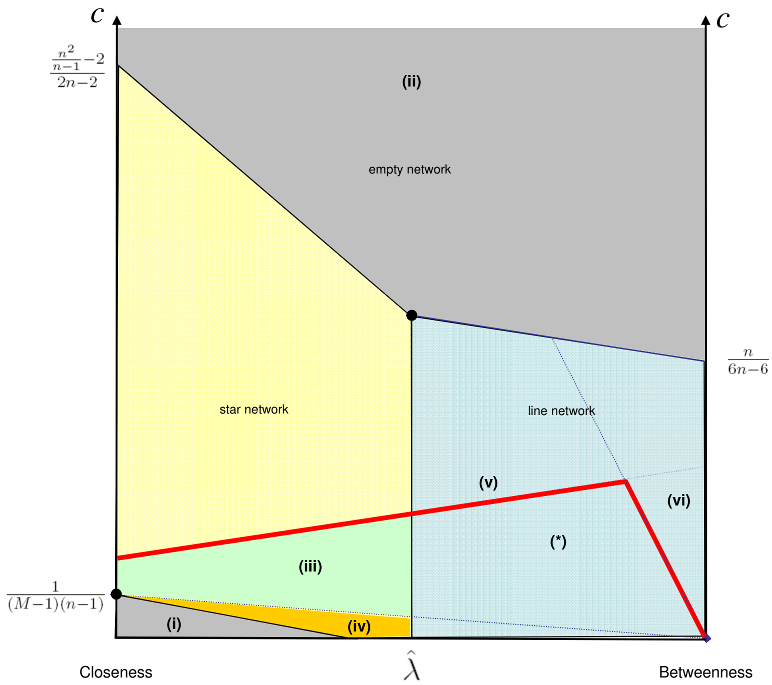

Figure 23: Stability of the efficient networks.

Considering a dynamic process that only leads to stable networks, such as the one we use in the simulation (see Subsection 3.2.3), the analysis above reveals the following: efficiency is guaranteed if $c < T1$ where the complete network is efficient and uniquely stable; and for high enough c, where the empty network is efficient and uniquely stable. In a small parameter range, an efficient outcome is possible but not sure; that is when the star network is efficient and stable but not uniquely so (and this might also happen for the line network in very special settings). In most other cases an inefficient outcome certainly occurs, either because, the line is efficient but not stable, or because the star is efficient but not stable.

In general, assessing the stability of the efficient networks only partially answers whether a dynamic process is efficient or not. First, it can neither guarantee efficient nor inefficient outcomes when some of the stable networks are efficient but not all. A second shortcoming is that, even if inefficient outcomes are certain, it does not quantify inefficiency, in the sense of measuring the gap of welfare between the efficient network and the set of stable networks. Several authors fill this gap by considering the welfare of the worst stable network ("price of anarchy") and the welfare of the best stable network ("price of stability").[60] Given the simulation method, we overcome those two shortcomings directly: we estimate welfare of emerging networks by weighting the stable networks by their frequency of emergence. So, while in this subsection we assessed whether the efficient networks can or cannot emerge; the next section tries to answer "how efficient" the emerging networks are.

4.3 Relative Efficiency of Emerging Networks

In this section we estimate the welfare of emerging networks for different settings by running simulations. We will use the simulation method introduced in Subsection 3.2.3. For the parameter settings where there is a unique stable network, there is no point in runnings simulations. For all other settings, the simulation can complement the equilibrium analysis of the model. However, we have to restrict ourselves to a few starting conditions. Taking the same settings as in the last chapter – those are depicted in Figure 6 and Figure 7 – we examine cases where the star network is efficient and stable and where the line network is efficient but unstable.

In the first subsection, we introduce the method and make a first prediction of relative efficiency for $n = 8$. Then, we recheck the findings for $n = 14$ and adding some more network statistics (aggregate closeness and aggregate betweenness). In Subsection 4.3.3 we reconsider the simulation for $n = 14$ and describe the emerging networks by the relevant network statistics according to W2.

[60] See Koutsoupias and Papadimitriou (2007), Corbo and Parkes (2005), Fabrikant et al. (2003), or Brandes et al. (2008).

4.3.1 How to Estimate Relative Efficiency? Simulation for Eight Agents

Recall that changing parameter settings λ, c, does not only affect the induced networks – the point we are interested in – but also the welfare of a given network (see the discussion at the end of Section 4.1). So, before comparing the welfare of different scenarios, one has to make sure not to deal with artifacts.

Table 11 shows the average welfare of all non-isomorphic networks for different levels of weight λ and marginal cost c, which will be the benchmark for the emerging networks (since these are the starting networks). Negative values of welfare simply mean that aggregate costs are larger than aggregate benefits. Since $w(g^{\emptyset}) = 0$ for any (λ, c), negative values can be interpreted as being worse than the empty network.

Table 11: Mean of welfare of starting networks (any non-isomorphic network for $n = 8$).

costs	epsilon	v_lo	lo	med	hi
$\lambda = 0$		6.7008	6.2612	5.3876	3.6376
$\lambda = 0.1$		6.0627	5.6231	4.7495	2.9995
$\lambda = 0.5$		3.5105	3.0709	2.1973	0.4473
$\lambda = 0.9$		0.9583	0.5187	-0.3549	-2.1049
$\lambda = 1$	0.7010	0.3202	-0.1194	-0.9930	-2.7430

Instead of writing the absolute values of welfare, let us express it in relation to the efficient network. We define the relative efficiency of a network g as a fraction of the maximal possible welfare $e(g) := \frac{w(g)}{w(g^*)}$, where g^* is efficient.[61] By definition of the term efficiency, a network either is efficient (a welfare maximizer) or is not efficient. In contrast, the relative efficiency allows us to state *"how efficient"* a network is.

The maximal welfare is taken from the efficient networks as described in the Table 12. For weights $\lambda = 0$ (and $\lambda = 0.1$), the complete network is efficient if $c = v\_lo$ (bold) and the star network if $c = lo, med, hi$ (in italics). For any other parameter combination the line network is efficient.

By Prop. 4.4 (i) we know that for the two settings where the complete network is efficient (bold) the complete network emerges, such that the simulation would have not been necessary. In the six settings where the star network is efficient (italic), it is also stable but not uniquely so. In the thirteen settings where the line is efficient, the line is not stable.

[61] Relative efficiency makes some implicit assumption on the cardinality of the utility function. E.g., it is not invariant to addition of constants in the welfare function (resp. the utility functions).

Table 12: Welfare of efficient networks ($n = 8$).

costs	epsilon	v_lo	lo	med	hi
$\lambda = 0$	x	**7.1824**	*6.7188*	*6.2820*	*5.4070*
$\lambda = 0.1$	x	**6.3824**	*6.1045*	*5.6677*	*4.7927*
$\lambda = 0.5$	x	3.9861	3.7663	3.3295	2.4545
$\lambda = 0.9$	x	2.7673	2.5475	2.1107	1.2357
$\lambda = 1$	2.6530	2.4626	2.2428	1.8060	0.9310

Simulation Results for Eight Agents

Table 13 shows the detailed simulation results for $n = 8$. The first and the second column describe the total benefits and the total costs. Their difference is the absolute welfare, written in the third column. The fourth column shows the relative efficiency of the emerging networks. The last column is the benchmark describing the relative efficiency of the starting networks. Presented is the mean (and the standard deviation) of these variables for all simulation runs. Since we started with any possible network multiple times, the mean serves as an estimator for starting with an arbitrary network.

Table 13: Estimation of relative efficiency for $n = 8$.

WEIGHT	COSTS	TOTAL BENEFITS	TOTAL COSTS	WELFARE	REL. EFFICIENCY	REL. EFFICIENCY STARTING
λ=0 (Closeness)						
	v_lo	8.0000	0.8193	**7.1807**	100.0%	93.3%
		0.00	0.00	0.00		
	lo	7.3590	0.7435	**6.6155**	98.5%	93.2%
		0.05	0.08	0.02		
	med	6.9729	1.0554	**5.9175**	94.2%	85.8%
		0.12	0.11	0.06		
	hi	6.4656	1.7591	**4.7065**	87.0%	67.3%
		0.18	0.07	0.17		
λ=0.5						
	v_lo	4.0360	0.5103	**3.5257**	88.4%	88.1%
		0.01	0.05	0.06		
	lo	4.0578	0.6958	**3.3621**	89.2%	81.5%
		0.01	0.07	0.08		
	med	4.0881	1.0360	**3.0521**	91.7%	66.0%
		0.01	0.10	0.11		
	hi	4.1217	1.7885	**2.3332**	95.1%	18.2%
		0.09	0.11	0.12		
λ=1 (Betweenness)						
	epsilon	0.4485	0.0344	**0.4141**	15.6%	26.4%
		0.13	0.01	0.13		
	v_lo	0.4529	0.3875	**0.0654**	2.7%	13.0%
		0.13	0.09	0.08		
	lo	0.2201	0.3911	**-0.1710**	-6.9%	-5.3%
		0.12	0.22	0.12		
	med	0.0008	0.0012	**-0.0004**	0.0%	-55.0%
		0.03	0.04	0.02		
	hi	0.0598	0.1037	**-0.0440**	-4.7%	-294.6%
		0.24	0.41	0.17		

First, one can note that the standard deviation is small compared to the absolute values.

Thus, differences across groups (i.e. settings) are interpretable.

For full weight on closeness incentives ($\lambda = 0$), relative efficiency of the emerging networks is around 90 percent, with slightly higher numbers for low costs c. For $c = v\_lo$, the complete network is uniquely stable such that this simulation would not have been necessary.[62] The efficiency of the starting networks is improved for any cost level. For a mixture of incentives ($\lambda = 0.5$), efficiency is also close to 100 percent. For low costs, the improvement of the starting networks is modest (because relative efficiency of the starting networks is already high, as will be explained later on). For $c = hi$, there is a strong improvement from 18 percent to 95 percent.

Finally, let us look at the case of full weight on betweenness ($\lambda = 1$). The welfare of the emerging networks is much lower than the efficient network's welfare. But the emerging networks are not only inefficient; they exhibit lower welfare than the starting networks for $c = lo, v\_lo, \epsilon$. That means that the dynamics of betweenness have decreased the welfare. For $c = med, hi$ this is not true anymore. For those cost levels the empty network emerges in almost all simulation runs,[63] while the line network would still be efficient. The empty network exhibits a welfare of zero, which means an improvement to the starting networks that all have negative welfare.

4.3.2 Simulation Results for Fourteen Agents

A second simulation is used to recheck the findings. The computation of relative efficiency works as above. In addition, we show the results for $\lambda = 0.1$, $\lambda = 0.9$ and also present the relevant network statistics according to W1. In this simulation the efficient networks are only the star (for $\lambda \leq 0.1$) and the line (for $\lambda \geq 0.5$) (see Figure 7 in Subsection 3.2.3 and the results of Section 4.2). In addition to the simulation results, the first two rows of Table 14 present the network statistics for these two networks.

Note first that the results on relative efficiency look very similar to the case of $n = 8$. Relative efficiency is between 90 percent and 100 percent for the weights 0, 0.5, and also for $\lambda = 0.1$; again, the dynamics of betweenness ($\lambda = 1$) lead to drastically suboptimal results. Now, reconsider W1:

$$w(g) = (1-\lambda)\sum_{i\in N} CLOSE_i(g) + \lambda \sum_{i\in N} BETW_i(g) - c\sum_{i\in N} DEGREE_i(g).$$

[62] For larger network size, e.g. $n = 14$, this is not true anymore, such that using very low costs in the simulation is informative.

[63] This point was already mentioned in Section 3.3.2. Here it can be seen by total costs and total benefits being so close to 0.

Table 14: Estimation of relative efficiency for $n = 14$ and properties determining W1.

WEIGHT	COST	AG'DEGREE	AG'CLOSE	AG'BETW	WELFARE	REL'EFF
line network $g^{|}$		26.0	9.692	4.667	depends	depends
star network $g^{\star}$		26.0	13.077	1	depends	depends
$\lambda = 0$	very low	63.2	13.297	0.762	12.796	99.4%
	low	45.0	13.014	1.068	12.254	97.0%
	medium	34.0	12.506	1.618	11.327	93.0%
	high	27.4	11.928	2.245	9.993	88.9%
$\lambda = 0.1$	very low	53.4	13.238	0.825	11.574	99.2%
	low	45.0	13.020	1.062	11.066	96.8%
	medium	33.4	12.483	1.643	10.241	93.4%
	high	27.4	11.906	2.269	9.017	89.8%
$\lambda = 0.5$	very low	64.6	13.305	0.753	6.517	93.5%
	low	35.8	12.993	1.091	6.439	95.5%
	medium	28.6	12.522	1.601	6.069	96.7%
	high	26.6	11.962	2.207	5.205	97.3%
$\lambda = 0.9$	very low	81.2	13.341	0.714	1.332	26.8%
	low	39.8	12.893	1.199	1.698	35.9%
	medium	29.2	12.512	1.612	1.687	39.5%
	high	25.4	11.577	2.053	1.215	36.4%
$\lambda = 1$	epsilon	142.2	13.143	0.199	0.056	1.2%
	very low	46.8	6.477	0.260	-0.111	-2.6%
	low	0.0	0.004	0.000	0.000	0.0%
	medium	1.6	0.337	0.030	-0.024	-0.7%
	high	0.2	0.049	0.007	-0.006	-0.2%

If $\lambda = 0$, aggregate closeness and aggregate degree determine the results. For $c = v\_lo$, the emerging networks have higher aggregate closeness than the efficient network (star) which implies higher total benefits, but they also exhibit higher aggregate degree implying higher total costs. In sum, their welfare is close to the efficient network. For higher c, the aggregate degree is close to the efficient network (the star has as any tree an aggregate degree of 26) such that the total costs are almost as in that network. However, aggregate closeness of the emerging networks is significantly lower than of the efficient network, implying that the relative efficiency is below 100 %. Similarly for $\lambda = 0.1$.

For weight $\lambda = 0.1$, aggregate closeness does not matter, but aggregate betweenness determines the total benefits. Aggregate betweenness is low in any of the simulation runs (compared to the line network with an aggregate betweenness of 4.67). Moreover, for very low c, total aggregate degree is high, implying high total costs. In sum the emerging networks are heavily inefficient. For $c = lo, med, hi$, the empty network emerges in most cases. This is indicated by the aggregate degree (and the aggregate closeness and aggregate betweenness) to be so close to zero.

For weight $\lambda = 0.9$, this is different. Most of the emerging networks are connected. Still aggregate betweenness is low in comparison to the efficient network. For $c = lo$, relative efficiency is lower than for $c = med$ by both reasons: lower total benefits (driven by aggregate betweenness), higher total costs (driven by aggregate degree).

Finally, for $\lambda = 0.5$, both aggregate closeness and aggregate betweenness determine total benefits. For any c, the emerging networks have lower aggregate betweenness and higher aggregate closeness than the efficient network. It can be shown that aggregate benefits are extremely similar in all emerging networks (see Table 13 above).[64] For high enough c, aggregate degree is close to 26, the aggregate degree of the efficient network(s). For low c, the average degree is higher such that networks are not as efficient.

Before addressing why emerging networks differ from efficient networks, let us describe in some more detail how they differ.

4.3.3 Comparison of Emerging Networks and Efficient Networks (Simulation for Fourteen Agents)

Table 15 depicts the same simulation results as Table 14, but this time showing some more basic network statistics (that are relevant for welfare according to W2). Instead of presenting ν, the number of unconnected pairs of a network, we indicate in column CONNECTED, the fraction of emerging networks that are connected. AV'RDIS is defined as the average distance between any two connected pairs, that is $AV'RDIS(g) := \frac{\sum_{j<k:connected} d_{jk}(g)}{\#\{j<k:connected\}}$.[65] This gives information about the typical length of a shortest path (not considering unconnected pairs). Finally, the statistics of the other columns are as in Table 14 above. In addition to the simulation results, the first two rows of Table 15 present the network statistics for the efficient networks.

Before interpreting the results, recall W2 (derived Section 4.1):

$$w(g) = (1-\lambda)\left[\frac{nM}{M-1} - \frac{2SD(g)}{(n-1)(M-1)}\right] + \lambda\left[\frac{2SD(g) - n(n-1) - 2\nu(g)(M-1)}{(n-1)(n-2)}\right] - c2l(g).$$

The term in the first brackets stands for aggregate closeness; the term in the second brackets stands for aggregate betweenness and $2l(g)$ equals aggregate degree. Recall that welfare is decreasing in l and decreasing in ν, while the effect of SD is ambivalent: for high weight on closeness ($\lambda < \hat{\lambda}$) welfare is decreasing in the sum of distances and for high weight on

[64]In fact, all connected networks (also the star and the line network) have the same total benefits for $\lambda \approx 0.48$, as discussed in Subsection 4.2.1.

[65]In a connected network, this network statistic is fully correlated to SD but not so in unconnected networks.

Table 15: Estimation of relative efficiency for $n = 14$ and properties determining W2.

WEIGHT	COST c	CONNECTED	$l(g)$	$SD(g)$	AV'RDIS	WELFARE	REL'EFF
line network $g^{\mid}$		yes	13.0	455	5.00	depends	depends
star network $g^{\star}$		yes	13.0	169.00	1.86	depends	depends
$\lambda = 0$	very_low	100%	31.6	150.42	1.65	12.796	99.4%
	low	100%	22.5	174.31	1.92	12.254	97.0%
	medium	100%	17.0	217.23	2.39	11.327	93.0%
	high	100%	13.7	266.10	2.92	9.993	88.9%
$\lambda = 0.1$	very_low	100%	26.7	155.33	1.71	11.574	99.2%
	low	100%	22.5	173.82	1.91	11.066	96.8%
	medium	100%	16.7	219.14	2.41	10.241	93.4%
	high	100%	13.7	267.95	2.94	9.017	89.8%
$\lambda = 0.5$	very_low	100%	32.3	149.75	1.65	6.517	93.5%
	low	100%	17.9	176.11	1.94	6.39	95.5%
	medium	100%	14.3	215.87	2.37	6.069	96.7%
	high	100%	13.3	263.19	2.89	5.205	97.3%
$\lambda = 0.9$	very_low	100%	40.6	146.69	1.61	1.332	26.8%
	low	100%	19.9	184.55	2.03	1.698	35.9%
	medium	100%	14.6	216.75	2.38	1.687	39.5%
	high	96%	12.7	295.70	2.83	1.215	36.4%
$\lambda = 1$	epsilon	83%	71.1	163.45	1.18	0.056	1.2%
	very_low	3%	23.4	726.72	1.45	-0.111	-2.6%
	low	0%	0.0	1273.63	1.71	0.000	0.0%
	medium	0%	0.8	1245.50	2.00	-0.024	-0.7%
	high	0%	0.1	1269.82	2.50	-0.006	-0.2%

betweenness ($\lambda > \hat{\lambda}$) welfare is increasing in the sum of distances. The effect of SD cancels out for $\lambda = \hat{\lambda}$. Thus, let us distinguish three cases from now on:

Case I: $\lambda << \hat{\lambda}$, i.e. $\lambda = 0$ *or* 0.1.

Total benefits are dominated by aggregate closeness, which is decreasing in the sum of distances SD. Total costs depend on the number of links. Thus, there is a trade-off between low number of links and short distances.[66] For the starting conditions used in the simulation, the star network handles this trade-off best[67], using 13 links to produce an aggregate closeness of 13.077 (where 14 is the maximum reached in the complete network), which corresponds to a sum of distance of $SD(g^{\star}) = (n-1)^2 = 169$. For very low c, networks emerge that even have a lower SD (implying higher aggregate closeness), but with using much more links: 31.6, resp. 26.7, on average. The higher c, the closer the number of links approaches the number 13 of the star network. However, the SD is not as small implying that the aggregate closeness is lower than the aggregate closeness of the efficient network.

[66]This trade-off is depicted in Figure 20.
[67]Neither the complete network nor the empty network is efficient, for $n = 14$ and $c = v\_lo, lo, med, hi$.

Observe that for $c = hi$ the predicted number of links of the emerging networks is 13.7 and all of the emerging networks have at least 13 links since they are connected. This means that usually only critical links will be present in the long run because minimally connected networks have exactly 13 links. The trees that emerge differ from the efficient network (the star which is a special case of a tree) in that they feature longer distances.[68]

Case II: $\lambda >> \hat{\lambda}$, i.e. $\lambda = 1$ *or* 0.9.

In this case, total benefits are dominated by aggregate betweenness, which depends on the sum of distances SD and the number of unconnected pairs ν. Keeping the number of unconnected pairs fixed, aggregate betweenness is an increasing function in SD. If both network statistics vary, the interpretation is not straightforward. Large distances are only valuable if they are real, that is if two agents are connected (recall Theorem 4.1 for the determinants of aggregate betweenness). Thus, for unconnected networks we do not only consider SD, but also the network statistic $AV'RDIS$, providing information about the distances within components of a network.

As always, total costs are determined by the number of links in a network. The efficient network, the line graph, has the maximal aggregate betweenness of all networks based on the maximal sum of real distances, i.e. $SD(g^{|}) = \frac{1}{6}n^3 - \frac{1}{6}n = 455$.[69] As the star, it is minimally connected with 13 links. Recall that with fewer links, it is not possible to produce the same aggregate betweenness and this is also true for more than $n - 1 = 13$ links (see Figure 21).

Now let us look at the emerging networks of the simulation. All of them (from $c = \epsilon$ to $c = hi$) have very small aggregate betweenness compared to the efficient network, and consequently, all of them are heavily inefficient.

One main driver of that is depicted in the first column : (1) Many of the emerging networks for pure betweenness incentives $\lambda = 1$ are not connected. As noted several times, for $\lambda = 1$ and not too low costs (for $n = 14$ that is $c = lo, med, hi$) the empty network emerges in most of the simulation runs.

The non-connectedness, however, can only partially be responsible for the inefficiency. For $\lambda = 0.9$ almost all runs lead to connected networks, but still relative efficiency is low. Moreover, for $c = \epsilon$, where 83 percent of the emerging networks are connected, relative efficiency is also

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[68] As argued in Section 3.3.1, the star network only occurs for a special sequence of link formation (the order of pairs who are allowed to change their relationship).

[69] The empty network has the maximal SD but aggregate betweenness of zero.

extremely low: While the efficient network exhibits an aggregate betweenness of $4.\bar{6}$, none of the emerging network is close to that value (not presented in Table 15, but in Table 14.

Thus, there is a second driver of insufficient aggregate betweenness: (2) the distances between connected agents are short. The average real distance of the efficient network (the line) is 5, while the emerging networks have a average distance between 1.6 (for $\lambda = 0.9$ and $c = v\_lo$) and 2.8 (for $\lambda = 0.9$ and $c = hi$). Thus, for low enough costs, in the emerging networks most pairs of agents are directly or indirectly linked, while brokerage opportunities increase with long distances.

Remark 4.3.1 (Role of distances)**.** *Since this is a crucial point of the analysis, let us check this issue in the simulation for $n = 8$. For $n = 8, \lambda = 1, c = \epsilon$ the efficient network has an average real distance of 3. The starting networks have an average real distance of* 1.563*. Finally, the emerging networks have an average real distances of* 1.36*. This corresponds to a predicted betweenness of* 0.0561 *that is only half of an arbitrary starting network (mean is 0.091, quartiles are 0.071 and 0.107). For $c = \epsilon, v\_lo, lo$, the estimated AV'RDIS of the emerging networks is smaller than the distances of the starting networks and of the star network.*

The third network statistic is the number of links: (3) We observe that the emerging networks can be quite dense. This is more surprising when considering the number of emerging networks that are not connected.[70] Recall Table 7 in Subsection 3.3.2 showing that for $c = lo, v\_lo, \epsilon$ and $\lambda = 1$, the dominant network architecture of emerging networks are complete bipartite networks allowing for isolates besides (for $n = 14$). Those networks exhibit all three indicators of insufficient welfare according to W2: (1) not connected, (2) high density, (3) short distances.

Case III: $\lambda \approx \hat{\lambda}$, i.e. $\lambda = 0.5$.

A weight of 0.5 is very close to $\hat{\lambda} = 0.48$ (for $n = 14 = M$). In Section 4.1 we have shown that for $\lambda = \hat{\lambda}$, SD cancels out of W2, implying that welfare is only determined by the number of links l and the connectedness ν. The tradeoff for efficiency is that the number of unconnected pairs reduces welfare, while it takes links to connect agents. Any connected network has the same total benefit, i.e. constant $\hat{v} = \frac{n^2-n}{2n-3}$. The efficient networks for not too high costs c are

[70] Of course, problem indicator (3) (too dense networks) and problem indicator (2) (too short distances) are heavily related to each other. While problem (1) occurs in high cost settings; problems (2) and (3) occur for low enough costs. Consider $c = v\_lo$ and let the weight change from $\lambda = 0$ to $\lambda = 0.1$. Before the change, we have all problems of inefficiency, after the change, we are in a situation where agents with zero betweenness keep their links because of closeness incentives. Ruling out problem (1) leads to a jump in relative efficiency from a negative value to 26.8 percent.

minimally connected (trees).

By looking at the simulation results, we observe that all emerging networks are connected for the parameters we used ($c = v\_lo, lo, med, hi$). For high costs c, most of the emerging networks are connected with the minimal number of links (13). In the simulation the average number of links is 13.3 and the minimum is 13 (since all of them are connected). However, for lower costs, i.e. $c = lo, v\_lo$, significantly more than 13 links emerge. Thus, there are non-critical links persisting in the long-run, although they are useless from a welfare perspective (they do not reduce the number of unconnected agents). Still, for any starting condition we use, the dynamic process leads to networks that are close to efficiency. Note that we did not run simulations with costs higher than the threshold of Prop. 4.4 (v).[71]

Remark 4.3.2 (Cost effects)**.** *In all the simulation runs we observed a strong effect of c on the density (number of links, aggregate degree) of the emerging networks: the higher the marginal costs c, the lower the density (as already shown in Table 3 in Subsection 3.2.3). Moreover, the density of a network is strongly correlated to many other network statistics including connectedness and sum of distances (that determine welfare according to W2).*

Thus, Table 15 also shows how settings of the parameters λ, c can induce high total benefits and total costs, respectively their determinants. The estimated density is heavily decreasing with the marginal costs c. Moreover, the lower the marginal costs c, the lower SD, and the higher aggregate closeness. For aggregate betweenness the effect of c goes in the other direction: increasing costs increases aggregate betweenness by inducing longer real distances. This is only true up to a certain level where issues of unconnectedness become dominant: for high enough c, the emerging networks exhibit small aggregate closeness and small aggregate betweenness, e.g. the empty network.

Although the set of stable networks for closeness incentives exhibits high aggregate betweenness (as shown in Figure 17), the effect of the parameter λ on the aggregate closeness and betweenness is not clear. Any weight – except full weight on betweenness – leads to roughly the same level of average closeness and betweenness. For full weight on betweenness the average closeness and betweenness is lower because it converges to the empty network in most cases.

Summing up the assessment of efficiency: For enough weight on betweenness ($\lambda \geq 0.5 > \hat{\lambda}$), the stable and emerging networks are inefficient, except for $c > T3$, where the empty network is efficient. First, the efficient network (line) is usually not stable. Secondly, the (stable and)

[71]In fact, our conjecture is that for $c > \frac{1+\lambda}{n-1}$ the empty network is the most frequent outcome.

emerging networks exhibit low welfare. Thirdly, dynamics of betweenness regularly lower the welfare of the starting networks or lead to the empty network. For the dynamics of closeness, the picture is not so clear. In some parts the efficient networks (complete, star, empty) are also stable. For settings of c where the star is efficient and stable, it does not often emerge, but the emerging networks exhibit almost as high welfare. No simulations were run where the star is efficient but not stable (higher c than the frontier for stability of loose ends). While this section described *how* the emerging networks differ from the efficient networks in some settings, the next section addresses *why* this happens.

4.4 The Sources of Inefficiency

We have observed that agents that try to improve their utility do not automatically induce a process that leads to networks with a high sum of utilities. There are even situations where agents lower the welfare of the starting networks. In this section we try to trace back the inefficient outcomes to a discrepancy between individual interests and collective interests. First, we analyze the collective consequences of individual linking actions. Then, we discuss in three examples why stable networks differ from the efficient networks. Finally, we summarize the tension between centrality and efficiency in this model.

4.4.1 Individual versus Collective Interest

Let the marginal utility of player i in network g be $\mu_i^l(g) := u_i(g \cup l) - u_i(g \setminus l)$. While linking actions are made by individual players, they might affect the utility of any agent in the network. Consider a network g and and a network $g' = g \cup ij$. Recall that $w(g') \geq w(g) \iff \sum_{h \in N} u_h(g \cup ij) - \sum_{h \in N} u_h(g) \geq 0 \iff \sum_{h \in N} \mu_h^{ij}(g) \geq 0$; a network g' has a higher welfare than a network g if the sum of marginal utilities (of a change from g to g') is positive. To emphasize the role of the active and passive players we can also write

$$w(g') \geq w(g) \iff \mu_i(g) + \mu_j(g) + \sum_{k \in N \setminus \{i,j\}} \mu_k(g) \geq 0. \tag{4.12}$$

To analyze how different actions affect the utility of the focal players (i, j) and of the players not involved in a link $(k \in N \setminus \{i, j\})$, we have to analyze the fundamentals of utility in our model: $u_i(g) = (1 - \lambda)CLOSE_i(g) + \lambda BETW_i(g) - cl_i(g)$. Clearly, two agents forming a link increase their degree by 1, while the degree of any other player stays constant. Moreover, the following Lemma shows that individuals establishing a link, strictly increase their closeness and weakly increase their betweenness. If the link is critical, it also weakly increases the closeness and betweenness of all other players. If the link is non-critical, it also weakly increases the closeness of all other players, but not so for betweenness.

Lemma 4.4.1. *By the definition of closeness and betweenness the following holds:*

(i) For all g, *and* $ij \notin g$, $CLOSE_i(g \cup ij) > CLOSE_i(g)$ *and* $CLOSE_j(g \cup ij) > CLOSE_j(g)$ *and for all* $k \in N \setminus \{i, j\}$ *it holds that* $CLOSE_k(g \cup ij) \geq CLOSE_k(g)$.

(ii) For all g, *and* $i, j : d_{ij}(g) = M$, $BETW_i(g \cup ij) \geq BETW_i(g)$ *and* $BETW_j(g \cup ij) \geq BETW_j(g)$ *and for all* $k \in N \setminus \{i, j\}$ *it holds that* $BETW_k(g \cup ij) \geq BETW_k(g)$.

(iii) For all g, *and* $i, j : 1 < d_{ij}(g) < M$, $BETW_i(g \cup ij) \geq BETW_i(g)$ *and* $BETW_j(g \cup ij) \geq BETW_j(g)$; *and* $\sum_{k \in N \setminus \{i,j\}} BETW_k(g \cup ij) < \sum_{k \in N \setminus \{i,j\}} BETW_k(g)$.

Part (i) is based on a fundamental property that additional links can only reduce distances between players but never increase them. For part (ii) and (iii) w.r.t. i and j, we use that any new path in $g \cup ij$ (that was not present in g) uses link ij, such that the brokerage of i and j cannot decrease. Part (ii) and (iii) w.r.t $k \in N \setminus \{i, j\}$ distinguishes between critical and non-critical links. Critical links establish new connections and can lead to additional brokerage. Non-critical links reduce aggregate betweenness by reducing the sum of distances (without changing the number of unconnected pairs, see Eq. 4.2). Since the betweenness of the focal players cannot decrease, there must be other players k who lose betweenness benefits. This is plausible, because the link ij first of all takes away the brokerage benefits for all agents that were on their geodesics before. Moreover i, j can now be on shortest paths were others were before.

Effects on Welfare

When individuals alter the network structure, they do not consider the consequences for other players. First of all, there might be diverging interests between two agents (i and j) about forming a link. While one of them might want to form, the other one can hinder him; respectively, if the link is already present (g'), one agent can cut the link, without considering the harm it does to the other player involved. Such a problem can be relaxed by allowing agents to pay transfers to other players whom they form a link with. In Chapter 5 we define the concept of pairwise stability with transfers. Our focus here will be on a more robust problem: The effect of an action (of one or two players, i, j) on the players not involved in a link $k \in N \setminus \{i, j\}$.[72]

Suppose two agents i and j agree to form a link. With use of Lemma 4.4.1, we can partially characterize whether this agreement increases or decreases welfare:

Proposition 4.5. *In the centrality model for any $g \in G$ and $i, j \in N : u_i(g \cup ij) > u_i(g)$ and $u_j(g \cup ij) \geq u_j(g)$, the following holds (according to the utilitarian welfare function):*
(i) If $d_{ij}(g) = M$, then $w(g \cup ij) > w(g)$.
(ii) If $d_{ij}(g) < M$ and $\lambda \geq \hat{\lambda}$, then $w(g \cup ij) < w(g)$.
(iii) If $\lambda = 0$, then $w(g \cup ij) > w(g)$.

Part (iii) obviously follows from Lemma 4.4.1 (i): by assumption, player i strictly increases his utility, player j weakly increases his utility, while for all other players the benefits weakly increase (based on closeness only), but the costs (based on degree) do not change. Similarly, part (i) follows from Lemma 4.4.1 (i) and (ii). In contrast, part (ii) is directly shown by using W2. It is based on the increase of total costs, while the total benefits decrease. In fact, (ii) does not require the condition that $u_i(g \cup ij) > u_i(g)$ and $u_j(g \cup ij) \geq u_j(g)$, that is: it holds for any $g \in G$ and $i, j \in N : 1 < d_{ij}(g) < M$. In words Prop. 4.5 shows the following effect of a new link: if the link is critical, it increases welfare. If the link is non-critical and $\lambda \geq \hat{\lambda}$, then it decreases welfare. For $\lambda = 0$ the link increases welfare, whether it is critical or not. The only case that is not covered by Prop. 4.5 is when ij is not critical (for g) and $0 < \lambda < \hat{\lambda}$. Then welfare may increase or decrease (because aggregate benefits increase, while aggregate costs decrease).

Note that the statements are formulated for the addition of links. For non-severance the proposition must be adapted by $u_i(g \cup ij) \geq u_i(g)$. Then, statement (i) and (iii) are not strict anymore ($w(g \cup ij) \geq w(g)$), since it might happen that everybody is indifferent.[73]

Remark 4.4.1. *This proposition does not make statements about severance of links! Consider a network g and a player $i : u_i(g \setminus ij) > u_i(g)$. If the link is critical (i), we can show that for all players $k \in N \setminus \{i, j\}$ it holds that $u_k(g \setminus ij) \leq u_k(g)$, but the effect for j is not clear. In the third case (iii) ($\lambda = 0$), it can be shown that for all $k \in N \setminus \{i, j\}$ it holds that $u_k(g \setminus ij) \leq u_k(g)$, but the effect for j is not clear (since his closeness decreases and his costs decrease). In the*

[72] We will informally call this effect "spillovers" throughout this section, while in the next chapter we formally define externalities and analyze their role in strategic network formation.
[73] A new link need not have spillovers, e.g. a link between two isolated players.

discussion of integrating an isolate in Subsection 3.3.3, we have already observed that marginal utilities of a new link can be very different. Only in the second case (ii) (if the link is non-critical and $\lambda \geq \hat{\lambda}$), the effect of cutting a link is clear: welfare increases. Considering W2, one can see that total benefits do not decrease, while total costs strictly decrease. Agent j then might have negative marginal utility, but he cannot block i's decision.

Let us study how the discrepancy between individual and collective interests drives inefficiency in three examples, one for each case of Prop. 4.5.

4.4.2 Example A: Betweenness Incentives and Very Low Costs

Consider the setting $\lambda = 1, c = v\_lo$, which is a subcase of Case II in Subsection 4.3.3. Recall that in that case the line network is uniquely efficient but not stable. In the simulation we observed heavily inefficient outcomes: Welfare is not only low compared to the efficient network, but also the welfare of the starting networks is lowered by the dynamics. Figure 24 depicts the most frequently emerging network in this setting for $n = 8$, the balanced complete bipartite network, emerging in 25 percent of all simulation runs.[74]

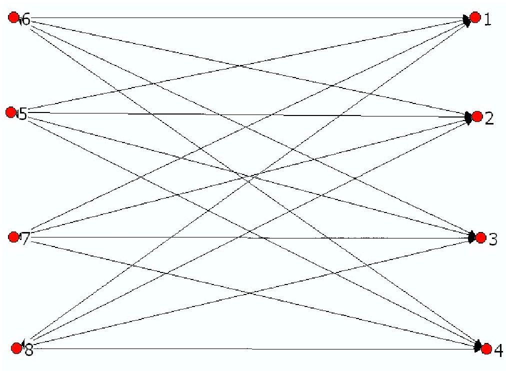

Figure 24: Most frequently emerging network for $\lambda = 1, c = v\_lo$ (sim. $n = 8$).

The balanced complete bipartite network (henceforth: BCB) g is connected with $\frac{n}{2} * \frac{n}{2}$ links. It has a density of $\frac{l(g)}{l(g^N)} = \frac{n}{2(n-1)}$ which is 57% for $n = 8$. All pairs of players are at distance one or two. Thus, the depicted network exhibits two of the three characteristic aspects of the inefficient networks identified in Subsection 4.3.3 for Case II: (3) the network is dense and (2) the network has short distances.

[74]A similar argumentation can be held for $(\lambda, c) = (0.9, v\_lo)$. In that case the BCB emerges in 49% of all simulation runs for $n = 8$.

For $\lambda = 1$ the addition of links can increase individual utility by increasing betweenness (Lemma 4.4.1 states that it cannot decrease it) such that for low c, agents do so. Consider the network of Figure 24 without one link, say $g \setminus 84$. This network is not stable, since both, player 8 and player 4, have an interest in reestablishing their link. They increase their benefits by $\frac{2[1-\frac{2}{n}]}{(n-1)(n-2)}$, which is larger than $c = v\_lo$. In the various simulation runs leading to this BCB network (exactly, 9'415 of 37'038), there must be several actions where agents add non-critical links, respectively do not sever them. However, by Prop. 4.5 (ii) the addition of any non-critical link decreases welfare, i.e. $w(g) < w(g \setminus 84)$.

The individual incentive to establish a link, as well as the collective benefit is dependent on distances. Theorem 4.1 shows that collectively it takes long real distance to produce high aggregate betweenness, which implies total benefits in this case. However, the presence of long distances is ground for establishing new links, as shown in the diameter result (Prop. 3.6) in Subsection 3.3.2.[75] In fact, the efficient network – the line network – is a subnetwork of the BCB network. Its long distances on one hand produce the highest welfare. At the same time they offer multiple possibilities to increase individual intermediation rents. Any two players that are at a distance larger than 3 (and less than M) are eager to form a link, each time increasing their betweenness, while reducing the welfare of a network. Respectively, if such agents have the possibility to sever their link, they will not do so. We conjecture that this situation of diverging interests – shortening distances increases individual utility but reduces social welfare – is the main source of inefficiency for c "not too high" and $\lambda >> \hat{\lambda}$.

Remark 4.4.2 (When does the argument apply)**.** *The individual benefits of shortening distances is not restricted to the case $\lambda = 1$, because it is also true for closeness that bridging distances increases individual centrality substantially. c not too high is used to make sure that benefits of adding non-critical links exceeds costs and to exclude issues of non-connectedness. $\lambda >> \hat{\lambda}$ assures that welfare steeply decreases in non-critical links.*

[75]Prop. 3.6 states that any network with a diameter (maximal real distance) of size ($p \geq 4$) or larger is not stable if $c < \frac{(\lfloor\frac{p}{2}\rfloor-1)\lfloor\frac{p}{2}\rfloor}{(n-1)(n-2)} := \tilde{c}$. Plugging in $p = 4$, the threshold becomes $\tilde{c} = \frac{2}{(n-1)(n-2)}$. For $c = v\_low = \frac{1}{2^3 n} - \epsilon$ and n not too large ($n \in \{3, 4, ..., 18\}$) it holds that $c < \tilde{c}$ such that the maximal real distance of any stable (and emerging) network is 3.

4.4.3 Example B: Closeness Incentives and High Costs

Consider the setting $(\lambda, c) = (0, hi)$ which is a subcase of Case I in Subsection 4.3.3. In that case the star network is uniquely efficient and also stable but not uniquely so. The emerging networks exhibit welfare close to efficiency. They can be characterized as connected and sparse (usually they are trees), but differ from the star network by having longer distances. Figure 25 depicts the most frequently emerging network (eleven percent of the simulation runs for $n = 8$), which illustrates those typical characteristics.

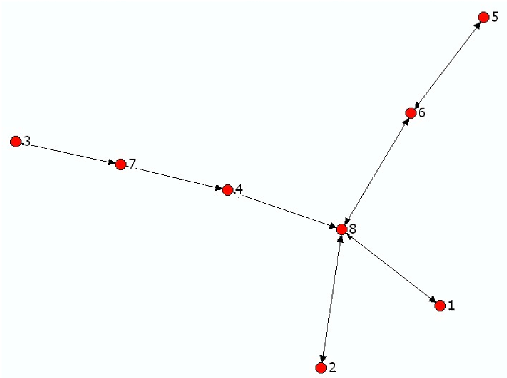

Figure 25: Most frequently emerging network for $\lambda = 0, c = hi$ (sim. $n = 8$).

The depicted network only consists of critical links. In fact, for $(\lambda, c) = (0, hi)$ it holds that: when two players have the opportunity to form a critical link, they will do so. They increase their benefits by at least $\frac{1}{n-1}$ (if they are isolates, see Prop. 3.7, setting $l = r = 0$ and $\lambda = 0$) while costs are lower than that $c = hi = \frac{1}{n} - \epsilon$; conversely, if two players are asked to cut a critical link, they will not. If two players have the opportunity to form a non-critical link, this looks different. Non-critical links only rarely persist in the long run (see Subsection 4.3.3). These dynamics lead to networks with slightly lower welfare than the efficient network (the star network).

One explanation of the occurring inefficiency is not based on the incentives of the players, but on the basic process of the simulation: the rule of link formation is to pick a pair of players at random which is allowed to change their relationship. Watts (2001) shows that such dynamics only occasionally leads to the star network. With high probability it is not possible to reach the star network, even if any selected pair of players would only act to improve social welfare. If this issue was the only source of inefficient outcomes, it would hold that any emerging network cannot be improved by the addition or severance of a link.

We will show that there is a second reason for inefficient outcomes, which is based on a conflict between individual and collective interest. Let us check in Figure 25 whether there is a pair of agents who are not directly linked, although this would increase welfare. This network g is stable if and only if $lb \approx 0.12245 \leq c \leq ub \approx 0.14268$. If the network is stable, a new link can increase total costs maximally by $2 * ub$ (once for each holder). By W1, it is socially beneficial to take these costs if the total benefits increase sufficiently. Total benefits are determined by aggregate closeness which is determined by SD (see Section 4.1). It is easy to compute that for $c = ub$ an additional link ij improves welfare if $2 * ub < \triangle \sum_{i \in N} CLOSE_i(g, ij) \Leftrightarrow \triangle SD(g, ij) > 7$. In network g there are multiple possibilities to shorten the sum of distances by more than 7, e.g. a link between agents "5" and "7" would reduce the sum of distances by 9. However, the network is stable. As this problem occurs even at the upper bound of the cost range, this network can be socially improved by addition of links whenever (that is for any c) it is stable.

A look at the simulation results shows that this problem frequently occurs: For network size $n = 14$ (and still $(\lambda, c) = (0, hi)$) a new link needs to change the sum of distances by at least 12 in order to increase social welfare. In the simulation with full weight on closeness 67.9% of all emerging networks have a distance of 6. If two players at distance 6 form a link, then the sum of distances reduces by at least 13 in any possible network.[76] That implies that at least the 67.9% of the emerging networks for this setting can be socially improved by the addition of a link. Thus, in many situations the individual interest to build links is not as strong as the collective interest.[77]

To explain this discrepancy reconsider the network in Figure 25. The costs of the link 57 is paid by player 5 and player 7 alone. However, link 57 not only contributes to their benefits, but also to the benefits of other players (i.e. players 3, 4, and 6). More generally, it can be shown that if $\lambda = 0$, then for any g and $\forall ij$ it holds that for any $k \in N \setminus \{i, j\}$, $u_k(g \cup ij) \geq u_k(g)$, that is: the formation of links can only have positive spillovers for $\lambda = 0$.[78] But there are also situations without spillovers: consider a network $g : g \supseteq g^\star$. Any two players are maximally at distance 2 in network g. As one consequence, the addition of a link only increases the closeness of the two players involved such that for $\lambda = 0$, the utility of any other player does not change.

[76] In the worst case there is a line of seven players.

[77] Since benefits of all agents are only based on distances and any newly formed link decreases distances, one can put it in the following way: The individual interest to shorten distances is present but not as strong as the collective interest.

[78] We show this result in the proof of Prop. 4.5 (iii).

We argue that the process described in this example is a general driver of inefficiency in settings with $\lambda = 0$ (not restricted to high c). For lower c, we observed that the emerging networks are much denser (see Subsection 4.3.3), consisting of critical and non-critical links. Still, it holds that any link in a stable network, $ij \in g$, contributes to its welfare, $w(g) \leq w(g \backslash ij)$. Prop. 4.5 (iii) shows even more generally, that any link (critical or not) that is build in the dynamic process increases welfare. Thus, inefficient outcomes must be based on agents refusing to build (respectively cutting) socially desirable links, besides some artifacts due to the sequence of link formation.

4.4.4 Example C: Very High Costs

Consider the setting $(\lambda, c) = (0.1, \tilde{c})$ with $\frac{1+0.1}{n-1} < \tilde{c} < T2 \;\; (= \frac{(1-\lambda)(Mn-2n+2)}{2(M-1)(n-1)} + \frac{\lambda}{2(n-1)})$, where the star network is efficient but not stable.

For this setting of c, the sequence of link formation will lead to sparse networks (any method showed a strong pattern that density is decreasing with c). For $c = hi$ already non-critical links only rarely persist in the long run. However, networks that only consist of critical links, contain loose ends and cannot emerge for $c > \frac{1+0.1}{n-1}$ by Prop. 3.3 (ii). In the enumeration for $M = n = 8$, 97.5 percent of the stable networks are not stable for $c > \frac{1+0.1}{n-1}$. There are only three candidates for emerging networks: the circle network, a network consisting of a 7-circle and one isolate, and the empty network. We have not run any simulation for such high costs, $c = v\_hi$, but our conjecture is that in the large majority of simulation runs the empty network would emerge.

Consider two players willing to form a link in the empty network. They would increase their benefits by $\frac{1-0.1}{n-1}$, which is not worth the costs c. Forming such a link would not have any spillovers to other players, thus individual interest and collective interest in this link coincide. However, consider the network $g = \{12, 34\}$ (for $n \geq 5$). Forming a link between player 2 and 3, would change their benefits by $\beta := \frac{(1-\lambda)[2M-3]}{(M-1)(n-1)} + \frac{2*2\lambda}{(n-1)(n-2)}$. Now consider that $\tilde{c} = \beta$, then 2 and 3 are indifferent about forming the link. For lower costs $\tilde{c} < \beta$ the link 23 will be formed, while also agent 1 and 4 benefit from that (their closeness increases by $\frac{(1-\lambda)[2M-5]}{(M-1)(n-1)}$, their degree and betweenness stays constant). By Prop. 4.5 (i) welfare increased. If, however, $\tilde{c}$ is slightly greater than β, then players 2 and 3 do not form the link, although it would still be socially beneficial (as long as $2c < 2\beta + 2\frac{(1-\lambda)[2M-5]}{(M-1)(n-1)}$).

As discussed in the network g and $g^{\emptyset}$ above, the formation of critical links might have positive spillovers or no spillovers. In fact, we can generally show that there are no negative spillovers: for any g and $\forall ij : d_{ij}(g) = M$ (ij is critical for g), it holds that for any $k \in N \setminus \{i, j\}$, $u_k(g \cup ij) \geq u_k(g)$.[79] Moreover, it can be can shown that if $\lambda < 1$ and $N_i(g) \cup N_j(g) \neq \emptyset$ (not both are isolates), it holds that $\sum_{k \in N \setminus i,j} u_k(g \cup ij) > \sum_{k \in N \setminus i,j} u_k(g)$.

While in the network discussed above ($g = \{12, 34\}$) both players agree not to form a critical link, although it would be socially desirable, it is generally sufficient that one player involved in the link does not accept it. For example, in the efficient network, the star network, the central player of the star is not willing to keep his links, since the costs $\tilde{c}$ exceed the benefit of a neighbor that does not lead to indirect connections $\frac{1+0.1}{n-1}$. This central player does not consider the harm he does to the peripheral player, who is willing to keep his link. Neither he considers the negative spillovers to the other peripheral players.

We argue that the issues discussed in this example can be generalized to any setting of λ, for sufficiently high c (c high enough that most non-critical links are not formed and loose ends are not stable, but below the thresholds where the empty network is efficient). First, it may happen that critical links typically have positive spillovers that are not internalized by the two players involved. At the same time a different issue seems to be predominant: A critical link that would increase welfare, is only accepted by one of the two players involved.[80] Thus, inefficient outcomes (empty network emerges, while trees, i.e. the star network and the line network, are efficient) are rather explained by an asymmetry of link formation – it takes two players to form a link but one to sever it – than by positive spillovers of critical links.

In the three examples of this section, we identified some basic effects leading to inefficient networks. Example C presents agents that decrease the welfare of a network by unilaterally severing critical links. Those links exhibit positive spillovers either for the second player involved or for other players or for both. A similar issue is illustrated in Example B, where critical and non-critical links have positive spillovers. Moreover, we discussed in that example that the sequence of link formation might also prevent the dynamic process from reaching the efficient outcome. Finally, Example A shows agents that form links that are socially harmful, not

[79] We show this result in the proof of Prop. 4.5 (i).

[80] We already discussed this issue in Subsection 3.3.3 where we examined the integration of an isolate to explain the emergence of unconnected networks for $\lambda = 1$ and $c = med, hi$. This was the only setting of the simulation where costs were sufficiently high to observe the phenomenon of unconnected networks frequently.

considering the negative spillovers.

Although the identified effects seem very important in determining the outcome of our model, it must be noted that they are not necessarily the only effects and that they do not occur in isolation. This makes the explanation of the emerging networks a non-trivial task. For example, the most frequently emerging network for $\lambda = 1, c = lo$ (in the simulation with $n = 8$), depicted in Figure 8, exhibits the issue of non-connectedness (discussed in Example C) and the issue of addition of non-critical links (discussed in Example A). Still, we try to trace back the emerging networks to those effects when summarizing the tension between stability and efficiency in the next subsection.

4.4.5 Summary of the Tension

The summary refers to Figure 23 and is organized according to the areas where a network is uniquely efficient (ignoring the frontiers of these areas, where we have a multitude of efficient networks).

Starting at the bottom left (i), in this area the complete network is efficient and uniquely stable. From both, the individual and the collective perspective, a link between any pair of players is worth its costs. Above that, there is the region where the star network is efficient. In the lowest area (iv) of this region the complete network is uniquely stable. Consider a network $g : g \supseteq g^{\star}$. As discussed in Example C, there are no positive spillovers on closeness from linking in this case. However, the addition of a link decreases the betweenness of the players who are directly linked to both (there is at least one such player, the center of the star). In this area (of the parameters), costs c are low enough such that the individual increase in closeness is worth the costs ($c < \frac{1-\lambda}{(n-1)(M-1)}$). Thus, each pair of players uses each opportunity to add a link. While this might also increase other player's utility in many situations, at some point, the network becomes a superset of the star network. Then, players continue to add links without considering the negative effects on the utility of other players (some loose betweenness benefits). That is why we observe for $\lambda > 0$ such an area where the star is efficient and the complete network is uniquely stable.

Above that, there is an area (iii), where the star network is efficient and also stable but not uniquely so. The emerging networks exhibit welfare close to efficiency. However, for $\lambda \approx 0$, there is systematic problem that was discussed in Example B. The addition of links not only increases the utility of the involved agents but might also increase closeness of other agents. If this increase in closeness is higher than the decrease in betweenness, then not involved agents benefit from the link. Since maintenance of links is costly, there are situations where agents refuse to build the links, although they would increase welfare. Thus, many emerging networks feature distances that would socially be worth bridging but no pair of agents is willing to do so. For $\lambda >> 0$ this effect need not be at work, since spillovers on betweenness can be negative.

Above that, there is a large area where the star network is still efficient but not stable. Prop. 4.4 (v) states that in this area any non-empty network is inefficient. Specifically, the central player of the star is not willing to keep his links, although they would be socially beneficial, as discussed in Example C. Moreover, we make the conjecture in Example C that for this setting of c, the sequence of link formation will most frequently lead to the empty network, either because the other stable networks (typically we found networks with isolates and large circles) are not very easily reached in the dynamic process, or because for large enough c the empty network is uniquely stable.[81] Thus, individual incentives to form some links break down for a level of c, where it can still be socially desirable to keep them.

For high enough c (ii), the empty network is efficient and uniquely stable (Prop. 4.4 (ii)). In that case the individual and collective incentives are aligned, since no link is worth its costs, neither collectively nor individually.

Finally, there is a region where the line network is efficient. Prop. 4.4 (vi) ruled out stability of the line for λ large enough.[82] Consider the line network as starting network in the dynamic process. With its distances, the line network offers a maximum of betweenness. However, it is not stable. When two players that are at a large enough distance have the opportunity to form a link, they will do so. For λ large enough, players with zero betweenness, i.e. at a loose end, do not keep their link. This sequence of shortening distances and cutting ties by the loose ends would lead to isolates and a dense component (such as in Figure 8 for $(\lambda, c) = (1, lo)$). Similarly, for given λ and c large enough, we expect the empty network to emerge in most of the cases as discussed in Example C.

Thus, in the area where the line network is efficient, for higher c there is the problem that unconnected networks, i.e. the empty network, emerge; for lower c, the problem is that non-critical links are added. Both problems can occur at the same time, e.g. for $\lambda = 1$ and $c = lo$.

[81] That is, we argue that the threshold of the empty network to be uniquely stable is lower than the threshold for the empty network to be efficient. See also, Prop. 3.3 (iii) and its discussion in Subsection 3.2.1.

[82] Moreover, there is the conjecture that the line network is never stable for $\lambda \geq \hat{\lambda}$ and $n \geq 7$, see Remark 4.5.1 in proof of Prop. 4.4.

As a consequence, relative efficiency is low, which becomes apparent for $\lambda >> \hat{\lambda}$. For λ close enough to $\hat{\lambda}$, say $\lambda = 0.5$, and c low enough to exclude the first issue, the emerging networks are much denser than efficient, but without exhibiting low relative efficiency. This is because short distances do not severely reduce total betweenness (for $\lambda \approx \hat{\lambda}$) and total costs are not dominating total benefits (for c low enough). However, it is still true in this region that networks emerge that are socially improvable by severance of ties.

Individuals in our model myopically increase their utility. What they do not consider are the consequences for other players. This basic problem between individual incentives and collective consequences is one of the main forces driving the discrepancy between the agents' goals and the social outcome – that is: the tension between stability and efficiency – in various settings of our model. We will analyze this problem in a much more general framework in the next chapter.

4.5 Proofs of Chapter 4

To prove Theorem 4.1 we need the following lemma:

Lemma 4.5.1. *$\forall g \in G$ and $\forall j \neq k$: $d_{jk}(g) < M$, it holds that*

$$\sum_{i \in N \setminus j,k} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)} = d_{jk}(g) - 1. \tag{4.13}$$

In words: fixing a connected pair of agents (i.e. $\{j, k\}$) and summing up all agents that are on one or more of their geodesics (weighted by the fraction they are on) results in counting the length of the shortest path between j and k.

Proof of Lemma 4.5.1. Take any g with a pair of connected agents j and k. Let $t (\neq 0)$ be the number of geodesics $\tau_{jk}(g) = t$ and let H be the set of agents who are on some geodesics, that is $H := \{i \in N \setminus j, k \mid \tau^i_{jk}(g) > 0\}$. Denote by h_x the number of (distinct) agents who are on x geodesics ($h_x := \#\{i \in N \setminus j, k \mid \tau^i_{jk}(g) = x\}$). The definitions imply that $|H| = h_1 + h_2 + ... + h_t$, which will be refered to as (*). Note first that if the t geodesics are independent (disjoint), then there are $t(d_{jk}(g) - 1)$ distinct agents in H. This number is reduced by any agent that is on more than one geodesic:

$$\begin{aligned} |H| &= t(d_{jk}(g)-1) - h_2 - 2h_3 - 3h_4 - ... - (t-1)h_t \\ \iff \quad t(d_{jk}(g)-1) &= |H| + h_2 + 2h_3 + 3h_4 + ... + (t-1)h_t \end{aligned} \tag{4.14}$$

On the other hand, recall that $\tau^i_{jk}(g) = x$ means that agent i is on x geodesics of j and k. By the definition of h_x we can write

$$\sum_{i \in N \setminus j,k} \tau^i_{jk}(g) = h_1 + 2h_2 + 3h_3 + ... + th_t \tag{4.15}$$

We show that the left-hand side (LHS) of (4.14) equals the LHS of (4.15) by subtracting the right-hand sides RHS (4.15)-(4.14):

$$\begin{aligned} \sum_{i \in N \setminus j,k} \tau^i_{jk}(g) - t(d_{jk}(g)-1) &= h_1 + h_2 + ... + h_t - |H| \overset{(*)}{=} 0 \\ \implies \quad \frac{\sum_{i \in N \setminus j,k} \tau^i_{jk}(g)}{t} &= d_{jk}(g) - 1 \end{aligned}$$

where the (*) part follows from the definitions.[83] □

Proof of Theorem 4.1 We have to show that $\forall g \in G$,

$$\sum_{i \in N} BETW_i(g) = \frac{2}{(n-1)(n-2)} \sum_{j<k: d_{jk}(g)<M} (d_{jk}(g) - 1). \tag{4.16}$$

By definition of betweenness (eq. 3.1)

$$\sum_{i \in N} BETW_i(g) = \sum_{i \in N} \left[\frac{2}{(n-1)(n-2)} \sum_{\substack{j<k \\ (j \neq i, k \neq i)}} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)} \right]. \tag{4.17}$$

[83] The same result was also found by Gago Alvarez (2007). To check the plausibility of the lemma just let the t geodesics be fully independent. Then $|H| = t(d_{jk}(g) - 1)$. Each agent $i \in H$ derives a betweenness of $\frac{1}{t}$. Hence $\sum_{i \in N \setminus j,k} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)} = \sum_{i \in H} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)} = t(d_{jk}(g) - 1) * \frac{1}{t} = d_{jk}(g) - 1$.

By changing summation we get:

$$\frac{2}{(n-1)(n-2)}\sum_{i\in N}\left[\sum_{\substack{j<k\\(j\neq i,k\neq i)}}\frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}\right]=\frac{2}{(n-1)(n-2)}\sum_{j<k}\left[\sum_{i\in N\backslash j,k}\frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}\right].$$

The fraction in brackets was defined to be zero if the denominator is zero. Since this is always true for unconnected pairs (i.e. $d_{jk}(g)=M\implies\tau_{j,k}(g)=0$), only connected pairs count for the sum before the brackets. Therefore we can apply lemma 4.5.1 which yields the result:

$$\sum_{i\in N}BETW_i(g)=\frac{2}{(n-1)(n-2)}\sum_{j<k:connected}(d_{jk}(g)-1).$$

□

Derivation of W3. We start with W2 and get W3:

$$\begin{aligned}
w(g) &= (1-\lambda)\left[\frac{nM}{M-1}-\frac{2SD(g)}{(n-1)(M-1)}\right]+\lambda\frac{2SD(g)-n(n-1)-2\nu(g)(M-1)}{(n-1)(n-2)}-c2l(g)\\
&= \frac{(1-\lambda)nM}{M-1}-\frac{(1-\lambda)2\sum_{j<k}d_{jk}(g)}{(n-1)(M-1)}+\frac{\lambda2\sum_{j<k}d_{jk}(g)}{(n-1)(n-2)}-\frac{\lambda n(n-1)}{(n-1)(n-2)}\\
&\quad -\frac{\lambda2\nu(g)(M-1)}{(n-1)(n-2)}-c2l(g)\\
&= \frac{(1-\lambda)nM}{M-1}-\frac{\lambda n(n-1)}{(n-1)(n-2)}+\sum_{j<k}d_{jk}(g)\left[-\frac{2(1-\lambda)}{(n-1)(M-1)}+\frac{2\lambda}{(n-1)(n-2)}\right]\\
&\quad -2c\psi^1(g)-\psi^M(g)\frac{\lambda2(M-1)}{(n-1)(n-2)}\\
&= \frac{(1-\lambda)nM}{M-1}-\frac{\lambda n}{n-2}+\psi^1(g)\left[\frac{1*2\lambda}{(n-1)(n-2)}-\frac{1*(1-\lambda)2}{(M-1)(n-1)}-2c\right]\\
&\quad +\psi^2(g)*2\left[\frac{2\lambda}{(n-1)(n-2)}-\frac{(1-\lambda)2}{(M-1)(n-1)}\right]\\
&\quad +\psi^3(g)*3\left[\frac{2\lambda}{(n-1)(n-2)}-\frac{(1-\lambda)2}{(M-1)(n-1)}\right]\\
&\quad +...+\psi^{n-1}(g)*(n-1)\left[\frac{2\lambda}{(n-1)(n-2)}-\frac{(1-\lambda)2}{(M-1)(n-1)}\right]\\
&\quad +\psi^M(g)\left[\frac{M*2\lambda}{(n-1)(n-2)}-\frac{M*(1-\lambda)2}{(M-1)(n-1)}-\frac{\lambda2(M-1)}{(n-1)(n-2)}\right],
\end{aligned}$$

where the expression in the very last brackets simplifies to $\alpha_M=\frac{2\lambda}{(n-1)(n-2)}-\frac{(1-\lambda)2M}{(M-1)(n-1)}$.

Proof of Prop. 4.1. The following lemma is helpful for this proof as well as for the proofs of Prop. 4.2 and Prop. 4.3.

Lemma 4.5.2. *For any network $g \in G$ with $l(g) = l$ it holds that $\psi^M(g) \geq \frac{1}{2}n(n-1) - \frac{1}{2}l(l+1)$. In words: the number of links of a network determines the minimal number of unconnected pairs.*

Proof. For $l \geq n-1$, the statement is not restrictive, because $\frac{1}{2}n(n-1) - \frac{1}{2}l(l+1) \leq 0$. For $l = 0$, it holds that $\psi^M(g) = \frac{1}{2}n(n-1)$. For $l \in \{1, 2, ..., n-1\}$ a network with l links can maximally connect $\frac{1}{2}l(l+1)$ pairs of agents. This is because such a network must be minimal and contain only one non-trivial component (we call such networks local trees). A local tree of size l connects $l+1$ agents, while $\frac{1}{2}n(n-1) - \frac{1}{2}l(l+1)$ pairs stay unconnected. *q.e.d.*

(i) Consider W3. For $\lambda < \hat{\lambda}$ and $c < \frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)}$ it holds that $\alpha_1 > \alpha_2 > \alpha_3 > ... > \alpha_{n-1} > \alpha_M$ (see inequalities (4.7) and (4.11)). g^N is the only network with all pairs at distance 1: $\psi^1(g^N) = \frac{1}{2}n(n-1)$, while $\psi^x(g^N) = 0$ for all $x \neq 1$.

(ii)-(iii) Consider W3. For $\lambda < \hat{\lambda}$ it holds that $\alpha_2 > \alpha_3 > ... > \alpha_{n-1} > \alpha_M$ (see inequalities (4.9) and (4.11)). Let us distinguish two cases:

- Consider a network g with $l(g) > n-1$. Necessarily: $\psi^1(g) = l(g)$. There are $\frac{1}{2}n(n-1) - \psi^1(g)$ other pairs. If all of them are at distance 2, the welfare function is maximized (because α_2 is the biggest coefficient in W3).[84] Let g be such a network and denote $t := l - (n-1) = l - n + 1$. Such a network differs from a star by having t pairs more at distance one and t pairs less at distance two. By W3 this implies that $w(g^\star) - w(g) = t(\alpha_2 - \alpha_1)$. Since $c > \frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)}$ it holds that $\alpha_2 > \alpha_1$ (see Inequality (4.7)), implying that $w(g^\star) > w(g)$.

- Consider a network g with $l(g) := l \leq n-1$. Note that having $l \leq n-1$ links implies that the number of connected pairs is at most $\frac{1}{2}l(l+1)$ (see Lemma 4.5.2). l of them are at distance 1: $\psi^1(g) = l(g)$. If all other connected pairs are at distance 2, the welfare of a network with l links is maximized (because α_2 is the biggest coefficient in W3). Such networks exist and can be described as local stars of size l (only local trees minimize the number of unconnected pairs; only stars have a maximal distance of 2 for all connected pairs). This shows that for $l \leq n-1$ the best networks are local stars.

  Because for local stars of size l it holds that $\psi^2(g) = \frac{1}{2}l(l+1) - l$ and $\psi^M(g) = \frac{1}{2}n(n-1) - \frac{1}{2}l(l+1)$, its welfare can be computed as $l\alpha_1 + (\frac{1}{2}l(l+1) - l)\alpha_2 + \frac{1}{2}n(n-1) - \frac{1}{2}l(l+1)\alpha_M$. straightforward computations show that for $c > \frac{(1-\lambda)(Mn-2n+2)}{2(M-1)(n-1)} + \frac{\lambda}{2(n-1)}$

[84] Indeed, for $l(g) > n-1$ there exist such networks (e.g. a star with other links in addition).

any local star has negative welfare such that the empty network is efficient. Moreover, if a local star of size l has positive welfare, then a local star of size $l+1$ has strictly higher welfare. For $c < \frac{(1-\lambda)(Mn-2n+2)}{2(M-1)(n-1)} + \frac{\lambda}{2(n-1)}$ the local star with $l = n-1$ links (that is $g^{\star}$) has strictly higher welfare than the empty network (and any other local star).

□

Proof of Prop. 4.2. The following Lemma is helpful.

Lemma 4.5.3. *For $\lambda \geq \hat{\lambda}$ any efficient network is minimal or empty.*

Proof. Let $g \in G \setminus g^{\emptyset}$ be a network that is not minimal. That means $\exists i, j : d_{ij}(g \setminus ij) < M$. Thus, $\nu(g \setminus ij) = \nu(g)$. Since cutting a link cannot shorten any distance, $SD(g \setminus ij) \geq SD(g)$. Finally, $l(g \setminus ij) < l(g)$. Consider W2 and its discussion in Section 4.1. For $\lambda \geq \hat{\lambda}$ W2 is non-decreasing in $SD(g)$, while it is always decreasing in $l(g)$ and $\nu(g)$. Therefore, $w(g \setminus ij) > w(g)$. *q.e.d.*

Lemma 4.5.3 implies that an efficient network does not have more than $n-1$ links (any network with more links is not minimal). For $\lambda > \hat{\lambda}$ it holds that $\alpha_{n-1} > \alpha_{n-2} > ... > \alpha_3 > \alpha_2 > \alpha_M$ (see inequalities (4.10) and (4.11)). Consider a network g with $l(g) \leq n-1$. Such a network can have at most $\frac{1}{2}l(l+1)$ connected pairs (by Lemma 4.5.2). The higher the distances between the connected pairs, the higher the welfare. Consider a network that forms a line of size l. For such a local line it holds that $\psi^l(g) = 1, \psi^{l-1}(g) = 2, \psi^{l-2}(g) = 3, ...$ Since a local line not only minimizes the number of unconnected pairs, but also uniquely maximizes the sum of distances, it is the network structure with highest aggregate betweenness of all networks with l links. By W1 any efficient network must be a local line.[85]

A local line network of size l has the following welfare: $l\alpha_1 + (l-1)\alpha_2 + (l-2)\alpha_3 + ... + 2\alpha_{l-1} + 1\alpha_l + (\frac{1}{2}n(n-1) - \frac{1}{2}l(l+1))\alpha_M$. straightforward computations show that for $c > \frac{1-\lambda}{2(n-1)}\left[\frac{Mn}{4(M-1)} - \frac{n^3-n}{3(M-1)(n-1)}\right] + \lambda\frac{n^2-4n+3}{4(n-1)(n-2)}$ any local line has negative welfare such that the empty network (the local line of length zero) is efficient. Moreover, if a local line of size l has positive welfare, then a local line of size $l+1$ has strictly higher welfare. □

[85] A local line of size 0 is the empty network.

Proof of Prop. 4.3. (i) By Inequality (4.7)

$\alpha_1 \geq \alpha_2 \iff c \leq (2-1)\left[\frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)}\right].$

Thus, for $\lambda = 0$ and $c = \frac{1}{(M-1)(n-1)}$, it holds that $\alpha_1 = \alpha_2$. Since $0 < \hat{\lambda}$, it holds that $\alpha_1 = \alpha_2 > \alpha_3 > \alpha_4 > ... > \alpha_{n-1} > \alpha_M$ (see Inequality (4.9)). Take any network g, with maximal distance of two.[86] This network is connected since $M > 2$. It holds that $\psi^1(g) + \psi^2(g) = \frac{1}{2}n(n-1)$. Considering W3 and the inequalities above, this implies that g is efficient.

(ii)-(iii) For $\lambda = \hat{\lambda}$ it holds that $\alpha_1 < \alpha_2 = \alpha_3 = \alpha_4 = ... = \alpha_{n-1} > \alpha_M$ (see inequalities (4.7) and (4.11)). By Lemma 4.5.3 only minimal networks can be efficient. For minimal networks it holds the $l(g) \leq n-1$. Consider a network g with $l(g) =: l \leq n-1$. Clearly, $\psi^1(g) = l$. Considering W3, the highest possible welfare for a network with l links occurs if ψ^M is at its minimum. This is only true for local trees of size l (see Lemma 4.5.2). So among all networks with l links a local tree has the highest welfare.

A local tree has welfare of $l\alpha_1 + \frac{1}{2}l(l+1)\alpha_2 + [\frac{1}{2}n(n-1) - \frac{1}{2}l(l+1)]\alpha_M$. It is straightforward to compute that for $c > \frac{n^2-n}{2n-3}$ any local tree has negative welfare, rendering the empty network efficient. Moreover, if a local tree of size l has positive welfare, then a local tree of size $l+1$ has strictly higher welfare. So, for $c < \frac{n^2-n}{2n-3}$ only local trees of size $l = n-1$ are efficient. Since $\alpha_2 = \alpha_3 = ... = \alpha_{n-1}$, any local tree is efficient.

□

Proof of Prop. 4.4. (i) This result is a direct corollary of Prop. 4.1 (i) and Prop. 3.3 (i). Just note that $(\frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)}) = T1 \leq \frac{1-\lambda}{(n-1)(M-1)}$.

(ii) This result is a triviality. Consider $\tilde{c} > \max\{T2, T3, \frac{n^2-n}{2n-3}, 1\}$. By Prop. 4.1 (iii), Prop. 4.2 (ii), and Prop. 4.3 (iii), $g^{\emptyset}$ is uniquely efficient. It remains to show that the empty network is uniquely stable for $\tilde{c} > 1$. This is done in the proof of Prop. 3.3.

(iii) The stability of the star network $g^{\star}$ follows directly from Prop. 3.1 (iii). For efficiency, we use Prop. 4.1 (ii). We have to show that for any c : $\frac{1-\lambda}{(n-1)(M-1)} < c < \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$, it holds that $T1 < c < T2$. The first part is obvious since $T1 \leq \frac{1-\lambda}{(n-1)(M-1)}$. The second

[86]Such networks exist. For example, a star with other links in addition ($g \supseteq g^{\star}$).

part requires some more computational effort. We show for $n \geq 6$ that $ub1 \leq T2$.

$$ub1 \leq T2 \iff \frac{1+\lambda}{n-1} \leq \frac{(1-\lambda)(Mn-2n+3)}{2(M-1)(n-1)} + \frac{\lambda}{2(n-1)} \tag{4.18}$$

$$\iff -Mn+2M+2n-4 \leq \lambda(-M+1-Mn+2n-2) \tag{4.19}$$

$$\iff \lambda \leq \frac{-Mn+2M+2n-4}{-Mn-M+2n-1} = \frac{Mn-2M-2n+4}{Mn+M-2n+1} \tag{4.20}$$

Eq. (4.18) is to show. Subtracting $\frac{\lambda}{2(n-1)}$ and after some simplifications (e.g. we multiply both sides by $(n-1)2(M-1)$) we get Eq. (4.20). The last step is to divide by the expression in the brackets on the LHS, which has a negative sign (because $Mn > 2n$). To show that Eq. (4.21) holds we make an approximation, showing that the fraction is larger than 0.5, while clearly $\lambda < \hat{\lambda} < 0.5$.

$$\frac{Mn-2M-2n+4}{Mn+M-2n+1} \leq \frac{1}{2} \iff 2[Mn-2M-2n+4] \geq Mn+M-2n+1 \tag{4.21}$$

$$\iff Mn-5M-2n+7 \geq 0 \tag{4.22}$$

$$\iff (n-1+\sigma)n-5(n-1+\sigma)-2n+7 \geq 0 \tag{4.23}$$

$$\iff \sigma(n-5)+n(n-8)+12 \geq 0 \tag{4.24}$$

From Eq. (4.22) to Eq. (4.23) we replaced M by $(n-1+\sigma)$ for some $\sigma > 0$. Eq. (4.24) holds for $n \geq 6$.

(iv) This result is a direct corollary of Prop. 3.3 (i) and Prop. 4.1 (ii).

(v) Let $c > \min\{\frac{1+\lambda}{n-1}; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)}\}$. By Prop. 3.3 (ii), any network containing loose ends is unstable. It remains to show that for c, any efficient network that is non-empty contains loose ends. Let us distinguish two cases. For $\lambda \geq \hat{\lambda}$ Lemma 4.5.3 in Proof of Prop. 4.2 shows that any efficient network is minimal. Clearly, a non-empty minimal network contains loose ends.

Now consider the case $\lambda < \hat{\lambda}$. We firstly show that $\min\{\frac{1+\lambda}{n-1} = ub1; \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)} = ub2\} > \frac{1-\lambda}{(M-1)(n-1)} - \frac{\lambda}{(n-1)(n-2)} = T1$.

With simple transformations we get $T1 < ub1 \iff \frac{1}{M-1} < \lambda(\frac{1}{M-1} + 1 + \frac{1}{n-1})$. The LHS must be strictly negative, while the RHS is a product of two non-negative factors. Similarly, $T1 < ub2 \iff \frac{1-\lambda}{(n-1)(M-1)}[1-(M(n-1)-2n+3)] < \frac{\lambda}{(n-1)(n-2)}$. The RHS is non-negative; the first factor of the LHS is strictly positive (because $\lambda < 1$). Thus, it remains to check that the second factor is strictly negative, that is $[1-(M(n-1)-2n+3)] < 0$. Since this expression is decreasing in M and since $M > n-1$ (as always), it holds that $[1-(M(n-1)-2n+3)] < [1-((n-1)^2-2n+3)]$ replacing M with $n-1$. Thus, it suffices to show that $[1-((n-1)^2-2n+3)] \leq 0$ which simplifies to $(n-1)[-n+3] \leq 0$.

Thus, by assumption $c > T1$. In the proof of Prop. 4.1 part (ii) and (iii) we show that for $c > T1$ any efficient network must be a local star (including the star network and the empty network).[87] Clearly, any non-empty local star contains loose ends.

(vi) Prop. 4.2 shows that the line network is uniquely efficient for $\lambda > \hat{\lambda}$ (and $c < T3$). Since the line network has loose ends, a necessary condition for stability comes from Prop. 3.3 (ii): $c \leq \min\{ub1; ub2 = (1-\lambda)\frac{M(n-1)-2n+3}{(n-1)(M-1)}\}$. Since the fraction has always a positive sign ($n \geq 3, M > n-1$), ub2 is a linearly decreasing function in λ, going to 0 for λ going to 1. Thus, for any cost $\tilde{c}$ we can find a $\tilde{\lambda}$ such that $\tilde{c} < ub2$, rendering the line network unstable.

Remark 4.5.1 (Stability of the line network)**.** *In fact, the line network is usually unstable. A second necessary condition for stability is that the players at the ends do not improve by adding a link between them. Let us make an approximation of the marginal benefits of a player i at the end linking with the other end j.*

His distances decrease by

$$\Delta d_i(n) = \begin{cases} n-2+n-4+n-6+...+3+1 = \frac{1}{4}(n-1)^2 \text{ , for odd } n \\ n-2+n-4+n-6+...+4+2 = \frac{1}{4}n(n-2) \text{ , for even } n \end{cases} \tag{4.25}$$

Thus, closeness of player i increased by at least $\frac{1}{(M-1)(n-1)}[\frac{1}{4}n(n-2)]$.

Betweenness increases (see also Proof of Prop. 3.6), too. Let χ be the number of pairs whose distance shortens ($\lfloor x \rfloor$ stands for the next lower integer):

$$\chi(n) \geq 1+2+3+4+...+\lfloor\frac{n-1}{2}\rfloor \geq \frac{1}{2}(\lfloor\frac{n-1}{2}\rfloor - 1)\lfloor\frac{n-1}{2}\rfloor \tag{4.26}$$

$$\Delta BETW_i(g) \geq \frac{(\lfloor\frac{n-1}{2}\rfloor - 1)\lfloor\frac{n-1}{2}\rfloor}{(n-1)(n-2)}. \tag{4.27}$$

Since $\lfloor\frac{n-1}{2}\rfloor - 1 \geq \frac{n-2}{2}$, *betweenness of player i increased by at least* $\frac{(\frac{n-2}{2}-1)\frac{n-2}{2}}{(n-1)(n-2)}$, *which further simplifies to* $\frac{n-4}{4(n-1)}$. *Let* $\beta_i := (1-\lambda)CLOSE_i(g^| \cup ij) + \lambda BETW_i(g^| \cup ij) - [(1-\lambda)CLOSE_i(g^|) + \lambda BETW_i(g^|)]$ *and similarly for j. Since the increase in closeness and betweenness is identical for j, a necessary condition for stability is:*

$$c \geq \beta_i = \beta_j \geq \frac{(1-\lambda)[n(n-2)]}{4(M-1)(n-1)} + \frac{\lambda(n-4)}{4(n-1)} \tag{4.28}$$

[87] Using proposition Prop. 4.1 (ii) and (iii), we see directly that the statement holds for the case $T1 < c < T2$ (star network is uniquely efficient) and for the case $c > T2$ (the empty network is uniquely efficient). However, this proposition does not make a statement about the case $c = T2$. There it holds that $w(g^\star) = w(g^\emptyset) = 0$.

On the other hand, a line can only be stable if its loose ends are stable. By Prop. 3.3 (ii) this means

$$c \leq ub2 = \frac{(1-\lambda)[M(n-1)-2n+3]}{(n-1)(M-1)} \tag{4.29}$$

The two conditions Eq. (4.28) and Eq. (4.29) can easily get in conflict with each other.

□

Proof of Lemma 4.4.1. The lemma follows from the definition of closeness and betweenness and some characteristics of shortest paths.

(i) Let us first show that for any $h \in N$, $\forall g \in G$, $\forall ij$ it holds that $CLOSE_h(g \cup ij) \geq CLOSE_h(g)$. Since any path that exists in g also exists in $g \cup ij$, no distance can be increased by addition of a link: For all $h, k \in N$, $\forall g \in G$ and $\forall ij$ it holds that $d_{hk}(g \cup ij) \leq d_{hk}(g)$. By definition closeness is decreasing in the distance to any other player, i.e. $CLOSE_h(g) = \frac{M}{M-1} - \frac{\sum_{k \in N} d_{hk}(g)}{(M-1)(n-1)}$, showing the last statement of the first part ($\forall k \in N \setminus \{i,j\}$, it holds that $CLOSE_k(g \cup ij) \geq CLOSE_k(g)$). It remains to show that two players involved in a link (i.e. i and j) strictly increase their closeness. For this end it suffices to find one distance for each of them that is shorter in $g \cup ij$ than in g. Consider, the distance between the focal players i and j themselves. Since, $ij \notin g$, $d_{ij}(g) > 1$, while $d_{ij}(g \cup ij) = 1$.

(ii) We shall first show that in any network g any new link ij does not decrease the betweenness of a player i (independently of $d_{ij}(g)$). Recall the definition of betweenness: $BETW_i(g) = \frac{2}{(n-1)(n-2)} \sum_{j<k(j \neq i, k \neq i)} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}$, (where $\tau_{jk}(g)$ is the number of shortest paths between j and k, and $\tau^i_{jk}(g)$ indicates the number of shortest paths between j and k that go through i; the fraction $\frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}$ is replaced by zero, when $\tau_{jk}(g) = 0$). We show that in any network g for all $h \neq p (\in N)$ and all $i, j : h \neq i, p \neq i$ it holds that $\frac{\tau^i_{hp}(g \cup ij)}{\tau_{hp}(g \cup ij)} \geq \frac{\tau^i_{hp}(g)}{\tau_{hp}(g)}$.
Consider a network g and any distinct $i, h, p \in N$ and some $j \neq i$. Since $g \cup ij \supseteq g$ any path that is present in g is also present in $g \cup ij$. Since the two networks only differ by the link ij, it must hold that any path that is not present in g, but is present in $g \cup ij$, uses the link ij. There are three possibilities: (a) None of the new paths is a shortest path between h and p,[88] (b) some of the new paths are shortest paths between h and p

[88] $ij \in g$ belongs to that case.

and those paths are shorter than their shortest path in g, (c) some of the new paths are shortest paths between h and p with the same length as the paths before.

In case (a) clearly the inequality holds with equality (the nominator and denominator on both sides are equal). In case (b) any shortest path in $g \cup ij$ uses i, thus $\frac{\tau^i_{jk}(g\cup ij)}{\tau_{jk}(g\cup ij)} = 1$ which is the maximal value this fraction can attain. (Thus, the inequality holds.) In case (c) let $a > 0$ be the number of new shortest paths. Since any new path uses i, it holds that $\frac{\tau^i_{jk}(g\cup ij)}{\tau_{jk}(g\cup ij)} = \frac{\tau^i_{jk}(g)+a}{\tau_{jk}(g)+a} \geq \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}$ (the last inequality holds because $\tau^i_{jk}(g) \leq \tau_{jk}(g)$). Therefore, for all g, and for all i, j, $BETW_i(g \cup ij) \geq BETW_i(g)$ and $BETW_j(g \cup ij) \geq BETW_j(g)$.

For the second statement of (ii) we use the condition that the new link is critical: We have to show that for all $k \in N \setminus \{i, j\}$ it holds that $BETW_k(g \cup ij) \geq BETW_k(g)$ for some $ij : d_{ij}(g) = M$. By the definition of betweenness, it is sufficient to show that in any network g for all distinct $h, p, k \in N$ and any $i(\neq k), j(\neq k) : d_{ij}(g) = M$ it holds that $\frac{\tau^k_{hp}(g\cup ij)}{\tau_{hp}(g\cup ij)} \geq \frac{\tau^k_{hp}(g)}{\tau_{hp}(g)}$.

Since $g \cup ij \supset g$ any path that is present in g is also present in $g \cup ij$. Since the two networks only differ by the link ij, it must hold that any path that is not present in g, but is present in $g \cup ij$, uses the link ij. Let C_i and C_j denote the two components of g that are bridged by ij. Since no (shortest) path uses an edge twice, any path present in $g \cup ij$ and not present in g must lead from C_i to C_j (connecting two agents of those two components). Thus, for any pair h, p such that neither $h \in C_i$ and $p \in C_j$ nor $p \in C_i$ and $h \in C_j$, the inequality above holds with equality, because g and $g \cup ij$ do not differ with respect to shortest paths between h and p. Otherwise – if either $h \in C_i$ and $p \in C_j$ or $p \in C_i$ and $h \in C_j$ – it holds that $\frac{\tau^k_{hp}(g)}{\tau_{hp}(g)} = 0$ because there is no shortest path.

(iii) The first statement is shown in the proof of part (ii).

The second statement is that $\sum_{k\in N\setminus i,j} BETW_k(g \cup ij) < \sum_{k\in N\setminus i,j} BETW_k(g)$ for some $\{i, j\} : 1 < d_{ij}(g) < M$. Since we have shown that $BETW_i(g \cup ij) \geq BETW_i(g)$ and $BETW_j(g \cup ij) \geq BETW_j(g)$, it suffices to establish that $\sum_{h\in N} BETW_h(g \cup ij) < \sum_{h\in N} BETW_h(g)$.[89]

As a consequence of Theorem 4.1, we have shown that Eq. (4.2):

$\sum_{k\in N} BETW_k(g) = \frac{SD(g)-\frac{1}{2}n(n-1)-\nu(g)(M-1)}{\frac{1}{2}(n-1)(n-2)}$. $d_{ij}(g) < M$ means that ij is a non-critical

[89]The argument follows from $\sum_{h\in N} BETW_h(g) = BETW_i(g) + BETW_j(g) + \sum_{k\in N\setminus i,j} BETW_k(g)$.

link. By definition a non-critical link does not connect different components, thus all pairs who are not connected in g are also not connected in $g \cup ij$: $\nu(g \cup ij) = \nu(g)$. SD was defined as the sum of distances, $SD(g) := \sum_{j<k} d_{jk}(g)$. Since any path that is present in g is also present in $g \cup ij$, there is no pair of players whose distance increases when establishing ij. However, there is at least one pair of players, i.e. $\{i, j\}$, whose distance strictly decreases ($d_{ij}(g) > 1$, while $d_{ij}(g \cup ij) = 1$). Thus, $SD(g) > SD(g \cup ij)$ implying that $\sum_{k \in N} BETW_k(g \cup ij) < \sum_{k \in N} BETW_k(g)$.

□

Proof of Prop. 4.5. Recall, that $w(g) = \sum_{h \in N} u_h(g)$. Thus, $w(g \cup ij) \geq w(g) \iff u_i(g \cup ij) - u_i(g) + u_j(g \cup ij) - u_j(g) + \sum_{k \in N \setminus i,j}(u_k(g \cup ij) - u_k(g)) \geq 0$.

(i) Since by assumption the utility of the i and j are increasing from g to $g \cup ij$, it remains to show that if $d_{ij}(g) = M$, then $\sum_{k \in N \setminus \{i,j\}}(u_k(g \cup ij) - u_k(g)) \geq 0$. For this purpose let us show that $\forall k \in N \setminus \{i, j\}$, it holds that $u_k(g \cup ij) - u_k(g)) \geq 0$. By definition $u_k(g) = (1 - \lambda)CLOSE_k(g) + \lambda BETW_k(g) - cl_k(g)$. Thus, $u_k(g \cup ij) - u_k(g) = (1 - \lambda)[CLOSE_k(g \cup ij) - CLOSE_k(g)] + \lambda[BETW_i(g \cup ij) - BETW_i(g)] - c[l_k(g \cup ij) - l_k(g)]$. Clearly, $l_k(g \cup ij) = l_k(g)$ (because $k \notin \{i, j\}$) such that the last term is zero. Since ij is critical ($d_{ij}(g) = M$), Lemma 4.4.1 part (i) applies, stating that the expression in the first brackets cannot be negative and by Lemma 4.4.1 part (ii), the second brackets cannot be negative. This implies that $u_k(g \cup ij) - u_k(g) \geq 0$.

(ii) We will show directly that $w(g \cup ij) < w(g)$. Consider W2: $w(g) = (1-\lambda)\left[\frac{nM}{M-1} - \frac{2SD(g)}{(n-1)(M-1)}\right] + \lambda\frac{2SD(g)-n(n-1)-2\nu(g)(M-1)}{(n-1)(n-2)} - c2l(g)$. For $\lambda \geq \hat{\lambda}$ this function is weakly increasing in $SD(g)$ and strictly decreasing in $l(g)$ and $\nu(g)$.

Note first that $\nu(g \cup ij) = \nu(g)$ since ij is not critical for g.[90] Secondly, recall that $SD(g) := \sum_{h<p} d_{hp}(g)$. It holds that $SD(g \cup ij) \leq SD(g)$ because an additional link cannot increase the distance between any pair (the shortest path between h and p in network g is still available in $g \cup ij$). Finally, $l(g \cup ij) = l(g) + 1$. Therefore, $w(g \cup ij) < w(g)$.

[90] The number of unconnected agents ν only changes by the addition of a link if the number of components decreases. If a link is not critical, it does by definition not change the number of components. That is, there is no pair of players h, p such that $d_{hp}(g) = M$ and $d_{hp}(g \cup ij) < M$ when ij is non-critical for g.

(iii) Let again $\mu_i^l(g) = u_i(g \cup l) - u_i(g \setminus l)$ denote marginal utility. We show that $w(g \cup ij) > w(g)$ $(\iff \sum_{h \in N} \mu_h^{ij}(g) \geq 0)$ by showing that for agent i, $\mu_i^{ij}(g) > 0$, while for all $j \neq i$ it holds that $\mu_j^{ij}(g) \geq 0$. For i and j this holds by assumption. For any $k \in N \setminus \{i, j\}$, it holds that $\mu_k^{ij}(g) = (1-0)[CLOSE_k(g \cup ij) - CLOSE_k(g)] + 0[BETW_k(g \cup ij) - BETW_k(g)] - c[l_k(g \cup ij) - l_k(g)]$. Clearly $l_k(g \cup ij) = l_k(g)$. Lemma 4.4.1 (i) states that the expression in the first brackets cannot be negative. Thus, $\mu_k^{ij}(g) \geq 0$.

□

5 Generalizations – The Role of Externalities

In Chapter 4 we observed in the centrality model that individual interest can be at odds with societal welfare. Such a tension between stability and efficiency is known to be a central problem in strategic network formation since the seminal contribution of Jackson and Wolinsky (1996). What has not been explicitly studied are the *sources of inefficiency.* In particular the question is *how* do stable networks generally differ from efficient networks? And, *why* does individual interest not always lead to efficient outcomes? We approach these questions by analyzing the role of externalities (or spillovers) of link formation. First, we address positive externalities and its examples; then we turn to situations with negative externalities (Subsection 5.2). Subsection 5.3 concludes.

Consider a society (N, G, u). Recall that a network g^* is called **efficient** with respect to the welfare function w if it is a welfare maximizing network, that is $w(g^*) \geq w(g) \ \ \forall g \in G$, where the utilitarian welfare function w^u is just one of the reasonable welfare functions satisfying monotonicity. To further characterize efficiency, we introduce the following two notions:

Definition 5.1. *A network g is called* **over-connected** *(with respect to the welfare function w) if $\exists g' \subset g$ such that $w(g') > w(g)$.*

Definition 5.2. *A network g is called* **under-connected** *(with respect to the welfare function w) if $\exists g' \supset g$ such that $w(g') > w(g)$.*

A network is over-connected if it is "too dense" in the sense that overall welfare can be improved by cutting links. Similarly, under-connected networks are "not dense enough". Efficient networks are neither over-connected nor under-connected. Note that for a given w, a network can satisfy both, one, or none of these two properties. To shed some light into the tension between stability and efficiency, we will ask whether and under what conditions stable networks are under-connected respectively over-connected. From the perspective of a social planner, this gives some insights whether to subsidize or to tax the formation of links in order to arrive in a socially preferred outcome.

5.1 Positive Externalities

5.1.1 Main Result on Positive Externalities

Recall that each (exogenously given) u defines a situation that is the basis for a strategic network formation game. Externalities describe the spillovers of link formation: Positive externalities simply capture that two players forming a link cannot mean harm for others.[91]

Definition 5.3. *A profile of utility functions u satisfies* **positive externalities** *if $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i, j\}$ it holds that*

$$u_k(g \cup ij) \geq u_k(g).$$

Being required for any network, any link, and any player, this property seems quite restrictive. However, we argue that there are many such contexts and we can easily find examples in the literature on strategic network formation that satisfy this property. Among them are "Provision of a pure public good" (Goyal and Joshi, 2006a), "Market sharing agreements" (Goyal and Joshi, 2006a), and the "Connections model" (Jackson and Wolinsky, 1996) which we discuss below. In case of a utility function that is additive separable into costs and benefits (where costs only depend on the own links), positive externalities are implied by a simple monotonicity property of the benefit function.

Since players who pay for certain links have to share their benefits with other agents, individual incentives to establish links can be lower than their collective value. The subsequent results, follow this intuition.

Theorem 5.1. *If a profile of utility functions u satisfies positive externalities, then no pairwise Nash stable network is over-connected with respect to any monotonic welfare function w, that is $\forall g \in [PNS(u)]$ it holds that $\nexists g' \subset g : w(g') > w(g)$.*

All proofs of this chapter can be found in Section 5.4. To prove this result we show that any player is worse off in a subnetwork g' of a PNS network g.[92] Because of PNS, a player cannot prefer a network $\tilde{g}(\subset g)$ that has only been reduced by some of his own links. Because of positive externalities, he cannot prefer a subnetwork $g' \subset \tilde{g}$.

[91] What we call positive externalities here, is called "non-negative externalities" in Jackson (2008). "Positive externalities" in the sense of Jackson (2008) require that the inequality is strict for at least one agent k. Any result that we state for positive externalities, also holds for "positive externalities" in the stronger sense.

[92] The result also implies that no subnetwork of a PNS network is Pareto better.

Extension to Pairwise Stability

Since (PS) is weaker than (PNS), we need an additional assumption to make a statement about the pairwise stable networks as well. Let $\mu_i^l(g) := u_i(g \cup l) - u_i(g \setminus l)$ denote the marginal utility. We employ a definition that was also used in Bloch and Jackson (2007):

Definition 5.4. *A profile of utility functions u is* **convex in own current links**, *if $\forall i \in N, \forall g \in G$, and $\forall l \subseteq L_i(g)$ it holds $\mu_i^l(g) \geq \sum_{ij \in l} \mu_i^{ij}(g)$.*

Because convexity in own current links[93] is sufficient for $[PNS(u)] = [PS(u)]$, the next result follows.

Corollary 5.1.1. *If u satisfies positive externalities and convexity in own current links, then a $g \in [PS(u)]$ is not over-connected (with respect to any monotonic welfare function).*

The non-over-connectedness results have trivial implications for the complete and empty network. As any network is a subnetwork of the complete network, it follows that (a) if the complete network is stable, then it must also be efficient. Since any network is a supernetwork of the empty network, it follows that (b) if the empty network is uniquely efficient, then no other network can be stable. More insights, can be won when studying specific examples.

Remark 5.1.1 (Convex or concave). *Hellmann (2009) shows that convexity in own* current *links is equivalent to "concavity in own* new *links" and equivalent to an intuitive property called "concavity (in own links)". Informally put, the third property (concavity) requires that the marginal contribution of a link for a player is decreasing in the set of links he already has. We use this equivalence in proof of Theorem 5.2 in the next section.*

5.1.2 The Connections Model Revisited

Recall the connections model (introduced in Jackson and Wolinsky, 1996) that models the flow of resources (like information or support) via the shortest paths in a network. In its general form the utility of each player can be written as

$$u_i^{CO}(g) = w_{ii} + \sum_{j \neq i} \delta^{d_{ij}(g)} w_{ij} - \sum_{j:ij \in g} c_{ij}, \tag{5.1}$$

where w_{ij} stands for the undiscounted worth of a connection to agent j and c_{ij} stands for the cost of maintaining a link with agent j, while $\delta \in (0, 1)$ and $M = \infty$.

[93]Recall that $L_i(g) \subseteq g$ such that $\mu_i^l(g) = u_i(g) - u_i(g \setminus l)$.

It is easy to see that the connections model satisfies positive externalities. If ij forms in some network g, then the utility of player $k \neq \{i,j\}$ either does not change, or increases as some of k's distances are shortened, because $d_{km}(g \cup ij) \leq d_{km}(g)$ for all ij and m. Moreover, Calvó-Armengol and Ilkiliç (2007) show that a specific form of the connections model (u^{SCO}), the symmetric connections model, satisfies convexity in own current links – a result that can be generalized to various distance-based utility functions including u^{CO} above.[94] Consequently (by the results Theorem 5.1 and Corollary 5.1.1 above), no pairwise (Nash) stable network can be over-connected w.r.t. any monotonic welfare function. While the stable networks depend on the dyadic specifications of value and costs (w_{ij}, c_{ij}), the non-over-connectedness results tell that the welfare of a stable network can never be improved by severing certain links – in fact, any player weakly prefers a stable network to any of its subnetworks.

There are more specific results for the connections model in its symmetric version, setting $w_{ij} = 1$, $c_{ij} = c$ ($\forall i \neq j$) and considering w^u. This was studied in Jackson and Wolinsky (1996), Jackson (2003), Hummon (2000), and in Section 2.3.3 of this work, among others. Jackson and Wolinsky (1996, Prop. 1 and Prop. 2) show that for low costs ($c < \delta - \delta^2$), the complete network is efficient (and uniquely pairwise stable); for medium costs ($\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$), the star network is efficient; while for very high costs ($c > \delta + \frac{n-2}{2}\delta^2$), the empty network is efficient. Their famous statement of inefficiency in the connections model is the following: "For $\delta < c$, any pairwise stable network which is non-empty is such that each player has at least two links and thus is inefficient."[95]

What does our result excluding over-connectedness add to their discussion of inefficiency? First, there are the trivial implications for the complete and the empty network: If the empty network is uniquely efficient, then it must be uniquely stable. Thus, the statement of inefficiency is restricted to $\delta < c < \delta + \frac{n-2}{2}\delta^2$. Secondly, the result on over-connectedness adds a new point of view on the flavor of inefficiency. This can be illustrated in the following example, which is also taken from Jackson and Wolinsky (1996, Ex. 1).

Example 5.1. *The network in Figure 26, called "Tetrahedron", is stable for costs $c > \delta$, where the star network is efficient.*[96] *The tetrahedron is "too dense" in the sense that it has* 18 *links, while the efficient network has* 15*. Accordingly, Jackson and Wolinsky (1996, p. 51) label it as "over-connected". However, it is not over-connected according to the definition used in this*

[94] The proof can be requested by the author.
[95] Jackson and Wolinsky (1996), p. 51.
[96] More precisely, g is pairwise stable iff $\delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4) \leq c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5)$.

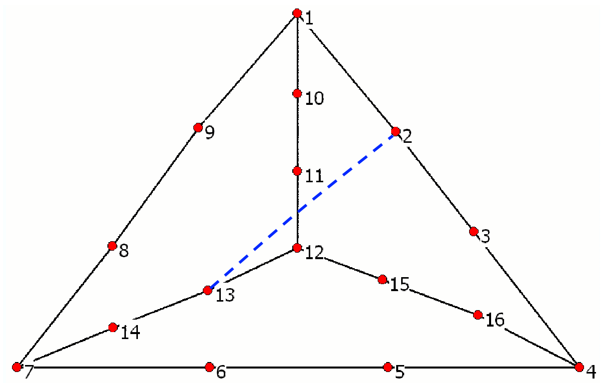

Figure 26: Example of an inefficient network ("Tetrahedron").

work. This means that the welfare of the tetrahedron cannot be improved by leaving out some of its links. Moreover, we claim that the tetrahedron is under-connected for the parameters where it is pairwise stable. In Section 5.4 we show that the addition of a link between the players "2" and "13" would strictly improve utilitarian welfare (Prop. 5.4). The same point as in the Tetrahedron can be illustrated in a circle graph of $n \geq 7$*: both networks are under-connected for any costs that they are pairwise stable.*

The example illustrates two different viewpoints on the inefficiency (in the connections model). From the viewpoint of a social planner that can unrestrictedly manipulate a given network, some stable networks are "too dense" in the sense that less links are needed to form the efficient one. From the viewpoint of a social planner that is either able to foster or to hinder the formation of links, many stable network in the connections model are "not dense enough" (under-connected), while none is "too dense" (over-connected).

5.1.3 Further Examples for Positive Externalities

Besides the connections model, it is easy to find further examples for positive externalities. Among them is the model of "market sharing agreements" described in Goyal and Joshi (2006a). Given n firms and n markets, each firm has one home market and can be active in all other markets, too. In $t = 0$ bilateral agreements can be made to stay out of each others home market. In $t = 1$ there is Cournot competition (with homogeneous goods) in each market. Let the Cournot profit of a firm i in market k be represented by $\tilde{\pi}_i = \lambda(n - l_k(g))$, where $\lambda(\cdot)$ is assumed to be decreasing and convex (in the number of active firms). This makes for one firm

$$u_i^{MSA}(g) = \sum_{j \notin N_i(g)} \lambda(n - l_j(g)) - cl_i(g), \tag{5.2}$$

counting the home market as well as all foreign markets for which there is no agreement (note that by definition $i \notin N_i(g)$).[97]

New agreements of a firm increase its profit on the home market, but reduce the number of other markets it is active on. To establish positive externalities only the assumption $\lambda(\cdot)$ decreasing is needed since a new link reduces the number of competitors in two markets. Any player (not paying for the link) is either active in both, one or none of them.

So, again, Theorem 5.1 applies. Goyal and Joshi (2006a) show that the pairwise Nash stable networks consist $m \in [1, n]$ fully connected groups and besides there may be some isolated players. While these networks can be quite dense, the non-over-connectedness result tells that the components are not "too dense", as welfare cannot be improved by deletion of links.

Remark 5.1.2 (Non-monotonic welfare)**.** *Efficiency here is only from the firms' point of view. Considering the consumers, the efficient networks look completely different.*

Before turning to negative externalities, let us have a closer look at one more example for positive externalities.

Provision of a Pure Public Good

The model "provision of a pure public good" is also taken from Goyal and Joshi (2006a), who extended a model of Bloch (1997). n players choose an output level x_i (second stage), which is valuable for everybody $\tilde{\pi}_i(x) = \sum_{i \in N} x_i$. Collaboration (knowledge sharing) between any two players costs c, but can reduce the cost of producing the output (first stage).[98] Assuming that any player chooses his output quantity optimally (in the second stage), the utility of a player is

$$u_i^{PG}(g) = \frac{1}{2}(l_i(g)+1)^2 + \sum_{j \in N \setminus i} (l_j(g)+1)^2 - c l_i(g), \tag{5.3}$$

where the first term is the difference of own output and the costs to produce it (the second term is the output of all others and the last part is the costs of collaborating).

Not surprisingly, the network formation situation of the first stage satisfies positive externalities, because other agents cooperating lowers their costs, increases their optimal output, and, hence, is beneficial. To see this, observe that the addition of foreign links increases the middle

[97] The notation of Goyal and Joshi (2006a) is inconsistent in this point.

[98] Agent i's cost of producing the output is $f_i(x_i, g) = \frac{1}{2}(\frac{x_i}{l_i(g)+1})^2$. So, fixing the number of collaborators $l_i(g)$, the utility maximizing output quantity of an agent i can be derived by $\max_{x_i \in \mathbb{R}_+} x_i + \sum_{j \in N \setminus i} x_j - \frac{1}{2}(\frac{x_i}{l_i(g)+1})^2 =: F(x)$. This yields $F'(x) = 0 \iff x_i^* = (l_i(g)+1)^2$. Then, plugging in the optimal output $(F(x^*))$ for any agent into the objective function subtracting the linking costs yields the utility of one agent.

term of the utility function. Note that in this example the externalities are strict in the sense that the addition of any link increases the utility of all agents that are not involved.

Consider very low costs $c \leq \frac{5n^2-6n+3}{2n-2} =: lb$ such that the complete network is Nash stable. By Theorem 5.1, no pairwise Nash stable network can be over-connected. Thus, the complete network must also be efficient for $c \leq lb$. In fact, since the externalities are strict, there exists an $\varepsilon > 0$ such that g^N is efficient for $c \leq lb + \varepsilon$. The tension can be observed for $lb < c < lb + \varepsilon$. In this cost range the complete network is efficient, but stable networks are not complete.

The model can be interpreted as a doubled public goods problem. In the second stage there is the classic public goods problem where individual output x_i is chosen "too low" (from a collective perspective). This problem persists, but in addition (in the first stage) players tend to choose "too few" links reducing the cost of provision, such that the overall outcome is even worse. In the same manner any network formation situation with positive externalities can be interpreted as a public goods problem. Utility maximizing agents do not internalize the positive effects that establishing a bilateral link means for other agents.

5.2 Negative Externalities

Negative externalities in network formation occur, when adding links is not beneficial for the players not involved in them. Formally, we speak of negative externalities[99] if the following holds:

Definition 5.5. *A profile of utility functions u satisfies* **negative externalities** *if* $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i,j\}$ *it holds that*

$$u_k(g \cup ij) \leq u_k(g).$$

Before, turning to the general results, let us study a prominent example.

5.2.1 The Co-author Model

The co-author model has been introduced by Jackson and Wolinsky (1996) and describes the utility of joint work. The nodes of the network are interpreted as researchers, who spend time writing papers. A link between two researchers i and j represents a collaboration between both researchers. The amount of time a researcher spends on a project is inversely related to the

[99] What we call negative externalities here, is called non-positive externalities in Jackson (2008).

number of projects he is involved in. The payoff function (consisting of the contribution of i, the contribution of j plus a term for synergies) is given by:

$$u_i^{CA}(g) = \sum_{j \in N_i(g)} \left(\frac{1}{l_i(g)} + \frac{1}{l_j(g)} + \frac{1}{l_i(g) l_j(g)} \right) = 1 + \left(1 + \frac{1}{l_i(g)}\right) \sum_{k \in N_i(g)} \frac{1}{l_j(g)},$$

and $u_i^{CA}(g) = 0$ if $l_i(g) = 0$. The utility only depends on own degree and neighbors' degree. Obviously, the functional form satisfies negative externalities as utility of players decrease, when neighbors add links, i.e. increase their degree. This is already mentioned in Jackson and Wolinsky (1996), who get the following result:

Proposition 5.1. *In this co-author model, if n is even, then (a) any efficient network (with respect to utilitarian welfare) consists of $n/2$ separate pairs. (b) Any pairwise stable network can be partitioned into fully intraconnected components, each of which has a different number of members (if a is the number of members of one such component and b is the next largest size, then $b > a^2$).*

The proposition shows that the stable networks are much denser than the efficient networks. Although not every stable network contains the efficient network as a subset, it can be shown that any stable network can be socially improved by the severance of some links.

Proposition 5.2. *In this co-author model, every stable network is over-connected w.r.t. the utilitarian welfare function.*

The proof uses the proposition of Jackson and Wolinsky (1996) component-wise (that is for any completely connected component of the pairwise stable networks) and is then straightforward. It is welfare better for any component of at least size three to be connected such as one of the efficient networks. Thus any component of any pairwise stable network contains a welfare better subcomponent, implying that any pairwise stable network contains a welfare better subnetwork.

Consider a dynamic process as described by the simulation in Section 3.2.3. Suppose that the empty network is the starting network. By linking two players can improve (Jackson and Wolinsky, 1996 show in their proof of Prop. 5.1 that if $l_i(g) = l_j(g)$, then they both improve by linking). If, by coincidence the sequence of drawing pairs does not draw the same player twice before any player was drawn, then the efficient network forms. However, the process is not finished. Further links will be added, despite the collective harm this action has. So, the co-author model is an example where the emerging networks are "heavily" over-connected.

5.2.2 Main Result on Negative Externalities

The co-author model is in line with the intuition on negative externalities: egoistic players do not internalize the negative effects of their linking actions such that the resulting networks tend to be over-connected. Despite the clear intuition, it is not that easy to generally show the relation between negative externalities and over-connectedness of the stable networks.

When using pairwise stability (or similar concepts), it can occur that a stable network is under-connected despite negative externalities. To see this, suppose that one player i gains a lot of utility from a link with player j, but the link is simply not formed because j looses a little bit of utility. Assuming that all others do not loose a lot of utility either, the addition of the link ij could produce higher welfare (e.g. according to the utilitarian welfare function).

A simple way out of this problem is to allow for side payments (first proposed in Jackson and Wolinsky, 1996). This refinement requires in addition to pairwise stability that $\forall ij \notin g, \quad u_i(g) + u_j(g) \geq u_i(g \cup ij) + u_j(g \cup ij)$, excluding networks from being stable if there are two players who in sum would benefit of establishing a link. This requirement is most sensible in a context where agents can pay transfers for others to make them willing to form and not to form a link. A basic stability concept in that context can be found in Bloch and Jackson (2007):

Definition 5.6. *A network g is* ***pairwise stable with transfers*** *(PS^t) if there does not exist any pair of players that can jointly benefit by adding, respectively cutting, their link:*

(i) $\forall ij \in g, \quad u_i(g) + u_j(g) \geq u_i(g \setminus ij) + u_j(g \setminus ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g) + u_j(g) \geq u_i(g \cup ij) + u_j(g \cup ij).$

Denote by $[PS^t(u)]$ the set of pairwise stable networks with transfers and by $[PS^{SP}(u)]$ the set of pairwise stable networks with side payments. While it holds that $[PS^{SP}(u)] = [PS^t(u)] \cap [PS(u)]$[100]; in general, neither $[PS(u)] \subseteq [PS^t(u)]$ nor $[PS(u)] \supseteq [PS^t(u)]$.

Preclusion of Under-connected Networks

Given the possibility of side payments (be it as a refinement of PS or in a context of transfers), it can be assured that the addition of a link to a stable network cannot increase the utilitarian welfare. To establish a non-under-connectedness result, we need one additional property, that

[100] PS^t is a stronger concept than PS for links that are not in g, whereas PS is stronger than PS^t for links in g. PS^{SP} captures both strong conditions.

can be described as "concavity in own new links" (see Calvó-Armengol and Ilkiliç, 2007). However, this property is shown (in Hellmann, 2009) to be equivalent to convexity in own current links, such that we get the following result:

Theorem 5.2. *Suppose a profile of utility functions u satisfies negative externalities and convexity in own current links, then no network $g \in G$, which is pairwise stable with transfers, is under-connected with respect to the utilitarian welfare function.*

In full analogy to non-over-connectedness for positive externalities, the non-under-connectedness result has trivial implications for the complete and empty network. As any network is a supernetwork of the empty network, it follows that (a) if the empty network is pairwise stable with transfers, then it must be efficient. Since any network is a subnetwork of the complete network, it follows that (b) if the complete network is uniquely efficient, then no other network is pairwise stable with transfers.

In contrast to the results on positive externalities, this result requires the possibility of transfers, while there is no exogenous justification, why contexts of negative externalities should typically be contexts of transfers. In Buechel and Hellmann (2008) it is shown, mainly due to Tim Hellmann, that by restricting attention either to a certain class of network formation games or to Pareto efficiency (instead of utilitarian welfare), it is possible to get non-underconnectedness results for the notion of pairwise stability as well. Moreover, in many specific models there are situations where two agents feel an identical change in utility by the formation or severance of a specific link, such that the requirements of pairwise stability and pairwise stability with transfers coincide. Conversely, we have not discussed a setting of transfers in the part on positive externalities. Theorem 5.3 in Section 5.4 shows that the analogue result to Theorem 5.2 holds for positive externalities.

The assumptions on the utility function, namely negative externalities and convexity in own current links, appears in a lot of networks formation models, some of which will be analyzed subsequently.

5.2.3 Examples for Negative Externalities

The model of free trade agreements (FTA) has been introduced by) (2006b). In each of n countries there is one firm producing a homogeneous good. The firm may sell the product in the domestic market, as well as in foreign markets. If two countries do not have a free trade agreement (FTA) the importing country charges tariffs. Given a network of FTAs, the firms then compete in each market by choosing quantities. In this model u^{FTA} satisfies negative externalities (this can be checked in the utility function for each country in Goyal and Joshi, 2006b). Suppose a free trade partner j of country i signs a free trade agreement with country k. Then firm k will enter market j and thus reduces the Cournot output of firm i, lowering the country's utility. If two non-trade partners of i sign a free trade agreement, then i's payoff remains unaffected. Let us study one final example.

Patent Races

Goyal and Joshi (2006a) derive this model as a variation of the classical patent race model.[101] In addition to the classical model, firms can join R&D collaborations to accelerate research. The first firm to develop the new product is awarded a patent.

With some assumptions on the probability of winning the patent race, the expected utility of a firm can be written as

$$u_i^{PR} = \frac{l_i(g)}{\rho + 2l(g)} - l_i(g)c \quad = \quad \frac{l_i(g)}{\rho + 2l_i(g) + 2l(g_{-i})} - l_i(g)c \tag{5.4}$$

where g_{-i} represents the network, obtained by deleting player i and all his links. This model satisfies negative externalities since links of other firms reduce the probability to innovate firstly. Moreover, u_i^{PR} is a concave function of $l_i(g)$ implying that it satisfies convexity in own current links. From Theorem 5.2 we can thus conclude that no pairwise stable network with transfers is under-connected.

In fact, it is straightforward to calculate the efficient networks since the utilitarian welfare is given by:

$$w^{PR}(g) = \sum_{i \in N} u_i^{PR}(g) = \sum_{i \in N} \left(\frac{l_i(g)}{\rho + 2l(g)} - l_i(g)c \right) = \frac{2l(g)}{\rho + 2l(g)} - 2l(g)c.$$

In this case the utilitarian welfare only depends on the total number of links and thus any network that contains the optimal number of total links is efficient. The distribution of links and the structure of the network do not matter for efficiency. We can easily calculate that for $\frac{\rho}{(\rho+2(k+1))(\rho+2k)} < c < \frac{\rho}{(\rho+2k)(\rho+2(k-1))}$ any network which contains exactly k links is efficient and no other networks are efficient.

It requires a little bit more to characterize stable networks. However, for this matter we can

[101] See Dasgupta and Stiglitz (1980) among others.

apply Theorem 5.2 in order to bound the total number of links.

Proposition 5.3. *Suppose that* $\frac{\rho}{(\rho+2k+2)(\rho+2k)} < c$. *Then any network* g *which is pairwise stable with transfers contains at least* k *links, that is* $l(g) \geq k$.

The example shows that Theorem 5.2 not only describes the tension between stability and efficiency, but it can also be applied to characterize the stable networks (resp. the efficient ones).

Clearly, the patent to win is a constant reward. Certain agents investing to win the race, cannot increase the reward but only their chances of winning. Consequently, they increase their utility by reducing other players utility, which is the interpretation of negative externalities, not restricted to this specific functional form.

5.3 Concluding Remarks

The results of this chapter are formalized for general classes of network formation models. We illustrated a few examples while the results can be applied to many others models. For instance, reconsider the models discussed throughout this work. In Section 2.2.1 there is a model (M1) without any externalities, since agents only care about degree. Formally, the model satisfies both negative and positive externalities (as defined here). Thus, both groups of results apply, excluding over-connectedness and under-connectedness in many settings. In Section 2.2.2 we study a model (M2) that obviously exhibits positive externalities: A new link can increase the number of indirect connections of agents not involved in the link but never decrease it. Such a model is studied by Bala and Goyal (2000) in a framework of unilateral link formation (considering directed links). Similarly, the closeness model analyzed in Section 2.3 exhibits positive externalities. Finally, the centrality model studied in Chapter 3 and Chapter 4 neither satisfies positive externalities nor negative externalities (see Lemma 4.4.1) – except for the setting $\lambda = 0$, which is a special case of the closeness model. Although the results of this chapter do not apply to the centrality model, their heuristic substance is of use. In Section 4.4 we describe how positive and negative "spillovers" drive the discrepancy between efficient networks and emerging networks.

For future work it seems helpful to adapt the results of this section, to requirements of special models. One can think of network formation models that fulfill the properties of Theorem 5.2 or Theorem 5.1 only on a certain domain $\tilde{G} \subset G$, e.g. the set of all connected networks. Moreover, when restricting attention to utilitarian welfare, the assumption of positive (resp. negative) externalities is stronger than necessary. It suffices that the sum of spillovers is always positive (or negative), i.e. the requirement $\forall g \in G, \forall ij \notin g$ it holds that $\sum_{k \in N \setminus \{i,j\}} u_k(g \cup ij) \geq u_k(g)$ is sufficient to assure that no pairwise Nash stable network is over-connected. Finally, there is a straightforward adaption of the results to a framework of formation of directed links.

Limitations

This work discusses some topics in strategic network formation as summarized in Section 1.2. It belongs to an approach of network dynamics where the current network structure affects the probability of the future network structure via individual assessment and action. The strengths of our modeling are also its shortcomings: it isolates single effects, e.g. the impact of centrality incentives on the network structure, out of a very complex social reality. Clearly, various other aspects influence network dynamics. First of all, there are institutional conditions that shape the opportunity and constraints of meeting. Secondly, we restrict the benefits of social networks to aspects that are reducible to network statistics. By ignoring actor specific and dyadic specific explanatory variables, we exclude hedonic effects such as homophily. Further effects are excluded by studying only unweighted and non-directed networks, e.g. effects of reciprocity.

Finally, we work with unrealistically high assumptions on rationality: Fully informed actors who are able to compute the change in centrality a new link (resp. a link less) would mean. But this only serves as an "as if" assumption. In their analysis Padgett and Ansell (1993) made very clear that Cosimo de' Medici did not purposefully plan to create his family's central position in the marriage network. However, the Medici's marriage behavior differed from the behavior of the other superelite families by frequently marrying outside of their quarters. The model of Burger and Buskens (2006) incorporates only local utility considerations that can be easily understood in laboratory experiments. Their results are not in contradiction with high rationality models. Thus, there are much simpler rules of thumb that lead to behavior "as if" an individual agent was optimizing his network position.

5.4 Proofs of Chapter 5

Proof of Theorem 5.1. Let $g \in [PNS(u)]$. We show that for all $g' \subset g$ it holds that $u_i(g') \leq u_i(g)$ for all $i \in N$. Let $l := l(g, g') = g \setminus g'$ for some $g' \subset g$, and denote $l^i := l^i(g, g') = l \cap L_i(g)$

and $l^{-i} := l \setminus l^i$. Since $g \in PNS(u)$, we get that $u_i(g) \geq u_i(g \setminus l^i)$. Since u satisfies positive externalities, it holds for all $\tilde{g} := g \setminus l^i$ that $u_i(\tilde{g}) \geq u_i(\tilde{g} \setminus l^{-i})$ (because player i does not own a link in l^{-i}), i.e. $l_{-1} \cap L_i(g) = \emptyset$). Thus: $u_i(g) \geq u_i(g \setminus l^i) \geq u((g \setminus l^i) \setminus l^{-i}) = u(g')$. The same argument holds for all $i \in N$, implying that $w(g) \geq w(g')$ for any welfare function satisfying monotonicity. □

Proof of Corollary 5.1.1. Calvó-Armengol and Ilkiliç (2007) show that (1-)convexity in own current links is sufficient for $[PS(u)] = [PNS(u)]$. Thus, Theorem 5.1 applies. □

Proposition 5.4. *In the symmetric connections model with parameters δ, c and the corresponding utilitarian welfare function, the following holds: g^{Tetra} is under-connected for any parameters that it is pairwise stable.*

Proof of Prop. 5.4. We have to show that if δ, c is such that $g^{Tetra} \in PS(u_{\delta,c})$, then $\exists g' \supset g^{Tetra}$ such that $w_{\delta,c}(g') > w_{\delta,c}(g^{Tetra})$. Specifically, we show that the condition

$$c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5) := ub \tag{5.5}$$

is necessary for stability, but sufficient for $w_{\delta,c}(g^{Tetra} \cup \{2, 13\}) > w_{\delta,c}(g^{Tetra})$. The labeling of the players corresponds to figure 26.

The first part was done in Jackson and Wolinsky (1996) already. Suppose that $c > ub$, then player 1 benefits from cutting $\{1, 2\}$ (because his change in benefits is just ub).

For the second part denote by $\beta_i := \sum_{j \neq i} \delta^{d_{ij}(g^{Tetra} \cup \{2,13\})} - \sum_{j \neq i} \delta^{d_{ij}(g^{Tetra})}$ the change in utility for player i and by $\Delta := \sum_{i \in N} \beta_i$ the aggregate change of benefits. This allows us to write

$$w_{\delta,c}(g^{Tetra} \cup \{2, 13\}) > w_{\delta,c}(g^{Tetra}) \iff \Delta > 2c. \tag{5.6}$$

It is straightforward to derive that

$$\begin{aligned}
\beta_1 &= \beta_{12} = \delta^2 - \delta^4 + \delta^3 - \delta^4 \\
\beta_2 &= \beta_{13} = \delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4) \\
\beta_3 &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5 \\
\beta_4 &= \beta_7 = \beta_9 = \beta_{15} = \delta^3 - \delta^4 \\
\beta_{14} &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5,
\end{aligned}$$

and for all other i: $\beta_i = 0$.

This yields

$$\Delta = 2(\delta - \delta^5) + 4(\delta^2 - \delta^4) + 4(\delta^2 - \delta^5) + 12(\delta^3 - \delta^4) + 2(\delta^3 - \delta^5). \tag{5.7}$$

To show that $\Delta > 2c$ under the condition $c \leq ub$, it is sufficient to show that $\Delta > 2ub$ holds. Recall that

$$2ub(g) = 2(\delta - \delta^8) + 2(\delta^2 - \delta^7) + 2(\delta^3 - \delta^6) + 4(\delta^4 - \delta^5). \tag{5.8}$$

Thus,

$$\Delta > 2ub \Longleftrightarrow 6\delta^2 + 12\delta^3 - 20\delta^4 - 4\delta^5 + 2\delta^6 + 2\delta^7 + 2\delta^8 > 0. \tag{5.9}$$

That the last equation holds for $\delta \in (0, 1)$ can be checked numerically (we used Maple). □

Proof of Prop. 5.2. All pairwise stable networks consist of completely connected components that can be ordered according to size such that each larger component of size m satisfies $m > n^2$, where n is the size of the smaller component. There cannot be singleton components, because each player is better off connecting to some player than to none, and each player i wants a link to a player j, for whom $l_j \leq l_i$. Note that this implies that there exists at least one component of size three, if $n \geq 3$. Since any even sized network of $n/2$ separate pairs and any odd sized network of $(n-2)/2$ pairs and the remaining three players being connected by 2 links is efficient (w.r.t. utilitarian welfare), it is also component efficient for any component of size n and, hence, strictly welfare better than any completely connected component of at least size three. For the exact calculations see Jackson and Wolinsky (1996). Hence, any completely connected component of size three or larger contains a welfare better subcomponent, whereas a completely connected component of size two is component welfare maximizing, implying the result. □

Proof of Theorem 5.2. Let us first transform the property convexity in own current links. Hellmann (2009) shows that this property is equivalent to the following property called concavity in own new links. Denote the marginal utility by $\mu_i^l(g) := u_i(g \cup l) - u_i(g \setminus l)$. Thus, for $l \notin g$, $\mu_i^l(g) = u_i(g \cup l) - u_i(g)$.

Definition 5.7. *A profile of utility functions u is* ***concave in own new links****, if for all $i \in N$, for all $g \in G$ and for all links l such that $l \subseteq L_i(g^N)$, and $l \cap g = \emptyset$ the following holds: $\mu_i^l(g) \leq \sum_{ij \in l} \mu_i^{ij}(g)$.*

Now, let g be pairwise stable with transfers. We show that for all $g' \supset g$ it holds that $\sum_{i\in N} u_i(g') \leq \sum_{i \in N} u_i(g)$. Suppose that u satisfies negative externalities and concavity in own new links. For $g' \supset g$, let $l := g' \setminus g$ and for each $i \in N$ let $l^i := l \cap L_i(g')$ and $l^{-i} := l \setminus l^i$. Since u satisfies negative externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \setminus l^{-i}). \tag{5.10}$$

Concavity in own new links implies for all $i \in N$:

$$u_i(g \cup l^i) - u_i(g) \leq \sum_{j:ij \in l^i} u_i(g \cup ij) - u_i(g). \tag{5.11}$$

Now, since g is pairwise stable with transfers, (5.10) and (5.11) imply:

$$\begin{aligned}
\sum_{i \in N}(u_i(g') - u_i(g)) &= \sum_{i\in N} \big(u_i(g \cup l^i \cup l^{-i}) - u_i(g)\big) \\
&\overset{(5.10)}{\leq} \sum_{i \in N} \big(u_i(g \cup l^i) - u_i(g)\big) \\
&\overset{(5.11)}{\leq} \sum_{i \in N} \Big(\sum_{j:ij \in l^i} [u_i(g \cup ij) - u_i(g)]\Big) \\
&\overset{(*)}{=} \sum_{ij \in l} u_i(g \cup ij) - u_i(g) + u_j(g \cup ij) - u_j(g) \overset{(5.6)}{\leq} 0,
\end{aligned}$$

where the equality (*) holds, because for each link $ij \in l$ it holds that $ij \in l^k$ if and only if $k \in \{i, j\}$ and only links in l are considered. □

Theorem 5.3. *If u satisfies positive externalities and convexity in own current links, then a $g \in [PS^t(u)]$ is not over-connected with respect to the utilitarian welfare function.*

Proof of Theorem 5.3. Let g be pairwise stable with transfers. We show that for all $g' \subset g$ it holds that $\sum_{i\in N} u_i(g') \leq \sum_{i\in N} u_i(g)$. Suppose that u satisfies positive externalities and convexity in own current links. For $g' \subset g$, let $l := l(g, g') := g \setminus g'$ and for each $i \in N$ let $l^i := l^i(g, g') := l(g, g') \cap L_i(g')$ and $l^{-i} := l^{-i}(g, g') := l(g, g') \setminus l^i(g, g')$. Given these definitions, we have to show that

$$\sum_{i\in N} u_i(g) - \sum_{i\in N} u_i(g') = \sum_{i\in N} \mu_i^l(g) \geq 0. \tag{5.12}$$

Since u satisfies positive externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \cup l^{-i}). \tag{5.13}$$

Convexity in own current links implies for all $i \in N$:

$$\mu_i^{l^i}(g) \geq \sum_{ij\in l^i} \mu_i^{ij}(g). \tag{5.14}$$

Now, since g is pairwise stable with transfers (5.6), (5.13), and (5.14) imply:

$$\begin{aligned}\sum_{i\in N} \mu_i^l(g) &\overset{(5.13)}{\geq} \sum_{i\in N} \mu_i^{l^i}(g) \\ &\overset{(5.14)}{\geq} \sum_{i\in N}\sum_{j:ij\in l^i} \mu_i^{ij}(g) \\ &\overset{(*)}{=} \sum_{ij\in l}[\mu_i^{ij}(g) + \mu_j(g, ij)] \overset{(5.6)}{\geq} 0,\end{aligned}$$

where the equality (*) holds, because for each link $ij \in l$ it holds that $ij \in l^k$ if and only if $k \in \{i, j\}$ and only links in l are considered. □

Proof of Prop. 5.3. Let $c < \frac{\rho}{(\rho+2k+2)(\rho+2k)}$, then it holds for the welfare maximizing number of links that $l^*(g) \geq k$. Since any network containing $l^*(g)$ links is welfare maximizing, any network, which has less than $l^*(g)$ links is under-connected. By Theorem 5.2 no network that is pairwise stable with transfers can be under-connected since u^{PR} satisfies negative externalities and convexity in own current links. Thus, any network $g \in [PS^t]$ has to contain at least $l^*(g) \geq k$ links. □

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