Some classes of Markov processes on configuration spaces and their applications

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Contents

1 Introduction

The field of interacting particle systems began as a branch of probability theory in the late 1960's. Much of the original impulse came from the works of Spitzer and Dobrushin. Since then, this area has grown and developed rapidly, establishing surprising connections with many other fields. The original motivation for this field came mainly from statistical mechanics. One of the aims was to analyze stochastic models which describe the time evolution of systems, whose equilibrium measures are classical Gibbs states. In particular, one wanted to get a better understanding of the phenomenon of phase transition in the dynamical framework. As time passed, it became clear that models with very similar mathematical structure appear naturally in other contexts – neural networks, spreading of infection, ecological systems, economical and sociological models, biology, demography, etc.

An interacting particle system usually consists of infinitely many particles, which interact with each other in some position space (for example lattice $\mathbb{Z}^d$, or continuum $\mathbb{R}^d$, or more general topological space X). As might be expected, the behavior of an interacting particles system depends in a rather sensitive way on the precise nature of the interaction. Thus most of the research deals with certain types of models in which the interaction is of a prescribed form. In most of the considered models it is assumed that the position space is a lattice. However, this assumption is not always suitable. Therefore, in many cases it is reasonable, and even necessary to consider interacting particle systems in continuum.

In this work we study some classes of Markov processes for interacting particle systems in continuum. More precisely, we deal with Glauber and Kawasaki dynamics and consider applications of certain birth-and-death processes to demography.

The Glauber dynamics was first studied on the lattice. In the classical d-dimensional Ising model with spin space $S = \{-1, 1\}$, the Glauber dynamics means that particles randomly change their spin value, which is called a spin-flip. In the Kawasaki dynamics, pairs of neighboring particles with different spins randomly exchange their spin values. Under appropriate conditions on the coefficients the corresponding dynamics has a Gibbs measure as a symmetrizing (and hence invariant) measure. We refer to [CMR02, Lig85, Mar99] for a discussion of the Glauber and Kawasaki dynamics of lattice spin systems.

Let us now interpret a lattice system with spin space $S = \{-1, 1\}$ as a model of a lattice gas. Then $\sigma(x) = 1$ means that there is a particle at site x, while $\sigma(x) = -1$ means that the site x is empty. The Glauber dynamics of such a system means that, at each site x, a particle randomly appears and disappears. Hence, this dynamics may be interpreted as a birth-and-death process on $\mathbb{Z}^d$. A corresponding interpretation of the Kawasaki dynamics yields that particles randomly jump from one site to another.

If we consider a continuous particle system, i.e., a system of particles which can take any position in the Euclidean space $\mathbb{R}^d$, then an analog of the Glauber dynamics should be a process in which particles randomly appear and disappear in the space, i.e., a spatial birth-and-death process. The generator of such a process is informally given by the formula

$$(H_{\mathrm{G}}F)(\gamma) = -\sum_{x\in\gamma} d(x,\gamma)(D_x^- F)(\gamma) - \int_{\mathbb{R}^d} b(x,\gamma)(D_x^+ F)(\gamma)\,dx, \tag{1.0.1}$$

where

$$(D_x^- F)(\gamma) = F(\gamma \setminus x) - F(\gamma), \quad (D_x^+ F)(\gamma) = F(\gamma \cup x) - F(\gamma).$$

The coefficient $d(x,\gamma)$ describes the rate at which the particle x of the configuration γ dies, while $b(x,\gamma)$ describes the rate at which, given the configuration γ, a new particle is born at x.

Furthermore, an analog of the Kawasaki dynamics of continuous particles should be a process in which particles randomly jump over the space $\mathbb{R}^d$. The generator of such a process is then informally given by

$$(H_{\mathrm{K}}F)(\gamma) = -2\sum_{x\in\gamma}\int_{\mathbb{R}^d} c(x,y,\gamma)(D_{xy}^{-+} F)(\gamma)\,dy, \tag{1.0.2}$$

where

$$(D_{xy}^{-+} F)(\gamma) = F(\gamma \setminus x \cup y) - F(\gamma)$$

and the coefficient $c(x,y,\gamma)$ describes the rate at which the particle x of the configuration γ jumps to y.

Further we describe the contents of the work chapter by chapter in more details.

Configuration spaces – general facts and notations

In this chapter we give the necessary definitions and facts, related to the configuration spaces, which are used in this thesis. These spaces can be constructed for a quite general underlying space X, we restrict ourselves to a locally compact topological spaces.

The subject of Section 2.1 are the space of finite configurations $\Gamma_0(X)$ and the configuration space $\Gamma(X)$, and their topological properties. The space of finite configurations $\Gamma_0(X)$ is given by

$$\Gamma_0(X) = \{\eta \subset X : |\eta| < \infty\},$$

(where $|\gamma|$ denotes the number of elements of the set γ) and the configuration space $\Gamma(X)$ is defined as

$$\Gamma(X) := \{\gamma \subset X : |\gamma \cap \Lambda| < \infty \quad \text{for all bounded} \quad \Lambda \subset X\}.$$

In the Section 2.2 we remind the definitions and basic facts about Lebesgue-Poisson and Poisson measures. There we also define the correlation functions, which can be regarded as a density of the correlation measure w.r.t. Lebesgue-Poisson measure. We also remind the notion of Gibbs measures through Georgii-Nguyen-Zessin equation, and quote some existence theorems for Gibbs measures, corresponding to pair potentials.

In Section 2.3 we recall the notions of marked configuration spaces and measures on them, in particular marked Lebesgue-Poisson and marked Poisson measures.

Glauber and Kawasaki equilibrium dynamics for determinantal point processes

Spatial birth-and-death processes were first discussed in bounded volume by Preston in [Pre75], see also [HS78]. By using the theory of Dirichlet forms, Glauber and Kawasaki dynamics of continuous particle systems in infinite volume, which have a Gibbs measure as symmetrizing measure, were constructed in [KL05, KLR07]. In [SY02] Shirai and Yoo investigate the Glauber dynamics on the lattice which has, instead of a Gibbs measure, a so-called determinantal point process (on the lattice) as an invariant measure. Thus we came to the problem of construction of Glauber and Kawasaki dynamics in continuum, which have a determinantal point process as an invariant measure. Below we define a determinantal point process.

Let X be a locally compact Polish space. Let ν be a Radon measure on X and let K be a linear, Hermitian, locally trace class operator on $L^2(X, \nu)$ for which $\mathbf{0} \le K \le \mathbf{1}$. Then K is an integral operator and we denote by $K(\cdot, \cdot)$ the integral kernel of K.

A determinantal (also called fermion) point process, abbreviated DPP, corresponding to K is a probability measure on Γ whose correlation functions are given by

$$k_\mu^{(n)}(x_1, \dots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n.$$

DPPs were introduced by Macchi [Mac75]. These processes naturally arise in quantum mechanics, statistical mechanics, random matrix theory, and representation theory, see e.g. [BO05, ST03, Sos00] and the references therein.

In [Spo87], Spohn investigated a diffusion dynamics on the configuration space $\Gamma(\mathbb{R})$ for which the DPP corresponding to the Dyson kernel $K(x,y) = \dfrac{\sin(x-y)}{x-y}$ is an invariant measure.

In the case where the operator K satisfies the condition $K < \mathbf{1}$, Georgii and Yoo [GY05] (see also [Yoo06]) investigated Gibbsianness of fermion point processes. In particular, they proved that every fermion process with K as above possesses Papangelou (conditional) intensity.

Using Gibbsianness of fermion point processes, Yoo [Yoo05] constructed an equilibrium diffusion dynamics on the configuration space over $\mathbb{R}^d$, which has a DPP as an invariant measure. This Markov process is an analog of the gradient stochastic dynamics which has the standard Gibbs measure corresponding to a potential of pair interaction as invariant measure (see e.g. [AKR98]).

On the other hand, in the case of an invariant Gibbs measure, one considers, as described above, also further classes of equilibrium processes on the configuration space, e.g. Glauber and Kawasaki dynamics in continuum.

Using the theory of Dirichlet forms (see e.g. [MR92]), we construct conservative Markov processes on Γ with cadlag paths which have a DPP μ as symmetrizing, hence invariant, measure. First we derive the properties of the bilinear forms, which correspond to Glauber and Kawasaki dynamics. We show that they are closable, Dirichlet, quasi-regular, and then apply the appropriate theorems from [MR92] which give the existence of the process. The main technical difficulty we have to deal with is the absence of a good explicit form of the Papangelou intensity $r(x,\gamma)$. Furthermore, we discuss the explicit form of the $L^2(\mu)$-generators of these processes on the set of cylinder functions, and give examples of Glauber and Kawasaki dynamics, for which the conditions of the existence theorems are satisfied. These generators will have the form (1.0.1) in the case of Glauber dynamics, and (1.0.2) in the case of Kawasaki dynamics (with $\mathbb{R}^d$ replaced by a general topological space X). Since we essentially use the Papangelou intensity of the fermion point process, our study here is restricted by the assumption that $K < \mathbf{1}$. We also obtain a sufficient condition for the existence of the spectral gap of the Glauber dynamics generator.

Spectral Gap for Glauber dynamics

Another question which arises in connection with different dynamics is the rate of convergence to equilibrium. As one of the characteristics which give us the information about the speed of convergence we can consider the spectral gap of the generator. Most commonly, the Poincaré inequality

$$c \cdot \mathrm{Var}_\mu(f) \le \mathcal{E}(f,f), \quad f \in D(\mathcal{E}),$$

where $\mathrm{Var}_\mu(f) = \int (f - \int f d\mu)^2 d\mu$, is used in the context of the spectral gap analysis. The largest c for which the inequality holds is the spectral gap of the generator L in $L^2(\mu)$, where L is the generator corresponding to the Dirichlet form $\mathcal{E}$.

For the Glauber dynamics in continuum the problem of the existence of the spectral gap was studied in [BCC02, BCDPP06, Wu04, KL05]. Furthermore, in [KMZ04] under certain conditions on the invariant measure the one-particle invariant subspace of the generator was constructed, the spectral gap and the second gap between the one-particle branch and the rest of the spectrum were estimated.

In [BCC02] the generator of the Glauber dynamics in a finite volume was studied. Precisely, the authors consider a non-negative finite range potential ϕ and activity z which satisfy the condition of the low activity-high temperature regime (LAHT). For any finite volume $\Lambda \subset \mathbb{R}^d$ and a boundary condition η outside Λ one can associate the finite volume Gibbs measure $\mu_{\Lambda,\eta}$. They showed the Poincaré inequality in bounded volume, which implies that the generator of the Dirichlet form has a spectral gap $(0, G_{\Lambda,\eta})$. Moreover, they proved that the infimum of $G_{\Lambda,\eta}$ over all finite volumes and boundary conditions η is positive.

This result was extended in [KL05] to the case of general non-negative potentials and the infinite volume dynamics, and, moreover, an explicit estimate of the spectral gap was shown. To produce this estimate, the coercivity identity approach was used. Similar results were obtained with other techniques in [Wu04], where also the hard core case was considered. In the aforementioned articles the existence of spectral gap was obtained by using different methods. However, in all of them the potential is assumed to be positive, and this assumption is crucial for the proof. Therefore there emerged a question, if the spectral gap can exists in the case when the potential has a negative part. In Chapter 4 we present an answer to this question. Precisely, we show the existence of the spectral gap for a certain class of pair potentials, which do not have to be positive. Namely, we consider the Glauber dynamics on $\mathbb{R}^d$ with corresponding invariant measure μ, for which the Papangelou intensity $r(x,\gamma)$ exists. The Markov generator of the process is given on cylinder functions by

$$(HF)(\gamma) = -\int_{\mathbb{R}^d} \gamma(dx)\,(F(\gamma\backslash x) - F(\gamma)) - \int_{\mathbb{R}^d} r(x,\gamma)(F(\gamma \cup x) - F(\gamma))dx, \quad \mu\text{-a.e.}$$

We define the "carré du champ" and the "carré du champ itéré" operators respectively as

$$\square(F,G) := \frac{1}{2}(H(FG) - FHG - GHF),$$

and

$$\square_2(F,G) := \frac{1}{2}(H\square(F,G) - \square(F,HG) - \square(G,HF)),$$

cf. [Bak85, BÉ85a, BÉ85b, BÉ86, Bak94]. If H is the Laplace operator on a n-dimensional Riemannian manifold, then $\square(f,f) = |\mathrm{grad} f|^2$ and $\square_2(f,f) = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)$ (Weitzenböck formula), where $|\mathrm{Hess} f|$ denotes the Hilbert-Schmidt norm of the Hessian of f and $\mathrm{Ric}(\cdot,\cdot)$ is the Ricci curvature tensor.

We calculate explicitly the "carré du champ" and the "carré du champ itéré" which correspond to the Glauber dynamics generator. Using these expressions we obtain in Theorem 4.2.7 the coercivity identity for the generator H.

We use the so-called coercivity inequality to investigate the spectral properties of the generator H. We say that the coercivity inequality holds for a positive essentially self-adjoint operator H with constant c if

$$\int_\Gamma (HF)^2(\gamma)\mu(d\gamma) \geq c\,\mathcal{E}(F,F), \quad c > 0.$$

If it is fulfilled then the interval $(0,c)$ does not belong to the spectrum of H. Note that the Poincaré inequality is slightly stronger and means that, in addition to the fact that $(0,c)$ does not belong to the spectrum of H, that the kernel of H consists only of constants. Using the coercivity identity we derive the following sufficient condition for the fact that the interval $(0,c)$ does not belong to the spectrum of H (Theorem 4.3.2). If for each fixed $\gamma \in \Gamma$ the kernel

$$r(x,\gamma)(r(y,\gamma) - r(y,\gamma\cup x)) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y) \tag{1.0.3}$$

is positive definite then the coercivity inequality holds for H with constant c.

As the main example we consider a Gibbs measures μ corresponding to a translation invariant pair potential ϕ and activity z. Writing the condition (1.0.3) for such a Gibbs measure μ and $c = 1$ we obtain the condition

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1 - e^{-\phi(x-y)})\psi(y)\psi(x)dxdy \geq 0 \tag{1.0.4}$$

for all $\psi \in C_0(\mathbb{R}^d)$. Note that this condition does not contain the activity z. When we speak about regular functions in the following, we have in mind regularity in the sense of pair potentials, cf. Section 2.2.2. Consider the class $\mathcal{K}$ of pair potentials ϕ of the form

$$\phi := -\ln(1-f),$$

where f is a continuous positive definite regular function such that $f(0) \leq 1$. Then we obtain (Theorem 4.4.5) that for a tempered Gibbs measure μ, for a pair potential $\phi \in \mathcal{K}$ and for all activities $z > 0$ the generator of the Glauber dynamics, operator H, fulfills the coercivity inequality for $c = 1$. The class $\mathcal{K}$ contains also non-positive potentials, see e.g. examples in Section 4.4.3. Further properties of potentials which belong to $\mathcal{K}$ are examined in Proposition 4.4.7. We mention that such potentials are positive definite in the sense of generalized functions, and integrable at 0.

Spatial Markov Processes in Mutation-Selection Models

In Chapter 5 we present an application of birth-and-death processes on configuration spaces to a generalized mutation-selection balance model. It is a generalization of a model presented in [SEK05]. The model describes aging of a population as a process of accumulation of mutations in a genotype. A rigorous treatment demands that mutations correspond to points in abstract spaces. Our model describes an infinite-population, infinite-sites model in continuum. The dynamical equation of Kimura-Maruyama type which describes the system is a fairly standard one for mathematical mutation-selection theory.

The problem can be posed in terms of evolution of states (differential equation) or, equivalently, represented in terms of Feynman-Kac formula. The questions of interest are existence of a solution, its asymptotic behavior and properties of the limiting state. In the non-epistatic case the problem was posed and solved in [SEK05]. The articles [KMP07], [KMZ07] were motivated by this work, and treat the case of a more general potential – the epistatic one. In both articles the space of the possible positions of mutations is $\mathbb{R}^d$. The generalization to a topological space X seems natural and necessary because the geometrical structure of the DNA is far from the geometrical structure of $\mathbb{R}^d$. In our model we consider a topological space X as the space of positions of mutations and the influence of an epistatic potential.

Let X be a Polish space, interpreted as the space of positions of possible mutations. The set of all genotypes γ is thus the configuration space $\Gamma(X)$. The emergence of mutant alleles is described by a stochastic process, the state of the population of genotypes at each fixed moment of time t is described by a probability measure μ_t on $\Gamma(X)$. The time development of the population is modelled by a Kimura-Maruyama type equation

$$\frac{d}{dt}\mu_t(F) = \mu_t\left(\int_X (F(\cdot \cup x) - F(\cdot))d\sigma(x)\right) - \mu_t(F \cdot \Phi) + \mu_t(F)\mu_t(\Phi).$$

Here $\Phi : \Gamma \longrightarrow \mathbb{R}_+$ is a selection cost function, which consists of two parts:

$$\Phi(\gamma) = \Phi_{ne}(\gamma) + \Phi_e(\gamma).$$

$\Phi_{ne}(\gamma)$ is the nonepistatic part, which describes the life costs of a mutation, is given by

$$\Phi_{ne}(\gamma) := \langle h, \gamma \rangle = \sum_{x\in\gamma} h(x), \;\; h(x) \geq c > 0.$$

$\Phi_e(\gamma)$ is the epistatic part, which describes the coexistence costs of mutations, is defined by

$$\Phi_e(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x;y), \quad \phi \geq 0.$$

As the configuration γ may contain, in general, infinite number of points, the above cost functions are well-defined only in a bounded region $\Lambda \subset X$.

The questions of interest for us are: existence of solution μ_t and asymptotic behavior μ_t for $t \to +\infty$. The useful choice of time parameterization is to start the process in the remote past, namely at time $t = -T < 0$, in the state μ_{-T}. Then we arrive at $t = 0$ in the state $\mu_{0,T}$. The limiting state for long time is then given by

$$\lim_{T\to+\infty} \mu_{0,T} = \mu_0.$$

Another representation of the model, which gives us the explicit solution of our Kimura-Maruyama type equation can be explicitly written as

$$\mu_t^T(f) = \frac{\mathbb{E}\left[f(\xi_t^T)e^{-\int_{-T}^t \Phi(\xi_\tau^T)d\tau}\right]}{\mathbb{E}\left[1\cdot e^{-\int_{-T}^t \Phi(\xi_\tau^T)d\tau}\right]},$$

where ξ_τ^T denotes the Markov process corresponding to the generator L, started in $\mu_{-T}^T = \mu$. Here L is given by

$$LF(\gamma) := \int_X (F(\gamma \cup x) - F(\gamma))d\sigma(x). \tag{1.0.5}$$

Performing the limit $T \longrightarrow +\infty$ gives us heuristically

$$\mu_0(f) = \int_{\Omega(\mathbb{R}_-\to E)} f(\xi(0))d\nu^\Phi(\xi(\cdot)),$$

where

$$d\nu^\Phi(\xi(\cdot)) = \frac{1}{Z}e^{-\int_{-\infty}^0 \Phi(\xi(\tau))d\tau}d\nu^0(\xi(\cdot)),$$

Z is the normalizing constant, and $E = \Gamma(X)$.

The aim of Chapter 5 is to give proper sense to ν^Φ, defining the measure first in a bounded volume and for finite time and then going to the limit. By means of ν^Φ we derive the large time asymptotic for μ_0^T. In the first section we consider the generator L as given above and in the subsequent section the more general case of the birth-and-death Markov generator.

Thus in Section 5.2 we consider the model, corresponding to the generator L given by (1.0.5). First we construct the pure birth Markov process $\xi_\tau(\hat{\gamma})$, $0 \leq \tau \leq T$, on $(\hat{\Gamma}(X,[0,T]),\ \nu_T^0)$, starting from an empty configuration at time $t = -T$, which corresponds to the generator L. Here ν_T^0 denotes a marked Poisson measure on $\hat{\Gamma}(X,[0,T])$ with intensity measure $\sigma(dx)dt$. Next, we take into account the influence of the cost function, what is for convenience done in two steps. First we consider only the influence of the nonepistatic part Φ_{ne} and then add the influence of the epistatic part Φ_e.

Denote by ν^h the path space measure on the space $\hat{\Gamma}(X,\mathbb{R}_+)$, obtained under the influence of Φ_{ne}. The restriction of ν^h to $\hat{\Gamma}(\Lambda,[0,T])$ is defined for bounded $\Lambda \subset X$ as

$$d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\Lambda,T}} \exp\left\{ -\int_0^T \Phi_{ne}^{T,\Lambda}(\xi_\tau(\hat{\gamma}_\Lambda))d\tau \right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda),$$

where $Z_{\Lambda,T}$ is the normalizing constant. Then we obtain in Theorem 5.2.4 the measure ν^h as the limit of measures $\nu^h_{\Lambda,T}$. The statement of the theorem is: there exists a weak limit ν^h of measures $\nu^h_{\Lambda,T}$ for $\Lambda \uparrow X$, $T \to +\infty$. The measure ν^h is a marked Poisson measure on $\hat{\Gamma}(X,\mathbb{R}_+)$ with intensity measure $e^{-sh(x)}\sigma(dx)ds$.

We are also interested in the final distribution of mutations μ^h. The restriction of μ^h to $\hat{\Gamma}(\Lambda,[0,T])$ is defined for bounded $\Lambda \subset X$ and $F(\eta) = e^{\langle f,\eta\rangle}$, $\eta \in \Gamma(X)$ as

$$\int_{\Gamma(X)} F(\gamma_\Lambda) d\mu^0_{\Lambda,T}(\gamma_\Lambda) = \int_{\hat{\Gamma}(\Lambda,[0,T])} F(\xi_0(\hat{\gamma}_\Lambda)) d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda)$$

We obtain in Theorem 5.2.6 the measure μ^h as the weak limit of measures $\mu^0_{\Lambda,T}$. The measure μ^h is a Poisson measure on $\Gamma(X)$ with intensity measure $\frac{1}{h(x)}\sigma$. This theorem was also proved in [SEK05].

Now we include the influence of the epistatic part of the potential $\Phi_e(\gamma)$. We consider the Gibbs perturbation $\nu^{\beta,\phi}$ of measure ν^h, obtained in Theorem 5.2.4, through Φ_e. Denote by $\nu^{\beta,\phi}_\Lambda$ the restriction of the measure $\nu^{\beta,\phi}$ to the space $\hat{\Gamma}(\Lambda,\mathbb{R}_+)$, which is given by

$$d\nu^{\beta,\phi}_\Lambda(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\beta,\Lambda}} \exp\left\{ -\beta \int_0^{+\infty} \Phi_e^\Lambda(\xi_\tau(\hat{\gamma}_\Lambda))d\tau \right\} d\nu^h_\Lambda(\hat{\gamma}_\Lambda).$$

To construct the weak limit of $\nu^{\beta,\phi}_\Lambda$ we use the cluster expansion method cf. [KKDS98, Kun99]. Under the following assumptions

- $h \equiv const$

- $\phi(x,y) \geq 0, \quad \forall x,y \in X.$
- $$C(\beta,h) = \operatorname*{esssup}_{y\in X} \int\limits_X \frac{\beta\phi(x,y)}{h(h+\beta\phi(x,y))}\sigma(dx) \leq \frac{1}{2e}$$

on the cost function

$$\Phi(\gamma) = \sum_{x\in\gamma} h(x) + \sum_{\{x,y\}\subset\gamma} \phi(x;y)$$

we obtain Theorem 5.2.11, which gives us the existence of the weak limit $\nu_\Lambda^{\beta,\phi} \to \nu^{\beta,\phi}$, $\Lambda \uparrow X$.

In the Section 5.3 we consider the process, corresponding to the birth-and-death generator

$$LF(\gamma) = \sum_{x\in\gamma} d(x)(F(\gamma \setminus x) - F(\gamma)) + \int b(y)(F(\gamma \cup y) - F(\gamma))\sigma(dy).$$

Analogously to the previous section we first construct the Markov birth-and-death process Y_t^T, $-T \leq t \leq 0$ on $(\Omega(\hat{X}_T), \pi_\rho)$, starting from an empty configuration at time $t = -T$, corresponding to L. Here π_ρ denotes the marked Poisson measure on $\Omega(\hat{X}_T)$ with intensity measure $\rho(dx,ds,dl) = b(x)d(x)e^{-d(x)l}\sigma(dx)dsdl$. As in the case of birth process we consider the influence of a selection cost function

$$\Phi(\gamma) := \langle h,\gamma\rangle = \sum_{x\in\gamma} h(x), \;\; h \geq c > 0.$$

Again, we are interested in the path space measure ν^h on the space $\hat{\Gamma}(X \times (-\infty,0], \mathbb{R}_+)$, the restriction of which to $\hat{\Gamma}(\Lambda \times [-T,0], \mathbb{R}_+)$ is defined for $\Lambda \subset X$ as

$$d\nu_{\Lambda,T}^h(\hat{\gamma}) = \frac{1}{Z_{\Lambda,T}} \exp\left\{-\int_{-T}^0 \Phi_\Lambda(Y_t^T(\hat{\gamma}))dt\right\} d\pi_\rho^{T,\Lambda}(\hat{\gamma}).$$

In Theorem 5.3.3 we obtain ν^h as the weak limit of $\nu_{\Lambda,T}^h$. The measure ν^h is a marked Poisson measure on $\Omega(\hat{X})$ with intensity measure τ,

$$\tau(dx,dl,ds) = \exp\{-((-s)\vee l)h(x)\}\, b(x)d(x)e^{-d(x)l}d\sigma(x)dlds.$$

We also calculate the final distribution of mutations μ^h. We show in Theorem 5.3.5 that μ^h is the weak limit of measures $\mu_{\Lambda,T}^0$, and μ^h is a Poisson measure on $\Gamma(X)$ with intensity measure $\dfrac{\sigma(dx)}{h(x)+d(x)}$.

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2 Configuration spaces

2.1 Configuration spaces

Let X be a locally compact topological space (describing the position space of particles). Denote by $\mathcal{B}(X)$ the corresponding Borel σ-algebra on X, and by $\mathcal{B}_c(X)$ the collection of all sets from $\mathcal{B}(X)$ which are relatively compact. $\mathcal{B}_b(X)$ is the collection of all bounded Borel sets.

For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we define the *space of n-point configurations* $\Gamma_0^{(n)}$ by

$$\Gamma_0^{(n)} = \Gamma_0^{(n)}(X) := \{\eta \subset X \mid |\eta| = n\}\,, \quad \Gamma_0^{(0)} := \{\emptyset\},$$

where $|\cdot|$ denotes the cardinality of a set. The space $\Gamma_0^{(n)}(\Lambda)$ for $\Lambda \in \mathcal{B}_c(X)$ is defined analogously.

For every $\Lambda \in \mathcal{B}_c(X)$ we define a mapping $N_\Lambda : \Gamma_0^{(n)} \to \mathbb{N}_0$; $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For short we write $\eta_\Lambda := \eta \cap \Lambda$. A topological structure on $\Gamma_0^{(n)}$ may be introduced through the natural projective mapping of

$$\widetilde{X^n} := \{(x_1, \ldots, x_n) \in X^n \mid x_k \neq x_l \text{ if } k \neq l\}\,,$$

onto $\Gamma_0^{(n)}$ defined by

$$\begin{aligned} \mathrm{sym}^n : \widetilde{X^n} &\to \Gamma_0^{(n)} \\ (x_1, \ldots, x_n) &\mapsto \{x_1, \ldots, x_n\}. \end{aligned}$$

Hence sym^n induces a topology on $\Gamma_0^{(n)}$. The corresponding Borel σ-algebra on $\Gamma_0^{(n)}$ we denote by $\mathcal{B}(\Gamma_0^{(n)})$, and it coincides with the σ-algebra generated by the mappings N_Λ, i.e.

$$\mathcal{B}(\Gamma_0^{(n)}) = \sigma\left(\{N_\Lambda \mid \Lambda \in \mathcal{B}_c(X)\}\right).$$

Finally, we define the *space of finite configurations* Γ_0

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

equipped with the topology of disjoint union $\mathcal{O}(\Gamma_0)$.

The *configuration space* $\Gamma := \Gamma(X)$ over X is defined as the set of all subsets of X which are locally finite:

$$\Gamma := \{\gamma \subset X : |\gamma_\Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_c(X)\}.$$

Elements $\gamma \in \Gamma$ we will call locally finite configurations. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(X)$, where δ_x is the Dirac measure with mass at x, $\sum_{x\in\varnothing} \delta_x$:=zero measure, and $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. The space Γ can be equipped with the vague topology, which is the relative topology as a subset of the space $\mathcal{M}(X)$, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_X f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x), \qquad f \in C_0(X),$$

are continuous. Here, $C_0(X)$ is the space of all continuous, real-valued functions on X with compact support. The corresponding Borel σ-algebra on Γ we denote by $\mathcal{B}(\Gamma)$, and it is equal to the σ-algebra generated by mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$, i.e.

$$\mathcal{B}(\Gamma) = \sigma\left(\{N_\Lambda \,|\Lambda \in \mathcal{B}_c(X)\}\right).$$

Given any $\Lambda \in \mathcal{B}(X)$ we can introduce the space $\Gamma(\Lambda)$ of configurations contained in Λ

$$\Gamma(\Lambda) := \left\{\gamma \in \Gamma |\ \gamma_{X\setminus\Lambda} = \emptyset\right\},$$

the σ-algebra $\mathcal{B}(\Gamma(\Lambda))$ may be introduced in a similar way:

$$\mathcal{B}(\Gamma(\Lambda)) = \sigma\left(\{N_Y \upharpoonright_{\Gamma_\Lambda} : Y \in \mathcal{B}_c(X)\}\right).$$

The following classes of function are used in the following: $L^0(\Gamma)$ is the set of all measurable functions on Γ. $\mathcal{F}L^0(\Gamma)$ is the set of cylinder functions, i.e. the set of all measurable functions $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}(\Gamma(\Lambda))$ for some $\Lambda \in \mathcal{B}_c(X)$. These functions are characterized by the following relation: there exists $\Lambda \in \mathcal{B}_c(X)$ such that

$$F(\gamma) = F \upharpoonright_{\Gamma(\Lambda)} (\gamma_\Lambda).$$

2.2 Measures on configuration spaces

For the construction of measures on Γ and Γ_0 we fix an intensity measure σ on the underlying space X. Assume σ is a non-atomic and locally finite measure on $(X, \mathcal{B}(X))$. Having in mind applications, we assume $\sigma(X) = \infty$.

We will call probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ also point processes.

We say that a measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ has *Papangelou (conditional) intensity* if there exists a measurable function $r : X \times \Gamma \to [0, +\infty]$ such that

$$\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) F(x, \gamma) = \int_\Gamma \mu(d\gamma) \int_X \sigma(dx) r(x, \gamma) F(x, \gamma \cup x) \tag{2.2.1}$$

for any measurable function $F : X \times \Gamma \to [0, +\infty]$. Here and below, for simplicity of notations we just write x instead of $\{x\}$.

2.2.1 Poisson and Lebesgue-Poisson measure

For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered as a measure restricted to the space $(\widetilde{X^n}, \mathcal{B}(\widetilde{X^n}))$. Let $\sigma^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}^n)^{-1}$ be the corresponding measure on $\Gamma_0^{(n)}$. For $n = 0$ we put $\sigma^{(0)}(\{\emptyset\}) := 1$.

The *Lebesgue-Poisson measure* $\lambda_{z\sigma}$ on Γ_0 is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^\infty \frac{z^n}{n!} \sigma^{(n)}.$$

Here $z > 0$ is the so-called activity parameter. For $\Lambda \in \mathcal{B}_c(X)$ we obtain $\lambda_{z\sigma}(\Gamma(\Lambda)) = e^{z\sigma(\Lambda)}$. The restriction of $\lambda_{z\sigma}$ to $\Gamma(\Lambda)$ will be also denoted by $\lambda_{z\sigma}$. Therefore, we can define a probability measure $\pi_{z\sigma}^\Lambda$ on $\Gamma(\Lambda)$ by

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}.$$

For every $\Lambda \in \mathcal{B}_c(X)$ define a projection $p_\Lambda : \Gamma \to \Gamma(\Lambda)$; $p_\Lambda(\gamma) := \gamma_\Lambda$. We notice that the family $\{\pi_{z\sigma}^\Lambda,\ \Lambda \in \mathcal{B}_c(X)\}$ is consistent and thus, by a version of Kolmogorov's theorem for projective limit spaces, such family uniquely determines a measure $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ such that $\pi_{z\sigma}^\Lambda = \pi_{z\sigma} \circ p_\Lambda^{-1}$. The measure $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is called *Poisson measure* with intensity measure $z\sigma$.

Calculation of Laplace transform of the Poisson measure π_σ yields

$$\int_\Gamma \exp(\langle \gamma, f \rangle) d\pi_\sigma(\gamma) = \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x) \right). \tag{2.2.2}$$

This characteristic can also be used to define the Poisson measure using the Minlos theorem, see e.g. [Oli], [GV64].

A measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is said to have *correlation functions* $k_\mu^{(n)}$, if for any $n \in \mathbb{N}$, there exists a non-negative, measurable, symmetric function $k_\mu^{(n)}$ on X^n such that

$$\begin{aligned} &\int_\Gamma \sum_{\{x_1, \dots, x_n\} \subset \gamma} f^{(n)}(x_1, \dots, x_n)\, \mu(d\gamma) \qquad (2.2.3) \\ &\quad = \frac{1}{n!} \int_{X^n} f^{(n)}(x_1, \dots, x_n) k_\mu^{(n)}(x_1, \dots, x_n)\, \sigma(dx_1) \cdots \sigma(dx_n). \end{aligned}$$

for any measurable, symmetric function $f^{(n)} : X^n \to [0, +\infty]$. These are well known correlation functions of statistical physics, see e.g [Rue69], [Rue70].

If there exists $\xi > 0$ independent of n such that

$$\forall (x_1, \dots, x_n) \in X^n : \quad k_\mu^{(n)}(x_1, \dots, x_n) \le \xi^n, \tag{2.2.4}$$

then we say that the correlation functions $k_\mu^{(n)}$ satisfy the *Ruelle bound.*

Note that any probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ satisfying the Ruelle bound has all local moments finite, i.e.,

$$\int_\Gamma \langle f, \gamma \rangle^n \, \mu(d\gamma) < \infty, \qquad f \in C_0(X), \ f \ge 0, \ n \in \mathbb{N}.$$

2.2.2 Gibbs measures

Fix a measure σ on $(X, \mathcal{B}(X))$. For $\gamma \in \Gamma$ and $x \in X$, we consider a relative energy $E(x, \gamma) \in (-\infty, +\infty]$ of interaction between a particle located at x and the configuration γ. We suppose that the mapping E is measurable.

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a (grand-canonical) Gibbs measure corresponding to activity $z > 0$ and the relative energy E if it satisfies the Georgii–Nguyen–Zessin identity ([NZ79, Theorem 2], see also [Kun99, Theorem 2.2.4]):

$$\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) F(x, \gamma) = \int_\Gamma \mu(d\gamma) \int_X z\sigma(dx) \exp\left[-E(x, \gamma)\right] F(x, \gamma \cup x) \tag{2.2.5}$$

for any measurable function $F : X \times \Gamma \to [0, +\infty]$. Let $\mathcal{G}(z, E)$ denote the set of all Gibbs measures corresponding to z and E. Note that in terms of Papangelou intensity it means just that $r(x, \gamma) = z \exp\left[-E(x, \gamma)\right]$. In particular, if $E(x, \gamma) \equiv 0$, then (2.2.5) is the Mecke identity, which holds if and only if μ is the Poisson measure $\pi_{z\sigma}$ with intensity measure $z\sigma(dx)$.

We assume that

$$E(x, \gamma) \in \mathbb{R} \quad \text{for } \sigma \otimes \mu\text{-a.e. } (x, \gamma) \in X \times \Gamma. \tag{2.2.6}$$

Consider the special case of Gibbs measures corresponding to translation invariant pair potentials ϕ. Here we assume that the position space of particles is $X = \mathbb{R}^d$.

For each $x \in X$ and $\gamma \in \Gamma$, we define

$$E(x, \gamma) := \begin{cases} \sum\limits_{y \in \gamma} \phi(x - y), & \text{if } \sum\limits_{y \in \gamma} |\phi(x - y)| < \infty, \\ +\infty, & \text{otherwise} \end{cases}$$

where ϕ is a pair potential, $\phi : \mathbb{R}^d \to (-\infty, \infty]$, which is a measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. We formulate some conditions on the pair potential ϕ under which the corresponding Gibbs measure exists.

(SS) (*Superstability*) For every $r \in \mathbb{Z}^d$ define a cube

$$\Delta_r = \left\{ x \in \mathbb{R}^d : r_i - \frac{1}{2} \leq x_i < r_i + \frac{1}{2} \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. We set $N_r(\gamma) = \gamma(\Delta_r)$. One says that ϕ is superstable if there exist $A > 0$, $B \geq 0$ such that, for all $\gamma \in \Gamma_0$ holds

$$\sum_{\{x,y\} \subset \gamma} \phi(x - y) \geq \sum_{r \in \mathbb{Z}^d} \left[A N_r^2(\gamma) - B N_r(\gamma) \right].$$

(S) (*Stability*) There exists $B \geq 0$ such that, for any $\gamma \in \Gamma$, $|\gamma| < \infty$,

$$\sum_{\{x,y\} \subset \gamma} \phi(x - y) \geq -B|\gamma|.$$

(I) (*Integrability*) We have

$$C := \int_{\mathbb{R}^d} | \exp[-\phi(x)] - 1 | \, \sigma(dx) < \infty.$$

(LR) (*Lower regularity*) We say ϕ is lower regular if there exists a positive decreasing function φ on $[0, +\infty)$ such that $\phi(x) \geq -\varphi(|x|)$ for all $x \in \mathbb{R}^d$ and

$$\int_0^\infty t^{d-1} \varphi(t) dt < \infty. \tag{2.2.7}$$

In Chapter 4 we will use the following regularity condition, from which conditions (LR) and (I) follow:

(R) (*Regularity*) We say that ϕ is regular if ϕ is bounded from below and there exists an $R > 0$ and a positive decreasing function φ on $[0, +\infty)$ such that $|\phi(x)| \leq \varphi(|x|)$ for all $x \in \mathbb{R}^d$ with $|x| \geq R$ and

$$\int_R^\infty t^{d-1} \varphi(t) dt < \infty. \tag{2.2.8}$$

For the notion of tempered Gibbs measure and the following theorem, see [Rue70].

Theorem 2.2.1. Assume that $X = \mathbb{R}^d$ and ϕ is translation invariant.

1) Let (S) and (I) hold and let $z > 0$ be such that

$$z < \frac{1}{e}(e^{2B}C)^{-1},$$

where B and C are as in (S) and (I), respectively. Then there exists a Gibbs measure $\mu \in \mathcal{G}(z, E)$ whose correlation functions exist and satisfy the Ruelle bound.

2) Let ϕ be a non-negative potential which fulfills (I). Then, for each $z > 0$, there exists a Gibbs measure $\mu \in \mathcal{G}(z, E)$ whose correlation functions exist and satisfy the Ruelle bound.

3) Let ϕ satisfy (I), (SS) and (LR). Then the set $\mathcal{G}_{\mathrm{temp}}(z, E)$ of all tempered Gibbs measures is non-empty and each measure from $\mathcal{G}_{\mathrm{temp}}(z, E)$ has correlation functions which satisfy the Ruelle bound.

2.3 Marked configurations

2.3.1 Marked configuration spaces

Let $(X, \mathcal{B}(X))$ be given as in Section 2.1. Additionally, let S be a complete separable metric space, the corresponding Borel σ-algebra we denote by $\mathcal{B}(S)$. The elements of this space we call *marks*.

The *space of marked n-point configurations* $\hat{\Gamma}_0^{(n)}(X \times S)$ for $n \in \mathbb{N}$, is defined by

$$\hat{\Gamma}_0^{(n)}(X \times S) := \left\{ \hat{\eta} = \{(x_1, s_1), \ldots, (x_n, s_n)\} \in \Gamma_0^{(n)}(X \times S) \middle| \; x_i \neq x_j \text{ if } i \neq j \right\} \quad (2.3.1)$$

and $\hat{\Gamma}_0^{(0)}(X \times S) := \{\emptyset\}$. We will denote $\hat{\Gamma}_0^{(n)}(X \times S)$ for short $\hat{\Gamma}_0^{(n)}$, if no confusion of underlying spaces is possible. For every $\hat{\eta} = \{(x_1, s_1), \ldots, (x_n, s_n)\} \in \hat{\Gamma}_0^{(n)}$ we may assign a configuration $\eta = \{x_1, \ldots, x_n\} \in \Gamma_0^{(n)}$. We define analogously the space $\hat{\Gamma}_0^{(n)}(Y \times S)$, $Y \in \mathcal{B}(X)$. Also we use the shorthand $\hat{\eta}_Y$ (resp. η_Y) for $\hat{\eta} \cap (Y \times S)$, $Y \in \mathcal{B}(X)$ (resp. $\eta \cap Y$) and $\hat{x} := (x, s) \in X \times S$.

In order to define a measurable structure on the marked configuration space we use the following family of sets $\mathfrak{I}$ (the "local" sets),

$$\mathfrak{I} := \{ B \in \mathcal{B}(X) \times \mathcal{B}(S) | \; \exists \Lambda \in \mathcal{B}_c(X) \text{ with } B \subset \Lambda \times S\} \quad (2.3.2)$$

and the mappings, defined for any $Y \in \mathcal{B}(X)$ and every $B \in \mathfrak{I}$ with $B \subset Y \times S$,

$$\begin{aligned} N_B : \hat{\Gamma}_0^{(n)}(Y \times S) &\to \mathbb{N}_0 \\ \hat{\eta} &\mapsto |\hat{\eta} \cap B|. \end{aligned} \quad (2.3.3)$$

To define more structure on $\hat{\Gamma}_0^{(n)}$, we may use the following natural mapping

$$\begin{aligned} \mathrm{sym}^n : \widetilde{(X \times S)}^n &\rightarrow \hat{\Gamma}_0^{(n)}, \quad n \in \mathbb{N}, \\ \mathrm{sym}^n(\hat{x}_1, \ldots, \hat{x}_n) &:= \{\hat{x}_1, \ldots, \hat{x}_n\}, \end{aligned} \tag{2.3.4}$$

where

$$\widetilde{(X \times S)}^n := \{ (\hat{x}_1, \ldots, \hat{x}_n) \in (X \times S)^n | \; x_k \neq x_j \text{ if } k \neq j \} . \tag{2.3.5}$$

Using this mapping we can identify the space of n-point marked configuration with the symmetrization of $\widetilde{(X \times S)}^n$, i.e., $\widetilde{(X \times S)}^n / S_n$, where S_n is the permutation group over $\{1, \ldots, n\}$. This gives us a measurable structure on $\hat{\Gamma}_0^{(n)}$. We denote the σ-algebra on $\hat{\Gamma}_0^{(n)}$ by $\mathcal{B}(\hat{\Gamma}_0^{(n)})$, and it coincides with the σ-algebra generated by the mappings N_B, i.e.,

$$\mathcal{B}(\hat{\Gamma}_0^{(n)}) = \sigma(N_B | \, B \in \mathfrak{I}). \tag{2.3.6}$$

Finally, we define the *space of finite marked configurations* $\hat{\Gamma}_0(X \times S)$

$$\hat{\Gamma}_0(X \times S) := \bigsqcup_{n \in \mathbb{N}_0} \hat{\Gamma}_0^{(n)}(X \times S) \tag{2.3.7}$$

equipped with the σ-algebra $\mathcal{B}(\hat{\Gamma}_0)$ of disjoint union.

The *space of marked configurations* $\hat{\Gamma}(X \times S) = \hat{\Gamma}$ is defined as

$$\hat{\Gamma}(X \times S) := \{ \hat{\gamma} := \{(x, s_x) | x \in \gamma\} \in \Gamma(X \times S) : \; \gamma \in \Gamma(X), s_x \in S \text{ for all } x \in \gamma \} .$$

Its measurable structure is given by

$$\mathcal{B}(\hat{\Gamma}) := \sigma(N_B | \, B \in \mathfrak{I}). \tag{2.3.8}$$

2.3.2 Marked Poisson and Lebesgue-Poisson measures

For the construction of the marked Lebesgue-Poisson measure on $\hat{\Gamma}_0$ we need, first of all, to fix an intensity measure σ on the underlying space X. Thus, let us assume that σ is a non-atomic Radon measure on X, i.e., $\sigma(\{x\}) = 0$ for all $x \in X$ and $\sigma(\Lambda) < \infty$ for all $\Lambda \in \mathcal{B}_c(X)$. Additionally, we need a kernel $\tau : X \times \mathcal{B}(S) \rightarrow \mathbb{R}^+$, i.e., $\forall x \in X \; \tau(x, \cdot)$ is a finite measure on $(S, \mathcal{B}(S))$ and $\tau(\cdot, A)$ is $\mathcal{B}(X)$-measurable for each $A \in \mathcal{B}(S)$. Moreover, we assume that the following condition is fulfilled for any $\Lambda \in \mathcal{B}_c(X)$

$$\int_\Lambda \tau(x, S) \sigma(dx) < \infty. \tag{2.3.9}$$

This condition reflects the different roles of mark and position variables.

In the product space $X \times S$ we define a σ-finite measure σ^τ by

$$\sigma^\tau(dx, ds) := \tau(x, ds)\sigma(dx), \tag{2.3.10}$$

that means for $A \times B \in \mathcal{B}(X \times S)$

$$\sigma^\tau(A \times B) = \int_A \tau(x, B)\sigma(dx), \tag{2.3.11}$$

which is a non-atomic Radon measure.

For any $Y \in \mathcal{B}(X)$ and $n \in \mathbb{N}$, the product measure $(\sigma^\tau)^{\otimes n}$ can be considered as a measure on $(\widetilde{Y \times S})^n$. Let

$$(\sigma^\tau)^{(n)} := (\sigma^\tau)^{\otimes n} \circ (\mathrm{sym}^n)^{-1} \tag{2.3.12}$$

be the corresponding measure on $\hat{\Gamma}_0^{(n)}$, define $(\sigma^\tau)^{(0)}(\{\emptyset\}) := 1$. Then we consider the so-called *marked Lebesgue-Poisson measure* $\lambda_{z\sigma^\tau}$ on $\mathcal{B}(\hat{\Gamma}_0)$, which coincides on each $\hat{\Gamma}_0^{(n)}$ with the measure $\frac{z^n}{n!}(\sigma^\tau)^{(n)}$, as follows

$$\lambda_{z\sigma^\tau} := \sum_{n=0}^{\infty} \frac{z^n}{n!}(\sigma^\tau)^{(n)}, \tag{2.3.13}$$

where $z > 0$ is the activity parameter. As a consequence $\lambda_{z\sigma^\tau}$ is σ-finite. For $\Lambda \in \mathcal{B}_c(X)$ we obtain $\lambda_{z\sigma^\tau}(\hat{\Gamma}_0(\Lambda \times S)) = e^{z\sigma^\tau(\Lambda \times S)}$. Therefore, we can define a probability measure $\pi_{z\sigma}^{\tau,\Lambda}$ on $\hat{\Gamma}_0(\Lambda \times S)$ by

$$\pi_{z\sigma}^{\tau,\Lambda} := e^{-z\sigma^\tau(\Lambda \times S)}\lambda_{z\sigma^\tau}.$$

In order to obtain the existence of unique probability measure $\pi_{z\sigma}^\tau$ on $(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ such that $\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1}$, $\Lambda \in \mathcal{B}_c(X)$ we notice that the family $\{\pi_{z\sigma}^{\tau,\Lambda}, \Lambda \in \mathcal{B}_c(X)\}$ is consistent. Thus, by a version of Kolmogorov's theorem for projective limit spaces such family determines uniquely a measure $\pi_{z\sigma}^\tau$ on $\mathcal{B}(\hat{\Gamma})$ such that $\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1}$. The measure $\pi_{z\sigma}^\tau$ is called *marked Poisson measure*.

3 Glauber and Kawasaki dynamics for determinantal point processes

3.1 Determinantal point processes

In this chapter we assume X to be a locally compact, second countable Hausdorff topological space. We fix a metric on X which generates the topology on X. For any $x \in X$ and $r > 0$, we denote by $\bar{B}(x,r)$ the closed ball in X with center at x and radius r, and by $B(x,r)$ the corresponding open ball. We fix a Radon, non-atomic measure ν on $(X, \mathcal{B}(X))$.

Let K be a linear Hermitian operator on the space $L^2(X,\nu)$ (real or complex) which satisfies the following assumptions:
(1) K is locally of trace class, i.e.,

$$\operatorname{Tr}(P_\Lambda K P_\Lambda) < \infty \quad \text{for all } \Lambda \in \mathcal{B}_c(X),$$

where P_Λ denotes the operator of multiplication by the indicator function $1\!\!1_\Lambda$ of the set Λ.
(2) We have $\mathbf{0} \le K \le \mathbf{1}$.

Under the above assumptions K is an integral operator, and its kernel can be chosen as

$$K(x,y) = \int_X K_1(x,z) K_1(z,y)\, \nu(dz),$$

where $K_1(\cdot,\cdot)$ is any version of the kernel of the integral operator $\sqrt{K}$, [LM07] (see also [GY05, Lemma A.4]).

A point process μ having correlation functions

$$k_\mu^{(n)}(x_1,\dots,x_n) = \det(K(x_i,x_j))_{i,j=1}^n$$

is called the fermion (or determinantal) point process corresponding to the operator K. Under the above assumptions on K, such a point process μ exists and is unique, see e.g. [Mac75, Sos00, ST03, LM07].

Using the definition of a fermion process, we see that μ has all local moments finite, i.e.,

$$\int_\Gamma \langle f,\gamma\rangle^n\, \mu(d\gamma) < \infty, \qquad f \in C_0(X),\ f \ge 0,\ n \in \mathbb{N}. \tag{3.1.1}$$

In what follows, we will always assume that the operator K is strictly less than $\mathbf{1}$, i.e., $\mathbf{1}$ does not belong to the spectrum of K. Then, as has been shown by Georgii and Yoo in [GY05], the fermion process μ has Papangelou (conditional) intensity $r(x,\gamma)$, defined by (2.2.1).

We explain the construction of the explicit formula for the Papangelou intensity $r(x,\gamma)$ in short, following [GY05]. For each $\Lambda \in \mathcal{B}_c(X)$, let $P_\Lambda : L^2(X,\nu) \to L^2(X,\nu)$ be the projection operator and $K_\Lambda := P_\Lambda K P_\Lambda$ the restriction of the operator K on $L^2(\Lambda,\nu)$. Define also $J_{[\Lambda]} := K_\Lambda(\mathbf{1}-K_\Lambda)^{-1}$. Denote by $J_{[\Lambda]}(\cdot,\cdot)$ the kernel of the operator $J_{[\Lambda]}$ (chosen analogously to the kernel of K). For any $\gamma \in \Gamma$, $\Lambda \in \mathcal{B}_c(X)$ set

$$\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda) := \det \left[J_{[\Lambda]}(x_i,x_j)\right]_{i,j=1}^m ,$$

with $\gamma_\Lambda = \{x_1,\dots,x_m\}$ being any numeration of points of γ_Λ (in the case $\gamma_\Lambda = \varnothing$, set $\det J_{[\Lambda]}(\varnothing,\varnothing) := 0$). Now, for any $x \in \Lambda$ and $\gamma \in \Gamma$, set

$$r_\Lambda(x,\gamma_\Lambda) := \frac{\det J_{[\Lambda]}(x \cup \gamma_\Lambda, x \cup \gamma_\Lambda)}{\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda)} , \tag{3.1.2}$$

where the expression on the right hand side is defined to be zero if $\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda) = 0$. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathcal{B}_c(X)$ that increases to X. Then $r(x,\gamma)$ is given by

$$r(x,\gamma) = \lim_{n\to\infty} r_{\Lambda_n}(x,\gamma_{\Lambda_n}) \text{ for } \nu\otimes\mu\text{-a.a. } (x,\gamma) \in X \times \Gamma. \tag{3.1.3}$$

Set $J := K(\mathbf{1}-K)^{-1}$. J is an integral operator and we choose its kernel $J(\cdot,\cdot)$ analogously to choosing the kernel of K. Note that

$$\mathrm{Tr}(P_\Lambda J P_\Lambda) = \int_\Lambda J(x,x)\,\nu(dx) < \infty \text{ for } \Lambda \in \mathcal{B}_c(X). \tag{3.1.4}$$

The following proposition is a direct corollary of Theorem 3.6 and Lemma A.1 in [GY05].

Proposition 3.1.1. We have, for $\nu\otimes\mu$-a.e. $(x,\gamma) \in X \times \Gamma$:

$$r(x,\gamma) \le J(x,x). \tag{3.1.5}$$

In what follows, we will consider a determinantal point process μ corresponding to an operator K as defined above. We introduce the set $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle),$$

where $N \in \mathbb{N}$, $\varphi_1,\dots,\varphi_N \in C_0(X)$ and $g \in C_{\mathrm{b}}(\mathbb{R}^N)$. Here, $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the set of all continuous, bounded functions on $\mathbb{R}^N$. Note that these functions have the following property: there exists a set $\Lambda \in \mathcal{B}_c(X)$ such that

$$F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda).$$

3.2 Dirichlet forms

In this section we will only cite the most important definitions. For a complete reference see [MR92].

Let $(E, \mathcal{B}, m)$ be a measure-space. Let $D := D(\mathcal{E})$ be a linear subspace of $L^2(E, m)$ and $\mathcal{E} : D \times D \to \mathbb{R}$ a bilinear map. We define its symmetric part and antisymmetric part $(\tilde{\mathcal{E}}, D)$, $(\check{\mathcal{E}}, D)$, respectively, by

$$\tilde{\mathcal{E}}(u, v) := \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u)); \quad \check{\mathcal{E}}(u, v) := \frac{1}{2}(\mathcal{E}(u, v) - \mathcal{E}(v, u)),$$

$u, v \in D$. Clearly $\mathcal{E}(u, v) = \tilde{\mathcal{E}}(u, v) + \check{\mathcal{E}}(u, v)$. For $\alpha \geq 0$ we set

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v); \quad u, v \in D.$$

Assume $(\mathcal{E}, D)$ is positive definite. Then $(\mathcal{E}, D)$ is said to satisfy the weak sector condition if there exists a constant $K > 0$ such that

$$|\mathcal{E}_1(u, v)| \leq K \mathcal{E}_1(u, u)^{1/2} \mathcal{E}_1(v, v)^{1/2} \text{ for all } u, v \in D.$$

Definition 3.2.1. Let $(\mathcal{E}, D)$ be a positive definite bilinear form on $L^2(E, m)$. $(\mathcal{E}, D)$ is called closable (on $L^2(E, m)$), if for all $u_n \in D$, $n \in \mathbb{N}$, such that $(u_n)_{n \in \mathbb{N}}$ is $\mathcal{E}$-Cauchy (i.e. $\mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow[n,m\to\infty]{} 0$) and $u_n \xrightarrow[n\to\infty]{} 0$ in $L^2(E, m)$, it follows that $\mathcal{E}(u_n, u_n) \xrightarrow[n\to\infty]{} 0$.

Definition 3.2.2. A pair $(\mathcal{E}, D)$ is called a symmetric closed form on $L^2(E, m)$, if D is a dense linear subspace of $L^2(E, m)$ and $\mathcal{E} : D \times D \to \mathbb{R}$ is a positive definite bilinear form which is symmetric (i.e. $\mathcal{E} = \tilde{\mathcal{E}}$) and closed on $L^2(E, m)$ (i.e. D is complete w.r.t. the norm $\mathcal{E}_1^{1/2}$).

Definition 3.2.3. A pair $(\mathcal{E}, D)$ is called a coercive closed form on $L^2(E, m)$ if D is a dense linear subspace of $L^2(E, m)$ and $\mathcal{E} : D \times D \to \mathbb{R}$ is a bilinear form such that the following two conditions hold:

(i) Its symmetric part $(\tilde{\mathcal{E}}, D)$ is a symmetric closed form on $L^2(E, m)$.

(ii) $(\mathcal{E}, D)$ satisfies the weak sector condition.

Definition 3.2.4. A coercive closed form $(\mathcal{E}, D)$ on $L^2(E, m)$ is called a *Dirichlet form* if for all $u \in D$ the following holds:

$$u^+ \wedge 1 \in D \qquad \text{and} \qquad \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$$
$$\text{and} \qquad \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0.$$

Remark 3.2.5. It is shown in [MR92, Theorem I.2.8.], that a coercive closed form $\mathcal{E}$ uniquely determines a pair of strongly continuous contraction resolvents $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ on $L^2(E,m)$, such that

$$\mathcal{E}(G_\alpha f, u) + (G_\alpha f, u) = (f, u) = \mathcal{E}(u, \hat{G}_\alpha f) + (u, \hat{G}_\alpha f).$$

Corresponding to these resolvents are two strongly continuous contraction semigroups $(T_t)_{t\geq 0}$ and $(\hat{T}_t)_{t\geq 0}$.

In order to define quasi-regularity, we have to introduce some potential theoretic notions.

Define for $F \subset E$, F closed

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) | u = 0 \ m\text{-a.e. on } E \setminus F\}.$$

Definition 3.2.6. (i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$*-nest*, if $\bigcup_{k\geq 1} D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$.

(ii) A subset $N \subset E$ is called $\mathcal{E}$*-exceptional*, if $N \subset \bigcap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$.

(iii) A property is said to hold $\mathcal{E}$*-quasi-everywhere* ($\mathcal{E}$-q.e.), if there exists an $\mathcal{E}$-exceptional set N, such that the property holds on $E \setminus N$.

(iv) An $\mathcal{E}$-q.e. defined function f on E is called $\mathcal{E}$*-quasi-continuous*, if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $f_{|F_k}$ is continuous for every $k \in \mathbb{N}$.

Definition 3.2.7. A Dirichlet form $(\mathcal{E}, D)$ on $L^2(E,m)$ is called *quasi-regular*, if the following conditions hold:

(i) There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(ii) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of D whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exists $u_n \in D$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$, such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

3.3 Glauber dynamics

3.3.1 Existence results

Fix a determinantal point process μ, corresponding to the operator K.

For a function $F : \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$, $x \in \gamma$, $y \in X \setminus \gamma$, we introduce the following notations

$$(D_x^- F)(\gamma) := F(\gamma \setminus x) - F(\gamma), \quad (D_y^+ F)(\gamma) = F(\gamma \cup y) - F(\gamma).$$

Consider a measurable mapping

$$X \times \Gamma \ni (x, \gamma) \mapsto d(x, \gamma) \in [0, \infty),$$

for which we assume that, for each $\Lambda \in \mathcal{B}_c(X)$

$$\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) d(x, \gamma \setminus x) < \infty. \tag{3.3.1}$$

The coefficient $d(x, \gamma \setminus x)$ describes the death rate, i.e. the rate at which the particle x of the configuration γ dies. We define the bilinear form

$$\mathcal{E}_{\mathrm{G}}(F, G) := \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) d(x, \gamma \setminus x)(D_x^- F)(\gamma)(D_x^- G)(\gamma), \tag{3.3.2}$$

where $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$. As we see later in Theorem 3.3.8, $\mathcal{E}_{\mathrm{G}}$ corresponds to the Glauber dynamics generator. We will write for short $\mathcal{E}_{\mathrm{G}}(F)$ instead of $\mathcal{E}_{\mathrm{G}}(F, F)$.

Note that for any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$, there exist $\Lambda \in \mathcal{B}_c(X)$ and $C > 0$ such that

$$|(D_x^- F)(\gamma)| \le C 1\!\!1_\Lambda(x). \tag{3.3.3}$$

In fact, for $F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle)$ denote supp $\varphi_k = \Lambda_k \in \mathcal{B}_c(X)$ and $\Lambda = \cup_{k=1}^n \Lambda_k$, and assume $|g(y)| \le C/2$ for all $y \in \mathbb{R}^N$. Then $(D_x^- F)(\gamma) = 0$ for $x \in X \setminus \Lambda$, otherwise $|(D_x^- F)(\gamma)| \le C$.

Therefore, by assumption (3.3.1) the right-hand side of the formula (3.3.2) is well-defined and finite.

Lemma 3.3.1. We have $\mathcal{E}_{\mathrm{G}}(F, G) = 0$ for all $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $F = 0$ μ-a.e.

Proof. It suffices to show that, for $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $F = 0$ μ-a.e., we have $(D_x^- F)(\gamma) = 0$ $\tilde{\mu}$-a.e. Here, $\tilde{\mu}$ is the measure on $X \times \Gamma$ defined by

$$\tilde{\mu}(dx, d\gamma) := \gamma(dx)\mu(d\gamma). \tag{3.3.4}$$

For any F as above, we evidently have that $F(\gamma) = 0$ $\tilde{\mu}$-a.e. Therefore, it is enough to show that $F(\gamma \setminus x) = 0$ $\tilde{\mu}$-a.e. By (2.2.1) we get for $\Lambda \in \mathcal{B}_c(X)$

$$\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) |F(\gamma \setminus x)| = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) r(x, \gamma) |F(\gamma)|. \tag{3.3.5}$$

Since F is bounded and by (3.1.1), the integral on the left-hand side of (3.3.5) is finite. Therefore,

$$r(x,\gamma)|F(\gamma)| < \infty \quad \text{for } \mu\otimes\nu\text{-a.a. } (x,\gamma)\in X\times\Gamma. \tag{3.3.6}$$

Because $F=0$ $\mu\otimes\nu$-a.e., from (3.3.5) and (3.3.6), we get $F(\gamma\setminus x)=0$ $\tilde{\mu}$-a.e. □

Lemma 3.3.2. The bilinear form $(\mathcal{E}_{\mathrm{G}}, \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ is closable on $L^2(\Gamma,\mu)$, and its closure will be denoted by $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$.

Proof. Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $\|F_n\|_{L^2(\Gamma,\mu)}\to 0$ as $n\to\infty$ and

$$\mathcal{E}_{\mathrm{G}}(F_n-F_k)\to 0 \quad \text{as } n,k\to\infty. \tag{3.3.7}$$

To prove the closability of $\mathcal{E}_{\mathrm{G}}$, it suffices to show that there exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$, such that $\mathcal{E}_{\mathrm{G}}(F_{n_k})\to 0$ as $k\to\infty$.

Let $\Lambda\in\mathcal{B}_c(X)$. By Cauchy inequality and (3.1.1), we have

$$\int_\Gamma \mu(d\gamma)\int_\Lambda \gamma(dx)|F_n(\gamma)| \le ||F_n||_{L^2(\mu)}\left(\int_\Gamma \langle 1\!\!1_\Lambda,\gamma\rangle^2\mu(d\gamma)\right)^{1/2} \to 0 \text{ as } n\to\infty.$$

Therefore, there exists a subsequence of $(F_n)_{n=1}^\infty$, denoted by $(F_n^{(1)})_{n=1}^\infty$, such that $F_n^{(1)}(\gamma)\to 0$ for $\gamma(dx)\mu(d\gamma)$-a.e. $(x,\gamma)\in\Lambda\times\Gamma$. Hence, there exists a subsequence $(F_n^{(2)})_{n=1}^\infty$ of $(F_n^{(1)})_{n=1}^\infty$ such that $F_n^{(2)}(\gamma)\to 0$ for $\gamma(dx)\mu(d\gamma)$-a.a. $(x,\gamma)\in X\times\Gamma$.

Next, by (2.2.1), (3.1.4) and (3.1.5)

$$\begin{aligned}\int_\Gamma \mu(d\gamma)\int_\Lambda \gamma(dx)|F_n^{(2)}(\gamma\setminus x)| &= \int_\Gamma \mu(d\gamma)\int_\Lambda \nu(dx)r(x,\gamma)|F_n^{(2)}(\gamma)|\\ &\le \int_\Gamma \mu(d\gamma)|F_n^{(2)}(\gamma)|\int_\Lambda J(x,x)\nu(dx)\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, there exists a subsequence $(F_n^{(3)})_{n=1}^\infty$ of $(F_n^{(2)})_{n=1}^\infty$ such that

$$(D_x^- F_n^{(3)})(\gamma)\to 0 \quad \text{for } \tilde{\mu}\text{-a.e. } (x,\gamma)\in X\times\Gamma. \tag{3.3.8}$$

Now, by (3.3.8) and Fatou's lemma,

$$\begin{aligned}\mathcal{E}_{\mathrm{G}}(F_n^{(3)}) &= \int_\Gamma \mu(d\gamma)\int_X \gamma(dx)d(x,\gamma\setminus x)(D_x^-F_n^{(3)})(\gamma)^2\\ &= \int_\Gamma \mu(d\gamma)\int_X \gamma(dx)d(x,\gamma\setminus x)\left((D_x^-F_n^{(3)})(\gamma)-\lim_{m\to\infty}(D_x^-F_m^{(3)})(\gamma)\right)^2\\ &\le \liminf_{m\to\infty}\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)d(x,\gamma\setminus x)\left((D_x^-F_n^{(3)})(\gamma)-(D_x^-F_m^{(3)})(\gamma)\right)^2\\ &= \liminf_{m\to\infty}\mathcal{E}_{\mathrm{G}}(F_n^{(3)}-F_m^{(3)}),\end{aligned}$$

which by (3.3.7) can be made arbitrarily small for n large enough. □

Lemma 3.3.3. $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$ is a Dirichlet form on $L^2(\Gamma, \mu)$.

Proof. On $D(\mathcal{E}_{\mathrm{G}})$ we consider the norm $\|F\|_{D(\mathcal{E}_{\mathrm{G}})} := (\|F\|^2_{L^2(\mu)} + \mathcal{E}_{\mathrm{G}}(F))^{1/2}$, $F \in D(\mathcal{E}_{\mathrm{G}})$. For any $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$, we define

$$S(F,G)(x,\gamma) := d(x, \gamma \setminus x)(D_x^- F)(\gamma)(D_x^- G)(\gamma), \qquad x \in \gamma,\ \gamma \in \Gamma.$$

Using the Cauchy inequality, we conclude that S extends to a bilinear continuous map from $(D(\mathcal{E}_{\mathrm{G}}), \|\cdot\|_{D(\mathcal{E}_{\mathrm{G}})}) \times (D(\mathcal{E}_{\mathrm{G}}), \|\cdot\|_{D(\mathcal{E}_{\mathrm{G}})})$ into $L^1(X \times \Gamma, \tilde{\mu})$. Let $F \in D(\mathcal{E}_{\mathrm{G}})$ and consider any sequence $(F_n)_{n=1}^\infty$ in $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $F_n \to F$ in $(D(\mathcal{E}_{\mathrm{G}}), \|\cdot\|_{D(\mathcal{E}_{\mathrm{G}})})$. In particular, $F_n \to F$ in $L^2(\mu)$. Then, step by step analogously to the proof of Lemma 3.3.2, we find some subsequence $(F_{n_k})_{k=1}^\infty$, such that

$$(D_x^- F_{n_k})(\gamma) \to (D_x^- F)(\gamma) \quad \text{for } \tilde{\mu}\text{-a.e. } (x,\gamma) \in X \times \Gamma.$$

Therefore, for any $F, G \in D(\mathcal{E}_{\mathrm{G}})$,

$$S(F,G)(x,\gamma) = d(x,\gamma\setminus x)(D_x^- F)(\gamma)(D_x^- G)(\gamma) \quad \text{for } \tilde{\mu}\text{-a.e. } (x,\gamma) \in X \times \Gamma \quad (3.3.9)$$

and

$$\mathcal{E}_{\mathrm{G}}(F,G) = \int_{X \times \Gamma} S(F,G)(x,\gamma)\, \tilde{\mu}(dx, d\gamma). \qquad (3.3.10)$$

Define $\mathbb{R} \ni x \mapsto g(x) := (0 \vee x) \wedge 1$. We again fix any $F \in D(\mathcal{E}_{\mathrm{G}})$ and let $(F_n)_{n=1}^\infty$ be a sequence of functions from $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $F_n \to F$ in $(D(\mathcal{E}_{\mathrm{G}}), \|\cdot\|_{D(\mathcal{E}_{\mathrm{G}})})$. Consider the sequence $(g(F_n))_{n \in \mathbb{N}}$. We evidently have: $g(F_n) \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ for each $n \in \mathbb{N}$ and, by the dominated convergence theorem, $g(F_n) \to g(F)$ as $n \to \infty$ in $L^2(\mu)$. Next, by the above argument, we have, for some subsequence $(F_{n_k})_{k=1}^\infty$, $(D_x^- g(F_{n_k}))(\gamma) \to (D_x^- g(F))(\gamma)$ as $n \to \infty$ for $\tilde{\mu}$-a.e. (x, γ).

For any $x, y \in \mathbb{R}$, we evidently have

$$|g(x) - g(y)| \le |x - y|. \qquad (3.3.11)$$

Therefore, the sequence $d(x, \gamma \setminus x)^{1/2}(D_x^- g(F_n))(\gamma)$, $n \in \mathbb{N}$, is $\tilde{\mu}$-uniformly square-integrable, since so is the sequence $d(x, \gamma \setminus x)^{1/2}(D_x^- F_n)(\gamma)$, $n \in \mathbb{N}$. Hence

$$d(x, \gamma \setminus x)^{1/2}(D_x^- g(F_{n_k}))(\gamma) \to d(x, \gamma \setminus x)^{1/2}(D_x^- F)(\gamma) \quad \text{as } k \to \infty \text{ in } L^2(\tilde{\mu}).$$

By (3.3.9) and (3.3.10), this yields: $g(F) \in D(\mathcal{E}_{\mathrm{G}})$.

Finally, by (3.3.9)–(3.3.11), $\mathcal{E}_{\mathrm{G}}(g(F)) \le \mathcal{E}_{\mathrm{G}}(F)$, which means that $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$ is a Dirichlet form. $\square$

We now need the bigger space $\ddot{\Gamma}$ consisting of all $\mathbb{Z}_+ \cup \{\infty\}$-valued Radon measures on X (which is Polish, see e.g. [Kal86]). Since $\Gamma \subset \ddot{\Gamma}$ and $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a bilinear form on $L^2(\ddot{\Gamma}, \mu)$.

Lemma 3.3.4. $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$ is a quasi-regular Dirichlet form on $L^2(\ddot{\Gamma}, \mu)$.

Proof. By [MR00, Proposition 4.1], it suffices to show that there exists a bounded, complete metric ρ on $\ddot{\Gamma}$ generating the vague topology such that, for all $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{G}})$ and

$$\int_X S(\rho(\cdot, \gamma_0))(x, \gamma)\gamma(dx) \le \eta(\gamma) \quad \mu\text{-a.e.}$$

for some $\eta \in L^1(\ddot{\Gamma}, \mu)$ (independent of γ_0). Here, $S(F) := S(F, F)$. The proof below is a modification of the proof of [MR00, Proposition 4.8] and the proof of [KL05, Proposition 3.2].

Fix any $x_0 \in X$, denote for short $B(r) := B(x_0, r)$. For each $k \in \mathbb{N}$, we define

$$g_k(x) := \frac{2}{3}\left(\frac{1}{2} - \operatorname{dist}(x, B(k)) \wedge \frac{1}{2}\right), \qquad x \in X,$$

where $\operatorname{dist}(x, B(k))$ denotes the distance from the point x to the ball $B(k)$. Next, we set

$$\phi_k(x) := 3g_k(x), \qquad x \in X,\ k \in \mathbb{N}.$$

Let ζ be a function in $C_{\mathrm{b}}^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $[0, \infty)$, $\zeta(t) = t$ on $[-1/2, 1/2]$, $\zeta' \in [0, 1]$ on $[0, \infty)$. For any fixed $\gamma_0 \in \ddot{\Gamma}$ and for any $k \in \mathbb{N}$, the restriction to Γ of the function

$$\zeta\left(|\langle \phi_k g_k, \cdot\rangle - \langle \phi_k g_k, \gamma_0\rangle|\right)$$

belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ (note that $\langle \phi_k g_k, \gamma_0\rangle$ is a constant). Furthermore, taking into account that $\zeta' \in [0, 1]$ on $[0, \infty)$, we get from the mean value theorem, for each $\gamma \in \Gamma$, $x \in \gamma$

$$\begin{aligned} S\left(\zeta\left(|\langle \phi_k g_k, \cdot\rangle - \langle \phi_k g_k, \gamma_0\rangle|\right)\right)(x, \gamma) &\le d(x, \gamma \setminus x)(\phi_k g_k)^2(x) \\ &\le d(x, \gamma \setminus x)\mathbb{1}_{B(k+1/2)}(x). \end{aligned} \tag{3.3.12}$$

Set

$$c_k := \left(1 + 2\int d(x, \gamma \setminus x)\mathbb{1}_{B(k+1/2)}(x)\tilde{\mu}(dx, d\gamma)\right)^{-1/2} 2^{-k/2}, \qquad k \in \mathbb{N},$$

which are finite positive numbers by (3.3.1), and furthermore, $c_k \to 0$ as $k \to \infty$.

We define

$$\rho(\gamma_1, \gamma_2) := \sup_{k\in\mathbb{N}} \left(c_k \zeta \left(\sup_{j\in\mathbb{N}} |\langle \phi_k g_j, \gamma_1\rangle - \langle \phi_k g_j, \gamma_2\rangle| \right)\right), \qquad \gamma_1, \gamma_2 \in \ddot{\Gamma}.$$

By [MR00, Theorem 3.6], ρ is a bounded, complete metric on $\ddot{\Gamma}$ generating the vague topology.

Analogously to the above, we now conclude that, for any fixed $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{G}})$ and

$$\int_X S(\rho(\cdot, \gamma_0))(x, \gamma)\gamma(dx) \le \eta(\gamma) \quad \mu\text{-a.e.}$$

where

$$\eta(\gamma) := \sup_{k\in\mathbb{N}} \left(c_k^2 \int_X d(x, \gamma \setminus x) 1\!\!1_{B(k+1/2)}(x)\gamma(dx)\right).$$

Finally,

$$\int_\Gamma \eta(\gamma)\, \mu(d\gamma) \le \sum_{k=1}^\infty c_k^2 \int d(x, \gamma\setminus x) 1\!\!1_{B(k+1/2)}(x)\tilde{\mu}(dx, d\gamma) \le \sum_{k=1}^\infty 2^{-k} = 1.$$

Thus, the lemma is proved. □

Lemma 3.3.5. The set $\ddot{\Gamma} \setminus \Gamma$ is exceptional for $\mathcal{E}_{\mathrm{G}}$.

Proof. We modify the proof of [RS98, Proposition 1 and Corollary 1] according to our situation. It suffices to prove the lemma locally, i.e., to show that, for any fixed $a \in X$ and $r > 0$

$$N_a := \{\gamma \in \ddot{\Gamma} \mid \sup_{x\in \bar{B}(a,r)} \gamma(\{x\}) \ge 2\}$$

is $\mathcal{E}_{\mathrm{G}}$-exceptional.

By [RS98, Lemma 1], we need to prove that there exists a sequence $u_n \in D(\mathcal{E}_{\mathrm{G}})$, $n \in \mathbb{N}$, such that each u_n is a continuous function on $\ddot{\Gamma}$, $u_n \to 1\!\!1_{N_a}$ pointwise as $n \to \infty$, and $\sup_{n\in\mathbb{N}} \mathcal{E}_{\mathrm{G}}(u_n) < \infty$.

Fix some $n \in \mathbb{N}$ such that

$$2/n < r. \tag{3.3.13}$$

Let

$$\{\bar{B}(a_k, 1/n) \mid k = 1, \dots, K_n\},$$

with $a_k \in \bar{B}(a, r)$, $k = 1, \dots, K_n$, be a finite covering of $\bar{B}(a, r)$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(u) := (1 - |u|) \vee 0$.

For each $k = 1, \dots, K_n$, we define a continuous function $f_k^{(n)} : X \to \mathbb{R}$ by

$$f_k^{(n)}(x) := f\big(n \operatorname{dist}(x, \bar{B}(a_k, 1/n))\big), \quad x \in X.$$

We evidently have:

$$\mathbb{1}_{\bar{B}(a_k,1/n)} \le f_k^{(n)} \le \mathbb{1}_{\bar{B}(a_k,2/n)}. \tag{3.3.14}$$

Let $\psi \in C_{\mathrm{b}}^1(\mathbb{R})$ be such that $\mathbb{1}_{[2,\infty)} \le \psi \le \mathbb{1}_{[1,\infty)}$ and

$$0 \le \psi' \le 2. \tag{3.3.15}$$

We define a continuous function

$$\ddot{\Gamma} \ni \gamma \mapsto u_n(\gamma) := \psi\left(\sup_{k\in\{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma\rangle\right),$$

whose restriction to Γ belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$. Evidently, $u_n \to \mathbb{1}_{N_a}$ pointwise as $n \to \infty$.

By (3.3.13)-(3.3.15) and the mean value theorem we have for each $\gamma \in \Gamma$, $x \in \gamma$,

$$\begin{aligned}(D_x^- u_n)^2(\gamma) &\le 2 \sup_{k\in\{1,\dots,K_n\}} |\langle f_k^{(n)}, \gamma\setminus x\rangle - \langle f_k^{(n)}, \gamma\rangle|^2 = 2 \sup_{k\in\{1,\dots,K_n\}} f_k^{(n)}(x)^2 \\ &\le 2 \sup_{k\in\{1,\dots,K_n\}} \mathbb{1}_{\bar{B}(a_k,2/n)}(x) \le 2\mathbb{1}_{\bar{B}(a,2r)}(x).\end{aligned}$$

Hence, by (3.3.1),

$$\sup_n \mathcal{E}_{\mathrm{G}}(u_n) < \infty,$$

which implies the lemma. □

We now have the main result of this section.

Theorem 3.3.6. Let (3.3.1) hold. Then we have:

1. There exists a conservative Hunt process

$$\mathbf{M} = \big(\boldsymbol{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge 0}, (\boldsymbol{\Theta}_t)_{t\ge 0}, (\mathbf{X}(t))_{t\ge 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma}\big)$$

on Γ (see e.g. [MR92, p. 92]), which is properly associated with $(\mathcal{E}_G, D(\mathcal{E}_G))$, i.e., for all ($\mu$-versions of) $F \in L^2(\Gamma, \mu)$ and all $t > 0$, the function

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\boldsymbol{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma \tag{3.3.16}$$

is an $\mathcal{E}_G$-quasi-continuous version of $\exp(-tH_G)F$, where $(H_G, D(H_G))$ is the generator of $(\mathcal{E}_G, D(\mathcal{E}_G))$. $\mathbf{M}$ is up to μ-equivalence unique. In particular, $\mathbf{M}$ is μ-symmetric (i.e., $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ for all $\mathcal{B}(\Gamma)$-measurable $F, G : \Gamma \to \mathbb{R}_+$), so has μ as an invariant measure.

2. $\mathbf{M}$ from 1. is up to μ-equivalence (cf. [MR92, Definition IV.6.3]) unique among all Hunt processes $\mathbf{M}' = \big(\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}'(t))_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\Gamma}\big)$ on Γ having μ as invariant measure and solving the martingale problem for $(-H_G, D(H_G))$, i.e., for all $G \in D(H_G)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (H_G G)(\mathbf{X}'(s))\, ds, \qquad t \geq 0,$$

is an $(\mathbf{F}'_t)$-martingale under $\mathbf{P}'_\gamma$ for $\mathcal{E}_G$-q.e. $\gamma \in \Gamma$. (Here, $\widetilde{G}$ denotes an $\mathcal{E}_G$-quasi-continuous version of G, cf. [MR92, Proposition IV.3.3].)

Remark 3.3.7. In Theorem 3.3.6, $\mathbf{M}$ can be taken canonical, which means that $\mathbf{\Omega}$ is the set of all *cadlag* functions $\omega : [0,\infty) \to \Gamma$ (i.e., ω is right continuous on $[0,\infty)$ and has left limits on $(0,\infty)$), $\mathbf{X}(t)(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, $(\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [FŌT94, Section 4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

Proof of Theorem 3.3.6. The first part of the theorem follows from Lemmas 3.3.4, 3.3.5, the fact that $1 \in D(\mathcal{E}_G)$ and $\mathcal{E}_G(1,1) = 0$, and [MR92, Theorem IV.3.5 and Proposition V.2.15]. The second part follows directly from the proof of [AR95, Theorem 3.5]. $\square$

Now we will derive an explicit formula for the generator of $\mathcal{E}_{\mathrm{G}}$. However, for this we have to demand stronger conditions on the coefficient $d(x,\gamma)$.

Theorem 3.3.8. Assume that, for each $\Lambda \in \mathcal{B}_c(X)$,

$$\int_\Lambda \gamma(dx) d(x, \gamma \setminus x) \in L^2(\Gamma, \mu), \tag{3.3.17}$$

$$\int_\Lambda \nu(dx) b(x, \gamma) \in L^2(\Gamma, \mu), \tag{3.3.18}$$

where

$$b(x,\gamma) := r(x,\gamma) d(x,\gamma), \qquad x \in \gamma,\ \gamma \in \Gamma. \tag{3.3.19}$$

Then, for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$, we have μ-a.e.

$$(H_{\mathrm{G}} F)(\gamma) = -\int_X \nu(dx)\, b(x,\gamma) (D_x^+ F)(\gamma) - \int_X \gamma(dx)\, d(x, \gamma\setminus x)(D_x^- F)(\gamma) \tag{3.3.20}$$

and $(H_{\mathrm{G}}, D(H_{\mathrm{G}}))$ is the Friedrichs extension of $(H_{\mathrm{G}}, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma))$ in $L^2(\Gamma, \mu)$.

Proof. By (2.2.1) we get:

$$\begin{aligned}\int_\Gamma (H_{\mathrm{G}}F)(\gamma)F(\gamma)\mu(d\gamma) &= -\int_\Gamma \int_X b(x,\gamma)(D_x^+ F)(\gamma)F(\gamma)\nu(dx)\mu(d\gamma) \\ &\quad -\int_\Gamma \int_X d(x,\gamma\setminus x)(D_x^- F)(\gamma)F(\gamma)\gamma(dx)\mu(d\gamma) \\ &= \int_\Gamma \int_X d(x,\gamma\setminus x)(D_x^- F)(\gamma)F(\gamma\setminus x)\gamma(dx)\mu(d\gamma) \\ &\quad -\int_\Gamma \int_X d(x,\gamma\setminus x)(D_x^- F)(\gamma)F(\gamma)\gamma(dx)\mu(d\gamma) \\ &= \mathcal{E}_{\mathrm{G}}(F,F).\end{aligned}$$

The expressions above are well-defined by assumptions (3.3.18). □

The coefficient $d(x,\gamma\setminus x)$ describes the rate at which the particle x of the configuration γ dies, while $b(x,\gamma)$ describes the rate at which, given the configuration γ, a new particle is born in x.

3.3.2 Examples

In this section we give examples of birth and death rates $d(x,\gamma)$ and $b(x,\gamma)$, for which the conditions of Theorem 3.3.8 are fulfilled. For each $s\in[0,1]$, we define

$$d(x,\gamma) := r(x,\gamma)^{s-1}\mathbb{1}_{\{r(x,\gamma)>0\}}, \tag{3.3.21}$$

then the coefficient $b(x,\gamma)$ from Theorem 3.3.8 is given by

$$b(x,\gamma) := r(x,\gamma)^{s}\mathbb{1}_{\{r(x,\gamma)>0\}}.$$

Proposition 3.3.9. Let the coefficient $d(x,\gamma)$ be given by (3.3.21). Then for each $s\in[0,1]$ the condition (3.3.1) is satisfied, and hence the conclusion of Theorem 3.3.6 holds for the corresponding Dirichlet form.

Furthermore, for $s=1$, conditions (3.3.17) and (3.3.18) are satisfied, and the Theorem 3.3.8 holds for the corresponding generator $(H_{\mathrm{G}}, D(H_{\mathrm{G}}))$.

Proof. We have, by (2.2.1) (3.3.21) and Proposition 3.1.1,

$$\begin{aligned}&\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)d(x,\gamma\setminus x)\mathbb{1}_\Lambda(x) \\ &\quad = \int_\Gamma \mu(d\gamma)\int_X \nu(dx)r(x,\gamma)^s\mathbb{1}_\Lambda(x) \\ &\quad \le \int_\Gamma \mu(d\gamma)\int_X \nu(dx)(1\wedge J(x,x))\mathbb{1}_\Lambda(x) < \infty,\end{aligned}$$

so the condition (3.3.1) is satisfied.

For $s = 1$ the condition (3.3.17) is equivalent to existence of the second local moment, cf. (3.1.1), and therefore fulfilled. It remains to check (3.3.18):

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \left(\int_\Lambda \nu(dx) b(x,\gamma) \right)^2 \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) r(x,\gamma) \int_\Lambda \nu(dy) r(y,\gamma) \\
&\quad \le \int_\Gamma \mu(d\gamma) \int_X \nu(dx)(1 \wedge J(x,x)) 1\!\!1_\Lambda(x) \int_X \nu(dy)(1 \wedge J(y,y)) 1\!\!1_\Lambda(y) < \infty,
\end{aligned}$$

Hence the proposition is proved. □

We finally note that all our assumptions are trivially satisfied in the case of bounded coefficients $d(x,\gamma)$, $b(x,\gamma)$.

3.3.3 Spectral gap of the generator

Here we consider the case $d \equiv 1$. We first show the coercivity identity for the gradient D^-. For any $\gamma \in \Gamma$ and $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, $(D^-)^2 F(\gamma)$ is the element of the Hilbert space $T_\gamma^{\otimes 2} = L^2((\mathbb{R}^d)^2, \gamma^{\otimes 2})$ given by $(D^-)^2 F(\gamma, x, y) = D_x^- D_y^- F(\gamma), x, y \in \gamma$. Moreover, for $x, y \in \gamma$:

$$D_x^- D_y^- F(\gamma) = \begin{cases} F(\gamma \setminus \{x,y\}) - F(\gamma \setminus x) - F(\gamma \setminus y) + F(\gamma), & x \neq y, \\ F(\gamma) - F(\gamma \setminus x) = -D_x^- F(\gamma), & x = y. \end{cases} \tag{3.3.22}$$

We also get

$$\mathrm{Tr}(D^-)^2 F(\gamma)((D^-)^2 F(\gamma))^* = \sum_{x,y\in\gamma} (D_x^- D_y^- F(\gamma))^2. \tag{3.3.23}$$

Lemma 3.3.10. (Coercivity identity) For any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, we obtain

$$\int_\Gamma (H_G F(\gamma))^2 \mu(d\gamma) = \int_\Gamma \Bigg[\mathrm{Tr}(D^-)^2 F(\gamma)((D^-)^2 F(\gamma))^* + \sum_{x,y\in\gamma, x\neq y} \left(\frac{r(y,\gamma\setminus\{x,y\})}{r(y,\gamma\setminus y)} - 1 \right) (D_y^- F)(\gamma \setminus x)(D_x^- F)(\gamma \setminus y) \Bigg] \mu(d\gamma)$$

Proof. We get from (3.3.20)

$$\begin{aligned}\int_\Gamma (H_G F(\gamma))^2 \mu(d\gamma) = &\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx)\,(D_x^- F)(\gamma)\right)^2 \\ &+ 2\int_\Gamma \mu(d\gamma) \int_X \nu(dx)\, r(x,\gamma)(D_x^+ F)(\gamma) \int_X \gamma(dy)\,(D_y^- F)(\gamma) \\ &+ \int_\Gamma \mu(d\gamma) \left(\int_X \nu(dx)\, r(x,\gamma)(D_x^+ F)(\gamma)\right)^2 .\end{aligned} \tag{3.3.24}$$

The first summand can be written as

$$\begin{aligned}&\int_\Gamma \mu(d\gamma) \left(\int_X \gamma(dx)\,(D_x^- F)(\gamma)\right)^2 \\ &\quad= \int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} (D_x^- F(\gamma))^2 + \int_\Gamma \mu(d\gamma) \sum_{x,y\in\gamma, x\neq y} D_x^- F(\gamma) D_y^- F(\gamma).\end{aligned} \tag{3.3.25}$$

To transform the next two we use (2.2.1):

$$\begin{aligned}&2\int_\Gamma \mu(d\gamma) \int_X \nu(dx)\, r(x,\gamma)(D_x^+ F)(\gamma) \int_X \gamma(dy)\,(D_y^- F)(\gamma) \\ &\quad= -2\int_\Gamma \mu(d\gamma) \int_X \gamma(dx)\,(D_x^- F)(\gamma) \int_X (\gamma\setminus x)(dy)(D_y^- F)(\gamma\setminus x)(dy) \\ &\quad= -\int_\Gamma \mu(d\gamma) \sum_{x,y\in\gamma, x\neq y} \big[(F(\gamma\setminus x) - F(\gamma))(F(\gamma\setminus\{x,y\}) - F(\gamma\setminus x)) \\ &\qquad\qquad + (F(\gamma\setminus y) - F(\gamma))(F(\gamma\setminus\{x,y\}) - F(\gamma\setminus y))\big] .\end{aligned} \tag{3.3.26}$$

The last summand is

$$\begin{aligned}&\int_\Gamma \mu(d\gamma) \left(\int_X \nu(dx)\, r(x,\gamma)(D_x^+ F)(\gamma)\right)^2 \\ &\quad= \int_\Gamma \mu(d\gamma) \sum_{x,y\in\gamma, x\neq y} \frac{r(y,\gamma\setminus\{x,y\})}{r(y,\gamma\setminus y)} (D_y^- F)(\gamma\setminus x)(D_x^- F)(\gamma\setminus y).\end{aligned} \tag{3.3.27}$$

By (3.3.23)-(3.3.27) the lemma follows. □

Theorem 3.3.11. Suppose the operator H_G is essentially selfadjoint in $L^2(\Gamma,\mu)$, and

$$\int (r(y,\gamma\setminus x) - r(y,\gamma))dy \leq \delta < 1. \tag{3.3.28}$$

Then $(0, 1-\delta)$ does not belong to the spectrum of the operator H_G.

Proof. We fix any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$. By (3.3.22) and (3.3.23) we have

$$\operatorname{Tr}(D^-)^2 F(\gamma)((D^-)^2 F(\gamma))^* \geq \sum_{x\in\gamma} (D_x^- D_x^- F(\gamma))^2 = \sum_{x\in\gamma} (D_x^- F(\gamma))^2. \qquad (3.3.29)$$

Using (2.2.1), (3.3.28) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left| \int_\Gamma \mu(d\gamma) \sum_{x,y\in\gamma, x\neq y} \left(\frac{r(y, \gamma \setminus \{x,y\})}{r(y, \gamma\setminus y)} - 1 \right) (D_y^- F)(\gamma \setminus x)(D_x^- F)(\gamma \setminus y) \right| \\
&\leq \int_\Gamma \mu(d\gamma) \sum_{x,y\in\gamma, x\neq y} \left| \left(\frac{r(y, \gamma \setminus \{x,y\})}{r(y, \gamma\setminus y)} - 1 \right) \right| (D_x^- F)^2(\gamma \setminus y) \\
&= \int_\Gamma \mu(d\gamma) \int_X \gamma(dy) \int_X (\gamma \setminus y)(dx) \left| \left(\frac{r(y, \gamma \setminus \{x,y\})}{r(y, \gamma\setminus y)} - 1 \right) \right| (D_x^- F)^2(\gamma \setminus y) \\
&= \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) (D_x^- F(\gamma))^2 \int_X dy \, |r(y, \gamma \setminus x) - r(y,\gamma)| \\
&\leq \delta(H_G F, F)_{L^2(\mu)} \qquad (3.3.30)
\end{aligned}$$

Note that according to [GY05, Theorem 3.1], $r(x,\gamma) \geq r(x,\eta)$ if $\gamma \subset \eta$. Therefore, we can omit the modulus sign in the integral. Using Lemma 3.3.10, (3.3.29) and (3.3.30), we get for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$

$$(H_G F, H_G F)_{L^2(\mu)} \geq (1-\delta)(H_G F, F)_{L^2(\mu)}. \qquad (3.3.31)$$

By assumption, (3.3.31) holds for each $F \in D(H_G)$. Thus $(0, 1-\delta)$ does not belong to the spectrum of operator H_G. $\square$

We have the following example of a DPP for which the condition of Theorem 3.3.11 is fulfilled.

Example 3.3.12. Let $X = \mathbb{R}^1$ and $K(f) := k * f$ the convolution operator for the function $k(x) = \rho e^{-a|x|}$, where $\rho, a > 0$, such that $\rho < a/2$ (the last condition means that $||k||_1 < 1$, and thus $||K|| < 1$ by Young's inequality). Then the Fourier transform of k is equal to $\hat{k}(t) = 2\rho a/(a^2+t^2)$. Hence

$$\hat{j}(t) = \frac{\hat{k}}{1-\hat{k}} = \frac{2\rho a}{\sigma^2 + t^2},$$

where $\sigma^2 = a^2 - 2\rho a$, and therefore $j(x) = \frac{\rho a}{\sigma} e^{-\sigma|x|}$. Hence $\hat{j} \geq 0$, and $j(x) \in L^1(\mathbb{R})$, and the corresponding operator K belongs to the class of operators considered in [GY05, Example 3.10]. Therefore the Papangelou intensity $r(x,\gamma)$ can be calculated through formulas (3.1.2), (3.1.3) but with J in place of $J_{[\Lambda]}$. The associated integral kernel can be written in the form

$$J(x,y) := j(x-y) = u(x\vee y)v(x\wedge y),$$

with $u(x) = e^{\sigma x}$ and $v(x) = \frac{\rho a}{\sigma}e^{-\sigma x}$. Therefore, if $\gamma = \{x_1,\dots,x_n\}$ with $x_1 < \dots < x_n$ then

$$\det J(\gamma,\gamma) = u(x_1)u(x_n)\prod_{i=1}^{n-1} d(x_{i+1}-x_i)$$

with $d(x_{i+1}-x_i) = \frac{2\rho a}{\sigma}\sinh(\sigma(x_{i+1}-x_i))$. For definition of $J(\gamma,\gamma)$ see Section 3.1. Then we obtain

$$r(x,\gamma) = \begin{cases} \dfrac{d(l_x(\gamma))d(r_x(\gamma))}{d(l_x(\gamma)+r_x(\gamma))}, & \text{if } l_x(\gamma), r_x(\gamma) < \infty, \\ \dfrac{u(x)d(r_x(\gamma))}{u(r_x(\gamma))}, & \text{if } x < x_1, \\ \dfrac{d(l_x(\gamma))v(x)}{v(l_x(\gamma))}, & \text{if } x > x_n, \end{cases}$$

where $l_x(\gamma)$ and $r_x(\gamma)$ are the distances from x to the closest point on the left, respectively on the right (which we set ∞ if there are no points from γ on the left, respectively, on the right of x). Consider first the case when $l_x(\gamma), r_x(\gamma) < \infty$. Denote the neighbour of x to the left by z_1, to the right by z_2. Then

$$\int_{\mathbb{R}} (r(y,\gamma\setminus x) - r(y,\gamma))dy = \int_{z_1}^{z_2} (r(y,\gamma\setminus x) - r(y,\gamma))dy.$$

After some elementary calculations we get

$$\begin{aligned}\int_{\mathbb{R}} (r(y,\gamma\setminus x) - r(y,\gamma))dy = \frac{\rho a}{\sigma^2} + \frac{\rho a}{\sigma}\,[&(z_2-z_1)\coth\sigma(z_2-z_1) \\ &- (z_2-x)\coth\sigma(z_2-x) \\ &-(x-z_1)\coth\sigma(x-z_1)]\,.\end{aligned} \tag{3.3.32}$$

Setting $x = z_1 + \varepsilon(z_2-z_1)$, $0 \le \varepsilon \le 1/2$ (or resp. $x = z_2 + \varepsilon(z_1 - z_2)$, $0 \le \varepsilon \le 1/2$) we rewrite the right-hand side of (3.3.32) as

$$\begin{aligned}\frac{\rho a}{\sigma}\Bigg[&\frac{1}{\sigma} + (z_2-z_1)(\coth\sigma(z_2-z_1) - \coth\sigma(1-\varepsilon)(z_2-z_1)) \\ &+ \varepsilon(z_2-z_1)(\coth\sigma(1-\varepsilon)(z_2-z_1) - \coth\sigma\varepsilon(z_2-z_1))\Bigg].\end{aligned}$$

Denote $z = z_2 - z_1$ and use

$$\coth x - \coth y = -\frac{\sinh(x-y)}{\sinh x \sinh y}.$$

Then we obtain that the right-hand side of (3.3.32) is equal to

$$\frac{\rho a}{\sigma}\frac{1}{\sigma} - \frac{\rho a}{\sigma}\left[\frac{z\sinh(\sigma\varepsilon z)}{\sinh(\sigma z)\sinh(\sigma(1-\varepsilon)z)} + \frac{\varepsilon z\sinh(\sigma(1-2\varepsilon)z)}{\sinh(\sigma\varepsilon z)\sinh(\sigma(1-\varepsilon)z)}\right].$$

The expression in brackets is bounded from below by 0 and from above by $\frac{1}{\sigma}$, hence we obtain that

$$\int_{\mathbb{R}} (r(y, \gamma \setminus x) - r(y,\gamma))dy \leq \frac{\rho a}{\sigma^2} < \frac{a^2}{2(a^2 - 2\rho a)}.$$

For ρ small enough the expression above is less than one, and therefore there exists a spectral gap of the corresponding generator.

3.4 Kawasaki dynamics

3.4.1 Existence results

In what follows, we will consider a determinantal point process μ corresponding to an operator K as defined in Section 3.1.

For a function $F : \Gamma \to \mathbb{R}$, $x \in \gamma$, $y \in X \setminus \gamma$, $\gamma \in \Gamma$, we introduce the following notation

$$(D_{xy}^{-+}F)(\gamma) = F(\gamma \setminus x \cup y) - F(\gamma).$$

We consider a measurable mapping

$$X \times X \times \Gamma \ni (x, y, \gamma) \mapsto c(x, y, \gamma \setminus x) \in [0, \infty).$$

Assume that

$$c(x, y, \gamma) = c(x, y, \gamma)1\!\!1_{\{r(x,\gamma)>0,\, r(y,\gamma)>0\}}, \qquad x, y \in X,\ \gamma \in \Gamma. \tag{3.4.1}$$

Remark 3.4.1. As we will see below, the coefficient $c(x, y, \gamma \setminus x)$ describes the rate of the jump of particle $x \in \gamma$ to y. If $r(y, \gamma \setminus x) = 0$, then the relative energy of interaction between the configuration $\gamma \setminus x$ and point y is $+\infty$, so that the particle x cannot jump to y, i.e., $c(x, y, \gamma \setminus x)$ should be equal to zero. A symmetry reason also implies that we should have $c(x, y, \gamma \setminus x) = 0$ if $r(x, \gamma \setminus x) = 0$.

Further, we assume that, for each $\Lambda \in \mathcal{B}_c(X)$,

$$\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) c(x, y, \gamma \setminus x)(\mathbb{1}_\Lambda(x) + \mathbb{1}_\Lambda(y)) < \infty. \tag{3.4.2}$$

We define the bilinear form

$$\mathcal{E}_\mathrm{K}(F, G) := \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) c(x, y, \gamma \setminus x)(D^{-+}_{xy} F)(\gamma)(D^{-+}_{xy} G)(\gamma), \tag{3.4.3}$$

where $F, G \in \mathcal{F}C_\mathrm{b}(C_0(X), \Gamma)$. As we see later, $\mathcal{E}_\mathrm{K}$ corresponds to the Kawasaki dynamics generator. We use the notation $\mathcal{E}_\mathrm{K}(F) := \mathcal{E}_\mathrm{K}(F, F)$.

We note that, for any $F \in \mathcal{F}C_\mathrm{b}(C_0(X), \Gamma)$, there exist $\Lambda \in \mathcal{B}_c(X)$ and $C > 0$ such that

$$|(D^{-+}_{xy} F)(\gamma)| \le C(\mathbb{1}_\Lambda(x) + \mathbb{1}_\Lambda(y)), \qquad \gamma \in \Gamma,\ x \in \gamma,\ y \in X \setminus \gamma.$$

Therefore, by assumptions (3.4.1), (3.4.2) the bilinear form $\mathcal{E}_\mathrm{K}$ in (3.4.3) is well-defined.

Using (2.2.1) and (3.4.1), we have, for any $F \in \mathcal{F}C_\mathrm{b}(C_0(X), \Gamma)$:

$$\begin{aligned} \mathcal{E}_\mathrm{K}(F) &= \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) r(x, \gamma) c(x, y, \gamma) \\ &\quad \times \mathbb{1}_{\{r(y,\gamma)>0)\}} \frac{r(y, \gamma)}{r(y, \gamma)} (F(\gamma \cup y) - F(\gamma \cup x))^2 \\ &= \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \gamma(dy) r(x, \gamma \setminus y) c(x, y, \gamma \setminus y) \\ &\quad \times \mathbb{1}_{\{r(y,\gamma\setminus y)>0\}} \frac{1}{r(y, \gamma \setminus y)} (D^{-+}_{yx} F)^2(\gamma). \end{aligned}$$

Therefore, for any $F, G \in \mathcal{F}C_\mathrm{b}(C_0(X), \Gamma)$,

$$\mathcal{E}_\mathrm{K}(F, G) = \int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) \tilde{c}(x, y, \gamma \setminus x)(D^{-+}_{xy} F)(\gamma)(D^{-+}_{xy} G)(\gamma),$$

where

$$\tilde{c}(x, y, \gamma) := c(y, x, \gamma) \mathbb{1}_{\{r(x,\gamma)>0\}} \frac{r(y, \gamma)}{r(x, \gamma)}.$$

Therefore, without loss of generality, in what follows we assume that $\tilde{c}(x, y, \gamma) = c(x, y, \gamma)$, i.e.,

$$r(x, \gamma) c(x, y, \gamma) = r(y, \gamma) c(y, x, \gamma). \tag{3.4.4}$$

Lemma 3.4.2. We have $\mathcal{E}_K(F, G) = 0$ for all $F, G \in \mathcal{F}C_\mathrm{b}(C_0(X), \Gamma)$ such that $F = 0$ μ-a.e.

Proof. It suffices to show that, for $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $F = 0$ μ-a.e., we have $(D^{-+}_{x,y}F)(\gamma) = 0$ $\tilde{\mu}$-a.e. Here, $\tilde{\mu}$ is the measure on $X \times X \times \Gamma$ defined by

$$\tilde{\mu}(dx, dy, d\gamma) := c(x, y, \gamma \setminus x)\gamma(dx)\nu(dy)\mu(d\gamma). \tag{3.4.5}$$

For any F as above, we evidently have that $F(\gamma) = 0$ $\tilde{\mu}$-a.e. Next, by (2.2.1) and (3.4.1)

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) \int_\Lambda \nu(dy) |F(\gamma \setminus x \cup y)| c(x, y, \gamma \setminus x) \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \nu(dy) r(x, \gamma) |F(\gamma \cup y)| c(x, y, \gamma) 1\!\!1_{\{r(y,\gamma)>0\}} \frac{r(y, \gamma)}{r(y, \gamma)} \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) |F(\gamma)| c(x, y, \gamma \setminus y) \, \frac{r(x, \gamma \setminus y)}{r(y, \gamma \setminus y)} \, 1\!\!1_{\{r(y,\gamma\setminus y)>0\}} \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) |F(\gamma)| c(x, y, \gamma \setminus y) \frac{r(x, \gamma \setminus y)}{r(y, \gamma \setminus y)}.
\end{aligned} \tag{3.4.6}$$

Since F is bounded, by (3.4.2) the integral in (3.4.6) is finite. Therefore,

$$|F(\gamma)| \, \frac{r(x, \gamma \setminus y)}{r(y, \gamma \setminus y)} < \infty \quad \text{for } \tilde{\mu}\text{-a.a. } (x, y, \gamma) \in X \times X \times \Gamma. \tag{3.4.7}$$

Because $F = 0$ $\tilde{\mu}$-a.e., by (3.4.6) and (3.4.7), $F(\gamma \setminus x \cup y) = 0$ $\tilde{\mu}$-a.e. □

Lemma 3.4.3. Assume that, for some $u \in \mathbb{R}$,

$$\int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) r(x, \gamma \setminus y) r(y, \gamma \setminus y)^u 1\!\!1_{\{r(y,\gamma\setminus y)>0\}} c(x, y, \gamma \setminus y) \in L^2(\Gamma, \mu) \tag{3.4.8}$$

for all $\Lambda \in \mathcal{B}_c(X)$. Then the bilinear form $(\mathcal{E}_{\mathrm{K}}, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma))$ is closable on $L^2(\Gamma, \mu)$, and its closure will be denoted by $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$.

Proof. Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ such that $\|F_n\|_{L^2(\Gamma,\mu)} \to 0$ as $n \to \infty$ and

$$\mathcal{E}_{\mathrm{K}}(F_n - F_k) \to 0 \quad \text{as } n, k \to \infty. \tag{3.4.9}$$

To prove the closability of $\mathcal{E}_{\mathrm{K}}$, it suffices to show that there exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$ such that $\mathcal{E}_{\mathrm{K}}(F_{n_k}) \to 0$ as $k \to \infty$.

Since $\|F_n\|_{L^2(\Gamma,\mu)} \to 0$ as $n \to \infty$, there exists a subsequence $(F_n^{(1)})_{n=1}^\infty$ of $(F_n)_{n=1}^\infty$ such that $F_n^{(1)}(\gamma) \to 0$ for $\tilde{\mu}$-a.a. $(x, y, \gamma) \in X \times X \times \Gamma$. Next, by (3.4.8), we have, for any $\Lambda \in \mathcal{B}_c(X)$,

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda \gamma(dx) \int_\Lambda \nu(dy) c(x,y,\gamma\setminus x) r(y,\gamma\setminus x)^{u+1} 1\!\!1_{\{r(y,\gamma\setminus x)>0\}} |F_n^{(1)}(\gamma\setminus x\cup y)| \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \nu(dy) r(x,\gamma) c(x,y,\gamma) r(y,\gamma)^{u+1} 1\!\!1_{\{r(y,\gamma)>0\}} |F_n^{(1)}(\gamma\cup y)| \\
&\quad = \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) r(x,\gamma\setminus y) r(y,\gamma\setminus y)^u \\
&\qquad \times 1\!\!1_{\{r(y,\gamma\setminus y)>0\}} c(x,y,\gamma\setminus y) |F_n^{(1)}(\gamma)| \\
&\quad \le \Big(\int_\Gamma \mu(d\gamma) |F_n^{(1)}(\gamma)|^2\Big)^{1/2} \Big(\int_\Gamma \mu(d\gamma) \Big(\int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) r(x,\gamma\setminus y) \\
&\qquad \times r(y,\gamma\setminus y)^u 1\!\!1_{\{r(y,\gamma\setminus y)>0\}} c(x,y,\gamma\setminus y)\Big)^2\Big)^{1/2} \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore, there exists a subsequence $(F_n^{(2)})_{n=1}^\infty$ of $(F_n^{(1)})_{n=1}^\infty$ such that $F_n^{(2)}(\gamma\setminus x\cup y)\to 0$ as $n\to\infty$ for

$$c(x,y,\gamma\setminus x) r(y,\gamma\setminus x)^u 1\!\!1_{\{r(y,\gamma\setminus x)>0\}} \gamma(dx)\nu(dy)\mu(d\gamma)\text{-a.e. } (x,y,\gamma)\in X\times X\times\Gamma.$$

By (3.4.1), the latter measure is equivalent to $\tilde\mu$, and therefore

$$(D_{x,y}^{-+} F_n^{(2)})(\gamma) \to 0 \quad \text{for } \tilde\mu\text{-a.e. } (x,y,\gamma)\in X\times X\times\Gamma. \tag{3.4.10}$$

Now, by (3.4.10) and Fatou's lemma,

$$\begin{aligned}
\mathcal{E}_{\mathrm{K}}(F_n^{(2)}) &= \int (D_{xy}^{-+} F_n^{(2)})(\gamma)^2\, \tilde\mu(dx,dy,d\gamma) \\
&= \int \Big((D_{xy}^{-+} F_n^{(2)})(\gamma) - \lim_{m\to\infty} (D_{xy}^{-+} F_m^{(2)})(\gamma)\Big)^2\, \tilde\mu(dx,dy,d\gamma) \\
&\le \liminf_{m\to\infty} \int ((D_{xy}^{-+} F_n^{(2)})(\gamma) - (D_{xy}^{-+} F_m^{(2)})(\gamma))^2\, \tilde\mu(dx,dy,d\gamma) \\
&= \liminf_{m\to\infty} \mathcal{E}_{\mathrm{K}}(F_n^{(2)} - F_m^{(2)}),
\end{aligned}$$

which by (3.4.9) can be made arbitrarily small for n large enough. □

Lemma 3.4.4. $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ is a Dirichlet form on $L^2(\Gamma,\mu)$.

The proof of Lemma 3.4.4 is analogous to that of Lemma 3.3.3, so we omit it.

Now, analogously to the previous section, we consider μ as a measure on $(\ddot\Gamma, \mathcal{B}(\ddot\Gamma))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a bilinear form on $L^2(\ddot\Gamma,\mu)$.

Lemma 3.4.5. Under the assumption of Lemma 3.4.3, $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$ is a quasi-regular Dirichlet form on $L^2(\ddot\Gamma,\mu)$.

Proof. Analogously to [MR00, Proposition 4.1], it suffices to show that there exists a bounded, complete metric ρ on $\ddot{\Gamma}$ generating the vague topology such that, for all $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{K}})$ and

$$\int_X \gamma(dx) \int_X \nu(dy)\, S(\rho(\cdot, \gamma_0))(x, y, \gamma) \leq \eta(\gamma) \quad \mu\text{-a.e.}$$

for some $\eta \in L^1(\ddot{\Gamma}, \mu)$ (independent of γ_0). Here,

$$S(F, G) := c(x, y, \gamma \setminus x)(D_{xy}^{-+} F)(\gamma)(D_{xy}^{-+} G)(\gamma),$$

and $S(F){:=}S(F, F)$. The proof below is a modification of the proof of [MR00, Proposition 4.8] and the proof of [KL05, Proposition 3.2].

For a fixed $x_0 \in X$, denote for short $B(r) := B(x_0, r)$. For each $k \in \mathbb{N}$, we define

$$g_k(x) := \frac{2}{3}\left(\frac{1}{2} - \operatorname{dist}(x, B(k)) \wedge \frac{1}{2}\right), \qquad x \in X.$$

Next, we set $\phi_k(x) := 3g_k(x)$, $x \in X,\ k \in \mathbb{N}$.

Let ζ be a function in $C_{\mathrm{b}}^1(\mathbb{R})$ such that $0 \leq \zeta \leq 1$ on $[0, \infty)$, $\zeta(t) = t$ on $[-1/2, 1/2]$, $\zeta' \in [0, 1]$ on $[0, \infty)$. For any fixed $\gamma_0 \in \ddot{\Gamma}$ and for any $k, n \in \mathbb{N}$, the restriction to Γ of the function

$$\zeta\left(\sup_{j \leq n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\right)$$

belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$ (note that $\langle \phi_k g_j, \gamma_0\rangle$ is a constant). Furthermore, taking into account that $\zeta' \in [0, 1]$ on $[0, \infty)$, we get from the mean value theorem, for each $\gamma \in \Gamma$, $x \in \gamma$, and $y \in X \setminus \gamma$,

$$\begin{aligned}
&S\left(\zeta\left(\sup_{j \leq n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\right)\right)(x, y, \gamma) \\
&\quad \leq c(x, y, \gamma \setminus x)\Bigg(\sup_{j \leq n} |\langle \phi_k g_j, \gamma\rangle - \langle \phi_k g_j, \gamma_0\rangle - (\phi_k g_j)(x) + (\phi_k g_j)(y)| \\
&\qquad - \sup_{j \leq n} |\langle \phi_k g_j, \gamma\rangle - \langle \phi_k g_j, \gamma_0\rangle|\Bigg)^2 \\
&\quad \leq c(x, y, \gamma \setminus x) \sup_{j \leq n} |-(\phi_k g_j)(x) + (\phi_k g_j)(y)|^2 \\
&\quad \leq 2c(x, y, \gamma \setminus x)\left(\sup_{j \leq n} (\phi_k g_j)(x)^2 + \sup_{j \leq n} (\phi_k g_j)(y)^2\right) \\
&\quad \leq 2c(x, y, \gamma \setminus x)(1\!\!1_{B(k+1/2)}(x) + 1\!\!1_{B(k+1/2)}(y)).
\end{aligned} \tag{3.4.11}$$

For each $k \in \mathbb{N}$, we define

$$F_k(\gamma, \gamma_0) := \zeta \left(\sup_{j \in \mathbb{N}} |\langle \phi_k g_j, \gamma \rangle - \langle \phi_k g_j, \gamma_0 \rangle| \right), \qquad \gamma, \gamma_0 \in \ddot{\Gamma}.$$

Then, for a fixed $\gamma_0 \in \ddot{\Gamma}$,

$$\zeta \left(\sup_{j \le n} |\langle \phi_k g_j, \gamma \rangle - \langle \phi_k g_j, \gamma_0 \rangle| \right) \to F_k(\gamma, \gamma_0)$$

as $n \to \infty$ for each $\gamma \in \ddot{\Gamma}$ and in $L^2(\mu)$. Hence, by (3.4.11) and the Banach–Alaoglu and the Banach–Saks theorems (see e.g. [MR92, Appendix A.2]), $F_k(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{K}})$ and

$$S(F_k(\cdot, \gamma_0))(x, y, \gamma) \le 2c(x, y, \gamma \setminus x)(1\!\!1_{B(k+1/2)}(x) + 1\!\!1_{B(k+1/2)}(y))$$

for $\gamma(dx)\nu(dy)\mu(d\gamma)$-a.a. $(x, y, \gamma) \in X \times X \times \Gamma$.

Define for $k \in \mathbb{N}$

$$c_k := \left(1 + 2 \int c(x, y, \gamma \setminus x)(1\!\!1_{B(k+1/2)}(x) + 1\!\!1_{B(k+1/2)}(y))\, \gamma(dx)\nu(dy)\mu(d\gamma) \right)^{-1/2} 2^{-k/2},$$

which are finite positive numbers by (3.4.2), and furthermore, $c_k \to 0$ as $k \to \infty$.

We define

$$\rho(\gamma_1, \gamma_2) := \sup_{k \in \mathbb{N}} \big(c_k F_k(\gamma_1, \gamma_2) \big), \qquad \gamma_1, \gamma_2 \in \ddot{\Gamma}.$$

By [MR00, Theorem 3.6], ρ is a bounded, complete metric on $\ddot{\Gamma}$ generating the vague topology.

Analogously to the above, we now conclude that, for any fixed $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot, \gamma_0) \in D(\mathcal{E}_{\mathrm{K}})$ and

$$\int_X \gamma(dx) \int_X \nu(dy) S(\rho(\cdot, \gamma_0))(x, y, \gamma) \le \eta(\gamma) \quad \mu\text{-a.e.},$$

where

$$\eta(\gamma) := 2 \sup_{k \in \mathbb{N}} \left(c_k^2 \int_X \gamma(dx) \int_X \nu(dy) c(x, y, \gamma \setminus x)(1\!\!1_{B(k+1/2)}(x) + 1\!\!1_{B(k+1/2)}(y)) \right).$$

Finally,

$$\begin{aligned}\int_\Gamma \eta(\gamma)\, \mu(d\gamma) &\le 2 \sum_{k=1}^{\infty} c_k^2 \iiint c(x, y, \gamma \setminus x)(1\!\!1_{B(k+1/2)}(x) + 1\!\!1_{B(k+1/2)}(y))\, \gamma(dx)\nu(dy)\mu(d\gamma) \\ &\le \sum_{k=1}^{\infty} 2^{-k} = 1.\end{aligned}$$

Thus, the lemma is proved. □

Lemma 3.4.6. The set $\ddot{\Gamma} \setminus \Gamma$ is exceptional for $\mathcal{E}_{\mathrm{K}}$.

Proof. It suffices to prove the lemma locally, i.e., to show that, for any fixed $a \in X$ and $r > 0$

$$N_a := \{\gamma \in \ddot{\Gamma} : \sup_{x \in \bar{B}(a,r)} \gamma(\{x\}) \geq 2\}$$

is $\mathcal{E}_{\mathrm{K}}$-exceptional.

By [RS98, Lemma 1], we need to prove that there exists a sequence $u_n \in D(\mathcal{E}_{\mathrm{K}})$, $n \in \mathbb{N}$, such that each u_n is a continuous function on $\ddot{\Gamma}$, $u_n \to 1\!\!1_{N_a}$ pointwise as $n \to \infty$, and $\sup_{n \in \mathbb{N}} \mathcal{E}_{\mathrm{K}}(u_n) < \infty$.

The proof is analogous to the proof of Lemma 3.3.5. So we choose n, define the functions f_k and u_n as in Lemma 3.3.5. Then by (3.3.13)-(3.3.15), and the mean value theorem, we obtain, for each $\gamma \in \Gamma$, $x \in \gamma$, $y \in X \setminus \gamma$,

$$\begin{aligned}
(D^{-+}_{xy} u_n)^2(\gamma) &\leq 4\biggl(\sup_{k \in \{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma \setminus x \cup y\rangle - \sup_{k \in \{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma\rangle\biggr)^2 \\
&\leq 4 \sup_{k \in \{1,\dots,K_n\}} |\langle f_k^{(n)}, \gamma \setminus x \cup y\rangle - \langle f_k^{(n)}, \gamma\rangle|^2 \\
&\leq 8\biggl(\sup_{k \in \{1,\dots,K_n\}} f_k^{(n)}(x)^2 + \sup_{k \in \{1,\dots,K_n\}} f_k^{(n)}(y)^2\biggr) \\
&\leq 8\biggl(\sup_{k \in \{1,\dots,K_n\}} 1\!\!1_{\bar{B}(a_k,2/n)}(x) + \sup_{k \in \{1,\dots,K_n\}} 1\!\!1_{\bar{B}(a_k,2/n)}(y)\biggr) \\
&\leq 8(1\!\!1_{\bar{B}(a,2r)}(x) + 1\!\!1_{\bar{B}(a,2r)}(y)).
\end{aligned}$$

Hence, by (3.4.2),

$$\sup_n \mathcal{E}_{\mathrm{K}}(u_n) < \infty,$$

which implies the lemma. □

We now have the main result of this section.

Theorem 3.4.7. Let (3.4.2) and (3.4.8) hold. Then we have:

1. There exists a conservative Hunt process

$$\mathbf{M} = \bigl(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t \geq 0}, (\mathbf{\Theta}_t)_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\gamma)_{\gamma \in \Gamma}\bigr)$$

on Γ (see e.g. [MR92, p. 92]), which is properly associated with $(\mathcal{E}_K, D(\mathcal{E}_K))$, i.e., for all ($\mu$-versions of) $F \in L^2(\Gamma, \mu)$ and all $t > 0$, the function

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma \qquad (3.4.12)$$

is an $\mathcal{E}_K$-quasi-continuous version of $\exp(-tH_K)F$, where $(H_K, D(H_K))$ is the generator of $(\mathcal{E}_K, D(\mathcal{E}_K))$. $\mathbf{M}$ is up to μ-equivalence unique (cf. [MR92, Chap. IV, Sect. 6]). In particular, $\mathbf{M}$ is μ-symmetric (i.e., $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ for all $F, G : \Gamma \to \mathbb{R}_+$, $\mathcal{B}(\Gamma)$-measurable), so has μ as an invariant measure.

2. $\mathbf{M}$ from 1) is up to μ-equivalence (cf. [MR92, Definition 6.3]) unique among all Hunt processes $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}'(t))_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\Gamma})$ on Γ having μ as invariant measure and solving the martingale problem for $(-H_K, D(H_K))$, i.e., for all $G \in D(H_K)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (H_K G)(\mathbf{X}'(s))\, ds, \qquad t \geq 0,$$

is an $(\mathbf{F}'_t)$-martingale under $\mathbf{P}'_\gamma$ for $\mathcal{E}_K$-q.e. $\gamma \in \Gamma$. (Here, $\widetilde{G}$ denotes an $\mathcal{E}_K$-quasi-continuous version of G, cf. [MR92, Ch. IV, Proposition 3.3].)

Remark 3.4.8. In Theorem 3.4.7, $\mathbf{M}$ can be taken canonical, i.e., $\mathbf{\Omega}$ is the set of all *cadlag* functions $\omega : [0, \infty) \to \Gamma$ (i.e., ω is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$), $\mathbf{X}(t)(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, $(\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [FŌT94, Section 4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

Proof of Theorem 3.4.7. The first part of the theorem follows from Lemmas 3.4.5, 3.3.5, the fact that $1 \in D(\mathcal{E}_K)$ and $\mathcal{E}_K(1, 1) = 0$, and [MR92, Chap. IV, Theorem 3.5 and Chap. V, Proposition 2.15]. The second part follows directly from the proof of [AR95, Theorem 3.5]. □

Now we will derive explicit formula for the generator of $\mathcal{E}_{\mathrm{K}}$. However, for this, we will demand stronger conditions on the coefficient $c(x, y, \gamma \setminus x)$.

Theorem 3.4.9. Assume that, for each $\Lambda \in \mathcal{B}_c(X)$,

$$\int_X \gamma(dx) \int_X \nu(dy)\, c(x, y, \gamma \setminus x)(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) \in L^2(\Gamma, \mu). \tag{3.4.13}$$

Then, for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma)$,

$$(H_{\mathrm{K}}F)(\gamma) = -2 \int_X \gamma(dx) \int_X \nu(dy) c(x, y, \gamma \setminus x)(D^{-+}_{xy} F)(\gamma) \qquad \mu\text{-a.e.} \tag{3.4.14}$$

and $(H_{\mathrm{K}}, D(H_{\mathrm{K}}))$ is the Friedrichs extension of $(H_{\mathrm{K}}, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma))$ in $L^2(\Gamma, \mu)$.

Proof. By (2.2.1) and (3.4.4), the theorem easily follows from our assumption (3.4.13). □

3.4.2 Examples

For each $s \in [0,1]$, we define

$$c(x,y,\gamma) := a(x,y)r(x,\gamma)^{s-1}r(y,\gamma)^s 1\!\!1_{\{r(x,\gamma)>0,\, r(y,\gamma)>0\}}. \tag{3.4.15}$$

Here, the function $a : X^2 \to [0,\infty)$ is measurable, symmetric (i.e., $a(x,y) = a(y,x)$), bounded, and satisfies

$$\sup_{x \in X} \int_X a(x,y)\, \nu(dy) < \infty. \tag{3.4.16}$$

Assume also that there exists $\Lambda \in \mathcal{B}_c(X)$ such that the integral operator J (for definition see Section 3.1) fulfills

$$\sup_{x \in X \setminus \Lambda} J(x,x) < \infty. \tag{3.4.17}$$

We remind that $c(x,y,\gamma)$ satisfies the balance condition (3.4.4). Analogously to the Proposition 3.3.9 we get

Proposition 3.4.10. Let the coefficient $c(x,y,\gamma)$ be given by (3.4.15), and let conditions (3.4.16), (3.4.17) hold. Then, for each $s \in [0,1]$, conditions (3.4.2) and (3.4.8) are satisfied, and therefore the conclusion of Theorem 3.4.7 holds for the corresponding Dirichlet form.

Furthermore, for $s = 1$, condition (3.4.13) is satisfied, and hence the conclusion of Theorem 3.4.9 holds for the corresponding generator $(H_{\mathrm{K}}, D(H_{\mathrm{K}}))$.

Proof. Let $s \in [0,1]$. We have, by (2.2.1), (3.4.16), (3.4.17) and Proposition 3.1.1,

$$\begin{aligned}
\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) c(x,y,\gamma \setminus x)(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) \\
&= \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x,y) r(x,\gamma)^s r(y,\gamma)^s \\
&\quad \times 1\!\!1_{\{r(x,\gamma)>0,\, r(y,\gamma)>0\}}(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) \\
&\le \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x,y) \\
&\quad \times (1 \wedge J(x,x))(1 \wedge J(y,y))(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y)) \\
&= 2 \int_\Gamma \mu(d\gamma) \int_X \nu(dx) 1\!\!1_\Lambda(x)(1 \wedge J(x,x)) \\
&\quad \times \int_X \nu(dy) a(x,y)(1 \wedge J(y,y)) < \infty,
\end{aligned}$$

so that condition (3.4.2) is satisfied.

Next, setting $u = -s$, we see that in order to show that (3.4.8) is satisfied, it suffices to prove that, for each $\Lambda \in \mathcal{B}_c(X)$,

$$\int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) a(x,y) r(x, \gamma \setminus y)^s \in L^2(\mu).$$

So, by Proposition 3.1.1, (2.2.1), (3.4.16), (3.4.17), and the boundedness of a, we have:

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma)\Big(\int_\Lambda \nu(dx)\int_\Lambda \gamma(dy) a(x,y) r(x,\gamma\setminus y)^s\Big)^2 \\
&\quad= \int_\Gamma \mu(d\gamma)\int_\Lambda \nu(dy) r(y,\gamma)\int_\Lambda \nu(dx_1)\int_\Lambda \nu(dx_2) a(x_1,y)a(x_2,y) r(x_1,\gamma)^s r(x_2,\gamma)^s \\
&\qquad+ \int_\Gamma \mu(d\gamma)\int_\Lambda \nu(dy_1)\int_\Lambda \nu(dy_2)\int_\Lambda \nu(dx_1)\int_\Lambda \nu(dx_2) r(y_2,\gamma) r(y_1,\gamma\cup y_2) \\
&\qquad\times a(x_1,y_1)a(x_2,y_2) r(x_1,\gamma\cup y_2)^s r(x_2,\gamma\cup y_1)^s \\
&\quad\le \int_\Lambda \nu(dy) J(y,y)\int_\Lambda \nu(dx_1)\int_\Lambda \nu(dx_2) a(x_1,y)a(x_2,y)(1+J(x_1,x_1))(1+J(x_2,x_2)) \\
&\quad+ \int_\Lambda \nu(dy_1)\int_\Lambda \nu(dy_2)\int_\Lambda \nu(dx_1)\int_\Lambda \nu(dx_2) a(x_1,y_1)a(x_2,y_2) \\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(1+J(x_1,x_1))(1+J(x_2,x_2)) < \infty.
\end{aligned}$$

Now, let $s = 1$. Analogously to the above, we have:

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma)\Big(\int_X \gamma(dx)\int_X \nu(dy) c(x,y,\gamma\setminus x)(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y))\Big)^2 \\
&\quad= \int_\Gamma \mu(d\gamma)\int_X \nu(dx) r(x,\gamma)\int_X \nu(dy_1)\int_X \nu(dy_2) a(x,y_1)a(x,y_2) \\
&\qquad\times r(y_1,\gamma) r(y_2,\gamma) 1\!\!1_{\{r(x,\gamma)>0,\, r(y_1,\gamma)>0,\, r(y_2,\gamma)>0\}}(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y_1))(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y_2)) \\
&\qquad+ \int_\Gamma \mu(d\gamma)\int_X \nu(dx_1)\int_X \nu(dx_2) r(x_2,\gamma) r(x_1,\gamma\cup x_2) \\
&\qquad\times \int_X \nu(dy_1)\int_X \nu(dy_2) a(x_1,y_1)a(x_2,y_2) r(y_1,\gamma\cup x_2) r(y_2,\gamma\cup x_1) \\
&\qquad\times 1\!\!1_{\{r(x_1,\gamma\cup x_2)>0,\, r(x_2,\gamma\cup x_1)>0,\, r(y_1,\gamma\cup x_2)>0,\, r(y_2,\gamma\cup x_1)>0\}} \\
&\qquad\times (1\!\!1_\Lambda(x_1) + 1\!\!1_\Lambda(y_1))(1\!\!1_\Lambda(x_2) + 1\!\!1_\Lambda(y_2)) \\
&\quad\le \int_X \nu(dx)\int_X \nu(dy_1)\int_X \nu(dy_2) a(x,y_1)a(x,y_2) \\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y_1))(1\!\!1_\Lambda(x) + 1\!\!1_\Lambda(y_2)) \\
&\qquad+ \int_X \nu(dx_1)\int_X \nu(dx_2)\int_X \nu(dy_1)\int_X \nu(dy_2) a(x_1,y_1)a(x_2,y_2) \\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(1\!\!1_\Lambda(x_1) + 1\!\!1_\Lambda(y_1))(1\!\!1_\Lambda(x_2) + 1\!\!1_\Lambda(y_2)).
\end{aligned} \tag{3.4.18}$$

We can write the first integral as

$$\begin{aligned}
&4\int_\Lambda \nu(dx)\int_\Lambda \nu(dy_1)\int_\Lambda \nu(dy_2)a(x,y_1)a(x,y_2)J(y_1,y_1)J(y_2,y_2)\\
&+4\int_\Lambda \nu(dx)\int_\Lambda \nu(dy_1)\int_X \nu(dy_2)a(x,y_1)a(x,y_2)J(y_1,y_1)J(y_2,y_2)\\
&+4\int_\Lambda \nu(dx)\int_X \nu(dy_1)\int_X \nu(dy_2)a(x,y_1)a(x,y_2)J(y_1,y_1)J(y_2,y_2)\\
&+\int_{\Lambda^c} \nu(dx)\int_\Lambda \nu(dy_1)\int_\Lambda \nu(dy_2)a(x,y_1)a(x,y_2)J(y_1,y_1)J(y_2,y_2).
\end{aligned}$$

The finiteness of the first integral follows immediately from our assumptions. To get a bound for the other ones note that

$$\int_X \nu(dy_2)a(x,y_2)J(y_2,y_2) \le c_a\int_\Lambda \nu(dy_2)J(y_2,y_2) + c_J\int_{\Lambda^c}\nu(dy_2)a(x,y_2)$$

and

$$\int_{\Lambda^c}\nu(dx)a(x,y_1)a(x,y_2) \le c_a\int_{\Lambda^c}\nu(dx)a(x,y_1).$$

Here we used the notations $|a| \le c_a$, and $\sup_{x\in X\setminus\Lambda} J(x,x) \le c_J$. The finiteness of the second summand in (3.4.18) we get analogously. $\square$

We finally note that all our assumptions are trivially satisfied in the case when the coefficient $c(x,y,\gamma)$ is bounded.

4 Spectral Gap for Glauber dynamics

Throughout this chapter we will consider the configuration space $\Gamma := \Gamma(\mathbb{R}^d)$ over $\mathbb{R}^d$.

Fix a point process μ with Papangelou intensity r, defined in (2.2.1), for which the correlation functions $k_\mu^{(1)}$ and $k_\mu^{(2)}$ exist and are locally integrable. This means that the first two local moments exist, i.e., for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$

$$\int_\Gamma \langle 1\!\!1_\Lambda, \gamma\rangle \mu(d\gamma) < \infty, \qquad \int_\Gamma \langle 1\!\!1_\Lambda, \gamma\rangle^2 \mu(d\gamma) < \infty. \tag{4.0.1}$$

We also assume the following local integrability condition

$$\int_{\mathbb{R}^d} r(x,\gamma)|r(y,\gamma) - r(y, \gamma \cup x)|dx \in L^1_{loc}(dy) \quad \text{for } \mu\text{-a.a. } \gamma \in \Gamma. \tag{4.0.2}$$

For Gibbs measures, corresponding to an integrable pair potential ϕ, the condition above is fulfilled, for example, if the second correlation function $k_\mu^{(2)}$ is bounded. In this case, by assumption (4.0.1), using (2.2.1) and the definition of k_μ we obtain that the following integral is finite:

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_\Lambda dy \int_{\mathbb{R}^d} dx\; e^{-E(x,\gamma)} e^{-E(y,\gamma)} |1 - e^{\phi(x-y)}| \\
&= \int_\Gamma \sum_{x\in\gamma, y\in\gamma_\Lambda} |1 - e^{\phi(x-y)}| \mu(d\gamma) \\
&= \int_\Gamma \sum_{\{x,y\}\subseteq\gamma} 1\!\!1_\Lambda(y)|1 - e^{\phi(x-y)}|\mu(d\gamma) + \int_\Gamma \int_\Lambda k_\mu^{(1)}(x) dx \mu(d\gamma) \\
&= \int_\Gamma \int_\Lambda dy \int_{\mathbb{R}^d} dx |1 - e^{\phi(x-y)}| k_\mu^{(2)}(x,y) \mu(d\gamma) + \int_\Gamma \int_\Lambda k_\mu^{(1)}(x) dx \mu(d\gamma) < \infty.
\end{aligned}$$

4.1 Glauber dynamics

Here we introduce the Dirichlet form which will be considered throughout this chapter, and the corresponding birth and death dynamics, which is called Glauber.

This Dirichlet form is a special case of the one considered in Section 3.3, namely we choose constant death rate $d(x,\gamma) \equiv 1$.

We recollect shortly the notations used in this chapter. Define two types of difference operators for $F : \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$, and $x, y \in \mathbb{R}^d$

$$(D_x^- F)(\gamma) := F(\gamma \setminus x) - F(\gamma), \quad (D_x^+ F)(\gamma) := F(\gamma) - F(\gamma \cup x). \tag{4.1.1}$$

Now we introduce the aforementioned bilinear form, cf. [KL05, KLR07]

$$\mathcal{E}(F,G) := \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)(D_x^- F)(\gamma)(D_x^- G)(\gamma), \tag{4.1.2}$$

for functions $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$. The following properties of the bilinear form $\mathcal{E}$, which are useful for our considerations, were proved in [KL05]. The bilinear form $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma))$ is closable on $L^2(\Gamma, \mu)$ and its closure is a Dirichlet form which we will denote by $(\mathcal{E}, D(\mathcal{E}))$. The generator $(L, D(L))$ of $(\mathcal{E}, D(\mathcal{E}))$ is given by

$$(LF)(\gamma) = \int_{\mathbb{R}^d} \gamma(dx)\,(D_x^- F)(\gamma) - \int_{\mathbb{R}^d} r(x,\gamma)(D_x^+ F)(\gamma)dx \qquad \mu\text{-a.e.} \tag{4.1.3}$$

for functions $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma) \subset D(L)$.

Theorem 4.1.1. ([KL05]) There exists a conservative Hunt process

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t \geq 0}, (\mathbf{\Theta}_t)_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\gamma)_{\gamma \in \Gamma})$$

on Γ (see e.g. [MR92, p. 92]) which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e., for all ($\mu$-versions of) $F \in L^2(\Gamma, \mu)$ and all $t > 0$ the function

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma$$

is an $\mathcal{E}$-quasi-continuous version of $\exp(-tH)F$, where $H = -L$. $\mathbf{M}$ is up to μ-equivalence unique (cf. [MR92, Chap. IV, Sect. 6]). In particular, $\mathbf{M}$ is μ-symmetric, and has μ as an invariant measure.

4.2 Coercivity identity for Glauber dynamics

4.2.1 Carré du champ

For $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$ we define the "carré du champ" corresponding to L as

$$\square(F,G) := \frac{1}{2}(L(FG) - FLG - GLF), \tag{4.2.1}$$

see, e.g. [BÉ85a, Bak94]. Splitting the generator L in its death and birth part

$$L^-F(\gamma) := \sum_{x\in\gamma} D_x^- F(\gamma), \qquad L^+F(\gamma) := \int_{\mathbb{R}^d} r(x,\gamma) D_x^+ F(\gamma) dx, \tag{4.2.2}$$

such that $L = L^- - L^+$ one obtains

$$\square(F,G) = \square^-(F,G) + \square^+(F,G),$$

where the “carré du champ” splits correspondingly into the death and birth part

$$\begin{aligned}\square^-(F,G) &:= \frac{1}{2}\sum_{x\in\gamma} D_x^- F(\gamma) D_x^- G(\gamma),\\ \square^+(F,G) &:= \frac{1}{2}\int_{\mathbb{R}^d} r(x,\gamma) D_x^+ F(\gamma) D_x^+ G(\gamma) dx.\end{aligned}$$

Iterating the definition of “carré du champ” one may introduce the so-called “carré du champ itéré” $\square_2$ as follows

$$\square_2(F,G) := \frac{1}{2}(L\square(F,G) - \square(F,LG) - \square(G,LF)), \tag{4.2.3}$$

see, e.g. [Bak85, BÉ85a, Bak94]. Using the splitting in birth and death part we rewrite $\square_2$ in the following way:

$$\begin{aligned}2\square_2(F,F) = &\big(L^-\square^-(F,F) - 2\square^-(F,L^-F)\big) \\ &- \big(L^+\square^+(F,F) - 2\square^+(F,L^+F)\big)\\ &+ \big(L^-\square^+(F,F) - L^+\square^-(F,F) - 2\square^+(F,L^-F) + 2\square^-(F,L^+F)\big)\end{aligned} \tag{4.2.4}$$

All summands will be treated separately, where the following product rule type formulas are at the base of further calculations.

Lemma 4.2.1. For cylindric function $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$

$$D_y^-\left[\sum_{x\in\cdot} D_x^- F(\cdot)\right](\gamma) = \sum_{x\in\gamma\setminus y} D_y^- F(\gamma\setminus x) - \sum_{x\in\gamma} D_y^- F(\gamma), \tag{4.2.5}$$

$$D_y^+\left[\sum_{x\in\cdot} D_x^- F(\cdot)\right](\gamma) = \sum_{x\in\gamma} D_y^+(\gamma\setminus x) - \sum_{x\in\gamma\cup y} D_y^+(\gamma). \tag{4.2.6}$$

For a function $H_y(\gamma) : \Gamma\times\mathbb{R}^d \to \mathbb{R}$ such that the expressions below are μ-a.s. well-defined we have

$$\begin{aligned}D_x^+\left(\int_{\mathbb{R}^d} r(y,\gamma) H_y(\gamma) dy\right) = &\int_{\mathbb{R}^d} r(y,\gamma) D_x^+ H_y(\gamma) dy\\ &+ \int_{\mathbb{R}^d} D_x^+ r(y,\gamma) H_y(\gamma\cup x) dy,\end{aligned} \tag{4.2.7}$$

$$D_x^- \left(\int_{\mathbb{R}^d} r(y,\gamma) H_y(\gamma) dy \right) = \int_{\mathbb{R}^d} r(y,\gamma) D_x^- H_y(\gamma) dy + \int_{\mathbb{R}^d} D_x^- r(y,\gamma) H_y(\gamma \setminus x) dy. \tag{4.2.8}$$

Proof. Using the definition of difference operators D_x^- and D_x^+

$$\begin{aligned} D_y^- \left[\sum_{x \in \cdot} D_x^- F(\cdot) \right] (\gamma) &= D_y^- \left[\sum_{x \in \cdot} (F(\cdot \setminus x) - F(\cdot)) \right] (\gamma) \\ &= \sum_{x \in \gamma \setminus y} (F(\gamma \setminus \{x,y\}) - F(\gamma \setminus y)) - \sum_{x \in \gamma} (F(\gamma \setminus x) - F(\gamma)) \\ &= \sum_{x \in \gamma \setminus y} (F(\gamma \setminus \{x,y\}) - F(\gamma \setminus x)) - F(\gamma \setminus y) \\ &\quad - \sum_{x \in \gamma} (F(\gamma \setminus y) - F(\gamma)) + F(\gamma \setminus y) \\ &= \sum_{x \in \gamma \setminus y} D_y^- F(\gamma \setminus x) - \sum_{x \in \gamma} D_y^- F(\gamma). \end{aligned}$$

Analogously

$$\begin{aligned} D_y^+ \left[\sum_{x \in \cdot} D_x^- F(\cdot) \right] (\gamma) &= D_y^+ \left[\sum_{x \in \cdot} [F(\cdot \setminus x) - F(\cdot)] \right] (\gamma) \\ &= \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)] - \sum_{x \in \gamma \cup y} [F(\gamma \cup y \setminus x) - F(\gamma \cup y)] \\ &= \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma \cup y \setminus x)] - F(\gamma) \\ &\quad - \sum_{x \in \gamma \cup y} [F(\gamma) - F(\gamma \cup y)] + F(\gamma) \\ &= \sum_{x \in \gamma} D_y^+ (\gamma \setminus x) - \sum_{x \in \gamma \cup y} D_y^+ (\gamma). \end{aligned}$$

By definition of D_x^- and D_x^+ we obtain

$$\begin{aligned} D_x^+ \left(\int r(y,\gamma) H_y(\gamma) dy \right) &= \int r(y,\gamma) H_y(\gamma) dy - \int r(y, \gamma \cup x) H_y(\gamma \cup x) dy \\ &= \int r(y,\gamma)(H_y(\gamma) - H_y(\gamma \cup x)) dy + \int r(y,\gamma) H_y(\gamma \cup x) dy \\ &\quad - \int r(y, \gamma \cup x) H_y(\gamma \cup x) dy \\ &= \int r(y,\gamma) D_x^+ H_y(\gamma) dy + \int D_x^+ r(y,\gamma) H_y(\gamma \cup x) dy. \end{aligned}$$

Analogously we get the following formula

$$
\begin{aligned}
&D_x^- \left(\int r(y,\gamma) H_y(\gamma) dy \right) \\
&= \int r(y,\gamma \setminus x) H_y(\gamma \setminus x) dy - \int r(y,\gamma) H_y(\gamma) dy \\
&= \int r(y,\gamma \setminus x) H_y(\gamma \setminus x) dy - \int r(y,\gamma) H_y(\gamma \setminus x) dy \\
&\quad + \int r(y,\gamma)(H_y(\gamma \setminus x) - H_y(\gamma)) dy \\
&= \int r(y,\gamma) D_x^- H_y(\gamma) dy + \int D_x^- r(y,\gamma) H_y(\gamma \setminus x) dy. \qquad \square
\end{aligned}
$$

We will use the lemma above only for $H_y(\gamma) = (D_y^+ F)(\gamma)$ or $(D_y^+ F)^2(\gamma)$, then all integrals are well-defined.

Now we compute step by step the summands of $2\square_2(F,F)$. The calculations in the subsequent lemmas are done μ-a.e. and for $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$. For the first summand in (4.2.4) we obtain the following statement:

Lemma 4.2.2.

$$
\begin{aligned}
&L^- \square^-(F,F)(\gamma) - 2\square^-(L^- F, F)(\gamma) \\
&= \frac{1}{2} \sum_{x\in\gamma} \sum_{y \in \gamma \setminus x} \left(D_x^- D_y^- F \right)^2 (\gamma) + \square^-(F,F)(\gamma).
\end{aligned}
$$

Proof. By definition of D_x^-

$$
\begin{aligned}
L^- \square^-(F) &= \frac{1}{2} \sum_{x\in\gamma} D_x^- \left[\sum_{y\in\cdot} (D_y^- F)^2(\cdot) \right](\gamma) \\
&= \frac{1}{2} \sum_{x\in\gamma} \left(\sum_{y\in\gamma\setminus x} (D_y^- F)^2(\gamma\setminus x) - \sum_{y\in\gamma} (D_y^- F)^2(\gamma) \right).
\end{aligned}
$$

Using (4.2.5) we get

$$
\begin{aligned}
2\square^-(F, L^- F) &= \sum_{y\in\gamma} D_y^- F(\gamma) D_y^- \left[\sum_{x\in\cdot} D_x^- F(\cdot) \right](\gamma) \\
&= \sum_{y\in\gamma} D_y^- F(\gamma) \left[\sum_{x\in\gamma\setminus y} D_y^- F(\gamma\setminus x) - \sum_{x\in\gamma} D_y^- F(\gamma) \right]
\end{aligned}
$$

$$= \sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x} D_y^- F(\gamma) D_y^- F(\gamma\setminus x) - \sum_{y\in\gamma}(D_y^- F)^2(\gamma)\right).$$

The difference of these two expressions is:

$$\begin{aligned}
&L^-\square^-(F) - 2\square^-(F, L^- F)\\
&= \sum_{x\in\gamma}\left(\frac{1}{2}\sum_{y\in\gamma\setminus x}(D_y^- F)^2(\gamma\setminus x) - \frac{1}{2}\sum_{y\in\gamma}(D_y^- F)^2(\gamma)\right)\\
&\quad - \sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x} D_y^- F(\gamma) D_y^- F(\gamma\setminus x) - \sum_{y\in\gamma}(D_y^- F)^2(\gamma)\right)\\
&= \frac{1}{2}\sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x}(D_y^- F)^2(\gamma\setminus x) - 2\sum_{y\in\gamma\setminus x} D_y^- F(\gamma) D_y^- F(\gamma\setminus x)\right.\\
&\quad \left. + \sum_{y\in\gamma}(D_y^- F)^2(\gamma)\right)\\
&= \frac{1}{2}\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}(D_y^- F(\gamma\setminus x) - D_y^- F(\gamma))^2 + \frac{1}{2}\sum_{x\in\gamma}(D_x^- F)^2(\gamma)\\
&= \frac{1}{2}\left(\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}(D_x^- D_y^- F)^2(\gamma) + \sum_{x\in\gamma}(D_x^- F)^2(\gamma)\right)
\end{aligned}$$

So, by using the definition of $\square^-$ we get the statement of the lemma. $\square$

For the second summand of $2\square_2^-$ in (4.2.4) we derive the following expression.

Lemma 4.2.3.

$$\begin{aligned}
&L^+\square^+(F,F)(\gamma) - 2\square^+(F, L^+ F)(\gamma)\\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma) r(y,\gamma)(D_x^+ D_y^+ F)^2(\gamma)\,dxdy\\
&\quad\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma) D_x^+ r(y,\cdot)(\gamma)(D_y^+ F)^2(\gamma\cup x)\,dxdy\\
&\quad\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma) D_x^+ F(\gamma) D_x^+ r(y,\cdot)(\gamma) D_y^+ F(\gamma\cup x)\,dxdy.
\end{aligned}$$

Proof. Using definitions of L^+ and $\square^+$, and Lemma 4.2.1, (4.2.7) we gain

$$
\begin{aligned}
L^+\square^+(F) &= \frac{1}{2}\int r(x,\gamma)D_x^+\left[\int r(y,\cdot)(D_y^+F)^2(\cdot)dy\right](\gamma)dx \\
&= \frac{1}{2}\left(\int r(x,\gamma)\int r(y,\gamma)D_x^+(D_y^+F)^2(\gamma)dydx\right. \\
&\quad \left.+\int r(x,\gamma)\int D_x^+r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\cup x)dydx\right). \\
2\square^+(F,L^+F) &= \int r(x,\gamma)D_x^+F(\gamma)D_x^+\left[\int r(y,\cdot)D_y^+F(\cdot)dy\right](\gamma)dx \\
&= \int r(x,\gamma)D_x^+F(\gamma)\int r(y,\gamma)D_x^+D_y^+F(\gamma)dydx \\
&\quad +\int r(x,\gamma)D_x^+F(\gamma)\int D_x^+r(y,\cdot)(\gamma)D_y^+F(\gamma\cup x)dydx.
\end{aligned}
$$

The difference of the two expressions above is

$$
\begin{aligned}
&L^+\square^+(F)-2\square^+(F,L^+F) \\
&\quad= \frac{1}{2}\left(\int r(x,\gamma)\int r(y,\gamma)D_x^+(D_y^+F)^2(\gamma)dydx\right. \\
&\qquad \left.+\int r(x,\gamma)\int D_x^+r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\cup x)dydx\right) \\
&\qquad -\left(\int r(x,\gamma)D_x^+F(\gamma)\int r(y,\gamma)D_x^+D_y^+F(\gamma)dydx\right. \\
&\qquad \left.+\int r(x,\gamma)D_x^+F(\gamma)\int D_x^+r(y,\cdot)(\gamma)D_y^+F(\gamma\cup x)dydx\right).
\end{aligned}
$$

We treat first the summands without $D_x^+r(y,\cdot)(\gamma)$, i.e. the first and the third.

$$
\begin{aligned}
&\frac{1}{2}\int r(x,\gamma)\int r(y,\gamma)D_x^+(D_y^+F)^2(\gamma)dydx \\
&\quad -\int r(x,\gamma)D_x^+F(\gamma)\int r(y,\gamma)D_x^+D_y^+F(\gamma)dydx \\
&= \int\int dxdy r(x,\gamma)r(y,\gamma)\left[\frac{1}{2}D_x^+(D_y^+F)^2(\gamma)-D_y^+F(\gamma)D_y^+D_x^+F(\gamma)\right] \\
&= \int\int dxdy r(x,\gamma)r(y,\gamma)\left[\frac{1}{2}\left((D_y^+F)^2(\gamma)-(D_y^+F)^2(\gamma\cup x)\right)\right. \\
&\qquad\qquad \left.-D_y^+F(\gamma)\left(D_y^+[F(\gamma)-F(\gamma\cup x)]\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \int\int dxdy r(x,\gamma)r(y,\gamma)\Big[-\frac{1}{2}(D_y^+F)^2(\gamma) - \frac{1}{2}(D_y^+F)^2(\gamma\cup x) \\
&\qquad\qquad + D_y^+F(\gamma)D_y^+F(\gamma\cup x)\Big] \\
&= -\frac{1}{2}\int\int r(x,\gamma)r(y,\gamma)(D_x^+D_y^+F)^2(\gamma)dydx.
\end{aligned}
$$

The latter gives the statement of the lemma. □

Finally we calculate the mixed terms of the splitting (4.2.4).

Lemma 4.2.4. For a cylinder function $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d),\Gamma)$ we have μ-a.e.

$$
\begin{aligned}
&(L^-\Box^+(F,F) - L^+\Box^-(F,F) - 2\Box^+(F,L^-F) + 2\Box^-(F,L^+F))(\gamma) \\
&\quad = \sum_{x\in\gamma}\int_{\mathbb{R}^d} r(y,\gamma)(D_x^-D_y^+F)^2(\gamma)dy + \frac{3}{2}\int_{\mathbb{R}^d} r(y,\gamma)(D_y^+F)^2(\gamma)dy \\
&\quad + \frac{1}{2}\sum_{x\in\gamma}\int_{\mathbb{R}^d} D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy \\
&\quad + \sum_{y\in\gamma} D_y^-F(\gamma)\int_{\mathbb{R}^d} D_y^- r(x,\cdot)(\gamma)D_x^+F(\gamma\setminus y)dx.
\end{aligned}
$$

Proof. Again, using (4.2.8) and definitions of D_x^- and D_x^+ we get

$$
\begin{aligned}
L^-\Box^+(F) &= \frac{1}{2}\sum_{x\in\gamma} D_x^-\left[\int r(y,\cdot)(D_y^+F)^2(\cdot)dy\right](\gamma) \\
&= \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)D_x^-(D_y^+F)^2(\gamma)dy \\
&\quad + \frac{1}{2}\sum_{x\in\gamma}\int D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy \\
&= \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)[(D_y^+F)^2(\gamma\setminus x) - (D_y^+F)^2(\gamma)]dy \\
&\quad + \frac{1}{2}\sum_{x\in\gamma}\int D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy.
\end{aligned}
$$

Using (4.2.6) we obtain

$$
\begin{aligned}
-2\Box^+(F,L^-F) &= -\int r(y,\gamma)D_y^+F(\gamma)D_y^+\left[\sum_{x\in\cdot} D_x^-F(\cdot)\right](\gamma)dy \\
&= -\int r(y,\gamma)D_y^+F(\gamma)\left[\sum_{x\in\gamma} D_y^+F(\gamma\setminus x) - \sum_{x\in\gamma\cup y} D_y^+F(\gamma)\right]dy.
\end{aligned}
$$

Calculating the difference of these two terms and reordering the summands

$$\begin{aligned}
&L^-\square^+(F) - 2\square^+(F, L^-F)\\
&\quad= \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)[(D_y^+F)^2(\gamma\setminus x) - (D_y^+F)^2(\gamma)]dy\\
&\qquad+ \frac{1}{2}\sum_{x\in\gamma}\int D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy\\
&\qquad- \int r(y,\gamma)D_y^+F(\gamma)\left[\sum_{x\in\gamma} D_y^+F(\gamma\setminus x) - \sum_{x\in\gamma\cup y} D_y^+F(\gamma)\right]dy\\
&\quad= \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)(D_y^+F)^2(\gamma\setminus x)dy + \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)(D_y^+F)^2(\gamma)dy\\
&\qquad+ \int r(y,\gamma)(D_y^+F)^2(\gamma)dy - \sum_{x\in\gamma}\int r(y,\gamma)D_y^+F(\gamma)D_y^+F(\gamma\setminus x)\\
&\qquad+ \frac{1}{2}\sum_{x\in\gamma}\int D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
&L^-\square^+(F) - 2\square^+(F, L^-F)\\
&\quad= \frac{1}{2}\sum_{x\in\gamma}\int r(y,\gamma)(D_x^-D_y^+F)^2(\gamma)dy + \int r(y,\gamma)(D_y^+F)^2(\gamma)dy\\
&\qquad+ \frac{1}{2}\sum_{x\in\gamma}\int D_x^- r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma\setminus x)dy.
\end{aligned}$$

Analogously, using (4.2.6)

$$\begin{aligned}
-L^+\square^-(F) &= -\frac{1}{2}\int r(x,\gamma)D_x^+\left[\sum_{y\in\cdot}(D_y^-F)^2(\cdot)\right](\gamma)dx\\
&= -\frac{1}{2}\int r(x,\gamma)\left[\sum_{y\in\gamma}(D_y^-F)^2(\gamma) - \sum_{y\in\gamma\cup x}(D_y^-F)^2(\gamma\cup x)\right]dx.
\end{aligned}$$

Using (4.2.8) and the definition of D_x^-

$$2\square^-(F, L^+F) = \sum_{y\in\gamma} D_y^-F(\gamma)D_y^-\left[\int r(x,\cdot)D_x^+F(\cdot)dx\right](\gamma)$$

$$
\begin{aligned}
&= \sum_{y\in\gamma} D_y^- F(\gamma) \int r(x,\gamma) D_y^- D_x^+ F(\gamma) dx \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int D_y^- r(x,\cdot)(\gamma) D_x^+ F(\gamma\setminus y) dx \\
&= \sum_{y\in\gamma} D_y^- F(\gamma) \int r(x,\gamma) D_y^- (F(\gamma) - F(\gamma\cup x)) dx \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int D_y^- r(x,\cdot)(\gamma) D_x^+ F(\gamma\setminus y) dx.
\end{aligned}
$$

The difference of these two expressions is

$$
\begin{aligned}
&- L^+\Box^-(F) + 2\Box^-(F, L^+F) \\
&\quad = -\frac{1}{2}\int r(x,\gamma)\left[\sum_{y\in\gamma}(D_y^- F)^2(\gamma) - \sum_{y\in\gamma\cup x}(D_y^- F)^2(\gamma\cup x)\right] dx \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int r(x,\gamma) D_y^- (F(\gamma) - F(\gamma\cup x)) dx \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int D_y^- r(x,\cdot)(\gamma) D_x^+ F(\gamma\setminus y) dx \\
&\quad = \frac{1}{2}\sum_{y\in\gamma}\int r(x,\gamma)(D_y^- F)^2(\gamma) dx + \frac{1}{2}\sum_{y\in\gamma\cup x}\int r(x,\gamma)(D_y^- F)^2(\gamma\cup x) dx \\
&\quad - \sum_{y\in\gamma}\int r(x,\gamma) D_y^- F(\gamma) D_y^- F(\gamma\cup x) dx \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int D_y^- r(x,\cdot)(\gamma) D_x^+ F(\gamma\setminus y) dx,
\end{aligned}
$$

which is the same as

$$
\begin{aligned}
-L^+\Box^-(F) + 2\Box^-(F, L^+F) &= \frac{1}{2}\sum_{y\in\gamma}\int r(x,\gamma)(D_x^+ D_y^- F)^2(\gamma) dx \\
&\quad + \frac{1}{2}\int r(y,\gamma)(D_y^- F)^2(\gamma\cup y) dy \\
&\quad + \sum_{y\in\gamma} D_y^- F(\gamma) \int D_y^- r(x,\cdot)(\gamma) D_x^+ F(\gamma\setminus y) dx.
\end{aligned}
$$

Exchanging x and y and using $D_y^- F(\gamma\cup y) = D_y^+ F(\gamma)$ for $y\in\gamma$, and the fact that $D_x^- D_y^+ F(\gamma) = D_y^+ D_x^- F(\gamma)$ for $x\in\gamma$, $y\notin\gamma$ we obtain the statement of the lemma. $\Box$

Adding the three parts of the splitting according to (4.2.4) we gain the following expression for $\Box_2$

$$\begin{aligned}
\Box_2(F,F)(\gamma) &= \frac{1}{2}\Box(F,F)(\gamma) + \Box^+(F,F)(\gamma) \\
&+ \frac{1}{4}\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\left(D_x^- D_y^- F\right)^2(\gamma) + \frac{1}{2}\sum_{y\in\gamma}\int_{\mathbb{R}^d} r(x,\gamma)\left(D_x^+ D_y^- F\right)^2(\gamma)dx \qquad (4.2.9)\\
&+ \frac{1}{4}\int_{\mathbb{R}^d}\sum_{x\in\gamma} D_x^- r(y,\cdot)(\gamma)\left[(D_y^+F)^2(\gamma\setminus x) + 2D_y^+F(\gamma\setminus x)D_x^-F(\gamma)\right]dy \\
&+ \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma)r(y,\gamma)(D_x^+D_y^+F)^2(\gamma)dxdy \qquad (4.2.10)\\
&+ \frac{1}{4}\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)\left[-(D_y^+F)^2(\gamma\cup x) + 2D_y^+F(\gamma\cup x)D_x^+F(\gamma)\right]dydx
\end{aligned}$$

This representation for $\Box_2$ will turn out to be not very convenient. There are three terms of fourth order in the difference operator, precisely lines (4.2.9) and (4.2.10). One should expect that in a natural representation all these three terms have the same integral w.r.t. the reversible measure μ, which is not the case for the summand (4.2.10). We rearrange the summands in order to obtain this property.

Theorem 4.2.5. For all $F, G \in \mathcal{F}C_\mathrm{b}(C_0(\mathbb{R}^d),\Gamma)$ it holds μ-a.e that

$$\begin{aligned}
\Box_2(F,F)(\gamma) &= \frac{1}{2}\Box(F,F)(\gamma) + \Box^+(F,F)(\gamma) \\
&+ \frac{1}{4}\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\left(D_x^- D_y^- F\right)^2(\gamma) + \frac{1}{2}\sum_{y\in\gamma}\int_{\mathbb{R}^d} r(x,\gamma)\left(D_x^+ D_y^- F\right)^2(\gamma)dx \qquad (4.2.11)\\
&+ \frac{1}{4}\int_{\mathbb{R}^d}\sum_{x\in\gamma} D_x^- r(y,\cdot)(\gamma)\left[(D_y^+F)^2(\gamma\setminus x) + 2D_y^+F(\gamma\setminus x)D_x^-F(\gamma)\right]dy \qquad (4.2.12)\\
&+ \frac{1}{4}\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} r(y,\gamma\cup x)(D_x^+D_y^+F)^2(\gamma)dydx \qquad (4.2.13)\\
&+ \frac{1}{4}\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)\left[-(D_y^+F)^2(\gamma) + 2D_y^+F(\gamma)D_x^+F(\gamma)\right]dydx. \qquad (4.2.14)
\end{aligned}$$

Proof. Using just the definition of D_x^+ the last two summands of $\Box_2$ can be rewritten as follows

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} r(y,\gamma\cup x)(D_x^+D_y^+F)^2(\gamma)dydx \\
&\quad+ \frac{1}{4}\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)\left[(D_x^+D_y^+F)^2(\gamma) - (D_y^+F)^2(\gamma\cup x)\right.\\
&\qquad\qquad\left. + 2D_y^+F(\gamma\cup x)D_x^+F(\gamma)\right]dydx \qquad (4.2.15)
\end{aligned}$$

It remains to simplify the last bracket. Expanding the first summand of the bracket

$$
\begin{aligned}
(D_x^+ D_y^+ F)^2(\gamma) &= (D_y^+ F)^2(\gamma \cup x) - 2(D_y^+ F)(\gamma)(D_y^+ F)(\gamma \cup x) + (D_y^+ F)^2(\gamma) \\
&= ((D_y^+ F)^2(\gamma \cup x) - (D_y^+ F)^2(\gamma)) + 2(D_y^+ F(\gamma) - D_y^+ F(\gamma \cup x)) D_y^+ F(\gamma)
\end{aligned}
$$

and using

$$
2 D_x^+ D_y^+ F(\gamma) D_y^+ F(\gamma) = 2(D_y^+ F(\gamma) - D_y^+ F(\gamma \cup x)) D_y^+ F(\gamma)
$$

we obtain

$$
(D_x^+ D_y^+ F)^2(\gamma) = ((D_y^+ F)^2(\gamma \cup x) - (D_y^+ F)^2(\gamma)) + 2 D_x^+ D_y^+ F(\gamma) D_y^+ F(\gamma). \tag{4.2.16}
$$

Inserting (4.2.16) in (4.2.15) and using $D_x^+ D_y^+ F(\gamma) = D_y^+ D_x^+ F(\gamma)$ for $x, y \notin \gamma$ we obtain

$$
\begin{aligned}
&(D_x^+ D_y^+ F)^2(\gamma) - (D_y^+ F)^2(\gamma \cup x) + 2 D_y^+ F(\gamma \cup x) D_x^+ F(\gamma) \\
&\quad = -(D_y^+ F)^2(\gamma) + 2 D_y^+ D_x^+ F(\gamma) D_y^+ F(\gamma) + 2 D_y^+ F(\gamma \cup x) D_x^+ F(\gamma) \\
&\quad = -(D_y^+ F)^2(\gamma) + 2 D_x^+ F(\gamma) D_y^+ F(\gamma) - 2 D_x^+ F(\gamma \cup y) D_y^+ F(\gamma) \\
&\qquad\qquad + 2 D_y^+ F(\gamma \cup x) D_x^+ F(\gamma).
\end{aligned}
$$

According to Lemma 4.2.6 the last two terms cancel each other. Thus (4.2.15) can be simplified to

$$
\int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) [-(D_y^+ F)^2(\gamma) + 2 D_x^+ F(\gamma) D_y^+ F(\gamma)] dy dx,
$$

what yields the result. □

Lemma 4.2.6. For μ-a.a. $\gamma \in \Gamma$ holds that

$$
r(x, \gamma) D_x^+ r(y, \cdot)(\gamma) dx dy = r(y, \gamma) D_y^+ r(x, \cdot)(\gamma) dy dx
$$

Proof. As the above equality has to be interpreted μ-a.s. it is sufficient to show that the expression below is symmetric under the interchange of x and y, what we check using (2.2.1):

$$
\begin{aligned}
&\int_\Gamma \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} D_x^+ r(y, \cdot)(\gamma) H(\gamma \cup x \cup y, x, y) dy dx \mu(d\gamma) \\
&\quad = \int_\Gamma \int_{\mathbb{R}^d} r(x, \gamma) \int_{\mathbb{R}^d} r(y, \gamma) H(\gamma \cup x \cup y, x, y) dy dx \mu(d\gamma) \\
&\qquad - \int_\Gamma \sum_{\substack{x, y \in \gamma \\ x \neq y}} H(\gamma, x, y) \mu(d\gamma).
\end{aligned}
$$

□

4.2.2 Coercivity identity

In this section we consider the integrals of $\square$ and $\square_2$ w.r.t. μ. Remember that L is symmetric w.r.t. μ. Denote $H := -L$. We will use the representation of $\square_2$ given by Theorem 4.2.5. The calculations below are based on the frequent use of the identity $D_x^+ F(\gamma \setminus x) = D_x^- F(\gamma)$ (for $x \in \gamma$) and the repeated application of the definition of the Papangelou intensity, cf. (2.2.1). First, one notes that for $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$

$$\frac{1}{2}\int_\Gamma \square(F,F)(\gamma)\mu(d\gamma) = \int_\Gamma \square^{\pm}(F,F)(\gamma)\mu(d\gamma) \tag{4.2.17}$$

$$= \frac{1}{2}\int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma)(D_x^+ F)^2(\gamma)dx\mu(d\gamma). \tag{4.2.18}$$

Thus we get the following representation for the Dirichlet form

$$\mathcal{E}(F,F) = \int_\Gamma F(\gamma)HF(\gamma)\mu(d\gamma) = \int_\Gamma \square(F,F)(\gamma)\mu(d\gamma). \tag{4.2.19}$$

Next we calculate the integral of the “carré du champ itéré” $\square_2$ w.r.t μ. We see that the integrals of all three forth order terms (given in lines (4.2.11) and (4.2.13)), coincide with

$$\int_\Gamma \sum_{x\in\gamma} \sum_{y\in\gamma\setminus x} \left(D_x^- D_y^- F\right)^2(\gamma)\mu(d\gamma).$$

Calculating the integral of (4.2.12)

$$\begin{aligned}
&\int_\Gamma \int_{\mathbb{R}^d} \sum_{x\in\gamma} D_x^- r(y,\cdot)(\gamma)\left[(D_y^+ F)^2(\gamma\setminus x) + 2D_y^+ F(\gamma\setminus x)D_x^- F(\gamma)\right] dy\mu(d\gamma)\\
&= \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^- r(y,\cdot)(\gamma\cup x)\left[(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma)D_x^- F(\gamma\cup x)\right] dydx\mu(d\gamma)\\
&= \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)\left[(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma)D_x^+ F(\gamma)\right] dydx\mu(d\gamma)
\end{aligned}$$

and comparing it with the integral of (4.2.14), one can find some cancellations. Summarizing, one obtains the coercivity identity

Theorem 4.2.7. For all $F \in \mathcal{F}C_{\mathrm{b}}(C_0(\mathbb{R}^d), \Gamma)$ holds

$$\begin{aligned}
\int_\Gamma (HF)^2(\gamma)\mu(d\gamma) &= \int_\Gamma \square_2(F,F)(\gamma)\mu(d\gamma) \qquad (4.2.20)\\
&= \int_\Gamma \square(F,F)(\gamma)\mu(d\gamma) + \int_\Gamma \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x}\left(D_x^- D_y^- F\right)^2(\gamma)\mu(d\gamma)\\
&\quad + \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)D_y^+ F(\gamma)D_x^+ F(\gamma)dydx\mu(d\gamma).
\end{aligned}$$

Proof. The first equality can be easily checked by symmetricity of L w.r.t μ and $(L,1)_{L^2(\Gamma,\mu)} = 0$, where 1 denotes the function constantly equal to 1; the integral of $\square_2$ w.r.t μ was calculated above. $\square$

4.3 Sufficient condition for the spectral gap

Most commonly in the context of spectral gap the Poincaré inequality

$$c\int\left(f-\int f d\mu\right)^2 d\mu \leq \mathcal{E}(f,f)$$

for operator H is used. We use the so-called coercivity inequality to investigate the spectral properties of H. We say that the coercivity inequality holds for a positive essentially self-adjoint operator H with constant c if

$$\int_\Gamma (HF)^2(\gamma)\mu(d\gamma) \geq c\mathcal{E}(F,F), \quad c>0. \tag{4.3.1}$$

If this inequality is fulfilled then the interval $(0,c)$ does not belong to the spectrum of H. Note that the Poincaré inequality is slightly stronger and means that, in addition to the fact that $(0,c)$ does not belong to the spectrum of H, it also implies that the kernel of H consists only of constants. Using the results of the previous subsection we can express the coercivity inequality in terms of the "carré du champ" and $\square_2$ as

$$\int_\Gamma \square_2(F,F)(\gamma)\mu(d\gamma) \geq c\int_\Gamma \square(F,F)(\gamma)\mu(d\gamma). \tag{4.3.2}$$

Inserting in (4.3.2) the representations derived in the previous sections we obtain the following inequality sufficient for (4.3.1)

$$\begin{aligned}(1-c)&\int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)(D_x^+F)^2(\gamma)dx\mu(d\gamma) \\ +\quad&\int_\Gamma\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x,\gamma)D_x^+ r(y,\cdot)(\gamma)D_y^+F(\gamma)D_x^+F(\gamma)dydx\mu(d\gamma) \geq 0.\end{aligned} \tag{4.3.3}$$

For fixed γ and introduce following notations

$$K_\gamma(x,y) = r(x,\gamma)(r(y,\gamma)-r(y,\gamma\cup x)), \quad \psi_\gamma(x) = D_x^+F(\gamma).$$

We write formally, for the convenience of representation

$$\begin{aligned}&\int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)(D_x^+F)^2(\gamma)dx\mu(d\gamma) \\ &\quad= \int_\Gamma\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}(D_x^+F)(\gamma)(D_y^+F)(\gamma)\delta(x-y)dxdy\mu(d\gamma).\end{aligned}$$

Hence it is sufficient for the inequality (4.3.1) that the following holds

$$\int_\Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (K_\gamma(x,y) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y))\psi_\gamma(y)\psi_\gamma(x)dxdy\mu(d\gamma) \geq 0. \tag{4.3.4}$$

In order to formulate the final theorem we need to introduce the following definition

Definition 4.3.1. A locally integrable function $B : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ is called a positive definite kernel if for all $\psi \in C_0(\mathbb{R}^d)$ holds

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} B(x,y)\psi(x)\overline{\psi(y)}dxdy \geq 0. \tag{4.3.5}$$

Theorem 4.3.2. If for each fixed $\gamma \in \Gamma$ the kernel

$$r(x,\gamma)(r(y,\gamma) - r(y,\gamma \cup x)) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y) \tag{4.3.6}$$

is positive definite then the coercivity inequality (4.3.1) holds for H with constant c.

4.4 Sufficient condition for the spectral gap for Gibbs measures

In this section we investigate the consequences we can draw from condition (4.3.6) in the case when μ is a Gibbs measure for a translation invariant pair potential ϕ and activity z. The aim is to derive sufficient conditions in terms of potential ϕ. We remind that in this case $r(x,\gamma) = z\exp\left[-E(x,\gamma)\right]$.

Corollary 4.4.1. Let μ be a Gibbs measure for a translation invariant pair potential ϕ and activity z. If for each fixed $\gamma \in \Gamma$ the kernel

$$e^{-E(x,\gamma)}e^{-E(y,\gamma)}z(1-e^{-\phi(x-y)}) + (1-c)e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}\delta(x-y) \tag{4.4.1}$$

is positive definite then the coercivity inequality (4.3.1) holds for H with constant c.

Applying Theorem 4.3.2 to $e^{-\frac{1}{2}E(x,\gamma)}\psi_\gamma(x)$ instead of the function ψ_γ gives

Corollary 4.4.2. Let μ be a Gibbs measure for a translation invariant pair potential ϕ and activity z. If for each fixed $\gamma \in \Gamma$ the kernel

$$e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}z(1-e^{-\phi(x-y)}) + (1-c)\delta(x-y) \tag{4.4.2}$$

is positive definite then the coercivity inequality (4.3.1) holds for H with constant c.

4.4.1 Spectral gap for a certain class of potentials

For the Poisson point process, i.e. the Gibbs measure for $\phi = 0$, one has spectral gap $c = 1$, which follows also immediately from condition (4.4.1). In order to check the condition (4.4.2) for $c = 1$ it is sufficient to prove non-negativity of the expression

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}(1 - e^{-\phi(x-y)})\psi(y)\psi(x)dxdy. \tag{4.4.3}$$

Hence considering $e^{-\frac{1}{2}E(x,\gamma)}\psi(x)$ instead of ψ one is led to the sufficient condition

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1 - e^{-\phi(x-y)})\psi(y)\psi(x)dxdy \geq 0 \tag{4.4.4}$$

for all $\psi \in C_0(\mathbb{R}^d)$. We remind the definition

Definition 4.4.3. A measurable function $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called positive definite if for all $\psi \in C_0(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x-y)\psi(x)\overline{\psi(y)}dxdy \geq 0.$$

So the condition (4.4.4) means that $x \mapsto 1-e^{-\phi(x)}$ is a positive definite function.

Remark 4.4.4. Note that the condition (4.4.4) does not contain the activity z, it is just a condition on potential.

When we speak about regular, stable or superstable functions we have in mind these properties in sense of pair potentials, cf. Section 2.2.2.

Theorem 4.4.5. Let f be a continuous positive definite function such that $f(0) \leq 1$, which is regular. Define

$$\phi := -\ln(1 - f). \tag{4.4.5}$$

Then ϕ fulfills (4.4.4), is superstable and regular. For any tempered Gibbs measure μ, corresponding to a pair potential ϕ and for all activities $z > 0$ the operator $H = -L$, where L is the associated generator of the Glauber dynamics, fulfills the coercivity inequality for $c = 1$.

Proof. Due to positive definiteness $|f(x)| \leq f(0) \leq 1$. Defining for $x \in [-1, 1]$ the function $h(x) := -\ln(1 - x)$ one can write $\phi = h \circ f$. By assumption on f there exists an $\tilde{R} > 0$ and a positive decreasing function φ on $[0, +\infty)$, which fulfills (2.2.8) and such that $|f(x)| \leq \varphi(|x|)$ for all $|x| \geq \tilde{R}$, $x \in \mathbb{R}^d$. Note that for $x \in [-1, 1/2]$ we have $|h(x)| \leq 2|x|$. Choose R bigger than $\tilde{R}$ and so large that $\varphi(R) \leq 1/2$. Then for all $|x| \geq R$, $x \in \mathbb{R}^d$ we have $|f(x)| \leq 1/2$ and hence

$$|\phi(x)| \leq 2|f(x)| \leq 2\varphi(|x|),$$

which implies that ϕ is regular.

Next, we show that ϕ is superstable. One easily sees that

$$h(x) \geq x + (-\ln(1-x) - x)\mathbb{1}_{[0,1]}(x).$$

Shorthanding $g(x) := -\ln(1-x) - x$ one gets $\phi(x) \geq f(x) + g(f(x))\mathbb{1}_{[c,1]}(f(x))$, where $0 < c < f(0)$. Hence, ϕ is bigger then the sum of a positive definite function and a function, which is nonnegative, at 0 strictly bigger then 0, and continuous at 0. Therefore due to Proposition 1.2 in [Rue70] ϕ is a superstable potential. $\square$

In the rest of the subsection we investigate which properties a potential necessarily has which fulfills condition (4.4.4). First of all we recall the following definition

Definition 4.4.6. A generalized function (distribution) $u \in \mathcal{D}(\mathbb{R}^d)$ is called positive definite if for all $\varphi \in \mathbb{C}_0(\mathbb{R}^d)$

$$\langle u, \tilde{\varphi} * \varphi \rangle \geq 0 \tag{4.4.6}$$

holds, where $*$ denotes the convolution and $\tilde{f}(x) := \overline{f(-x)}$.

Proposition 4.4.7. Let ϕ be a potential which fulfills condition (4.4.4) and is stable, regular, and continuous. Then it is of the form (4.4.5) and hence also superstable. Furthermore, ϕ is integrable at 0, itself positive definite in the sense of generalized functions, and

$$\lim_{x \to 0} \frac{\phi(x)}{-2\ln(x)} \leq 1. \tag{4.4.7}$$

Proof. (4.4.4) implies that the function $f := 1 - e^{-\phi}$ is positive definite. As ϕ is stable it is non-negative at 0 and hence $f(x) \leq 1$. Due to the positive definiteness of f one has $|f(x)| \leq f(0) \leq 1$. The representation (4.4.5) is obtained by inverting the definition of f. So the function f is continuous and positive definite. To apply Theorem 4.4.5 we check the regularity of f. Defining $g(x) := 1 - e^{-x}$ one can write $f = g \circ \phi$. By assumption on ϕ there exists an $\tilde{R} > 0$ and a positive decreasing function φ on $[0, +\infty)$ which fulfills (2.2.8) and such that $|\phi(x)| \leq \varphi(|x|)$ for all $|x| \geq \tilde{R}$, $x \in \mathbb{R}^d$. Choose R bigger than $\tilde{R}$ and so large that $\varphi(R) \leq \ln 2$. Note that for x such that $|x| \leq \ln 2$ we have $|g(x)| \leq 2|x|$. Then for all $|x| \geq R$, $x \in \mathbb{R}^d$ we have $|\phi(x)| \leq \ln 2$ and hence

$$|f(x)| \leq 2|\phi(x)| \leq 2\varphi(|x|),$$

which implies that f is regular. Applying Theorem 4.4.5 we get that ϕ is also superstable.

Denote the limit in (4.4.7) by c. Assume that $c > 1$, then for an $\varepsilon > 0$ such that $c - \varepsilon > 1$ there exists an $R > 0$ such that for $x \in \mathbb{R}^d, |x| < R$ we have $\phi(x) \geq -(c-\varepsilon)2\ln(x)$. As

$$0 \leq \frac{1-f(x)}{x^2} = e^{-\phi(x)-2\ln(x)} \leq e^{((c-\varepsilon)-1)2\ln(x)}$$

we get that $\lim\limits_{x\to 0} \frac{1-f(x)}{x^2} = 0$. According to cf. [Jac01, Proposition 3.5.21] this yields that f is constant.

In particular, from (4.4.7) follows that ϕ is integrable at 0.

Writing again $\phi = h \circ f$, where $h(x) = -\ln(1-x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ with radius of convergence 1. Approximate ϕ by the functions $\phi_\delta(x) := h \circ ((1-\delta)f(x))$ for $0 < \delta < 1$. Since $|(1-\delta)f(x)| < 1$ and h has a Taylor series with non-negative coefficients, for all $0 < \delta < 1$ the function ϕ_δ is positive definite, cf. e.g. [Jac01, Proposition 3.5.17]. As h is monotone increasing $|\phi_\delta| \leq |\phi|$ and the latter function is integrable. Hence ϕ_δ is also positive definite in the sense of generalized functions. ϕ_δ converge pointwisely to ϕ for $\delta \to 0$, and are uniformly bounded by ϕ. Therefore by Lebesgue's dominated convergence ϕ is also positive definite in the sense of generalized functions. □

4.4.2 Parameter dependence

Motivated by statistical mechanics we introduce two parameters: the inverse temperature $\beta > 0$ and the activity $z > 0$. One says that μ is a Gibbs measure for ϕ, β, z if the corresponding Papangelou intensity is $r(x,\gamma) = ze^{-\beta E(x,\gamma)}$. Instead of condition (4.4.4) one has to consider the following sufficient condition

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 - e^{-\beta\phi(x-y)})\psi(y)\psi(x)dxdy \geq 0 \tag{4.4.8}$$

for all $\psi \in C_0(\mathbb{R}^d)$. Note that this condition is independent of the activity z. If (4.4.8) is fulfilled for all $\psi \in C_0(\mathbb{R}^d)$ then we say that ϕ fulfills condition (4.4.8) for β. In this case the generator of the Glauber dynamics corresponding to the measure $\mu = \mu(\phi, \beta, z)$ has spectral gap at least 1.

Proposition 4.4.8. Let ϕ be a potential which fulfills condition (4.4.8) for a $\bar{\beta} > 0$ and is stable, regular, and lower semi-continuous at zero. Then ϕ fulfills condition (4.4.8) for all β such that $0 < \beta \leq \bar{\beta}$.

Proof. Denote by $f = 1 - e^{-\bar{\beta}\phi}$ the positive definite function given by condition (4.4.8). Then on the one hand, $f_\beta(x) := 1 - e^{-\beta\phi(x)}$ is also continuous and regular. On the other hand, $f_\beta(x) = 1 - (1 - f(x))^{\beta/\bar{\beta}}$ and therefore can be written in the following way as a power series expansion

$$f_\beta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \cdot \frac{\beta}{\bar{\beta}} \left(\frac{\beta}{\bar{\beta}} - 1 \right) \dots \left(\frac{\beta}{\bar{\beta}} - n + 1 \right) (f(x))^n$$

with radius of convergence 1. All the coefficients of the series are nonnegative, if $\beta/\bar{\beta} \leq 1$. Proceeding as in Proposition 4.4.7 one proves that f_β is the pointwise limit of positive definite functions. As f_β is bounded it is itself positive definite in the sense of functions. $\square$

Corollary 4.4.9. Let $f : \mathbb{R} \to [0, 1]$ be a two times differentiable even function, which is decreasing and convex on $\mathbb{R}_+$. Denote $\phi(x) = -\ln(1 - f(x))$. Then the function $f_\beta = 1 - e^{-\beta\phi(x)}$ is also positive definite for all β such that $0 \leq \beta \leq 1$.

Proof. Obviously $f_\beta \geq 0$. Using the representation $f_\beta(x) = 1 - (1 - f(x))^\beta$ we obtain

$$\frac{d}{dx} f_\beta(x) = \beta(1 - f(x))^{\beta-1} f'(x) \leq 0,$$

$$\frac{d^2}{dx^2} f_\beta(x) = -\beta(\beta - 1)(1 - f(x))^{\beta-2}(f'(x))^2 + \beta(1 - f(x))^{\beta-1} f''(x) \geq 0.$$

By Polya's theorem f_β is positive definite. $\square$

4.4.3 Examples

For concreteness we collect some examples of potentials which fulfill the condition of Theorem 4.4.5. Especially interesting is that among them there are examples of potentials, which take also negative values. All these examples are constructed by choosing a regular positive definite function f and expressing $\phi(x) = -\ln(1 - f(x))$.

$\phi(x)$	$f(x)$	Parameters
$-\ln(1-e^{-tx^2}\cos(ax))$	$e^{-tx^2}\cos(ax)$	$t>0, a\in\mathbb{R}$
$-\ln(1-e^{-t\lvert x\rvert}\cos(ax))$	$e^{-t\lvert x\rvert}\cos(ax)$	$t>0, a\in\mathbb{R}$
$-\ln\left(1-\dfrac{\cos(ax)}{1+\sigma^2x^2}\right)$	$\dfrac{1}{1+\sigma^2x^2}\cos(ax)$	$\sigma>0,\ a\in\mathbb{R}$
$-\ln\big(1-(1-\frac{\lvert x\rvert}{a})\mathbb{1}_{[-a,a]}(x)\cos(bx)\big)$	$(1-\frac{\lvert x\rvert}{a})\mathbb{1}_{[-a,a]}(x)\cos(bx)$	$a>0, b\in\mathbb{R}$

In all examples above one can exchange $\cos(ax)$ by $\dfrac{\sin(ax)}{ax}$.

Regularity of $f(x)=e^{-tx^2}\cos(ax)$ follows immediately from $|e^{-tx^2}\cos(ax)|\le e^{-tx^2}$, which is integrable. Analogously one checks the regularity of all the other functions from the table above.

Figure 4.1: A sample of a potential which takes negative values

In the d-dimensional case we give the following examples:

$\phi(x)$	$f(x)$	Parameters
$-\ln(1-e^{-t\lvert x\rvert^2}\cos(a\cdot x))$	$e^{-t\lvert x\rvert^2}\cos(a\cdot x)$	$x\in\mathbb{R}^d, t>0, a\in\mathbb{R}^d$
$-\ln\left(1-e^{-t\lvert x\rvert^2}\prod_{j=1}^{d}\frac{\sin(a_jx_j)}{a_jx_j}\right)$	$e^{-t\lvert x\rvert^2}\prod_{j=1}^{d}\frac{\sin(a_jx_j)}{a_jx_j}$	$x\in\mathbb{R}^d, t>0$
$-\ln\left(1-\left(\frac{r}{\lvert x\rvert}\right)^{n/2}\cdot J_{n/2}(r\lvert x\rvert)\right)$	$\left(\frac{r}{\lvert x\rvert}\right)^{n/2}\cdot J_{n/2}(r\lvert x\rvert)$	$r\geq 0,\ n>2d-1$
$-\ln\left(1-\frac{2^{n/2}t\Gamma(\frac{n+1}{2})}{\sqrt{\pi}(\lvert x\rvert^2+t^2)^{\frac{n+1}{2}}}\right)$	$\frac{2^{n/2}t\,\Gamma(\frac{n+1}{2})}{\sqrt{\pi}(\lvert x\rvert^2+t^2)^{\frac{n+1}{2}}}$	$t>0,\ n>d-1$

where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$.

Regularity of the examples in the first two lines is checked analogously to the one-dimensional case.

Consider the third example $f(x)=\left(\frac{r}{|x|}\right)^{n/2}\cdot J_{n/2}(r|x|)$. For $x\gg|\nu^2-\frac{1}{4}|$ we have the following asymptotic expansion (see cf. [Bat53])

$$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(\sum_{m=0}^{M-1}(-1)^m(\nu,2m)(2x)^{-2m}+O(|x|^{-2M})\right)\right.$$
$$\left.-\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(\sum_{m=0}^{M}(-1)^m(\nu,2m+1)(2x)^{-2m-1}+O(|x|^{-2M-1})\right)\right]$$

where (ν,m) is Hankel's symbol defined by $(\nu,m):=\dfrac{\Gamma(1/2+\nu+m)}{m!\Gamma(1/2+\nu-m)}$.

Then $\left|\left(\frac{r}{|x|}\right)^{n/2}\cdot J_{n/2}(r|x|)\right|\leq\varphi(x)$, where

$$\varphi(x)=\sqrt{\frac{2}{\pi}}\cdot\frac{r^{n/2}}{x^{\frac{n+1}{2}}}\left[\sum_{m=0}^{M-1}(-1)^m(n/2,2m)(2r|x|)^{-2m}+O(|x|^{-2M})\right.$$
$$\left.+\sum_{m=0}^{M}(-1)^m(n/2,2m+1)(2r|x|)^{-2m-1}+O(|x|^{-2M-1})\right].$$

The condition (2.2.8) on φ is fulfilled if

$$\int_R^\infty t^{d-1}t^{-\frac{n+1}{2}}<\infty,$$

which is equivalent to $n > 2d - 1$.

Consider the last example

$$f(x) = \frac{2^{n/2} t\, \Gamma(\frac{n+1}{2})}{\sqrt{\pi}(|x|^2 + t^2)^{\frac{n+1}{2}}}.$$

For large x we have $f(x) \sim x^{-(n+1)}$, therefore condition (2.2.8) on φ is fulfilled if the integral $\displaystyle\int_R^\infty t^{d-1} t^{-(n+1)}$ is finite, which is equivalent to $n > d - 1$.

5 Spatial Markov Processes in Mutation-Selection Models

5.1 The model

Recall first some genetical concepts and notions, see e.g. [Bür00]. A gene represents a (contiguous) region of DNA coding. It may have different forms, called alleles. Thus an allele is one of the variant forms of a gene that occupies a given locus (position) on a chromosome, i.e. alleles are DNA sequences that code a gene. An individual's genotype for a certain gene is the collection of alleles it consists of. A change of genetic material is called a mutation, and the affected allele is called mutant allele. We call the "null genotype" the one which has wild-type alleles at every locus and carries none of mutant alleles. So a wild-type allele is an allele which is considered to be "normal" for the organism in question, as opposed to a mutant allele which appears due to mutation. In this chapter we will use the word "genotype" in a sense which somewhat differs from the mentioned above: a genotype represents a set of mutant alleles that an individual may carry. So in contrast to the usual definition we are interested only in the set of mutant alleles, but not in the whole information about all alleles.

In this section we describe a model introduced by [SEK05], which describes the aging of a population. Let X be a Polish space, interpreted as the space of **loci** (i.e. positions of possible mutations). Denote the Borel $\sigma-$algebra on X by $\mathcal{B}(X)$, and fix a Borel σ-finite measure σ on $(X, \mathcal{B}(X))$ – interpreted as **mutation rate**. For simplicity, we assume that at each locus at most one mutation may occur. A locally finite configuration of points in X (defined as usual) is interpreted as a **genotype**. Then $\gamma = \emptyset$ plays the role of the null genotype (wild-type genotype). The set of all genotypes γ is thus the configuration space $\Gamma := \Gamma(X)$. We assume that genotypes are influenced by a **selection cost** Φ, which is a continuous function $\Phi : \Gamma \longrightarrow \mathbb{R}$, e.g. $\Phi(\emptyset) = 0,\ \Phi(\gamma) > 0$, for $\gamma \neq \emptyset$.

The emergence of mutant alleles is described by a stochastic process, the **state of the population** of genotypes at each fixed moment of time t is described by a probability measure μ_t on Γ. The time development of the population is modelled by a Kimura-Maruyama type equation

$$\frac{d}{dt}\mu_t(F) = \mu_t\left(\int_X (F(\cdot \cup x) - F(\cdot))d\sigma(x)\right) - \mu_t(F \cdot \Phi) + \mu_t(F)\mu_t(\Phi), \quad (5.1.1)$$

where $\mu_t(F) := \int_\Gamma F d\mu_t$, $F : \Gamma \longrightarrow \mathbb{R}$ is a bounded cylindric function. The questions of interest for us are: existence of solution μ_t, convergence of $\mu_t \to \mu$ for $t \to +\infty$ and properties of the obtained limiting state μ. A useful choice of time parameterization is to start the process in the remote past, namely at time $t = -T < 0$. Consequently, the initial state is denoted by μ_{-T}. After the state develops for the time T we arrive (at time $t = 0$) at a state which we denote by $\mu_{0,T}$. The limiting state for a long time can be conveniently described by

$$\lim_{T\to+\infty} \mu_{0,T} = \mu_0.$$

Next, using the Feynman-Kac formula, we give another representation of the model, an explicit solution of equation (5.1.1). Remind that $\Gamma(X)$ is a Polish space. Let L be a Markov generator, in the case studied in this subsection L is given by

$$LF(\gamma) = \int_X (F(\gamma \cup x) - F(\gamma))d\sigma(x)$$

for bounded cylindric functions $F : \Gamma(X) \longrightarrow \mathbb{R}$. The continuous function $\Phi : \Gamma(X) \longrightarrow \mathbb{R}$ will play the role of potential in Feynman-Kac formula. Rewriting (5.1.1) in terms of these notations we obtain

$$\frac{d}{dt}\mu_t^T(F) = \mu_t^T(LF) - \mu_t^T(F \cdot \Phi) + \mu_t^T(F)\mu_t^T(\Phi). \quad (5.1.2)$$

Denote by $(\mu_t^T, -T \le t \le 0)$ the measure-valued dynamical system which is the solution of (5.1.2) for each bounded cylindric function $F : \Gamma(X) \longrightarrow \mathbb{R}$, started in $\mu_{-T}^T = \mu$.

The solution μ_t^T of (5.1.2) can be explicitly written as

$$\mu_t^T = \frac{1}{Z_t(\Phi)} e^{(t+T)(L-\Phi)^*}\mu,$$

where Z_t is the normalizing constant. Via Feynman-Kac formula we can represent μ_t^T as

$$\mu_t^T(f) = \frac{\mathbb{E}\left[f(\xi_t^T)e^{-\int_{-T}^t \Phi(\xi_\tau^T)d\tau}\right]}{\mathbb{E}\left[1 \cdot e^{-\int_{-T}^t \Phi(\xi_\tau^T)d\tau}\right]},$$

where ξ^T_τ denotes the Markov process corresponding to the generator L, started in $\mu^T_{-T} = \mu$. Performing the limit $T \longrightarrow +\infty$ gives us heuristically

$$\mu_0(f) = \int_{\Omega(\mathbb{R}_- \rightarrow \Gamma(X))} f(\xi(0)) d\nu^\Phi(\xi(\cdot)), \tag{5.1.3}$$

where

$$d\nu^\Phi(\xi(\cdot)) = \frac{1}{Z} e^{-\int_{-\infty}^0 \Phi(\xi(\tau))d\tau} d\nu^0(\xi(\cdot)), \tag{5.1.4}$$

Z is the normalizing constant.

The aim of the following sections is to give proper sense to ν^Φ, defining the measure first in a bounded volume and for finite time and then going to the limit. By means of ν^Φ we are able to derive the large time asymptotic for μ^T_0. In the first section we consider the generator L as given above and in the subsequent section the more general case of the birth-and-death Markov generator.

Finally, note that we can relate (5.1.2) to the equation

$$\frac{d}{dt}\rho^T_t(F) = \rho^T_t((L - \Phi)F), \tag{5.1.5}$$

describing the time development of the density of the population. From (5.1.5) we obtain the equation (5.1.2) through normalization, namely $\mu^T_t(F) := \dfrac{\rho^T_t(F)}{\rho^T_t(1)}$ Conversely if we have a solution μ^T_t of (5.1.2) then $\rho^T_t := c_0 e^{-\int_0^t \mu_s(\Phi)ds} \mu^T_t$ solves (5.1.5). Note that if the generator L and the potential Φ are bounded, then (5.1.5) has a unique solution for a given initial condition. In other cases the question of uniqueness is a more complicated problem, which beyond the aim of our considerations.

5.2 Pure Birth Process

We define the pure birth Markov process ξ^T_t, $-T \leq t \leq 0$, on $\Gamma(X)$, starting from an empty configuration at time $t = -T$, via the generator

$$L_B F(\gamma) = \int_X (F(\gamma \cup y) - F(\gamma))\sigma(dy) \tag{5.2.1}$$

for bounded cylinder functions $F(\gamma)$. In our interpretation this means that there were no mutant alleles at the beginning, in other words we start from the null genotype. As the time passes, the mutations gradually appear in some points $x_i \in X$ at times t_i, $-T < t_i \leq 0$, and then they stay there forever.

We can describe the paths of this process in the following way: the mutations together with all their history in time can be considered as a collection of bars located in space-time $X \times \mathbb{R}_-$ and directed along the time axis t, where we denote by $(x_i, t_i) \in X \times [-T, 0]$ the starting points of the bars. These bars extend till the time $t = 0$, i.e. they end at the points $(x_i, 0) \in X \times \mathbb{R}_-$.

For each $T > 0$ define

$$\Gamma_T := \Gamma(X \times [-T, 0]) = \{\eta = \{(x_i, t_i)\} | x_i \in X, t_i \in [-T, 0]\}.$$

Then the space of starting points (x_i, t_i) of our process, described above, can be identified with the so-called space of marked configurations $\Omega_T \subset \Gamma_T$

$$\Omega_T = \{\eta = \{(x_i, t_i)\} \in \Gamma_T | \{x_i\} \in \Gamma(X)\}.$$

Denote the elements of Ω_T by $\eta = (\gamma, t(\gamma)) = \{(x, t_x)\}_{x \in \gamma}, t(\gamma) = \{t_x | x \in \gamma\}$. For more details about marked configuration spaces see cf. [GZ93, Kin93, MM91, Kun99, KKDS98].

Figure 5.1: A sample path of the pure birth process ξ_t^T on Ω_T, where "$\circ$" denotes the points of the configuration $\xi_t^T(\eta)$.

To any $\eta \in \Omega_T$ corresponds a path $(\xi_t^T(\eta), t \in [-T, 0])$ of the process. Denote by $\Omega(\mathbb{R}_- \to \Gamma)$ the set of all such paths, i.e. the image of Ω_T under ξ_t^T. Note that $\Omega(\mathbb{R}_- \to \Gamma) \subset D(\mathbb{R}_-, \Gamma)$, where $D(\mathbb{R}_-, \Gamma)$ is the Skorokhod space of right-continuous functions $f : \mathbb{R}_- \to \Gamma$ with left limits. The space $\Omega(\mathbb{R}_- \to \Gamma)$ is isomorphic to Ω_T, hence a distribution on Ω_T can be regarded as a distribution on $\Omega(\mathbb{R}_- \to \Gamma)$. From now on when we speak of paths and path measures we have in mind Ω_T and measures on it.

We assume that the path measure of the process ξ_t^T is the following: the starting points of the bars – points (x, t_x) – are distributed according to a marked Poisson measure $\tilde{\nu}_T^0$ on Γ_T with intensity measure $\sigma(dx)dt$. It is well known that the marked Poisson measure $\tilde{\nu}_T^0$ can be characterized by its Laplace transform

$$\int_{\Gamma_T} e^{\langle f,\eta\rangle} d\tilde{\nu}_T^0(\eta) = \exp\left\{\int_X \int_{-T}^0 (e^{f(x,t)} - 1) dt d\sigma(x)\right\}, \quad f \in C_0(X \times [-T,0]),$$

where $\langle f,\eta\rangle := \sum_{(x,t_x)\in\eta} f(x,t_x)$. The measure $\tilde{\nu}_T^0$ is concentrated on the space of marked configurations, i.e. $\tilde{\nu}_T^0(\Omega_T) = 1$.

A process X_t indexed by time $-T \leq t \leq 0$, $X_t : \Omega_T \longrightarrow \Gamma(X)$ fulfills the Markov property if for $-T \leq r < \tau \leq 0$

$$\mathbb{E}(e^{\langle\varphi,\, X_\tau\rangle} e^{\langle\psi,\, X_r\rangle}) = \mathbb{E}(\mathbb{E}_{X_r}[e^{\langle\varphi, X_{\tau-r-T}\rangle}] e^{\langle\psi,\, X_r\rangle}). \tag{5.2.2}$$

According to a monotone class argument and the integrability of exponentials it is sufficient to consider only exponential functions. In the following $*$ denotes the convolution of measures.

Lemma 5.2.1. The process $((\xi_t^T)_{-T\leq t\leq 0}, \Omega_T, P_{\gamma_0})$ on $(\Omega_T, \tilde{\nu}_T^0)$, where $P_{\gamma_0} = \tilde{\nu}_T^0 * \delta_{\gamma_0}$ and

$$\xi_t^T : \Omega_T \longrightarrow \Gamma(X),$$
$$\Omega_T \ni \eta = (\gamma, t(\gamma)) \mapsto \xi_t^T(\eta) = \int_{-T}^t \eta(ds) := \sum_{(x,t_x)\in\eta:\ t_x\leq t} \delta_x = \{x | x \in \gamma, t_x \leq t\}.$$

is a Markov process with generator L_B giben by (5.2.1).

Proof. To check that ξ_t^T is a Markov process generated by L_B first we need to show for $\varphi \in C_0(X)$

$$\frac{d}{dt}\mathbb{E}_{\gamma_0}(e^{\langle\varphi,\, \xi_t^T\rangle}) = \mathbb{E}_{\gamma_0}[(L_B e^{\langle\varphi,\cdot\rangle})(\xi_t^T)]. \tag{5.2.3}$$

(by monotone class arguments it is sufficient to consider exponential functions). Due to the definition of P_{γ_0}

$$\mathbb{E}_{\gamma_0}(e^{\langle\varphi,\, \xi_t^T\rangle}) = \mathbb{E}(e^{\langle\varphi,\, \xi_t^T\rangle}) e^{\langle\varphi,\, \gamma_0\rangle}. \tag{5.2.4}$$

Here $\mathbb{E}_{\gamma_0}$ denotes the expectation w.r.t. P_{γ_0}, and $\mathbb{E}$ the expectation with respect to $\tilde{\nu}_T^0$. Calculating the expectation of $e^{\langle\varphi,\, \xi_t^T\rangle}$ we get

$$\begin{aligned}\mathbb{E}(e^{\langle\varphi,\, \xi_t^T\rangle}) = \int e^{\langle\varphi,\, \xi_t^T(\eta)\rangle} d\tilde{\nu}_T^0(\eta) &= \exp\left\{\int_{-T}^t \int_X (e^{\varphi(x)} - 1) d\sigma(x) ds\right\} \\ &= \exp\left\{(T+t)\int_X (e^{\varphi(x)} - 1) d\sigma(x)\right\}. \end{aligned} \tag{5.2.5}$$

Then its derivative is

$$\frac{d}{dt}\mathbb{E}_{\gamma_0}(e^{\langle\varphi,\,\xi_t^T\rangle}) = \mathbb{E}_{\gamma_0}(e^{\langle\varphi,\,\xi_t^T\rangle})\int_X (e^{\varphi(x)}-1)d\sigma(x).$$

The right-hand side of the equation (5.2.3) is given by

$$L_B e^{\langle\varphi,\cdot\rangle}(\xi_t^T(\eta)) = \int_X (e^{\varphi(x)}-1)e^{\langle\varphi,\,\xi_t^T(\eta)\rangle}d\sigma(x) = e^{\langle\varphi,\,\xi_t^T(\eta)\rangle}\int_X (e^{\varphi(x)}-1)d\sigma(x).$$

$$\mathbb{E}_{\gamma_0}(L_B e^{\langle\varphi,\cdot\rangle}(\xi_t^T)) = \mathbb{E}_{\gamma_0}(e^{\langle\varphi,\,\xi_t^T\rangle})\int_X (e^{\varphi(x)}-1)d\sigma(x).$$

Therefore ξ_t^T corresponds to L_B.

Second, we have to check the Markov property (5.2.2) for the process ξ_t^T, i.e. for $-T \le r < \tau \le 0$

$$\mathbb{E}_{\gamma_0}(e^{\langle\varphi,\,\xi_\tau^T\rangle}e^{\langle\psi,\,\xi_r^T\rangle}) = \mathbb{E}_{\gamma_0}(\mathbb{E}_{\xi_r^T}[e^{\langle\varphi,\xi_{\tau-r-T}^T\rangle}]e^{\langle\psi,\,\xi_r^T\rangle}). \tag{5.2.6}$$

Hence for the right-hand side we obtain

$$\mathbb{E}_{\xi_r^T}[e^{\langle\varphi,\xi_{\tau-r-T}^T\rangle}] = \mathbb{E}[e^{\langle\varphi,\xi_{\tau-r-T}^T\rangle}]e^{\langle\varphi,\xi_r^T\rangle}.$$

Using (5.2.5) we get the following expression for the right-hand side:

$$\begin{aligned}&\mathbb{E}_{\gamma_0}(\mathbb{E}_{\xi_r^T}[e^{\langle\varphi,\xi_{\tau-r-T}^T\rangle}]e^{\langle\psi,\,\xi_r^T\rangle}) = \mathbb{E}[e^{\langle\varphi,\xi_{\tau-r-T}^T\rangle}]\mathbb{E}_{\gamma_0}[e^{\langle\varphi+\psi,\xi_r^T\rangle}]\\ &= \exp\left\{(\tau-r)\int_X (e^{\varphi(x)}-1)d\sigma(x)\right\}\exp\left\{(r+T)\int_X (e^{\varphi(x)+\psi(x)}-1)d\sigma(x)\right\}e^{\langle\varphi+\psi,\gamma_0\rangle}.\end{aligned}$$

Using the definition of Poisson measure $\tilde{\nu}_T^0$ we calculate the left-hand side of (5.2.6)

$$\begin{aligned}&\mathbb{E}(e^{\langle\varphi,\,\xi_\tau^T\rangle}e^{\langle\psi,\,\xi_r^T\rangle})\\ &= e^{\langle\varphi+\psi,\gamma_0\rangle}\int \exp\left\{\sum_{(x,t_x)\in\eta:\ t_x\le\tau}\varphi(x)\right\}\cdot\exp\left\{\sum_{(x,t_x)\in\eta:\ t_x\le r}\psi(x)\right\}d\tilde{\nu}_T^0(\eta)\\ &= e^{\langle\varphi+\psi,\gamma_0\rangle}\int \exp\left\{\langle\varphi+\psi 1\!\!1_{[-T,r]},\,\xi_\tau^T\rangle\right\}d\tilde{\nu}_T^0(\eta)\\ &= e^{\langle\varphi+\psi,\gamma_0\rangle}\exp\left\{\int_X\int_{-T}^{\tau}(e^{\varphi(x)+\psi(x)1\!\!1_{[-T,r]}(t)}-1)dtd\sigma(x)\right\}\\ &= e^{\langle\varphi+\psi,\gamma_0\rangle}\exp\left\{\int_X\int_{-T}^{r}(e^{\varphi(x)+\psi(x)}-1)dtd\sigma(x)+\int_X\int_r^{\tau}(e^{\varphi(x)}-1)dtd\sigma(x)\right\}\\ &= e^{\langle\varphi+\psi,\gamma_0\rangle}\exp\left\{(r+T)\int_X (e^{\varphi(x)+\psi(x)}-1)d\sigma(x)\right\}\exp\left\{(\tau-r)\int_X (e^{\varphi(x)}-1)d\sigma(x)\right\}.\end{aligned}$$

Thus condition (5.2.6) is satisfied, what implies the lemma. □

Notational convention: we prefer for readability reasons to consider in following positive times. Nevertheless, we would like to consider 0 as the final time. Therefore, we reflect the time w.r.t. to the origin. So we consider our pure birth process on the space of marked configurations $\hat{\Gamma}(X, \mathbb{R}_+)$, which is defined by

$$\hat{\Gamma}(X, \mathbb{R}_+) = \{\hat{\gamma} = (\gamma, s(\gamma)) |\ \gamma \in \Gamma(X),\ s(\gamma) = \{s_x | x \in \gamma\}, s_x \in \mathbb{R}_+\}.$$

Analogously, we define the spaces $\hat{\Gamma}(\Lambda, \mathbb{R}_+)$ and $\hat{\Gamma}(\Lambda, [0, T])$. Denote the marked Poisson measure on $\hat{\Gamma}(X, [0, T])$ by ν_T^0. Its Laplace transform is given by

$$\int_{\hat{\Gamma}(X,[0,T])} e^{\langle f, \hat{\gamma} \rangle} d\nu_T^0(\hat{\gamma}) = \exp\left\{ \int_X \int_0^T (e^{f(x,t)} - 1) dt d\sigma(x) \right\}, \quad f \in C_0(X \times [0, T]).$$

Denote our process on $(\hat{\Gamma}(X, [0, T]),\ \nu_T^0)$ by $\xi_\tau(\hat{\gamma})$. Lemma 5.2.1 yields that the birth process $\xi_\tau(\hat{\gamma}), 0 \leq \tau \leq T$ (time is considered as going backwards, i.e. the process starts at T and ends at 0) on $(\hat{\Gamma}(X, [0, T]),\ \nu_T^0)$ is realized by

$$\xi_\tau : \hat{\Gamma}(X, [0, T]) \to \Gamma(X), \quad \xi_\tau(\hat{\gamma}) = \{x \in \gamma |\ \tau \leq s_x(\gamma)\} = \int_\tau^T \hat{\gamma}(\cdot, ds). \qquad (5.2.7)$$

Here $\gamma \in \hat{\Gamma}(X, [0, T])$, $\gamma = \hat{\gamma}(dx, ds)$ describes a collection of birth places of mutations and corresponding birth times, whereas $\xi_\tau(\hat{\gamma})(dx) = \int_\tau^T \hat{\gamma}(dx, ds)$, $0 \leq \tau \leq T$ is a path of the process starting in the empty configuration at time T and developing backwards in time.

Figure 5.2: A sample path of the pure birth process ξ_τ on $\hat{\Gamma}(X, [0, T])$, where " $\circ$" denotes the points of the configuration $\xi_\tau(\hat{\gamma})$.

Denote the restriction of ν_T^0 to $\hat{\Gamma}(\Lambda, [0, T])$ by $\nu_{\Lambda,T}^0$. The restriction of the process $\xi_\tau(\hat{\gamma})$ to $(\hat{\Gamma}(\Lambda, [0, T]), \nu_{\Lambda,T}^0)$ describes the same kind of system but restricted to the volume $\Lambda \subset X$.

Furthermore it is important to take into account the influence of a selection cost function $\Phi : \Gamma \longrightarrow \mathbb{R}_+$. We split the cost function in two parts:

$$\Phi(\gamma) = \Phi_{ne}(\gamma) + \Phi_e(\gamma).$$

$\Phi_{ne}(\gamma)$ is the nonepistatic part, which describes the life costs of a mutation, is given by

$$\Phi_{ne}(\gamma) := \langle h, \gamma \rangle = \sum_{x \in \gamma} h(x), \; h(x) \geq c > 0.$$

$\Phi_e(\gamma)$ is the epistatic part, which describes the coexistence costs of mutations. Here we consider only pairwise suppression of mutations defined by

$$\Phi_e(\gamma) := \sum_{\{x,y\} \subset \gamma} \phi(x; y),$$

conditions on ϕ are specified later. More complicated epistatic cost functions could be treated with the same technique.

As the configuration γ may contain, in general, an infinite number of points, the above cost functions are well-defined only in a bounded region $\Lambda \subset X$.

The strategy is to construct the path measure in two steps: first we consider only the influence of the nonepistatic part of the cost function. In this case the model is still explicilty solvable. Then we take into consideration the influence of the epistatic part, which is much more involved.

5.2.1 Influence of the nonepistatic part of the potential

First we construct the path space measure ν^h on the space $\hat{\Gamma}(X, \mathbb{R}_+)$, obtained under the influence of Φ_{ne}. The restriction of ν^h to $\hat{\Gamma}(\Lambda, [0,T])$ is denoted by $\nu^h_{\Lambda,T}$, and defined for bounded $\Lambda \subset X$ as

$$d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\Lambda,T}} \exp\left\{ - \int_0^T \Phi^{T,\Lambda}_{ne}(\xi_\tau(\hat{\gamma}_\Lambda)) d\tau \right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda), \tag{5.2.8}$$

where $Z_{\Lambda,T}$ is the normalizing constant

$$Z_{\Lambda,T} = \int_{\hat{\Gamma}(\Lambda,[0,T])} \exp\left\{ - \int_0^T \Phi^{T,\Lambda}_{ne}(\xi_\tau(\hat{\gamma}_\Lambda)) d\tau \right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda). \tag{5.2.9}$$

Then we construct the measure ν^h as the limit of the measures $\nu^h_{\Lambda,T}$, which are defined in a bounded volume Λ and for finite time T, for $T \to +\infty, \Lambda \uparrow X$. The measure $\nu^h_{\Lambda,T}$ is also called the Gibbs perturbation of marked Poisson measure ν^0_T.

First we will show that $\nu^h_{\Lambda,T}$ still remains a Poisson measure. For this we calculate its intensity measure by computing the Laplace transform of $\nu^h_{\Lambda,T}$.

Lemma 5.2.2. Let $F(\hat{\gamma}) = e^{\langle f,\hat{\gamma}\rangle}$, $f \in C_0(X \times [-T, 0])$ where

$$\langle f, \hat{\gamma}\rangle := \sum_{(x,t_x)\in\hat{\gamma}} f(x, t_x) = \int_0^T \int_X f(x, s)\hat{\gamma}(dx, ds), \quad \hat{\gamma} \in \hat{\Gamma}(X, \mathbb{R}_+).$$

Then we have

$$\int_{\hat{\Gamma}(\Lambda,[0,T])} F(\hat{\gamma}_\Lambda) \exp\left\{-\int_0^T \Phi_{ne}^{T,\Lambda}(\xi_\tau(\hat{\gamma}_\Lambda))d\tau\right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda)$$
$$= \exp\left\{\int_\Lambda \int_0^T (\exp\{f(x, s) - sh(x)\} - 1)dsd\sigma(x)\right\}.$$

Proof. By definition of F, Φ_{ne} and (5.2.7) we have that

$$\Phi_{ne}^{T,\Lambda}(\xi_\tau(\hat{\gamma}_\Lambda)) = \int_\tau^T \int_\Lambda h(x)\hat{\gamma}_\Lambda(dx, ds)$$

and that

$$\int F(\hat{\gamma}_\Lambda) \exp\left\{-\int_0^T \Phi_{ne}^{T,\Lambda}(\xi_\tau(\hat{\gamma}_\Lambda))d\tau\right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda) \tag{5.2.10}$$
$$= \int \exp\left\{\int_0^T \int_\Lambda f(x, s)\hat{\gamma}_\Lambda(dx, ds) - \int_0^T \int_\tau^T \int_\Lambda h(x)\hat{\gamma}_\Lambda(dx, ds)d\tau\right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda).$$

Using

$$\int_0^T \int_\tau^T \hat{\gamma}_\Lambda(\cdot, ds)d\tau = \int_0^T \int_0^T 1\!\!1_{[\tau,T]}(s)\hat{\gamma}_\Lambda(\cdot, ds)d\tau$$
$$= \int_0^T \hat{\gamma}_\Lambda(\cdot, ds) \int_0^T 1\!\!1_{[0,s]}(\tau)d\tau = s \int_0^T \hat{\gamma}_\Lambda(\cdot, ds)$$

we get the required result using the Laplace transform of the marked Poisson measure $d\nu^0_{\Lambda,T}$. □

Then the normalizing constant $Z_{\Lambda,T}$ is

$$Z_{\Lambda,T} = \exp\left\{\int_\Lambda \int_0^T (\exp\{-sh(x)\} - 1)dsd\sigma(x)\right\}. \tag{5.2.11}$$

Calculating the integral of $F = e^{\langle f,\hat{\gamma}\rangle}$ w.r.t the measure $\nu^h_{\Lambda,T}$ we obtain

$$\int F(\hat{\gamma}_\Lambda)d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda) = \frac{\exp\left\{\int_\Lambda \int_0^T (\exp\{f(x, s) - sh(x)\} - 1)dsd\sigma(x)\right\}}{\exp\left\{\int_\Lambda \int_0^T (\exp\{-sh(x)\} - 1)dsd\sigma(x)\right\}}$$
$$= \exp\left\{\int_\Lambda \int_0^T (e^{f(x,s)} - 1)e^{-sh(x)}dsd\sigma(x)\right\}.$$

Thus $\nu^h_{\Lambda,T}$ is a marked Poisson measure on $\hat{\Gamma}(\Lambda,[0,T])$ with intensity measure $e^{-sh(x)}d\sigma(x)ds$.

Definition 5.2.3. We say that a sequence of measures $(\rho_\Lambda)_\Lambda$ converges "weakly" to ρ for $\Lambda \uparrow X$ on $\hat{\Gamma}(X,[0,T])$ if

$$\int F(\hat{\gamma})d\rho_\Lambda(\hat{\gamma}) \xrightarrow[\Lambda\nearrow X]{} \int F(\hat{\gamma})d\rho(\hat{\gamma}).$$

for all cylinder functions $F \in \mathcal{F}L^0(\hat{\Gamma}(X,[0,T]))$. Recall that the set of cylinder functions $\mathcal{F}L^0(\hat{\Gamma}(X,[0,T]))$ is defined as the set of all measurable bounded F such that there exists a $\Lambda \in \mathcal{B}_c(X)$ with

$$F(\hat{\gamma}) = F(\hat{\gamma}\restriction_{\Lambda\times[0,T]}).$$

We are interested in the "weak" limit of $\nu^h_{\Lambda,T}$ for $\Lambda \uparrow X$, $T \to +\infty$. In the case considered here the limit does not depend on the order in which the limits are taken. As result we get the following statement:

Theorem 5.2.4. 1) There exists the "weak" limit

$$\lim_{\Lambda\uparrow X} \nu^h_{\Lambda,T} = \nu^h_T,$$

where ν^h_T is a marked Poisson measure on $\hat{\Gamma}(X,[0,T])$ with intensity measure $e^{-sh(x)}\sigma(dx)ds$.

2) There exists the "weak" limit

$$\lim_{T\to+\infty} \nu^h_T = \nu^h,$$

where ν^h is a marked Poisson measure on $\hat{\Gamma}(X,\mathbb{R}_+)$ with the same intensity measure $e^{-sh(x)}\sigma(dx)ds$.

3) There exists the "weak" limit

$$\lim_{T\to+\infty} \nu^h_{\Lambda,T} = \nu^h_\Lambda,$$

where ν^h_Λ is a marked Poisson measure on $\hat{\Gamma}(\Lambda,\mathbb{R}_+)$ with intensity measure $e^{-sh(x)}\sigma(dx)ds$.

4) There exists the "weak" limit

$$\lim_{\Lambda\uparrow X} \nu^h_\Lambda = \nu^h,$$

where ν^h is a marked Poisson measure on $\hat{\Gamma}(X,\mathbb{R}_+)$ with intensity measure $e^{-sh(x)}\sigma(dx)ds$. The measure ν^h can alternatively also be described as a marked point field $\hat{\gamma} = (\gamma, s_\gamma)$, where γ is distributed according to $\pi_{\sigma/h}$ – Poisson measure on $\Gamma(X)$ – with marks $s_x \in \mathbb{R}_+$ distributed independently with probability $p(ds) = h(x)e^{-h(x)s}ds$.

The main object of our interest is the final distribution at the time infinity of mutations μ^h, i.e. the distribution of end points of bars. Recall that we have chosen the time range so that the final time is 0. We obtain μ^h, similar to the construction above, as the limit of final distributions $\mu^0_{\Lambda,T}$ given in bounded volume and for finite time. The measure $\mu^0_{\Lambda,T}$ on $\Gamma(X)$ is defined for $F(\eta) = e^{\langle f,\eta\rangle}$, $\eta \in \Gamma(X)$ by

$$\int_{\Gamma(X)} F(\gamma_\Lambda) d\mu^0_{\Lambda,T}(\gamma_\Lambda) := \int_{\hat{\Gamma}(\Lambda,[0,T])} F(\xi_0(\hat{\gamma}_\Lambda)) d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda) \tag{5.2.12}$$
$$= \frac{\int F(\xi_0(\hat{\gamma}_\Lambda)) \exp\{-\int_0^T \Phi^{T,\Lambda}_{ne}(\xi_t(\hat{\gamma}_\Lambda))dt\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda)}{\int \exp\{-\int_0^T \Phi^{T,\Lambda}_{ne}(\xi_t(\hat{\gamma}_\Lambda))dt\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda)}.$$

By definition of $\mu^h_{\Lambda,T}$ and $\nu^h_{\Lambda,T}$ we have

$$\int_{\hat{\Gamma}(\Lambda,[0,T])} F(\hat{\gamma}) d\mu^h_{\Lambda,T}(\hat{\gamma}) = \frac{\int_{\hat{\Gamma}(\Lambda,[0,T])} e^{T(L_\Lambda - \Phi^{T,\Lambda}_{ne})} F(\hat{\gamma}) d\mu^0_{\Lambda,T}(\hat{\gamma})}{\int_{\hat{\Gamma}(\Lambda,[0,T])} e^{T(L_\Lambda - \Phi^{T,\Lambda}_{ne})} 1 d\mu^0_{\Lambda,T}(\hat{\gamma})}, \tag{5.2.13}$$

thus $\mu^h_{\Lambda,T}$ is a solution of (5.1.1) in the bounded volume Λ for finite time T, where $\Phi(\gamma) := \langle h, \gamma \rangle$.

Note that for $f \in C_0(X)$, $\hat{\gamma} \in \hat{\Gamma}(X, \mathbb{R}_+)$ we have by (5.2.7) that

$$\langle f, \xi_0(\hat{\gamma}) \rangle = \int_X \int_0^T f(x) \hat{\gamma}(dx, ds) = \langle g, \hat{\gamma} \rangle,$$

where $g(x,s) = f(x) \mathbb{1}_{[0,T]}(s)$. Therefore the following lemma is just a corollary of Lemma 5.2.2.

Lemma 5.2.5. Let $F(\eta) = e^{\langle f,\eta\rangle}$, where $\eta \in \Gamma(X)$, $f \in C_0(X)$. Then

$$\int F(\xi_0(\hat{\gamma}_\Lambda)) \exp\left\{-\int_0^T \Phi^{T,\Lambda}_{ne}(\xi_t(\hat{\gamma}_\Lambda))dt\right\} d\nu^0_{\Lambda,T}(\hat{\gamma}_\Lambda)$$
$$= \exp\left\{\int_\Lambda \int_0^T (\exp\{f(x) - sh(x)\} - 1) ds d\sigma(x)\right\}.$$

Thus the integral in (5.2.12) for $F(\eta) = e^{\langle f,\eta\rangle}$ is given by

$$\int F(\gamma_\Lambda) d\mu^0_{\Lambda,T}(\gamma_\Lambda) = \frac{\exp\left\{\int_\Lambda \int_0^T (\exp\{f(x) - sh(x)\} - 1) ds d\sigma(x)\right\}}{\exp\left\{\int_\Lambda \int_0^T (\exp\{-sh(x)\} - 1) ds d\sigma(x)\right\}}$$
$$= \exp\left\{\int_\Lambda (e^{f(x)} - 1) \frac{(1 - \exp\{-Th(x)\})}{h(x)} d\sigma(x)\right\}.$$

Again, as before, we are interested in the "weak" limit of $\mu^0_{\Lambda,T}$ for $\Lambda \uparrow X$, $T \to +\infty$. Note that under the "weak" limit of a sequence of measures $(\rho_\Lambda)_\Lambda$ on $\Gamma(X)$ for $\Lambda \uparrow X$ we understand that for all bounded cylinder functions $F \in \mathcal{F}L^0(\Gamma(X))$

$$\int F(\hat{\gamma}) d\mu^0_{\Lambda,T}(\hat{\gamma}) \xrightarrow[\Lambda \nearrow X]{} \int F(\hat{\gamma}) d\mu^h_T(\hat{\gamma}).$$

The limit does also not depend on the order in which the limits are taken. As result we get as corollary of Theorem 5.2.4. the following statement:

Theorem 5.2.6. (cf. [SEK05])

1) There exists the "weak" limit

$$\lim_{\Lambda \uparrow X} \mu^0_{\Lambda,T} = \mu^h_T,$$

where μ^h_T is a Poisson measure on $\Gamma(X)$ with intensity measure

$$\frac{(1 - \exp\{-Th(x)\})}{h(x)} d\sigma(x).$$

2) According to Lebesgues dominated convergence theorem there exists the "weak" limit

$$\lim_{T \to +\infty} \mu^h_T = \mu^h,$$

where μ^h is a Poisson measure on $\Gamma(X)$ with intensity measure $\frac{1}{h(x)} d\sigma(x)$.

3) There exists the "weak" limit

$$\lim_{T \to +\infty} \mu^0_{\Lambda,T} = \mu^h_\Lambda,$$

where μ^h_Λ is a Poisson measure on $\Gamma(\Lambda)$ with intensity measure $\frac{1}{h(x)} d\sigma(x)$.

4) There exists the "weak" limit

$$\lim_{\Lambda \uparrow X} \mu^h_\Lambda = \mu^h,$$

where μ^h is a Poisson measure on $\Gamma(X)$ with intensity measure $\frac{1}{h(x)} d\sigma(x)$.

5.2.2 Influence of the epistatic part of the potential

In this subsection we include the influence of the epistatic part of the potential, namely $\Phi_e(\gamma)$. We do so by a Gibbs perturbation of the measure ν^h from Theorem 5.2.4 through Φ_e, i.e.

$$d\nu^{\beta,\phi}(\hat{\gamma}) = \frac{1}{Z_\beta} \exp\left\{-\beta \int_0^{+\infty} \Phi_e(\xi_\tau(\hat{\gamma})) d\tau\right\} d\nu^h(\hat{\gamma}), \quad \beta > 0.$$

Again such a construction is well-defined only for a bounded region $\Lambda \subset X$ and we consider first the restriction of measures to the space $\hat{\Gamma}(\Lambda, \mathbb{R}_+)$:

$$d\nu_\Lambda^{\beta,\phi}(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\beta,\Lambda}} \exp\left\{-\beta \int_0^{+\infty} \Phi_e^\Lambda(\xi_\tau(\hat{\gamma}_\Lambda))d\tau\right\} d\nu_\Lambda^h(\hat{\gamma}_\Lambda).$$

We will define the measure $\nu^{\beta,\phi}$ as the weak limit of $\nu_\Lambda^{\beta,\phi}$. The main technique is cluster expansion. Note that

$$\int_0^{+\infty} \Phi_e(\xi_\tau(\hat{\gamma}))d\tau = \sum_{\{x,y\}\subseteq\gamma} \phi(x;y)\min(s_x, s_y), \quad \hat{\gamma} = (\gamma, s(\gamma)).$$

To check the convergence of the cluster expansion we have to make some assumptions on ϕ and ψ, where

$$\psi(\hat{x}, \hat{y}) := \phi(x;y)\min(s_x, s_y), \quad \hat{x} = (x, s_x), \; \hat{y} = (y, s_y).$$

(S) Stability of ϕ: $\quad \exists B \geq 0$ such that $\forall \gamma \in \Gamma_0(X)$

$$\sum_{\{x,y\}\subseteq\gamma} \phi(x;y) \geq -B|\gamma|. \tag{5.2.14}$$

(I_ψ) Integrability of ψ

$$C(\beta, h) := \operatorname*{esssup}_{y\in X,\, t\in\mathbb{R}_+} \int_X \int_0^{+\infty} |e^{-\beta\psi((x,s),(y,t))} - 1| e^{2\beta Bs - hs} ds\sigma(dx) < \infty. \tag{5.2.15}$$

Let us state the following consequence of stability assumption before we start.

Lemma 5.2.7. Let ϕ fulfill (S). Then $\forall \hat{\gamma} = (\gamma, s(\gamma)) \in \hat{\Gamma}_0(X, \mathbb{R}_+)$ there exists $x_0 \in \gamma$ such that

$$\sum_{x\in\gamma\setminus\{x_0\}} \phi(x;x_0)\min(s_x, s_{x_0}) \geq -2Bs_{x_0}. \tag{5.2.16}$$

5.2.3 Cluster expansion

By the definition of $d\nu_\Lambda^{\beta,\phi}$

$$d\nu_\Lambda^{\beta,\phi}(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\beta,\Lambda}} \exp\left\{-\beta \sum_{\{\hat{x},\hat{y}\}\subseteq\hat{\gamma}_\Lambda} \psi(\hat{x};\hat{y})\right\} d\nu_\Lambda^h(\hat{\gamma}_\Lambda). \tag{5.2.17}$$

Denote by $\hat{\sigma}(dx, ds) = e^{-sh(x)}\sigma(dx)ds$. Theorem 5.2.4 says that ν^h_Λ is the Poisson measure on $\hat{\Gamma}(\Lambda, \mathbb{R}_+)$ with intensity $\hat{\sigma}(dx, ds)$. By the definition of Poisson and the Lebesgue-Poisson measure

$$d\nu^h_\Lambda = \exp\{-\hat{\sigma}(\Lambda \times [0, +\infty))\} d\lambda_{\hat{\sigma}}.$$

Then (5.2.17) can be written as

$$d\mu^{\beta,\phi}_\Lambda(\hat{\gamma}_\Lambda) = \frac{1}{\hat{Z}_{\beta,\Lambda}} \exp\left\{ -\beta \sum_{\{\hat{x},\hat{y}\}\subseteq\hat{\gamma}_\Lambda} \psi(\hat{x};\hat{y}) \right\} d\lambda_{\hat{\sigma}}(\gamma_\Lambda),$$

where $\hat{Z}_{\beta,\Lambda} = Z_{\beta,\Lambda} \cdot \exp\{\hat{\sigma}(\Lambda \times [0, +\infty))\}$.

Cluster expansion is a tool which provides us with an effective perturbation theory of the Gibbs factor $e^{-\beta E(\gamma)}$ for small parameters, see e.g. [MM91] and references therein. We follow here the presentation given in [Kun99, KKDS98]. There the cluster expansion was generalized to a general metric space, i.e. no translation invariant structure is present. In our case the factor which we are going to expand is

$$p_{\Lambda,\beta}(\hat{\gamma}_\Lambda) := \exp\left\{ -\beta \sum_{\{\hat{x},\hat{y}\}\subseteq\hat{\gamma}_\Lambda} \psi(\hat{x};\hat{y}) \right\}. \tag{5.2.18}$$

From [Kun99, KKDS98] we know that the cluster decomposition of (5.2.18) is the following:

$$p_{\Lambda,\beta}(\hat{\gamma}_\Lambda) = \sum^{(\hat{\gamma}_\Lambda)}_{(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_m)} k(\hat{\gamma}_1)k(\hat{\gamma}_2)\ldots k(\hat{\gamma}_m),$$

$p_{\Lambda,\beta}(\emptyset) = 1$. Here $\sum\limits^{(\hat{\gamma}_\Lambda)}_{(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_m)}$ means the summation over all partitions of the configuration $\hat{\gamma}_\Lambda$ into non-empty subconfigurations $\hat{\gamma}_i \subseteq \hat{\gamma}_\Lambda$, i.e. over all non-ordered sets $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_m\}$, $m = 1, 2, \ldots, |\hat{\gamma}_\Lambda|$ of subconfigurations $\gamma_i \subseteq \gamma_\Lambda$ of $\hat{\gamma}_\Lambda$ which have mutually disjoint supports such that $\cup_{i=1}^m \gamma_i = \gamma_\Lambda$. $k(\hat{\gamma})$ is defined for a finite non-empty configuration $\hat{\gamma}$ by

$$k(\hat{\gamma}) = \sum_{G\in\mathcal{G}(\hat{\gamma})} \prod_{\{x,y\}\in G} (e^{-\beta\phi(x;y)\min\{s_x,s_y\}} - 1),$$

$k(\hat{\gamma}) = 1$ if $|\hat{\gamma}| = 1$. By $\mathcal{G}(\hat{\gamma})$ we denote the set of all connected graphs with the set of vertices γ, and the product $\prod\limits_{\{x,y\}\in G}$ is taken over all edges of the graph G.

In order to estimate k we have to introduce a function $\bar{k}$, which is related to k by the formula

$$\bar{k}(\{\hat{x}\}, \hat{\gamma} \setminus \{\hat{x}\}) := k(\hat{\gamma}). \tag{5.2.19}$$

The main idea of cluster expansion is to find some function Q dominating $\bar{k}$. This function Q is defined as the solution of the following iterative equation

$$Q(\hat{\gamma},\hat{\zeta}) = e^{2\beta B s_{x_0(\hat{\gamma})}} \sum_{\eta\subset\zeta} e_\lambda(e^{-\beta\psi(x_0(\hat{\gamma}),\cdot)}-1,\hat{\eta})Q(\hat{\eta}\cup\hat{\gamma}\setminus\{x_0(\hat{\gamma})\},\hat{\zeta}\setminus\hat{\eta}),$$

where $x_0(\hat{\gamma})$ is given by condition (5.2.16). Define

$$\tilde{Q}(\hat{\gamma},\hat{\zeta}) := \prod_{y\in\gamma\cup\zeta} e^{-2\beta B s_y} Q(\hat{\gamma},\hat{\zeta}). \tag{5.2.20}$$

Then $\tilde{Q}$ fulfills an equation as (3.107) from [Kun99] for $B = 0$. The later equation has a unique solution, cf. [Kun99, Proposition 3.3.10], given by

$$\tilde{Q}(\hat{\gamma},\hat{\zeta}) := \sum_{T\in\mathcal{T}(\hat{\gamma}\cup\hat{\zeta})} \prod_{\{y,y'\}\in T} |e^{-\beta\phi(y,y')\min(s_y,s_{y'})}-1|.$$

Here $\mathcal{T}(\hat{\gamma})$ denotes the set of all trees with set of vertices γ; recall that a tree is a connected graph without loops. Expressing Q via (5.2.20) we get

$$Q(\hat{\gamma},\hat{\zeta}) = \prod_{y\in\gamma\cup\zeta} \exp\{2\beta B s_y\} \sum_{T\in\mathcal{T}(\hat{\gamma}\cup\hat{\zeta})} \prod_{\{y,y'\}\in T} |e^{-\beta\phi(y,y')\min(s_y,s_{y'})}-1|. \tag{5.2.21}$$

Recall that Q gives an upper bound for $\bar{k}$, thus we have

$$|\bar{k}(\hat{x},\hat{\eta})| \le Q(\hat{x},\hat{\eta}). \tag{5.2.22}$$

To prove our main statement – the convergence of measures $\nu_\Lambda^{\beta,\phi}$ – we need first to derive an upper bound on a certain integral of Q, which is a consequence of the following analog of a standard estimate, c.f. e.g. [Kun99, Lemma 3.3.12].

Lemma 5.2.8. Let Y be in $\mathcal{B}(X)$, and $n \ge 1$ fixed. Then for $\hat{\sigma}$-a.a. $\hat{x} \in X \times \mathbb{R}_+$ we have

$$\begin{aligned}&\int_{(Y\times\mathbb{R}_+)^n} Q(\hat{x},(\hat{y})_1^n)\hat{\sigma}(d\hat{y})_1^n \qquad (5.2.23)\\ &\quad\le e^{2\beta B s_x} C(\beta,h)^{n-1}(n+1)^{n-1} \int_{Y\times\mathbb{R}_+} |e^{-\beta\phi(x,y)\min\{s_x,s_y\}}-1| e^{2\beta B s_y}\hat{\sigma}(d\hat{y}),\end{aligned}$$

where $(\hat{y})_1^n$ is an abbreviation for $(\hat{y}_1,\dots,\hat{y}_n)$, and $C(\beta,h)$ is given by (5.2.15). Here $\hat{\sigma}(\hat{y}) = \hat{\sigma}(dy, ds_y)$ for $\hat{y} = (y, s_y)$.

Proof. In the following we denote $\hat{y}_{n+1} := \hat{x}$, $\mathcal{T}([n]) := \mathcal{T}((\hat{y})_1^n)$. The equality (5.2.21) implies that (5.2.23) is equal to

$$e^{2\beta B s_x} \sum_{T\in\mathcal{T}([n+1])} \int_{(Y\times\mathbb{R}_+)^n} \prod_{k=1}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1|\hat{\sigma}(d\hat{y})_1^n. \quad (5.2.24)$$

Now we estimate by induction in n the term

$$\int_{(Y\times\mathbb{R}_+)^n} \prod_{k=1}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1|\hat{\sigma}(d\hat{y})_1^n. \quad (5.2.25)$$

For $n = 1$ the only tree T is $\{\{\hat{x}\},\{\hat{y}_1\}\}$ and (5.2.25) is reduced to

$$\int_{Y\times\mathbb{R}_+} e^{2\beta B s_{y_1}}|e^{-\beta\phi(x,y_1)\min(s_x,s_{y_1})} - 1|\hat{\sigma}(d\hat{y}_1).$$

Let us assume that for $n = N-1$ we have for all trees $T \in \mathcal{T}([n+1])$

$$\int_{(Y\times\mathbb{R}_+)^n} \prod_{k=1}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1|\hat{\sigma}(d\hat{y})_1^n$$
$$\leq C(\beta,h)^{n-1} \int_{Y\times\mathbb{R}_+} |e^{-\beta\phi(x,y)\min\{s_x,s_y\}} - 1|e^{2\beta B s_y}\hat{\sigma}(dy, ds_y).$$

For the case $n = N$ proceed as follows: fix a tree $T \in \mathcal{T}([n+1])$ and choose $\hat{y}_{n+1}$ as a root point of T. Then there exists a final pair $\{j_1, j_2\} \in T$ where $\hat{y}_{j_1}$ is the final vertex and $y_{j_1} \neq \hat{y}_{n+1}$. This implies the following estimate

$$\int_{(Y\times\mathbb{R}_+)^n} \prod_{k=1}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1|\hat{\sigma}(d\hat{y})_1^n$$
$$= \int_{(Y\times\mathbb{R}_+)^{n-1}} \prod_{\substack{k=1\\k\neq j_1}}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T\setminus\{j_1,j_2\}} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1|$$
$$\times \int_{Y\times\mathbb{R}_+} e^{2\beta B s_{y_{j_1}}}|e^{-\beta\phi(y_{j_1},y_{j_2})\min(s_{y_{j_1}},s_{y_{j_2}})} - 1|\hat{\sigma}(d\hat{y}_{j_1}) \prod_{\substack{l=1\\l\neq j_1}}^n \hat{\sigma}(d\hat{y}_l)$$
$$\leq C(\beta,h) \int_{(Y\times\mathbb{R}_+)^{n-1}} \prod_{\substack{k=1\\k\neq j_1}}^n e^{2\beta B s_{y_k}} \prod_{(i,j)\in T\setminus\{j_1,j_2\}} |e^{-\beta\phi(y_i,y_j)\min(s_{y_i},s_{y_j})} - 1| \prod_{\substack{l=1\\l\neq j_1}}^n \hat{\sigma}(d\hat{y}_l)$$
$$\leq C(\beta,h)^{n-1} \int_{Y\times\mathbb{R}_+} e^{2\beta B s_{y_{n+1}}}|e^{-\beta\phi(y_{j_{n+1}},y)\min(s_{y_{n+1}},s_y)} - 1|\hat{\sigma}(d\hat{y}),$$

where in the last inequality we used the induction step. Therefore (5.2.24) can be bounded by

$$e^{2\beta B s_x} C(\beta,h)^{n-1} \int\limits_{Y\times\mathbb{R}_+} e^{2\beta B s_x} |e^{-\beta\phi(x,y)\min(s_x,s_y)} - 1| \hat{\sigma}(d\hat{y}) \sum_{T\in\mathcal{T}([n+1])} 1.$$

Using the fact that $|\mathcal{T}([n+1])| = (n+1)^{n-1}$ we obtain the statement of the lemma. □

Now we prove the analog of [Kun99, Proposition 3.3.13].

Theorem 5.2.9. Let $\Lambda \in \mathcal{B}_c(\hat{\Gamma}(X,\mathbb{R}_+))$ be given. Then for any parameters β and h such that $2\beta B - h < 0$ and

$$C(\beta,h) < \frac{1}{2e}, \tag{5.2.26}$$

where $C(\beta,h)$ is given by the integrability condition (5.2.15), we have

$$\int_{\hat{\Gamma}(\Lambda,\mathbb{R}_+)\setminus\{\emptyset\}} \int_{\hat{\Gamma}_0(X,\mathbb{R}_+)} |k(\hat{\gamma}\cup\hat{\eta})| \lambda_{\hat{\sigma}}(d\hat{\gamma})\lambda_{\hat{\sigma}}(d\hat{\eta}) < \infty. \tag{5.2.27}$$

Proof. Using the definition of $\lambda_{\hat{\sigma}}$ and the relation between k and $\bar{k}$ given by (5.2.19) we may write (5.2.27) as

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!m!} \int\limits_{(\Lambda\times\mathbb{R}_+)^n} \int\limits_{(X\times\mathbb{R}_+)^m} |\bar{k}(\{\hat{x}_n\}, \{\hat{x}\}_1^{n-1} \cup \{\hat{y}\}_1^m)| \hat{\sigma}(d\hat{x})_1^n \hat{\sigma}(d\hat{y})_1^m.$$

According to (5.2.22) and Lemma 5.2.8 we can bound the above term by

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!m!} C(\beta,h)^{n+m-2}(n+m)^{n+m-2}$$
$$\times \int\limits_{\Lambda\times\mathbb{R}_+} e^{2\beta B s_x} \int\limits_{X\times\mathbb{R}_+} e^{2\beta B s_y} |e^{-\beta\phi(x,y)\min(s_x,s_y)} - 1| \hat{\sigma}(d\hat{y})\hat{\sigma}(d\hat{x}).$$

The integral above can be bounded by

$$C(\beta,h) \int\limits_{\Lambda} \int\limits_{0}^{+\infty} e^{2\beta B s - hs} ds \sigma(dx).$$

This expression is finite for $2\beta B - h < 0$, and in this case equal to $\dfrac{C(\beta,h)\sigma(\Lambda)}{h-2\beta B}$. Using this and the fact that $(n+m)^{n+m-2} \le e^{n+m}(n+m)!$ we can estimate (5.2.27) by

$$\begin{aligned}
&\frac{\sigma(\Lambda)}{C(\beta,h)(h-2\beta B)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(n+m)!}{n!m!}(eC(\beta,h))^{n+m}\\
&\quad=\frac{\sigma(\Lambda)}{C(\beta,h)(h-2\beta B)}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}\frac{l!}{m!(l-m)!}\right)(eC(\beta,h))^{n+m}\\
&\quad\le\frac{\sigma(\Lambda)}{C(\beta,h)(h-2\beta B)}\sum_{l=0}^{\infty}(2eC(\beta,h))^{n+m}.
\end{aligned}$$

This yields the result of the lemma. □

From this theorem follows our main result, analogously of [Kun99, Theorem 3.3.23].

Theorem 5.2.10. Let conditions (S), (I_ψ) be fulfilled, $2\beta B - h < 0$, and

$$C(\beta,h) < \frac{1}{2e}.$$

Then the "weak" limit $\nu_\Lambda^{\beta,\phi} \to \nu^{\beta,\phi}$, $\Lambda \uparrow X$ exists.

We intend to find some sufficient conditions on ϕ which implies the conditions of Theorem 5.2.10. First, let us derive another expression for $C(\beta,h)$. We start with the part of which the esssup is taken. Denote for short by $k := 2\beta B - h$, $k < 0$ and by $\{\phi > 0\} := \{x \in X : \phi(x,y) > 0\}$ for a fixed y. Then

$$\begin{aligned}
&\int_X\int_0^{+\infty}|e^{-\beta\phi(x,y)\min\{s,t\}}-1|e^{2\beta Bs-hs}ds\sigma(dx)\\
&\quad=\int\limits_{\{\phi>0\}}\int_0^t e^{ks}(1-e^{-\beta\phi(x,y)s})ds\sigma(dx)+\int\limits_{\{\phi>0\}}\int_t^\infty e^{ks}(1-e^{-\beta\phi(x,y)t})ds\sigma(dx)\\
&\quad\quad+\int\limits_{\{\phi<0\}}\int_0^t e^{ks}(e^{-\beta\phi(x,y)s}-1)ds\sigma(dx)+\int\limits_{\{\phi<0\}}\int_t^\infty e^{ks}(e^{-\beta\phi(x,y)t}-1)ds\sigma(dx)\\
&\quad=\int\limits_{\{\phi>0\}}\frac{\beta\phi(x,y)}{k(k-\beta\phi)}(1-e^{(k-\beta\phi(x,y))t})\sigma(dx)+\int\limits_{\{\phi<0\}}\frac{-\beta\phi(x,y)}{k(k-\beta\phi)}(1-e^{(k-\beta\phi(x,y))t})\sigma(dx).
\end{aligned}$$

Thus we obtain

$$C(\beta,h)=\operatorname*{esssup}_{y\in X,\ t\in\mathbb{R}_+}\int_X\frac{\beta|\phi(x,y)|(1-\exp\{t[2\beta B-h(x)-\beta\phi(x,y)]\})}{(2\beta B-h(x))(2\beta B-h(x)-\beta\phi(x,y))}\sigma(dx).$$

Having in mind the applications in genetics, it is reasonable to assume that

$$\phi(x,y) \geq 0, \quad \forall x, y \in X.$$

In this case the stability condition (5.2.14) holds for $B = 0$ and

$$C(\beta, h) = \operatorname*{esssup}_{y \in X,\ t \in \mathbb{R}_+} \int_X \frac{\beta\phi(x,y)}{h(h+\beta\phi(x,y))}(1 - e^{-(h+\beta\phi(x,y))t})\sigma(dx).$$

From now on we assume for simplicity that $h(x) \equiv \text{const}$. The integrand $g(t,x)$ is nonnegative and monotone increasing in t. Therefore according to Beppo Levi's theorem we may interchange sup and the integral deriving

$$C(\beta, h) = \operatorname*{esssup}_{y \in X} \int\limits_X \frac{\beta\phi(x,y)}{h(h+\beta\phi(x,y))}\sigma(dx) \leq \frac{1}{h} \operatorname*{esssup}_{y \in X} \int\limits_X \left(\frac{\beta}{h}\phi(x,y) \wedge 1\right) \sigma(dx).$$

We may reformulate Theorem 5.2.10 for nonnegative $\phi(x,y)$.

Theorem 5.2.11. Let $\phi(x,y)$ be nonnegative, and

$$\operatorname*{esssup}_{y \in X} \int\limits_X \frac{\beta\phi(x,y)}{h(h+\beta\phi(x,y))}\sigma(dx) \leq \frac{1}{2e}.$$

Then the "weak" limit $\nu_\Lambda^{\beta,\phi} \to \nu^{\beta,\phi}$, $\Lambda \uparrow X$ exist.

5.3 Birth-and-Death Process

We consider the birth-and-death Markov process Y_t^T, $-T \leq t \leq 0$ on $\Gamma(X)$, starting from an empty configuration at time $t = -T$, corresponding to the generator

$$LF(\gamma) = \sum_{x \in \gamma} d(x)(F(\gamma \setminus x) - F(\gamma)) + \int b(y)(F(\gamma \cup y) - F(\gamma))\sigma(dy) \tag{5.3.1}$$

where F is a bounded cylinder function. The death part of the generator is

$$L_D(\gamma) := \sum_{x \in \gamma} d(x)(F(\gamma \setminus x) - F(\gamma)),$$

and the birth part of the generator is

$$L_B(\gamma) := \int b(y)(F(\gamma \cup y) - F(\gamma))\sigma(dy),$$

where $d(x) \geq 0$ is the death rate (intensity), $b(x) \geq 0$ is the birth rate. In our interpretation this means that there were no mutant alleles at the beginning. As the time passes, the mutations gradually appear in points $x \in X$ at times s_x, $-T < s_x \leq 0$, and exists for some time l_x. A path of the process can be described as a collection of bars located in space-time $X \times [-T, 0]$ and directed along the time axis t, where the points $(x, s_x) \in X \times [-T, 0]$ denote the starting points of the bars, and the length of the bar is denoted by l_x.

Figure 5.3: A path of birth-and-death process Y_t^T on $\Omega(\hat{X}_T)$, where " $\circ$ " denotes the points of the configuration $Y_t^T(\hat{\gamma})$.

We introduce some further notations: denote by

$$X_T := X \times [-T, 0] = \{y = (x, s), x \in X, s \in [-T, 0]\}$$

and let

$$\hat{X}_T = \{(y, l) | y = (x, s) \in X \times [-T, 0], l \in \mathbb{R}_+\}.$$

Then the path space of the process Y_t^T, described above, can be identified with the marked configuration space

$$\Omega(\hat{X}_T) = \{\hat{\gamma} = (y, l_y)_{y \in \gamma} |\ \gamma \in \Gamma(X_T),\ l_y \in \mathbb{R}_+\}.$$

Assume that the path measure of the process Y_t^T is the following: the starting points of the bars – points (x, s_x) – are distributed according to a marked Poisson measure on $X \times [-T, 0]$ with intensity measure $b(x)\sigma(dx)ds$, and for a given point configuration $\xi \in \Gamma(X \times [-T, 0])$ the conditional distribution of marks l w.r.t. this configuration is conditionally independent and exponentially distributed with density $d(x)e^{-d(x)l}$. Equivalently, it can be described as the marked

Poisson measure π_ρ with intensity measure ρ, where the intensity measure ρ on $\hat{X}_T$ is given by

$$\rho(dx, ds, dl) = b(x)d(x)e^{-d(x)l}\sigma(dx)dsdl.$$

It is well known that the marked Poisson measure π_ρ can be characterized by its Laplace transform

$$\int_{\Omega(\hat{X}_T)} e^{\langle g,\hat{\gamma}\rangle} d\pi_\rho(\hat{\gamma}) = \exp\left\{\int_{-T}^{0}\int_0^\infty\int_X (e^{g(y,l,s)} - 1)b(y)d(y)e^{-d(y)l}\sigma(dy)dlds\right\},$$

where $\langle g, \hat{\gamma}\rangle := \sum_{(x,s,l)\in\hat{\gamma}} g(x, s, l)$.

We define the distribution P_{D,γ_0} by

$$\begin{aligned}
\mathbb{E}_{D,\gamma_0}(e^{\langle h,Y_s\rangle}) :=& \int e^{\langle h,Y_s\rangle} dP_{D,\gamma_0}(\gamma) = \mathbb{E}_{D,\gamma_0}\left(\prod_{x\in Y_s} e^{h(x)}\right)\\
=& \prod_{x\in\gamma_0} \mathbb{E}_X[\mathbb{1}_{(0,s+T)}(r) + \mathbb{1}_{(s+T,\infty)}(r)e^{h(x)}]\\
=& \prod_{x\in\gamma_0}\left(\int_0^{s+T} e^{-rd(x)}d(x)dr + \int_{s+T}^\infty e^{-rd(x)}d(x)dre^{h(x)}\right)\\
=& \prod_{x\in\gamma_0}(1 - e^{-(s+T)d(x)} + e^{-(s+T)d(x)}e^{h(x)})\\
=& \exp\{\langle \ln[1 + e^{-(s+T)d(x)}(e^{h(x)} - 1)], \gamma_0\rangle\}.
\end{aligned} \tag{5.3.2}$$

Here X is an exponentially distributed random variable with density $d(x)e^{-rd(x)}$.

Lemma 5.3.1. The Markov birth-and-death process Y_t^T, $-T \le t \le 0$, starting from an empty configuration, is realized on $(\Omega(\hat{X}_T), P_\gamma)$, where $P_{\gamma_0} = \pi_\rho * P_{D,\gamma}$, by

$$\begin{aligned}
&Y_t^T : \Omega(\hat{X}_T) \longrightarrow \Gamma(X),\\
&\Omega(\hat{X}_T) \ni \hat{\gamma} \mapsto Y_t^T(\hat{\gamma}) = \eta(t) = \sum_{(x,s,l)\in\gamma} \delta_x \mathbb{1}_{[s,s+l]}(t).
\end{aligned}$$

Proof. First we show that the expectation of $g : X \to \mathbb{R}$ is

$$\mathbb{E}^{\pi_\rho}[e^{\langle g,Y_t^T\rangle}] = \exp\left\{\int_X \frac{b(y)}{d(y)}(e^{g(y)} - 1)(1 - e^{-(T+t)d(y)})\sigma(dy)\right\}. \tag{5.3.3}$$

By definition

$$\mathbb{E}^{\pi_\rho}[e^{\langle g,Y_t^T\rangle}] = \mathbb{E}^{\pi_\rho}\left[\exp\left\{\int g(y)\eta(t, dy)\right\}\right].$$

This is the Laplace transform of π_ρ, therefore the expression above is equal to

$$\begin{aligned}&\exp\left\{\int_{-T}^{t}\int_0^\infty\int_X (e^{g(y)\mathbb{1}_{[s,s+l]}(t)}-1)b(y)d(y)e^{-d(y)l}\sigma(dy)dlds\right\}\\&\quad=\exp\left\{\int_{-T}^{t}\int_0^\infty\int_X (e^{g(y)}-1)\mathbb{1}_{[s,s+l]}(t)b(y)d(y)e^{-d(y)l}\sigma(dy)dlds\right\}.\end{aligned}$$

Calculating the integrals w.r.t. dl and ds we obtain the required result.

Next, in order to check that the process Y_t^T has the generator given by (5.3.1) we should verify that for $\varphi \in C_0(X)$

$$\frac{d}{dt}\mathbb{E}^{\pi_\rho}(e^{\langle\varphi,Y_t^T\rangle}) = \mathbb{E}^{\pi_\rho}[(Le^{\langle\varphi,\cdot\rangle})(Y_t^T)] \tag{5.3.4}$$

(by monotone class arguments it is sufficient to consider only exponential functions). By (5.3.3) the left-hand side of (5.3.4) is

$$\frac{d}{dt}\mathbb{E}^{\pi_\rho}(e^{\langle g,Y_t^T\rangle}) = \mathbb{E}^{\pi_\rho}(e^{\langle g,Y_t^T\rangle})\int b(y)(e^{g(y)}-1)e^{-(T+t)d(y)}\sigma(dy). \tag{5.3.5}$$

Calculating the right-hand side of (5.3.4) we obtain for the birth part straightforward

$$\mathbb{E}^{\pi_\rho}[(L_B e^{\langle g,\cdot\rangle})(Y_t^T)] = \mathbb{E}^{\pi_\rho}(e^{\langle g,Y_t^T\rangle})\int b(y)(e^{g(y)}-1)\sigma(dy). \tag{5.3.6}$$

The action of the death part of the operator is equal to

$$(L_D e^{\langle g,\cdot\rangle})(Y_t^T) = \langle d(e^{-g}-1), Y_t^T\rangle e^{\langle g,Y_t^T\rangle}. \tag{5.3.7}$$

Using Mecke formula we get

$$\int\langle f,\hat\gamma\rangle e^{\langle g,\hat\gamma\rangle}\pi_\rho(d\hat\gamma) = \int\sum_{\hat x\in\hat\gamma} f(\hat x)e^{g(\hat x)}e^{\langle g,\hat\gamma\setminus\hat x\rangle}\pi_\rho(d\hat\gamma) = \int f(\hat x)e^{g(\hat x)}d\rho(\hat x)\int e^{\langle g,\hat\gamma\rangle}\pi_\rho(d\hat\gamma).$$

and thus for the death part we get

$$\begin{aligned}&\mathbb{E}^{\pi_\rho}[(L_D e^{\langle g,\cdot\rangle})(Y_t^T)] = \mathbb{E}^{\pi_\rho}[\langle d(e^{-g}-1), Y_t^T\rangle e^{\langle g,Y_t^T\rangle}]\\&\quad= \mathbb{E}^{\pi_\rho}(e^{\langle g,Y_t^T\rangle})\cdot\int_{-T}^{t}\int_{t-s}^\infty\int_X d(x)(e^{-g(x)}-1)e^{g(x)}e^{-d(x)l}d(x)b(x)\sigma(dx)dlds\\&\quad= \mathbb{E}^{\pi_\rho}(e^{\langle g,Y_t^T\rangle})\int b(x)(1-e^{g(x)})(1-e^{-(T+t)d(x)})\sigma(dx).\end{aligned} \tag{5.3.8}$$

Adding (5.3.6) and (5.3.8) we see that (5.3.4) is fulfilled. Therefore Y_t^T corresponds to the generator L.

Next, we will check the Markov property (5.2.2) for the process Y_t^T, i.e. for $-T \le r < \tau \le 0$

$$\mathbb{E}^{\pi_\rho}(e^{\langle \varphi, Y_\tau^T\rangle} e^{\langle \psi, Y_r^T\rangle}) = \mathbb{E}^{\pi_\rho}(\mathbb{E}_{Y_r^T}[e^{\langle \varphi, Y_{\tau-r-T}^T\rangle}] e^{\langle \psi, Y_r^T\rangle}). \tag{5.3.9}$$

Using the definition of the process Y_t^T and the properties of the marked Poisson measure π_ρ we calculate the left-hand side of (5.3.9)

$$\begin{aligned}&\mathbb{E}^{\pi_\rho}(e^{\langle \varphi, Y_\tau^T\rangle} e^{\langle \psi, Y_r^T\rangle})\\ &\quad = \exp\left\{\int_{-T}^{\tau}\int_0^\infty\int_X (e^{\varphi(y)1\!\!1_{[s,s+l]}(\tau)+\psi(y)1\!\!1_{[s,s+l]}(r)} - 1) b(y) d(y) e^{-d(y)l}\sigma(dy)dlds\right\}.\end{aligned}$$

By definition of the indicator function we obtain

$$\begin{aligned}&\int_{-T}^{\tau}\int_0^\infty\int_X (e^{\varphi(y)1\!\!1_{[s,s+l]}(\tau)+\psi(y)1\!\!1_{[s,s+l]}(r)} - 1) b(y) d(y) e^{-d(y)l}\sigma(dy)dlds\\ &\quad = \int_{-T}^{r}\int_{r-s}^{t-s}\int_X (e^{\psi(y)} - 1) b(y) d(y) e^{-d(y)l}\sigma(dy)dlds\\ &\quad\quad + \int_{-T}^{r}\int_{t-s}^{\infty}\int_X (e^{\varphi(y)+\psi(y)} - 1) b(y) d(y) e^{-d(y)l}\sigma(dy)dlds\\ &\quad\quad + \int_{r}^{t}\int_{t-s}^{\infty}\int_X (e^{\varphi(y)} - 1) b(y) d(y) e^{-d(y)l}\sigma(dy)dlds.\end{aligned}$$

Calculating the integrals w.r.t. dl and ds we get

$$\begin{aligned}\int_X \frac{b(y)}{d(y)}\Big[&(e^{\psi(y)} - 1)(e^{-d(y)r} - e^{-d(y)\tau})(e^{d(y)r} - e^{-d(y)T})\\ &+ (e^{\varphi(y)+\psi(y)} - 1)e^{-d(y)\tau}(e^{d(y)r} - e^{-d(y)T})\\ &+ (e^{\varphi(y)} - 1)e^{-d(y)\tau}(e^{-d(y)\tau} - e^{-d(y)r})\Big]\sigma(dy).\end{aligned} \tag{5.3.10}$$

Now calculate the right-hand side of (5.3.9). According to (5.3.2)

$$\mathbb{E}_{D,Y_r^T} e^{\langle \varphi, Y_{\tau-r-T}^T\rangle} = \exp\{\langle \ln[1 + e^{-(\tau-r)d(x)}(e^{\varphi(x)} - 1)], Y_r^T\rangle\}.$$

Then we get the following expression for the right-hand side of (5.3.9):

$$\mathbb{E}^{\pi_\rho}(\mathbb{E}_{Y_r^T}[e^{\langle \varphi, Y_{\tau-r-T}^T\rangle}] e^{\langle \psi, Y_r^T\rangle}) = \mathbb{E}^{\pi_\rho}[e^{\langle \varphi, Y_{\tau-r-T}^T\rangle}]\mathbb{E}^{\pi_\rho}[e^{\langle v+\psi, Y_r^T\rangle}], \tag{5.3.11}$$

where

$$v = \ln[1 + e^{-(\tau-r)d(x)}(e^{\varphi(x)} - 1)]. \tag{5.3.12}$$

Using (5.3.3) we obtain

$$\mathbb{E}[e^{\langle\varphi,Y^T_{\tau-r-T}\rangle}] = \exp\left\{\int_X \frac{b(y)}{d(y)}(e^{\varphi(y)}-1)(1-e^{-(\tau-r)d(y)})\sigma(dy)\right\}, \tag{5.3.13}$$
$$\mathbb{E}[e^{\langle v+\psi,Y^T_r\rangle}] = \exp\left\{\int_X \frac{b(y)}{d(y)}(e^{v(y)+\psi(y)}-1)(1-e^{-(T+r)d(y)})\sigma(dy)\right\}.$$

Inserting the expression for v, given by (5.3.12) in the last term can be rewritten as

$$\exp\left\{\int_X \frac{b(y)}{d(y)}(e^{\psi(y)}[1+e^{-(\tau-r)d(x)}(e^{\varphi(x)}-1)]-1)(1-e^{-(T+r)d(y)})\sigma(dy)\right\}. \tag{5.3.14}$$

Inserting (5.3.13) and (5.3.14) in (5.3.11) we obtain

$$\exp\left\{\int_X \frac{b(y)}{d(y)}[(e^{\varphi(y)}-1)(1-e^{-(\tau-r)d(y)})\right.$$
$$\left.+(e^{\psi(y)}[1+e^{-(\tau-r)d(x)}(e^{\varphi(x)}-1)]-1)(1-e^{-(T+r)d(y)})]\sigma(dy)\right\}.$$

After elementary calculations we see that this expression coincides with (5.3.10), what implies the lemma. □

Now, analogously to the pure birth process, we will take into account the influence of a (nonepistatic) selection cost function $\Phi : \Gamma \longrightarrow \mathbb{R}_+$ given by

$$\Phi(\gamma) := \langle h,\gamma\rangle = \sum_{x\in\gamma} h(x),\ \ h \geq c > 0.$$

Because the configuration γ contains, in general, an infinite number of points, the cost function Φ is well-defined only in a bounded region $\Lambda \subset X$.

We are interested in the measure ν^h on the space $\hat{\Gamma}(X\times(-\infty,0],\mathbb{R}_+)$, the so-called Gibbs perturbation of marked Poisson measure π_ρ. We denote by $\nu^h_{\Lambda,T}$ the restriction of ν^h to $\hat{\Gamma}(\Lambda\times[-T,0],\mathbb{R}_+)$, which is defined for $\Lambda\subset X$ as

$$d\nu^h_{\Lambda,T}(\hat{\gamma}) = \frac{1}{Z_{\Lambda,T}}\exp\left\{-\int_{-T}^0 \Phi_\Lambda(Y^T_t(\hat{\gamma}))dt\right\} d\pi^{T,\Lambda}_\rho(\hat{\gamma}),$$

where $Z_{\Lambda,T}$ is the normalizing constant

$$Z_{\Lambda,T} = \int_{\Omega(\hat{\Lambda}_T)} \exp\left\{-\int_{-T}^0 \Phi_\Lambda(Y^T_t(\hat{\gamma}))dt\right\} d\pi^{T,\Lambda}_\rho(\hat{\gamma}).$$

First we will show that $\nu^h_{\Lambda,T}$ still remains a Poisson measure. For this we calculate its intensity measure by computing the Laplace transform of $\nu^h_{\Lambda,T}$.

Lemma 5.3.2. For $F(\hat{\gamma}) = e^{\langle f,\hat{\gamma}\rangle}$, $\langle f,\hat{\gamma}\rangle := \sum_{(x,s,l)\in\hat{\gamma}} f(x,s,l)$, where $\hat{\gamma} \in \Omega(\hat{X}_T)$, we have

$$\int F(\hat{\gamma}_\Lambda)\exp\left\{-\int_{-T}^{0}\Phi_\Lambda(Y_t^T(\hat{\gamma}_\Lambda))dt\right\}d\pi_\rho^{T,\Lambda}(\hat{\gamma}_\Lambda)$$
$$= \exp\left\{\iiint\left(\exp\left\{f(x,s,l) - h(x)\int_{-T}^{0}1\!\!1_{[s,s+l]}(u)du\right\} - 1\right)d\rho\right\}.$$

Proof. By definition of F and Φ

$$\int F(\hat{\gamma}_\Lambda)\exp\left\{-\int_{-T}^{0}\Phi_\Lambda(Y_t^T(\hat{\gamma}_\Lambda))dt\right\}d\pi_\rho^{T,\Lambda}(\hat{\gamma}_\Lambda)$$
$$= \int \exp\left\{\sum_{(x,s,l)\in\gamma_\Lambda}\left(f(x,s,l) - h(x)\int_{-T}^{0}1\!\!1_{[s,s+l]}(u)du\right)\right\}d\pi_\rho.$$

From this we get the required result using the Laplace transform of π_ρ. □

Hence the integral w.r.t. $\nu_{\Lambda,T}^h$ is given by

$$\int F(\hat{\gamma}_\Lambda)d\nu_{\Lambda,T}^h(\hat{\gamma}_\Lambda) = \frac{1}{Z_{\Lambda,T}}\int \exp\left\{-\int_{-T}^{0}\Phi_\Lambda(Y_t^T(\hat{\gamma}))dt\right\}d\pi_\rho^{T,\Lambda}(\hat{\gamma})$$
$$= \exp\left\{\iiint \exp\left\{-h(x)\int_{-T}^{0}1\!\!1_{[s,s+l]}(u)du\right\}(e^{f(x,s,l)} - 1)d\rho\right\}.$$

Using the definition of the measure ρ and calculating the integral in u, we see that the above expression can be represented as

$$\exp\left\{\int_{-T}^{0}\int_0^\infty\int_\Lambda (e^{f(x,s,l)} - 1)\exp\{-h(x)(0\wedge(s+l) - s)\}\,b(x)d(x)e^{-d(x)l}d\sigma(x)dlds\right\}$$
$$= \exp\left\{\int_{-T}^{0}\int_0^\infty\int_\Lambda (e^{f(x,s,l)} - 1)\exp\{-((-s)\vee l)h(x)\}\,b(x)d(x)e^{-d(x)l}d\sigma(x)dlds\right\}.$$

Thus $\nu_{\Lambda,T}^h$ is a Poisson measure on $\Omega(\hat{\Lambda}_T)$ with intensity measure

$$\tau(dx,dl,ds) = \exp\{-((-s)\vee l)h(x)\}\,b(x)d(x)e^{-d(x)l}d\sigma(x)dlds.$$

Again, as before, we are interested in the "weak" limit of $\nu_{\Lambda,T}^h$ for $\Lambda \uparrow X$, $T \to +\infty$. The limit does also not depend on the order in which the limits are taken.

Theorem 5.3.3. 1) There exists the "weak" limit

$$\lim_{\Lambda\uparrow X} \nu^h_{\Lambda,T} = \nu^h_T,$$

where ν^h_T is the marked Poisson measure on the space $\Omega(\hat{X}_T)$ with the same intensity measure τ, but now as a measure on $\hat{X}_T$.

2) There exists the "weak" limit

$$\lim_{T\to+\infty} \nu^h_T = \nu^h,$$

where ν^h is the marked Poisson measure on the space $\Omega(\hat{X})$ with the same intensity measure τ, but now as a measure on $\hat{X}$.

3) There exists the "weak" limit

$$\lim_{T\to+\infty} \nu^h_{\Lambda,T} = \nu^h_\Lambda,$$

where ν^h_T is the marked Poisson measure on the space $\Omega(\hat{\Lambda})$ with the same intensity measure τ, but now as a measure on $\hat{X}_\Lambda$.

4) There exists the "weak" limit

$$\lim_{\Lambda\uparrow X} \nu^h_\Lambda = \nu^h,$$

where ν^h is the marked Poisson measure on the space $\Omega(\hat{X})$ with the same intensity measure τ, but as a measure on $\hat{X}$.

The main object of our interest is the final distribution of mutations μ^h, i.e. is the distribution of end points of bars. Recall that we have chosen the time range so that the final time is 0. We obtain μ^h, similar to the above construction, as the limit of final distributions $\mu^0_{\Lambda,T}$ for given bounded volume and finite time. The measure $\mu^0_{\Lambda,T}$ on $\Gamma(\Lambda)$ is defined for $F(\eta) = e^{\langle f,\eta\rangle}$, $\eta \in \Gamma(X)$ by

$$\begin{aligned}\int_{\Gamma(X)} F(\eta_\Lambda) d\mu^0_{\Lambda,T}(\eta_\Lambda) &:= \int_{\Omega(\hat{X}_T)} F(Y_0^T(\hat{\gamma}_\Lambda)) d\nu^h_{\Lambda,T}(\hat{\gamma}_\Lambda) \\ &= \frac{\int F(Y_0^T(\hat{\gamma}_\Lambda)) \exp\{-\int_{-T}^0 \Phi_\Lambda(Y_t^T(\hat{\gamma}_\Lambda))dt\} d\pi_\rho^{T,\Lambda}(\hat{\gamma}_\Lambda)}{\int \exp\{-\int_{-T}^0 \Phi_\Lambda(Y_t^T(\hat{\gamma}_\Lambda))dt\} d\pi_\rho^{T,\Lambda}(\hat{\gamma}_\Lambda)}.\end{aligned} \tag{5.3.15}$$

Lemma 5.3.4. Let $F(\eta) = \exp\{\int f(x)\eta(dx)\}$, where $\eta \in \Gamma(X)$, $f \in C_0(X)$. Then

$$\begin{aligned}&\int F(Y_0^T(\hat{\gamma}_\Lambda)) \exp\left\{-\int_{-T}^0 \Phi_\Lambda(Y_t^T(\hat{\gamma}_\Lambda))dt\right\} d\pi_\rho^{T,\Lambda}(\hat{\gamma}_\Lambda) \\ &\quad = \exp\left\{\iiint \left(\exp\left\{f(x)1\!\!1_{[s,s+l]}(0) - h(x)\int_{-T}^0 1\!\!1_{[s,s+l]}(u)du\right\} - 1\right) d\rho\right\}.\end{aligned}$$

The statement is a corollar of Lemma 5.3.2. Using the observation that for $f \in C_0(X)$, $\hat{\gamma} \in \Omega(\hat{X}_T)$ we have that $\langle f, Y_0^T(\hat{\gamma})\rangle = \langle F, \hat{\gamma}\rangle$, where $F(x,s,l) = f(x)1\!\!1_{[s,s+l]}(0)$.

Now we can calculate (5.3.15)

$$\int_{\Gamma(X)} F(\eta_\Lambda) d\mu^0_{\Lambda,T}(\eta_\Lambda)$$
$$= \exp\left\{\iiint \exp\left\{-h(x)\int_{-T}^0 1\!\!1_{[s,s+l]}(u)du\right\}(e^{f(x)1\!\!1_{[s,s+l]}(0)}-1)\rho(dx,dl,ds)\right\}.$$

By definition of ρ the expression above can be written as

$$\exp\left\{\int_{-T}^0\int_0^\infty\int_\Lambda \exp\left\{-h(x)(0\wedge(s+l)-s)\right\}(e^{f(x)}-1)1\!\!1_{[s,s+l]}(0)\rho(dx,dl,ds)\right\}$$
$$= \exp\left\{\int_{-T}^0\int_0^\infty\int_\Lambda \exp\left\{sh(x)\right\}(e^{f(x)}-1)1\!\!1_{[s,s+l]}(0)b(x)d(x)e^{-d(x)l}d\sigma(x)dlds\right\}.$$

Calculating the integrals w.r.t dl and ds we obtain

$$\int_{\Gamma(X)} F(\eta_\Lambda)d\mu^0_{\Lambda,T}(\eta_\Lambda) = \exp\left\{\int_\Lambda (e^{f(x)}-1)b(x)\frac{1-e^{-T(h(x)+d(x))}}{h(x)+d(x)}d\sigma(x)\right\}.$$

Again, as before, we are interested in the "weak" limit of $\mu^0_{\Lambda,T}$ for $\Lambda \uparrow X$, $T \to +\infty$. The limit does also not depend on the ordern which the limits are taken.

Theorem 5.3.5. 1) There exists the "weak" limit

$$\lim_{\Lambda\uparrow X}\mu^0_{\Lambda,T} = \mu^h_T,$$

where μ^h_T is the Poisson measure on $\Gamma(X)$ with intensity measure

$$b(x)\frac{1-e^{-T(h(x)+d(x))}}{h(x)+d(x)}d\sigma(x).$$

2) By the dominated convergence theorem there exists the "weak" limit

$$\lim_{T\to+\infty}\mu^h_T = \mu^h,$$

where μ^h is the Poisson measure on $\Gamma(X)$ with intensity measure $\frac{\sigma(dx)}{h(x)+d(x)}$.

3) There exists the "weak" limit

$$\lim_{T\to+\infty}\mu^0_{\Lambda,T} = \mu^h_\Lambda,$$

where μ_T^h is the Poisson measure on $\Gamma(\Lambda)$ with intensity measure $\frac{\sigma(dx)}{h(x)+d(x)}$.

4) By the dominated convergence theorem there exists the "weak" limit

$$\lim_{\Lambda \uparrow X} \mu_\Lambda^h = \mu^h,$$

where μ^h is the Poisson measure on $\Gamma(X)$ with intensity measure $\frac{\sigma(dx)}{h(x)+d(x)}$.

Remark 5.3.6. Obviously Theorem 5.2.6 of the previous section is a special case of the theorem above for death rate $d(x) \equiv 0$.

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