The Dynamic Approach to Heterogeneous Innovations

(Anton Bondarev)

The author thanks H. Dawid, F. Riedel, J.-M. Bonniseau and I. G. Pospelov.

Abstract. In this work the dynamical framework which combines different aspects of innovative activity is analyzed. First the basic model with finite time horizon is constructed where the single agent (planner) is optimizing his stream of investments into the process of creation of new products together with investments into the improvement of already invented products. The range of products which might be invented is given by the bounded real interval. Next the role of heterogeneity of the investment characteristics of these new products is analyzed and it is demonstrated that this heterogeneity plays the essential role in the dynamics of the model. Further on the analysis is extended to account for long-run behavior of the planner on the infinite-time horizon. Steady-states of the system are derived and their stability is analyzed. Two following chapters of the work deal with two different extensions of the basic model. Possibilities for further analysis of the given approach and the difference in conclusions and policy implications with earlier approaches to innovations' analysis are demonstrated. First the effect of patenting policy on the innovative activity is taken into account. There the diversity of possible outcomes with respect to the patent's length is demonstrated and it is argued that this effect may not be captured without the precense of heterogeneity of innovative products under analysis. In the last chapter the extension which introduces several innovating agents is considered. There the subsequent optimal control problem is transformed into the differential game in infinite-dimensional space. The set of piecewise-constant strategies is derived and it is shown to be the only one stable set in the class of at most linear feedback strategies. In an effect the specialization of innovative activity between agents is observed and it is demonstrated that this specialization has a foundation in internal characteristics of these agents. The suggested work provides several prospects for further enrichment and development in all the areas being considered.

Contents

Preface

One of the basic sources of economic growth is technological progress, as it is argued by economic growth theory. Technological progress emerges as a result of the innovative activity of economic agents. That's why modeling innovations is one of the key areas of modern economics. Starting from 1960's there have been a lot of attempts of such modeling and incorporation of innovations into macroeconomic models. This strand of literature is concentrated on the effects of technological progress on the economic growth, rather then on the nature and source of this progress. That's why nowadays these theories are referred to as theories of exogenous technological progress and/or exogenous economic growth. It was soon recognized, that it is important to model explicitly the process of innovations themselves to endogenize the technological progress. First such attempts have been made in the era of classical growth theory in 1960's. All these theories may be divided into three groups: disembodied technological change, embodied technological change and induced technological change.

In the framework of neoclassical growth theory the technological change has been treated as the only source of economic growth and this source of growth received much of the attention of economists. First explicit models of technological change were that of disembodied one. They assumed that technological progress is reflected in the growth of productivity of labour and capital in the aggregate production function. There was no explicit formulation of micro-foundations of such growth of productivity of factors. They have been assumed to grow with some exogenous rate. Such an approach was not satisfying to explain technological change as it was not compatible with stylized facts and moreover, such a technological progress had to be Harrod-neutral all the time (that is, only the productivity of capital may increase over time, not that of labour) which is not the case.

Another approach, that of induced technological change, endogenized technological progress and made it the object of profit maximization on the firms' level. However, this rate of progress does not depend on any resources and is bounded from above by some invention frontier which is not allowed to drift in time. This approach is much closer to present-day ones and bears signs of the microfoundation of technological progress. Examples of such models may be found in [**47**], [**48**]. It is this strand of literature where the research sector of economy was first modeled as a separate one and the role of human capital has been recognized, [**50**], [**51**]. At that time the interest in patent's length has risen. This is closely related to the discussion of the role of innovative entities in the economy , see [**20**].

Last neoclassical concept is that of embodied technological change. There the technological growth is embodied in factors of production and is distributed over time when these factors have been produced. This is one of the foundations for recent vintage models and methods being used in this literature are also employed in the suggested work for other purposes. One of the scarce works on this is [**53**]. In summary, some of the neoclassical models already contained ideas of product varieties, developed further on and that of distributed nature of technological progress. However these concepts have been developed in full only later on.

At the beginning of 1990's two new approaches to innovations in growth theory emerged, namely, Romer's (1990) model of expanding variety of products and Aghion&Howitt's (1993) model of quality ladders. Each of them addressed different aspects of innovative activity, but they both have been built on the idea of endogenizing technological change through means of modeling innovative activity.

First of them explained technological progress as the process of invention of new goods. This idea originated from works of Dixit&Stigliz, Ethier, Spence [**45**], [**44**], [**46**]. The final widely accepted form of this approach is represented by works of Romer (1990) and Grossman&Helpman (1991). In these works the technological progress is modeled as the process that expands the variety of products available on the market, thus stimulating growth through increase in the consumer demand (Dixit-Stigliz theory) or through increase in the productivity (Romer's model). However, quality (or productivity in the case of investment goods) was assumed to be constant. This idea lacked the precense of capital or other durable goods which would grant rise in productivity and this approach was sensitive to scale effects. To overcome these difficulties a number of extensions to this approach has been considered in last two decades. Namely the assumption of rising costs of R&D has been employed to weaken the scaling effect and this is also one of the reasons of adoption of the similar assumption in the given work. Some competitive effects has been considered also, as well as introduction of knowledge as one of the factors of production [**54**], [**55**], [**26**]. Current work benefits from some of the ideas used in these extensions, namely the initial knowledge about varieties is one of the key factors of the dynamics as well as rising costs of innovations at the quality side (decreasing efficiency of investments).

Second approach explained technological change as the process of creative destruction of products, based on the idea of Schumpeter, [**7**]. Every product is assumed to have varying quality, which may be increased through investments, while the outcome of these investments is assumed to be uncertain. However, quantity or variety of goods available on the market is assumed to be constant and every new 'better' product destroys the preceding one upon its invention. This approach gave birth to a vast strand of literature starting from early 1990's. Here main subject of study is competition between innovative firms and thus it is related to IO literature too. Among further extensions of this strand of literature are those considering the possibility of imitation of the leading technology as well as some more dense structure of quality ladders allowing for intermediate levels of quality to be achieved, [**27**], [**28**]. One of the extensions allowed for subsequent competition on the products' market between firms. The basic idea of the current work also follows this line. The suggested research uses the main idea of quality ladders' literature of allowing for quality growth of given products as one of the sources of innovative activity.

It is argued, that both these approaches are complementary in nature, describing two aspects of the same single process, which are going on simultaneously. At the same time, there is no unified model, which would take into account both these aspects in the dynamical framework and allow for heterogeneity of innovations. Current work uses these ideas as guidelines. Namely the process of creative destruction has its place in the growth of quality of products, as it is in the Aghion's approach and every increase in quality of a given product does not increase the overall variety of products but rather this marginally improved product replaces its predecessor. However, in the Aghion's approach there are explicit discrete generations of products while in the suggested work the whole process of quality improvement is the smooth one without jumps. On the other hand the whole range of products does not need to remain constant but rather is expanding governed by the dynamical law in the nature of Romer's work. Unlike the latter, quality of all these newly invented products is subject to change and this change is described for each product by the quality improving process.

Innovative activity has also been considered in IO literature. The process of innovations on industrial level got attention of economists rather early, since the end of 1960's. First works in this area were concerned with patenting problems and patent races. The patent is viewed as the necessary mechanism to protect the innovator and to create necessary level of incentives to innovate for the economic agents. Too long patents would create less stimuli for new inventions and hence the notion of optimal length of the patent was born. First it was introduced by Nordhaus, [**20**]. This seminal paper gave birth to a wide strand of literature on patents in IO literature where the main emphasis was on single-agent (stand alone) models of innovations. At the same time the literature on patent races assumed the precense of several competing firms in the innovative sector of the economy and this gave birth to the notion of competition in innovations. Then the question whether the competitive environment should boost innovative activity or not has been raised. One of the first formal models with patent races is of Loury, [**19**]. In all this early literature the process of innovations themselves remained something like a black box, that is, it was assumed that the process of innovations, although depending on some exogenous factors is just 'emerging'. Later on it has been noticed, that the market structure, such as the number of competing innovative firms present, may substantially influence the speed of innovations also. See [**18**] as an example of such influence of the number of competitors on the innovative activity. Even more later on the nature of interactions between competing innovative agents has been taken into account and it has been modeled explicitly by means of various static games as well as of differential games as in [**22**], [**21**], [**17**], where the notion of imitation (costly or costless) as well as R&D cooperations have been used to describe the nature of strategic interactions between different innovative agents.

Later on some uncertainty has been considered as one of the fundamental features of the innovative process and formal models of patent races under uncertainty have been constructed. However, till not that long ago the process of innovations itself even if assumed to be the dynamic one, has been viewed upon as a monotonic single-shot process of 'investing something' like it is in the papers on patent races, where the race is going on for receiving one single patent for a given innovation. In the middle of 1990's this has been replaced by the widely acknowledged notion of sequential or cumulative innovations framework. In this approach there is a sequence of different innovations going on in the same market (economy) one after another and which are based on preceding innovations. See [**16**] as one of the first examples of such a framework. Still, all these innovations have been of the same nature and have been built up one on the base of another. At the same time it was widely acknowledged that from the IO point of view there are at least two types of innovative activity, namely, cost-reducing innovations and product-introducing ones. It has been noted, that the given R&D firm may have several different research projects at a time and they may have different nature and/or complexity. Hence not long ago the notion of heterogeneous innovations has been born. It is this framework in which the suggested work belongs.

One of the first models in such a framework may be considered the work of Hopenhayn, [**15**] which is mainly devoted to the problem of patenting in the precense of multiple research projects for a single agent. He considers as examples both previously referred basic models of quality ladders and variety expansion, but still he does not unify them. Moreover, his framework is more or less static in nature as are later works in the field [**12**], [**13**], [**14**]. Current work suggests the unified model of heterogeneous innovations in dynamic context.

Proceedings in this work are related to these strands of literature in several ways. It benefits from ideas of innovative activity as it is represented in New Growth theories. At the same time it is based on the idea of heterogeneous innovations which belongs to the IO literature on Economics of Innovations. Ideas of patents as limited life-cycles of products, as well as ideas of imitation and cooperation are taken from literature on innovations. Dynamic framework being constructed follows mainly the guidelines of patent literature of the 1980's with extension on distributed systems.

The basic formulation does not include any uncertainty, competition or notions of patents, there are no notions of consumer, economy, social planner as well. It is mainly concentrated on the analysis of the technological side of innovative process. However, it is demonstrated that mere technological constraints may govern much of the behavior of innovating agents on the industry level. The framework does not include profitability, prices, or supply-demand interaction. Nevertheless, notions of patent and strategic interaction of agents may be included rather naturally in it. At the same time since the suggested framework is free from market-specific mechanisms, it also may be considered as a prototype for the extension of literature on technological change in the sense of generalizing results of Aghion and Romer. All these define the area of current research as in between standard literature on innovations at IO level and New Growth theories.

After constructing such a unified dynamical framework the role of heterogeneity in the characteristics of innovative products is studied in the first chapter of the work. Then the basic analysis is extended to infinite-time horizon to obtain information on the steady-states and their stability. This part of the work is inspired mainly by the New Growth Theory rather then by the recent findings in Economics of Innovations. However due to the restrictions being made in the basic model it belongs to the literature on heterogeneous innovations. The importance of differences between homogeneous and heterogeneous innovations is explored and the role of dynamic framework is also illustrated there.

In two subsequent parts this basic model is modified to consider effects of patents and their length as well as of competition in the space of innovations. For that the Hamilton-Jacobi-Bellman dynamic programming approach together with the Maximum Principle are used and then a differential game in the space of innovations is formulated. To our knowledge it is one of the few examples of differential game with explicit solution in infinite-dimensional space. The role of products' diversity and dynamics in the results obtained and their difference from previous works in the field is discussed at the end of each chapter.

In the chapter with patents the main point of interest is how the limited life-cycle for all invented products may affect the behavior of the innovator. Notice that in the framework with only one agent and in the absence of any regulating mechanism and consumers this is equivalent to the limited length of patents for newly invented products. It is pointed out, that in the developed framework the question of limited optimal patent's length in the precense of sequential and heterogeneous innovations is also resolved positively and yield the finite length of a patent and this is in agreement with the literature. However the full rigorous proof of that has still to be obtained under the framework discussed, as it turns out to be much more dificult then in homogeneous case. At the same time, there is no need to employ notions of social welfare functions or to model the consumer side of the market to make this conclusion and this distinguishes the approach of the chapter from being previously used. It is shown that it is the heterogeneity of innovations which stimulates innovative activity with limited patent's length even without any competition being present. Note also that this framework may be easily extended to the variable length of patents by choosing some specification of this patent's length as a function of the product's index.

In the last part of the given work effects of strategic interactions in the dynamic context of two multiproduct innovative agents are considered. This work may be considered as an extension of results of Lin and Lambertini, [**14**], [**13**] into the dynamic environment and uncountable number of products. From the other hand it complements the recent model of Lambertini, [**56**] by allowing for the infinite-dimensional space of products and not only for the dynamic interaction. However such an extension proofs to be essentially different from previous findings. In particular, it supports and extends the literature on joint R&D ventures as the nature of strategic interaction but combines it with imitative behavior in quality growth. It has been noted in the beginning of 1990's that in real world economies innovative firms do not compete with each other in the invention of new products, if this invention requires substantial amount of efforts. They instead cooperate in the creation of new products and compete only in product markets further on. See [**23**], [**24**]. However due to the more general framework adopted in the suggested work it is possible to extend this idea and allow for imitation in quality (cost-reducing) innovations on the second phase of innovative activity. It is demonstrated, that oligopolistic market with such a structure may be at least as efficient in terms of the rate of innovations as it is in the case of the monopoly. It is analyzed, what conditions make oligopolistic environment more productive and what conditions make monopoly to produce more inventions per unit of time. In this way the given work contributes to the discussion on the optimal market structure for innovations as in late [**25**] where the differential-game approach is also used for this. Unlike this work, no ambiguity is found in the incentives and behavior of the agents due to the precense of different kinds of innovations in the model. Instead, it is shown that the equilibrium with imitation and cooperation may exist only if both kinds of innovations (quality improving and variety enhancing) are considered withing the unified framework. In such a situation incentives to invest less because of the imitation in quality and the incentive to invest more in variety enhancing while being the imitator are mutually balanced and this gives a possibility for the desired outcome for both agents. This framework may be also extended to the arbitrary number of agents.

In the following chapters first the basic model is constructed and then both of these extensions are considered.

CHAPTER 1

Product and Quality Innovations: A Unified Approach

1. Introduction

In this chapter the construction of basic finite-time dynamical model of quality&product innovations is carried out. The main purpose is to demonstrate the importance of such an approach when comparing to separate analysis of these two processes. It has to be noted, that to our knowledge there are no models which permit the simultaneous dynamic optimization of product and quality innovations of heterogeneous type. However some attempts has been made to bring together heterogeneous process and product innovations in a static context.

Frequently this purpose is achieved through the construction of the 2-stage static game, where on the first stage the decision upon the introduction of new good is being made and on the second - how much investment to put into the development of quality of this newly introduced product (conditional upon the successful introduction of it on the first stage). One example of such papers is [**10**] which is mainly devoted not to the interaction between both types of innovations themselves, but to the relation between organizational structure of the firm and its innovative decisions. It is shown, that the complementarity between process and product innovations is the direct consequence of the complementarity between firm's manufacturing capabilities and its research capabilities. Current work correlates with this kind of literature in the idea of simultaneous decision making upon innovations of both types. However, here the situation with multiple products to be introduced on the market with some speed, which is controlled by the innovating firm is considered. One other paper which corresponds to some extent to the suggested analysis is of Boone, [**11**]. Their the process of innovations is also formulated as the 2-stage game, but the author tackles mainly with incentives to innovate and their relation to the particular characteristics of the profit function. The current work abstracts from the market characteristics of the firm (particularly from profit function characteristics). It is sufficient to assume some linear and constant return from the increase in quality and range of products to yield the results of the model. Both these examples are static in nature and they do not handle multiproduct situations.

Later on it has been noted, that real innovative companies are often multiproduct monopolies. Papers by Lambertini, [**12**], [**13**] study the equilibrium characteristics of investments of such a monopoly. He allows for multiproduct investments, and the number of existing products may also increase in the result of product innovations. However the whole model is static because it handles only the equilibrium points of innovative policy of a monopolist. Author does not study any dynamical characteristics of product and process innovations but only the equilibrium distribution of investments. In the second paper Lambertini claims that the equilibrium level of quality investments is higher for the monopolist then the social optimum. However, more recent paper by Lin [**14**] suggests that this heavily depends on the level of economies of scope for the monopolist. In general to be able to answer this question one has to account for dynamical perspective of multiple products development and the evolution of the product space.

Methodologically the current model is closer to the recent literature on vintage capital models although it concentrates on another type of questions. It is this strand of literature where the distributed parameter optimal control models are extensively used to describe the investment policy of an agent which has capital with different dates of appearance at hand. Then his policy should depend on the distribution of the mass of his capital in past time and hence the dynamic problem the agent has to solve is of distributed parameter optimal control type. Examples of such models are [**29**], [**31**] and others. This strand of literature uses vintage capital idea to describe policy of investments on industrial level, like in [**31**], [**43**] and also to contribute to the growth theories with embodied technological progress of the neoclassical type. Somehow different approach of Hritonenko, [**32**] uses integral equations of Lotka-Volterra type to describe the dynamics of the overall mass of capital instead of partial differential equations but also in the optimal control framework. To our purpose the distributed optimal control method is more revelant. The difference of the suggested work methods is that it does not use delayed structure of capital as in vintage capital models. For purposes of this work it is sufficient to assume the distribution over the products' space. This significantly simplifies the analysis while the delayed time structure of the mass of innovations have little of interest at this stage.

One of the few dynamic approaches to modeling heterogeneous innovations is the work of Hopenhayn&Mitchell, [**15**] which handles the innovative process in a rather general way using operators in a Banach space. However their work is mainly concentrated on the patent policy and handles innovative process in a sense of previous theories, namely of Shapiro, [**34**]. There is no underlying process of generation of new products, instead there is a static space of ideas from which an innovator is making a random draw of an idea to develop. At the same time all these ideas may differ from each other in there value and thus the heterogeneity of innovations is observed. But this heterogeneity is not that important for the dynamics of the overall model, since only one innovation is developed at every given time so the innovative agent does not have a choice between different investment/innovative strategies.

The suggested approach combines ideas of Hopenhayn&Mitchell and of Lambertini and Lin in a way that innovations are assumed to differ in their characteristics from each other as in [**15**] and in the same time the appearance of the new products on the market as in [**12**], [**14**] is allowed in the dynamic context.

So it is argued, that to handle the innovative activity in multiproduct situations one have to consider the dynamic context. At the same time, there is no such a model, which would take into account these dynamic interactions between different types of investments into multiple products simultaneously in the literature on economics of innovations. This chapter tries to fill this gap. It concentrates on two questions, namely:

- How expansion of products' variety influences quality innovations of all different already existing and new products?
- What is the role of structural characteristics of these products by themselves in innovative process?

To answer these and related questions the model is built using optimal control theory methods. The basic framework discussed below allows for a very general formulation of innovative activities. However in the current chapter the simplified approach is chosen as the main goal is just to demonstrate the importance of such a unified approach to innovations. For that uncertainty is neglected in the model despite the importance of uncertainty in innovations which is widely acknowledged now. More then this, only one agent is modeled and there are no possible strategic interactions here. It is also assumed that all products are present since their invention till the terminal time of the overall model. Two last simplifying assumptions are relaxed in subsequent chapters of the work.

The analysis of the introduced model reveals the importance of the interaction between variety expansion process and the quality innovations. It turns out, that the most essential characteristic of products to be invented ('product space') is the heterogeneity of some parameters across products. If all products have similar properties, then the overall innovative process may be described by a simple optimal control model with only two state variables as in [**56**]. However, if these potential products are not similar to each other in their investments efficiencies or some other technological parameters, explicit formulation of the infinite-dimensional model is the natural way to capture the link between investment strategies and choices between variety expansion intensity and speed of quality growth. This link has essential influence both on variety expansion and quality growth processes. It is demonstrated that overall dynamics of innovations is different under the assumption of the absence of such a link and with the explicit introduction of it into the system. Such more complete system has much more complicated dynamics. Despite of the number of simplifying assumptions (linearity, single agent, compactness, etc.) the model introduced below is capable of reproducing the number of well known facts concerning innovations and reacts on the change of parameters in a very intuitive way. Due to the simplicity of the structure, the model has plenty of useful future extensions.

The structure of the chapter is as following. First the basic framework suggested and importance of the assumptions being made in the process are discussed. Second the general model in the form of distributed parameter optimal control one is constructed. After that some necessary theoretical properties of such a construction, namely the structure of state-space and control space are analyzed as well as the result on the existence of non-zero adjoint variables is proved. In the subsequent section general solution and its properties are described. It turns out that not much may be stated on this stage concerning the model's dynamics, except of the general nature of interactions between different types of innovations. Specifically, it turns out that only the characteristics of the boundary product's quality dynamics are revelant for the variety expansion dynamics, not those of all the mass of products invented. This justifies the choice of distributed parameter optimal control method in the form of differential equations rather then integral ones. However, the explicit solution may not be achieved at this stage. That's why in the following section the model is simplified by allowing all products to be identical, thus turning the model into the homogeneous one. This simple case helps to analyze more general and complicated properties of the model. It turns out that even in homogeneous case the link between different types of innovations influences the dynamics substantially although due to the similar nature of all products this link is rather static. Rest of the chapter is devoted to the heterogeneous case. However, only some special type of heterogeneity which yields linear differential system as the result is allowed. Namely decreasing efficiency of investments in quality growth across products is assumed. The overall dynamics of quality investments is discussed and the dynamics of the whole model is reconstructed, combining results of previous sections. Then the analysis of influence of various parameters changes on the dynamics of the system is carried out. In the last section results and understanding achieved upon the unified process of heterogeneous innovations are described and some possible future extensions and refinements of the model suggested here are mentioned.

2. Assumptions and Basic Framework

To model the process of expansion of products' variety and quality growth simultaneously the notion of the products' space is introduced. This space contains as elements all products which are already invented as well as potential products that may be invented in the future. Every product has its own characteristic - its quality. This characteristic is not constant but is the function of time and investments. From this point of view, space of products is the functional space and its elements are quality functions for every product i. Then products themselves (both already invented and potential ones) are dimensions of such a space. Assume that number of products (that is, dimensions of the product space) is infinite. More than this, assume that this number is the real number, so dimensionality of the product space is uncountable. In such a framework it is no longer correct to speak about number of products but rather about the range of them. This range is assumed to be bounded from above by some maximal range of products which can be invented in a given system (economy, market).

The space of products thus consists of infinite-dimensional vector-valued functions $Q(t)$, which describe evolution of all products' qualities over time. Such a space is hard to analyze in general and additional structure is put onto it to make in manageable. For this assume that quality growth of every product does not explicitly depend on other products. Then every function $Q(t)$ may be represented by the infinite dimensional system of real-valued functions $q_i(t)$ or, equivalently, by the function of two arguments $q(i,t)$, where i is the index of a product and t is time. In the last representation it may be shown using standard arguments from functional analysis, [**1**], that the space of functions $q(i,t)$ is isomorphic to the $\mathbf{L}^2$ space, provided it has a compact support. So assume that the range of product as well as time are compact subsets of $\mathbb{R}$:

$$\begin{aligned} t &\in [0,..,T] = \mathbf{T} \subset \mathbb{R}_+; \\ i &\in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+. \end{aligned} \tag{1.1}$$

Boundedness of Lebesgue integrals over that space is assumed but this requirement is not essential for the analysis, as the problem may be casted in a Banach space. Denote the space of such functions by $\mathbf{L}^2(\mathbf{T} \times \mathbf{I}; \mathbf{Q})$, where $\mathbf{Q} \subset \mathbb{R}^+$ as well.

Process of the expansion of variety of actual products is described by a one-dimensional function of time, $n(t)$, which takes values in the space $\mathbf{I}$. So range of existing products is constantly changing over time. It is then natural to require that quality for products which are not yet invented, cannot change from its initial level. Then $n(t)$ dynamics represents the motion in the space of potential products along the subspace represented by $q(i,t)$ functions. It describes maximal index i of this function with non-zero value. Analytically the last requirement can be expressed as a constraint upon the function $q(i,t)$:

$$\begin{aligned} q(\dot{i,t}) &= 0 \mid_{i \geq n(t)}; \\ &\forall i \in \mathbf{I}. \end{aligned} \tag{1.2}$$

The whole process of innovations is then described by continuous expansion of the range of products available and by simultaneous growth of quality of all products which are already invented in some infinite dimensional product space $\mathbf{Q}$, while quality growth process for each product is independent from other processes and is launched at the time when this product is actually invented, (1.2).

To impose control over these processes denote investments being made in the variety expansion and in quality growth of every product separately, $u(t)$ and $g_i(t)$ respectively. To ensure existence of optimal behavior assume these two process to be positive real valued and bounded from above. Dynamics of variety expansion and quality growth are determined through investment policy, that is the actual choice of investments at each point in time in both directions. No uncertainty is present in the model for simplification purposes.

3. Model

Under the basic framework described in previous section to cast the model into the optimal control framework, the scheme of so-called 'planned' innovations is adopted: there is only one agent (social planner or, alternatively, the monopolist) who maximizes the output of innovations in any given period of time over the fixed time horizon according to some objective functional. It is defined as:

$$J \stackrel{\text{def}}{=} \int_0^T e^{-rt} \left(\int_0^{n(t)} \left[q(i,t) - \frac{1}{2} g(i,t)^2 \right] di - \frac{1}{2} u(t)^2 \right) dt \to \mathbf{max} \tag{1.3}$$

Planner is maximizing integral sum of qualities of all products invented until each time t minus investments being made to every invented product's quality and to the overall expansion process over the planning horizon. There is no sign of prices or profit in this formulation. Market clearing mechanism and all the mechanics behind the market structure are neglected. One way to motivate such a formulation of the objective functional is to assume linear in every product profit function and unitary price of each product. However, the main point of interest here is not the profit maximization, but the maximization of the output of innovations in every given moment of time. It is equivalent to the linearity of profit function which is a standard assumption in innovation literature (consider Lambertini, [**12**] as one of later examples). One may treat the planner here either as the central authority in some centralized planned economy or as the monopolist on some market.

Dynamics of quality growth and variety expansion processes are governed by subsequent dynamic equations:

$$\begin{aligned} \dot{n(t)} &= \alpha u(t); \\ \dot{q(i,t)} &= \gamma(i) g(i,t) - \beta(i) q(i,t); \\ \forall i &\in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+; \\ \forall t &\in [0,..,T] = \mathbf{T} \subset \mathbb{R}_+. \end{aligned} \tag{1.4}$$

and static constraints:

$$\begin{aligned} u(t) &\geq 0; \\ g(i,t) &\geq 0; \\ 0 \leq n(t) &\leq N; \\ q(i,t) \mid_{i=n(t)} &= 0; \\ q(i,0) &= 0, \forall i \in \mathbf{I}; \\ n(0) &= n_0 \geq 0. \end{aligned} \tag{1.5}$$

This assumes zero initial quality for all products and some fixed initial range of products available. Observe, that the fourth constraint in (1.5) is equivalent to (1.2), provided investments to quality growth are nonnegative. Next observation concerns $\gamma(i)$ and $\beta(i)$ functions. These are functions of efficiency of investments to every product's quality and rate of quality decay in the absence of investments depending on the product's index i respectively. These two functions represent structural characteristics of the products' space being considered as a whole, as they define relative differences in products as functions.

Expressions (1.3), (1.4), (1.5) together constitute a distributed parameter control system. In general such systems may be hard to solve, but making use of the assumption of the independence of quality growth processes for every i it can be equivalently written in the form of infinite-dimensional optimal control system with respect to $q_i(t), g_i(t)$ functions.

Infinite dimensional representation of the system (1.3), (1.4), (1.5) allows for the application of Maximum Principle, while each $q_i(t)$ function represents a separate state variable. As a result, one have system of $N+1$ first order conditions as well as adjoint equations. Some details of the theoretical treatment of the introduced model are presented below.

4. Theoretical Results

In this section some theoretical results concerning the model (1.3), (1.4), (1.5) are presented. One of the reasons to transform the model from distributed parameter form to infinite dimensional one is that maximum principle is easier to prove for infinite dimensional problem then for the distributed system. Note, that any distributed parameter system may be transformed into the infinite dimensional ODE system. However, preservation of optimality results is not granted. That is, solution to infinite dimensional model may be not an optimal solution to the subsequent distributed parameter problem due to the form of objective functional (1.3). In formal terms this means that convergence with respect to $\mathbf{L}^2(\mathbf{T} \times \mathbf{I}; \mathbf{Q})$ norm implies but is not implied by convergence in components (projection spaces for each $i \in \mathbf{I}$) in general. However in the case of the model (1.3), (1.4), (1.5) one may show such an equivalence. For this it is sufficient to show the continuity of all projections of the functional (1.3) along the $\mathbf{I}$ index space.

PROPOSITION 1. *Objective functional (1.3) is continuous with respect to $\boldsymbol{I}$ and $\boldsymbol{T}$ spaces.*

PROOF. One need only to show the continuity of $\int_0^{n(t)} q(i,t) - \frac{1}{2} g(i,t)^2 di$ term in (1.3). For every given $n(t)$ it is a function of the interval in the subsequent $\mathbf{L}^2$ space, generated by $q(i,t)$ and $g(i,t)$ functions. This function of an interval is continuous if the generating function of a point has countable number of discontinuities. This is true for the function given, as $q(i,t)$ is a continuous function (since its a state variable) and $g(i,t)$ is assumed to have finite number of discontinuities in t for every i as any control function. These implies continuity with respect to $\mathbf{T}$ space. The only problem is possible discontinuity along $\mathbf{I}$ space, since this index space is a subset of real numbers.
Consider projections of the form $g_t(i), q_t(i)$. These are functions of the index for every t. Note that functions $q_t(i)$ are continuous as the last condition in (1.5) is exactly the continuity requirement. Regularity of $g_t(i)$ functions depends on the regularity of parameter functions $\gamma(i), \beta(i)$ only which are arbitrary at the moment. It is sufficient to assume some regularity conditions on functions $\gamma(i), \beta(i)$ to grant objective functional's continuity. For purposes of the rest of the work assume them to be continuous functions of i. Then functions $g_t(i)$ can have at most countable number of discontinuities and this is sufficient for functions of interval $\int_0^{n(t)} q(i,t) - \frac{1}{2} g(i,t)^2 di$ to be continuous. □

With continuity of the objective functional one may freely transform the problem (1.3), (1.4), (1.5) to the infinite dimensional framework.

First note that (1.5) imply compactness of state space for every $q_i(t)$ and for $n(t)$ both. Second observation is that the whole system of differential equations (1.4) may be decoupled into equations for $n(t)$ and for $q(i,t)$. The only link between the two components of the system is through the (1.5). The system for quality growth is then written in the form of infinite dimensional system of ODEs:

$$\dot{q_i}(t) = \gamma_i g_i(t) - \beta_i q_i(t), \qquad \forall i \in [0, .., N] = \mathbf{I} \subset \mathbb{R}_+. \tag{1.6}$$

Which may be written in operator form:

$$\dot{q}(t) = Aq(t) + Bg(t). \tag{1.7}$$

This system of controlled equations is linear and as such is object of Hille-Yosida theory. To grant implementation of Maximum Principle one have to be sure that the uncontrolled part of the system (1.6) is a well-posed Cauchy problem. This can be done through Hille-Yosida theorem [**3**]. For the system (1.7) it is particularly simple, since operator A has diagonal form and does not depend on t explicitly. The formal proof is not given here as it follows standard textbooks [**3**]. It is sufficient to note that this relies on the proof that operator A has a full rank and its invert is bounded. This obviously depends on the choice of $\gamma(i), \beta(i)$ functions, so assume these to be not zero everywhere and with bounded inverses. Note, that this is the second regularity assumption imposed on parameter functions.

Next observe that optimal controls exist. First observe that control space possesses product topology and may be decoupled into controls over variety expansion process and quality growth. First of these is one dimensional and bounded, second is infinite dimensional and bounded in each coordinate. Define control space as

$$\mathcal{J} = \mathcal{U} \times \mathcal{G}. \tag{1.8}$$

Taking into account constraints (1.5) one may define admissible control set as

$$\mathcal{J}_{ad} = \mathcal{U}_{ad} \times \mathcal{G}_{ad}. \tag{1.9}$$

where both subspaces are compacts since boundedness of investments. Then the whole admissible control space is compact also. This yields the existence result.

PROPOSITION 2. *Optimal controls $u(t)^*$, $g(t)^*$ exist.*

Proof follows from compactness of admissible control space. And one obtains a useful corollary:

COROLLARY 1. *Admissible control space is spike-complete.*

Proof follows directly from compactness of the admissible control space. Spike-completeness of the control space is the necessary condition for the formulation of Maximum Principle and does not coincide in general with compactness for infinite-dimensional spaces. In plain words spike-completeness means that control space is closed under the operation of spike-perturbations of any control trajectory which belongs to the admissible space. As spike-completeness is necessary for the existence of optimal controls, in this case this important property is just a direct consequence of the existence result above. For more details see [**3**].

Now one may make use of Maximum Principle approach to obtain optimal controls. For that some standard properties of control space and control system have to be fulfilled, including regularity of the B operator in (1.7) and completeness of the control space with respect to perturbations. These two are straightforward to show and rigorous proofs are not given here. It is sufficient to note that completeness of the control space with respect to spike perturbations is the consequence of compactness of admissible control space and regularity of operator B follows from the fact that it is diagonal and does not depend on time. Then the only regularity requirement is on the functions $\gamma(i)$, which is already assumed above.

The difficulty of infinite dimensional problem is that the existence of non zero optimal set of adjoint variables is not granted. It has to be proved separately. For this make use of the following lemma:

LEMMA 1. [**3**] *Let $\{t_n\}, \{\tilde{j}^n\}, \{\tilde{y}^n\}$ be the sequences of time, controls and states converging to optimal solution of the control problem. Assume that there exists such $\rho > 0$ and a precompact sequence $\{Q_n\}, Q_n \subseteq E$, such that:*

$$\bigcap_{n=n_0}^{\infty} \{t_n^{-1} R(0, t_n; \boldsymbol{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho) + Q_n\} \tag{1.10}$$

contains an interior point for n_0 large enough. Then the multiplier z is not zero.

It can be shown that requirements of this lemma hold for the given problem due to the special structure of the control and state spaces (they are compact and possess product topology). Full proof may be found in Appendix. In conclusion note that all existence and regularity results presented here directly depend on the regularity assumptions on $\gamma(i), \beta(i)$ functions. Up to now they are not specified explicitly but it is assumed that:

- These are continuous functions of i;
- Inverse functions $\gamma(i)^{-1}, \beta(i)^{-1}$ exist.

No monotonicity or differentiability requirements are necessary. To ensure economically meaningful values, $\alpha, \beta(i), \gamma(i)$ are restricted to be positive:

$$\begin{aligned} \alpha &> 0; \\ \beta(i) &> 0, \forall i \in [0, N]\,; \\ \gamma(i) &> 0, \forall i \in [0, N)\,. \end{aligned} \tag{1.11}$$

Note that $\gamma(i)$ is allowed to be zero for the last product to be invented. This is done to allow for those investment efficiency functions which yield decreasing down to zero efficiency of investments.
Observe, that these functions represent measure of heterogeneity of the product space **Q**: the more regular these functions are, the more close different products are to each other in their structural characteristics.

5. General Solution

General solution to the problem (1.3), (1.4), (1.5) is obtained through application of the Maximum Principle to the subsequent infinite dimensional system. Hamiltonian function is given by

$$\begin{aligned} \mathcal{H} = \int_0^{n(t)} \{q_i(t) - \frac{1}{2} g_i(t)^2\} di - \frac{1}{2} u(t)^2 + \\ \lambda(t) \times \alpha u(t) + \int_0^N \{\psi_i(t) \times (\gamma_i g_i(t) - \beta_i q_i(t))\} di. \end{aligned} \tag{1.12}$$

For each i one may derive first order conditions for optimal control and state trajectories for quality of every product:

$$\begin{aligned} g_i(t) &= \gamma_i \psi_i(t); \\ \dot{\psi_i}(t) &= (r + \beta_i) \psi_i(t) - 1; \\ \dot{q_i}(t) &= \gamma_i g_i(t) - \beta_i q_i(t). \end{aligned} \tag{1.13}$$

This system yield optimal control and state trajectories as functions of γ_i and β_i:

$$g_i(t) = \gamma_i \left(\frac{1 - e^{(r+\beta_i)(t-T)}}{r+\beta_i} \right);$$
$$q_i(t) = \frac{\left((e^{-rT-\beta_i(t+T)} - e^{(r+\beta_i)(T-t)})\beta_i + (1-e^{-\beta_i t})(2\beta_i + r)\right)\gamma_i^2}{(r+\beta_i)(2\beta_i+r)\beta_i}. \tag{1.14}$$

However, taking into account (1.5) these solutions are effective only for $i > n(t)$ as investments cannot be positive until that time and quality level remains zero. This means quality dynamics has a piecewise form:

$$g_i(t) = \begin{cases} \gamma_i \left(\frac{1-e^{(r+\beta_i)(t-T)}}{r+\beta_i} \right), n(t) \geq i; \\ 0, n(t) < i; \end{cases}$$
$$q_i(t) = \begin{cases} \frac{\left((e^{-rT-\beta_i(t+T)} - e^{(r+\beta_i)(T-t)})\beta_i + (1-e^{-\beta_i t})(2\beta_i+r)\right)\gamma_i^2}{(r+\beta_i)(2\beta_i+r)\beta_i}, n(t) \geq i; \\ 0, n(t) < i. \end{cases} \tag{1.15}$$

Note, that all product's qualities has some maximal attainable level, which is never reached in the finite time. This level is computed as a fixed point of a given dynamical system (1.13):

$$q_i^* = \frac{\gamma_i^2}{(r+\beta_i)\beta_i}. \tag{1.16}$$

First order conditions for $n(t)$ yield the system of two differential equations on $n(t)$ and its costate:

$$\dot{\lambda(t)} = r\lambda(t) + \frac{1}{2} g_{n(t)}(t)^2 - \psi_{n(t)}(t)\gamma_{n(t)} g_{n(t)}(t);$$
$$\dot{n(t)} = \alpha\lambda(t). \tag{1.17}$$

with boundary conditions

$$n(0) = n_0;$$
$$\lambda(T) = 0. \tag{1.18}$$

Variety expansion process however, is the process across all the states q_i. Hence, to obtain dynamics of $n(t)$ one should aggregate quality dynamics across states to return to the distributed parameter form of the problem. In that way system (1.17) depends on functions $\gamma(i)$ and $\beta(i)$. After substitution for $g_{n(t)}(t)$ from (1.15) it can be seen that system (1.17) is not time invariant for any specification of $\gamma(i)$ and $\beta(i)$ functions. Dynamics of this system depends at each point in time from the value of investments efficiency to the growth of quality of the next product to be invented and do not depend on any other ones. This system represents a recurrence relation in the space of products $\mathbf{Q}$, as $\gamma(i)$ function depends on i, which is also the value of $n(t)$ function. That means that at $i = n(t)$ $\gamma(i)$ function is the function of $n(t)$. Actual shape of dynamics of the process of variety expansion heavily depends on the shape of this function then.

Through direct integration one may obtain general equation of motion for $n(t)$, as an integro-differential equation:

$$n(t) = \int_0^t \int_T^s \gamma(n(\tau)) * e^{\beta(n(\tau)) \times f(\tau,T)} d\tau ds. \tag{1.19}$$

This is hard to analyze and existence of the solution is not granted for arbitrary $\gamma(\bullet), \beta(\bullet)$ functions.

Some general properties of the system may be captured from the form of the

general solution though. Qualities' growths for all products have similar form and differ from each other only by values of $\gamma(i)$ and $\beta(i)$ functions. Starting from the time when the product is invented, its growth process is independent of any other variables of the system. However, time when this process starts is defined through the expansion process, $n(t)$.
This last one depends heavily on the parameters' functions of the product space. This means that no single product nor its quality process affect variety expansion. Instead this last one depends on some aggregate characterization of the product space, which is given by $\gamma(i)$ and $\beta(i)$ functions. Mere existence of solution to the problem itself depends on the properties of these functions which may be viewed as fundamental characteristics of the product space itself. That is, the more regularity requirements one puts on them, the more homogeneous space of products in terms of their diversity one is considering. Note that this diversity is different from the range of products N, which simply denotes some distance between different products. Parameter functions are measures of efficiency of investments across products. They may or may not depend on the maximal range of potential products. Here the simplest case when they do depend on N is considered. That means the products' space has only one efficient characteristic of its diversity - that is range of potential products. However the general framework is not limited to these particular spaces and more general structure may be considered.

To obtain some particular solution to the problem one have to specify the form of these parameter functions. Consider as two examples the simplest case with constant functions (transforming the problem to the homogeneous one) and the case which linearizes the system (1.17) in the rest of the chapter.

6. Homogeneous Products

To observe the role of diversity of the product space, first account for the homogeneous version of it. For that assume both $\gamma(i), \beta(i)$ functions to be constant across different products:

$$\begin{aligned} \gamma(i) &= \gamma; \\ \beta(i) &= \beta; \\ &\forall i \in \mathbf{I}. \end{aligned} \tag{1.20}$$

Note, that this would not mean that products are identical in their consumption characteristics, but only that efficiency of investments into the qualities of different products do not change with the expansion of variety of products. This is the reason to refer to such a product's space as 'homogeneous'.

In such a homogeneous case every product's quality has essentially the same shape of dynamics. The only difference is in the starting date of investments (being defined from variety expansion). Then the only source for identifying different products and separating them from each other lies in the $n(t)$ space, since the piecewise form of quality functions (1.15). Solutions for quality growth in homogeneous case then are of the form (for any product i):

$$g_i^{homo}(t) = \begin{cases} \gamma_i \left(\frac{1-e^{(r+\beta)(t-T)}}{r+\beta} \right), n(t) \geq i; \\ 0, n(t) < i; \end{cases}$$

$$q_i^{homo}(t) = \begin{cases} \frac{\left((e^{-rT-\beta(t+T)} - e^{-(r+\beta)(T-t)})\beta + (1-e^{-\beta t})(2\beta+r)\right)\gamma^2}{(r+\beta)(2\beta+r)\beta}, n(t) \geq i; \\ 0, n(t) < i. \end{cases} \tag{1.21}$$

All product's qualities have the same maximal attainable level, which is never reached in the finite time. This level is:

$$q_i^* = \frac{\gamma^2}{(r+\beta)\beta}; \quad \forall i \in \mathbf{I}. \tag{1.22}$$

It may be obtained by equating to zero lefthandside of the system (1.13) with homogeneous efficiency functions. With constant $\beta(\bullet), \gamma(\bullet)$ functions this level is unique and the same for all products being invented. This level is also the saddle-point of the system, since the characteristic equation has two distinct real roots of different signs. This means that asymptotically all products' qualities would reach that point and stay there in infinite-time horizon with given initial conditions. With the assumption of homogeneity between products' investment characteristics the system (1.17) is reduced to

$$\dot{\lambda(t)} = r\lambda(t) - \frac{1}{2(r+\beta)^2}\gamma^2 \times (1 - e^{(r+\beta)(t-T)})^2; \quad \dot{n(t)} = \alpha\lambda(t). \tag{1.23}$$

This is still a non-autonomous system but may be solved in elementary functions. Homogeneous solution for variety expansion process is then:

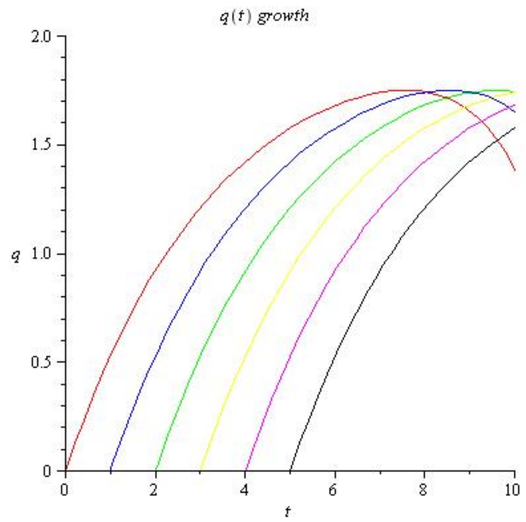
$$n^{homo}(t) = \\ = \frac{1}{(\beta+r)^3(2\beta+r)\beta r^2} \times \left(C_1 - C_2 e^{2(r+\beta)(t-T)} + C_3 e^{(r+\beta)(t-T)} - C_3 e^{r(t-T)} + C_4 t\right);$$

(1.24)

$$\lambda^{homo}(t) = \frac{\gamma^2}{(\beta+r)^2(2\beta+r)\beta r} \times \left(C_5 - C_6 e^{2(r+\beta)(t-T)} + C_7 e^{(r+\beta)(t-T)} - C_8 e^{r(t-T)}\right).$$

where $C_1, .., C_8$ denote some combinations of parameters of the model.
This solution is an increasing function of time in $n(t)$ and decreasing in $\lambda(t)$ parts. With solution at hand, one can easily analyze the influence of different parameters on the system's behavior.

Note first, that the qualitative dynamics of every product's quality does not change irrespectively of the chosen specification of the parameter functions. This happens because for any given quality function, $q_i(t)$ with i fixed, these paramaters are constants. The only requirement is their positivity. Thus, it is exactly the same as in the general model with exception that $\gamma(i), \beta(i)$ functions are now constants across indices i also. Actual projections of quality functions in $q - t$ plane are plotted below for some arbitrary underlying process of variety expansion (it defines the $q = 0$ point at t axis for all solution curves).



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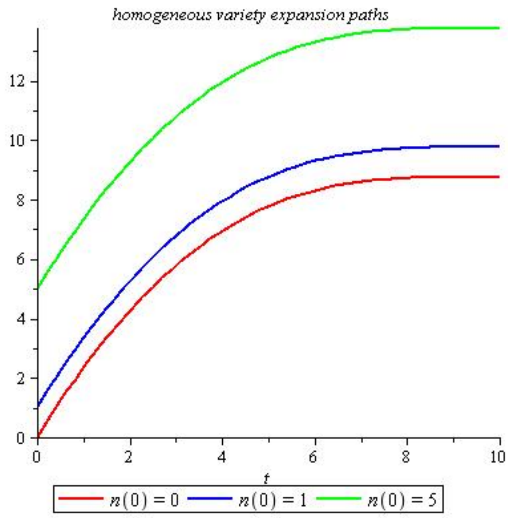
This set of quality functions is evaluated for the set of parameters:

$$SETH := [\beta = 0.2, \gamma = 0.4, r = 0.05, T = 10]\,. \tag{1.25}$$

Below several solution curves corresponding to (1.24) with the same parameter set plus $\alpha = 0.5$ and varying initial range values are plotted.



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This solution does not have any fixed points, as it can be readily seen from system (1.23) and it's characteristic equation, which has one real positive root and one zero root. One has the unbounded growth with decreasing speed of $n(t)$ function.

Next observe the parameter influence:

- Range of products N does not influence dynamics of the system in the case of interior solution as long as parameter values do not depend on it. However if $\gamma = \gamma(N), \beta = \beta(N)$, as it was assumed earlier, its influence is equivalent to the influence of these parameters;
- Length of the planning horizon, T, positively influences the range of products invented until this horizon. This is the direct consequence of the monotonicity of solution with respect to time. Shadow price of investments, $\lambda(t)$ has lesser initial values for shorter horizons;

- Rate of decay of the quality, β which is here constant across products, significantly reduces the speed of variety expansion. However, even for $\beta > \gamma$ variety expansion has a positive dynamics;
- Efficiency of investments into products' qualities, γ positively influences the dynamics of variety expansion. The same is true for efficiency of investments into variety expansion itself. The last one has more significant influence.

All the parameters influence dynamics in a quiet intuitive way. Note however, that as long as γ, β are constants, N, the maximal range, disappears from the system completely.

Now observe the character of interdependence between variety expansion and quality growth processes. In this homogeneous case the only link is one noted above, that is, quality investments to every product starts only when it is invented. At the same time all investments to quality after they start, are identical in their speed. Variety expansion process does depend on the characteristics of the product space. However this dependence is also quiet fragile: the only difference between the independent variety expansion process and the one accounted here is in constants γ, β. Obviously, this does not change dynamical characteristics of the process, but only the mass of inventions, acquired at each point in time. So one may conclude, that qualitative behavior of the system (1.23) does not differ substantially from independent development of quality and product innovations. Observe that the variety expansion process will not evolve if all products' qualities would be zero all the time, that is if $\gamma = 0$. However variety expansion process does not depend on frontier product investment characteristics as it is in the general case, (1.17). So there is some but very weak dependence of variety expansion from investments to quality growth.

This observation leads to the conclusion that some sufficient degree of heterogeneity of the product space is essential for non-trivial interdependence of quality growth and variety expansion processes. On the other hand, the simple version of the model studied in this section demonstrates, that standard models of innovations which treat both processes independently may be casted into the suggested framework as special cases for homogeneous (in the sense defined above) space of products. If one wants to describe evolution of range and quality of products which are not that identical in their investment characteristics, one has to model both processes simultaneously.

7. Linear Model

From now on some specific form of parameter functions is assumed. Assume that $\gamma(\bullet)$ function does depend on N and is monotonic and decreasing in i, while $\beta(\bullet)$ function is constant for simplicity:

$$\begin{aligned} \beta(i) &= \beta; \\ \gamma(i) &= \sqrt{N-i} \times \gamma. \end{aligned} \tag{1.26}$$

With such a form of parameter functions system (1.17) is a linear non-autonomous system:

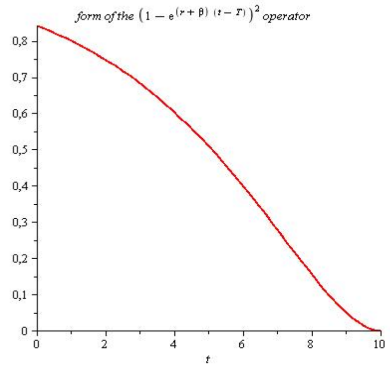
$$\begin{aligned} \dot{\lambda(t)} &= r\lambda(t) - \frac{(N-n(t))(1-e^{(\beta+r)(t-T)})^2}{(r+\beta)^2}; \\ \dot{n(t)} &= \alpha\lambda(t). \end{aligned} \tag{1.27}$$

This system does not have solutions in elementary functions (see [**4**]), but it does have real-valued solutions according to the general Sturm-Luiville theory, [**5**]. So one can analyze dynamics of this system. It can be also noted that this system provides monotonic motions in $n(t)$ direction given initial and terminal conditions: change of $n(t)$ cannot be negative and is decreasing to zero at the terminal time.

These observations give one the possibility to analyze the properties of the system (1.27). For that one may employ the method of analysis, which is valid for linear dynamical systems (non-autonomous) only, [**4**]. That is, one may introduce some artificial variable, $x(t)$, which would make the system autonomous:

$$\dot{\lambda(t)} = r\lambda(t) - \frac{(N-n(t))x(t)}{(r+\beta)^2};$$
$$\dot{n(t)} = \alpha\lambda(t);$$
$$x(t) = (1-e^{(\beta+r)(t-T)})^2. \tag{1.28}$$

In this way by fixing certain levels of $x(t)$ variable one obtains autonomous linear system in $\lambda(t), n(t)$ which has usual solution and may be analyzed by standard techniques. The $x(t)$ variable is decreasing function of time, varying from zero to one. With $r = 0.05, \beta = 0.2, T = 10$ it looks like:



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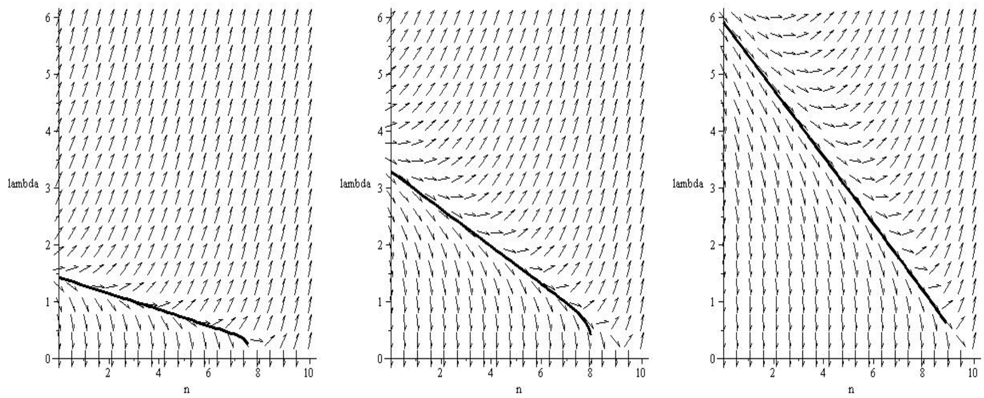
It may be treated as a contraction operator of the system (1.28), which transforms the phase space of the system. For a given system this operator spans the phase space along the $\lambda(t)$ direction due to the negative sign of it. To observe this, consider gradient fields for different levels of x (0.2, 0.5, 1) for the same parameter values:



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The only locus of the phase space which is consistent with boundary value of the costate variable ($\lambda(T) = 0$) is the locus constrained by the solid black line from above. During the evolution of a system, this locus is spanned along the lambda axis.

Then one may move this system and its resulting solutions along the $x(t)$ axis with the given speed to reconstruct the dynamics of initial non-autonomous system. Observe also, that in terminal time, $t = T$, $x(t)$ operator vanishes making the system (1.28) autonomous. Given arguments reveal, that the action of the mapping $x(t)$ on the system preserves its fixed points and changes the dynamics with respect to shadow price movements. So one may obtain stability results through investigation of the (1.28).

Note, that by simple change of variables, letting $(N - n(t)) = y(t)$, the (1.28) can be made homogeneous. Subsequent autonomous system has the fundamental system of solutions of the following form:

$$n(t) = C_1 \times e^{\frac{1}{2}\frac{r^2+r\beta+\sqrt{r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2}}{(r+\beta)}} +$$
$$+C_2 \times e^{\frac{1}{2}\frac{r^2+r\beta-\sqrt{r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2}}{(r+\beta)}} + N;$$
$$\lambda(t) = \dot{n(t)}. \tag{1.29}$$

Where C_1, C_2 are constants of integration.

This system of linearly independent solutions has rather simple dynamics in the $\lambda(t) - n(t)$ phase space. Given boundary conditions (1.18) one has monotonic motion of the system. The $n(t)$ variable is growing steadily with decreasing speed, while $\lambda(t)$ - shadow price of investments - is decreasing until zero. This is the standard dynamics of the capital accumulation problem. Observe, that the given autonomous system has a saddle type dynamics with the fixed point at $n(t) = N$. This can be checked through computing eigenvalues of the homogeneous autonomous problem, resulting from (1.28) for fixed $x(t)$:

$$|J - \delta I| = \delta(\delta - r) - \frac{\alpha}{(r+\beta)^2}x = \lambda^2 - r\lambda - \frac{\alpha}{(r+\beta)^2}x;$$
$$\delta_{1,2} = \frac{r}{2} \pm \frac{\sqrt{r^2 + 4\frac{\alpha}{(r+\beta)^2}x}}{2}. \tag{1.30}$$

It is clear, that both eigenvalues are real and of different signs, since $x \geq 0$.

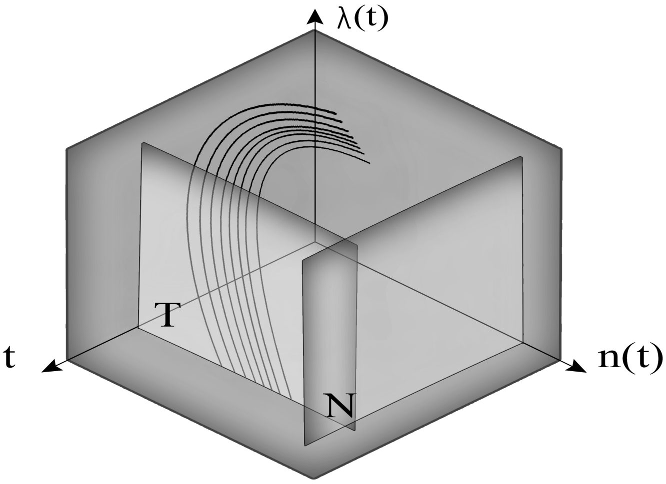
It is also clear, that this autonomous system has the unique fixed point at

$$\bar{\lambda(t)} = 0;$$
$$\bar{n(t)} = N. \tag{1.31}$$

which may be reached only at the terminal time.

Observe , that the original non-autonomous system (1.27) will reach this level at $t = T$ also, because the $x(t)$ function goes to zero with $t \to T$ and at time close to the terminal one the original system's behavior is the same as of the autonomous one. Since the $x(t)$ is the decreasing function of time from one to zero, it would push $n(t)$ level constantly away from its steady-state level, as shadow price of investments will decrease much slower, then in the (1.28), due to the action of $x(t)$. However, at $T \to \infty$ this $x(t)$ term would eventually go close to 1 for all t and hence asymptotically at long time horizons $n(t)$ dynamics may be described by the means of autonomous system (1.28) with $x(t) = 1$ for all t, not only at the end period.

It also has to be noted, that the type of dynamics is different if all products are identical in terms of investments efficiency (that is, when $\gamma(\bullet)$ function is also constant across products). It has been shown, that the homogeneous system (1.23) do not have any fixed points and $n(t)$ growth is unbounded. However our main interest is to analyze the differences in the behavior of the system in the presence of heterogeneity of products being developed. This is what $\gamma(\bullet)$ function accounts for as well as $x(t)$ term in the (1.28). In the simplest case being studied here with parameters specification like in (1.26) there is no very much structural difference in the system (1.28) behavior in comparison to autonomous system. Movement along the $x(t)$ axis with some exponential speed brings possibility for temporary shadow price increases, while expansion process is speeding up in comparison with homogeneous case. Shadow price of investments to $n(t)$ may have temporary growth period in the beginning of the planning horizon. At times close to $t = \frac{1}{2}T$ shadow price reaches its maximum and begins to decrease steadily until zero. With longer planning horizons there might be longer fluctuations of shadow price of investments. Below is the schematical reconstruction of the 3-dimensional system movement.



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It is more important that the heterogeneity of products brings heterogeneity of quality investments into the model and through that some significant changes to the overall system's behavior.

8. Quality Investments

Now take a closer look on the 2nd part of the dynamical system - investments to the quality growth. For any given i dynamics of quality is of the saddle-type. This can be demonstrated both graphically and analytically.

Observe that the system (1.13) takes the form:

$$\dot{\psi_i}(t) = (r + \beta_i)\psi_i(t) - 1;$$
$$\dot{q_i}(t) = \gamma_i^2\psi_i(t) - \beta_i q_i(t). \tag{1.32}$$

after substitution of optimal controls $g_i(t)$ for each i into the system.
This is the usual system of two 1-st order ODE's, which may be analyzed through conventional methods, [**4**]. For that change variables in such a way as to make the system homogeneous:

$$\hat{\psi_i}(t) = (r + \beta_i)\psi_i(t) - 1;$$
$$\hat{q_i}(t) = \beta_i q_i(t) - \frac{\gamma_i^2}{(r + \beta_i)}. \tag{1.33}$$

The homogeneous system defined in such a way has two eigenvalues which are real, distinct and have different signs:

$$|J - \lambda_i I| = \lambda_i^2 - r\lambda_i - \beta_i(\beta_i + r);$$
$$\lambda_i = \frac{r}{2} \pm \frac{\sqrt{r^2 + \beta_i(\beta_i + r)}}{2} \tag{1.34}$$

Obviously, these roots are of different signs, since expression under square root is bigger then r. This means exactly the saddle - type dynamics of the system. One can easily compute singular points of this system for each i. For that just equate lefthand side of the system (1.32) to zero to get fixed point values of quality level and shadow price of investments:

$$\bar{\psi_i}(t) = \frac{1}{r+\beta_i};$$

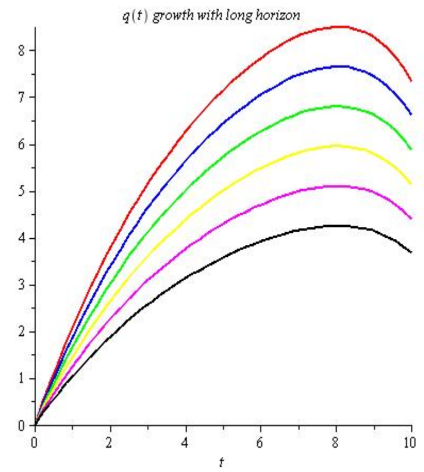
$$\bar{q_i}(t) = \frac{\gamma_i^2}{(r+\beta_i)\beta_i}. \tag{1.35}$$

For every product i there is a maximal attainable level of quality corresponding to the saddle point of the dynamical system in quality growth and its shadow price of investments, represented by the co-state variables. This maximal level is defined from $\gamma(i)$ function specification. Following our assumption concerning the structure of the space of products $\mathbf{Q}$, one comes to the conclusion that range of products available, N, and the product's index, i, influences directly this maximal quality. Solution curves for every i has 3 distinct stages: initial rapid growth of investments, then asymptotic approach to the maximal quality level and some decrease of the quality in the end. Note, that in finite time system never reaches its fixed point at the maximal quality level.

Specification of the parameter functions defines relative positions of qualities of different products. With constant parameters, as in homogeneous case, maximal quality is the same for all products and they are identical in this aspect. However, any other specification of parameter functions generates specific distribution of fixed points for different products. Here only for the linearizing specification (1.26) is analyzed as it is sufficient to demonstrate the role of these functions in the system. In this specification $\gamma(i)$ is the decreasing function of the product's position in the potential products' space. Then maximal attainable quality for every next product is lower, then for the preceding one. Moreover, distribution of these fixed points across products is the decreasing straight line with the angle proportional to the $\gamma(i)$. So, the choice of parameter functions not only defines the type of $n(t)$ dynamics, but also the exact form of 2-dimensional attractor for the distributed $q(i,t)$ system. With decreasing efficiency function specification one have the decrease in maximal qualities across products and the greater is the extent of $\gamma(i)$ function decrease over i's, the greater is the decrease of $q(i,t)$ function across products. Below solution curves for different products' qualities with two lengths of time period, namely, $T = 10, T = 1$ and other parameters' settings

$$SETHET := [N = 10, \alpha = 0.5, \beta = 0.2, \gamma = 0.4] \tag{1.36}$$

which are equal to $SETH$ parameters, are plotted with time of emergence normalized to 0 for all products.



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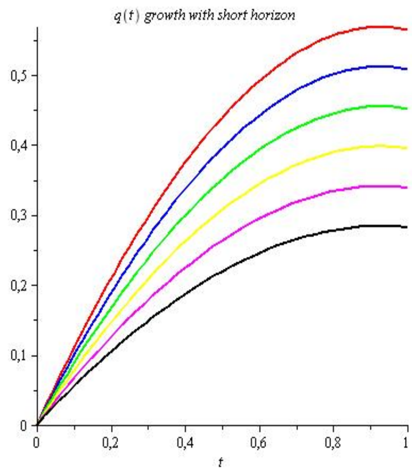

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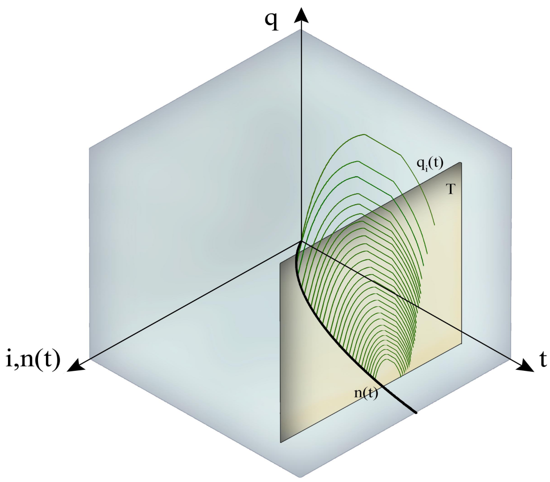
In the case of shorter time horizon ($T = 1$ in the example given) qualities reach the neighborhood of their maximal level only in the end of planning horizon and actual solution curves are increasing almost all the time, as in the last picture. In the case of longer planning horizon the solution curves reach the neighborhood of the steady-state and stay there for some time before the decline. Quality levels in this case are parabolas.

Observe also, that due to the construction of the system, start of investments to some product's quality corresponds to some certain point on the $n(t)$ solution curve. Unfortunately to obtain the exact starting point one have to obtain the exact closed-form solution for $n(t)$ process, which is not easy. However, to be able to account for qualitative behavior of the whole $n(t) - q(i,t)$ system one may resort to numerical integration procedures. This is not done in this chapter since in the infinite-horizon extension the comparison of finite and infinite time horizons is performed.

Equipped with the knowledge about the behavior of $q(i,t)$ function and about the general properties of the $n(t)$ solution curves, discussed in previous section, it is possible to reconstruct the final combined process. One have a monotonic increase of the variety of products in $n(t) - t$ plane, and this increase is going with the decreasing speed. This, in turn, means that speed of emergence of new products is decreasing upon the approach of the system to the terminal time, but it never reaches zero with the linearizing specification of parameter function $\gamma(i)$. From every point along the $n(t)$ solution curve there is a corresponding process of quality growth for the product i. Altogether these quality growth curves generate a generalized function $q(i,t)$ in 3-dimensional space $n - q - t$. Note, that the decreasing speed of expansion means that density of products is higher in the beginning of the process, then in the end. Below is the schematic reconstruction of the system's behavior.



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9. Parametric Analysis

Now consider influence of the system's parameters on the dynamics.

First observe that the most important parameter is the $\gamma(i)$ function. It governs qualitative features of the system:

- It defines the exact form of the variety expansion process - whether it is monotonic, concave or convex;

- It also defines the distribution of maximal quality levels across the mass of products - the form of 2-dimensional attractor of $q(i,t)$ dynamical system;
- Existence of solution to the general system depends on the regularity of this function.

Economically $\gamma(i)$ is the function of relative efficiency of investments into qualities of different products. Then if this efficiency is decreasing along the movement to the boundary of the potential products' space, this means the increasing difficulty of the quality buildup due to the increase in the complexity of products. It may also be defined the other way around as the assumed ordering in the products' space may be introduced in different ways.

Note, that the above points support the conjecture about $\gamma(i)$ function as the measure of homogeneity of the products' space. The more smooth is this function, the closer are neighbor products to each other in terms of investments efficiency and thus, in their technological nature.

Another technological parameter, β, is assumed to be constant, as it is difficult to analyze the system in the other case. However this assumption seems to be quiet natural, as it is enough to assume varying investments efficiency across products, while β is the rate of decay of qualities in the absence of investments. It can be assumed to be independent on the exact technological characteristics of the given product and is defined by the market.

Increase in this rate of decay shifts down the maximal quality level for all products. This has a clear interpretation. Maximal level of quality is the level at which investments are equal to the depreciation level. Increase in this depreciation without changing investment incentives will slow down each product's development. What is more interesting, increase in this rate of decay also slows down the process of variety expansion, $n(t)$. This fact may be explained by the change in the overall investments policy. Increased rate of the quality decay for all products means that more investment efforts have to be allocated to maintain quality level for all invented products. This means there are less funds available for the ongoing increase of the range of products available. It is now less profitable to increase this range, as the final payoff is obtained in this model only from developed qualities.

Increase in the efficiency of investments to the variety expansion process, α, boosts the speed of this expansion and through that, the speed of emergence of new products. This means the higher density of the $q(i,t)$ function withing the whole planning horizon. As the result one would have broader range of products available, but with less developed quality for each of them, since more resources should be allocated to the variety expansion and final payoff increase happens due to the increase in the density of products without actual increase in their quality levels.

Increase in the discounting rate, r, shifts the maximal quality level down for each i and also limits possibilities for $n(t)$ growth. This last happens due to the fact of the shrinkage of feasible set of trajectories for the variety expansion process. The increase in the discounting factor just leads to the decrease of the value of future payoff and thus limits investment capabilities of the planner.

Two parameters left to analyze are the range restriction, N, and length of the time horizon, T. The first one is the measure of the power of potential products' space, $\mathbf{Q}$. Increase in the range of products to be invented leads to the increase in this power. Note, that actual variety expansion process reaches it's boundary only at $t = T$ and from formal point of view this restriction is not binding. However it does influence the system dynamics with the given parameter functions specification. As it has been noted, for every i maximal attainable quality is closely related to this range restriction. The broader is this range, the higher is maximal quality for every product in this range, provided the positive dependence of relative investments efficiency on this range. In the other case, if one assumes negative influence (e.g. due to the fact of coordination problem - the higher is the number of products, the harder it is to manage their development), influence of the range of products to be invented on the maximal attainable quality would be negative. With the assumed linearizing specification note, that asymptotically with $\mathbf{N} \to \infty$, the maximal quality for each product will also increase in that direction. In the limiting case then quality growth will become unconstrained in the sense, that it's dynamics will always remain in the first stage of rapid increase and will never reach the asymptotic approach stage, as the fixed point for each product's i quality level would become infinite. So the saddle-type dynamics of qualities is the direct consequence of range restriction. Economically this means that the broader is the range of potential products one is accounting for, the broader is the set of possibilities for quality development of every product. However, even in the limiting case the ordering of qualities is preserved: every next product has lower quality at each given moment of time, then the preceding one.

One may also account for the influence of the time horizon given. Note, that maximal attainable level of quality for every product is also the steady state level of quality in the long-run dynamics. Long-run behavior of $q(i,t)$ function is slightly different from the case of the finite time. In the long-run, each product's quality will eventually reach it's steady-state level and, as it is the fixed point of the subsequent dynamical system, will stay there infinitely long. There will not be the final stage of decrease in quality then. In finite time system will not reach its steady state level, but the longer is the time horizon given, the longer is the period of close proximity of quality to it's steady-state level for all i's, while periods of initial rapid growth and final decay would remain unchanged. All these give the possibility to claim that for the majority of time system of qualities' dynamics will stay near it's steady state. For T small enough (lesser then 3 in the $SETHET$ case) qualities' trajectories will reach the proximity of their maximal values only near the terminal time and start to decline only after that. With longer horizons ($T > 50$ for $SETHET$) more then half of the whole time the quality would remain relatively unchanged, staying in the neighborhood of the steady-state. Note, that it will never reach this steady-state in final time though.

10. Discussion

The main goal of this chapter is to demonstrate the importance of unified approach to quality and products' innovations modeling in heterogeneous context. Although such a model must be more complicated in structure and methods involved, as it requires infinite-dimensional or distributed parameters control methods, it is clear from the analysis, that this may reveal a lot about interdependencies between these two types of innovative activities. Some comparisons with earlier results in the field might also be obtained.

Consider the model of heterogeneous innovations of [**15**]. First and most basic difference is that Honephayn's model is postulating that patent is the only stimuli for the innovator to tackle with innovating activity while the given model does not concentrate on stimulus and incentives to innovate. Very abstract framework where any innovation which is being made will provide profit to the innovator proportional to the 'size' of innovation is adopted here. Notion of size or breadth may be considered as a measure of a mass of qualities being achieved by given time. Presented model is concentrated on the influence of structural characteristic of the space of potential products (space of ideas) on the rate of innovations. Next turn to the quality ladders model. In described model quality is continuously changing for each product after the invention of this given product. It might be considered as a limiting version of quality ladders approach where every next generation of a given product is marginal improvement over the preceding one. At the same time one have the generating process of variety expansion which is in the nature of Romer's model but in the absence of consumer preferences. One may assume some simple form of preferences here to close the model in a sense of Romer. Assume for example monotonic preferences for variety of products. One may also ask about the social optimality of the level of provision of quality and/or variety in a sense of Lambertini and Lin. To discuss this question first note that the given model has a finite time horizon and thus cannot be analyzed in a sense of social optimum. For that one first have to extend the model to infinite-time horizon and obtain steady states which is done in the next chapter of the work. However, some preliminary conclusions may be made. Consider the argument of Lin [**14**] about the rigorous dependence of the level of provision of innovations on the level of economies of scale of the innovator. In the given model the specification of investment efficiency function, $\gamma(i)$ is the only parameter that may correspond to the scale economies. In the given formulation it provides diminishing returns to every next product's investments. However one may easily change the specification of this function to obtain the opposite effect. This will not change the overall bounded nature of dynamics of variety expansion but may change the relative growth of qualities of products. In this sense the model is robust in the provision of the range of products to the efficiency function specification, but the level of quality achieved may differ. This may be demonstrated by accounting for the homogeneous products case in the chapter, where the variety expansion is different but still bounded although there are no decrease in the efficiency of quality investments. So there is a dependence of variety expansion on the specification of efficiency parameter function but in the end this level will never exceed the boundary N for heterogeneous case which also may be interpreted as the desired level of provision of variety provided by the authority.

The most important conceptual feature of the model being discussed is that it allows to reveal the key characteristic of innovations which is related to the distance between different innovative products. This characteristic in the current setting is reflected by relative efficiency of investments into qualities of different products, $\gamma(i)$. Note, that this is just a feature of interpretation of the model above, but the characteristic of the potential products' space would play the key role in any practical model which would combine two aspects of the process of innovations in heterogeneous setting. To support this claim, just consider the overall picture of the process being described above. As long as one allows for uncountable number of possible products, the process of variety expansion may be treated as the generating function of the space of products, while quality investments as waves, being generated by this function. The overall process of aggregate quality growth is then the distribution of these waves. Then it is straightforward that exact form of this distribution and it's separate waves would depend on the structure of the space, where this process is going on. And the measure of heterogeneity of products themselves is exactly such a characteristic. With constant β coefficient it may be claimed also, that this is the only relevant characteristic of this space. Formally this argument follows from the fact, that exact form of generalized distribution (in Schwarzian sense) would depend on the way of definition of measure and distance in the space of products. Clearly, the given $\gamma(i)$ function in the current model is such a measure and it measures the distance between products in terms of investment efficiency. One may employ any other kind of measure of difference between products, such that the difference in their consumption properties, in their closeness in terms of industry, etc. What is important, that such type of a model would require by it's mere construction some kind of such a measure. What kind of such measures may be chosen to pertain the model's consistency and solvability is the interesting question for further research in the field.

Another question of theoretical interest is concerning topological properties of the general products' space $\mathbf{Q}$, if one would drop the assumption of the diagonal nature of operators there. In that case, without any restrictions on the nature of interdependence between different products after their invention, little may be said concerning the model structure without imposing some (hopefully weak) restrictions on the topology of the products' space. As it has been seen here, assumption of independence of products after their invention from each other immediately leads to the separable Hilbert space structure. Intuitively, however, uncountable number of possible products may lead to the endogenous uncertainty in the model and this is one of the reasons of not treating uncertainty here. In general model without assumption of independent $q_i(t)$'s in the $q(i,t)$ distribution would have much more complicated form, and the space of products may turn to be non-separable. In that case one have to treat explicitly probabilistic nature of any action of the planner in such a space. This also may give some formal theoretical foundation for the uncertainty of the innovative process.

There are a lot of more applied questions that may be posed within the model's framework. For example, one may try to account for effects of competition between innovative agents, thus applying differential games theory instead of optimal control as in [**56**]. Such an extension would give one a possibility to distinguish between notions of 'new to the market' and 'new to the firm' products and separate different types of innovative agents according to their internal parameters. In such a model as the discussed one, there is at least one more possible degree of freedom for innovative policy: some agents may concentrate more on the process of invention, while others on the quality growth. Such an extension is considered in the last chapter of this work and indeed some conditions for specializing of agents in different types of innovating activities are obtained.

Another immediate extension is to account for some patent policy effects in the model. In first approximation this would mean every invented product would have it's own life-cycle and then all quality innovations would not be limited by the same terminal time condition, but would possess different ones. This may or may not have a stimulating effect on innovative activities. This extension is considered in the third chapter of the current work and the difference of patenting effects between homogeneous and heterogeneous products is analyzed.

However to tackle with the problems posed by such modifications one first has to consider possible simplifications of the basic model discussed here to allow for further enrichment of applied and economic meaning of it. This is done in the next chapter through transition of the model to infinite time horizon. As it will be seen this simplifies model a lot and permits in turn to employ HJB principle fruitfully.

CHAPTER 2

Infinite Time Horizon Problem

1. Introduction

In this chapter the extension of the the basic model to infinite-time horizon is considered. The goal of such an extension is twofold. First it allows for steady-state analysis which is not available in the basic model since its bounded time-horizon. Second, the infinite-time horizon version of the model provides a benchmark for further development of patent and competition effects, since the solution achieved is much simpler and tractable. Hence the comparison of performance is easier.

Development of infinite-time horizon extension allows for formal analysis of limiting behavior of the agent in the basic model.

2. Monopolist Problem Formulation

Consider the same general framework as before, with one agent, no uncertainty and bounded state variables as well as with irreversible investments of both types. Basic finite-time model is described above where the reader may refer for detailed treatment. One point to mention is that compactness in proof of existence of optimal controls is no longer valid, since the infinite-time horizon. This problem is addressed through implementation of HJB approach which yields sufficient conditions for optimum in contrast with Maximum Principle employed above which gives only necessary conditions for that and where the existence of optimal controls pose problems. However, HJB approach cannot yield fruitful results concerning solutions and optimal controls in limited-time case. That's why it is not used in the previous chapter. At the same time rigorous treatment of the nature and internal characteristics of the suggested approach is easier within the Maximum Principle framework. In HJB approach any candidate for optimal controls being found is automatically the optimal solution provided it delivers maximum to the assumed value function and transversality conditions hold. For infinite time the formulation becomes simpler and thus the explicit solution is possible here. To summarize, one has not to stop on theoretical proofs of optimal controls existence and/or the structure of state-space (the latter is the same as in the basic model except for compactness in time).

In real economies it is hard to estimate the exact planning horizon of given agents, so it seems plausible to assume infinite-time horizon for the purposes of this chapter. The first step to introduce competition and patents into the basic model should be its extension to the infinite-time horizon. The scheme of so-called 'planned' innovation is assumed in this chapter also: there is only one agent (social planner or monopolist since one cannot distinguish between both in the absence of social utility notion) who maximizes the output of innovations in any given period of time over the infinite time horizon according to some objective functional. For the purposes of this chapter one may consider the agent to be the monopolist in some market where innovations are being made. Since there is no notions of demand, profit and utility functions here, the market is understood in some general sense: it can be an economy or an industry. The objective functional of the monopolist is defined as:

$$J \overset{\text{def}}{=} \int_0^\infty e^{-rt} \left(\int_0^{n(t)} \Big[q(i,t) - \frac{1}{2} g(i,t)^2 \Big] di - \frac{1}{2} u(t)^2 \right) dt \to \mathbf{max} \tag{2.1}$$

Monopolist is maximizing integral sum of qualities of all products invented until each time t minus investments being made to every invented product's quality and to the overall expansion process over the planning horizon. There is no sign of prices or profit in this formulation. The market clearing mechanism and all the mechanics behind the market structure are also omitted. Lines of reasoning follow the previous chapter of the work. Such a specification would give the independent from prices and profit dynamical system as it is presented here. However, the main concern here is not in the profit maximization, but in the maximization of the output of innovations in every given moment of time. It is equivalent to the linearity of profit function which is a standard assumption in innovation literature, [**12, 14**].

Dynamics of quality growth and expansion process are governed by subsequent dynamic equations:

$$\begin{aligned} \dot{n(t)} &= \alpha u(t); \\ \dot{q(i,t)} &= \gamma(i) g(i,t) - \beta(i) q(i,t); \\ \forall i &\in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+; \\ \forall t &\in [0,..,\infty) = \mathbf{T} \subseteq \mathbb{R}_+. \end{aligned} \tag{2.2}$$

and static constraints:

$$\begin{aligned} u(t) &\geq 0; \\ g(i,t) &\geq 0; \\ n(t) &\leq N; \\ q(i,t) \mid_{i=n(t)} &= 0. \end{aligned} \tag{2.3}$$

Assume zero initial quality for all products and some fixed initial range of products available (the same as before). Here the dependence of model results upon the form of these functions is not considered and the same form as in (1.26) is adopted. It is sufficient to note, that dependence of the given specification on the maximal range of products, N, is not crucial for the boundedness of solution. If one would consider products homogeneous in their investment characteristics (hence no dependence on the maximal range) results still remain bounded. See previous chapter. So further on assume the simplest possible form of these functions which would linearize subsequent dynamic equations as before:

$$\begin{aligned} \beta(i) &= \beta; \\ \gamma(i) &= \sqrt{N-i} \times \gamma. \end{aligned} \tag{2.4}$$

The HJB approach is employed to resolve the problem.

3. Quality Growth Problem

With infinite time horizon, the straightforward application of HJB approach yields the explicit solution for quality growth system and for variety expansion.

To construct the value function of the model, first observe, that problem for quality growth may be solved independently of the variety expansion process except for the time of emergence of the new product which is defined through $n(t)$

dynamics. Starting from this time (denoted by $t(0)_i$) value of each product's quality growth is independent of variety expansion process:

$$V(i) = \int_{t(0)_i}^{\infty} e^{-r(t-t(0)_i)}(q_i(t) - \frac{1}{2}g_i(t)^2)dt \tag{2.5}$$

Motion of quality of each product i is governed by the system:

$$\begin{aligned} \frac{dq_i(t)}{dt} &= \gamma\sqrt{N-i}g_i(t) + \beta q_i(t); \\ &\forall t \in [t_i(0), .., \infty); \\ &t_i(0) = i^{-1}(n); \\ &q_i(t_i(0)) = 0. \end{aligned} \tag{2.6}$$

Observe also, that due to infinite-time horizon, this value function does not explicitly depend on time. One may construct the Hamilton-Jacobi-Bellman equation for every q_i dynamics and first-order conditions for optimal controls as following:

$$\begin{aligned} rV(i) = max\{q_i(t) - \frac{1}{2}g_i(t)^2 + \frac{dV(i)}{dq_i}(\gamma\sqrt{N-i}g_i(t) - \beta_i q_i(t))\}; \\ g_i(t)^{opt} = \frac{dV(i)}{dq_i} \times \gamma\sqrt{N-i}. \end{aligned} \tag{2.7}$$

These yield the system of algebraic equations on the coefficients of value function. As the given model is linear quadratic one, one may assume the value function of polynomial form with degree of the polynomial not greater then two, [**6**]. It turns out that only the linear value function fits the above HJB equation:

$$V(i)^{ass} = A_i q_i(t) + B_i. \tag{2.8}$$

Then one may solve the resulting algebraic equations on value function coefficients which result from equalizing to zero all coefficients at time-varying variable q_i:

$$\begin{cases} (r+\beta)A_i - 1 = 0; \\ rB_i - \frac{1}{2}\gamma^2(N-i)A_i^2 = 0. \end{cases} \tag{2.9}$$

Subsequent value function is then:

$$V(i) = \frac{1}{r+\beta} \times q_i + \frac{1}{2}\frac{\gamma^2(N-i)}{r(r+\beta)^2}. \tag{2.10}$$

The resulting optimal investments are constant and do not depend on time, but differ only across products:

$$g_i(t)^{opt} = \frac{\gamma * \sqrt{(N-i)}}{(r+\beta)}. \tag{2.11}$$

Substituting this to the quality growth dynamics yields the ODE for $q_i(t)$ dynamics:

$$\dot{q_i}(t) = \frac{1}{r+\beta} \times \gamma^2(N-i) - \beta q_i(t). \tag{2.12}$$

which yields the solution for $q_i(t)$:

$$q_i(t) = \frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times (1 - e^{-\beta t}). \tag{2.13}$$

Note that quality and control are of the piecewise form due to the constraint (2.6) and are zero before $t(0)_i$.

Subsequent value function representation:

$$V(q_i)^*_i = \frac{1}{r+\beta} \times \frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times (1 - e^{-\beta t}) + \frac{\gamma^2 \times (N-i)}{2r(r+\beta)^2};$$

$$V(q_i = 0)^*_{i=n(t)} = \frac{\gamma^2 \times (N - n(t))}{2r(r+\beta)^2} = V(q). \tag{2.14}$$

It has to be noted, that no other optimal controls in linear feedback form which satisfy all nonnegativity constraints exist except the piecewise constant one. This fact comes from the formulation of the objective functional (2.1) which includes all q_i's only linearly. This has been done to simplify the formal side of the model.

4. Variety Expansion Problem

Second part of the overall value generation consists of the intensity of addition of new products at every time given the expected value of the stream of profit derived from the quality of the newly introduced products. This part may be represented by the integral over all potential stream of quality of the product over it's lifecycle. At the same time this information is already contained in the value function of the quality problem above, so it suffices to integrate over all potential products at initial time. Last observation to be made is that at the moment of the emergence of the new product it's quality is zero, as it is required by (2.3). These yield the value function for variety expansion problem in the following form:

$$V_{n_0} = max_{u(\bullet)} \int_0^\infty e^{-rt}\Big(\alpha u(t) \times V(q) - \frac{1}{2}u(t)^2\Big)dt. \tag{2.15}$$

subject to dynamic and boundary constraints on variety expansion process:

$$\dot{n(t)} = \alpha u(t);$$

$$n(0) = n_0. \tag{2.16}$$

Here the term $V(q)$ is the current value of the quality growth problem for the next product to be invented ($i = n(t)$) estimated at zero quality level.
These yield the Hamilton-Jacobi-Bellman equation for the variety expansion problem and subsequent first-order condition:

$$rV_{n(t)} = max\{\alpha u(t) \times V(q) - \frac{1}{2}u(t)^2 + \alpha u(t) \times \frac{\partial V_{n(t)}}{\partial n(t)}\};$$

$$u(t)^{opt} = \alpha \frac{\partial V_{n(t)}}{\partial n(t)} V(q). \tag{2.17}$$

Since the problem is formulated in infinite time, the coefficients of value function do not depend on time also. The HJB equation then yields simple enough system of algebraic equations on coefficients of the value function (the same assumption of polynomial value function form is adopted as in quality problem part). Due to the presence of $n(t)$ variable in value function for quality growth on the right hand side, the minimal power of the polynomial here is 2, yielding quadratic value function:

$$V^{ass}_{n(t)} = Cn(t)^2 + Fn(t) + E \tag{2.18}$$

Solution of the system of equations on value function coefficients yields value function representation. System for value function coefficients:

$$\begin{cases} -2\alpha^2 C^2 + (\frac{\alpha^2\gamma^2}{r(r+\beta)^2} + r)C - \frac{1}{8}\frac{\alpha^2\gamma^4}{r^2(r+\beta)^4} = 0; \\ (\frac{1}{2}\frac{\alpha^2\gamma^2}{r(r+\beta)^2} + r)F - 2\alpha^2 C * F - \frac{\alpha^2\gamma^2}{r(r+\beta)^2}C - \frac{1}{4}\frac{\alpha^2\gamma^4}{r^2(r+\beta)^4}N = 0; \\ rE - \frac{1}{2}F^2 - \frac{1}{2}\frac{\alpha^2\gamma^2}{r(r+\beta)^2}F - \frac{1}{8}\frac{\alpha^2\gamma^4}{r^2(r+\beta)^4}N^2 = 0. \end{cases} \tag{2.19}$$

Substitution of the solution of (2.19) to the assumed value function, (2.18), yield the explicit form of this value function:

$$V_{n(t)} = \frac{1}{2}\frac{\alpha^2\gamma^4 r(N-n(t))^2}{(r(r+\beta)\sqrt{r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2}+r^4+2r^3\beta+r^2\beta^2)^2}. \tag{2.20}$$

Optimal investments to the variety expansion then are:

$$u(t)^{opt} = \frac{\alpha r\gamma^2}{r(r+\beta)(\sqrt{r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2}+r(r+\beta))}(N-n(t)). \tag{2.21}$$

which are in fact different from optimal investments formulation in initial problem with bounded time. It may be shown however, that the resulting dynamics is the same in its shape. Moreover, due to the extension of the model to the infinite horizon, the explicit solution for $n(t)$ dynamics is achievable. Inserting expressions for value function coefficients into the optimal investments equation and then into the dynamic constraint for $n(t)$ (2.2) yields ODE for the variety expansion process of the first order (instead of second for the basic finite time model):

$$\dot{n(t)} = \frac{\alpha^2\gamma^2 r(N-n(t)}{\sqrt{(r+\beta)^2r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2r^2}. \tag{2.22}$$

This is the equation with constant coefficients which may be solved by conventional methods, yielding optimal variety expansion path:

$$n^*(t) = N + e^{-\frac{\alpha^2\gamma^2 rt}{\sqrt{(r+\beta)^2r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2r^2}}(n_0-N). \tag{2.23}$$

This expansion dynamics is bounded from above by the maximal range of potential products' space, N, and is approaching it at the decreasing rate. Note also that N is the steady-state level of the range of products. Irrespectively of the chosen particular value for this maximal range, it is achieved by the variety expansion process only in infinite time. It can be easily demonstrated by taking limit for $t\to\infty$ in the above solution.

With explicit formulation of variety expansion process one may derive the emergence time for every product as a function of this product's index. This is the inverse function of $n(t)$, with $n(t)$ substituted by the product's index, i:

$$t(0)_i = f^{-1}(n^*(t))|_{n(t)=i};$$

(2.24)

$$t(0)_i = -\frac{\sqrt{r^2(r+\beta)^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+r^2(r+\beta)^2}{\alpha^2\gamma^2 r}\times\ln(\frac{N-i}{N-n_0})>0.$$

With this at hand the quality growth for any product is fully defined as a piecewise function:

$$q_i(t) = \begin{cases} \frac{\gamma^2(N-i)}{\beta(r+\beta)^2}\times(1-e^{-\beta t}), t > -\frac{\sqrt{r^2(r+\beta)^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+r^2(r+\beta)^2}{\alpha^2\gamma^2 r}\times\ln(\frac{N-i}{N-n_0}); \\ 0, t\le -\frac{\sqrt{r^2(r+\beta)^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+r^2(r+\beta)^2}{\alpha^2\gamma^2 r}\times\ln(\frac{N-i}{N-n_0}). \end{cases}$$

(2.25)

5. Steady States

With infinite-time horizon solution at hand one may analyze the steady states of qualitites and variety expansion. For that first consider the fixed points of both dynamical systems by finding those levels of state-variables for which their growth is zero:

$$\frac{\alpha^2\gamma^2 r(N-n(t)}{\sqrt{(r+\beta)^2 r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}=0;$$

$$n^{fix}(t)=N;$$

$$\frac{1}{r+\beta}\times\gamma^2(N-i)-\beta q_i(t)=0;$$

$$q_i^{fix}(t)=\frac{\gamma^2}{\beta(r+\beta)}(N-i). \tag{2.26}$$

There is only one unique fixed point for every dynamical variable. These fixed points are exactly the same as for basic time-constrained model and they depend on the maximal variety N. This happens due to the choice of efficiency functions in (2.4). With other choices it would be possible (provided that the solution exists) to obtain other fixed points. For example if one would consider the homogeneous case with identical parameters across products, all quality levels would have identical fixed points while the fixed point for variety expansion would not depend on maximal range (if it exists).

Next consider steady states of state variables:

$$n^*(\infty)=lim_{t\to\infty}\Big(N+e^{-\infty}(n_0-N)\Big)=N;$$

$$q^*(\infty)=lim_{t\to\infty}\Big(\frac{\gamma^2(N-i)}{\beta(r+\beta)^2}\times(1-e^{-\beta\infty})\Big)=\frac{\gamma^2}{\beta(r+\beta)}(N-i). \tag{2.27}$$

They appear to be exactly the same as fixed points of the system. One may conclude that fixed points are reached only in infinite time and system variables stay at these levels forever after reaching them.

The last point to consider is the stability of the given set of steady states. For that take values of quality and variety higher and lower then steady states and account for the rate of their changes. First consider the variety expansion:

$$n(t)^{up}=N+\epsilon;$$

$$\dot{n(t)}|_{n(t)=N+\epsilon}=\frac{\alpha^2\gamma^2 r(N-(N+\epsilon))}{\sqrt{(r+\beta)^2 r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}=$$

$$-\frac{\alpha^2\gamma^2 r\epsilon}{\sqrt{(r+\beta)^2 r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}<0,\epsilon>0. \tag{2.28}$$

It can be seen that the variety expansion steady state is the stable one then: $n(t)$ grows up to this level when its current level is lower and decreases down to it, when variety level is higher. Now consider the quality growth rates:

$$\dot{q_i(t)}|_{q_i(t)=q_i^{fix}(t)+\epsilon}=\frac{1}{r+\beta}\times\gamma^2(N-i)-\beta\times(\frac{\gamma^2}{\beta(r+\beta)}(N-i)+\epsilon)=$$

$$=-\beta\epsilon<0,\epsilon>0. \tag{2.29}$$

So steady states for all qualities' dynamics are also stable and are reached at infinite time. Both system variables have maximal levels achieved at infinite time. This is the main difference of infinite-time dynamics from the basic model, where the same fixed points are not achieved at all for qualities while variety expansion reaches its fixed point at the end of planning horizon - which is finite in that case. One may also check the type of dynamics of a system through the same means of analyzing the eigenvalues of both systems. Results do not differ from those in basic finite-time case.

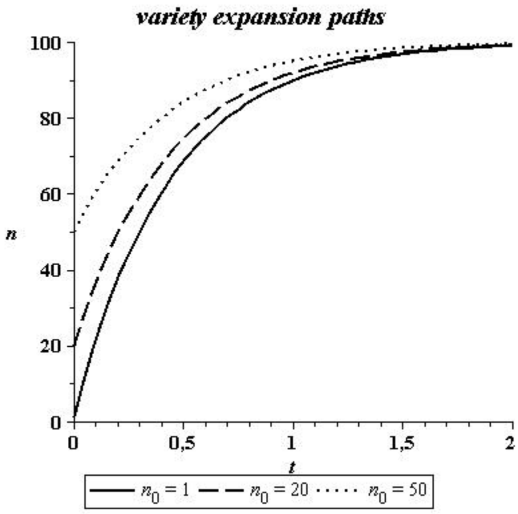
6. Parameter Influence

Qualitative behavior of both quality and range growth processes is the same as in the basic model which shows that the infinite-time horizon expansion is natural for the model. Moreover, application of HJB approach permits in this case the explicit solution for variety expansion process also. Availability of explicit solutions provides the possibility of more detailed comparative analysis of parameter's influence both analytically and graphically. In this section the influence of initial conditions and various parameters on quality growth and variety expansion is analyzed and the resulting dynamics is compared with basic finite-time model. This is done to illustrate differences which occur due to the change in planning horizons.

Variety expansion is rapidly increasing toward the boundary N and reaches 90 percents of available range in finite time. However the actual reach of the steady state cannot happen in finite time and the process is slowing down then as it can be seen from (2.23). To illustrate the influence of initial conditions and exogenous parameters of the system the following set of parameters is used:

$$SETM1 := [N = 100, \alpha = 0.7, \beta = 0.1, \gamma = 0.7, r = 0.05.] \tag{2.30}$$

First consider variation in initial conditions:



It can be seen, that for any initial range of products the variety expansion process is converging to the same steady state. Analytically it follows from the fact that initial variety level has weakening influence on the variety expansion dynamics in time since the power of subsequent exponent in (2.23) is negative. Formally one may consider the measure of change of distance between different solution paths in time:

$$\frac{d(n(t)_{n(0)+\epsilon} - n(t)_{n(0)})}{dt} = -\frac{\alpha^2\gamma^2 rt}{\sqrt{(r+\beta)^2 r^2 (r^4 + 2r^3\beta + r^2\beta^2 + 2\alpha^2\gamma^2)} + (r+\beta)^2 r^2} \times e^{-\frac{\alpha^2\gamma^2 rt}{\sqrt{(r+\beta)^2 r^2 (r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}}, \quad \epsilon > 0. \tag{2.31}$$

It can be noted that the same result holds for the basic model although it can be demonstrated only numerically since the explicit solution for the basic variety expansion process is not available.

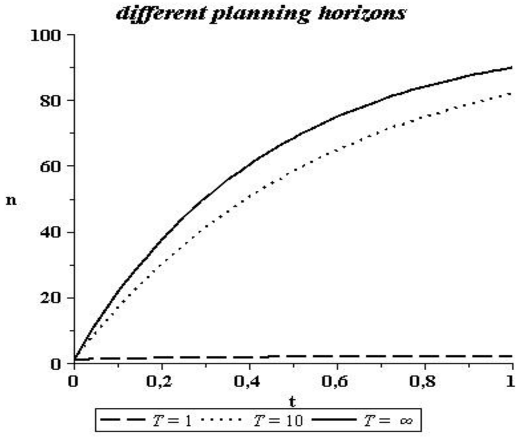
Now recall the results of T influence for the basic model above. It can be shown that the longer is the horizon, the faster is the increase in variety expansion for the system (1.27). For short horizons the variety expansion never reaches the close proximity of the fixed point N, while for longer ones it does. It can be seen from the form of this dynamical system:

$$\dot{\lambda(t)} = r\lambda(t) - \frac{(N - n(t))(1 - e^{(\beta+r)(t-T)})^2}{(r+\beta)^2};$$
$$\dot{n(t)} = \alpha\lambda(t). \tag{2.32}$$

Increase in T slows down the rate of change in shadow cost of investments $\lambda(t)$ thus increasing the growth rate of variety expansion:

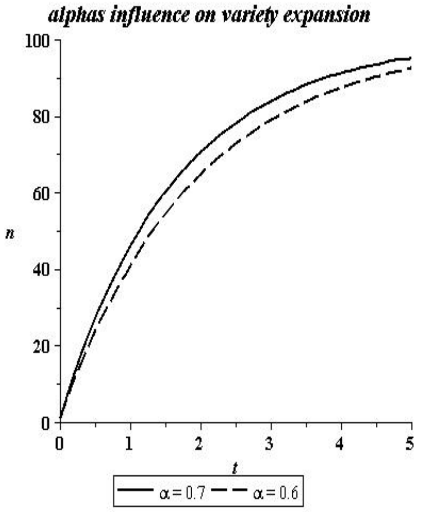
$$\frac{d(\dot{\lambda(t)})}{dT} = -\frac{\gamma^2}{r+\beta} \times (e^{-(r+\beta)(T-t))} - e^{-2(r+\beta)(T-t))}) \times (N - n(t)) < 0. \tag{2.33}$$

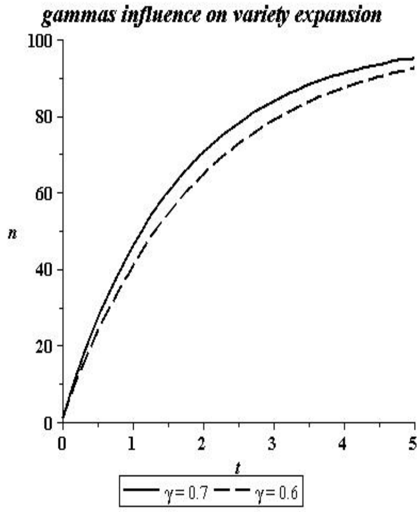
It can be demonstrated by graphical comparisons also:



Then with $T \to \infty$ one have the limiting case of variety expansion which is given exactly by (2.23). Economically longer planning horizons mean higher potential profit from expected variety expansion and this creates more stimulus for inventions. One may conclude that it is essential for innovation-stimulating policy to ensure long horizons for agents if one would like to stimulate variety expansion. With too short horizons(e.g., $T = 1$ for the $SETM1$) not all possibillities for inventions and variety expansion are used and hence the innovative activity is dampened. At the same time it is not that necessary to grant infinite-horizons. It is sufficient to ensure some sufficiently long horizons to provide the opportunity of usage of all given investment possibilities in that field. Given that the innovative activity by itself is not what creates profits, but the sales of innovative products on the market, it is not optimal from social point of view to provide possibillity for infinite or very long planning horizons to innovative agents since the same result in terms of generation of innovative products may be reached within shorter period. However to answer precisely on the question what is the optimal time horizon in terms of social welfare some explicit representation of social welfare function is needed which is absent from the given framework. Here only the conclusion that starting from some sufficiently long planning horizon there is no crucial difference between fixed and infinite terminal times may be made.

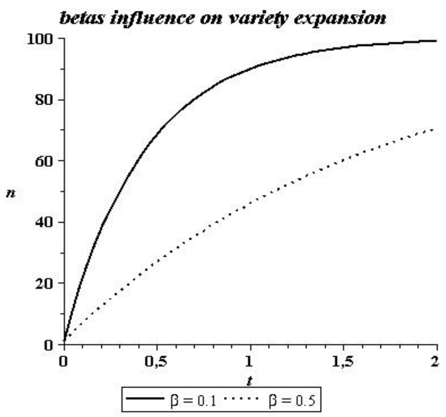
Efficiency of investments positively influences the variety expansion process. It has to be noted, that the sensitivity of variety expansion to the changes in investments efficiency is roughly the same for changes in quality investments efficiency γ and own investments efficiency α.

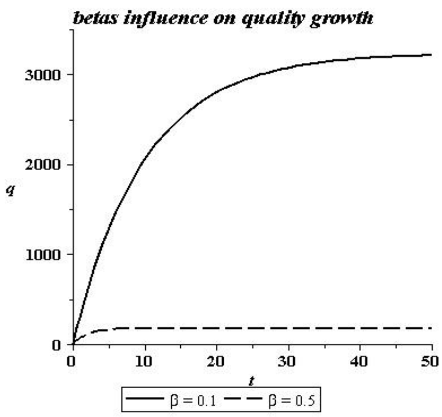




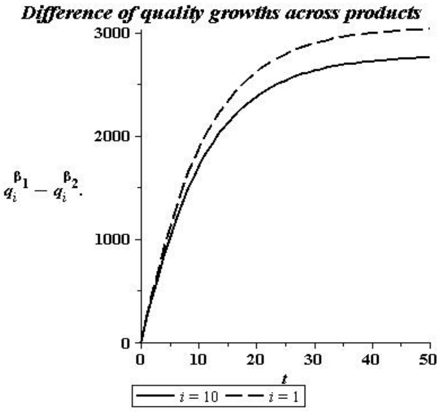
Economically rise in investment efficiencies leads to the increase in potential growth rates keeping the amount of investment the same. At the same time, investments themselves also depend on these parameters. Observe that quality growth investments depend positively on efficiency of quality investments and do not depend on efficiency of variety expansion investments, since variety expansion process influences quality growth only through defining emergence times of products. At the same time variety expansion investments depend on both efficiency parameters and dependence on γ is roughly the same as on α in its scope. This illustrates the fact that variety expansion is driven by the potential of the profit generation by products which still are to be invented and not by the process of inventions by itself. In the given framework the variety expansion cannot grow in the absence of associated stream of qualities' growths.
The same result holds for finite-time horizon. Solution has the same degree of sensitivity to investment efficiency changes.

Increase in the rate of quality decay significantly changes the rate of variety expansion. It slows down not only quality growth itself which is an obvious result, but also the rate of emergence of new products:





It can be seen that changes in quality decay rate change qualities' growth rates and steady states of qualities also. Variety expansion is much less responsive to this parameter changes. As rate of quality decay increases, investments into quality growth are falling for all products since in this infinite-time version of the basic model investments into all products' qualitites depend on γ constant in the same way. As a result, steady-state level of qualities are also decreased. Variety expansion is then less attractive since the potential stream of gain from quality growth of new products is much lower. This leads to the reduction of investments in variety expansion also. Again, as with investment efficiencies one may observe that characteristics of products which are to be invented affect the investment decisions on variety expansion level. At the same time, the higher is the position (index) of the given product,the lesser is the influence of quality decay rates on the quality growth path.



Formally observe that the difference in quality levels is defined by:

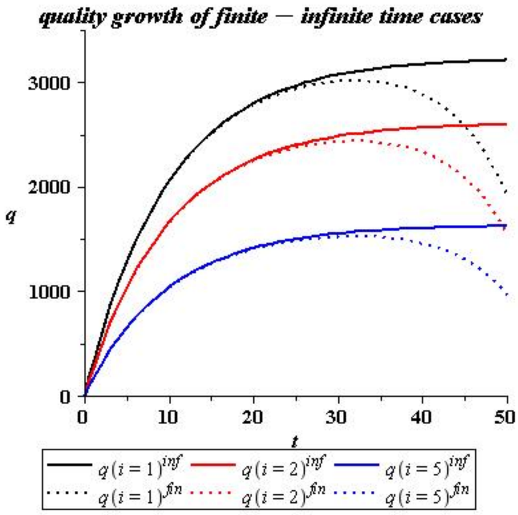
(2.34)

$$q_i^{\beta_1} - q_i^{\beta_2} = \frac{(\beta_1(r+\beta_1)e^{-\beta_2 t} - \beta_2(r+\beta_2)e^{-\beta_1 t} - (\beta_1-\beta_2)(r+\beta_1+\beta_2))\gamma^2}{\beta_1\beta_2(r+\beta_1)(r+\beta_2)} \times (N-i).$$

This difference depends negatively from the product's index, i. The main formal reason behind this is the specific form of the investment efficiency function which gives diminishing returns with growing complexity of a product. Economic intuition behind this is as following. Each next product is harder to invest in in terms of quality and thus the intensity of growth is falling across products. Then the rate of decay which is directly proportional to the quality level achieved is also lesser. Then the sensitivity of quality to the parameter of decay is lesser with growing complexity of investments. So the chosen specification of parameter functions yields the twofold effect: on one hand, the efficiency of investments is falling down with growing complexity, but on the other hand the sensitivity of growth to the decay rate is also falling down.

Next account for discount rate influence on growth rates. Given the form of solutions, (2.23), (2.13), it is straightforward to notice that the higher is the discount rate, the lower are the growth paths. One may check this immediately by taking respective derivatives. Sensitivity of quality innovations to the changes in discount rate is higher then that of variety expansion. Taking into account the role this parameter plays as the factor of time planning it means that quality growth process is 'faster' then the expansion process which is the generator of innovations.

One also may account for the influence of N parameter which is the maximal achievable variety level. It's influence is the same in direction and scale as for the basic model above and so it is not discussed here. Much more interesting is the comparison of solution paths of the basic time-limited model and the current infinite-time extension under discussion. It has been discussed already that the infinite-time extension is the natural limiting case of time-limited model in this aspect. One may also compare quality growths for limited and infinite time horizons. It turns out, that both models have the same steady states and shape of qualities' dynamics. In finite-time case as it has been discussed already, solutions never reach this steady-state levels which appear to be the maximal qualities' levels for each product also. In infinite-time case these steady states are achieved and solutions stay there unchanged infinitely long. This does not mean that investments to quality stop at this time; instead the steady state is achieved when the level of investments is counterbalanced by the decay rate of quality of the product. It is the main reason of difference of steady states of qualities across products: while investments are different according to the chosen form of the investment efficiency function, decay rates are chosen to be constant across products. If one would assume the same rate of change of decay rates as for efficiency functions, steady states for qualities of all products would appear to be the same as it is in the case of homogeneous products. In conclusion consider difference in qualitites' growth for different products with finite and infinite time horizons (for finite time $T = 50$ is assumed as a horizon length while other parameters are from $SETM1$). Here emergence times for all products are normalized to zero for comparisons.



7. Discussion

Development of infinite-time horizon extension provided the possibility for more detailed analysis of the structure of the model and the role of different exogenous parameters. It became clear that the quality growth problem has only the solution which is piecewise constant across states. It depends on time in the case of finite-time horizon but this dependence comes only from the non-zero discount factor. For infinite-time horizon this dependence cancels out and thus the optimal investment in quality becomes constant since the product emergence.

The exact form of variety expansion dynamics is also much more clear after the analysis undertaken in this chapter as with infinite-time horizon the explicit solution is obtined for this part of the model. It turns out to be the limiting case of the basic model's dynamics (with $T \to \infty$). Then it is possible to consider the role of the length of planning horizon in the model. For longer horizons growth rates of product qualities and of the variety expansion are higher. This can be explained easily: with longer horizons there are more stimulus for agents to invest in innovations.

Now with solution for variety expansion problem at hand one may answer the question what drives the process of product innovations. The answer that it is the heterogeneity of products characteristics appears not to be the full one. Indeed, in the absence of differences between products from the point of view of investments the growth rate of variety expansion will be constant as it is for quality growth. Then the variety expansion process would be described by some constant growth rate because there are no stimulus to change the investment policy across time: all products are essentially the same hence there is no reason to invest more or less in their creation. But the chosen specification of investment efficiency assumes the decreasing efficiency of investment across products. Then in homogeneous case the growth rate of variety expansion should be higher. Apparently this is not the case: for the chosen form of heterogeneity growth rate is higher initially but it negatively depends on the level of variety already achieved. This happens because of the form of $\gamma(i)$ function. This function enters the dynamics of variety expansion. If it is constant, growth rate does not change over time and is proportional to γ. But consider the current form of this function. It decreases in i but initially has a higher value then the constant γ in homogeneous case, namely $\gamma \times \sqrt{N - n_0}$ provided N is high enough. Now recall that investments in variety expansion depends only on the boundary product's investment efficiency and thus from $n(t)$. Given that new products appear continuously, the multiplier of gamma is falling down and thus the rate of variety expansion also. Observe also, that for $\sqrt{N - n_0} \leq 1$ the dependence is reversed: variety expansion in homogeneous products case would be faster starting from initial time and along all the planning horizon. Hence one may observe the role of the N parameter. This one defines the 'richness' of the potential innovations' space. As actual variety level approaches this boundary the potential for new products' creation, $N - n(t)$, is decreasing thus making it less attractive to invest in this process. This may partially be explained by the fact, that every next product has lower steady-state level of quality which may be obtained and thus yields less potential profit. Then incentives to invest in the creation of products are decreasing with time. At the same time initial incentives to invest are higher for higher dimension of the potential products' space. This initial rate of investments is typically higher for heterogeneous case then for homogeneous case. This corresponds to this 'richness' of the space: for homogeneous products case the potential products' space always has dimensionality of 1, as all products are identical and thus their range cannot influence the behavior of the innovator. On the other hand, heterogeneous products do differ from each other and this gives weight to the size of products' space by itself.

The other question of interest is the direction of influence of planning horizon T. Variety expansion process reaches its boundary, N, only in infinite time. Quality levels do the same, while in finite-time case they never reach their steady states. The question is what happens with variety expansion process when it is time-limited. As it can be seen from the form of dynamical system (1.27), the growth rate of variety is proportional to the shadow price of investments, which, in finite time, depends on the chosen time horizon and thus is time-varying parameter. In infinite time the role of shadow price of investments is carried out by the coefficients of the value function and they are constant across time. Returning to the finite time case, one would expect two different effects of time horizon on variety expansion. First, when the time-horizon is shortening this creates stimulus for the investor to increase variety faster as he has less time to use his innovative opportunities. At the same time shorter time horizons mean that the investor has less time to use these new created products for deriving profits out from them. It turns out that one may formally represent the existence of both these effects.

For that consider the derivative of shadow price of investment changes with respect to T:

$$\frac{d\dot{\lambda}(t)}{dT} = -\frac{\gamma^2}{r+\beta} \times (e^{(r+\beta)(t-T)} - e^{2(r+\beta)(t-T)})(N - n(t)) < 0. \tag{2.35}$$

This expression is always negative, as $t - T$ is negative. Observe however that this derivative has two different parts, corresponding to two exponents in brackets, one of them being positive and the other being negative. They correspond to those effects being mentioned above. As general sign of derivative is negative, rate of change of shadow price of variety expansion investments would rise with shortening horizon and fall for longer horizon. Higher rate of change of the shadow price causes the decrease in investments themselves as the general theory tells us. So it is straightforward to conclude that longer horizons stimulate variety expansion. In the limiting case of infinite time only the positive effect of T remains as there are no limitations on the qualities' development as well as on the variety expansion. This effect is dealt with in more details in the next chapter where limited life-cycles for products' qualities are considered.

CHAPTER 3

Patents in Heterogeneous Innovations Framework

1. Introduction

In this chapter the notion of patents (which are equivalent to finite lifecycles in the framework presented below) of products is introduced into the basic framework. Then the influence of the length of these lifecycles on the dynamics of qualities and that of variety expansion is considered.

The question of how patents and patenting policy influence the dynamics of innovations and more generally, rates of technological change has a long history in economics. As it has been mentioned in Preface, this question has been considered already by Nordhaus, [**20**]. In this paper the first formal model of the optimal patent's length has been considered. It has been argued that the patent need not to be of infinite length to stimulate innovative activity. The basic idea behind this statement was that one needs patents to protect and stimulate innovators, but these patents need not to be very long to stimulate further innovative activity. In this chapter some progress towards establishing the similar argument for heterogeneous innovations is made. Similar to Nordhaus' paper one has a stand alone model of innovator in this framework since there are no competitors or potential entrants into the industry. On the other hand, one has the stream of two types of innovations here not the single one and patent has to be granted to every single product.

This form of innovative activity has also been considered in the patent literature under the name of sequential or cumulative innovations, where every next innovation is built up on the results of the previous one, [**33**]. Then the question of optimal patent's length becomes more complicated as the sequential character of innovations rises questions not only of the length but also of the breadth of the patent. These questions have been extensively studied by such authors as Shapiro, [**34**] and [**38**], Scotchmer, [**36**] and [**35**], and others.

One of the other approaches to the patenting problem is known under the name of patent races. Under this approach two or more agents are competing to be the first one to invent some product in order to obtain a prize which is the patent on this product and associated stream of profits from it, [**39**], [**40**]. Since the current framework assumes a stand alone innovator up to now, the model does not have any notion of patent races in it. Rather it is concentrated on optimal patent length for cumulative streams of innovations. Concerning this last there is also literature on variable length of patents for different products, as in [**37**]. The suggested framework although assuming identical length of patents for all products may be modified to consider this also.

The model has an uncountable number of such cumulative streams of innovations represented by every single product's quality growth process. More then this the underlying process of variety expansion is also modeled. So the question arises what is the level (scope) of patents one would like to consider in such a framework? One variant is patenting of every level of quality of every product. Such an approach would generate non-smooth quality dynamics since the agent would not have stimulus to increase quality of a given product until the patent on the preceding level of quality will not expire. Then one has to assume patenting on the level of variety expansion process. Such a modification to the basic model has a clear interpretation, since every level of $n(t)$ is associated with invention of the new product. Every such product is then granted a patent, or, alternatively has a limited life-cycle within each the agent is free to develop its quality without external pressure from the market. Such a formulation of patenting problem is in line with works on cumulative innovations with patents, as in [**16**], [**34**], but has some differences from it. In the suggested model one has not a single stream of innovations each of which is then patented or not, as in cumulative innovations literature, but rather one has stream of patented innovations and additional parameter of quality growth for each of this patented newly invented products. One also may account for the role of heterogeneity of these products, as it is done in [**15**], but in a given model it would influence not only the rate of innovations (which is the rate of variety expansion in suggested framework) but also the growth rate of qualities of all these products. This makes the suggested approach richer then the preceding ones.

It can also be noted in advance that the logic which pushes patents to be limited instead of infinite in their length is not the same as in the literature above. Since there are no any government authority or profit and competition incentives, the only thing that is analyzed is how the length of the patent influences the rate of innovations. Although it turns out that the infinite-time patents would maximize the output of innovations of both types there is an argument of distribution of resources between different types of innovative activity. It is this argument that constellates the limited nature of a desired patent. One last observation concerns the concept of a breadth of a patent, that is, how broad must be the category of products covered by each patent. In the given framework the variable breadth of patent may also be considered by allowing patent to cover not only the product being invented but some neighborhood of this product. Again this would change conclusions of the chapter but it is not analyzed here and left for future extensions.

In the rest of this chapter necessary modifications are made to the basic framework developed above to allow for limited life-cycles of products. The infinite-time horizon version of the model from chapter 3 is used as a benchmark and the same HJB approach with decomposition of quality growth and variety expansion problems is used. These problems are solved sequentially and then the results are combined. After this the effects of changes in patents' lengths on the rate of innovations of both types in the absence of other agents (stand alone model) are described and distribution of additional investments between both types of innovative activities is analyzed. It turns out that direct effect of increase in the patents' length is positive for both types of innovations, but the majority of the additional investments generated by the increase in patent's length is devoted to the growth of quality rather then to the variety expansion. The results obtained for heterogeneous and homogeneous setting are also compared. In conclusion the analysis of parameters' influence on the dynamics of a model is undertaken.

2. Finite Time Patents

In the framework established previously it is very difficult to introduce the notion of patent into the original finite-time horizon model. On the other hand with the infinite-time horizon the notion of patent might be introduced in a rather natural way.

The model abstracts from any competition as well as from patent races. However, the developed framework permits the analysis of the following question: whether the introduction of limited time patents will stimulate or depress innovative activity.

For that purpose one may handle patents as products' lifecycles: after the expiration of the patent's time the agent cannot use his achieved quality level of the given product for profitable activities (e.g. sell this product). This of course is not true in real economies, but one can imagine the high degree of competition on the product market which approaches the perfect one. As soon as the patent expires, all quality development of this product becomes the common knowledge to all the competitors and hence the agent in the model is no longer able to derive non-zero economic profit from it and thus he is no longer interested in quality investments in this product. In terms of the model this means that every product from $n(t)$ range has a limited time lifecycle, τ (determined by the patent length) during which its quality is developed by the agent. After this time development stops. At the same time such a setup would make sense only if at any given time the agent may switch his activities to the variety expansion investments thus earning himself another portion of patents. For this one would need the infinite-time horizon model with respect to variety expansion process. Observe also that in such a framework the variety expansion process naturally has a notion of effective range of products, defined by $n(t) - n(t-\tau)$. This may also refer to the process of disappearing of old products from the market: in basic model any product, once invented will never disappear from the economy. One may argue that it may be not the case, since many products which have been in use some time since their invention gradually disappear from daily life with invention of new 'better' products. This would in effect mean that the range of products' variety is not only widening, but it is also decreasing from the side of the oldest products introduced (it terms of the model with the lowest i's). The approach suggested here may be interpreted as an attempt to model this situation as well, as the effective range of products now is not only widening but may decrease at later stages.
Now consider all these in formal terms.
The objective functional of the agent is almost the same as in the initial model:

$$J^{patent} \stackrel{\text{def}}{=} \int_0^{\infty} e^{-rt}\left(\int_{n(t-\tau)}^{n(t)} \{q(i,t) - \frac{1}{2}g(i,t)^2\}di - \frac{1}{2}u(t)^2\right)dt \to \mathbf{max} \tag{3.1}$$

The main difference from the basic model here in the objective functional is that only those products which are in the effective patents length (or, alternatively, products within the lifecycle time period) are considered as positive inputs. At the same time variety expansion process is going on in infinite time. Consider this model as an extension of monopolist infinite-time horizon problem above with limited time of quality growth for each product being invented. For the purpose of this chapter one may change the integration variable of qualities. Instead of integrating over products' space one may integrate over the time-length of the patent:
(3.2)

$$J^{patent} = \int_0^{\infty} e^{-rt}\left(\alpha u(t)\int_0^{\tau} \{e^{-rs}q(n,s+t) - \frac{1}{2}g(n,s+t)^2\}ds - \frac{1}{2}u(t)^2\right)dt \to \mathbf{max}$$

This expression is equivalent to the preceding one but illustrates the decomposition method employed in previous as well as forthcoming chapters. It may be deduced from (3.1) by rearranging sequence of integration and renormalizing time periods of quality development to τ. The original formulation is used to illustrate the correspondence between current model and the basic one. However, the last one may be used to demonstrate the current value effect. In fact, the agent is maximizing the output of quality growth process minus investments into the boundary product ($i = n(t)$). The term $\alpha u(t)$ measures the intensity of the process of new products creation and is equivalent to maximizing over all range of products being invented in the time period not exceeding the length of the patent, τ.

All dynamic constraints (2.2) remain the same as in the original model with subsequent change of the area of definition of time: q_i is now defined on the time domain $t \in [t(0)_i; t(0)_i + \tau]$, while $n(t)$ is defined on the time domain $t \in [0; \infty)$. To solve such a model the Hamilton-Jacobi-Bellman approach is used in the same manner as in the previous chapter. The decomposition of the given problem into quality growth problem and variety ezpansion problem is used in the similar way also. Note, however, that this time quality growth is time-dependent and time horizon for this problem should be finite, thus yielding system of differential equations on coefficients of value function instead of algeabric equations.

3. Quality Growth in Patent Model

Value function representation for the quality growth problem:

$$V(i,\tau) = \int_{t(0)_i}^{\tau+t(0)_i} e^{-rs}\{q(i,s) - \frac{1}{2}g(i,s)^2\}ds \rightarrow \max \tag{3.3}$$

Note that integration limits may omit $t(0)_i$ term as well. One may translate all quality functions to the same time for simplicity (time-translation invariance). It won't change anything in results since outside the area of definition all quality functions are zero. Note also, that value function now depends not only on the index of the product (which implicitly defines dependence on the time of emergence $t(0)_i$ as the inverse function of i) but also on the length of patent which is assumed to be the same for all products. In the effect value function for quality growth depends only on the position of the product in the products' space, i, and on the length of the patent, τ. It is the form of time-dependence: quality growth now depends on time as compared to infinite-time monopolist problem above, but this 'time' is essentially different from the general time of the variety expansion process which distinguishes this model from the basic time-limited case. It has to be noted, that for that basic model the same decomposition method employed here would yield systems of differential equations for quality growth and variety expansion in the same time domain which makes the overall problem rather difficult. That's why wthe simplified version with infinite time is considered. Here one step further is made by allowing of time constraints for qualities but separating time domains of state variables.

The corresponding HJB equation for the quality growth:

$$rV(q(t)_i,t) + \frac{dV(q(t)_i,t)}{dt} = \max\Big\{q(t)_i - \frac{1}{2}g(t)_i^2 + \frac{\partial V(q(t)_i,t)}{\partial q(t)_i} \times (\gamma\sqrt{(N-i})g(t)_i - \beta q(t)_i)\Big\}. \tag{3.4}$$

with resulting first-order condition for optimal investments:

$$g(t)_i^{opt} = \frac{\partial V(q(t)_i,t)}{\partial q(t)_i} \times (\gamma\sqrt{(N-i}). \tag{3.5}$$

Assuming the same linear form of value function for this problem as before one have a system of 2 differential equations on value function coefficients:

$$
\begin{aligned}
\dot{A(t)} &= (r+\beta)A(t) - 1;\\
\dot{B(t)} &= rB(t) - \frac{1}{2}\gamma^2(N-i)A(t)^2;\\
A(\tau + t(0)_i) &= 0;\\
B(\tau + t(0)_i) &= 0;\\
t &\in [t(0)_i; \tau + t(0)_i].
\end{aligned}
\tag{3.6}
$$

Boundary conditions on coefficients follow from transversality (value function is zero at the end of the problem as well as its derivative with respect to the state variable). This is a system of first order equations which may be solved. First the solution for $A(t)$ coefficient is obtained:

$$
A(t) = \frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t(0)_i-\tau)}). \tag{3.7}
$$

Substituting this to the equation for $B(t)$ yields:

$$
\begin{aligned}
\dot{B(t)} &= rB(t) - \frac{1}{2}\gamma^2(N-i)\Big(\frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t(0)_i-\tau)})\Big)^2;\\
B(t) &= \frac{\gamma^2(N-i)}{2(r+\beta)^2}e^{rt}\Big(\frac{1}{(r+2\beta)}(e^{(r+\beta)(t-2(\tau+t(0)_i))} + e^{-r(\tau+t(0)_i)})\\
&\quad -\frac{1}{r}(e^{-rt} + e^{-r(\tau+t(0)_i)}) - \frac{1}{\beta}(e^{\beta t-(r+\beta)(t(0)_i+\tau)} + e^{-r(\tau+t(0)_i})\Big).
\end{aligned}
\tag{3.8}
$$

These calculations provide the form of the value function for quality growth:

$$
\begin{aligned}
V(q(t)_i, t) &= \frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t(0)_i-\tau)}) \times q(t)_i +\\
&\frac{\gamma^2(N-i)}{2(r+\beta)^2}e^{rt}\Big(\frac{1}{(r+2\beta)}(e^{(r+\beta)(t-2(\tau+t(0)_i))} + e^{-r(\tau+t(0)_i)})\\
&-\frac{1}{r}(e^{-rt} + e^{-r(\tau+t(0)_i)}) - \frac{1}{\beta}(e^{\beta t-(r+\beta)(t(0)_i+\tau)} + e^{-r(\tau+t(0)_i})\Big).
\end{aligned}
\tag{3.9}
$$

The resulting coefficients being inserted into the first order condition (3.5) yield optimal investments into quality growth which now do depend on time but only within the limits of the patent's length τ ($t \in [t(0)_i; t(0)_i + \tau]$):

$$
g(t)_i^{opt} = \gamma\sqrt{(N-i)}\Big(\frac{1 - e^{(r+\beta)(t-t(0)_i-\tau)}}{(r+\beta)}\Big). \tag{3.10}
$$

Observe that optimal investments to quality growth now are time-varying but they still do not depend on states themselves.
Finally one obtain ODE for quality growth:

$$
\begin{aligned}
\dot{q(t)}_i &= \gamma^2(N-i)\Big(\frac{1 - e^{(r+\beta)(t-t(0)_i-\tau)}}{(r+\beta)}\Big) - \beta q(t)_i;\\
q(t(0)_i)_i &= 0.
\end{aligned}
\tag{3.11}
$$

which is the first-order linear ODE with the solution:

$$
\begin{aligned}
q^*(t)_i &= \frac{\gamma^2(N-i)}{(r+\beta)(r+2\beta)\beta}\times\\
&\Big(\beta(e^{-(r+\beta)(t(0)_i+\tau-t)} - e^{-(r+\beta)(t(0)_i+\tau-t)}) + (r+2\beta)(e^{-\beta t} - 1)\Big).
\end{aligned}
\tag{3.12}
$$

It may be seen from (3.12) that quality growth for each product now depends on time as it is in the original finite-time model, but only within the boundaries of the patent's length and from the patent's length itself. It also depends on the time of emergence, $t(0)_i$. To calculate this last one one needs to solve the variety expansion problem. This does not lead to the inconsistency, since the variety expansion depends on the value function of quality growth only for $t = t(0)_i$ and thus this term is canceling out in the formulation of variety expansion problem. This justifies the normalization of time.

The final step of the solution of quality growth problem is the calculation of the value function. Then this value function with zero quality level is used as an input for variety expansion problem in the same way as it has been done for infinite-time horizon extension. One more thing to note here is that value function for quality growth in general depends on time within the patent length. Applying the same logic as for infinite-time horizon model, one may note that for variety expansion problem only the value of quality growth model at time of emergence, $t(0)_i$, is revelant because the agent estimates his potential profit from the expansion of the range of products available for him to develop only and this is done at the moment of the emergence of this good, $t(0)_i$. Since the terminal time for every product's quality growth problem is given by $T_i = t(0)_i + \tau$, the value function at $t(0)_i$ depends only on the paten's length, τ which is the same for all products. Hence one need to know only $V(i|i = n(t), \tau)_{|q_i=0,t=t(0)_i} = V(n(t), \tau)$, which is:

$$V(n(t), \tau) = \frac{\gamma^2(N - n(t))}{r\beta(r + 2\beta)(r + \beta)^2}\times$$

$$\times\Big(r(r + 2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r + \beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r + 2\beta)\Big). \tag{3.13}$$

Also note that the time horizon for variety expansion model is infinite. Then the HJB equation for this part is of the same form as in previous chapter (2.17).
Now deduce the form of quality growth problem value function which is used in variety expansion problem. Denote

$$V(\tau) = \frac{V(n(t), \tau)}{(N - n(t))} = \frac{\gamma^2}{r\beta(r + 2\beta)(r + \beta)^2}\times$$

$$\times\Big(r(r + 2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r + \beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r + 2\beta)\Big). \tag{3.14}$$

This does not depend on $n(t)$ then.

4. Variety Expansion Process in the Patent Model

To formulate the HJB equation for variety expansion problem make use of (3.14). Problem at this stage is to find optimal investments into variety expansion given that the quality growth of each next product yields the accumulated payoff of (3.14) during the time τ but not during the infinite-time horizon as in the preceding model.
Now the HJB equation for variety expansion problem takes the form:

$$rV_{n(t)} = max\{\alpha u(t) \times V(\tau)(N - n(t)) - \frac{1}{2}u(t)^2 + \alpha u(t) \times \frac{dV_{n(t)}}{dn(t)}\}. \tag{3.15}$$

Assuming quadratic form of the value function for this problem one obtains first order condition for the optimal control which depends on value function for quality problem:

$$u(t)^{opt} = \alpha\Big(V(\tau)(N - n(t)) + 2Cn(t) + F\Big). \tag{3.16}$$

with $V^{ass}_{n(t)} = Cn(t)^2 + Fn(t) + E$
The subsequent system of algebraic equations on value function coefficients is almost the same as for infinite-time case:

$$\begin{cases} rC - 2\alpha^2(C + V(\tau)N)(F - \frac{1}{2}V(\tau)) = 0; \\ rF - 2\alpha^2(F - \frac{1}{2}V(\tau))^2 = 0; \\ rE - \frac{1}{2}\alpha^2(C + V(\tau)N)^2 = 0. \end{cases} \tag{3.17}$$

Solving this yields coefficients of the variety expansion value function as functions of $V(\tau)$:

$$F = \frac{1}{4}\frac{2\alpha^2 V(\tau) + r - \sqrt{4\alpha^2 rV(\tau) + r^2}}{\alpha^2};$$
$$C = \frac{r - \sqrt{4\alpha^2 rV(\tau) + r^2}}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}V(\tau)N;$$
$$E = \frac{2\alpha^2 rV(\tau)^2 N^2}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}. \tag{3.18}$$

Now substitute this into the first order condition (3.16) to obtain optimal investments into variety expansion process as a function of $V(\tau)$:

$$u(t)^{opt} = \frac{2\alpha r(N - n(t))V(\tau)}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}. \tag{3.19}$$

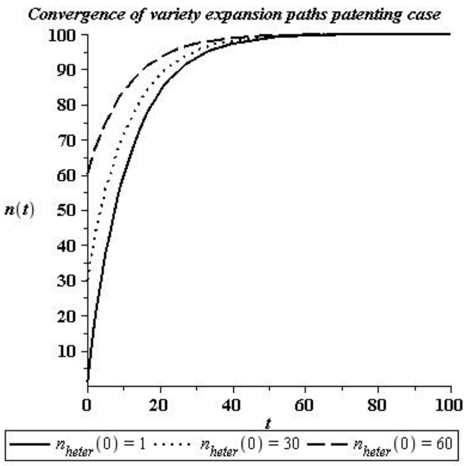
It may be seen that optimal investments into variety expansion are formulated in a feedback form and do depend negatively on state, $n(t)$, but unlike the infinite-time case, also on the patent length and the associated quality growth value function, τ and $V(\tau)$. Then (2.2) yield the first-order ODE for $n(t)$:

$$\dot{n(t)} = \frac{2\alpha^2 rV(\tau)}{r + \sqrt{4\alpha^2 r \times V(\tau) + r^2}}(N - n(t)). \tag{3.20}$$

which has the solution

$$n^*(t) = N + e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t}(n_0 - N). \tag{3.21}$$

This is of the same structure as the solution for infinite-time horizon model, but now includes additional parameter, τ. Then the influence of parameters is roughly the same (except for τ) and expansion paths with different initial conditions converge to the same trajectory:



If not stated otherwise for the rest of this chapter the following set of parameters is used for illustration:

$$SETP := [\alpha = 0.7, \beta = 0.1, \gamma = 0.7, r = 0.05, N = 100, \tau = 1]\,. \tag{3.22}$$

In the last picture the same set except for $\tau = 2$ is used for illustration.

Observe that in finite-time patent model quality dynamics is dependent on the time of emergence of new product. The solution for quality growth above abstracted from this fact. This does not influence variety expansion dynamics, since it depends only on $V(\tau)$ which is invariant for emergence times. The general dynamics of all products' qualities follows the form of (3.12), with equal lengths of the assoicated time horizon, but since time of emergence is different and life-cycle is finite (unlikely the infinite-time model), the exact form of dynamics is different. It is closer to the dynamics of qualities in the initial basic model from chapter 1. There the further is the index of a product in the products' range N, the lesser is the quality growth of that product. The same is true for the patents' model with the difference that terminal times are different for all products. Hence to get the final explicit form of quality dynamics one have to derive explicitly the expression for the time of emergence of the product as a function of the product's index, as this defines the overall form of dynamics then. This can be easily done in the given model since one has the explicit solution for variety expansion at hand. The derivation is essentially the same as for infinite-time monopolist problem.

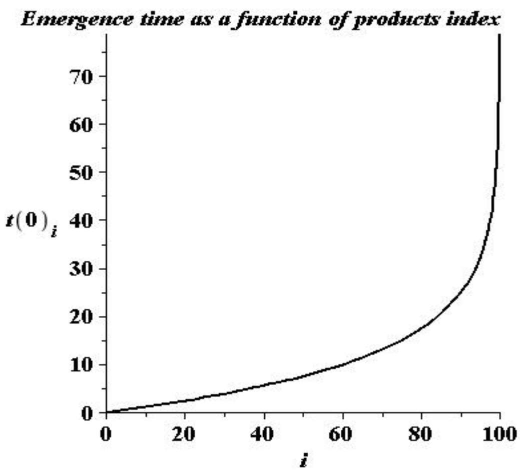
To calculate this time of emergence, $t(0)_i$, first observe that this is the inverse function of variety expansion process, $n(t)$:

$$t(0)_i = i^{-1}(n(t)). \tag{3.23}$$

Here one has the explicit expression for $n(t)$ and this inverse function may be derived:

$$i = N + e^{-\frac{2\alpha r V(\tau)}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}}t}(n_0 - N);$$
$$t = t(0)_i;$$
$$t(0)_i = -\ln(N-i) - \ln(N-n_0) \times \frac{1}{\frac{2\alpha r V(\tau)}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}}} = f(i,\tau). \tag{3.24}$$

This last expression is just the inverse function of $n(t)$. Since the $n(t)$ function is continuous and differentiable, this last exists always.



It is clear that emergence intensity is decreasing over time and new products appear relatively more rare when variety expansion is already at mature stage (close to N). Now consider the final form of dynamics for qualities, with time of emergence taken into consideration:

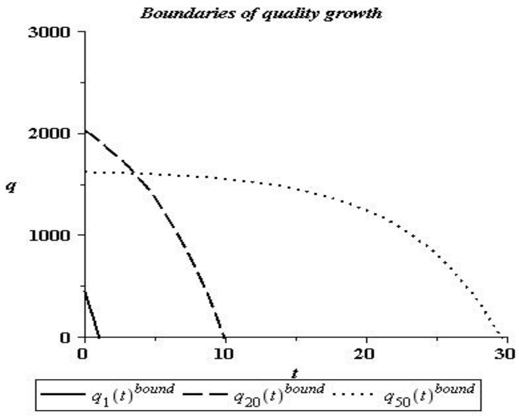
$$q_i(t) = \frac{\gamma^2(N-i)}{(r+\beta)(r+2\beta)\beta} \times$$
$$\times\Big(\beta(e^{\beta(f(i,\tau)-\tau-t)-r\tau} - e^{(r+\beta)(t-\tau-f(i,t))}) - (r+2\beta)(e^{-\beta(t-f(i,\tau))} - 1)\Big);$$
$$t \in [f(i,\tau), .., \tau + f(i,\tau)]. \tag{3.25}$$

Where $f(i,\tau)$ is given by (3.24). All products' qualities have the same length of time horizon, but they emerge at different times. Then the resulting dynamics is not invariant to time-translation, as it is in the basic model. There are no fixed points in this model for quality levels also. Instead, there are boundaries of growth. These boundaries depend not only on the product's index, i as in the basic model, but also on time. They are derived from the condition of zero growth for quality. Formally this boundary is given by:

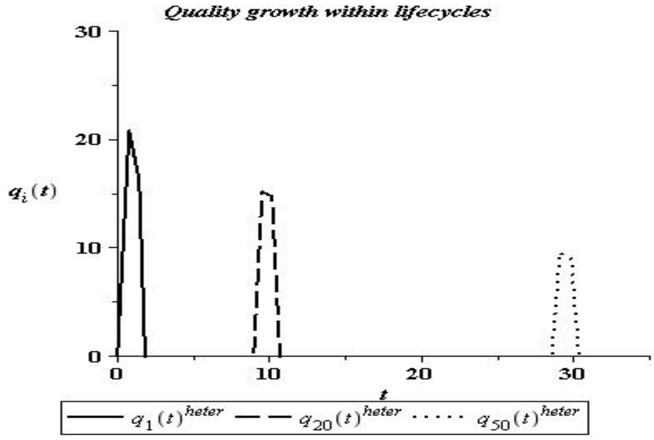
$$\dot{q(t)}_i = 0;$$
$$q(t)_i^{bound} = \frac{\gamma^2(N-i)}{\beta(r+\beta)} \times (1 - e^{(r+\beta)(t-\tau-f(i,t))}). \tag{3.26}$$

It is clear that this boundary differs from product to product (according to i changes) as it is in the basic model, but also it changes over time for every product. The quality growth stops after reaching this boundary and decrease of quality begins until the quality level falls down to zero at the time of expiration of patent. Note that the model permits for automatic disappearance of product after its' patent expiration.

Now consider variation and domain of these boundaries:



It can be seen that for higher products' indices the boundary admits lower qualities but longer time. This boundary reaches zero exactly at time of patent expiration thus pushing quality level to zero at that time. In the effect quality growth is single-peaked as it is in the basic time-limited model, but qualities always reach zero after the end of their life-cycles. Observe however, that at the end of the life-cycle the quality of any given product is not zero, but reaches zero afterwards since after investments stop some additional time is needed for the quality decay to push the level of quality to zero. Hence actual qualities reach zero level some time after the patent ends. This has to be kept in mind ofr the rest of the chapter while analyzing illustrative numeric examples.



5. Homogeneous vs Heterogeneous Products in Finite-Time Patents Setting

To illustrate the role of heterogeneity in such a model one may consider the homogeneous case also. For that one has just to assume investment efficiency function $\gamma(i) = \gamma = const$. Then all quality growth paths are identical and products differ only in the time of their emergence. For such a modification of the model all calculations above are held the same except that value function for quality growth problem now does not depend on $n(t)$ in any way. This results in linear growth of variety expansion process with much slower rate then for heterogeneous case.

Quality growth solution and optimal investment for homogeneous case:

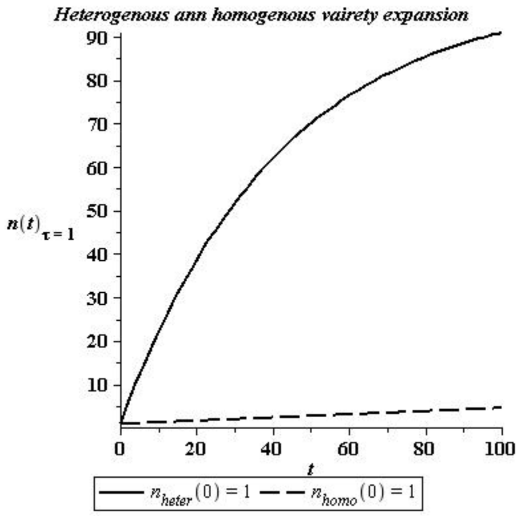
$$g(t)^{opt}_{i,homo} = \gamma\Big(\frac{1 - e^{(r+\beta)(t-t(0)_i-\tau)}}{(r+\beta)}\Big);$$

$$q(t)^*_{i,homo} = \frac{\gamma^2}{(r+\beta)(r+2\beta)\beta} \times \Big(\beta(e^{-(r+\beta)(\tau-t)} - e^{-(r+\beta)\tau - t}) + (r+2\beta)(e^{-\beta t} - 1)\Big). \tag{3.27}$$

which is the straightforward result from heterogeneous case with the term $N - i$ canceled out. Much more difference is in variety expansion:

$$u(t)^{opt}_{homo} = \alpha \times V(\tau);$$
$$n(t)^*_{homo} = \alpha^2 \times V(\tau)t + n_0. \tag{3.28}$$

which is the simple linear growth. It then has to be noted that the variety expansion rate is always slower for homogeneous case, since it is a linear growth versus exponential one for heterogeneous case, while relative quality growth depends on the scale of γ. If γ is higher than $(N - i)$ term for heterogeneous case, the quality growth may be higher in homogeneous case and vice versa. The conclusion should be that the heterogeneity of products stimulates the variety expansion but not necessarily stimulates the quality growth.



This illustrates relative growth rate for heterogeneous and homogeneous versions of the model. It can be seen that heterogeneous solution displays much higher growth rate then the homogeneous one with the same parameter set.

Now consider the effective range of products at the agents disposal, $n(t)-n(t-\tau)$. This is exactly the range of products at each t which lifecycles did not expire yet (or, alternatively, patents did not expire). Observe, that in homogeneous products case this range remains constant over time, since the variety expansion is linear:

$$n(t)_{homo} - n(t-\tau)_{homo} =$$
$$\alpha^2\gamma^2\tau\Big(\frac{(\frac{1}{2}r+\beta)}{r(r+2\beta)(r+\beta)^2} + \frac{r(r+2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r+\beta)^2}{r\beta(r+2\beta)(r+\beta)^2}\Big);$$
$$\frac{d[n(t)_{homo} - n(t-\tau)_{homo}]}{dt} = 0. \tag{3.29}$$

For heterogeneous products the situation is more complicated since the variety expansion does not have the constant speed:

$$n(t)_{hetero} - n(t-\tau)_{hetero} =$$
$$(n_0 - N)\times e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t}\times(1-e^{\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}\tau});$$
$$\frac{d[n(t)_{hetero} - n(t-\tau)_{hetero}]}{dt} = (n_0 - N)\times(1-e^{\frac{2\alpha rtV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}\tau})\times$$
$$\frac{d(e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t})}{dt} =$$
$$(n_0 - N)\times(1-e^{\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}\tau})\times$$
$$\times\Big(-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t)e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t})\Big) < 0. \tag{3.30}$$

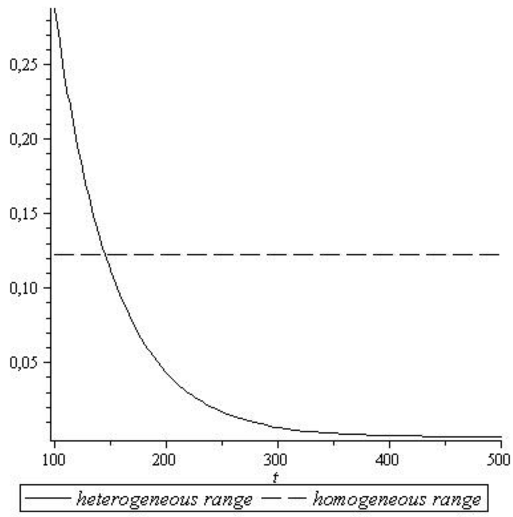
So one may observe that the effective range of products is decreasing over time for heterogeneous case due to slowing rates of variety expansion in time. At the same time the initial rate of growth is higher in heterogeneous case as it is in the basic model also. Then there is the initial period of time when effective range of products is higher for heterogeneous model then that for the homogeneous case. The exact time when this relation reverses may be defined from expressions above by equalizing effective ranges and defining the time t_{eq} when they are equal from this expression:

$$t_{eq} : n(t)_{hetero} - n(t-\tau)_{hetero} = n(t)_{homo} - n(t-\tau)_{homo}. \tag{3.31}$$

The exact expression is cumbersome and is not displayed here. Observe however, that this time corresponds to the point in the variety expansion path for heterogeneous case which is closer to the boundary N that to the initial variety n_0. Homogeneous range for most parameter values is lesser than one and is constant while the heterogeneous range initially is greater than one. To illustrate the relation between these two effective ranges, consider the numerical example with parameter values

$$SETR := [n_0 = 0, N = 100, \alpha = 0.7, \beta = 0.2, \gamma = 0.5, r = 0.05, \tau = 1]. \tag{3.32}$$

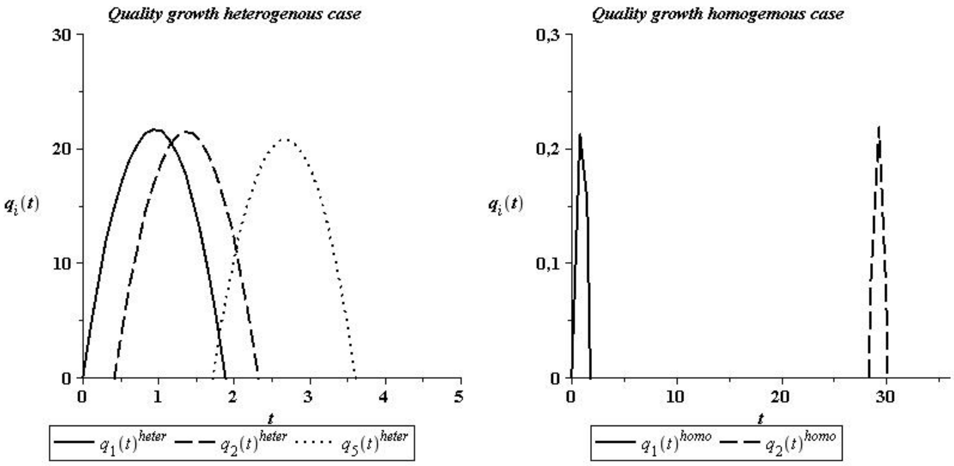
With such a set of parameters the effective variety range for heterogeneous products becomes lesser than that for the homogeneous one at $t_{eq} = 145.05$ which corresponds to the range $n(t) - n(t-\tau) = 0.125$ while the variety of the heterogeneous process reaches the value $n(t)_{hetero} = 85.39$ which is above 50 percents of the whole available range:



Economically this means that the agent dealing with more diverse set of possible innovations has to invest in a more intelligent way into variety expansion. Instead of constantly expanding the product space at a constant rate he modifies his policy in accordance with the variety already reached and the more variety he already has at hand the slower he is expanding this variety even if that means he would have smaller effective range of products. So one cannot unambiguously say that heterogeneous space of potential products yield more stimulus for innovations in the limited life-cycle case as it is in the basic model above: with the flow of time the innovative agent first would invest much more in variety expansion then that in homogeneous case, but then his investments gradually shrink down to zero. Note also that the more rapid is the increase in the complexity of new products (e.g. assume $(N-i)^2$ instead of $\sqrt{(N-i)}$ rate of efficiency decrease) the faster is the decay in the effective range of products for the agent.

Finally consider the consequences of such differences in variety expansion modes for the quality growth. In homogeneous model the variety expansion is linear and thus new products emerge relatively rare before the t_{eq}. The result should be the lower density of quality growth trajectories in time domain. At the same time absence of diversity lowers the boundary of growth for homogeneous products and this boundary is the same for all of them. In fact, it is straightforward that

$q(t)^{bound}_{homo} \leq \min\{q(t)^{bound}_{hetero}\}$. This follows from the aforementioned fact of relative rareness of new products in homogeneous setting: even when investments in new products are always less effective then into preceding ones, the diversity stimulates quality growth rates just by means of higher speed of variety expansion and thus higher density of new products.



Note the period of time taken in the graph and maximal quality levels. Solution paths are plotted for identical parameter sets $SETP$ with change $\tau = 2$. It has to be noted that in homogeneous case quality growth is almost zero if comparing to heterogeneous case and products appear much more rarely also. So one may again conclude that for patenting model the homogeneous model does not create enough incentives for innovations due to the similarity of products' characteristics. This effect is similar to the one being observed in previous chapters.

6. Compensation Effect vs Potential Profit Effect

Now one may ask whether the introduction of patents stimulates quality growth and variety expansion processes or not. First observe effects on the variety expansion. One may observe two opposite effects here. The first one is negative: the shorter is the length of the patent, the lesser is the effective range of products at the agent's disposal at each point in time. At the same time short patent length means that the agent is able to develop quality of the given product to the smaller extent which in turn lowers his incentives to invest in the variety expansion. These effects are refered to as compensation effect and potential profit effect accordingly, the last due to implicit assumption of linear profit being derived from qualities.

The effective range is given by $n(t) - n(t-\tau)$ since only products introduced during this time are covered by patents at the time t. Observe that for homogeneous and for heterogeneous cases the effective range's response to the patents' length changes may be of the different sign:

$$\frac{d[n(t)_{homo} - n(t-\tau)_{homo}]}{d\tau} = \frac{d(\alpha^2 V(\tau)\tau)}{d\tau} = \alpha^2 V(\tau) + \alpha^2 \tau \frac{dV(\tau)}{d\tau} > 0;$$

$$\frac{d[n(t)_{heter} - n(t-\tau)_{heter}]}{d\tau} = -\frac{2\alpha r^2 (r + \sqrt{4\alpha^2 r V(\tau) + r^2} + 2\alpha^2 V(\tau))}{\sqrt{4\alpha^2 r V(\tau) + r^2} \times (r + \sqrt{4\alpha^2 r V(\tau) + r^2})^2} \times$$

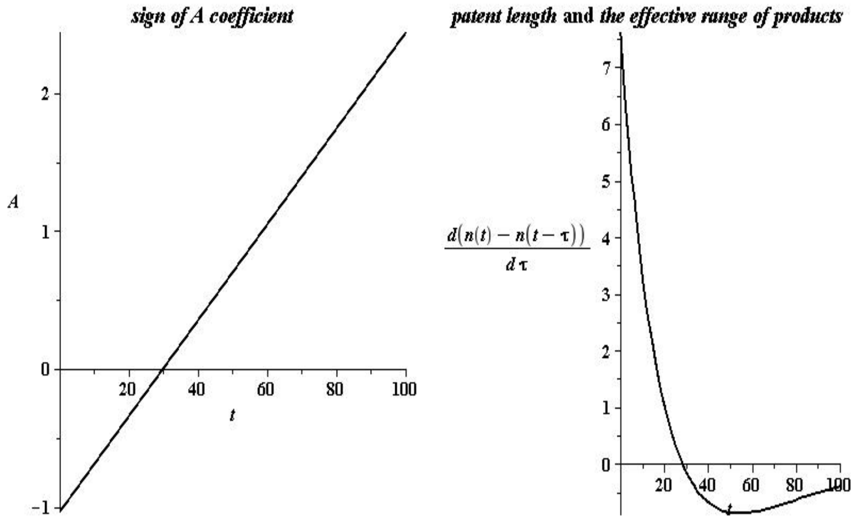
$$(N - n_0)\frac{dV(\tau)}{d\tau} e^{-\frac{2\alpha r V(\tau)}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}} t}\left((t-1)e^{\frac{2\alpha r V(\tau)\tau}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}}} - 1 - t\right). \tag{3.33}$$

The first derivative is always positive since $\frac{dV(\tau)}{d\tau}$ is positive, as value of the quality growth may only increase with the increase in the patent's length.

At the same time the sign of the last derivative depends on the sign of expression

$$\mathbf{A} = ((t-1)e^{\frac{2\alpha r V(\tau)\tau}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}}} - 1 - t). \tag{3.34}$$

This last may be positive or negative depending on relative size of the value function $V(\tau)$. It depends on the length of the patent, τ. For long patents it is greater then one and the subsequent expression (3.34) is positive for almost all t's yielding negative derivative sign, for short patents it is negative for most t's yielding positive derivative sign. Observe also that since this expression depends on time there is always some initial period when it is negative for $t \to 0$ and always positive for $t \to \infty$. In effect this means that changes in patents' length have always a positive effect on the effective range for homogeneous case, but the effect is of changing sign for heterogeneous case. This last phenomena is illustrated on the following graph.



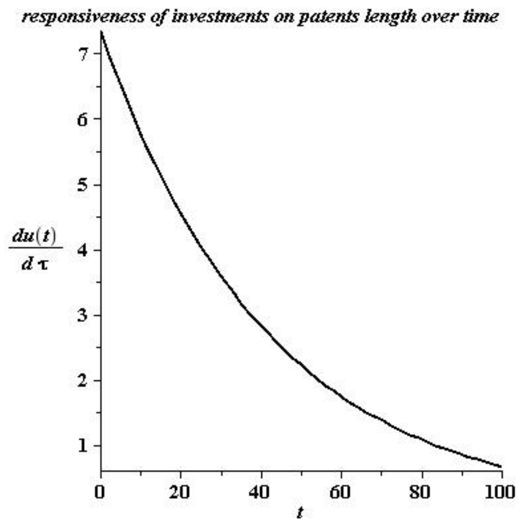
Further on this effect is analyzed in more details to demonstrate that it exactly corresponds to the interplay between compensation and potential profit effects.

First observe that changes in the sign of the aforementioned effect happen because of the changes in the response of variety expansion investments to the patent's length. At the initial period the effect of changing patent's length is positive for this variable and that means domination of potential profit effect in comparison with compensation effect. As time goes on, the compensation effect eventually becomes the dominating one. One may demonstrate that there is a period of time at $t \to \infty$, when the impact of patent's length on the variety expansion is negative. This happens because the more possibilities for innovations the agent has at hand, e.g. $n(t)$ is small enough, the more attractive the variety expansion is for him as it grants more opportunities for profit generation through the subsequent quality growth. On the other hand with potential products space almost exhausted the effective range of products at the agent's disposal is decreasing down to zero and the shorter length of products' life-cycles would stimulate him more on widening it while the potential profit effect would be small enough as the increased complexity of quality investments make it less attractive to invest in quality innovations and this in turn means less profit generation. Then shorter patents' lengths would not decrease the investor's incentives to increase the variety in a degree enough to outweight his incentive to invest more to increase the effective range at his disposal. This is an important feature of the model under discussion as it provides ground for finiteness of the optimal patent. Note that this interplay of two different effects cannot be captured in the standard patenting model. This happens not only because of the dynamic character of the model under consideration but also due to the unified approach one has here. The compensation effect appears only in heterogeneous innovations model and exactly due to the difference in products' investment characteristics. This may be seen from the constancy of the effect of patents on the homogeneous model above, where the compensation effect is absent. Consider also the impact of patent's length on the variety investment and therefore on the variety expansion itself (since it is proportional to investments):

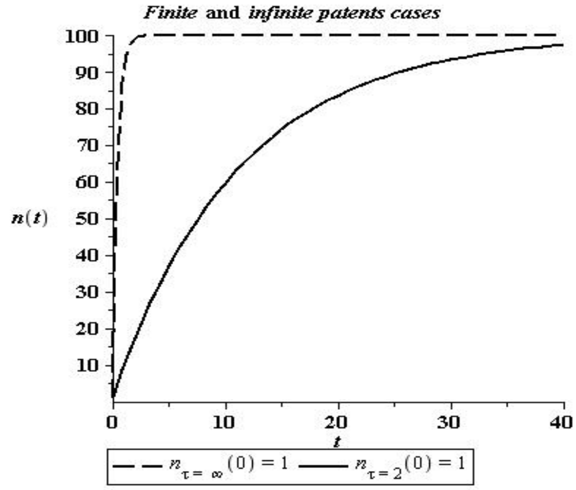
$$\frac{du(t)_{homo}}{d\tau} = \alpha \times \frac{dV(\tau)}{d\tau} > 0;$$
$$\frac{du(t)_{heter}}{d\tau} = 2\alpha r \frac{dV(\tau)}{d\tau} \times$$
$$\times\Big(\frac{1}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}} - \frac{2\alpha^2 V(\tau)}{(r+\sqrt{4\alpha^2 rV(\tau)+r^2})^2\sqrt{4\alpha^2 rV(\tau)+r^2}}\Big). \tag{3.35}$$

As it may be seen from the expressions above, there is no ambiguity of patent's impact on investments in homogeneous case, but in heterogeneous case there are two effects of opposite direction. Therefore the variety expansion itself, not only the effective range of products may shrink because of the increase in patent's length, provided that this length is not very big.

However it turns out that the investments in variety expansion always react positively on the increase in patent's length which means that the potential profit effect outweights the compensation effect all the time. It does not mean, that the effective range of products always increases with respect to patents prolongation as it is discussed above. This controversy is resolved if one would consider changes in investments responsiveness to the patents' lengths over time. It turns out, that second (negative) term is increasing while the first one is decreasing. That's why the positive impact of patents on investments is decreasing over time. Then effective range of products response becomes negative, as it is decreasing over time since the overall dynamics of variety expansion and it is decreasing because of decreasing investments given increase in patent's length.



Because the patent's length always has a positive effect on growth rates, the infinite-time horizon model may be considered as a patent model with infinite patent length in this respect. It can be shown that the patent model is equivalent to the infinite-time horizon model with $\tau \to \infty$. To provide some graphical illustrations below again the set $SETP$ with $\tau = 2$ of parameters is used: Below is the comparison of $n(t)$ dynamics for infinite-time horizon and patent models.



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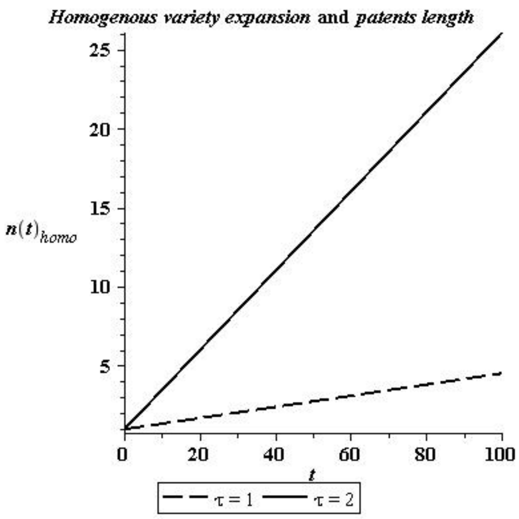
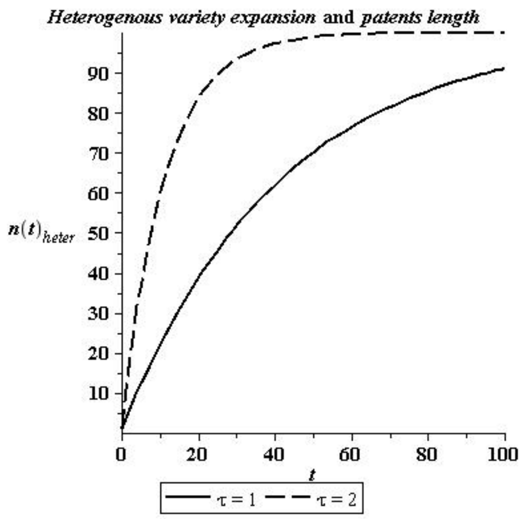
It is clear that the variety expansion process of infinite-time horizon model is the limiting case of variety expansion process of the patent model.

It may be observed that due to the precense of the compensation effect in the model the variety expansion response to the increase in the patent's length is less then in homogeneous model. This would be clear from the fact that in homogeneous case the compensation effect is zero and potential profit effect is constant over time. As a result the increase in patent's length has much more stimulating effect on variety expansion rate in homogeneous case.

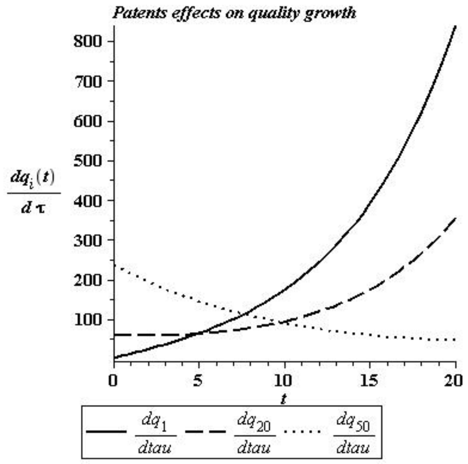




Quality growth essentially depends on the patent's length also. The longer the patent, the closer patent model quality dynamics is to the infinite-time one. The quality growth displays only the potential profit effect if one would encounter single product investments. For that consider the respective derivative of (3.25):

$$\frac{dq(t)_i}{d\tau} = \frac{\gamma^2}{(r+\beta)\beta(r+2\beta)} \times \frac{1}{G(\tau)^2} \times \Big(\beta(r+\beta)(N-i)(\frac{N-i}{N-n_0})^{-\frac{r+\beta}{G(\tau)}}(G(\tau)^2 - \ln(\frac{N-i}{N-n_0})\frac{dG(\tau)}{d\tau})\Big) \times (e^{(-t-\tau)\beta-rt} - e^{(t-\tau)(r+\beta)}) > 0, \forall i, \forall t. \tag{3.36}$$

Where $G(\tau)$ is the power of exponent in (3.21) which positively depends on τ. Hence the sign of the expression above remains positive all the time and there is no compensation effect for quality growth here.

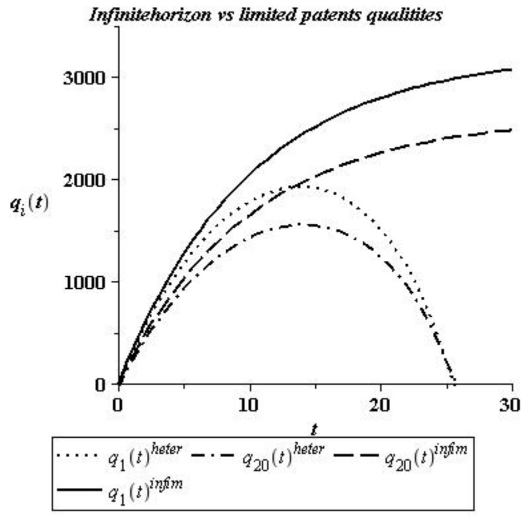


Observe that at any given time t there is the effective mass of products which qualities are to be developed. This mass is

$$\int_{n(t-\tau)}^{n(t)} q_i(t)di. \tag{3.37}$$

The compensation effect if present must be observed in the changes of this quantity: changes in patent's length change the number of products which qualities are to be invested in. Shortening patents decrease the range of qualities to develop at each point in time. Unlikely the variety expansion, quality growth is not stimulated by the decrease in the range of qualities, since the decrease in patent's length unambiguously decrease the maximal quality for every product within this range. Then the compensation effect concerning qualities should be of the same sign as the potential profits effect: the longer is the patent, the bigger is the quantity (3.37). The difference with variety expansion dynamics is that the agent cannot switch from lower to higher trajectory of quality growth in the sense of aggregate investments into quality with shortening patent's length. This means quality investments are more responsive to the changes in patents' length, then the variety expansion, where responsiveness is lowered by the different directions of two effects.

Observe however, that in the patent model quality level never reaches its maximal level for any product. The distance from the maximal level to the actual trajectory increases with the decrease in the patent length. This maximal level is the level of steady-state quality in infinite-time horizon model. Below several trajectories for different product indices with patent length $\tau = 20$ are plotted.



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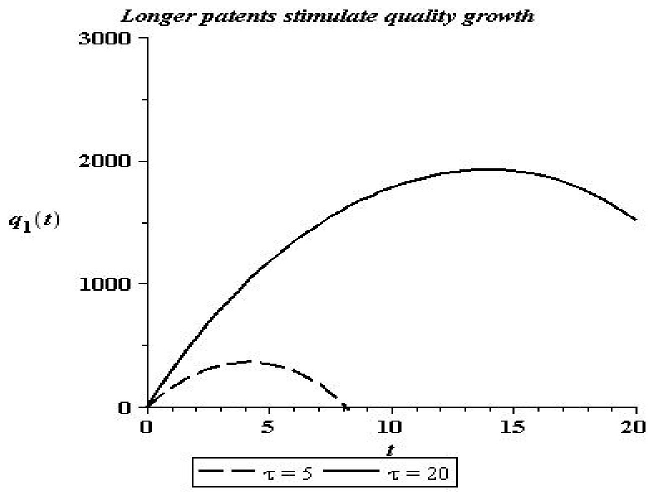
Observe that for long lifecycles the variety expansion is faster and that's why the density of new products' qualities is higher. Then the differences in terminal and emergence times across products is smaller. Compare this last picture with the situation where $\tau = 2$ while all other parameters are from $SETP$. Nevertheless, the infinite-time qualities appear to be the limiting case for qualities of limited patent model as this picture shows.

At the same time increase in patents' lengths (the same for all products) shifts up the trajectory of quality growth for any given product. It should be noted, that for any $\tau < \infty$ quality growth is less then the maximal level for infinite-time case:



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7. Distributional Effects

In the previous section it has been noted that the relative responsiveness of quality investments to the changes in patent's length is stronger then that of variety expansion investments. In this section this effect is investigated in further details. This distributional effect is rather important because it may give an argument why too long patents are not efficient from the social welfare viewpoint. In its absence there is a straightforward conclusion upon the patents' lengths effects:

longer patents just imply bigger investments both into quality growth of each product and into the variety expansion.

Now consider the relative growth of quality and variety expansion investments conditional upon the change in patent's length. Note also that to compare the distribution of additional investments into products' variety and quality growth one would take into account the total amount of additional investments into the effective range of products, (3.37):

$$\frac{\frac{d\dot{n}(t)}{d\tau}}{n(t)};$$
$$\frac{\frac{\int_{n(t-\tau)}^{n(t)} \dot{q}_i(t)di}{d\tau}}{\int_{n(t-\tau)}^{n(t)} q_i(t)di}. \tag{3.38}$$

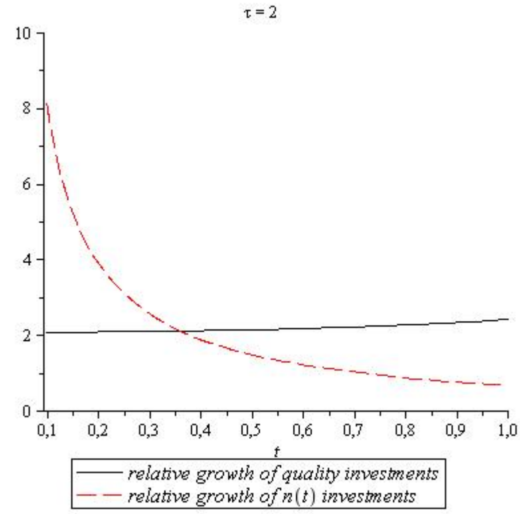
Using these expressions relative increase in investments being normalized by the level of state variable in the first case and by the integral (sum) of all states for the case of quality investments may be compared. There is little interest in comparing investments to variety expansion and single product quality investments, since these products are constantly changing governed by the variety expansion which is one of the parameters in the comparison. So the only way should be to search for some invariant measure of quality investments. This would be the integral sum of relative investments to all products' qualities which are still in existence (covered by patents).

Since solution for $n(t)$ is known, (3.21), the last expression may be computed as well. It turns out, that analytically and numerically the second expression is bigger then the first one with longer patents lengths:

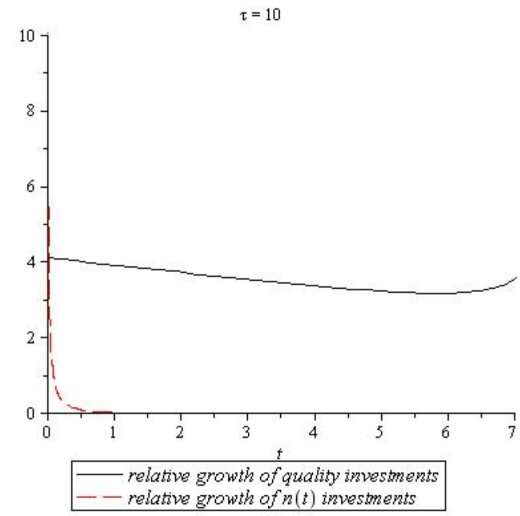
$$\frac{\frac{d\dot{n}(t)}{d\tau}}{n(t)} << \frac{\frac{\int_{n(t-\tau)}^{n(t)} \dot{q}_i(t)di}{d\tau}}{\int_{n(t-\tau)}^{n(t)} q_i(t)di}. \tag{3.39}$$

Formal derivation of this result is not displayed here as it must be clear from the previous discussion that the relative responsiveness of quality investments should be higher. Nevertheless it turns out that for short patents ($\tau \leq 5$ with $SETP$) it is possible that variety expansion investments grow more than the quality investments with respect to increase in patents lengths. This may happen only at initial period of development when the achieved variety is low. In fact it must be lower then the effective range. After achieving the variety level of $n(t) > n(\tau)$ this effect disappears.

Graphical illustration of this effect follows.



It is clear that responsiveness of variety expansion investments is quickly falling down while response of quality investments is almost constant. If one would take a look on the formulation of variety expansion investments response, (3.35) this changes are clear enough: the compensation effect plays bigger role after variety expansion reaches mature stage (e.g., when $n(t) > n(\tau)$), while quality investments always react on patents in unambiguous way.
At the same time with longer patents ($\tau \geq 5$ with $SETP$) increase of quality investments is always higher then that of variety expansion:



It also has to be noted that exact value of the patent's length at which this partial domination of variety expansion investments response happens may vary depending on the exact value of other parameters of the system. This means that longer patents mean potential harm to the society because they may induce underinvestments into the variety expansion and overinvestments into quality growth of products. This creates a potential gap between the desired growth of the variety of products and the optimal one for the monopolistic agent. Hypothetically that would create the possibility for open source innovations which are mainly concentrated on the variety expansion thus compensating the lack of variety growth due to longer then optimal patents. It has to be noted that this argument is not formally well established here. To discuss the social optimality of variety expansion provision and quality provision one would introduce some form of social welfare function, which is absent in the given framework. Yet it would be rather difficult to justify one or another specification of such a function in the dynamic context of the model. There is some discussion in the literature on dynamic social preferences and welfare functions yet there is no consensus on the correct form of its representation. As one example consider the current setup where the number (variety) of products on the market is not constant but changes over time. The main complication comes from the introduction of patents themselves: in a given model products' variety may not only increase as before, but also the effective range may decrease. This decrease would then decrease the social welfare if formulated in e.g., Dixit-Stiglitz form which is not the desired outcome. However even in the absence of well-established social welfare objective function it is possible to discuss the optimal length of the patent in the model. The optimality criterion would be the maximization of both kinds of investments. It turns out, that the growth of both kinds of state variables is maximized for $\tau \to \infty$ as it has been already noted. However, quality growth is much faster then that of variety expansion in relative measure and what is the desired ratio is an open question.

One may also conclude that the potential profit effect of the increase in patents' lengths dominates the compensation effect of substitution of investments. Introduction of several competing agents may help to enrich the set of possible outcomes. The distinguishing feature of the suggested model is the precense of these compensation effects which are due to the introduction of heterogeneous technologies (products) and simultaneous modeling of multiple streams of innovative activities. Yet the single-agent model possibly underestimates the scale of these effects. The incentive to invest more in variety with shortening patents is not very big and always is outweighted by the perspective of more profits with longer patents. With competitive pressure this may not be the case, since there is another additional incentive to increase the variety expansion if other innovators are trying to achieve the same goal. Yet such a form of competition is extremely difficult to model and moreover one has to provide some theoretical ground to demonstrate that such a competition on the level of fundamental research associated with new products' creation is possible. This question is not addressed in current work. Instead, more specific form of competition of innovative agents in the same dynamic context is considered in the next chapter. There the question why the competition on the variety expansion level is not a realistic choice is also discussed in more details.

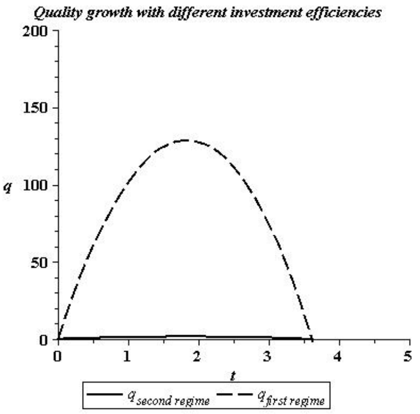
8. Parameters' Influence in the Patenting Model

In this section the direction and degree of influence of model's parameters are analyzed. The exception is the patent's length, τ, which is already extensively discussed previously. Main point of interest in such an analysis would be the influence of different market conditions which are represented by products' investment efficiencies as well as by the decay rate β. This set defines the nature of products available to the innovator. N defines the size or scale of possible innovations instead. First note that direct influence of parameters on the growth rates is exactly the same as in the infinite-time horizon model discussed previously. Hence this is not discussed here. It is sufficient to remind that increase in efficiency parameters α, γ stimulates variety expansion while quality growth remains unchanged with alpha's changes. Increase in quality decay rates has a negative effect on both state variables and increase in the size of the products' space, N, stimulates both variables.

Consider the relative effect of investment efficiencies. One would expect that different values of investment efficiencies would be able to provide the switch of the main part of investments from one kind of innovations (e.g. variety expansion) to another one (quality growth). In this section only the heterogeneous version of the model is analyzed as the most interesting one. To test relative influence of these parameters consider the set with high quality investment efficiencies and low variety expansion efficiency and the reversed variant:

$$\gamma_1 = 0.9; \alpha_1 = 0.1;$$
$$\alpha_2 = 0.9; \gamma_2 = 0.1. \tag{3.40}$$

Keeping all other parameters constant as in $SETP$ and $\tau = 2$. Numeric computation demonstrates that these variations is investment efficiencies have a tremendous effect on quality growth while variety expansion remains unchanged. Formally this happens because variety expansion investments depend equally from investments efficiency into the variety expansion and quality, while quality growth does not depend on variety expansion efficiency. In effect variations in these efficiencies if zero in total have no effect on variety expansion. Yet with given changes quality level changes in nearly 100 times (for $i = 1$).



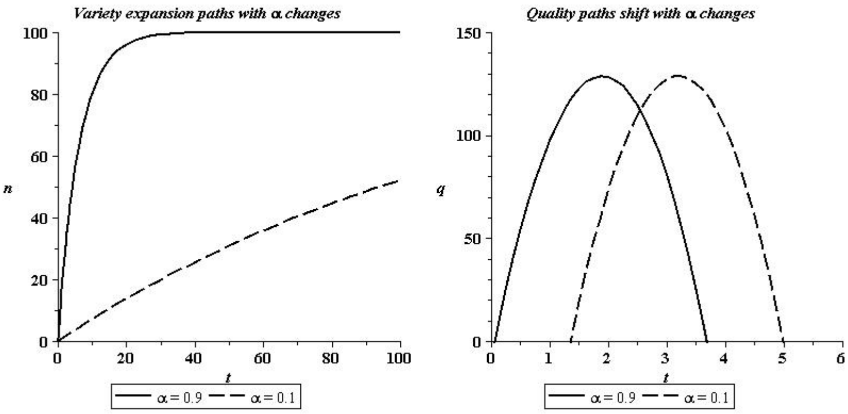
This numeric argument may be supported formally also. Consider the response of qualities and of variety expansion to these parameters changes. For that recall the solutions of this model, given by (3.21) for variety expansion and (3.25) for quality growth. It is then clear that:

$$\frac{dq(t)_i}{d\alpha} = 0;$$
$$\frac{\frac{dn(t)}{d\alpha}}{\frac{dn(t)}{d\gamma}} = \frac{\alpha}{\gamma}. \tag{3.41}$$

So that changes in variety expansion in response to increase in one or another of the efficiency parameters is the same.

Observe that any changes in efficiency parameters cannot increase variety expansion investments at the cost of quality growth. The only situation when variety expansion would increase while quality growth will not, is if α parameter increases while γ remains unchanged. But then quality growth remains unchanged, since (3.41) and it is clear that increase in variety expansion comes from some additional investments. This highlights one of the shortcomings of the given framework: since there are no resource constraints, the agent would be able to increase investments of one kind without reducing investments of the other. Such an extension of the basic framework may prove itself useful.

At the same time observe that α does not influence the maximal reachable quality, but does influence the actual choice of the trajectory for any given products except for $i \leq n_0$. Change in α shifts the time of emergence of any product preserving the shape of quality growth path. This effect comes directly from the dependence of quality trajectory on the time of emergence in the patent model. For products with $i \leq n_0$ there is no such dependence since this product is already known at the beginning and its emergence time cannot be shifted (it is always zero).



Because of such a shift in emergence times one may claim that changes in variety expansion efficiency affects the average level of quality across products at each point in time, but not quality levels of individual products. This parameter variation changes the structure of the space of products itself without changing each product separately. It is obvious that shift of the form demonstrated here would decrease number of products in existence at each point in time and thus increase the average quality level. Denote:

$$q(t)^{AV} = \frac{\int_{n(t-\tau)}^{n(t)} q(t)_i di}{n(t) - n(t-\tau)}, \tag{3.42}$$

the average quality level of products within the coverage of the patent. Now consider the response of this variable to changes in investment efficiency. If investment efficiency of variety expansion would grow, then $n(t)$ will increase and thus the emergence of new products is accelerated. As discussed above, this will increase the effective range $n(t) - n(t-\tau)$ also. At the same time quality levels of these products (all of them) will remain unchanged. As a result, the average quality will decrease:

$$\frac{dn(t)}{d\alpha} > 0;$$
$$\frac{dq(t)^{AV}}{d\alpha} < 0. \tag{3.43}$$

Then one may conclude that there is an interplay between quality and variety expansion investments in a given model, but this interplay cannot be observed on the level of individual products' qualities. Rather one has to consider the average quality level. At this level of analysis there is a natural interplay: the higher is the relative efficiency of investments into qualities or variety expansion, the more intensive is the growth of this kind of investments, while the other's intensity is slowing down. At the same time note that γ's influence is not that symmetric, because its growth stimulates variety expansion in the same direction as quality growth. In this case the sign of change of average quality may be different, because there is simultaneous growth of numerator and denominator of the (3.42). One may only note that this should be positive, since the responsiveness of quality growth to γ's changes is much higher then that of variety expansion.

9. Discussion

In this chapter an extension of the basic model which allows to consider finite-time lifecycles of products together with the infinite-time process of products' generation is developed. Such a setting is closer to real economic activities then the basic one, since from the one hand, it is not realistic to assume finite planning horizons of innovating agents as typical innovative activity is performed with long and uncertain time horizons. On the other hand it is also not realistic to assume infinite life of all products whenever they have been invented. Every product has an effective life-time within which it has a substantial demand associated with it and hence is capable of generating profit for the innovator. Although not very much products literally disappear from the market if to take some reasonable scope of analysis, rather big portion of existing variety of products is renewed within some periods of time. This means one has to formulate a logic of behavior of an innovating agent which would combine infinite planning horizon with finite life-time of products. It can be done obviously through the framework of cumulative or sequential innovations. In some respect the current version of the model is closer to this strand of literature then the basic one, since it allows for finite life-times of products and thus explicitly the 'sequence' of innovations is introduced in a continuous way. Apart from the patenting problem this may be viewed as a stream of finitely living innovations, as in [**16**] which is the closest paper. In this paper Chang assumes two competing firms which are innovators and in two periods of time where in the first period firm 1 is introducing some new product and in period two the other firm is introducing another product which is an improved version of the first product, introduced previously. In this respect this describes cumulative innovations pattern. In current model the process of variety expansion is going in a cumulative way, since the next product cannot be introduced without introducing the previous one. At the same time smooth time structure and infinite-time horizon for this process are assumed. Moreover every of the introduced products has its own new dimension of quality. Despite all these differences the general idea of cumulative innovations looks like the same. With finite lifetimes of this products one has some notion of creative destruction here in the sense of Schumpeter but with the difference that it is not the creation of new product which destroys the previous one, but the time-structure itself. New products do not replace the old ones automatically, rather with the flow of time some older products are scrapped due to the end of their life-cycles. At the same time the basic idea of the creative destruction as an engine of technological progress remains intact: the finite life-cycles of invented products stimulate the innovator to invest more in variety expansion process. Introduction of quality growth into this problem provides the possibility to analyze the influence of the length of life of developed products on the stream of innovations without taking into account any government, consumers or competitors. From this point of view it is not important whether one has single or multiple innovating agents. One may just assume that as a whole they provide this given dynamics of innovations. Since there are no any notion of prices/profits in the model, this would be a correct assumption, although some strategic interactions may change this result as it will be discussed further on. Then to answer the question of what is the extent of influence of the life-time of products on the innovating activity it turns to be sufficient to consider mere technological restrictions and incentives of this set of agents. It turns out that the more heterogeneous (rich) is the set of potential products, the more stimulus are created by the longer lifecycles for quality growth of these products in comparison with the expansion of this set of products, while with homogeneous set more investments are flowing into variety expansion. This may be interpreted from the point of view of social planner as a rule of thumb: to define the effective patenting policy, one first has to consider the nature of the industry (economy) under question and goals of the policy. If the goal is to provide the maximal possible variety of products to the market, one has to increase the length of patent (life-cycle) of products up to infinity in homogeneous case, while for heterogeneous case it is not that simple and one has first to consider the resources available to innovators. The more tight is this resource constraint (which is not explicitly modeled in the given framework but might be considered as an extension) the less effect the prolongation of patents would have on variety expansion while boosting quality growth.

Now consider the literature on patents and optimal patents. To incorporate the results of this chapter over there consider τ as the true patent's length, although identical across products. In the literature on patent's length two different distinctive notions came into being since Nordhaus work [**20**]. They are the scope (breadth) of the patent and effective length of patent versus its statutory length.

Concerning the notion of effective and statutory length of the patent in the literature, like [**35**], there is a distinction between these two notions. Statutory length of the patent is the length of patent explicitly granted to the innovator whereas effective length is the actual time during which the patent is effective - that is prevents others from using or producing the same product, [**36**]. In the given model this distinction is not that explicit, as there is only one parameter, τ, which is the statutory patent's length. At the same time one has the notion of effective mass of products and this is varying over time and does depend on the patent's length. This effective mass, $n(t)-n(t-\tau)$, might be treated as the effective protection. It has been seen that while the rate of variety expansion is slowing down over time this effective mass is decreasing and statutory patent's length may have effects of different directions on it. It is not true that the stronger is the protection the larger is this effective mass as shorter patents stimulate more investments to be redirected from quality improvements into variety expansion. Then when considering the effect of patents on innovations of this type one has to take into account not the general speed of increase of variety, but the rate of change of this effective mass with respect to time and to patent's length changes. One may argue that with sufficiently heterogeneous products (more then given in the model) there might be the case whith shorter patents stimulating innovations into variety expansion. Such an argument is following [**41**], where some discussion on the effect of patenting for software industry is given. It is argued there that for such industries where innovations are essentially cumulative and complementary, stronger patents may inhibit innovative activity instead of stimulating it. The results here are partially in line with that of Maskin: depending on the nature of products in the industry (the degree of heterogeneity), effect of patents' lengths may vary. Possible shortcomings of excessively long patents for industries with heterogeneous products have been discussed already before. It turns out that this may cause underprovision of variety expansion in favor of quality growth, while for some industries variety expansion is more important. As an example, one may consider the open software development, where quality is much more underdeveloped in favor of the great variety of products and this may actually happen because of the lack of patents which would stimulate the opposite distribution of investments.

The other feature of patenting literature is taking into account not only the length (duration) but also the breadth of the patent, e.g. how much close products it may cover. In the given framework it would mean that the single patent is granted upon the invention of the product $n(t_0)$ to all products $n(t_0) + \epsilon$. This would make no difference in the given model since patents are granted automatically and with no costs. But observe that in the absence of competitors the notion of the breadth of patent cannot be revelant here. Alternatively one may tract the given patenting pattern as that of a maximal possible breadth, since every of the products in $n(t)$ range are not substitutes to each other. Close enough products are already grouped withing the single $q_i(t)$ dynamics and thus the patent on the product i covers all these products already. If one would like to consider lesser breadth of patents that may be done by granting shorter length of patents for each i since then not all the range of possible products associated with this invention should be covered. In an effect it would mean lesser range of qualities (substitutable products) to be developed by the innovator. In this respect length of the patent, τ, implicitly defines the breadth of the patent also.

To conclude note that the suggested model allows for some extensions into the field of patenting. Namely, its rather easy to consider patents of variable length, as in [**37**]. For that one has to replace τ parameter by some continuous function to obtain the same kind of modification with respect to patents as one has with respect to investment efficiencies in $\gamma(\bullet)$ function. This function might be considered even non-monotonic. The only difficulty one would have then is with obtaining explicit solution for variety expansion dynamics. Another immediate extension is to allow for some patenting costs and include them into the variety expansion dynamics like a constant (for constant patents) or a function of a number of product (for variable patents' lengths). Last extension which might be of interest is to consider the response of the dynamics under patents to different market structures but to do this one has first to consider the model of strategic interactions in heterogeneous innovations framework, which is done in the last chapter of the current work.

CHAPTER 4

Strategic Interactions in Heterogeneous Innovations Framework

1. Introduction

In this chapter the analysis of heterogeneous multiproduct innovations is further extended. Now several competing agents in the innovative industry are assumed while preserving the dynamic framework of previous chapters. The main goal of this suggested extension is the consideration of the role of strategic interactions between agents involved in innovative activity. The differential games framework is employed for the analysis in this chapter. Differential games are mainly used in the literature for analysis of strategic interactions in oligopolistic markets, for modeling patent races and some other applications.

To our knowledge one of the first attempts to model the strategic interactions of oligopolistic agents via the differential games approach is given in Reinganum, [**17**]. In this paper the author combined static games approach with optimal control to obtain the dynamic game of R&D competition in a n-firm industry. However this was not the first paper on the influence of the market structure on the outcome of R&D competition. One of the first works in the field is that of Loury, [**19**]. In this pioneering work the discrete single innovation is assumed and n firms compete for being the first to introduce the new product. The first firm which would introduce such a product obtains an exclusive right for its production and hence receives the perpetual stream of profits associated with this product. This model lacks the explicit formulation of strategic interactions and consists of identical optimization problems for all the firms. However the equilibrium outcome does depend on the number of firms in the industry. Another basic approach to modeling R&D competition consists mainly in static game formulation, as in the work of Dasgupta&Stigliz, [**18**] where no explicit dynamical interactions appear. In their model they mainly tackle with the question how the market structure (e.g. monopoly versus oligopoly) would influence the equilibrium level and price of innovative products. They come to the conclusion that it is the elasticity of the market demand function which defines the optimal structure of the industry.

In the work of Reinganum these two basic approaches are combined in a single model and this is the basic paper one would compare the suggested framework with, since the same differential games approach is used here. The Reinganum's model make use of differential games approach to explicitly model such strategic interactions between firms. On the other hand he pertains the general structure of Loury, namely there is a single innovative process and every player seeks to introduce this given product first to the market. What differs in Reinganum's model is that in this paper players do not only optimize their own value functions independently from each other but account for possible actions of other players.

To our knowledge there are no examples of applying differential game to the analysis of heterogeneous innovations. Moreover, due to the distributed character of the basic framework being used in the current work the differential game which would describe strategic interactions in such a model necessarily would be the infinite-dimensional one. As far as we are concerned there are no examples of such extensions of differential games approach although theoretically it does not yield very much difficulties. The only point is the existence of equilibrium but this question on the general level is not resolved even for finite-dimensional games. However as long as one has the linear-quadratic game, the infinite-dimensional extension does not change main results of existence for this class of games, [**6**]

Concerning the particular form of interactions between players, the imitation effect at no cost in quality dynamics is assumed. There is some literature on such imitation also. One of the examples is the work of Gallini, [**42**], although his approach is different from the one assumed here. Namely, he analyzes the effect of imitation of the patented product which is costly, while in the suggested model it is assumed to be costless. This is done mainly due to technical reasons as the dynamic game of the given form cannot be resolved in the precense of additional control parameter which would be the costs of imitation. The only possible variant is to assume some exogenously given costs of a constant nature, but this would not change any of the results of the model below. Another difference is that the Gallini's model is static in nature whereas here the imitation effect is analyzed in the dynamic game-theoretic context. The whole space of potential products each of which possesses its own quality characteristic is modeled instead of the 1-dimensional sequence of innovations. It will be demonstrated that in the given framework the imitation effect may constitute the equilibrium only if one take into account the underlying process of variety expansion also. Another more recent work on dynamic interactions of R&D firms is that of Judd, [**25**]. In this paper the author analyzes the multistage innovative race between multiple agents with multiproduct situation and this is rather close to current approach. Nevertheless he assumes the multistage structure of the game and hence a static situation with some transition between stages whereas here the dynamic game with continuous time is modeled. He finds out, that there is an ambiguity in the results of a game, namely a given player may increase his expenditure (investments in our case) when the other agent is ahead of him, while this is not profitable for him as an imitator. It is demonstrated that in the suggested model this is not the case and any ambiguity disappears if one would consider both aspects of innovation.

The most recent paper on product and process innovations in differential game framework is the work of Lambertini&Mantovani, [**56**]. This paper assumes fully dynamical model of the duopoly competition of innovating firms. The suggested model differs from the work of Lambertini&Mantovani in two significant aspects. First, in their model authors assume uncertainty of innovations in the form of Poisson arrival rates, while the suggested model does not contain uncertainty in any form. This is done for simplification of analysis. Next, the discussed paper does not handle heterogeneity of innovations and hence is reduced to the differential game with two states, while the suggested model allows for distributed nature of innovations and all products differ from each other in their investment characteristics. This is more in line with the setup of Lin, [**14**], but with fully dynamic context. It will be discussed later on how the results of the suggested model would differ from these two papers due to dynamic context and heterogeneity being modeled simultaneously.

The last feature of the suggested model is the R&D cooperation on the level of variety expansion. It is argued that such a situation is more typical for R&D firms then the full-scale competition on both levels. First such a type of strategic interactions has been considered in D'Aspremaunt&Jacquemin, [**22**] where it is argued that in real economies the majority of R&D activities is performed in the form of joint R&D ventures if one would consider innovations of big enough size. As long as one assumes that the variety expansion represents the process of introduction of completely new products to the market it is rather natural to assume the precense of joint cooperative efforts on this level. This would mean that agents share common knowledge and efforts concerning this part of their innovative activity.

To conclude one may view the suggested model of strategic interactions as an extension and combination of the results from different directions. First, it contributes to the line of literature on strategic interactions between innovating agents in the spirit of [**17**], [**25**] and late [**56**]. These approaches are extended by considering the distributed parameter model and formulating the fully dynamical differential game with richer strategic sets for all players. Next, approaches to imitation , [**42**] and R&D joint ventures, [**22**] are combined in a single model and it is demonstrated that these two effects are complementary in nature and resulting strategies cannot be optimal while taking into account only one of them. At last, everything is put together to obtain the model of strategic interactions of innovative agents in heterogeneous multiproduct framework which might be considered more general in its nature then previous findings in the field. However note that the current model abstracts from patenting effects as well as from any uncertainty which is essential part of innovative activity. One may modify the model to allow for patents in the form as it was introduced in the previous chapter, but this would not change the results since every product is simultaneously available for all competing agents and thus the only effect will be of the same nature as considered previously in the preceding chapter. The uncertainty also may be included in some form into the model but this would complicate the results to the great extent and hence this model neglects it. It may be considered as future extension of the suggested framework. Only the duopoly case with number of agents $n = 2$ is considered. However the possibility of inclusion of arbitrary finite number of players into the model is rather straightforward modification of the model predented here. It turns out that for the main results to hold the requirement of finiteness of the number of players is essential and hence the suggested framework may not be used to model the perfectly competitive environment but anything close to oligopolistic or monopolistic competition may be easily considered.

First the overall problem for both agents is formulated and then it is solved sequentially through employing the Hamilton-Jacobi-Bellman approach and Maximum Principle whenever necessary. After obtaining these results the dynamic of the given strategies is considered and the nature and form of the resulting strategic interactions as well as some practical implications of the findings are discussed.

2. Strategic Interactions in Heterogeneous Innovations Framework

As long as one has two aspects of innovative activity in this framework, there are several different ways of modeling strategic interactions of innovative agents. Assume for simplicity only 2 agents. Both of them have the possibility to develop variety expanding innovations and quality innovations. Then they may interact with each other in different ways.

First, if they are operating on the same markets for all products there must be some sort of interaction on the level of every product's quality. At the same time this would also mean they have the same potential products' space to develop variety expansion process in. On the other hand, if one would assume existence of different varieties for these agents, there would be no possibility of their interaction on quality level provided they are inventing different products' varieties. In this case there will be no interaction between agents at all. So the first possibility is considered.

Second, the nature of interactions may differ also. Agents may choose to cooperate with each other on one of the levels of innovative activity or to compete with each other on both levels. One has exactly two different directions of innovative activity, so there are no more then 4 different pure variants of strategic interactions between agents.

Consider first the full cooperation variant, where both agents choose to join their efforts in developing the variety expansion over the given range of products N. This will mean that in an effect they will have the emergence time of all products coinciding for both agents. Additionally they may choose to cooperate in the development of resulting products' qualities. This would be possible since for both agents any product would then emerge in the same time. As long as the model do not take into consideration market competition and profit functions, their actions in this case may be described by the single-agent stand alone model, considered above. To see that, just consider the model of the previous chapters where investment efficiencies are given as averages from investment efficiencies of both agents. Then, provided they have the same space of products, $\mathbf{Q}$ and same discount factors, r, this model would describe the dynamics of such a joint venture.

Now consider the variant where agents would choose competition on both levels of innovations. Such a system looks like quite natural and is often considered as a frequent example of competition in innovative activities, although there are no models which would account for both kinds of competition simultaneously in heterogeneous innovations framework. However, given the dynamic aspect of the framework under consideration there are certain theoretical and technical difficulties in modeling such a situation. Provided both agents are operating over the same potential products' space it is unclear how the competition in dynamical form may be described for variety expansion process. Let one agent be inventing some product earlier then the other one. Then, if there is a competition on this product innovations stage, the information will be closed for the other agent and he would invest into the development of this same product by himself. If there is some patenting law in the economy under consideration he will not have the right to use this product and develop its quality. Then the only other variant will be some royalty payments to the first agent which invented that product as it is done in many cases in real markets. But then there are no competition and strategic interaction on the product innovations level between agents, since it is just described by the royalties. Note also that since the potential products' space $\mathbf{Q}$ is a partially-ordered set and variety expansion process is a continuous one, there cannot be the situation with one of the agents inventing some products and the other inventing other ones, at least within boundaries of the framework under consideration. The only variant when it may happen is when agents' potential spaces are not coinciding and their products' indices are different from each other. This may be the case, but to model such a situation one must weaken requirements on this products' space structure, namely, variety expansion cannot be continuous in a usual sense then. Hence, variety level competition cannot be modeled within the given framework and this may be considered as one of serious limitations of the suggested approach. So both variants of competition in both levels of innovative activity and competition on the level of products' innovations and cooperation on the level of quality innovations may not be considered here.

At last, consider the variant of competition on quality innovations level and cooperation on the level of variety expansion. In such a variant both agents would join there efforts in developing the range of products and share open knowledge between them concerning this fundamental technologies. At the same time they will separately develop qualities of resulting products and in effect would obtain different but close to each other in characteristics product sets over which they will be able to compete afterwards. Such a variant of strategic interactions on the market of innovations is not the most common one but it is observed sometimes in real economies. Consider for example R&D alliances between large multinational ventures in development of some key and fundamental technologies which are too expensive for any separate company to develop and too costly to keep the private information concerning the technology out of reach of competing companies. At the same time there is competition and private information concerning less expensive and less fundamental technologies. In the current setting it seems plausible given considerations above to stop on this variant of strategic interactions to model.

3. Basic Problem Formulation

Assume 2 agents, $\{j, l\}$, continuum of products, and infinite time horizon (as it has been mentioned, in finite time HJB approach is not fruitful even in one-agent case). Both agents have identical dynamic problems to solve. Assume also the same form of parameter functions, as in previous chapters of the work. Suppose the given agents join their efforts in the development of variety expansion over the same potential products' space $\mathbf{Q}$, while having possibly different investment efficiencies. Then variety dynamics should be:

$$\dot{n(t)} = \alpha_{[j]} u^{[j]}(t) + \alpha_{[l]} u^{[l]}(t). \tag{4.1}$$

This means both agents have the same underlying variety expansion process while freely choosing the level of efforts they would devote to the development of this variety. Note that this does not exclude the possibility for one or the other agent to have zero investments while benefiting from the investments being made by the other through using achieved variety level. This is close to what is known as the free-rider property in public goods provision. In fact this property is essential to the model and it will be shown that instead of having the negative impact on the resulting overall growth of innovations it has a positive impact.

At the same time let them have different quality growth processes for each product. Strategic interaction is modeled in the form of imitation on this level. Agents do not benefit from investments of each other, but they may copy the progress of the other in the case their own quality level is lower. In the effect one has two dynamic processes linked with each other and two different independent streams of investments:

$$\begin{aligned} \dot{q^{[j]}(i,t)} &= \gamma_{[j]}\sqrt{N-i}\,g^{[j]}(i,t) - \beta_{[j]} q^{[j]}(i,t) + \theta \times max\{0, (q^{[l]}(i,t) - q^{[j]}(i,t))\}; \\ \dot{q^{[l]}(i,t)} &= \gamma_{[l]}\sqrt{N-i}\,g^{[l]}(i,t) - \beta_{[j]} q^{[l]}(i,t) + \theta \times max\{0, (q^{[j]}(i,t) - q^{[l]}(i,t))\}. \end{aligned} \tag{4.2}$$

As a result the agent's quality level may grow in two different ways. As long as the given agent is the leader in quality of a given product, e.g. $t : q_i^{[j]}(t) > q_i^{[l]}(t)$, his quality grows only due to his own investments in quality of this product, $g_i^{[j]}(t)$, as it is in the basic model above. At the same time, the other agent's quality is lower and he benefits not only from his own investments but also from the 'imitation effect': he benefits from the difference between the leader's quality level and his own one. Clearly this effect will boost the second agent's quality growth rates but will wear down eventually while the follower's quality approaches that of the leader.

Both agents are maximizing their objective functionals which are of the same form as in previous chapters except with the difference that they have common variety expansion process but distinct quality levels:

$$J^{[j,l]} \stackrel{\text{def}}{=} \int_0^\infty e^{-rt}\Big[\int_0^{n(t)} \Big(q^{[j,l]}(i,t) - \frac{1}{2}g^{[j,l]}(i,t)^2\Big)di - \frac{1}{2}u^{[j,l]}(t)^2\Big]dt \to \mathbf{max}. \tag{4.3}$$

Each agent therefore is maximizing the integral sum of qualities of all products being invented up to time t minus investments in all these qualities and into the variety expansion process at each time t. Observe that this time the infinite-time horizon is assumed both for variety expansion and quality growth processes. This is done mainly to provide ground for comparisons of results of the current chapter with those of infinite-time benchmark model of chapter 2. However, it may be demonstrated that the same type of results as those following may be formally obtained for finite-time horizon also and for the case of patents. At the same time the chosen variant is the simplest one in terms of its formal exposition while all the effects of patents' lengths and time-horizon length would not differ in their direction (although they may differ in their extent) from single-agent model.
Static constraints remain the same as in the basic model also:

$$\begin{aligned} u^{[j,l]}(t) &\geq 0; \\ g^{[j,l]}(i,t) &\geq 0; \\ 0 \leq n(t) \leq N;\ q^{[j,l]}(i,t) &\geq 0; \\ q^{[j,l]}(i,t)\ |_{i=n(t)} &= 0. \end{aligned} \tag{4.4}$$

Observe that the only form of strategic interaction in quality innovations is the imitation effect which influences the dynamic of state variables but does not influence directly the objective function. This seems to be a weak link between agents, but it turns out that this is the only kind of interaction one may take into account in the absence of market competition in the model. As long as the whole framework under consideration is concentrated on the technological side of innovative activity, that is on growth rates of different kinds of innovations, one may not consider any market effects over here. At the same time even such simple introduction of interdependencies between agents' strategies allow for the rich enough set of outcomes as it will be demonstrated further on.

4. Decomposition Method in Differential Game Setting

Given the basic formulation of dynamical problems of both agents above, it is clear that the optimal solution has to be found in the form of the equilibrium pair of strategies in the differential game framework. From the general point of view the model considered here is the infinite-dimensional one as long as one have the continuum of quality growth variables for every player. This may provide some difficulties in formal construction of the game. However due to the special structure of the dynamic framework being used it is possible to decompose the problem into quality growth problem and variety expansion problem. This can be done due to the fact that quality growth does not depend on the variety expansion except for the time of emergence of new products. Then every such a problem should be the finite-dimensional one and as long as it is of the linear-quadratic form, one may be assured that equilibrium exists for each such a game of quality growth under the same conditions as in standard linear-quadratic differential game, [**6**]. Then the results obtained for quality growth may be used for solution of variety expansion problem which is also the differential game but with only one state, $n(t)$. This may be done through applying the same decomposition method as has been employed previously for single-agent models with the difference that now this decomposition constitutes not the pair of sequential optimal control problems but a pair of interdependent differential games.

Formally applying HJB approach directly to the basic formulation above would yield a pair of HJB equations with integral sum of dynamic constraints in them. It has been shown that such type of dynamic problems for one agent can be formulated and solved, but since here the goal is equilibrium strategies of a game rather then pure optimal controls, the existence of such a pair of fixed mapping in the strategy space when strategies are of infinite dimensions may constitute a problem. Although this problem itself is an interesting one here the main focus is more applied results of modeling. Therefore the decomposition method is employed to represent this strategy set as just a pair of dependent strategies. Observe that such a decomposition is still a correct method since there is no competition on the level of variety expansion. If it would not be the case not only conceptual difficulties discussed above arise, but also formal ones. It is rather easy to see. If both agents have different variety expansion processes but given on the same range of products N, there might be situations when one of the players invents the product first and the other is the follower. Then the follower's value function for variety expansion would depend not only on the potential value created by the quality of next product, as it is in the single-agent model, but also on the time, when this product has been invented by the other agent and the quality level achieved by him by the time of its invention by the second. Then variety expansion process for the follower would depend on other agent's quality level of the boundary product and on time of its emergence, thus making the whole problem for variety expansion not invariant to the position of the product to be invented in the product space. But then decomposition is impossible since it heavily relies on the fact, that the quality growth processes have the same shape irrespective of the exact time of emergence of a product. As it has been seen before in patenting model, the only thing that may happen to quality growth within the single agent framework is the shift in time which however does not influence the shape of dynamics itself and thus does not influence optimal strategies of the agent. Then an agent may estimate the value of variety expansion (which has the invention of next product as a positive output) based on the potential value accumulated by this product irrespective of the exact position in time. In differential game considered here this is not the case, since quality of the product for an agent is now dependent not only on his own investment strategy for which the invariance argument is still valid, but also on the quality of the other agent and thus on his strategic choice. Then variety expansion process cannot be separated from the quality growth in such a way by considering only the potential value accumulation, because this last in turn would depend on relative speed of variety expansion of two agents.

However the suggested competitive-cooperative model avoids this difficulty and this is one of the reasons of modeling such a specific form of competition. Observe that common variety expansion process yields the coincidence of emergence times of products for both agents. There is no dependence of value creation on the quality level on the relative speed of variety expansion. Moreover, every agent then is able to estimate the potential accumulated value from the quality of each product, because he may estimate it at zero quality levels not only for himself but for the other agent also, since the time of emergence is the same. Then the value function for the variety expansion still does not depend on qualities or quality investments themselves, but only from this potential value. In the effect this value function although different for both agents (as their qualities' value functions are different) are invariant to the position of the boundary product within the product space

Q. Then one may apply the decomposition method as before. This is done in subsequent sections of the chapter.

5. Quality Growth Problem

Consider the problem of quality growth for each product i for both players j, l. This problem is of maximizing the value of quality level of product i:

$$V^{[j]}(q_i^{[j]}(t)) = \int_0^\infty e^{-rs}\Big(q_i^{[j]}(s) - \frac{1}{2}g_i^{[j]}(s)\Big)ds \to \max;$$

$$V^{[l]}(q_i^{[l]}(t)) = \int_0^\infty e^{-rs}\Big(q_i^{[l]}(s) - \frac{1}{2}g_i^{[l]}(s)\Big)ds \to \max. \tag{4.5}$$

Observe that as now quality growth happens along the same infinite-time horizon as that of variety expansion the associated value functions do not depend explicitly on time. Observe also that the form of the value function for both agents follows that of the value function for quality growth in infinite-time monopolist problem from chapter 2 and depends only on the state variable and control of the same player. The difference is in the dynamic constraints which now include the imitation effect:

$$\dot{q_i^{[j]}}(t) = \gamma_{[j]}\sqrt{N-i}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t) + \theta \times \max\{0, (q_i^{[l]}(t) - q_i^{[j]}(t))\};$$

$$\dot{q_i^{[l]}}(t) = \gamma_{[l]}\sqrt{N-i}g_i^{[l]}(t) - \beta_{[j]}q_i^{[l]}(t) + \theta \times \max\{0, (q_i^{[j]}(t) - q_i^{[l]}(t))\}. \tag{4.6}$$

This results in a differential game with two states, $\{q_i^{[j]}(t), q_i^{[l]}(t)\}$ and two controls which are strategies of the players, $\{g_i^{[j]}(t), g_i^{[l]}(t)\}$ for every i. Note that this formulation is of the same form across all products' qualities and they are independent of each other. So solution of this game is valid for any i.

Now formulate the associated pair of HJB equations:

$$rV_i^{[j]} = max\Bigg\{ q_i^{[j]}(t) - \frac{1}{2}g_i^{[j]}(t)^2 +$$
$$\frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)}\bigg(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t) + \theta \times max\{0, \big(q_i^{[l]}(t) - q_i^{[j]}(t)\big)\}\bigg)$$
$$+\frac{\partial V_i^{[j]}}{\partial q_i^{[l]}(t)}\bigg(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) + \theta \times max\{0, \big(q_i^{[j]}(t) - q_i^{[l]}(t)\big)\}\bigg)\Bigg\};$$

$$rV_i^{[l]} = max\Bigg\{ q_i^{[l]}(t) - \frac{1}{2}g_i^{[l]}(t)^2 +$$
$$\frac{\partial V_i^{[l]}}{\partial q_i^{[j]}(t)}\bigg(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t) + \theta \times max\{0, \big(q_i^{[l]}(t) - q_i^{[j]}(t)\big)\}\bigg)$$
$$+\frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)}\bigg(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) + \theta \times max\{0, \big(q_i^{[j]}(t) - q_i^{[l]}(t)\big)\}\bigg)\Bigg\}. \tag{4.7}$$

Where $V_i^{[j,l]} = V^{[j,l]}(q_i^{[j]}(t), q_i^{[l]}(t))$. Observe that these equations are symmetric and include partial derivatives of the associated value function with respect to both states, as it is required by the form of HJB equation for a 2-state game. At the same time first-order conditions for optimal investments depend only on the

own player's value function but not on the other's.
First-order conditions for investments:

$$g_i^{[j]}(t) = \gamma_{[j]}\sqrt{(N-i)}\frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)};$$

$$g_i^{[l]}(t) = \gamma_{[l]}\sqrt{(N-i)}\frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)}. \tag{4.8}$$

Assume the quadratic form of the value functions for both players since it is standard for linear-quadratic games:

$$V_i^{[j,l]} = A_1^{[j,l]}(q_i^{[j]})^2 + A_2^{[j,l]}(q_i^{[l]})^2 + A_3^{[j,l]}(q_i^{[j]}q_i^{[l]}) + A_4^{[j,l]}q_i^{[j]} + A_5^{[j,l]}q_i^{[l]} + A_6^{[j,l]}. \tag{4.9}$$

Observe, that depending on the realization of the $max\{0, \big(q_i^{[j,l]}(t) - q_i^{[l,j]}(t)\big)\}$ functions in both derivatives one has 3 different formulations of the value function which correspond to the leadership of one or the other player and symmetric dynamics with no leadership. Considering that all HJB formulations are symmetric it is sufficient to consider only two variants instead of three: leader-follower and symmetric case.

5.1. Leader-Follower Case. Assume for certainty player j is the leader and the other is the follower. First consider the situation when this player is the leader in quality during all the game and no switch of leadership may occur. As long as one of the agents has the leadership in the quality growth, that is, $t : q_i^{[j]}(t) > q_i^{[l]}(t)$, his dynamics does not depend on the imitation effect, while the other's does. Then subsequent HJB equations are reduced to:

$$\begin{aligned} rV_i^{[j]} = max\Bigg\{ & q_i^{[j]}(t) - \frac{1}{2}g_i^{[j]}(t)^2 + \\ & \frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)}\Bigg(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t)\Bigg) \\ & + \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}(t)}\Bigg(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) - \theta \times \big(q_i^{[j]}(t) - q_i^{[l]}(t)\big)\Bigg)\Bigg\};\end{aligned}$$

$$\begin{aligned} rV_i^{[l]} = max\Bigg\{ & q_i^{[l]}(t) - \frac{1}{2}g_i^{[l]}(t)^2 + \\ & \frac{\partial V_i^{[l]}}{\partial q_i^{[j]}(t)}\Bigg(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t)\Bigg) \\ & + \frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)}\Bigg(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) + \theta \times \big(q_i^{[j]}(t) - q_i^{[l]}(t)\big)\Bigg)\Bigg\}.\end{aligned} \tag{4.10}$$

Value functions are polynomials of degree not bigger then 2 of 2 variables, with the general form given by (4.9). However there is the interdependence of value functions of players from the quality of other player and the simplification to polynomial of one variable cannot be done. Observe also that this time problems to solve are different for both players. They are solved sequentially assuming fixed optimal controls of the other player. To condense notation expression for other player's optimal control is not inserted into the expressions for value functions above on this stage. It is sufficient to note that with given assumed form of the value function, (4.9), the other player's optimal control is the function of his value function coefficients. This implies that equations for value functions' coefficients are interrelated.

Consider the HJB equation for the leader. The leader does not benefit from the imitation effect but his problem is still different from the symmetric case since the imitation effect present for the other player. The position of the leader is characterized by the condition:

$$\forall t : q_i^{[j]}(t) > q_i^{[l]}(t).$$

As long as one of the players is the leader in quality investments, the other is the follower. For certainty assume player j as the leader above and thus let player l be the follower. The condition for being the follower is strictly the opposite of that being the leader:

$$\forall t : q_i^{[l]}(t) < q_i^{[j]}(t).$$

Follower is solving the dynamic problem which is different from that of the leader, since he benefits from the imitation effect as long as his quality is low enough. However, the form of the value function remains the same as for the leader and does not include quadratic cross-effects but only linear ones. Substituting these two value functions of the form (4.9) into the (4.10) yield the system of equations on coefficients of value functions of both players. In general this is a non-linear system with 12 equations:

$$\begin{cases}
-\frac{1}{2}A_4^{[j]}(N-i)\gamma_{[j]}^2 - A_5^{[j]}A_4^{[l]}(N-i)\gamma_{[l]}^2 + rA_6^{[j]} = 0;\\
A_4^{[j]}(\beta_{[j]} + r + 2\gamma_{[j]}^2 A_1^{[j]}(N-i)) - \gamma_{[l]}^2(N-i)(A_3^{[j]}A_6^{[l]} + A_5^{[j]}A_3^{[l]})\\
-1 - A_5^{[j]}\theta = 0;\\
A_3^{[j]}\left(r + \beta_{[l]} + \beta_{[j]} + \theta + 2\gamma_{[j]}(N-i)A_1^{[j]} - \gamma_{[l]}^2(N-i)A_1^{[l]}\right) -\\
-2A_2^{[j]}(A_3^{[l]}\gamma_{[l]}^2(N-i) + \theta) = 0;\\
A_1^{[j]}(r + 2\beta_j) - 2\gamma_{[j]}^2(N-i)(A_1^{[j]})^2 - (A_3^{[l]}\gamma_{[l]}^2(N-i) + \theta)A_3^{[j]} = 0;\\
(r + \beta_{[l]} + \theta - 2\gamma_{[l]}^2(N-i)A_1^{[l]})A_5^{[j]} - (\gamma_{[j]}^2 A_3^{[j]}A_4^{[j]} + 2\gamma_{[l]}^2 A_2^{[j]}A_4^{[l]})(N-i) = 0;\\
(r + 2\beta_{[l]} + 2\theta - 4\gamma_{[l]}^2(N-i)A_1^{[l]})A_2^{[j]} - \frac{1}{2}\gamma_{[j]}^2(N-i)A_3^{[j]} = 0;\\
rA_6^{[l]} - \frac{1}{2}\gamma_{[l]}^2(N-i)(A_4^{[l]})^2 - \gamma_{[j]}^2(N-i)A_4^{[j]}A_5^{[l]} = 0;\\
(r + \theta + \beta_{[j]} + \beta_{[l]} - 2\gamma_{[j]}^2(N-i)A_1^{[j]} -\\
-2\gamma_{[l]}^2(N-i)A_1^{[l]})A_3^{[l]} - 2\gamma_{[j]}^2(N-i)A_3^{[j]}A_4^{[l]} - 2\theta A_1^{[l]} = 0;\\
(\beta_{[j]} + r)A_5^{[l]} - 2\gamma_{[j]}^2(N-i)(A_4^{[j]}A_2^{[l]} + A_3^{[j]}A_5^{[l]}) - (\theta + \gamma_{[l]}^2(N-i)A_2^{[l]})A_4^{[l]} = 0;\\
(r + 2\beta_{[j]} - 4\gamma_{[j]}^2(N-i)A_1^{[j]})A_2^{[l]} - \frac{1}{2}(2\theta + \gamma_{[l]}^2(N-i)A_3^{[l]})A_3^{[l]} = 0;\\
(r + 2\beta_{[l]} + 2\theta)A_1^{[l]} - 2\gamma_{[l]}^2(N-i)(A_1^{[l]})^2 - \gamma_{[j]}^2(N-i)A_3^{[j]}A_3^{[l]} = 0;\\
(\theta + r + \beta_{[l]} + 2\gamma_{[l]}^2(N-i)A_1^{[l]})A_4^{[l]} - \gamma_{[j]}^2(N-i)(A_4^{[j]}A_3^{[l]} + A_3^{[j]}A_5^{[l]}) - 1 = 0.
\end{cases}$$

(4.11)

This system has up to 12 different solutions. However, one of these solutions yield the linear form of subsequent value functions and piecewise-constant optimal controls of both players, while other solutions provide quadratic value functions and linear feedback controls for players. They are analyzed sequentially, starting with piecewise-constant form of strategies.

5.1.1. *Piecewise - Constant MPE.* First consider the solution of the system (4.11) which yields linear value functions. In such a case coefficients for the leader's value function are:

$$\begin{cases} A_1^{[j]} = 0; \\ A_2^{[j]} = 0; \\ A_3^{[j]} = 0; \\ A_4^{[j]} = \frac{1}{\beta_{[j]}+r}; \\ A_5^{[j]} = 0; \\ A_6^{[j]} = \frac{1}{2}\frac{\gamma_{[j]}^2(N-i)}{(\beta_{[j]}+r)^2 r}. \end{cases} \tag{4.12}$$

These do not depend on the optimal control of the other player, hence the value function of the leader and his optimal control may be derived independently. This set of coefficients corresponds to the linear value function of the leader with the absence of cross-effect and hence the optimal strategy is constant as long as $\forall t : q_i^{[j]}(t) > q_i^{[l]}(t)$. The subsequent value function for the leader in the constant MPE case:

$$V_i^{L,CON} = \frac{q_i^{[j]}}{r+\beta_j} + \frac{1}{2}\frac{\gamma_j^2(N-i)}{r(r+\beta_j)^2}. \tag{4.13}$$

This together with first-order conditions on controls constitute optimal (constant) control for the leader:

$$g_{L,CON}^{[j]} = \frac{\gamma_{[j]}\sqrt{(N-i)}}{r+\beta_{[j]}} = const; \tag{4.14}$$

Now turn to the problem of the follower. The resulting set of value function's coefficients for the constant strategies case:

$$\begin{cases} A_1^{[l]} = 0; \\ A_2^{[l]} = 0; \\ A_3^{[l]} = 0; \\ A_4^{[l]} = \frac{1}{\theta+\beta_{[l]}+r}; \\ A_5^{[l]} = \frac{\theta}{(\beta_{[j]}+r)(\beta_{[l]}+\theta+r)}; \\ A_6^{[l]} = \frac{1}{2}\frac{(r^2\gamma_{[l]}^2+\beta_{[j]}^2\gamma_{[l]}^2+2\gamma_{[j]}^2\theta^2+2\theta\beta_{[l]}\gamma_{[j]}^2+(2\theta\gamma_{[j]}^2+2\beta_{[j]}\gamma_{[l]}^2)r)}{r(\beta_{[j]}+r)^2(r+\beta_{[l]}+\theta)^2}(N-i). \end{cases} \tag{4.15}$$

These ones do not include cross-effect, but depend on the quality of the leader. Observe the resulting value function of the follower in the constant MPE case:

$$\begin{aligned} V_i^{F,CON} &= 2\left(\frac{q_i^{[l]}}{\theta+r+\beta_l} + \frac{\theta}{r+\beta_j} \times \frac{q_i^{[j]}}{\theta+r+\beta_l}\right) + \\ &+ \frac{\theta\gamma_j\sqrt{(N-i)}}{r(r+\beta_j)(r+\theta+\beta_l)}g_i^{[j]} + \frac{1}{2}\frac{\gamma_l^2(N-i)}{r(r+\theta+\beta_l)} = \\ &\quad 2\left(\frac{q_i^{[l]}}{\theta+r+\beta_l} + \frac{\theta}{r+\beta_j} \times \frac{q_i^{[j]}}{\theta+r+\beta_l}\right) + \\ &+ \frac{\theta\gamma_{[j]}^2(N-i)}{r(r+\beta_j)(r+\theta+\beta_l)}\frac{1}{r+\beta_{[j]}} + \frac{1}{2}\frac{\gamma_l^2(N-i)}{r(r+\theta+\beta_l)}. \end{aligned} \tag{4.16}$$

Observe that in constant MPE strategies case the leader's value function does not depend on the state and control of the follower, that is, q_i^l, but only on the own leader's quality level, while the follower's value function depends on the state of quality and control of the leader. Since this last is known already, one has the explicit formulation of the value function for the follower as it is shown.

The constant strategy does not depend on the follower's or the leader's quality levels except to the fact that this strategy is effective only for follower's quality being smaller then that of the leader:

$$g^{[l]}_{F,CON} = \frac{\gamma_{[l]}\sqrt{N-i}}{r+\theta+\beta_{[l]}} = const;$$
$$\forall t : q_i^{[l]}(t) < q_i^{[j]}(t). \tag{4.17}$$

This is a constant rate of investments but lesser then that in the leadership case. It differs from the latter by the term θ in the denominator. Provided θ is the imitation speed and is defined from zero to one, this decreases the overall investment rate for the follower in this case.
Provided formulation of optimal controls, the dynamic system for qualities' growth in the constant leader-follower regime is:

$$\begin{cases} \dot{q_i^{[j]}}(t) = \frac{\gamma_{[j]}^2(N-i)}{\beta_{[j]}+r} - \beta_{[j]}q_i^{[j]}(t); \\ \dot{q_i^{[l]}}(t) = \frac{\gamma_{[l]}^2(N-i)}{\beta_{[l]}+\theta+r} + \theta q_i^{[j]}(t) - (\beta_{[l]}+\theta)q_i^{[l]}(t). \end{cases} \tag{4.18}$$

From this system one may observe that the growth of quality of the follower is faster for higher quality level of the leader while his investments' rate is lesser then for the leader. Now turn to the case of linear feedback strategies.

5.1.2. *Linear Feedback MPE.* The system of equations on value functions' coefficients yield 11 solutions which include non-zero quadratic terms and hence correspond to linear-feedback strategies. All of these solution sets may be derived easily from the system (4.11) and they are not displayed here due to their excessive length. The selection of proper candidates for optimal strategies defined by these sets of value functions' coefficients is required.

First note that 3 of these solutions yield negative controls for zero initial quality and hence they cannot constitute optimal strategies in this model. These are solutions for value functions' coefficients which contain positive quadratic terms and negative linear ones and hence the first-order conditions for optimal controls yield negative values. Combined with the initial condition $q_i^{[j]}(t_i^0) = 0, q_i^{[l]}(t_i^0) = 0$ this yield negative value functions and negative controls in the neighborhood of initial quality levels. These are solution sets which yield value functions of the form:

$$V_i^{[j]}|_{q_i^{[j]}=0} = A_1^{[j]}(q_i^{[j]})^2 - A_4^{[j]}q_i^{[j]} + A_6^{[j]} < 0;$$
$$V_i^{[l]}|_{q_i^{[l]}=0} = A_1^{[l]}(q_i^{[l]})^2 - A_4^{[l]}q_i^{[l]} + A_6^{[l]} < 0; \tag{4.19}$$

with other coefficients being zero. Now observe that the control is defined as being non-negative and hence it is set to zero for such value functions. Then it is zero along all the trajectory as quality may not grow above zero in the abcense of investments. These candidates then are trivial and excluded from analysis.

These considerations leave 8 candidates for optimal strategies' pair for linear-feedback case. The stability of resulting quality dynamics for these candidates is explored. First analyze the general stability of dynamical systems. It has to be noted that 6 of them are of the type:

$$\begin{cases} \dot{q_i^{[j]}}(t) = C_1^{[j]} q_i^{[j]}(t) + C_2^{[j]} q_i^{[l]}(t) + C_3^{[j]}; \\ \dot{q_i^{[l]}}(t) = C_1^{[l]} q_i^{[j]}(t) + C_2^{[l]} q_i^{[l]}(t) + C_3^{[l]}. \end{cases} \tag{4.20}$$

that is, include both states as variables in both equations and are of linear type. Here $\{C_1^{[j]}, .., C_3^{[l]}\}$ are some combinations of exogenous parameters of the model, $\{\alpha_{j,l}, \beta_{j,l}, \gamma_{j,l}, r, \theta\}$.

For these 6 remaining variants of the full linear-feedback type the same method of stability analysis as before is employed. The quality dynamics appears to be of saddle-type stable for all the candidates with some additional conditions on parameters of the system. It has to be noted that this time it does not mean that the stability result is invariant to the positions of players (e.g. which of them is the leader and which is the follower) since it includes conditions on parameters. Then some of these conditions for the stability of dynamical system in both leader-follower modes may appear to be incompatible. This is not discussed in details for now as there is another argument for rejection of these candidates.

Now observe that the method employed above does not say anything about the actual trajectories of qualities which have to be compatible with the requirement of zero initial quality. It turns out that zero initial quality condition leads to the set of solutions which are not bounded at infinity and hence the steady state with finite qualities does not exist for any of these 6 candidates. This means that those trajectories of dynamical systems of type presented above which start at the point $(0, 0)$ in state space do not lie in the region of convergence to the saddle point.

To demonstrate this compute corresponding pairs of strategies and solutions for the subsequent dynamic systems on quality growth. All of these solutions contain exponents with positive powers which are functions of time and hence yield infinite growth of the quality in the long run. As the model under consideration is infinite-time horizon model, any growth with positive exponential speed would be unbounded. Formally it means that all the resulting solutions for qualities in leader-follower case are of the type:

$$\begin{aligned} q_i^{[j]}(t) &= \pm F_1^{[j]} e^{\delta_1 t} \pm F_2^{[j]} e^{-\delta_2 t} + F_3^{[j]}; \\ q_i^{[l]}(t) &= \pm F_1^{[l]} e^{\delta_3 t} \pm F_2^{[l]} e^{-\delta_4 t} + F_3^{[l]}. \end{aligned} \tag{4.21}$$

where $F^{[j,l]}$ and $\delta_{1,.,4}$ are some parameters' combinations and δ's are positive. Hence for any combination of parameters one have the growth upto plus or minus infinity for both players without reaching any finite steady-state level. One may conclude that the pair of strategies of the linear-feedback type with constant leadership of one of the players may not constitute stable steady state of the system. Hence these solutions may not be optimal. Formally this argument is supported by the transversality condition, which is not fullfilled for these candidates. As an example take one of these candidates and check transversality conditions for it. The coefficients of the value function for player j, which is the leader for linear-feedback case (one of the candidates) are (one of the solutions for (4.11)):

$$\begin{cases} A_1^{[j]} = \frac{\beta_{[j]}+\beta_{[l]}+r+\theta}{\gamma_{[j]}^2(N-i)}; \\ A_2^{[j]} = \frac{1}{2}\frac{(2\theta+2\beta_{[l]}+r)(\beta_{[j]}+\beta_{[l]}+r+\theta)^2}{\gamma_{[j]}^2\theta^2(N-i)}; \\ A_3^{[j]} = -\frac{(2\theta+2\beta_{[l]}+r)(\beta_{[j]}+\beta_{[l]}+r+\theta)}{\gamma_{[j]}^2\theta(N-i)}; \\ A_4^{[j]} = \frac{\theta\gamma_{[j]}^2(\beta_{[j]}+\theta)(\theta+r+\beta_{[l]})}{\gamma_{[j]}^2\beta_{[j]}\theta(\beta_{[l]}+\theta)^2} - \\ -\frac{(2\beta_{[l]}+2\theta+r)(\beta_{[j]}+\beta_{[l]}+r+\theta)(2\beta_{[l]}+2\theta+r+\beta_{[j]})}{\gamma_{[j]}^2\beta_{[j]}\theta(\beta_{[l]}+\theta)^2}; \\ A_5^{[j]} = \frac{(2\beta_{[l]}+2\theta+r)(\beta_{[j]}+\beta_{[l]}+r+\theta)}{\gamma_{[j]}^2\beta_{[j]}\theta^2(\beta_{[l]}+\theta)^2} \times \\ \times((2\gamma_{[l]}^2-\gamma_{[j]}^2)\theta^2+((3r+3\beta_{[j]}+4\beta_{[l]})\gamma_{[l]}^2-\gamma_{[j]}^2\beta_{[l]})\theta+ \\ +(r+\beta_{[j]}+2\beta_{[l]})(\beta_{[j]}+\beta_{[l]}+r)\gamma_{[l]}^2); \\ A_6^{[j]} = A. \end{cases} \tag{4.22}$$

$A_6^{[j]}$ is not presented since it is irrelevant for the analysis here.
Transversality conditions for player j for the inifnite-time horizon case are:

$$lim_{t\to\infty}\frac{\partial V_i^{[j]}(q_i^{[j]},q_i^{[l]})}{\partial q_i^{[j]}} \times e^{-rt} = 0;$$
$$lim_{t\to\infty}\frac{\partial V_i^{[j]}(q_i^{[j]},q_i^{[l]})}{\partial q_i^{[l]}} \times e^{-rt} = 0. \tag{4.23}$$

It is sufficient to demonstrate that at least one of them is not fullfilled to prove that the corresponding candidate strategy is not optimal. For that consider the subsequent derivative of value function for player j with respect to his own quality multiplied by the discount factor. This may be easily deduced from the fisrt-order condition for controls, (4.8), substituted into quality dynamics, (4.6), and obtaining the respective solution for qualities of the form (4.21). This last is then substituted into the value function derivative. It has the form:

$$lim_{t\to\infty}\frac{\partial V_i^{[j]}(q_i^{[j]},q_i^{[l]})}{\partial q_i^{[j]}} \times e^{-rt} =$$
$$= S_1 \times e^{-rt} - S_2 \times e^{(\beta_{[l]}+\theta)t} + S_3 \times e^{\beta_{[j]}t} \neq 0. \tag{4.24}$$

This cannot be equal to zero but is diverging to plus/minus infinity with time. Hence transversality condition is not fullfilled for this candidate strategy. In general observe that this is true for any quality dynamics of the form (4.21), since it's growth is unbounded and the corresponding value function derivative contains state variable since it is not linear as in the case of piecewise-constant MPE considered above, but quadratic in states. At the same time the linear value function corresponding to the piecewise-constant strategies has constant derivatives in states and thus the transversality condition for it is fullfilled. Some additional arguments concerning linear-feedback strategies are considered in the end of this section.

Two remaining candidates result in constant strategy of the leader, as in piecewise-constant case and linear-feedback strategy for the follower of the type described above. Such a pair of strategies cannot constitute the equilibrium since the imitator will always catch up with the leader in multiple (countable but infinite) number of points and either this strategy is not optimal for the follower, either for the leader it is not optimal to undertake constant investment policy without adapting to the feedback strategy of the other player.

Formal argument for dismissing these candidates is the same as for linear-feedback strategies: the transversality condition for the player l, which is the follower and whose strategy is of linear-feedback type, is not fullfilled, since the solution for quality for this player is of type (4.21), and his value function appears to be quadratic. Hence repeating arguments given above the transversality conditions (at least one of the two) for this player cannot be fullfilled.

5.2. Symmetric Case. The symmetric case is characterized by the condition

$$q_i^{[j]}(t) = q_i^{[l]}(t). \tag{4.25}$$

This condition defines the situation when the imitation effect is absent for both players. Hence, in this case the HJB equations (4.7) are reduced to:

$$\begin{aligned} rV_i^{[j]} = max\Bigg\{ & q_i^{[j]}(t) - \frac{1}{2}g_i^{[j]}(t)^2 + \\ & \frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)}\left(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t)\right) \\ & + \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}(t)}\left(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t)\right)\Bigg\} \end{aligned} \tag{4.26}$$

$$\begin{aligned} rV_i^{[l]} = max\Bigg\{ & q_i^{[l]}(t) - \frac{1}{2}g_i^{[l]}(t)^2 + \\ & \frac{\partial V_i^{[l]}}{\partial q_i^{[j]}(t)}\left(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t)\right) \\ & + \frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)}\left(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t)\right)\Bigg\}. \end{aligned} \tag{4.27}$$

as there are no dominating in quality levels. Observe also that such a situation may happen only if parameter sets of both players are identical since if one of the players has an advantage related to investment efficiency or to the rate of decay, eventually the symmetric position will be left turning the game into leader-follower mode discussed earlier. Formally, symmetric outcome takes place only if:

$$q_i^{[j]}(t) = q_i^{[l]}(t); \forall t \in [0, .., \infty) \Longleftrightarrow \begin{cases} \gamma_j = \gamma_l; \\ \beta_j = \beta_l. \end{cases} \tag{4.28}$$

Since the HJB equation for each player does not depend explicitly on time this value function coefficients are constant across time and may be found through the system of algebraic equations as it is done for the leader-follower case above.

However this standard method is not applicable here. Observe that due to the precense of *max* function in the righthandside of the HJB equation for both players in general case, (4.7), the value functions of both players are not differentiable with respect to states at the point $q_i^{[j]} = q_i^{[l]}$, and this reveals a kink in the value functions of players themselves such that first-order conditions for optimal strategies are undefined as respective derivatives of the value functions are undefined. To see this account for one-sided derivatives of value functions. Only the constant strategies case is considered since all of the linear-feedback rules yield unstable dynamics wihtout steady states and transversality conditions are violated for them:

$$\frac{\partial V_i^{[j]}}{\partial q_i^{[j]}}|_{q_i^{[j]}>q_i^{[l]}} = \frac{1}{r+\beta_j}, \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}}|_{q_i^{[j]}>q_i^{[l]}} = 0;$$
$$\frac{\partial V_i^{[j]}}{\partial q_i^{[j]}}|_{q_i^{[j]}<q_i^{[l]}} = \frac{1}{r+\beta_j+\theta}, \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}}|_{q_i^{[j]}<q_i^{[l]}} = \frac{\theta}{(r+\beta_l)(r+\theta+\beta_j)};$$
$$\frac{\partial V_i^{[j]}}{\partial q_i^{[j]}}|_{q_i^{[j]}\xrightarrow{+}q_i^{[l]}} < \frac{\partial V_i^{[j]}}{\partial q_i^{[j]}}|_{q_i^{[j]}\xrightarrow{-}q_i^{[l]}};$$
$$\frac{\partial V_i^{[j]}}{\partial q_i^{[l]}}|_{q_i^{[j]}\xrightarrow{+}q_i^{[l]}} > \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}}|_{q_i^{[j]}\xrightarrow{-}q_i^{[l]}} \tag{4.29}$$

This discontinuity corresponds to the symmetric position of players, $q_i^{[j]} = q_i^{[l]}$. Note that these one-sided derivatives are valid only for the situation when there is only one stable steady state on each side off the diagonal, $q_i^{[j]} = q_i^{[l]}$, and hence qualities of both players do not return to the symmetric position once they left it. However, for piecewise-constant strategies case being considered here this is always the case: there are at most two stable steady states on different sides of the diagonal in state space. These steady states are derived further on. In such a case any value in between the two one-sided derivatives may be inserted into the HJB equation, [**6**]. There is a continuum spectrum of these values and hence only extreme values of these derivatives are derived.

First consider the case with derivative at symmetric position to be equal to the derivative in the position of the leader. Then the corresponding strategies in the symmetric case will be the same as in the case of a leader, (4.14). In the other case when setting the derivative equal to that in the follower's mode, the same line of reasoning yields strategies equal to the follower's one for both players in the symmetric case, (4.17). More precisely, the form of the gradient of the value function of given player in the symmetric mode is given by the range of linear combinations of gradients in the leader and follower cases:

$$\nabla V_i^{[j]}|_{q_i^{[j]}=q_i^{[l]}} = \alpha \nabla V_i^{[j]}|_{q_i^{[j]}\xrightarrow{+}q_i^{[l]}} + (1-\alpha)\nabla V_i^{[j]}|_{q_i^{[j]}\xrightarrow{-}q_i^{[l]}};$$
$$\alpha \in [0,..,1]. \tag{4.30}$$

For the constant strategies case this range constitutes the range of MPE constant strategies according to the first-order conditions. Observe that all these strategies are constant but yield different rates of investments and hence the resulting quality levels are different. They may be as low as for both players being imitators and as high as for both players being leaders. The subsequent set of possible MPE strategies and associated quality levels is then:

$$g_i^{SYM,CON} \in [\frac{\gamma\sqrt{N-i}}{r+\theta+\beta}; \frac{\gamma\sqrt{N-i}}{r+\beta}];$$
$$q_i^{SYM,CON}(t) \in [\frac{(1-e^{-\beta t})}{\beta(r+\beta+\theta)}\gamma^2(N-i), \frac{(1-e^{-\beta t})}{\beta(r+\beta)}\gamma^2(N-i)]. \tag{4.31}$$

Optimal strategies do not depend on state variables except for the condition that this set is valued only along the diagonal in the state-space. Observe that the upper boundary for quality development in symmetric regime coincides with the quality dynamics of the leader, while the lower bound of quality growth in this case is lower then the quality of the imitator since the absence of the imitation effect.

However, there is a way to select the unique equilibrium path in the symmetric case. For that employ the condition of continuity of both value functions at the point of the kink and the observation that this (identical for both players) value function has to be generated by the pair of optimal controls. Since this value function is unique, the pair of controls which generate this value function has also to be unique.

Continuity of the value function implies (assuming player j is the leader and l is the follower):

$$lim_{q_i^{[j]} \xrightarrow{+} q_i^{SYM,CON}} V_i^{[j]} = \\ = lim_{q_i^{[j]} \xrightarrow{-} q_i^{[SYM]}} V_i^{[j]} = V_i^{SYM,CON};$$

$$\begin{aligned} lim_{q_i^{[j]} \xrightarrow{+} q_i^{SYM,CON}} V_i^{[j]} &= \frac{1}{r+\beta_{[j]}} q^{SYM,CON} + \frac{1}{2} \frac{\gamma_{[j]}^2 (N-i)}{r(r+\beta_{[j]})}; \\ lim_{q_i^{[j]} \xrightarrow{-} q_i^{[SYM]}} V_i^{[j]} &= \frac{\beta_{[l]} + r + \theta}{(r+\beta_{[l]})((\beta_{[j]} + \theta + r))} q_i^{[SYM,CON]} + \\ + \frac{1}{2} & \frac{((r^2 + \beta_{[l]}^2)\gamma_{[j]}^2 + 2\theta\gamma_{[l]}^2(\beta_{[j]} + \theta) + 2r(\gamma_{[j]}^2 \beta_{[l]} + \gamma_{[l]}^2 \theta))(N-i)}{r(r+\beta_{[l]})^2 (\beta_{[j]} + \theta + r)^2}. \end{aligned} \tag{4.32}$$

It is converging to the same value from the leader and from the follower modes in symmetric case. It turns out that this condition may be fulfilled only in two cases. First, when value functions of both players coincide. In this case only the symmetric outcome may happen and value functions are not interrelated at all:

$$\begin{cases} \beta_{[j]} = \beta_{[l]} = \beta; \\ \gamma_{[j]} = \gamma_{[l]} = \gamma; \\ \theta = 0. \end{cases} \tag{4.33}$$

which coincides with symmetry conditions above except for $\theta = 0$ condition. With such values of coefficients value functions coincide and are equal to the value function of the leader:

$$V^{SYM,CON} = \frac{1}{r+\beta} q^{SYM,CON} + \frac{1}{2} \frac{\gamma^2(N-i)}{r(r+\beta)}. \tag{4.34}$$

Now observe that the optimal control in symmetric case has to generate the value function of a given form, namely it should satisfy the HJB equation:

$$\begin{aligned} rV_i^{SYM,CON} &= q_i^{SYM,CON} - \frac{1}{2}(g_i^{SYM,CON})^2 + \\ + \frac{\partial V_i^{SYM,CON}}{\partial q_i^{SYM,CON}} & \times (\gamma\sqrt{N-i} g_i^{SYM,CON} - \beta q_i^{SYM,CON}). \end{aligned} \tag{4.35}$$

From this one may derive the expression for controls in the symmetric case. They turn to coincide with the upper limit of the range of optimal controls specified earlier:

$$g_i^{SYM,CON} = \frac{\gamma\sqrt{N-i}}{r+\beta}. \tag{4.36}$$

With such controls qualitites of both players are equal to

$$q_i^{SYM,CON} = \frac{(1-e^{-\beta t})}{\beta(r+\beta)} \gamma^2 (N-i). \tag{4.37}$$

Observe that the requirement of equal parameters and the absence of imitation $\theta = 0$ transforms the whole differential game in qualities into two independent optimal control problems, since there is no even potential interaction between players in the absence of imitation.

Next consider the case with non-zero imitation speed, $\theta \neq 0$, but equal parameters of both players. In such a case the limits of the linear value functions in (4.32) cannot coincide and hence value functions of both players are not continious in the point of changing position from leader to the follower and vice versa. However both players start and zero quality levels and choose their strategies at this point without altering it afterwards since piecewise-constant strategies are of the open-loop type. Then with positive θ both players will have intentions to minimize their investment to benefit from the imitation effect. At the same time, possible range of strategies with linear value functions is given by (4.31). The minimum amount of investments and subsequent quality levels are then given for both players by

$$g_i^{SYM,\theta,CON} = \frac{\gamma\sqrt{N-i}}{r+\beta+\theta};$$
$$q_i^{SYM,\theta,CON}(t) = \frac{(1-e^{-\beta t})}{\beta(r+\beta+\theta)}\gamma^2(N-i). \tag{4.38}$$

which yield the lowest possible level of investments and qualities for both players which are symmetric. Both players then have value functions of the follower's form. Observe also, that with such parameters' configuration the only steady state is where both players' qualities are equal to that of the follower. Then one may conclude that the only case when the symmetric outcome may occur at more then one point in time and yield continuous investment strategy is the case when no leader-follower situation may occur. At the same time this yields another observation, namely, that when the leader-follower situation occurs in the constant strategies case, the symmetry of states (qualities) may be observed at most at some points in time which do not follow each other. To see this consider the form of given value functions. They are of linear type. If they do not coincide along the whole diagonal $q_i^{[j]} = q_i^{[l]}$, the only way the given value function may be continuous is that there is some single transition point between the two regimes for each player. This is the point, where $\exists! t_i^{trig} \neq t_i^0 : q_i^{[j]}(t_i^{trig}) = q_i^{[l]}(t_i^{trig})$. Equivalently one may find the threshold value of quality from the condition of convergence of value functions:

$$\exists! q_i^j = q_i^l = q_i^{SYM,CON} > 0 :$$
$$lim_{q_i^{[j]} \xrightarrow{+} q_i^{SYM,CON}} V_i^{[j]} =$$
$$= lim_{q_i^{[j]} \xrightarrow{-} q_i^{[SYM]}} V_i^{[j]} = V_i^{SYM,CON}. \tag{4.39}$$

This condition yields the threshold value of quality for both players as a function of parameters. Then making use of the HJB equation as above one may define optimal controls for this case. It turns to be negative and since investments are irreversible, it should be zero at this point. Observe however, that zero investments for both players while their qualities are equal lead to different consequences since the difference in decay rates, β between them (as with equal parameters only the symmetric outcome may occur). Then it cannot be optimal for both players to make the same zero investments. Hence, such a situation with equal continuous across symmetric states value functions is not revelant. This points to the fact that convergence condition is violated here for any non-zero imitation speed and linear value functions of both players:

$$lim_{q_i^{[j]} \xrightarrow{+} q_i^{SYM,CON}} V_i^{[j]} \neq \tag{4.40}$$
$$\neq lim_{q_i^{[j]} \xrightarrow{-} q_i^{[SYM]}} V_i^{[j]}$$

Then as value function has to be continuous across all ranges of state variables, it cannot be of linear form. Hence at the catching-up situation the value function has to be computed together with optimal strategies separately from the case of constant leadership.

5.3. Piecewise Formulation of Strategies for Constant Leadership. Consider the set of possible outcomes of the quality game with the restriction of picewise-constant strategies only. The resulting strategies must be of a piecewise form, depending on the regime of the game in qualities. Specifically, there are 4 different regimes for both players: any player can be the leader, the follower or there are two possible symmetric strategies:

$$g_{CON}^{[j]} = \begin{cases} g_{SYM,CON}^{[j]}, q_i^{[j]}(t) = q_i^{[l]}(t), \theta = 0; \\ g_{SYM,\theta,CON}^{[j]}, q_i^{[j]}(t) = q_i^{[l]}(t), \theta > 0; \\ g_{F,CON}^{[j]}, q_i^{[j]}(t) < q_i^{[l]}(t), \theta > 0; \\ g_{L,CON}^{[j]}, q_i^{[j]}(t) > q_i^{[l]}(t), \theta > 0. \end{cases} \tag{4.41}$$

One have a piecewise formulation of optimal controls and quality growth, depending on the leadership. This leadership, in turn, due to the similar initial conditions, $q_{[j]}(i,0) = q_{[l]}(i,0) = 0, \forall i$ depends on the ratio of relative investment efficiencies and rates of quality decay.
In both cases optimal control is constant and this is the same as in the monopolistic case, where the value function also has the linear form.

Subsequent quality levels:

$$q_{CON}^{[j]}(i,t) = \begin{cases} \frac{\gamma_{[j]}^2 (N-i)}{\beta_{[j]}(r+\beta_{[j]})} \times (1 - e^{-\beta_{[j]}t}, q^{[j]}(i,t) > q^{[l]}(i,t); \\ \Big(1 + (N-i) \times \\ \times \big(\frac{E_1 (e^{(\beta_{[j]}+\theta)t}+1)}{\beta_{[j]}+\theta} - \frac{E_2 (e^{(\beta_{[j]}-\beta_{[l]}+\theta)t}+1)}{\beta_{[j]}-\beta_{[l]}+\theta} \Big) \times e^{-(\beta_{[j]}+\theta)t}, q^{[j]}(i,t) < q^{[l]}(i,t); \\ \frac{(1-e^{-\beta t})}{\beta(r+\beta)} \gamma^2 (N-i), q^{[j]}(i,t) = q^{[l]}(i,t), \theta = 0; \\ \frac{(1-e^{-\beta t})}{\beta(r+\beta+\theta)} \gamma^2 (N-i), q^{[j]}(i,t) = q^{[l]}(i,t), \theta > 0. \end{cases} \tag{4.42}$$

Here $E_1, E_2 = f(\gamma_{[j],[l]}, \beta_{[j],[l]}, \theta)$. The same form of solutions is true for the other player.

Observe that this set of solutions yields two different steady-state levels of quality for every player in leader-follower mode, one corresponding to being the leader (and this one coincide with the steady-state level of quality of monopolist from chapter 2) and the other with being the follower. The first steady-state level of quality is not always higher then the second one, but the actual steady state in which trajectories of qualitites will be eventually depends on the relation between efficiency parameters of the players. To be more specific for any player steady-state level of quality is given by (j player is taken for certainty):

$$q_i^{STL}(t) = \frac{\gamma_{[j]}^2 (N-i)}{\beta_{[j]}(\beta_{[j]}+r)};$$
$$q_i^{STF}(t) = \frac{\gamma_{[l]}^2\theta^2 + \gamma_{[l]}^2\theta(\beta_{[j]}+r) + \gamma_{[j]}^2\beta_{[l]}(\beta_{[l]}+r)}{\beta_{[l]}(\beta_{[j]}+\theta)(r+\theta+\beta_{[j]})(\beta_{[l]}+r)} \times (N-i). \tag{4.43}$$

and of the same form with change of j by l and vice versa for the other player. The first one is corresponding to the steady-state level of quality for the leader and is computed by the same way as (2.26), while the second one corresponds to the situation of being the follower and this value depends on the steady-state level of quality of the leader, given in turn by the first expression. It cannot be said, that for the given player the steady-state level of quality of being the leader is always higher then of being the follower. The exact relation depends on the relation between the parameters of both players. It may be the case that for the given player the steady-state level in the follower mode is higher then of being in the leader mode.
Formal conditions for the leadership may be easily derived from the steady states. They are:

$$\beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2. \tag{4.44}$$

This is the condition for the system to end up in the steady state with player j being the leader in quality ($q_i^{[j]} > q_i^{[l]}$). If additionally,

$$(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > (\beta_{[j]}+r+\theta)\gamma_{[l]}^2, \tag{4.45}$$

then player j is always the leader and no catching up may occur. Observe however, that conditions for player j or player l to end up as a leader are not exclusive: it may be the case that simultaneously

$$\begin{aligned} &\beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2; \\ &\beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2 > \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2. \end{aligned} \tag{4.46}$$

Then there are two different steady states of the game with player j being the leader and vice versa. Hence one have the problem of equilibrium selection in the case of open-loop equilibrium. It is solved by comparing the resulting steady-state levels of qualities and associated value functions for both players. If for player j steady-state level of quality as the leader is higher then the steady-state level of quality for player l while he is the leader, then the equilibrium with player j being the leader is selected. Then one have the following system of conditions for defining the steady state in which the system will eventually arrive:

$$\begin{cases} \beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2, q_i^{STL}=q_i^{[j]}; \\ \\ \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2 > \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2, q_i^{STL}=q_i^{[l]}; \\ \\ \beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2, and \\ \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2 > \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2, but \\ \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2, q_i^{STL}=q_i^{[j]}; \\ \\ \beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2, and \\ \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2 > \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2, but \\ \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2 < \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2, q_i^{STL}=q_i^{[l]}. \end{cases} \tag{4.47}$$

Observe that the given set of piecewise-constant strategies may constitute the closed-loop equilibrium only in two first cases, where only one steady state of the system is possible. If there are two possible steady states, the piecewise-constant solution still would constitute the open-loop equilibrium, but not the closed-loop one, since the equilibrium selection problem cannot be resolved in a way it is done here. Namely, due to the fact that follower's position is always more attractive then that of the leader, there are incentives for both players to deviate from the given steady state to the other one where he is the follower. Hence both players would underinvest in quality growth and no equilibrium closed-loop strategies of at most linear-feedback type exist. This argument is considered in more details further on in the work.

Analysis of these conditions yields following observations. With close values of quality decay levels the player whose investment efficiency is lower will never catch up with the leader irrespective of the scale of imitation effect. If his rate of decay is much lower then that of the leader he will catch up even if his investment efficiency is lower then that of the leader. After the trigger point is reached (the follower equalizes his quality level with that of the leader), the former leader will remain the follower until the end of the game. This is exactly due to the fact that the catching up process may be successful only if the follower's rate of decay is much lower then that of the leader. This result may be proved analytically also. With linear value functions of players after the trigger point (which is the one corresponding to open-loop equilibrium) there might be at most one trigger point for the quality. This corresponds to the case when catching-up may occur.

For catching-up situation to occur it is necessary that two stable steady states exist in the system. This happens when two last sets of conditions are fullfilled in the last system: there exist two stable steady states with the leadership of one or another player but only one of them is realized. Exact formal conditions for catching-up are as following:

$$\begin{aligned} \beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 &> \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2; \\ \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2 &> \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2; \\ \beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2 &> \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2; \\ (\beta_{[l]}+r+\theta)\gamma_{[j]}^2 &< (\beta_{[j]}+r+\theta)\gamma_{[l]}^2. \end{aligned} \tag{4.48}$$

They follow from the expressions for steady-state levels of quality for both players. The meaning of these conditions is rather straigthforward: player j will end up

as a leader if his steady-state level of quality as a leader is higher then that of the player l, (hence the third condition). Then the steady state with him as a leader is eventually selected. At the same time this does not mean per se that the catching-up will occur. For that it is also necessary that two steady states exist in the system, hence two first conditions. The last condition denotes that investments of the player j in the follower mode are higher then those of player l in the follower mode. Then the system will end up in the steady-state with player j as a leader but there is some initial period of time, when qualitites are low enough, and player j's investment are lower then those of player l.
That is, the catching-up phenomena may occur only if the investment efficiency of one player is higher then that of the other (so he is initially the leader in quality) but his rate of decay is sufficiently higher then that of the follower. Then the follower will eventually catch-up with the leader because of accumulated quality level. The rate of decay becomes more significant then efficiency of investments while follower's growth rate is higher due to the imitation effect. This catching-up may occur only one time at most, because after the follower will catch-up with the leader, due to the condition above, the former leader will be unable to catch-up again because his ratio of growth would be lower. As a result, the player whose steady-state level of quality in the leader mode is higher will end up as the leader while the second one will remain the follower. This means in turn that there are unique q_i^{trig} and t_i^{trig} which denote the point of the catching-up.

Observe that this uniqueness of the trigger point is the direct consequence of the constant nature of given strategies after the trigger point. There is no possibility for any of the players to adapt their strategies with accordance with the quality achieved already by both players after the strategy is selected (that of the leader or the follower). This demonstrates the limited nature of information being used by the players but at the same time this gives the possibility for stable equilibrium as it will be discussed further. With more complicated profile of strategies (other then at most linear) multiple trigger points may be possible. One may conclude that there are exactly three possible outcomes in quality growth game with two players in constant MPE case: either one of the players is constantly the leader, either both players have equal qualities all the time or there is the catching-up process. Observe also, that positions of players are constant across products due to the monotonicity of assumed investment efficiencies. That is, if one of the players is the leader in the quality of one of the products, he is the leader in all products' qualities and vice versa. This is the direct consequence of the form of $\gamma(\bullet)$ functions being adopted: every next product is harder to invest in, but this decrease in efficiency of investments is the same for both player and the term $(N - i)$ symmetrically enters strategies of both players. Hence the rate of investments differences across players does not depend on i, but only on the relations between constant parameters of the system.

5.4. Catching Up. Consider the situation when one of the players is initially the leader in quality level but eventually due to the configuration of parameters the other player catches up and becomes the leader. In such a situation the analysis above is no longer complete, since the catching up situation cannot be described by polynomial value functions. To observe this consider the results of the section on the symmetric case. It has been shown, that the only case when linear value functions may be continuous across the diagonal in the state space is the case of the absence of imitation, $\theta = 0$. Symmetric outcome is possible for $\theta > 0$, but not the transition of value function across the diagonal from the leader's regime to the follower's and vice versa. Such a condition destroys the differential game completely and transforms it into two independent problems of quality investments which would have exactly the same solutions as for infinite-horizon one agent problem described earlier.

Hence one has to conclude that in the situation when quality levels cross the diagonal in the state space at non-zero quality level the linear value functions obtained previously are boundary conditions for value functions before the catching up occurs. It means that value function of each of the players have to be equal to the linear value function derived above at the time when catching up occurs and at the respective quality level. It is demonstrated that such a condition may not be fulfilled by linear value functions themselves, that is,

$$V_i^{L,CON}(q_i(t_i^{trig}))|_{q_i^{[j]}=q_i^{[l]}} \neq V_i^{F,CON}(q_i(t_i^{trig}))|_{q_i^{[j]}=q_i^{[l]}}; \quad \theta \neq 0. \tag{4.49}$$

To obtain the value function for each player for the case of catching up as well as equilibrium strategies and quality dynamics one have to solve separately the problem of leader-follower along the time period $t \in [t(0)_i, .., t_i^{trig}]$ with $V^{L,F,CON}(q_i(t_i^{trig}))$ as the boundary conditions. This solution will depend on the time of the trigger then.

Only the open-loop case is considerd here as the closed-loop strategies of at most linear-feedback type cannot exist in this framework and other types of closed-loop strategies are hard to find. This is left to future extensions of the model. In open-loop case one may cast the problem into the maximum principle framework and make use of the maximum principle approach instead of the HJB one. This approach is constructive by its nature and hence it would be possible to derive explicitly the form of the value function. HJB approach is not useful here since one may not assume any form of the value function in advance (the linear value function does not fit the problem as any value function of polynomial form as well).

For simplicity assume that player j is the leader before the trigger point and then he is the follower while player l is the follower before the switch and vice versa. The problem for the leader before the trigger point is:

$$\begin{aligned}
J_i^{[j]} = \int_{t(0)_i}^{t_i^{trig}} e^{-rt}[q_i^{[j]}(t) - \frac{1}{2}g_i^{[j]}(t)^2]dt + e^{-rt_i^{trig}} \times V_i^{F,CON}(q_i^{[j]}(t_i^{trig}), q_i^{[l]}(t_i^{trig}))|_{q_i^{[j]}=q_i^{[l]}} \\
s.t. \\
\dot{q_i^{[j]}}(t) = \gamma_{[j]}\sqrt{N-i}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]})(t); \\
\dot{q_i^{[l]}}(t) = \gamma_{[l]}\sqrt{N-i}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) + \theta(q_i^{[j]}(t) - q_i^{[l]}(t)); \\
q_i^{[j]}(t) - q_i^{[l]}(t) > 0, \forall t < t_i^{trig}; \\
q_i^{[j]}(t_i^{trig}) = q_i^{[l]}(t_i^{trig}); \\
q_i^{[j]}(t(0)_i) = 0; \\
q_i^{[l]}(t(0)_i) = 0.
\end{aligned} \tag{4.50}$$

Problem to solve by the second player who is the follower before the trigger is:

$$J_i^{[l]} = \int_{t(0)_i}^{t_i^{trig}} e^{-rt}[q_i^{[l]}(t) - \frac{1}{2}g_i^{[l]}(t)^2]dt + e^{-rt_i^{trig}} \times V_i^{L,CON}(q_i^{[j]}(t_i^{trig}), q_i^{[l]}(t_i^{trig}))|_{q_i^{[j]}=q_i^{[l]}}$$

$$s.t.$$

$$\dot{q_i^{[j]}}(t) = \gamma_{[j]}\sqrt{N-i}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]})(t);$$

$$\dot{q_i^{[l]}}(t) = \gamma_{[l]}\sqrt{N-i}g_i^{[l]}(t) - \beta_{[l]}q_i^{[l]}(t) + \theta(q_i^{[j]}(t) - q_i^{[l]}(t));$$

$$q_i^{[j]}(t) - q_i^{[l]}(t) > 0, \forall t < t_i^{trig};$$

$$q_i^{[j]}(t_i^{trig}) = q_i^{[l]}(t_i^{trig});$$

$$q_i^{[j]}(t(0)_i) = 0;$$

$$q_i^{[l]}(t(0)_i) = 0. \tag{4.51}$$

Observe that the trigger time, t_i^{trig} is endogenous here and is influenced by actions of both players. Both problems include both quality levels as state variables also. Hence the problem has to be solved in two stages. First optimal strategies and quality dynamics are derived as functions of the trigger time t_i^{trig}, and then this last one is computed. In an effect one would have this trigger time as the function of the model's exogenous parameters.

There is an issue concerning this trigger time though. Both agents are optimizing their investments with respect to other player's optimal strategy as given. Formally trigger time is found from the boundary condition (dealt with later on) for each player separately and this means that this trigger time might be different for both players. Denote them $t_i^{trig,[j]}, t_i^{trig,[l]}$. First observe that since player j is the follower after the trigger, his optimal control before the trigger depends on the optimal control of the other player through costate equations while this is not true for the player l. This may be observed from costate equations for both players which are derived further on. Suppose player's j optimal trigger time, $t_i^{trig,[j]}$ is greater then the trigger time for player l, $t_i^{trig,[j]} \geq t_i^{trig,[l]}$. Then his control will switch from that of the current problem to the follower's control in constant strategies case being described earlier in the work. This last is constant and does not depend on the control of the other player at all. At the same time the control of player l will change earlier on, at time $t_i^{trig,[l]}$ to the constant control of the leader. Then at time $t_i^{trig,[j]}$ the scrap value of player j is indeed given by $V_i^{F,CON}(q_i(t_i^{trig,[j]}))$. However, if $t_i^{trig,[j]} < t_i^{trig,[l]}$, player j will switch to the follower's strategy before player l will change his control. Since the follower's value function is always higher then that of the leader in the constant leader-follower mode, the actual scrap value of player j would be smaller then $V_i^{F,CON}(q_i(t_i^{trig,[j]}))$ when $t_i^{trig,[j]} < t_i^{trig,[l]}$. It is not explicitly defined here. It is sufficient to note that then the value generated by the strategic choice $\{g_i^{[j]}, t_i^{trig,[j]}\}$ is no longer optimal for the player j. Then one may conclude that the optimal time of the trigger for player j should not be less then $t_i^{trig,[l]}$, while with $t_i^{trig,[j]} \geq t_i^{trig,[l]}$ his scrap value function (value function after the trigger point) is indeed given by $V_i^{F,CON}(q_i(t_i^{trig}))$ where $t_i^{trig} = \{t_i^{trig,[j]}, t_i^{trig,[j]} \geq t_i^{trig,[l]}; t_i^{trig,[l]}, t_i^{trig,[j]} < t_i^{trig,[l]}\}$. In an effect this means that one may assume equal trigger times of both players since otherwise the strategic choice of player j under the system (4.50) would not be optimal. Observe that this is not so for the player l, since his control before the trigger does not depend on the control of the other player.

To save on notation normalize time of emergence to zero for all i and hence the trigger time is also normalized to T_i which may be in general different across products. Indices i of all other variables are condensed.

First construct hamiltonians of both problems:

$$\mathcal{H}^{[j]} = q^{[j]}(t) - \frac{1}{2}g^{[j]}(t)^2 + \psi_1^{[j]}(t)\Big(\gamma_{[j]}\sqrt{N-i}g^{[j]}(t) - \beta_{[j]}q^{[j]})(t)\Big) + \psi_2^{[j]}(t)\Big(\gamma_{[l]}\sqrt{N-i}g^{[l]}_{opt}(t) - \beta_{[l]}q^{[l]}(t) + \theta(q^{[j]}(t) - q^{[l]}(t))\Big);$$

$$\mathcal{H}^{[l]} = q^{[l]}(t) - \frac{1}{2}g^{[l]}(t)^2 + \psi_1^{[l]}(t)\Big(\gamma_{[j]}\sqrt{N-i}g^{[j]}_{opt}(t) - \beta_{[j]}q^{[j]})(t)\Big) + \psi_2^{[l]}(t)\Big(\gamma_{[l]}\sqrt{N-i}g^{[l]}(t) - \beta_{[l]}q^{[l]}(t) + \theta(q^{[j]}(t) - q^{[l]}(t))\Big). \tag{4.52}$$

One also have to take into account state constraints of the problems. For that construct lagrangian functions:

$$\mathcal{L}^{[j]} = \mathcal{H}^{[j]} + \mu^{[j]}(t)(q^{[j]}(t) - q^{[l]}(t)) + \eta^{[j]}(T_i)(q^{[j]}(T_i) - q^{[l]}(T_i));$$

$$\mathcal{L}^{[j]} = \mathcal{H}^{[j]} + \mu^{[l]}(t)(q^{[j]}(t) - q^{[l]}(t)) + \eta^{[l]}(T_i)(q^{[j]}(T_i) - q^{[l]}(T_i)). \tag{4.53}$$

Now apply maximum principle to this system.
Optimal controls are given by:

$$\frac{\partial \mathcal{L}^{[j]}}{\partial g^{[j]}} = 0 : g^{[j]}_{opt} = \gamma_{[j]}\sqrt{N-i}\psi_1^{[j]};$$
$$\frac{\partial \mathcal{L}^{[l]}}{\partial g^{[l]}} = 0 : g^{[l]}_{opt} = \gamma_{[l]}\sqrt{N-i}\psi_2^{[l]} \tag{4.54}$$

Costate equations:

$$\dot{\psi_1^{[j]}}(t) = r\psi_1^{[j]}(t) - \frac{\partial \mathcal{L}^{[j]}}{\partial q^{[j]}(t)};$$
$$\dot{\psi_2^{[j]}}(t) = r\psi_2^{[j]}(t) - \frac{\partial \mathcal{L}^{[j]}}{\partial q^{[l]}(t)};$$
$$\dot{\psi_1^{[l]}}(t) = r\psi_1^{[l]}(t) - \frac{\partial \mathcal{L}^{[l]}}{\partial q^{[j]}(t)};$$
$$\dot{\psi_2^{[l]}}(t) = r\psi_2^{[l]}(t) - \frac{\partial \mathcal{L}^{[l]}}{\partial q^{[l]}(t)}. \tag{4.55}$$

These equations form two independent systems of ODEs which may be solved separately. Due to the precense of boundary constraints in the form of the scrap value functions and terminal time state constraints one has specific transversality conditions on costate variables. The system on costates for the player j is the boundary value problem:

$$\begin{aligned}\dot{\psi_1^{[j]}}(t) &= (r+\beta_{[j]})\psi_1^{[j]}(t)+\theta\psi_2^{[j]}(t)-(1+\mu^{[j]}(t));\\ \dot{\psi_2^{[j]}}(t) &= (r+\beta_{[l]}+\theta)\psi_2^{[j]}(t)+\mu^{[j]}(t);\\ \psi_1^{[j]}(T_i) &= \frac{\partial V_i^{F,CON}(q^{[j]}(T_i),q^{[l]}(T_i))|_{q^{[j]}=q^{[l]}}}{\partial q^{[j]}(T_i)}+\eta^{[j]};\\ \psi_2^{[j]}(T_i) &= \frac{\partial V_i^{F,CON}(q^{[j]}(T_i),q^{[l]}(T_i))|_{q^{[j]}=q^{[l]}}}{\partial q^{[l]}(T_i)}-\eta^{[j]}\end{aligned} \tag{4.56}$$

Here $\mu[j],\eta^{[j]}$ are the lagrange multipliers of the state constraints. This system has a parametric solution with respect to $\mu^{[j]},\eta^{[j]}$.

$$\begin{aligned}\psi_1^{[j]}(t)^* &= C_1+C_2e^{-(r+\beta_{[j]})(T_i-t)}+C_3e^{-(r+\beta_{[l]}+\theta)(T_i-t)};\\ \psi_2^{[j]}(t)^* &= C_4e^{-(r+\beta_{[l]}+\theta)(T_i-t)}-C_5.\end{aligned} \tag{4.57}$$

Here $\{C_1,..,C_5\}$ are functions of $\{\mu^{[j]},\eta^{[j]}\}$ and exogenous parameters of the model. The system on costates for player l is:

$$\begin{aligned}\dot{\psi_1^{[l]}}(t) &= (r+\beta_{[j]})\psi_1^{[l]}(t)-\theta\psi_2^{[l]}(t)-\mu^{[l]}(t);\\ \dot{\psi_2^{[l]}}(t) &= (r+\beta_{[l]}+\theta)\psi_2^{[l]}(t)+\mu^{[l]}-1;\\ \psi_1^{[l]}(T_i) &= \frac{\partial V_i^{L,CON}(q^{[j]}(T_i),q^{[l]}(T_i))|_{q^{[j]}=q^{[l]}}}{\partial q^{[j]}(T_i)}+\eta^{[l]};\\ \psi_2^{[l]}(T_i) &= \frac{\partial V_i^{L,CON}(q^{[j]}(T_i),q^{[l]}(T_i))|_{q^{[j]}=q^{[l]}}}{\partial q^{[l]}(T_i)}-\eta^{[l]}\end{aligned} \tag{4.58}$$

which has the solution

$$\begin{aligned}\psi_1^{[l]}(t)^* &= F_1+F_2e^{-(r+\beta_{[j]})(T_i-t)}+F_3e^{-(r+\beta_{[l]}+\theta)(T_i-t)};\\ \psi_2^{[l]}(t)^* &= F_4e^{-(r+\beta_{[l]}+\theta)(T_i-t)}-F_5.\end{aligned} \tag{4.59}$$

where $\{F_1,..,F_5\}$ are functions of $\{\mu^{[l]},\eta^{[l]}\}$ and exogenous parameters.
Observe that in parametric form this solution is identical to that of the costate system for player j. However, the set of parameters $\{F_1,..,F_5\}$ is different here. This happens due to different boundary conditions of two systems.

Now with both costates and optimal controls at hand one may derive the system for states evolution:

$$\begin{aligned}\dot{q^{[j]}}(t) &= \gamma_{[j]}^2(N-i)\psi_1^{[j]}(t,\mu^{[j]}(t),\eta^{[j]})-\beta_{[j]}q^{[j]}(t);\\ \dot{q^{[l]}}(t) &= \gamma_{[l]}^2(N-i)\psi_2^{[l]}(t,\mu^{[l]}(t),\eta^{[l]})-\beta_{[l]}q^{[l]}(t)+\theta(q^{[j]}(t)-q^{[l]}(t));\\ & q^{[j]}(0)=0, q^{[l]}(0)=0.\end{aligned} \tag{4.60}$$

This system includes two still undefined functions of time, $\mu^{[j]}(t),\mu^{[l]}(t)$. They are to be found from complementary slackness conditions:

$$t: q^{[j]}(t) > q^{[l]}(t), \mu^{[j]}(t)=0, \mu^{[l]}(t)=0 \tag{4.61}$$

By the construction of the problem this is always true, hence, $\mu^{[j]}(t)=0,\mu^{[l]}(t)=0,\forall t<T_i$. The one may solve the system (4.60) parametrically as functions of $\eta^{[j]},\eta^{[l]}$. Note, that these lagrange multipliers are defined from the system

$$\begin{aligned}\eta^{[j]} &: q^{[j]}(T_i^{[j]},\eta^{[j]}) = q^{[l]}(T_i^{[j]},\eta^{[j]},\eta^{[l]});\\ \eta^{[j]} &: q^{[j]}(T_i^{[l]},\eta^{[j]}) = q^{[l]}(T_i^{[l]},\eta^{[j]},\eta^{[l]}).\end{aligned} \tag{4.62}$$

In general then one have a continuum of possible values of η's. However, observe that the trigger time, T_i should be the same for both players while there are two different conditions for that:

$$\begin{aligned} T_i^{[j]} : \mathcal{H}^{[l]}(q^{[j]}(T_i, \eta^{[j]})^*, q^{[l]}(T_i, \eta^{[j]}, \eta^{[l]})^*, \psi_1^{[j]}(T_i, \eta^{[j]})^*, \psi_2^{[j]}(T_i, \eta^{[j]})^*) = \\ = V_i^{L,CON}(q^{[j]}(T_i, \eta^{[j]})^*, q^{[l]}(T_i, \eta^{[j]}, \eta^{[l]})^*)|_{q^{[j]}=q^{[l]}} \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} T_i^{[l]} : \mathcal{H}^{[j]}(q^{[j]}(T_i, \eta^{[j]})^*, q^{[l]}(T_i, \eta^{[j]}, \eta^{[l]})^*, \psi_1^{[l]}(T_i, \eta^{[l]}, \eta^{[j]})^*, \psi_2^{[l]}(T_i, \eta^{[l]}, \eta^{[j]})^*) = \\ = V_i^{F,CON}(q^{[j]}(T_i, \eta^{[j]})^*, q^{[l]}(T_i, \eta^{[j]}, \eta^{[l]})^*)|_{q^{[j]}=q^{[l]}}. \\ T_i^{[j]} = T_i^{[l]}. \end{aligned} \tag{4.64}$$

where $T_i^{[l]}$ is the time of the trigger being defined from the dynamic problem of the player l and $T_i^{[j]}$ is the trigger time being defined from the dynamic problem for player j.

These two expressions must yield the same trigger time, T_i, following the discussion in the beginning of this subsection. The parametric solution for qualities is then:

$$\begin{aligned} q^{[j]}(t)^{BTR} = (N-i)\times \\ \times\Bigg(Q_1(1-e^{-\beta_{[j]}t}) - (Q_3\eta^{[j]} + Q_4)e^{-\beta_{[j]}(T_i+t)-rT_i} + \\ +(Q_7\eta^{[j]} + Q_8)e^{-\beta_{[j]}t-(r+\beta_{[l]}+\theta)T_i} + (Q_8 - Q_7\eta^{[j]})e^{(r+\theta+\beta_{[l]})(t-T_i)} \Bigg); \end{aligned}$$

$$\begin{aligned} q^{[l]}(t)^{BTR} = (N-i)\times \\ \times\Bigg(Q_9 + (Q_{10}\eta^{[l]} + Q_{11})e^{(r+\beta_{[l]}+\theta)(t-T_i)} + Q_{12}e^{-\beta_{[j]}t} + \\ +(Q_{13}\eta^{[l]} + Q_{14})e^{(r+\beta_{[j]})(t-T_i)} + (Q_{15}\eta^{[l]} + Q_{16})e^{-(\beta_{[l]}+\theta)(t+T_i)-rT_i} + \\ +(Q_{17}\eta^{[l]} + Q_{18})e^{-\beta_{[j]}(t+T_i)-rT_i} + (Q_{19}\eta^{[l]} + Q_{20})e^{-\beta_{[j]}t-(r+\beta_{[l]}+\theta)T_i} + \\ +(Q_{21}\eta^{[l]} + Q_{22})e^{-(\beta_{[l]}+\theta)t-(r+\beta_{[j]})T_i} + Q_{23}e^{-(\beta_{[l]}+\theta)t} \Bigg). \end{aligned} \tag{4.65}$$

where $\{Q_1, .., Q_{23}\}$ are functions of exogenous parameters of the model only. To achieve the explicit solution for qualities one have to use the complementarity slackness condition which is the same for both players:

$$\begin{aligned} \eta^{[j]} : q^{[l]}(T_i) = q^{[j]}(T_i), \\ \eta^{[l]} : q^{[l]}(T_i) = q^{[j]}(T_i). \end{aligned} \tag{4.66}$$

As long as parametric solutions for qualitites are the same in both expressions as well as the trigger time, $T_i^{[j]} = T_i^{[l]} = T_i$, these two expressions would yield the same value of $\eta = \eta^{[j]} = \eta^{[l]}$.

Note also that so far T_i is treated as exogenous and independent on parameters and i. Then this last condition means that qualities of both players have to be equal at the terminal time whatever this terminal time might be. Solving equations for

η yields

$$\eta^* = \Bigg(Q_{16}e^{-(2\beta_{[l]}+2\theta+r)T_i} + (Q_{20}+Q_{22}-Q_8)e^{-(\beta_{[j]}+\beta_{[l]}+\theta+r)T_i} + \\ +(Q_{18}-Q_4)e^{-(2\beta_{[j]}+r)T_i} + (Q_1+Q_{12})e^{-\beta_{[j]}T_i} + Q_{23}e^{-(\theta+\beta_{[l]})T_i} + \\ +Q_4+Q_9+Q_{11}+Q_{14}-Q_1-Q8 \Bigg) \times \\ \times \Bigg((Q_7-Q_{19}-Q_{21})e^{-(\beta_{[j]}+\beta_{[l]}+\theta+r)T_i} + (Q_3-Q_{17})e^{-(2\beta_{[j]}+r)T_i} - \\ -Q_{15}e^{-(2\beta_{[l]}+2\theta+r)T_i} - Q_3 - Q_7 - Q_{10} - Q_{13} \Bigg)^{-1} \tag{4.67}$$

The explicit solution for qualities then is obtained with substitution of η^* function, given by the last expression into parametric solutions for $q^{[j],[l]}$ given by (4.65).
The last step in the solution of this problem is to define the time of the trigger, T_i as function of parameters of the system. For that make use of the condition on optimal time of the trigger,

$$\mathcal{H}^{[l]}(q^{[j]}(T_i)^*, q^{[l]}(T_i)^*, \psi_1^{[j]}(T_i)^*, \psi_2^{[j]}(T_i)^*) = \\ = rV_i^{L,CON}(q^{[j]}(T_i)^*, q^{[l]}(T_i)^*)|_{q^{[j]}=q^{[l]}} \tag{4.68}$$

or

$$\mathcal{H}^{[j]}(q^{[j]}(T_i)^*, q^{[l]}(T_i)^*, \psi_1^{[j]}(T_i)^*, \psi_2^{[j]}(T_i)^*) = \\ = rV_i^{F,CON}(q^{[j]}(T_i)^*, q^{[l]}(T_i)^*)|_{q^{[j]}=q^{[l]}} \tag{4.69}$$

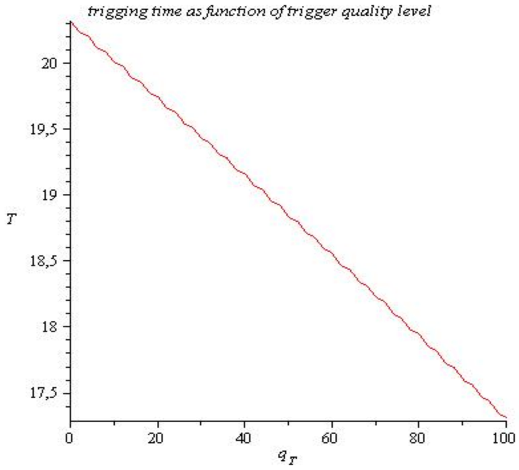
as these two yield exactly the same expression. This is just the condition that optimized hamiltonian has to be equal to the scrap value at the terminal time. In turn, scrap value is given by the value function of the leader-follower problem with qualities' values at the time of the trigger (equal to each other). This expression gives time of the trigger as implicit function of qualities and costates at the trigger time (derived from the expression for player l, (4.69)):

$$T_i : \frac{(1-\beta_{[j]}\psi_1^{[j]}(T_i)^* - \frac{r}{\theta+\beta_{[j]}+r})q^{[j]}(T_i)^*}{(N-i)} - \\ - \frac{((\beta_{[l]}+\theta)\psi_2^{[j]}(T_i)^* + \frac{\theta r}{(r+\beta_{[l]})(\theta+\beta_{[j]}+r)})q^{[l]}(T_i)^*}{(N-i)} + \\ + \frac{1}{2}\gamma_{[j]}^2\psi_1^{[j]}(T_i)^* + \gamma_{[l]}^2\psi_2^{[j]}(T_i)^*\psi_2^{[l]}(T_i)^* - \\ - \frac{1}{2}\frac{(\gamma_{[j]}^2 r^2 + \beta_{[l]}^2\gamma_{[j]}^2 + 2\gamma_{[l]}^2\theta^2 + 2\gamma_{[l]}^2\theta\beta_{[j]} + 2r(\gamma_{[j]}^2\beta_{[l]} + \gamma_{[l]}^2\theta))}{(r+\beta_{[l]})^2(\theta+\beta_{[j]}+r)^2} = 0. \tag{4.70}$$

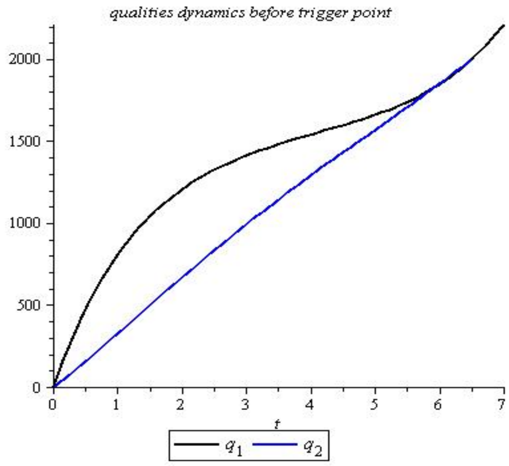
Observe that as $(N-i)$ enters terminal states linearly, as it follows from (4.65), this expression does not depend on $(N-i)$ and hence on i. Then trigger time is the same for all products except for translation time due to the difference in the time of emergence, $t(0)_i$. Hence trigger time is the implicit function of exogenous model's parameters. It cannot be defined explicitly in analytic form, however one may find it for any numeric set of parameters and define the dynamics of system's variables. As an example, take the set of parameters:

(4.71)
$$SET0 := [r=0.01, \theta=0.15, \gamma_{[j]}=0.9, \gamma_{[l]}=0.25, \beta_{[j]}=0.7, \beta_{[l]}=0.1, N=1000, i=1]$$

For this set of parameters the trigger time is the function of trigger quality level:



Quality dynamics are the function of time of the trigger also. With trigger time equal 6.5 (which is the optimal trigger time for this set of parameters) the dynamics is:



This dynamics is produced by the solution set (4.65). First the optimal trigger time from (4.70) is defined and then substitute this trigger time into (4.65) which is defined as a function of the trigger time. This picture demonstrates that initially player's j quality is growing faster then that of the player l (who is the follower before the trigger time) but then it slows down, while followers quality is growing with almost constant speed. This decrease in the leader's speed of quality growth illustrates the increasing influence of quality decay rate in comparison to efficiency of investments which is higher for player j in $SET0$.
Next the characterization of the quality game taking into account the possibility of catching up effect is given.

5.5. Complete Characterization of the Quality Game. When the catching-up occurs in the quality game, the dynamics of qualities after the switch are different from those described in the constant leader-follower case. All value functions and optimal strategies of the players remain the same since they do not depend on time. However, due to the change in initial conditions the quality dynamics is different.

Now one have the following system for qualities:

$$\begin{cases} \dot{q_i^{[j]}}(t) = \frac{\gamma_{[j]}^2(N-i)}{\beta_{[j]}+\theta+r} + \theta q_i^{[l]}(t) - (\beta_{[j]}+\theta) q_i^{[j]}(t); \\ \dot{q_i^{[l]}}(t) = \frac{\gamma_{[l]}^2(N-i)}{\beta_{[l]}+r} - \beta_{[l]} q_i^{[l]}(t). \end{cases}$$

$$q_i^{[j]}(T_i) = q_i^{[l]}(T_i) = Q^{trig} \tag{4.72}$$

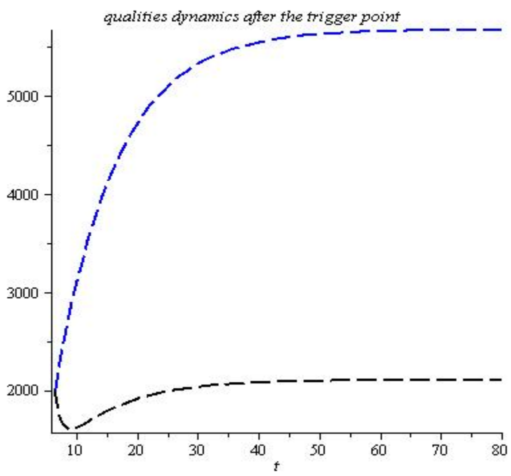
where the differential system is the same as (4.18) with change of leadership from player j to player l and Q^{trig} is given by the quality level achieved by (4.65) at time of the trigger, T_i. This means that the solution for this system will depend not on the time of emergence, $t(0)_i$ but on the trigger time.

Then the quality dynamics after the trigger are:

$$q_i^{[j]}(t)^{ATR} = \frac{(\gamma_{[l]}^2\theta^2 + (r+\beta_{[j]})\gamma_{[l]}^2\theta + (r+\beta_{[l]})\gamma_{[j]}^2\beta_{[l]})}{\beta_{[l]}(r+\beta_{[l]})(\theta+\beta_{[j]}+r)(\beta_{[j]}+\theta)}(N-i) -$$
$$-\frac{\theta(\gamma_{[l]}^2(N-i) - (\beta_{[l]}^2+\beta_{[l]}r)Q^{trig})}{\beta_{[l]}(r+\beta_{[l]})(\theta+\beta_{[j]}-\beta_{[l]})}e^{\beta_{[l]}(T_i-t)} +$$
$$+\Big(\frac{(\beta_{[j]}-\beta_{[l]})Q^{trig}}{\theta+\beta_{[j]}-\beta_{[l]}} -$$
$$-\frac{((\gamma_{[j]}^2(r+\beta_{[l]}) - \gamma_{[l]}^2(r+\beta_{[j]}))\theta + \gamma_{[j]}^2(\beta_{[j]}-\beta_{[l]})(r+\beta_{[l]}) - \gamma_{[l]}^2\theta^2)}{(r+\beta_{[l]})(\theta+\beta_{[j]}+r)(\theta+\beta_{[j]})(\theta+\beta_{[j]}-\beta_{[l]})}(N-i)\Big)\times$$
$$\times e^{(\beta_{[j]}+\theta)(T_i-t)};$$

(4.73)
$$q_i^{[l]}(t)^{ATR} = \frac{\gamma_{[l]}^2}{\beta_{[l]}(r+\beta_{[l]})}(N-i) + \frac{(r+\beta_{[l]})\beta_{[l]}Q^{trig} - \gamma_{[l]}^2(N-i)}{\beta_{[l]}(r+\beta_{[l]})}e^{\beta_{[l]}(T_i-t)}.$$

Here by q^{ATR} the solution for quality after the trigger time when catching-up occurs is denoted. Observe that Q^{trig} is just a constant, provided the T_i is known. This last is found from (4.70) and $t > T_i$ in the last system, hence the solution is stable as all exponents have negative powers. Eventually both solutions reach their respective steady states, q^{STL}, q^{STF} which might be seen from the following illustration (with parameters from SET0):



The complete dynamics for the case of catching-up then consists in two different parts before and after the trigger time. Optimal controls are defined as:

$$g_i^{[j]}(t)^{LF} = \begin{cases} \gamma_{[j]}\sqrt{N-i}\psi_1^{[j]}(t), 0 \le t \le T_i; \\ \frac{\gamma_{[j]}\sqrt{N-i}}{r+\theta+\beta_{[j]}}, t > T_i; \end{cases} \quad g_i^{[l]}(t)^{FL} = \begin{cases} \gamma_{[l]}\sqrt{N-i}\psi_2^{[l]}(t), 0 \le t \le T_i; \\ \frac{\gamma_{[l]}\sqrt{N-i}}{r+\beta_{[l]}}, t > T_i. \end{cases}$$

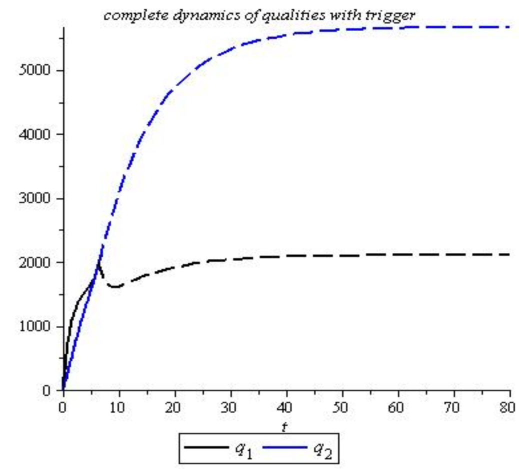
(4.74)

where $\psi_1^{[j]}(t), \psi_2^{[l]}(t)$ are given by (4.57,4.59). Indices FL, LF mark the order of change in positions of players: from leader to the follower and vice versa.
Evolution of qualities:

$$q_i^{[j]}(t)^{LF} = \begin{cases} q_i^{[j]}(t)^{BTR}, 0 \le t \le T_i; \\ q_i^{[j]}(t)^{ATR}, t > T_i. \end{cases} \quad q_i^{[l]}(t)^{FL} = \begin{cases} q_i^{[l]}(t)^{BTR}, 0 \le t \le T_i; \\ q_i^{[l]}(t)^{ATR}, t > T_i. \end{cases}$$

(4.75)

Here q^{ATR} are given by (4.73), solution for qualities after the trigger time and q^{BTR} are given by (4.65), solution for qualities before the trigger time. The complete dynamics of qualities with catching-up is then given by the following picture (with parameters from SET0 as before)



Now recall that the occurrence of the catching-up depends on the combination of parameters. Hence the complete characterization of the game would include all possible variants of quality dynamics and investment strategies. For open-loop equilibria there are exactly four different possible situations for this game. Namely, constant leadership without the switch of one of the players and switch of leadership with player j being the initial leader and player l being the initial leader. The set of possible equilibrium strategies is given by:

$$g_i^{[j]}(t)^{opt} = \begin{cases} g_i^{[j]}(t)^{SYM,CON}, q_i^{[j]}(t) = q_i^{[l]}(t), \theta = 0 \forall t; \\ g_i^{[j]}(t)^{SYM,\theta,CON}, q_i^{[j]}(t) = q_i^{[l]}(t), \theta > 0 \forall t; \\ g_i^{[j]}(t)^{F,CON}, q_i^{[j]}(t) < q_i^{[l]}(t) \forall t; \\ g_i^{[j]}(t)^{L,CON}, q_i^{[j]}(t) > q_i^{[l]}(t) \forall t; \\ g_i^{[j]}(t)^{LF}, \\ g_i^{[j]}(t)^{FL}. \end{cases}$$

(4.76)

Here two last options are given by (4.74) and three others being defined from the leader-follower problem and the symmetric case. Note, that the symmetric case strategy for $\theta = 0$ is not the equilibrium one, since there are no imitation in there

and no game occurs. The same set of strategies is valid for the second player with his parameters inserted.
Associated with this set of equilibrium strategies are the following quality evolution paths:

$$q_i^{[j]}(t)^{opt} = \begin{cases} q_i^{[j]}(t)^{SYM,CON}, q_i^{[j]}(t) = q_i^{[l]}(t), g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{SYM,CON}, \theta = 0 \forall t; \\ q_i^{[j]}(t)^{SYM,\theta,CON}, q_i^{[j]}(t) = q_i^{[l]}(t), g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{SYM,\theta,CON}, \theta > 0 \forall t; \\ q_i^{[j]}(t)^{L,CON}, q_i^{[j]}(t) > q_i^{[l]}(t), g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{L,CON}, \forall t > 0; \\ q_i^{[j]}(t)^{F,CON}, q_i^{[j]}(t) < q_i^{[l]}(t), g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{F,CON}, \forall t > 0; \\ q_i^{[j]}(t)^{LF,CON}, g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{LF}; \\ q_i^{[j]}(t)^{FL,CON}, g_i^{[j]}(t)^{opt} = g_i^{[j]}(t)^{FL}. \end{cases} \tag{4.77}$$

Here $q_i^{[j]}(t)^{SYM,CON}$ is given by (4.37) and $q_i^{[j]}(t)^{SYM,\theta,CON}$ is given by (4.38), both from the section on symmetric outcome of the game. In these cases both qualities are equal all the time and the first regime yields higher quality levels. The exact formal conditions for this type of dynamics to happen are (4.28): efficiency and decay parameters of both players have to be similar. If imitation speed is zero, the first type of dynamics takes place, if the imitation speed is positive - the second one.

Those types of dynamics being labeled as $q_i^{[j]}(t)^{L,CON}, q_i^{[j]}(t)^{F,CON}$ are defined by (4.18) for cases of follower's and leader's constant positions. Note that this does say anything concerning the uniqueness of steady-states. It may be the case that two steady-states exist on both sides off the diagonal $q_i^{[j]}(t) = q_i^{[l]}(t)$ while there is a constant leader in the quality dynamics. The condition for constant leadership of the player j with no catching-up is given by the set

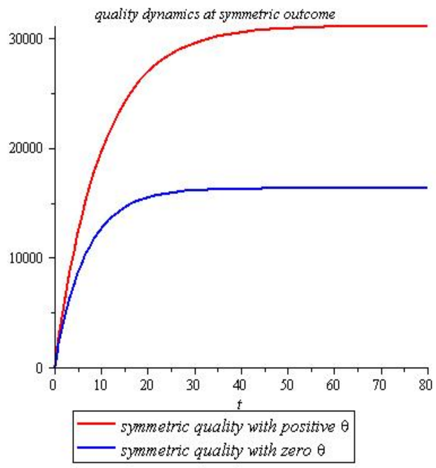
$$\begin{aligned} \beta_{[l]}(\beta_{[l]} + r + \theta)\gamma_{[j]}^2 &> \beta_{[j]}(\beta_{[j]} + r)\gamma_{[l]}^2; \\ (\beta_{[l]} + r + \theta)\gamma_{[j]}^2 &> (\beta_{[j]} + r + \theta)\gamma_{[l]}^2. \end{aligned} \tag{4.78}$$

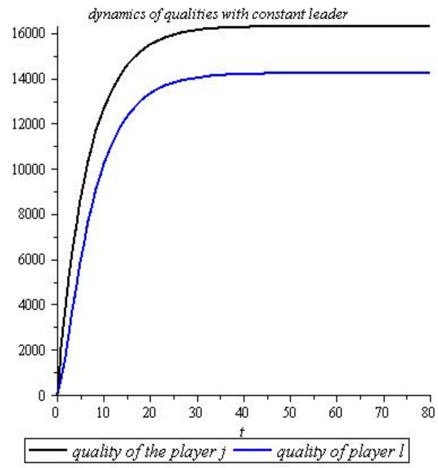
which is derived in the section on piecewise formulation of strategies.
At last, $q_i^{[j]}(t)^{LF,CON}, q_i^{[j]}(t)^{FL,CON}$ are defined by (4.65, 4.73) with the order of change from leader's to follower's or vice versa position defined by the initial leadership as the type of dynamics in the precense of the catching-up effect. Conditions on parameters for catching-up to happen are given in (4.48).

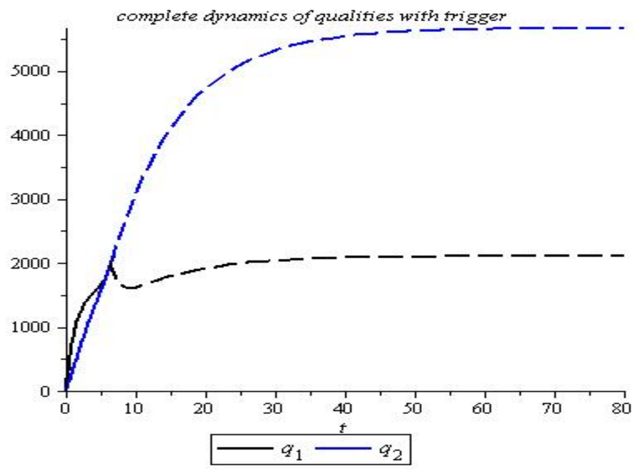
Below cases for catching-up and no catching-up as well as symmetric outcome are displayed for parameter values given below:

$$\begin{aligned} SETLF &:= [r = 0.01, \theta = 0.15, \gamma_{[j]} = 0.9, \gamma_{[l]} = 0.25, \beta_{[j]} = 0.7, \beta_{[l]} = 0.1, N = 1000, i = 1] \\ SETSYM1 &:= [r = 0.01, \theta = 0.15, \gamma_{[j]} = 0.9, \gamma_{[l]} = 0.9, \beta_{[j]} = 0.1, \beta_{[l]} = 0.1, N = 1000, i = 1] \\ SETSYM2 &:= [r = 0.01, \theta = 0, \gamma_{[j]} = 0.9, \gamma_{[l]} = 0.9, \beta_{[j]} = 0.1, \beta_{[l]} = 0.1, N = 1000, i = 1] \\ SETCON &:= [r = 0.01, \theta = 0.15, \gamma_{[j]} = 0.9, \gamma_{[l]} = 0.5, \beta_{[j]} = 0.15, \beta_{[l]} = 0.1, N = 1000, i = 1] \end{aligned} \tag{4.79}$$

The first set is the same set of parameters as $SET0$ where the catching-up occurs. The second one corresponds to the symmetric case where both players behave themselves as followers and differs from the first by lower β. The third set is the same as the second but with zero imitation speed and the last one concerns the constant leadership of player j case.







Note, that due to the special form of efficiency of investments across products, $\gamma(\bullet) = \gamma_{[j],[l]} \times \sqrt{N-i}$, there is no possibillity for catching-up to occur only in some products' qualities and not to occur in others. This happens due to the fact that the term $(N-i)$ enters controls of both players in the same way, hence they are modified in accordance to the product's index in the same way. Also consider the conditions for leadership and catching-up in the subsection Piecewise Formulation of Strategies. These conditions do not depend on the product index, i as well as shadow costs of investments (costate variables in catching-up situation and value function coefficients with no catching-up). Hence, once it is determined whether the catching-up or constant leadership of one of the players occur for some i, it occurs for any i also.

The main observation coming from the comparison of dynamics of qualities in different modes is following: the steady-state level of qualitites being reached by both players is lower in the catching-up mode then in others. This happens because the efficiency of investments of the final leader (which is player l on the picture) is much less then in other cases. At the same time note, that only the significant difference in efficiency functions may lead to the catching-up situation. Hence the player who is less efficient as an investor, may become a leader because of the imitation effect. Then both players would achieve steady-state levels of quality less then in constant leadership or symmetric modes. However this does not mean per se that this mode is less efficient in terms of the achieved quality levels, as the formulation of steady-state values remains the same as in constant leadership mode and is given by (4.43). It is the combination of parameters' values which yields such a result. Exact conditions on parameters for catching-up or constant modes to realize are derived above. In economic terms one may say that catching-up happens when the initial leader in the development of qualities cannot sustain the imitation from the follower because starting from some point in product development he cannot increase his quality further on due to increased costs of preserving the achieved level of quality.

Observe the drastic difference in levels of quality for symmetric case with positive and zero imitation speed. The incentive to benefit from imitation pushes both players into the position of minimizing their investments. One may observe that steady-state levels of qualities for the symmetric case with positive θ are less then steady-state level of quality of the follower in the constant leadership mode. This would not happen if the imitation may be costly. If it is costless as in the current model, it threatens the development of quality of the product as long as the difference between players is not substantial (they are close to symmetric situation) as well as if there is no clear dominance in terms of efficiency parameters (γ, β) of one player in comparison to another. However this imitation effect stimulates the variety expansion as it will be seen later on.

To proceed to the solution of $n(t)$ problem value functions $V_{[j]}(0,0)\mid_{(i=n(t))}$, $V_{[l]}(0,0)\mid_{(i=n(t))}$ are needed. They result from open-loop solution for the quality growth subgame which differ only in parameters and are constant in time. So to complete the characterization of the catching-up mode of the quality game value functions of both players for catching-up case are also needed. Observe, that the value function computed at zero quality level at the time of emergence of product i includes all the evolution of qualities and associated controls. Here only with the open-loop case is considered, so the value function for each player may be computed as the optimized hamiltonian at time of emergence (or, equivalently, at zero time level). In what follows player j is assumed to be the leader before the catching-up occurs:

$$V_{[j]}^{LF}(0,0)_{i=n(t)} = \frac{1}{r}\mathcal{H}_{[j]}^{LF}(0,0) =$$
$$= \frac{1}{r}\Big(\frac{1}{2}\gamma_{[j]}^2(N-i)(\psi_1^{[j]}(0))^2 + \gamma_{[l]}^2(N-i)\psi_2^{[j]}(0)\psi_2^{[l]}(0)\Big);$$
$$V_{[j]}^{FL}(0,0)_{i=n(t)} = \frac{1}{r}\mathcal{H}_{[j]}^{FL}(0,0) =$$
$$= \frac{1}{r}\Big(\frac{1}{2}\gamma_{[j]}^2(N-i)(\psi_2^{[j]}(0))^2 + \gamma_{[l]}^2(N-i)\psi_1^{[j]}(0)\psi_1^{[l]}(0)\Big);$$
$$V_{[l]}^{LF}(0,0)_{i=n(t)} = \frac{1}{r}\mathcal{H}_{[l]}^{LF}(0,0) =$$
$$= \frac{1}{r}\Big(\frac{1}{2}\gamma_{[l]}^2(N-i)(\psi_1^{[l]}(0))^2 + \gamma_{[j]}^2(N-i)\psi_2^{[l]}(0)\psi_2^{[j]}(0)\Big);$$
$$V_{[l]}^{FL}(0,0)_{i=n(t)} = \frac{1}{r}\mathcal{H}_{[l]}^{FL}(0,0) =$$
$$= \frac{1}{r}\Big(\frac{1}{2}\gamma_{[l]}^2(N-i)(\psi_2^{[l]}(0))^2 + \gamma_{[j]}^2(N-i)\psi_1^{[l]}(0)\psi_1^{[j]}(0)\Big). \tag{4.80}$$

Provided the costate values are defined from (4.57,4.59) with $t = 0$, these last two expressions are functions of system's parameters and the time of the trigger, T_i which is in turn defined from (4.70). Observe that value functions for both players include i and thus $n(t)$ only linearly. Hence the structure of these value functions may be described by

$$V_{[j]}^{CATCH}(0,0)_{i=n(t)} = C_{[j]}^{CATCH} \times (N-i);$$
$$V_{[l]}^{CATCH}(0,0)_{i=n(t)} = C_{[l]}^{CATCH} \times (N-i) \tag{4.81}$$

where $CATCH$ is in $\{FL, LF\}$. Then the value function for each of the players is given in 4 possible variants:

$$V_{[j]}(0,0)\mid_{(i=n(t))}= \begin{cases} \frac{\gamma_{[j]}}{(r+\beta_{[j]})^2 r}(N-n(t)), q^{[j]}(i,t) \geq q^{[l]}(i,t), \forall t>0; \\ \frac{1}{2}\frac{r^2\gamma_{[j]}^2 + r(2\gamma_{[j]}^2\beta_{[l]} + 2\gamma_{[l]}^2\theta) + \gamma_{[j]}^2\beta_{[l]}^2 + 2\theta\gamma_{[l]}^2\beta_{[j]} + 2\theta^2\gamma_{[l]}^2}{r(r+\beta_{[l]})^2(r+\theta+\beta_{[j]})^2} \times \\ \times (N-n(t)), q^{[j]}(i,t) < q^{[l]}(i,t), \forall t>0; \\ C_{[j]}^{LF} \times (N-n(t)); \\ C_{[j]}^{FL} \times (N-n(t)). \end{cases}$$

$$V_{[l]}(0,0)\mid_{(i=n(t))}= \begin{cases} \frac{\gamma_{[l]}}{(r+\beta_{[l]})^2 r}(N-n(t)), q^{[l]}(i,t) \geq q^{[j]}(i,t), \forall t>0; \\ \frac{1}{2}\frac{r^2\gamma_{[l]}^2 + r(2\gamma_{[l]}^2\beta_{[j]} + 2\gamma_{[j]}^2\theta) + \gamma_{[l]}^2\beta_{[j]}^2 + 2\theta\gamma_{[j]}^2\beta_{[l]} + 2\theta^2\gamma_{[j]}^2}{r(r+\beta_{[j]})^2(r+\theta+\beta_{[l]})^2} \times \\ \times (N-n(t)), q^{[l]}(i,t) < q^{[j]}(i,t), \forall t>0; \\ C_{[j]}^{LF} \times (N-n(t)); \\ C_{[j]}^{FL} \times (N-n(t)). \end{cases} \tag{4.82}$$

Note that the value functions, although constant in time, differ between modes.

5.6. On Closed-Loop Strategies. Now consider closed-loop strategies in the quality game once again. To demonstrate that there are no closed-loop strategies of at most linear-feedback form except for those piecewise-constant, derived above the discussion proceeds in two stages. First it is demonstrated that in the situation with unique steady state without catching-up no linear-feedback strategy is stable and hence cannot be the equilibrium. Next the situation with two different steady states is considered and it is demonstrated that there are no stable linear-feedback strategies which would solve the equilibrium selection problem. And at last the catching-up situation is considered and it is demonstrated that there are no closed-loop strategy of liner-feedback type before the trigger point. Hence the absence of linear-feedback strategies in all cases of the game is demonstrated.

Observe that the quality game has 3 different regimes which are defined by the relation between parameters of the players. First regime corresponds to the case when (4.44) is fulfilled,

$$\beta_{[l]}(\beta_{[l]} + r + \theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]} + r)\gamma_{[l]}^2 \tag{4.83}$$

but the relation

$$\beta_{[l]}(\beta_{[l]} + r)\gamma_{[j]}^2 < \beta_{[j]}(\beta_{[j]} + r + \theta)\gamma_{[l]}^2. \tag{4.84}$$

does not hold. Then there is only one steady state in the game with player j being the leader and player l being the follower. If in addition,

$$(\beta_{[l]} + r + \theta)\gamma_{[j]}^2 > (\beta_{[j]} + r + \theta)\gamma_{[l]}^2 \tag{4.85}$$

holds, then no catching-up is observed. In such a situation the game is resolved by the means of the HJB approach as it is done in the subsection on constant leader-follower case. It has been demonstrated that if one restricts himself to at most linear-feedback strategies, the only stable pair of strategies is

$$g^{[j]}_{L,CON} = \frac{\gamma_{[j]}\sqrt{(N-i)}}{r+\beta_{[j]}} = const;$$
$$g^{[l]}_{F,CON} = \frac{\gamma_{[l]}\sqrt{N-i}}{r+\theta+\beta_{[l]}} = const;$$
$$\forall t : q^{[l]}_i(t) < q^{[j]}_i(t). \tag{4.86}$$

which is of the piecewise-constant form. Any set of linear-feedback strategies, derived from HJB equations is unstable and violates transversality conditions as it has been shown in that section. This pair constitutes the open-loop equilibrium as well as the closed-loop one, but the last only in the case of unique steady state.

Consider now the situation with two steady states, one with player j being the leader and the other with player l being the leader without the catching up. It has been argued that in such a situation the player whose steady-state level of quality while being the leader is higher will be the leader. So then one may compare steady-state levels of quality to select the leader. Observe however, that such an approach is viable only for open-loop equilibrium, but not for the closed-loop one. If one would consider the same pair of strategies as before in the closed-loop context, one has to account for possible deviations of both players from their strategies. Observe that it is always more profitable in terms of additional value being generated to be the follower in the quality game. To see that assume both players start at the $(0, 0)$ position in qualities as it is required by initial conditions of the model. Then both of them will have incentives to invest less then the other player to become the follower. This illustrates the fact that value functions of both players in linear form are not continuous along the diagonal in the state-space unless speed of imitation is equal zero. Now assume any other position in the state-space, $(q^{[j]}_i \neq q^{[l]}_i)$ out off the diagonal. There are two possible steady states present, one with $q^{[j]}_i = q^{STL}_i$ and the other with $q^{[l]}_i = q^{STL}_i$ on different sides from the diagonal. Then irrespective of the given position in the state space any given player will try to reach the steady state where he is the follower as long as $V^{F,CON} > V^{L,CON}$. Then eventually quality levels will reach the diagonal again, where linear value functions are not continuous.

Hence the situation with two steady states leads to the catching-up situation in the closed-loop case which cannot be described by the linear value functions and hence by the piecewise-constant strategy set. The same is true for the catching-up case in general. At any given quality level on the diagonal it is profitable for any player to deviate from the given piecewise-constant strategy into the region where this player is the follower and the same is true for the other player. Hence the piecewise-constant strategy pair, which constitutes the open-loop equilibrium cannot constitute the closed-loop one in the situation with two stable steady states.

Now consider the situation with catching-up. In open-loop case one may find the set of strategies before the trigger through maximum principle as it has been done in the subsection on catching-up. Observe that the closed-loop situation would require the HJB approach for the strategy pair before the trigger. Even if such a pair of strategies may be found, the point of the trigger, $q^{[j]}_i = q^{[l]}_i = q^{TRIG}$ lies on the diagonal and there are two different possible steady states from then further on. In open-loop case one may choose one of these, while in closed-loop situation it is not possible, since the value functions of the leader and the follower are never equal unless the $\theta = 0$. Hence the strategy of the form (4.74) cannot constitute the closed-loop equilibrium.

One may conclude that in the given quality game there is only one case where the piecewise-constant strategies set may constitute closed-loop equilibrium of Markovian Nash type: this is the case of unique steady-state. Observe now, that the catching-up may occur even if there is only one steady-state. This corresponds to the set of conditions:

$$\begin{aligned} &\beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2; \\ &\beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2; \\ &(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 < (\beta_{[j]}+r+\theta)\gamma_{[l]}^2. \end{aligned} \tag{4.87}$$

In this case some closed-loop equilibrium may occur, but they are not defined here since it is unclear what form value functions of players may have in this case. One may conclude, that closed-loop equilibrium of the game is defined in the form of piecewise-constant strategies for the case of unique steady-state levels of qualities and is not defined for the case of two different ones. Note that this does not mean that there do not exist closed-loop strategies at all, but only that closed-loop equilibrium in the form of at most linear-feedback strategies is defined only for the case of single steady state of the game.

6. Variety Expansion Problem

Variety expansion problem is the differential game with one state and two controls. Both players invest simultaneously in the variety expansion and benefit from the resulting variety on common base thus sharing all the information on this level of innovations. The dynamic problem for both players is to maximize the potential output of innovations given the costs of investments. Note that the potential profit on this level consists only from the future accumulated profit from development of quality of newly invented products so the same logic as in the monopolist case applies here. The quality game is described by means of the open-loop strategies, but there is also the case where the set of piecewise-constant strategies constitutes the closed-loop equilibrium. Hence in this section first the open-loop equilibrium for variety expansion game which coressponds to open-loop equilibrium of the quality game above is considered. For that the maximum principle approach is used. After this the HJB approach is employed to derive the closed-loop equilibrium for the variety expansion game which is associated with the closed-loop equilibrium of the quality gamee (picewise-constant strategies with unique stable steady state).

6.1. Open-Loop Solution. This subsection is devoted to the characterization of the open-loop solution for the variety expansion game. For that the maximum principle method is convenient. The reformulation of the objective functional is used in the same way as in the patenting case, (3.2), except for now one have two objective functionals and infinite-time horizon for qualities also. One may rewrite the problem of variety expansion as following:

$$\begin{aligned} J^{[j]} &= \int_0^\infty e^{-rt}\Big((\alpha_{[j]}u_{[j]}(t)+\alpha_{[l]}u_{[l]}(t))V_{[j]}(0,0)\mid_{(i=n(t))} -\frac{1}{2}u_{[j]}(t)^2\Big)dt \to max; \\ J^{[l]} &= \int_0^\infty e^{-rt}\Big((\alpha_{[j]}u_{[j]}(t)+\alpha_{[l]}u_{[l]}(t))V_{[l]}(0,0)\mid_{(i=n(t))} -\frac{1}{2}u_{[l]}(t)^2\Big)dt \to max; \\ &s.t. \\ \dot{n}(t) &= \alpha_{[j]}u_{[j]}(t)+\alpha_{[l]}u_{[l]}(t); \\ &u_{[j]}(t), u_{[l]}(t) \geq 0, \forall t \geq 0. \end{aligned} \tag{4.88}$$

where $V_{[j]}(0,0)\mid_{(i=n(t))}, V_{[l]}(0,0)\mid_{(i=n(t))}$ are given by (4.82) and depend on $n(t)$ linearly.

Denote the value functions from the quality game by

$$
\begin{aligned}
V_{[j]}(0,0) \mid_{(i=n(t))} &= C_v^{[j]} \times (N - n(t));\\
V_{[l]}(0,0) \mid_{(i=n(t))} &= C_v^{[l]} \times (n - n(t)).
\end{aligned} \tag{4.89}
$$

The constant part may vary depending on the leadership in the quality game, but the variety expansion is analyzed parametrically and then the dynamics corresponding to different regimes of the quality game is compared. This may be done since these constant parts of value functions above do not depend on the state variable and controls or time. This constitutes the one-state differential game with common state constraint which may be solved using standard techniques. First construct hamiltonians of the given problem:

$$
\begin{aligned}
\mathcal{H}^{[j]} &= (\alpha_{[j]}u_{[j]}(t) + \alpha_{[l]}u_{[l]}(t))C_v^{[j]}(N - n(t)) - \frac{1}{2}u_{[j]}(t)^2 + \lambda_{[j]}(t)(\alpha_{[j]}u_{[j]}(t) + \alpha_{[l]}u_{[l]}(t));\\
\mathcal{H}^{[l]} &= (\alpha_{[j]}u_{[j]}(t) + \alpha_{[l]}u_{[l]}(t))C_v^{[l]}(N - n(t)) - \frac{1}{2}u_{[l]}(t)^2 + \lambda_{[j]}(t)(\alpha_{[j]}u_{[j]}(t) + \alpha_{[l]}u_{[l]}(t)).
\end{aligned} \tag{4.90}
$$

Now derive first-order conditions on controls:

$$
\begin{aligned}
\frac{\partial \mathcal{H}^{[j]}}{\partial u_{[j]}} &= 0 : u^*_{[j]} = \alpha_{[j]}\lambda_{[j]}(t) + C_v^{[j]}(N - n(t));\\
\frac{\partial \mathcal{H}^{[l]}}{\partial u_{[l]}} &= 0 : u^*_{[l]} = \alpha_{[l]}\lambda_{[l]}(t) + C_v^{[l]}(N - n(t)).
\end{aligned} \tag{4.91}
$$

Substituting these into hamiltonian functions and writing down costate equations yield the canonical system for the variety expansion game:

$$
\begin{aligned}
\dot{\lambda_{[j]}} &= r\lambda_{[j]} - \frac{\partial \mathcal{H}^{[j]}}{\partial n(t)} =\\
&= (r + \alpha^2_{[j]}C_v^{[j]})\lambda_{[j]}(t) + \alpha^2_{[l]}C_v^{[j]}\lambda_{[l]}(t) + (\alpha^2_{[j]}(C_v^{[j]})^2 + \alpha^2_{[l]}C_v^{[j]}C_v^{[l]})(N - n(t));\\
\dot{\lambda_{[l]}} &= r\lambda_{[l]} - \frac{\partial \mathcal{H}^{[l]}}{\partial n(t)} =\\
&= (r + \alpha^2_{[l]}C_v^{[l]})\lambda_{[l]}(t) + \alpha^2_{[j]}C_v^{[l]}\lambda_{[j]}(t) + (\alpha^2_{[l]}(C_v^{[l]})^2 + \alpha^2_{[j]}C_v^{[j]}C_v^{[l]})(N - n(t));\\
\dot{n(t)} &= \alpha^2_{[j]}\lambda_{[j]}(t) + \alpha^2_{[l]}\lambda_{[l]}(t) + (\alpha^2_{[j]}(C_v^{[j]})^2 + \alpha^2_{[l]}(C_v^{[l]})^2)(N - n(t));\\
n(0) &= n_0;\\
lim_{t\to\infty}e^{-rt}\lambda_{[j]}(t) &= 0;\\
lim_{t\to\infty}e^{-rt}\lambda_{[l]}(t) &= 0.
\end{aligned} \tag{4.92}
$$

This constitutes the system of linear ODEs with one initial condition and two boundary conditions (transversal ones) which is then solved. The solution is:

$$n(t)^* = N - (N-n_0)e^{\frac{1}{2}(r-\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})t};$$

$$\lambda_{[j]}(t)^* = -\frac{C^{[j]}_v(\alpha^2_{[j]}C^{[j]}_v+\alpha^2_{[l]}C^{[j]}_v)}{2\alpha^2_{[j]}C^{[j]}_v+r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)}}\times$$
$$\times 2(N-n_0)e^{\frac{1}{2}(r-\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})t};$$

$$\lambda_{[l]}(t)^* = -\frac{C^{[l]}_v(\alpha^2_{[j]}C^{[j]}_v+\alpha^2_{[l]}C^{[j]}_v)}{2\alpha^2_{[j]}C^{[j]}_v+r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)}}\times$$
$$\times 2(N-n_0)e^{\frac{1}{2}(r-\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})t}. \tag{4.93}$$

Due to the nature of the problem analyzed here the costates' dynamics is negative. This happens because in the form the problem is reformulated in this section, every agent cares only about future investments into variety expansion. From this point of view shadow price of investments is negative since every marginal addition to investments reduces future possibilities to invest. This happens because one has bounded product space in the model and inventions reduce the dimensionality of this space. Agents take into account the profit generated only by the next potential product but neglect all the products which are already invented before. Hence the shadow price of investing into the expansion of products variety is negative. Still, investments are positive for both agents as well as the growth of variety. Explicit formulation of investments into variety expansion for both players hence is:

$$u_{[j]}(t)^* = \frac{\alpha_{[j]}C^{[j]}_v(r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})}{2\alpha^2_{[j]}C^{[j]}_v+2\alpha^2_{[l]}C^{[l]}_v+r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)}}\times$$
$$\times(N-n_0)e^{\frac{1}{2}(r-\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})t};$$

$$u_{[l]}(t)^* = \frac{\alpha_{[l]}C^{[l]}_v(r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})}{2\alpha^2_{[j]}C^{[j]}_v+2\alpha^2_{[l]}C^{[l]}_v+r+\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)}}\times$$
$$\times(N-n_0)e^{\frac{1}{2}(r-\sqrt{r(r+4\alpha^2_{[j]}C^{[j]}_v+4\alpha^2_{[l]}C^{[l]}_v)})t}. \tag{4.94}$$

These are completely symmetric except for the term $\alpha_{[j,l]}C^{[j,l]}_v$ which depends on investment efficiencies and value generated by the game in quality of the next product to be invented of both players. Hence relative scale of investments into the variety expansion depends on the outcome of the quality game.

One also may compute the value function of the variety expansion game as the optimized hamiltonian function. Observe that this value function does not exhaust all the value generated for each of the players within the model as it includes only the value of the quality game for the next product to be invented but not all of the products which are already invented for any time t. The value of the variety expansion game for each of the players is the respective hamiltonian function, (4.90) at time $t = 0$ and optimal costate and variety values, given by (4.93):

$$V_{[j]}(n) = \frac{1}{r}\mathcal{H}^{[j]}(n(0)^*, \lambda_{[j]}(0)^*) =$$
$$= C_v^{[j]} \times \frac{(N-n_0)^2(2\alpha_{[l]}^2 C_v^{[l]} + \alpha_{[j]}^2 C_v^{[j]})}{2\alpha_{[l]}^2 C_v^{[l]} + 2\alpha_{[j]}^2 C_v^{[j]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}};$$

$$V_{[l]}(n) = \frac{1}{r}\mathcal{H}^{[l]}(n(0)^*, \lambda_{[l]}(0)^*) =$$
$$= C_v^{[l]} \times \frac{(N-n_0)^2(2\alpha_{[j]}^2 C_v^{[j]} + \alpha_{[l]}^2 C_v^{[l]})}{2\alpha_{[l]}^2 C_v^{[l]} + 2\alpha_{[j]}^2 C_v^{[l]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}}. \tag{4.95}$$

It can be seen that value function of the variety expansion game is just the combination of efficiencies of investments into the variety expansion, $\alpha_{[j],[l]}$ and value functions of the quality game of both players. Observe that these are independent of the index of the product, i, as they include estimation of value generation of all products to be invented at the point $i = n_0$. Hence these two functions give the total value of the combined game of quality growth and variety expansion. However, they may take different values depending on the leadership regime in quality game. Due to the special and symmetric form of investment efficiencies $\gamma(\bullet)$ the same regime is preserved for all product indices i and values of $C_v^{[j]}, C_v^{[l]}$ are independent of i. As long as shadow costs of investments in quality game are independent of i, optimal investments for both players depend on i in the same way. Then value functions and quality dynamics will depend on i also in the same way for all regimes and hence conditions for realisation of one or another regime are independent on i also.

This of course is not necessarily the case with more general (e.g. defined differently for different i) specification of efficiency functions $\gamma_{j,l}(i)$. No claims concerning general properties of these functions are made here. One would stop on the conclusion that with the adopted specification of $\gamma(\bullet)$ functions the regime of leadership in quality game is constant across products and hence the value function for the variety expansion part may be defined independently on i or $n(t)$.
Now turn to the analysis of the influence of regime of the quality game onto the dynamics of variety expansion investments. As it has been mentioned the quality game may have four different solutions depending on the combination of parameters. Variety expansion policies of both players depend on value functions generated by the quality game. Hence one has four possible outcomes in the variety expansion game and in the overall model also.

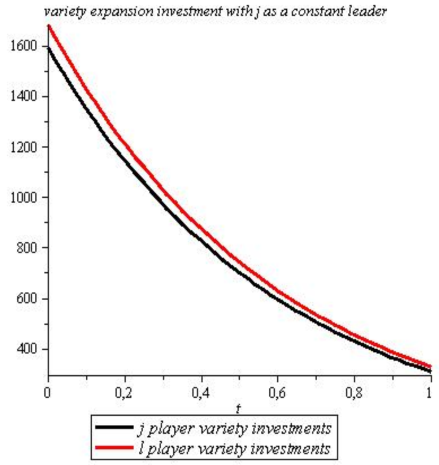
For illustration of the difference in investment policies caused by different leader-follower patterns take 4 different sets of parameters which correspond to leadership of player j and l and to the catching-up situation with player j and player l as initial leaders in quality growth respectively. Efficiency of investments into variety expansion is assumed to be equal for both players:

$$SETJL := [\gamma_{[j]} = 0.4, \gamma_{[l]} = 0.7, \beta_{[j]} = 0.2, \beta_{[l]} = 0.2];$$
$$SETLL := [\gamma_{[j]} = 0.7, \gamma_{[l]} = 0.4, \beta_{[j]} = 0.2, \beta_{[l]} = 0.2];$$
$$SETJIL := [\gamma_{[j]} = 0.9, \gamma_{[l]} = 0.25, \beta_{[j]} = 0.7, \beta_{[l]} = 0.2];$$
$$SETLIL := [\gamma_{[j]} = 0.25, \gamma_{[l]} = 0.9, \beta_{[j]} = 0.2, \beta_{[l]} = 0.7].$$

with

$$[n_0 = 1, \alpha_{[j]} = \alpha_{[l]} = 0.5, r = 0.01, \theta = 0.15]$$

for all variants.
With constant leadership in qualities variety expansion investments are rather large and the rate of investments decreases rapidly until zero. The player who is the leader in quality growth invests less then the follower:



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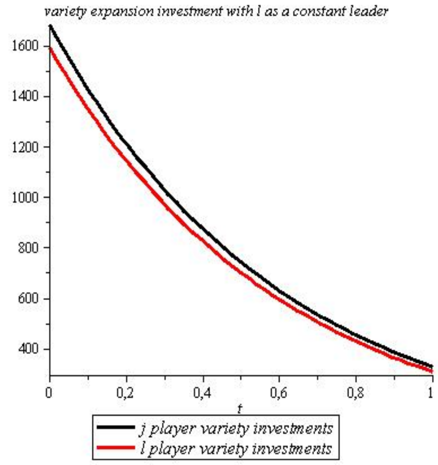

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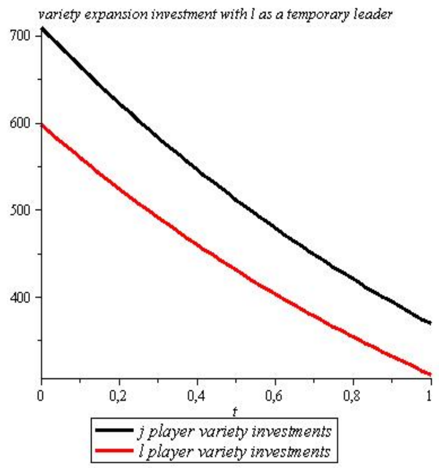
With the catching-up situation in quality growth investments into variety expansion are much slower then with constant leadership. The player which is the initial leader but is the follower in the long-term invests less then the other.



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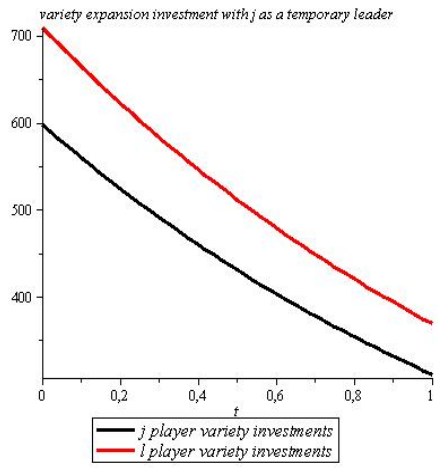

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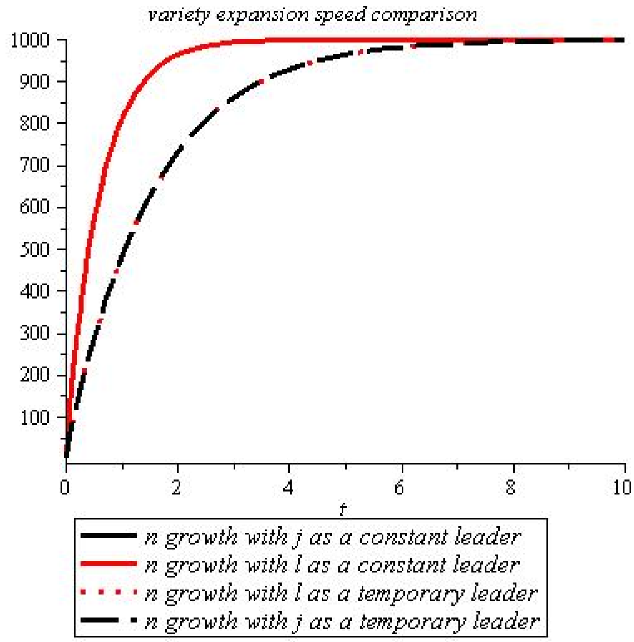
As a result variety expansion is much slower with the catching-up situation then with constant leadership:



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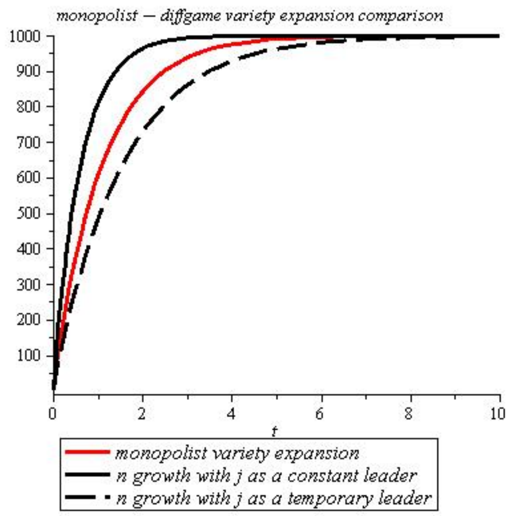
Here one observes only two lines instead of four because of the symmetric efficiencies of investments into variety expansion being assumed for both players. Hence irrespective of the leadership of one or the other player the total expansion of variety is the same. Then one may observe that with constant leadership of one or the other player the dynamics of variety expansion coincide. The same is true for the catching-up situation. One may observe only the difference between variety expansion speed caused by catching-up and constant leadership situations.

Such a significant difference in variety expansion rates follows from the fact that value functions of quality game are much lower for the catching-up situation for the constant leadership case for both players. This might be interpreted as an effect of competition between players. Hence when the catching-up is possible, the variety of products is expanding much slower and qualities grow to lower steady-state levels for all invented products. Less resources are devoted to both types of investments. If both players anticipate that no catching-up is possible, they devote more resources to the quality growth and eventually reach their respective steady states while expanding the available variety of products faster. One may claim that with open-loop strategies imitation in quality growth may have ambigious effect on the variety expansion, depending on the intensity of competition between players. It is also interesting to compare the resulting variety expansion paths of the differential game with that of the monopolist. For that take the set of parameters for a monopolist as an average from parameters for both of the players from the SETJL:

$$SETM := [\alpha = 0.5, \gamma = 0.55, \beta = 0.2, r = 0.01, N = 1000, n_0 = 1].$$

This parameter set is an average of efficiencies and decay rates of both players. It may be seen that the monopolist's variety expansion is slower then for differential game with constant leadership. However, it is faster then for the case of catching-up:



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Then one may conclude that the cooperation between players in variety expansion yields higher aggregate growth of variety expansion then that of monopolist only if the players preserve their leader-follower positions. In this case there is substantial difference between their investments into quality growth and hence the imitation effect is high enough to boost variety expansion investments of the player which is the follower. At the same time if the catching-up effect is observed, the imitation does not provide enough incentives to boost variety expansion as this effect is present for both players (before or after the trigger T_i) and the whole variety expansion process is slowing down due to the problem of free-riding. Every player does not benefit enough from the imitation effect and in turn, has less incentives to invest in variety. At the same time it has to be noted that due to the open-loop nature of the investment strategies analyzed here both players invest non-zero amounts into variety expansion irrespective of their positions in the quality game. It will be shown that this is not the case in closed-loop situation.

6.2. Closed-Loop Solution. One may consider the closed-loop strategies for variety expansion. Observe that this may be valid only for the case where single steady state exists in the quality game, that one with the leadership of one or another player. In such a situation the given set of piecewise-constant strategies constitutes closed-loop equilibrium of the Markovian Nash type in the quality game. This follows from the discussion on closed-loop equilibria existence in the end of the section on quality game. The situation when there is only one stable steady state in the quality game is described by the set of conditions on steady-state levels of qualitites for both players. Observe however, that even the situation with the single stable steady state in piecewise strategies for the quality game permits catching-up strategies. Hence there might be some closed-loop equilibria of the quality game with piecewise-constant strategies of both players without catching-up but with some other type of investments policy in the precense of catching-up. However, this is not considered in this work because there is no clear way to define closed-loop strategies with catching-up which are not of linear-feedback type for the quality game. More discussion on this may be found in the end of section 5 of this chapter. Hence consider for closed-loop equilibrium of the variety expansion game associated with the closed-loop (Markovian Nash) equilibrium of the quality game only the situation with single steady state and no catching-up. This is characterized by the set of conditions:

$$\beta_{[l]}(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]}+r)\gamma_{[l]}^2;$$
$$\beta_{[l]}(\beta_{[l]}+r)\gamma_{[j]}^2 \geq \beta_{[j]}(\beta_{[j]}+r+\theta)\gamma_{[l]}^2;$$
$$(\beta_{[l]}+r+\theta)\gamma_{[j]}^2 > (\beta_{[j]}+r+\theta)\gamma_{[l]}^2. \tag{4.96}$$

Which describes the situation of single steady state with no catching-up. Above player j is assumed to be the (constant) leader of the quality game.
Hence, the value function of the quality game for any player may take only two possible values:

$$V_{[j]}(0,0)\mid_{(i=n(t))}=\begin{cases}\frac{\gamma_{[j]}}{(r+\beta_{[j]})^2 r}(N-n(t)), q^{[j]}(i,t)\geq q^{[l]}(i,t)\forall t;\\ \\ \frac{1}{2}\frac{r^2\gamma_{[j]}^2+r(2\gamma_{[j]}^2\beta_{[l]}+2\gamma_{[l]}^2\theta)+\gamma_{[j]}^2\beta_{[l]}^2+2\theta\gamma_{[l]}^2\beta_{[j]}+2\theta^2\gamma_{[l]}^2}{r(r+\beta_{[l]})^2(r+\theta+\beta_{[j]})^2}(N-n(t)),\\ q^{[j]}(i,t)<q^{[l]}(i,t)\forall t.\end{cases}$$

$$V_{[l]}(0,0)\mid_{(i=n(t))}=\begin{cases}\frac{\gamma_{[l]}}{(r+\beta_{[l]})^2 r}(N-n(t)), q^{[l]}(i,t)\geq q^{[j]}(i,t)\forall t;\\ \\ \frac{1}{2}\frac{r^2\gamma_{[l]}^2+r(2\gamma_{[l]}^2\beta_{[j]}+2\gamma_{[j]}^2\theta)+\gamma_{[l]}^2\beta_{[j]}^2+2\theta\gamma_{[j]}^2\beta_{[l]}+2\theta^2\gamma_{[j]}^2}{r(r+\beta_{[j]})^2(r+\theta+\beta_{[l]})^2}(N-n(t)),\\ q^{[l]}(i,t)<q^{[j]}(i,t)\forall t.\end{cases}$$

given by first two lines for both players in (4.82). Then one is able to construct HJB equations for this part of the problem in the same way as it has been done for the monopolist's case (chapter 2):

$$rV^{[j]}(n(t))=$$
$$\max\Bigg\{(\alpha u^{[j]}(t)+\alpha_{[l]}u^{[l]}(t))\times V_{[j]}(0,0)\mid_{(i=n(t))}-\frac{1}{2}(u^{[j]}(t))^2+$$
$$+\frac{\partial V^{[j]}(n(t))}{\partial n(t)}(\alpha_{[j]}u^{[j]}(t)+\alpha_{[l]}u^{[l]}(t))\Bigg\};$$
$$rV^{[l]}(n(t))=$$
$$\max\Bigg\{(\alpha u^{[j]}(t)+\alpha_{[l]}u^{[l]}(t))\times V_{[l]}(0,0)\mid_{(i=n(t))}-\frac{1}{2}(u^{[l]}(t))^2+$$
$$+\frac{\partial V^{[l]}(n(t))}{\partial n(t)}(\alpha_{[l]}u^{[l]}(t)+\alpha_{[j]}u^{[j]}(t))\Bigg\}. \tag{4.97}$$

As it can be seen this time HJB equations for both players are almost identical with respect to variety expansion investments. The main difference lies in different value functions of quality problem which enter these HJB equations as arguments. Observe that these last have constant parts multiplied by the same factor $(N-n(t))$ and these constant parts do not change the general shape of solutions for variety expansion problem. First his general solution is obtained and then the discussion of the influence of these quality problem value functions on the exact form of dynamics of variety investments follows.
First-order conditions and associated controls:

$$u^{[j]}(t)=\alpha_{[j]}\times\Big(V_{[j]}(0,0)\mid_{(i=n(t))}+\frac{\partial V^{[j]}(n(t))}{\partial n(t)}\Big);$$
$$u^{[l]}(t)=\alpha_{[l]}\times\Big(V_{[l]}(0,0)\mid_{(i=n(t))}+\frac{\partial V^{[l]}(n(t))}{\partial n(t)}\Big). \tag{4.98}$$

Note that quality problem value functions given by (4.82) do depend on the variety expansion, $n(t)$. Denote

$$V(q)^{[j]} = V_{[j]}(0,0)\,|_{(i=n(t))} \times \frac{1}{(N-n(t))};$$
$$V(q)^{[l]} = V_{[l]}(0,0)\,|_{(i=n(t))} \times \frac{1}{(N-n(t))}. \tag{4.99}$$

These two are therefore constants and do not influence the general form of solution.

Value functions of variety expansion problem for both players are dependent on one state variable only and the game is linear-quadratic. Then one may assume 2-degree polynomial of one variable as the form of the value function for both players:

$$V^{[j]}(n(t)) = K_1^{[j]} n(t)^2 + K_2^{[j]} n(t) + K_3^{[j]};$$
$$V^{[l]}(n(t)) = K_1^{[l]} n(t)^2 + K_2^{[l]} n(t) + K_3^{[l]}. \tag{4.100}$$

Observe that despite there is only one state variable and common dynamic constraint in this problem, value functions of players are different since the difference in their quality problems.
Substituting these into the HJB equation for each player yields system of 6 algebraic equations for value functions' coefficients:

$$\begin{cases} (r + 2\alpha_{[j]} V(q)^{[j]}) K_1^{[j]} - 2\alpha_{[j]} (K_1^{[j]})^2 - \frac{1}{2}\alpha_{[j]}^2 (V(q)^{[j]})^2 = 0; \\ \\ (r + \alpha_{[j]}^2 V(q)^{[j]} - 2\alpha_{[j]}^2 K_1^{[j]}) K_2^{[j]} + \alpha_{[l]} u^{[l]} (V(q)^{[j]} - 2K_1^{[j]}) + \\ +N(\alpha_{[j]}^2 (V(q)^{[j]})^2 - 2\alpha_{[j]}^2 V(q)^{[j]} K_1^{[j]}) = 0; \\ \\ rK_3^{[j]} - (\alpha_{[l]} u^{[l]} + \alpha_{[j]}^2 V(q)^{[j]} N) K_2^{[j]} - \frac{1}{2}\alpha_{[j]}^2 ((V(q)^{[j]} N)^2 + (K_2^{[j]})^2) = 0. \end{cases}$$
$$\begin{cases} (r + 2\alpha_{[l]} V(q)^{[l]}) K_1^{[l]} - 2\alpha_{[l]} (K_1^{[l]})^2 - \frac{1}{2}\alpha_{[l]}^2 (V(q)^{[l]})^2 = 0; \\ \\ (r + \alpha_{[l]}^2 V(q)^{[l]} - 2\alpha_{[l]}^2 K_1^{[l]}) K_2^{[l]} + \alpha_{[j]} u^{[j]} (V(q)^{[l]} - 2K_1^{[l]}) + \\ +N(\alpha_{[l]}^2 (V(q)^{[l]})^2 - 2\alpha_{[l]}^2 V(q)^{[l]} K_1^{[l]}) = 0; \\ \\ rK_3^{[l]} - (\alpha_{[j]} u^{[j]} + \alpha_{[l]}^2 V(q)^{[l]} N) K_2^{[l]} - \frac{1}{2}\alpha_{[l]}^2 ((V(q)^{[l]} N)^2 + (K_2^{[l]})^2) = 0. \end{cases}$$
(4.101)

Observe that the system of equations for every player includes optimal control of the other player. Then the resulting solution for coefficients is dependent from coefficients of value function of the other player. In general, this system yield four different sets of solutions corresponding to different roots of quadratic equations included in the system. All 4 sets define strategies of both players in the form of reaction functions. As a result one have 4 different sets of candidate strategies and it is hard to know in advance which of these strategies are optimal. However with the help of numeric simulations on all these sets of strategies it becomes clear that only one pair of strategies yields positive variety expansion process of a stable type. 2 other sets yield negative results and hence are not admissible, while the third one yield unlimited growth of variety and thus corresponds to the unstable branch of solution. This is discussed in more details further on.

Specifically one has 2 different possible reaction functions for each player:

$$u^{[j]}(t)=\begin{cases}\frac{2r\alpha_{[j]}V(q)^{[j]}(N-n(t))}{r+\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}+\\+\frac{\alpha_{[l]}r-\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}{\alpha_{[j]}(r+\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2})}u^{[l]}(t);\\ \frac{2r\alpha_{[j]}V(q)^{[j]}(N-n(t))}{r-\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}+\\+\frac{\alpha_{[l]}r+\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}{\alpha_{[j]}(r-\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2})}u^{[l]}(t);\end{cases}$$

$$u^{[l]}(t)=\begin{cases}\frac{2r\alpha_{[l]}V(q)^{[l]}(N-n(t))}{r+\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}+\\+\frac{\alpha_{[j]}r-\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}{\alpha_{[l]}(r+\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2})}u^{[j]}(t);\\ \frac{2r\alpha_{[l]}V(q)^{[l]}(N-n(t))}{r-\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}+\\+\frac{\alpha_{[j]}r+\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}{\alpha_{[l]}(r-\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2})}u^{[j]}(t).\end{cases}$$

(4.102)

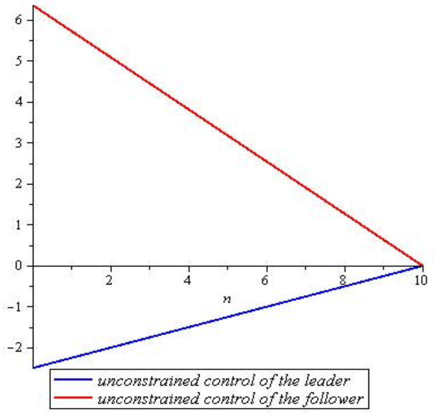
Here one may observe that first of the pair of candidate reaction functions for each of the players contains positive constant $\frac{2r\alpha_{[j,l]}V(q)^{[j,l]}N}{r+\sqrt{4\alpha_{[j,l]}^2rV(q)^{[j,l]}+r^2}}$ and negative dependence on the control of the other player and on the state variable, while the second one has negative sign of the constant and positive dependence on the state variable. Clearly, the second form of reaction function may lead only to the set of zero strategies, namely, $\{u^{[j]}(t)=0,u^{[l]}(t)=0\},\forall t$. Indeed, positive dependence on the state variable does not help since it is exactly outweighted by the constant N which is the maximal reachable state variable level. Such an equilibrium is trivial and hence the second pair of reaction functions is neglected further on. Hence one has only one pair of reaction functions left, namely

$$u^{[j]}(t)=\frac{2r\alpha_{[j]}V(q)^{[j]}(N-n(t))}{r+\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}++\frac{\alpha_{[l]}r-\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2}}{\alpha_{[j]}(r+\sqrt{4\alpha_{[j]}^2rV(q)^{[j]}+r^2})}u^{[l]}(t);$$

$$u^{[l]}(t)=\frac{2r\alpha_{[l]}V(q)^{[l]}(N-n(t))}{r+\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}++\frac{\alpha_{[j]}r-\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2}}{\alpha_{[l]}(r+\sqrt{4\alpha_{[l]}^2rV(q)^{[l]}+r^2})}u^{[j]}(t). \quad (4.103)$$

As a result of resolving this system one obtains the set of candidate strategies for every player.

Now consider the general structure of these strategies. Observe first the negative relation between strategies of the players. If one player is increasing investments this will push the other to decrease them and vice-versa.This is illustrated on the following picture:



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The picture illustrates the case $V(q)^{[j]} < V(q)^{[l]}$. Control of the player j reaches non-negative level with $n(t) \geq N$ only. However due to the requirement of nonnegative investments the strategy of given player has to be not less then zero. Then at any given time only one of the players invests positive amount in variety expansion while the other one does not. Exact form of candidate strategies for both players being displayed for some reasonable parameter set on the picture above is:

$$u^{[j]}(t) = -\frac{1}{4}\frac{\left(S(r-R_{[j]})R_{[l]} + (-SR_{[j]} + rS + 8\alpha_{[j]}^2\alpha_{[l]}^2 V(q)_{[j]}V(q)_{[l]})r\right)}{\alpha_{[j]}rS_-} \times (N - n(t));$$
(4.104)
$$u^{[l]}(t) = \frac{1}{4}\frac{\left(-S(r+R_{[j]})R_{[l]} + (SR_{[j]} + rS + 8\alpha_{[j]}^2\alpha_{[l]}^2 V(q)_{[j]}V(q)_{[l]})r\right)}{\alpha_{[l]}rS_-} \times (N - n(t)).$$

where $S = (\alpha_{[j]}^2 V(q)^{[j]} + \alpha_{[l]}^2 V(q)^{[l]}), S_- = (-\alpha_{[j]}^2 V(q)^{[j]} + \alpha_{[l]}^2 V(q)^{[l]})$ and $R_{[j],[l]} = \sqrt{4\alpha_{[j],[l]} r V(q)^{[j],[l]} + r^2}$.

Further on, consider set of candidate strategies with the condition of non-negativeness of the investments. This condition gives the set with zero j's player investments and non-zero investments of player l:

$$\{u^{[j]}(t), u^{[l]}(t)\} \geq 0 \forall t;$$
$$\begin{cases} u^{[j]}(t) = 0, \\ u^{[l]}(t) > 0. \end{cases}$$
(4.105)

where $u^{[l]}(t)$ is given by the second line in (4.104) with $u^{[j]}(t) = 0$ consequently. Now the natural question is what pushes the j's player investments to be non-positive? To answer this one has to turn to the value functions of quality problem which are entering these equations. It turns out that this parameter defines who will invest in variety expansion and who will not.

Consider the dependence of variety investments on perspectives of quality growth. It has been discussed that the player whose quality is lower and who appears to be the imitator benefits more from quality growth then the player who is the leader. This happens because imitation effect is costless and the imitator may reach any given quality level with less investments then the leader. As a result his value of quality problem would be greater. Formally,

$$\frac{\gamma_{[j]}}{(r+\beta_{[j]})^2 r}(N-n(t)) \leq$$

$$\leq \frac{1}{2}\frac{r^2\gamma_{[j]}^2 + r(2\gamma_{[j]}^2\beta_{[l]} + 2\gamma_{[l]}^2\theta) + \gamma_{[j]}^2\beta_{[l]}^2 + 2\theta\gamma_{[l]}^2\beta_{[j]} + 2\theta^2\gamma_{[l]}^2}{r(r+\beta_{[l]})^2(r+\theta+\beta_{[j]})^2}(N-n(t)). \tag{4.106}$$

This means that the follower in the quality of a boundary product will have greater incentives for investments in variety expansion then the leader. Because the strategic behavior in variety expansion game depends on the difference of values generated by the quality game, S_-, this will define who is the active investor.

Observe now, that the leadership only in the boundary product is what matters for the selection of active investor on the variety expansion level. Note that this 'boundary product' term does not define some specific product and its quality. Rather this notion refers to the boundary of the products' range as an aggregate parameter of the system. This boundary product then at any given time coincides with different products as long as variety expansion process is a monotonically increasing one. One may think of this notion as a measure of relative profitability or attractiveness of the whole range of products which are already in existence. Then the rule of selection is defined by:

$$\begin{cases} u^{[j]}(t) > 0, \forall t : q^{[j]}(t,n(t)) < q^{[l]}(t,n(t)); \\ u^{[j]}(t) = 0, \forall t : q^{[j]}(t,n(t)) > q^{[l]}(t,n(t)). \end{cases}$$

Provided the assumed form of quality investment efficiencies this means that the player who is the leader in one product's quality is the leader in all the others also. Thus the leadership mode is defined for all products and depends only on relations between parameters $\gamma_{[j,l]}, \beta_{[j,l]}$ which are independent on the products index. The argument here should follow the same lines as for the open-loop case described above.

With given quality investments efficiencies it is then clear that relative positions of players in correspondence to the boundary product are defined exogenously from the beginning through parameters' relations. Without this consideration it is not that clear how the regime of investments in variety expansion may be defined, since the quality which then contributes to the value generation does not appear yet in the moment of defining variety investments. With parameter functions as being adopted in the model players know in advance about leadership in all products and the notion of boundary product becomes redundant. Observe that this may not be the case for other forms of parameter functions. In such a case the leadership relation may change across products and thus the resulting variety investments are not constantly zero or positive as in monotonic case. It might be the case that for some time variety is increased due to the efforts of one player and for some time - by the other player. The simplest case of such a change in regimes of investments may be considered within the framework of monotonic efficiencies also. That will be the case when the catching-up in qualities occurs. However the closed-loop strategies for quality game in catching-up case is not derived and hence this situation is not analyzed here.

The player which is the follower in qualities is investing positive amount into variety expansion which is given by

$$u^F(t) = \alpha_{[F]} V(q)^{[F]}(N - n(t)) + \frac{1}{2}\frac{2\alpha^2_{[F]}V(q)^{[F]} + r - \sqrt{4\alpha^2_{[F]}rV(q)^{[F]} + r^2}}{\alpha_{[F]}} n(t) - \frac{\alpha^2_{[F]}NV(q)^{[F]}}{\alpha_{[F]}} \tag{4.107}$$

with

$$V(q)^F = \frac{1}{2}\frac{r^2\gamma^2_{[F]} + r(2\gamma^2_{[F]}\beta_{[L]} + 2\gamma^2_{[L]}\theta) + \gamma^2_{[F]}\beta^2_{[L]} + 2\theta\gamma^2_{[L]}\beta_{[F]} + 2\theta^2\gamma^2_{[L]}}{r(r+\beta_{[L]})^2(r+\theta+\beta_{[F]})^2}(N - n(t)) \tag{4.108}$$

where L denoted the leader in qualities and F - the follower, and

$$u^L(t) = 0. \tag{4.109}$$

Then the final formulation of strategies in variety expansion is given by:

$$u^{[j]}(t) = \begin{cases} 0, q^{[j]}_{n(t)} > q^{[l]}_{n(t)}; \\ \\ \alpha_{[j]}V(q)^{[j]}(N - n(t)) + \frac{1}{2}\frac{2\alpha^2_{[j]}V(q)^{[j]} + r - \sqrt{4\alpha^2_{[j]}rV(q)^{[j]} + r^2}}{\alpha_{[j]}} n(t) - \\ \frac{\alpha^2_{[j]}NV(q)^{[j]}}{\alpha_{[j]}}; q^{[j]}_{n(t)} < q^{[l]}_{n(t)}. \end{cases} \tag{4.110}$$

And the other way around for the second player. Now one may fully solve the system of equations on value functions coefficients substituting zero for one of the players' controls. Assume for certainty the zero control for the player j (who is then the constant leader in quality growth). Then the pair of HJB equations is reduced to:

$$\begin{aligned} rV^{[j]}(n(t)) &= max\Big(\alpha_{[l]}u^{[l]}V^{[j]}(q)(N - n(t)) + \frac{\partial V^{[j]}(n(t))}{\partial u^{[l]}(t)}\alpha_{[l]}u^{[l]}\Big); \\ rV^{[l]}(n(t)) &= max\Big(\alpha_{[l]}u^{[l]}V^{[l]}(q)(N - n(t)) + \frac{\partial V^{[l]}(n(t))}{\partial u^{[l]}(t)}\alpha_{[l]}u^{[l]}\Big). \end{aligned} \tag{4.111}$$

Hence both expressions depend only on the control of the second player. Then one may solve for the value function of the second player first and derive his optimal control and then use it for calculation of the value function of the other. The control of the player l who is the investor in the variety expansion is then reduced to:

$$u^{[l]}(t)^* = \frac{2\alpha_{[l]}rV^{[l]}(q)(N - n(t))}{r + \sqrt{r^2 + 4V^{[l]}(q)\alpha^2_{[l]}r}}. \tag{4.112}$$

Substitute this into the HJB equation of the first player to derive his value function. The resulting value functions of both players are:

$$\begin{aligned} V^{[j]}(n) = & \frac{2\alpha_{[l]}^2 V^{[j]}(q) V^{[l]}(q)}{4\alpha_{[l]}^2 V^{[l]}(q) + r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}} n^2 - \\ & - \frac{4\alpha_{[l]}^2 V^{[l]}(q) V^{[j]}(q) N}{4\alpha_{[l]}^2 V^{[l]}(q) + r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}} n + \\ & + \frac{2\alpha_{[l]}^2 V^{[j]}(q) V^{[l]}(q)}{4\alpha_{[l]}^2 V^{[l]}(q) + r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}} N^2; \end{aligned}$$

$$\begin{aligned} V^{[l]}(n) = & \frac{2\alpha_{[l]} V^{[l]}(q) + r - \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}}{4\alpha_{[l]}^2} n^2 + \\ & + \frac{V^{[l]}(q) N (r - \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r})}{r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}} n + \\ & + \frac{2\alpha_{[l]}^2 r V^{[l]}(q)^2}{(r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r})^2} N^2. \end{aligned} \tag{4.113}$$

With given optimal controls one have the evolution of variety expansion:

$$\dot{n(t)} = \begin{cases} \frac{2\alpha_{[l]} r V^{[l]}(q)(N - n(t))}{r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}}, q_{n(t)}^{[j]} > q_{n(t)}^{[l]}; \\ \frac{2\alpha_{[j]} r V^{[j]}(q)(N - n(t))}{r + \sqrt{r^2 + 4V^{[j]}(q)\alpha_{[j]}^2 r}}, q_{n(t)}^{[l]} > q_{n(t)}^{[j]}. \end{cases} \tag{4.114}$$

which yields the variety expansion path of the form:

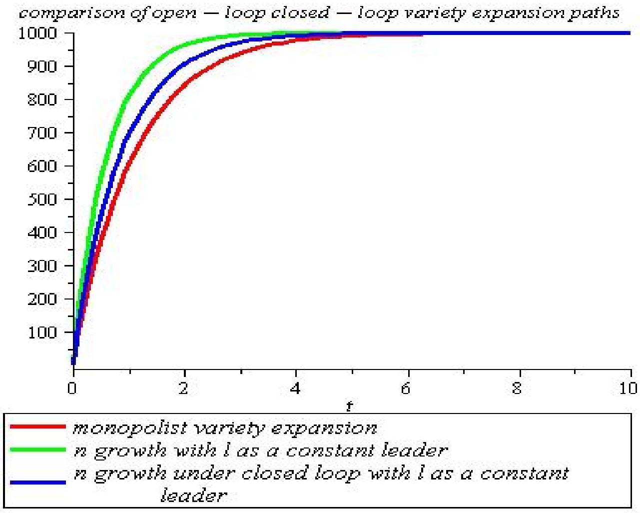
$$n(t)^{opt} = \begin{cases} N - e^{-\frac{2\alpha_{[l]}^2 r V^{[l]}(q)}{r + \sqrt{r^2 + 4V^{[l]}(q)\alpha_{[l]}^2 r}} t} (N - n_0), q_{n(t)}^{[j]} > q_{n(t)}^{[l]}; \\ N - e^{-\frac{2\alpha_{[j]}^2 r V^{[j]}(q)}{r + \sqrt{r^2 + 4V^{[j]}(q)\alpha_{[j]}^2 r}} t} (N - n_0), q_{n(t)}^{[l]} > q_{n(t)}^{[j]}. \end{cases} \tag{4.115}$$

depending on the leadership in quality growth. This closes the variety expansion game for the case of closed-loop (Markovian Nash) strategies.

One may compare the rate of variety expansion between open-loop and closed-loop strategies. Observe that closed-loop solution is effective only for constant leadership with single steady state. To compare the solutions take the set of parameters corresponding to the single steady state in the quality game, where player l is the constant leader. As such the set SETLL is relevant, since the condition

$$\beta_{[l]}(\beta_{[l]} + r + \theta)\gamma_{[j]}^2 > \beta_{[j]}(\beta_{[j]} + r)\gamma_{[l]}^2 \tag{4.116}$$

is fulfilled and the opposite does not hold. Then the variety expansion for closed-loop strategy profile is slower then for the open-loop but higher then for the monopolist's problem:



The obvious reason for such a difference is that the leader does not invest into variety expansion under closed-loop at all while in open-loop he is investing positive amount although lesser then the follower.

7. Discussion

In this chapter the model of strategic interactions between innovating agents has bee introduced. It combines the imitation effect and the R&D cooperation into the single model while allowing for heterogeneity of products' characteristics. One may compare the results of this model with other papers in the field. First of all consider the paper of Judd, [**25**] where the competition of multiproduct innovating agents has been considered also. In this paper the distinction between two types of innovation is made, namely the author considers leap investments and partial jump investments which closely correlate to the structure of quality growth and variety expansion in the given framework. The paper of Judd differs in two aspects from current approach. First he accounts for uncertainty of innovations while here it is absent and second he analyzes the static framework with multistage innovations while the given model allows for a dynamic game. The results of the given model substantially differ from that of Judd's. This happens mainly because although Judd allows for leaps in his model, this leaps are characterized by the same space as ordinary jumps and thus there is no explicit distinction between both types of innovations. Both types of investments are defined in the same 1-dimensional space and change only relative positions of the agents. At the same time in current model two types of investments are explicitly distinguished and moreover it allows for the distributed parameter type of the quality growth innovations. That's why the terms of excessive investments, the relative positions and strategies of the leader and the follower are much more clearly defined in the current model. Namely one may observe the incentives which push each of the players to over- or underinvestment in comparison with the other player. In Judd's paper it is claimed that competitive environment yield investments higher then the social optimum and moreover that it is possible for the follower to invest more then the leader. In the current case it is also possible, but the distinction is more explicit: the follower will invest more then the leader but only in variety expansion (which corresponds to leap investments in Judd's) while investing less in quality growth. Meanwhile this last effect depends on the precense of imitation effect and is fully defined by it. At the same time observe, that in the absence of this effect it is not possible to define the strategic interaction between players on the quality growth stage in the given framework. Also observe that only the equilibrium of open-loop type is defined for all outcomes of the model, since the strategies in quality growth are of open-loop type and the same in variety expansion. The closed-loop type strategies for the variety expansion path are also defined as well as for quality growth. However the area of existence of piecewise constant closed-loop strategies as equilibrium ones is limited by the situation with unique steady state of the quality growth game. In this case closed-loop equilibrium coincides with the open-loop in the quality growth game but not in the variety expansion game.

Concerning the comparison with social optimum it cannot be done within the framework of the suggested model. Partial compensation for it is the comparison with the monopolist's case. In the absence of any other structures in the game this monopolist's outcome may be considered as the action of social planner as well. This is in agreement with Judd in the conclusion that the competitive environment yield higher investments rates in almost all cases. Yet there is a room for the domination of the monopolist's outcome as it has been demonstrated. This takes place in the situation when the monopolist's efficiency of investments into variety expansion is in between the efficiencies of both players and there is a catching-up situation in quality growth. In this case the overall level of investments of the monopolist may be higher then that of multiple agents. It may be concluded that the suggested model allows for more heterogeneous outcomes then that of Judd.

Another paper in the field to consider is the work of Boone, [**11**]. The main claim of this paper is that the rise in competitive pressure may not increase innovations of both types but only of the one of at the cost of reduction of the other. First note that this model is completely static and the author considers the competition of static Cournot type in the industry. Next, he has the distinction by firm types with respect to their behavior which is dependent on the efficiency parameters of their investments/costs. In the suggested model such a distinction has not been made although depending on the mutual relationship of efficiencies and decay rates of both agents there are 4 different types of variety expansion dynamics and 4 types of quality dynamics and one may label this types according to types of agents. The main difference is that again the current model is dynamic in nature. Concerning the results the model does not support the claim of Boone, as it turns out that in the majority of cases the rise in the number of players from 1 to 2 yields rise in both types of innovations as the result. At the same time the further increase in the number of players does not necessarily mean the further increase in the level of innovations of both types in comparison with the 2-player case. Observe also that Boone accounts for competitive pressure as an exogenous parameter which may differ across players while in the suggested model it would be defined just by the number of players as there is no effect of potential entrants at all.

One may argue that it is not revelant to compare the results of a given model with static models above. Consider then the literature on dynamic R&D competition which dates back to Reinganum, [**17**]. Current model substantially differs from this literature also in that it has the simplified framework which abstracts from uncertainty in any form which is substantial for this kind of literature while allows for continuous stream of heterogeneous innovations, which has not been made there. Thus the performed analysis may be compared with that of Reinganum in a rather limited way. Namely, one may compare the effect of R&D competition in rather general terms only. In Reinganum's paper necessary and sufficient conditions for an optimum constitute optimal controls of piecewise form ranging from zero to some max value depending on the potential marginal profit. This is the same to the variety expansion strategies in the suggested model, where every agent invests different amounts depending on what is his position in the quality game with minimum investments being achieved when the given player is the initial leader in quality but looses this position eventually. At the same time observe that the imitator in current model does not invest zero in the quality growth in all cases, but some strictly less amount then the leader while investing more in variety expansion.

To conclude with comparison one may observe that the suggested model combines the dynamic game-theoretic approach of Reinganum, [**17**] with multiproduct innovations framework introduced in [**12**]. One then obtains effects which may not be described within both of these approaches. The recent paper by Lambertini&Montavani,[**56**] has somewhat similar differential games approach which combines product and process innovations. However, they do not handle multiple products with different investment characteristics which is the essential part of the suggested heterogeneous framework. Effects captured by this heterogeneous approach are discussed below.

First effect is the interaction between two different types of investments for both agents. It has been demonstrated that neither of the effects being considered (imitation and cooperation) may constitute the equilibrium in the absence of the other. Consider first the imitation effect. As long as it is costless (or, alternatively, its costs are neglectible), every of the players has stimuli to deviate from the strategy of being the leader as it is more profitable in terms of the value function to be the imitator. The most clear this effect is in the symmetric case where both players have identical parameters' sets. In this situation any marginal deviation for both players would be profitable and hence the pair of symmetric strategies $\{g^{[j]}(i,t)_{SYM}, g^{[i]}(i,t)_{SYM}\}$ may not be the equilibrium of the quality growth game in the sense of closed-loop. Moreover, the catching-up effect has no logical justification then: as it is more profitable to be the follower, this follower has no incentives to catch up with the leader. Instead he would decrease his investments by some marginal amount to remain the follower. Now consider the cooperation effect. In the absence of the associated quality growth problem even for a stand alone innovator there are no stimulus to invest at all. Now suppose the game in the precense of quality innovations but in the absence of imitation effect. As long as one of the players has higher efficiency of investments then the other, he would have stronger incentives to invest in variety expansion because this would generate more value for him. Then the second player will not invest in variety expansion at all. But then the first player's incentives to invest in variety expansion would also decrease. In the precense of imitation effect the situation is the opposite: it is more profitable to invest in variety for the follower, who benefits from imitation in the quality growth game. As it has been shown, the leader's investments in variety expansion should be lower then. So in the absence of imitation there is a free-rider problem concerning variety expansion where both agents have incentives to invest less then the competitor and some external mechanism is required as it is usual with public good provision. At the same time if the imitation effect is present this problem not only disappears, but helps to constitute the equilibrium, as incentives to invest less then the other agent into variety expansion is counterbalanced by incentives to invest less in the quality growth. As a result, there is a specialization of innovative activities of agents, with that one which has higher efficiency of investments in quality growth investing in that field and the other investing in variety expansion. Such an outcome of the game seems rather plausible since it selects areas of specialization in accordance with efficiency parameters.

Second effect is the effect of heterogeneity of products being considered. This effect is common for all versions of the model being considered within the work but in this last chapter it is even more drastic. Here one may observe that if one would assume homogeneous products but pertain the difference in efficiency characteristics of players this would not change basic results of the model concerning imitation and cooperation. On the other hand it is the special form of differences between products which is important. If one would assume non-monotonic or non-symmetric functions of investment efficiencies the domination of one player or the other may change across products while in the given specification this is not the case. As such it is not surprising that the quality growth problem admits only piecewise-constant type strategies, since there cannot be any changes in mutual positions of players across different products. One may argue that with non-monotonic functions of investment efficiencies the linear-feedback strategies may appear to be stable in quality growth problem also. This may be considered as one of future extensions of the current work. Another immediate extension should be to account for limited life-cycles of products under consideration although one may predict that with cooperative structure of variety expansion dynamics this would not change the results in a substantial way.

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Appendix

Proposition 3. *The set (1.10) for the problem (4.3),(2.2),(4.4) contains an interior point for n_0 large enough.*

Proof. First note, that

$$t_n^{-1} R(0, t_n; \mathbf{J}, \tilde{j}^n) \subseteq \mathbf{Der}\ f_n(\tilde{j}^n) \tag{4.117}$$

This can be shown by writing arbitrary directional derivative $t_n^{-1}\xi(t_n, \tilde{j}^n, v)$ by means of variation-of-constants formula and comparing it with the reachable set. This means that reachable space multiplied by some convergent sequence $\{t_n^{-1}\}$ is just a subset of a set of directional derivatives (which are variations in some sense). Note further, that since $t_n \to \bar{t} > 0$ and $K_Y(\tilde{y}^n)$ is a cone, one may get rid of t_n^{-1} term in (1.10). Then, Q_n is a precompact sequence and otherwise arbitrary, while being multiplied by a convergent sequence (t_n) remains precompact. Then (1.10) may be rewritten:

$$\begin{aligned} &\{t_n^{-1} R(0, t_n; \mathbf{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho) + Q_n\} = \\ &\{R(0, t_n; \mathbf{J}, \tilde{j}^n) - K_Y(\tilde{y}^n) \cap B(0, \rho_2) + X_n\} \end{aligned} \tag{4.118}$$

where X_n is another precompact sequence.
Consider first two terms of (4.118) for two fixed n:

$$\begin{aligned} &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2), \\ &R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2). \end{aligned} \tag{4.119}$$

What we need first is to show that intersection of this sets is not empty. Obviously this depends on the reachable space, since intersection of the cone and ρ-ball is not empty, and intersection of two such sets is also not empty:

$$\begin{aligned} &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2) = \\ &R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap K_Y(\tilde{y}^{n_2}) \cap B(0, \rho_2) = \\ &\qquad R(0, t_{n_1}; \mathbf{J}, \tilde{j}^{n_1}) \cap R(0, t_{n_2}; \mathbf{J}, \tilde{j}^{n_2}) - K_Y(\tilde{y}^{n_1}) \cap B(0, \rho_2) \cap K_Y(\tilde{y}^{n_2}). \end{aligned}$$

So, this depends on the "size" of reachable space for every n. Reachable space includes all variations, possible at time t_n and control sequence value $\tilde{j}^n$. But we know, that admissible control space is spike complete, that is, it includes all possible spike variations of $j(\bullet)$. What we also know, is that $\mathbf{J}_{ad}$ is "big" enough.
Now take sequences $\{t_n\}, \{\tilde{j}^n\}, \{\tilde{y}^n\}$ such that:

$$\begin{aligned} t_n \to \bar{t}, \bar{t} - t_n \le \delta_n, \\ d_n(\tilde{j}^n, \bar{j}) \to 0, d_n(\tilde{j}^n, \bar{j}) \le \delta_n, \\ \|\tilde{y}^n - \bar{y}\| \to 0, \|\tilde{y}^n - \bar{y}\| \le \delta_n, \\ \delta_n \to 0, n \in \mathbb{N}. \end{aligned} \tag{4.120}$$

Now note, that $\mathbf{J}_{opt}$, set of optimal controls, is contained in $\mathbf{J}_{ad}$ by existence of optimal controls. Then taking $\bar{j} = j_{opt}$ one may ensure that there is at least one converging sequence of $\{\tilde{j}^n\}$. Since $\delta_n \to 0$, starting from some n_0 all members of this sequence are contained in $\mathbf{J}_{ad}$. The same is then true for $\{\tilde{y}^n\}$ sequences.

Then, starting at least from some number n_0, reachable space is also there. Then intersection for different close numbers, $n_1 > n_0, n_2 > n_0$ is not empty.
The last question is of intersection of contingent cones. Is it big enough or close to reachable space?
Note, that these cones are decreasing in size with growing n, as y converges to $\bar{y}$. Then there intersections are also decreasing in size. At the same time intersections of reachable spaces is growing in size, so starting from some another n, $n_0^2 \geq n_0$, the difference in (4.120) is not empty.
The same argument may be repeated for all $n \geq n_0^2$. Adding members of precompact sequence to these sets will not decrease them, so their intersection will remain not empty, thus containing interior point starting from some number n_{00}. □