Universal Beliefs Structures

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Contents

Preface

Nowhere in this thesis are "Beliefs Structures" defined. This term is meant to serve as a generic term for mathematical structures that describe uncertainty in an interactive context, such as those that will appear in the sequel, namely, Kripke Structures, (κ-) Type Spaces, and (Conditional) Possibility Structures. One of the central questions of this thesis is as to whether, for a class of beliefs structures, there exists a "largest" structure in this class that "contains" all the structures of that class. Such a structure is said to be "universal". Formulated in category theoretic terms, we are looking for a terminal object in a category (of beliefs structures). Apart from the introduction, the chapters are quite independent of each other, hence they can be read in any order the reader might find convenient.

The work contained in this thesis owes much to the pioneering work of John C. Harsanyi (1967/68) and Robert J. Aumann (1976, 1995, 1999a, 1999b), as well as to the work of Mertens and Zamir (1985), Brandenburger and Dekel (1993), Aviad Heifetz (1993, 1997), Mertens, Sorin and Zamir (1994), Brandenburger and Keisler (1999), Heifetz and Mongin (2001) and Battigalli and Siniscalchi (1999a). There are two names that one will find almost everywhere in this thesis, namely Aviad Heifetz and Dov Samet, whose articles (1998a, 1998b) greatly influenced much of this work, and I am most grateful for their willingness to referee this thesis. I am also very grateful to Adam Brandenburger, who inspired the work of the last chapter.

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Martin Meier

Chapter 0

Introduction

Consider a group of players facing uncertainty about a set S, called the *space of states of nature*, each element of which can be thought of as a complete list of the players' strategy sets and payoff functions[1] (i.e. a complete specification of the "rules" of the game that depend on the state of nature). In such a situation, following a Bayesian approach, each player will base his choice of a strategy on his subjective beliefs (i.e. probability measure) on S. Since a player's payoff depends also on the choices of the other players, and these are based on their beliefs as well, each player must also have beliefs on the other players' beliefs on S. But he must, by the same argument, also have beliefs on other players' beliefs on his beliefs on S, beliefs on other players' beliefs on his beliefs on their beliefs on S, and so on. So, in analyzing such a situation, it seems to be unavoidable to work with infinite hierarchies of beliefs. Thus, the resulting model is complicated and cumbersome to handle. In fact, this was the reason that prevented for a long time the analysis of games of incomplete information.

A major breakthrough took place with three articles of Harsanyi (1967/68), where he succeeded in finding another, more workable model to describe interactive uncertainty. He invented the notions of *type*[2] and *type space*: With each point in a type space, called a *state of the world*, are associated a state of nature and, for each player, a probability measure on the type space itself (i.e. that player's *type* in this state of the world). Usually it is assumed that the players "know their own type", that is, a type of a player in a state assigns probability one to the set of those states where this player is of this type. This is the formalization of the idea that the players should be self conscious. Since each state of the world is associated with a state of nature, each player's type in a state

[1]Other interpretations are possible, for example, if a game of complete information is given, s could be the strategy profile that the players are actually going to chose (see the analysis of epistemic conditions for Nash equilibrium by Aumann and Brandenburger (1995)).

[2]This notion of 'type' should not be confused with the notion of 'type' used in the model theory of first-order logic, where a type is a (maximal) consistent set of first-order formulas with one free variable.

of the world induces a probability measure on S. But also, since with each state of the world there is associated a type for each player (and hence indirectly a probability measure on S for this player), the type of a player in a state of the world induces a probability measure on the other players' probability measures on S. Proceeding like this, one obtains in each state of the world a hierarchy of beliefs for each player, in the sense described above.

The advantages of Harsanyi's model are obvious: Since we have in each state of the world just one probability measure for each player, contrary to the hierarchical description of beliefs, this model fits in the classical Bayesian framework of describing beliefs by one probability measure, and provides therefore all its advantages (for example, it allows for integration with respect to beliefs).

However, there are also several serious questions that arise with the use of this model: Although each state of the world in a type space induces a hierarchy of beliefs for each player, the converse is not obvious: Does each profile of hierarchies of beliefs arise from a state of the world in some type space, and if so, is there a type space such that every profile of belief hierarchies is generated by some state in this type space? What "are" the states of the world, and what justifies using one particular type space and not another? In particular, contrary to the case of the hierarchical description of beliefs, it is not clear what "all possible types" (resp. "all possible states of the world") are? More precisely, given a game of incomplete information, working in one fixed type space to analyze that game could be restrictive in the sense that we might miss some possible types (resp. possible states of the world) that are just present in a bigger type space that contains the one we use. If this were the case for every type space, then the use of type spaces would be problematic from a theoretical point of view, because of the restrictive character of this concept, but it would be problematic also from a more practical point of view: In their contributions to the recent debate on epistemic conditions for backward induction in perfect information games, Stalnaker (1998) and Battigalli and Siniscalchi (1999b) have pointed out that the players do "their best" to rationalize their opponents' behavior if the backward-induction outcome is to obtain. This translates into using a type space where a player can find the needed types he has to attribute to the other players, if he has to explain (i.e. rationalize) the others' behavior.

The question concerning "all possible types" can be answered and the related problems can be solved if there is a type space to which every type space (on the same space of states of nature and for the same set of players, of course) can be mapped, preferably always in a unique way, by a map that preserves the structure of the type space, i.e. the manner in which types and states of nature are associated with states of the world, so-called *type morphisms*. Such a type space would be called a *universal type space*. If such a space always exists, one could, in principle, carry out the analysis of a game of incomplete information in the corresponding universal type space without any risk of missing a relevant state of the world. On a technical level, the type spaces - on a fixed set of states of nature and for a fixed player set - as objects and the type morphisms as morphisms form a category. If we always require the map from a type space to the universal type space to be unique, then, if it exists, such a universal type space is a *terminal object* of this category. By the Yoneda Lemma, a terminal object of a category is known to be unique up to isomorphism. Hence, we are justified to talk about *the* universal type space.

The existence of a universal type space was proved by Mertens and Zamir (1985) under the assumption that the underlying space of states of nature is a compact Hausdorff space and all involved functions are continuous. Moreover it turned out that in this case the universal type space consists of all hierarchies of beliefs in the above sense, that satisfy some prespecified *coherency conditions*. That topological assumption was relaxed by Brandenburger and Dekel (1993), Heifetz (1993), Mertens, Sorin and Zamir (1994/2000) to more general topological assumptions, but the general idea of the proof was always the same: To use this assumption to show, by the Kolmogorov extension theorem, say, that each hierarchy of beliefs induces in a unique way a σ-additive probability measure on the whole space of (profiles of) belief hierarchies.

The general measure theoretic case was solved by Heifetz and Samet (1998b). They showed that there also exists a universal type space in this case, except that it is not the space of all coherent hierarchies in the above sense, but rather a subspace of that space. In another article (1999) they showed that in the general measure theoretic case there are coherent hierarchies that give rise just to a finitely additive but not σ-additive measure on the whole space of coherent hierarchies (although every measure that makes up some level of such a hierarchy is σ-additive!).

Heifetz and Samet constructed the universal type space by collecting all those coherent hierarchies that are generated by some state of the world in some type space. (They offered also another proof, where they collected the "descriptions" of all the states by means of some kind of modal language the formulas of which they called *expressions*.) Their method is nice in the sense that it leads to a very clear and relatively short proof. But unlike the proofs in the other cases, this method has the disadvantage that the universal type space is not characterized in an independent way (for example, it cannot be characterized as the space of all coherent hierarchies of beliefs).

Most of the authors consider just those type spaces that we will call here *product type spaces*. For each player $i \in I$, there is a measurable space (M_i, Σ_i), *i's component space*, such that with each point $m_i \in M_i$ there is associated a probability measure (i's type in m_i) on the space $S \times \Pi_{i\in I} M_i$ endowed with the product σ-field. The players' self consciousness is expressed in the following way: The marginal of the type of player i in state $m_i \in M_i$ on M_i is δ_{m_i}, the delta measure at m_i.

Given such a product type space, one can define the notion of "all possible types in that space", namely the requirement that each probability measure on

$S \times \Pi_{j \in I \setminus \{i\}} M_j$ is the marginal of a type of player i in some state $m_i \in M_i$. We call this property *beliefs completeness*. Although just applicable to product type spaces, this notion of "all possible types (in a space)" has the advantage that no notion of "structure preserving maps" is needed to define it. Intuitively, it should be the case that a universal type space, as a space of "all possible types", should be beliefs complete, i.e. contain in particular "all possible types in the universal type space". However, since the mathematical formalization of "all possible types" and "all possible types in a given space" are quite different, this is far from being straightforward and has to be proved. In fact, in all the above mentioned type space literature, the universal type space is a product space. (This follows either from the authors' a priori restriction to product type spaces, or as an outcome of the construction of the universal type space). In all the topological cases the universal type space has the following property (which, of course, implies beliefs completeness): The component space of each player is homeomorphic to the space of probability measures on the product of the space of states of nature and the other players' component spaces (endowed with the weak* topology). The converse is not in general true: Mertens, Sorin and Zamir (2000, Comment 1.5 Ch. III) gave an example of a beliefs complete type space that is not the universal type space. In the general measure theoretic case, Heifetz and Samet, although working exclusively with product type spaces, did not consider beliefs completeness. So, up to now, it remained open whether in this case the universal type space is beliefs complete.

Another approach to uncertainty in a group of players (or agents) used in game-theory, economics, and also computer science, are purely set theoretic models. A player's beliefs are not described by probability measures, but rather by the set of those states that he considers possible. Again, the hierarchical approach is conceptually straightforward, but the resulting models are quite impractical to use, while the use of the "state of the world models" - they are in fact *Kripke structures* known from modal logic - raises similar questions as in the case of type spaces.

Knowledge (see, for example, Aumann (1976, 1995 and 1999a), or Heifetz (1997)) is usually described by partition spaces, or equivalently by Kripke structures, where all the relations are equivalence relations. For each player, there is a partition of the underlying space of states of the world. In each state of the world, a player knows exactly those events which contain the partition member that contains that state. For an event, one can define the event that a player knows this event as the set of those states in which he knows this event. Defined in that way, knowledge has the following properties: What is known by a player is true, a player knows what he knows and he knows what he does not know. So, in these models the assumptions on the epistemic attitudes of the players are rather strong.

Heifetz and Samet (1998a) showed that - unfortunately - for at least two players and at least two states of nature there is no universal knowledge space.

What happens if we consider other (weaker) epistemic attitudes?

In the remaining part of this introduction, we give a summary of - and a motivation for - each of the chapters of this thesis.

0.1 On the Nonexistence of Universal Kripke Structures

Kripke structures can be seen as generalizations of partition spaces to describe other, in particular weaker, epistemic attitudes. Instead of a partition, a binary relation over the underlying space of states of the world is ascribed to each player. In a state of the world, a player considers all the states that are in relation with the actual state as being possible. He knows, in a state of the world, all those events which contain all states that he considers to be possible there. If the relation is not reflexive, it no longer holds that what is known is true and we should rather talk about *belief* than knowledge. Imposing different restrictions on the relations of the players leads to different epistemic attitudes of the players (see Fagin et al. (1995) for more on this issue). For example, if the relations are equivalence relations, we are, in fact, in the case of knowledge spaces (it is perfectly equivalent to work either with equivalence relations or with the equivalence classes of the relations).

We mentioned above that there is a universal type space for probabilistic beliefs but no universal knowledge space. Where does this difference come from? Does it come from the strong assumptions on the epistemic behavior of the players in a knowledge space, especially from the truth axiom ("what a player knows in a state is true there")? Note that in a type space it is possible that the type of a player in state of the world assigns probability one to an event, and yet this event is not true in that state of the world. If this difference were the reason why there is no universal knowledge space, one might hope that there is a universal Kripke structure, at least for classes with weak conditions on the epistemic attitudes of the players (especially abandoning the truth axiom).

We show in Chapter 1 that this is - unfortunately - not true: Given at least two players and at least two states of nature, for every class of Kripke structures that contains all knowledge spaces, there is no universal Kripke structure (Corollary 1.1), even if we do not require the morphisms from the Kripke structures to the universal Kripke structure to be unique (Theorem 1.1).

0.2 Finitely Additive Beliefs and Universal Type Spaces

σ-additivity, though desirable, is quite a strong assumption on the players' beliefs. "Savage's (1954, 1972) framework for modelling choice under uncertainty provides a theory of subjective probability, and has been called the 'crowning glory of choice theory.' (Kreps (1988))... Savage chose to work in a framework that implies that the subjective probabilities fail to be countably additive." (Stinchcombe (1997)).

Now, game theory can be viewed as multi-agent decision theory, and in this view, one is naturally led to consider type spaces where the players' beliefs are described by *finitely* additive probability measures on the space of states of the world.

We show in Chapter 2, for a variety of measurability conditions (*κ-measurability*, for every regular cardinal κ) that include usual measurability as a special case, that universal type spaces exist even if the players beliefs are finitely additive. If we require that all subsets of the type spaces should always be measurable (called here *$*$-measurability*), then, for at least two states of nature and at least two players, there is no universal type space. This means that in the case of finitely additive beliefs, the existence of a universal type space depends crucially on the measurability conditions that we require.

We provide also a characterization of the universal type space (for finitely additive beliefs). In the case of κ-measurability, the universal type space is the space of coherent hierarchies (Theorem 2.5), but, for $\kappa > \aleph_0$, all the finite levels of the hierarchies do not suffice, one has to take the hierarchies up to (but excluding) level κ. This is so because for every $\alpha < \kappa$ there are coherent hierarchies of (κ-measurable) beliefs up to (but excluding) level α that have at least two different extensions to level α. (This follows from our Theorem 2.2, where we formulate this fact for sets of *κ-expressions* and not for hierarchies of measures, but it is straightforward to see that the structure constructed there generates coherent hierarchies with these properties.)

Furthermore, the space of (finitely additive) coherent hierarchies (and hence the universal type space (for finitely additive beliefs)) is a product type space (Theorem 2.4) and it is beliefs complete (Theorem 2.6). In the case of $\kappa = \aleph_0$, the component space of each player is - up to isomorphism of measurable spaces - the space of finitely additive probability measures on the product of the space of states of nature and the other players' component spaces.

0.3 Infinitary Probability Logic for Type Spaces

As mentioned above, there is no universal knowledge space (and no universal Kripke structure). But still, in order to justify using knowledge spaces, we would like to have a meaningful notion of "all (knowledge) types" (resp. "all states of the world") and a knowledge space that contains "all (knowledge) types" (resp. "all states of the world") in that sense. If we restricted our attention to "important events" (just as one does in the probabilistic context, where one considers just measurable events and not all subsets of the structure), maybe then there would be such a space of "all types" (resp. of "all states of the world").

This is where another idea comes into the picture, namely that of a *language*. The basic "philosophy" behind this idea is that a player cannot think about events he cannot "describe". And to formalize what "describe" means, a *mathematical language* is defined. The language of *(propositional) modal logic* is considered to be the appropriate language (i.e. the *syntax*) for Kripke structures. One starts with a set of *atomic formulas* or *primitive propositions*, that we interpret as statements about nature, and forms more complicated formulas by closing off under *negation*, *conjunction* and, for each player, a *modal operator*, that expresses that player's knowledge. The language is then just the set of these formulas. Then, one adds *axioms*, i.e. some of these formulas that we require to be always true, and *inference rules*, i.e. rules that allow to infer true formulas from other true formulas. This leads to the notions of *theorem* as a formula that follows from the axioms by applying some of the inference rules, and of *consistent set of formulas* as a set of formulas such that there is no formula such that this formula and its negation follow from that set of formulas (by means of the inference rules). In fact, the importance of the idea of using a language is illustrated by the fact that the language of modal logic was there before Kripke invented Kripke structures as the *semantics* for modal logic.

Since we have syntax and semantics, we would like to relate the two approaches. This is done by the *model relation* "$\models$", that tells us, given a Kripke structure and a state in this structure, if a formula holds in that state. An axiom system (together with inference rules) is *sound* with respect to a class of (Kripke) structures if every theorem is valid with respect to that class (i.e. true in every state in every structure of that class) and it is *complete* with respect to a class if every valid formula is a theorem. But there are also the stronger notions of *strong soundness* and *strong completeness*. Although they are usually defined differently, these properties are equivalent to the following: An axiom system is strongly sound with respect to a class of structures iff every set of formulas that has a *model* (i.e. a structure in this class and a state in this structure such that every formula of that set is true in this state) is consistent, and it is strongly complete with respect to a class of structures iff every consistent set of formulas has a model.

Following several predecessors (Kripke (1959,1963), Hintikka (1962), and Fagin et al. (1995)), Aumann (1995, 1999a) for the finitary version and Heifetz (1997) for infinitary versions, showed that the multiplayer S5 axiom system of modal logic is strongly sound and strongly complete with respect to knowledge spaces. Moreover, to prove that theorem they were able to construct a *canonical knowledge space*, the states of which consist of the maximal consistent sets of formulas in that language such that a formula is true in a state iff it is in the set of formulas that constitute that state. So, this gives a well-defined notion of "all states" as the set of all maximal consistent sets of formulas, and a space of "all types", i.e. the canonical space. Also, in the case that we have strong soundness and strong completeness, the axioms and inference rules give a deeper insight in our assumptions on the structure of the spaces in the corresponding class. If we reject these axioms and inference rules, then, since they reflect the structural properties of the corresponding class of structures, we should consequently also work with a different class of structures at the semantic level.

However, as Heifetz (1997) has shown, the canonical knowledge space depends on the language chosen: If we allow for infinitary formulas of larger size (without changing anything essential), then the canonical space is also larger (i.e. it contains more states).

Now, type spaces in the sense of Harsanyi can be viewed as the probabilistic analog of Kripke structures: It is not just specified what - in a state - each player believes, but also the intensity of these beliefs. Hence, one wonders if such a strong soundness and strong completeness theorem also holds for type spaces and, if so, how the corresponding axioms and inference rules, and - if it can be constructed - the corresponding canonical space would look like.

Heifetz and Mongin (2001) and before Fagin, Halpern and Meggido (1990), for a richer syntax than the one used by Heifetz and Mongin, axiomatized the class of type spaces in a finitary logic similar to the modal logic mentioned above, where the modal operators expressing knowledge are replaced by operators of the form p_i^α, read as "player i assigns probability at least α to", for each player i and each rational $\alpha \in [0,1]$. They were able to show that their axioms and inference rules are sound and complete with respect to the class of type spaces. But a finitary axiom system cannot be used to get strong soundness and strong completeness, and hence it is also not possible to construct a canonical type space using such an axiom system. To see that, consider the set of formulas $\left\{p_i^{\frac{n}{n+1}}(x) : n \in \mathbb{N}\right\} \cup \{\neg(p_i^1(x))\}$, where x is some primitive proposition. It is easy to see that each finite subset has a model in the class of type spaces, while the whole set has no model. Since in a finitary logic a set of formulas is consistent iff each finite subset is consistent, strong soundness and completeness would imply that this set of formulas would have a model, which is impossible.

In Chapter 3, we consider an infinitary extension of the language of Heifetz and Mongin (2001) and invent an infinitary axiom system as well as inference rules which we prove to be strongly sound and strongly complete with respect to type spaces (Theorem 3.1). To the best of our knowledge this is the first strong completeness theorem for a probability logic of this kind. In fact, we prove this theorem by constructing the canonical type space - in the sense explained above - (Proposition 3.3 and Corollary 3.1). It is worth remarking that, like Heifetz and Samet (1998b), we need no topological assumptions on the spaces, just measurability. Furthermore, it turns out that our canonical type space is a product type space (Proposition 3.4 and Theorem 3.3).

As already mentioned, there is a space of "all types" defined in purely semantic terms, namely the universal type space, in the measure-theoretic case constructed by Heifetz and Samet (1998b). In a fortunate coincidence, our canonical type space is the universal type space (Theorem 3.2). Therefore we provide a novel characterization of the universal type space in the general measure theoretic case - as the space of maximal consistent (with respect to our axioms and inference rules) sets of formulas.

Heifetz and Mongin (2001) are the only ones that considered also type spaces where the players are allowed to be ignorant of their own type.[3] We construct our canonical type space with the introspection property, i.e. the property that the players "know their own types", as well as without the introspection property. Therefore, we provide the first proof of the existence of a universal type space without the introspection property.

We mentioned above that Heifetz and Samet (1998b) did not consider beliefs completeness. We show that, in the introspective case, the component space of each player is - up to isomorphism of measurable spaces - the space of probability measures on the product of the space of states of nature and the other players' component spaces, and in the non-introspective case, each player's component space is - up to isomorphism of measurable spaces - the space of probability measures on the whole space of states of the world (Theorem 3.4).

0.4 Conditional Possibility Structures

Brandenburger and Keisler (1999) invented *possibility structures* as a set-theoretic analog of product type spaces. Given a nonempty space of states of nature[4] S and a nonempty player set I, for each player $i \in I$ there is a nonempty component space M_i (the set of "i's types") and a relation R_i (*i's possibility relation*) on $M_i \times S \times \Pi_{j \in I \setminus \{i\}} M_j$ such that $R_i(m_i)$, the m_i-section of R_i, is nonempty for each $m_i \in M_i$. $R_i(m_i)$, the *possibility set of* m_i, is the set of those nature-others' types pairs considered possible by the type m_i of player i. Another way of describing the possibility structure is to replace each R_i by a function ρ_i from

[3] Though Heifetz and Samet (1998b) mentioned it in their discussion.

[4] In fact, Brandenburger and Keisler (1999) allow for different spaces of states of nature for the different players.

M_i to $\text{Pow}_\emptyset\left(S \times \Pi_{j\in I\setminus\{i\}} M_j\right)$, the set of nonempty subsets of $S \times \Pi_{j\in I\setminus\{i\}} M_j$. It is easy to see that a possibility structure "is" (i.e. can easily be turned into) a Kripke structure.

The initial motivation of Brandenburger and Keisler (1999) was to provide a framework for analyzing the epistemic conditions for the backward-induction algorithm. They wrote on the issue of the recent debate on the backward-induction algorithm: "In significant contributions to this debate, Stalnaker (1998) and Battigalli and Siniscalchi (1999b) have pointed out that the use of the BI algorithm implicitly assumes that players do their 'best' to rationalize the moves they see other players make. They further observe that in an epistemic analysis, this translates into the need to work in a 'rich' probability structure, in which player a, say, can find the beliefs he needs to attribute to player b, if he is to explain player b's behavior. Indeed, Battigalli and Siniscalchi (1999b) work in a complete probability structure, where they are able to establish epistemic conditions for extensive-form rationalizability (Pearce 1984), a solution concept for extensive games that reduces to the BI algorithm in the special case of perfect-information (PI) games."

They proceeded: "Now, there has been a long-standing intuition that probabilities play an inessential role in PI games. Recall, for example, that every two-person zero-sum PI is strictly determined (Zermelo 1913, von Neumann and Morgenstern 1944) and that every n-person PI game possesses a pure-strategy equilibrium (Kuhn 1950, 1953). Thus, the particular context of PI games invites a non-probabilistic epistemic analysis. Paralleling Battigalli and Siniscalchi (1999b), the key to a set-theoretic epistemic analysis of the BI algorithm will be finding a sufficiently rich possibility structure."

Unlike most of the above mentioned literature, they use the formalization of the notion of "all types" (resp. "all states") in the sense of beliefs completeness and not in the sense of universality. A straightforward cardinality-argument shows that - with at least two states of nature and at least two players - there is no beliefs complete possibility structure in the purely set-theoretic sense. I.e. there is no possibility structure such that for each player i and every nonempty subset $Y \subseteq S \times \Pi_{j\in I\setminus\{i\}} M_j$ there is a $m_i \in M_i$ with $R_i(m_i) = Y$.

So, to have a chance at all to get some kind of beliefs completeness one has to restrict attention to "interesting" or "important" events, as one does in the probabilistic context, where one considers just measurable events and not all subsets of the structure or in the case of Kripke structures (resp. knowledge spaces) where one restricts attention to events described (i.e. defined) by (sets of) formulas. (Above, we gave some justification for doing so.) As one does in the case of Kripke structures, Brandenburger and Keisler (1999) went on to use a mathematical language to get another notion of "all possible types" (resp. "all states") relative to that language, but as said above, in the sense of beliefs completeness and not in the sense of universality.

From a game theoretic point of view, the main conceptual invention of Brandenburger and Keisler (1999) was the use of another language than the (modal) languages mentioned above: They considered possibility structures as models of a (many-sorted) first-order logic. They defined the "*language induced by*" a possibility structure, which is, in model-theoretic terms, *the language of the elementary diagram* of the structure.

Their motivation to do so was as follows (Brandenburger 2001): "The language Brandenburger-Keisler set up is a first-order logic, with symbols for elements of the belief system. This seems like the natural choice, given that first-order logic is widely considered to be the basic language of mathematics (see e.g. Barwise 1977). A game is a mathematical structure, so a player uses first-order logic to reason about it."

The new notion of beliefs completeness they were defining is that of *first-order definably beliefs completeness*: A possibility structure is definably beliefs complete if for each player i and every nonempty subset $Y \subseteq S \times \Pi_{j \in I \setminus \{i\}} M_j$ that is defined by a formula of the language induced by the possibility structure there is a $m_i \in M_i$ with $R_i(m_i) = Y$.

Now, does a cardinality-argument still work to show that such a possibility structure cannot exist? Assume that I is finite, S is finite or countable and M_i is countable, for $i \in I$. Then it is easy to see that the language of the elementary diagram of the possibility structure is countable, hence there are at most countably many definable subsets of $S \times \Pi_{j \in I \setminus \{i\}} M_j$, for $i \in I$. Therefore it seems to be possible to have definably beliefs completeness. However, Brandenburger and Keisler (1999) showed that - unfortunately - it is still not possible, with at least two states of nature and at least two players, to have a (now: definably) beliefs complete possibility structure. The reason is that the language allows for too much self-reference. To see that, consider the following configuration of beliefs (Brandenburger and Keisler (1999) and Brandenburger (2001)):

"*Ann believes that Bob believes that Ann believes that Bob has a false belief about* (*the beliefs of*) *Ann.*"

Given this configuration, Ann believes that Bob has a false belief (about the beliefs of Ann) if and only if Ann does not believe that Bob has a false belief (about the beliefs of Ann). Of course, this argument is somewhat ambiguous and just meant to provide some intuition of what was going on in Brandenburger and Keisler (1999) on the precise mathematical level. Brandenburger and Keisler (1999) developed not just this verbal argument, but also the precise formal counterpart. For example (and this is the essence of their proof), they showed that (the formal counterparts of) all the beliefs of the above configuration are definable in the language induced by the structure. A definably beliefs complete possibility structure would have to contain this configuration, but this is impossible, hence a definably beliefs complete possibility structure cannot exist.

Although interesting in its own right (and illuminating a lack of our understanding of the players' reasoning), this result is not as desired. What was wished was a (positive) beliefs completeness result in order to get a non-probabilistic framework to analyze backward induction. Given the result of Brandenburger and Keisler (1999), there are two ways how this could be done. One possibility is to make topological assumptions on the space of states of nature as well as on the component spaces of the players and to require the possibility relations of the players to respect this structure in an appropriate way to produce a positive beliefs completeness result. Mariotti and Piccione (2000) followed this direction. Another possibility is to use another language than the one used by Brandenburger and Keisler (1999): "Completeness of a belief system is defined relative to a language. (We have to say how the players think, before we can say whether everything they can think of is present.) If we choose a different language from the one we just looked at, perhaps we can get a belief system that is complete relative to that language." (Brandenburger (2001)).

Another ingredient, not mentioned yet, is conditionality. We have to take into account how players would revise their beliefs should they observe new information. For example, by observing the previous actions taken by the other players during the play of a multistage game (with observed actions), a player learns something about their strategies. True, the traditional Harsanyi-like framework suffices to describe what a player would believe should he observe an event to which he ascribed a positive probability before (one could work then with the conditional probability distribution). But should he observe an event he assigned probability 0 before, the traditional framework does not specify his beliefs thereafter. Among others, Battigalli and Siniscalchi (1999a) and Brandenburger and Keisler (2000) developed extensions of Harsanyi type spaces to deal with this problem by using conditional probability systems in the sense of Rênyi (1955), resp. lexicographic probability systems developed by Blume, Brandenburger and Dekel (1991).

In Chapter 4, we follow both of the above mentioned avenues, the topological approach, as well as the definition of a language other than the one used by Brandenburger and Keisler to produce (positive) existence results of (definably) *beliefs complete* (respectively: universal) *conditional possibility structures.* Like Brandenburger and Keisler (1999), we allow for different spaces of states of nature S_i for the different players $i \in I$. For the application we have in mind, i.e. backward induction, since we assume a player knows his own strategy and is uncertain of the strategy-profile played by the other players, S_i would be the product of the strategy spaces of the other players. For each $i \in I$, we add a collection of nonempty subsets $\mathcal{B}_i \subseteq S_i$, *"observable events"*, which would reflect the knowledge of i about the others' strategies at the different stages of the game, such that $S_i \in \mathcal{B}_i$ (reflecting i's information before he makes any observation (ex ante)).

We define a *conditional possibility structure* on $(S_i, \mathcal{B}_i)_{i\in I}$ as a tuple $\left\langle (U^i)_{i\in I}, (\rho^i)_{i\in I} \right\rangle$, where each U^i is a nonempty set and each ρ^i is a function from U^i to $\prod_{B_i\in\mathcal{B}_i} \mathrm{Pow}_\emptyset \left(B_i \times \Pi_{j\neq i} U^j\right)$, where we impose some restrictions on how the players update their beliefs, given some new information. The set $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\big(S_i\times \Pi_{j\neq i}U^j\big)$ is the subset that satisfies these restrictions. Hence, each ρ^i is actually a function from U^i to $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i\times \Pi_{j\neq i}U^j\right)$. An element of $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i\times \Pi_{j\neq i}U^j\right)$ is called a *conditional i-event*, or a *(nonempty) conditional subset of* $S_i\times \Pi_{j\neq i}U^j$. A conditional i-event that is the image of a $u^i\in U^i$ under the mapping ρ^i is called a *conditional possibility set for* i, resp. *the conditional possibility set of* u^i.

In Section 4.3 of Chapter 4 we define *topological conditional possibility structures,* where we assume in addition that all the U^i and S_i are compact Hausdorff, all the $B_i\in\mathcal{B}_i$ are clopen and we replace each of the $\mathrm{Pow}_{\emptyset}\left(B_i\times \Pi_{j\neq i}U^j\right)$ by $\mathcal{V}\left(B_i\times \Pi_{j\neq i}U^j\right)$, the space of nonempty compact subsets of $B_i\times \Pi_{j\neq i}U^j$, which we endow with the *Vietoris topology*,[5] and accordingly we replace $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\big(S_i\times \Pi_{j\neq i}U^j\big)$ by a topological space denoted by $\mathcal{V}^{\mathcal{B}_i}\left(S_i\times \Pi_{j\neq i}U^j\right)$. We also require all the ρ^i to be continuous. Then, we define structure preserving maps (*morphisms*) between topological conditional possibility structures. By using a projective limit construction, we are able to construct a universal topological conditional possibility structure $\langle (T^i)_{i\in I}, (\delta^i)_{i\in I}\rangle$ such that every topological conditional possibility structure (for the same set of players and on the same sets of states of nature, of course) can be mapped to it by a unique morphism (Theorem 4.1 and Corollary 4.1). In addition, each δ^i is a homeomorphism. Hence the universal topological conditional possibility structure is beliefs complete (again Theorem 4.1 and Corollary 4.1). We note here that in the special unconditional (i.e. $\mathcal{B}_i=\{S_i\}$, for $i\in I$) two player (i.e. $I=\{a,b\}$) case, where both players are uncertain about the same space of states of nature (i.e. $S_a=S_b$), Mariotti and Piccione (2000) obtained the results of Theorem 4.1 and Corollary 4.1.

In the last section of Chapter 4 we restrict ourselves to a finite player set and a finite set of states of nature for each player. In a similar construction than the one used in the topological approach, we define inductively structures $\mathcal{A}_n$, for $n\in\mathbb{N}\cup\{-1\}$, and projections from the component spaces of each player in $\mathcal{A}_n$ onto those in $\mathcal{A}_m$, for $-1\leq m<n$, such that for each player i and every $-1\leq m<n$ the inverse image of every nonempty conditional subset of the product of i's space of states of nature and the other players' component spaces in $\mathcal{A}_m$ is the conditional possibility set of some type of player i in $\mathcal{A}_n$. The $\mathcal{A}_n$'s contain also the inverse images of the possibility relations of the $\mathcal{A}_m$'s, for $-1\leq m<n$, as well as sorts, the elements of which interpret the inverse images of the elements of the sorts of the $\mathcal{A}_m$'s, for $-1\leq m<n$. So, the structures $\mathcal{A}_n$, for $n\in\mathbb{N}$, are "extended" possibility structures that "contain" also the inverse images of the structures of lower index.

The many-sorted language $\mathcal{L}_n$ is the language induced by the structure $\mathcal{A}_n$. The language $\mathcal{L}_{n+1}$ contains the language $\mathcal{L}_n$ and is a kind of mixture between first-order and second-order logic, where the $\mathcal{L}_n$- part of $\mathcal{L}_{n+1}$ is the "second-order

[5]Some authors call this topology the *Hausdorff topology.*

logic part" of $\mathcal{L}_{n+1}$. The $\mathcal{L}_n$- part of $\mathcal{L}_{n+1}$ "talks" about the inverse image of $\mathcal{A}_n$ in $\mathcal{A}_{n+1}$. The definition of $\mathcal{L}_n$ and the construction of $\mathcal{A}_n$ are done in such a way that for each player i all nonempty $\mathcal{L}_{n-1}$-definable conditional subsets of the product of i's space of states of nature and the other players' component spaces in $\mathcal{A}_n$ are possibility sets for player i.

We construct the structure $\mathcal{A}$ as the "projective limit" of the structures $\mathcal{A}_n$, for $n \in \mathbb{N} \cup \{-1\}$. In particular, the possibility relation of each player in $\mathcal{A}$ is the projective limit of the finite-step possibility relations of that player. As above, $\mathcal{A}$ contains also the inverse images of the possibility relations of the $\mathcal{A}_n$'s, for $n \in \mathbb{N} \cup \{-1\}$, as well as sorts, the elements of which interpret the inverse images (and are therefore subsets) of the elements of the sorts of the $\mathcal{A}_n$'s, for $n \in \mathbb{N} \cup \{-1\}$. So, the structure $\mathcal{A}$ is also an "extended" possibility structure that "contains" also the inverse images of the structures of finite index.

The syntactic counterpart of $\mathcal{A}$ is the limit language $\mathcal{L}$, which is defined in the spirit of the neocompact language invented by Keisler (1998), but as the $\mathcal{L}_n$, and unlike Keisler's original definition, it is a kind of mixture between first and second-order logic. The definition of $\mathcal{L}$ is done in such a way that the union of all the $\mathcal{L}_n$'s is the "second-order part" of $\mathcal{L}$. Again, the $\mathcal{L}_n$- part of $\mathcal{L}$ "talks" about the inverse image of $\mathcal{A}_n$ in $\mathcal{A}$.

Finally, we are able to prove that the possibility correspondence pos_i, that sends $t^i \in T^i$ to the t^i-section of R^i, where R^i is the possibility relation of player i in $\mathcal{A}$, is a bijection from T^i to $\mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j)$, the space of nonempty $\mathcal{L}$-definable conditional i-events. It follows that $\mathcal{A}$ is $\mathcal{L}$-definably beliefs complete (Theorem 4.2, Corollary 4.2 and Corollary 4.3).

Chapter 1

On the Nonexistence of Universal Kripke Structures

1.1 Introduction

Nonprobabilistic beliefs in a group of players or agents are usually modeled by Kripke structures. Each player's beliefs are described by a binary relation on the set of states of the world. In a state of the world, a player believes an event if the event contains all those states of the world that are in relation with that state of the world. As in the case of type spaces, with each state of the world there is associated a state of nature[1] which specifies the values of all the relevant objective parameters of the players' interaction, for example the payoff functions, signals, or initial endowments. In this manner, one has described what each player believes about nature: In a state of the world, a player believes an event (i.e. a subset) in the space of states of nature if every state of nature that is associated with some state of the world that is in relation with the given state of the world, is an element of that event. Thus, it is possible to describe the event that a player believes some event in the space of states of nature, and hence the event that another player believes that player to believe this, and so on. In this way, all levels of mutual beliefs of the players can be described.

We pointed already out that, as in the case of type spaces, to justify the use of Kripke structures, it would be desirable to establish the existence of a universal Kripke structure to which every Kripke structure (on the same space of states of nature and for the same set of players) could be mapped in a beliefs-preserving manner. We also mentioned that in the special case of knowledge spaces, which are Kripke structures where for every player the relation ascribed to that player is an equivalence relation, there is - given at least two states of nature and two players - no universal knowledge space, while for type spaces there is a universal

[1]When a Kripke structure is used as semantics for modal logic, we have a valuation function instead, that describes which primitive propositions are true in a state of the world.

type space.

The question arises whether this difference comes from the different mathematical frameworks - sets and binary relations (respectively partitions) on one side, and measurable spaces and probability measures on the other side - or from the strong assumptions on the players' epistemic attitudes in the case of knowledge - especially the truth axiom ("what a player knows in a state of the world is true there", corresponding on a technical level to reflexivity of the players' binary relations), in contrary to the possibility that a player's type in some state of the world in a type space might be sure of (i.e. ascribing probability 1 to) some wrong event.

In the latter case, it would be possible that there is a universal Kripke structure for classes of Kripke structures with relaxed assumptions (in comparison to the case of knowledge) on the epistemic behavior of the players. Of course, "relaxed" here means also that knowledge spaces would satisfy these assumptions. In fact, almost all classes of Kripke structures considered in the literature contain the class of knowledge spaces.

To treat the question of the existence of a universal Kripke structure rigorously, we define structure preserving maps between Kripke structures, so-called *morphisms*. Then we show that, unfortunately, we have to abandon the hope of having a universal Kripke structure: For every class of Kripke structures that contains all knowledge spaces - given at least two players and two states of nature - there is no universal Kripke structure (Corollary 1.1), even if we do not require the morphisms from the Kripke structures to the universal Kripke structure to be unique (Theorem 1.1).

The result is not proved by adapting the methods of the proof of Heifetz and Samet (1998a). Instead, we use their result. The underlying idea is simple: Assuming the existence of a universal Kripke structure for a class of Kripke structures that contains all knowledge spaces, we "collect" all those states lying in the images of knowledge spaces under morphisms, thus constructing a universal knowledge space, which, by the result of Heifetz and Samet, cannot exist.

1.2 The Nonexistence Theorem

In all what follows in this chapter, let S be a nonempty *set of states of nature* and I a nonempty *set of players*.

Definition 1.1 A Kripke structure on S for player set I ("Kripke structure", for short) is a triplet

$$\underline{M} := \langle M, (\mathcal{K}_i)_{i \in I}, \theta \rangle$$

where

- M is a nonempty set,
- for $i \in I : \mathcal{K}_i \subseteq M \times M$,
- θ is function from M to S.

θ relates the states of M, the so-called *states of the world*, to the states of nature. $\theta(m)$ is the state of nature that corresponds to the state of the world m. $\mathcal{K}_i$ is the *possibility relation of player* i. $(m, n) \in \mathcal{K}_i$ means that in the state m player i considers the state n to be possible.

Definition 1.2 A Kripke structure $\underline{M}$ on S for player set I is called a *knowledge space* if for every $i \in I : \mathcal{K}_i$ is an equivalence relation.

The following definition captures the idea of mapping one Kripke structure to another in a way that preserves the structure of the spaces.

Definition 1.3 Let $\underline{M}$ and $\underline{M}'$ be Kripke structures on S for player set I. A *morphism from* $\underline{M}$ *to* $\underline{M}'$ is a function $f : M \to M'$ such that:

1. for all $m \in M$:

$$\theta'(f(m)) = \theta(m),$$

2. for all $i \in I$ and $m \in M$:

$$f(\mathcal{K}_i(m)) = \mathcal{K}'_i(f(m)),$$

where

$$\begin{aligned} \mathcal{K}_i(m) &:= \{n \in M \mid (m, n) \in \mathcal{K}_i\} \quad \text{and} \\ f(N) &:= \{f(n) \mid n \in N\}, \quad \text{for } N \subseteq M. \end{aligned}$$

Definition 1.4 Let $\underline{M}$ be a Kripke structure on S for player set I. For an event $E \subseteq M$ define

$$K_i(E) := \{m \in M \mid \mathcal{K}_i(m) \subseteq E\}.$$

$$K_i : \text{Pow}(M) \to \text{Pow}(M)$$

is called *i's belief operator*.

The following proposition, which is analogous to Proposition 2.1. of Heifetz and Samet (1998a), shows that Condition 2 of Definition 1.3 is equivalent to the preservation of beliefs in terms of the belief operators:

Proposition 1.1 *Let $\underline{M}$ and $\underline{M}'$ be Kripke structures on S for player set I. For a function $f : M \to M'$ the following are equivalent:*

1. *For all $i \in I$ and $m \in M$: $f(\mathcal{K}_i(m)) = \mathcal{K}'_i(f(m))$.*
2. *For all $i \in I$ and $E' \subseteq M'$: $f^{-1}(K'_i(E')) = K_i(f^{-1}(E'))$.*

This shows in particular that the morphisms defined by Heifetz and Samet (1998a), although defined for partition spaces, are the same as the ones defined here in the special case of knowledge spaces.

Proof The proposition is implied by the following facts, which follow from the definitions:

$$\begin{aligned} m \in K_i(f^{-1}(E')) \quad &\text{iff} \quad \mathcal{K}_i(m) \subseteq f^{-1}(E') \\ &\text{iff} \quad f(\mathcal{K}_i(m)) \subseteq E', \end{aligned}$$

and

$$\begin{aligned} m \in f^{-1}(K'_i(E')) \quad &\text{iff} \quad f(m) \in K'_i(E') \\ &\text{iff} \quad \mathcal{K}'_i(f(m)) \subseteq E'. \end{aligned}$$

■

Definition 1.5 Let $\mathcal{C}$ be a nonempty class of Kripke structures on S for player set I. A Kripke structure on S for player set I

$$\underline{\Omega}^{\mathcal{C}} := \langle \Omega, (\mathcal{K}_i^{\Omega})_{i \in I}, \theta^{\Omega} \rangle$$

is

- *weak-universal for $\mathcal{C}$* if for every Kripke structure $\underline{M} \in \mathcal{C}$ there is a morphism from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$,
- *universal for $\mathcal{C}$* if $\underline{\Omega}^{\mathcal{C}} \in \mathcal{C}$ and for every Kripke structure $\underline{M} \in \mathcal{C}$ there is a **unique** morphism from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$.

Definition 1.6 Let M be a nonempty set. A binary relation $\mathcal{K}$ on M is

- *serial* if for every $m \in M$ there is a $n \in M$ such that $(m, n) \in \mathcal{K}$,
- *Euclidean* if for all $m_1, m_2, m_3 \in M$: $(m_1, m_2) \in \mathcal{K}$ and $(m_1, m_3) \in \mathcal{K}$ imply $(m_2, m_3) \in \mathcal{K}$.

Definition 1.7 Let $\mathcal{M}_I$ (resp., $\mathcal{M}_I^r$; $\mathcal{M}_I^{rt}$; $\mathcal{M}_I^{est}$; $\mathcal{P}_I$) be the class of all Kripke structures on S for player set I (resp. the class of Kripke structures on S for player set I where all the relations are reflexive; reflexive and transitive; Euclidean, serial and transitive; reflexive, symmetric and transitive (i.e. knowledge spaces)).

These classes correspond to different axiom systems of modal logic, namely the systems K_I (resp., T_I; $\mathrm{S4}_I$; $\mathrm{KD45}_I$; $\mathrm{S5}_I$), see Fagin et al. (1995) for the definitions of these axiom systems. It is easy to see that all the above classes contain the class $\mathcal{P}_I$ (i.e. the class of knowledge spaces).

Theorem 1.1 *Let $\mathcal{C}$ be a class of Kripke structures on S for player set I that contains all knowledge spaces. If S and I have each at least two elements, then there is no weak-universal Kripke structure for $\mathcal{C}$.*

Of course, a universal Kripke structure for $\mathcal{C}$ is also a weak-universal Kripke structure for $\mathcal{C}$, hence:

Corollary 1.1 *Let $\mathcal{C}$ be a class of Kripke structures on S for player set I that contains all knowledge spaces. If S and I have each at least two elements, then there is no universal Kripke structure for $\mathcal{C}$.*

Corollary 1.2 *There is no weak-universal Kripke structure for the class of all Kripke structures on S for player set I, the class $\mathcal{M}_I^r$, the class $\mathcal{M}_I^{rt}$, the class $\mathcal{M}_I^{est}$, and the class $\mathcal{P}_I$.*

Proof As already remarked, every knowledge space on S for player set I belongs to each of these classes. ∎

Corollary 1.3 *There is no universal Kripke structure for the class of all Kripke structures on S for player set I, the class $\mathcal{M}_I^r$, the class $\mathcal{M}_I^{rt}$, the class $\mathcal{M}_I^{est}$, and the class $\mathcal{P}_I$.*

1.3 The Proof

Proof We use the nonexistence result of Heifetz and Samet, where it was shown that for the class of knowledge spaces there is no weak-universal Kripke structure which is itself a knowledge space.[2] Now, let us assume on the contrary that there is a weak-universal Kripke structure $\underline{\Omega}^{\mathcal{C}} = \langle \Omega, (\mathcal{K}_i^{\Omega})_{i \in I}, \theta^{\Omega} \rangle$ for $\mathcal{C}$. According to the assumptions of the theorem, $\mathcal{P}_I$ is contained in $\mathcal{C}$. Let

$$\Omega^{\mathcal{P}} := \left\{ \omega \in \Omega \mid \exists \underline{M} \in \mathcal{P}_I \ \exists \text{ morphism } h : \underline{M} \to \underline{\Omega}^{\mathcal{C}} \text{ such that } \omega \in h(M) \right\}.$$

Now define

$$\underline{\Omega}^{\mathcal{P}} := \left\langle \Omega^{\mathcal{P}}, (\mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}}))_{i \in I}, \theta^{\Omega} \lceil \Omega^{\mathcal{P}} \right\rangle.$$

We have to show:

1. $\underline{\Omega}^{\mathcal{P}}$ is a knowledge space,
2. $\underline{\Omega}^{\mathcal{P}}$ is weak-universal for $\mathcal{P}_I$.

This implies the desired contradiction.

1. $\Omega^{\mathcal{P}}$ is nonempty, because $\mathcal{P}_I$ is nonempty. That $\theta \lceil \Omega^{\mathcal{P}}$ is a function from $\Omega^{\mathcal{P}}$ to S is clear. Fix a player $i \in I$.
   Let $\omega_1 \in \Omega^{\mathcal{P}}$. Then, there is a knowledge space $\underline{M}$ in $\mathcal{P}_I$, $m_1 \in M$, and a morphism h from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$ such that $h(m_1) = \omega_1$. But then, $(m_1, m_1) \in \mathcal{K}_i^M$, hence $(h(m_1), h(m_1)) \in \mathcal{K}_i^{\Omega}$, so $(h(m_1), h(m_1)) \in \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$, i.e. $\mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$ is reflexive.
   Let $\omega_1, \omega_2 \in \Omega^{\mathcal{P}}$ and $(\omega_1, \omega_2) \in \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$. Then, there is a knowledge space $\underline{M}$ in $\mathcal{P}_I$, $m_1 \in M$, and a morphism h from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$ such that $h(m_1) = \omega_1$. We have $\omega_2 \in \mathcal{K}_i^{\Omega}(h(m_1)) = h(\mathcal{K}_i^M(m_1))$. So there is a $m_2 \in M$ such that $\omega_2 = h(m_2)$ and $(m_1, m_2) \in \mathcal{K}_i^M$. But then $(m_2, m_1) \in \mathcal{K}_i^M$ and therefore $(\omega_2, \omega_1) \in \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$, so $\mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$ is symmetric.
   Let $\omega_1, \omega_2, \omega_3 \in \Omega^{\mathcal{P}}$ and $(\omega_1, \omega_2), (\omega_2, \omega_3) \in \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$. Then, as above, there is a knowledge space $\underline{M}$ in $\mathcal{P}_I$, $m_1, m_2 \in M$, and a morphism h from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$ such that $(m_1, m_2) \in \mathcal{K}_i^M$, $h(m_1) = \omega_1$, and $h(m_2) = \omega_2$. We have $\omega_3 \in \mathcal{K}_i^{\Omega}(h(m_2)) = h(\mathcal{K}_i^M(m_2))$. So there is a $m_3 \in M$ such that $\omega_3 = h(m_3)$ and $(m_2, m_3) \in \mathcal{K}_i^M$. Then, $(m_1, m_3) \in \mathcal{K}_i^M$, so it follows that $\omega_3 \in h(\mathcal{K}_i^M(m_1)) = \mathcal{K}_i^{\Omega}(\omega_1)$. Therefore $(\omega_1, \omega_3) \in \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$, so $\mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})$ is transitive.

[2]Heifetz and Samet (1998a) state just that there is no universal knowledge space. However, in that paper their use of "universal" is identical with our use of "weak-universal" here.

2. Let $\underline{M} \in \mathcal{P}_I$. Then there is a morphism h from $\underline{M}$ to $\underline{\Omega}^{\mathcal{C}}$. For $m \in M$ is $h(m) \in \Omega^{\mathcal{P}}$ and

$$\theta^{\Omega}\lceil\Omega^{\mathcal{P}}(h(m)) = \theta^{\Omega}(h(m)) = \theta^{M}(m).$$

We have $h(\mathcal{K}_i^M(m)) \subseteq \Omega^{\mathcal{P}}$ and $h(m) \in \Omega^{\mathcal{P}}$, so

$$h(\mathcal{K}_i^M(m)) = \mathcal{K}_i^{\Omega}(h(m)) = \mathcal{K}_i^{\Omega} \cap (\Omega^{\mathcal{P}} \times \Omega^{\mathcal{P}})(h(m)).$$

Hence h is a morphism from $\underline{M}$ to $\underline{\Omega}^{\mathcal{P}}$.

■

Chapter 2

Finitely Additive Beliefs and Universal Type Spaces

2.1 Introduction

The seminal result of Mertens and Zamir (1985), namely the existence of a universal type space in the sense of Haransyi (1967/68) under the assumption that the underlying space of states of nature is compact and Hausdorff, has been generalized in various ways (mainly by relaxing the topological assumptions made there) by Brandenburger and Dekel (1993), Heifetz (1993), Mertens, Sorin and Zamir (1994), and finally to the general measure-theoretic case by Heifetz and Samet (1998)[1]. However, it has always been assumed that the players' beliefs are σ-additive. This seems to be a rather strong assumption on the epistemic attitudes of the players.

As we mentioned in the general introduction, Savage's Postulates (1954, 1972) imply subjective probabilities that are finitely but not countably additive. Given the importance of Savage's theory within decision theory (i.e. "one-player game theory"), it is natural to ask - and desirable to know - what happens if we describe the beliefs of players accordingly in an interactive context (games) by finitely additive probability measures? Does there still exist a universal type space? And if so, what does it look like? Is it, for example, the space of coherent hierarchies?

Still, we are dealing with (now finitely additive) measures, so we have to define a field of events. The question arises as to which measurability condition is the right one. Should the field of events be just a field, a σ-field or should we assume that all subsets of the space of states of the world are events, i.e. that the field is simply the power set? As this question does not seem to have a clear-cut answer, we analyze the existence/nonexistence of a universal type space

[1] Relevant are also Vassilakis (1992), Battigalli-Siniscalchi (1999) and the preference-based approach of Epstein and Wang (1996).

for finitely additive beliefs for a whole class of different measurability conditions that include the three above-mentioned cases.

We introduce the κ-fields, where κ is an (often regular) infinite cardinal number, as fields that are closed under the intersection of every set of events (i.e. subset of the field) that has cardinality strictly less than κ. It follows that $\aleph_0$-fields are fields in the usual sense and $\aleph_1$-fields are σ-fields in the usual sense. Then, we define ∞-fields as fields that are closed under the intersection of every set of events (of cardinality whatsoever). (∞-fields were also defined in Aumann (1999a) as "universal fields".)

We define κ-type spaces as type spaces where the set of measurable events in the set of states of the world, as well as the set of measurable events in the set of states of nature, is a κ-field, and ∞-type spaces as type spaces where the set of measurable events in the set of states of nature is the full power set and the set of measurable events in the set of states of the world is a ∞-field. Also, we define $*$-type spaces as type spaces where the set of measurable events in the set of states of the world as well as the set of measurable events in the set of states of nature is the full power set. Furthermore, we define type morphisms, i.e. structure preserving maps from one κ-type space (resp. ∞-type space or $*$-type space) to another (not necessarily different) one.

Given a nonempty set of players I, a nonempty set of states of nature S, and a κ-field Σ_S on S, we define, similar to Heifetz and Samet (1998b), a kind of modal language, the formulas of which we call κ-expressions. But if κ is uncountable, contrary to Heifetz and Samet (1998b), we allow also for formulas of infinite length (but strictly less than κ). Then, we collect the κ-descriptions (by means of κ-expressions) of all states of the world in all κ-type spaces on S for player set I. In this way, we construct in Section 2.3 a universal κ-type space on S for player set I to which every κ-type space on S for player set I can be mapped by a unique type morphism (Theorem 2.1). However, since we use all the κ-type spaces (on the same space of states of nature and for the same player set) to construct the universal κ-type space, this construction tells us little about the structure of the universal κ-type space. (As for example, in the Mertens-Zamir (1985) case, where we know in addition that the universal type space is the space of coherent hierarchies.)

As Heifetz (2001) has shown, there are coherent hierarchies of finitely additive (in fact even σ-additive) beliefs up to - but excluding - level ω (i.e. the first infinite level), that have at least two different finitely additive extensions to level ω. Does a similar phenomenon holds also on the higher transfinite levels of coherent hierarchies? Or, put differently in terms of expressions rather than hierarchies, is there, on the contrary, a regular cardinal $\widehat{\kappa}$ such that for all (regular) cardinals $\kappa > \widehat{\kappa}$, the $\widehat{\kappa}$-description of a state in a κ-type space determines already the κ-description? If this were the case, it would be unnecessary to consider κ-type spaces for $\kappa > \widehat{\kappa}$ and we could restrict ourselves to κ-type spaces for $\kappa \leq \widehat{\kappa}$. We show in Theorem 2.2, by using a probabilistic adaptation of the "sober-drunk"-

example of Heifetz and Samet (1998a), that - with at least two players and two states of nature - this is not the case. Hence, it makes sense to consider κ-type spaces for every (regular) infinite cardinal κ.

Also, this example implies that - again, with at least two players and two states of nature - there is no (weak) universal ∞-type space and no (weak) universal $*$-type space (Theorem 2.3 and Corollary 2.1).

In section 2.5, given a regular cardinal κ, a nonempty set of players I, a nonempty set of states of nature S, and a κ-field Σ_S on S, we construct the space of coherent hierarchies up to (but excluding) level κ and show that this space constitutes a product κ-type space (Theorem 2.4). On top of that, this space turns out to be the universal κ-type space (Theorem 2.5). Therefore we provide a characterization of the universal κ-type space that we were missing in section 2.3, which is independent of the property of being universal.

In section 2.6, we show that the universal κ-type space is beliefs complete. In addition, in the case $\kappa = \aleph_0$, the component space of each player is - up to isomorphism of measurable spaces - the space of finitely additive probability measures on the product of the space of states of nature and the other players' component spaces.

2.2 Preliminaries

First, we will define κ-measurable spaces, where κ-denotes here an (often regular) infinite cardinal number. For an introduction to ordinal and cardinal numbers, see Devlin (1993) or any other textbook on set theory. Then, we will develop parts of the theory of κ-measurable spaces needed in the sequel, collect some known facts about finitely additive (probability) measures and define the main objects of our study in this chapter, the κ-, ∞- and $*$-type spaces.

An infinite cardinal number κ is called *regular*, if it is not the supremum of a set of less than κ-many ordinal numbers which are all strictly smaller than κ. For example, $\aleph_0$ and all the $\aleph_{\alpha+1}$ are regular, while $\aleph_\omega$ is *singular* (i.e. infinite and not regular) ($\aleph_\omega = \sup\{\aleph_n \mid n < \omega\}$), where ω denotes here the first infinite ordinal number. For a set M, denote by $|M|$ the cardinality of M.

In this chapter, unless otherwise stated, $\alpha, \beta, \gamma, \zeta, \eta, \xi$ denote ordinal numbers, δ delta-measures, θ functions from the set of states of the world to the set of states of nature, κ cardinal numbers, λ limit ordinal numbers, μ and ν measures, π projections, φ, χ, ψ expressions, and ω, apart form above, sets of expressions.

Definition 2.1 Let κ be an infinite cardinal number and M a nonempty set.
A *κ-field on M* is a set $\Sigma \subseteq \mathrm{Pow(M)}$ such that:

1. $M \in \Sigma$,
2. $A \in \Sigma$ implies $M \setminus A \in \Sigma$,
3. $\mathcal{E} \subseteq \Sigma$ and $|\mathcal{E}| < \kappa$ imply $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E \in \Sigma$.

It follows that $\mathcal{E} \subseteq \Sigma$ and $|\mathcal{E}| < \kappa$ imply $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E \in \Sigma$.

Remark 2.1 *According to our definition, a set of subsets of a nonempty set is a $\aleph_0$-field iff it is a field (in the usual sense) and a $\aleph_1$-field iff it is a σ-field (in the usual sense). If $\kappa' < \kappa$, then every κ-field is a κ'-field.*

Definition 2.2 Let M be a nonempty set.
A *∞-field*[2] *on M* is a set $\Sigma \subseteq \mathrm{Pow(M)}$ such that:

1. $M \in \Sigma$,
2. $A \in \Sigma$ implies $M \setminus A \in \Sigma$,
3. $\mathcal{E} \subseteq \Sigma$ implies $\bigcap \mathcal{E} \in \Sigma$.

It follows that $\mathcal{E} \subseteq \Sigma$ implies $\bigcup \mathcal{E} \in \Sigma$.

Definition 2.3 A *κ-measurable space* is a pair (M, Σ), where M is a nonempty set and Σ is a κ-field on M.

Remark 2.2 *Let κ be a singular cardinal number and (M, Σ) be a κ-measurable space. Then Σ is already a κ^+-field, where κ^+ denotes the successor cardinal of κ.*

Proof Let $\mathcal{E} \subseteq \Sigma$ such that $|\mathcal{E}| \leq \kappa$. So, $\mathcal{E}$ has the form $\{E_\alpha \mid \alpha < \kappa\}$. Let $\widehat{\kappa} < \kappa$ be the cofinality of κ. Then there is a function $f : \widehat{\kappa} \to \kappa$, such that $\bigcup_{\beta < \widehat{\kappa}} f(\beta) = \kappa$. Note that $|f(\beta)| < \kappa$, for $\beta < \widehat{\kappa}$. It follows that $\bigcap_{\alpha < \kappa} E_\alpha = \bigcap_{\beta < \widehat{\kappa}} \left(\bigcap_{\alpha < f(\beta)} E_\alpha \right) \in \Sigma$. Since Σ is a field, it follows that it is a κ^+-field. ■

Since κ^+ is always regular, the above remark shows that it is redundant to consider κ-fields (κ-measurable spaces, respectively) if κ is a singular cardinal.

[2] What we call here "∞-field", is called "universal field" in Aumann (1999a).

Definition 2.4 A ∞-*measurable space* is a pair (M, Σ), where M is a nonempty set and Σ is a ∞-field on M.

Definition 2.5 A $*$-*measurable space* is a pair $(M, \mathrm{Pow}(M))$, where M is a nonempty set and $\mathrm{Pow}(M)$ is the power set (i.e. the set of all subsets) of M.

Note that every $*$-measurable space is a ∞-measurable space and every ∞-measurable space is a κ-measurable space for every infinite ordinal κ. A ∞-measurable space (M, Σ) is a $*$-measurable space iff for all $m \neq m' \in M$ there is a $E \in \Sigma$ such that $m \in E$ and $m' \notin E$.

Example 2.1 Let $M = \{0, 1\}$, $\Sigma = \{\emptyset, M\}$. (M, Σ) is ∞-measurable, but not $*$-measurable.

Remark 2.3 *If M is a nonempty set and $\mathcal{E} \subseteq \mathrm{Pow(M)}$, then the intersection of all κ-fields that contain $\mathcal{E}$ (as a subset) is a κ-field and it is the smallest one that contains $\mathcal{E}$. We denote this κ-field by $\kappa(\mathcal{E})$ and call it the κ-field generated by $\mathcal{E}$.*

Note that if κ is greater than κ', then $\kappa(\mathcal{E}) \supseteq \kappa'(\mathcal{E})$ and $\kappa(\mathcal{E})$ might strictly contain $\kappa'(\mathcal{E})$. If Σ is a κ-field and $\mathcal{E} \subseteq \Sigma$ such that $\Sigma = \kappa'(\mathcal{E})$, then also $\Sigma = \kappa(\mathcal{E})$, but there might be $\mathcal{E}' \subseteq \Sigma$ such that $\kappa(\mathcal{E}') = \Sigma$ and $\kappa'(\mathcal{E}')$ is strictly contained in Σ.

The following lemma shows that to prove the measurability of a function between two κ-measurable spaces, it is enough to proof the measurability for a generating set of the κ-field of the image space.

Lemma 2.1 *Let (M', Σ') and (M, Σ) be κ-measurable spaces, where κ is a regular cardinal number. Let $\mathcal{E} \subseteq \Sigma$ such that $\kappa(\mathcal{E}) = \Sigma$. Then $f : M' \to M$ is $\Sigma' - \Sigma$ measurable iff for all $E \in \mathcal{E} : f^{-1}(E) \in \Sigma'$.*

Proof Define $\mathcal{E}_0 := \mathcal{E} \cup \{M\}$. If $0 < \alpha < \kappa$ and if $\mathcal{E}_\beta$ is already defined for all $\beta < \alpha$, then define

$$\mathcal{E}_\alpha := \left\{ \bigcap_{A \in \Lambda} A \;\middle|\; \Lambda \subseteq \bigcup_{\beta<\alpha} \mathcal{E}_\beta \text{ s.t. } |\Lambda| < \kappa \right\} \cup \left\{ M \backslash A \;\middle|\; A \in \bigcup_{\beta<\alpha} \mathcal{E}_\beta \right\} \cup \bigcup_{\beta<\alpha} \mathcal{E}_\beta.$$

By transfinite induction, it is obvious that $\bigcup_{\beta<\kappa}\mathcal{E}_\beta \subseteq \Sigma$. We show that $\bigcup_{\beta<\kappa}\mathcal{E}_\beta$ is a κ-field, which implies that $\bigcup_{\beta<\kappa}\mathcal{E}_\beta = \Sigma$. We have to show that $\bigcup_{\beta<\kappa}\mathcal{E}_\beta$ is closed under the intersection of less than κ-many elements. The other properties are clear. So, let $\widehat{\kappa}$ be a cardinal number with $\widehat{\kappa} < \kappa$ and $E_\gamma \in \bigcup_{\beta<\kappa}\mathcal{E}_\beta$ for $\gamma < \widehat{\kappa}$. Let $f(\gamma) := \min\{\beta < \kappa \mid E_\gamma \in \mathcal{E}_\beta\}$. By the regularity of κ, we have $\zeta := \sup\{f(\beta) \mid \beta < \widehat{\kappa}\} < \kappa$, hence $\bigcap_{\gamma<\widehat{\kappa}} E_\gamma \in \mathcal{E}_{\zeta+1} \subseteq \bigcup_{\beta<\kappa}\mathcal{E}_\beta$.

Having now established that form of Σ, the lemma follows from the fact that inverse images of maps commute with arbitrary intersections and complements.

■

Definition 2.6 Let $(M_j, \Sigma_j)_{j\in J}$ be a family of κ-measurable spaces. Let $\mathcal{E}$ be the set of sets of the form $\Pi_{j\in J}E_j$, such that $E_j \in \Sigma_j$, for $j \in J$, and $E_j = M_j$ for all but finitely many[3] $j \in J$.
We define the *product κ-field* of the Σ_j, $j \in J$, on $\Pi_{j\in J}M_j$ as the κ-field $\kappa(\mathcal{E})$.

Convention 2.1 Let κ be a (regular) infinite cardinal number, let J and L be nonempty disjoint sets and let (M_i, Σ_i) be a κ-measurable space, for $i \in J \cup L$. Let the product space $M_J := \Pi_{j\in J}M_j$ be endowed with the product κ-field Σ_J of the Σ_j, $j \in J$, let the product space $M_L := \Pi_{l\in L}M_l$ be endowed with the product κ-field Σ_L of the Σ_l, $l \in L$, and let the product space $M_J \times M_L$ be endowed with the product κ-field $\Sigma_{\{J,L\}}$ of Σ_J and Σ_L. Consider the product space $M_{J\cup L} := \Pi_{i\in J\cup L}M_i$ and let $M_{J\cup L}$ be endowed with the product κ-field $\Sigma_{J\cup L}$ of the Σ_i, $i \in J \cup L$. Now, let

$$h : M_J \times M_L \to M_{J\cup L}$$

be defined by

$$h((m_j)_{j\in J}, (m_l)_{l\in L}) := (m_i)_{i\in J\cup L}.$$

Then, h is clearly one-to-one and onto, but also, by looking at the generators of the product κ-fields (and applying Lemma 2.1 to h and h^{-1}), one sees that h and h^{-1} are measurable. Hence h is an isomorphism of measurable spaces and we are justified to identify these two κ-measurable spaces, in all what follows (with some abuse of notation).

Lemma 2.2 *Let M be a nonempty set, let κ be a regular cardinal number and, for $\alpha < \kappa$, let Σ_α be a κ-field on M such that $\alpha < \beta < \kappa$ implies $\Sigma_\alpha \subseteq \Sigma_\beta$.*
Then $\Sigma := \bigcup_{\alpha<\kappa}\Sigma_\alpha$ is a κ -field on M.

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[3]It is easily seen that if we would change the definition, also "for $j \in J$: $E_j \in \Sigma_j$ and $E_j = M_j$ for all but one $j \in J$" and "for $j \in J$: $E_j \in \Sigma_j$ and $E_j = M_j$ for all but less than κ-many $j \in J$" would yield the same product κ-field as the one defined above.

Proof We have just to show that the intersection of less than κ-many Σ-measurable sets is a Σ-measurable set, the other properties are clear. So, let $\widehat{\kappa} < \kappa$ and $E_\beta \in \Sigma$, for $\beta < \widehat{\kappa}$. Let $f(\beta) := \min\{\alpha < \kappa \mid E_\beta \in \Sigma_\alpha\}$. By the regularity of κ, it follows that $\gamma := \sup\{f(\beta) \mid \beta < \widehat{\kappa}\} < \kappa$, hence $\bigcap_{\beta<\widehat{\kappa}} E_\beta \in \Sigma_\gamma \subseteq \Sigma$. ■

Definition 2.7 Let M be a nonempty set and $\mathcal{F}$ a field on M.
A *finitely additive measure on* $(M, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$, such that

- $0 \leq \mu(F)$, for all $F \in \mathcal{F}$,
- $\mu(E \cup F) = \mu(E) + \mu(F)$, for all disjoint $E, F \in \mathcal{F}$.

μ is a *finitely additive probability measure on* $(M, \mathcal{F})$, if in addition

- $\mu(M) = 1$.

Definition 2.8 Let M be a nonempty set, $\mathcal{F}$ a field on M, μ a finitely additive measure on $(M, \mathcal{F})$, and $E \subseteq M$.
We define the *outer measure of* E *induced by* μ as

$$\mu^*(E) := \inf\{\mu(F) \mid F \in \mathcal{F} \text{ such that } E \subseteq F\},$$

and the *inner measure of* E *induced by* μ as

$$\mu_*(E) := \sup\{\mu(F) \mid F \in \mathcal{F} \text{ such that } F \subseteq E\}.$$

If not stated otherwise, we keep in this chapter the following

Convention 2.2

- Products of κ-measurable spaces are endowed with the product κ-field.
- If (M, Σ) is a κ-measurable space, then $\Delta^\kappa(M, \Sigma)$ denotes the space of finitely additive probability measures on (M, Σ). We consider this space itself as a κ-measurable space endowed with the κ-field Σ_{Δ^κ} generated by all the sets $\{\mu \in \Delta^\kappa(M, \Sigma) \mid \mu(E) \geq p\}$, where $E \in \Sigma$ and $p \in [0, 1]$.

Of course, this convention depends on the particular κ chosen.

Remark 2.4 *Let (M', Σ') and (M, Σ) be κ-measurable spaces and let $f : M' \to M$ be measurable.*

1. *If μ' is a finitely additive probability measure on (M', Σ'), then $\mu'(f^{-1}(\cdot))$ (that is $\mu'(f^{-1}(E))$, for $E \in \Sigma$) is a finitely additive probability measure on (M, Σ).*
2. *If $\Delta^\kappa_f : \Delta^\kappa(M', \Sigma') \to \Delta^\kappa(M, \Sigma)$ is defined by $\Delta^\kappa_f(\mu') := \mu'(f^{-1}(\cdot))$, for $\mu' \in \Delta^\kappa(M', \Sigma')$, then Δ^κ_f is measurable, since we have $\Delta^\kappa_f(\mu')(E) \geq p$ iff $\mu'(f^{-1}(E)) \geq p$, for $E \in \Sigma$.*

Remark 2.5 *Let (M', Σ') and (M, Σ) be κ-measurable spaces and let $f : M' \to M$ be measurable and onto.*
Then:

1. *$f^{-1}(\Sigma) := \{f^{-1}(E) \mid E \in \Sigma\}$ is a κ -field on M' and a subset of Σ'.*
2. *If μ is a finitely additive measure on (M, Σ), then μ induces a finitely additive measure μ' on $(M', f^{-1}(\Sigma))$ defined by $\mu'(f^{-1}(E)) := \mu(E)$. Furthermore, if μ is a finitely additive probability measure, then μ' is a finitely additive probability measure.*

Lemma 2.3 *Let $\gamma < \alpha$ be ordinal numbers. For $\gamma \leq \beta < \alpha$ let $(M^\beta, \mathcal{F}^\beta)$ be a $\aleph_0$-measurable space (i.e. M^β is a nonempty set and $\mathcal{F}^\beta$ is a field on M^β) and μ^β a finitely additive probability measure on $(M^\beta, \mathcal{F}^\beta)$, let M^α be a nonempty set, and for $\gamma \leq \xi < \zeta \leq \alpha$ let $f_{\xi,\zeta} : M^\zeta \to M^\xi$ be onto and, if $\zeta < \alpha$, let $f_{\xi,\zeta}$ be $\mathcal{F}^\zeta$-$\mathcal{F}^\xi$-measurable, such that:*

1. *$f_{\xi,\beta} \circ f_{\beta,\zeta} = f_{\xi,\zeta}$, for all $\xi < \beta < \zeta$ such that $\gamma \leq \xi < \beta < \zeta \leq \alpha$,*
2. *$\mu^\beta\left(f^{-1}_{\xi,\beta}\left(E^\xi\right)\right) = \mu^\xi\left(E^\xi\right)$, for all $\xi < \beta$ such that $\gamma \leq \xi < \beta < \alpha$ and all $E^\xi \in \mathcal{F}^\xi$.*

Then:

- *$\bigcup_{\gamma \leq \beta < \alpha} f^{-1}_{\beta,\alpha}\left(\mathcal{F}^\beta\right)$ is a field on M^α,*
- *$\left(\mu^\beta\right)_{\gamma \leq \beta < \alpha}$ induces a well-defined finitely additive probability measure $\mu^{<\alpha}$ on $\left(M^\alpha, \bigcup_{\gamma \leq \beta < \alpha} f^{-1}_{\beta,\alpha}\left(\mathcal{F}^\beta\right)\right)$, defined by $\mu^{<\alpha}\left(f^{-1}_{\beta,\alpha}\left(E^\beta\right)\right) := \mu^\beta\left(E^\beta\right)$, for $E^\beta \in \mathcal{F}^\beta$.*

Proof That $\mu^{<\alpha}$ is well-defined follows from the above conditions 1 and 2 and the fact that the $f_{\beta,\alpha}$'s are onto. In light of the preceding remark, the rest is clear. ■

Notation 2.1 Let M be a nonempty set, $\mathcal{F}$ a field on M, and $E \subseteq M$. Then denote by $[\mathcal{F}, E]$ the set of all subsets of M of the form $(L \cap E) \cup (N \cap (M \setminus E))$, where $L, N \in \mathcal{F}$. It is easy to check that $[\mathcal{F}, E]$ is the smallest field that extends $\mathcal{F}$ and contains E as an element.

For further reference, we cite the following two lemmas (in a somewhat different form), which are theorems by Loś and Marczewski (1949) and Horn and Tarski (1948).

Lemma 2.4 *Let M be a nonempty set, $\mathcal{F}$ a field on M, $E \subseteq M$, μ a finitely additive probability measure on $(M, \mathcal{F})$, $\mu_*(E)$ the inner measure of E, $\mu^*(E)$ the outer measure of E, and p a real number such that $\mu_*(E) \leq p \leq \mu^*(E)$. Then there exists a finitely additive probability measure ν that extends μ to the field $[\mathcal{F}, E]$ such that $\nu(E) = p$.*

Proof Follows directly from Theorem 2 of Loś and Marczewski (1949). ■

Sometimes, we will refer to the above lemma as the "Loś-Marczewski Theorem".

Lemma 2.5 *Let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be fields on the nonempty set M and let μ be a finitely additive probability measure on $(M, \mathcal{F}_1)$. Then there exists an extension of μ to a finitely additive probability measure ν on $(M, \mathcal{F}_2)$.*

Proof Follows from 4 (i) of Loś and Marczewski (1949) and also from Horn and Tarski (1948), p. 477, Theorem 1.21. ■

For the rest of this section, unless otherwise stated, we fix a regular cardinal κ. Furthermore we fix a nonempty set of players I, a nonempty set of states of nature S, and, unless otherwise stated, a κ-field Σ_S on S, such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$. Without loss of generality, we assume that $0 \notin I$ and set $I_0 := I \cup \{0\}$.

We define now κ-type spaces, ∞-type spaces and $*$-type spaces, i.e. the objects which we will study in the rest of this chapter.

Definition 2.9 A *κ-type space on S for player set I* is a 4-tuple

$$\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- Σ is a κ-field on M,
- for $i \in I$: T_i is a $\Sigma - \Sigma_{\Delta^\kappa}$-measurable function form M to $\Delta^\kappa (M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), such that for all $m \in M$ and $A \in \Sigma$: $[T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, where $[T_i (m)] := \{m' \in M \mid T_i (m') = T_i (m)\}$,
- θ is a Σ-Σ_S-measurable function from M to S.

We will refer to the property that for all $i \in I$, $m \in M$ and $A \in \Sigma : [T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, as the "*introspection property*" of κ-type spaces (∞-type spaces and $*$-type spaces, resp., see below). Doing obvious changes, proofs (of the theorems in this chapter) would went through, if we were to abandon this property, in fact, things would be easier then.

Definition 2.10 A *∞-type space on S for player set I* is a 4-tuple

$$\underline{M} := \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- Σ is a ∞-field on M,
- for $i \in I : T_i$ is a measurable function form (M, Σ) to $\Delta^\infty (M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), endowed with the ∞-field generated by all the sets $\{\mu \in \Delta^\infty (M, \Sigma) \mid \mu (E) \geq p\}$, where $E \in \Sigma$ and $p \in [0, 1]$, such that for all $m \in M$ and $A \in \Sigma$: $[T_i (m)] \subseteq A$ implies $T_i (m) (A) = 1$, where $[T_i (m)] := \{m' \in M \mid T_i (m') = T_i (m)\}$,
- θ is a Σ-$\text{Pow}(S)$-measurable function from M to S.

Note that for $\mu \neq \nu \in \Delta^{\infty}(M, \Sigma)$ there is a $E \in \Sigma$ and a $p \in [0,1]$ such that $\mu(E) \geq p$ and $\nu(E) < p$. This implies that the ∞-field of $\Delta^{\infty}(M, \Sigma)$ is in fact $\mathrm{Pow}(\Delta^{\infty}(M, \Sigma))$, the full power set. Hence, by the measurability of T_i, we have $[T_i(m)] \in \Sigma$. So, in fact, the condition that $[T_i(m)] \subseteq A$ implies $T_i(m)(A) = 1$ reduces to $T_i(m)([T_i(m)]) = 1$.

By the definitions, it is obvious that every ∞-type space on S is a κ-type space on S, for every regular κ. (Set $\Sigma_S := \mathrm{Pow}(S)$ in the κ-type space.)

Definition 2.11 A *$*$-type space on S for player set I* is a 3-tuple

$$\underline{M} := \left\langle M, (T_i)_{i \in I}, \theta \right\rangle,$$

where

- M is a nonempty set,
- for $i \in I : T_i$ is a function form M to $\Delta(M, \mathrm{Pow}(M))$, the space of finitely additive probability measures on $(M, \mathrm{Pow}(M))$, such that for all $m \in M$: $T_i(m)([T_i(m)]) = 1$, where $[T_i(m)] := \{m' \in M \mid T_i(m') = T_i(m)\}$
- θ is a function from M to S.

It is obvious from the definitions, that every $*$-type space on S is a ∞-type space on S. (Set $\Sigma := \mathrm{Pow}(M)$ in the ∞-type space.) And hence, it is also a κ-type space on S.

We define now the beliefs preserving maps between type spaces.

Definition 2.12 Let $\underline{M}' = \left\langle M', \Sigma', (T_i')_{i \in I}, \theta' \right\rangle$ and $\underline{M} = \left\langle M, \Sigma, (T_i)_{i \in I}, \theta \right\rangle$ be κ-type spaces (∞-type spaces, $*$-type spaces, respectively) on S for player set I.

A function $f : M' \to M$ is a *type morphism* if it satisfies the following conditions:

1. f is $\Sigma' - \Sigma$-measurable,
2. for all $m' \in M'$:

$$\theta'(m') = \theta(f(m')),$$

3. for all $m' \in M'$, $E \in \Sigma$, and $i \in I$:

$$T_i(f(m'))(E) = T_i'(m')\left(f^{-1}(E)\right).$$

Note that the above definition of a type morphism does not depend on κ, that is, if $\kappa < \kappa'$, and $\underline{M}$ and $\underline{M}'$ are κ'-type spaces (∞-type spaces, $*$-type spaces, respectively), then $f : M' \to M$ is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as κ'-type spaces (∞-type spaces, $*$-type spaces, respectively) iff it is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as κ-type spaces. Similarly, if $\underline{M}'$ and $\underline{M}$ are $*$-type spaces, then $f : M' \to M$ is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as $*$-type spaces iff it is a type morphism from $\underline{M}'$ to $\underline{M}$ viewed as ∞-type spaces. (Note that in the case of $*$-type spaces, every function $f : M' \to M$ is measurable.)

Definition 2.13 A type morphism is a *type isomorphism*, if it is one-to-one, onto, and the inverse function is also a type morphism.

It is easy to see that a function $f : M' \to M$ is a type isomorphism iff it is a type morphism and isomorphism of the measurable spaces (M', Σ') and (M, Σ).

Definition 2.14 A *product κ-type space on S for player set I* is a 4-tuple

$$\underline{M} := \langle M, \Sigma, (T_i)_{i \in I}, \theta \rangle$$

such that there are κ-measurable spaces (M_j, Σ_j), for $j \in I_0$, such that (up to type isomorphism):

- $M_0 = S$,
- $\Sigma_0 = \Sigma_S$,
- $M = \Pi_{j \in I_0} M_j$,
- Σ is the product κ-field on M which is generated by the Σ_j, $j \in I_0$,
- for $i \in I$: T_i is a $\Sigma_i - \Sigma_{\Delta^\kappa}$-measurable function from M_i to $\Delta^\kappa(M, \Sigma)$, the space of finitely additive probability measures on (M, Σ), such that for all $m_i \in M_i : \operatorname{marg}_{M_i} (T_i (m_i)) = \delta_{m_i}$,
- $\theta : M_0 \to S$ is the identity on S.

Obviously, T_i, for $i \in I$, can be viewed as a $\Sigma - \Sigma_{\Delta^\kappa}$-measurable function from M, to $\Delta^\kappa(M, \Sigma)$ and θ can be viewed as a $\Sigma - \Sigma_S$-measurable function θ from M to S. (Take first the projection to M_i, for $i \in I$, (resp. to M_0) and then the original function.) So, every product κ-type space on S is a κ-type space on S. Note that if $i \in I$ and if m_i is the ith coordinate of m, then we have by the fifth point of the definition that $[T_i (m)] = \{m_i\} \times \Pi_{j \in I_0 \setminus \{i\}} M_j$.

An easy check shows:

Remark 2.6 *κ-type spaces on S for player set I (∞-type spaces, $*$-type spaces, respectively), as objects, and type morphisms, as morphisms, form a category.*

Definition 2.15

- A κ-type space $\underline{\Omega}$ on S for player set I (∞-type space, $*$-type space, respectively) is *weak-universal* if for every κ-type space $\underline{M}$ on S for player set I (∞-type space, $*$-type space, respectively) there is a type morphism from $\underline{M}$ to $\underline{\Omega}$.
- A κ-type space $\underline{\Omega}$ on S for player set I (∞-type space, $*$-type space, respectively) is *universal*[4] if for every κ-type space $\underline{M}$ on S for player set I (∞-type space, $*$-type space, respectively) there is a *unique* type morphism from $\underline{M}$ to $\underline{\Omega}$.

Remark 2.7 *Universal κ-type spaces on S for player set I (∞-type spaces, $*$-type spaces, respectively) are unique up to type isomorphism.*

Proof If $\underline{\Omega}$ and $\underline{U}$ are universal κ-type spaces (∞-type spaces, $*$-type spaces, respectively) (on the same space of states of nature and for the same player set, of course), then there are type morphisms $f : U \to \Omega$ and $g : \Omega \to U$. It is easy to check, that the composite of two type morphisms is also a type morphism and that the identity is always a type morphism form a κ-type space $\underline{\Omega}$ (∞-type space, $*$-type space, respectively) to itself. By the uniqueness , it follows that $g \circ f = \mathrm{id}_U$ and therefore f is one-to-one and g is onto, and $f \circ g = \mathrm{id}_\Omega$ and therefore g is one-to-one and f is onto. f and g are type morphisms and $f = g^{-1}$ and $g = f^{-1}$. ■

2.3 The Universal κ-Type Space in Terms of Expressions

Again, for this section, unless otherwise stated, we fix a regular cardinal κ, a nonempty player set I, and a κ-measurable space of states of nature (S, Σ_S) such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$.

Given these data, we define κ-*expressions* (allowing also for infinite conjunctions) which are natural generalizations of the *expressions* defined by Heifetz and

[4] We use here the term "universal type space" although, in terms of category theory the term "terminal type space" would be the adequate one, since the universal type space is a terminal object in the category of type spaces. However, we take the former notion to keep the terms of the already existing type space literature.

Samet (1998b). Expressions are defined in a similar fashion as, for example, the formulas of the probability logic of Heifetz and Mongin (2001) (or ours in the next chapter). Analogous to Heifetz and Samet (1998b), given a κ-type space on S for player set I and a state of the world in this type space, we define the *κ-description* of this state as the set of those *κ-expressions* that are true in this state of the world. Then, we show that the set of all κ-descriptions constitutes a κ-type space (Proposition 2.3) and that this κ-type space is the universal κ-type space (Theorem 2.1).

Definition 2.16 For a κ-type space $\left\langle M, \Sigma, (T_i)_{i\in I}, \theta\right\rangle$ on S for player set I, $i \in I$, $E \in \Sigma$, and $p \in [0,1]$ define

$$\overline{B}_i^p(E) := \{m \in M \mid T_i(m)(E) \geq p\}.$$

Note that $\overline{B}_i^p(E) = T_i^{-1}(\{\mu \in \Delta^\kappa(M,\Sigma) \mid \mu(E) \geq p\})$ and that $\overline{B}_i^p(E) \in \Sigma$, if $E \in \Sigma$.

Definition 2.17 Given a κ-measurable space of states of nature (S, Σ_S) and a nonempty player set I, the set Φ^κ of *κ-expressions* is the least set such that:

1. every $E \in \Sigma_S$ is a κ-expression,
2. if φ is a κ-expression, then $\neg\varphi$ is a κ-expression,
3. if φ is a κ-expression, then $B_i^p(\varphi)$ is a κ-expression, for $i \in I$ and $p \in [0,1]$,
4. if Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$, then $\bigwedge_{\varphi\in\Psi}\varphi$ is a κ-expression.

If Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$, then we set $\bigvee_{\varphi\in\Psi}\varphi := \neg\bigwedge_{\varphi\in\Psi}\neg\varphi$.

Since we work here with a fixed regular κ, we omit sometimes, in this section, the superscript κ.

Definition 2.18 Let $\underline{M} := \left\langle M, \Sigma, (T_i)_{i\in I}, \theta\right\rangle$ be a κ-type space on S for player set I. Define

1. $E^{\underline{M}} := \theta^{-1}(E)$, for $E \in \Sigma_S$,
2. $(\neg\varphi)^{\underline{M}} := M \setminus \varphi^{\underline{M}}$, for $\varphi \in \Phi^\kappa$,
3. $(B_i^p(\varphi))^{\underline{M}} := \overline{B}_i^p\left(\varphi^{\underline{M}}\right)$, for $\varphi \in \Phi^\kappa$, $i \in I$ and $p \in [0,1]$,

4. $\left(\bigwedge_{\varphi\in\Psi}\varphi\right)^{\underline{M}} := \bigcap_{\varphi\in\Psi}\varphi^{\underline{M}}$, for Ψ such that $\emptyset \neq \Psi \subseteq \Phi^{\kappa}$ and $|\Psi| < \kappa$.

So, defined as above, κ-expressions define measurable subsets of M. It is easy to check that $\left(\bigvee_{\varphi\in\Psi}\varphi\right)^{\underline{M}} := \bigcup_{\varphi\in\Psi}\varphi^{\underline{M}}$, for Ψ such that $\emptyset \neq \Psi \subseteq \Phi^{\kappa}$ and $|\Psi| < \kappa$.

If no confusion may arise, we omit - with some abuse of notation - the superscript $\underline{M}$.

Definition 2.19 For a κ-type space $\underline{M} := \left\langle M, \Sigma, (T_i)_{i\in I}, \theta\right\rangle$ on S for player set I and $m \in M$ define $D^{\kappa}(m)$, the *κ-description* of m, as

$$D^{\kappa}(m) := \left\{\varphi \in \Phi^{\kappa} \mid m \in \varphi^{\underline{M}}\right\}.$$

Again, we omit sometimes, in this section, the superscript κ.

Proposition 2.1 *Let $\left\langle M, \Sigma, (T_i)_{i\in I}, \theta\right\rangle$ and $\left\langle N, \Sigma^N, \left(T_i^N\right)_{i\in I}, \theta^N\right\rangle$ be κ-type spaces on S for player set I and let $f : M \to N$ be a type morphism.*
Then, for all $m \in M$:

$$D^{\kappa}(f(m)) = D^{\kappa}(m).$$

Proof We show by induction on the formation of the expressions that $m \in \varphi^M$ iff $f(m) \in \varphi^N$:

- Let $E \in \Sigma_S$. We have $\theta^N(f(m)) = \theta(m)$, so $f(m) \in E^N$ iff $m \in E^M$.
- We have
$$f(m) \in (\neg\varphi)^N \text{ iff } f(m) \notin \varphi^N \text{ iff } m \notin \varphi^M \text{ iff } m \in (\neg\varphi)^M.$$
- Let Ψ be a nonempty set of expressions with $|\Psi| < \kappa$. Then:
$$f(m) \in (\bigwedge_{\varphi\in\Psi}\varphi)^N \text{ iff for all } \varphi \in \Psi: \ f(m) \in \varphi^N,$$
which is by induction hypothesis the case iff for all $\varphi \in \Psi : m \in \varphi^M$, which is the case iff $m \in (\bigwedge_{\varphi\in\Psi}\varphi)^M$.

- We have

$$f(m) \in \left(B_i^p(\varphi)\right)^N \text{ iff } T_i^N(f(m))\left(\varphi^N\right) \geq p \text{ iff } T_i(m)\left(f^{-1}\left(\varphi^N\right)\right) \geq p.$$

  By the induction hypothesis: $f^{-1}\left(\varphi^N\right) = \varphi^M$. Hence $T_i(m)\left(f^{-1}\left(\varphi^N\right)\right) = T_i(m)\left(\varphi^M\right)$. We have $T_i(m)\left(\varphi^M\right) \geq p$ iff $m \in \left(B_i^p(\varphi)\right)^M$. It follows that

$$f(m) \in \left(B_i^p(\varphi)\right)^N \text{ iff } m \in \left(B_i^p(\varphi)\right)^M.$$

■

Definition 2.20 Define Ω^κ to be the set of all κ-descriptions of states of the world in κ-type spaces on S for player set I. For $\varphi \in \Phi^\kappa$ define

$$[\varphi] := \{\omega \in \Omega^\kappa \mid \varphi \in \omega\}.$$

Again, we omit sometimes, in this section, the superscript κ.

Obviously, we have $\Omega \setminus [\varphi] = [\neg\varphi]$ and $\bigcap_{\psi\in\Psi}[\psi] = \left[\bigwedge_{\psi\in\Psi}\psi\right]$, where φ is an κ-expression and Ψ is a nonempty set of κ-expressions with $|\Psi| < \kappa$. It follows that:

Remark 2.8 *The set*

$$\Sigma_\Omega := \{[\varphi] \mid \varphi \in \Phi^\kappa\}$$

is a κ-field on Ω.

Lemma 2.6 *For every κ-type space $\underline{M}$ on S for player set I and for every $\varphi \in \Phi^\kappa$, the κ-description map $D : M \to \Omega$ satisfies*

$$D^{-1}([\varphi]) = \varphi^{\underline{M}}.$$

Proof Clear by the definition of $[\varphi]$. ■

Note that Lemma 2.6 implies that D is measurable.

Proposition 2.2 *For every $i \in I$ there exists a function*

$$T_i^* : \Omega \to \Delta^\kappa (\Omega, \Sigma_\Omega)$$

such that for every κ-type space $\underline{M}$ on S for player set I with κ-description map D and every $m \in M$:

$$T_i^* (D (m)) = T_i (m) \circ D^{-1}.$$

Proof For $\omega \in \Omega$ chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D (m) = \omega$. For $[\varphi] \in \Sigma_\Omega$ define

$$T_i^* (\omega) ([\varphi]) := T_i (m) \circ D^{-1} ([\varphi]) .$$

We have

$$T_i (m) \circ D^{-1} ([\varphi]) = T_i (m) \left(\varphi^{\underline{M}}\right) = \sup \{p \mid B_i^p (\varphi) \in D (m)\} ,$$

so $T_i^* (\omega) ([\varphi])$ depends just on $D (m)$ and is well-defined. By Remark 2.4, we have

$$T_i (m) \circ D^{-1} \in \Delta^\kappa (\Omega, \Sigma_\Omega) .$$

■

Lemma 2.7 *There is a measurable function $\theta^* : \Omega \to S$ such that for every κ-type space $\underline{M}$ on S for player set I and every $m \in M$:*

$$\theta^* (D (m)) = \theta (m) .$$

Proof Let

$$d_0 (m) := \left\{E \in \Sigma_S \mid m \in \theta^{-1} (E)\right\} .$$

Obviously, $d_0 (m) = D (m) \cap \Sigma_S$. By the properties of (S, Σ_S), we have for all $s \in S : \{s\} = \bigcap_{s \in E \in \Sigma_S} E$. It follows for every κ-type space $\underline{M}'$ on S for player set I and $m' \in M'$ that

$$\theta (m') = s \text{ iff } d_0 (m') = \{E \mid s \in E\} .$$

For $\omega \in \Omega$ chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$, such that $D (m) = \omega$. Define now $\theta^* (\omega) := \theta (m)$. Since $\theta (m)$ just depends on $D (m)$, $\theta^* (\omega)$ is well-defined.

It remains to show that θ^* is measurable: Let $E \in \Sigma_S$. We have

$$\theta^* (D (m)) \in E \text{ iff } m \in \theta^{-1} (E) \text{ iff } E \in D (m) \text{ iff } D (m) \in [E] .$$

It follows that $\theta^{*-1} (E) = [E]$. ■

Proposition 2.3

$$\left\langle \Omega, \Sigma_\Omega, \left(T_i^*\right)_{i\in I}, \theta^*\right\rangle$$

is a κ-type space on S for player set I.

Proof It remains to show:

1. Ω is nonempty.
2. For every $i \in I$: T_i^* is measurable as a function from Ω to $\Delta^\kappa\left(\Omega, \Sigma_\Omega\right)$.
3. For every $i \in I$, $\omega \in \Omega$ and $A \in \Sigma_\Omega$: If

$$\left\{\omega' \in \Omega \mid T_i^*\left(\omega'\right) = T_i^*\left(\omega\right)\right\} \subseteq A,$$

then $T_i^*\left(\omega\right)\left(A\right) = 1$.

To:

1. Let $M := \{m\}$ and chose $s \in S$. Set $\Sigma := \operatorname{Pow}\left(M\right)$, $T_i\left(m\right) := \delta_m$, for $i \in I$, and $\theta\left(m\right) := s$. Then

$$\left\langle M, \Sigma, \left(T_i\right)_{i\in I}, \theta\right\rangle$$

is a κ-type space (even a $*$-type space) on S for player set I.

2. Since inverse images commute with unions, intersections and complements, it is enough to show that $T_i^{*-1}\left(b^p\left(E\right)\right) \in \Sigma_\Omega$, for

$$b^p\left(E\right) := \left\{\mu \in \Delta^\kappa\left(\Omega, \Sigma_\Omega\right) \mid \mu\left(E\right) \geq p\right\},$$

where $E \in \Sigma_\Omega$ and $p \in [0,1]$. We have

$$T_i^{*-1}\left(b^p\left(E\right)\right) = \left\{\omega \in \Omega \mid T_i^*\left(\omega\right)\left(E\right) \geq p\right\}.$$

Since $E \in \Sigma_\Omega$, there is a κ-expression φ such that $E = [\varphi]$. Note that if $p \in [0,1]$ and $p = \sup\left\{q \mid B_i^q\left(\varphi\right) \in \omega\right\}$, then $B_i^p\left(\varphi\right) \in \omega$. This implies that

$$\omega \in T_i^{*-1}\left(b^p\left([\varphi]\right)\right) \text{ iff } B_i^p\left(\varphi\right) \in \omega.$$

It follows that $T_i^{*-1}\left(b^p\left(E\right)\right) = \left[B_i^p\left(\varphi\right)\right]$.

3. Let φ be a κ-expression and
$$\{\omega' \in \Omega \mid T_i^*(\omega') = T_i^*(\omega)\} \subseteq [\varphi].$$
Chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D(m) = \omega$. Let $m' \in M$. If $T_i^*(D(m')) \neq T_i^*(D(m))$, then there is a κ-expression ψ such that $T_i(m')(\psi^M) \neq T_i(m)(\psi^M)$. It follows that
$$D(\{m' \in M \mid T_i(m') = T_i(m)\}) \subseteq \{\omega' \in \Omega \mid T_i^*(\omega') = T_i^*(\omega)\},$$
which implies
$$\{m' \in M \mid T_i(m') = T_i(m)\} \subseteq D^{-1}([\varphi]) = \varphi^M.$$
So we have
$$1 = T_i(m)(\varphi^M) = T_i(m) \circ D^{-1}([\varphi]) = T_i^*(\omega)([\varphi]).$$

■

Lemma 2.8 *The κ-description map*
$$D : \Omega \to \Omega$$
is the identity.

Proof For $\omega \in \Omega$, we have
$$\omega = \{\varphi \in \Phi \mid \omega \in [\varphi]\}.$$
We have to show that for every κ-expression φ and every $\omega \in \Omega : \omega \in \varphi^{\underline{\Omega}}$ iff $\omega \in [\varphi]$. We know this already if $\varphi = E$, where $E \in \Sigma_S$. It is obvious that $\Omega \setminus [\varphi] = [\neg\varphi]$, and that if Ψ is a nonempty set of κ-expressions of cardinality $< \kappa$, then
$$\bigcap_{\varphi \in \Psi} [\varphi] = \left[\bigwedge_{\varphi \in \Psi} \varphi\right].$$
So it remains to show that $[\varphi] = \varphi^{\underline{\Omega}}$ implies $[B_i^p(\varphi)] = \overline{B}_i^p([\varphi])$. For $\omega \in \Omega$, chose a κ-type space $\underline{M}$ on S for player set I and $m \in M$ such that $D(m) = \omega$. We have
$$D(m) \in [B_i^p(\varphi)] \text{ iff } B_i^p(\varphi) \in D(m) \text{ iff } T_i(m)\left(\varphi^{\underline{M}}\right) \geq p.$$
But we have
$$T_i^*(\omega)([\varphi]) = T_i(m) \circ D^{-1}([\varphi]) = T_i(m)\left(\varphi^{\underline{M}}\right).$$
This implies that $[B_i^p(\varphi)] = \overline{B}_i^p([\varphi])$. ■

Theorem 2.1 *The space*

$$\left\langle \Omega, \Sigma_\Omega, (T_i^*)_{i\in I}, \theta^* \right\rangle$$

is a universal κ-type space on S for player set I.

Proof According to Lemma 2.6, for every κ-type space $\underline{M}$ on S for player set I, the κ-description map $D : M \to \Omega$ is measurable, and according to Proposition 2.2 and Lemma 2.7, D is a type morphism. It remains to show that it is the unique type morphism from $\underline{M}$ to $\underline{\Omega}$. But this clear by Proposition 2.1 and Lemma 2.8. ■

2.4 Spaces of Arbitrary Complexity

Is there a cardinal κ, such that the κ-descriptions determine already the κ'-descriptions, for all cardinal numbers $\kappa' > \kappa$? In the sequel, using a probabilistic adaptation of the elegant "sober-drunk" example of Heifetz and Samet (1998a) (see that paper also for the "story" interpreting the mathematical structure), we construct, for every regular cardinal κ', a κ'-type space (in fact even a $*$-type space), such that for every ordinal $\alpha < \kappa'$ there are at least two states of the world such that for every κ'-expression of depth $\leq \alpha$ this κ'-expression is true either in both states or in none of the two, and yet there is a κ'-expression of depth $\alpha + 1$ that is true in one state and not in the other. Thus, we answer the above question in the negative. Hence, it makes sense to consider κ-type spaces for every regular cardinality κ whatsoever.

In addition, this example will imply that, for at least two players and two states of nature, there is no universal $*$-type space and no universal ∞-type space (Theorem 2.3 and Corollary 2.1).

For this section, let $I := \{a, b\}$ be the set of players (the following analysis can be trivially extended to more than two players). We fix a set of states of nature $S = \{h, t\}$, consisting of the two possible outcomes of tossing a coin, h(ead) and t(ail).

To simplify the notation let us make the following

Convention 2.3 $\{i, j\} := \{a, b\}$, that is $j = \begin{cases} a, & \text{if} \quad i = b, \\ b, & \text{if} \quad i = a. \end{cases}$

The following three definitions of are taken form Heifetz and Samet (1998a).

Definition 2.21 Let α be an ordinal. A *record of length* α is a sequence $r^\alpha = (r(\beta))_{\beta<\alpha}$ of numbers "0" and "1" such that for every limit ordinal $\lambda \leq \alpha$ there is an ordinal $\gamma < \lambda$ such that $r(\beta) = 0$ for all ordinals β that satisfy $\gamma \leq \beta < \lambda$.

For every infinite ordinal γ there are a unique natural number n and a unique limit ordinal $\widehat{\lambda}$ such that $\gamma = \widehat{\lambda} + n$. We say γ is *even* or *odd* according to wether n is even or odd. If γ is a finite ordinal, i.e. a natural number, we take the usual notion of being even or odd.

Definition 2.22 Let α be an ordinal, r^α a record of length α, and λ a limit ordinal $\leq \alpha$. By the definition of a record, there is a minimal ordinal $o^\lambda(r^\alpha) < \lambda$ such that $r^\alpha(\beta) = 0$, for all β with $o^\lambda(r^\alpha) \leq \beta < \lambda$.
Define λ-par(r^α), the λ-*parity* of r^α, as

$$\lambda - \text{par}(r^\alpha) := \begin{cases} \text{even, if } o^\lambda(r^\alpha) \text{ is even,} \\ \text{odd, if } o^\lambda(r^\alpha) \text{ is odd.} \end{cases}$$

Note that by the definition of a record, $o^\lambda(r^\alpha)$ must be either 0 or a successor ordinal (i.e. $o^\lambda(r^\alpha) = \gamma + 1$, for some ordinal γ).

Definition 2.23 Let α be an ordinal. Define the spaces W^α by

- $$W^0 := \{h, t\},$$

- $$W^\alpha := \{(w_0, w_a^\alpha, w_b^\alpha) \mid w_0 \in \{h, t\}, w_a^\alpha \text{ and } w_b^\alpha \text{ are records of length } \alpha\},$$

  if $\alpha \geq 1$.

Definition 2.24

- If $0 < \beta \leq \alpha$ and $r^\alpha = (r(\xi))_{\xi<\alpha}$ is a record of length α, then denote by $r^\alpha\lceil\beta$ the record $(r(\xi))_{\xi<\beta}$ of length β.
- If $0 \leq \alpha$ and $w^\alpha \in W^\alpha$, then define $w^\alpha\lceil 0 := w_0$.
- If $0 < \beta \leq \alpha$ and $w^\alpha \in W^\alpha$, then define $w^\alpha\lceil\beta := (w_0, w_a^\alpha\lceil\beta, w_b^\alpha\lceil\beta)$.

By the definition, it is obvious that $w^{\alpha}\lceil\beta \in W^{\beta}$, for every $\beta < \alpha$.

Definition 2.25 Let $0 \leq \beta \leq \alpha$. Define

$$\pi_{\beta,\alpha} : W^{\alpha} \to W^{\beta}$$

by $\pi_{\beta,\alpha}(w^{\alpha}) := w^{\alpha}\lceil\beta$.

It is obvious that $\pi_{\xi,\beta}(\pi_{\beta,\alpha}(w^{\alpha})) = \pi_{\xi,\alpha}(w^{\alpha})$, for $0 \leq \xi \leq \beta \leq \alpha$.

Remark 2.9 *Let $0 \leq \beta < \alpha$, $w^{\beta} \in W^{\beta}$, and $i \in \{a, b\}$. Then there are $w^{\alpha}, u^{\alpha} \in W^{\alpha}$ such that*

$$w^{\alpha}\lceil\beta = u^{\alpha}\lceil\beta = w^{\beta} \text{ and } 0 = w_i^{\alpha}(\beta) \neq u_i^{\alpha}(\beta) = 1.$$

And in particular, it follows that $\pi_{\beta,\alpha} : W^{\alpha} \to W^{\beta}$ is onto.

Let $w^{\alpha} = (w_0, w_a^{\alpha}, w_b^{\alpha})$ be a state in W^{α}. We define the element of i's partition that contains this element:

Definition 2.26 Let α be an ordinal > 0, and $w^{\alpha} \in W^{\alpha}$. We define:

$$P_i(w^{\alpha}) := \left\{ (v_0, v_a^{\alpha}, v_b^{\alpha}) \in W^{\alpha} \;\middle|\; \begin{array}{l} v_i^{\alpha} = w_i^{\alpha}, \\ w_i^{\alpha}(0) = 1 \text{ implies } v_0 = w_0, \\ \text{for all } \beta \text{ such that } \beta + 1 < \alpha : \\ \ w_i^{\alpha}(\beta+1) = 1 \text{ implies } v_j^{\alpha}(\beta) = w_j^{\alpha}(\beta), \\ \text{for every limit ordinal } \lambda < \alpha : \\ \ w_i^{\alpha}(\lambda) = 1 \text{ implies } \lambda - \text{par}\left(v_j^{\alpha}\right) = \lambda - \text{par}\left(w_j^{\alpha}\right). \end{array} \right\}.$$

Remark 2.10
- *Let α be an ordinal > 0. The set $\{P_i(w^{\alpha}) \mid w^{\alpha} \in W^{\alpha}\}$ is a partition of W^{α} and $w^{\alpha} \in P_i(w^{\alpha})$.*
- *Let $0 < \beta < \alpha$ and $u^{\alpha} \in P_i(w^{\alpha})$. Then $u^{\alpha}\lceil\beta \in P_i(w^{\alpha}\lceil\beta)$, and hence $\pi_{\beta,\alpha}^{-1}(P_i(w^{\alpha}\lceil\beta)) \supseteq P_i(w^{\alpha})$.*

It is easy to see that if α is an infinite ordinal number, then the cardinality of W^{α} is the same as the cardinality of α. (To see that, in the case of an infinite α, the cardinality of W^{α} does not exceed that of α, note that the definition of a record implies that there are only finitely many $\beta < \alpha$ such that $r^{\alpha}(\beta) = 1$. (Consider, assuming the contrary, the minimal $\gamma \leq \alpha$ such that there are infinitely many $\beta < \gamma$ with $r^{\alpha}(\beta) = 1$.))

Lemma 2.9 *Let γ be an ordinal > 0, $\alpha = \gamma + 1$, $w^{\alpha} \in W^{\alpha}$, and*

$$E \in \left[\left[\bigcup_{\beta<\gamma} \pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^{\beta}\right)\right), \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right], P_i\left(w^{\alpha}\right)\right].$$

Then there are a $\beta < \gamma$ and $A_{\beta}, C_{\beta}, D_{\beta} \in \mathrm{Pow}\left(W^{\beta}\right)$ such that

$$\begin{aligned} E &= \left(\pi^{-1}_{\beta,\alpha}\left(A_{\beta}\right) \cap P_i\left(w^{\alpha}\right)\right) \cup \left(\pi^{-1}_{\beta,\alpha}\left(C_{\beta}\right) \cap \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right) \cap \left(W^{\alpha} \setminus P_i\left(w^{\alpha}\right)\right)\right) \\ &\quad \cup \left(\pi^{-1}_{\beta,\alpha}\left(D_{\beta}\right) \cap \left(W^{\alpha} \setminus \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right)\right). \end{aligned}$$

Proof By the definition of

$$\left[\left[\bigcup_{\beta<\gamma} \pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^{\beta}\right)\right), \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right], P_i\left(w^{\alpha}\right)\right],$$

E has the form

$$\begin{aligned} E = &\Bigg(\Big(\left(\pi^{-1}_{\beta,\alpha}\left(A_{\beta}\right) \cap \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right) \cup \left(\pi^{-1}_{\eta,\alpha}\left(B_{\eta}\right) \cap \left(\pi^{-1}_{\gamma,\alpha}\left(W^{\gamma} \setminus P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right)\right)\Big) \\ &\cap P_i\left(w^{\alpha}\right)\Bigg) \cup \\ &\Bigg(\Big(\left(\pi^{-1}_{\xi,\alpha}\left(C_{\xi}\right) \cap \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right) \cup \left(\pi^{-1}_{\zeta,\alpha}\left(D_{\zeta}\right) \cap \left(\pi^{-1}_{\gamma,\alpha}\left(W^{\gamma} \setminus P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right)\right)\Big) \\ &\cap \left(W^{\alpha} \setminus P_i\left(w^{\alpha}\right)\right)\Bigg), \end{aligned}$$

where $\beta, \eta, \xi, \zeta < \gamma$ and $A_{\beta} \subseteq W^{\beta}$, $B_{\eta} \subseteq W^{\eta}$, $C_{\xi} \subseteq W^{\xi}$, $D_{\zeta} \subseteq W^{\zeta}$.

The lemma follows from the following facts: If $\eta < \beta$, then $\pi^{-1}_{\eta,\beta}\left(B_{\eta}\right) \subseteq W^{\beta}$ and $\pi^{-1}_{\eta,\alpha}\left(B_{\eta}\right) = \pi^{-1}_{\beta,\alpha}\left(\pi^{-1}_{\eta,\beta}\left(B_{\eta}\right)\right)$, so we can assume without loss of generality that $\beta = \eta = \xi = \zeta$. We have

$$\pi^{-1}_{\gamma,\alpha}\left(W^{\gamma} \setminus P_i\left(w^{\alpha}\lceil\gamma\right)\right) = W^{\alpha} \setminus \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)$$

and $P_i\left(w^{\alpha}\right) \subseteq \pi^{-1}_{\gamma,\alpha}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)$. ■

Notation 2.2 • For $0 \leq \alpha$ and $w_0 \in \{h, t\}$, we denote by

$$[X_0^\alpha = w_0] \text{ the set } \{u^\alpha \in W^\alpha \mid u_0 = w_0\}.$$

- For $\beta < \alpha$, $i \in \{a, b\}$ and $w_i^\alpha(\beta) \in \{0, 1\}$, we denote by

$$[X_i^\alpha(\beta) = w_i^\alpha(\beta)] \text{ the set } \{u^\alpha \in W^\alpha \mid u_i^\alpha(\beta) = w_i^\alpha(\beta)\}.$$

- For a limit ordinal $\lambda \leq \alpha$ and $\lambda\text{-par}(w_i^\alpha) \in \{\text{even}, \text{odd}\}$, we denote by

$$[\lambda\text{-par}(X_i^\alpha) = \lambda\text{-par}(w_i^\alpha)] \text{ the set } \{u^\alpha \in W^\alpha \mid \lambda\text{-par}(u_i^\alpha) = \lambda\text{-par}(w_i^\alpha)\}.$$

Remark 2.11 *Let $0 \leq \alpha \leq \gamma$ and $w_0 \in \{h, t\}$.*
Then:

- $$\pi_{\alpha,\gamma}^{-1}([X_0^\alpha = w_0]) = [X_0^\gamma = w_0].$$

- *If $\beta < \alpha$, $i \in \{a, b\}$, and $w^\gamma \in W^\gamma$, then*

$$\begin{array}{rcl} w_i^\gamma(\beta) & = & (w_i^\gamma\lceil\alpha)(\beta) \quad \textit{and} \\ \pi_{\alpha,\gamma}^{-1}([X_i^\alpha(\beta) = (w_i^\gamma\lceil\alpha)(\beta)]) & = & [X_i^\gamma(\beta) = w_i^\gamma(\beta)]. \end{array}$$

- *If $i \in \{a, b\}$, $w^\gamma \in W^\gamma$, and if λ is a limit ordinal such that $\lambda \leq \alpha$, then*

$$\begin{array}{rcl} \lambda - \text{par}(w_i^\gamma) & = & \lambda - \text{par}(w_i^\gamma\lceil\alpha) \quad \textit{and} \\ \pi_{\alpha,\gamma}^{-1}([\lambda\text{-par}(X_i^\alpha) = \lambda\text{-par}(w_i^\gamma\lceil\alpha)]) & = & [\lambda\text{-par}(X_i^\gamma) = \lambda\text{-par}(w_i^\gamma)]. \end{array}$$

For further reference, we cite here (in a slightly changed formulation and in our notation) Lemma 3.2. of Heifetz and Samet (1998a):

Lemma 2.10 *Let $v^\alpha, w^\alpha \in W^\alpha$, where $v^\alpha\lceil\gamma{+}1 \in P_i(w_i^\alpha\lceil\gamma + 1)$, for some $\gamma < \alpha$. Then there is a $u^\alpha \in P_i(w^\alpha)$ such that $u^\alpha\lceil\gamma = v^\alpha\lceil\gamma$.*

Lemma 2.11 *Let* λ *be a limit ordinal,* $\alpha = \lambda + 1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\lambda) = 0$, *and* $E = \pi_{\beta,\alpha}^{-1}(E_\beta)$, *where* $E_\beta \subseteq W^\beta$ *for a* $\beta < \lambda$.
Then:

- *If* $v^\alpha \in E \cap P_i(w^\alpha)$, *then there is a* $u^\alpha \in E \cap P_i(w^\alpha)$ *such that*

$$\lambda - \text{par}\left(u_j^\alpha\right) \neq \lambda - \text{par}\left(v_j^\alpha\right).$$

- *If*

$$v^\alpha \in E \cap \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha\lceil\lambda)) \cap (W^\alpha \setminus P_i(w^\alpha)),$$

  then there is a

$$u^\alpha \in E \cap \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha\lceil\lambda)) \cap (W^\alpha \setminus P_i(w^\alpha))$$

  such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.

- *If*

$$v^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha\lceil\lambda))\right),$$

  then there is a

$$u^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\lambda,\alpha}^{-1}(P_i(w^\alpha\lceil\lambda))\right)$$

  such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.

Proof Let $v^\alpha \in E$. Since $\beta, o^\lambda(v_i^\alpha), o^\lambda\left(v_j^\alpha\right) < \lambda$, there is an ordinal ξ such that $\max\left\{\beta, o^\lambda(v_i^\alpha), o^\lambda\left(v_j^\alpha\right)\right\} \leq \xi < \lambda$ and such that the parity of $\xi + 1$ is different from $\lambda\text{-par}\left(v_j^\alpha\right)$. Define now $u^\alpha \in W^\alpha$ by

$$\begin{array}{lcl} u_0 & := & v_0, \\ u_i^\alpha & := & v_i^\alpha, \\ u_j^\alpha(\gamma) & := & v_j^\alpha(\gamma), \quad \text{for all } \gamma < \alpha \text{ with } \gamma \neq \xi, \\ u_j^\alpha(\xi) & := & 1. \end{array}$$

It follows that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$ and $u^\alpha\lceil\beta = v^\alpha\lceil\beta$, which implies $u^\alpha \in E$.

- If $v^\alpha \in P_i(w^\alpha)$, then it is easy to check that $u^\alpha \in P_i(v^\alpha) = P_i(w^\alpha)$.

- If

$$v^\alpha \in \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right) \cap \left(W^\alpha \setminus P_i\left(w^\alpha\right)\right),$$

then it follows that $v^\alpha \in \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)$ and $v^\alpha_i(\lambda) = 1$. It is again easy to check that

$$u^\alpha \in \pi^{-1}_{\lambda,\alpha}\left(P_i\left(v^\alpha\lceil\lambda\right)\right) = \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)$$

and since $u^\alpha_i(\lambda) = 1$, we have $u^\alpha \in \left(W^\alpha \setminus P_i\left(w^\alpha\right)\right).$

- If $v^\alpha \notin \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)$, then there are four cases:

1. $v^\alpha_i\lceil\lambda \neq w^\alpha_i\lceil\lambda:$
From $u^\alpha_i\lceil\lambda = v^\alpha_i\lceil\lambda$ it follows that $u^\alpha \notin \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right).$

2. There is a $\gamma < \lambda$ such that

$$\left(v^\alpha_i\lceil\lambda\right)(\gamma+1) = \left(w^\alpha_i\lceil\lambda\right)(\gamma+1) = 1 \text{ and } \left(v^\alpha_j\lceil\lambda\right)(\gamma) \neq \left(w^\alpha_j\lceil\lambda\right)(\gamma):$$

Since $\gamma + 1 < \lambda$ and $\max\left\{\beta, o^\lambda\left(v^\alpha_i\right), o^\lambda\left(v^\alpha_j\right)\right\} > \gamma + 1$, it follows that $u^\alpha\lceil\gamma+2 = v^\alpha\lceil\gamma+2$ and therefore $u^\alpha \notin \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right).$

3. There is a limit ordinal $\widehat{\lambda} < \lambda$ such that

$$\left(v^\alpha_i\lceil\lambda\right)(\widehat{\lambda}) = \left(w^\alpha_i\lceil\lambda\right)(\widehat{\lambda}) = 1 \text{ and } \widehat{\lambda}-\text{par}\left(v^\alpha_j\right) \neq \widehat{\lambda}-\text{par}\left(w^\alpha_j\right):$$

We have $\xi > \widehat{\lambda}$, and therefore

$$\left(u^\alpha_i\lceil\lambda\right)(\widehat{\lambda}) = \left(v^\alpha_i\lceil\lambda\right)(\widehat{\lambda}) \text{ and } \widehat{\lambda}-\text{par}\left(u^\alpha_j\right) = \widehat{\lambda}-\text{par}\left(v^\alpha_j\right).$$

It follows that $u^\alpha \notin \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right).$

4. $\left(v^\alpha_i\lceil\lambda\right)(0) = \left(w^\alpha_i\lceil\lambda\right)(0) = 1$ and $v_0 \neq w_0:$
We have

$$\left(u^\alpha_i\lceil\lambda\right)(0) = \left(v^\alpha_i\lceil\lambda\right)(0) = 1 \text{ and } u_0 = v_0.$$

It follows that $u^\alpha \notin \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right).$

■

Lemma 2.12 *Let β be an ordinal, $\alpha = (\beta+1)+1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\beta+1) = 0$, and $E = \pi_{\beta,\alpha}^{-1}(E_\beta)$, such that $E_\beta \subseteq W^\beta$.*
Then:

- *If $v^\alpha \in E \cap P_i(w^\alpha)$, then there is a $u^\alpha \in E \cap P_i(w^\alpha)$ such that*

$$u_j^\alpha(\beta) \neq v_j^\alpha(\beta).$$

- *If*

$$v^\alpha \in E \cap \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1)) \cap (W^\alpha \setminus P_i(w^\alpha)),$$

  then there is a

$$u^\alpha \in E \cap \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1)) \cap (W^\alpha \setminus P_i(w^\alpha))$$

  such that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.

- *If*

$$v^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1))\right),$$

  then there is a

$$u^\alpha \in E \cap \left(W^\alpha \setminus \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1))\right)$$

  such that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.

Proof Let $v^\alpha \in E$. Define $u^\alpha \in W^\alpha$ by

$$\begin{array}{lcll} u_0 & := & v_0, & \\ u_i^\alpha & := & v_i^\alpha, & \\ u_j^\alpha(\gamma) & := & v_j^\alpha(\gamma), & \text{for all } \gamma < \alpha \text{ with } \gamma \neq \beta, \\ u_j^\alpha(\beta) & := & 1 - v_j^\alpha(\beta). & \end{array}$$

It follows that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$ and $u^\alpha \lceil \beta = v^\alpha \lceil \beta$, which implies $u^\alpha \in E$.

- If $v^\alpha \in P_i(w^\alpha)$, then it is easy to check that $u^\alpha \in P_i(v^\alpha) = P_i(w^\alpha)$.

- If

$$v^\alpha \in \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1)) \cap (W^\alpha \setminus P_i(w^\alpha)),$$

  then it follows that $v^\alpha \in \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1))$ and $v_i^\alpha(\beta+1) = 1$. It is again easy to check that

$$u^\alpha \in \pi_{\beta+1,\alpha}^{-1}(P_i(v^\alpha \lceil \beta+1)) = \pi_{\beta+1,\alpha}^{-1}(P_i(w^\alpha \lceil \beta+1))$$

  and since $u_i^\alpha(\beta+1) = 1$, we have $u^\alpha \in (W^\alpha \setminus P_i(w^\alpha))$.

- If $v^\alpha \notin \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)$, then there are four cases:

1. $v_i^\alpha\lceil\beta+1 \neq w_i^\alpha\lceil\beta+1$:
From $u_i^\alpha\lceil\beta+1 = v_i^\alpha\lceil\beta+1$, it follows that $u^\alpha \notin \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)$.

2. There is a $\gamma < \beta$ such that

$$\begin{array}{ll} \left(v_i^\alpha\lceil\beta+1\right)(\gamma+1) = \left(w_i^\alpha\lceil\beta+1\right)(\gamma+1) = 1 & \text{and} \\ \left(v_j^\alpha\lceil\beta+1\right)(\gamma) \neq \left(w_j^\alpha\lceil\beta+1\right)(\gamma) & : \end{array}$$

By the definition of u^α, $\left(u_i^\alpha\lceil\beta+1\right)(\gamma+1) = 1$ and, since $\gamma < \beta$,

$$\left(v_j^\alpha\lceil\beta+1\right)(\gamma) = \left(u_j^\alpha\lceil\beta+1\right)(\gamma),$$

hence $u^\alpha \notin \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)$.

3. There is a limit ordinal $\lambda < \beta+1$ such that

$$\left(v_i^\alpha\lceil\beta+1\right)(\lambda) = \left(w_i^\alpha\lceil\beta+1\right)(\lambda) = 1 \text{ and } \lambda-\text{par}\left(v_j^\alpha\right) \neq \lambda-\text{par}\left(w_j^\alpha\right) :$$

We have $\left(u_i^\alpha\lceil\beta+1\right)(\lambda) = 1$ and, since $\lambda \leq \beta$, $\lambda\text{-par}\left(u_j^\alpha\right) = \lambda\text{-par}\left(v_j^\alpha\right)$. It follows that $u^\alpha \notin \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)$.

4. $\left(v_i^\alpha\lceil\beta+1\right)(0) = \left(w_i^\alpha\lceil\beta+1\right)(0) = 1$ and $v_0 \neq w_0$:
We have

$$\left(u_i^\alpha\lceil\beta+1\right)(0) = \left(v_i^\alpha\lceil\beta+1\right)(0) = 1 \text{ and } u_0 = v_0.$$

It follows that $u^\alpha \notin \pi^{-1}_{\beta+1,\alpha}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)$.

■

Lemma 2.13 *Let λ be a limit ordinal, $\alpha = \lambda+1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\lambda) = 0$, and*

$$E \in \left[\left[\bigcup_{\beta<\lambda} \pi^{-1}_{\beta,\alpha}\left(\text{Pow}\left(W^\beta\right)\right), \pi^{-1}_{\lambda,\alpha}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)\right], P_i\left(w^\alpha\right)\right].$$

Then:

- *If $v^\alpha \in E$, then there is a $u^\alpha \in E$ such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.*
- *If $v^\alpha \in W^\alpha \setminus E$, then there is a $u^\alpha \in W^\alpha \setminus E$ such that $\lambda\text{-par}\left(u_j^\alpha\right) \neq \lambda\text{-par}\left(v_j^\alpha\right)$.*

- *If*

$$E \supseteq \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right],$$

then $E = W^\alpha$.

- *If*

$$E \subseteq \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right],$$

then $E = \emptyset$.

Proof The first point of the Lemma follows from Lemma 2.9 and Lemma 2.11.

The second point follows from the first and the fact that

$$\left[\left[\bigcup_{\beta<\lambda} \pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\lambda\right)\right)\right], P_i\left(w^\alpha\right)\right]$$

is a field on W^α (and therefore it is closed under complements).

The last two points of the Lemma follow directly from the first two points. ■

Lemma 2.14 *Let β be an ordinal, $\alpha = (\beta+1)+1$, $w^\alpha \in W^\alpha$, $w_i^\alpha(\beta+1) = 0$, and*

$$E \in \left[\left[\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha\lceil\beta+1\right)\right)\right], P_i\left(w^\alpha\right)\right].$$

Then:

- *If $v^\alpha \in E$, then there is a $u^\alpha \in E$ such that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.*
- *If $v^\alpha \in W^\alpha \setminus E$, then there is a $u^\alpha \in W^\alpha \setminus E$ such that $u_j^\alpha(\beta) \neq v_j^\alpha(\beta)$.*
- *If*

$$E \supseteq \left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right],$$

then $E = W^\alpha$.

- *If*

$$E \subseteq \left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right],$$

then $E = \emptyset$.

Proof Note that if $\alpha = \beta + 1$, then

$$\bigcup_{\xi<\beta+1} \pi_{\xi,\alpha}^{-1}\left(\text{Pow}\left(W^{\xi}\right)\right) = \pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^{\beta}\right)\right).$$

The proof is now analogous to the proof of Lemma 2.13, just replace λ by $\beta + 1$ and Lemma 2.11 by Lemma 2.12. ■

Lemma 2.15 *Let γ be an ordinal > 0, $\alpha = \gamma + 1$, $w^{\alpha} \in W^{\alpha}$, and*

$$E \in \left[\bigcup_{\beta<\gamma} \pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^{\beta}\right)\right), \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right]$$

such that $E \supseteq P_i\left(w^{\alpha}\right)$.

Then

$$E \supseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right).$$

Proof Since $P_i\left(w^{\alpha}\right) \subseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)$, it follows from the definition of

$$\left[\bigcup_{\beta<\gamma} \pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^{\beta}\right)\right), \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)\right],$$

that there is a $\beta < \gamma$ and a $E^{\beta} \subseteq W^{\beta}$ such that

$$E \cap \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right) = \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right) \cap \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right) \supseteq P_i\left(w^{\alpha}\right).$$

<u>Claim</u>**:** $\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right) \supseteq \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right)$.
Assume to the contrary, that there is a

$$v^{\alpha} \in \pi_{\gamma,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\gamma\right)\right) \setminus \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right).$$

Since $\beta + 1 \leq \gamma$, we have $v^{\alpha}\lceil\beta + 1 \in P_i\left(w^{\alpha}\lceil\beta + 1\right)$. By Lemma 2.10, there is a $u^{\alpha} \in P_i\left(w^{\alpha}\right)$ such that $u^{\alpha}\lceil\beta = v^{\alpha}\lceil\beta$. Since $E^{\beta} \subseteq W^{\beta}$ and $v^{\alpha} \notin \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)$, it follows that $u^{\alpha} \notin \pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)$, a contradiction to $\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right) \supseteq P_i\left(w^{\alpha}\right)$. ■

Lemma 2.16 *Let λ be a limit ordinal, $w^{\lambda} \in W^{\lambda}$, $\beta < \lambda$, and $E^{\beta} \subseteq W^{\beta}$ such that $\pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right) \supseteq P_i\left(w^{\lambda}\right)$.*

Then

$$\pi_{\beta,\lambda}^{-1}\left(E^{\beta}\right) \supseteq \pi_{\beta+1,\lambda}^{-1}\left(P_i\left(w^{\lambda}\lceil\beta + 1\right)\right).$$

Proof Assume that there is a

$$v^{\lambda} \in \pi^{-1}_{\beta+1,\lambda}\left(P_i\left(w^{\lambda}\lceil\beta+1\right)\right) \setminus \pi^{-1}_{\beta,\lambda}\left(E^{\beta}\right).$$

By Lemma 2.10, there is a $u^{\lambda} \in P_i\left(w^{\lambda}\right)$ such that $u^{\lambda}\lceil\beta = v^{\lambda}\lceil\beta$. Therefore $u^{\lambda} \notin \pi^{-1}_{\beta,\lambda}\left(E^{\beta}\right)$, a contradiction to $\pi^{-1}_{\beta,\lambda}\left(E^{\beta}\right) \supseteq P_i\left(w^{\lambda}\right)$. ■

2.4.1 The Construction

Now, let κ be a fixed regular cardinal.

For every $w^{\kappa} \in W^{\kappa}$ and $i \in \{a,b\}$, we will define finitely additive probability measures $T_i\left(w^{\kappa}\right)$ on $\left(W^{\kappa}, \mathrm{Pow}\left(W^{\kappa}\right)\right)$ such that:

- $T_i\left(u^{\kappa}\right) = T_i\left(w^{\kappa}\right)$, for $u^{\kappa} \in P_i\left(w^{\kappa}\right)$,
- $T_i\left(w^{\kappa}\right)\left(P_i\left(w^{\kappa}\right)\right) = 1$,
- $T_i\left(w^{\kappa}\right)\left(\left[X_0^{\kappa} = w_0\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^{\kappa}(0) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^{\kappa}(0) = 0, \end{cases}$
- for $\beta < \kappa$:
  $T_i\left(w^{\kappa}\right)\left(\left[X_j^{\kappa}(\beta) = w_j^{\kappa}(\beta)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^{\kappa}(\beta+1) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^{\kappa}(\beta+1) = 0, \end{cases}$
- for $\lambda < \kappa$ such that λ is a limit ordinal :
  $T_i\left(w^{\kappa}\right)\left(\left[\lambda\text{-par}\left(X_j^{\kappa}\right) = \lambda\text{-par}\left(w_j^{\kappa}\right)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^{\kappa}(\lambda) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^{\kappa}(\lambda) = 0. \end{cases}$

If we define

$$\theta\left(w^{\kappa}\right) := w_0,$$

it follows from the first two points and the fact that $T_i\left(w^{\kappa}\right)$ will be defined on $\left(W^{\kappa}, \mathrm{Pow}\left(W^{\kappa}\right)\right)$, that

$$\left\langle W^{\kappa}, \left(T_i\right)_{i\in\{a,b\}}, \theta\right\rangle$$

will be a $*$-type space on $S = \{h,t\}$ for player set $\{a,b\}$.

The construction will not be carried out at once. By a transfinite induction on $1 \le \alpha \le \kappa$, we will endow W^{α} with fields $\mathcal{F}\left(i, w^{\alpha}\right)$ and finitely additive probability measures $T_i^{\alpha}\left(w^{\alpha}\right)$ on $\left(W^{\alpha}, \mathcal{F}\left(i, w^{\alpha}\right)\right)$ such that:

Induction Hypothesis

1. $\mathcal{F}(i, w^\alpha) := \left[\bigcup_{\beta<\alpha} \left(\pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^\beta\right)\right)\right), P_i(w^\alpha)\right],$

2. for every $\beta < \alpha$ and $E^\beta \in \mathcal{F}(i, w^\alpha \lceil \beta) :$
   $T_i^\alpha(w^\alpha)\left(\pi^{-1}_{\beta,\alpha}\left(E^\beta\right)\right) = T_i^\beta(w^\alpha \lceil \beta)\left(E^\beta\right),$
   that is $\mathrm{marg}_{\left(W^\beta, \mathcal{F}(i, w^\alpha \lceil \beta)\right)} T_i^\alpha(w^\alpha) = T_i^\beta(w^\alpha \lceil \beta),$

3. $T_i^\alpha(w^\alpha) = T_i^\alpha(u^\alpha)$, for $u^\alpha \in P_i(w^\alpha)$,

4. $T_i^\alpha(w^\alpha)(P_i(w^\alpha)) = 1,$

5. $T_i^\alpha(w^\alpha)([X_0^\alpha = w_0]) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(0) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(0) = 0, \end{cases}$

6. for β such that $\beta + 1 < \alpha$:
   $T_i^\alpha(w^\alpha)\left(\left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(\beta+1) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(\beta+1) = 0, \end{cases}$

7. for $\lambda < \alpha$ such that λ is a limit ordinal :
   $T_i^\alpha(w^\alpha)\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = \begin{cases} 1, & \text{if} \quad w_i^\alpha(\lambda) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^\alpha(\lambda) = 0. \end{cases}$

Since inverse images commute with complements, arbitrary unions and intersections, it follows:

Remark 2.12 *Let $\beta + 1 \leq \alpha$. Then*:

$$\pi^{-1}_{\beta+1,\alpha}\left(\mathcal{F}(i, w^\alpha \lceil \beta+1)\right) = \left[\pi^{-1}_{\beta,\alpha}\left(\mathrm{Pow}\left(W^\beta\right)\right), \pi^{-1}_{\beta+1,\alpha}\left(P_i(w^\alpha \lceil \beta+1)\right)\right].$$

Remark 2.13 *Let $1 \leq \beta \leq \alpha \leq \kappa$ and let $u^\alpha, w^\alpha \in W^\alpha$ such that $u^\alpha \in P_i(w^\alpha)$ (and hence $P_i(u^\alpha) = P_i(w^\alpha)$). Then*:

- *We have*

$$\begin{array}{rcll} \mathcal{F}(i, w^\alpha) & = & \mathcal{F}(i, u^\alpha) & \textit{and} \\ \mathcal{F}(i, w^\alpha \lceil \beta) & = & \mathcal{F}(i, u^\alpha \lceil \beta). & \end{array}$$

- *If*

$$\begin{array}{lll} T_i^{\alpha}(w^{\alpha}) & = & T_i^{\alpha}(u^{\alpha}), \\ T_i^{\beta}(w^{\alpha}\lceil\beta) & = & T_i^{\beta}(u^{\alpha}\lceil\beta) \quad \textit{and} \\ \operatorname{marg}_{\left(W^{\beta},\mathcal{F}(i,w^{\alpha}\lceil\beta)\right)} T_i^{\alpha}(w^{\alpha}) & = & T_i^{\beta}(w^{\alpha}\lceil\beta), \end{array}$$

then we also have

$$\operatorname{marg}_{\left(W^{\beta},\mathcal{F}(i,u^{\alpha}\lceil\beta)\right)} T_i^{\alpha}(u^{\alpha}) = T_i^{\beta}(u^{\alpha}\lceil\beta).$$

Definition and Construction

(By transfinite induction on $1 \leq \alpha \leq \kappa$.)

Step $\alpha = 1$:

We have $W^0 = \{h, t\}$. Let $w^1 \in W^1$.
Define

$$\begin{array}{lll} T_i^{<1}(w^1)([X_0^1 = w_0]) & := & \begin{cases} 1, & \text{if} \quad w_i^1(0) = 1, \\ \frac{1}{2}, & \text{if} \quad w_i^1(0) = 0, \end{cases} \\ T_i^{<1}(w^1)([X_0^1 \neq w_0]) & := & 1 - T_i^{<1}(w^1)([X_0^1 = w_0]), \\ T_i^{<1}(w^1)(\emptyset) & := & 0, \\ T_i^{<1}(w^1)(W^1) & := & 1. \end{array}$$

It is clear, that by this definition, $T_i^{<1}(w^1)$ is a probability measure on

$$\left(W^1, \pi_{0,1}^{-1}\left(\operatorname{Pow}\left(W^0\right)\right)\right).$$

Let $E^0 \subseteq W^0$ such that $P_i(w^1) \subseteq \pi_{0,1}^{-1}(E^0)$.

1. <u>case:</u> $w_i^1(0) = 1$: Then $E^0 = W^0$ or $E^0 = [X_0^0 = w_0]$, hence the outer measure $T_i^{<1}(w^1)^*(P_i(w^1))$ is equal to 1.

2. <u>case:</u> $w_i^1(0) = 0$: Then $E^0 = W^0$ and the outer measure $T_i^{<1}(w^1)^*(P_i(w^1))$ is equal to 1.

For $u^1 \in P_i(w^1)$, we have in both cases that $P_i(u^1) = P_i(w^1)$ and $T_i^{<1}(u^1) = T_i^{<1}(w^1)$. For each $P_i(u^1)$ such that $u^1 \in W^1$, chose a representing element $w^1 \in P_i(u^1) = P_i(w^1)$. By the Łoś-Marczewski Theorem, we can extend $T_i^{<1}(w^1)$ to a finitely additive probability measure $T_i^1(w^1)$ on the field

$$\mathcal{F}\left(i, w^1\right) = \left[\pi_{0,1}^{-1}\left(\text{Pow}\left(W^0\right)\right), P_i\left(w^1\right)\right]$$

such that $T_i^1(w^1)(P_i(w^1)) = 1$. Define $T_i^1(u^1) := T_i^1(w^1)$, for all $u^1 \in P_i(w^1)$. (Note that for $u^1 \in P_i(w^1)$, it is the case that $\mathcal{F}(i, u^1) = \mathcal{F}(i, w^1)$.) $T_i^1(u^1)$ and $\mathcal{F}(i, u^1)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha = (\beta + 1) + 1$:

For each $P_i(u^\alpha)$ such that $u^\alpha \in W^\alpha$, chose a representing element $w^\alpha \in P_i(u^\alpha) = P_i(w^\alpha)$.

Let $T_i^{<\alpha}(w^\alpha)$ be the finitely additive probability measure defined on the field

$$\pi_{\beta+1,\alpha}^{-1}\left(\mathcal{F}\left(i, w^\alpha \lceil \beta + 1\right)\right) = \left[\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \beta + 1\right)\right)\right],$$

which is induced by $T_i^{\beta+1}(w^\alpha \lceil \beta + 1)$ (as defined in Lemma 2.3). According to Lemma 2.15 and the induction hypothesis, we have for the outer measure of $P_i(w^\alpha)$: $T_i^{<\alpha}(w^\alpha)^*(P_i(w^\alpha)) = 1$. So, by the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}(w^\alpha)$ to a finitely additive probability measure $\widetilde{T}_i^\alpha(w^\alpha)$ defined on the field

$$\widetilde{\mathcal{F}}\left(i, w^\alpha\right) := \left[\left[\pi_{\beta,\alpha}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), \pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^\alpha \lceil \beta + 1\right)\right)\right], P_i\left(w^\alpha\right)\right]$$

such that $\widetilde{T}_i^\alpha(w^\alpha)(P_i(w^\alpha)) = 1$.

1. <u>case:</u> $w_i^\alpha(\beta + 1) = 1$: Then

$$\pi_{\beta+1,\alpha}^{-1}\left(\left[X_j^{\beta+1}(\beta) = \left(w_j^\alpha \lceil \beta + 1\right)(\beta)\right]\right) = \left[X_j^\alpha(\beta) = w_j^\alpha(\beta)\right] \supseteq P_i\left(w^\alpha\right).$$

By Lemma 2.5, extend $\widetilde{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on the field

$$\left[\pi_{\beta+1,\alpha}^{-1}\left(\text{Pow}\left(W^{\beta+1}\right)\right), P_i\left(w^\alpha\right)\right] = \mathcal{F}\left(i, w^\alpha\right).$$

By the above, we have

$$T_i^\alpha (w^\alpha) \left(\left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right] \right) = 1.$$

Define now $T_i^\alpha (u^\alpha) := T_i^\alpha (w^\alpha)$, for all $u^\alpha \in P_i (w^\alpha)$. (Note that, for $u^\alpha \in P_i (w^\alpha)$, we have $\widetilde{\mathcal{F}} (i, u^\alpha) = \widetilde{\mathcal{F}} (i, w^\alpha)$, $\mathcal{F} (i, u^\alpha) = \mathcal{F} (i, w^\alpha)$, and $u_i^\alpha (\beta + 1) = w_i^\alpha (\beta + 1) = 1$ and hence, $u_j^\alpha (\beta) = w_j^\alpha (\beta)$.) It is now easy to check that $T_i^\alpha (u^\alpha)$ and $\mathcal{F} (i, u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

2. <u>case:</u> $w_i^\alpha (\beta + 1) = 0$: By Lemma 2.14 and the induction hypothesis, we have for the outer measure of $\left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right]$:

$$\widetilde{T}_i^\alpha (w^\alpha)^* \left(\left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right] \right) = 1,$$

and for the inner measure

$$\widetilde{T}_i^\alpha (w^\alpha)_* \left(\left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right] \right) = 0.$$

By the Łoś-Marczewski Theorem, we can extend $\widetilde{T}_i^\alpha (w^\alpha)$ to a finitely additive probability measure $\widehat{T}_i^\alpha (w^\alpha)$ on the field

$$\left[\widetilde{\mathcal{F}} (i, w^\alpha), \left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right] \right]$$

such that

$$\widehat{T}_i^\alpha (w^\alpha) \left(\left[X_j^\alpha (\beta) = w_j^\alpha (\beta) \right] \right) = \frac{1}{2}.$$

Finally, by Lemma 2.5, extend $\widehat{T}_i^\alpha (w^\alpha)$ to a finitely additive probability measure $T_i^\alpha (w^\alpha)$ on $\mathcal{F} (i, w^\alpha)$. Define now $T_i^\alpha (u^\alpha) := T_i^\alpha (w^\alpha)$, for all $u^\alpha \in P_i (w^\alpha)$. It is easy to check that $T_i^\alpha (u^\alpha)$ and $\mathcal{F} (i, u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha = \lambda$, λ limit ordinal.

For each $P_i (u^\alpha)$ such that $u^\alpha \in W^\alpha$, chose a representing element $w^\alpha \in P_i (u^\alpha) = P_i (w^\alpha)$.

Let $T_i^{<\alpha} (w^\alpha)$ be the finitely additive probability measure defined on the field

$$\bigcup_{\beta < \alpha} \pi_{\beta,\alpha}^{-1} \left(\mathcal{F} (i, w^\alpha \lceil \beta) \right) = \bigcup_{\beta < \alpha} \pi_{\beta,\alpha}^{-1} \left(\mathrm{Pow} \left(W^\beta \right) \right)$$

which is induced by $\left(T_i^{\beta}\left(w^{\alpha}\lceil\beta\right)\right)_{1\leq\beta<\alpha}$ (as defined in Lemma 2.3).

Let $\beta<\alpha$ and $E^{\beta}\subseteq W^{\beta}$ such that $\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)\supseteq P_i\left(w^{\alpha}\right)$. By Lemma 2.16, we have

$$\pi_{\beta,\alpha}^{-1}\left(E^{\beta}\right)\supseteq\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right).$$

(Note that $\beta<\alpha$ implies $\beta+1<\alpha$.) Since, by the definition of $T_i^{<\alpha}\left(w^{\alpha}\right)$,

$$T_i^{<\alpha}\left(w^{\alpha}\right)\left(\pi_{\beta+1,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)\right)=T_i^{\beta+1}\left(w^{\alpha}\lceil\beta+1\right)\left(P_i\left(w^{\alpha}\lceil\beta+1\right)\right)=1,$$

the outer measure $T_i^{<\alpha}\left(w^{\alpha}\right)^{*}\left(P_i\left(w^{\alpha}\right)\right)$ is equal to 1. By the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}\left(w^{\alpha}\right)$ to a finitely additive probability measure $T_i^{\alpha}\left(w^{\alpha}\right)$ on the field

$$\mathcal{F}\left(i,w^{\alpha}\right)=\left[\bigcup_{\beta<\alpha}\pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\beta}\right)\right),P_i\left(w^{\alpha}\right)\right]$$

such that $T_i^{\alpha}\left(w^{\alpha}\right)\left(P_i\left(w^{\alpha}\right)\right)=1$. For $u^{\alpha}\in P_i\left(w^{\alpha}\right)$ define $T_i^{\alpha}\left(u^{\alpha}\right):=T_i^{\alpha}\left(w^{\alpha}\right)$. It is easy to check that $T_i^{\alpha}\left(u^{\alpha}\right)$ and $\mathcal{F}\left(i,u^{\alpha}\right)$ satisfy the conditions 1.-7. of the induction hypothesis.

Step $\alpha=\lambda+1$, λ limit ordinal:

For each $P_i\left(u^{\alpha}\right)$ such that $u^{\alpha}\in W^{\alpha}$, chose a representing element $w^{\alpha}\in P_i\left(u^{\alpha}\right)=P_i\left(w^{\alpha}\right)$.

Let $T_i^{<\alpha}\left(w^{\alpha}\right)$ be the finitely additive probability measure defined on the field

$$\pi_{\lambda,\alpha}^{-1}\left(\mathcal{F}\left(i,w^{\alpha}\lceil\lambda\right)\right)=\left[\bigcup_{\beta<\lambda}\pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\beta}\right)\right),\pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\lambda\right)\right)\right],$$

which is induced by $T_i^{\lambda}\left(w^{\alpha}\lceil\lambda\right)$ (as defined in Lemma 2.3). According to Lemma 2.15 and the induction hypothesis, we have for the outer measure of $P_i\left(w^{\alpha}\right)$: $T_i^{<\alpha}\left(w^{\alpha}\right)^{*}\left(P_i\left(w^{\alpha}\right)\right)=1$. So, by the Łoś-Marczewski Theorem, we can extend $T_i^{<\alpha}\left(w^{\alpha}\right)$ to a finitely additive probability measure $\widetilde{T}_i^{\alpha}\left(w^{\alpha}\right)$ defined on the field

$$\widetilde{\mathcal{F}}\left(i,w^{\alpha}\right):=\left[\left[\bigcup_{\beta<\lambda}\pi_{\beta,\alpha}^{-1}\left(\mathrm{Pow}\left(W^{\beta}\right)\right),\pi_{\lambda,\alpha}^{-1}\left(P_i\left(w^{\alpha}\lceil\lambda\right)\right)\right],P_i\left(w^{\alpha}\right)\right],$$

such that $\widetilde{T}_i^{\alpha}\left(w^{\alpha}\right)\left(P_i\left(w^{\alpha}\right)\right)=1$.

1. <u>case:</u> $w_i^\alpha(\lambda) = 1$. Then

$$\begin{aligned} \pi_{\lambda,\alpha}^{-1}\left(\left[\lambda\text{-par}\left(X_j^\lambda\right) = \lambda\text{-par}\left(w_j^\alpha \lceil \lambda\right)\right]\right) &= \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right] \\ &\supseteq P_i\left(w^\alpha\right). \end{aligned}$$

By Lemma 2.5, extend $\widetilde{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on the field

$$\left[\pi_{\lambda,\alpha}^{-1}\left(\text{Pow}\left(W^\lambda\right)\right), P_i\left(w^\alpha\right)\right] = \mathcal{F}\left(i, w^\alpha\right).$$

By the above, we have

$$T_i^\alpha\left(w^\alpha\right)\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = 1.$$

Define now $T_i^\alpha(u^\alpha) := T_i^\alpha(w^\alpha)$, for all $u^\alpha \in P_i(w^\alpha)$. (Note that $\mathcal{F}(i, u^\alpha) = \mathcal{F}(i, w^\alpha)$, $u_i^\alpha(\lambda) = w_i^\alpha(\lambda) = 1$ and hence, $\lambda\text{-par}\left(w_j^\alpha\right) = \lambda\text{-par}\left(u_j^\alpha\right)$.) It is now easy to check that $T_i^\alpha(u^\alpha)$ and $\mathcal{F}(i, u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

2. <u>case:</u> $w_i^\alpha(\lambda) = 0$. By Lemma 2.13, we have for the outer measure of $\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]$:

$$\widetilde{T}_i^\alpha\left(w^\alpha\right)^*\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = 1,$$

and for the inner measure

$$\widetilde{T}_i^\alpha\left(w^\alpha\right)_*\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = 0.$$

By the Łoś-Marczewski Theorem, we can extend $\widetilde{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $\widehat{T}_i^\alpha(w^\alpha)$ on the field

$$\left[\widetilde{\mathcal{F}}\left(i, w^\alpha\right), \left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right]$$

such that

$$\widehat{T}_i^\alpha\left(w^\alpha\right)\left(\left[\lambda\text{-par}\left(X_j^\alpha\right) = \lambda\text{-par}\left(w_j^\alpha\right)\right]\right) = \frac{1}{2}.$$

Finally, by Lemma 2.5, extend $\widehat{T}_i^\alpha(w^\alpha)$ to a finitely additive probability measure $T_i^\alpha(w^\alpha)$ on $\mathcal{F}(i, w^\alpha)$. Define now $T_i^\alpha(u^\alpha) := T_i^\alpha(w^\alpha)$, for all $u^\alpha \in P_i(w^\alpha)$. It is easy to check that $T_i^\alpha(u^\alpha)$ and $\mathcal{F}(i, u^\alpha)$ satisfy the conditions 1.-7. of the induction hypothesis.

Remaining Step:

We have to extend $T_i^\kappa(u^\kappa)$ to a finitely additive probability measure defined on the field $\text{Pow}(W^\kappa)$:

By the inductive construction, $T_i^\kappa(u^\kappa)$ is defined on

$$\left[\bigcup_{\beta<\kappa} \pi_{\beta,\kappa}^{-1}\left(\text{Pow}\left(W^\beta\right)\right), P_i(w^\kappa)\right]$$

such that 1.-7. of the induction hypothesis are satisfied. For each $P_i(u^\kappa)$ such that $u^\kappa \in W^\kappa$, chose a representing element

$$w^\kappa \in P_i(u^\kappa) = P_i(w^\kappa).$$

Now, by Lemma 2.5, extend $T_i^\kappa(w^\kappa)$ to a finitely additive probability measure $T_i(w^\kappa)$ on the field $\text{Pow}(W^\kappa)$ and define

$$T_i(u^\kappa) := T_i(w^\kappa),$$

for $u^\kappa \in P_i(w^\kappa)$.

Finally, let

$$\theta(w^\kappa) := w_0 \in S,$$

for $w^\kappa \in W^\kappa$.

$$\underline{W^\kappa} := \langle W^\kappa, T_a, T_b, \theta\rangle$$

has now all the desired properties.

Next, by induction on the formation of κ-expressions, we define the *depth* of a κ-expression:

Definition 2.27

- If $E \in \Sigma_S$, then

$$\text{dp}(E) := 0,$$

- if $0 \leq p \leq 1$, $i \in I$ and if φ is a κ-expression, then

$$\text{dp}\left(B_i^p(\varphi)\right) := \text{dp}(\varphi) + 1,$$

- if φ is a κ-expression, then

$$\mathrm{dp}(\neg\varphi) := \mathrm{dp}(\varphi),$$

- if Ψ is a set of κ-expressions such that $|\Psi| < \kappa$, then

$$\mathrm{dp}\Big(\bigwedge_{\varphi\in\Psi}\varphi\Big) := \sup\{\mathrm{dp}(\varphi) \mid \varphi\in\Psi\}.$$

It is easy to see that, since κ is regular, the depth of a κ-expression is always strictly smaller than κ.

Lemma 2.17 *Let $\alpha \leq \kappa$, $w^\kappa, u^\kappa \in W^\kappa$ and $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$.*
Then, for all κ-expressions φ such that $\mathrm{dp}(\varphi) \leq \alpha$:

$$w^\kappa \in \varphi^{W^\kappa} \text{ iff } u^\kappa \in \varphi^{W^\kappa}.$$

Proof We prove the Lemma by induction on the formation of κ-expressions.

- Let $\varphi = E$, where $E \in \Sigma_{\{h,t\}} = \mathrm{Pow}(\{h,t\})$, and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil 0 = u^\kappa\lceil 0$. By definition, $v^\kappa \in E^{W^\kappa}$ iff $\theta(v^\kappa) \in E$, for $v^\kappa \in W^\kappa$. But we have $\theta(v^\kappa) = v_0 = v^\kappa\lceil 0$, for $v^\kappa \in W^\kappa$. It follows that $u^\kappa \in E^{W^\kappa}$ iff $w^\kappa \in E^{W^\kappa}$.

- Let $\varphi = \neg\psi$ such that $\mathrm{depth}(\varphi) \leq \alpha$ and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. It follows that $\mathrm{depth}(\psi) \leq \alpha$ and $u^\kappa \in \varphi^{W^\kappa}$ iff $u^\kappa \notin \psi^{W^\kappa}$ iff- by the induction assumption - $w^\kappa \notin \psi^{W^\kappa}$, which is the case iff $w^\kappa \in \varphi^{W^\kappa}$.

- Let $p \in [0,1]$, $i \in \{a,b\}$, $\varphi = B_i^p(\psi)$, $\mathrm{dp}(\psi) + 1 = \beta + 1 \leq \alpha$, and $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. By the induction assumption, there is a $E^\beta \subseteq W^\beta$ such that $\psi^{W^\kappa} = \pi_{\beta,\kappa}^{-1}(E^\beta)$. By the definition,

$$\begin{aligned} T_i(u^\kappa)\left(\psi^{W^\kappa}\right) &= T_i^\alpha(u^\kappa\lceil\alpha)\left(\pi_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) \\ &= T_i^\alpha(w^\kappa\lceil\alpha)\left(\pi_{\beta,\alpha}^{-1}\left(E^\beta\right)\right) \\ &= T_i(w^\kappa)\left(\psi^{W^\kappa}\right). \end{aligned}$$

  It follows that $u^\kappa \in (B_i^p(\psi))^{W^\kappa}$ iff $w^\kappa \in (B_i^p(\psi))^{W^\kappa}$.

- Let $|\Psi| < \kappa$, $\varphi = \bigwedge_{\psi\in\Psi}\psi$ such that $\mathrm{depth}(\varphi) \leq \alpha$, and let $w^\kappa, u^\kappa \in W^\kappa$ such that $w^\kappa\lceil\alpha = u^\kappa\lceil\alpha$. Then $\mathrm{depth}(\psi) \leq \alpha$, for $\psi \in \Psi$. By the induction assumption, $u^\kappa \in \psi^{W^\kappa}$ iff $w^\kappa \in \psi^{W^\kappa}$, for $\psi \in \Psi$. It follows that $u^\kappa \in \varphi^{W^\kappa}$ iff $w^\kappa \in \varphi^{W^\kappa}$.

■

Lemma 2.18 *In the $*$-type space $\left\langle W^{\kappa}, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$, we have:*

$$\begin{array}{lcl}
[X_i^{\kappa}(0)=1] & = & \overline{B}_i^1\left([X_0^{\kappa}=h]\right)\cup\overline{B}_i^1\left([X_0^{\kappa}=t]\right). \\
[X_i^{\kappa}(\beta+1)=1] & = & \overline{B}_i^1\left(\left[X_j^{\kappa}(\beta)=1\right]\right)\cup\overline{B}_i^1\left(\left[X_j^{\kappa}(\beta)=0\right]\right), \\
 & & \quad \textit{for all ordinals } \beta<\kappa. \\
[X_i^{\kappa}(\lambda)=1] & = & \overline{B}_i^1\left(\left[\lambda\text{-par}\left(X_j^{\kappa}\right)=\textit{even}\right]\right)\cup\overline{B}_i^1\left(\left[\lambda\text{-par}\left(X_j^{\kappa}\right)=\textit{odd}\right]\right), \\
 & & \quad \textit{for all limit ordinals } \lambda<\kappa.
\end{array}$$

Proof Follows directly from the construction of $\left\langle W^{\kappa}, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$. ■

Lemma 2.19 *In the $*$-type space $\left\langle W^{\kappa}, (T_i)_{i\in\{a,b\}}, \theta\right\rangle$, we have:*

- $$\begin{array}{lclcl}
\{h\}^{W^{\kappa}} & = & [X_0^{\kappa}=h], & & \\
\{t\}^{W^{\kappa}} & = & [X_0^{\kappa}=t], & & \\
\mathrm{dp}(\{h\}) & = & \mathrm{dp}(\{t\}) & = & 0.
\end{array}$$

- *For every $i\in\{a,b\}$ and β such that $0\le\beta<\kappa$, there are κ-expressions $\varphi_i^0(\beta)$ and $\varphi_i^1(\beta)$ with*

$$\begin{array}{lcll}
\mathrm{dp}\left(\varphi_i^0(\beta)\right) & = & \mathrm{dp}\left(\varphi_i^1(\beta)\right) & \\
 & = & \beta+1 & \textit{such that} \\
\left(\varphi_i^1(\beta)\right)^{W^{\kappa}} & = & [X_i^{\kappa}(\beta)=1] & \textit{and} \\
\left(\varphi_i^0(\beta)\right)^{W^{\kappa}} & = & [X_i^{\kappa}(\beta)=0]. &
\end{array}$$

- *For every $i\in\{a,b\}$ and limit ordinal $\lambda<\kappa$, there are κ-expressions $\varphi_i^{even}(\lambda)$ and $\varphi_i^{odd}(\lambda)$ with*

$$\begin{array}{lcll}
\mathrm{dp}(\varphi_i^{even}(\lambda)) & = & \mathrm{dp}(\varphi_i^{odd}(\lambda)) & \\
 & = & \lambda & \textit{such that} \\
(\varphi_i^{even}(\lambda))^{W^{\kappa}} & = & [\lambda\text{-par}\left(X_i^{\kappa}\right)=\textit{even}] & \textit{and} \\
(\varphi_i^{odd}(\lambda))^{W^{\kappa}} & = & [\lambda\text{-par}\left(X_i^{\kappa}\right)=\textit{odd}]. &
\end{array}$$

Proof The first point is clear. We show the second and the third point by a transfinite induction on $0\le\beta<\kappa$.

- $\beta = 0$: According to Lemma 2.18 and the first point of this Lemma, we

$$\begin{array}{lrcl} \text{have} & [X_i^\kappa(0)=1] & = & \left(B_i^1(\{h\}) \vee B_i^1(\{t\})\right)^{W^\kappa} \\ \text{and} & [X_i^\kappa(0)=0] & = & \left(\neg\left(B_i^1(\{h\}) \vee B_i^1(\{t\})\right)\right)^{W^\kappa}. \end{array}$$

- $\beta = \gamma + 1$: According to the induction assumption, there are κ-expressions $\varphi_j^0(\gamma)$ and $\varphi_j^1(\gamma)$ such that

$$\begin{array}{rcll} \left(\varphi_j^0(\gamma)\right)^{W^\kappa} & = & \left[X_j^\kappa(\gamma)=0\right], & \\ \left(\varphi_j^1(\gamma)\right)^{W^\kappa} & = & \left[X_j^\kappa(\gamma)=1\right] & \text{and} \\ \mathrm{dp}\left(\varphi_j^0(\gamma)\right) & = & \mathrm{dp}\left(\varphi_j^1(\gamma)\right) & \\ & = & \beta. & \end{array}$$

  Define

$$\varphi_i^1(\beta) := B_i^1\left(\varphi_j^0(\gamma)\right) \vee B_i^1\left(\varphi_j^1(\gamma)\right)$$

  and

$$\varphi_i^0(\beta) := \neg\varphi_i^1(\beta).$$

  $\varphi_i^1(\beta)$ and $\varphi_i^0(\beta)$ are κ-expressions of depth $\beta + 1$. We have

$$[X_i^\kappa(\beta)=0] = W^\kappa \setminus [X_i^\kappa(\beta)=1].$$

  By Lemma 2.18 and the induction assumption, it follows that

$$\begin{array}{rcll} [X_i^\kappa(\beta)=1] & = & \left(\varphi_i^1(\beta)\right)^{W^\kappa} & \text{and} \\ [X_i^\kappa(\beta)=0] & = & \left(\varphi_i^0(\beta)\right)^{W^\kappa}. & \end{array}$$

- Let $\lambda < \kappa$ be a limit ordinal: For $i \in \{a, b\}$ and $\beta < \lambda$ define in $\underline{W}^\kappa$:

$$\begin{array}{rcl} \left[Y_i^\lambda(\beta)\right] & := & [X_i^\kappa(\beta)=1] \cap \bigcap_{\beta<\alpha<\lambda} [X_i^\kappa(\alpha)=0], \\ \left[Z_i^\lambda\right] & := & \bigcap_{0\le\alpha<\lambda} [X_i^\kappa(\alpha)=0]. \end{array}$$

  According to the induction assumption and the fact that $|\lambda| < \kappa$, it follows that

$$\begin{array}{rcll} \psi_i^\lambda(\beta) & := & \varphi_i^1(\beta) \wedge \bigwedge_{\beta<\alpha<\lambda} \varphi_i^0(\alpha) & \text{and} \\ \chi_i^\lambda & := & \bigwedge_{0\le\alpha<\lambda} \varphi_i^0(\alpha) & \end{array}$$

  are κ-expressions such that

$$\begin{array}{rcl} \mathrm{dp}\left(\psi_i^\lambda(\beta)\right) & = & \max\left\{\beta+1, \sup\left\{\mathrm{dp}\left(\varphi_i^0(\alpha)\right) \mid \beta<\alpha<\lambda\right\}\right\} \\ & = & \sup\{\alpha+1 \mid \alpha<\lambda\} \\ & = & \lambda, \end{array}$$

and, similarly,

$$\mathrm{dp}\left(\chi_i^\lambda\right) = \lambda.$$

It follows from the induction assumption that

$$\left[Y_i^\lambda(\beta)\right] = \left(\psi_i^\lambda(\beta)\right)^{W^\kappa} \text{ and } \left[Z_i^\lambda\right] = \left(\chi_i^\lambda\right)^{W^\kappa}.$$

Since $o^\lambda(w_i^\kappa)$, for $w_i^\kappa \in W_i^\kappa$, can never be a limit ordinal, we have

$$\begin{array}{rcl} [\lambda\text{-par}(X_i^\kappa) = \text{even}] & = & \left[Z_i^\lambda\right] \cup \bigcup_{\beta<\lambda,\ \beta \text{ odd}} \left[Y_i^\lambda(\beta)\right] \quad \text{and} \\ [\lambda\text{-par}(X_i^\kappa) = \text{odd}] & = & \bigcup_{\beta<\lambda,\ \beta\text{even}} \left[Y_i^\lambda(\beta)\right]. \end{array}$$

Again, since $|\lambda| < \kappa$, it follows from the above that

$$\begin{array}{rcl} \varphi_i^{\text{even}}(\lambda) & := & \chi_i^\lambda \vee \bigvee_{\beta<\lambda,\ \beta\text{odd}} \psi_i^\lambda(\beta) \quad \text{and} \\ \varphi_i^{\text{odd}}(\lambda) & := & \bigvee_{\beta<\lambda,\ \beta\text{even}} \psi_i^\lambda(\beta) \end{array}$$

are κ-expressions such that

$$\mathrm{dp}(\varphi_i^{\text{even}}(\lambda)) = \max\left\{\mathrm{dp}(\chi_i^\lambda), \sup\left\{\mathrm{dp}\left(\psi_i^\lambda(\beta)\right) \mid \beta < \lambda,\ \beta \text{ odd}\right\}\right\} = \lambda,$$

and $\quad \mathrm{dp}(\varphi_i^{\text{odd}}(\lambda)) = \lambda.$

By the definitions and the induction assumption, we have

$$\begin{array}{rcl} (\varphi_i^{\text{even}}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}(X_i^\kappa) = \text{even}] \quad \text{and} \\ (\varphi_i^{\text{odd}}(\lambda))^{W^\kappa} & = & [\lambda\text{-par}(X_i^\kappa) = \text{odd}]. \end{array}$$

- $\beta = \lambda$, λ limit ordinal $< \kappa$: By Lemma 2.18, and the above we have

$$\begin{array}{rcl} \left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)^{W^\kappa} & = & [X_i^\kappa(\lambda) = 1], \\ \left(\neg\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)\right)^{W^\kappa} & = & [X_i^\kappa(\lambda) = 0], \end{array}$$

and

$$\begin{array}{rcl} \mathrm{dp}\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)^{W^\kappa} & = & \\ \mathrm{dp}\left(\neg\left(B_i^1(\varphi_j^{\text{even}}(\lambda)) \vee B_i^1(\varphi_j^{\text{odd}}(\lambda))\right)\right)^{W^\kappa} & = & \lambda + 1. \end{array}$$

■

Theorem 2.2 *For every ordinal $\alpha < \kappa$ there are $u^\kappa, w^\kappa \in W^\kappa$ such that:*

1. *For all κ-expressions φ with $\mathrm{dp}(\varphi) \le \alpha$:*

$$u^\kappa \in \varphi^{W^\kappa} \text{ iff } w^\kappa \in \varphi^{W^\kappa}.$$

2. *There is a κ-expression ψ with $\mathrm{dp}(\psi) = \alpha + 1$ such that*

$$u^\kappa \in \psi^{W^\kappa} \text{ and } w^\kappa \in (\neg\psi)^{W^\kappa}.$$

Proof Let $\alpha < \kappa$ and $i \in \{a, b\}$. By the definition of W^κ, there are $u^\kappa, w^\kappa \in W^\kappa$ such that $u^\kappa \lceil \alpha = w^\kappa \lceil \alpha$ and $1 = u_i^\kappa(\alpha) \neq w_i^\kappa(\alpha) = 0$. The first point follows now by Lemma 2.17. By Lemma 2.19, it follows that $u^\kappa \in \varphi_i^1(\alpha)^{W^\kappa}$, $w^\kappa \in (\neg\varphi_i^1(\alpha))^{W^\kappa}$, and $\mathrm{dp}(\varphi_i^1(\alpha)) = \alpha + 1$. ■

Note that Lemma 2.17 and the proof of Theorem 2.2 show that the α-th levels of u_i^κ and u_j^κ determine the κ-expressions of depth $\alpha + 1$.

Theorem 2.3 *Let $|I| \ge 2$ and $|S| \ge 2$. Then, there is no weak-universal ∞-type space on S for player set I and there is no weak-universal $*$-type space on S for player set I.*

Proof Assume there is a weak-universal ∞-type space (a weak-universal $*$-type space, respectively)

$$\underline{U} = \left\langle U, (T_i)_{i \in I}, \Sigma, \theta^U \right\rangle$$

on S for player set I. Then, the underlying set U has a cardinality $|U|$. There is a regular cardinal number $\kappa > |U|$.

$$\underline{W^\kappa} = \left\langle W^\kappa, (T_i)_{i \in \{a,b\}}, \mathrm{Pow}(W^\kappa), \theta \right\rangle$$

is a $*$-type space on $\{h, t\}$ (and therefore a ∞-type space). Since $|\{h, t\}| = 2$, we can assume without loss of generality that $\{h, t\} \subseteq S$, and since $\Sigma_S = \mathrm{Pow}(S)$, that $\Sigma_S \supseteq \mathrm{Pow}(\{h, t\})$. Also, since $|I| \ge 2$, we can assume without loss of generality that $\{a, b\} \subseteq I$. For $i \in I \setminus \{a, b\}$ define $T_i(w^\kappa) := \delta_{w^\kappa}$, and view θ as a function from W^κ to S. Then

$$\underline{W}_I^\kappa := \left\langle W^\kappa, (T_i)_{i\in I}, \text{Pow}(W^\kappa), \theta \right\rangle$$

is a $*$-type space on S (with player set I). Since every $*$-type space is a ∞-type space, $\underline{W}_I^\kappa$ is also a ∞-type space. According to the assumption, there is a type morphism $f : W^\kappa \to U$. Since both spaces are in particular κ-type spaces, this morphism preserves κ-descriptions. If φ is a κ -expression in the 'language' corresponding to the set of states of nature $\{h, t\}$ and the player set $\{a, b\}$, then φ is also a κ-expression in the 'language' corresponding to the set of states of nature S and the player set I, and it is easy to check that for $w^\kappa \in W^\kappa$ we have $w^\kappa \in \varphi^{\underline{W}_I^\kappa}$ iff $w^\kappa \in \varphi^{\underline{W}^\kappa}$. So, by Lemma 2.19, it is still the case that two different states of $\underline{W}_I^\kappa$ have different κ-descriptions. Hence, since by Proposition 2.1, f preserves κ-descriptions, f is one-to-one. It follows that $|U| \geq |\underline{W}_I^\kappa| \geq \kappa$, which is a contradiction to $|U| < \kappa$. ■

Corollary 2.1 *Let $|I| \geq 2$ and $|S| \geq 2$. Then there is no universal ∞-type space on S for player set I and there is no universal $*$-type space on S for player set I.*

2.5 The Universal κ-Type Space as a Space of Coherent Hierarchies

For the rest of this chapter, fix a nonempty player set I (of arbitrary cardinality), a nonempty set of states of nature S, and a κ-field Σ_S on S such that for all $s, s' \in S$ with $s \neq s'$ there is a $E \in \Sigma_S$ such that $s \in E$ and $s' \notin E$. We assume, without loss of generality, that $0 \notin I$ and define $I_0 := I \cup \{0\}$, where "0" stands for nature. For $i \in I$, we define $-i := I_0 \setminus \{i\}$.

Given the above data, we construct the space of coherent hierarchies up to (but excluding) level κ. We show that the set of all such coherent hierarchies generates a product κ-type space (Theorem 2.4) and that this space is the universal κ-type space (Theorem 2.5). It is easy to see that levels of the coherent hierarchies correspond to depths of κ-expressions. So, Theorem 2.2 (together with Proposition 2.1) shows that the hierarchies up to some lower level than κ would not suffice to construct the universal κ-type space.

The remarkable fact of the construction in this chapter is that, unlike the construction in section 2.3, it does not use other κ-type spaces. Therefore, it provides an independent characterization of the universal κ-type space as the space of coherent hierarchies. So, we learn here what the states of the world in the universal κ-type space 'are'.

Definition 2.28 Define

- $C_0^\alpha := S$, for $\kappa \geq \alpha \geq 0$,
- $\Sigma_0^\alpha := \Sigma_{S,}$ for $\kappa \geq \alpha \geq 0$,
- C_i^0 to be a singleton, for $i \in I$,
- $\Sigma_i^0 := \{\emptyset, C_i^0\}$, for $i \in I$,
- $C^\alpha := \prod_{i \in I_0} C_i^\alpha$, for $\kappa \geq \alpha \geq 0$,
- Σ^α to be the product κ-field of the Σ_i^α, $i \in I_0$, on C^α, for $\kappa \geq \alpha \geq 0$,
- $C_{-i}^\alpha := \prod_{j \in -i} C_j^\alpha$, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- Σ_{-i}^α to be the product κ-field of the Σ_j^α, $j \in -i$, on C_{-i}^α, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- $C_i^\alpha :=$
  $$\left\{ (t_i^\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} \Delta^\kappa\left(C^\beta\right) \;\middle|\; \begin{array}{lllll} \forall \beta < \gamma < \alpha & : & \operatorname{marg}_{C^\beta}\left(t_i^\gamma\right) & = & t_i^\beta \quad \text{and} \\ \forall 0 < \gamma < \alpha & : & \operatorname{marg}_{C_i^\gamma}\left(t_i^\gamma\right) & = & \delta_{\left((t_i^\beta)_{\beta<\gamma}\right)} \end{array} \right\},$$
  for $\kappa \geq \alpha \geq 1$ and $i \in I$,
- Σ_i^α to be the κ-field on C_i^α inherited from the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \alpha$, on $\prod_{\beta<\alpha} \Delta^\kappa\left(C^\beta\right)$, for $\kappa \geq \alpha \geq 1$ and $i \in I$,
- $\pi_{\gamma,\alpha}^0 : C_0^\alpha \to C_0^\gamma$ to be the identity on S, for $\kappa \geq \alpha \geq \gamma \geq 0$,
- $\pi_{0,\alpha}^i : C_i^\alpha \to C_i^0$ to be the obvious map, for $\kappa \geq \alpha \geq 0$ and $i \in I$,
- $\pi_{\gamma,\alpha}^i : C_i^\alpha \to C_i^\gamma$ to be the canonical projection $\pi_{\gamma,\alpha}^i((t_i^\beta)_{\beta<\alpha}) := (t_i^\beta)_{\beta<\gamma}$, for $\kappa \geq \alpha \geq \gamma \geq 1$ and $i \in I$,
- $\pi_{\gamma,\alpha} := (\pi_{\gamma,\alpha}^i)_{i \in I_0}$, for $\kappa \geq \alpha \geq \gamma \geq 0$,
- $\pi_{\gamma,\alpha}^{-i} := (\pi_{\gamma,\alpha}^j)_{j \in -i}$, for $\kappa \geq \alpha \geq \gamma \geq 0$ and $i \in I$.

Notation 2.3 Let $0 \leq \gamma < \alpha \leq \kappa$ and $c_i^\alpha = (t_i^\beta)_{\beta<\alpha} \in C_i^\alpha$. Then we define

$$\begin{array}{lll} c_i^\alpha(\gamma) & := & t_i^\gamma, \\ c_i^\alpha\lceil 0 & := & c_i^0, \qquad \text{where } C_i^0 = \{c_i^0\}, \\ c_i^\alpha\lceil\gamma & := & (t_i^\beta)_{\beta<\gamma}, \quad \text{for } \gamma > 0. \end{array}$$

Lemma 2.20 *1. For every $i \in I_0$ and for every ordinal α such that $0 \leq \alpha \leq \kappa :$*

$$C_i^\alpha \quad \textit{is nonempty.}$$

2. For every $i \in I_0$ and for all ordinals α, β such that $0 \leq \beta < \alpha \leq \kappa :$

$$\pi^i_{\beta,\alpha} : C_i^\alpha \to C_i^\beta$$

is onto and measurable.

Proof For $i = 0$, 1. and 2. are clear by the definitions.

For $i \in I$ and every $\alpha \leq \kappa$, the measurability of $\pi^i_{0,\alpha} : C_i^\alpha \to C_i^0$ is clear, since C_i^0 is a singleton. For $i \in I$ and $1 \leq \gamma < \alpha \leq \kappa$, the measurability of $\pi^i_{\gamma,\alpha}$ follows from the fact that the inverse images of the generators of the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \gamma$, on $\Pi_{\beta<\gamma}\Delta^\kappa\left(C^\beta\right)$ are among the generators of the product κ-field of the $\Sigma_{\Delta^\kappa\left(C^\beta\right)}$, $\beta < \alpha$, on $\Pi_{\beta<\alpha}\Delta^\kappa\left(C^\beta\right)$. Note that by the nonemptyness of C_i^0, for $i \in I$, 2. implies 1. Note furthermore that, if $0 \leq \beta < \gamma \leq \kappa$ and if $\pi^i_{\beta,\gamma} : C_i^\gamma \to C_i^\beta$, for $i \in I_0$, is onto and measurable, then $\pi_{\beta,\gamma} : C^\gamma \to C^\beta$ is onto and measurable.

It remains to show by transfinite induction on $\alpha \leq \kappa$ that $\pi^i_{\beta,\alpha} : C_i^\alpha \to C_i^\beta$ is onto, for $i \in I$ and $0 \leq \beta < \alpha \leq \kappa :$

- $\alpha = 0 :$ Since there is no ordinal $\beta < 0$, there is nothing to show.
- $\alpha = 1 :$ Since S and the C_i^0, for $i \in I$, are nonempty, C^0 is nonempty. It follows that $\Delta^\kappa\left(C^0\right)$ is nonempty: Take for example $c^0 \in C^0$, and define for $E \in \Sigma^0$:

$$\delta_{c^0}(E) := \begin{cases} 1 & \text{if} \quad c^0 \in E, \\ 0 & \text{if} \quad c^0 \notin E. \end{cases}$$

  δ_{c^0} is the so-called δ-measure at c^0 and it is a (even σ-additive) probability measure on (C^0, Σ^0). Since C_i^0, for $i \in I$, is a singleton, we have $\text{marg}_{C_i^0}\left(\delta_{c^0}\right) = \delta_{c_i^0}$, for $i \in I$, where $\{c_i^0\} = C_i^0$. By definition, we have $C_i^1 = \Delta^\kappa\left(C^0\right)$, for $i \in I$. Since C_i^0, for $i \in I$, is a singleton, it follows that $\pi^i_{0,1}$ is onto, for $i \in I$.
- Let $i \in I$ and let $\alpha < \kappa$ be a successor ordinal, i.e. $\alpha = \gamma + 1$, for an ordinal $\gamma > 0$. Since for every $\zeta < \gamma$ and for every $c_i^\zeta \in C_i^\zeta$ there is, by the induction assumption, a $c_i^\gamma \in C_i^\gamma$ with $\pi^i_{\zeta,\gamma}\left(c_i^\gamma\right) = c_i^\zeta$, it is enough to show that for every $c_i^\gamma \in C_i^\gamma$ there is a $c_i^\alpha \in C_i^\alpha$ with $\pi^i_{\gamma,\alpha}\left(c_i^\alpha\right) \in c_i^\gamma$.

  So, let $c_i^\gamma = (t_i^\beta)_{\beta<\gamma} \in C_i^\gamma$. We have to find a $t_i^\gamma \in \Delta^\kappa\left(C^\gamma\right)$ such that

$\operatorname{marg}_{C^\beta}(t_i^\gamma) = t_i^\beta$, for all $\beta < \gamma$ and $\operatorname{marg}_{C_i^\gamma}(t_i^\gamma) = \delta_{\left((t_i^\beta)_{\beta<\gamma}\right)}$. By the induction assumption, and in the case of $j = 0$, by the definition, $\pi^j_{\beta,\xi} : C_j^\xi \to C_j^\beta$ is onto and measurable, for all $j \in I_0$ and all ordinals β, ξ such that $0 \leq \beta \leq \xi \leq \gamma$. This and the fact that $\operatorname{marg}_{C^\beta}(t_i^\xi) = t_i^\beta$, for $\beta < \xi < \gamma$, imply by Lemma 2.3 that c_i^γ induces a well-defined finitely additive probability measure $t_i^{<\gamma}$ on the field

$$\mathcal{F} := \left\{E \subseteq C^\gamma \mid \exists \beta < \gamma : E = \pi^{-1}_{\beta,\gamma}(F) \text{ for a } F \in \Sigma^\beta\right\} \subseteq \Sigma^\gamma$$

by $t_i^{<\gamma}\left(\pi^{-1}_{\beta,\gamma}(F)\right) := t_i^\beta(F)$. We show now that the outer measure $\left(t_i^{<\gamma}\right)^*\left(\{c_i^\gamma\} \times C^\gamma_{-i}\right)$ of $\{c_i^\gamma\} \times C^\gamma_{-i}$ is equal to 1. Clearly, it cannot be greater, since $t_i^\beta\left(C^\beta\right) = 1$, for $\beta < \gamma$. Let now $\beta < \gamma$, $E^\beta \in \Sigma^\beta$ and $\pi^{-1}_{\beta,\gamma}\left(E^\beta\right) \supseteq \{c_i^\gamma\} \times C^\gamma_{-i}$. Then $E^\beta \supseteq \left\{\pi^i_{\beta,\gamma}(c_i^\gamma)\right\} \times C^\beta_{-i}$ and therefore $t_i^\beta\left(E^\beta\right) = 1$. This implies that $t_i^{<\gamma}\left(\pi^{-1}_{\beta,\gamma}\left(E^\beta\right)\right) = 1$. By the Łoś-Marczewski Theorem, we can extend $t_i^{<\gamma}$ to a finitely additive probability measure μ_0 on the field $\left[\mathcal{F}, \{c_i^\gamma\} \times C^\gamma_{-i}\right]$ on C^γ such that $\mu_0\left(\{c_i^\gamma\} \times C^\gamma_{-i}\right) = 1$. By Lemma 2.5, we can extend μ_0 to a finitely additive probability measure μ_1 on the field $\left[\Sigma^\gamma, \{c_i^\gamma\} \times C^\gamma_{-i}\right]$. Define now t_i^γ as the restriction of μ_1 to Σ^γ. t_i^γ has the desired properties.

- Let $i \in I$, let α be a limit ordinal $\leq \kappa$, let $\gamma < \alpha$, and let $c_i^\gamma \in C_i^\gamma$. Consider the set

$$Y := \left\{a_i^\zeta \in \bigcup_{\gamma \leq \zeta \leq \alpha} C_i^\zeta \;\middle|\; \pi^i_{\gamma,\zeta}(a_i^\zeta) = c_i^\gamma\right\}.$$

Y is nonempty, because it contains c_i^γ. Y is partially ordered by:

$$a_i^\zeta \sqsubseteq a_i^\eta \text{ iff } \eta \geq \zeta \text{ and } \pi^i_{\zeta,\eta}(a_i^\eta) = a_i^\zeta.$$

Note that $a_i^\zeta \sqsubseteq a_i^\eta$ and $a_i^\eta \sqsubseteq a_i^\zeta$ imply $a_i^\zeta = a_i^\eta$. Now, let A be a nonempty and totally $\sqsubseteq$-ordered subset of Y. Define

$$\alpha(A) := \sup\{\zeta \mid \exists a_i^\zeta \in A \cap C_i^\zeta\}.$$

Obviously, we have $\alpha(A) \leq \alpha$. For every $\beta < \alpha(A)$ there is a $\zeta > \beta$ such that there is a $a_i^\zeta \in A \cap C_i^\zeta$. Define

$$\sqcup A := (t_i^\beta)_{\beta<\alpha(A)},$$

such that $t_i^\beta = a_i^\zeta(\beta)$ for a $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$. Note that if $\beta < \zeta$ and $\beta < \eta$, and if $a_i^\zeta \in A \cap C_i^\zeta$ and $a_i^\eta \in A \cap C_i^\eta$, then we have $\eta \leq \zeta$ and $\pi^i_{\eta,\zeta}(a_i^\zeta) = a_i^\eta$ or $\zeta \leq \eta$ and $\pi^i_{\zeta,\eta}(a_i^\eta) = a_i^\zeta$. In both cases, we have $a_i^\eta(\beta) = a_i^\zeta(\beta)$, and hence $\sqcup A$ is well-defined. Note furthermore that, for $0 < \beta < \alpha(A)$ and $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$, we have

$$\operatorname{marg}_{C_i^\beta}(\sqcup A(\beta)) = \operatorname{marg}_{C_i^\beta}(t_i^\beta) = \operatorname{marg}_{C_i^\beta}(a_i^\zeta(\beta)) = \delta_{a_i^\zeta \lceil \beta} = \delta_{\sqcup A \lceil \beta},$$

and for $\xi < \beta < \alpha(A)$ and $a_i^\zeta \in A \cap C_i^\zeta$ with $\beta < \zeta$, we have

$$\operatorname{marg}_{C^\xi}(\sqcup A(\beta)) = \operatorname{marg}_{C^\xi}(t_i^\beta) = \operatorname{marg}_{C^\xi}(a_i^\zeta(\beta)) = a_i^\zeta(\xi) = \sqcup A(\xi).$$

Hence, $\sqcup A \in C_i^{\alpha(A)}$, and by construction (since $a_i^\zeta \lceil \gamma = c_i^\gamma$, for $a_i^\zeta \in A$): $\pi^i_{\gamma,\alpha(A)}(\sqcup A) = c_i^\gamma$. Therefore, $\sqcup A \in Y$ and, by construction, $\sqcup A$ is an $\sqsubseteq$-upper bound of the set A in Y.

Therefore we can apply Zorn's Lemma, hence there is a $\sqsubseteq$-maximal element $a \in Y$. Assume $a \in C_i^\zeta$, for a $\zeta < \alpha$. Since α is a limit number, there is a η with $\zeta < \eta < \alpha$. The induction assumption implies that there is a $a_i^\eta \in C_i^\eta$, such that $\pi^i_{\zeta,\eta}(a_i^\eta) = a$. Since $a \in Y$, it follows that $\pi^i_{\gamma,\eta}(a_i^\eta) = \pi^i_{\gamma,\zeta}(a) = c_i^\gamma$, and hence $a_i^\eta \in Y$. Therefore a is not maximal, which is a contradiction. It follows that $a \in C_i^\alpha$ and $\pi^i_{\gamma,\alpha}(a) = c_i^\gamma$, as desired.

■

Lemma 2.21

$$\Sigma^\kappa = \bigcup_{\beta<\kappa} \pi^{-1}_{\beta,\kappa}\left(\Sigma^\beta\right).$$

Proof By Lemma 1 and Remark 2.5, it follows that $\pi^{-1}_{\beta,\kappa}\left(\Sigma^\beta\right)$ is a κ-field on C^κ, for $\beta < \kappa$. If $\gamma < \alpha < \kappa$, then, by Lemma 1 it follows that $\Sigma^\kappa \supseteq \pi^{-1}_{\alpha,\kappa}(\Sigma^\alpha) \supseteq \pi^{-1}_{\gamma,\kappa}(\Sigma^\gamma)$ and therefore $\Sigma^\kappa \supseteq \bigcup_{\beta<\kappa} \pi^{-1}_{\beta,\kappa}\left(\Sigma^\beta\right)$. By Lemma 2.2, $\bigcup_{\beta<\kappa} \pi^{-1}_{\beta,\kappa}\left(\Sigma^\beta\right)$ is a κ-field on C^κ.

Σ^κ is the product κ-field of the Σ_i^κ, $i \in I_0$. By Lemma 1, Remark 2.5, and Lemma 2.2, it follows that $\bigcup_{\beta<\kappa} \left(\pi^i_{\beta,\kappa}\right)^{-1}\left(\Sigma_i^\beta\right) = \Sigma_i^\kappa$, since every generator of Σ_i^κ is in $\left(\pi^i_{\beta,\kappa}\right)^{-1}\left(\Sigma_i^\beta\right)$ for a $\beta < \kappa$, and since the left hand side is already a κ-field. It follows that the sets of the form $\Pi_{j\in I_0} E_j$ such that $E_j = C_j^\kappa$, for all but one $i \in I_0$, and $E_i = \left(\pi^i_{\beta,\kappa}\right)^{-1}\left(E_i^\beta\right)$, for a $\beta < \kappa$ and $E_i^\beta \in \Sigma_i^\beta$, are already a generating set of Σ^κ. This implies that $\bigcup_{\beta<\kappa} \pi^{-1}_{\beta,\kappa}\left(\Sigma^\beta\right) \supseteq \Sigma^\kappa$. ■

And in the same way, with obvious changes, one proves:

Lemma 2.22 *Let $i \in I$. Then,*

$$\Sigma^{\kappa}_{-i} = \bigcup_{\beta<\kappa} \left(\pi^{-i}_{\beta,\kappa}\right)^{-1} \left(\Sigma^{\beta}_{-i}\right).$$

Definition 2.29
- Define θ^{κ} to be the identity on $C^{\kappa}_0 = S$.
- For $i \in I$, define $T^{\kappa}_i : C^{\kappa}_i \to \Delta^{\kappa}(C^{\kappa})$ by:

$$T^{\kappa}_i((t^{\beta}_i)_{\beta<\kappa})(E) := t^{\alpha}_i(E^{\alpha}),$$

  for $(t^{\beta}_i)_{\beta<\kappa} \in C^{\kappa}_i$ and for $E = \pi^{-1}_{\alpha,\kappa}(E^{\alpha})$, where $E^{\alpha} \in \Sigma^{\alpha}$, for a $\alpha < \kappa$.

Since for $\beta < \gamma < \kappa$: $\operatorname{marg}_{C^{\beta}}(t^{\gamma}_i) = t^{\beta}_i$, and since $\pi_{\beta,\alpha}$ is onto, for $\beta < \alpha \leq \kappa$, this definition is independent of the particular E^{α} chosen to define $T^{\kappa}_i(c^{\kappa}_i)(E)$.

Theorem 2.4

$$\underline{C}^{\kappa} := \langle C^{\kappa}, \Sigma^{\kappa}, (T^{\kappa}_i)_{i\in I}, \theta^{\kappa}\rangle$$

is a product κ-type space on S for player set I.

Proof By the definition, Lemma 1, and Lemma 2.3, $T^{\kappa}_i(c^{\kappa}_i)$ is a finitely additive probability measure on $\bigcup_{\beta<\kappa} \pi^{-1}_{\beta,\kappa}(\Sigma^{\beta}) = \Sigma^{\kappa}$.

$T^{\kappa}_i : C^{\kappa}_i \to \Delta^{\kappa}(C^{\kappa})$ is measurable: Let $c^{\kappa}_i = (t^{\beta}_i)_{\beta<\kappa} \in C^{\kappa}_i$, $E = \pi^{-1}_{\alpha,\kappa}(E^{\alpha})$, for a $\alpha < \kappa$ and a $E^{\alpha} \in \Sigma^{\alpha}$, and let $p \in [0,1]$. We have $T^{\kappa}_i(c^{\kappa}_i)(E) \geq p$ iff $t^{\alpha}_i(E^{\alpha}) \geq p$. But $\{t^{\alpha}_i \in \Delta^{\kappa}(C^{\alpha}) \mid t^{\alpha}_i(E^{\alpha}) \geq p\}$ is measurable in $\Delta^{\kappa}(C^{\alpha})$, so by the definition of Σ^{κ}_i, the set

$$\left\{(t^{\beta}_i)_{\beta<\kappa} \in C^{\kappa}_i \middle| \; T^{\kappa}_i((t^{\beta}_i)_{\beta<\kappa})(E) \geq p\right\}$$

is measurable.

It remains to prove the introspection property. (The measurability of θ^{κ} is clear anyway.) Let $c^{\kappa}_i, a^{\kappa}_i \in C^{\kappa}_i$, with $c^{\kappa}_i \neq a^{\kappa}_i$. Then, by the definitions, there is a $\alpha < \kappa$ and a $E^{\alpha} \in \Sigma^{\alpha}$ such that

$$T^{\kappa}_i(c^{\kappa}_i)\left(\pi^{-1}_{\alpha,\kappa}(E^{\alpha})\right) \neq T^{\kappa}_i(a^{\kappa}_i)\left(\pi^{-1}_{\alpha,\kappa}(E^{\alpha})\right).$$

It follows that

$$[T_i^\kappa(c_i^\kappa)] := \{a^\kappa \in C^\kappa \mid T_i^\kappa(c_i^\kappa) = T_i^\kappa(a_i^\kappa)\} = \{c_i^\kappa\} \times C_{-i}^\kappa.$$

Let $A \in \Sigma^\kappa$ with $A \supseteq \{c_i^\kappa\} \times C_{-i}^\kappa$. Since $A = \pi_{\alpha,\kappa}^{-1}(A^\alpha)$ for a $\alpha < \kappa$ and a $A^\alpha \in \Sigma^\alpha$, it follows that $A^\alpha \supseteq \{c_i^\kappa \lceil \alpha\} \times C_{-i}^\alpha$. By the definition of $c_i^\kappa = (t_i^\beta)_{\beta<\kappa}$, we have $\operatorname{marg}_{C_i^\alpha}(t_i^\alpha) = \delta_{(t_i^\beta)_{\beta<\alpha}} = \delta_{c_i^\kappa \lceil \alpha}$. Therefore $t_i^\alpha(A^\alpha) = 1$, and by the definition, $T_i^\kappa(c_i^\kappa)(A) = 1$ and therefore $\operatorname{marg}_{C_i^\kappa}(T_i^\kappa(c_i^\kappa)) = \delta_{c_i^\kappa}$. ■

Definition 2.30 Let

$$\langle M, \Sigma, (T_i)_{i\in I}, \theta\rangle$$

be a κ-type space on S for player set I. By transfinite induction on $\alpha \leq \kappa$, we define:

1. Define

   $$h_0^\alpha : M \to C_0^\alpha = S \text{ by } h_0^\alpha := \theta,$$

   for $0 \leq \alpha \leq \kappa$. Note that θ is measurable.

2. Define

   $$h_i^0 : M \to C_i^0$$

   in the obvious way, for $i \in I$. Note that h_i^0 is uniquely defined and measurable, since C_i^0 is a singleton.

3. If $h_i^\alpha : M \to C_i^\alpha$ is already defined and measurable, for $i \in I$, then define

   $$h^\alpha : M \to C^\alpha \text{ by } h^\alpha(m) := \left(h_j^\alpha(m)\right)_{j\in I_0}.$$

   h^α is measurable, since by 1. h_0^α is measurable and each h_i^α, for $i \in I$, is measurable.

4. If $h^\alpha : M \to C^\alpha$ is already defined and measurable and if $\alpha < \kappa$, then define

   $$g_i^\alpha(m)(\cdot) := T_i(m)\left((h^\alpha)^{-1}(\cdot)\right), \quad \text{for } i \in I \text{ and } m \in M.$$

   Note that (in the notation of Remark 2.4) $g_i^\alpha = \Delta_{h^\alpha}^\kappa \circ T_i$, for $i \in I$. By Remark 2.4, it follows that

   $$g_i^\alpha : M \to \Delta^\kappa(C^\alpha), \quad \text{for } i \in I.$$

   By the measurability of h^α and of T_i, for $i \in I$, and by Remark 2.4, we have that g_i^α is measurable, for $i \in I$.

5. Define

$$h_i^1 : M \to C_i^1 = \Delta^\kappa\left(C^0\right) \quad \text{by} \quad h_i^1 := g_i^0, \quad \text{for } i \in I.$$

6. Let $\alpha = \gamma + 1$, where $0 < \gamma < \kappa$, and let $h_i^\gamma : M \to C_i^\gamma$, for $i \in I$, be already defined and measurable such that $h_i^\gamma(m) = (g_i^\beta(m))_{\beta<\gamma}$, for all $m \in M$ and for $i \in I$, where $g_i^\beta : M \to \Delta^\kappa\left(C^\beta\right)$ is already defined and measurable, for $i \in I$ and $\beta < \gamma$. By 3. and 4. from above, $g_i^\gamma : M \to \Delta^\kappa(C^\gamma)$ is defined and measurable, for $i \in I$. Define

$$h_i^{\gamma+1}(m) := (g_i^\beta(m))_{\beta<\gamma+1}, \quad \text{for } m \in M \text{ and } i \in I.$$

We have to show that $h_i^{\gamma+1} : M \to C_i^{\gamma+1}$, for $i \in I$. It suffices to show that

$$\operatorname{marg}_{C_i^\gamma}(g_i^\gamma(m)) = \delta_{\left(g_i^\beta(m)\right)_{\beta<\gamma}}, \quad \text{for } m \in M \text{ and } i \in I,$$

and for all $\beta < \gamma$:

$$\operatorname{marg}_{C^\beta}(g_i^\gamma(m)) = g_i^\beta(m), \quad \text{for } m \in M \text{ and } i \in I.$$

Let $i \in I$ and $E^\gamma \in \Sigma^\gamma$ such that $E^\gamma \supseteq \{(g_i^\beta(m))_{\beta<\gamma}\} \times C_{-i}^\gamma$. We have to show that

$$g_i^\gamma(m)(E^\gamma) = 1.$$

First observe that for all $m, m' \in M$ and $i \in I$: if $T_i(m) = T_i(m')$, then, by the definition, $g_i^\beta(m) = g_i^\beta(m')$, for all $\beta < \gamma$. So it follows that $h_i^\gamma(m) = h_i^\gamma(m')$. Obviously, we have $h^\gamma(m) \in \{h_i^\gamma(m)\} \times C_{-i}^\gamma$. This implies that $(h^\gamma)^{-1}(E^\gamma) \supseteq [T_i(m)]$ and therefore

$$T_i(m)\left((h^\gamma)^{-1}(E^\gamma)\right) = 1.$$

By the definitions of h_i^β and h_i^η, for $i \in I$, we have

$$h_i^\beta = \pi_{\beta,\eta}^i \circ h_i^\eta \quad \text{and} \quad h_0^\beta = \theta = \mathrm{id}_S \circ \theta = \pi_{\beta,\eta}^0 \circ h_0^\eta, \quad \text{for } \beta < \eta \leq \gamma \text{ and } i \in I.$$

Therefore, $h^\beta = \pi_{\beta,\eta} \circ h^\eta$, for $\beta < \eta \leq \gamma$. It follows that

$$\left(h^\beta\right)^{-1}\left(E^\beta\right) = (h^\eta)^{-1}\left(\pi_{\beta,\eta}^{-1}\left(E^\beta\right)\right),$$

for $\beta < \eta \leq \gamma$ and $E^\beta \in \Sigma^\beta$.
Let $E^\beta \in \Sigma^\beta$, $\beta < \gamma$, $m \in M$ and $i \in I$. It follows now:

$$\begin{aligned} g_i^\gamma(m)\left(\pi_{\beta,\gamma}^{-1}\left(E^\beta\right)\right) &= T_i(m)\left((h^\gamma)^{-1}\left(\pi_{\beta,\gamma}^{-1}\left(E^\beta\right)\right)\right) \\ &= T_i(m)\left(\left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ &= g_i^\beta(m)\left(E^\beta\right). \end{aligned}$$

We note that h_i^α, for $i \in I$, is measurable, since every g_i^β, for $i \in I$ and $\beta < \alpha$, is measurable.

7. Let α be a limit ordinal $\leq \kappa$ and let, for every $i \in I$ and every $\gamma < \alpha$, $h_i^\gamma : M \to C_i^\gamma$ be already defined and measurable such that $h_i^\gamma(m) = (g_i^\beta(m))_{\beta<\gamma}$, for all $m \in M$, where $g_i^\beta : M \to \Delta^\kappa\left(C^\beta\right)$ is already defined and measurable, for $\beta < \gamma$.
Since α is a limit ordinal, it follows for every $\beta < \alpha$ that $\beta + 1 < \alpha$. Therefore $g_i^\beta : M \to \Delta^\kappa\left(C^\beta\right)$ is already defined and measurable, for $\beta < \alpha$ and $i \in I$. It follows that

$$h_i^\alpha(m) := (g_i^\beta(m))_{\beta<\alpha}$$

is well-defined and measurable, for $m \in M$ and $i \in I$. For $i \in I$, we have to show that $h_i^\alpha : M \to C_i^\alpha$ and that h_i^α is measurable:
Let $\beta < \gamma < \alpha$. Since α is a limit ordinal, we have $\gamma + 1 < \alpha$. By the induction assumption, we have that $h_i^{\gamma+1} : M \to C_i^{\gamma+1}$, and it follows that g_i^ξ is measurable, for $0 \leq \xi \leq \gamma$ and that

$$\mathrm{marg}_{C_i^\gamma}(g_i^\gamma(m)) = \delta_{\left(g_i^\xi(m)\right)_{\xi<\gamma}} \quad \text{and} \quad \mathrm{marg}_{C^\beta}(g_i^\gamma(m)) = g_i^\beta(m),$$

for all $m \in M$. This shows that

$$h_i^\alpha : M \to C_i^\alpha, \quad \text{for } i \in I.$$

Finally, we note that h_i^α, for $i \in I$, is measurable, since every g_i^β, for $i \in I$ and $\beta < \alpha$, is measurable.

Proposition 2.4 *Let $\langle M, \Sigma, (T_i)_{i\in I}, \theta\rangle$ be a κ-type space on S for player set I. Then, the κ-hierarchy description map*

$$h^\kappa : M \to C^\kappa$$

is a type morphism.

Proof We showed already in the above definition that h^κ is measurable.
We have

$$\theta^\kappa\left(h^\kappa\left(m\right)\right)=h_0^\kappa\left(m\right)=\theta\left(m\right),\quad\text{for } m\in M.$$

Let $i\in I$ and $E\in\Sigma^\kappa$. Then there is an ordinal $\alpha<\kappa$ and $E^\alpha\in\Sigma^\alpha$ such that $E=\pi_{\alpha,\kappa}^{-1}\left(E^\alpha\right)$. It follows by the definitions that for $m\in M$:

$$\begin{aligned} T_i^\kappa\left(h^\kappa\left(m\right)\right)\left(E\right) &= T_i^\kappa((g_i^\beta\left(m\right))_{\beta<\kappa})\left(E\right) \\ &= g_i^\alpha\left(m\right)\left(E^\alpha\right) \\ &= T_i\left(m\right)\left(\left(h^\alpha\right)^{-1}\left(E^\alpha\right)\right) \\ &= T_i\left(m\right)\left(\left(h^\kappa\right)^{-1}\left(\pi_{\alpha,\kappa}^{-1}\left(E^\alpha\right)\right)\right) \\ &= T_i\left(m\right)\left(\left(h^\kappa\right)^{-1}\left(E\right)\right). \end{aligned}$$

■

The next Lemma shows that type morphisms between κ-type spaces preserve the κ-hierarchy descriptions.

Lemma 2.23 *Let $\langle M,\Sigma,(T_i)_{i\in I},\theta\rangle$ and $\langle\widehat{M},\widehat{\Sigma},(\widehat{T}_i)_{i\in I},\widehat{\theta}\rangle$ be κ-type spaces on S for player set I, let $h^\kappa:M\to C^\kappa$ and $\widehat{h}^\kappa:\widehat{M}\to C^\kappa$ be the κ-hierarchy description maps, and let $f:M\to\widehat{M}$ be a type morphism.*
Then, for all $m\in M$:

$$h^\kappa\left(m\right)=\widehat{h}^\kappa\left(f\left(m\right)\right).$$

Proof Since C_i^0 is a singleton, for $i\in I$, we have that $h_i^0\left(m\right)=\widehat{h}_i^0\left(f\left(m\right)\right)$, for $i\in I$ and $m\in M$.
For $0\leq\alpha\leq\kappa$ and $m\in M$ we have

$$h_0^\alpha\left(m\right)=\theta\left(m\right)=\widehat{\theta}\left(f\left(m\right)\right)=\widehat{h}_0^\alpha\left(f\left(m\right)\right).$$

Let $\alpha=\beta+1<\kappa$ and $h^\beta\left(m\right)=\widehat{h}^\beta\left(f\left(m\right)\right)$, for $m\in M$. For $i\in I$ and $m\in M$, it follows that

$$\begin{aligned} g_i^\beta\left(m\right) &= T_i\left(m\right)\circ\left(h^\beta\right)^{-1} \\ &= T_i\left(m\right)\circ(f^{-1}\circ(\widehat{h}^\beta)^{-1}) \\ &= \widehat{T}_i\left(f\left(m\right)\right)\circ(\widehat{h}^\beta)^{-1} \\ &= \widehat{g}_i^\beta\left(f\left(m\right)\right). \end{aligned}$$

This implies that $h_i^{\beta+1}(m) = \widehat{h}_i^{\beta+1}(f(m))$, for $i \in I$ and $m \in M$.

Let $i \in I$, let α be a limit ordinal $\leq \kappa$, and let $h_i^{\beta}(m) = \widehat{h}_i^{\beta}(f(m))$, for all $\beta < \alpha$ and $m \in M$. To show that $h_i^{\alpha}(m) = \widehat{h}_i^{\alpha}(f(m))$, for $i \in I$ and $m \in M$, it suffices to verify that $g_i^{\beta}(m) = \widehat{g}_i^{\beta}(f(m))$, for all $\beta < \alpha$. But this follows from the fact that $\beta < \alpha$ implies that $\beta + 1 < \alpha$ and from

$$(g_i^{\zeta}(m))_{\zeta \leq \beta} = h_i^{\beta+1}(m) = \widehat{h}_i^{\beta+1}(f(m)) = (\widehat{g}_i^{\zeta}(f(m)))_{\zeta \leq \beta}.$$

■

Proposition 2.5 *The κ-hierarchy description map*

$$h^{\kappa} : C^{\kappa} \to C^{\kappa}$$

is the identity.

Proof Let $c^{\kappa} \in C^{\kappa}$, let $0 \leq \alpha \leq \kappa$ and, for $i \in I_0$, let π_i be the projection

$$\pi_i : C^{\kappa} \to C_i^{\kappa}.$$

Then

- $$h_0^{\alpha}(c^{\kappa}) = \theta^{\kappa}(c^{\kappa}) = c_0^{\kappa} = \pi_0(c^{\kappa}) = \pi_{\alpha,\kappa}^0(\pi_0(c^{\kappa})).$$

- We have $h_i^0(c^{\kappa}) = c_i^0$, for $i \in I$, where $\{c_i^0\} = C_i^0$, and therefore $h_i^0(c^{\kappa}) = \pi_{0,\kappa}^i(\pi_i(c^{\kappa}))$.

- Let $\alpha = \beta + 1 < \kappa$ and let $h_i^{\beta} = \pi_{\beta,\kappa}^i \circ \pi_i$, for $i \in I$. Let furthermore $c_i^{\kappa} = (t_i^{\zeta})_{\zeta < \kappa}$, for $i \in I$, and let $E^{\beta} \in \Sigma^{\beta}$. For $i \in I$, we have

$$\begin{aligned} g_i^{\beta}(c^{\kappa})\left(E^{\beta}\right) &= T_i^{\kappa}(c^{\kappa})\left(\left(h^{\beta}\right)^{-1}\left(E^{\beta}\right)\right) \\ &= T_i^{\kappa}(c^{\kappa})\left(\left(\left(\pi_{\beta,\kappa}^i \circ \pi_i\right)_{i \in I_0}\right)^{-1}\left(E^{\beta}\right)\right) \\ &= T_i^{\kappa}(c^{\kappa})\left((\pi_{\beta,\kappa})^{-1}\left(E^{\beta}\right)\right) \\ &= t_i^{\beta}\left(E^{\beta}\right). \end{aligned}$$

It follows that $g_i^\beta(c^\kappa) = t_i^\beta$ and therefore $h_i^\alpha(c^\kappa) = \pi^i_{\alpha,\kappa}(\pi_i(c^\kappa))$. (Note that by the induction assumption, $h_i^\beta(c^\kappa) = \pi^i_{\beta,\kappa}(\pi_i(c^\kappa)) = (t_i^\zeta)_{\zeta<\beta}$, for all $i \in I$.)

- Let now α be a limit ordinal $\leq \kappa$ and let for all $\beta < \alpha$, for all $i \in I$, and all $c_i^\kappa = (t_i^\zeta)_{\zeta<\kappa}$:

$$h_i^\beta(c^\kappa) = (g_i^\zeta(c^\kappa))_{\zeta<\beta} = (t_i^\zeta)_{\zeta<\beta}.$$

Then

$$h_i^\alpha(c^\kappa) = (g_i^\beta(c^\kappa))_{\beta<\alpha} = (t_i^\beta)_{\beta<\alpha} = \pi^i_{\alpha,\kappa} \circ \pi_i(c^\kappa).$$

(Note that $\beta < \alpha$ implies $\beta + 1 < \alpha$.)

Altogether, it follows that $h_i^\kappa(c^\kappa) = \pi_i(c^\kappa)$, for $i \in I_0$, and therefore $h^\kappa = \mathrm{id}_{C^\kappa}$. ∎

Theorem 2.5 *$\underline{C}^\kappa$ is a universal κ-type space on S for player set I.*

Proof Let $\underline{M} = \langle M, \Sigma, (T_i)_{i\in I}, \theta\rangle$ be a κ-type space on S. By Proposition 2.4, there is a type morphism from $\underline{M}$ to $\underline{C}^\kappa$. By Proposition 2.5 and Lemma 2.23, this type morphism is unique. ∎

2.6 Beliefs Completeness

In this section, we show that the universal κ-type space is beliefs complete. In the case $\kappa = \aleph_0$, we can say much more: The component space of each player is - up to isomorphism of measurable spaces - the space of finitely additive probability measures on the product of the space of states of nature and the other players' component spaces.

Theorem 2.6 *Let $\langle C^\kappa, \Sigma^\kappa, (T_i^\kappa)_{i\in I}, \theta^\kappa\rangle$ be the universal κ-type space on S for player set I constructed in the previous section.*
Then, for every $i \in I$:

- $$T_i^\kappa : C_i^\kappa \to \Delta^\kappa\left(C^\kappa\right)$$

  is one-to-one and measurable.

- $$\mathrm{marg}_{C_{-i}^\kappa} \circ T_i^\kappa : C_i^\kappa \to \Delta^\kappa\left(C_{-i}^\kappa\right)$$

  is onto and measurable.

- $$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)$$

  is an isomorphism of measurable spaces.[5]

Proof The first point was already proved in the proof of Theorem 2.4.

Fix $i \in I$. According to the first point of this theorem, $T_i^\kappa : C_i^\kappa \to \Delta^\kappa\left(C^\kappa\right)$ is measurable. Since all the inverse images (under the projection) of the generators of Σ_{-i}^κ are among the generators of Σ^κ, the projection from C^κ to C_{-i}^κ is measurable. Remark 2.4 implies now that

$$\mathrm{marg}_{C_{-i}^\kappa} : \Delta^\kappa\left(C^\kappa\right) \to \Delta^\kappa\left(C_{-i}^\kappa\right)$$

is measurable, and hence

$$\mathrm{marg}_{C_{-i}^\kappa} \circ T_i^\kappa : C_i^\kappa \to \Delta^\kappa\left(C_{-i}^\kappa\right)$$

is measurable.

[5]Although, for $\kappa > \aleph_0$, it is no longer true (contrary to the case $\kappa = \aleph_0$) that the values of a finitely additive probability measure on the measurable rectangles in a product of two κ -measurable spaces determine this finitely additive probability measure, we believe that this is true in the special case where the marginal on one of the factors is a delta-measure. Therefore we tend to conjecture that the third point of Theorem 2.6 holds also for $\kappa > \aleph_0$, but it seems to be considerably harder to prove.

Let $\mu \in \Delta^{\kappa}\left(C_{-i}^{\kappa}\right)$. Now, we construct a product κ-type space on S for player set I

$$\underline{U} := \left\langle \left(U_j\right)_{j\in I_0}, \left(\Sigma_j^U\right)_{j\in I_0}, \left(T_j^U\right)_{j\in I_0}, \theta^U \right\rangle$$

such that there is a point $u \in U_i$ with $\operatorname{marg}_{C_{-i}^{\kappa}} \circ T_i^{\kappa}\left(h_i^{\kappa}(u)\right) = \mu$.
Without loss of generality, let $u \notin C_i^{\kappa}$ and define

$$\begin{array}{lll}
U_i & := & C_i^{\kappa} \cup \{u\}, \\
U_j & := & C_j^{\kappa}, \quad \text{for } j \in I_0 \backslash \{i\}, \\
\Sigma_i^U & := & \Sigma_i^{\kappa} \cup \{E \cup \{u\} \mid E \in \Sigma_i^{\kappa}\}, \\
\Sigma_j^U & := & \Sigma_j^{\kappa}, \quad \text{for } j \in I_0 \backslash \{i\}, \\
U & := & \Pi_{j\in I_0} U_j, \\
\Sigma^U & := & \text{the product } \kappa\text{-field of the } \Sigma_j^U,\ j \in I_0, \\
U_{-i} & := & \Pi_{j\in I_0\backslash\{i\}} U_j, \\
\Sigma_{-i}^U & := & \text{the product } \kappa\text{-field of the } \Sigma_j^U,\ j \in I_0 \backslash \{i\}.
\end{array}$$

It is obvious that Σ_i^U is a κ-field on U_i. Note that $\Sigma^{\kappa} \subseteq \Sigma^U$.

Let $E = \Pi_{j\in I_0} E_j$ such that $E_j = U_j$ for all but one $j \in I_0$. The sets of this form generate Σ^U. If E is of this form, we have $E \cap C^{\kappa} \in \Sigma^{\kappa}$. Since $\left(\bigcap_{F\in\Psi} F\right) \cap C^{\kappa} = \bigcap_{F\in\Psi}\left(F \cap C^{\kappa}\right)$, for $\Psi \subseteq \Sigma^U$, and since $\left(U \backslash F\right) \cap C^{\kappa} = C^{\kappa} \backslash \left(F \cap C^{\kappa}\right)$, for $F \subseteq U$, it follows by the proof of Lemma 2.1 that $F \cap C^{\kappa} \in \Sigma^{\kappa}$, for $F \in \Sigma^U$.

Now define for $j \in I$, $c_j^{\kappa} \in C_j^{\kappa}$ and $E \in \Sigma^U$:

$$T_j^U(c_j^{\kappa})(E) := T_j^{\kappa}(c_j^{\kappa})(E \cap C^{\kappa}).$$

It is obvious that $T_j^U\left(c_j^{\kappa}\right)$ is a finitely additive probability measure on $\left(U, \Sigma^U\right)$, for $j \in I$ and $c_j^{\kappa} \in C_j^{\kappa}$. Clearly, $T_j^U : U_j \to \Delta^{\kappa}(U)$ is Σ_j^U-$\Sigma_{\Delta^{\kappa}(U)}$ measurable, for $j \in I \backslash \{i\}$.
Let $j \in I$, $c_j^{\kappa} \in C_j^{\kappa}$, $E \in \Sigma^U$, and $E \supseteq \left\{c_j^{\kappa}\right\} \times U_{-j}$. We have to show that $T_j^U(c_j^{\kappa})(E) = 1$. We have $\left\{c_j^{\kappa}\right\} \times U_{-j} \supseteq \left\{c_j^{\kappa}\right\} \times C_{-j}^{\kappa}$, so, $C^{\kappa} \cap E \supseteq \left\{c_j^{\kappa}\right\} \times C_{-j}^{\kappa}$ and it follows that

$$T_j^U(c_j^{\kappa})(E) = T_j^{\kappa}(c_j^{\kappa})(C^{\kappa} \cap E) = 1.$$

Σ_i^U and Σ_{-i}^U are fields on U_i and $U_{-i} := \Pi_{j\in I_0\backslash\{i\}} U_j = C_{-i}^{\kappa}$. The finite disjoint unions of sets of the form $E_i \times E_{-i}$, where $E_i \in \Sigma_i^U$ and $E_{-i} \in \Sigma_{-i}^U$, form a field $\mathcal{F}$ on U that is contained in Σ^U. Note that for every finitely additive probability measure ν on $\left(U, \Sigma^U\right)$, the values of ν on $\mathcal{F}$ determine already the marginal of ν on $\left(U_i, \Sigma_i^U\right)$ and on $\left(U_{-i}, \Sigma_{-i}^U\right)$. Now, define for disjoint $E_i^1 \times E_{-i}^1, \ldots, E_i^n \times E_{-i}^n \in \mathcal{F}$:

$$\widehat{\mu}\left(\bigcup_{l=1}^{n} E_i^l \times E_{-i}^l\right) := \delta_u\left(E_i^1\right) \cdot \mu\left(E_{-i}^1\right) + \delta_u\left(E_i^2\right) \cdot \mu\left(E_{-i}^2\right) + \ldots + \delta_u\left(E_i^n\right) \cdot \mu\left(E_{-i}^n\right).$$

Obviously $\widehat{\mu}$ is well-defined (because of the additivity of δ_u and μ) and it is a finitely additive probability measure on $\mathcal{F}$. Next, by the Łoś-Marczewski Theorem, extend $\widehat{\mu}$ to a finitely additive probability measure $\widetilde{\mu}$ on Σ^U. By the definitions, it is clear that $\operatorname{marg}_{U_i}(\widetilde{\mu}) = \delta_u$ and that $\operatorname{marg}_{C^\kappa_{-i}}(\widetilde{\mu}) = \mu$. Note also that

$$\widetilde{\mu}\left(\{u\} \times C^\kappa_{-i}\right) = 1 \neq T_i^U\left(c_i^\kappa\right)\left(\{u\} \times C^\kappa_{-i}\right) = 0, \quad \text{for } c_i^\kappa \in C_i^\kappa.$$

(By definition, we have $\{u\} \times C^\kappa_{-i} \in \Sigma^U$.) Define now

$$T_i^U(u) := \widetilde{\mu}.$$

It follows that $T_i^U(u)(\{u\} \times U_{-i}) = 1$.

Finally, define

$$\theta^U\left(c_0^\kappa\right) := c_0^\kappa.$$

To check that $\underline{U}$ is a κ-type space, it remains to show that

$$T_i^U : U_i \to \Delta^\kappa(U)$$

is measurable. We have for $p \in [0,1]$ and $E \in \Sigma^U$:

$$\left(T_i^U\right)^{-1}\left(\{\nu \in \Delta^\kappa(U) \mid \nu(E) \geq p\}\right) =$$
$$\begin{cases} \left(T_i^\kappa\right)^{-1}\left(\{\nu \in \Delta^\kappa(C^\kappa) \mid \nu(E \cap C^\kappa) \geq p\}\right) \in \Sigma_i^\kappa \subseteq \Sigma_i^U, & \text{if } \quad \widetilde{\mu}(E) < p, \\ \left(T_i^\kappa\right)^{-1}\left(\{\nu \in \Delta^\kappa(C^\kappa) \mid \nu(E \cap C^\kappa) \geq p\}\right) \cup \{u\} \in \Sigma_i^U, & \text{if } \quad \widetilde{\mu}(E) \geq p. \end{cases}$$

Since $\underline{U}$ is a product κ-type space on S for player set I, g_j^β is defined on U_j and is Σ_j^U-$\Sigma_{\Delta^\kappa(C^\beta)}$-measurable, for $j \in I_0$ and $0 \leq \beta < \kappa$, and h_j^α is defined on U_j and is Σ_j^U-Σ_j^α-measurable, for $j \in I_0$ and $0 \leq \alpha \leq \kappa$.

Since $U_0 = S$ and $\theta^U = \mathrm{id}_S$, we have $h_0^\alpha = \theta^U = \pi^0_{\alpha,\kappa}$, for $0 \leq \alpha \leq \kappa$.

For $j \in I$, h_j^0 is uniquely defined, since the C_j^0 are singletons. It follows that $h_j^0 = \pi^j_{0,\kappa}$, for $j \in I \setminus \{i\}$ and $h_i^0 \lceil C_i^\kappa = \pi^i_{0,\kappa}$.

Let $j \in I$, $\alpha = \beta + 1 < \kappa$, and $h_l^\beta \lceil C_l^\kappa = \pi^l_{\beta,\kappa}$, for all $l \in I_0$, and furthermore, let $c_j^\kappa = (t_j^\zeta)_{\zeta<\kappa} \in C_j^\kappa$ and $E^\beta \in \Sigma^\beta$. Then

$$\begin{aligned} g_j^\beta(c_j^\kappa)\left(E^\beta\right) &= T_j^U(c_j^\kappa)\left(\left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ &= T_j^\kappa(c_j^\kappa)\left(C^\kappa \cap \left(h^\beta\right)^{-1}\left(E^\beta\right)\right) \\ &= T_j^\kappa(c_j^\kappa)\left(\left(\pi_{\beta,\kappa}\right)^{-1}\left(E^\beta\right)\right) \\ &= t_j^\beta\left(E^\beta\right). \end{aligned}$$

It follows that $g_j^\beta(c_j^\kappa) = t_j^\beta$ and therefore $h_j^\alpha \lceil C_j^\kappa = \pi^j_{\alpha,\kappa}$.

Let $j \in I$, let $\alpha = \lambda \leq \kappa$ be a limit ordinal and for all $\beta < \alpha$: let $h_j^\beta \lceil C_j^\kappa = \pi_{\beta,\kappa}^j$, and let $c_j^\kappa = (t_j^\zeta)_{\zeta<\kappa}$. By the definition and the induction assumption (note that $\beta < \alpha$ implies $\beta + 1 < \alpha$),

$$h_j^\alpha(c_j^\kappa) = (g_j^\beta(c_j^\kappa))_{\beta<\alpha} = (t_j^\beta)_{\beta<\alpha} = \pi_{\alpha,\kappa}^j(c_j^\kappa).$$

Altogether, we showed now that $h_j^\kappa \lceil C_j^\kappa = \mathrm{id}_{C_j^\kappa}$, for $j \in I_0$.

Let now $E_{-i} \in \Sigma_{-i}^\kappa$. By the definition and Proposition 2.4,

$$\begin{aligned} T_i^\kappa\left(h_i^\kappa\left(u\right)\right)\left(C_i^\kappa \times E_{-i}\right) &= T_i^U\left(u\right)\left(\left(h^\kappa\right)^{-1}\left(C_i^\kappa \times E_{-i}\right)\right) \\ &= T_i^U\left(u\right)\left(U_i \times E_{-i}\right) \\ &= \delta_u\left(U_i\right) \cdot \mu\left(E_{-i}\right) \\ &= \mu\left(E_{-i}\right) \end{aligned}$$

and the second part of the theorem is proved.

For the third point, it remains to prove that

1.

$$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)$$

is one-to-one, and

2. for every $E_i \in \Sigma_i^{\aleph_0}$:

$$\mathrm{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(E_i\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}.$$

To 1 : Let $\left(t_i^n\right)_{n<\aleph_0}, \left(r_i^n\right)_{n<\aleph_0} \in C_i^{\aleph_0}$ and

$$\mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\right) = \mathrm{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left(\left(r_i^n\right)_{n<\aleph_0}\right)\right).$$

We show by induction on $n < \aleph_0$ that $t_i^n = r_i^n$:

Step $n = 0$: Let $E^0 \in \Sigma^0$. Since C_i^0 is a singleton, there is a $E_{-i}^0 \in \Sigma_{-i}^0$ such that $E^0 = C_i^0 \times E_{-i}^0$. We have $\pi_{0,\aleph_0}^{-1}\left(E^0\right) = C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)$. Since $\pi_{0,\aleph_0}^j$ is measurable, for $j \in I_0$, we have, by definition,

$$\begin{aligned} t_i^0\left(E^0\right) &= T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)\right) \\ &= T_i^{\aleph_0}\left(\left(r_i^n\right)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right)\right) \\ &= r_i^0\left(E^0\right). \end{aligned}$$

Step $n \to n+1$: Since Σ^{n+1} is the product field of Σ_i^{n+1} and Σ_{-i}^{n+1} and since finite direct sums of measurable rectangles form already the product field of Σ_i^{n+1} and Σ_{-i}^{n+1}, it is enough to show that $t_i^{n+1}\left(F^{n+1}\right) = r_i^{n+1}\left(F^{n+1}\right)$, for sets of the form $F^{n+1} = E_i^{n+1} \times E_{-i}^{n+1}$, where $E_i^{n+1} \in \Sigma_i^{n+1}$ and $E_{-i}^{n+1} \in \Sigma_{-i}^{n+1}$. Since $\left(t_i^k\right)_{0\leq k\leq n} = \left(r_i^k\right)_{0\leq k\leq n}$, we have $\operatorname{marg}_{C_i^{n+1}}\left(\left(t_i^{n+1}\right)\right) = \operatorname{marg}_{C_i^{n+1}}\left(\left(r_i^{n+1}\right)\right) = \delta_{\left(\left(t_i^k\right)_{0\leq k\leq n}\right)}$, and since

$$\begin{aligned} t_i^{n+1}\left(C_i^{n+1} \times E_{-i}^{n+1}\right) &= T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right)\right) \\ &= T_i^{\aleph_0}\left(\left(r_i^n\right)_{n<\aleph_0}\right)\left(C_i^{\aleph_0} \times \left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right)\right) \\ &= r_i^{n+1}\left(C_i^{n+1} \times E_{-i}^{n+1}\right), \end{aligned}$$

we have

$$\begin{aligned} & t_i^{n+1}\left(E_i^{n+1} \times E_{-i}^{n+1}\right) \\ &= \delta_{\left(\left(t_i^k\right)_{0\leq k\leq n}\right)}\left(E_i^{n+1}\right) \cdot \left(\operatorname{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\right)\right)\left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right) \\ &= \delta_{\left(\left(r_i^k\right)_{0\leq k\leq n}\right)}\left(E_i^{n+1}\right) \cdot \left(\operatorname{marg}_{C_{-i}^{\aleph_0}}\left(T_i^{\aleph_0}\left(\left(r_i^n\right)_{n<\aleph_0}\right)\right)\right)\left(\left(\pi_{n+1,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^{n+1}\right)\right) \\ &= r_i^{n+1}\left(E_i^{n+1} \times E_{-i}^{n+1}\right). \end{aligned}$$

It follows that $t_i^{n+1} = r_i^{n+1}$.

To 2 : Since $\Sigma_i^{\aleph_0}$ is the field on $C_i^{\aleph_0}$ inherited from the product field of the $\Sigma_{\Delta^{\aleph_0}(C^n)}$, $n < \aleph_0$, and since

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0} : C_i^{\aleph_0} \to \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)$$

is one-to-one and onto, it remains to show - by an induction on $m < \aleph_0$ - that

$$(*) \qquad \operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{\left(t_i^n\right)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^m \in A^m\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for $A^m \in \Sigma_{\Delta^{\aleph_0}(C^m)}$:

Again, since $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ is one-to-one and onto and since the sets $\left\{\mu \in \Delta^{\aleph_0}\left(C^m\right) \mid \mu\left(E^m\right) \geq p\right\}$, where $E^m \in \Sigma^m$ and $p \in [0,1]$, generate $\Sigma_{\Delta^{\aleph_0}(C^m)}$, to show $(*)$, it is enough to verify that

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{\left(t_i^n\right)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^m\left(E^m\right) \geq p\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for all $E^m \in \Sigma^m$ and $p \in [0,1]$.

Step $m = 0$: Let $E^0 \in \Sigma^0$ and $p \in [0,1]$. Since C_i^0 is a singleton, there is a $E_{-i}^0 \in \Sigma_{-i}^0$ such that $E^0 = C_i^0 \times E_{-i}^0$. We have

$$\pi_{0,\aleph_0}^{-1}\left(E^0\right) = C_i^{\aleph_0} \times \left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right) \right)$$

and by definition, for $(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0}$,

$$\begin{aligned} t_i^0\left(E^0\right) &= T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\left(\pi_{0,\aleph_0}^{-1}\left(E^0\right)\right) \\ &= \left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left((t_i^n)_{n<\aleph_0}\right)\right)\left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} &\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^0\left(E^0\right) \geq p\right\}\right) = \\ &\left\{\mu \in \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right) \,\middle|\, \mu\left(\left(\pi_{0,\aleph_0}^{-i}\right)^{-1}\left(E_{-i}^0\right)\right) \geq p\right\} \quad \in \quad \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}. \end{aligned}$$

(Note that $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ was onto.)

Step $m \to m+1$: Let $E^{m+1} \in \Sigma^{m+1}$ and $p \in [0,1]$. Σ^{m+1} is the product field of Σ_i^{m+1} and Σ_{-i}^{m+1}. E^{m+1} can be written as a finite direct sum of measurable rectangles. Furthermore, E^{m+1} can be written as

$$E^{m+1} = F_i^1 \times F_{-i}^1 + \ldots + F_i^k \times F_{-i}^k,$$

for some $k \geq 1$ and $F_{-i}^l \in \Sigma_{-i}^{m+1}$, for $l = 1, \ldots, k$, and $F_i^l \in \Sigma_i^{m+1}$, for $l = 1, \ldots, k$, such that the F_i^l are disjoint.
According to the induction assumption, we have

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, t_i^q \in A^q\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)},$$

for $q = 0, \ldots, m$ and $A^q \in \Sigma_{\Delta^{\aleph_0}(C^q)}$. Recall that Σ_i^{m+1} is the field on C_i^{m+1} inherited from the product field of the $\Sigma_{\Delta^{\aleph_0}(C^q)}$, $q = 0, \ldots, m$, on $\Pi_{q \leq m}\Delta^{\aleph_0}\left(C^q\right)$. In particular, the projection from C_i^{m+1} to $\Delta^{\aleph_0}\left(C^q\right)$ is measurable, for $q = 0, \ldots, m$. Since $\pi_{m+1,\aleph_0}^i$ is onto and measurable, and since $\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}$ is one-to-one and onto, it follows that

$$\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q \leq m} \in F_i^l\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}$$

and that

$$\begin{aligned} &\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q \leq m} \in F_i^{l_0}\right\}\right) \\ &\cap \operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{(t_i^n)_{n<\aleph_0} \in C_i^{\aleph_0} \,\middle|\, (t_i^q)_{q \leq m} \in F_i^{l_1}\right\}\right) = \emptyset, \end{aligned}$$

for $0 \leq l_0 \neq l_1 \leq k$. As argued in 1, we have

$$t_i^{m+1}\left(F_i^l \times F_{-i}^l\right) = \delta_{\left(\left(t_i^q\right)_{q\leq m}\right)}\left(F_i^l\right) \cdot \left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\right)\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right).$$

Since all the F_i^l are disjoint, we have $t_i^{m+1}\left(E^{m+1}\right) \geq p$ iff there is a l such that $\left(t_i^q\right)_{q\leq m} \in F_i^l$ and $\left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left(t_i^n\right)_{n<\aleph_0}\right)\right)\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right) \geq p$. Hence

$$\begin{aligned}
&\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{\left(t_i^n\right)_{n<\aleph_0} \in C_i^{\aleph_0} \middle|\, t_i^{m+1}\left(E^{m+1}\right) \geq p\right\}\right) = \\
&\bigcup_{l=1}^k \left(\left(\operatorname{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\left(\left\{\left(t_i^n\right)_{n<\aleph_0} \in C_i^{\aleph_0} \middle|\, \left(t_i^q\right)_{q\leq m} \in F_i^l\right\}\right)\right)\right. \\
&\qquad \left.\cap \left\{\mu \in \Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right) \middle|\, \mu\left(\left(\pi_{m+1,\aleph_0}^{-i}\right)^{-1}\left(F_{-i}^l\right)\right) \geq p\right\}\right) \in \Sigma_{\Delta^{\aleph_0}\left(C_{-i}^{\aleph_0}\right)}.
\end{aligned}$$

■

Chapter 3

An Infinitary Probability Logic for Type Spaces

3.1 Introduction

It is well-known that Kripke structures (and in particular Knowledge spaces introduced by Aumann (1976)) can be axiomatized in terms of modal logic (See for example Kripke (1963), Aumann (1995), Fagin et al. (1995), Heifetz (1997), and Aumann (1999a)). In this chapter we aim to do the same for type spaces in the sense of Harsanyi (1967/68), which can be considered as the probabilistic analog of Kripke structures. Type spaces are the predominant structures to describe incomplete information in an interactive context in game theory. (See Aumann and Heifetz (2001) for a nice and well-accessible introduction to the subject.)

We define an infinitary modal language with operators p_i^α, "individual i assigns probability at least α" for rational $\alpha \in [0,1]$, and then a system of infinitary axioms and inference rules, which we prove to be strongly sound and strongly complete with respect to the class of (Harsanyi) type spaces (Theorem 3.1). To the best of our knowledge, this is the very first strong completeness theorem for a probability logic of the present kind. Strongly complete means that, if a formula φ holds whenever a (possibly infinite) set of formulas Γ holds, then there is a proof of φ from Γ.

Heifetz and Mongin (2001) - and before Fagin, Halpern and Megiddo (1990) for a much richer syntax also expressing valuations for linear combinations of formulas - axiomatized the class of type spaces in terms of a purely finitary logic. They showed that their axiomatization is sound and complete with respect to the class of (Harsanyi) type spaces.

However, a purely finitary axiomatization cannot be used to get strong soundness and strong completeness for this class of models. This was noted by Heifetz and Mongin (2001) and Aumann (1999b). They argue that for the set of formulas $\left\{p_i^{\frac{1}{2}-\frac{1}{n}}(\varphi) : \ n \geq 2, \ n \in \mathbb{N}\right\} \cup \left\{\neg p_i^{\frac{1}{2}}(\varphi)\right\}$, each finite subset has a model (a type space and a state in it, such that each formula in this finite subset is true in that state), while the whole set itself has no model. This implies that, although we prove a strong completeness theorem, our logic is not compact.

We construct (Proposition 3.3) a canonical model whose points consist of the maximal consistent sets of formulas. This construction requires strong soundness and strong completeness. In a very natural way, the maximal consistent sets of formulas determine already the structure of this space. The spirit of this construction follows the constructions of the canonical models for the infinitary versions of the S5-epistemic logics, proposed by Heifetz (1997) (see Aumann (1999a) for the finitary version). Such constructions show that there is a space which contains "all states of the world" in the syntactic sense. A state of the world in the syntactic sense is a maximal consistent set of formulas of the language. The existence of a canonical model guarantees that, without loss of generality, a game of incomplete information can be modeled by a type space.

Type spaces cannot be axiomatized by an infinitary logic in the sense of Heifetz (1997): An example by Karp (1964) in a purely propositional setting shows already that, in the presence of $\aleph_\gamma$ many formulas (where $\aleph_\gamma$ denotes the γth infinite cardinal number) whose truth values can be chosen independently of one another, if one allows for infinite conjunctions of $\aleph_\gamma$ many formulas, then one must also allow for conjunctions of $2^{\aleph_\gamma}$ many formulas and proofs of length of cardinality $\leq 2^{\aleph_\gamma}$ to get strong completeness. This collides with measurability conditions that must be met: When we want to define the validity relation "$\models$" for a type space τ and some point (i.e. state) ω in τ, then, for a formula φ in our language, $(\tau, \omega) \models p_i^\alpha(\varphi)$ can be defined, if $[\varphi]^\tau$, the set of points in τ where φ is true, is a measurable set. Since conjunctions of formulas correspond to intersections of subsets of the structure, uncountable conjunctions cannot be guaranteed to interpret measurable sets unless we do assume that the σ-fields of the type spaces are closed under uncountable intersections (i.e. they would be κ-fields for some $\kappa > \aleph_1$), which, of course, would strongly restrict the class of type spaces we could consider.

We resolve this problem by defining a language which takes the advantages and avoids the disadvantages of both the finitary and the infinitary languages. We start with a finitary language $\mathcal{L}_0$ à la Aumann (1995) and Heifetz and Mongin (2001) with operators p_i^α, "individual i assigns probability at least α". Then, we define an infinitary propositional language $\mathcal{L}$, the primitive propositions of which are the formulas in $\mathcal{L}_0$. So, $\mathcal{L}_0$ is a sublanguage of $\mathcal{L}$.

For the class of type spaces - contrary to the class of knowledge spaces, as was shown by Heifetz and Samet (1998a) - there is also a unique (up to isomorphism) special type space, defined in purely semantic terms, namely the universal type space, a space to which every type space can be mapped by a unique structure preserving map, a so called "type morphism". The existence of such a space was first proved - under certain topological assumptions - by Mertens and Zamir (1985), followed by many others. Recently Heifetz and Samet (1998b) proved the general measure-theoretic case. In their proof, they - as well as Aumann (1999b) for the proof of the existence of a canonical knowledge-belief system - used a language similar to ours, with one important difference: They did not have a purely syntactic definition of what the maximal consistent sets of formulas are. Instead, such sets were obtained semantically by collecting the sets of formulas that hold true in some state in some type space, thus constructing the universal space (resp. the canonical knowledge-belief system).

It is not too surprising then that the canonical model and the universal space are one and the same. This is stated in our Theorem 3.2. Hence, we provide here a (up to now missing) characterization of the universal type space (as the space of maximal consistent sets of formulas).

In the literature, "type spaces" usually are what we call here "product type spaces". Other authors who considered the more general version are Heifetz and Mongin (2001), who called it also "type spaces" and Mertens and Zamir (1985), who called these spaces "beliefs spaces". As it turned out in their topological setting, the universal type space of Mertens and Zamir is a product type space.

In Theorem 3.3 we prove that this is still true in our topology-free setting, namely our canonical model is a product type space.

Furthermore, everywhere in the literature except in (Heifetz and Mongin (2001)), only type spaces are considered where the players know their own beliefs (we call these spaces, in this chapter only, like Heifetz and Mongin (2001), "Harsanyi type spaces").

We construct our canonical model with and without this property and establish the first proof of the existence of a universal type space for the class of type spaces without introspection.

In the topological cases of Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993) and Mertens, Sorin and Zamir (1994) it was shown that the universal type space is beliefs complete. However, the general measure-theoretic case was left open up to now.

We show here in Theorem 3.4 that this is also still true in the general measure-theoretic setting, in the introspective as well as in the non-introspective case. Moreover the component space of each player is - as a measurable space - isomorphic to the space of probability measures on the whole space in the nonintrospective case, and in the introspective case, the component space of each player is isomorphic to the space of probability measures on the product of the space of states of nature and the other players' component spaces.

3.2 Preliminaries

For this chapter, we fix a nonempty set X of primitive propositions (to be interpreted as statements about nature, i.e. the primary source of uncertainty for the players)[1] and a nonempty set I of players, and we assume without loss of generality that $0 \notin I$ and define $I_0 := I \cup \{0\}$. For a set M, denote by $|M|$ the cardinality of M.

In this chapter, α and β denote rational numbers $\in [0,1]$, φ, χ, ψ formulas, and ω formulas that are conjunctions of maximal consistent sets of finitary formulas.

Definition 3.1 We define

$$\aleph_\gamma := \max\{|I|, |X|, \aleph_0\}.$$

Definition 3.2 The set $\mathcal{L}_0$ of *finitary formulas* is the least set such that:

1. each $x \in X \cup \{\top\}$ is a finitary formula,
2. if φ is a finitary formula, then $(\neg\varphi)$ is a finitary formula,
3. if φ and ψ are finitary formulas, then $(\varphi \wedge \psi)$ is a finitary formula,
4. if φ is a finitary formula, then for every $i \in I$ and rational $\alpha \in [0,1]$: $(p_i^\alpha(\varphi))$ is a finitary formula.

Remark 3.1

$$|\mathcal{L}_0| = \max\{|I|, |X|, \aleph_0\} = \aleph_\gamma.$$

Definition 3.3 The set $\mathcal{L}$ of *formulas* is the least set such that:

1. each $\varphi \in \mathcal{L}_0$ is a formula,
2. if φ is a formula, then $(\neg\varphi)$ is a formula,
3. if Φ is a set of formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $(\bigwedge_{\varphi\in\Phi}\varphi)$ is a formula.[2]

[1] If we define things in this way, the S of the other chapters corresponds to $\mathrm{Pow}(X)$ and Σ_S to the σ-field on $\mathrm{Pow}(X)$ generated by the sets $\{s \subseteq X \mid x \in s\}$, where $x \in X$.

[2] By convention, we set: $\bigwedge_{\varphi\in\emptyset}\varphi := \top$, and accordingly: $\bigvee_{\varphi\in\emptyset}\varphi := \neg\top$. Futhermore, if we write "$\varphi \wedge \psi$", where φ or $\psi \in \mathcal{L} \setminus \mathcal{L}_0$, we mean implicitly the formula $\bigwedge_{\chi\in\{\varphi,\psi\}}\chi$.

Convention 3.1
- As usual, "$\leftrightarrow$", "$\rightarrow$", "$\vee$" and "$\bigvee$" are abbreviations, defined in the usual way:

$$\begin{aligned}(\bigvee_{\varphi\in\Phi}\varphi) &:= (\neg(\bigwedge_{\varphi\in\Phi}(\neg\varphi))),\\ (\varphi\rightarrow\psi) &:= ((\neg\varphi)\vee\psi),\\ (\varphi\leftrightarrow\psi) &:= ((\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)).\end{aligned}$$

- To avoid the use of too many brackets, we apply the usual convention of decreasing priority: $\neg$, $\bigwedge$, $\wedge$, $\bigvee$, $\vee$, $\rightarrow$, $\leftrightarrow$. This means, for example, that "$\neg\varphi\wedge\psi$" is an abbreviation for "$((\neg\varphi)\wedge\psi)$".

Definition 3.4 The set $\mathcal{L}^0$ of 0-*formulas* is the set of (infinitary) propositional formulas in $\mathcal{L}$. More formally, it is the least set of formulas (and obviously: subset of $\mathcal{L}$) such that:

1. each $x \in X \cup \{\top\}$ is a 0-formula,
2. if φ is a 0-formula, then $\neg\varphi$ is a 0-formula,
3. if Φ is a set of 0-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is a 0-formula.

Definition 3.5 Let $i \in I$. The set $\mathcal{L}^i$ of i-*formulas* is the least set of formulas (and obviously: subset of $\mathcal{L}$) such that:

1. if $\varphi \in \mathcal{L}_0$, then for every rational $\alpha \in [0,1]$: $p_i^\alpha(\varphi)$ is an i-formula,
2. if φ is an i-formula, then $\neg\varphi$ is an i-formula,
3. if Φ is a set of i-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is an i-formula.

Definition 3.6 For $i \in I_0$ define

$$\mathcal{L}_0^i := \mathcal{L}_0 \cap \mathcal{L}^i.$$

Definition 3.7 Let M be a nonempty set and let Σ be a σ-field on M. We denote by $\Delta(M,\Sigma)$ - or short: $\Delta(M)$ - the set of all σ-additive probability measures on (M,Σ). Unless stated differently, we consider $\Delta(M,\Sigma)$ as a measurable space with the σ-field Σ_Δ generated by all the sets of the form $b^\alpha(E) := \{\mu \in \Delta(M,\Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and $\alpha \in [0,1] \cap \mathbb{Q}$.

Note that if $r \in [0,1]$ and $E \in \Sigma$, then $b^r(E) = \bigcap_{\alpha \in [0,r] \cap \mathbb{Q}} b^\alpha(E) \in \Sigma_\Delta$. Therefore Σ_Δ is also generated by all the sets $b^r(E)$, where $E \in \Sigma$ and $r \in [0,1]$.

Definition 3.8 A *type space on X for player set I* is a 4-tupel

$$\underline{M} := \langle M, \Sigma, (T_i)_{i \in I,} v \rangle,$$

where

- M is a nonempty set,
- Σ is a σ-field on M,
- for $i \in I : T_i$ is a $\Sigma - \Sigma_\Delta$-measurable function from M to $\Delta(M, \Sigma)$, the space of probability measures on (M, Σ),
- v is a function from $M \times (X \cup \{\top\})$ to $\{0,1\}$, such that $v(\cdot, x)$ is $\Sigma - \mathrm{Pow}(\{0,1\})$-measurable, for every $x \in X$, and such that $v(m, \top) = 1$, for all $m \in M$.

Definition 3.9 For a type space $\langle M,\ \Sigma,\ (T_i)_{i \in I,}\ v \rangle$ on X for player set I define

$$[T_i(m)] := \{m' \in M \mid T_i(m') = T_i(m)\},$$

for $m \in M$ and $i \in I$.

$$\langle M, \Sigma, (T_i)_{i \in I,} v \rangle$$

is called a *Harsanyi type space on X for player set I* iff for all $A \in \Sigma$, $m \in M$ and $i \in I : A \supseteq [T_i(m)]$ implies $T_i(m)(A) = 1$.[3]

The following lemma, which will be needed in the proof of the Completeness Theorem, is a slightly changed version of Lemma 2.1. of Heifetz and Samet (1999):

Lemma 3.1 *Let M be a nonempty set, let $\mathcal{F}$ be a field on M that generates the σ-field Σ on M and let $\mathcal{F}_\Delta$ be the σ-field on $\Delta(M, \Sigma)$ generated by the sets of the form*

$$b^p(E) := \{\mu \in \Delta(M, \Sigma) \mid \mu(E) \geq p\},$$

where $E \in \mathcal{F}$ and $p \in [0,1] \cap \mathbb{Q}$. Then

$$\mathcal{F}_\Delta = \Sigma_\Delta.$$

[3]Note that if $[T_i(m)]$ is measurable, then this condition reduces to: $T_i(m)([T_i(m)]) = 1$.

Proof The proof is the same as the proof of Lemma 2.1. of Heifetz and Samet (1999), if we replace there "such that $\beta^p(F) \in \mathcal{F}_\Delta$ for all $0 \le p \le 1$" by "such that $\beta^p(F) \in \mathcal{F}_\Delta$ for all $p \in [0,1] \cap \mathbb{Q}$". ■

Definition 3.10 Let $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,}\ v \rangle$ be a type space on X for player set I. We define:

- $(\underline{M}, m) \models \top$ in any case,
- for every $x \in X \cup \{\top\}$:
  $(\underline{M}, m) \models x$ iff $v(m, x) = 1$,
- for all $\varphi, \psi \in \mathcal{L}$:

  $(\underline{M}, m) \models \varphi \wedge \psi$ iff $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \psi$,
- for every $\varphi \in \mathcal{L}$:

  $(\underline{M}, m) \models \neg\varphi$ iff $(\underline{M}, m) \not\models \varphi$,
- for $\varphi \in \mathcal{L}_{0,}$ such that $[\varphi]^{\underline{M}} := \{m \in M \mid (\underline{M}, m) \models \varphi\} \in \Sigma$, and for $i \in I$ and rational $\alpha \in [0,1]$:

  $(\underline{M}, m) \models p_i^\alpha(\varphi)$ iff $T_i(m)([\varphi]^{\underline{M}}) \ge \alpha$.

It is easy to show by induction on the formation of the formulas in $\mathcal{L}_0$ that $[\varphi]^{\underline{M}} \in \Sigma$, for every $\varphi \in \mathcal{L}_0$ (in particular, since $T_i : M \to \Delta(M)$ is $\Sigma - \Sigma_\Delta$-measurable, it follows that $[\varphi]^{\underline{M}} \in \Sigma$ implies $[p_i^\alpha(\varphi)]^{\underline{M}} \in \Sigma$), so the relation "$(\underline{M}, m) \models \varphi$" is well-defined for every type space $\underline{M}$ on X for player set I, every $m \in M$, and every $\varphi \in \mathcal{L}_0$.

- If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \le 2^{\aleph_\gamma}$, then:

  $(\underline{M}, m) \models \bigwedge_{\varphi \in \Phi} \varphi$ iff for every $\varphi \in \Phi : (\underline{M}, m) \models \varphi$.

It is now easy to show, that the relation "$(\underline{M}, m) \models \varphi$" is well-defined for every type space $\underline{M}$ on X for player set I, every $m \in M$, and every $\varphi \in \mathcal{L}$.

Definition 3.11 A formula $\varphi \in \mathcal{L}$ is *valid* in the class of type spaces (resp. Harsanyi type spaces) on X for player set I, iff for every type space (resp. Harsanyi type space) $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,} v \rangle$ on X for player set I and every $m \in M$:

$$(\underline{M}, m) \models \varphi.$$

Notation 3.1

1. Let $\Gamma \subseteq \mathcal{L}$, let $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,} v \rangle$ be a type space (resp. Harsanyi type space) on X for player set I, and let $m \in M$. We write

$$(\underline{M}, m) \models \Gamma,$$

iff for every $\psi \in \Gamma$:

$$(\underline{M}, m) \models \psi.$$

2. Let $\Gamma \subseteq \mathcal{L}$. We say Γ *has a model* in the class of type spaces (resp. Harsanyi type spaces) on X for player set I, iff there is a type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and a $m \in M$ such that $(\underline{M}, m) \models \Gamma$.

3. Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. We write

$$\Gamma \models \varphi,$$

iff for every type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and every $m \in M$:

$$(\underline{M}, m) \models \Gamma \text{ implies } (\underline{M}, m) \models \varphi.$$

3.3 Strong Completeness

In this section we define our axioms and inference rules, our notion of "proof" (in the sense of our logic) and prove strong soundness and, by constructing the canonical model, strong completeness. As already said, in this chapter, α and β denote rational numbers in $[0, 1]$.

The List of Axioms

(A0) $\top$

(A1) $\varphi \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A2) $(\varphi \to (\psi \to \varrho)) \to ((\varphi \to \psi) \to (\varphi \to \varrho))$, for $\varphi, \psi, \varrho \in \mathcal{L}$,

(A3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A4) $\bigwedge_{\varphi\in\Phi}(\psi \to \varphi) \to (\psi \to \bigwedge_{\varphi\in\Phi}\varphi)$, for $\psi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$,

(A5) $\bigwedge_{\varphi\in\Phi}\varphi \to \psi$, for $\psi \in \Phi$, where $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$,

(A6) $\bigwedge_{a\in A}(\bigvee_{b\in A}\varphi_{a,b}) \to \bigvee_{g\in A^A}(\bigwedge_{a\in A}\varphi_{a,g(a)})$, whenever $|A| \leq \aleph_\gamma$,[4]

(P1) $p_i^0(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P2) $p_i^1(\top)$,

(P3) $\bigwedge_{\alpha<\beta} p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P4) $\Big(p_i^\alpha(\varphi \wedge \psi) \wedge p_i^\beta(\varphi \wedge \neg\psi)\Big) \to p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \leq 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P5) $\Big(\neg p_i^\alpha(\varphi \wedge \psi) \wedge \neg p_i^\beta(\varphi \wedge \neg\psi)\Big) \to \neg p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \leq 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P6) $p_i^\alpha(\varphi) \to \neg p_i^\beta(\neg\varphi)$, for α, β with $\alpha + \beta > 1$ and $\varphi \in \mathcal{L}_0$,

(P7) $p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for α, β with $\beta < \alpha$ and $\varphi \in \mathcal{L}_0$,

(P8) $p_i^1(\varphi \to \psi) \to (p_i^\alpha(\varphi) \to p_i^\alpha(\psi))$, for $\varphi, \psi \in \mathcal{L}_0$,

(I1) $p_i^\alpha(\varphi) \to p_i^1(p_i^\alpha(\varphi))$, for $\varphi \in \mathcal{L}_0$,

(I2) $\neg\, p_i^\alpha(\varphi) \to p_i^1\ (\neg\, p_i^\alpha(\varphi))$, for $\varphi \in \mathcal{L}_0$.

Except (A0) and (P2), all the above axioms are in fact axiom schemes, i.e. lists of axioms.

[4] A^A denotes here the set of all functions from A to A.

We adopt the following *inference rules*:

- *Modus Ponens*: From φ and $\varphi \rightarrow \psi$ infer ψ.
- *Conjunction*: From Φ infer $\bigwedge_{\varphi \in \Phi} \varphi$, if $\Phi \subseteq \mathcal{L}$ such that $\mid \Phi \mid \leq 2^{\aleph_\gamma}$.
- *Necessitation*: From φ infer $p_i^1(\varphi)$, if $\varphi \in \mathcal{L}_0$.
- *Continuity at $\emptyset$* : From $\bigwedge_{n \in \mathbb{N}} \varphi_n \rightarrow \neg\top$, where $\varphi_n \in \mathcal{L}_0$, for all $n \in \mathbb{N}$,

  infer $\bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} \neg p_i^{\frac{1}{k}}(\bigwedge_{n \leq l} \varphi_n)$.
- *Uncountable Introspection*: From $\varphi \rightarrow \bigvee_{n \in \mathbb{N}} \varphi_n$, where $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$ for all $n \in \mathbb{N}$,

  infer $\varphi \rightarrow \bigwedge_{k \in \mathbb{N}\setminus\{0\}} \bigvee_{l \in \mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n \leq l} \varphi_n)$.

(A0) – (A6) are the axioms and "Modus Ponens" and "Conjunction" are the inference rules for infinitary propositional logic, where the language is the propositional part, $\mathcal{L}^0$, of our infinitary language $\mathcal{L}$. Karp (1964), has proved strong soundness and strong completeness (Karp (1964, Theorem 5.5.4)) for this logic. We will use this result, sometimes without referring to it explicitly.

Most of the axioms (P1) - (P8), (I1), (I2) above can be found in Aumann (1995) and Heifetz and Mongin (2001).

Note that the above set of axioms is not minimal:

- (P1) follows from (P3), (A0) and Modus Ponens, if we adopt the usual convention that $\bigwedge_{\varphi \in \emptyset} \varphi := \top$.
- (P2) follows from (A0) and Necessitation.
- Heifetz and Mongin (2001) proved that (P7) follows from (A0) - (A6), (P3) - (P6) and (P8).
- The proof of the Completeness Theorem will also show that (A0) - (A6), (P3) - (P6), (P8) and (I1) imply (I2).
- It is easy to see that the inference rule "Uncountable Introspection" implies (together with (A0) - (A6), (P3) - (P6) and (P8) and the other inference rules) the axioms (I1) and (I2).
- The proof of the Completeness Theorem shows that, in the case of $\aleph_\gamma = \aleph_0$, (I1) (together with (A0) - (A6), (P3) - (P6) and (P8) and the other inference rules) implies the inference rule "Uncountable Introspection".

Definition 3.12 1. The *system P* consists of the axioms (A0) - (A6), (P3) - (P6), (P8), and the inference rules "Modus Ponens", "Conjunction", "Necessitation", and "Continuity at $\emptyset$".

2. The *system H* is the system P together with the additional axiom (I1) if $\aleph_\gamma = \aleph_0$ and the system H is the system P together with the inference rule "Uncountable Introspection" otherwise.

Definition 3.13 1. The *set of theorems of the system P* is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\emptyset$".

2. The *set of theorems of the system H* is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), (I1), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\emptyset$", in the case $\aleph_\gamma = \aleph_0$.
And if $\aleph_\gamma > \aleph_0$: The set of theorems of the system H is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation", "Continuity at $\emptyset$" and "Uncountable Introspection".

In fact we have here two papers (or chapters) in one: Given a nonempty set of players I and a nonempty set of primitive propositions X, if nothing else is said, we do all what follows for the system P on the syntactic side and for the class of type spaces on X for player set I on the semantic side. And we also do all what follows for the system H on the syntactic side and for the class of Harsanyi type spaces on X for player set I on the semantic side. We only specify the system, if there is a difference between the two cases in the proofs or in the statements of the Lemmas, Propositions or Theorems.

Definition 3.14 Let Γ be a set of formulas in $\mathcal{L}$. A *proof of φ from Γ* in the system P (resp. in the system H) is a sequence whose length is smaller than $(2^{\aleph_\gamma})^+$ and whose last formula is φ, such that each formula in the proof is in Γ, a theorem of the system P (resp. of the system H), or inferred from the previous formulas by "Modus Ponens" or "Conjunction".[5]

If there is a proof of φ from Γ, we write $\Gamma \vdash \varphi$. In particular, "$\vdash \varphi$" means that φ is a theorem.

[5]Of course, whether φ is a theorem of the system, resp., whether there is a proof of φ from Γ, depends on the system under consideration, i.e. there might be a proof of φ from Γ in the system H, but not in the system P. It follows also that the notion of consistency depends on the system.

Definition 3.15
- The system P (resp. H) is strongly sound iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L}$:
$$\Gamma \vdash \varphi \text{ implies } \Gamma \models \varphi.$$
- The system P (resp. H) is strongly complete iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L}$:
$$\Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi.$$

Definition 3.16 Γ is *consistent* in the system P (resp. in the system H), if there is no formula $\varphi \in \mathcal{L}$ such that there are proofs of φ and $\neg\varphi$ from Γ in the system P (resp. in the system H).

Lemma 3.2 *Let $\varphi, \psi, \widetilde{\psi} \in \mathcal{L}$. Then:*

1. *If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\Phi \vdash \varphi$ iff $\left\{\bigwedge_{\chi\in\Phi} \chi\right\} \vdash \varphi$.*
2. *$\{\psi\} \vdash \varphi$ iff $\vdash \psi \to \varphi$.*
3. *If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \widetilde{\psi}$, then $\Gamma \vdash \varphi \to \widetilde{\psi}$.*
4. *$\vdash \varphi \to \neg(\neg\varphi)$.*
5. *If $\psi \in \Phi$, $\Phi \subseteq \mathcal{L}$, and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\vdash \neg\psi \to \neg \bigwedge_{\chi\in\Phi} \chi$.*

Proof

1. "If" follows by applying the inference rule "Conjunction" to Φ. "Only if" follows by replacing in the proof of φ from Φ every occurrence of a $\chi \in \Phi$ by the sequence $\bigwedge_{\chi'\in\Phi} \chi'$, $\bigwedge_{\chi'\in\Phi} \chi' \to \chi$, χ. This yields then a proof of φ from $\left\{\bigwedge_{\chi\in\Phi} \chi\right\}$.
2. "If" follows immediately by Modus Ponens. "Only if" follows by induction on the length of the proof of φ from $\{\psi\}$. There are four cases:
   (a) $\varphi = \psi$: By (A5) applied to $\{\psi\}$, it follows that $\vdash \psi \to \psi$.
   (b) φ is a theorem: By (A1), $\vdash \varphi \to (\psi \to \varphi)$, and by Modus Ponens it follows that $\vdash \psi \to \varphi$.

(c) φ follows by Modus Ponens: Then there is a χ such that χ and $\chi \to \varphi$ occur in the proof of φ. The sequences up to (and including) χ and $\chi \to \varphi$ are proofs of χ and $\chi \to \varphi$ from $\{\psi\}$ of shorter length. Hence, by the induction hypothesis, $\vdash \psi \to \chi$ and $\vdash \psi \to (\chi \to \varphi)$.

$$(\psi \to (\chi \to \varphi)) \to ((\psi \to \chi) \to (\psi \to \varphi))$$

is a theorem (A2), so by applying Modus Ponens two times we get $\vdash \psi \to \varphi$.

(d) φ follows by Conjunction: Then $\varphi = \bigwedge_{\chi \in \Phi} \chi$ with $|\Phi| \leq 2^{\aleph_\gamma}$. By the induction hypothesis (since each χ must occur before φ in the proof), we have $\vdash \psi \to \chi$, for every $\chi \in \Phi$. By conjunction, we get $\vdash \bigwedge_{\chi \in \Phi} (\psi \to \chi)$ and by applying Modus Ponens to (A4), $\vdash \psi \to \bigwedge_{\chi \in \Phi} \chi$.

3. We have $\Gamma \vdash \psi \to \widetilde{\psi}$. By (A1), $(\psi \to \widetilde{\psi}) \to (\varphi \to (\psi \to \widetilde{\psi}))$ is an axiom. Modus Ponens yields $\Gamma \vdash \varphi \to (\psi \to \widetilde{\psi})$. By (A2),

$$(\varphi \to (\psi \to \widetilde{\psi})) \to ((\varphi \to \psi) \to (\varphi \to \widetilde{\psi}))$$

is an axiom. Modus Ponens yields $\Gamma \vdash (\varphi \to \psi) \to (\varphi \to \widetilde{\psi})$. Together with $\Gamma \vdash \varphi \to \psi$, Modus Ponens yields now $\Gamma \vdash \varphi \to \widetilde{\psi}$.

4. and 5. are well-known tautologies of Propositional Calculus, so, according to the Completeness Theorem of Karp (1964, Theorem 5.5.4), theorems of our system.

■

Proposition 3.1 • *The system P is strongly sound with respect to the class of type spaces on X for player set I.*

- *The system H is strongly sound with respect to the class of Harsanyi type spaces on X for player set I.*

Proof

1. For $\varphi \in \mathcal{L}$, we have to show that if $\vdash \varphi$ (i.e. φ is a theorem) in the system P (resp. in the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I and $m \in M$, then $(\underline{M}, m) \models \varphi$.

2. And for $\varphi \in \mathcal{L}$ and a nonempty set $\Gamma \subseteq \mathcal{L}$, we have to show that if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$, $(\underline{M}, m) \models \Gamma$ and if $\Gamma \vdash \varphi$ in the system P (resp. in the system H), then $(\underline{M}, m) \models \varphi$.

To:

1. It suffices to show for $\varphi, \psi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$:

   (a) If φ is an axiom of the system P (resp. of the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I and $m \in M$, then $(\underline{M}, m) \models \varphi$.

   (b) If φ is valid and $\varphi \to \psi$ is valid, then ψ is valid.

   (c) If $\varphi \in \mathcal{L}_0$ and φ is valid, then $p_i^1(\varphi)$ is valid.

   (d) If $|\Phi| \leq 2^{\aleph_\gamma}$ and each $\varphi \in \Phi$ is valid, then $\bigwedge_{\varphi\in\Phi} \varphi$ is valid.

   (e) If $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and $\bigwedge \varphi_n \to \neg\top$ is valid, then

$$\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}(\bigwedge_{n\leq l} \varphi_n)$$

   is valid.

   (f) If $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, then the validity of $\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$ in the class of Harsanyi type spaces on X for player set I implies that

$$\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$$

   is valid in the class of Harsanyi type spaces on X for player set I.

To:

   (a) That the axioms are valid is an easy check, (A6) is valid, provided we include the axiom of choice (like always) in our underlying set theory. (P1) – (P8) correspond to well-known properties of probability measures.
   (I1): Let $\underline{M}$ be a Harsanyi type space on X for player set I, $\varphi \in \mathcal{L}_0$ and $m \in M$. Then, $(\underline{M}, m) \models \neg p_i^\alpha(\varphi) \vee p_i^1(p_i^\alpha(\varphi))$ iff $(\underline{M}, m) \models \neg p_i^\alpha(\varphi)$ or $(\underline{M}, m) \models p_i^1(p_i^\alpha(\varphi))$. Let $(\underline{M}, m) \models p_i^\alpha(\varphi)$. This means that $T_i(m)([\varphi]^{\underline{M}}) \geq \alpha$. But then $[T_i(m)]^{\underline{M}} \subseteq [p_i^\alpha(\varphi)]^{\underline{M}}$ and hence $T_i(m)([p_i^\alpha(\varphi)]^{\underline{M}}) = 1$ and (I1) is valid. (I2) follows in the same manner.

   (b) - (d) above are clear,

(e) corresponds to the continuity at $\emptyset$, a well-known property of σ-additive probability measures: Let $\underline{M}$ be a type space (resp. Harsanyi type space) on X for player set I, $m \in M$ and $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$. By the definition of "$\models$", we have

$$\begin{aligned}\left[\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top\right]^{\underline{M}} &= \left(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\right) \cup \left(M \setminus [\top]^{\underline{M}}\right) \\ &= \left(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\right).\end{aligned}$$

If

$$\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$$

is valid, then $\bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}} = \emptyset$. In this case, we have for

$$E_l := \bigcap_{n\le l} [\varphi_n]^{\underline{M}} = \left[\bigwedge_{n\le l} \varphi_n\right]^{\underline{M}},$$

that $E_l \downarrow \emptyset$. So, for every $m \in M$ and $k \in \mathbb{N}\backslash\{0\}$ there is a $l\,(k,m) \in \mathbb{N}$ such that $T_i\,(m)\left(E_{l(k,m)}\right) < \frac{1}{k}$. By definition of "$\models$", we have

$$(\underline{M}, m) \models \neg p_i^{\frac{1}{k}}\big(\bigwedge_{n\le l(k,m)} \varphi_n\big).$$

Again by definition of "$\models$", it follows that

$$(\underline{M}, m) \models \bigwedge_{k\in\mathbb{N}\backslash\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\big(\bigwedge_{n\le l} \varphi_n\big).$$

Hence

$$\bigwedge_{k\in\mathbb{N}\backslash\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\big(\bigwedge_{n\le l} \varphi_n\big)$$

is valid.

(f) Let $\underline{M}$ be a type space (resp. Harsanyi type space) on X for player set I and $m \in M$. Then, it is easy to see by induction on the formation of formulas $\varphi \in \mathcal{L}^i$: Either $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$ or $[T_i(m)]^{\underline{M}} \cap [\varphi]^{\underline{M}} = \emptyset$, for $\varphi \in \mathcal{L}^i$. Let $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$. Then,

$$(\underline{M}, m) \models \bigwedge_{k\in\mathbb{N}\backslash\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\big(\bigvee_{n\le l} \varphi_n\big) \quad \text{iff} \quad \lim_{l\to\infty} T_i(m)\big([\bigvee_{n\le l} \varphi_n]\big) = 1,$$

which is by σ-additivity the case iff

$$T_i(m)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1.$$

Let $\varphi \in \mathcal{L}^i$ and assume that

$$\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$$

is valid in the class of Harsanyi type spaces on X for player set I. Assume that $(\underline{M}, m) \models \varphi$. By the above, $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$, since φ is an i-formula. This implies that $[T_i(m)]^{\underline{M}} \subseteq [\bigvee_{n\in\mathbb{N}} \varphi_n]^{\underline{M}}$ and by the introspection property of the Harsanyi type spaces $T_i(m)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1$. The above observation implies now that

$$(\underline{M}, m) \models \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\le l} \varphi_n),$$

hence

$$\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\le l} \varphi_n)$$

is valid in the class of Harsanyi type spaces on X for player set I.

2. Given 1., we have to show:

   (a) If $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$, $\varphi, \psi \in \mathcal{L}$, $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \varphi \to \psi$, then $(\underline{M}, m) \models \psi$. But

   $$(\underline{M}, m) \models \varphi \to \psi \text{ iff } m \in [\neg\varphi \vee \psi]^{\underline{M}} = (M \setminus [\varphi]^{\underline{M}}) \cup [\psi]^{\underline{M}},$$

   so $m \in [\varphi]^{\underline{M}}$ and $(\underline{M}, m) \models \varphi \to \psi$ imply $m \in [\psi]^{\underline{M}}$.

   (b) If $\underline{M}$ is a type space (resp. Harsanyi type space) on X for player set I, $m \in M$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \le 2^{\aleph_\gamma}$ and such that for all $\varphi \in \Phi$: $(\underline{M}, m) \models \varphi$, then $(\underline{M}, m) \models \bigwedge_{\varphi\in\Phi} \varphi$, but this is clear by the definition of "$\models$".

■

Definition 3.17

- $\Omega :=$

$$\left\{ \bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0\setminus\Phi_0} \neg\psi \;\middle|\; \Phi_0 \subseteq \mathcal{L}_0, \text{ s.t. } \Phi_0 \cup \{\neg\psi \mid \psi \in \mathcal{L}_0\setminus\Phi_0\} \text{ is consistent} \right\},$$

- for $\psi \in \mathcal{L}$, define

$$[\psi] := \{\omega \in \Omega \mid \vdash \omega \to \psi\},$$

- for $\Gamma \subseteq \mathcal{L}$, define

$$[\Gamma] := \bigcap_{\psi\in\Gamma} [\psi],$$

- for $\omega \in \Omega$, such that $\omega = \bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0\setminus\Phi_0} \neg\psi$, define

$$\Psi_\omega := \Phi_0 \cup \{\neg\psi \mid \psi \in \mathcal{L}_0\setminus\Phi_0\}.$$

Note that although we write "Ω", we define in fact two Ω's, one corresponding the to system P, and one corresponding to the system H. By the definitions of the system P and of the system H, it follows that a set of $\mathcal{L}$-formulas that is consistent in the system H is also consistent in the system P. Hence the Ω corresponding to the system H is a subset of the Ω corresponding to the system P.

Remark 3.2 *1. The class of Harsanyi type spaces on X for player set I is nonempty. And hence, class of type spaces on X for player set I is nonempty.*

2. The set Ω is nonempty.

Proof

1. Set

$$\begin{array}{lcll} M & := & \{m\}, & \\ \Sigma & := & \mathrm{Pow}\,(M)\,, & \text{(i.e. the power set of } M), \\ T_i\,(m) & := & \delta_m, & \text{for every } i \in I, \quad \text{(i.e. the delta-measure at } m), \\ v(m,x) & := & 1, & \text{for every } x \in X \cup \{\top\}. \end{array}$$

   Then,

$$\underline{M} := \langle M, \Sigma, (T_i)_{i\in I}, v\rangle$$

   forms a Harsanyi type space on X for player set I.

2. Let $\underline{M}$ be the Harsanyi type space on X for player set I constructed above. Consider the set

$$\Phi_0 := \{\varphi \in \mathcal{L}_0 \mid (\underline{M}, m) \models \varphi\}\,.$$

   By the definition of "$\models$", we have $(\underline{M}, m) \models \neg\psi$, for $\psi \in \mathcal{L}_0 \backslash \Phi_0$, and hence

$$\Phi_0 = \Phi_0 \cup \{\neg\psi \in \mathcal{L}_0 \mid \psi \in \mathcal{L}_0 \backslash \Phi_0\}\,.$$

   We claim that Φ_0 is consistent in the system H, (and hence also in the system P). Otherwise, $\Phi_0 \vdash \chi$ and $\Phi_0 \vdash \neg\chi$, for some $\chi \in \mathcal{L}$. But then, by Proposition 3.1, we have $(\underline{M}, m) \models \chi$ and $(\underline{M}, m) \models \neg\chi$. By the definition of the relation "$\models$", this is impossible. Hence,

$$\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0\backslash\Phi_0} \neg\psi \in \Omega \neq \emptyset.$$

■

Proposition 3.2 *1.*

$$\vdash \bigvee_{\omega \in \Omega} \omega.$$

2. *For every formula* $\psi \in \mathcal{L}$ *and for every* $\omega \in \Omega$:

$$\textit{Either} \vdash \omega \rightarrow \psi \textit{ or } \vdash \omega \rightarrow \neg\psi,$$

but not both.

3. *For every formula* $\psi \in \mathcal{L}$:

$$\vdash \psi \leftrightarrow \bigvee_{\omega \in [\psi]} \omega.$$

4. *If* $\Phi \subseteq \mathcal{L}$ *such that* $|\Phi| \leq 2^{\aleph_\gamma}$, *then*

$$\vdash \bigwedge_{\varphi \in \Phi} \varphi \leftrightarrow \bigvee_{\omega \in [\Phi]} \omega.$$

5. *For every formula* $\psi \in \mathcal{L}$:

$$\vdash \neg\psi \leftrightarrow \bigvee_{\omega \in \Omega \setminus [\psi]} \omega.$$

6. *For every formula* $\psi \in \mathcal{L}$:

$$[\neg\psi] = \Omega \setminus [\psi] \, .$$

7. *If* $\Phi \subseteq \mathcal{L}$ *such that* $|\Phi| \leq 2^{\aleph_\gamma}$, *then*

$$[\Phi] = \left[\bigwedge_{\varphi \in \Phi} \varphi \right].$$

Proof

1. By (A5), $\vdash \varphi \vee \neg\varphi$, for $\varphi \in \mathcal{L}_0$. Since $|\mathcal{L}_0| \leq \aleph_\gamma$, it follows by Conjunction that $\vdash \bigwedge_{\varphi\in\mathcal{L}_0} (\varphi \vee \neg\varphi)$. By (A6) and Modus Ponens, it follows that

$$\vdash \bigvee_{\Phi_0\subseteq\mathcal{L}_0} \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big).$$

If $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0\backslash\Phi_0\}$ is inconsistent (i.e. not consistent), then it follows by 1 and 2 of the Lemma 3.2 that

$$\vdash \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big) \rightarrow \psi \text{ and } \vdash \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big) \rightarrow \neg\psi,$$

for a $\psi \in \mathcal{L}$. By Conjunction, (A4) and Modus Ponens, we get

$$\vdash \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big) \rightarrow (\psi \wedge \neg\psi)\,.$$

Since $(\chi \rightarrow \rho) \rightarrow (\neg\rho \rightarrow \neg\chi)$ is a tautology of the Propositional Calculus, we get, by Modus Ponens,

$$\vdash \neg\,(\psi \wedge \neg\psi) \rightarrow \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big).$$

$\neg\,(\psi \wedge \neg\psi)$ is a tautology of the Propositional Calculus, hence Modus Ponens yields

$$\vdash \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big).$$

Let $\mathbf{C}_0$ be the set of all $\Phi_0 \subseteq \mathcal{L}_0$ such that $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0\backslash\Phi_0\}$ is inconsistent. By Conjunction,

$$\bigwedge_{\Phi_0\in\mathbf{C}_0} \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big)$$

is a theorem. By the definition of "$\bigvee$", "$\vee$" and "$\rightarrow$",

$$\Big(\bigvee_{\Phi_0\subseteq\mathcal{L}_0} \Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big)\Big) \rightarrow \Big(\Big(\bigwedge_{\Phi_0\in\mathbf{C}_0} \neg\Big(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\backslash\Phi_0} \neg\varphi \Big)\Big) \rightarrow \bigvee_{\omega\in\Omega} \omega \Big),$$

is a tautology of Propositional Calculus, hence a theorem. Applying Modus Ponens two times yields now

$$\vdash \bigvee_{\omega\in\Omega} \omega.$$

2. Follows by induction on the formation of the formulas in $\mathcal{L}$. Let $\omega \in \Omega$. If $\vdash \omega \to \varphi$ and $\vdash \omega \to \neg\varphi$ for some $\varphi \in \mathcal{L}$, then by 1 and 2 of Lemma 3.2, Ψ_ω is not consistent, a contradiction.
For every $\psi \in \mathcal{L}_0$ we have $\psi \in \Psi_\omega$ or $\neg\psi \in \Psi_\omega$. Again by 1 and 2 of Lemma 3.2 it follows that $\vdash \omega \to \psi$ or $\vdash \omega \to \neg\psi$.
If $\varphi \in \mathcal{L}$ and $\psi = \neg\varphi$, then, by the induction hypothesis, either $\vdash \omega \to \neg\varphi$, or $\vdash \omega \to \varphi$. In the second case, since $\varphi \to \neg(\neg\varphi)$ is a tautology of the Propositional Calculus, we have $\vdash \varphi \to \neg\psi$ and by 3 of Lemma 3.2 it follows that $\vdash \omega \to \neg\psi$.
If $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$ and $\psi = \bigwedge_{\varphi\in\Phi} \varphi$, then, by the induction hypothesis, either $\vdash \omega \to \varphi$ for all $\varphi \in \Phi$, or there is a $\chi \in \Phi$ such that $\vdash \omega \to \neg\chi$. In the first case, by Conjunction, it follows that $\vdash \bigwedge_{\varphi\in\Phi}(\omega \to \varphi)$ and by (A4) and Modus Ponens,

$$\vdash \omega \to \bigwedge_{\varphi\in\Phi} \varphi.$$

In the second case, since $\neg\chi \to \neg\bigwedge_{\varphi\in\Phi} \varphi$ is a tautology of the Propositional Calculus, hence a theorem, we conclude by 3 of Lemma 3.2 that

$$\vdash \omega \to \neg\bigwedge_{\varphi\in\Phi} \varphi.$$

3. Let $\omega \in \Omega$ and $\psi \in \mathcal{L}$. If $\omega \notin [\psi]$, then $\vdash \omega \to \neg\psi$. But then, since $(\omega \to \neg\psi) \to (\psi \to \neg\omega)$ is a tautology of the Propositional Calculus, we have $\vdash \psi \to \neg\omega$. By Conjunction, (A4) and Modus Ponens, we conclude $\vdash \psi \to \bigwedge_{\omega\in\Omega\setminus[\psi]} \neg\omega$.

$$\bigwedge_{\omega\in\Omega\setminus[\psi]} \neg\omega \to \neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega$$

is a tautology of the Propositional Calculus, so we conclude $\vdash \psi \to \neg\bigvee_{\omega\in\Omega\setminus[\psi]} \omega$.

$$\bigvee_{\omega\in\Omega} \omega \to \Big(\neg \bigvee_{\omega\in\Omega\setminus[\psi]} \omega \to \bigvee_{\omega\in[\psi]} \omega\Big)$$

is a tautology of the Propositional Calculus, hence a theorem, so we infer by 1 and Modus Ponens that $\vdash \neg \bigvee_{\omega \in \Omega \setminus [\psi]} \omega \to \bigvee_{\omega \in [\psi]} \omega$. By 3 of Lemma 3.2, it follows that

$$\vdash \psi \to \bigvee_{\omega \in [\psi]} \omega.$$

If $\omega \in [\psi]$, then $\vdash \omega \to \psi$. Since $(\omega \to \psi) \to (\neg\psi \to \neg\omega)$ is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that $\vdash \neg\psi \to \neg\omega$. By Conjunction, (A4) and Modus Ponens, we get $\vdash \neg\psi \to \bigwedge_{\omega \in [\psi]} \neg\omega$. Then, since

$$\Big(\neg\psi \to \bigwedge_{\omega \in [\psi]} \neg\omega\Big) \to \Big(\bigvee_{\omega \in [\psi]} \omega \to \psi\Big)$$

is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that

$$\vdash \bigvee_{\omega \in [\psi]} \omega \to \psi,$$

so, by Conjunction, we conclude

$$\vdash \psi \leftrightarrow \bigvee_{\omega \in [\psi]} \omega.$$

4. Let $\omega \in \Omega$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$. By 3 it suffices to show that $\omega \in \left[\bigwedge_{\varphi \in \Phi} \varphi\right]$ iff $\omega \in [\Phi]$. If $\omega \in [\Phi]$, then, for every $\varphi \in \Phi$, we have $\vdash \omega \to \varphi$. By Conjunction, (A4) and Modus Ponens it follows that $\vdash \omega \to \bigwedge_{\varphi \in \Phi} \varphi$, so $\omega \in \left[\bigwedge_{\varphi \in \Phi} \varphi\right]$.
   If $\omega \in \left[\bigwedge_{\varphi \in \Phi} \varphi\right]$, then $\vdash \omega \to \bigwedge_{\varphi \in \Phi} \varphi$. For every $\psi \in \Phi$ we have, by (A5), $\vdash \bigwedge_{\varphi \in \Phi} \varphi \to \psi$, so, by 3 of Lemma 3.2, it follows that $\vdash \omega \to \psi$, and hence $\omega \in [\Phi]$.

5. By 2. we have $\Omega \setminus [\psi] = [\neg\psi]$. 5. follows now from 3.

6. See the Proof of 5.

7. See the Proof of 4.

■

Now, we are going to build the canonical model. The first step is to define a measurable space:

Definition 3.18 Let Σ be the $\sigma-$Field on Ω generated by the set

$$\{[\psi] \mid \psi \in \mathcal{L}_0\}.$$

By (A0) and 2 of Lemma 3.2, it follows that $\Omega = [\top]$ and by 2 of Proposition 3.2, it follows that $\Omega \backslash [\psi] = [\neg\psi]$, for $\psi \in \mathcal{L}_0$. By Conjunction, (A4) and Modus Ponens, and by (A5) and 3 of Lemma 3.2, it follows that $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, for $\varphi, \psi \in \mathcal{L}_0$. Hence:

Remark 3.3 *The set*

$$\mathcal{F} := \{[\psi] \mid \psi \in \mathcal{L}_0\}$$

is a field on Ω.

Definition 3.19

- For $\omega \in \Omega$ and $\psi \in \mathcal{L}_0$, define

$$T_i'(\omega)([\psi]) := \sup\{\alpha \in [0,1] \cap \mathbb{Q} \mid \vdash \omega \rightarrow p_i^{\alpha}(\psi)\}.$$

- For $\omega \in \Omega$ and $x \in X \cup \{\top\}$, define

$$v(\omega, x) := \begin{cases} 1, & \text{if } \omega \in [x], \\ 0, & \text{if } \omega \notin [x]. \end{cases}$$

Obviously, we have:

Remark 3.4 $v(\cdot, x)$ *is* $\mathcal{F} - \mathrm{Pow}(\{0,1\})$*-measurable, for every* $x \in X$.

Lemma 3.3 *Let* $\psi \in \mathcal{L}_0$, $\omega \in \Omega$, *and* $\alpha \in [0,1] \cap \mathbb{Q}$ *such that* $\vdash \omega \to \neg p_i^{\alpha}(\psi)$. *Then,*

$$T_i'(\omega)([\psi]) < \alpha.$$

Proof Assume that $T_i'(\omega)([\psi]) \geq \alpha$. Then, for every $\beta' < \alpha$, there is a $\beta > \beta'$ with $\vdash \omega \to p_i^{\beta}(\psi)$. By (P7), $\vdash p_i^{\beta}(\psi) \to p_i^{\beta'}(\psi)$. Hence, by 3 of Lemma 3.2, $\vdash \omega \to p_i^{\beta'}(\psi)$. By Conjunction, (A4) and Modus Ponens, we have $\vdash \omega \to \bigwedge_{\beta<\alpha} p_i^{\beta}(\psi)$. By 3 of Lemma 3.2 and (P3), it follows that $\vdash \omega \to p_i^{\alpha}(\psi)$, a contradiction to 2 of Proposition 3.2. ■

Lemma 3.4 *For every* $i \in I$ *and* $\omega \in \Omega$:

$$T_i'(\omega)(\cdot)$$

is well-defined and a countably additive measure on $\mathcal{F}$.
Furthermore, for every $i \in I$ *and* $\omega \in \Omega$:

$$T_i'(\omega)(\Omega) = 1.$$

Proof Let $\varphi \in \mathcal{L}_0$. By (P1), $p_i^0(\varphi)$ is an axiom and by (A1),

$$p_i^0(\varphi) \to \left(\omega \to p_i^0(\varphi)\right)$$

is an axiom. By Modus Ponens, it follows that $\vdash \omega \to p_i^0(\varphi)$. Hence $T_i'(\omega)([\varphi]) \geq 0$.

Let $\varphi,\ \psi \in \mathcal{L}_0$ with $[\varphi] = [\psi]$. (Of course we, have by (A5) and Modus Ponens that $\vdash \varphi \leftrightarrow \varphi'$ implies $\vdash \varphi \to \varphi'$ and $\vdash \varphi' \to \varphi$, and by Conjunction follows the opposite direction.) Then, $\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]} \omega$ and $\vdash (\bigvee_{\omega\in[\psi]} \omega) \leftrightarrow \varphi$, so by Lemma 3.2, $\vdash \varphi \leftrightarrow \psi$. By Necessitation, (P8) and Modus Ponens, it follows that $\vdash p_i^{\alpha}(\varphi) \to p_i^{\alpha}(\psi)$ and $\vdash p_i^{\alpha}(\psi) \to p_i^{\alpha}(\varphi)$, so

$$\sup\{\alpha \mid \vdash \omega \to p_i^{\alpha}(\varphi)\} = \sup\{\alpha \mid \vdash \omega \to p_i^{\alpha}(\psi)\},$$

and $T_i'(\omega)$ is well-defined.

Let $\omega \in \Omega$. To show that $T_i'(\omega)$ is countably additive it is enough to show that it is finitely additive and continuous at $\emptyset$ (see Dudley (1989), Theorem 3.1.1). Let φ, $\psi \in \mathcal{L}_0$ with $[\varphi] \cap [\psi] = \emptyset$. Then, $[\varphi] \subseteq \Omega \setminus [\psi]$. It follows that

$$[\varphi] = ([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi]) \text{ and } [\psi] = ([\varphi] \cup [\psi]) \cap [\psi].$$

By 6 and 7 of Proposition 3.2, it follows that

$$([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi]) = [(\varphi \vee \psi) \wedge \neg\psi] \text{ and } ([\varphi] \cup [\psi]) \cap [\psi] = [(\varphi \vee \psi) \wedge \psi].$$

Now, let

$$T_i'(\omega)([\varphi]) = r \text{ and } T_i'(\omega)([\psi]) = r'.$$

Assume that $r + r' > 1$. Then there are rationals $\alpha, \beta \in [0,1]$, such that $\alpha \leq r$, $\beta \leq r'$ and $\alpha + \beta > 1$. But then,

$$\vdash \omega \rightarrow p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \rightarrow p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

We have

$$\vdash (\varphi \vee \psi) \wedge \psi \rightarrow \neg((\varphi \vee \psi) \wedge \neg\psi),$$

because this is a tautology of the Propositional Calculus. Necessitation, (P8) and Modus Ponens yield now

$$\vdash p_i^{\beta}((\varphi \vee \psi) \wedge \psi) \rightarrow p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)).$$

By Lemma 3.2, we conclude that

$$\vdash \omega \rightarrow p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)).$$

But since $\alpha + \beta > 1$, we have that

$$p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \rightarrow \neg p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi))$$

is an axiom (P6), hence

$$\vdash \omega \rightarrow \neg p_i^{\beta}(\neg((\varphi \vee \psi) \wedge \neg\psi)),$$

which is by 2 of Proposition 3.2 a contradiction. So, it follows that $r + r' \leq 1$.

For every $\varepsilon > 0$, there are rational $\alpha, \beta \in [0,1]$ with $\alpha \leq r$ and $\beta \leq r'$ such that $\alpha \geq r - \frac{\varepsilon}{2}$ and $\beta \geq r' - \frac{\varepsilon}{2}$. For such α and β we have

$$\vdash \omega \rightarrow p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \rightarrow p_i^{\beta}((\varphi \vee \psi) \wedge \psi),$$

so, by Conjunction, (A4) and Modus Ponens,

$$\vdash \omega \rightarrow p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \wedge p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

Together with (P4) and Lemma 3.2, we conclude that $\vdash \omega \rightarrow p_i^{\alpha+\beta}(\varphi \vee \psi)$. This implies that

$$T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \geq r + r'.$$

If $r+r' = 1$, then we have $T_i'(\omega)([\varphi \vee \psi]) = 1$, since by definition $T_i'(\omega)([\varphi \vee \psi]) \leq 1$. If $r + r' < 1$, then for all $\varepsilon > 0$ such that $\varepsilon + r + r' \leq 1$, there are rationals $\alpha, \beta \in [0,1]$, such that $\alpha > r$, $\beta > r'$ and $\alpha + \beta \leq \varepsilon + r + r'$. For such α, β we have

$$\vdash \omega \rightarrow \neg p_i^{\alpha}((\varphi \vee \psi) \wedge \neg\psi) \text{ and } \vdash \omega \rightarrow \neg p_i^{\beta}((\varphi \vee \psi) \wedge \psi).$$

This implies (like above, but with the use of (P5)) that $\vdash \omega \rightarrow \neg p_i^{\alpha+\beta}(\varphi \vee \psi)$. So, by Lemma 3.3, we have

$$T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \leq r + r'.$$

Altogether, this shows that $T_i'(\omega)$ is finitely additive.

Since $\top$ is an axiom, we have for every $\omega \in \Omega$ that $\{\omega\} \vdash \top$. Therefore, by Lemma 3.2, $\vdash \omega \to \top$, and hence $[\top] = \Omega$. Since $\top$ is a theorem, Necessitation yields $\vdash p_i^1(\top)$, so, as above, we have for every $\omega \in \Omega$ that $\vdash \omega \to p_i^1(\top)$. This implies that $T_i'(\omega)(\Omega) = 1$, for every $\omega \in \Omega$.

Note that we have $[\neg\top] = \emptyset$, and since, for $\omega \in \Omega$ and $i \in I$, $T_i'(\omega)$ is finitely additive, we have

$$T_i'(\omega)(\emptyset) = T_i'(\omega)(\emptyset \cup \emptyset) = T_i'(\omega)(\emptyset) + T_i'(\omega)(\emptyset),$$

and hence $T_i'(\omega)(\emptyset) = 0$.

For $\omega \in \Omega$, it remains to show that $T_i'(\omega)$ is continuous at $\emptyset$: For $n \in \mathbb{N}$, let $E_n = [\varphi_n]$ with $\varphi_n \in \mathcal{L}_0$ and let $E_n \downarrow \emptyset$, that is, for all $n \in \mathbb{N} : E_{n+1} \subseteq E_n$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Then, by 7 of Proposition 3.2, we have

$$[\varphi_n] = \Big[\bigwedge_{m\le n} \varphi_m\Big] \text{ and } \Big[\bigwedge_{n\in\mathbb{N}} \varphi_n\Big] = \bigcap_{n\in\mathbb{N}} [\varphi_n] = \emptyset.$$

It follows that

$$\Omega = \bigg(\Big(\Omega \setminus \Big[\bigwedge_{n\in\mathbb{N}} \varphi_n\Big]\Big) \cup [\neg\top]\bigg) = \Big[\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top\Big],$$

by 6 and 7 of Proposition 3.2. By 3 and 1 of Proposition 3.2 and by Modus Ponens, it follows that

$$\vdash \bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top.$$

So, by the inference rule "Continuity at $\emptyset$", we have

$$\vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\le l} \varphi_n\Big).$$

Hence,

$$\{\omega\} \vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\le l} \varphi_n\Big)$$

and, by 2 of Lemma 3.2,

$$\vdash \omega \rightarrow \bigwedge_{k\in\mathbb{N}\backslash\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l}\varphi_n\Big).$$

For $\varepsilon > 0$ fix $k \in \mathbb{N}\backslash\{0\}$ with $\frac{1}{k} \leq \varepsilon$. By (A5) and 3 of Lemma 3.2, it follows that

$$\vdash \omega \rightarrow \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l}\varphi_n\Big).$$

But then there is a $l \in \mathbb{N}$ such that

$$\vdash \omega \rightarrow \neg p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l}\varphi_n\Big),$$

for if not, it follows by 2 of Proposition 3.2, Conjunction, (A4) and Modus Ponens that

$$\vdash \omega \rightarrow \bigwedge_{l\in\mathbb{N}} p_i^{\frac{1}{k}}\Big(\bigwedge_{n\leq l}\varphi_n\Big),$$

a contradiction. (Note that $\bigvee_{l\in\mathbb{N}} \neg\psi_l \rightarrow \neg\bigwedge_{l\in\mathbb{N}}\psi_l$ is a tautology of the Propositional Calculus). By Lemma 3.3, it follows that

$$T_i'(\omega)\Big(\Big[\bigwedge_{n\leq l}\varphi_n\Big]\Big) < \frac{1}{k} \leq \varepsilon.$$

The additivity implies then that

$$T_i'(\omega)\Big(\Big[\bigwedge_{n\leq m}\varphi_n\Big]\Big) < \frac{1}{k} \leq \varepsilon,$$

for $m \geq l$. So, we have

$$\lim_{n\rightarrow\infty} T_i'(\omega)(E_n) = 0.$$

■

Proposition 3.3 *1. For every $i \in I$ and $\omega \in \Omega$, there is a unique extension of $T_i'(\omega)$ to a σ-additive probability measure $T_i(\omega)$ on (Ω, Σ).*

2. *For every $i \in I$ and $\omega \in \Omega$, $T_i(\omega)$ is a $\Sigma - \Sigma_\Delta$-measurable function from Ω to $\Delta(\Omega, \Sigma)$, the space of probability measures on (Ω, Σ), which is endowed with the σ-field Σ_Δ generated by the sets $\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and rational $\alpha \in [0, 1]$.*

3. $$\underline{\Omega} := \left\langle \Omega, \Sigma, (T_i)_{i \in I,} v \right\rangle$$

   is a type space on X for player set I.

4. *For every $\psi \in \mathcal{L}$ and $\omega \in \Omega$:*

   $$\left(\left\langle \Omega, \Sigma, (T_i)_{i \in I,} v \right\rangle, \omega\right) \models \psi \quad \textit{iff} \quad \omega \in [\psi].$$

5. *If the axiom $(I1)$ is added in case of $\aleph_\gamma = \aleph_0$, and if the inference rule Uncountable Introspection is added in case of $\aleph_\gamma > \aleph_0$ (i.e. in the H-system case), then*

   $$\left\langle \Omega, \Sigma, (T_i)_{i \in I,} v \right\rangle$$

   is a Harsanyi type space on X for player set I.

Proof

1. Follows directly from Lemma 3.4 and Caratheodory's extension Theorem.

2. Follows from Lemma 3.1. Since $\mathcal{F}$ is a field that generates Σ, by that Lemma, the σ-field on $\Delta(\Omega, \Sigma)$ generated by the sets

   $$\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(F) \geq \alpha\},$$

   with $F \in \mathcal{F}$ and rational $\alpha \in [0, 1]$, is equal to the $\sigma-$field on $\Delta(\Omega, \Sigma)$ generated by the sets

   $$\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\},$$

   with $E \in \Sigma$ and rational $\alpha \in [0, 1]$. Inverse images commute with arbitrary intersections and unions and with complements. So, it suffices to show that

$\{\omega \mid T_i(\omega)([\psi]) \geq \alpha\} \in \Sigma$, for all $\psi \in \mathcal{L}_{0,}$ $i \in I$ and rational $\alpha \in [0,1]$. By Lemma 3.3, 2 of Proposition 3.2 and the definition of $T_i(\omega)$, it follows that

$$T_i(\omega)(\psi) \geq \alpha \text{ iff } \vdash \omega \rightarrow p_i^\alpha(\psi).$$

But we have that

$$\vdash \omega \rightarrow p_i^\alpha(\psi) \text{ iff } \omega \in [p_i^\alpha(\psi)],$$

and $[p_i^\alpha(\psi)] \in \mathcal{F} \subseteq \Sigma$.

3. Follows from Remark 3.4, 2 of Remark 3.2, and 1. and 2. of this proposition.

4. We proceed by induction on the formation of the formulas in $\mathcal{L}$: Let $\omega \in \Omega$. Then:

   (a) For $x \in X \cup \{\top\}$:

   $$\begin{aligned}(\underline{\Omega}, \omega) \models x \quad &\text{iff} \quad v(\omega, x) = 1 \\ &\text{iff} \quad \omega \in [x].\end{aligned}$$

   (b) For $\varphi \in \mathcal{L}$:

   $$\begin{aligned}(\underline{\Omega}, \omega) \models \neg\varphi \quad &\text{iff} \quad (\underline{\Omega}, \omega) \not\models \varphi \\ &\text{iff} \quad \omega \notin [\varphi] \\ &\text{iff} \quad \omega \in [\neg\varphi],\end{aligned}$$

   where the last equivalence follows from 2 of Proposition 3.2.

   (c) For $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$:

   $$\begin{aligned}(\underline{\Omega}, \omega) \models \textstyle\bigwedge_{\varphi\in\Phi} \varphi \quad &\text{iff} \quad (\underline{\Omega}, \omega) \models \varphi, \quad \text{for all } \varphi \in \Phi, \\ &\text{iff} \quad \omega \in \textstyle\bigcap_{\varphi\in\Phi} [\varphi] \\ &\text{iff} \quad \omega \in \left[\textstyle\bigwedge_{\varphi\in\Phi} \varphi\right],\end{aligned}$$

   where the last equivalence follows from 7 of Proposition 3.2.

   (d) For $i \in I$, $\alpha \in [0,1] \cap \mathbb{Q}$, and $\varphi \in \mathcal{L}_0$:

   $$\begin{aligned}(\underline{\Omega}, \omega) \models p_i^\alpha(\varphi) \quad \text{iff} \quad &\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \models \varphi\} \in \Sigma \text{ and} \\ &T_i(\omega)(\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \models \varphi\}) \geq \alpha.\end{aligned}$$

   But, by the induction hypothesis,

   $$\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \models \varphi\} = [\varphi] \in \Sigma.$$

So,

$$(\underline{\Omega}, \omega) \models p_i^{\alpha}(\varphi) \quad \text{iff} \quad \sup \left\{ \beta \in [0,1] \cap \mathbb{Q} \mid \vdash \omega \to p_i^{\beta}(\varphi) \right\} \geq \alpha$$
$$\text{iff} \quad \vdash \omega \to p_i^{\alpha}(\varphi),$$

where the last equivalence follows by Lemma 3.3 and Proposition 3.2.

5.

(a) Case $\aleph_\gamma = \aleph_0$: We show that, for all $\omega \in \Omega$ and $i \in I$: $[T_i(\omega)]$ is measurable and $T_i(\omega)([T_i(\omega)]) = 1$. Since $\mathcal{L}_0$ is countable and since

$$\{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\} = [p_i^{\alpha}(\varphi)],$$

for $\varphi \in \mathcal{L}_0$ (by 4. of this proposition), the set

$$[T_i(\omega)]_0 := \bigcap_{\alpha \in [0,1] \cap \mathbb{Q},\ \varphi \in \mathcal{L}_0,\ \text{s.t. } T_i(\omega)([\varphi]) \geq \alpha} \{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\}$$

is measurable. Since $\mathcal{F}$ is closed under complements, every $\omega' \in [T_i(\omega)]_0$ satisfies for all $A \in \mathcal{F}$:

$$T_i(\omega)(A) = T_i(\omega')(A).$$

Since $\mathcal{F}$ is a field which generates Σ, by Caratheodory's extension Theorem, it follows that $T_i(\omega) = T_i(\omega')$. Hence,

$$[T_i(\omega)] = [T_i(\omega)]_0 \in \Sigma.$$

By (I1), $p_i^{\alpha}(\varphi) \to p_i^1(p_i^{\alpha}(\varphi))$ is an axiom, so, by 3. of Lemma 3.2, $\vdash \omega \to p_i^{\alpha}(\varphi)$ implies $\vdash \omega \to p_i^1(p_i^{\alpha}(\varphi))$. Hence, by the definition of $T_i(\omega)$, it follows that $T_i(\omega)([\varphi]) \geq \alpha$ implies $T_i(\omega)([p_i^{\alpha}(\varphi)]) = 1$. Since $T_i(\omega)$ is a $\sigma-$additive probability measure, it follows that $T_i(\omega)([T_i(\omega)]) = 1$.

(b) Case $\aleph_\gamma > \aleph_0$: We have to show that, for all $\omega \in \Omega$, $i \in I$ and $A \in \Sigma$: $[T_i(\omega)] \subseteq A$ implies $T_i(\omega)(A) = 1$.
By the definition of $T_i(\omega)$ in the proof of Caratheodory's Theorem, it is enough to show that for $(\varphi_n)_{n \in \mathbb{N}}$, where $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$:

$$\bigcup_{n \in \mathbb{N}} [\varphi_n] \supseteq A \quad \text{implies} \quad \sum_{n \in \mathbb{N}} T_i(\omega)([\varphi_n]) \geq 1.$$

We can assume without loss of generality that the $[\varphi_n]$ are pairwise disjoint. (That

$$\inf\left\{ \sum_{n\in\mathbb{N}} T_i'(\omega)([\varphi_n]) \,\middle|\, \varphi_n \in \mathcal{L}_0, n \in \mathbb{N} \text{ such that } \bigcup_{n\in\mathbb{N}} [\varphi_n] \supseteq A \right\} \leq 1$$

is clear, because $A \subseteq \Omega$ and $T_i'(\omega)(\Omega) = 1$.)
For $i \in I$ and $\omega \in \Omega$ define

$\varphi_i(\omega) :=$

$$\bigwedge_{\alpha\in[0,1]\cap\mathbb{Q},\ \chi\in\mathcal{L}_0,\ \text{s.t. } \vdash\omega\to p_i^\alpha(\chi)} p_i^\alpha(\chi) \wedge \bigwedge_{\beta\in[0,1]\cap\mathbb{Q},\ \psi\in\mathcal{L}_0,\ \text{s.t. } \vdash\omega\to\neg p_i^\beta(\psi)} \neg p_i^\beta(\psi).$$

By the definition of $T_i(\omega)$, we have $[T_i(\omega)] = [\varphi_i(\omega)]$, where the formula on the right hand side is an i-formula. From

$$[T_i(\omega)] \subseteq \bigcup_{n\in\mathbb{N}} [\varphi_n] = \Big[\bigvee_{n\in\mathbb{N}} \varphi_n\Big]$$

it follows that

$$\Omega = \Big[\varphi_i(\omega) \to \bigvee_{n\in\mathbb{N}} \varphi_n\Big].$$

By 1 and 3 of Proposition 3.2 and Modus Ponens, it follows that $\varphi_i(\omega) \to \bigvee_{n\in\mathbb{N}} \varphi_n$ is a theorem. By the inference rule "Uncountable Introspection", we can conclude that

$$\varphi_i(\omega) \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\Big(\bigvee_{n\leq l} \varphi_n\Big)$$

is a theorem. Since $\vdash \omega \to \varphi_i(\omega)$, it follows that

$$\vdash \omega \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\Big(\bigvee_{n\leq l} \varphi_n\Big),$$

which implies that

$$1 = \lim_{l\to\infty} T_i(\omega)\Big(\Big[\bigvee_{n\leq l} \varphi_n\Big]\Big) \leq \sum_{n\in\mathbb{N}} T_i(\omega)([\varphi_n]).$$

■

Theorem 3.1 1. *The system P is strongly sound and strongly complete with respect to the class of type spaces on X for player set I.*

2. *The system H is strongly sound and strongly complete with respect to the class of Harsanyi type spaces on X for player set I.*

Proof The "strongly sound" follows from Proposition 3.1.

According to Proposition 3.3, $\underline{\Omega} = \langle \Omega,\ \Sigma,\ (T_i)_{i\in I,}\ v\rangle$ is a type space (resp. Harsanyi type space) on X for player set I.

Let $\Gamma \models \varphi$ in the class of type spaces (resp. Harsanyi type spaces) on X for player set I. Then, for all $\omega \in \Omega$:

$$(\underline{\Omega}, \omega) \models \Gamma \text{ implies } (\underline{\Omega}, \omega) \models \varphi.$$

So, by 4 of Proposition 3.3, $[\Gamma] \subseteq [\varphi]$, which implies, by 3 of Proposition 3.2, $\vdash \bigvee_{\omega\in[\Gamma]} \omega \rightarrow \varphi$, because for sets of formulas $A \subseteq B$ with $|B| \leq 2^{\aleph_\gamma}$: $\bigvee_{\chi\in A} \chi \rightarrow \bigvee_{\chi\in B} \chi$ is a tautology of the Propositional Calculus. Let $\omega \in \Omega$ and $\omega \notin [\Gamma]$. Then there is $\psi \in \Gamma$ with $\omega \notin [\psi]$, so $\vdash \omega \rightarrow \neg\psi$. But then, $\vdash \psi \rightarrow \neg\omega$ (since $(\chi \rightarrow \neg\widetilde{\chi}) \rightarrow (\widetilde{\chi} \rightarrow \neg\chi)$ is a tautology of the Propositional Calculus). By Modus Ponens, it follows that $\Gamma \vdash \neg\omega$. By Conjunction and the fact that $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\Gamma \vdash \bigwedge_{\omega\notin[\Gamma]} \neg\omega$. Then, since (as in the proof of 3. of Proposition 3.2) $\vdash \bigwedge_{\omega\notin[\Gamma]} \neg\omega \rightarrow \bigvee_{\omega\in[\Gamma]} \omega$ is a theorem, it follows that $\Gamma \vdash \bigvee_{\omega\in[\Gamma]} \omega$, hence, by Modus Ponens, we have $\Gamma \vdash \varphi$. ■

Corollary 3.1 *Let $\Gamma \subseteq \mathcal{L}$.*

- *Then Γ is consistent in the system P (resp. in the system H) iff Γ has a model in the class of type spaces (resp. Harsanyi type spaces) on X for player set I.*
- *Furthermore, if Γ is consistent in the system P (resp. in the system H), then there is a $\omega \in \Omega$, where Ω is the Ω corresponding to the system P (resp. to the system H), such that $(\underline{\Omega}, \omega) \models \Gamma$.*

Proof Assume that $(\underline{\Omega}, \omega) \not\models \Gamma$, for every $\omega \in \Omega$, where Ω is the Ω corresponding to the system P (resp. to the system H). Hence, for every ω there is a $\varphi_\omega \in \Gamma$ such that $(\underline{\Omega}, \omega) \models \neg\varphi_\omega$. By 4 of Proposition 3.3, it follows that $\omega \in [\neg\varphi_\omega]$, that is $\vdash \omega \rightarrow \neg\varphi_\omega$, hence, $\vdash \varphi_\omega \rightarrow \neg\omega$. Since $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\vdash \bigwedge_{\omega'\in\Omega} \varphi_{\omega'} \rightarrow \neg\omega$ (because $\vdash \bigwedge_{\omega'\in\Omega} \varphi_{\omega'} \rightarrow \varphi_\omega$ is an axiom). It follows (by Conjunction, (A4) and Modus Ponens) that $\vdash \bigwedge_{\omega'\in\Omega} \varphi_{\omega'} \rightarrow \bigwedge_{\omega\in\Omega} \neg\omega$. By 4 of Proposition 3.3 and the definition of "$\models$", we have $[\neg\,(x \wedge \neg x)] = \Omega$, for $x \in X$. So, by Proposition 3.2, it follows that $\vdash \neg(x \wedge \neg x) \to \bigvee_{\omega\in\Omega} \omega$ and hence $\vdash \bigwedge_{\omega\in\Omega} \neg\omega \to x \wedge \neg x$, which implies $\vdash \bigwedge_{\omega'\in\Omega} \varphi_{\omega'} \to x \wedge \neg x$, and we conclude $\Gamma \vdash x \wedge \neg x$. By (A5) and Modus Ponens, it follows that $\Gamma \vdash x$ and $\Gamma \vdash \neg x$, so Γ is inconsistent in the system P (resp. in the system H).

If Γ is not consistent in the system P (resp. in the system H), then there is a φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ in the system P (resp. in the system H). So, by the strong soundness, for every type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I and every $m \in M$: If $(\underline{M}, m) \models \Gamma$, then $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \neg\varphi$. By the definition of the relation "$\models$", there is no $(\underline{M}, m)$ such that $(\underline{M}, m) \models \varphi$ and $(\underline{M}, m) \models \neg\varphi$. So Γ has no model in the class of type spaces (resp. Harsanyi type spaces) on X for player set I. ■

3.4 Universality of the Canonical (Harsanyi) Type Space

In this section we prove that, by a fortunate coincidence, the canonical (Harsanyi) type space on X for player set I is (up to type isomorphism) the universal (Harsanyi) type space on X for player set I. This gives a characterization of the universal (Harsanyi) type space on X for player set I and shows that our language is rich enough to describe the states in the universal (Harsanyi) type space on X for player set I. (So, in some sense, the language is rich enough to capture "all relevant information".)

Definition 3.20 Let $\underline{M} = \langle M, \Sigma, (T_i)_{i\in I,} v\rangle$ and $\underline{N} = \langle N, \Sigma^N, \left(T_i^N\right)_{i\in I,} v^N\rangle$ be type spaces on X for player set I. A function $f : M \to N$ is a *type morphism* if it satisfies the following conditions:

1. f is $\Sigma - \Sigma^N$-measurable,
2. for all $m \in M$ and $x \in X$:
$$v(m, x) = v(f(m), x),$$
3. for all $m \in M$, $E \in \Sigma^N$, and $i \in I$:
$$T_i^N(f(m))(E) = T_i(m)\left(f^{-1}(E)\right).$$

Definition 3.21 A type morphism f is a *type isomorphism* if it is one-to-one, onto, and the inverse of f is also a type morphism.

Lemma 3.5 *Type morphisms preserve the validity of formulas, i.e. if f is a type morphism from $\underline{M}$ to $\underline{N}$, $m \in M$, and $\varphi \in \mathcal{L}$, then*

$$(\underline{M}, m) \models \varphi \text{ iff } (\underline{N}, f(m)) \models \varphi.$$

Proof By induction on the formation of the formulas in $\mathcal{L}$:

1. Let $x \in X \cup \{\top\}$. Then,

$$(\underline{M}, m) \models x \text{ iff } v(m, x) = 1 \text{ iff } v(f(m), x) = 1 \text{ iff } (\underline{N}, f(m)) \models x.$$

2. Let $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$. Then, by the induction hypothesis,

$$\begin{aligned} (\underline{M}, m) \models \textstyle\bigwedge_{\varphi\in\Phi} \varphi \quad & \text{iff} \quad (\underline{M}, m) \models \varphi, && \text{for all } \varphi \in \Phi, \\ & \text{iff} \quad (\underline{N}, f(m)) \models \varphi, && \text{for all } \varphi \in \Phi, \\ & \text{iff} \quad (\underline{N}, f(m)) \models \textstyle\bigwedge_{\varphi\in\Phi} \varphi. \end{aligned}$$

3. Let $\varphi \in \mathcal{L}$. Then, by the induction hypothesis,

$$(\underline{M}, m) \models \neg\varphi \text{ iff } (\underline{M}, m) \not\models \varphi \text{ iff } (\underline{N}, f(m)) \not\models \varphi \text{ iff } (\underline{N}, f(m)) \models \neg\varphi.$$

4. Let $\psi \in \mathcal{L}_0$. As remarked in the definition of the relation "$\models$", $[\psi]^{\underline{M}}$ is measurable in $\underline{M}$ and $[\psi]^{\underline{N}}$ is measurable in $\underline{N}$. By the induction hypothesis, we have $f^{-1}([\psi]^{\underline{N}}) = [\psi]^{\underline{M}}$, so we have

$$\begin{aligned} (\underline{M}, m) \models p_i^\alpha(\psi) \quad & \text{iff} \quad \alpha \leq T_i(m)([\psi]^{\underline{M}}) \\ & \text{iff} \quad \alpha \leq T_i^N(f(m))([\psi]^{\underline{N}}) \\ & \text{iff} \quad (\underline{N}, f(m)) \models p_i^\alpha(\psi). \end{aligned}$$

■

An easy check shows:

Remark 3.5 • *The type spaces on X for player set I - as the objects - together with the type morphisms - as the morphisms - form a category.*

- *The Harsanyi type spaces on X for player set I - as the objects - together with the type morphisms - as the morphisms - form a category.*

Definition 3.22 A type space (resp. Harsanyi type space) $\underline{M}$ on X for player set I is *universal,* if for every type space (resp. Harsanyi type space) $\underline{N}$ there is exactly one type morphism from $\underline{N}$ to $\underline{M}$.

It is obvious that a type morphism $f : \underline{M} \to \underline{N}$ is a type isomorphism iff there is a type morphism $g : \underline{N} \to \underline{M}$ such that $g \circ f = \mathrm{id}_{\underline{M}}$ and $f \circ g = \mathrm{id}_{\underline{N}}$. Hence, type isomorphisms coincide with the isomorphisms of the category of type spaces on X. In category theoretic terms, a universal (Harsanyi) type space on X for player set I is a terminal object of the category of (Harsanyi) type spaces on X for player set I. Terminal objects, if they exist, are known to be unique up to isomorphism, hence (but it is also easily seen directly):

Remark 3.6 *If there exists a universal type space (resp. Harsanyi type space) on X for player set I, then it is unique up to type isomorphism.*

Theorem 3.2 *The (Harsanyi) type space*

$$\underline{\Omega} = \left\langle \Omega, \Sigma, (T_i)_{i \in I,} v \right\rangle$$

on X for player set I is universal.

Proof Let $\underline{M} = \langle M, \Sigma^M, \left(T_i^M\right), v^M \rangle$ be a type space (resp. Harsanyi type space) on X for player set I. By Lemma 3.5, 2 of Proposition 3.2, and 4 of Proposition 3.3, it follows that there is at most one type morphism from $\underline{M}$ to $\underline{\Omega}$.

Let

$$\Phi_m := \{\varphi \in \mathcal{L}_0 \mid (\underline{M}, m) \models \varphi\},$$

for $m \in M$, and define

$$f(m) := \bigwedge_{\varphi \in \Phi_m} \varphi.$$

By Corollary 3.1, Φ_m is consistent and by the definition of the relation "$\models$", it follows for $\psi \in \mathcal{L}_0$, that

$$\psi \in \Phi_m \text{ iff } \neg\psi \notin \Phi_m.$$

This implies that $f(m) \in \Omega$. It remains to show that $f : M \to \Omega$ is a type morphism:

1. It is enough to show that for every $\psi \in \mathcal{L}_0 : f^{-1}([\psi]) \in \Sigma^M$, since the set $\{[\varphi] \mid \varphi \in \mathcal{L}_0\}$ is a field that generates Σ. We have

$$f(m) \in [\psi] \text{ iff } \vdash f(m) \to \psi \text{ iff } \psi \in \Phi_m \text{ iff } m \in [\psi]^{\underline{M}}.$$

   But $[\psi]^{\underline{M}} \in \Sigma^M$ (see the definition of "$\models$").

2. Let $x \in X \cup \{\top\}$. Then,

$$v^M(m,x) = 1 \text{ iff } x \in \Phi_m \text{ iff } \vdash f(m) \to x \text{ iff } f(m) \in [x] \text{ iff } v(f(m), x) = 1.$$

3. Let $i \in I$ and $m \in M$. Since $f : M \to \Omega$ is Σ^M–Σ-measurable, $T_i^M(m)(f^{-1}(\cdot))$ is a σ-additive probability measure on (Ω, Σ). Since $\mathcal{F}$ is a field that generates Σ, by Caratheodory's Extension Theorem,

$$T_i^M(m)\left(f^{-1}([\varphi])\right) = T_i(f(m))([\varphi]),$$

   for all $\varphi \in \mathcal{L}_0$, implies

$$T_i^M(m)\left(f^{-1}(E)\right) = T_i(f(m))(E),$$

   for all $E \in \Sigma$. As shown in 1, we have $f^{-1}([\varphi]) = [\varphi]^{\underline{M}}$, for $\varphi \in \mathcal{L}_0$, and hence

$$\begin{aligned} T_i^M(m)(f^{-1}([\varphi])) &= T_i^M(m)([\varphi]^{\underline{M}}) \\ &= \sup\{\alpha \mid (\underline{M}, m) \models p_i^\alpha(\varphi)\} \\ &= \sup\{\alpha \mid p_i^\alpha(\varphi) \in \Phi_m\} \\ &= \sup\{\alpha \mid \vdash f(m) \to p_i^\alpha(\varphi)\} \\ &= T_i(f(m))([\varphi]). \end{aligned}$$

■

3.5 Product Type Spaces

The aim of this section is to show that the canonical (Harsanyi) type space on X for player set I is (up to isomorphism) a product (Harsanyi) type space. Since this space is then a universal (Harsanyi) type space in the category of product (Harsanyi) type spaces on X for player set I, this implies that - in the case of Harsanyi type spaces (i.e. the H-system case) - our canonical model is, up to isomorphism, the universal Harsanyi type space on X for player set I constructed by Heifetz and Samet (1998b).

We define (in the same fashion as in the previous chapter):

Definition 3.23 A *product type space on X for player set I* is a 4-tupel

$$\underline{M} := \langle M, \Sigma, (T_i)_{i\in I}, v\rangle$$

such that there are measurable spaces (M_j, Σ_j), for $j \in I_0$, such that (up to type isomorphism):

- $M_0 = \mathrm{Pow}(X)$,
- Σ_0 is the σ-field on $\mathrm{Pow}(X)$ generated by the sets $[x]^0 := \{m_0 \subseteq X \mid x \in m_0\}$, where $x \in X$,
- $M = \mathrm{Pow}(X) \times \Pi_{i\in I} M_i$, where all the M_i are nonempty,
- Σ is the product σ-field on M which is generated by the σ-fields Σ_j, $j \in I_0$,
- for $i \in I$: T_i is a $\Sigma_i - \Sigma_\Delta$-measurable function from M_i to $\Delta(M, \Sigma)$, the space of probability measures on (M, Σ),
- for $x \in X$: $v(m_0, x) = \begin{cases} 1, & \text{if } x \in m_0, \\ 0, & \text{if } x \notin m_0, \end{cases}$
  and $v(m_0, \top) = 1$ in any case.

Obviously, T_i, for $i \in I$, can be viewed as a $\Sigma - \Sigma_\Delta$-measurable function from M to $\Delta(M, \Sigma)$ and $v(., x)$ can be viewed as a $\Sigma - \mathrm{Pow}(\{0, 1\})$-measurable function from M to $\{0, 1\}$, for every $x \in X \cup \{\top\}$. So every product type space on X for player set I is a type space on X for player set I.

Definition 3.24
- For $j \in I_0$, define

$$\Omega_j := \left\{ \bigwedge_{\varphi \in \Phi_0^j} \varphi \wedge \bigwedge_{\psi \in \mathcal{L}_0^j \setminus \Phi_0^j} \neg\psi \,\middle|\, \Phi_0^j \subseteq \mathcal{L}_0^j, \text{ s.t. } \Phi_0^j \cup \left\{\neg\psi \mid \psi \in \mathcal{L}_0^j \setminus \Phi_0^j\right\} \text{ is consistent} \right\}.$$

- For $j \in I_0$ and $\psi_j \in \mathcal{L}_0^j$, define

$$\left[\psi_j\right]^j := \left\{\omega_j \in \Omega_j \mid\vdash \omega_j \to \psi_j\right\}.$$

- For $j \in I_0$, denote by Σ_j the σ-field on Ω_j generated by the all the sets $\left[\psi_j\right]^j$, where $\psi_j \in \mathcal{L}_0^j$.
- Define

$$\Omega^* := \Pi_{j \in I_0} \Omega_j.$$

- Denote by Σ^* the product σ-field of the σ-fields Σ_j, $j \in I_0$, on Ω^*.
- For $i \in I$, define

$$\Omega_{-i} := \Pi_{j \in I_0 \setminus \{i\}} \Omega_j.$$

- For $i \in I$, denote by Σ_{-i} the product σ-field of the σ-fields Σ_j, $j \in I_0 \setminus \{i\}$, on Ω_{-i}.

Remark 3.7 *Let $j \in I_0$. By 4 of Proposition 3.3 and Corollary 3.1, for every $\omega_j \in \Omega_j$, there is a $\omega \in \Omega$ such that $\vdash \omega \to \omega_j$.*
For $\omega \in \Omega$, we have,

$$\vdash \omega \to \bigwedge_{\varphi_j \in \mathcal{L}_0^j \cap \Psi_\omega} \varphi_j$$

and by the definitions and the consistency of ω, we have $(\bigwedge_{\varphi_j \in \mathcal{L}_0^j \cap \Psi_\omega} \varphi_j) \in \Omega_j$. By definition of the $\omega_j \in \Omega_j$, two such formulas contradict each other, i.e. for $\omega_j \neq \omega_j' \in \Omega_j$ there is a $\varphi \in \mathcal{L}_0^i$ such that $\vdash \omega_j \to \varphi$ and $\vdash \omega_j' \to \neg\varphi$. Hence, for every $\omega \in \Omega$, there is exactly one $\omega_j \in \Omega_j$ such that $\vdash \omega \to \omega_j$. We denote this ω_j by $\omega(j)$.
Since Ω is nonempty and since $\omega(j) \in \Omega_j$, for $\omega \in \Omega$ and $j \in I_0$, we have that each Ω_j, for $j \in I_0$, is nonempty.

Lemma 3.6 *Let $j \in I_0$, $\omega_j \in \Omega_j$ and $\psi_j \in \mathcal{L}^j$. Then,*

$$\textit{either} \vdash \omega_j \to \psi_j \textit{ or } \vdash \omega_j \to \neg\psi_j,$$

but not both.

Proof The "either" follows by the consistency of ω_j, while the "or" follows by an easy induction on the formation of the formulas in $\mathcal{L}^j$, which is done in the same way as in the proof of 2 of Proposition 3.2. ■

Definition 3.25 For $i \in I_0$ and $E_i \in \Sigma_i$ define

$$E_i^* := \Pi_{j \in I_0} U_j,$$

where $U_j = \Omega_j$, for $j \neq i$ and $U_i = E_i$.
We have $E_i^* \in \Sigma^*$. Observe that $\mathcal{L}_0^i \cap \mathcal{L}_0^j = \emptyset$, for $i \neq j \in I_0$. Hence, for $\varphi_i \in \mathcal{L}_0^i$, the following is well-defined:

$$[\varphi_i]^* := ([\varphi_i]^i)^*$$

and by the definition and the consistency of the $\omega_i \in \Omega_i$, we have

$$\Omega^* \setminus [\varphi_i]^* = [\neg\varphi_i]^* \text{ and } [\varphi_i]^* \cap [\psi_i]^* = [\varphi_i \wedge \psi_i]^*,$$

for $\varphi_i, \psi_i \in \mathcal{L}_0^i$.
By starting with the i-formulas, for $i \in I_0$, we define now recursively:

$$[\neg\varphi]^* := \Omega^* \setminus [\varphi]^* \text{ and } [\varphi \wedge \psi]^* := [\varphi]^* \cap [\psi]^*,$$

for $\varphi, \psi \in \mathcal{L}_0$.
This is still well-defined for the finitary i -formulas, for $i \in I_0$, and it is well-defined for the other finitary formulas by the unique readability of finitary formulas as finite Boolean combinations of finitary formulas $\varphi \in \bigcup_{i \in I_0} \mathcal{L}_0^i$, which can be proved in the usual way (what we don't know at the moment is that logically equivalent finitary formulas define the same sets in Ω^*).

It is obvious that these sets form a field $\mathcal{F}^*$ on Ω^* which generates Σ^*.

Remark 3.8 • *Σ_0 is generated by the sets $[x]^0$, where $x \in X$.*

- *For every $m_0 \subseteq X$, there is exactly one $\omega_0 \in \Omega_0$ such that for every $x \in X$:*

$$\vdash \omega_0 \to x \quad \textit{iff} \quad x \in m_0.$$

Proof For $\omega_0 \in \Omega_0$ and $\varphi_0, \psi_0 \in \mathcal{L}_0^0$, we have by Conjunction, (A4) , and Modus Ponens that

$$\omega_0 \in [\varphi_0]^0 \cap [\psi_0]^0 \text{ implies } \omega_0 \in [\varphi_0 \wedge \psi_0]^0 ,$$

and by (A5) and 3 of Lemma 3.2

$$\omega_0 \in [\varphi_0 \wedge \psi_0]^0 \text{ implies } \omega_0 \in [\varphi_0]^0 \cap [\psi_0]^0 ,$$

and hence

$$[\varphi_0 \wedge \psi_0]^0 = [\varphi_0]^0 \cap [\psi_0]^0 .$$

By Lemma 3.6, we have

$$\Omega_0 \setminus [\varphi_0]^0 = [\neg\varphi_0]^0 ,$$

for $\varphi_0 \in \mathcal{L}_0^0$. It follows that the generators of Σ_0 are finite Boolean combinations of the sets $[x]^0$, where $x \in X$. Hence Σ_0 is generated by the sets $[x]^0$, where $x \in X$.

For the second point: Existence: Take a type space $\underline{M}$ on X for player set I consisting of one point (i.e. $M = \{m\}$, note that $\underline{M}$ is necessarily a Harsanyi type space) such that $v(m,x) = 1$ iff $x \in m_0$. By Corollary 3.1

$$\omega_0 := \bigwedge_{\varphi^0 \in \mathcal{L}_0^0 : (\underline{M},m) \models \varphi^0} \varphi^0$$

is consistent in the system H (and hence in the system P) and by the definition of the relation "$\models$", $\omega_0 \in \Omega_0$. By (A5) and Lemma 3.6, we have

$$\vdash \omega_0 \to x \text{ iff } x \in m_0.$$

Furthermore, we have by Lemma 3.6, Conjunction, (A4), and Modus Ponens,

$$\vdash \omega_0 \rightarrow \bigwedge_{x \in X : \vdash \omega_0 \rightarrow x} x \wedge \bigwedge_{y \in X : \vdash \omega_0 \rightarrow \neg y} \neg y.$$

Uniqueness: An easy induction on the formation of the formulas in $\mathcal{L}_0^0$ shows that for all $\varphi^0 \in \mathcal{L}_0^0$:

$$\vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \rightarrow \varphi^0 \ \text{ or } \ \vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \rightarrow \neg \varphi^0.$$

By 3 of Lemma 3.2 and the consistency of ω_0, this implies

$$\vdash \omega_0 \leftrightarrow \bigwedge_{x \in X : \vdash \omega_0 \rightarrow x} x \wedge \bigwedge_{y \in X : \vdash \omega_0 \rightarrow \neg y} \neg y.$$

■

Lemma 3.7 *Let $(\omega_j)_{j \in I_0} \in \Omega^*$ and let $\{\omega_j \mid j \in I_0\}$ be consistent. Then there is exactly one $\omega \in \Omega$ such that*

$$\vdash \omega \leftrightarrow \bigwedge_{j \in I_0} \omega_j,$$

and furthermore $\omega_j = \omega(j)$, for all $j \in I_0$. Conversely, for every $\omega \in \Omega$:

$$\vdash \omega \leftrightarrow \bigwedge_{j \in I_0} \omega(j).$$

Proof An easy induction on the formation of the formulas shows that for all $\varphi \in \mathcal{L}$: Either

$$\vdash \Big(\bigwedge_{j \in I_0} \omega_j \Big) \rightarrow \varphi \ \text{ or } \ \vdash \Big(\bigwedge_{j \in I_0} \omega_j \Big) \rightarrow \neg \varphi,$$

but not both. By the consistency of $\bigwedge_{j \in I_0} \omega_j$, Corollary 3.1 and 4 of Proposition 3.3, the rest is now obvious. ■

The above lemma shows that $h : \Omega \rightarrow \Omega^*$, defined by $h(\omega) := (\omega(j))_{j \in I_0}$ is one to one. We are now justified to identify with some abuse of notation $h(\Omega)$ with Ω.

Lemma 3.8 *Let* $\varphi \in \mathcal{L}_0$. *Then*:

- $$[\varphi]^* \cap h(\Omega) = h([\varphi]),$$

- $$h^{-1}([\varphi]^*) = [\varphi].$$

Proof The first assertion is true for i-formulas, according to the above two Lemmas 3.6 and 3.7. The rest of the first assertion follows from the definition of $[\cdot]^*$ and 6 and 7 of Proposition 3.2.

To the second assertion: Since

$$[\varphi] \subseteq h^{-1}([\varphi]^*) \quad \text{and} \quad \Omega \setminus [\varphi] = [\neg\varphi] \subseteq h^{-1}([\neg\varphi]^*),$$

and since

$$h^{-1}([\varphi]^*) \cap h^{-1}([\neg\varphi]^*) = h^{-1}([\varphi]^*) \cap h^{-1}(\Omega^* \setminus [\varphi]^*) = \emptyset,$$

we have

$$[\varphi] = h^{-1}([\varphi]^*).$$

■

Definition 3.26 • For $i \in I$, $\omega_i \in \Omega_i$, and $\psi \in \mathcal{L}_0$, define

$$T_i^*(\omega_i)([\psi]^*) := \sup\left\{\alpha \in [0,1] \cap \mathbb{Q} \mid \vdash \omega_i \to p_i^\alpha(\psi)\right\}.$$

- For $\omega_0 \in \Omega_0$ and $x \in X$, define

$$v^*(\omega_0, x) := \begin{cases} 1, & \text{if} \quad \omega_0 \in [x]^0, \\ 0, & \text{if} \quad \omega_0 \notin [x]^0, \end{cases}$$

  and

$$v^*(\omega_0, \top) := 1 \quad \text{in any case.}$$

Obviously, for every $x \in X$, $v^*(\cdot, x)$ is $\Sigma_0 - \mathrm{Pow}(\{0,1\})$-measurable, hence viewed as a function from Ω^* to $\{0,1\}$, it is $\Sigma^* - \mathrm{Pow}(\{0,1\})$-measurable.

Lemma 3.9 *For every $i \in I$ and $\omega_i \in \Omega_i$:*

$$T_i^* (\omega_i) (\cdot)$$

is well-defined and a countably additive measure on $\mathcal{F}^*$.
Furthermore, for every $i \in I$ and $\omega_i \in \Omega_i$:

$$T_i^* (\omega_i) (\Omega^*) = 1.$$

Proof Let $i \in I$. For $\omega_i \in \Omega_i$, choose $\omega \in \Omega$ such that $\omega(i) = \omega_i$. By the definitions and Lemma 3.6,

$$T_i^*(\omega_i)([\varphi]^*) = T_i'(\omega)([\varphi]),$$

for all $\varphi \in \mathcal{L}_0$. This implies in particular that $T_i^* (\omega_i) (\cdot)$ is non-negative.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* = [\psi]^*$, then $[\varphi] = [\psi]$, by Lemma 3.8. It follows that

$$T_i^*(\omega_i)([\varphi]^*) = T_i'(\omega)([\varphi]) = T_i'(\omega)([\psi]) = T_i^*(\omega_i)([\psi]^*),$$

hence $T_i^* (\omega_i) (\cdot)$ is well-defined.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* \cap [\psi]^* = \emptyset$, then $[\varphi] \cap [\psi] = \emptyset$, by Lemma 3.8. By the definition of $[\cdot]^*$, it follows that $[\varphi]^* \cup [\psi]^* = [\varphi \vee \psi]^*$. Hence,

$$\begin{aligned} T_i^*(\omega_i)([\varphi]^*) + T_i^*(\omega_i)([\psi]^*) &= T_i'(\omega)([\varphi]) + T_i'(\omega)([\psi]) \\ &= T_i'(\omega)([\varphi \vee \psi]) \\ &= T_i^*(\omega_i)([\varphi \vee \psi]^*) \\ &= T_i^*(\omega_i)([\varphi]^* \cup [\psi]^*). \end{aligned}$$

Therefore, $T_i^*(\omega_i) (\cdot)$ is additive on $\mathcal{F}^*$.

Since $\top$ and $p_i^1 (\top)$ are theorems, we have $\Omega_0 = [\top]^0$ and, by the definition of the $\omega_i \in \Omega_i$, $\vdash \omega_i \to p_i^1(\top)$. It follows that $T_i^* (\omega_i) (\Omega^*) = 1$.

Let $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and let $[\varphi_n]^* \downarrow \emptyset$. It follows by Lemma 3.8 that $[\varphi_n] \downarrow \emptyset$ and therefore

$$\lim_{n\to\infty} T_i^* (\omega_i) ([\varphi_n]^*) = \lim_{n\to\infty} T_i' (\omega) ([\varphi_n]) = 0.$$

■

Proposition 3.4 *1. For every $i \in I$ and $\omega_i \in \Omega_i$, there is a unique extension of $T_i^*(\omega_i)$ to a σ-additive probability measure on (Ω^*, Σ^*), which we denote also by $T_i^*(\omega_i)$.*

2. *For every $i \in I$ and $\omega_i \in \Omega_i$, this extension is a $\Sigma_i - \Sigma_\Delta^*$-measurable function from Ω_i to $\Delta(\Omega^*, \Sigma^*)$, the space of probability measures on (Ω^*, Σ^*), which is endowed with the σ-field Σ_Δ^* generated by the sets $\{\mu \in \Delta(\Omega^*, \Sigma^*) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma^*$ and rational $\alpha \in [0,1]$.*

3. $$\underline{\Omega}^* := \langle \Omega^*, \Sigma^*, (T_i^*)_{i\in I}, v^* \rangle$$

   is a product type space on X for player set I.

4. *For every $\psi \in \mathcal{L}_0$ and $(\omega_j)_{j\in I_0} \in \Omega^*$:*

   $$(\langle \Omega^*, \Sigma^*, (T_i^*)_{i\in I}, v^* \rangle, (\omega_j)_{j\in I_0}) \models \psi \quad \textit{iff} \quad (\omega_j)_{j\in I_0} \in [\psi]^*.$$

5. *In the H-system case, i.e. if the axiom $(I1)$ is added in case of $\aleph_\gamma = \aleph_0$, and if the inference rule Uncountable Introspection is added in case of $\aleph_\gamma > \aleph_0$, then*

   $$\langle \Omega^*, \Sigma^*, (T_i^*)_{i\in I}, v^* \rangle$$

   is a Harsanyi product type space on X for player set I.

Proof

1. Follows from Caratheodory's extension Theorem and Lemma 3.9.

2. Follows from Lemma 3.1: It suffices to show that for every $\psi \in \mathcal{L}_0$, rational $\alpha \in [0,1]$ and $i \in I$:

   $$\{\omega_i \mid T_i^*(\omega_i)([\psi]^*) \geq \alpha\} \in \Sigma_i.$$

   Chose a $\omega \in \Omega$ such that $\omega(i) = \omega_i$. By Lemma 3.6, we have

   $$\vdash \omega \to p_i^\alpha(\psi) \quad \text{iff} \quad \vdash \omega_i \to p_i^\alpha(\psi).$$

   By Lemma 3.3, it follows that

   $$T_i^*(\omega_i)([\psi]^*) \geq \alpha \quad \text{iff} \quad \vdash \omega_i \to p_i^\alpha(\psi).$$

   Hence,

   $$\{\omega_i \mid T_i^*(\omega_i)([\psi]^*) \geq \alpha\} = [p_i^\alpha(\psi)]^i \in \Sigma_i.$$

3. Follows from Remark 3.8 and 1 and 2 of this proposition.

4. We proceed by induction on the formation of the formulas in $\mathcal{L}_0$:

   - For $x \in X \cup \{\top\}$ and $(\omega_j)_{j \in I_0} \in \Omega^*$:

$$\begin{aligned}(\underline{\Omega}^*, (\omega_j)_{j\in I_0}) \models x \quad & \text{iff} \quad v^*(\omega_0, x) = 1 \\ & \text{iff} \quad \omega_0 \in [x]^0 \\ & \text{iff} \quad (\omega_j)_{j\in I_0} \in [x]^*.\end{aligned}$$

   - The induction steps "$\neg\varphi$", for $\varphi \in \mathcal{L}_0$, and "$\varphi \wedge \psi$", for φ and $\psi \in \mathcal{L}_0$, are clear by the definition of "$\models$" and "$[\cdot]^*$".
   - So it remains the step "$p_i^\alpha(\varphi)$", for $i \in I$, $\alpha \in [0,1] \cap \mathbb{Q}$, and $\varphi \in \mathcal{L}_0$. By the induction hypothesis, we have

$$[\varphi]^* = \left\{ (\omega'_j)_{j\in I_0} \in \underline{\Omega}^* \middle| \; (\underline{\Omega}^*, (\omega'_j)_{j\in I_0}) \models \varphi \right\}.$$

   It follows that, for $(\omega_j)_{j\in I_0} \in \underline{\Omega}^*$:

$$\begin{aligned}(\underline{\Omega}^*, (\omega_j)_{j\in I_0}) \models p_i^\alpha(\varphi) \quad & \text{iff} \quad T_i^*((\omega_j)_{j\in I_0})([\varphi]^*) \geq \alpha \\ & \text{iff} \quad \sup\left\{ \beta \in [0,1]\cap\mathbb{Q} \middle| \; \vdash \omega_i \to p_i^\beta(\varphi) \right\} \geq \alpha \\ & \text{iff} \quad \vdash \omega_i \to p_i^\alpha(\varphi) \\ & \text{iff} \quad \omega_i \in [p_i^\alpha(\varphi)]^i \\ & \text{iff} \quad (\omega_j)_{j\in I_0} \in [p_i^\alpha(\varphi)]^*,\end{aligned}$$

   where the third equivalence follows by the axioms (P7) and (P3)).

5. We have to show that, for $i \in I$ and $\omega_i \in \Omega_i$:

$$A \in \Sigma^* \;\text{ and }\; [T_i^*(\omega_i)]^* := \{(\omega'_j)_{j\in I_0} \mid T_i^*(\omega'_i) = T_i^*(\omega_i)\} \subseteq A$$

   imply

$$T_i^*(\omega_i)(A) = 1.$$

   By definition of $\mathcal{F}^*, \Sigma^*$, and $T_i^*(\omega_i)$, it suffices to show that $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{L}_0$ and $\bigcup_{n\in\mathbb{N}}[\varphi_n]^* \supseteq A$ imply

$$\lim_{l\to\infty} T_i^*(\omega_i)\left(\left[\bigvee_{n\leq l}\varphi_n\right]^*\right) = 1.$$

Choose $\omega \in \Omega$ with $\vdash \omega \rightarrow \omega_i$.
By an easy induction on the formation of the i-formulas $\varphi^i \in \mathcal{L}^i$, we have that for $\omega', \widetilde{\omega} \in \Omega$ and $i \in I : T_i(\omega') = T_i(\widetilde{\omega})$ implies

$$(\underline{\Omega}, \omega') \models \varphi^i \quad \text{iff} \quad (\underline{\Omega}, \widetilde{\omega}) \models \varphi^i.$$

Hence, by 4 of Proposition 3.3, $T_i(\omega') = T_i(\widetilde{\omega})$ implies

$$\vdash \omega' \rightarrow \omega_i \quad \text{iff} \quad \vdash \widetilde{\omega} \rightarrow \omega_i.$$

It follows that $h^{-1}\left([T_i^*(\omega_i)]^*\right) \supseteq [T_i(\omega)]$. Hence, by Lemma 3.8, $\bigcup_{n\in\mathbb{N}}[\varphi_n] \supseteq [T_i(\omega)]$. This implies

$$\lim_{l\rightarrow\infty} T_i^*(\omega_i)\left(\left[\bigvee_{n\leq l}\varphi_n\right]^*\right) = \lim_{l\rightarrow\infty} T_i(\omega)\left(\left[\bigvee_{n\leq l}\varphi_n\right]\right) = 1.$$

■

Theorem 3.3

$$h : \Omega \rightarrow \Omega^*$$

defined by

$$h(\omega) := (\omega(j))_{j\in I_0}, \quad \text{for } \omega \in \Omega,$$

is a type isomorphism from $\underline{\Omega}$ *to* $\underline{\Omega}^*$.

Proof By Lemma 3.7, h is one-to-one.

Let $(\omega_j)_{j\in I_0} \in \Omega^*$. By 4 of Proposition 3.4, Lemma 3.6 and the definition of $[\cdot]^*$, we have for $i \in I_0$ and $\varphi^i \in \mathcal{L}_0^i$:

$$\vdash \omega_i \rightarrow \varphi^i \quad \text{iff} \quad (\omega_j)_{j\in I_0} \in \left[\varphi^i\right]^* \quad \text{iff} \quad \langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \varphi^i.$$

By definition of "$\models$", this implies that $\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \omega_i$, for $i \in I_0$. Hence, we have

$$\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \bigwedge_{j\in I_0} \omega_j.$$

So $\bigwedge_{j\in I_0} \omega_j$ is consistent for every $(\omega_j)_{j\in I_0} \in \Omega^*$, hence, by Lemma 3.7, h is onto.

For $(\omega_j)_{j\in I_0} \in \Omega^*$ let $\omega \in \Omega$ such that $\vdash \omega \leftrightarrow \bigwedge_{j\in I_0} \omega_j$. Then it follows that

$$\langle \underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \models \omega.$$

Hence, h^{-1} is the type morphism from the proof of Theorem 3.2.

h is a type morphism:
For $\varphi \in \mathcal{L}_0$, we have by Lemma 3.8:

$$h^{-1}([\varphi]^*) = [\varphi] \in \Sigma.$$

Since $\mathcal{F}^*$ is a field that generates Σ^*, it follows that h is measurable.

Let $\omega \in \Omega$. For $x \in X \cup \{\top\}$, we have by Lemma 3.6, Lemma 3.7 and the definitions

$$v^*(h(\omega), x) = v^*((\omega(j))_{j\in I_0}, x) = v^*(\omega(0), x),$$

and

$$\begin{aligned} v^*(\omega(0), x) = 1 \quad & \text{iff} \quad \vdash \omega(0) \to x \\ & \text{iff} \quad \vdash \omega \to x \\ & \text{iff} \quad v(\omega, x) = 1. \end{aligned}$$

By Caratheodory's extension Theorem, it is enough to show that for $\omega \in \Omega$, $i \in I$ and $\varphi \in \mathcal{L}_0$:

$$T_i^* \left(\omega(j)_{j\in I_0}\right) ([\varphi]^*) = T_i\left(\omega\right) \left(h^{-1}([\varphi]^*)\right).$$

Since $h^{-1}([\varphi]^*) = [\varphi]$, this is clear by Lemma 3.6, Lemma 3.7 and the definitions. ■

That the canonical space is a product space implies the following: For states $u_i \in \Omega$, where $i \in I_0$, there is one state $u \in \Omega$ such that: $v\left(u_0, x\right) = v\left(u, x\right)$, for all $x \in X$, and $T_i\left(u_i\right) = T_i\left(u\right)$, for $i \in I$. This fact is reflected by the axioms in the following way: There is no axiom and also no inference rule that relates the believes of one player with the beliefs of other players or with nature. So, whatever a player in a state of the world believes about other players or nature might be wrong (as long as this is nothing tautological, of course). This is not the case for the canonical knowledge space and the corresponding $S5$ axiom system, where there is an axiom "$k_i\varphi \to \varphi$". So, if, for example, $\varphi = k_j x$ and if $k_i\varphi$ is true in a state, then the fact that i knows that j knows x forces j to know x, and this forces x to be true in that state.

3.6 Beliefs Completeness

The aim of this section is to prove the following - somewhat surprising - theorem of appealing measure-theoretic taste, which, in some topological cases, was proved by Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993) and Mertens, Sorin and Zamir (1994). The general measure-theoretic case proved here is original. The theorem says that, in the P-system case, the component space of each player is - up to isomorphism of measurable spaces - the space of probability measures on the space of states of the world, and in the H-system case, for each player $i \in I$, the component space of i is - up to isomorphism of measurable spaces - the space of probability measures on Ω_{-i}.

Theorem 3.4
- *In the P-system, let $\mu \in \Delta(\Omega^*, \Sigma^*)$. For every $i \in I$, there is exactly one $\omega_i \in \Omega_i$ such that $T_i^*(\omega_i) = \mu$. Furthermore, for every $i \in I$,*

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is an isomorphism of the measurable spaces (Ω_i, Σ_i) and $(\Delta(\Omega^*, \Sigma^*), \Sigma_\Delta^*)$.

- *In the H-system, let $i \in I$ and $\mu_i \in \Delta(\Omega_{-i}, \Sigma_{-i})$. Then there is exactly one $\omega_i \in \Omega_i$ such that the marginal of $T_i^*(\omega_i)$ on Ω_{-i} is μ_i. Furthermore, for every $i \in I$,*

$$\mathrm{marg}_{\Omega_{-i}} \circ T_i^* : \Omega_i \to \Delta(\Omega_{-i}, \Sigma_{-i})$$

is an isomorphism of the measurable spaces (Ω_i, Σ_i) and $(\Delta(\Omega_{-i}, \Sigma_{-i}), (\Sigma_{-i})_\Delta)$.

Proof We prove the P-system case and sketch the differences for the proof of the H-system case. For the P-system case:

Let $i \in I$. For $\omega_i \in \Omega_i$ define

$$\varphi_i(\omega_i) := \left(\bigwedge_{\chi \in \mathcal{L}_0,\ \alpha \in [0,1] \cap \mathbb{Q},\ \text{s.t. } \vdash \omega_i \to p_i^\alpha(\chi)} p_i^\alpha(\chi) \right) \wedge \left(\bigwedge_{\psi \in \mathcal{L}_0,\ \beta \in [0,1] \cap \mathbb{Q},\ \text{s.t. } \vdash \omega_i \to \neg p_i^\beta(\psi)} \neg p_i^\beta(\psi) \right).$$

An easy induction on the formation of the i-formulas shows that for every i-formula $\chi_i \in \mathcal{L}^i$:

$$\text{Either} \quad \vdash \varphi_i(\omega_i) \to \chi^i \quad \text{or} \quad \vdash \varphi_i(\omega_i) \to \neg\chi^i.$$

Since $\vdash \omega_i \to \varphi_i(\omega_i)$, by the consistency of ω_i, it follows that $\vdash \omega_i \leftrightarrow \varphi_i(\omega_i)$. Hence, $\omega_i' \neq \omega_i'' \in \Omega_i$ implies that $\varphi_i(\omega_i') \neq \varphi_i(\omega_i'')$. Therefore, by the definitions of $T_i^*(\omega_i')$ and $T_i^*(\omega_i'')$, we have $T_i^*(\omega_i')([\varphi]^*) \neq T_i^*(\omega_i'')([\varphi]^*)$, for some $\varphi \in \mathcal{L}_0$. We conclude that

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is one-to-one.

It follows - and in the same manner, also in the H-system case - that for $i \in I$ and $\omega_i \in \Omega_i : [T_i^*(\omega_i)]^* = \{\omega_i\} \times \Omega_{-i}$. Hence, in the H-system case, the introspection property of the canonical Harsanyi type space on X for player set I implies that

$$\mathrm{marg}_{\Omega_i} \circ T_i^*(\omega_i) = \delta_{\omega_i},$$

for $i \in I$ and $\omega_i \in \Omega_i$.

By 2 of Proposition 3.4, T_i^* is measurable, for $i \in I$.

Let $\mu \in \Delta(\Omega^*, \Sigma^*)$ and fix $i \in I$. Consider the following set of formulas:

$$\begin{aligned} \Phi^\mu \;:=\; & \{p_i^\alpha(\varphi) \mid \varphi \in \mathcal{L}_0,\ \alpha \in [0,1] \cap \mathbb{Q} \text{ s.t. } \mu([\varphi]^*) \geq \alpha\} \\ & \cup \{\neg p_i^\beta(\psi) \mid \psi \in \mathcal{L}_0,\ \beta \in [0,1] \cap \mathbb{Q} \text{ s.t. } \mu([\psi]^*) < \beta\}. \end{aligned}$$

If this set of formulas is consistent in the system P, then by Corollary 3.1, there is a $\omega \in \Omega$, where Ω is the Ω belonging to the system P, such that

$$(\underline{\Omega}, \omega) \models \Phi^\mu.$$

But then, from 4 of Proposition 3.3, the definition of $\omega(i)$, the fact that $\vdash \omega \to \omega(i)$, and the consistency of ω, it follows that $\vdash \omega(i) \to \chi$, for all $\chi \in \Phi^\mu$. The definition of $T_i^*(\omega(i))$ implies then that

$$T_i^*(\omega(i))([\varphi]^*) = \mu([\varphi]^*),$$

for all $\varphi \in \mathcal{L}_0$. Hence, since $T_i^*(\omega(i))$ and μ are σ-additive probability measures on Σ^* that coincide on the field $\mathcal{F}^*$, and since $\mathcal{F}^*$ generates Σ^*, Caratheodory's extension Theorem implies then that

$$T_i^*(\omega(i)) = \mu.$$

In the following, we show that Φ^μ is consistent in the system P.
Let $u \notin \Omega_i$. Define

$$\begin{array}{lll} \Omega_i^\mu & := & \Omega_i \cup \{u\}, \\ \Omega_j^\mu & := & \Omega_j, \quad \text{for } j \in I_0 \setminus \{i\}, \\ \Sigma_i^\mu & := & \Sigma_i \cup \{E \cup \{u\} \mid E \in \Sigma_i\}, \\ \Sigma_j^\mu & := & \Sigma_j, \quad \text{for } j \in I_0 \setminus \{i\}, \\ \Omega^\mu & := & \Pi_{j \in I_0} \Omega_j^\mu, \\ \Sigma^\mu & := & \text{the product } \sigma\text{-field of the } \Sigma_j^\mu,\ j \in I_0. \end{array}$$

Note that Σ_i^μ is a σ-field, $\Sigma^* \subseteq \Sigma^\mu$, and $E \cap \Omega^* \in \Sigma^*$, for $E \in \Sigma^\mu$. Note furthermore that, since each Ω_j, for $j \in I_0$, is nonempty, each Ω_j^μ, for $j \in I_0$, is nonempty.

For $j \in I_0 \setminus \{i\}$, chose $u_j \in \Omega_j$ and set $u_i := u$ and define

$$\overline{u} := (u_j)_{j \in I_0}.$$

For $j \in I$, $\omega_j \in \Omega_j^\mu$ and $E \in \Sigma^\mu$ define

$$\begin{array}{lll} T_j^\mu(\omega_j)(E) & := & T_j^*(\omega_j)(E \cap \Omega^*), \quad \text{if} \quad j \neq i \text{ or if } i = j \text{ and } \omega_i \neq u, \\ T_i^\mu(\omega_i)(E) & := & \mu(E \cap \Omega^*), \qquad \text{if} \quad \omega_i = u. \end{array}$$

By this definition, $T_j^\mu(\omega_j)$ is a σ-additive probability measure on (Ω^μ, Σ^μ), for $j \in I$ and $\omega_j \in \Omega_j^\mu$.

For $x \in X$ and $\omega_0 \in \Omega_0^\mu$ define:

$$\begin{array}{lll} v^\mu(\omega_0, x) & := & v^*(\omega_0, x), \\ v^\mu(\omega_0, \top) & := & 1, \qquad \text{in any case.} \end{array}$$

By this definition, it is clear that $v^\mu(\cdot, x)$ is $\Sigma_0 - \mathrm{Pow}(\{0,1\})$-measurable, for $x \in X \cup \{\top\}$.

Let $E \in \Sigma^\mu$, $j \in I \setminus \{i\}$, $\alpha \in [0,1] \cap \mathbb{Q}$, and $b^\alpha(E) := \{\nu \in \Delta(\Omega^\mu, \Sigma^\mu) \mid \nu(E) \geq \alpha\}$. Then, by the definitions:

$$(T_j^\mu)^{-1}(b^\alpha(E)) = (T_j^*)^{-1}(b^\alpha(E \cap \Omega^*)) \in \Sigma_j.$$

Hence,

$$T_j^\mu : \Omega_j^\mu \to \Delta(\Omega^\mu, \Sigma^\mu)$$

is $\Sigma_j^\mu - \Sigma_\Delta^\mu$-measurable, for $j \in I \setminus \{i\}$.

Note that

$$T_i^{\mu}(u)(E) = \mu(E \cap \Omega^*) = T_i^{\mu}(u)(E \cap \Omega^*),$$

for $E \in \Sigma^{\mu}$. So, for all $j \in I$, $\omega_j \in \Omega_j^{\mu}$ and $E \in \Sigma^{\mu}$:

$$T_j^{\mu}(\omega_j)(E) = T_j^{\mu}(\omega_j)(E \cap \Omega^*).$$

By definition, we have

$$(T_i^{\mu})^{-1}(b^{\alpha}(E)) = \begin{cases} (T_i^*)^{-1}(b^{\alpha}(E \cap \Omega^*)) \in \Sigma_i \subseteq \Sigma_i^{\mu}, & \text{if} \quad \mu(E \cap \Omega^*) < \alpha, \\ \{u\} \cup (T_i^*)^{-1}(b^{\alpha}(E \cap \Omega^*)) \in \Sigma_i^{\mu}, & \text{if} \quad \mu(E \cap \Omega^*) \geq \alpha. \end{cases}$$

Hence,

$$T_i^{\mu} : \Omega_i^{\mu} \to \Delta(\Omega^{\mu}, \Sigma^{\mu})$$

is $\Sigma_i^{\mu} - \Sigma_{\Delta}^{\mu}$-measurable.

Now, we have proved that

$$\underline{\Omega}^{\mu} := \left\langle \Omega^{\mu}, \Sigma^{\mu}, (T_j^{\mu})_{j \in I}, v^{\mu} \right\rangle$$

is a product type space on X for player set I.

Next, we show by induction on the formation of the formulas in $\varphi \in \mathcal{L}_0$ that, for $\omega \in \Omega^*$:

$$(\underline{\Omega}^{\mu}, \omega) \models \varphi \;\text{ iff }\; (\underline{\Omega}^*, \omega) \models \varphi.$$

An equivalent statement is:

$$[\varphi]^* = [\varphi]^{\mu} \cap \Omega^*, \quad \text{where} \;\; [\varphi]^{\mu} := \{\omega \in \Omega^{\mu} \mid (\underline{\Omega}^{\mu}, \omega) \models \varphi\}.$$

(Recall that, by 4 of Proposition 3.4, $[\varphi]^{\underline{\Omega}^*} = [\varphi]^*$, for $\varphi \in \mathcal{L}_0$.)

Since

$$T_i^{\mu}(u)([\varphi]^{\mu}) = \mu([\varphi]^{\mu} \cap \Omega^*),$$

it follows then that

$$(\underline{\Omega}^{\mu}, \overline{u}) \models \Phi^{\mu}.$$

Let $\omega \in \Omega^*$. By definition, we have for $x \in X \cup \{\top\}$:

$$\begin{array}{rll}(\underline{\Omega}^\mu, \omega) \models x & \text{iff} & v^\mu(\omega(0), x) = 1 \\ & \text{iff} & v^*(\omega(0), x) = 1 \\ & \text{iff} & (\underline{\Omega}^*, \omega) \models x.\end{array}$$

The steps "$\wedge$" and "$\neg$" are trivial.

Let $j \in I$. For $\varphi \in \mathcal{L}_0$, we have by the induction hypothesis $[\varphi]^* = [\varphi]^\mu \cap \Omega^*$, and hence, for $\alpha \in [0,1] \cap \mathbb{Q}$:

$$\begin{array}{rcl}[p_j^\alpha(\varphi)]^* & = & \{\omega \in \Omega^* \mid T_j^*(\omega(j))([\varphi]^*) \geq \alpha\} \\ & = & \{\omega \in \Omega^* \mid T_j^*(\omega(j))([\varphi]^\mu \cap \Omega^*) \geq \alpha\} \\ & = & \{\omega \in \Omega^* \mid T_j^\mu(\omega(j))([\varphi]^\mu) \geq \alpha\} \\ & = & \{\omega \in \Omega^\mu \mid T_j^\mu(\omega(j))([\varphi]^\mu) \geq \alpha\} \cap \Omega^* \\ & = & [p_j^\alpha(\varphi)]^\mu \cap \Omega^*.\end{array}$$

Now, we have shown that

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is onto, for $i \in I$.

For $i \in I$, it remains to prove that $(T_i^*)^{-1}$ is measurable:
The sets $[p_i^\alpha(\varphi)]^i$, where $\varphi \in \mathcal{L}_0$ and $\alpha \in [0,1] \cap \mathbb{Q}$, generate the σ-field Σ_i. So it is enough to show that $T_i^*([p_i^\alpha(\varphi)]^i)$ is a measurable set in $\Delta(\Omega^*, \Sigma^*)$. But we have

$$T_i^*([p_i^\alpha(\varphi)]^i) = \{\nu \in \Delta(\Omega^*, \Sigma^*) \mid \nu([\varphi]^*) \geq \alpha\} \in \Sigma_\Delta^*.$$

The the H-system case is proved similarly, but for the "onto part" one starts with $\mu \in \Delta(\Omega_{-i}, \Sigma_{-i})$ and defines $T^\mu := \delta_u \times \mu$, where "$\times$" denotes here the product of measures and δ_u is the delta measure at $u \in \Omega_i^\mu$.

And for the "one-to-one" part, one uses the following fact, which holds in general for σ-additive probability measures on product spaces (endowed with the product σ-field): If $\nu \in \Delta(\Omega^*, \Sigma^*)$ and for $\omega_i \in \Omega_i$: $\text{marg}_{\Omega_i}(\nu) = \delta_{\omega_i}$, then $\nu = \delta_{\omega_i} \times \text{marg}_{\Omega_{-i}}(\nu)$. This fact together with Caratheodory's extension Theorem is then used to show by induction on the formation of the i-formulas that $\text{marg}_{\Omega_{-i}} \circ T_i^*$ is one-to-one. The measurability of $\text{marg}_{\Omega_{-i}} \circ T_i^*$ is straightforward, while, using Lemma 3.1, the measurability of $\left(\text{marg}_{\Omega_{-i}} \circ T_i^*\right)^{-1}$ follows in a similar fashion as the measurability of $\left(\text{marg}_{C_{-i}^{\aleph_0}} \circ T_i^{\aleph_0}\right)^{-1}$ in third point of Theorem 2.6 in the preceding chapter. ■

Chapter 4

Conditional Possibility Structures

4.1 Introduction

In the general introduction, we gave a short exposition of the nonexistence result of Brandenburger and Keisler (1999), who showed that, given at least two players and two states of nature, there is no first-order definably beliefs complete possibility structure. We also mentioned two possible ways of modifying the framework that could help to get a positive beliefs completeness (resp. universality) result. The first one, to impose topological restrictions, we will follow in Section 4.3, the second one, considering another language than the one used by Brandenburger and Keisler (1999), we will follow in Section 4.4.

We start with a set I of at least two players and, for each $i \in I$, a nonempty set S_i, called *"i's" space of states of nature,* and $\mathcal{B}_i$, the set of *observable events in* S_i, i.e. a fixed set of nonempty subsets of S_i such that $S_i \in \mathcal{B}_i$. The interpretation is that player i is uncertain about the "true" state of nature $s_i \in S_i$ and $\mathcal{B}_i$ stands for a fixed collection of possible observations about the true state of nature player i might make. For example, given a perfect information game, S_i could be the space of strategy profiles (of the players $j \in I \setminus \{i\}$) and each $B_i \in \mathcal{B}_i$ could be the set of those strategy profiles that are consistent with some partial history of play (given some strategy of i).

But again, as in the previous chapters, being in a strategic environment, apart from having beliefs on "his" space of states of nature, player i must also have beliefs on the other players' beliefs on "their" spaces of states of nature and beliefs on other players' beliefs on his beliefs on "his" space of states of nature and so on. So, S_i is only the primary source of uncertainty for player i.

Similarly to the case of Harsanyi type spaces or the case of Kripke structures, the different levels of the players' hierarchies of beliefs are described implicitly by introducing, for each $i \in I$, a nonempty set U^i, *i's component space* (i.e. the set of "*i's types*"), and mappings $\rho^i : U^i \to \prod_{B_i \in \mathcal{B}_i} \mathrm{Pow}_\emptyset \left(B_i \times \Pi_{j \neq i} U^j\right)$, for $i \in I$, where $\mathrm{Pow}_\emptyset (M)$ denotes the set of nonempty subsets of the nonempty set M. The interpretation is that $(s_i, (u^j)_{j \neq i}) \in \left(\rho^i \left(u^i\right)\right)_{B_i}$ iff being of type u^i and should he observe that the true state of nature is in B_i, i would believe it possible that s_i is the true state of nature and player j's type is u^j, for $j \in I \setminus \{i\}$. The fact that the sets on which the players condition their beliefs are of the form $B_i \times \Pi_{j \neq i} U^j$, expresses the view that only observations about nature are possible, while there is no way to "look into other players' minds", i.e. there is no way to receive certain information about other players' beliefs, and hence there is no way to exclude for sure any type of another player.

There is no explicit time in this framework. However, in the interpretation, we have in mind that more informative observations about nature are made, should they be made at all, later than less informative ones. Otherwise the less informative ones would be irrelevant and could be ignored. Furthermore, if $C_i \in \mathcal{B}_i$ is observed after $D_i \in \mathcal{B}_i$, should C_i be observed at all, then we assume that $C_i \subseteq D_i$. Otherwise C_i could be replaced by $C_i \cap D_i$, which is nonempty since C_i and D_i have some state in common that would be the "true" state of nature if C_i should be observed after D_i. Also, just like a strategy in an extensive-form game is a plan to act at different information sets, a conditional belief, i.e. a conditional possibility set, is a plan to hold beliefs at different information sets. (See Battigalli and Siniscalchi (1999a).)

After some very basic definitions, we define the two versions of how the players would revise their beliefs should they make some new observation about the true state of nature.

In the A-version, should a player make a new observation, i.e. becoming sure that the true state of nature is in $C_i \subseteq D_i$, for some $C_i, D_i \in \mathcal{B}_i$, where D_i stands for an observation "made before", he would keep his beliefs "as long as possible", while taking the new information into account: $A_{C_i} = A_{D_i} \cap \left(C_i \times \Pi_{j \neq i} U^j\right)$, if $A_{D_i} \cap \left(C_i \times \Pi_{j \neq i} U^j\right) \neq \emptyset$, where A_{C_i}, resp. A_{D_i}, stands for the player's belief, i.e. his possibility set, should he observe $C_i \subseteq S_i$, resp. $D_i \subseteq S_i$, for $C_i, D_i \in \mathcal{B}_i$.

In the B-version, a player is allowed to change his beliefs in a more free manner. He would just be forced to keep his beliefs as long as they would be "completely confirmed", i.e. if $A_{D_i} \subseteq C_i \times \Pi_{j \neq i} U^j$, then $A_{C_i} = A_{D_i}$, and he could change his beliefs as soon as he would have "a good reason", i.e. if there is at least one state $(s_i, (u^j)_{j \neq i}) \in A_{D_i} \setminus C_i \times \Pi_{j \neq i} U^j$. The "philosophy" here is that, although there are no probabilities in this framework, the player could consider this state "much more likely" than the other states he believes to be possible in case he observes D_i. The need to exclude this state would be such a "surprise" that he wishes to change his beliefs completely in case he observes C_i.

An element $\left(A_{B_i}\right)_{B_i \in \mathcal{B}_i} \in \prod_{B_i \in \mathcal{B}_i} \mathrm{Pow}_\emptyset \left(B_i \times \Pi_{j \neq i} U^j\right)$ that satisfies the A-version (resp. B-version) of conditionality is said to be A-*nice* (resp. B-*nice*). We define $\mathrm{Pow}_\emptyset^{\mathcal{B}_i} \left(S_i \times \Pi_{j \neq i} U^j\right)$ as the set of A-nice (resp. B-nice) elements of $\prod_{B_i \in \mathcal{B}_i} \mathrm{Pow}_\emptyset \left(B_i \times \Pi_{j \neq i} U^j\right)$ in the A-version of conditionality (resp. in the B-

version of conditionality), and we define a *conditional possibility structure on* $(S_i, \mathcal{B}_i)_{i\in I}$ as a tuple $\left\langle (U^i)_{i\in I}, (\rho^i)_{i\in I} \right\rangle$, where each U^i is a nonempty set and each ρ^i is a mapping from U^i to $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$. The conditional possibility structure is *beliefs complete* if each ρ^i is onto. An element of $\mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$ is called a *conditional i-event* (or a (*nonempty*) *conditional subset of* $S_i \times \Pi_{j\neq i} U^j$). A conditional i-event that is the image of a $u^i \in U^i$ under the mapping ρ^i is called a *conditional possibility set for* i (or *the conditional possibility set of* u^i).

In section 4.3 we follow the topological avenue to produce a beliefs completeness and universality result. There, we make the additional assumption that all the S_i, for $i \in I$, and all the U^i, for $i \in I$, are compact and Hausdorff, and all the $B_i \in \mathcal{B}_i$, for $i \in I$, are closed *and* open (clopen). We denote the space of nonempty compact subsets of $B_i \times \Pi_{j\neq i} U^j$, where $B_i \times \Pi_{j\neq i} U^j$ is endowed with the product topology, by $\mathcal{V}\left(B_i \times \Pi_{j\neq i} U^j\right)$ and endow this space with the *Vietoris topology*. Then, we endow $\prod_{B_i\in\mathcal{B}_i} \mathcal{V}\left(B_i \times \Pi_{j\neq i} U^j\right)$ with the product topology and define $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$ to be the set of A-nice (resp. B-nice) elements of $\prod_{B_i\in\mathcal{B}_i} \mathcal{V}\left(B_i \times \Pi_{j\neq i} U^j\right)$ in the A-version of conditionality (resp. in the B-version of conditionality), and endow this space with the topology inherited from the topology of $\prod_{B_i\in\mathcal{B}_i} \mathcal{V}\left(B_i \times \Pi_{j\neq i} U^j\right)$. We require in addition that $\rho^i : U^i \to \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$ and that ρ^i is continuous, for $i \in I$.

We also define structure preserving maps (*morphisms*) between topological conditional possibility structures $\left\langle (V^i)_{i\in I}, (\sigma^i)_{i\in I} \right\rangle$ on $(S_i, \mathcal{B}_i)_{i\in I}$ and $\left\langle (U^i)_{i\in I}, (\rho^i)_{i\in I} \right\rangle$ on $(S_i, \mathcal{B}_i)_{i\in I}$ as an I-tuple $(v^i)_{i\in I}$ of continuous maps $v^i : V^i \to U^i$, for $i \in I$, such that

$$\mathcal{V}^{\mathcal{B}_i}\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} v^j\right) \circ \sigma^i = \rho^i \circ v^i,$$

for every $i \in I$, where $\mathcal{V}^{\mathcal{B}_i}\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} v^j\right)$ is the mapping from $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} V^j\right)$ to $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$ that is induced by $\mathrm{id}_{S_i} \times \Pi_{j\neq i} v^j$.

Under these topological assumptions we construct, by a projective limit construction, a *universal topological conditional possibility structure* $\langle (T^i)_{i\in I}, (\delta^i)_{i\in I} \rangle$ on $(S_i, \mathcal{B}_i)_{i\in I}$ such that for every topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$ there is a unique morphism to the universal topological conditional possibility structure (Theorem 4.1 and Corollary 4.1). The universal topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$ is unique up to isomorphism of topological conditional possibility structures on $(S_i, \mathcal{B}_i)_{i\in I}$ and each δ^i, for $i \in I$, is a homeomorphism. This implies that the universal topological conditional possibility structure is also beliefs complete. (Again, Theorem 4.1 and Corollary 4.1). In the special unconditional, i.e. $\mathcal{B}_i = \{S_i\}$, for $i \in I$, two player case $I = \{a, b\}$, where both players are uncertain about the same space of states of nature (i.e. $S_a = S_b$), the results of Theorem 4.1 and Corollary 4.1 were already obtained by Mariotti and Piccione (2000).

In Section 4.4 we follow the other avenue to produce a (definably) beliefs completeness result, namely to use a language other than the one used by Brandenburger and Keisler (1999), but similar to it. Here, we restrict ourselves to a finite player set N and a finite set of states of nature S_i, for each player $i \in N$.

We define inductively structures

$$\mathcal{A}_n = \left\langle (S_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \left(T_k^i\right)_{i\in N, -1\leq k\leq n}, \left(R_l^{i,n}\right)_{i\in N, 0\leq l\leq n}, \left(q_{k,l}^i\right)_{i\in N, -1\leq k<l\leq n} \right\rangle,$$

for $n \in \mathbb{N} \cup \{-1\}$. On level 0, we define (up to isomorphism) T_0^i, the component space of player i, to be $\mathrm{Pow}_\emptyset^{\mathcal{B}_i}(S_i)$, i.e. the set of conditional i-events in S_i. On higher levels $n \geq 1$, we define each player's component space to be $T_n^i := \mathrm{Pow}_\emptyset^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T_{n-1}^j\right)$, i.e. the set of conditional i-events in the product space of S_i and the other players' component spaces on the level $n-1$. $q_{k,l}^i$ is a projection from T_l^i onto T_k^i. In $\mathcal{A}_n$, the $t_k^i \in T_k^i$, for $k < n$, serve as names for the subsets of T_n^i that are of the form $\left(q_{k,n}^i\right)^{-1}(\{t_k^i\})$. The relations $R_n^{i,n}$ are defined in a way such that the conditional possibility set of t_n^i, i.e. the t_n^i-section of $R_n^{i,n}$, is the inverse image of t_n^i under $\mathcal{V}^{\mathcal{B}_i}\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q_{n-1,n}^j\right)$.

$\left\langle (T_n^i)_{i\in N}, (R_n^{i,n})_{i\in N}\right\rangle$ is the conditional possibility structure of level n on $(S_i, \mathcal{B}_i)_{i\in N}$, where the usual mapping ρ^i is replaced by it's "graph" $R_n^{i,n}$, S_i stands for player i's space of states of nature and $\mathcal{B}_i$ for the observable events in S_i, and $(T_l^i)_{i\in N}$ and $\left(R_l^{i,n}\right)_{i\in N}$, for $0 \leq l < n$, stand for (the inverse images of) the conditional possibility structures of lower levels $l < n$ in $\left\langle (T_n^i)_{i\in N}, (R_n^{i,n})_{i\in N}\right\rangle$. As mentioned above, $q_{k,l}^i$ is a projection from T_l^i onto T_k^i. (The T_{-1}^i are singletons and just needed for a technical purpose.)

The counterpart of $\mathcal{A}_n$ on the syntactic side is $\mathcal{L}_n$, the *language induced by* $\mathcal{A}_n$, for $n \in \mathbb{N} \cup \{-1\}$, which is defined in a similar way as the language induced by a possibility structure defined by Brandenburger and Keisler. However, $\mathcal{L}_n$, for $n \in \mathbb{N}$, is not a first-order language: The language $\mathcal{L}_n$ contains the language $\mathcal{L}_{n-1}$ and is a kind of mixture between first-order and second-order logic, where the $\mathcal{L}_{n-1}$-part of $\mathcal{L}_n$ (that "talks" about the inverse image of $\mathcal{A}_{n-1}$ in $\mathcal{A}_n$) is the "second-order part" of $\mathcal{L}_n$. The definition of $\mathcal{L}_n$ is done in a way such that all subsets of $S_i \times \Pi_{j\neq i} T_n^j$, for $i \in N$, are $\mathcal{L}_n$-definable, i.e. describable by some $\mathcal{L}_n$-formula.

It is easy to see that the Brandenburger-Keisler Impossibility Theorem implies that $\mathcal{A}_n$ cannot be $\mathcal{L}_n$-definably beliefs complete, but at least we manage to get $\mathcal{L}_{n-1}$-definably beliefs completeness of $\mathcal{A}_n$. Since we will show that the subset defined by a $\mathcal{L}_{n-1}$-formula in $S_i \times \Pi_{j\neq i} T_n^j$ is the inverse image of the subset defined by the same formula in $S_i \times \Pi_{j\neq i} T_{n-1}^j$, the definition of the $t_n^i \in T_n^i$ and the fact that all subsets of $S_i \times \Pi_{j\neq i} T_{n-1}^j$ are $\mathcal{L}_{n-1}$-definable imply that the conditional possibility sets of player i in $\mathcal{A}_n$ are exactly the $\mathcal{L}_{n-1}$-definable conditional i-events.

The structure $\mathcal{A}$ we aim to construct, is the limit structure

$$\left\langle (S_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \left(T^i\right)_{i\in N}, \left(T^i_m\right)_{i\in N,-1\leq m}, \left(R^i\right)_{i\in N}, \left(R^i_n\right)_{i\in N,0\leq n}, \left(p^i_m\right)_{i\in N,-1\leq m}\right\rangle .$$

The structure $\mathcal{A}$ can be viewed as the "projective limit" of the $\mathcal{A}_n$, $n \in \mathbb{N}\cup\{-1\}$. Player i's component space T^i is the projective limit of the T^i_m, $m \in \mathbb{N}\cup\{-1\}$, and p^i_m is the projection from T^i to T^i_m, for $m \in \mathbb{N}\cup\{-1\}$. The elements $t^i_m \in T^i_m$ serve as names for their inverse images under p^i_m. R^i_n, for $n \in \mathbb{N}$, is defined to be the inverse image of $R^{i,n}_n$ under $p^i_n \times \mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n$, and R^i, the "projective limit" of the $R^{i,n}_n$, $n \in \mathbb{N}$, is defined as $R^i := \bigcap_{n\in\mathbb{N}} R^i_n$. $\left\langle (T^i)_{i\in N}, (R^i)_{i\in N}\right\rangle$ is a conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in N}$ and the $(T^i_m)_{i\in N,0\leq m}, (R^i_m)_{i\in N,0\leq m}$ stand for the inverse images of the finite-step conditional possibility structures $\left\langle (T^i_m)_{i\in N}, (R^{i,m}_m)_{i\in N}\right\rangle$ in $\left\langle (T^i)_{i\in N}, (R^i)_{i\in N}\right\rangle$.

The syntactic counterpart of $\mathcal{A}$ is the limit language $\mathcal{L}$, the *"neocompact" language induced by* $\mathcal{A}$, which is defined in the spirit of the neocompact language invented by Keisler (1998), but as the $\mathcal{L}_n$ (and unlike Keisler's original definition), it is a kind of mixture between first and second-order logic, where $\mathcal{L}$ is defined in such a way that the union of the $\mathcal{L}_n$, $n \in \mathbb{N}\cup\{-1\}$, is the "second-order part" of $\mathcal{L}$. The $\mathcal{L}_n$-part of $\mathcal{L}$ "talks"about the inverse image of $\mathcal{A}_n$ in $\mathcal{A}$. Similarly to the case of it's semantic counterpart, $\mathcal{L}$ can be viewed as the "projective limit" of the $\mathcal{L}_n$, $n \in \mathbb{N}\cup\{-1\}$.

Up to now, in Section 4.4 no topological assumptions on the structures are made (except in some proofs). However, for the proof of the main result of Section 4.4, we endow each of the spaces S_i, for $i \in N$, and T^i_m, for $i \in N$ and $m \in \mathbb{N}\cup\{-1\}$, with the discrete topology and the T^i, for $i \in N$, with the projective limit topology. This allows us to apply Theorem 4.1. Essentially, by proving that the possibility correspondence pos_i, that sends $t^i \in T^i$ to the t^i-section of R^i, is equal to the δ^i of Theorem 4.1, and that the $\mathcal{L}$-definable nonempty subsets of finite products of sorts of the structure $\mathcal{A}$ are closed if considered to be endowed with the corresponding product topology, we show that pos_i is a bijection from T^i to $\mathrm{Def}^{\mathcal{B}_i}_{\mathcal{L}}\left(S_i \times \Pi_{j\neq i} T^j\right)$, the space of nonempty $\mathcal{L}$-definable conditional i-events. It follows in particular that $\mathcal{A}$ is $\mathcal{L}$-definably beliefs complete (Theorem 4.2, Corollary 4.2 and Corollary 4.3).

4.2 Preliminaries

After some basic definitions, we define the two versions of how the players update their beliefs, should they make some new observation (about nature). Then, we make some definitions and prove some lemmas (most of them of topological nature), before we define the main objects of our study in this chapter, namely *conditional possibility structures.*

Unless stated otherwise, in this chapter k, l, m, n denote integers in $\mathbb{N} \cup \{-1\}$.

Notation 4.1 For a nonempty set M, denote by $\mathrm{Pow}_\emptyset(M)$ the set of nonempty subsets of M.

Definition 4.1 Let S be a nonempty set and $\mathcal{B}$ a set of nonempty subsets of S such that $S \in \mathcal{B}$. For a nonempty set X, define $\mathcal{B}(X) := \{B \times X \mid B \in \mathcal{B}\}$. It follows that (up to isomorphism of product spaces) $(\mathcal{B}(X))(Y) = \mathcal{B}(X \times Y)$[1], for nonempty sets X and Y.

Note that $\mathcal{B}(X)$ consists of nonempty subsets of $S \times X$ and that $S \times X \in \mathcal{B}(X)$. In the following, when no confusion may arise, we denote $\mathcal{B}(X)$ also by $\mathcal{B}$.

Definition 4.2 Let S be a nonempty set and $\mathcal{B}$ a set of nonempty subsets of S such that $S \in \mathcal{B}$.

$$(A_B)_{B \in \mathcal{B}} \in \prod_{B \in \mathcal{B}} \mathrm{Pow}_\emptyset(B)$$

is called:

- *A-nice* if $C, D \in \mathcal{B}$, $C \subseteq D$, and $A_D \cap C \neq \emptyset$ imply $A_C = A_D \cap C$.
- *B-nice* if $C, D \in \mathcal{B}$, $C \subseteq D$, and $C \supseteq A_D$ imply $A_C = A_D$.

In fact, we write here two papers (or chapters) in one. In one version (henceforth "*A-version*") the players are supposed to revise their beliefs according to the A-version of conditionality (see the definition below), and in the other version (henceforth "*B-version*") the players are supposed to revise their beliefs according to the B-version of conditionality (see also the definition below).

We will indicate if there are any differences in the formulations of the theorems, propositions, lemmas, or definitions. Furthermore, we will indicate if there are any differences between the two versions in the proofs.

Definition 4.3 Let S be a nonempty set and $\mathcal{B}$ a set of nonempty subsets of S such that $S \in \mathcal{B}$. Define

$$\mathrm{Pow}_\emptyset^{\mathcal{B}}(S)$$

to be the set of all

[1]With some abuse of notation, we identify $(S \times X) \times Y$ with $S \times (X \times Y)$.

- A-nice elements of $\prod_{B\in\mathcal{B}} \mathrm{Pow}_{\emptyset}(B)$ in the *A-version of conditionality*,
- B-nice elements of $\prod_{B\in\mathcal{B}} \mathrm{Pow}_{\emptyset}(B)$ in the *B-version of conditionality*.

Note that if $(A_B)_{B\in\mathcal{B}} \in \prod_{B\in\mathcal{B}} \mathrm{Pow}_{\emptyset}(B)$ is A-nice, then it is also B-nice.

Definition 4.4 For a nonempty topological space X let $\mathcal{V}(X)$ denote the space of *nonempty compact subsets* of X. We consider, in all what follows, $\mathcal{V}(X)$ to be endowed with the *Vietoris topology*, i.e. the topology with the basis consisting of the sets

$$\langle U_0, \dots, U_m\rangle := \{K \in \mathcal{V}(X) \mid K \subseteq U_0 \cup \ldots \cup U_m \wedge K \cap U_0 \neq \emptyset \wedge \ldots \wedge K \cap U_m \neq \emptyset\},$$

for $U_0, \dots, U_m$ open in X. (See Michael (1951)).

Convention 4.1 Products of topological spaces are considered to be endowed with the product topology.
Note that the product topology of finitely many discrete topological spaces is the discrete topology.

Definition 4.5 A *compactum* is a compact Hausdorff space.

Definition 4.6 Let X be a nonempty compactum and $\mathcal{B}$ a set of nonempty clopen subsets of X such that $X \in \mathcal{B}$. Define $\mathcal{V}^{\mathcal{B}}(X)$, the set of *nonempty conditional compact subsets* in the A-version as the set of all A-nice elements of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$. We endow $\mathcal{V}^{\mathcal{B}}(X)$ with the relative topology inherited from the product topology of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$.

Note that $\prod_{B\in\mathcal{B}} \mathcal{V}(B) \subseteq \prod_{B\in\mathcal{B}} \mathrm{Pow}_{\emptyset}(B)$.

Definition 4.7 Let X be a nonempty compactum and $\mathcal{B}$ a set of nonempty clopen subsets of X such that $X \in \mathcal{B}$. Define $\mathcal{V}^{\mathcal{B}}(X)$, the set of *nonempty conditional compact subsets* in the B-version as the set of all B-nice elements of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$. We endow $\mathcal{V}^{\mathcal{B}}(X)$ with the relative topology inherited from the product topology of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$.

Note that if $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(X)$ in the A-version, then $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(X)$ in the B-version.

Definition 4.8 Let $D \in \mathcal{B}$ and let $\mathcal{O}_D$ be an open subset of $\mathcal{V}(D)$. Then we denote by $\mathcal{O}'_D$ the following open subset of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$:

$$\mathcal{O}'_D := \prod_{B\in\mathcal{B}} \mathcal{U}_B,$$

where

$$\mathcal{U}_B = \begin{cases} \mathcal{V}(B), & \text{if} \quad B \neq D, \\ \mathcal{O}_D, & \text{if} \quad B = D. \end{cases}$$

Lemma 4.1 *Let X be a nonempty compactum and $\mathcal{B}$ a set of nonempty clopen subsets of X such that $X \in \mathcal{B}$.*
Then $\mathcal{V}^{\mathcal{B}}(X)$ is a closed subset of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$.

Proof

- ***A*-version:** Let $D, E \in \mathcal{B}$, $D \supseteq E$, $(A_B)_{B\in\mathcal{B}} \in \prod_{B\in\mathcal{B}} \mathcal{V}(B)$ and let $A_D \cap E \neq \emptyset$ and $A_D \cap E \neq A_E$.
  We have to show that there exists an open subset of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$ that has empty intersection with $\mathcal{V}^{\mathcal{B}}(X)$ and that contains $(A_B)_{B\in\mathcal{B}}$ as an element.

1. case $A_D \cap E \not\subseteq A_E$: Then, since E and D are compact Hausdorff, there are open subsets U_D, U_E of E (and hence of D) such that $U_D \cap U_E = \emptyset$ and $A_D \cap E \cap U_D \neq \emptyset$ and $A_E \subseteq U_E$.
   $(A_B)_{B\in\mathcal{B}}$ is an element of the open set
   $$\{K_D \mid K_D \cap (E \cap U_D) \neq \emptyset\}'_D \cap \{K_E \mid K_E \subseteq U_E\}'_E.$$
   Let $(C_B)_{B\in\mathcal{B}}$ be an element of this open set. Then $C_D \cap (E \cap U_D) \neq \emptyset$ and $C_E \subseteq U_E$. This implies $C_D \cap E \not\subseteq C_E$, hence this open subset has empty intersection with $\mathcal{V}^{\mathcal{B}}(X)$.

2. case $A_D \cap E \not\supseteq A_E$: Then there are open subsets U_D, U_E of E (and hence of D) such that $U_D \cap U_E = \emptyset$ and $A_E \cap U_E \neq \emptyset$ and $A_D \cap E \subseteq U_D$. $D \setminus E$ is open in D, hence also $U_D \cup (D \setminus E)$, and it follows that $A_D \subseteq U_D \cup (D \setminus E)$.
   $(A_B)_{B\in\mathcal{B}}$ is an element of the open set
   $$\{K_D \mid K_D \cap E \neq \emptyset\}'_D \cap \{K_D \mid K_D \subseteq U_D \cup (D \setminus E)\}'_D \cap \{K_E \mid K_E \cap U_E \neq \emptyset\}'_E.$$
   Let $(C_B)_{B\in\mathcal{B}}$ be an element of this open set. Then $C_D \cap E \neq \emptyset$ and $C_E \not\subseteq C_D$. This implies $C_E \not\subseteq C_D \cap E$, hence this open subset has empty intersection with $\mathcal{V}^{\mathcal{B}}(X)$.

- B**-version:** Let $D, E \in \mathcal{B}$, $D \supseteq E$, $(A_B)_{B\in\mathcal{B}} \in \prod_{B\in\mathcal{B}} \mathcal{V}(B)$ and let $A_D \subseteq E$ and $A_D \neq A_E$.

1. case $A_D \nsubseteq A_E$: Then, since E and D are compact Hausdorff, there are open subsets U_D, U_E of E (and hence of D) such that $U_D \cap U_E = \emptyset$ and $A_D \cap U_D \neq \emptyset$ and $A_E \subseteq U_E$.
   $(A_B)_{B\in\mathcal{B}}$ is an element of the open set
   $$\{K_D \mid K_D \subseteq E\}'_D \cap \{K_D \mid K_D \cap U_D \neq \emptyset\}'_D \cap \{K_E \mid K_E \subseteq U_E\}'_E.$$
   Let $(C_B)_{B\in\mathcal{B}}$ be an element of this open set. Then $C_D \subseteq E$, $C_D \cap U_D \neq \emptyset$ and $C_E \subseteq U_E$. This implies $C_D \nsubseteq C_E$, hence this open subset has empty intersection with $\mathcal{V}^{\mathcal{B}}(X)$.

2. case $A_D \nsupseteq A_E$: Then there are open subsets U_D, U_E of E (and hence of D) such that $U_D \cap U_E = \emptyset$ and $A_E \cap U_E \neq \emptyset$ and $A_D \subseteq U_D$.
   $(A_B)_{B\in\mathcal{B}}$ is an element of the open set
   $$\{K_D \mid K_D \subseteq E\}'_D \cap \{K_D \mid K_D \subseteq U_D\}'_D \cap \{K_E \mid K_E \cap U_E \neq \emptyset\}'_E.$$
   Let $(C_B)_{B\in\mathcal{B}}$ be an element of this open set. Then $C_D \subseteq E$ and $C_E \nsubseteq C_D$, hence this open subset has empty intersection with $\mathcal{V}^{\mathcal{B}}(X)$.

■

Lemma 4.2 *Let X and Y be nonempty compacta and let $f : X \to Y$ be continuous.*
Then
$$\mathcal{V}(f)(K) := \{f(x) \mid x \in K\} \in \mathcal{V}(Y),$$
for $K \in \mathcal{V}(X)$, and
$$\mathcal{V}(f) : \mathcal{V}(X) \to \mathcal{V}(Y)$$
is continuous.

Proof Images of compact sets under continuous maps are compact.

The $\{K \in \mathcal{V}(Y) \mid K \subseteq U\}$ and $\{K \in \mathcal{V}(Y) \mid K \cap U \neq \emptyset\}$, for U open in Y, form a subbase of the topology on $\mathcal{V}(Y)$. Therefore it is enough to show that $\mathcal{V}(f)^{-1}(\{K \in \mathcal{V}(Y) \mid K \subseteq U\})$ and $\mathcal{V}(f)^{-1}(\{K \in \mathcal{V}(Y) \mid K \cap U \neq \emptyset\})$ are open. But we have
$$\begin{aligned} \mathcal{V}(f)^{-1}(\{K \in \mathcal{V}(Y) \mid K \subseteq U\}) &= \{K \in \mathcal{V}(X) \mid K \subseteq f^{-1}(U)\} \text{ and} \\ \mathcal{V}(f)^{-1}(\{K \in \mathcal{V}(Y) \mid K \cap U \neq \emptyset\}) &= \{K \in \mathcal{V}(X) \mid K \cap f^{-1}(U) \neq \emptyset\}. \end{aligned}$$

■

Remark 4.1 *Let S and X be nonempty compact topological spaces, and $\mathcal{B}$ a set of nonempty clopen subsets of S such that $S \in \mathcal{B}$. Then*

$$\mathcal{B}(X) = \{B \times X \mid B \in \mathcal{B}\}$$

consists of nonempty clopen subsets of $S \times X$ and we have $S \times X \in \mathcal{B}(X)$. Again, when no confusion may arise, we denote $\mathcal{B}(X)$ also by $\mathcal{B}$. If Y is another nonempty compact topological space, then we have, up to isomorphism, $(\mathcal{B}(X))(Y) = \mathcal{B}(X \times Y)$, and we identify these two spaces (with some abuse of notation).

Lemma 4.3 *Let S, X and Y be nonempty compacta and let $\mathcal{B}$ be a set of nonempty clopen subsets of S such that $S \in \mathcal{B}$. Then:*

1. *$\mathcal{V}^{\mathcal{B}}(S)$ is a nonempty compactum.*
2. *Let $f : X \to Y$ be continuous, and define*

   $$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)((K_B)_{B\in\mathcal{B}}) := (\{(s, f(x)) \mid (s, x) \in K_B\})_{B\in\mathcal{B}},$$

   for $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}(X)}(S \times X)$.
   Then

   $$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathcal{V}^{\mathcal{B}(X)}(S \times X) \to \mathcal{V}^{\mathcal{B}(Y)}(S \times Y),$$

   and $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)$ is continuous.
3. *If $f : X \to Y$ is continuous and onto, then*

   $$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathcal{V}^{\mathcal{B}(X)}(S \times X) \to \mathcal{V}^{\mathcal{B}(Y)}(S \times Y)$$

   is continuous and onto.
4. *If $f : X \to Y$ is continuous and one-to-one, then*

   $$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathcal{V}^{\mathcal{B}(X)}(S \times X) \to \mathcal{V}^{\mathcal{B}(Y)}(S \times Y)$$

   is continuous and one-to-one.
5. *If $f : X \to Y$ is a homeomorphism, then*

   $$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathcal{V}^{\mathcal{B}(X)}(S \times X) \to \mathcal{V}^{\mathcal{B}(Y)}(S \times Y)$$

   is a homeomorphism.

Proof

1. The $B \in \mathcal{B}$ are nonempty closed subspaces of a compactum, hence nonempty compacta and therefore the $\mathcal{V}(B)$ are compacta (Michael (1951, Theorem 4.9.6)). This implies that $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$ is a compactum. Since $\mathcal{V}^{\mathcal{B}}(S)$ is a closed subset of $\prod_{B\in\mathcal{B}} \mathcal{V}(B)$ (by Lemma 4.1), it is a compactum, and since (in both versions of conditionality) $(B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S)$, $\mathcal{V}^{\mathcal{B}}(S)$ is nonempty.

2. $\mathrm{id}_S \times f : B \times X \to B \times Y$ is continuous, so by Lemma 4.2, $\mathcal{V}(\mathrm{id}_B \times f) : \mathcal{V}(B \times X) \to \mathcal{V}(B \times Y)$ is continuous, for $B \in \mathcal{B}$. Since the product of an arbitrary set of continuous functions is a continuous function, $\prod_{B\in\mathcal{B}} \mathcal{V}(\mathrm{id}_B \times f) : \prod_{B\in\mathcal{B}} \mathcal{V}(B \times X) \to \prod_{B\in\mathcal{B}} \mathcal{V}(B \times Y)$ is continuous. $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) = (\prod_{B\in\mathcal{B}} \mathcal{V}(\mathrm{id}_B \times f)) \lceil \mathcal{V}^{\mathcal{B}}(S \times X) : \mathcal{V}^{\mathcal{B}}(S \times X) \to \prod_{B\in\mathcal{B}} \mathcal{V}(B \times Y)$ is continuous, because $\mathcal{V}^{\mathcal{B}}(S \times X)$ is endowed with the relative topology.
It remains to show that the image of $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)$ is a subset of $\mathcal{V}^{\mathcal{B}}(S \times Y)$:

   - A-version: Let $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S \times X)$, let $D, E \in \mathcal{B}$, $D \supseteq E$ and let $(E \times Y) \cap ((\mathrm{id}_S \times f)(K_D)) \neq \emptyset$. This implies that there is a $(s, x) \in K_D$ such that $s \in E$. But then $K_D \cap (E \times X) \neq \emptyset$, hence $K_E = K_D \cap (E \times X)$. It follows that $(\mathrm{id}_S \times f)(K_E) \subseteq ((\mathrm{id}_S \times f)(K_D)) \cap (E \times Y)$. Let $(s, y) \in ((\mathrm{id}_S \times f)(K_D)) \cap (E \times Y)$. Then there is $x \in X$ such that $(s, x) \in K_D$ and $f(x) = y$. But since $s \in E$ and $K_E = K_D \cap (E \times X)$, we have $(s, x) \in K_E$ and so $(s, y) \in (\mathrm{id}_S \times f)(K_E)$. It follows that $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)((K_B)_{B\in\mathcal{B}}) \in \mathcal{V}^{\mathcal{B}}(S \times Y)$.
   - B-version: Let $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S \times X)$, let $D, E \in \mathcal{B}$, $D \supseteq E$ and let $(E \times Y) \supseteq (\mathrm{id}_S \times f)(K_D)$. It follows that $K_D \subseteq E \times X$, so $K_D = K_E$ and hence $(\mathrm{id}_S \times f)(K_D) = (\mathrm{id}_S \times f)(K_E)$. It follows that $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)((K_B)_{B\in\mathcal{B}}) \in \mathcal{V}^{\mathcal{B}}(S \times Y)$.

3. Let f be continuous and onto and let $(A_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S \times Y)$. Since A_B is a compact subset of the Hausdorff space $B \times Y$, it is closed, and since $(\mathrm{id}_S \times f)$ is continuous, $(\mathrm{id}_S \times f)^{-1}(A_B)$ is compact as a closed subset of the compact space $B \times X$. Since A_B is not empty and $(\mathrm{id}_S \times f)$ is onto, $(\mathrm{id}_S \times f)^{-1}(A_B)$ is not empty. So $((\mathrm{id}_S \times f)^{-1}(A_B))_{B\in\mathcal{B}} \in \prod_{B\in\mathcal{B}} \mathcal{V}(B \times X)$. We have to show that $((\mathrm{id}_S \times f)^{-1}(A_B))_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S \times X)$.

   - A-version: Let $D, E \in \mathcal{B}$, $D \supseteq E$ and let $(E \times X) \cap ((\mathrm{id}_S \times f)^{-1}(A_D)) \neq \emptyset$. Then $A_D \cap (E \times Y) \neq \emptyset$. This implies that $A_E = A_D \cap (E \times Y)$. It follows that $(\mathrm{id}_S \times f)^{-1}(A_E) \subseteq (E \times X) \cap ((\mathrm{id}_S \times f)^{-1}(A_D))$. Let $(s, x) \in (E \times X) \cap ((\mathrm{id}_S \times f)^{-1}(A_D))$. So, $(s, f(x)) \in A_D$, but also $s \in E$, hence $(s, f(x)) \in (E \times Y) \cap A_D = A_E$. It follows that $(\mathrm{id}_S \times f)^{-1}(A_E) = (E \times X) \cap ((\mathrm{id}_S \times f)^{-1}(A_D))$.
   - B-version: Let $D, E \in \mathcal{B}$, $D \supseteq E$ and let $(E \times X) \supseteq (\mathrm{id}_S \times f)^{-1}(A_D)$. This implies $A_D \subseteq E \times Y$ because $(\mathrm{id}_S \times f)$ is onto. But then $A_D = A_E$ and $(\mathrm{id}_S \times f)^{-1}(A_D) = (\mathrm{id}_S \times f)^{-1}(A_E)$.

4. is clear by 2. and the definitions.

5. By 3. and 4. of this lemma, $\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)$ is continuous, 1 to 1 and onto from a compact space to a Hausdorff space. Hence it is a homeomorphism.

■

Lemma 4.4 *Let X and Y be nonempty compacta and $\mathcal{B}$ a set of nonempty clopen subsets of X such that $X \in \mathcal{B}$.*
If $(K_{B\times Y})_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}(Y)}(X \times Y)$, then $(\mathrm{proj}_X(K_{B\times Y}))_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(X)$.

Proof Follows directly from Lemma 4.3, since the projections are continuous.

■

Lemma 4.5 *Let S, X, Y and Z be nonempty compacta, let $\mathcal{B}$ be a set of nonempty clopen subsets of S such that $S \in \mathcal{B}$, and let $g : X \to Y$ and $f : Y \to Z$ be continuous maps.*
Then

$$\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times (f \circ g)) = \mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) \circ \mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times g).$$

Proof Note that $(\mathrm{id}_S \times (f \circ g)) = (\mathrm{id}_S \times f) \circ (\mathrm{id}_S \times g)$.
For $(K_B)_{B\in\mathcal{B}} \in \mathcal{V}^{\mathcal{B}}(S \times X)$ we have

$$\begin{aligned}
\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times (f \circ g))((K_B)_{B\in\mathcal{B}}) &= (\{(s, f \circ g(x)) \mid (s,x) \in K_B\})_{B\in\mathcal{B}} \\
&= (\{(s, f(g(x))) \mid (s, g(x)) \in (\mathrm{id}_S \times g)(K_B)\})_{B\in\mathcal{B}} \\
&= \mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)\,(((\mathrm{id}_S \times g)(K_B))_{B\in\mathcal{B}}) \\
&= \mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f)\left(\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times g)((K_B)_{B\in\mathcal{B}})\right) \\
&= \left(\mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times f) \circ \mathcal{V}^{\mathcal{B}}(\mathrm{id}_S \times g)\right)((K_B)_{B\in\mathcal{B}}).
\end{aligned}$$

■

Lemma 4.6

- *Let X be nonempty, finite, and endowed with the discrete topology. Then*

  $$\mathcal{V}(X) = \mathrm{Pow}_{\emptyset}(X)$$

  and the topology of $\mathcal{V}(X)$ is the discrete topology.

- *If I is finite, $i \in I$, and if S_i and T^j, for $j \in I\setminus\{i\}$, are nonempty, finite and endowed with the discrete topology, and if $\mathcal{B}_i$ is a set of nonempty subsets of S_i such that $S_i \in \mathcal{B}_i$, then*

  $$\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i}T^j\right) = \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j)$$

  and the topology of $\mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j)$ is the discrete topology.

Proof Of course, we have $\mathcal{V}(X) = \mathrm{Pow}_\emptyset(X)$, since every subset of X is compact. Let $\emptyset \neq Y \subseteq X$ and $Y = \{x_1, \ldots, x_n\}$. It follows that

$$\{Y\} = \{Z \in \mathcal{V}(X) \mid Z \subseteq Y \wedge Z \cap \{x_1\} \neq \emptyset \wedge \ldots \wedge Z \cap \{x_n\} \neq \emptyset\},$$

hence $\{Y\}$ is open in $\mathcal{V}(X)$.

Since the product topology of finitely many spaces each of which is endowed with the discrete topology is also the discrete topology, and since the relative topology of a subspace of a topological space with the discrete topology is the discrete topology, the second part of the lemma follows from the first part. ■

Lemma 4.7 *Let S, X and Y be nonempty finite sets and let $\mathcal{B}$ be a set of nonempty subsets of S such that $S \in \mathcal{B}$. Then:*

1. $\mathrm{Pow}_\emptyset^{\mathcal{B}}(S)$ *is a nonempty set.*
2. *If $f : X \to Y$ is onto, then*
$$\mathrm{Pow}_\emptyset^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathrm{Pow}_\emptyset^{\mathcal{B}(X)}(S \times X) \to \mathrm{Pow}_\emptyset^{\mathcal{B}(Y)}(S \times Y)$$
is onto, where
$$\mathrm{Pow}_\emptyset^{\mathcal{B}}(\mathrm{id}_S \times f)((A_B)_{B \in \mathcal{B}}) := (\{(s, f(x)) \mid (s, x) \in A_B\})_{B \in \mathcal{B}}.$$
3. *If $f : X \to Y$ is one-to-one, then*
$$\mathrm{Pow}_\emptyset^{\mathcal{B}}(\mathrm{id}_S \times f) : \mathrm{Pow}_\emptyset^{\mathcal{B}(X)}(S \times X) \to \mathrm{Pow}_\emptyset^{\mathcal{B}(Y)}(S \times Y)$$
one-to-one.

Proof Endow all the spaces with the discrete topology. (Note that since S is finite, $\mathcal{B}$ is finite.) The lemma follows now from the above lemma and Lemma 4.3. ■

Definition 4.9 Let I be a nonempty player set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty set and $\mathcal{B}_i$ be a set of nonempty subsets of S_i such that $S_i \in \mathcal{B}_i$.

A *conditional possibility structure on* $(S_i, \mathcal{B}_i)_{i \in I}$ is a tuple

$$\underline{U} := \left\langle \left(U^i\right)_{i \in I}, \left(\rho^i\right)_{i \in I} \right\rangle,$$

where U^i is a nonempty set, for $i \in I$, and

$$\rho^i : U^i \to \mathrm{Pow}_\emptyset^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} U^j\right),$$

for $i \in I$.

$\left(A_{B_i}\right)_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_\emptyset^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} U^j\right)$ is called a *conditional i-event*, and a conditional i-event that is the image of some $u^i \in U^i$ under ρ^i is called a *conditional possibility set for player i*.

4.3 Topological Conditional Possibility Structures

In this section, we impose topological restrictions on the conditional possibility structures. We define structure preserving maps, called *morphisms*, between *topological conditional possibility structures* and we define the notions of *universal topological conditional possibility structure* and *beliefs complete topological conditional possibility structure*. Next, given a fixed set I of at least two players and, for each player i, a nonempty compact Hausdorff space S_i, i.e. "i's" space of states of nature, and a set $\mathcal{B}_i$ of clopen subsets of S_i, i.e. observable events in nature, such that $S_i \in \mathcal{B}_i$, we construct a universal topological conditional possibility structure $\left\langle (T^i)_{i\in I}, \left(\delta^i\right)_{i\in I} \right\rangle$ on $(S_i, \mathcal{B}_i)_{i\in I}$ to which every topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$ can be mapped by a unique morphism. The universal topological conditional possibility structure is unique up to isomorphism and each δ^i is a homeomorphism. Therefore, the universal topological conditional possibility structure is also beliefs complete.

Definition 4.10 Let I be a nonempty player set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ be a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$.

A *topological conditional possibility structure on* $(S_i, \mathcal{B}_i)_{i\in I}$ is a tuple

$$\underline{U} := \left\langle \left(U^i\right)_{i\in I}, \left(\rho^i\right)_{i\in I} \right\rangle,$$

where U^i is a nonempty compactum, for $i \in I$, and

$$\rho^i : U^i \to \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$$

is continuous, for $i \in I$.

$\left(A_{B_i}\right)_{B_i\in\mathcal{B}_i} \in \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} U^j\right)$ is called a *topological*[2] *conditional i-event*, and a conditional i-event that is the image of some $u^i \in U^i$ under ρ^i is called a *topological conditional possibility set for player i*.

[2]In the following, when no confusion may arrise, we skip the "topological" in "topological conditional i-event" and in "topological conditional possibility set".

Definition 4.11 Let I be a nonempty set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ be a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$. Let $\underline{U} := \left\langle (U^i)_{i\in I}, (\rho^i)_{i\in I} \right\rangle$ and $\underline{V} := \left\langle (V^i)_{i\in I}, (\sigma^i)_{i\in I} \right\rangle$ be topological conditional possibility structures on $(S_i, \mathcal{B}_i)_{i\in I}$.

A I-tuple $(v^i)_{i\in I}$ of continuous maps $v^i : V^i \to U^i$, $i \in I$, is a *morphism* of topological conditional possibility structures, if the following diagram commutes for every $i \in I$:

$$\begin{array}{ccc} V^i & \xrightarrow{\quad\sigma^i\quad} & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} V^j) \\ {\scriptstyle v^i}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{S_i}} \qquad \downarrow{\scriptstyle v^j} \\ U^i & \xrightarrow{\quad\rho^i\quad} & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} U^j) \end{array}$$

The morphism $(v^i)_{i\in I}$ is a *isomorphism* of topological conditional possibility structures, if v^i is a homeomorphism, for every $i \in I$.

By 5 of Lemma 4.3 and Lemma 4.5, $(v^i)_{i\in I}$ is a isomorphism of topological conditional possibility structures iff each v^i is one-to-one and onto and $((v^i)^{-1})_{i\in I}$ is also a morphism.

Definition 4.12 Let I be a nonempty set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ be a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$.

A topological conditional possibility structure $\underline{U}$ on $(S_i, \mathcal{B}_i)_{i\in I}$ is *universal* if for every topological conditional possibility structure $\underline{V}$ on $(S_i, \mathcal{B}_i)_{i\in I}$ there is a unique morphism from $\underline{V}$ to $\underline{U}$.

Definition 4.13 Let I be a nonempty set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ be a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$.

A topological conditional possibility structure $\left\langle (U^i)_{i\in I}, (\rho^i)_{i\in I} \right\rangle$ on $(S_i, \mathcal{B}_i)_{i\in I}$ is *beliefs complete* if ρ^i is onto, for every $i \in I$.

We are now ready to formulate and prove the main result of this section (i.e. the topological part of this chapter), namely Theorem 4.1 and Corollary 4.1. In order to see the analogy with the classical type space literature, Theorem 4.1 is formulated in a way that is similar to Theorem 1.1. of Chapter III of Mertens, Sorin and Zamir (2000) in the classical Bayesian setting.

Theorem 4.1 • *Let I be a nonempty player set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$.*

Then there exist nonempty compacta T^i, for $i \in I$, and homeomorphisms $\delta^i : T^i \to \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j)$, for $i \in I$, such that:
$(*)$ given nonempty compacta U^i, for $i \in I$, and continuous maps $\rho^i : U^i \to \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} U^j)$, for $i \in I$, there is a unique I-tuple of continuous maps $(\mu^i)_{i \in I}$ such that the following diagram commutes for every $i \in I$:

$$\begin{array}{ccc} U^i & \xrightarrow{\rho^i} & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} U^j) \\ \downarrow \mu^i & & \downarrow \mathrm{id}_{S_i} \quad \downarrow \mu^j \\ T^i & \xrightarrow{\delta^i} & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j) \end{array}$$

- *The above property $(*)$ characterizes the spaces T^i and the maps δ^i : If $(*)$ is satisfied by compacta $\hat{T}^i$, $i \in I$, and continuous maps $\hat{\delta}^i$, $i \in I$, then the $\mu^i : \hat{T}^i \to T^i$, $i \in I$, are canonical homeomorphisms.*

- *Let $|T^i_{-1}| = 1$, for $i \in I$, and define inductively continuous maps q^i_n, for $i \in I$ and $n \in \mathbb{N}$, and nonempty compacta T^i_m, for $i \in I$ and $m \in \mathbb{N} \cup \{-1\}$, by the commutativity of the lower parts of the following diagrams:*

$$\begin{array}{ccc} T^i & \xrightarrow{\delta^i} & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j) \\ \downarrow p^i_{n+1} & & \downarrow \mathrm{id}_{S_i} \quad \downarrow p^j_n \\ T^i_{n+1} & = & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j_n) \\ \downarrow q^i_{n+1} & & \downarrow \mathrm{id}_{S_i} \quad \downarrow q^j_n \\ T^i_n & = & \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j_{n-1}) \end{array}$$

Then the (T^i_n, q^i_n) form a projective system with limit T^i and continuous projections $p^i_n : T^i \to T^i_n$, for $n \in \mathbb{N} \cup \{-1\}$, such that we have $q^i_n \circ p^i_n = p^i_{n-1}$, for $n \in \mathbb{N}$. Using the universal property of the projective limit, we define δ^i by the commutativity of the upper parts of the above diagrams (for all $n \in \mathbb{N}$). δ^i is then a homeomorphism, for $i \in I$. The maps p^i_n are onto and have continuous selections r^i_n (i.e. $p^i_n \circ r^i_n = id_{T^i_n}$) : Defining inductively c^i_n, for $n \in \mathbb{N}$, c^i_0 arbitrary, by the commutativity of the diagrams:

$$\begin{array}{ccccc} T^i_{n+1} & = & \mathcal{V}^{\mathcal{B}_i}(S_i & \times \Pi_{j\neq i} & T^j_n) \\ c^i_{n+1}\Big\uparrow & & \mathrm{id}_{S_i}\Big\uparrow & & \Big\uparrow c^j_n \\ T^i_n & = & \mathcal{V}^{\mathcal{B}_i}(S_i & \times \Pi_{j\neq i} & T^j_{n-1}) \end{array}$$

we get inductively $q^i_n \circ c^i_n = id_{T^i_{n-1}}$ and we can define r^i_n as the projective limit (i.e. $r^i_{n-1} = r^i_n \circ c^i_n$).

Proof ot the theorem For every $i \in I$, $S_i \times \Pi_{j\neq i} T^j_{-1}$ is a nonempty compactum since S_i is a nonempty compactum and since $|T^j_{-1}| = 1$, for all $j \in I$. (The topology on the T^j_{-1} is the obvious one). So $T^i_0 = \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j_{-1})$ is a nonempty compactum (Lemma 4.3). q^j_0, for $j \in I$, is then continuous and onto, and so is $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_0)$, so $q^i_1 := \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_0)$ is continuous and onto (Lemma 4.3). Inductively, by repeating this argumentation in the induction step, we get, for every $i \in I$, a projective system $(T^i_n, q^i_n)_{n\geq 0}$ of nonempty compacta T^i_m, for $m \in \mathbb{N} \cup \{-1\}$, and continuous maps $q^i_n : T^i_n \to T^i_{n-1}$ that are onto. By Kurosh's Theorem (Arkhangel'skij et al. (1990), Proposition 1, p. 75), the projective limit exists and is a nonempty compactum $T^i \subseteq \Pi_{m\geq -1} T^i_m$ with continuous projections $p^i_m : T^i \to T^i_m$. By the definition of the projective limit, we have $q^i_n \circ p^i_n = p^i_{n-1}$, for $n \in \mathbb{N}$.

Start, for $i \in I$, with c^i_0 arbitrary. Since $|T^i_{-1}| = 1$, c^i_0 is clearly continuous, hence $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} c^j_0)$ is continuous and it is one-to-one, and of course, we have $q^i_0 \circ c^i_0 = \mathrm{id}_{T^i_{-1}}$. Let now, for $j \in I$, c^j_n be continuous and one-to-one and $q^j_n \circ c^j_n = \mathrm{id}_{T^j_{n-1}}$. It follows that $c^i_{n+1} := \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} c^j_n)$ is continuous and one-to-one, and by Lemma 4.5,

$$\begin{aligned} q^i_{n+1} \circ c^i_{n+1} &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} c^j_n) \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n \circ c^j_n) \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mathrm{id}_{T^j_{n-1}}) \\ &= \mathrm{id}_{T^i_n}. \end{aligned}$$

Fix $n \in \mathbb{N}$.

$$(\hat{c}^i(n)_m)_{m\geq -1} := (q^i_0 \circ \ldots \circ q^i_{n-1}, q^i_1 \circ \ldots \circ q^i_{n-1}, \ldots, q^i_{n-1}, \mathrm{id}_{T^i_{n-1}}, c^i_n, c^i_{n+1} \circ c^i_n, \ldots)$$

forms a family of continuous maps $\hat{c}^i(n)_m : T^i_{n-1} \to T^i_m$ such that $q^i_{m+1} \circ \hat{c}^i(n)_{m+1} = \hat{c}^i(n)_m$. By the universal property of the projective limit there exists a unique continuous map $r^i_{n-1} : T^i_{n-1} \to T^i$ such that $\hat{c}^i(n)_m = p^i_m \circ r^i_{n-1}$, and in particular $\mathrm{id}_{T^i_{n-1}} = p^i_{n-1} \circ r^i_{n-1}$. So, r^i_{n-1} is one-to-one and p^i_{n-1} is onto. Given r^i_{n-1}, it follows from the above definitions that $r^i_n \circ c^i_n$ has also the desired properties of r^i_{n-1}, hence by the uniqueness, $r^i_n \circ c^i_n = r^i_{n-1}$.

For $m \in \mathbb{N} \cup \{-1\}$, the map

$$\mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m) : \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j) \to \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j_m) = T^i_{m+1}$$

is continuous and, for $n \in \mathbb{N}$, we have

$$\begin{aligned} q^i_{n+1} \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n) &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n) \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}). \end{aligned}$$

Since $|T^i_{-1}| = 1$, there is a unique continuous map

$$\pi^i_{-1} : \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j) \to T^i_{-1}.$$

By the universal property of the projective limit, there is a unique continuous map

$$\gamma_i : \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j) \to T^i$$

such that for all $n \in \mathbb{N}$: $p^i_n \circ \gamma_i = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1})$ and $p^i_{-1} \circ \gamma_i = \pi^i_{-1}$.

Lemma 4.8 *γ_i is a homeomorphism.*

Proof of the lemma γ_i is one-to-one: Let $(A_{B_i})_{B_i\in\mathcal{B}_i}, (K_{B_i})_{B_i\in\mathcal{B}_i} \in \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j)$ and $(A_{B_i})_{B_i\in\mathcal{B}_i} \neq (K_{B_i})_{B_i\in\mathcal{B}_i}$. Then there is a $D_i \in \mathcal{B}_i$ such that $A_{D_i} \neq K_{D_i}$. Without loss of generality, it follows that there is a $(s_i, (t^j)_{j\neq i}) \in A_{D_i} \setminus K_{D_i}$. We show now that there is a $m \in \mathbb{N} \cup \{-1\}$ such that $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)(A_{D_i}) \neq (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)(K_{D_i})$:

Claim: There is a $m \in \mathbb{N}\cup\{-1\}$ such that $(s_i, ((p^j_m(t^j))_{j\neq i}) \notin (\mathrm{id}_{S_i}\times\Pi_{j\neq i} p^j_m)(K_{D_i})$.

Assume the contrary. $\{(s_i, (p^j_m(t^j))_{j\neq i})\}$ is closed in $S_i \times \Pi_{j\neq i} T^j_m$. Hence

$$P^{-1}_m := \big((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(\{(s_i, (p^j_m(t^j))_{j\neq i})\})\big) \cap K_{D_i}$$

is closed in $S_i \times \Pi_{j\neq i} T^j$ and nonempty, for every $m \in \mathbb{N} \cup \{-1\}$. Furthermore $P^{-1}_m \supseteq P^{-1}_{m+1}$, for all $m \in \mathbb{N} \cup \{-1\}$. Since $S_i \times \Pi_{j\neq i} T^j$ is compact, it follows that $\bigcap_{m\geq -1} P^{-1}_m \neq \emptyset$. Let $(s_i, (\hat{t}^j)_{j\neq i}) \in \bigcap_{m\geq -1} P^{-1}_m$. But then $p^j_m(\hat{t}^j) = p^j_m(t^j)$, for every $m \in \mathbb{N} \cup \{-1\}$ and every $j \neq i$. This implies that $\hat{t}^j = t^j$, for all $j \neq i$, hence $(s_i, (t^j)_{j\neq i}) \in K_{D_i}$, a contradiction and the claim is proved and therefore γ_i is one-to-one.

γ_i is onto: Let $(t^i_m)_{m\geq -1} \in T^i$. Then, we have

$$t^i_{m+1} = (A^m_{B_i})_{B_i\in\mathcal{B}_i} \in \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j_m)$$

and $q^i_{m+1}(t^i_{m+1}) = t^i_m$, for $m \in \mathbb{N} \cup \{-1\}$.
By the definitions, we have $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_{m+1})(A^{m+1}_{B_i}) = A^m_{B_i}$, for $B_i \in \mathcal{B}_i$ and $m \in \mathbb{N} \cup \{-1\}$.
Since

$$(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m) : S_i \times \Pi_{j\neq i} T^j \to S_i \times \Pi_{j\neq i} T^j_m$$

is continuous and onto, for all $m \in \mathbb{N} \cup \{-1\}$, and the $A^m_{B_i} \subseteq B_i \times \Pi_{j\neq i} T^j_m$ are nonempty and closed, and since

$$(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n) \circ (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n) = (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}),$$

for all $n \in \mathbb{N}$, it follows that

$$(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(A^m_{B_i}) \supseteq (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m+1})^{-1}(A^{m+1}_{B_i})$$

and that $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(A^m_{B_i})$ is closed in $B_i \times \Pi_{j\neq i} T^j$ and nonempty, for all $m \in \mathbb{N} \cup \{-1\}$.

$$A_{B_i} := \bigcap_{m \geq -1} (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(A^m_{B_i})$$

is closed and nonempty, since $B_i \times \Pi_{j\neq i} T^j$ is compact. (Since $A^m_{B_i} \subseteq B_i \times \Pi_{j\neq i} T^j_m$, for $m \in \mathbb{N} \cup \{-1\}$, it follows that $A_{B_i} \subseteq B_i \times \Pi_{j\neq i} T^j$.)

Let $m \in \mathbb{N} \cup \{-1\}$ and $(s_i, (t^j_m)_{j\neq i}) \in A^m_{B_i}$. Then, for $n \in \mathbb{N}$:

$$Q^{-1}_{n+m} := \\ ((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(\{(s_i, (t^j_m)_{j\neq i})\})) \cap \left((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m+n})^{-1}(A^{m+n}_{B_i})\right) \supseteq \\ ((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(\{(s_i, (t^j_m)_{j\neq i})\})) \cap \left((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m+n+1})^{-1}(A^{m+n+1}_{B_i})\right).$$

Each of the sets Q^{-1}_{n+m} is nonempty and a closed subset of $B_i \times \Pi_{j\neq i} T^j$. Note that for $-1 \leq l < m$: $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_l)^{-1}(A^l_{B_i}) \supseteq Q^{-1}_{n+m}$. It follows by the compactness of $B_i \times \Pi_{j\neq i} T^j$, that

$$\left(\bigcap_{l \geq -1} (\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_l)^{-1}(A^l_{B_i})\right) \cap \left((\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)^{-1}(\{(s_i, (t^j_m)_{j\neq i})\})\right)$$

is nonempty. This implies that $(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m)(A_{B_i}) = A^m_{B_i}$.

It remains to show that $(A_{B_i})_{B_i \in \mathcal{B}_i} \in \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j)$:

A-version: Let $D_i \supseteq E_i$ and $A_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j) \neq \emptyset$. Then, for all $m \in \mathbb{N} \cup \{-1\}$, $A^m_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j_m) \neq \emptyset$ and $A^m_{E_i} = A^m_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j_m)$. It follows that $A_{E_i} \subseteq A_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j)$. Let $(s_i, (t^j)_{j\neq i}) \in A_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j)$. Then is

$(s_i, (p^j_m(t^j))_{j\neq i}) \in A^m_{E_i}$, for $m \in \mathbb{N}\cup\{-1\}$, hence $(s_i, (t^j)_{j\neq i}) \in A_{E_i}$ and it follows that $A_{E_i} = A_{D_i} \cap (E_i \times \Pi_{j\neq i} T^j)$.

B-version: This is clear.

γ_i is one-to-one, onto and continuous from a compact space to a Hausdorff space, hence γ_i is a homeomorphism and the lemma is proved. ∎

We proceed now with the **proof of the theorem:**

Define now $\delta^i := (\gamma_i)^{-1}$. It remains to show for $t^i \in T^i$, that for all $n \in \mathbb{N}$: $p^i_n(t^i) = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1})(\delta^i(t^i))$. But we have

$$p^i_n(t^i) = p^i_n(\gamma_i(\delta^i(t^i))) = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1})(\delta^i(t^i)).$$

We establish **property** $(*)$**:** For $j \in I$, define $\mu^j_{-1} : U^j \to T^j_{-1}$ in the obvious way. μ^j_{-1} is continuous. Let now $\mu^j_{n-1} : U^j \to T^j_{n-1}$ be defined and continuous, for $j \in I$. Define

$$\mu^i_n : U^i \to T^i_n \quad \text{by} \quad \mu^i_n := \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j_{n-1}) \circ \rho^i, \quad \text{for } i \in I.$$

The continuity of the μ^j_{n-1} implies the continuity of μ^i_n (Lemma 4.3).

We show now for all $n \in \mathbb{N}$ and $j \in I$, that $q^j_n \circ \mu^j_n = \mu^j_{n-1}$: Since $|T^j_{-1}| = 1$, $q^j_0 \circ \mu^j_0 = \mu^j_{-1}$. Let now for all $j \in I$: $q^j_n \circ \mu^j_n = \mu^j_{n-1}$. It follows that for $i \in I$:

$$\begin{aligned} q^i_{n+1} \circ \mu^i_{n+1} &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j_n) \circ \rho^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_n \circ \mu^j_n) \circ \rho^i \\ &= \mu^i_n. \end{aligned}$$

By the universal property of the projective limit, there is, for every $i \in I$, a unique continuous map $\mu^i : U^i \to T^i$, such that for all $m \in \mathbb{N}\cup\{-1\}$: $\mu^i_m = p^i_m \circ \mu^i$. It follows that, for all $n \in \mathbb{N}$:

$$\begin{aligned} p^i_n \circ \mu^i &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1} \circ \mu^j) \circ \rho^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j) \circ \rho^i \\ &= p^i_n \circ \gamma_i \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j) \circ \rho^i. \end{aligned}$$

By the uniqueness part of the universal property of the projective limit (the statement for p^i_{-1} is clear anyway), we get

$$\mu^i = \gamma_i \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j) \circ \rho^i,$$

which implies

$$\delta^i \circ \mu^i = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} \mu^j) \circ \rho^i.$$

We have to show the uniqueness of the $(\mu^i)_{i\in I}$: Let, for $j \in I$,

$$\tilde{\mu}^j : U^j \to T^j$$

be continuous, such that for every $i \in I$:

$$\delta^i \circ \tilde{\mu}^i = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\tilde{\mu}^j) \circ \rho^i.$$

Since T^i_{-1} is a singleton, we have $p^i_{-1} \circ \tilde{\mu}^i = p^i_{-1} \circ \mu^i$, for $i \in I$. Let $m \in \mathbb{N} \cup \{-1\}$ and for all $j \in I$: $p^j_m \circ \tilde{\mu}^j = p^j_m \circ \mu^j$. It follows that for $i \in I$:

$$\begin{aligned} p^i_{m+1} \circ \tilde{\mu}^i &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}p^j_m) \circ \delta^i \circ \tilde{\mu}^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}p^j_m) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\tilde{\mu}^j) \circ \rho^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}p^j_m \circ \tilde{\mu}^j) \circ \rho^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}p^j_m \circ \mu^j) \circ \rho^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}p^j_m) \circ \delta^i \circ \mu^i \\ &= p^i_{m+1} \circ \mu^i. \end{aligned}$$

By the uniqueness property of the projective limit, it follows that $\tilde{\mu}^i = \mu^i$, for $i \in I$.

Now, it remains just to prove the second point of the theorem: If $U^i = T^i$ (resp. $U^i = \hat{T}^i$) and $\rho^i = \delta^i$ (resp. $\rho^i = \hat{\delta}^i$) for all $i \in I$, then $\mu^i := \mathrm{id}_{T^i}$ (resp. $\hat{\mu}^i := \mathrm{id}_{\hat{T}^i}$), for all $i \in I$, makes the diagram for the property $(*)$ commutative, because - by the definition - we have

$$\mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mathrm{id}_{T^j}) = \mathrm{id}_{\mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j)}.$$

Now let $\hat{\mu}^i : T^i \to \hat{T}^i$ and $\mu^i : \hat{T}^i \to T^i$, for $i \in I$, be the continuous maps with the property $(*)$. We have for $i \in I$:

$$\begin{aligned} \delta^i \circ \mu^i \circ \hat{\mu}^i &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mu^j) \circ \hat{\delta}^i \circ \hat{\mu}^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mu^j) \circ \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\hat{\mu}^j) \circ \delta^i \\ &= \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mu^j \circ \hat{\mu}^j) \circ \delta^i, \end{aligned}$$

hence $\mu^i \circ \hat{\mu}^i$, for $i \in I$, makes the diagram with $U^i = T^i$ and $\rho^i = \delta^i$, for $i \in I$, commutative. It follows that for $i \in I$: $\mu^i \circ \hat{\mu}^i = \mathrm{id}_{T^i}$ and in the same way $\hat{\mu}^i \circ \mu^i = \mathrm{id}_{\hat{T}^i}$. This implies that μ^i and $\hat{\mu}^i$ are homeomorphisms, for $i \in I$. Furthermore is

$$\hat{\delta}^i = \mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mu^j)^{-1} \circ \delta^i \circ \mu^i, \quad \text{for } i \in I.$$

Hence, since $\mathcal{V}^{\mathcal{B}_i}(\mathrm{id}_{S_i} \times \Pi_{j\neq i}\mu^j)$, δ^i and μ^i are a homeomorphisms, for $i \in I$, $\hat{\delta}^i$ is also a homeomorphism, for $i \in I$, and the theorem is proved. ∎

The above theorem implies immediately:

Corollary 4.1 *Let I be a nonempty set with $|I| > 1$, and, for every $i \in I$, let S_i be a nonempty compactum and $\mathcal{B}_i$ be a set of nonempty clopen subsets of S_i such that $S_i \in \mathcal{B}_i$. Let $\underline{T} := \langle (T^i)_{i\in I}, (\delta^i)_{i\in I} \rangle$ be the topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$ constructed in Theorem 4.1.*
Then:

1. *$\underline{T}$ is a universal topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$,*

2. *$\underline{T}$ is - up to isomorphism of topological conditional possibility structures on $(S_i, \mathcal{B}_i)_{i\in I}$ - the unique universal topological conditional possibility structure on $(S_i, \mathcal{B}_i)_{i\in I}$,*

3. *$\underline{T}$ is beliefs complete.*

4.4 Definable Completeness for 'Neocompact' Formulas

For the rest of this chapter, we fix a finite set N of at least two players and, for every $i \in N$, a nonempty finite set S_i, "i's" space of states of nature, and a set of nonempty subsets $\mathcal{B}_i$ of S_i, observable events in S_i, such that $S_i \in \mathcal{B}_i$.

In this section we define a language other than the one used by Brandenburger and Keisler (1999), but similar to it, in order to produce a positive definably beliefs completeness result.

4.4.1 Finitary Structures

In this subsection we define the finite-step structures $\mathcal{A}_n$, for $n \in \mathbb{N} \cup \{-1\}$, which are "extended" conditional possibility structures. Each structure $\mathcal{A}_n$, for $n \in \mathbb{N}$, contains (the inverse images of) the finite-step structures $\mathcal{A}_m$, for $m < n$.

Definition 4.14 For $i \in N$ define (inductively):

$$
\begin{aligned}
T^i_{-1} &:= \{i\}, \\
T^i_n &:= \mathrm{Pow}^{\mathcal{B}_i}_{\emptyset}(S_i \times \Pi_{j\neq i} T^j_{n-1}), \qquad \text{for } n \in \mathbb{N}, \\
q^i_{-1,0} : T^i_0 \to T^i_{-1}, & \quad \text{such that} \\
q^i_{-1,0}(t^i_0) &:= i, \qquad \text{for } t^i_0 \in T^i_0, \\
q^i_{n-1,n} : T^i_n \to T^i_{n-1}, & \quad \text{such that} \\
q^i_{n-1,n} &:= \mathrm{Pow}^{\mathcal{B}_i}_{\emptyset}(\mathrm{id}_{S_i} \times \Pi_{j\neq i} q^j_{n-2,n-1}), \quad \text{for } n \geq 1, \\
q^i_{k,n} &:= q^i_{k,n-1} \circ q^i_{n-1,n}, \qquad \text{for } -1 \leq k < n-1, \\
R^{i,n}_l &\subseteq T^i_n \times \mathcal{B}_i \times S_i \times \Pi_{j\neq i} T^j_n, \qquad \text{such that} \\
(t^i_n, B_i, s_i, (t^j_n)_{j\neq i}) \in R^{i,n}_l & \text{ iff } \big(s_i, (q^j_{l-1,n}(t^j_n))_{j\neq i}\big) \in \big(q^i_{l,n}(t^i_n)\big)(B_i), \\
& \qquad \text{for } 0 \leq l < n, \\
R^{i,n}_n &\subseteq T^i_n \times \mathcal{B}_i \times S_i \times \Pi_{j\neq i} T^j_n, \qquad \text{such that} \\
(t^i_n, B_i, s_i, (t^j_n)_{j\neq i}) \in R^{i,n}_n & \text{ iff } \big(s_i, (q^j_{n-1,n}(t^j_n))_{j\neq i}\big) \in t^i_n(B_i), \text{ for } 0 \leq n.
\end{aligned}
$$

Definition 4.15 For integers $n \in \mathbb{N} \cup \{-1\}$ define

$$\mathcal{A}_n := \left\langle (S_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \left(T^i_k\right)_{i\in N, -1\leq k\leq n}, \left(R^{i,n}_l\right)_{i\in N, 0\leq l\leq n}, \left(q^i_{k,l}\right)_{i\in N, -1\leq k<l\leq n} \right\rangle.$$

Proposition 4.1 *$\mathcal{A}_n$ has the following properties:*

1. $(t^i_n, B_i, s_i, (t^j_n)_{j\neq i}) \in R^{i,n}_k$ *iff* $(q^i_{n-1,n}(t^i_n), B_i, s_i, (q^j_{n-1,n}(t^j_n))_{j\neq i}) \in R^{i,n-1}_k$, *for* $i \in N$ *and* $0 \leq k \leq n-1$.
2. T^i_k *is nonempty and finite,* *for* $i \in N$ *and* $-1 \leq k \leq n$.
3. $q^i_{k,l} : T^i_l \to T^i_k$ *is onto,* *for* $i \in N$ *and* $-1 \leq k < l \leq n$.
4. $q^i_{m,k} \circ q^i_{k,l} = q^i_{m,l}$, *for* $i \in N$ *and* $-1 \leq m < k < l \leq n$.

Proof 4. follows from the definition, 1. follows from 4. and the definition. 2: That T^i_k is nonempty and finite, for $k \in \mathbb{N} \cup \{-1\}$, follows from the definition and 1. of Lemma 4.7. 3: $q^i_{-1,0}$ is onto, since T^i_0 is nonempty. That $q^i_{k,l}$ is onto follows now inductively by 2. of Lemma 4.7. ■

4.4.2 Finitary Languages

If not stated otherwise, let n be an integer in $\mathbb{N} \cup \{-1\}$. We define $\mathcal{L}_n$, i.e. the *language induced by* $\mathcal{A}_n$. This is done in such a way that the language $\mathcal{L}_n$, for $n \in \mathbb{N}$, contains the languages $\mathcal{L}_m$, for $m < n$.

Definition 4.16 The language $\mathcal{L}_n$ has the following **big** *sorts*:

- S_i, for $i \in N$,
- T_n^i, for $i \in N$.

$\mathcal{L}_n$ has the following **small** *sorts*:

- T_k^i, for $i \in N$, $k = -1, 0, 1, \ldots, n-1$.

Usually, one distinguishes between the sort $\mathbf{S}_i$ ($\mathbf{T}_n^i$, resp.) and the set S_i (T_n^i, resp.) that interprets this sort in a structure corresponding to the language. Since in all what follows, for each sort there will be just one fixed set interpreting this sort, we will not make this distinction and we will denote a sort with the same symbol as the corresponding set.

In the following, Z will be one of the sorts (resp. sets) S_i, where $i \in N$, or T_k^i, where $i \in N$ and $k = -1, 0, 1, \ldots, n$.

Definition 4.17 For a sort Z of $\mathcal{L}_n$, we define

$$\mathrm{big}_n(Z) := \begin{cases} Z, & \text{if } Z \text{ is a big sort of } \mathcal{L}_n, \\ T_n^i, & \text{if } Z = T_k^i \text{ for a } k < n \text{ and } i \in N. \end{cases}$$

Definition 4.18 The *alphabet of* $\mathcal{L}_n$ contains the following symbols:

1. $v_0^Z, v_1^Z, \ldots$, for each sort Z of $\mathcal{L}_n$ (variables of sort Z),
2. a constant $\underline{z}$ of sort Z, for each sort Z of $\mathcal{L}_n$ and each element $z \in Z$,
3. $\neg, \vee$[3] (not, or),
4. $\doteq$ (equality symbol),
5. (,) (parentheses),

[3] $\wedge, \rightarrow, \leftrightarrow$ are abbreviations defined in the usual way by $\neg$ and $\vee$.

6. $\underline{B}_i$, for $B_i \in \mathcal{B}_i$ and $i \in N$ (Relation symbols (of sort S_i)),

7. $\underline{R}^i_k$, for $i \in N$ and $0 \leq k \leq n$ (Relation symbols),

8. $\exists$, $\forall$ (existential and universal quantifier).

Note that the alphabet of $\mathcal{L}_n$ contains the alphabet of $\mathcal{L}_m$, for $-1 \leq m < n$.

Definition 4.19 Let Z be a sort of $\mathcal{L}_n$. A *term of sort* Z is a variable of sort Z or a constant of sort Z.

Definition 4.20 The set $\mathcal{L}_n$ of $\mathcal{L}_n$*-formulas* is the least set such that:

1. If Z is a sort of $\mathcal{L}_n$ and if x^Z and y^Z are terms of sort Z, then $x^Z \doteq y^Z$ is a $\mathcal{L}_n$-formula.

2. If $-1 \leq k < l \leq n$ and $i \in N$, and if $x^{T^i_k}$ and $y^{T^i_l}$ are terms of sort T^i_k and T^i_l, then $x^{T^i_k} y^{T^i_l}$ is a $\mathcal{L}_n$-formula.

3. If $i \in N$, and if x^{S_i} is a term of sort S_i, then $\underline{B}_i x^{S_i}$ is a $\mathcal{L}_n$-formula.

4. If $0 \leq k \leq n$ and $i \in N$, if x^{S_i} is a term of sort S_i, and if $x^{T^j_k}$ is a term of sort T^j_k, for $j \in N$, then $\underline{R}^i_k(x^{T^i_k}, \underline{B}_i, x^{S_i}, (x^{T^j_k})_{j \neq i})$ is a $\mathcal{L}_n$-formula.

5. If φ is a $\mathcal{L}_n$-formula, then $\neg\varphi$ is a $\mathcal{L}_n$-formula.

6. If φ and ψ are $\mathcal{L}_n$-formulas, then $(\varphi \vee \psi)$[4] is a $\mathcal{L}_n$-formula.

7. If Z is a sort of $\mathcal{L}_n$, if φ is a $\mathcal{L}_n$-formula, and if v^Z is a variable of sort Z, then $\exists v^Z \varphi$ and $\forall v^Z \varphi$ are $\mathcal{L}_n$-formulas.

Note that $\mathcal{L}_n$ contains $\mathcal{L}_m$, for $-1 \leq m < n$.

Definition 4.21 For a term x^Z of sort Z, where Z is a sort of $\mathcal{L}_n$, define

$$\mathrm{var}(x^Z) := \begin{cases} \{x^Z\}, & \text{if } x^Z \text{ is a variable,} \\ \emptyset, & \text{if } x^Z \text{ is a constant.} \end{cases}$$

[4] $(\varphi \wedge \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi)$ are defined by the use of $\neg$ and $\vee$ in the usual way. We omit the parentheses whenever possible.

Definition 4.22 We define now the set of *free variables in a* $\mathcal{L}_n$-formula by induction on the formation of the formulas in $\mathcal{L}_n$:

$$\begin{array}{rcl} \mathrm{free}(x^Z \doteq y^Z) & := & \mathrm{var}(x^Z) \cup \mathrm{var}(y^Z), \\ \mathrm{free}(x^{T^i_k} y^{T^i_l}) & := & \mathrm{var}(x^{T^i_k}) \cup \mathrm{var}(y^{T^i_l}), \\ \mathrm{free}(\underline{B}_i x^{S_i}) & := & \mathrm{var}(x^{S_i}), \\ \mathrm{free}(\underline{R}^i_k(x^{T^i_k}, \underline{B}_i, x^{S_i}, (x^{T^j_k})_{j\neq i})) & := & \mathrm{var}(x^{S_i}) \cup \bigcup_{j\in N} \mathrm{var}(x^{T^j_k}), \\ \mathrm{free}(\neg\varphi) & := & \mathrm{free}(\varphi), \\ \mathrm{free}(\varphi \vee \psi) & := & \mathrm{free}(\varphi) \cup \mathrm{free}(\psi), \\ \mathrm{free}(\exists v^Z_n \varphi) & := & \mathrm{free}(\varphi) \setminus \{v^Z_n\}, \\ \mathrm{free}(\forall v^Z_n \varphi) & := & \mathrm{free}(\varphi) \setminus \{v^Z_n\}. \end{array}$$

Definition 4.23 A $\mathcal{L}_n$-*assignment* is a map

$$\beta_n : \{v^Z_l \mid l \in \mathbb{N},\ Z \text{ is a sort of } \mathcal{L}_n\} \to \bigcup_{i\in N} \left(S_i \cup T^i_n \cup \mathrm{Pow}_\emptyset(T^i_n)\right)$$

such that

1. $\beta_n(v^Z_l) \in Z$, if Z is a big sort of $\mathcal{L}_n$, and
2. $\beta_n(v^{T^i_k}_l) \in \mathrm{Pow}_\emptyset(T^i_n)$, where $\beta_n(v^{T^i_k}_l) = (q^i_{k,n})^{-1}(\{t^i_k\})$ for a $t^i_k \in T^i_k$, if $-1 \leq k < n$.

Definition 4.24

- If $v^{Z'}_m$ is a variable of sort Z', where Z' is a big sort of $\mathcal{L}_n$, and if $c \in Z'$, then we define

$$\beta_{n\frac{c}{v^{Z'}_m}}(v^Z_l) := \begin{cases} \beta_n(v^Z_l), & \text{if } v^{Z'}_m \neq v^Z_l, \\ c, & \text{if } v^{Z'}_m = v^Z_l. \end{cases}$$

- If $k < n$, if $v^{T^i_k}_m$ is a variable of sort T^i_k, and if $t^i_k \in T^i_k$, then we define

$$\beta_{n\frac{t^i_k}{v^{T^i_k}_m}}(v^Z_l) := \begin{cases} \beta_n(v^Z_l), & \text{if } v^{T^i_k}_m \neq v^Z_l, \\ (q^i_{k,n})^{-1}(\{t^i_k\}), & \text{if } v^{T^i_k}_m = v^Z_l. \end{cases}$$

Definition 4.25

- We associate with every pair $(\mathcal{A}_n, \beta_n)$ and every term x^Z of sort Z, where Z is a big sort of $\mathcal{L}_n$, an element $\mathcal{I}_n(x^Z) \in Z$ such that:

$$\mathcal{I}_n(x^Z) := \begin{cases} \beta_n(x^Z), & \text{if } x^Z \text{ is a variable}, \\ c, & \text{if } x^Z = \underline{c} \text{ for a } c \in Z. \end{cases}$$

- We associate with every pair $(\mathcal{A}_n, \beta_n)$ and every term $x^{T^i_k}$ of sort T^i_k, where $-1 \leq k < n$, a nonempty subset $\mathcal{I}_n(x^{T^i_k})$ of T^i_n such that:

$$\mathcal{I}_n(x^{T^i_k}) := \begin{cases} \beta_n(x^{T^i_k}), & \text{if } \; x^{T^i_k} \text{ is a variable,} \\ (q^i_{k,n})^{-1}(\{t^i_k\}), & \text{if } \; x^{T^i_k} = \underline{t^i_k} \text{ for a } t^i_k \in T^i_k. \end{cases}$$

Definition 4.26 We define the *model relation* "$\models$" by induction on the formation of the formulas in $\mathcal{L}_n$:

$$
\begin{array}{lll}
(\mathcal{A}_n, \beta_n) \models x^Z \doteq y^Z & \text{iff} & \mathcal{I}_n(x^Z) = \mathcal{I}_n(y^Z), \\
(\mathcal{A}_n, \beta_n) \models x^{T^i_k} y^{T^i_m} & \text{iff} & \mathcal{I}_n(y^{T^i_m}) \subseteq \mathcal{I}_n(x^{T^i_k}), \\
 & & \quad \text{for } -1 \leq k < m < n, \\
(\mathcal{A}_n, \beta_n) \models x^{T^i_k} y^{T^i_n} & \text{iff} & \mathcal{I}_n(y^{T^i_n}) \in \mathcal{I}_n(x^{T^i_k}), \\
 & & \quad \text{for } -1 \leq k < n, \\
(\mathcal{A}_n, \beta_n) \models \underline{B}_i x^{S_i} & \text{iff} & \mathcal{I}_n(x^{S_i}) \in B_i, \\
(\mathcal{A}_n, \beta_n) \models \underline{R}^i_k(x^{T^i_k}, \underline{B}_i, x^{S_i}, (x^{T^j_k})_{j \neq i}) & \text{iff} & \\
\quad \mathcal{I}_n(x^{T^i_k}) \times \{B_i\} \times \{\mathcal{I}_n(x^{S_i})\} \times \Pi_{j \neq i} \mathcal{I}_n(x^{T^j_k}) & \subseteq & R^{i,n}_k, \text{ for } 0 \leq k < n, \\
(\mathcal{A}_n, \beta_n) \models \underline{R}^i_n(x^{T^i_n}, \underline{B}_i, x^{S_i}, (x^{T^j_n})_{j \neq i}) & \text{iff} & \\
\quad (\mathcal{I}_n(x^{T^i_n}), B_i, \mathcal{I}_n(x^{S_i}), (\mathcal{I}_n(x^{T^j_n}))_{j \neq i}) & \in & R^{i,n}_n, \text{ for } 0 \leq n, \\
(\mathcal{A}_n, \beta_n) \models \neg\varphi & \text{iff} & (\mathcal{A}_n, \beta_n) \not\models \varphi, \\
(\mathcal{A}_n, \beta_n) \models \varphi \vee \psi & \text{iff} & (\mathcal{A}_n, \beta_n) \models \varphi \text{ or} \\
 & & (\mathcal{A}_n, \beta_n) \models \psi, \\
(\mathcal{A}_n, \beta_n) \models \exists v^Z_l \varphi & \text{iff} & \text{there is a } c \in Z \text{ such} \\
 & & \text{that } (\mathcal{A}_n, \beta_n \frac{c}{v^Z_l}) \models \varphi, \\
(\mathcal{A}_n, \beta_n) \models \forall v^Z_l \varphi & \text{iff} & (\mathcal{A}_n, \beta_n) \models \neg\exists v^Z_l \neg\varphi.
\end{array}
$$

Convention 4.2 If we write $\varphi(v^{Z_1}_1, \ldots, v^{Z_l}_l)$, for (not necessary different) sorts $Z_1, \ldots, Z_l$ of $\mathcal{L}_n$, we indicate that $\text{free}(\varphi) \subseteq \{v^{Z_1}_1, \ldots, v^{Z_l}_l\}$.

Definition 4.27 Let Z be a sort of $\mathcal{L}_n$, let v^Z be a variable of sort Z, and let β_n be a $\mathcal{L}_n$-assignment. Define

$$\underline{\beta_n}(v^Z) := \begin{cases} \beta_n(v^Z), & \text{if } \ Z \text{ is a small sort of } \mathcal{L}_n, \\ \{\beta_n(v^Z)\}, & \text{if } \ Z \text{ is a big sort of } \mathcal{L}_n. \end{cases}$$

Definition 4.28 For $m = 1, \dots, l$ let Z_m be a sort of $\mathcal{L}_n$ and $v_m^{Z_m}$ a variable of sort Z_m, and let β_n be a $\mathcal{L}_n$-assignment. Define

$$\underline{\beta_n}(v_1^{Z_1}, \dots, v_l^{Z_l}) := \underline{\beta_n}(v_1^{Z_1}) \times \dots \times \underline{\beta_n}(v_l^{Z_l}).$$

By definition, a straightforward check shows:

Lemma 4.9 *Let $n \in \mathbb{N} \cup \{-1\}$, let β_n, β_n' be $\mathcal{L}_n$-assignments and let $\varphi \in \mathcal{L}_n$.*

1. *If $\beta_n \restriction \mathrm{free}(\varphi) = \beta_n' \restriction \mathrm{free}(\varphi)$, then:*

$$(\mathcal{A}_n, \beta_n) \models \varphi \text{ iff } (\mathcal{A}_n, \beta_n') \models \varphi.$$

2. *Let $Z_1, \dots, Z_l$ be sorts of $\mathcal{L}_n$ and $v_m^{Z_m}$ be a variable of sort Z_m, for $m = 1, \dots, l$, then:*

$$\begin{aligned} \underline{\beta_n}(v_1^{Z_1}, \dots, v_l^{Z_l}) &= \underline{\beta_n'}(v_1^{Z_1}, \dots, v_l^{Z_l}), \text{ iff} \\ \beta_n(v_m^{Z_m}) &= \beta_n'(v_m^{Z_m}), \quad \text{for } m = 1, \dots, l, \\ \underline{\beta_n}(v_1^{Z_1}, \dots, v_l^{Z_l}) \cap \underline{\beta_n'}(v_1^{Z_1}, \dots, v_l^{Z_l}) &= \emptyset, \qquad \text{else.} \end{aligned}$$

Definition 4.29 Let $Z_1, \dots, Z_l$ be big sorts of $\mathcal{L}_n$ and let $n \geq k \geq -1$.
A subset $A \subseteq Z_1 \times \dots \times Z_l$ is *defined by* $\varphi(v_1^{\hat{Z}_1}, \dots, v_l^{\hat{Z}_l}) \in \mathcal{L}_k$, iff

1. $\mathrm{big}_n(\hat{Z}_m) = Z_m$, for $m = 1, \dots, l$,
2. $A = \bigcup \left\{ \underline{\beta_n}(v_1^{\hat{Z}_1}, \dots, v_l^{\hat{Z}_l}) \middle| \ \beta_n \ \mathcal{L}_n\text{-assignment such that } (\mathcal{A}_n, \beta_n) \models \varphi \right\}.$

A is *$\mathcal{L}_k$-definable*, if there is a $\varphi \in \mathcal{L}_k$ such that A is defined by φ.

Notation 4.2 Let $\varphi(v_1^{Z_1}, \dots, v_l^{Z_l}) \in \mathcal{L}_n$. Denote by $[\varphi]^{\mathcal{A}_n}$ the subset of $\prod_{m=1}^{l} \mathrm{big}_n(Z_m)$ that is defined by φ.

Note that, by Lemma 4.9 and the fact that the possible values of a variable of a small sort of $\mathcal{L}_n$ under $\mathcal{L}_n$-assignments partition the corresponding big sort of $\mathcal{L}_n$, it follows that:

Remark 4.2 *Let* $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})$, $\psi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}_n$. *Then:*

$$[\neg\varphi]^{\mathcal{A}_n} = (big_n(Z_1) \times \ldots \times big_n(Z_l)) \setminus [\varphi]^{\mathcal{A}_n}$$

and

$$[\varphi \vee \psi]^{\mathcal{A}_n} = [\varphi]^{\mathcal{A}_n} \cup [\psi]^{\mathcal{A}_n}.$$

Note that all sorts of $\mathcal{L}_n$ are finite and for each element of each big sort of $\mathcal{L}_n$ there is a constant symbol interpreted exactly by this element. It follows:

Remark 4.3 *All subsets of all finite products of big sorts of $\mathcal{L}_n$ are $\mathcal{L}_n$-definable.*

Proof Let $Z_1, \ldots, Z_l$ be big sorts of $\mathcal{L}_n$ and $\emptyset \neq A \subseteq Z_1 \times \ldots \times Z_l$. (The case $A = \emptyset$ is trivial).

$$\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) := \bigvee_{(c_1,\ldots,c_l)\in A} \bigwedge_{m=1}^{l} v_m^{Z_m} \doteq \underline{c_m}$$

defines A. ∎

Definition 4.30 1. Let $n \geq k \geq -1$ and $Z_1, \ldots, Z_l$ be big sorts of $\mathcal{L}_n$. Denote by

$$\mathrm{Def}_{\mathcal{L}_k}(Z_1 \times \ldots \times Z_l)$$

the set of *nonempty $\mathcal{L}_k$-definable subsets of* $Z_1 \times \ldots \times Z_l$.

2. Let $-1 \leq k \leq n$, $i \in N$ and $B_i \in \mathcal{B}_i$. Denote by

$$\mathrm{Def}_{\mathcal{L}_k}\left(B_i \times \Pi_{j\neq i} T_n^j\right)$$

the set of nonempty subsets

$$A \subseteq B_i \times \Pi_{j\neq i} T_n^j$$

such that there is a $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}_k$ with $\mathrm{big}_n(Z_j) = T_n^j$, for $j \neq i$, such that $[\varphi]^{\mathcal{A}_n} = A$.

3. Let $n \geq k \geq -1$ and $i \in N$. Denote by

$$\mathrm{Def}^{\mathcal{B}_i}_{\mathcal{L}_k}\left(S_i \times \Pi_{j \neq i} T^j_n\right)$$

the set of A-nice (B-nice) elements of

$$\prod_{B_i \in \mathcal{B}_i} \mathrm{Def}_{\mathcal{L}_k}\left(B_i \times \Pi_{j \neq i} T^j_n\right)$$

in the A-version (in the B-version).

Note that if $C_i, D_i \in \mathcal{B}_i$, $C_i \subseteq D_i$, $-1 \leq k \leq n$, and $[\varphi]^{\mathcal{A}_n} \subseteq D_i \times \Pi_{j \neq i} T^j_n$, for $\varphi(v^{S_i}, (v^{Z_j})_{j \neq i}) \in \mathcal{L}_k$, where $\mathrm{big}_n(Z_j) = T^j_n$, for $j \neq i$, then

$$[\varphi]^{\mathcal{A}_n} \cap \left(C_i \times \Pi_{j \neq i} T^j_n\right) = \left[\varphi(v^{S_i}, (v^{Z_j})_{j \neq i}) \wedge \underline{C}_i v^{S_i}\right]^{\mathcal{A}_n}.$$

Note also that $\varphi(v^{S_i}, (v^{Z_j})_{j \neq i}) \in \mathcal{L}_k$ implies that $\varphi(v^{S_i}, (v^{Z_j})_{j \neq i}) \wedge \underline{C}_i v^{S_i} \in \mathcal{L}_k$.

Definition 4.31 Let $-1 \leq k < n$ and let Z be a sort of $\mathcal{L}_k$. Define

$$\pi^Z_{k,n} := \begin{cases} q^j_{k,n}, & \text{if } \mathrm{big}_k(Z) = T^j_k, \text{ for a } j \in N, \\ \mathrm{id}_{S_j}, & \text{if } \mathrm{big}_k(Z) = S_j, \text{ for a } j \in N. \end{cases}$$

Lemma 4.10 *Let $-1 \leq k < n$ and let β_n be a $\mathcal{L}_n$-assignment. Define for $j \in N$:*

$$\begin{aligned} (\beta_n)_k(v^{T^j_m}) &:= (q^j_{m,k})^{-1}(\{t^j_m\}), \quad \text{for the unique } t^j_m \in T^j_m \text{ such that} \\ &\qquad \beta_n(v^{T^j_m}) = (q^j_{m,n})^{-1}(\{t^j_m\}), \text{ if } -1 \leq m < k, \\ (\beta_n)_k(v^{T^j_k}) &:= \text{the unique } t^j_k \in T^j_k \text{ such that } \beta_n(v^{T^j_k}) = (q^j_{k,n})^{-1}(\{t^j_k\}), \\ (\beta_n)_k(v^{S_j}) &:= \beta_n(v^{S_j}). \end{aligned}$$

Then:

1. *$(\beta_n)_k$ is a $\mathcal{L}_k$-assignment.*
2. *For every $\mathcal{L}_k$-assignment β_k, there is a $\mathcal{L}_n$-assignment β_n such that $\beta_k = (\beta_n)_k$.*
3. *If $(\beta_n)_k = (\beta'_n)_k$, then β_n and β'_n coincide on the variables of $\mathcal{L}_k$-sorts.*

Proof The first and the third point are clear, for the second:
Define for $j \in N$:

$$\begin{array}{rcl} \beta_n(v^{T_m^j}) & := & (q_{k,n}^j)^{-1}(\beta_k(v^{T_m^j})), \text{ if } -1 \leq m < k, \\ \beta_n(v^{T_k^j}) & := & (q_{k,n}^j)^{-1}(\{\beta_k(v^{T_k^j})\}), \\ \beta_n(v^{S_j}) & := & \beta_k(v^{S_j}). \end{array}$$

It is obvious that β_n can be extended to a $\mathcal{L}_n$-assignment. ∎

Lemma 4.11 *Let* $-1 \leq k < n$ *and* $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}_k$. *Then*:

$$\left(\Pi_{m=1}^{l} \pi_{k,n}^{Z_m}\right)^{-1} \left([\varphi]^{\mathcal{A}_k}\right) = [\varphi]^{\mathcal{A}_n}.$$

Proof By Lemma 4.9 and Lemma 4.10 it is enough to show by induction on the formation of the formulas $\varphi \in \mathcal{L}_k$ that for every $\mathcal{L}_n$-assignment β_n :

$$(\mathcal{A}_k, (\beta_n)_k) \models \varphi \text{ iff } (\mathcal{A}_n, \beta_n) \models \varphi.$$

Note that for terms $\mathcal{I}_n(x^Z) = \left(\pi_{k,n}^Z\right)^{-1}\left(\mathcal{I}_k(x^Z)\right)$, if Z is a small sort of $\mathcal{L}_k$ or if $Z = S_j$ for a $j \in N$, and $\mathcal{I}_n(x^Z) = \left(\pi_{k,n}^Z\right)^{-1}\left(\{\mathcal{I}_k(x^Z)\}\right)$, if $Z = T_k^j$ for a $j \in N$.

- For atomic formulas φ (i.e. formulas as in 1., 2., 3. and 4. of Definition 4.20), by the above observation, it is easily seen from (an iterated application of) 1. of Proposition 4.1 and from the definition of "$\models$" that

$$(\mathcal{A}_k, (\beta_n)_k) \models \varphi \text{ iff } (\mathcal{A}_n, \beta_n) \models \varphi.$$

- The cases "$\varphi = \neg\psi$" and "$\varphi = \psi_0 \vee \psi_1$" follow from Lemma 4.9, Lemma 4.10 and the fact that inverse images commute with unions and complements.

- Let $\varphi = \exists v^Z \psi$, where Z is a sort of $\mathcal{L}_k$: According to the induction hypothesis, we have for $c \in Z$:

$$\left(\mathcal{A}_k, (\beta_n \frac{c}{v^Z})_k\right) \models \psi \text{ iff } \left(\mathcal{A}_n, \beta_n \frac{c}{v^Z}\right) \models \psi.$$

  The assertion follows now from the fact that $(\beta_n)_k \frac{c}{v^Z} = (\beta_n \frac{c}{v^Z})_k$.

- The case $\varphi = \forall v^Z \psi$, where Z is a sort of $\mathcal{L}_k$ follows from the " $\neg$ "-case and the " $\exists$ "-case.

∎

Proposition 4.2 *Let $n \in \mathbb{N}$ and $i \in N$.*

1. *Let*

$$\varphi^{B_i}(v_{B_i}^{S_i}, (v_{B_i}^{T^j_{k(j,B_i)}})_{j\neq i}) \in \mathcal{L}_{n-1}, \quad \textit{for} \quad B_i \in \mathcal{B}_i,$$

such that

$$\left([\varphi^{B_i}]^{\mathcal{A}_{n-1}}\right)_{B_i\in\mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j_{n-1}).$$

Then there is exactly one $t^i_n \in T^i_n$ such that for all $B_i \in \mathcal{B}_i$:

$$\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v^{S_i}_{B_i}, (v^{T^j_n}_{B_i})_{j\neq i})\right]^{\mathcal{A}_n} = [\varphi^{B_i}]^{\mathcal{A}_n}.$$

2. *For every $t^i_n \in T^i_n$ there are $\varphi^{B_i}(v_{B_i}^{S_i}, (v_{B_i}^{T^j_{k(j,B_i)}})_{j\neq i}) \in \mathcal{L}_{n-1}$, for $B_i \in \mathcal{B}_i$, such that*

$$\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v^{S_i}_{B_i}, (v^{T^j_n}_{B_i})_{j\neq i})\right]^{\mathcal{A}_n} = [\varphi^{B_i}]^{\mathcal{A}_n} \quad \textit{for all} \quad B_i \in \mathcal{B}_i,$$

and such that

$$\left([\varphi^{B_i}]^{\mathcal{A}_{n-1}}\right)_{B_i\in\mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j_{n-1}).$$

3. *For every $t^i_n \in T^i_n$:*

$$\left(\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v^{S_i}_{B_i}, (v^{T^j_n}_{B_i})_{j\neq i})\right]^{\mathcal{A}_n}\right)_{B_i\in\mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i}T^j_n).$$

Proof

1. By the definition is

$$\left[\underline{R}^i_n\left(\underline{t}^i_n, \underline{B}_i, v^{S_i}_{B_i}, (v^{T^j_n}_{B_i})_{j\neq i}\right)\right]^{\mathcal{A}_n} = \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i}q^j_{n-1,n}\right)^{-1}\left(t^i_n(B_i)\right).$$

The existence of a t^i_n like in the first point of the lemma follows now from Lemma 4.11 and the definitions of T^i_n and $R^{i,n}_n$. The uniqueness follows from the above equation, the definition of T^i_n and the fact that $\mathrm{id}_{S_i} \times \Pi_{j\neq i}q^j_{n-1,n}$ is onto.

2. Follows from the definition of T^i_n and $R^{i,n}_n$, the above equation, and Remark 4.3 and Lemma 4.11.

3. Since $\mathrm{id}_{S_i} \times \Pi_{j\neq i}q^j_{n-1,n}$ is onto and since inverse images commute with unions, intersections and complements, it follows (in the A-version and in the B-version) that:

$$\left(\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i}q^j_{n-1,n}\right)^{-1}\left(t^i_n(B_i)\right)\right)_{B_i\in\mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i}T^j_n\right).$$

Now apply the above equation.

■

4.4.3 Limit Structure

We are now in a position to define the main object of our study in this section, namely the limit structure $\mathcal{A}$, which is also an "extended" conditional possibility structure that contains (the inverse images of) the finite-step structures $\mathcal{A}_n$.

Definition 4.32 For $i \in N$ define:

$$\begin{aligned}
T^i &\subseteq \textstyle\prod_{-1\leq n} T^i_n, && \text{such that} \\
T^i &:= \left\{ (t^i_n)_{-1\leq n} \,|\, q^i_{n,n+1}(t^i_{n+1}) = t^i_n, \text{ for all } n \geq -1 \right\}, \\
p^i_m : T^i \to T^i_m, & \quad \text{such that} \\
p^i_m\left((t^i_n)_{-1\leq n}\right) &:= t^i_m, && \text{for } m \geq -1, \\
R^i_n &\subseteq T^i \times \mathcal{B}_i \times S_i \times \Pi_{j\neq i} T^j, && \text{such that} \\
(t^i, B_i, s_i, (t^j)_{j\neq i}) \in R^i_n &\text{ iff } (p^i_n(t^i), B_i, s_i, (p^j_n(t^j))_{j\neq i}) \in R^{i,n}_n, && \text{for } n \in \mathbb{N}, \\
R^i &\subseteq T^i \times \mathcal{B}_i \times S_i \times \Pi_{j\neq i} T^j, && \text{such that} \\
(t^i, B_i, s_i, (t^j)_{j\neq i}) \in R^i &\text{ iff } \text{for all } n \in \mathbb{N} : (t^i, B_i, s_i, (t^j)_{j\neq i}) \in R^i_n, \\
\text{i.e. } R^i &:= \textstyle\bigcap_{n\in\mathbb{N}} R^i_n \,.
\end{aligned}$$

Definition 4.33 Define $\mathcal{A} :=$

$$\left\langle (S_i)_{i\in N}, (\mathcal{B}_i)_{i\in N}, \left(T^i\right)_{i\in N}, \left(T^i_m\right)_{i\in N,-1\leq m}, \left(R^i\right)_{i\in N}, \left(R^i_n\right)_{i\in N,0\leq n}, \left(p^i_m\right)_{i\in N,-1\leq m} \right\rangle .$$

Proposition 4.3 *$\mathcal{A}$ has the following properties:*

1. *T^i is nonempty,* *for $i \in N$.*
2. *$p^i_m : T^i \to T^i_m$ is onto,* *for $i \in N$ and $m \in \mathbb{N} \cup \{-1\}$.*
3. *$q^i_{m,n} \circ p^i_n = p^i_m$,* *for $i \in N$, $n \in \mathbb{N}$ and $-1 \leq m < n$.*

Proof For every $i \in N$ and $m \in \mathbb{N} \cup \{-1\}$, endow S_i and T^i_m with the discrete topology. Since each of these spaces is nonempty and finite, each of them is then a nonempty compactum. By induction, it follows from Lemma 4.6 that for every $i \in N$ and every $n \in \mathbb{N}$:

1. $T_n^i = \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T_{n-1}^j\right)$.

2. The topology of $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T_{n-1}^j\right)$ is the discrete topology, hence the topologies on T_n^i and $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T_{n-1}^j\right)$ are the same, and they are compact and Hausdorff.

3. T^i is the underlying set of the projective limit of the projective system $\left(T_n^i, q_{n,n+1}^i\right)_{n\in\mathbb{N}\cup\{-1\}}$.

4. For every $m \in \mathbb{N}\cup\{-1\}$, the T_m^i here is then the T_m^i in the third point of Theorem 4.1, the $q_{m,m+1}^i$ here is the q_{m+1}^i there and the T^i here is the T^i there.

The proposition follows now from Theorem 4.1. ∎

4.4.4 Infinitary Languages

We define the *languages* $\mathcal{L}^*$ and $\mathcal{L}^-$, which are needed to define the limit language $\mathcal{L}$. Each of the languages $\mathcal{L}^*$, $\mathcal{L}^-$and $\mathcal{L}$ contains all the languages $\mathcal{L}_n$, for $n \in \mathbb{N}\cup\{-1\}$. $\mathcal{L}^*$ contains $\mathcal{L}^-$ and $\mathcal{L}$, and $\mathcal{L}^-$ contains $\mathcal{L}$.

Definition 4.34 The languages $\mathcal{L}^*$, $\mathcal{L}^-$ and $\mathcal{L}$ have the following **big** *sorts*:

- S_i, for $i \in N$,
- T^i, for $i \in N$.

$\mathcal{L}^*$, $\mathcal{L}^-$ and $\mathcal{L}$ have the following **small** *sorts*:

- T_k^i, for $i \in N$, $k \in \mathbb{N}\cup\{-1\}$.

In the following, Z will be one of the sorts (resp. sets) S_i, where $i \in N$, T^i, where $i \in N$, or T_k^i, where $i \in N$ and $k \in \mathbb{N}\cup\{-1\}$.

Definition 4.35 For a sort Z of $\mathcal{L}^*$, $\mathcal{L}^-$ or $\mathcal{L}$, we define

$$\operatorname{big}(Z) := \begin{cases} Z, & \text{if } Z \text{ is a big sort of } \mathcal{L}^*, \mathcal{L}^- \text{ or } \mathcal{L}, \\ T^i, & \text{if } Z = T_k^i \text{ for a } k \in \mathbb{N}\cup\{-1\} \text{ and } i \in N. \end{cases}$$

Definition 4.36 The *alphabet of* $\mathcal{L}^*$, $\mathcal{L}^-$ and $\mathcal{L}$ contains the following symbols:

1. $v_0^Z, v_1^Z, \ldots$, for each sort Z of $\mathcal{L}^*$, $\mathcal{L}^-$ or $\mathcal{L}$ (variables of sort Z),
2. a constant $\underline{z}$ of sort Z, for each sort Z of $\mathcal{L}^*$, $\mathcal{L}^-$ or $\mathcal{L}$ and each element $z \in Z$,
3. $\neg, \vee$[5] (not, or),
4. $\bigwedge$ (conjunction),
5. $\doteq$ (equality symbol),
6. (,) (parentheses),
7. $\underline{B}_i$, for $B_i \in \mathcal{B}_i$ and $i \in N$ (Relation symbols (of sort S_i)),
8. $\underline{R}^i$, for $i \in N$,
9. $\underline{R}^i_k$, for $i \in N$ and $k \in \mathbb{N}$ (Relation symbols),
10. $\exists, \forall$ (existential and universal quantifier).

Note that the alphabet of $\mathcal{L}^*$, $\mathcal{L}^-$ and $\mathcal{L}$ contains the alphabet of $\mathcal{L}_n$, for every $n \in \mathbb{N} \cup \{-1\}$.

Definition 4.37 Let Z be a sort of $\mathcal{L}^*$, $\mathcal{L}^-$ or $\mathcal{L}$. A *term of sort* Z is a variable of sort Z or a constant of sort Z.

Definition 4.38 The class $\mathcal{L}^*$ of $\mathcal{L}^*$*-formulas* is the least class such that:

1. If $\varphi \in \bigcup_{n \geq -1} \mathcal{L}_n$, then φ is a $\mathcal{L}^*$-formula.
2. If $i \in N$, and if x^{T^i} and y^{T^i} are terms of sort T^i, then $x^{T^i} \doteq y^{T^i}$ is a $\mathcal{L}^*$-formula.
3. If $k \in \mathbb{N} \cup \{-1\}$ and $i \in N$, and if $x^{T^i_k}$ and y^{T^i} are terms of sort T^i_k and T^i, then $x^{T^i_k} y^{T^i}$ is a $\mathcal{L}^*$-formula.
4. If $i \in N$, if x^{S_i} is a term of sort S_i, and if x^{T^j} is a term of sort T^j, for $j \in N$, then $\underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j \neq i})$ is a $\mathcal{L}^*$-formula.
5. If φ is a $\mathcal{L}^*$-formula, then $\neg\varphi$ is a $\mathcal{L}^*$-formula.

[5] $\wedge, \rightarrow, \leftrightarrow$ are abbreviations defined in the usual way by $\neg$ and $\vee$.

6. If φ and ψ are $\mathcal{L}^*$-formulas, then $(\varphi \vee \psi)$[6] is a $\mathcal{L}^*$-formula.

7. If Φ is a set of formulas, then $\bigwedge_{\varphi\in\Phi} \varphi$ is a $\mathcal{L}^*$-formula.

8. If Z is a sort of $\mathcal{L}^*$, if φ is a $\mathcal{L}^*$-formula, and if v^Z is a variable of sort Z, then $\exists v^Z \varphi$ and $\forall v^Z \varphi$ are $\mathcal{L}^*$-formulas.

Definition 4.39 For a term x^Z of sort Z, where Z is a sort of $\mathcal{L}^*$ ($\mathcal{L}^-$ or $\mathcal{L}$, resp.), define

$$\mathrm{var}(x^Z) := \begin{cases} \{x^Z\}, & \text{if } x^Z \text{ is a variable,} \\ \emptyset, & \text{if } x^Z \text{ is a constant.} \end{cases}$$

Definition 4.40 We define now the set of *free variables in* $\mathcal{L}^*$-formulas (and hence *in* $\mathcal{L}^-$-formulas and *in* $\mathcal{L}$-formulas) by induction on the formation of the formulas in $\mathcal{L}^*$:

If $\varphi \in \mathcal{L}_n$ for some $n \in \mathbb{N} \cup \{-1\}$, then, by Definition 4.22, free(φ) is already defined.

$$\begin{array}{rcl}
\mathrm{free}(x^{T^i} \doteq y^{T^i}) & := & \mathrm{var}(x^{T^i}) \cup \mathrm{var}(y^{T^i}), \\
\mathrm{free}(x^{T^i_k} y^{T^i}) & := & \mathrm{var}(x^{T^i_k}) \cup \mathrm{var}(y^{T^i}), \\
\mathrm{free}(\underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j\neq i})) & := & \mathrm{var}(x^{S_i}) \cup \bigcup_{j\in N} \mathrm{var}(x^{T^j}), \\
\mathrm{free}(\neg\varphi) & := & \mathrm{free}(\varphi), \\
\mathrm{free}(\varphi \vee \psi) & := & \mathrm{free}(\varphi) \cup \mathrm{free}(\psi), \\
\mathrm{free}(\bigwedge_{\varphi\in\Phi} \varphi) & := & \bigcup_{\varphi\in\Phi} \mathrm{free}\,(\varphi)\,, \\
\mathrm{free}(\exists v^Z_n \varphi) & := & \mathrm{free}(\varphi) \setminus \{v^Z_n\}, \\
\mathrm{free}(\forall v^Z_n \varphi) & := & \mathrm{free}(\varphi) \setminus \{v^Z_n\}.
\end{array}$$

Since the rules for "$\neg\varphi$", "$\varphi \vee \psi$", "$\exists v^Z_n \varphi$", and "$\forall v^Z_n \varphi$" coincide in this definition and in Definition 4.22, free(φ) is well-defined, for all $\varphi \in \mathcal{L}^*$ (and hence for all $\varphi \in \mathcal{L}^-$ (for all $\varphi \in \mathcal{L}$, resp.)).

Definition 4.41 The class $\mathcal{L}^-$ of $\mathcal{L}^-$*-formulas* is the least class such that:

1. If $\varphi \in \bigcup_{n\geq -1} \mathcal{L}_n$, then φ is a $\mathcal{L}^-$-formula.

2. If $i \in N$, and if x^{T^i} and y^{T^i} are terms of sort T^i, then $x^{T^i} \doteq y^{T^i}$ is a $\mathcal{L}^-$-formula.

[6] Also in the languages $\mathcal{L}^*$, $\mathcal{L}^-$ and $\mathcal{L}$, $(\varphi \wedge \psi), (\varphi \to \psi), (\varphi \leftrightarrow \psi)$ are defined by the use of $\neg$ and $\vee$ in the usual way. We omit the parentheses whenever possible.

3. If $k \in \mathbb{N} \cup \{-1\}$ and $i \in N$, and if $x^{T^i_k}$ and y^{T^i} are terms of sort T^i_k and T^i, then $x^{T^i_k} y^{T^i}$ is a $\mathcal{L}^-$-formula.

4. If $i \in N$, if x^{S_i} is a term of sort S_i, and if x^{T^j} is a term of sort T^j, for $j \in N$, then $\underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j \neq i})$ is a $\mathcal{L}^-$-formula.

5. If φ is a $\mathcal{L}^-$-formula, then $\neg\varphi$ is a $\mathcal{L}^-$-formula.

6. If φ and ψ are $\mathcal{L}^-$-formulas, then $(\varphi \vee \psi)$ is a $\mathcal{L}^-$-formula.

7. If Φ is a set of formulas, and if there is a finite set V of variables of $\mathcal{L}^-$ such that for all $\varphi \in \Phi$: free$(\varphi) \subseteq V$, then $\bigwedge_{\varphi \in \Phi} \varphi$ is a $\mathcal{L}^-$-formula.

8. If Z is a sort of $\mathcal{L}^-$, if φ is a $\mathcal{L}^-$-formula, and if v^Z is a variable of sort Z, then $\exists v^Z \varphi$ and $\forall v^Z \varphi$ are $\mathcal{L}^-$-formulas.

As already mentioned, $\mathcal{L}^-$ is contained in $\mathcal{L}^*$.

4.4.5 Limit Language

We define *the "neocompact" language $\mathcal{L}$ induced by $\mathcal{A}$*. This language contains all the languages $\mathcal{L}_n$ and can be viewed as the "projective limit" of the languages $\mathcal{L}_n$, $n \in \mathbb{N} \cup \{-1\}$. $\mathcal{L}$ is defined in the spirit of the neocompact language invented by Keisler (1998). After some preparations, we prove that the limit structure $\mathcal{A}$ is $\mathcal{L}$-definably beliefs complete. Furthermore we show that the possibility correspondence that sends $t^i \in T^i$ to $R^i(t^i)$, i.e. the t^i-section of i's possibility relation R^i, is a bijection from T^i to $\mathrm{Def}^{\mathcal{B}_i}_{\mathcal{L}}(S_i \times \Pi_{j \neq i} T^j)$, i.e. the space of nonempty $\mathcal{L}$-definable conditional i-events.

Definition 4.42 The class $\mathcal{L}$ of *$\mathcal{L}$-formulas* is the least class such that:

1. If $\varphi \in \bigcup_{n \geq -1} \mathcal{L}_n$, then φ is a $\mathcal{L}$-formula.

2. If $i \in N$, and if x^{T^i} and y^{T^i} are terms of sort T^i, then $x^{T^i} \doteq y^{T^i}$ is a $\mathcal{L}$-formula.

3. If $k \in \mathbb{N} \cup \{-1\}$ and $i \in N$, and if $x^{T^i_k}$ and y^{T^i} are terms of sort T^i_k and T^i, then $x^{T^i_k} y^{T^i}$ is a $\mathcal{L}$-formula.

4. If $i \in N$, if x^{S_i} is a term of sort S_i, and if x^{T^j} is a term of sort T^j, for $j \in N$, then $\underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j \neq i})$ is a $\mathcal{L}$-formula.

5. If φ and ψ are $\mathcal{L}$-formulas, then $(\varphi \vee \psi)$ is a $\mathcal{L}$-formula.

6. If Φ is a set of formulas, and if there is a finite set V of variables of $\mathcal{L}$ such that for all $\varphi \in \Phi$: free$(\varphi) \subseteq V$, then $\bigwedge_{\varphi\in\Phi} \varphi$ is a $\mathcal{L}$-formula.

7. If Z is a sort of $\mathcal{L}$, if φ is a $\mathcal{L}$-formula, and if v^Z is a variable of sort Z, then $\exists v^Z \varphi$ is a $\mathcal{L}$-formula.

8. If ψ is a $\mathcal{L}^-$-formula such that free$(\psi) \subseteq \left\{v_1^{Z_1}, \ldots, v_l^{Z_l}\right\}$, where $v_1^{Z_1}, \ldots, v_l^{Z_l}$ are variables of sorts $Z_1, \ldots, Z_l$, and if φ is a $\mathcal{L}$-formula, then $\forall v_1^{Z_1} \ldots \forall v_l^{Z_l} (\psi \rightarrow \varphi)$[7] is a $\mathcal{L}$-formula.

Note that $\mathcal{L}$ is contained in $\mathcal{L}^-$ (and hence in $\mathcal{L}^*$).

Definition 4.43 A *$\mathcal{L}$-assignment* (or equivalently: a *$\mathcal{L}^-$-assignment*, resp. *$\mathcal{L}^*$-assignment*) is a map

$$\beta : \{v_l^Z \mid l \in \mathbb{N},\ Z \text{ sort of } \mathcal{L}\} \rightarrow \bigcup_{i\in N} \left(S_i \cup T^i \cup \mathrm{Pow}_\emptyset(T^i)\right)$$

such that

1. $\beta(v_l^Z) \in Z$, if Z is a big sort of $\mathcal{L}$, and

2. $\beta(v_l^{T_k^i}) \in \mathrm{Pow}_\emptyset(T^i)$, where $\beta(v_l^{T_k^i}) = (p_k^i)^{-1}(\{t_k^i\})$ for a $t_k^i \in T_k^i$, if $k \in \mathbb{N} \cup \{-1\}$.

Definition 4.44

- If $v_m^{Z'}$ is a variable of sort Z', where Z' is a big sort of $\mathcal{L}$, and if $c \in Z'$, then we define

$$\beta_{\frac{c}{v_m^{Z'}}}(v_l^Z) := \begin{cases} \beta(v_l^Z), & \text{if } v_m^{Z'} \neq v_l^Z, \\ c, & \text{if } v_m^{Z'} = v_l^Z. \end{cases}$$

- If $k \in \mathbb{N} \cup \{-1\}$, if $v_m^{T_k^i}$ is a variable of sort T_k^i, and if $t_k^i \in T_k^i$, then we define

$$\beta_{\frac{t_k^i}{v_m^{T_k^i}}}(v_l^Z) := \begin{cases} \beta(v_l^Z), & \text{if } v_m^{T_k^i} \neq v_l^Z, \\ (p_k^i)^{-1}(\{t_k^i\}), & \text{if } v_m^{T_k^i} = v_l^Z. \end{cases}$$

[7] $(\psi \rightarrow \varphi)$ stands for $((\neg\psi) \vee \varphi)$.

Definition 4.45 • We associate with every pair $(\mathcal{A}, \beta)$ and every term x^Z of sort Z, where Z is a big sort of $\mathcal{L}$, an element $\mathcal{I}(x^Z) \in Z$ such that:

$$\mathcal{I}(x^Z) := \begin{cases} \beta(x^Z), & \text{if } \ x^Z \text{ is a variable,} \\ c, & \text{if } \ x^Z = \underline{c} \text{ for a } c \in Z. \end{cases}$$

- We associate with every pair $(\mathcal{A}, \beta)$ and every term $x^{T^i_k}$ of sort T^i_k, where $k \in \mathbb{N} \cup \{-1\}$, a nonempty subset $\mathcal{I}(x^{T^i_k})$ of T^i such that:

$$\mathcal{I}(x^{T^i_k}) := \begin{cases} \beta(x^{T^i_k}), & \text{if } \ x^{T^i_k} \text{ is a variable,} \\ (p^i_k)^{-1}(\{t^i_k\}), & \text{if } \ x^{T^i_k} = \underline{t^i_k} \text{ for a } t^i_k \in T^i_k. \end{cases}$$

Definition 4.46 We define the *model relation* "$\models$" by induction on the formation of the formulas in $\mathcal{L}^*$ (and hence also in $\mathcal{L}^-$ and in $\mathcal{L}$):

$(\mathcal{A}, \beta) \models x^Z \doteq y^Z$ iff $\mathcal{I}(x^Z) = \mathcal{I}(y^Z)$,

$(\mathcal{A}, \beta) \models x^{T^i_k} y^{T^i_m}$ iff $\mathcal{I}(y^{T^i_m}) \subseteq \mathcal{I}(x^{T^i_k})$, for $-1 \leq k < m$,

$(\mathcal{A}, \beta) \models x^{T^i_k} y^{T^i}$ iff $\mathcal{I}(y^{T^i}) \in \mathcal{I}(x^{T^i_k})$, for $k \in \mathbb{N} \cup \{-1\}$,

$(\mathcal{A}, \beta) \models \underline{B}_i x^{S_i}$ iff $\mathcal{I}(x^{S_i}) \in B_i$,

$(\mathcal{A}, \beta) \models \underline{R}^i_k(x^{T^i_k}, \underline{B}_i, x^{S_i}, (x^{T^j_k})_{j \neq i})$ iff
$\mathcal{I}(x^{T^i_k}) \times \{B_i\} \times \{\mathcal{I}(x^{S_i})\} \times \Pi_{j \neq i} \mathcal{I}(x^{T^j_k}) \subseteq R^i_k$, for $k \in \mathbb{N}$,

$(\mathcal{A}, \beta) \models \underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j \neq i})$ iff
$(\mathcal{I}(x^{T^i}), B_i, \mathcal{I}(x^{S_i}), (\mathcal{I}(x^{T^j}))_{j \neq i}) \in R^i$,

$(\mathcal{A}, \beta) \models \neg\varphi$ iff $(\mathcal{A}, \beta) \not\models \varphi$,

$(\mathcal{A}, \beta) \models \varphi \vee \psi$ iff $(\mathcal{A}, \beta) \models \varphi$ or $(\mathcal{A}, \beta) \models \psi$,

$(\mathcal{A}, \beta) \models \bigwedge_{\varphi \in \Phi} \varphi$ iff for all $\varphi \in \Phi$: $(\mathcal{A}, \beta) \models \varphi$,

$(\mathcal{A}, \beta) \models \exists v^Z_n \varphi$ iff there is a $c \in Z$ such that $(\mathcal{A}, \beta\frac{c}{v^Z_n}) \models \varphi$,

$(\mathcal{A}, \beta) \models \forall v^Z_n \varphi$ iff $(\mathcal{A}, \beta) \models \neg\exists v^Z_n \neg\varphi$.

Convention 4.3 If we write $\varphi(v^{Z_1}_1, \ldots, v^{Z_l}_l)$, for (not necessary different) sorts $Z_1, \ldots, Z_l$ of $\mathcal{L}$ ($\mathcal{L}^-$, resp.), we indicate that $\text{free}(\varphi) \subseteq \{v^{Z_1}_1, \ldots, v^{Z_l}_l\}$.

Definition 4.47 Let Z be a sort of $\mathcal{L}$ ($\mathcal{L}^-$, resp.), let v^Z be a variable of sort Z, and let β be a $\mathcal{L}$-assignment. Define

$$\underline{\beta}(v^Z) := \begin{cases} \beta(v^Z), & \text{if } \; Z \text{ is a small sort of } \mathcal{L}, \\ \{\beta(v^Z)\}, & \text{if } \; Z \text{ is a big sort of } \mathcal{L}. \end{cases}$$

Definition 4.48 For $m = 1, \ldots, l$ let Z_m be a sort of $\mathcal{L}$ and $v_m^{Z_m}$ a variable of sort Z_m, and let β be a $\mathcal{L}$-assignment. Define

$$\underline{\beta}(v_1^{Z_1}, \ldots, v_l^{Z_l}) := \underline{\beta}(v_1^{Z_1}) \times \ldots \times \underline{\beta}(v_l^{Z_l}).$$

Again, by definition, a straightforward check shows:

Lemma 4.12 *Let β, β' be $\mathcal{L}$-assignments and let $\varphi \in \mathcal{L}$ ($\varphi \in \mathcal{L}^-$, resp.).*

1. *If $\beta \restriction \mathrm{free}(\varphi) = \beta' \restriction \mathrm{free}(\varphi)$, then:*

$$(\mathcal{A}, \beta) \models \varphi \textit{ iff } (\mathcal{A}, \beta') \models \varphi.$$

2. *Let $Z_1, \ldots, Z_l$ be sorts of $\mathcal{L}$ and $v_m^{Z_m}$ be a variable of sort Z_m, for $m = 1, \ldots, l$, then:*

$$\begin{array}{rcll} \underline{\beta}(v_1^{Z_1}, \ldots, v_l^{Z_l}) & = & \underline{\beta'}(v_1^{Z_1}, \ldots, v_l^{Z_l}), \textit{ iff} & \\ \beta(v_m^{Z_m}) & = & \beta'(v_m^{Z_m}), \quad \textit{for } m = 1, \ldots, l, & \\ \underline{\beta}(v_1^{Z_1}, \ldots, v_l^{Z_l}) \cap \underline{\beta'}(v_1^{Z_1}, \ldots, v_l^{Z_l}) & = & \emptyset, & \textit{else.} \end{array}$$

Definition 4.49 Let $Z_1, \ldots, Z_l$ be big sorts of $\mathcal{L}$ and let $k \in \mathbb{N} \cup \{-1\}$.
A subset $A \subseteq Z_1 \times \ldots \times Z_l$ is *defined by* $\varphi(v_1^{\hat{Z}_1}, \ldots, v_l^{\hat{Z}_l}) \in \mathcal{L}$ ($\varphi(v_1^{\hat{Z}_1}, \ldots, v_l^{\hat{Z}_l}) \in \mathcal{L}^-$, resp.), iff

1. $\mathrm{big}(\hat{Z}_m) = Z_m$, for $m = 1, \ldots, l$,
2. $A = \bigcup \left\{ \underline{\beta}(v_1^{\hat{Z}_1}, \ldots, v_l^{\hat{Z}_l}) \,\middle|\, \beta \; \mathcal{L}\text{-assignment such that } (\mathcal{A}, \beta) \models \varphi \right\}.$

A is *$\mathcal{L}$-definable* (*$\mathcal{L}^-$-definable*, resp.), if there is a $\varphi \in \mathcal{L}$ ($\varphi \in \mathcal{L}^-$, resp.) such that A is defined by φ.
A is *$\mathcal{L}_k$-definable*, if there is a $\varphi \in \mathcal{L}_k$ such that A is defined by φ.

Notation 4.3 Let $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}$ ($\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}^-$, resp.). Denote by $[\varphi]^{\mathcal{A}}$ the subset of $\prod_{m=1}^{l} \text{big}(Z_m)$ that is defined by φ.

Note that, by Lemma 4.12 and the fact that the possible values of a variable of a small sort of $\mathcal{L}$ under $\mathcal{L}$-assignments partition the corresponding big sort of $\mathcal{L}$, it follows that:

Remark 4.4 *Let* $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})$, $\psi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}$ *(*$\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})$, $\psi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}^-$*, resp.), and let* $\Phi \subseteq \mathcal{L}$ *(*$\Phi \subseteq \mathcal{L}^-$*, resp.) such that there is a finite set of variables* V *such that for all* $\varphi \in \Phi$ *: free*$(\varphi) \subseteq V$*. Then:*

$$\begin{aligned} [\neg\varphi]^{\mathcal{A}} &= (\text{big}(Z_1) \times \ldots \times \text{big}(Z_l)) \setminus [\varphi]^{\mathcal{A}}, \\ [\varphi \vee \psi]^{\mathcal{A}} &= [\varphi]^{\mathcal{A}} \cup [\psi]^{\mathcal{A}}, \\ \left[\textstyle\bigwedge_{\varphi\in\Phi} \varphi\right]^{\mathcal{A}} &= \textstyle\bigcap_{\varphi\in\Phi} [\varphi]^{\mathcal{A}}. \end{aligned}$$

Definition 4.50

1. Let $k \in \mathbb{N} \cup \{-1\}$ and $Z_1, \ldots, Z_l$ be big sorts of $\mathcal{L}$. Denote by

   (a)

   $$\text{Def}_{\mathcal{L}_k}(Z_1 \times \ldots \times Z_l)$$

   the set of *nonempty* $\mathcal{L}_k$*-definable subsets of* $Z_1 \times \ldots \times Z_l$, and by

   (b)

   $$\text{Def}_{\mathcal{L}}(Z_1 \times \ldots \times Z_l)$$

   the set of *nonempty* $\mathcal{L}$*-definable subsets of* $Z_1 \times \ldots \times Z_l$.

2. Let $k \in \mathbb{N} \cup \{-1\}$, $i \in N$ and $B_i \in \mathcal{B}_i$. Denote by

   (a)

   $$\text{Def}_{\mathcal{L}_k}\left(B_i \times \Pi_{j\neq i} T^j\right)$$

   the set of nonempty subsets

   $$A \subseteq B_i \times \Pi_{j\neq i} T^j$$

   such that there is a $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}_k$ with $\text{big}(Z_j) = T^j$, for $j \neq i$, such that $[\varphi]^{\mathcal{A}} = A$, and by

(b)

$$\mathrm{Def}_{\mathcal{L}}\left(B_i \times \Pi_{j\neq i}T^j\right)$$

the set of nonempty subsets

$$A \subseteq B_i \times \Pi_{j\neq i}T^j$$

such that there is a $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}$ with $\mathrm{big}(Z_j) = T^j$, for $j \neq i$, such that $[\varphi]^{\mathcal{A}} = A$.

3. Let $k \in \mathbb{N} \cup \{-1\}$ and $i \in N$. Denote by

(a)

$$\mathrm{Def}^{\mathcal{B}_i}_{\mathcal{L}_k}\left(S_i \times \Pi_{j\neq i}T^j\right)$$

the set of A-nice (B-nice) elements of

$$\prod_{B_i \in \mathcal{B}_i} \mathrm{Def}_{\mathcal{L}_k}\left(B_i \times \Pi_{j\neq i}T^j\right)$$

in the A-version (in the B-version), and by

(b)

$$\mathrm{Def}^{\mathcal{B}_i}_{\mathcal{L}}\left(S_i \times \Pi_{j\neq i}T^j\right)$$

the set of A-nice (B-nice) elements of

$$\prod_{B_i \in \mathcal{B}_i} \mathrm{Def}_{\mathcal{L}}\left(B_i \times \Pi_{j\neq i}T^j\right)$$

in the A-version (in the B-version).

Note that if $C_i, D_i \in \mathcal{B}_i$, $C_i \subseteq D_i$, $k \in \mathbb{N} \cup \{-1\}$, and $[\varphi]^{\mathcal{A}} \subseteq D_i \times \Pi_{j\neq i}T^j$, for $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}_k$ (resp. $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}$), where $\mathrm{big}(Z_j) = T^j$, for $j \neq i$, then

$$[\varphi]^{\mathcal{A}} \cap \left(C_i \times \Pi_{j\neq i}T^j\right) = \left[\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \wedge \underline{C}_i v^{S_i}\right]^{\mathcal{A}}.$$

Note furthermore that $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}_k$ (resp. $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \in \mathcal{L}$) implies that $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \wedge \underline{C}_i v^{S_i} \in \mathcal{L}_k$ (resp. $\varphi(v^{S_i}, (v^{Z_j})_{j\neq i}) \wedge \underline{C}_i v^{S_i} \in \mathcal{L}$).

Definition 4.51 Let $k \in \mathbb{N} \cup \{-1\}$ and let Z be a sort of $\mathcal{L}_k$. Define

$$\pi_k^Z := \begin{cases} p_k^j, & \text{if } \operatorname{big}_k(Z) = T_k^j, \text{ for a } j \in N, \\ \mathrm{id}_{S_j}, & \text{if } \operatorname{big}_k(Z) = S_j, \text{ for a } j \in N. \end{cases}$$

Lemma 4.13 *Let $k \in \mathbb{N} \cup \{-1\}$ and let β be a $\mathcal{L}$-assignment. Define for $j \in N$:*

$$\begin{array}{rcl} (\beta)_k(v^{T_m^j}) & := & (q_{m,k}^j)^{-1}(\{t_m^j\}), \text{ for the unique } t_m^j \in T_m^j \text{ such that} \\ & & \qquad \beta(v^{T_m^j}) = (p_m^j)^{-1}(\{t_m^j\}), \text{ if } -1 \leq m < k, \\ (\beta)_k(v^{T_k^j}) & := & \text{the unique } t_k^j \in T_k^j \text{ such that } \beta(v^{T_k^j}) = (p_k^j)^{-1}(\{t_k^j\}), \\ (\beta)_k(v^{S_j}) & := & \beta(v^{S_j}). \end{array}$$

Then:

1. $(\beta)_k$ *is a* $\mathcal{L}_k$*-assignment.*
2. *For every* $\mathcal{L}_k$*-assignment* β_k*, there is a* $\mathcal{L}$*-assignment* β *such that* $\beta_k = (\beta)_k$.
3. *If* $(\beta)_k = (\beta')_k$*, then* β *and* β' *coincide on the variables of* $\mathcal{L}_k$*-sorts.*

Proof The first and the third point are clear, for the second:
Define for $j \in N$:

$$\begin{array}{rcl} \beta(v^{T_m^j}) & := & (p_k^j)^{-1}(\beta_k(v^{T_m^j})), \text{ if } -1 \leq m < k, \\ \beta(v^{T_k^j}) & := & (p_k^j)^{-1}(\{\beta_k(v^{T_k^j})\}), \\ \beta(v^{S_j}) & := & \beta_k(v^{S_j}). \end{array}$$

It is obvious that β can be extended to a $\mathcal{L}$-assignment. ■

Lemma 4.14 *Let $k \in \mathbb{N} \cup \{-1\}$ and $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}_k$. Then*:

$$\left(\Pi_{m=1}^l \pi_k^{Z_m}\right)^{-1} \left([\varphi]^{\mathcal{A}_k}\right) = [\varphi]^{\mathcal{A}}.$$

Proof The proof is analogous to the proof of Lemma 4.11: Lemma 4.12 plays now the role of Lemma 4.9 and Lemma 4.13 plays the role of Lemma 4.10. ■

Proposition 4.4 *Let $n \in \mathbb{N}$ and $i \in N$.*

1. *Let*

$$\varphi^{B_i}(v_{B_i}^{S_i}, (v_{B_i}^{T^j_{k(j,B_i)}})_{j\neq i}) \in \mathcal{L}_{n-1}, \quad \textit{for} \quad B_i \in \mathcal{B}_i,$$

such that

$$\left([\varphi^{B_i}]^{\mathcal{A}_{n-1}}\right)_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j_{n-1}).$$

Then there is exactly one $t^i_n \in T^i_n$ such that for all $B_i \in \mathcal{B}_i$:

$$\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j_n})_{j\neq i})\right]^{\mathcal{A}} = [\varphi^{B_i}]^{\mathcal{A}}.$$

2. *For every $t^i_n \in T^i_n$ there are $\varphi^{B_i}(v_{B_i}^{S_i}, (v_{B_i}^{T^j_{k(j,B_i)}})_{j\neq i}) \in \mathcal{L}_{n-1}$, for $B_i \in \mathcal{B}_i$, such that*

$$\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j_n})_{j\neq i})\right]^{\mathcal{A}} = [\varphi^{B_i}]^{\mathcal{A}} \quad \textit{for all} \ \ B_i \in \mathcal{B}_i,$$

and such that

$$\left([\varphi^{B_i}]^{\mathcal{A}_{n-1}}\right)_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j_{n-1}).$$

3. *For every $t^i_n \in T^i_n$:*

$$\left(\left[\underline{R}^i_n(\underline{t}^i_n, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j_n})_{j\neq i})\right]^{\mathcal{A}}\right)_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}(S_i \times \Pi_{j\neq i} T^j).$$

Proof

1. The existence of such a t^i_n follows from Lemma 4.14 and 1 of Proposition 4.2. The uniqueness follows from the uniqueness of the t^i_n in 1 of Proposition 4.2, Lemma 4.14, and the fact that all the $\pi_n^{Z_m}$ are onto (2 of Proposition 4.3).

2. Follows from Lemma 4.14 and 2 of Proposition 4.2.

3. Since all the $\pi_n^{Z_m}$ are onto and since inverse images commute with unions, intersections and complements, it follows (in the A-version and in the B-version) for

$$(A_{B_i})_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j_n\right)$$

that

$$\left(\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n\right)^{-1}(A_{B_i})\right)_{B_i \in \mathcal{B}_i} \in \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j\right).$$

The result follows now from 3 of Proposition 4.2 and Lemma 4.14.

■

Lemma 4.15 *Let X and Y be compacta and let $A \subseteq X \times Y$ be closed in $X \times Y$ with respect to the product topology. Then $\mathrm{proj}_X(A)$, the image of A under the projection to X, is closed in X.*

Proof The projection is continuous. Since A is closed and $X \times Y$ is compact, A is compact (with respect to the relative topology). Therefore $\mathrm{proj}_X(A)$ is compact. Since X is Hausdorff, $\mathrm{proj}_X(A)$ is closed in X. ∎

Remark and Convention 4.1 *As in the proof of Proposition 4.3, endow, for every $i \in N$ and $m \in \mathbb{N} \cup \{-1\}$, S_i and T^i_m with the discrete topology. Since each of these spaces is nonempty and finite, each of them is then a nonempty compactum. By induction, it follows from Lemma 4.6 that for every $i \in N$ and every $n \in \mathbb{N}$:*

1. $T^i_n = \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j_{n-1}\right)$.
2. *The topology of $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j_{n-1}\right)$ is the discrete topology, hence the topologies on T^i_n and $\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j_{n-1}\right)$ are the same, and they are compact and Hausdorff.*
3. *T^i is the underlying set of the projective limit of the projective system $\left(T^i_n, q^i_{n,n+1}\right)_{n\in\mathbb{N}\cup\{-1\}}$.*
4. *For every $m \in \mathbb{N} \cup \{-1\}$, the T^i_m here is then the T^i_m in the third point of Theorem 4.1, the $q^i_{m,m+1}$ here is the q^i_{m+1} there and the T^i here is the T^i there.*

For the rest of this chapter, we consider T^i_m, for $i \in N$ and $m \in \mathbb{N} \cup \{-1\}$, as topological space endowed with the discrete topology, and T^i, for $i \in N$, as topological space endowed with the projective limit topology induced by the projective system $\left(T^i_n, q^i_{n,n+1}\right)_{n\in\mathbb{N}\cup\{-1\}}$. This enables us to apply, in all what follows, Theorem 4.1. In particular, the p^i_m, for $i \in N$ and $m \in \mathbb{N}\cup\{-1\}$, are then continuous and onto, and the T^i, for $i \in N$, are then nonempty compacta.

Lemma 4.16 *Let $Z_1, \ldots, Z_l$ be sorts of $\mathcal{L}$ and consider each of the spaces $\mathrm{big}\,(Z_1), \ldots, \mathrm{big}\,(Z_l)$ as topological space endowed with the topology of Remark and Convention 4.1.*
Let $\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l}) \in \mathcal{L}^-$ ($\in \mathcal{L}$, resp.) be such that $[\varphi]^{\mathcal{A}}$ is closed in $\mathrm{big}\,(Z_1) \times \ldots \times \mathrm{big}\,(Z_l)$ endowed with the product topology, and let $c_m \in Z_m$, for $m = \hat{l}+1, \ldots, l$, where $\hat{l} < l$. Set

$$\psi(v_1^{Z_1}, \ldots, v_{\hat{l}}^{Z_{\hat{l}}}) := \varphi(v_1^{Z_1}, \ldots, v_{\hat{l}}^{Z_{\hat{l}}}, \underline{c_{\hat{l}+1}}, \ldots, \underline{c_l}).$$

Then $[\psi]^{\mathcal{A}}$ is closed in $\mathrm{big}\,(Z_1) \times \ldots \times \mathrm{big}\,(Z_{\hat{l}})$ endowed with the product topology.

Proof We show the lemma for $\hat{l}+1 = l$, the rest follows immediately by induction. By the definitions of assignments and of the relation "$\models$", it follows that:

- If Z_l is a big sort of $\mathcal{L}$, then $[\psi]^{\mathcal{A}}$ is the image of

$$[\varphi]^{\mathcal{A}} \cap (\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}}) \times \{c_l\})$$

  under the projection to $\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}})$.

- If Z_l is a small sort of $\mathcal{L}$, i.e. if $Z_l = T_k^j$, for a $k \in \mathbb{N} \cup \{-1\}$ and a $j \in N$, then $[\psi]^{\mathcal{A}}$ is the image of

$$[\varphi]^{\mathcal{A}} \cap \left(\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}}) \times (p_k^j)^{-1}\,(\{c_l\})\right)$$

  under the projection to $\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}})$.

$$[\varphi]^{\mathcal{A}} \cap (\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}}) \times \{c_l\})\,,$$

respectively

$$[\varphi]^{\mathcal{A}} \cap \left(\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}}) \times (p_k^j)^{-1}\,(\{c_l\})\right)$$

is closed in $\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_l)$. By the above lemma, it follows now that $[\psi]^{\mathcal{A}}$ is closed in $\text{big}\,(Z_1) \times \ldots \times \text{big}\,(Z_{\hat{l}})$. ∎

Notation 4.4 For $i \in N$, let δ^i be the homeomorphism defined in Theorem 4.1. If

$$\delta^i\left(t^i\right) = (A_{B_i})_{B_i \in \mathcal{B}_i} \in \mathcal{V}^{\mathcal{B}_i}(S_i \times \Pi_{j \neq i} T^j)$$

and $C_i \in \mathcal{B}_i$, then define

$$\left(\delta^i\left(t^i\right)\right)(C_i) := A_{C_i}.$$

Definition 4.52 For $i \in N$ and $t^i \in T^i$ define

$$\text{pos}_i\left(t^i\right) := \left(\left[\underline{R}^i(\underline{t}^i, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j})_{j \neq i})\right]^{\mathcal{A}}\right)_{B_i \in \mathcal{B}_i}.$$

Now, we state and prove the main results of Section 4.4, Theorem 4.2, Corollary 4.2, and Corollary 4.3.

Theorem 4.2 *For every $i \in N$:*

- $$\mathrm{pos}_i : T^i \to \mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j\right),$$

- $$\mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j\right) = \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j\neq i} T^j\right),$$

- $$\mathrm{pos}_i = \delta^i,$$

  where δ^i is the δ^i defined in Theorem 4.1.

Proof ot the theorem We apply Remark and Convention 4.1. In particular, we endow S_i, for $i \in N$, and T_m^i, for $i \in N$ and $m \in \mathbb{N} \cup \{-1\}$, with the discrete topology, and T^i, for $i \in N$, with the projective limit topology. In all what follows let δ^i be the δ^i defined in Theorem 4.1. By definition, we have for every $t^i \in T^i$:

$$\begin{array}{lcll} \delta^i(t^i) & \in & \prod_{B_i \in \mathcal{B}_i} \mathcal{V}\left(B_i \times \Pi_{j\neq i} T^j\right) & \subseteq \\ & & \prod_{B_i \in \mathcal{B}_i} \mathrm{Pow}_\emptyset\left(B_i \times \Pi_{j\neq i} T^j\right) & \text{and} \\ \left(\left[\underline{R}^i\left(\underline{t}^i, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j})_{j\neq i}\right)\right]^{\mathcal{A}}\right)_{B_i \in \mathcal{B}_i} & \in & \prod_{B_i \in \mathcal{B}_i} \mathrm{Def}_{\mathcal{L}}\left(B_i \times \Pi_{j\neq i} T^j\right) & \subseteq \\ & & \prod_{B_i \in \mathcal{B}_i} \mathrm{Pow}_\emptyset\left(B_i \times \Pi_{j\neq i} T^j\right). & \end{array}$$

We show:

Lemma 4.17 *For every $i \in N$, $t^i \in T^i$, and $B_i \in \mathcal{B}_i$:*

$$\left(\delta^i(t^i)\right)(B_i) = \left[\underline{R}^i\left(\underline{t}^i, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j})_{j\neq i}\right)\right]^{\mathcal{A}}.$$

Proof of the lemma By the definitions:

$$\begin{array}{lll} (t^i, B_i, s_i, (t^j)_{j\neq i}) \in R^i & \text{iff} \quad \text{for all } n \in \mathbb{N}: & (p_n^i(t^i), B_i, s_i, (p_n^j(t^j))_{j\neq i}) \in R_n^{i,n} \\ & \text{iff} \quad \text{for all } n \in \mathbb{N}: & (s_i, (p_{n-1}^j(t^j))_{j\neq i}) \in (p_n^i(t^i))(B_i). \end{array}$$

This shows that

$$\bigcap_{n\in\mathbb{N}} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}\right)^{-1} \left((p^i_n(t^i))(B_i)\right) = \left[\underline{R}^i \left(\underline{t}^i, \underline{B}_i, v^{S_i}_{B_i}, (v^{T^j}_{B_i})_{j\neq i}\right)\right]^{\mathcal{A}}.$$

We have to verify the following
Claim:

$$(\delta^i(t^i))(B_i) = \bigcap_{n\in\mathbb{N}} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}\right)^{-1} \left((p^i_n(t^i))(B_i)\right).$$

By the third point of Theorem 4.1, we have for $n \in \mathbb{N}$:

$$\mathcal{V}^{\mathcal{B}_i} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}\right) (\delta^i(t^i)) = p^i_n(t^i).$$

It follows that

$$\begin{aligned} (\delta^i(t^i))(B_i) &\subseteq \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{n-1}\right)^{-1} ((p^i_n(t^i))(B_i)), \\ \text{hence } (\delta^i(t^i))(B_i) &\subseteq \textstyle\bigcap_{m\in\mathbb{N}} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m-1}\right)^{-1} ((p^i_m(t^i))(B_i)). \end{aligned}$$

The right-hand-side above is closed, since each of the $\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m-1}\right)^{-1} ((p^i_m(t^i))(B_i))$ is closed in $B_i \times \Pi_{j\neq i} T^j$ as an inverse image of a closed set under a continuous map, and the left-hand-side is closed by the definition of δ^i.

From the proof of Lemma 4.8 (the part " γ^i is one-to-one") follows for $B_i \in \mathcal{B}_i$, that, if A_{B_i} and K_{B_i} are closed in $B_i \times \Pi_{j\neq i} T^j$ and if for all $m \in \mathbb{N} \cup \{-1\}$

$$\left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m\right) (A_{B_i}) = \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_m\right) (K_{B_i}),$$

then $A_{B_i} = K_{B_i}$.

For $n \in \mathbb{N} \cup \{-1\}$, we have

$$\begin{aligned} &\left(\mathcal{V}^{\mathcal{B}_i} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n\right) (\delta^i(t^i))\right) (B_i) \\ &\quad = \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n\right) \left((\delta^i(t^i))(B_i)\right) \\ &\quad \subseteq \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n\right) \left(\textstyle\bigcap_{m\in\mathbb{N}} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_{m-1}\right)^{-1} ((p^i_m(t^i))(B_i)) \right) \\ &\quad \subseteq \left(p^i_{n+1}(t^i)\right) (B_i) \\ &\quad = \left(\mathcal{V}^{\mathcal{B}_i} \left(\mathrm{id}_{S_i} \times \Pi_{j\neq i} p^j_n\right) (\delta^i(t^i))\right) (B_i). \end{aligned}$$

So, the Claim and therefore also the Lemma is proved. ■

We proceed with the **proof of the theorem.** We showed now that $\delta^i = \mathrm{pos}_i$, for $i \in N$. Since, for $i \in N$ and $t^i \in T^i$, $\delta^i(t^i)$ is A-nice in the A-version (resp. B-nice in the B-version), if follows that, for $i \in N$:

$$\mathrm{pos}_i : T^i \to \mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right).$$

Since

$$\delta^i : T^i \to \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right)$$

is onto, for $i \in N$, it follows that, for $i \in N$:

$$\mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right) \subseteq \mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right).$$

It remains to show that

$$\mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right) \subseteq \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right), \quad \text{for } i \in N.$$

Since, by definition, for $i \in N$:

$$\begin{array}{lcl} \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right) \cap \prod_{B_i \in \mathcal{B}_i} \mathcal{V}\left(B_i \times \Pi_{j \neq i} T^j\right) & = & \mathcal{V}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right) \quad \text{and} \\ \mathrm{Pow}_{\emptyset}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right) \cap \prod_{B_i \in \mathcal{B}_i} \mathrm{Def}_{\mathcal{L}}\left(B_i \times \Pi_{j \neq i} T^j\right) & = & \mathrm{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right), \end{array}$$

it is enough to show (by an induction on the formation of formulas in $\mathcal{L}$) that all $\mathcal{L}$-definable subsets of finite products of big sorts of $\mathcal{L}$ are closed (where the topologies are the ones of Remark and Convention 4.1).

First we note a helpful observation:

Remark 4.5 *Let* $\mathrm{free}(\varphi) \subseteq \{v_1^{Z_1}, \ldots, v_l^{Z_l}\} \subsetneq \{v_1^{Z_1}, \ldots, v_l^{Z_l}, v_{l+1}^{Z_{l+1}}, \ldots, v_m^{Z_m}\}$. *If we denote by* $[\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})]^{\mathcal{A}}$ *the subset of* $\mathrm{big}(Z_1) \times \ldots \times \mathrm{big}(Z_l)$ *defined by* φ *and by* $[\varphi(v_1^{Z_1}, \ldots, v_m^{Z_m})]^{\mathcal{A}}$ *the subset of* $\mathrm{big}(Z_1) \times \ldots \times \mathrm{big}(Z_m)$ *defined by* φ, *it follows that*

$$[\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})]^{\mathcal{A}} \times \mathrm{big}(Z_{l+1}) \times \ldots \times \mathrm{big}(Z_m) = [\varphi(v_1^{Z_1}, \ldots, v_m^{Z_m})]^{\mathcal{A}}.$$

By the definition of the product topology, $[\varphi(v_1^{Z_1}, \ldots, v_l^{Z_l})]^{\mathcal{A}}$ *is closed in* $\mathrm{big}(Z_1) \times \ldots \times \mathrm{big}(Z_l)$ *iff* $[\varphi(v_1^{Z_1}, \ldots, v_m^{Z_m})]^{\mathcal{A}}$ *is closed in* $\mathrm{big}(Z_1) \times \ldots \times \mathrm{big}(Z_m)$. *Therefore, to show that a subset* $[\varphi]^{\mathcal{A}}$ *defined by* $\varphi \in \mathcal{L}$ *is closed, it does not matter which set* $\{v_1^{Z_1}, \ldots, v_l^{Z_l}\} \supseteq \mathrm{free}(\varphi)$ *we consider.*

We start with the induction on the formation of the formulas in $\mathcal{L}$:

1. If, $\varphi(v_1^{Z_1}, \dots, v_l^{Z_l}) \in \mathcal{L}_n$, for a $n \in \mathbb{N} \cup \{-1\}$, then $[\varphi]^{\mathcal{A}}$ is closed (even clopen), since by Lemma 4.14, $[\varphi]^{\mathcal{A}}$ is the inverse image of the clopen set $[\varphi]^{\mathcal{A}_n}$ under the continuous map $\Pi_{m=1}^{l} \pi_n^{Z_m}$.

2. Let $i \in N$ and let $v_m^{T^i}$ and $v_l^{T^i}$ be variables of sort T^i.

   Case a: $m \neq l$: $\left[v_m^{T^i} \doteq v_l^{T^i}\right]^{\mathcal{A}}$ is the diagonal of $T^i \times T^i$, which is closed, since T^i is Hausdorff.

   Case b: $m = l$: Then, $\left[v_m^{T^i} \doteq v_l^{T^i}\right]^{\mathcal{A}} = T^i$.

   The case $\varphi = x^{T^i} \doteq y^{T^i}$, where x^{T^i} and y^{T^i} are terms of sort T^i, follows now by Lemma 4.16. (If $x^{T^i} = \underline{t}^i \neq \widehat{\underline{t}}^i = y^{T^i}$, then $\left[x^{T^i} \doteq y^{T^i}\right]^{\mathcal{A}} = \emptyset$.)

3. Let $i \in N$ and $k \in \mathbb{N} \cup \{-1\}$, and let $v^{T_k^i}$ be a variable of sort T_k^i and v^{T^i} be a variable of sort T^i.

   Then it is easy to check that $\left[v^{T_k^i} v^{T^i}\right]^{\mathcal{A}} = \left[v_m^{T_k^i} \doteq v_l^{T_k^i}\right]^{\mathcal{A}}$, where $m \neq l$. But $v_m^{T_k^i} \doteq v_l^{T_k^i} \in \mathcal{L}_k$, hence by 1., this set is closed.

   The case $\varphi = x^{T_k^i} y^{T^i}$, where $x^{T_k^i}$ is a term of sort T_k^i and y^{T^i} is a term of sort T^i, follows again by Lemma 4.16.

4. Let $i \in N$, let v^{S_i} be a variable of sort S_i and, for $j \in N$, let v^{T^j} be a variable of sort T^j. By the definition, we have

   $$\left[\underline{R}^i(v^{T^i}, \underline{B}_i, v^{S_i}, (v^{T^j})_{j \neq i})\right]^{\mathcal{A}} =$$
   $$\bigcap_{n \geq 0} \left(p_n^i \times \mathrm{id}_{S_i} \times \Pi_{j \neq i} p_n^j\right)^{-1}\Big(\{(t_n^i, s^i, (t_n^j)_{j \neq i}) \mid (t_n^i, B_i, s^i, (t_n^j)_{j \neq i}) \in R_n^{i,n}\} \Big).$$

   For $n \in \mathbb{N}$, the sets $\{(t_n^i, s^i, (t_n^j)_{j \neq i}) \mid (t_n^i, B_i, s^i, (t_n^j)_{j \neq i}) \in R_n^{i,n}\}$ are clopen, the mappings $p_n^i \times \mathrm{id}_{S_i} \times \Pi_{j \neq i} p_n^j$ are continuous, hence $\left[\underline{R}^i(v^{T^i}, \underline{B}_i, v^{S_i}, (v^{T^j})_{j \neq i})\right]^{\mathcal{A}}$ is closed in $T^i \times S_i \times \Pi_{j \neq i} T^j$.

   The case $\varphi = \underline{R}^i(x^{T^i}, \underline{B}_i, x^{S_i}, (x^{T^j})_{j \neq i})$, where x^{S_i} is a term of sort S_i and x^{T^j} is a term of sort T^j, for $j \in N$, follows by Lemma 4.16.

5. Let $\varphi = \psi_0 \vee \psi_1$. According to Remark 4.5, we can assume that $\psi_0 = \psi_0(v_1^{Z_1}, \dots, v_l^{Z_l})$ and $\psi_1 = \psi_1(v_1^{Z_1}, \dots, v_l^{Z_l})$. By Remark 4.4, we have $[\varphi]^{\mathcal{A}} = [\psi_0]^{\mathcal{A}} \cup [\psi_1]^{\mathcal{A}}$, which is closed, since $[\psi_0]^{\mathcal{A}}$ and $[\psi_1]^{\mathcal{A}}$ are closed by the induction assumption.

6. Let $\varphi = \bigwedge_{\psi \in \Psi} \psi$ such that $\text{free}(\psi) \subseteq \{v_1^{Z_1}, \dots, v_l^{Z_l}\}$, for all $\psi \in \Psi$. We can assume that $\psi = \psi(v_1^{Z_1}, \dots, v_l^{Z_l})$, for all $\psi \in \Psi$. By Remark 4.4, we have $[\varphi]^{\mathcal{A}} = \bigcap_{\psi \in \Psi} [\psi]^{\mathcal{A}}$. Each $[\psi]^{\mathcal{A}}$ is closed by the induction assumption, hence $[\varphi]^{\mathcal{A}}$ is closed, since the intersection of an arbitrary family of closed sets is closed.

7. Let $\varphi = \exists v_l^{Z_l} \psi(v_1^{Z_1}, \dots, v_l^{Z_l})$.

   Case a: Z_l is a big sort of $\mathcal{L}$. Then, $[\varphi]^{\mathcal{A}}$ is the projection of $[\psi]^{\mathcal{A}}$ to $\text{big}(Z_1) \times \dots \times \text{big}(Z_{l-1})$. $[\psi]^{\mathcal{A}}$ is closed by the induction assumption, hence $[\varphi]^{\mathcal{A}}$ is closed by Lemma 4.15.

   Case b: Z_l is a small sort of $\mathcal{L}$. Then, Z_l is finite and

$$[\varphi]^{\mathcal{A}} = \bigcup_{c \in Z_l} \left[\psi(v_1^{Z_1}, \dots, v_{l-1}^{Z_{l-1}}, \underline{c}) \right]^{\mathcal{A}}.$$

   By the induction assumption, $[\psi]^{\mathcal{A}}$ is closed, hence by Lemma 4.16, the right-hand side is closed, since it is a finite union of closed sets.

8. Let χ be a $\mathcal{L}^-$-formula and $\text{free}(\chi) \subseteq \{v_{l+1}^{Z_{l+1}}, \dots, v_m^{Z_m}\}$, let $\psi(v_1^{Z_1}, \dots, v_m^{Z_m})$ be a $\mathcal{L}$-formula and let

$$\varphi(v_1^{Z_1}, \dots, v_l^{Z_l}) = \forall v_{l+1}^{Z_{l+1}} \dots \forall v_m^{Z_m} (\chi(v_{l+1}^{Z_{l+1}}, \dots, v_m^{Z_m}) \rightarrow \psi(v_1^{Z_1}, \dots, v_m^{Z_m})).$$

   By the definition of "$\models$", it follows that

$$[\varphi]^{\mathcal{A}} = \bigcap_{(c_{l+1}, \dots, c_m) \in Z_{l+1} \times \dots \times Z_m : \mathcal{A} \models \chi(c_{l+1}, \dots, c_m)} \left[\psi(v_1^{Z_1}, \dots, v_l^{Z_l}, \underline{c_{l+1}}, \dots, \underline{c_m}) \right]^{\mathcal{A}}.$$

   $[\varphi]^{\mathcal{A}}$ is closed, since by the induction assumption and Lemma 4.16, each

$$\left[\psi(v_1^{Z_1}, \dots, v_l^{Z_l}, \underline{c_{l+1}}, \dots, \underline{c_m}) \right]^{\mathcal{A}}$$

   is closed in $\text{big}(Z_1) \times \dots \times \text{big}(Z_l)$ and $[\varphi]^{\mathcal{A}}$ is the intersection of these closed sets,

and the theorem is proved. ■

Corollary 4.2 *For every $i \in N$, pos_i is one-to-one and onto.*

Proof For every $i \in N$, δ^i is a homeomorphism. ■

The following corollary, a reformulation of the above Theorem 4.2, could be viewed, in one sense, as a strong quantifier elimination theorem. In another sense, it is weaker as it does not talk about definable subsets of all finite products of sorts of $\mathcal{L}$, although ordinary quantifier elimination can be shown easily. The corollary says that all (A-nice in the A-version, resp. B-nice in the B-version) $\mathcal{L}$-definable conditional i-events are t^i-sections of the relation R^i. Thus, we get the desired $\mathcal{L}$-definably beliefs completeness result.

Corollary 4.3

- *For every $i \in N$ and every $t^i \in T^i$:*

$$\left(\left[\underline{R}^i(\underline{t}^i, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j})_{j \neq i})\right]^{\mathcal{A}}\right)_{B_i \in \mathcal{B}_i} \in \text{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right).$$

- *Let $i \in N$ and let, for every $B_i \in \mathcal{B}_i$, $\varphi^{B_i}(v_{B_i}^{S_i}, (v_{B_i}^{Z_j})_{j \neq i}) \in \mathcal{L}$ such that $\text{big}(Z_j) = T^j$, for $j \neq i$, and such that*

$$\left([\varphi^{B_i}]^{\mathcal{A}}\right)_{B_i \in \mathcal{B}_i} \in \text{Def}_{\mathcal{L}}^{\mathcal{B}_i}\left(S_i \times \Pi_{j \neq i} T^j\right).$$

  Then there is exactly one $t^i \in T^i$ such that for all $B_i \in \mathcal{B}_i$:

$$\left[\underline{R}^i(\underline{t}^i, \underline{B}_i, v_{B_i}^{S_i}, (v_{B_i}^{T^j})_{j \neq i})\right]^{\mathcal{A}} = [\varphi^{B_i}]^{\mathcal{A}}.$$

Proof Follows directly from Theorem 4.2 and Corollary 4.2. ■

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