Doctoral Thesis in Theoretical Physics

Cosmological Backreaction

From the local Hubble expansion rate to dark energy

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Ihr naht euch wieder, schwankende Gestalten,
Die früh sich einst dem trüben Blick gezeigt.
Versuch ich wohl, euch diesmal festzuhalten?
Fühl ich mein Herz noch jenem Wahn geneigt?

Johann Wolfgang von Goethe [1]

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[1] J.W. von Goethe, *Faust*.

往古来今谓之宙
四方上下谓之宇
道在其间
而莫知其所

刘安[2]

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[2] 刘安,《淮南子 · 齐俗训》(Liu An, *Huainanzi, Qisuxu*). These sentences mean: *Entire time is named "Zhou". Whole space is named "Yu". Inside the Universe ("Yuzhou") lie the laws of the Nature. But agnostic.* In Chinese language, "space-time" and "Universe" are the same word. This ancient coincidence from thousands of years ago must be appreciated by modern cosmologists.

Abstract

Despite the glorious successes of modern cosmology, our understanding of the cosmic substitution is still limited to a tiny fraction (a few per cents only). Accelerated expansion of the Universe, caused by the mysterious dark energy is currently the most severe crisis in cosmology, even in physics. In this dissertation, we argue that light may be shed on this crisis by means of the cosmological backreaction mechanism in the averaging problem in inhomogeneous and anisotropic space-time.

Due to the non-commutation of temporal evolution and spatial averaging, the averaged Einstein tensor as the function of the perturbed metric is not trivially equal to the Einstein tensor of the averaged metric. Consequently, inhomogeneities and anisotropies (cosmic structures) influence the evolution of the background Universe.

In order to obtain the quantitative information of this mechanism, we combine Buchert's non-perturbative framework with cosmological perturbation theory, calculate the relevant averaged physical observables up to third order in the comoving synchronous gauge (both temporal and spatial dependence) and discuss their gauge dependence. With the help of an integrability condition, the leading higher order contributions follow from the lower order calculations. We demonstrate that the leading contributions to all the averaged physical observables under consideration are specified completely on the boundary of the averaged domain. For any finite domain, these surface terms are nonzero in general, and thus backreaction is for real.

We map the backreaction effect on an effectively homogeneous and isotropic (fluid) model and prove that a cosmological constant can be obtained at third order. We further identify the backreaction effects to be observable up to scales of 200 Mpc. The cosmic variance of the local Hubble expansion rate is 10% for spherical regions of radius 45 Mpc and 5% for 60 Mpc. We compare our results to the data from the Hubble Space Telescope Key Project and the simulations in Newtonian cosmology and find excellent agreement.

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1 Preface

No idea in physics has ever astonished me more than the evolution of our very Universe, which eliminated all static, obstinate and fatuous viewpoints of the habitat that we are living in. Just as the scientific theory of biological evolution overthrew the religious doctrines on the development of our society, the scientific theory of the Universe – *modern cosmology* established the framework, in which every era during the evolution can be understood consistently and correctly, based on firm physical laws. We, human being, for the first time grasped the behavior of our Universe as a whole, in which we are just a tiny dust. Five decades ago, even theoretical physicists were not able to explain the abundance of chemical elements in galaxies and how matter are woven into cosmic structures. Whereas, nowadays, ten-year-old children are accustomed to the so-called "Big Bang" and discuss the origin of the Universe on their way home. Modern cosmology not only simply reformed the knowledge within the physics community, but has also thoroughly altered the world view of our mankind. People of our generation are lucky to live in this special epoch to be possible to witness all these exciting processes with our own eyes, experience all these fruitful progresses with our own feelings and maybe even contribute to this grand and magnificent palace of science with our own hands, if we devote our life to it.

The significance and importance of cosmology lie in the following reason: the Universe is the vastest laboratory that we can imagine and almost uniquely presents the fields, where *all* the four fundamental interactions exhibit their effects. Strong interaction operates at the epoch of the phase transition from quark-gluon plasma to hadron matter. In the primordial nucleosysthesis, the decay rate of neutron, which significantly influences the abundances of light nucleons, is governed by weak force. In the process of recombination of protons and electrons to form neutral hydrogen atom, electromagnetic interaction plays the main role. And last, the study of cosmology is one of the most suitable places to test the different theories of gravity.

Nevertheless, cosmology should not only be regarded as a subject, in which we merely apply our known physical laws, it also feeds back to our basic theories. One early example is the fact that cosmologists could constrain efficiently the upper bound of the number of the generation of leptons from the research on nucleosysthesis long before the experimental particle physicists were capable to directly detect them from the Z decay. Modern examples that cosmology brings forward fundamental questions to particle physics are even too numerous to be mentioned one by one. Everyday, we hear inflation, baryogenesis, dark matter and dark energy in seminars, conferences, workshops and coffee rooms. If we claimed that half of the current research on particle physics is stimulated by the study of cosmology, this would not be too excessive. Especially at present, when there is little contact between particle experiments and new theoretical ideas, cosmology is offering the passion that particle physicists once experienced in the 1960s to 70s of their golden days.

However, in spite of its splendid glories, cosmology is also suffering some intricate mysteries, amongst which the accelerated expansion of the Universe (or the dark energy problem) may be the most severe one. Observations from the supernovae of Type Ia and cosmic microwave background indicate that the majority of the constituents in the

Universe comprises some nonluminous matter that only show their effects gravitationally (called dark matter) and some strange composition with negative pressure inducing the acceleration (named dark energy). To explain this dark energy mystery, various attempts have been delivered, and this dissertation is also on the attempt to demystify it. We investigate the inhomogeneities and anisotropies in the Universe, exploring their effects on the expansion of the averaged Universe and trying to link the large scale acceleration to small scale structure via the so-called backreaction mechanism. Our quantitative calculations show that this backreaction mechanism is not sufficient to solve the global acceleration of our Universe. Thus, we turn to study the effects of this mechanism on the local Hubble expansion rate. To say the least, although cosmological perturbations are deficient to account for the dark energy problem at large scales, it is responsible for fluctuations of the expansion rate of the Universe at small scales.

This dissertation is organized as follows. In Sec. 2, we briefly give an introduction to the standard model of cosmology. Afterward, we discuss the three problems in cosmology: dark energy, coincidence and averaging problems, which are all related to the inhomogeneities and anisotropies in the Universe, and then explain the meaning of the cosmological backreaction mechanism. We formulate the averaging problem in the comoving synchronous gauge and arrive at the averaged Einstein equations for an irrotational dust universe in Sec. 3. An integrability condition and an effective morphon field approach to the backreaction problem are also investigated there. Next, we proceed to cosmological perturbation theory in Sec. 4, both at linear and higher orders. We establish the gauge transformations for metric perturbations and solve these metric perturbations up to second order. In the following three sections, we calculate the temporal and spatial dependence of the averaged physical observables: kinematical backreaction term $\langle Q \rangle_D$, averaged spatial curvature $\langle \mathcal{R} \rangle_D$, averaged volume expansion rate $\langle \theta \rangle_D$, averaged energy density $\langle \rho \rangle_D$, effective equation of state w_{eff} and square of the speed of sound c^2_{eff} to first, second and third orders, respectively. In Sec. 8, we consider the ensemble averages and variances of these averaged quantities and compare them with the experimental data from the Hubble Space Telescope Key Project and simulations in Newtonian gravity. Summary and Outlook are given in the last two sections.

2 Standard model of cosmology and its problems

Modern cosmology traces back to Einstein's application of his renowned theory of General Relativity (GR) [1] to the whole Universe [2]. Due to the severe insufficiency of experimental observations of cosmic structures at that time, two simplest working hypotheses were adopted by him: the Universe is homogeneous and isotropic in space and static in time.

Although short of experimental supports and theoretical explanations, these two assumptions seem quite natural, even at present. Since there is no preexistent reason that our Earth, Solar System or Galaxy are in a specially favored position, and humans are privileged observers in the Universe (Copernican principle), why not just follow Kopernik and Bruno to admit the mediocrity of ourselves? This admission directly brings on the spatial homogeneity and isotropy of the Universe. As for the temporal steadiness, this is also not surprising that people agreed so ninety years ago. Actually, before the discovery of Cepheids in the Andromeda Galaxy and the calibration of the Cepheid period-luminosity relation, astronomers were not even capable of distinguishing vicinal nebulae and remote galaxies, say nothing of the expansion of space-time itself.

One century has passed, and when we retrospect these two hypotheses, we find that the first one, bearing the name the *cosmological principle*, has stood various astronomical and cosmological tests, and turned to be experimental fact, at least at large spatial scales, say, above a few hundred Mpc [3]; while the second one, although abandoned pretty well immediately by Einstein himself after the epoch-making discovery of the recession of distant galaxies by Hubble [3], opened Pandora's box via the introduction of the cosmological constant, and has been bothering particle physicists and cosmologists since the establishment of quantum theory.

This dissertation is concerned with these two aspects. We will examine the validity of the spatial homogeneity and isotropy of the Universe at small scales and discuss their influences on the accelerated expansion of the Universe and the local Hubble expansion rate. But before doing so, we pause a moment and first review the standard model of cosmology. Then, three related problems, namely those of dark energy, coincidence and averaging, are discussed in order. We show that these three aspects can be linked by the cosmological backreaction mechanism, *i.e.*, the inhomogeneities and anisotropies of cosmic structures influence the evolution of the background Universe.

2.1 Dynamics of the expanding Universe

In this subsection, we first derive the basic dynamical equations for the expanding Universe and then briefly discuss the Hubble law.

2.1.1 Friedmann-Lemaître-Robertson-Walker model

The *standard model of cosmology* is the *inflationary* Λ*CDM (*Λ *cold dark matter) model*. To understand it, we need first go through the basic knowledge of GR.

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[3] Megaparsec (Mpc) is a distance unit in astronomy. 1 Mpc=3.086×10^{22} m=3.262×10^{6} ly.

The governing relativistic covariant theory of the gravitational field is encoded in the *Einstein equations*,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{1}$$

where $g_{\mu\nu}$ is the *metric* of space-time, $R_{\mu\nu}$ is the *Ricci tensor*, R is the *Ricci scalar*, $G_{\mu\nu}$ is the *Einstein tensor*, $T_{\mu\nu}$ is the *energy-momentum tensor* of the cosmic medium, G is Newton's gravitational constant, and finally Λ is the famous *cosmological constant*, which is the intrinsic freedom in the Lagrangian of the gravitational field. [4]

In contrast to the special relativity, the metric $g_{\mu\nu}$ is a dynamical variable in GR and is determined the motion of matter, *i.e.*, the energy-momentum tensor. For the simplest case, as Einstein suggested, spatial homogeneity and isotropy uniquely fix the metric (up to a coordinate transformation): the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric* [5, 6, 7, 8] [5],

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \right) \right], \tag{2}$$

where $\mathrm{d}s$ is the line element, t is the *cosmic time*, r, θ, ϕ are radial and angular coordinates, $a(t)$ is the *scale factor*, characterizing the dynamical evolution of the Universe, and $k = -1, 0, +1$ is the *curvature parameter* for hyperbolic, Euclidean and spherical spaces, corresponding to open, flat and closed universes. Spatial isotropy is directly shown in the angular line element. To indicate spatial homogeneity manifestly, we may rescale the radial coordinate as $r \equiv \bar{r}/(1 + k\bar{r}^2/4)$, and the metric is thus transformed to $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2)/[1 + k(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)/4]^2$, with $\bar{x} \equiv \bar{r}\sin\theta\cos\varphi$, $\bar{y} \equiv \bar{r}\sin\theta\sin\varphi$ and $\bar{z} \equiv \bar{r}\cos\theta$.

At the right hand side of the Einstein equations, for a perfect fluid, [6] the energy-momentum tensor reads

$$T^{\mu}{}_{\nu} = (\rho + p) u^{\mu} u_{\nu} + p g^{\mu}{}_{\nu}, \tag{3}$$

where ρ and p are the *energy density* and *pressure* of the perfect fluid. In the FLRW context, we attain the nontrivial components of the energy-momentum tensor, $T^0_{\ 0} = -\rho$ and $T^i_{\ j} = p\delta^i_{\ j}$.

Substituting the FLRW metric and the energy-momentum tensor into the Einstein equations, we yield the prestigious *Friedmann equations* [5],

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{5}$$

where $H \equiv \dot{a}/a$ is the *Hubble expansion rate.*

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[4] For the introduction to GR, we refer to the standard textbooks in [4], and for the definitions of these geometrical quantities, see App. A.

[5] The speed of light c is taken to be 1 throughout this dissertation.

[6] For imperfect fluid, a brief discussion will be presented in Sec. 10.1.

The dynamics of the evolution of the Universe is stored in these equations, but to solve them, we require further knowledge on the energy-momentum tensor. The budget of the energy density of cosmic medium consists of

1. *Radiation*: Any constituent particle with its rest mass much smaller than its kinetic energy can be regarded as radiation if thermolized, and the equation of state for radiation is $p_{\rm r} = \rho_{\rm r}/3$. For example, in the early Universe, at the era of the quantum chromodynamics (QCD) phase transition when the temperature of the Universe is about 200 MeV, electron, with rest mass of 0.511 MeV, can surely be viewed as radiation; but when the temperature of the Universe cools down to the order of eV at the epoch of recombination, we cannot consider electrons as radiation any longer. In the present Universe, the unique remainder of radiation are photons, forming the *cosmic microwave background* (CMB).

2. *Matter*: Contrary to radiation, matter refers to the constituent particle with its rest mass much larger than its kinetic energy, *i.e.*, $\rho_{\rm m} \gg p_{\rm m} = 0$. Matter can be further classified into ordinary baryonic matter, which comprise heavy elements (building our colorful Earth, luminous stars in galaxies and free Hydrogen and Helium as interstellar medium) and *dark matter*, which can only be detected via its gravitational effects, *e.g.*, the gravitational lensing, the shape of the rotation curves of galaxies and the positions and heights of the peaks of the baryonic acoustic oscillations (BAO) in the CMB spectra. In most modern cosmological models, dark matter is thought to be cold (CDM), as hot dark matter usually prevents large scale structure from forming effectively.

3. *Neutrinos*: Neutrinos are the ghosts in our Universe. The mysteries are their tiny, but not inappreciable, masses. Therefore, in the early radiation-dominated (RD) Universe, *e.g.*, at the eras of QCD phase transition and Big Bang nucleosynthesis (BBN), they behave as radiation and affect the dynamical evolution of the Universe. Whereas, at late times, when radiation, *e.g.*, photons, contribute a negligible fraction to the energy density of cosmic medium, they, due to their small masses, convert themselves into a form of matter and continue to influence the behavior of the Universe. Hence, neutrinos are active actors, playing an important role *throughout* the whole process of the evolution of our Universe. The current upper bound on the sum of neutrino masses is 0.61 eV (95% confidence level) [10].

4. *Dark energy*: The repulsive force, with negative pressure, which leads to the current accelerated expansion of the Universe, is believed to be caused by dark energy. It might be a simple geometrical parameter: the cosmological constant, or some dynamical field. If only so, it is not very troublesome, the most astonishing is that this uncanny dark energy is now the *dominating* ingredient in our very Universe! All these aspects will be carefully discussed in Sec. 2.2.

5. Other cosmological components: There may be other possible forms of cosmic medium, *e.g.*, topological defects: monopoles, cosmic strings and domain walls, but to summarize these components far oversteps the aim of this dissertation, and a good review can be found in [11].

Having surveyed the cosmological composition, we normalize Eq. (4) by the parametrization as [7]

$$\Omega_{\rm r} + \Omega_{\rm b} + \Omega_{\rm CDM} + \Omega_\nu + \Omega_k + \Omega_\Lambda = 1, \tag{6}$$

by introducing the *critical energy density* $\rho_{\rm c}$

$$\rho_{\rm c} \equiv \frac{3H^2}{8\pi G},$$

and defining the following *energy density parameters* for the various components of the Universe, [8]

$$\Omega_{\rm r} \equiv \frac{\rho_{\rm r}}{\rho_{\rm c}}, \quad \Omega_{\rm m} \equiv \frac{\rho_{\rm m}}{\rho_{\rm c}}, \quad \Omega_{\rm CDM} \equiv \frac{\rho_{\rm CDM}}{\rho_{\rm c}}, \quad \Omega_\nu \equiv \frac{\rho_\nu}{\rho_{\rm c}}, \quad \Omega_k \equiv -\frac{k}{a^2H^2}, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}.$$

In the following, we do not distinguish baryonic matter and CDM but simply call them matter, [9] and the contributions from radiation and neutrinos are also neglected here, if we concentrate our attention to the evolution of the Universe in the matter-dominated (MD) era or later. Thus, Eq. (6) is reduced to the *cosmic triangle* [12],

$$\Omega_{\rm m} + \Omega_k + \Omega_\Lambda = 1.$$

In Sec. 3.2.4, we will see that when taking into account the inhomogeneities and anisotropies in the Universe, this cosmic triangle is extended to a cosmic quartet.

Now, let us turn to the expansion behavior of the Universe. The two Friedmann equations: Eqs. (4) and (5) are certainly not closed, as there are three unknown variables: a, ρ and p. The equation of state of the cosmic medium provides the third necessity. For a perfect fluid and in the FLRW model, we have

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{7}$$

If we introduce the *equation of state* as $p \equiv w\rho$, Eq. (7) reduces to $\dot{\rho}+3(1+w)H\rho = 0$, and we directly get $\rho a^{3(1+w)} = \text{const}$. At different eras in the Universe, the equation of motion for the cosmic medium takes different forms, resulting in different modes of expansion, summarized in the following Tab. (1).

2.1.2 Hubble law

Once we have obtained the dynamics of the expanding Universe, the propagation of light in space-time is just a mathematical exercise. Following the idea of Hubble, we try to find the relation between the distance of the remote galaxy and its redshift. First, we introduce the *redshift* as $1 + z \equiv \lambda_{\rm o}/\lambda_{\rm e}$, with $\lambda_{\rm o}$ and $\lambda_{\rm e}$ being the wavelengths of the light

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[7] The subscripts stand for radiation, baryon, CDM, neutrinos, curvature parameter and dark energy (now in the form of cosmological constant), respectively.

[8] The current values of these parameters can be found in App. D.

[9] This is allowed on scales above 10 Mpc, where baryonic pressure is unimportant. See Sec. 3.1.1 for further discussion.

eras	w	$a(t) \propto$	$H(t)$	$a(\eta) \propto$	$\mathcal{H}(\eta)$
RD	$1/3$	$t^{1/2}$	$1/(2t)$	η	$1/\eta$
MD	0	$t^{2/3}$	$2/(3t)$	η^2	$2/\eta$
k-dominated	$-1/3$	t	$1/t$	e^{η}	const.
Λ-dominated	-1	$\exp\left(\sqrt{\frac{\Lambda}{3}}t\right)$	$\sqrt{\frac{\Lambda}{3}}$	$-1/\eta$	$-1/\eta$

Table 1: Expansion behaviors of the Universe in different eras.

The equation of state, scale factor and Hubble expansion rate for different eras of the Universe are listed with w decreasing. For the k-dominated era, the right hand side of Eq. (5) vanishes, formally resulting $\rho_k + 3p_k = 0$, *i.e.*, $w_k = -1/3$. Similarly, combining Eqs. (4) and (5) induces $\rho_\Lambda = -p_\Lambda$, *i.e.*, $w_\Lambda = -1$. We also list the results expressed in terms of the conformal time η, which is explained in Sec. 4.

observed by us and emitted by the light source. In terms of the scale factor $a(t)$, the redshift is given by [10]

$$1 + z = \frac{a_0}{a(t)}.$$

Without specified initial conditions, the Friedmann equations cannot indicate the increasing or decreasing of the scale factor $a(t)$. This information must come directly from observations. Thanks to the great discovery, made by Hubble [3], that light from distant galaxies are redshifted, we know the Universe is expanding. At small redshifts, for small recession velocities v and distances d, we arrive at the Hubble law,

$$H_0 d = v \approx z \ll 1,$$

where H_0 is the Hubble expansion rate at present, *i.e.*, the *Hubble constant.* Its value is the most crucial number in cosmology. After five years of operation, the Wilkinson Microwave Anisotropy Probe (WMAP5) experiment measures [10]

$$H_0 = 100h \text{ km/s/Mpc}, \quad \text{with} \quad h = 0.701 \pm 0.013.$$

Furthermore, the inverse of H_0 defines the *Hubble radius*, or equivalently, the *Hubble time*,

$$R_\mathrm{H} \equiv \frac{1}{H_0} = 2.998h^{-1} \times 10^3 \text{ Mpc}, \quad t_\mathrm{H} \equiv \frac{1}{H_0} = 9.778h^{-1} \text{ Gyr}.$$

At large cosmological distances, we define the luminosity distance of the light source as $d_\mathrm{L} \equiv \sqrt{L/(4\pi F)}$, with L being the luminosity of the light source, and F the observed flux (energy passed per unit area per time). The luminosity distance d_L and redshift z satisfy the relation

$$H_0 d_\mathrm{L} = z + (1 - q_0)\,\frac{z^2}{2} + \mathcal{O}(z^3), \tag{8}$$

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[10] We use the script 0 to denote the present values of physical quantities, *e.g.*, a_0 and q_0. Of course, this "present" time can be chosen freely.

where $q \equiv -\ddot{a}/(aH^2)$ is defined as the *deceleration parameter*. Here, we only expand this relation to second order in the series of z at small redshifts $z \ll 1$. [11] In the FLRW model, the coefficients at each order, *e.g.*, q_0, are the functions of the energy density parameters. Thus, measuring the Hubble law to large redshifts enables us to determine these cosmological parameters.

2.1.3 Standard model of cosmology

Three centuries ago, if some scholar claimed that he was doing something called cosmology, people must equate him with an astrologist. Three decades ago, if some physicist claimed that he was doing something called cosmology, scientists might think that he was working in a branch of astronomy. But now, cosmology is already a precise science on its own, and this precision is increasing faster and faster. Five years ago, when I first started my study of cosmology, people were always excited that the cosmological observations had got across the threshold of 1% accuracy, whereas currently up to three or four significant numbers can frequently be found in the literature, *e.g.*, see the WMAP5 experiment [10]. Cosmology will and is becoming the second particle physics with its standard model the so-called inflationary ΛCDM model.

According to the cosmological inflation theory [14], our Universe undergoes an era of accelerated expansion at the very early times. The idea of inflation gives rise to an excellent causal theory, which solves the horizon problem and is responsible for the homogeneity and isotropy of the present Universe at large scales. In this dissertation, we stick to the inflation theory and set the curvature parameter k to be zero throughout. On top of that, cosmological inflation also predicts the existence of tiny primordial fluctuations [15], which are observed at a level of 10^{-5} in the CMB [10].

Experimental observations and numerical simulations show that the majority of matter in the Universe is nonluminous and cold, *i.e.*, the cold dark matter, as it would be too late for large scale structures to form, if this dark matter were hot. The currently best candidates for CDM are the weakly interacting massive particles (WIMPs) [16]. Besides, the supernova (SN) and CMB experiments allow us to infer that most of the energy budget of the cosmological composition is dark energy, with negative pressure and thus constituting a repulsive force. Among the models of this dark energy, the simplest one, the cosmological constant, is still the best candidate. This aspect will be discussed in detail in the next subsection.

Good theories are always aiming to encapsulate different phenomena and physical laws in a unified set of equations and to depend on a minimal set of physical parameters as few as possible. If so, cosmology is already one of those. We can summarize our present knowledge with only *seven* free parameters [10]: $\Omega_{\mathrm{b}}^0 = 0.0462 \pm 0.0015$, $\Omega_{\mathrm{CDM}}^0 = 0.233 \pm 0.013$, $h = 0.701 \pm 0.013$, the current CMB temperature $T_0 = 2.725 \pm 0.001$ K, the power spectrum index $n_{\mathrm{s}} = 0.960^{+0.014}_{-0.013}$, the optical depth $\tau = 0.084 \pm 0.016$ and the fluctuation amplitude (defined at the scale of 8 Mpc) $\sigma_8 = 0.817 \pm 0.026$. [12] With

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[11] The expansion of d_{L} in z up to fourth order can be found in [13].

[12] This number seven interestingly reminds us of another number six in the popular book *Just six numbers* by M. Rees [17]. Of course, six numbers are still too many for a final theory, as it is said that, with five parameters, we can even mimic an elephant.

these parameters, the standard inflationary ΛCDM model of modern cosmology is firmly established, and it has been precisely tested in various astronomical and cosmological observations.

Although the standard model of cosmology has gained great glorious successes, it has been encountering a great trouble: the dark energy problem, since the cosmology revolution ten years ago. In the next three subsections, we will review this dark energy problem and show how it can be linked to the averaging problem in the perturbed Universe.

2.2 Dark energy problem

"Physics thrives on crisis", [13] and every elimination of a crisis thoroughly revolutionizes the fundamental understanding of our Universe. If there are some unpleasant clouds over our heads, dark energy is definitely one of those. To cover every relevant aspect of this crisis, we have to write volumes of books, as thick and heavy as bricks, instead of one single dissertation. Actually, people really did so: hundreds of review articles and thousands of papers have already been devoted to this problem, in which we can amazedly experience the "creativity" of our lovely physicists. But before going through these fantastic proposals, let us first look at the experimental evidences for the strange dark energy.

2.2.1 Experimental evidences for dark energy

Modern cosmology should owe their rapid development to the improvements of experimental devices and equipments in recent decades. Ground based, balloon borne (*e.g.*, BOOMERanG and MAXIMA) and satellite (*e.g.*, COBE, WMAP and Planck) instruments make it possible to precisely measure cosmological parameters to 1%, or even 0.1% accuracy [14]. Actually, in the WMAP5 experiment, some physical quantities, *e.g.*, $\Omega^0_{\rm de}$ has been measured to three significant digits, and $\Omega^0_{\rm b}h^2$ and $\Omega^0_{\rm CDM}h^2$ even to four. These continuously improved experimental technologies pushed us in a position to achieve a global picture of our Universe, and eventually lead to the revolutionary discoveries at the beginning of the new millennium that our Universe is in an *accelerated expansion* phase, calling for the necessity of a repulsive force, induced by the so-called *dark energy.*

We are not able to list all the previous and ongoing cosmological experiments. They are so many that we can even build a high-dimensional Cosmological Parameters Model/Data Set Matrix to combine the different results [18]. Here, we address only two kinds of experiments: the SN and CMB experiments.

1. Ten years ago, the science community, was astounded by the conclusions drawn from the observations of Supernovae of Type Ia (SN Ia) by two groups independently [19, 20]. [15] In 1988, the Supernova Cosmology Project [20] was set up.

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[13] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[14] In fact, in the CMB experiments, the temperature fluctuation is about 10^{-5}, so 1% accuracy means 10^{-7} precision!

[15] An SN Ia is a cataclysmic variable star that results from the violent explosion of a white dwarf star. The suggestion of SNe as standard candles for cosmological measurements dated back to Baade [21]

Ten years of efforts enabled them to discover 75 SNe Ia at redshifts $z = 0.18 - 0.86$ spectroscopically, using the Cerro Tololo Inter-American Observatory 4 m telescope. Based on 42 SNe Ia and jointly fitted with a set of SNe from the Calán/Tololo Supernova Survey at redshifts below 0.1 [22], they yielded $\Omega_{\rm m}^0 = 0.28^{+0.09}_{-0.08}$ (1σ statistical) $^{+0.05}_{-0.04}$ (identified systematics), under the assumption that the Universe is flat. These data indicated that the cosmological constant is nonzero and positive, with a confidence of 99%, including the identified systematic uncertainties (see Fig. (1) for detailed explanations).

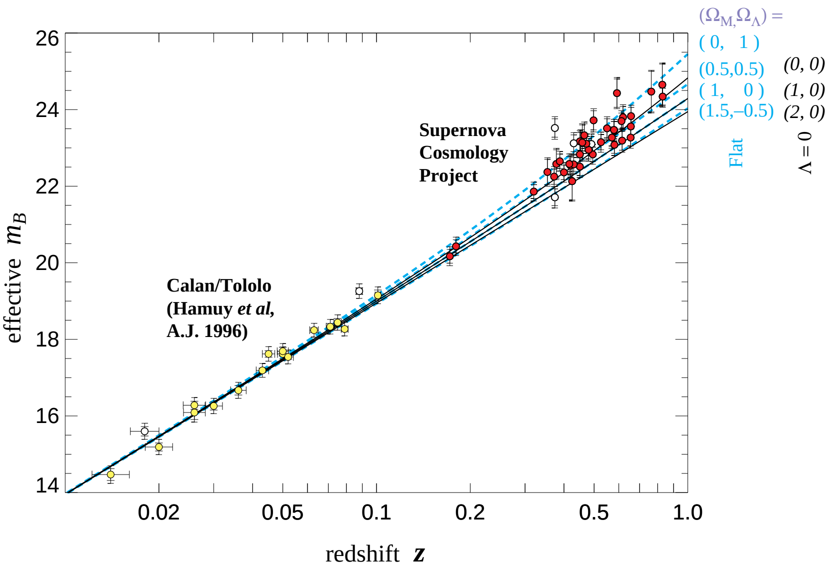

Figure 1: First evidence for dark energy from the Supernova Cosmology Project.

Hubble diagram for 42 high redshift SNe Ia [20] and 18 low redshift SNe Ia [22]. The solid curves are the theoretical brightness $m_B^{\rm eff}(z)$ for a range of cosmological models with $\Lambda = 0$: $(\Omega_{\rm m}^0, \Omega_\Lambda) = (0,0)$, $(1,0)$ and (2,0) from top to bottom. The dashed curves are for a range of cosmological models with $k = 0$: $(\Omega_{\rm m}^0, \Omega_\Lambda) = (0,1)$, $(0.5, 0.5)$, $(1,0)$ and $(1.5, -0.5)$ from top to bottom. We see that the up-bending fitting curve clearly favors $\Omega_\Lambda > 0$, and from Eq. (8), we find $q_0 < 0$, indicating the accelerated expansion of the Universe.

2. The evidence for dark energy can also be obtained from the temperature fluctuations and polarizations in the CMB experiment. In Fig. (2), we show the temperature-temperature correlation power spectrum from the WMAP5 experiment [23]. In the ΛCDM model, the positions and heights of the peaks in the power spectrum are functions of the cosmological parameters, *e.g.*, $\Omega_{\rm b}^0$, $\Omega_{\rm CDM}^0$, Ω_Λ and Ω_k^0. Briefly speaking, for example, the height of the first peak is enhanced if $\Omega_{\rm b}^0$ is larger, but

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seventy years ago. They can be used to measure the Hubble constant at small distances and determine cosmological parameters, *e.g.*, the deceleration parameter, at higher redshifts.

the height of the second peak is suppressed if so; the positions of these peaks move to smaller l if Ω_Λ increases. [16] Therefore, from the positions and heights of these peaks can we derive the cosmological parameters, and many useful quantities are listed in Tab. (3) in App. D. We again obtain a nonzero and positive cosmological constant, with $\Omega_\Lambda = 0.721 \pm 0.015$. But of course, we should mention here that these results rely on the assumption of the ΛCDM model. If we would try other possibilities, things may change.

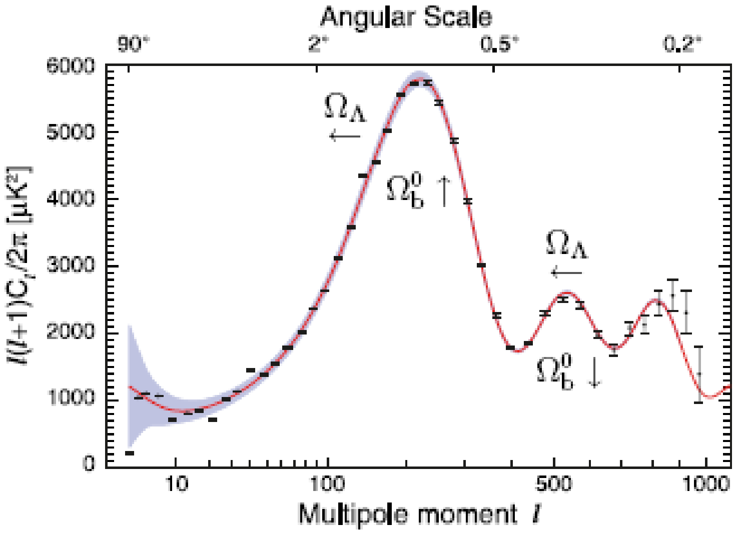

Figure 2: Evidence for dark energy from the WMAP5 experiment.

Temperature-temperature correlation power spectrum from the WMAP5 experiment [23]. The dependence of the shape of the power spectrum on the cosmological parameters, *e.g.*, Ω_b^0 and Ω_Λ, are shown in the figure. The shadow shows the cosmic variance.

Besides the methods above, we may also seek the evidence for dark energy in another way, *i.e.*, we ask at what level of confidence we can reject the null hypothesis that the Universe never accelerated [27]. In a spatially flat model, based on two different SN Ia data sets, two different fitting methods and two different calibration methods, this null hypothesis is rejected at $> 5\sigma$ [27].

2.2.2 Theoretical candidates for dark erergy

Although the accelerated expansion of the Universe has been confirmed, its reason is still in the dark. Perhaps that is why we call it dark energy. In fact, this is just a substitute to the seemingly existing repulsive force. We pass the buck to the magical dark

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[16] These dependence can be found in [24, 25], and an excellent intuitive movie can be downloaded from [26].

energy, whereas it is nothing but a name. However, naming is not explaining. We are still hungry for brilliant theoretical ideas to retrieve us from this catastrophe [29].

Dark energy can be further categorized into two types: geometrical and dynamical dark energies, whose energy density is constant in time or varies as the Universe evolves. We discuss them below, respectively.

1. For the geometrical constant dark energy, the cosmological constant is absolutely the best candidate. Unfortunately, this cosmological constant is indistinguishable from the vacuum energy (zero-point energy) of quantum fields. Let us simply pick a scalar field with mass m for example. Its vacuum energy reads

   $$\rho_{\text{vac}} = \sum_{\mathbf{k}} \frac{1}{2}\omega_{\mathbf{k}} = \int_0^{\Lambda} \frac{\mathrm{d}k}{4\pi^2} k^2 \sqrt{k^2 + m^2} \sim \frac{\Lambda^4}{16\pi^2}.$$

   To regularize this integral, we have introduced a cutoff Λ. [17] Definitely, in quantum field theories, where gravitation is always neglected, we can safely put aside this vacuum energy, as only energy difference is observable in experiments, so adding a constant, even infinity does not harm our practical calculations. [18] But be aware, energy is not like potential, in which we can arbitrarily add or subtract some value. This non-arbitrariness turns to be more distinct when gravitation is involved, because from the theory of relativity, energy is equivalent to mass, mass induces gravity, and this gravitational effect is observable in experiments! [19] So if there were vacuum energy, any conscientious physicist should ponder it seriously.

   Now let us compare the dark energy contributed by the vacuum energies of quantum fields and that from cosmological observations. If we believe that our quantum field theory works well up to the Planck scale $\Lambda_{\text{P}} = 1.221 \times 10^{19}$ GeV, $\rho_{\text{vac}} \sim 1.407 \times 10^{74}$ GeV^4. However, the WMAP5 experiment measures $\rho_{\text{de}}^0 = 2.869 \times 10^{-47}$ GeV^4, *120 orders of magnitude* smaller than that from then vacuum energy of quantum field theory! Even though we are more honest and decrease Λ from the Planck scale to the scales of the electroweak symmetry breaking $\sim$ 100 GeV, or nothing more than the nuclear physics level $\sim$ 1 MeV, this huge gap is still about 70 and 30 order of magnitude, *i.e.*, a catastrophe can only be turned into a disaster, unfortunately. Remember, here we only try a scalar field, if more quantum fields are taken into account, who knows what will happen. Actually, ρ_{de}^0 is about a few protons/m^3, while ρ_{vac} is approximately 10^{90} kg/m^3, denser than any form of matter that we can even dream! Does this vast number remind you of the abandoned aether, which is very rigid but allows our Earth to pass through without any difficulty? It is quite unnatural to image that so dense matter would flee from any known experiment. To help us from this dilemma, it is suggested that the cosmological constant in the Einstein equations would cancel very well (to 120 significant digits!) with the vacuum energy of quantum field theory, and the remainder is what we are observing in cosmological experiments. But any rational mind will discard this cancelation,

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[17] Not be confused with the cosmological constant.

[18] Frankly speaking, I was deeply disappointed by these arguments from physicists or in textbooks.

[19] This is the well-known Casimir effect [28].

as the vacuum energy and the cosmological constant intrinsically has nothing to do with each other within our present understanding.

2. The problem above is obviously beyond the scope of what we can hope to understand currently. Most of the time, we admit that the vacuum energy vanishes due to some still unknown mechanism without further ado and leave this mechanism for future research. What we can hope to understand is that why the dark energy is so small but not zero. For this purpose, dynamical approaches seem to be appreciated.

   The basic principle lying behind the suggestions of dynamical dark energy is to add extra source terms into the energy-momentum tensor, so that these extra terms play the role of dark energy, as the cosmological constant does. The discussions below follow an excellent review article by Copeland *et al.* [30].

   Let us first list the 23 different kinds of dynamical dark energy summarized in [30] and elsewhere, not for study, but just for fun: (1) quintessence, (2) quintessential inflation, (3) pseudo-Nambu-Goldstone boson, (4) Chameleon fields, (5) k-essence, (6) tachyon field, (7) $f(R)$ theory, (8) repulsive gravity at scales of Gpc, (9) Chaplygin gas, (10) feedback from nonlinearities, (11) dark energy from trans-Planckian regime, (12) de Sitter vacua in string theory (KKLT scenario), (13) DGP model, (14) braneworld modified gravity, (15) very light Kaluza-Klein graviton, (16) string landscapes, (17) anthropic principle, (18) phantom dark energy, (19) dilatonic ghost condensation, (20) a network of frustrated topological defects, (21) cyclic universe, (22) tired graviton, (23) causal sets in quantum gravity. Below we only spend several sentences for some of them.

   (a) *Quintessence fields*: Quintessence is a scalar field ϕ minimally coupled to gravity, but with some particular potential $V(\phi)$ [31]. The energy density and potential of this scalar field is $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi)$. So if $V(\phi) \gg \dot{\phi}^2$, we approximately mimic a cosmological constant, which leads to the late time acceleration. However, the problems with quintessence field is that first the mass of the corresponding particle of this scalar field is extremely small, about 10^{-33} eV, which is quite tiny compared with any known particle; second we have never observed any scalar field till now. Thus, quintessence is sometimes ironized as "quit tes sense". But the quintessential inflation [32] seems interesting, at least from my view of point.

   (b) $f(R)$ *theory*: This approach is to modify the traditional Einstein-Hilbert action for the gravitational field from the simple Ricci scalar R to some function of it, $f(R)$, *e.g.*, in [33], it is proposed $R \to R - \mu^4/R$. Actually, a cosmological constant Λ can also be regarded as an $f(R)$ theory, with $R \to R - 2\Lambda$. Whereas, the trouble is that these $f(R)$ theories not only modify gravity at large scales, but also small ones, *e.g.*, they are hard to evade the tests with the Solar System [34]. [20]

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[20] In fact, any $f(R)$ theory can be transformed into a scalar field theory, with $\phi \propto \ln \partial f/\partial R$ and $V(\phi) \propto (f - R\partial f/\partial R)/(\partial f/\partial R)^2$.

(c) *Chaplygin gas*: The basic idea of this model is to introduce an exotic equation of state, which has never be applied to any form of matter, *i.e.*, $\rho p =$ a negative constant. The advantage of this model is that first, although pressure now is negative, the speed of sound $\partial p/\partial \rho$ is still positive; second, this strange equation of state can lead to dark matter at early times and to a cosmological constant at late times, so mimic these two dark sectors simultaneously; third, this model, albeit adopted from aerodynamics, has some wonderful relation with the Nambu-Goto action in string theory! But the disadvantage is also obvious, *i.e.*, how to realize this equation of state from field theory? I once asked one of its authors how they excogitate this interesting model, the answer is as usual, "no reason".

(d) *Degravitation*: Motivated by the deelectrification in a uniformly distributed field of electric charges, people extend this idea to the gravitational field, and argue that the cosmological constant, as a uniform source, is degravitated no matter how large it is [36]. While, if this advice succeeded, we could somehow get rid of the cosmological constant totally, but I cannot see how a small amount of dark energy would remain to support the accelerated expansion.

Let us stop here: other theoretical approaches are largely identical with only minor differences. The general impression to us is that these suggestions can be helpful to some aspect of the dark energy problem, but at the same time, helpless to the rest. Moreover, these suggestions themselves are usually full of troubles, *i.e.*, the price we must pay for these suggestions are generally too high: we have to suffer from the new scalar field with tiny mass, extra long range forces, modification of GR, modification of gravity at both small and large scales, Lorentz violation, extra dimensions or anthropic principle these faith-based reasonings.

Science research is like the jigsaw puzzle, if we are on the right way, everything should fit automatically, not just as a makeshift. So if we merely want to present a new suggestion, it does not deserve to accomplish a long dissertation, as it is nothing but a multiplication of the ideas listed above. To the author himself, he solves a problem, but to people else, he only arouses new confusions.

Case now is a little bit analogous to that of the aether crisis. Instead of constructing various fantastic models for this imaginary medium, Einstein simply abandoned all the artificial substitutes and swept all the unnecessary substances off the clean vacuum. Situation nowadays is somewhat alike: we insert too much manmade particles and fields into vacuum, and frankly speaking, these particles and fields are always wedged into equations by hand; even if based on some fundamental field theories, these theories are always some hand-waving ones. These unpleasant circumstances cannot satisfy many of us, and we think that this status should not continue forever. So before playing with the above brave ideas, creative or deceptive, why not consider a *conservative* alternative *within* the framework of Einstein's gravity, *i.e.*, *dark energy from structure formation*?

To comprehend the basic principle of this alternative, we need to pause for a moment to discuss another aspect of dark energy crisis, *i.e.*, the coincidence problems.

2.3 Coincidence problems

The coincidence problems, however, are pure cosmological questions, and there are two ways to formulate them.

The first formulation is direct and easy to understand, so it widely appears in the literature. In one word, the question is why the energy densities of matter and dark energy happen to be the same order of magnitude at present? We show the temporal dependence of the energy density ρ_i and energy density parameter Ω_i of different cosmological compositions in Fig. (3), in which we clearly see $\Omega^0_{\rm de} \sim \Omega^0_{\rm m}$. This coincidence is indeed surprising, as one energy density parameter $\Omega_{\rm m}$ evolves with time and decays as $1/a^3$, while the other one $\Omega_{\rm de}$ (in terms of Λ) is a pure constant! For example, suppose we would have lived when the Universe had half or twice of its present linear size (at $z = 1$ or $-1/2$), the ratio $\Omega_{\rm de}/\Omega_{\rm m}$ would be 8 times smaller or larger than its current value, almost an order of magnitude! It is therefore quite strange that we are living at a very special epoch, observing $\Omega^0_{\rm de}/\Omega^0_{\rm m} = 2.584$ [10].

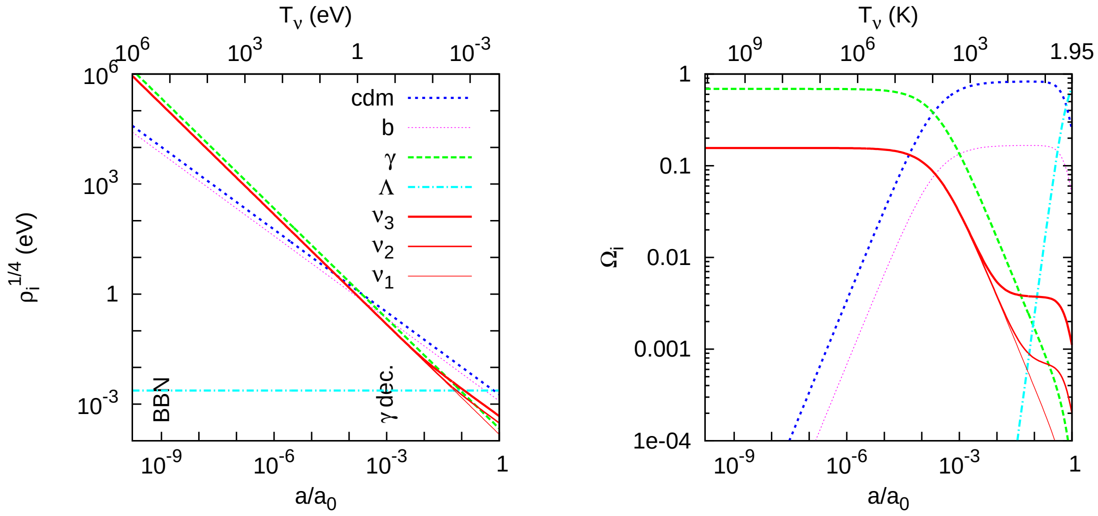

Figure 3: Coincidence problem.

Temporal dependence of the energy density ρ_i and energy density parameter Ω_i of different cosmological components, from the time when the neutrino temperature $T_\nu = 1$ MeV (soon after neutrino decoupling) until now [37]. Here, the index i stands for CDM, baryon, radiation, cosmological constant and three types of neutrinos. Data are obtained from the flat ΛCDM model with the inputs $h = 0.7$ and current energy density parameters $\Omega^0_{\rm CDM} = 0.25$, $\Omega^0_{\rm b} = 0.05$, $\Omega^0_\nu = 0.0013$ and $\Omega_\Lambda = 0.70$. The three neutrino masses are distributed according to the normal hierarchy scheme with $m_1 = 0$ eV, $m_2 = 0.009$ eV and $m_3 = 0.05$ eV. The coincidence of the energy densities of matter and dark energy is clearly shown at the right sides of both plots.

If people complain that there is some anthropic taste from this argument, we may reformulate the above situation via replacing our mankind by large scale structures on the typical scale for structure formation [21] and ask why the $z \sim 1$ is the same epoch when dark energy, *i.e.*, accelerated expansion emerges and the hierarchical structure formation starts to evolve nonlinear on the matter-radiation equality scale, *i.e.*, it decouples from the overall cosmological expansion and becomes self-gravitational system? This second coincidence problem may be even more fundamental, as the two aspects here: dark energy and structure formation, have entirely different physical essences. One might be related to the zero-point energy in quantum field theories and the other is a pure gravitational problem. So a natural question is: what does this coincidence imply? As we know there are many special epochs during the evolution of the Universe, why does not the onset of dark energy coincide with the epochs of the BBN or matter-radiation equality, but only with structure formation? A straightforward answer could be that dark energy is triggered by structure formation, *i.e.*, the gravitational amplification of inhomogeneities and anisotropies in the evolution of the Universe.

The next step certainly is how to realize this possibility. We will show that the backreaction mechanism arising from the averaging problem in perturbed space-time is an interesting candidate.

2.4 Averaging problem

The averaging problem in cosmology is another long-term question. In perturbed space-time, *i.e.*, for a curved manifold, averaging is quite different from that in flat space-time, but much more complicated. In this subsection, we first indicate the reason calling for averaged quantities in cosmology, then briefly look back at the different approaches in this problem and finally end with a discussion of the backreaction mechanism.

2.4.1 Why averaging?

Similar with the case in thermodynamics and statistical mechanics, tracing the motion of one peculiar atom is of no practical use for our understanding of the behavior of a physical system, focusing on one peculiar event in space-time does not make sense in cosmology, either. Indeed, many of our observables in cosmology are averaged quantities. Two important examples are the power spectrum $\mathcal{P}(k)$, which is a Fourier transform and thus a volume average weighted by a factor $e^{i\mathbf{k}\cdot\mathbf{x}}$, and the most important cosmological parameter, the Hubble constant H_0.

Let us pick H_0 to discuss this issue in some detail. The idealized measurement of the Hubble constant proceeds as follows [38]. Take a set of N standard candles (in reality the SNe Ia) that sample a local physical volume V homogeneously (*e.g.*, the Milky Way's neighborhood out to $\sim$ 100 Mpc), and measure their distances d_i (via magnitudes) and recession velocities $v_i = z_i$ and take the average

$$H_0 \equiv \frac{1}{N}\sum_{i=1}^{N}\frac{v_i}{d_i}.$$

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[21] This typical scale is set by the matter-radiation equality: roughly speaking, 100 Mpc for the flat ΛCDM model and 30 Mpc for the Einstein-de Sitter (EdS) model.

In the limit of a very big sample ($N \to \infty$), this turns into a volume average

$$H_0 = \frac{1}{V} \int \frac{v}{d} \mathrm{d}V.$$

In the second step, we neglect the effect of the light cone, as for $z \ll 1$, the spatial average is a good approximation for the average over the past light cone, because the expansion rate of the Universe is not changing significantly at time scales much shorter than the Hubble time. [22]

On the other hand, we have a theoretical object that we call the expansion rate, defined as $H_0^{\mathrm{th}} \equiv \dot{a}/a$. The issue in the averaging problem now is to establish the connection between H_0 and H_0^{th}. In linear theory, they agree by construction if the volume V becomes large enough. However, due to the nonlinearity of the Einstein equations, cosmological perturbations affect the evolution of the averaged (which we often identify with the "background") Universe. This is the so-called backreaction mechanism to be discussed in Sec. 2.4.3.

2.4.2 A short history of the averaging problem

Before we go into the details of the backreaction mechanism, we first briefly retrospect the history of the averaging problem in cosmology, which will be helpful for the next subsection.

The study of the averaging problem was initiated by Shirokov and Fisher [39] and further emphasized in great detail by Ellis [40]. They realized that in the traditional approach in cosmology, the metric that we use (the FLRW metric) in the left hand side of the Einstein equations is the averaged one, but what we insert in the right hand side is the averaged energy-momentum tensor, corresponding to a continuous matter distribution, *i.e.*, we usually equate two things $G_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi G \langle T_{\mu\nu}\rangle$. However, the nonlinear nature of the Einstein equations forbids us to simply write $G_{\mu\nu}(\langle g_{\mu\nu}\rangle) = \langle G_{\mu\nu}(g_{\mu\nu})\rangle$. Thus, the Friedmann equations should be only regarded as an oversimplified description of the real Universe, once the fluctuations become negligible, and the correct dynamical equations for the perturbed Universe should now be modified to $G_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi G \langle T_{\mu\nu}\rangle + 8\pi G T^{\mathrm{g}}_{\mu\nu}$, where $T^{\mathrm{g}}_{\mu\nu}$ is some effective energy-momentum tensor, with purely geometrical origin. In [39], the authors called them "polarization terms", from which they tried to get repulsive forces to prevent the Big Bang singularity. Their idea was carried forward by Noonan in [41] to define the average of a physical quantity O as [23]

$$\langle O \rangle = \frac{\int O(x)\sqrt{-g(x)}\mathrm{d}x}{\int \sqrt{-g(x)}\mathrm{d}x}.$$

But the definition of the average in inhomogeneous space-time is not so easy as in the equation above, especially for tensors. As we know, tensors cannot be compared directly

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[22] As we know, all information that we obtain of our Universe is encoded on the past light cone, so a general treatment of the averaging problem in cosmology should be performed on the past light cone, *i.e.*, the study of the propagation of light in perturbed space-time is strongly needed, which is already a big subject of itself, and we will not cover it in this dissertation.

[23] Here x is the 4-coordinate.

at different points. To define the covariant derivative, we have to introduce the parallel transport to keep its tensor character. Contrary to the case of the covariant derivative, in which we subtract tensors at different points, in the averaging problem, we should sum up them, but the trouble remains the same, *i.e.*, how to define this sum *at different points*?

This problem was pioneered by Issacson [42], and carefully addressed by Zalaletdinov [43] recently. The basic ideas of their works are to utilize a *bivector* $V_{\mu}{}^{\nu'}(x,x')$, which is a vector at both x and x', to link tensors at different points. Supposing $A_{\nu'}$ is a given vector defined at x', then the bivector $V_{\mu}{}^{\nu'}(x,x')$ defines a unique vector $A_{\mu} = V_{\mu}{}^{\nu'}(x,x')A_{\nu'}$ at x by parallel transporting $A_{\nu'}$ from x' to x along geodesic. For example, in [42], the average of a tensor is defined as

$$\langle T_{\mu\nu}(x)\rangle = \int V_{\mu}{}^{\lambda'}(x,x')V_{\nu}{}^{\rho'}(x,x')T_{\lambda'\rho'}(x,x')f(x,x')\mathrm{d}x',$$

where $f(x,x')$ is a weighting function, satisfying the normalization condition $\int f(x,x')\mathrm{d}x' = 1$. In [43], Zalaletdinov proposed a covariant, non-perturbative, geometrical approach for macroscopic gravity, according to which, the average of a tensor $T^{\mu\cdots}{}_{\nu\ldots}(x)$ in a domain D is defined as

$$\langle T^{\mu\cdots}{}_{\nu\ldots}(x)\rangle_D = \frac{1}{V_D}\int_D (\mathcal{V}^{-1})^{\mu}{}_{\lambda'}(x,x')\mathcal{V}_{\nu}{}^{\rho'}(x,x')T^{\lambda'\cdots}{}_{\rho'\ldots}(x,x')\mathrm{d}x',$$

with V_D the volume of the averaged domain. Here the bivector $\mathcal{V}_{\mu}{}^{\nu'}$ Lie drags the averaging region from x' to x along the integral lines, making the comparison of tensors at different points possible. For these reasons, the bivector $\mathcal{V}_{\mu}{}^{\nu'}$ is required to be

$$\begin{aligned}
&\lim_{x'\to x}\mathcal{V}_{\mu}{}^{\nu'}(x,x') = \delta_{\mu}{}^{\nu'},\\
&\mathcal{V}_{\mu}{}^{\nu'}(x,x') = \delta_{\mu}{}^{\nu'} \quad \text{for the Minkowski metric},\\
&\mathcal{V}_{\mu}{}^{\nu'}{}_{;\nu'}(x,x') = 0,\\
&\mathcal{V}_{\mu}{}^{\lambda'}{}_{,\nu}(x,x') + \mathcal{V}_{\mu}{}^{\lambda'}{}_{,\rho'}(x,x')\mathcal{V}_{\nu}{}^{\rho'}(x,x') = \mathcal{V}_{\nu,}{}^{\lambda'}{}_{\mu}(x,x') + \mathcal{V}_{\nu}{}^{\lambda'}{}_{,\rho'}(x,x')\mathcal{V}_{\mu}{}^{\rho'}(x,x').
\end{aligned}$$

In this dissertation, we will not consult this averaging procedure in detail, and some relevant references for this approach can be found in [44].

In this dissertation, we make use of the averaging procedure by Buchert, *i.e.*, we focus our attention on the average of *scalars* only. As the comparison of scalars is well-defined, no confusion will arise, and we further limit our approach to scales much smaller the horizon, *i.e.*, at redshifts $z \ll 1$. Thus, we are allowed to foliate the space-time manifold into constant time hypersurfaces and set the time axes orthogonal to these hypersurfaces, as we assume the Universe is irrotational. The details of this averaging process will be formulated in Sec. 3, and now we move on to the backreaction mechanism and see how the inhomogeneities and anisotropies react back on the evolution of the background Universe.

2.4.3 Backreaction mechanism

In one word, the essence of the backreaction mechanism is the *non-commutation* of temporal evolution and spatial averaging in inhomogeneous space-time. Much better than exhibiting tens of equations, this non-commutation can be clearly illustrated in the following Fig. (4). [24]

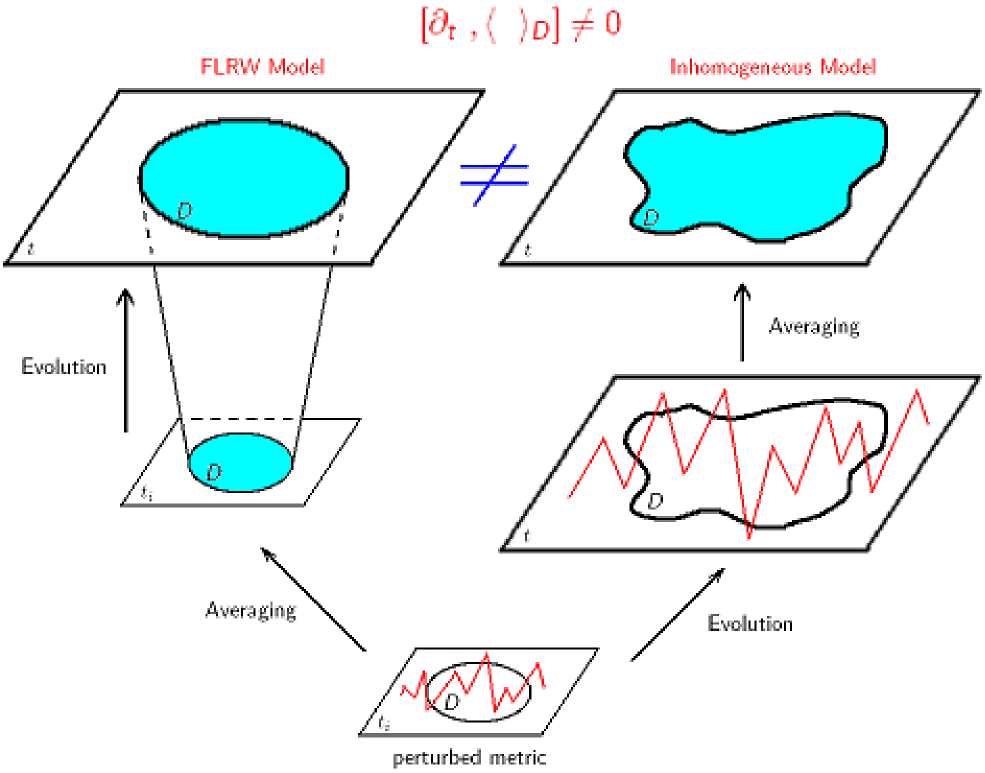

Figure 4: Noncommutation of temporal evolution of spatial averaging.

> We start from a domain D at the bottom, with perturbed metric (indicated with the red lines), at the time $t_{\rm i}$. On the left, we first smooth out the fluctuations in the metric at $t_{\rm i}$, and thus arrive at the simple FLRW model. Then, this averaged model evolves to the time t (nothing but an expansion). On the right, we exchange the order of these two operations: we first follow the evolution of the perturbed space-time from $t_{\rm i}$ to t and then average in the resulting domain at t. We clearly find the difference between the two models from the two upper panels. This indicates the non-commutation $[\partial_t, \langle \ \rangle_D] \neq 0$.

On the left part of Fig. (4), we first average the perturbed metric in a domain D at time $t_{\rm i}$, *i.e.*, we smooth out the fluctuations and obtain an unperturbed averaged metric (the FLRW context). Next, we follow the trivial evolution of this averaged domain, which is nothing but the simplest FLRW solution. However, if we *exchange the order of these two operations*, *i.e.*, we first trace the evolution of the perturbed Universe on the right part of Fig. (4) and then take the average at time t, we immediately reach a totally different result in the inhomogeneous models: during the evolution, the initial fluctuations are amplified, and finally we cannot smooth them entirely at t. Altogether, this non-commutation means

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[24] This figure is taken from the talk given by J. Larena in the second Kosmologietag in Bielefeld, April 26 - 27.

that inhomogeneities and anisotropies in perturbed space-time show their influence during the expansion of the Universe. This is the so-called *backreaction mechanism*. Therefore, an effective energy-momentum tensor is introduced into the dynamical equations, and it might thus play the role of dark energy. [25] The exploration of this possibility forms the main body of the following sections.

Now, let us reformulate this backreaction mechanism more mathematically. From the non-commutation, we have

$$[\partial_t, \langle\ \ \rangle_D] \neq 0.$$

This means that the averaged Einstein tensor of the perturbed metric does not coincide with the Einstein tensor calculated from the averaged metric (the FLRW one),

$$\langle G_{\mu\nu}(g_{\mu\nu})\rangle_D \neq G_{\mu\nu}(\langle g_{\mu\nu}\rangle_D).$$

Furthermore, this non-commutation means that generally speaking, in an inhomogeneous and anisotropic universe, we cannot deduce the *global* Friedmann equations for the averaged background Universe from the *local* Einstein equations, [26]

$$\begin{cases} \frac{1}{2}\left(\mathcal{R} + \theta^2 - \theta^i{}_j\theta^j{}_i\right) = 8\pi G\rho + \Lambda, \\ \dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi G\rho + \Lambda, \\ \dot{\rho} + \theta\rho = 0, \end{cases} \qquad \text{cannot} \quad \Rightarrow \quad \begin{cases} H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}, \\ \dot{\rho} + 3H\rho = 0. \end{cases}$$

Till now, from the last three subsections, we have shown in sequence the dark energy problem, coincidence problems and how the backreaction mechanism is related to these problems. In summary, all these three aspects are entangled with the issues of inhomogeneities and anisotropies in the local Universe. The beginning of the domination of dark energy coincides with the onset of structure formation. Consequently, light may be shed on the dark energy crisis, *i.e.*, the accelerated expansion of the Universe, by studying the averaging problem in perturbed space-time. We will explore this alternative in the coming subsection.

2.5 An alternative from inhomogeneities and anisotropies

In this subsection, we review the study of the possibility that dark energy is triggered by structure formation. But before so, let us first perform a survey over the inhomogeneities and anisotropies in our Universe to get some sensible knowledge of this problem.

2.5.1 Inhomogeneities and anisotropies in the Universe

Instead of the *a priori* assumption of homogeneity and isotropy of our Universe in the cosmological principle, the inhomogeneities and anisotropies in our Universe, *i.e.*, large scale structures, are indeed observed. Of course, if the cosmic medium were really

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[25] This happens no matter the Einstein equations are linear of nonlinear, *i.e.*, also in Newtonian gravity.

[26] The meanings of the equations (for irrotational dust) in the left column will be explained in the next section.

a homogeneous and isotropic soup, there would be no the very person who is writing this dissertation. It is the inhomogeneities and anisotropies that make our colorful world possible. Compared with the current $\rho_{\rm c}^0$, the contrast of the energy density $\delta\rho/\rho_{\rm c}^0 \equiv (\rho - \rho_{\rm c}^0)/\rho_{\rm c}^0$ of a typical star, *e.g.*, the Sun, is 10^{30}! For our Galaxy, which is of the scale 10 kpc, this ratio is 10^5, and for the Local Group with the scale of 1 Mpc, the ratio decreases to 10. Only till scales of 100 Mpc, *e.g.*, the Virgo Supercluster, the energy density of the cosmic medium approaches its average value.

We cannot exhaust all kinds of cosmic structures in this dissertation, but only briefly list some important and interesting ones: the main purpose here is just to demonstrate that homogeneity and isotropy of the Universe should not be optionally assumed as a matter of course.

1. *Great Walls*: It is a filament of galaxies, *e.g.*, the 100 Mpc long CfA2 Great Wall, observed by the CfA Redshift Survey in 1989 [47]. The more exciting discovery, maybe also the largest structure in the Universe known presently, is the famous Sloan Great Wall, based on the Sloan Digital Sky Survey [48], which extends 400 Mpc in length and is located 300 Mpc from the Earth.

2. *Voids*: Besides the clustered matter in those great walls, in our Universe, we also discover some nearly empty places, which are devoid of galaxies. For instance, the Boötes supervoid, almost 100 Mpc in diameter, is an approximately spherically shaped, tremendously large void [45]. It is amongst the largest voids in the Universe, and until recently only two dozen galaxies have been found in it. Recall that the Virgo Supercluster contains 100 groups and clusters of galaxies! Therefore, this void is so void that it is said in [46] that “If the Milky Way had been in the center of the Boötes void, we wouldn’t have known there were other galaxies until the 1960s.”

3. *Anisotropies of the CMB*: Needless to say, this is the most central topic of the ongoing research on cosmology, *e.g.*, the upcoming Planck satellite experiment. Different from the great walls and voids, these anisotropies are of primordial origin, *i.e.*, they are generated at the very early era (epoch of recombination) and reflect the large scale shapes of our Universe. Thanks to the heroic efforts in the Cosmic Background Explorer (COBE) [49] and WMAP [50] experiments, we have now been able to obtain a global image of the whole observable Universe. On top of the background temperature of the CMB, 10^{-5} fluctuations are found. Studying the anisotropies of the CMB provides us the cleanest way to understand the gravitational fluctuations, which are imprinted on the CMB in the early Universe.

4. *Anisotropies of the SN Ia Hubble diagram*: The homogeneity and isotropy of the Universe can also be tested with the help of SNe Ia. A simple test can be done by fitting the Hubble diagram ($d_{\rm L}$ as the function of z) for SNe from just one hemisphere and comparing the result of the fit to the opposite half of the sky. In [51], four different SN data sets were analyzed, and a statistically significant deviation from the isotropy of the Hubble diagram was reported. However, it seems that these anisotropies are aligned with the equatorial coordinate system (the Earth’s axis of rotation) and are thus likely to reflect systematic problems in the SN Ia

observations or analyses. Another different analysis revealed anisotropies in the Hubble diagram obtained by the Hubble Space Telescope (HST) Key Project [52]. If these anisotropies would be confirmed by future SN surveys, and a systematic effect could be excluded, this would pose a serious challenge to the interpretation of the Hubble diagram in the FLRW context. A systematic review of the measurement of the Hubble constant can be found in [54].

In summary, the homogeneity and isotropy of the Universe, introduced one century ago as a simple working hypothesis, although stood various inspections, is still far from being a cornerstone of cosmology. What we can safely say today is only that beyond scales of 100 Mpc, this homogeneity and isotropy is a good approximation. [27] Although good, it is after all a rough approximation. Actually, we will see later that the 100 Mpc scales are exactly the very scales, where strong backreaction effects show up. Thus, inside 100 Mpc, we can no longer trust them confidently, and different kinds of inhomogeneous models, which are closer to reality than the oversimplified FLRW one, are strongly needed.

2.5.2 A status review of the backreaction mechanism

The theoretical aspect of the study of inhomogeneities and anisotropies is also a large branch. Hundreds of papers, even books [56] have been published in this hot area. Passionate agreements and tough criticisms can easily be found in the literature for this ardent debate. It is fairly interesting to read these papers, but to totally review them is certainly impossible here. Several excellent review articles are available in the literature, in which the best one is of course the long status report by Buchert himself [57], with references therein.

Backreaction was maybe first used to study the quantum behavior of black holes. For instance, we obtain the Schwarzschild solution of a point mass source by setting $T_{\mu\nu} = 0$ in vacuum. There is no trouble with this solution in pure GR. However, once the quantum effects are evolved, things become more complicated. A black hole evaporates via the Hawking radiation, *i.e.*, when one of the two virtual particles generated by quantum fluctuation near the horizon of a black hole is absorbed into the black hole, the other one may escape and thus leads to a real particle outside the black hole's horizon. If this process could happen, the black hole is not so black and the vacuum is also not so vacuum. Therefore, an effective energy-momentum tensor arises in vacuum, and it must backreact to the classical Schwarzschild solution.

The first application of the idea of backreaction to cosmology, to my knowledge, are the works by Mukhanov, Abramo and Brandenberger. In [58], they considered the backreaction effects in the inflation era. The first investigation of the backreaction mechanism in the late Universe was carried out by Russ *et al.* [59], though scarcely known by the physics community, in which the basic equations, currently mainly referred as the Buchert equations were largely obtained, albeit not as fundamental and manifest as the elegant works by Buchert.

Then came the wonderful works by Buchert [60, 61, 62, 63], on which this dissertation is based. Before getting these relativistic results, he has worked on backreaction issues in

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[27] Further recent experiments about this problem can be found in, *e.g.*, [55].

Newtonian gravity for many years (see [64]). The details of his works and the equations now named after him will be carefully investigated in Sec. 3.

The diverse research topics in this area are already summarized in [57]. Generally speaking, recent research on the backreaction mechanism explored two directions:

1. The first one is to study the properties of the averaged physical quantities in the perturbed Universe. In Buchert's approach, the averaged Einstein equations were derived in the synchronous coordinates with two fluctuation terms. The behavior of the perturbed Universe thus depends on the properties of these averaged quantities. Räsänen [65] extended his idea to the Lemaître-Tolman-Bondi (LTB) model, *i.e.*, a model only demanding the spherical isotropy of the metric (with the FLRW model as a special case), and Biswas and Notari further considered a more realistic model, the swiss-cheese model, *i.e.*, a universe dominated by many voids. However, from my point of view, these model dependent works can only save us from the *over*-simplified FLRW model to some *very*-simplified ones, as our Universe is much more complex than those described in these toy models. So although sometimes these works provide us some exact solutions, we are still far from the final results.

2. The second direction is to utilize cosmological perturbation theory to study the evolution of the perturbed Universe. For example, in the papers by Kolb *et al.* [67, 68], the Hubble expansion rate was calculated to second order in a MD Universe. However, in [67], they defined a very strange Hubble expansion rate, which cannot be compared with the experimental data. In [68], they finally turned to the Buchert equations, but their perturbative approach then was not easy to understand, at least for me.

In this dissertation, we synthesize these two lines of research, and try to fill the gap between [67, 68]. Because doing perturbative calculations without averaging, we cannot obtain the *global* property of the Universe; whereas, averaging without perturbative calculations, we cannot get the *quantitative* information of the Universe. Therefore, we calculate the averaged physical quantities in cosmological perturbation theory up to third order. Following the previous studies, we also calculate the ensemble means and variances of spatial averages. Our interest is to quantify the backreaction effects in the Milky Way's neighborhood, *i.e.*, the domain used to measure H_0. As we will show in this work, the knowledge of the peculiar gravitational potential on the boundary of a physically comoving domain at some initial time allows us to predict the temporal evolution of the spatially averaged quantities, as long as these effects are small.

This dissertation includes the results of our previous three papers [69, 70, 71].

3 Dynamics of the averaged Universe

Having understood the entanglement of dark energy, structure formation and averaging problems, we turn our steps to the dynamics of the averaged Universe and cosmological perturbation theory. For these two aspects, we should never attach importance to one but neglect the other. Because doing perturbative calculations without averaging, we cannot obtain the global property of the Universe. Whereas, averaging without perturbative calculations, we cannot get quantitative information about our results. Therefore, any ignorance of one side raises insufficiency of our analyses, and we thus tightly incorporate them in this and the next sections. These two sections are the preconditions for the perturbative calculations of the averaged physical observables in Secs. 5 - 7.

3.1 Decomposition of the Einstein equations in the comoving synchronous gauge

The standard FLRW model is based on the *a priori* assumption of spatial homogeneity and isotropy in the Universe. However, these assumptions are not valid, even approximately, at the scales on which structure formation happens, *i.e.*, the scales much smaller with respect to the Hubble radius and adequately long after the matter-radiation equality. So necessarily one must consider not only the expansion, but also the shear and rotation of the Universe in order to understand its kinematics thoroughly. Thus, before discussing the averaging problem in the perturbed Universe, in this subsection, we first list some basic geometrical definitions and preparations and decompose the Einstein equations by means of these geometrical quantities.

3.1.1 Expansion, shear and rotation

To describe the kinematics of the Universe, we need to calculate the gradient field of the 4-velocity $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$ of observers, where τ is their proper time. Assuming only a dust universe (possibly with a cosmological constant), we may set these observers to be *comoving* [28] with the cosmic medium and thus have $u^\mu u^\nu{}_{;\mu} = 0$ (the geodesic flow of the comoving observers, locally only). We further introduce the *projection operator* onto the spatial hypersurface defined by the comoving observers $h^\mu{}_\nu \equiv g^\mu{}_\nu + u^\mu u_\nu$. Therefore, the component of the gradient field of the 4-velocity is equal to the *expansion tensor* [29]

$$u^\mu{}_{;\nu} \equiv \theta^\mu{}_\nu = h^\mu{}_\alpha h^\beta{}_\nu u^\alpha{}_{;\beta} = \omega^\mu{}_\nu + \sigma^\mu{}_\nu + \frac{1}{3}h^\mu{}_\nu \theta, \tag{9}$$

where

$$\omega^\mu{}_\nu \equiv \frac{1}{2}h^\mu{}_\alpha h^\beta{}_\nu (u^\alpha{}_{;\beta} - u_\beta{}^{;\alpha}), \tag{10}$$

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[28] For an irrotational dust universe it is possible to use the comoving coordinates, *i.e.*, the observer is at rest with respect to the cosmic medium. The comoving coordinate system is well adapted to the situation of a real observer, if we are allowed to neglect the difference between baryons and dark matter. On scales ≥ 10 Mpc, the baryonic pressure is insignificant, and a real observer comoves with matter, uses her own clock and regards space to be time-orthogonal.

[29] The expansion tensor is equivalent to the extrinsic curvature tensor $K^\mu{}_\nu \equiv -h^\mu{}_\alpha h^\beta{}_\nu u^\alpha{}_{;\beta} = -\theta^\mu{}_\nu$, *i.e.*, the second fundamental form of the hypersurface of constant time t.

$$\sigma^{\mu}{}_{\nu} \equiv h^{\mu}{}_{\alpha}h^{\beta}{}_{\nu}\left[\frac{1}{2}(u^{\alpha}{}_{;\beta} + u_{\beta}{}^{;\alpha}) - \frac{1}{3}h^{\alpha}{}_{\beta}u^{\lambda}{}_{;\lambda}\right], \tag{11}$$

$$\theta \equiv u^{\lambda}{}_{;\lambda}, \tag{12}$$

are the *rotation tensor*, *shear tensor* and *expansion scalar*, respectively. It is easy to see that $\sigma^{\mu}{}_{\nu}$ is symmetric and $\omega^{\mu}{}_{\nu}$ anti-symmetric.

In the following, we restrict our attention to an irrotational universe, *i.e.*, $\omega^{\mu}{}_{\nu} = 0$. Neglecting rotations seems to be a reasonable assumption in the context of inflationary cosmology, as there existed no seeds for vector perturbations and the conservation of angular momentum also implies that only nonlinear effects could lead to a generation of rotation.

Neglecting the rotation, we are allowed to utilize the synchronous coordinate system, in which we foliate the space-time to constant time hypersurfaces and set the time axis orthogonal to them. Then the metric of the inhomogeneous and anisotropic Universe may be expressed in terms of synchronous coordinates, [30]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}(t, \mathbf{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{13}$$

The corresponding nontrivial Christoffel symbols are

$$\Gamma^0_{ij} = \frac{1}{2}g_{ij,0}, \quad \Gamma^i_{0j} = \frac{1}{2}g^{ik}g_{kj,0}, \quad \Gamma^i_{jk} = \frac{1}{2}g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l}).$$

Moreover, the 4-velocity u^μ becomes [31]

$$u^\mu = (1, \mathbf{0}), \quad u_\mu = (-1, \mathbf{0}).$$

Therefore, $h^i{}_j = \delta^i{}_j$, and the nontrivial components in Eq. (9) are

$$\theta^i{}_j = \sigma^i{}_j + \frac{1}{3}\theta\delta^i{}_j, \tag{14}$$

and we straightforwardly get

$$\theta^i{}_j = u^i{}_{;j} = \Gamma^i_{0j} = \frac{1}{2}g^{ik}\dot{g}_{kj}, \quad \theta = u^i{}_{;i} = \Gamma^i_{0i} = \frac{1}{2}g^{ij}\dot{g}_{ij}. \tag{15}$$

It is direct to see from Eq. (14) that $\sigma^i{}_i = 0$, so we introduce the shear scalar in the following way,

$$\sigma^2 \equiv \frac{1}{2}\sigma^{\mu}{}_{\nu}\sigma^{\nu}{}_{\mu} = \frac{1}{2}\sigma^i{}_j\sigma^j{}_i = \frac{1}{2}\left(\theta^i{}_j\theta^j{}_i - \frac{1}{3}\theta^2\right). \tag{16}$$

Equations. (15) and (16) present all the necessary kinematical quantities that we need in the averaging problem below.

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[30] We stress that there is no predetermined symmetry in $g_{ij}(t, \mathbf{x})$, and it can thus represent the metric perturbed in any way.

[31] $\mathrm{d}\tau = \mathrm{d}t$ in the synchronous coordinate system due to Eq. (13).

Before proceeding to the averaging procedure, we prove a helpful relation. We define the measure of the integral in the averaging problem in the next subsection as the square root of the determinant of the 3-dimensional spatial metric,

$$J(t, \mathbf{x}) \equiv \sqrt{\det(g_{ij}(t, \mathbf{x}))}. \tag{17}$$

Then, we have

$$\frac{\dot{J}}{J} = \frac{1}{2}\frac{(\det g_{ij})^{\bullet}}{\det g_{ij}} = \frac{1}{2}g^{ij}\dot{g}_{ij} = \theta \quad \text{and} \quad \dot{J} = \theta J. \tag{18}$$

where $^{\bullet}$ is the derivative with respect to the cosmic time. Equation (18) will be useful for defining the effective Hubble expansion rate.

3.1.2 Arnowitt-Deser-Misner decomposition

Having obtained all the geometrical quantities describing the kinematics of the perturbed Universe, we proceed to its dynamics. For the ideal fluid in the dust Universe, in the comoving synchronous coordinate system, the unique nontrivial component of the energy-momentum tensor $T^{\mu}_{\ \nu} = (\rho + p)u^{\mu}u_{\nu} + pg^{\mu}_{\ \nu}$ is $T^{0}_{\ 0} = -\rho$, *i.e.*, the energy density of dust.

According to Arnowitt, Deser and Misner [72], the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ in the present situation can be decomposed into:

1. energy constraint

   For the $^{0}{}_{0}$-component of the the Einstein equations, $R^{0}_{\ 0} - \frac{1}{2}R + \Lambda = 8\pi G T^{0}_{\ 0} = -8\pi G\rho$, we have

   $$\frac{1}{2}\left(-R^{0}_{\ 0} + R^{i}_{\ i}\right) = 8\pi G\rho + \Lambda, \tag{19}$$

   where we have used $R = R^{0}_{\ 0} + R^{i}_{\ i}$, and

   $$\begin{aligned} R^{0}_{\ 0} &= -R_{00} = -\Gamma^{i}_{0i,0} - \Gamma^{j}_{0i}\Gamma^{i}_{0j}, & (20) \\ R^{i}_{\ i} &= g^{ij}R_{ij} \\ &= g^{ij}\left(\Gamma^{k}_{ji,k} - \Gamma^{k}_{ik,j} + \Gamma^{k}_{ij}\Gamma^{l}_{kl} - \Gamma^{l}_{ik}\Gamma^{k}_{jl}\right) + g^{ij}\left(\Gamma^{0}_{ij,0} + \Gamma^{0}_{ij}\Gamma^{k}_{0k} - \Gamma^{k}_{0i}\Gamma^{0}_{jk} - \Gamma^{0}_{ik}\Gamma^{k}_{0j}\right) \\ &\equiv \mathcal{R} + g^{ij}\left(\Gamma^{0}_{ij,0} + \Gamma^{0}_{ij}\Gamma^{k}_{0k} - \Gamma^{k}_{0i}\Gamma^{0}_{jk} - \Gamma^{0}_{ik}\Gamma^{k}_{0j}\right), & (21) \end{aligned}$$

   with

   $$\mathcal{R}^{i}_{\ j} \equiv g^{ik}\left(\Gamma^{l}_{kj,l} - \Gamma^{l}_{kl,j} + \Gamma^{l}_{kj}\Gamma^{m}_{lm} - \Gamma^{m}_{kl}\Gamma^{l}_{jm}\right) \tag{22}$$

   being the 3-dimensional spatial Ricci tensor, and $\mathcal{R} \equiv \mathcal{R}^{i}_{\ i}$ the spatial curvature.

   Substituting Eqs. (20) and (21) into Eq. (19) and using the facts $\theta^{i}_{\ j} = \Gamma^{i}_{0j}$ and $\theta = \Gamma^{i}_{0i}$ from Eq. (15), we obtain the *energy constraint*, [32]

   $$\frac{1}{2}\left(\mathcal{R} + \theta^{2} - \theta^{i}_{\ j}\theta^{j}_{\ i}\right) = 8\pi G\rho + \Lambda. \tag{23}$$

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[32] To prove Eq. (23), we need a small trick $\dot{g}^{ij}\dot{g}_{ij} + g^{ik}g^{jl}\dot{g}_{il}\dot{g}_{jk} = 0$, which can be obtained from $(g^{ik}g_{kj})^{\bullet} = 0$.

2. momentum constraint

   For the ${}^0{}_i$-component of the Einstein equations, $R^0{}_i = 8\pi G T^0_i = 0$, we have

$$R^0{}_i = \left(\Gamma^j_{0j,i} - \Gamma^j_{0i,j} + \Gamma^k_{0j}\Gamma^j_{ik} - \Gamma^j_{0i}\Gamma^k_{jk}\right) = 0.$$

   Again using Eq. (15), we directly attain the *momentum constraint*,

$$\theta_{,i} = \theta^j{}_{i;j}. \tag{24}$$

3. evolution equation

   For the ${}^i{}_j$-component of the the Einstein equations, $R^i{}_j - \frac{1}{2}R\delta^i{}_j + \Lambda\delta^i{}_j = 8\pi G T^i{}_j = 0$, we have

$$R^i{}_j = \mathcal{R}^i{}_j + g^{ik}\left(\Gamma^0_{kj,0} + \Gamma^0_{kj}\Gamma^l_{0l} - \Gamma^l_{0k}\Gamma^0_{jl} - \Gamma^0_{kl}\Gamma^l_{0j}\right).$$

   Using Eqs. (20), (21) and the energy constraint Eq. (23), we get

$$\theta^i{}_{j,0} = -\theta\theta^i{}_j - \mathcal{R}^i{}_j + \frac{1}{2}\left(\dot{\theta} + g^{kl}\Gamma^0_{kl,0} - 8\pi G\rho\right)\delta^i{}_j.$$

   Taking the trace of the above equation and combing it with the energy constraint Eq. (23), we finally arrive at the *evolution equation*, [33]

$$\dot{\theta}^i{}_j = -\theta\theta^i{}_j - \mathcal{R}^i{}_j + (4\pi G\rho + \Lambda)\delta^i{}_j. \tag{25}$$

Combining the trace of the evolution equation Eq. (25) with Eqs. (23) and (16) leads to the Raychaudhuri equation [73], which links the expansion and shear scalars together,

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi G\rho + \Lambda. \tag{26}$$

So far, we have not made any approximations apart from neglecting rotation and restricting matter to dust. These equations are satisfied at any point in space-time. However, our observations do not allow us to measure all the data that would be necessary to put a well posed Cauchy problem, as realistic observations deliver averaged quantities. In the next subsection, we discuss the averaged properties of these equations and in Sec. 5 - 7, we use cosmological perturbation theory to evaluate the averaged physical observables to first, second and third orders, respectively. [34]

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[33] To prove Eq. (25), we need another small trick $\dot{\theta} = g^{ij}\Gamma^0_{ij,0} - 2\theta^i{}_j\theta^j{}_i$, which can be proven straightforwardly by using Eq. (15) and the previous trick $\dot{g}^{ij}\dot{g}_{ij} + g^{ik}g^{jl}\dot{g}_{il}\dot{g}_{jk} = 0$ again.

[34] Strictly speaking, in the derivations above, the covariant and partial derivatives with respect to only spatial coordinates should be denoted differently from those to space-time coordinates: *e.g.*, using $_{||}$ and $_{|}$, instead of $_{;}$ and $_{,}$, as frequently adopted in the literature. However, in the synchronous gauge, spatial and temporal coordinates are fortunately separated, and $_{||}$ and $_{|}$ coincide with $_{;}$ and $_{,}$, so we do not distinguish them and no confusion will be caused by doing so.

3.2 Dynamics of finite domains

Based on the previous preparations, the dynamics of the perturbed Universe emerges naturally. Here, we first define the averaging procedure for scalar quantities, then yield the effective Friedmann equations (Buchert equations) for the perturbed dust Universe and finally obtain an integrability condition for backreaction terms, which is crucially important for our perturbative calculations at higher orders.

3.2.1 Averaging procedure

The spatial average of a scalar physical observable $O(t, \mathbf{x})$ in a comoving domain D at a fixed time t is defined [35] as [60]

$$\langle O \rangle_D \equiv \frac{1}{V_D(t)} \int_D O(t, \mathbf{x}) J(t, \mathbf{x}) \mathrm{d}\mathbf{x}, \tag{27}$$

with $V_D(t) \equiv \int_D J(t, \mathbf{x}) \mathrm{d}\mathbf{x}$ denoting the volume of the domain. Following this definition of V_D, we introduce an effective scale factor a_D [60]

$$\frac{a_D}{a_{D_0}} \equiv \left(\frac{V_D}{V_{D_0}} \right)^{1/3}, \tag{28}$$

where a_{D_0} and V_{D_0} are the values of a_D and V_D at the present time.

As an example of the above averaging procedure, we calculate the averaged volume expansion rate $\langle \theta \rangle_D$. With the help of Eqs. (18) and (28) we find

$$\langle \theta \rangle_D = \frac{1}{V_D} \int_D \theta J \mathrm{d}\mathbf{x} = \frac{1}{V_D} \int_D \dot{J} \mathrm{d}\mathbf{x} = \frac{\dot{V}_D}{V_D} = 3 \frac{\dot{a}_D}{a_D}. \tag{29}$$

The effective Hubble expansion rate is thus defined as

$$H_D \equiv \frac{\dot{a}_D}{a_D} = \frac{1}{3} \langle \theta \rangle_D. \tag{30}$$

The effective scale factor a_D and the Hubble expansion rate H_D reduce to a and $\dot{a}/a$ in the homogeneous and isotropic case doubtlessly.

An important consequence of the definition in Eq. (27) is that the spatial average and the temporal derivative do not commute with each other. It is straightforward to prove a corresponding Lemma (commutation rule) [60] [36]

$$\langle O \rangle_D^{\bullet} - \langle \dot{O} \rangle_D = \langle O \theta \rangle_D - \langle O \rangle_D \langle \theta \rangle_D. \tag{31}$$

This Lemma indicates the intrinsic non-commutation of the averaging problem in the perturbed space-time. Irrespectively of the explicit forms the fundamental dynamics (linear or nonlinear), spacial average and temporal evolution cannot be exchanged, and the backreaction effects must show up!

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[35] This operation is the so-called Riemannian volume integration.

[36] This Lemma will be used to calculate the second order term of the averaged expansion rate $\langle \theta \rangle_D$ in Sec. 6.1.3.

3.2.2 Buchert equations

With the spatial averaging procedure Eq. (27) and the Lemma Eq. (31), we yield the effective Friedmann equations (Buchert equations) [60] [37] by averaging the energy constraint Eq. (23) and the Raychaudhuri equation Eq. (26), [38]

$$\begin{aligned} H_D^2 &= \frac{8\pi G}{3}\langle\rho\rangle_D - \frac{\langle Q\rangle_D + \langle\mathcal{R}\rangle_D}{6} + \frac{\Lambda}{3}, \\ \frac{\ddot{a}_D}{a_D} &= -\frac{4\pi G}{3}\langle\rho\rangle_D + \frac{\langle Q\rangle_D}{3} + \frac{\Lambda}{3}. \end{aligned}$$

We may further recast these equations to an isotropic fluid, with effective energy density $\rho_{\rm eff}$ and pressure $p_{\rm eff}$, [39]

$$H_D^2 = \frac{8\pi G}{3}\rho_{\rm eff} + \frac{\Lambda}{3}, \tag{32}$$

$$\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3}(\rho_{\rm eff} + 3p_{\rm eff}) + \frac{\Lambda}{3}, \tag{33}$$

where

$$\rho_{\rm eff} \equiv \langle\rho\rangle_D - \frac{\langle Q\rangle_D + \langle\mathcal{R}\rangle_D}{16\pi G}, \tag{34}$$

$$p_{\rm eff} \equiv -\frac{1}{16\pi G}\left(\langle Q\rangle_D - \frac{1}{3}\langle\mathcal{R}\rangle_D\right). \tag{35}$$

The expression $\langle Q\rangle_D$ is the *kinematical backreaction* term, [40]

$$\langle Q\rangle_D \equiv \frac{2}{3}\langle(\theta - \langle\theta\rangle_D)^2\rangle_D - 2\langle\sigma^2\rangle_D = \frac{2}{3}(\langle\theta^2\rangle_D - \langle\theta\rangle_D^2) - 2\langle\sigma^2\rangle_D, \tag{36}$$

which consists of the variance of the averaged expansion rate and shear scalar, meaning that the more matter distribution is structured, with collapsed regions and voids, the more this term may contribute to dynamics. Furthermore, the averaged spatial curvature $\langle\mathcal{R}\rangle_D$ no longer behaves like a constant term in the FLRW model.

Equations (32) and (33) express a highly nontrivial result! They closely resemble the traditional Friedmann equations, successfully preserve their main features, and at the same time explicitly formulate the deviation from the standard FLRW model, but all these issues have been obtained without the *a priori* assumption of homogeneity and isotropy. What has been shown is that *any* irrotational dust universe, spatially averaged over comoving domains appears to the observers to be a FLRW-like Universe.

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[37] See also [61, 62, 63] for detailed discussions on various cases of these equations, and [64] for their Newtonian counterparts.

[38] These equations had been derived by Russ *et al.* in [59], although not in the present forms.

[39] These effective energy density and pressure can also be linked by the effective continuity equation $\dot{\rho}_{\rm eff} + 3H_D(\rho_{\rm eff} + p_{\rm eff}) = 0$.

[40] This term $\langle Q\rangle_D$ is sometimes just written as Q_D in the literature to show that it as a whole is the kinematical backreaction, but not the average of some physical quantity Q. While, we see from Eq. (36) that it is indeed the average of a term $\frac{2}{3}(\theta - \langle\theta\rangle_D)^2 - 2\sigma^2$, though this term is of no interest in practice.

From the Buchert equations, we see that the evolution of the inhomogeneous and anisotropic Universe depends not only on the energy density $\langle\rho\rangle_D$, but also on the kinematical backreaction term $\langle Q\rangle_D$ and the averaged spatial curvature $\langle\mathcal{R}\rangle_D$. So it is quite significant to know quantitatively the values of $\langle Q\rangle_D$ and $\langle R\rangle_D$. For instance, we find from Eq. (33) that if $\rho_{\rm eff}+3p_{\rm eff}<0$, *i.e.*, $\langle Q\rangle_D > 4\pi G\langle\rho\rangle_D$, the averaged expansion of the perturbed Universe accelerates. In other words, the averaged Universe can expand in an accelerating way in the dust era, even if the local expansion is decelerating everywhere. Accelerated expansion of the averaged expansion rate does not violate the strong energy condition.

Furthermore, we define the effective equation of state as

$$w_{\rm eff} \equiv \frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{\langle R\rangle_D - 3\langle Q\rangle_D}{2\langle\theta\rangle_D^2}, \tag{37}$$

and the square of an effective speed of sound as

$$c_{\rm eff}^2 \equiv \frac{\dot{p}_{\rm eff}}{\dot{\rho}_{\rm eff}}. \tag{38}$$

This effective speed of sound is the characteristic speed at which a small perturbation propagates through the effective fluid. An example would be a deformation of the boundary of the domain, or a perturbation caused by the introduction of some extra mass into the domain. Our effective speed of sound is different from the isentropic speed of sound.

Calculations of these averaged physical observables: $\langle Q\rangle_D$, $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$, H_D, $\langle\rho\rangle_D$, $w_{\rm eff}$ and $c_{\rm eff}^2$ form the main part of this dissertation in Secs. 5 - 7.

3.2.3 Integrability condition

The Buchert equations contain two averaged quantities, $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$, which influence the evolution of the inhomogeneous and anisotropic Universe. However, these two terms are not independent, but can be related by an *integrability condition.*

In the irrotational dust universe, from the covariant conservation of the time-like piece of the energy-momentum tensor, we find the continuity equation

$$\dot{\rho} = -\theta\rho. \tag{39}$$

Taking the spatial average of Eq. (39) and applying the Lemma Eq. (31), we have

$$\langle\rho\rangle_D^{\bullet} + \langle\theta\rangle_D\langle\rho\rangle_D = 0. \tag{40}$$

This result expresses the conservation of mass in the comoving synchronous gauge setting. From Eqs. (32), (33) and (40), we obtain the integrability condition for $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$ [60]

$$\left(a_D^6\langle Q\rangle_D\right)^{\bullet} + a_D^4\left(a_D^2\langle R\rangle_D\right)^{\bullet} = 0. \tag{41}$$

We should point out here that in contrast to the Buchert equations (32) and (33), this integrability condition has no analogue in Newtonian cosmology.

The integrability condition is an essential equation for the following calculations. Its advantage is that it can be applied to any order in perturbative calculations, as it is an *exact* result. This advantage will be shown in Sec. 6, where we make use of the integrability condition to derive the second order terms of $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$, but *without* using the metric perturbations of second order!

3.2.4 Cosmic quartet

Since the cosmological backreaction terms $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$ enter the right hand side of the effective Friedmann equations, they can thereby be thought as effective constituents of the energy-momentum tensor. Thus, together with the energy density $\langle\rho\rangle_D$ and cosmological constant, *four* ingredients play a role in the dynamics of the perturbed Universe, extending the cosmic triangle to a *cosmic quartet*.

We introduce the corresponding density parameters for the averaged Universe as

$$\Omega_{\mathrm{m}}^D \equiv \frac{8\pi G\langle\rho\rangle_D}{3H_D^2}, \quad \Omega_\Lambda^D \equiv \frac{\Lambda}{3H_D^2}, \quad \Omega_{\mathcal{R}}^D \equiv -\frac{\langle\mathcal{R}\rangle_D}{6H_D^2}, \quad \Omega_Q^D \equiv -\frac{\langle Q\rangle_D}{6H_D^2}. \tag{42}$$

and therefore the normalization condition reads

$$\Omega_{\mathrm{m}}^D + \Omega_\Lambda^D + \Omega_{\mathcal{R}}^D + \Omega_Q^D = 1. \tag{43}$$

We may reinterpret the cosmic quartet in another form, including the curvature term. To do so, we rewrite the integrability condition as $4a_D\dot{a}_D\langle Q\rangle_D + (a_D^2\langle Q\rangle_D)^{\bullet} + (a_D^2\langle\mathcal{R}\rangle_D)^{\bullet} = 0$, and its first integral is

$$\begin{aligned}\frac{\langle Q\rangle_D + \langle\mathcal{R}\rangle_D}{6} &= \frac{a_{D_0}^2(\langle Q\rangle_{D_0} + \langle\mathcal{R}\rangle_{D_0})}{6a_D^2} - \frac{1}{3a_D^2}\int_{t_0}^{t} \mathrm{d}t_1 \langle Q\rangle_D \frac{\mathrm{d}}{\mathrm{d}t_1}a_D^2 \\ &= \frac{a_{D_0}^2(\langle Q\rangle_{D_0} + \langle\mathcal{R}\rangle_{D_0})}{6a_D^2} - \frac{2}{3a_D^2}\int_{a_{D_0}}^{a_D} \mathrm{d}a_{D1}\langle Q\rangle_D a_{D1} \\ &\equiv \frac{k_{D_0}}{a_D^2} - \frac{2}{3a_D^2}\int_{a_{D_0}}^{a_D} \mathrm{d}a_{D1}\langle Q\rangle_D a_{D1}. \end{aligned} \tag{44}$$

From Eq. (44), we can have a constant curvature term k_{D_0}/a_D^2 in the perturbed Universe, making the Buchert equations more similar to the most general Friedmann equations,

$$H_D^2 = \frac{8\pi G}{3}\langle\rho\rangle_D - \frac{k_{D_0}}{a_D^2} + \frac{\Lambda}{3} + \frac{2}{3a_D^2}\int_{a_{D_0}}^{a_D} \mathrm{d}a_{D1}\langle Q\rangle_D a_{D1},$$

where the last term indicates the effect from cosmological backreaction in the whole history of the Universe since the beginning of the MD phase.

We introduce again two new density parameters,

$$\Omega_k^D \equiv -\frac{k_{D_0}}{a_D^2 H_D^2}, \quad \Omega_{\mathrm{N}}^D \equiv \frac{2}{3a_D^2 H_D^2}\int_{a_{D_0}}^{a_D} \mathrm{d}a_{D1}\langle Q\rangle_D a_{D1}.$$

Therefore, the normalization condition varies from Eq. (43) to

$$\Omega_{\mathrm{m}}^D + \Omega_\Lambda^D + \Omega_k^D + \Omega_{\mathrm{N}}^D = 1. \tag{45}$$

Equation (45) is formally equivalent to its Newtonian counterpart in [64].

3.3 Mapping the effective fluid on models with dark energy or morphon field

From this subsection on, we set the cosmological constant to be 0 and ask whether the backreaction terms can mimic a cosmological constant; even if not, to what degree these backreaction terms influence the evolution of the background Universe.

3.3.1 Mimicking dark energy by cosmological backreaction terms

We may map this effective fluid on a model with dust and dark energy. Let n be the number density of dust particles, and m be their mass. For any comoving domain $\langle n\rangle_D = \langle n\rangle_{D_0}(a_{D_0}/a_D)^3$. In the dust Universe, $\rho(t, \mathbf{x}) = mn(t, \mathbf{x})$, and we thus identify $\rho_{\mathrm{m}} \equiv \langle \rho\rangle_D = m\langle n\rangle_D$. From Eq. (34), the energy density of "dark energy" is consequently

$$\rho_{\mathrm{de}} = -\frac{\langle Q\rangle_D + \langle \mathcal{R}\rangle_D}{16\pi G},$$

with the relevant equation of state reading

$$w_{\mathrm{de}} \equiv \frac{p_{\mathrm{de}}}{\rho_{\mathrm{de}}} = \frac{p_{\mathrm{eff}}}{\rho_{\mathrm{de}}} = -\frac{1}{3} + \frac{4\langle Q\rangle_D}{3(\langle Q\rangle_D + \langle \mathcal{R}\rangle_D)}. \tag{46}$$

We discover that iff

$$\langle Q\rangle_D = -\frac{1}{3}\langle R\rangle_D, \tag{47}$$

$w_{\mathrm{de}} = -1$, corresponding to a cosmological constant $\Lambda = \langle Q\rangle_D$ [74]! [41] This would be an exciting result, if $\langle Q\rangle_D$ and $\langle \mathcal{R}\rangle_D$ really satisfied Eq. (47). While, in Secs. 6 and 7, we will see that this could only happen at third order of the perturbative series, which means cosmological backreaction could induce a cosmological constant, but lead to some extra terms at lower orders simultaneously. But we should emphasize that the conclusion above is only valid for the perturbative approach. If we consider the non-perturbative regime of cosmological backreaction, $\langle Q\rangle_D$ and $\langle \mathcal{R}\rangle_D$ would be strongly coupled. This is the power-law solutions for the backreaction terms in the morphon field.

3.3.2 Mapping cosmological backreaction on a minimally coupled scalar field

Morphon field An excellent reinterpretation of the cosmological backreaction in a mean field description, which can play the role of the quintessence field, of the morphology of the structures in the Universe, was presented in [74], and the relevant scalar field is named as a *morphon field*.

The basic idea of the morphon field is to reformulate the quintessence scenario not by routing the origin of the scalar field to additional sources arising from fundamental field theory, but to physical inhomogeneities and anisotropies in the perturbed Universe. We introduce a homogeneous scalar field ϕ_D evolving in an effective potential $U_D \equiv U(\phi_D)$,

$$\rho_{\mathrm{eff}} \equiv \langle \rho\rangle_D + \rho_{\phi_D}, \quad p_{\mathrm{eff}} \equiv p_{\phi_D},$$

where ρ_{ϕ_D} and p_{ϕ_D} are the energy density and pressure of this morphon field. They are both scale dependent and parameterized as

$$\rho_{\phi_D} = \frac{\epsilon}{2}\dot{\phi}_D^2 + U(\phi_D), \quad p_{\phi_D} = \frac{\epsilon}{2}\dot{\phi}_D^2 - U(\phi_D),$$

---

[41] The value of w_{eff} will be carefully discussed in Sec. 6.2.

where $\epsilon = \pm 1$ corresponds to either the standard or phantom scalar field. [42] Comparing with Eqs. (34) and (35), the cosmological backreaction terms are now reexpressed in terms of the kinetic and potential energy densities of the morphon field,

$$\langle Q \rangle_D = -8\pi G \left(\epsilon \dot{\phi}_D^2 - U(\phi_D) \right), \quad \langle \mathcal{R} \rangle_D = -24\pi G U(\phi_D). \tag{48}$$

We find that the potential of the morphon field now acquires a clear physical source: it is nothing but the averaged spatial curvature of the perturbed Universe. This means that the extra scalar field that we always inserting in the energy-momentum tensor can be identified as the perturbations in the Universe naturally, not plugged in optionally by hand! [43]

Substituting Eq. (48) into Eq. (41), the integrability condition then reads (for $\dot{\phi}_D \neq 0$)

$$\ddot{\phi}_D + 3H_D \dot{\phi}_D + \epsilon \frac{\partial}{\partial \phi_D} U(\phi_D) = 0.$$

The morphon field exactly obeys the Klein-Gordon equation in the expanding Universe! This perfect obeying justifies that the cosmological backreaction is formally equivalent to a regionally homogeneous and minimally coupled scalar field.

Scaling solutions Having mapped the effective fluid on models with dark energy or morphon field, the immediate question is to understand quantitatively how these cosmological backreaction terms affect the evolution of the background Universe. Are they large enough to give rise to the accelerated expansion of the averaged Universe or nothing more than subdominant modifications?

To answer these questions, we must solve the Buchert equations. However, these equations are not closed, as there are four unknown variables: a_D, $\langle \rho \rangle_D$, $\langle Q \rangle_D$ and $\langle \mathcal{R} \rangle_D$, but with only three independent constraints: two from the Buchert equations and one from the integrability condition. In order to close them, it is then necessary to introduce another relation, which either relies on some mathematical Ansatz, or origins form some physical statement. For the first, we give the power-law scaling solutions below, and for the second, we consult cosmological perturbation theory, forming the main body of this dissertation in Secs. 5 - 7.

The scaling solutions for the scale dependence of $\langle Q \rangle_D$ and $\langle \mathcal{R} \rangle_D$ are assumed for on particular reason to obey

$$\langle Q \rangle_D = \mathcal{Q} a_D^n, \quad \langle \mathcal{R} \rangle_D = \mathcal{R} a_D^p, \tag{49}$$

where $\mathcal{Q}$ and $\mathcal{R}$ are constants. Substituting them into the integrability condition, we have

---

[42] A phantom scalar field has a negative kinetic energy term and its equation of state w is smaller than -1.

[43] The Buchert equations are therefore rewritten as

$$H_D^2 = \frac{8\pi G}{3} \left[\langle \rho \rangle_D + \frac{\epsilon}{2} \dot{\phi}_D^2 + U(\phi_D) \right], \quad \frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} \left[\langle \rho \rangle_D + 2\epsilon \dot{\phi}_D^2 - 2U(\phi_D) \right].$$

1. for $n \neq p$, the only solutions are $\langle Q \rangle_D = \mathcal{Q} a_D^{-6}$ and $\langle \mathcal{R} \rangle_D = \mathcal{R} a_D^{-2}$. These are the near-Friedmannian solutions, as $\langle \mathcal{R} \rangle_D$ reduces to a constant curvature. It corresponds to the case where the backreacion and mean curvature evolve independently.

2. for $n = p$, we have $\langle Q \rangle_D = r_D \langle \mathcal{R} \rangle_D = r_D \mathcal{R} a_D^n$, ($r_D$ is constant in time, but of course a function of scale). From the integrability condition, the power index n is fixed to be
$$n = -2 \frac{1 + 3r_D}{1 + r_D}.$$
This result indicates the strong coupling between the kinematical backreaction and averaged spatial curvature. [44]

In the Secs. 5 - 7, we will close the Buchert equations by means of cosmological perturbation theory, and the scale dependence of the backreaction terms will go beyond the simple power-law solutions. We will show that they can be expanded as Laurent series of a_D. In order to do so, we concisely review cosmological perturbation theory in the next section, and provide the solutions for metric perturbations up to second order.

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[44] Further meanings on these scaling solutions will be briefly discussed in Sec. 9, and we also refer to [74] for deeper investigations.

4 Cosmological perturbation theory

Cosmological perturbation theory is one the most successful branches in modern cosmology, which is widely applied to study the aspects of: *e.g.*, the anisotropies of the CMB [25] and structure formation [75]. Strong observational evidences indicate that the Universe was highly homogeneous and isotropic at early times, and the primordial fluctuations were amplified by gravitational instabilities. When the scales of these fluctuations exceeded the Hubble scale during the phase of inflation, they lost causal correlations and got "frozen". Later on, in the RD and MD eras, these fluctuations reentered the Hubble radius, seeding the large scale structures that we are observing today.

Albeit mathematically the problems in cosmological perturbation theory are in principle just solving the perturbed Einstein equations around an expanding background, issues are highly nontrivial and essentially different from those in the Newtonian gravity, where perturbations can usually be separated transparently from the background by physical reasons. Due to the gauge freedom in GR, *i.e.*, no reference system is preferred and the intrinsic freedom to choose the coordinate system, what we could call perturbation obviously depends on which coordinate system we work with, and these ambiguities cannot be eliminated completely. As a consequence, there are two methods to handle cosmological perturbation theory in GR: one is that we arbitrarily pick a coordinate system and carefully keep track of the physical and nonphysical perturbation modes; the other is that we firstly distinguish between the physical perturbation modes, which indeed cause observable signatures, and fictitious perturbation modes, which can be gauged away artificially, and secondly seek the rules how these perturbation modes behave under *gauge transformations* and identify *gauge invariant* variables. These situations are analogous with classical electrodynamics, where we may either use the gauge dependent scalar and vector potentials ϕ and $\mathbf{A}$ or the gauge invariant electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

The first exposition on cosmological perturbation theory was pioneered by Lifshitz [76] in the synchronous gauge sixty years ago (also subsequent work together with Khalatnikov [77]). Although with the name "On the gravitational stability of the expanding Universe" and the conclusion that cosmological perturbations "cannot serve as centers of formation of separate nebulae and stars", this paper is still considered as the initial step to understand gravitational instabilities and structure formations in the context of GR. However, at the early times of the research on cosmological perturbations, different people favored different gauges, causing their results inconvincible. The turning point for this long term obstacle was the elegant work by Bardeen [78], in which gauge invariant cosmological perturbation theory was firmly founded. From then on, people have been able to tell confidently which are the real physical perturbation modes that we should be concerned with, and fruitful accomplishments on cosmological perturbation theory have been achieved since. [45]

Cosmological perturbations consist of two parts: metric perturbations and matter field perturbations. In this section, for a start we discuss the rules for transformations of metric perturbations between different gauges at different orders and then show the corresponding gauge invariant variables. Furthermore, we involve the matter field perturbations (ideal fluid perturbations), solve the perturbed Einstein equations in the synchronous

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[45] For some excellent review articles on cosmological perturbation theory, we refer to [79].

gauge and give the solutions for metric perturbations up to second order. With these mathematical preparations, we can calculate the averaged physical observables to third order in the next three sections.

From this section on, we utilize the *conformal time* η, which commonly appears in the literature when dealing with perturbative calculations, [46]

$$\mathrm{d}\eta \equiv \frac{\mathrm{d}t}{a(t)}, \quad i.e., \quad \eta - \eta_0 = \int_{t_0}^{t} \frac{\mathrm{d}t}{a(t)}. \tag{50}$$

So $\eta - \eta_0$ is the maximal comoving distance that a particle could have traveled since t_0 to t, which is referred as the comoving particle horizon, with the physical particle horizon multiplied by the scale factor a. We use $'$ to denote the derivative with respect to η, *i.e.*, $' \equiv \frac{\partial}{\partial\eta} = a(t)\frac{\partial}{\partial t}$. The cosmic time t will reappear for comparisons with experimental data and simulations in the Newtonian gravity.

4.1 Linear (first order) cosmological perturbation theory

In this subsection, we decompose linear metric perturbations to scalar, vector and tensor types, establish the rules for their transformations and show the gauge invariant variables. We restrict the presentation to spatially flat FLRW space-time. A survey on different gauges is presented finally.

4.1.1 Decomposition of linear metric perturbations

At linear order, the perturbed metric is written as $\mathrm{d}s^2 = (g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu$, with $g^{(1)}_{\mu\nu}$ the linear metric perturbations about the background $g^{(0)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j)$. [47]

The metric perturbations can be categorized into three distinct types: scalar, vector and tensor perturbations, and different components of $g^{(1)}_{\mu\nu}$ are thus decomposed into

1. $g^{(1)}_{00} = -2a^2\phi^{(1)}$, with $\phi^{(1)}$ a 3-scalar.
2. $g^{(1)}_{0i} = -a^2\left(B^{(1)}_{,i} + S^{(1)}_i\right)$, with $B^{(1)}$ a 3-scalar, and $S^{(1)}_i$ a transverse 3-vector, *i.e.*, ${S^{(1)i}}_{,i} = 0$.
3. $g^{(1)}_{ij} = -a^2\left(2\psi^{(1)}\delta_{ij} + 2E^{(1)}_{,ij} + F^{(1)}_{i,j} + F^{(1)}_{j,i} + h^{(1)}_{ij}\right)$, with $\psi^{(1)}$ and $E^{(1)}$ two 3-scalars, $F^{(1)}_i$ a transverse 3-vector, *i.e.*, ${F^{(1)i}}_{,i} = 0$, and $h^{(1)}_{ij}$ a symmetric, transverse and traceless 3-tensor, *i.e.*, $h^{(1)}_{ij} = h_{ji}$, ${h^{(1)i}}_{j,i} = 0$ and ${h^{(1)i}}_{i} = 0$.

---

[46] The advantage of the introduction of η is that it makes the temporal and spatial components in the metric more symmetric. So although it does not simplify analyses and derivations significantly for our purpose, we stick to it in order that our results can easily be compared with those in references. Furthermore, we should stress that any equation that can be analytically solved with the conformal time η can also be solved with the cosmic time t. For cosmological perturbation theory with the cosmic time, we refer to the textbook by Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008).

[47] The scale factor a in terms of η is certainly not the same as that in t, but we do not distinguish them here, as this causes no confusion in the context.

Altogether, we have four 3-scalars: $\phi^{(1)}$, $B^{(1)}$, $\psi^{(1)}$ and $E^{(1)}$, with four degrees of freedom; two 3-vectors: $S_i^{(1)}$ and $F_i^{(1)}$, with one constraint each, resulting in four degrees of freedom; one 3-tensor: $h_{ij}^{(1)}$, with seven constraints, leaving two degrees of freedom. In total, we have ten degrees of freedom in the linear metric perturbations, which coincides with the number of the independent components in $g_{\mu\nu}^{(1)}$. The spatial indices are raised and lowered by the 3-$\delta^i{}_j$.

In linear cosmological perturbation theory, different types of metric perturbations evolve independently and therefore can be analyzed separately, as any coupling of two linear metric perturbations causes barely a second order term. The four scalar modes: $\phi^{(1)}$, $B^{(1)}$, $\psi^{(1)}$ and $E^{(1)}$ are induced by energy density perturbations, so they are relevant for structure formation. The four vector modes in $S_i^{(1)}$ and $F_i^{(1)}$ are related to the rotation of cosmic medium, decaying quickly (just as $1/a$) as in the Newtonian gravity, and thus do not contribute to matter concentrations. The two tensor modes in $h_{ij}^{(1)}$ have no analogue in Newtonian gravity; they are the degrees of freedom of the gravitational field *itself* and do not lead to any perturbation in the perfect fluid, but only gravitational waves.

4.1.2 Linear gauge transformations

After decomposing metric perturbations into these different types, we now further discuss the transformations of these perturbation modes under the changes of the coordinate systems, *i.e.*, different *gauges*.

Gauge transformations can be explored from two equivalent viewpoints: the active one (diffeomorphisms) and the passive one (coordinate transformations). In the modern point of view of diffeomorphisms, the changing of coordinates of one point is identified as a mapping from this point to another in the same manifold. This viewpoint is a little bit mathematical, and to explain it exactly, we need to introduce a series of new concepts and notations, which deviates from the main route of this dissertation. [48] So here, we only investigate the traditional point of view of coordinate transformations, *i.e.*, the passive one. Thus, by "gauge" we mean a coordinate system and refer to gauge transformations and coordinate transformations exchangeably.

Let us consider the linear infinitesimal coordinate transformation

$$\tilde{x}^\mu = x^\mu + \xi^{(1)\mu}(x).$$

The Jacobian of the inverse coordinate transformation is

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = \delta^\alpha{}_\mu - \xi^{(1)\alpha}{}_{,\mu}.$$

Hence, we get the metric transformation at linear order as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) = \left(\delta^\alpha{}_\mu - \xi^{(1)\alpha}{}_{,\mu}\right)\left(\delta^\beta{}_\nu - \xi^{(1)\beta}{}_{,\nu}\right) g_{\alpha\beta}(x). \tag{51}$$

---

[48] For details of diffeomorphisms, we refer to [85, 86] and to the textbook by S.M. Carroll, *Space-time and geometry: an introduction to general relativity* (Addison-Wesley, San Francisco, 2004).

We expand the metric to linear order on both sides, [49]

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g^{(0)}_{\mu\nu}(\tilde{x}) + \tilde{g}^{(1)}_{\mu\nu}(\tilde{x}), \quad g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + g^{(1)}_{\mu\nu}(x). \tag{52}$$

To get the rules for gauge transformations, we must rewrite the different orders of $\tilde{g}_{\mu\nu}(\tilde{x})$ as a function of x, [50]

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) &= g^{(0)}_{\mu\nu}(\tilde{x}) + \tilde{g}^{(1)}_{\mu\nu}(\tilde{x}) \\ &= g^{(0)}_{\mu\nu}(x) + g^{(0)}_{\mu\nu,\lambda}(x)\left(\tilde{x}^\lambda - x^\lambda\right) + \tilde{g}^{(1)}_{\mu\nu}(x) \\ &= g^{(0)}_{\mu\nu}(x) + g^{(0)}_{\mu\nu,\lambda}(x)\xi^{(1)\lambda}(x) + \tilde{g}^{(1)}_{\mu\nu}(x).\end{aligned} \tag{53}$$

Substituting Eqs. (52) and (53) into Eq. (51), we get the rules for gauge transformations at linear order,

$$\tilde{g}^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} - g^{(0)}_{\mu\nu,\alpha}\xi^{(1)\alpha} - g^{(0)}_{\mu\alpha}\xi^{(1)\alpha}{}_{,\nu} - g^{(0)}_{\alpha\nu}\xi^{(1)\alpha}{}_{,\mu} = g^{(1)}_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} g^{(0)}_{\mu\nu}, \tag{54}$$

where $\mathcal{L}_{\xi^{(1)}}$ is the *Lie derivative* generated by $\xi^{(1)\mu}$. We should point out that generally speaking, the result in Eq. (54) is valid for not only the transformations of the metric perturbations, but for any tensor (scalar and vector included) at linear order. The general conclusion can thus be written as [51]

$$\tilde{T}^{(1)} = T^{(1)} - \mathcal{L}_{\xi^{(1)}} T^{(0)}. \tag{55}$$

Using Eq. (54), we are able to attain the rules for transformations of metric perturbations directly. But before doing so, we first reexpress the infinitesimal coordinate transformation as $\xi^{(1)\mu} \equiv (\xi^{(1)0}, \xi^{(1)i})$, with $\xi^{(1)0}$ a 3-scalar and $\xi^{(1)i}$ a 3-vector, which is further decomposed into the derivative of a 3-scalar and a transverse 3-vector, *i.e.*, $\xi^{(1)i} = \zeta^{(1),i} + \xi^{(1)i}_{\perp}$, with $\xi^{(1)i}_{\perp}{}_{,i} = 0$. Thus, we have the transformations of components of the perturbed metric in Eq. (54),

$$\tilde{g}^{(1)}_{00} = g^{(1)}_{00} + 2a\left(a\xi^{(1)0}\right)',$$

---

[49] We point out that $\tilde{g}^{(0)}_{\mu\nu}$ is the same as $g^{(0)}_{\mu\nu}$, as the background metric is $a^2(\eta)(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j)$. It is the same in any coordinate system.

[50] We can change $\tilde{g}^{(1)}_{\mu\nu}(\tilde{x})$ to $\tilde{g}^{(1)}_{\mu\nu}(x)$ without any harm at linear order, as $\tilde{g}^{(1)}_{\mu\nu}$ is already a first order term. This changing can no longer hold for second order gauge transformations, which will be discussed in Sec. 4.2.1.

[51] The explicit results for any scalar A, covariant vector A_μ, contravariant vector A^μ, covariant tensor $A_{\mu\nu}$, mixing tensor $A^\mu{}_\nu$ and contravariant tensor $A^{\mu\nu}$ at linear order are

$$\begin{aligned}\tilde{A}^{(1)} &= A^{(1)} - A^{(0)}_{,\alpha}\xi^{(1)\alpha}, \\ \tilde{A}^{(1)}_\mu &= A^{(1)}_\mu - A^{(0)}_{\mu,\alpha}\xi^{(1)\alpha} - A^{(0)}_\alpha\xi^{(1)\alpha}{}_{,\mu}, \\ \tilde{A}^{(1)\mu} &= A^{(1)\mu} - A^{(0)\mu}{}_{,\alpha}\xi^{(1)\alpha} + A^{(0)\alpha}\xi^{(1)\mu}{}_{,\alpha}, \\ \tilde{A}^{(1)}_{\mu\nu} &= A^{(1)}_{\mu\nu} - A^{(0)}_{\mu\nu,\alpha}\xi^{(1)\alpha} - A^{(0)}_{\mu\alpha}\xi^{(1)\alpha}{}_{,\nu} - A^{(0)}_{\alpha\nu}\xi^{(1)\alpha}{}_{,\mu}, \\ \tilde{A}^{(1)\mu}{}_\nu &= A^{(1)\mu}{}_\nu - A^{(0)\mu}{}_{\nu,\alpha}\xi^{(1)\alpha} - A^{(0)\mu}{}_\alpha\xi^{(1)\alpha}{}_{,\nu} + A^{(0)\alpha}{}_\nu\xi^{(1)\mu}{}_{,\alpha}, \\ \tilde{A}^{(1)\mu\nu} &= A^{(1)\mu\nu} - A^{(0)\mu\nu}{}_{,\alpha}\xi^{(1)\alpha} + A^{(0)\mu\alpha}\xi^{(1)\nu}{}_{,\alpha} + A^{(0)\alpha\nu}\xi^{(1)\mu}{}_{,\alpha}.\end{aligned}$$

$$
\begin{aligned}
\tilde{g}^{(1)}_{0i} &= g^{(1)}_{0i} + a^2\left[\left(\xi^{(1)0} - \zeta^{(1)\prime}\right)_{,i} - \xi^{(1)\prime}_{\perp i}\right], \\
\tilde{g}^{(1)}_{ij} &= g^{(1)}_{ij} - a^2\left(2\frac{a'}{a}\xi^{(1)0}\delta_{ij} + 2\zeta^{(1)}_{,ij} + \xi^{(1)}_{\perp i,j} + \xi^{(1)}_{\perp j,i}\right). \qquad (56)
\end{aligned}
$$

Substituting the metric perturbations $\phi^{(1)}$, $B^{(1)}$, $\psi^{(1)}$, $E^{(1)}$, $S^{(1)}_i$, $F^{(1)}_i$ and $h^{(1)}_{ij}$ into Eq. (56), we finally obtain the transformation rules for linear metric perturbations,

$$
\begin{aligned}
\tilde{\phi}^{(1)} &= \phi^{(1)} - \frac{1}{a}\left(a\xi^{(1)0}\right)', && (57)\\
\tilde{B}^{(1)} &= B^{(1)} + \zeta^{(1)\prime} - \xi^{(1)0}, && (58)\\
\tilde{\psi}^{(1)} &= \psi^{(1)} + \frac{a'}{a}\xi^{(1)0}, && (59)\\
\tilde{E}^{(1)} &= E^{(1)} + \zeta^{(1)}, && (60)\\
\tilde{S}^{(1)}_i &= S^{(1)}_i + \xi^{(1)\prime}_{\perp i}, && (61)\\
\tilde{F}^{(1)}_i &= F^{(1)}_i + \xi^{(1)}_{\perp i}, && (62)\\
\tilde{h}^{(1)}_{ij} &= h^{(1)}_{ij}. && (63)
\end{aligned}
$$

4.1.3 Gauge invariant variables

Equations (57) - (63) show the transformation rules for linear metric perturbations. We find that except the tensor piece $h^{(1)}_{ij}$, all metric perturbations depend on the gauge transformations. Therefore, the main goal in cosmological perturbation theory is to dig out the perturbation modes like $h^{(1)}_{ij}$, which are not influenced by gauge transformations, namely *gauge invariant variables.*

We see that there are ten constraints from Eqs. (57) - (63), but with only four coordinate transformations: $\xi^{(1)0}$, $\zeta^{(1)}$ and $\xi^{(1)}_{\perp i}$, so there remain altogether six gauge invariant variables. From Eqs. (58) and (60), we have $\zeta^{(1)} = \tilde{E}^{(1)} - E^{(1)}$ and $\xi^{(1)0} = (B^{(1)} - \tilde{B}^{(1)}) - (E^{(1)} - \tilde{E}^{(1)})'$. Substituting these relations into Eqs. (57) and (59), we get two gauge invariant variables,

$$
\Phi^{(1)\mathrm{inv}} \equiv \phi^{(1)} - \frac{1}{a}\left[a(B^{(1)} - E^{(1)\prime})\right]', \quad \Psi^{(1)\mathrm{inv}} \equiv \psi^{(1)} + \frac{a'}{a}(B^{(1)} - E^{(1)\prime}). \qquad (64)
$$

Similarly, there are two gauge invariant variables for the vector metric perturbations,

$$
V^{(1)\mathrm{inv}}_i \equiv S^{(1)}_i - F^{(1)\prime}_{\ \ i}, \qquad (65)
$$

and two gauge invariant variables from the tensor metric perturbations (they themselves),

$$
h^{(1)\mathrm{inv}}_{ij}. \qquad (66)
$$

Till now, we have gathered all the six gauge invariant variables: $\Phi^{(1)\mathrm{inv}}$, $\Psi^{(1)\mathrm{inv}}$, $V^{(1)\mathrm{inv}}_i$ and $h^{(1)\mathrm{inv}}_{ij}$. But of course, this does not mean they are the only gauge invariant variables that we have exhausted. Any combination of these six gauge invariant variables is automatically gauge invariant.

4.1.4 Survey on different gauges

Having generally investigated linear gauge transformations, we now briefly explore two most frequently used gauges, *i.e.*, the longitudinal (or Newton, Poisson [52]) gauge, which has gauge invariant variables in itself and synchronous gauge, which we will use for our perturbative calculations of averaged physical observables, and show how we transform from one gauge to the other. [53] In the linear perturbed metric, we can safely neglect the vector and tensor perturbations in the metric, which are so tiny that they can only exhibit their effects at higher orders (at least the second order). Thus, we concentrate on the scalar perturbation modes here.

For the four scalar metric perturbations: $\phi^{(1)}$, $B^{(1)}$, $\psi^{(1)}$ and $E^{(1)}$, their gauge transformations are characterized by the two coordinate transformations $\xi^{(1)0}$ and $\zeta^{(1)}$, which can be chosen freely, *i.e.*, the freedom of choosing coordinate systems. Different choosing methods correspond to different gauges.

Longitudinal gauge If we set $B^{(1)} = E^{(1)} \equiv 0$, we arrive at the *longitudinal gauge* and we have $\xi^{(1)0} = \zeta^{(1)} = 0$. So there is no residual coordinate freedom, and the linear perturbed metric is fixed as

$$\mathrm{d}s^2 = a^2 \left[- \left(1 + 2\phi_\mathrm{l}^{(1)} \right) \mathrm{d}\eta^2 + \left(1 - 2\psi_\mathrm{l}^{(1)} \right) \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \right] . \tag{67}$$

We enumerate some remarks for the longitudinal gauge,

1. Because $\xi^{(1)0} = \zeta^{(1)} = 0$, we cannot find any new longitudinal system.
2. In the longitudinal gauge, $\Phi^{(1)\mathrm{inv}} = \phi_\mathrm{l}$ and $\Psi^{(1)\mathrm{inv}} = \psi_\mathrm{l}$ as $B^{(1)} = E^{(1)} = 0$, so the gauge invariant variables now have clear physical interpretations, they are just the metric perturbations. Thus, working in the longitudinal gauge is equivalent to using Bardeen's variables directly.
3. $\phi_\mathrm{l}^{(1)}$ is the generalization of the Newtonian potential, and that is the reason why it bears the name "Newton gauge".
4. If the spatial part of the perturbative energy-momentum tensor is diagonal, $\phi_\mathrm{l}^{(1)} = \psi_\mathrm{l}^{(1)}$. This simplifies the perturbative calculations for isotropic fluids.

Synchronous gauge Similarly, if we set $\phi^{(1)} = B^{(1)} \equiv 0$, we get the *synchronous gauge*, and the linear perturbed metric reads

$$\mathrm{d}s^2 = a^2 \left\{ -\mathrm{d}\eta^2 + \left[\left(1 - 2\psi_\mathrm{s}^{(1)} \right) \delta_{ij} + E_{\mathrm{s},ij}^{(1)} \right] \mathrm{d}x^i \mathrm{d}x^j \right\} . \tag{68}$$

We again enumerate some remarks for the synchronous gauge,

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[52] For the Poisson gauge, we further pose a condition on the vector modes.

[53] For general discussions of various useful gauges, we refer to [80].

1. There are *residual* degrees of freedom for choosing coordinates, *i.e.*, in one coordinate system $\phi_{\mathrm{s}}^{(1)} = B_{\mathrm{s}}^{(1)} = 0$, then under the coordinate transformation

$$\xi^{(1)0} = \frac{\alpha(\mathbf{x})}{a}, \quad \zeta^{(1)} = \int^{\eta} \frac{\alpha(\mathbf{x})}{a} \mathrm{d}\eta_1 + \beta(\mathbf{x}),$$

with $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ being arbitrary functions of spatial coordinates only, the new coordinate system is also a synchronous system, so we can find a class of synchronous systems, *i.e.*, the synchronous gauge cannot eliminate all the gauge degrees of freedom, and it complicates the interpretation of results obtained from it, especially on superhorizon scales. So every quantity expressed in the synchronous gauge should be carefully examined on its gauge dependence, unless the residual degree of freedom is fixed by an extra condition.

2. In the synchronous gauge, the gauge invariable variables are

$$\Phi^{(1)\mathrm{inv}} = \frac{1}{a}\left(aE_{\mathrm{s}}^{(1)\prime}\right)^{\prime}, \quad \Psi^{(1)\mathrm{inv}} = \psi_{\mathrm{s}}^{(1)} - \frac{a^{\prime}}{a} E_{\mathrm{s}}^{(1)\prime}. \tag{69}$$

Clearly, they unfortunately have no explicit physical meaning.

Due to these two basic shortcomings, the synchronous gauge is criticized by some cosmologists. However, it does have its own advantages. Especially because the coefficient of time interval is set to be $-a^2$ by definition, its temporal coordinate is directly linked to the conformal time, whereas in the longitudinal gauge, the meaning of the temporal coordinate is not very clear. Also the synchronous gauge is very beneficial for the perturbative calculations in the averaging problem. At least, this might be one of the reasons why it has been applied to cosmological perturbation theory ever since the initial work by Lifshitz. Therefore, in the following part of this dissertation, we stick to the synchronous gauge. Of course, the gauge dependence of the quantities calculated in this gauge must be paid careful attention to.

In the following, we do not directly use the synchronous gauge in Eq. (68), but one of its variations,

$$\mathrm{d}s^2 = a^2 \left\{-\mathrm{d}\eta^2 + \left[\left(1 - 2\Psi^{(1)}\right)\delta_{ij} - D_{ij}\chi^{(1)}\right] \mathrm{d}x^i \mathrm{d}x^j\right\}, \tag{70}$$

where $\Psi^{(1)}$ and $\chi^{(1)}$ are the scalar metric perturbations at first order, $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta$ is the traceless derivative and Δ denotes the Laplace operator in a three-dimensional Euclidean space.

Transformations between different gauges Facing these various gauges, people naturally ask what are the relations between different metric perturbation modes. To know that, we do not need to solve the perturbed Einstein equations for each gauge. What we need is just to express the gauge invariant variables $\Phi^{(1)\mathrm{inv}}$ and $\Psi^{(1)\mathrm{inv}}$ in different gauges, and solving these equations enables us to link the different metric perturbation modes in different gauges.

For example, the transformations between those two kinds of synchronous gauges (Eqs. (68) and (70)) are straightforward,

$$E_{\rm s}^{(1)} = \chi^{(1)}, \quad \psi_{\rm s}^{(1)} = \Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}.$$

Similarly, from Eq. (69), we have the relation between the metric perturbations in the synchronous gauge in Eq. (68) and the longitudinal one,

$$E_{\rm s}^{(1)} = \int^{\eta} \frac{1}{a}\left(\int^{\eta_1} a\Phi^{(1){\rm inv}} {\rm d}\eta_2\right){\rm d}\eta_1, \quad \psi_{\rm s}^{(1)} = \Psi^{(1){\rm inv}} + \frac{a'}{a^2}\int^{\eta} a\Phi^{(1){\rm inv}}{\rm d}\eta_1.$$

Finally, the transformations of metric perturbations between the gauge in Eq. (70) and the longitudinal one is

$$\begin{aligned}\chi^{(1)} &= \int^{\eta} \frac{1}{a}\left(\int^{\eta_1} a\Phi^{(1){\rm inv}} {\rm d}\eta_2\right){\rm d}\eta_1, \\ \Psi^{(1)} &= \Psi^{(1){\rm inv}} + \frac{a'}{a^2}\int^{\eta} a\Phi^{(1){\rm inv}}{\rm d}\eta_1 - \int^{\eta}\frac{1}{6a}\left(\int^{\eta_1} a\Delta\Phi^{(1){\rm inv}}{\rm d}\eta_2\right){\rm d}\eta_1.\end{aligned}$$

4.2 Higher order cosmological perturbation theory

The motivations calling for higher order cosmological perturbation theory are that the Einstein equations are intrinsically nonlinear, so concerning only about the linearized equations of motion loses the essences of GR. Furthermore, there exist some physical observables, which cannot be self-consistently characterized within the framework of linear perturbation theory. One prominent example of the *non-Gaussianity* [81, 82], which has aroused great enthusiasm in physical community recently and is being accurately detected and constrained in the WMAP [83] and the forthcoming Planck experiments. If the primordial perturbations were gaussian, we obviously only need the two-point correlation function and its Fourier transform, *i.e.*, the power spectrum, of the scalar perturbations; but if not, to describe the deviations from Gaussianity, a three-point correlation function and also its Fourier transform, *i.e.*, the bispectrum are definitely required. For all these reasons, in this subsection, we generalize cosmological perturbation theory beyond linear order and briefly discuss the gauge invariance problem at higher orders.

Higher order cosmological perturbation theory can be formulated in two ways. First, one just work with the linear longitudinal or synchronous gauge, and any quantity made up of more than one metric perturbations is considered as higher order terms [84]. However, in this way, we can neither have the perturbed Christoffel connection of order higher than two, nor the perturbed Einstein tensor of order higher than four. This intrinsic disadvantage limits its application to higher order perturbation theory, at least from pure theoretical perspective. The other way out is to decompose the metric perturbations to different order, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \sum_{n=1} g_{\mu\nu}^{(n)}/n!$, and any quantity comprising the metric perturbations with the sum of their order being k is regarded as a k-th order one, *e.g.*, the product of a first and a second order quantities build up a term of third order. This method has unambiguous interpretation and allows us to go to any high order as we wish. Hence, in the following discussions, we only apply this approach.

4.2.1 Second order gauge transformations

As in Sec. 4.1.2, we consider the infinitesimal coordinate transformation to second order,

$$\tilde{x}^{\mu}=x^{\mu}+\xi^{(1)\mu}+\frac{1}{2}\left(\xi^{(1)\mu}{}_{,\nu}\xi^{(1)\nu}+\xi^{(2)\mu}\right),$$

where $\xi^{(1)\mu}$ is again the first order infinitesimal coordinate transformation. Because we have new degrees of freedom at new orders, we must introduce another infinitesimal coordinate transformation $\xi^{(2)\mu}$, which itself is a second order quantity, *i.e.*, $\xi^{(2)\mu}\sim(\xi^{(1)\mu})^{2}$. Then, the Jacobian of the inverse coordinate transformation is [54]

$$\begin{aligned}
\frac{\partial x^{\alpha}}{\partial\tilde{x}^{\mu}} &= \frac{\partial}{\partial\tilde{x}^{\mu}}\left[\tilde{x}^{\alpha}-\xi^{(1)\alpha}-\frac{1}{2}\left(\xi^{(1)\alpha}{}_{,\lambda}\xi^{(1)\lambda}+\xi^{(2)\alpha}\right)\right]\\
&= \delta^{\alpha}{}_{\mu}-\frac{\partial x^{\lambda}}{\partial\tilde{x}^{\mu}}\xi^{(1)\alpha}{}_{,\lambda}-\frac{1}{2}\left(\xi^{(1)\alpha}{}_{,\lambda}\xi^{(1)\lambda}+\xi^{(2)\alpha}\right)_{,\mu}\\
&= \delta^{\alpha}{}_{\mu}-\left(\delta^{\lambda}{}_{\mu}-\xi^{(1)\lambda}{}_{,\mu}\right)\xi^{(1)\alpha}{}_{,\lambda}-\frac{1}{2}\left(\xi^{(1)\alpha}{}_{,\lambda}\xi^{(1)\lambda}+\xi^{(2)\alpha}\right)_{,\mu}\\
&= \delta^{\alpha}{}_{\mu}-\xi^{(1)\alpha}{}_{,\mu}+\frac{1}{2}\left(\xi^{(1)\alpha}{}_{,\lambda}\xi^{(1)\lambda}{}_{,\mu}-\xi^{(1)\alpha}{}_{,\mu,\lambda}\xi^{(1)\lambda}\right)-\frac{1}{2}\xi^{(2)\alpha}{}_{,\mu}.
\end{aligned}$$

So the metric transformation is

$$\begin{aligned}
\tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^{\alpha}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}g_{\alpha\beta}(x)\\
&= \left[\delta^{\alpha}{}_{\mu}-\xi^{(1)\alpha}{}_{,\mu}+\frac{1}{2}\left(\xi^{(1)\alpha}{}_{,\lambda}\xi^{(1)\lambda}{}_{,\mu}-\xi^{(1)\alpha}{}_{,\mu,\lambda}\xi^{(1)\lambda}\right)-\frac{1}{2}\xi^{(2)\alpha}{}_{,\mu}\right]\times\\
&\quad\left[\delta^{\beta}{}_{\nu}-\xi^{(1)\beta}{}_{,\nu}+\frac{1}{2}\left(\xi^{(1)\beta}{}_{,\lambda}\xi^{(1)\lambda}{}_{,\nu}-\xi^{(1)\beta}{}_{,\nu,\lambda}\xi^{(1)\lambda}\right)-\frac{1}{2}\xi^{(2)\beta}{}_{,\nu}\right]g_{\alpha\beta}(x). \qquad (71)
\end{aligned}$$

We now must expand the metric as a perturbative series to second order on both sides,

$$\tilde{g}_{\mu\nu}(\tilde{x})=g^{(0)}_{\mu\nu}(\tilde{x})+\tilde{g}^{(1)}_{\mu\nu}(\tilde{x})+\frac{1}{2}\tilde{g}^{(2)}_{\mu\nu}(\tilde{x}),\quad g_{\mu\nu}(x)=g^{(0)}_{\mu\nu}(x)+g^{(1)}_{\mu\nu}(x)+\frac{1}{2}g^{(2)}_{\mu\nu}(x). \qquad (72)$$

Then, we again rewrite the different orders of $\tilde{g}_{\mu\nu}(\tilde{x})$ as the function of x,

$$\begin{aligned}
\tilde{g}_{\mu\nu}(\tilde{x}) &= g^{(0)}_{\mu\nu}(\tilde{x})+\tilde{g}^{(1)}_{\mu\nu}(\tilde{x})+\frac{1}{2}\tilde{g}^{(2)}_{\mu\nu}(\tilde{x})\\
&= g^{(0)}_{\mu\nu}+g^{(0)}_{\mu\nu,\lambda}\left(\tilde{x}^{\lambda}-x^{\lambda}\right)+\frac{1}{2}g^{(0)}_{\mu\nu,\lambda,\rho}\xi^{(1)\lambda}\xi^{(1)\rho}+\tilde{g}^{(1)}_{\mu\nu}+\tilde{g}^{(1)}_{\mu\nu,\lambda}\xi^{(1)\lambda}+\frac{1}{2}\tilde{g}^{(2)}_{\mu\nu}\\
&= g^{(0)}_{\mu\nu}+g^{(0)}_{\mu\nu,\lambda}\left[\xi^{(1)\lambda}+\frac{1}{2}\left(\xi^{(1)\lambda}{}_{,\rho}\xi^{(1)\rho}+\xi^{(1)\lambda}\right)\right]+\frac{1}{2}g^{(0)}_{\mu\nu,\lambda,\rho}\xi^{(1)\lambda}\xi^{(1)\rho}\\
&\quad+\tilde{g}^{(1)}_{\mu\nu}+\left(g^{(1)}_{\mu\nu}-g^{(0)}_{\mu\nu,\rho}\xi^{(1)\rho}-g^{(0)}_{\mu\rho}\xi^{(1)\rho}{}_{,\nu}-g^{(0)}_{\rho\nu}\xi^{(1)\rho}{}_{,\mu}\right)_{,\lambda}\xi^{(1)\lambda}+\frac{1}{2}\tilde{g}^{(2)}_{\mu\nu}. \qquad (73)
\end{aligned}$$

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[54] In the derivation, we should notice that the derivative of $\xi^{(1)\mu}$ also contributes to second order terms.

Substituting Eqs. (72) and (73) into Eq. (71), we obtain the rules for gauge transformations up to second order, [55]

$$\begin{aligned} \tilde{g}^{(1)}_{\mu\nu} &= g^{(1)}_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} g^{(0)}_{\mu\nu}, \\ \tilde{g}^{(2)}_{\mu\nu} &= g^{(2)}_{\mu\nu} - 2\mathcal{L}_{\xi^{(1)}} g^{(1)}_{\mu\nu} - \left(\mathcal{L}_{\xi^{(2)}} - \mathcal{L}^2_{\xi^{(1)}}\right) g^{(0)}_{\mu\nu}. \end{aligned} \tag{74}$$

It is straightforward to generate these results to even higher orders, and the third order gauge transformation was shown in [85] as

$$\tilde{g}^{(3)}_{\mu\nu} = g^{(3)}_{\mu\nu} - 3\mathcal{L}_{\xi^{(1)}} g^{(2)}_{\mu\nu} - 3\left(\mathcal{L}_{\xi^{(2)}} - \mathcal{L}^2_{\xi^{(1)}}\right) g^{(1)}_{\mu\nu} - \left(\mathcal{L}_{\xi^{(3)}} - 3\mathcal{L}_{\xi^{(1)}}\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^3_{\xi^{(1)}}\right) g^{(0)}_{\mu\nu}.$$

4.2.2 Gauge invariance at higher orders

Similar with Eq. (55), we summarize the general rules for transformations of tensor up to third order as

$$\begin{aligned} \tilde{T}^{(0)} &= T^{(0)}, \\ \tilde{T}^{(1)} &= T^{(1)} - \mathcal{L}_{\xi^{(1)}} T^{(0)}, \\ \tilde{T}^{(2)} &= T^{(2)} - 2\mathcal{L}_{\xi^{(1)}} T^{(1)} - \left(\mathcal{L}_{\xi^{(2)}} - \mathcal{L}^2_{\xi^{(1)}}\right) T^{(0)}, \\ \tilde{T}^{(3)} &= T^{(3)} - 3\mathcal{L}_{\xi^{(1)}} T^{(2)} - 3\left(\mathcal{L}_{\xi^{(2)}} - \mathcal{L}^2_{\xi^{(1)}}\right) T^{(1)} - \left(\mathcal{L}_{\xi^{(3)}} - 3\mathcal{L}_{\xi^{(1)}}\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^3_{\xi^{(1)}}\right) T^{(0)}. \end{aligned}$$

We see that a tensor T is gauge invariant to order n *if and only if* $\tilde{T}^{(k)} = T^{(k)}$ *for every* $k \leq n$. For instance, a tensor T is gauge invariant to second order if and only if $\tilde{T}^{(2)} = T^{(2)}$ and $\tilde{T}^{(1)} = T^{(1)}$, which implies $\mathcal{L}_{\xi^{(1)}} T^{(0)} = \mathcal{L}_{\xi^{(2)}} T^{(0)} = \mathcal{L}_{\xi^{(1)}} T^{(1)} = 0$, because $\xi^{(1)}$ and $\xi^{(2)}$ are arbitrary generators at first and second orders. Consequently, a tensor T is gauge invariant to second order strongly demands both $T^{(0)}$ and $T^{(1)}$ vanish in any gauge, except the trivial cases: it is a constant scalar field, or a linear combination of products of Kronecker deltas with constant coefficients on the background [86, 87]. Here, we omit the detailed forms of the transformations of metric perturbations in different gauges, as they are of little interest for this dissertation, and these results can easily be found in [86].

4.3 Solutions for linear order metric perturbations

Following the general explorations of cosmological perturbation theory, in this subsection, we solve the linearized Einstein equations in the spatially flat dust Universe without a cosmological constant and give the solutions of scalar metric perturbations $\Psi^{(1)}$ and $\chi^{(1)}$ in terms of the conformal time η and peculiar gravitational potential $\varphi(\mathbf{x})$. These two solutions are extremely important for our perturbative calculations of the averaged physical observables in the next sections.

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[55] These rules have already been shown in [85, 86], but in the active way. Frankly speaking, their treatments are rather mathematical and difficult to understand, so we now change to passive way and hope our treatments more pedagogical. We point out that the calculation of $\mathcal{L}^2_{\xi^{(1)}} g^{(0)}_{\mu\nu}$ is complicated, but trivial.

4.3.1 Perturbed Einstein equations to linear order

At linear order, the perturbed metric in the synchronous gauge reads

$$\mathrm{d}s^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \left[\left(1-2\Psi^{(1)}\right)\delta_{ij} + D_{ij}\chi^{(1)}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}, \tag{75}$$

where $\Psi^{(1)}$ and $\chi^{(1)}$ are the scalar metric perturbations at linear order. The scale factor a in Eq. (75) is certainly not the same as the effective scale factor a_D defined in Eq. (28), and their relation will be shown in Eq. (137) in Sec. 5.3.

From the line element in Eq. (75), we straightforwardly obtain the nontrivial components of the perturbed Christoffel connection and Einstein tensor to first order, [56]

$$\begin{aligned}
\Gamma^0_{00} &= \frac{a'}{a}, &(76)\\
\Gamma^0_{ij} &= \frac{a'}{a}\delta_{ij} - 2\frac{a'}{a}\Psi^{(1)}\delta_{ij} - \Psi^{(1)\prime}\delta_{ij} + \frac{a'}{a}D_{ij}\chi^{(1)} + \frac{1}{2}D_{ij}\chi^{(1)\prime}, &(77)\\
\Gamma^i_{0j} &= \frac{a'}{a}\delta^i{}_j - \Psi^{(1)\prime}\delta^i{}_j + \frac{1}{2}D^i{}_j\chi^{(1)\prime}, &(78)\\
\Gamma^i_{jk} &= -\partial_j\Psi^{(1)}\delta^i{}_k - \partial_k\Psi^{(1)}\delta^i{}_j + \partial^i\Psi^{(1)}\delta_{jk}\\
&\quad +\frac{1}{2}\partial_j D^i{}_k\chi^{(1)} + \frac{1}{2}\partial_k D^i{}_j\chi^{(1)} - \frac{1}{2}\partial^i D_{jk}\chi^{(1)}, &(79)
\end{aligned}$$

and

$$\begin{aligned}
G^0{}_0 &= -\frac{3}{a^2}\left(\frac{a'}{a}\right)^2 + \frac{1}{a^2}\left(6\frac{a'}{a}\Psi^{(1)\prime} - 2\Delta\Psi^{(1)} - \frac{1}{2}\partial_k\partial^i D^k{}_i\chi^{(1)}\right), &(80)\\
G^0{}_i &= \frac{1}{a^2}\left(-2\partial_i\Psi^{(1)\prime} - \frac{1}{2}\partial_k D^k{}_i\chi^{(1)\prime}\right), &(81)\\
G^i{}_j &= \frac{1}{a^2}\Bigg\{\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\delta^i{}_j + \left(4\frac{a'}{a}\Psi^{(1)\prime} + 2\Psi^{(1)\prime\prime} - \Delta\Psi^{(1)} - \frac{1}{2}\partial_k\partial^m D^k{}_m\chi^{(1)}\right)\delta^i{}_j\\
&\quad +\partial^i\partial_j\Psi^{(1)} + \frac{a'}{a}D^i{}_j\chi^{(1)\prime} + \frac{1}{2}D^i{}_j\chi^{(1)\prime\prime}\\
&\quad +\frac{1}{2}\partial_k\partial^i D^k{}_j\chi^{(1)} + \frac{1}{2}\partial_k\partial_j D^{ik}\chi^{(1)} - \frac{1}{2}\Delta D^i{}_j\chi^{(1)}\Bigg\}. &(82)
\end{aligned}$$

The unique nontrivial component of the energy-momentum tensor for the dust Universe is [57]

$$T^0{}_0 = -\rho = -\rho^{(0)} - \rho^{(1)}, \tag{83}$$

with $\rho^{(0)}$ and $\rho^{(1)}$ the energy density at the background and first order. With $\rho^{(1)}$, we may further define the *peculiar gravitational potential* $\varphi(\mathbf{x})$ from the cosmological Poisson equation as

$$\Delta\varphi(\mathbf{x}) \equiv 4\pi G\rho^{(1)}a^2. \tag{84}$$

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[56] These components can also be found in the App. C.2 (Eqs. (217) - (220)) and C.5 (Eqs. (235) - (239)).

[57] Since we use the comoving gauge throughout this dissertation, T^0_i and $T^i{}_j$ vanish automatically.

We are now ready to obtain the linearized equations of motion in the ADM decomposition. The different components at different orders are

1. the energy constraint at zeroth order

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho^{(0)}a^2, \tag{85}$$

2. the energy constraint at first order

$$\Delta\left(\Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}\right) - 3\frac{a'}{a}\Psi^{(1)\prime} = \Delta\varphi, \tag{86}$$

3. the momentum constraint

$$\partial_i\left(\Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}\right)' = 0, \tag{87}$$

4. the evolution equation at zeroth order

$$\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a} = 0, \tag{88}$$

5. the diagonal $(i = j)$ piece of the evolution equation at first order

$$\Delta\left(\Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}\right) - 6\frac{a'}{a}\Psi^{(1)\prime} - 3\Psi^{(1)\prime\prime} = 0. \tag{89}$$

6. the off-diagonal $(i \neq j)$ piece of the evolution equation at first order

$$\partial^i\partial_j\left(\Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)} + \frac{a'}{a}\chi^{(1)\prime} + \frac{1}{2}\chi^{(1)\prime\prime}\right) = 0. \tag{90}$$

From the covariant energy-momentum conservation $T^{\mu}_{\ \nu;\mu} = 0$, we find

1. at zeroth order

$$\rho^{(0)\prime} + 3\frac{a'}{a}\rho^{(0)} = 0, \tag{91}$$

2. at first order

$$\rho^{(1)\prime} + 3\frac{a'}{a}\rho^{(1)} - 3\Psi^{(1)\prime}\rho^{(0)} = 0. \tag{92}$$

Equation (92) has a first integral,

$$\bar{\zeta}(\mathbf{x}) \equiv \frac{\rho^{(1)}}{3\rho^{(0)}} - \Psi^{(1)}, \tag{93}$$

which resembles the famous hypersurface invariant variable, or Bardeen parameter (here for dust, expressed in the synchronous gauge)

$$\zeta(\eta, \mathbf{x}) \equiv \bar{\zeta}(\mathbf{x}) - \frac{1}{6}\Delta\chi^{(1)}(\eta, \mathbf{x}), \tag{94}$$

commonly used to characterize the primordial power spectrum [88]. At large (*e.g.*, superhorizon) scales, $\Delta\chi^{(1)}$ is negligible, so $\bar{\zeta} \approx \zeta$.

4.3.2 Solution for the scale factor a

Now solving Eqs. (85) - (92), we can get the solutions for first order metric perturbations. Firstly, from Eq. (91), we have $\rho^{(0)}a^3 = \rho_0^{(0)}a_0^3$, where $\rho_0^{(0)}$ and a_0 denote the values of $\rho^{(0)}$ and a at the present time. By means of Eq. (85), we have

$$\frac{a}{a_0} = \left(\frac{\eta}{\eta_0}\right)^2 . \tag{95}$$

We find that $a(\eta)$ grows as η^2, which is the same result as that for a spatially flat FLRW dust cosmology. But this does not mean that the perturbed Universe expands in the same way as the unperturbed one, because in the perturbed Universe the meaningful scale factor is the effective scale factor a_D defined in Eq. (28), which, however, is not simply equal to a, and their relation will be shown in Eq. (137). So we could not know the evolution of the perturbed Universe merely from the behavior of the scale factor a. In the following discussions, we set $a_0 = 1$.

4.3.3 Solutions for linear order metric perturbations

Let us move on to the solutions for linear order metric perturbations.

Solution for $\Psi^{(1)}$ We first eliminate $\rho^{(1)}$ with the help of the first integral $\bar{\zeta}$ from Eq. (93),

$$\rho^{(1)} = \frac{3\rho_0^{(0)}a_0^3}{a^3}\left(\Psi^{(1)} + \bar{\zeta}(\mathbf{x})\right). \tag{96}$$

This allows us to obtain an equation for $\Psi^{(1)}$. Namely, from Eqs. (86), (89) and (96), we have

$$\Psi^{(1)\prime\prime} + \frac{a'}{a}\Psi^{(1)\prime} = \frac{4\pi G\rho_0^{(0)}a_0^3}{a}\left(\Psi^{(1)} + \bar{\zeta}(\mathbf{x})\right),$$

and using Eq. (95), we obtain

$$\Psi^{(1)\prime\prime} + \frac{2}{\eta}\Psi^{(1)\prime} - \frac{6}{\eta^2}\Psi^{(1)} = \frac{6}{\eta^2}\bar{\zeta}.$$

So we get the solution for $\Psi^{(1)}$

$$\Psi^{(1)}(\eta, \mathbf{x}) = A_1(\mathbf{x})\eta^2 + \frac{A_2(\mathbf{x})}{\eta^3} - \bar{\zeta}(\mathbf{x}), \tag{97}$$

where $A_1(\mathbf{x})$ and $A_2(\mathbf{x})$ are constants of integration, *i.e.*, functions of the spatial coordinates only, which are free parameters to be fixed by the initial conditions. We see from Eq. (97) that $\Psi^{(1)}$ consists of one growing mode $A_1(\mathbf{x})\eta^2$, one decaying mode $A_2(\mathbf{x})/\eta^3$ and one constant mode $-\bar{\zeta}(\mathbf{x})$. In the next sections, we will see that only the time derivatives $\Psi^{(1)\prime}$ and $\Psi^{(1)\prime\prime}$ show up in the averaged physical observables that are of interest to us; besides, we are only concerned with the evolutions of perturbations at late times. So we can neglect the decaying and constant modes safely: only the growing mode $A_1(\mathbf{x})\eta^2$ is of importance for the following calculations.

Solution for $\chi^{(1)}$ From $G^{i(1)}_{j} = 0$, we have

$$\left[2\Psi^{(1)\prime\prime} + 4\frac{a'}{a}\Psi^{(1)\prime} - \frac{2}{3}\Delta\left(\Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}\right)\right]\delta^{i}{}_{j} + D^{i}{}_{j}\left(\frac{1}{2}\chi^{(1)\prime\prime} + \frac{a'}{a}\chi^{(1)\prime} + \Psi^{(1)} + \frac{1}{6}\Delta\chi^{(1)}\right) = 0.$$

Taking the time derivative and inserting Eq. (87), we have

$$\left(2\Psi^{(1)\prime\prime} + 4\frac{a'}{a}\Psi^{(1)\prime}\right)' + D^{i}{}_{j}\left(\frac{1}{2}\chi^{(1)\prime\prime} + \frac{a'}{a}\chi^{(1)\prime}\right)' = 0.$$

From Eqs. (95) and (97), the first part in the above equation vanishes, and we yield

$$D^{i}{}_{j}\chi^{(1)\prime\prime\prime} + \frac{4}{\eta}D^{i}{}_{j}\chi^{(1)\prime\prime} - \frac{4}{\eta^2}D^{i}{}_{j}\chi^{(1)\prime} = 0,$$

and thus

$$D^{i}{}_{j}\chi^{(1)} = C^{i}_{1j}(\mathbf{x})\eta^2 + \frac{C^{i}_{2j}(\mathbf{x})}{\eta^3} + f(\mathbf{x}),$$

where $C^{i}_{1j}(\mathbf{x})$ and $C^{i}_{2j}(\mathbf{x})$ are functions of spatial coordinates, too. So we write $\chi^{(1)}$ as

$$\chi^{(1)} = C_1(\mathbf{x})\eta^2 + \frac{C_2(\mathbf{x})}{\eta^3} + g(\mathbf{x}), \tag{98}$$

with $D^{i}{}_{j}C_1(\mathbf{x}) = C^{i}_{1j}(\mathbf{x})$, $D^{i}{}_{j}C_2(\mathbf{x}) = C^{i}_{2j}(\mathbf{x})$ and $D^{i}{}_{j}f(\mathbf{x}) = g(\mathbf{x})$. In the following calculations, we neglect the decaying mode $C^{i}_{2j}(\mathbf{x})\eta^3$. Furthermore, the constant mode $g(\mathbf{x})$ does not carry physical information, as it can be fixed by the residual spatial gauge transformation in the comoving synchronous gauge, and therefore we utilize this freedom to set $g(\mathbf{x}) = 0$.

Final results for $\Psi^{(1)}$ and $\chi^{(1)}$ In this paragraph, we give the relations between $A_1(\mathbf{x})$, $C_1(\mathbf{x})$, $\bar{\zeta}(\mathbf{x})$ and $\varphi(\mathbf{x})$, and reexpress $\Psi^{(1)}$ and $\chi^{(1)}$ in terms of η and $\varphi(\mathbf{x})$.

First, from Eqs. (86) and (89), we have

$$\Psi^{(1)\prime\prime} + \frac{a'}{a}\Psi^{(1)\prime} = \frac{1}{3}\Delta\varphi(\mathbf{x}),$$

and using Eq. (97), we get the relation between $A_1(\mathbf{x})$ and $\varphi(\mathbf{x})$ as

$$A_1(\mathbf{x}) = \frac{1}{18}\Delta\varphi(\mathbf{x}). \tag{99}$$

Second, the solutions for $\Psi^{(1)}$ and $\chi^{(1)}$ are not independent due to Eq. (87). From Eqs. (87), (97) and (98), we have $\partial_i(A_1(\mathbf{x}) + \Delta C_1(\mathbf{x})/6) = 0$. Because both $A_1(\mathbf{x})$ and $C_1(\mathbf{x})$ are functions of the spatial coordinates, we have

$$A_1(\mathbf{x}) + \frac{1}{6}\Delta C_1(\mathbf{x}) = K, \tag{100}$$

where K is a constant. In the spatially flat universe $K = 0$, which will be shown in Sec. 5.2.2, and we find the relation between $A_1(\mathbf{x})$ and $C_1(\mathbf{x})$,

$$A_1(\mathbf{x}) = -\frac{1}{6}\Delta C_1(\mathbf{x}). \tag{101}$$

Third, with the help of Eqs. (89) and (90), we have $\Delta(\bar{\zeta}(\mathbf{x}) - 5C_1(\mathbf{x})) = \partial^i\partial_j(\bar{\zeta}(\mathbf{x}) - 5C_1(\mathbf{x})) = 0, (i \neq j)$. Since both $\bar{\zeta}(\mathbf{x})$ and $C_1(\mathbf{x})$ are functions of spatial coordinates, it is straightforwardly to get

$$\bar{\zeta}(\mathbf{x}) = 5C_1(\mathbf{x}). \tag{102}$$

Let us note that at superhorizon scales $\bar{\zeta} \approx \zeta$. As the amplitude of ζ at superhorizon scales is measured by the CMB experiments, the magnitudes of the time independent functions $\bar{\zeta}$, A_1 and C_1 are thus determeined.

Combining the relations in Eqs. (99), (101) and (102), we obtain the expressions of $A_1(\mathbf{x})$, $C_1(\mathbf{x})$ and $\bar{\zeta}(\mathbf{x})$ as the functions of the peculiar gravitational potential $\varphi(\mathbf{x})$,

$$A_1(\mathbf{x}) = \frac{1}{18}\Delta\varphi(\mathbf{x}), \quad C_1(\mathbf{x}) = -\frac{1}{3}\varphi(\mathbf{x}), \quad \bar{\zeta}(\mathbf{x}) = -\frac{5}{3}\varphi(\mathbf{x}). \tag{103}$$

Therefore, the final results for $\Psi^{(1)}$ and $\chi^{(1)}$ are

$$\Psi^{(1)} = \frac{\eta^2}{18}\Delta\varphi(\mathbf{x}) + \frac{5}{3}\varphi(\mathbf{x}), \tag{104}$$

$$\chi^{(1)} = -\frac{\eta^2}{3}\varphi(\mathbf{x}), \tag{105}$$

and their time derivatives are

$$\Psi^{(1)\prime} = \frac{\eta}{9}\Delta\varphi(\mathbf{x}), \quad \Psi^{(1)\prime\prime} = \frac{1}{9}\Delta\varphi(\mathbf{x}), \tag{106}$$

$$\chi^{(1)\prime} = -\frac{2\eta}{3}\varphi(\mathbf{x}), \quad \chi^{(1)\prime\prime} = -\frac{2}{3}\varphi(\mathbf{x}). \tag{107}$$

In summary, we see from Eqs. (104) and (105) that both $\Psi^{(1)}$ and $\chi^{(1)}$ grow as η^2 at late times. Because $a(\eta) \propto \eta^2$, $\Psi^{(1)}$ and $\chi^{(1)}$ grow linearly as the scale factor $a(\eta)$ in the perturbed dust Universe. We know that in a non-expanding background, linearized gravity would always lead to exponential instabilities as there is no counteraction against the attractive gravitational force. However, in the MD era, gravitational attraction is partially diluted by the expansion of the Universe, and the growth of the first order scalar metric perturbations is reduced to the power-law η^2, rather than an exponential one in Newtonian gravity. So, if cosmological perturbation theory is valid, *i.e.*, the perturbative terms $\Psi^{(1)}$ and $\chi^{(1)}$ are small, the scale factor should not be too large. In other words, a perturbative analysis (at any order) is restricted to the "linear" regime.

4.4 Solutions for second order metric perturbations

Following the work above, we can extend the solutions for metric perturbations to second order. This sounds a trivial task, but in fact extraordinarily tedious. As we are not so concerned with the detailed derivations here, we only list their solutions directly. [58]

The perturbative metric for spatially flat space-time up to second order in the synchronous gauge reads

$$\mathrm{d}s^2 \;=\; a^2(\eta)\left[-\mathrm{d}\eta^2+\left(\delta_{ij}+\gamma^{(1)}_{ij}+\gamma^{(2)}_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right], \tag{108}$$

with

$$\begin{aligned}\gamma^{(1)}_{ij} &= -2\Psi^{(1)}\delta_{ij}+D_{ij}\chi^{(1)},\\ \gamma^{(2)}_{ij} &= -\Psi^{(2)}\delta_{ij}+\frac{1}{2}\left(D_{ij}\chi^{(2)}+\partial_i\chi^{(2)}_j+\partial_j\chi^{(2)}_i+\chi^{(2)}_{ij}\right).\end{aligned}$$

where $\Psi^{(1)}$ and $\chi^{(1)}$ are first order scalar metric perturbations as before; $\Psi^{(2)}$ and $\chi^{(2)}$ are scalar metric perturbations at second order; $\chi^{(2)}_i$ is the second order transverse vector perturbation, *i.e.*, $\chi^{(2)i}{}_{,i}=0$; $\chi^{(2)}_{ij}$ is the second order transverse and traceless tensor perturbation, *i.e.*, $\chi^{(2)i}{}_{j,i}=\chi^{(2)i}{}_{i}=0$. Similarly, the energy density ρ is also expanded to second order as $\rho=\rho^{(0)}+\rho^{(1)}+\rho^{(2)}/2$.

Now, again with the same procedure as for the linear calculations, we first calculate the perturbed Einstein tensor to second order [59] and then equal them to the second order perturbed energy-momentum tensor. Solving these perturbed equations, we get the solutions for second order metric perturbations as

1. For the second order scalar perturbation $\Psi^{(2)}$,

$$\Psi^{(2)} \;=\; \frac{\eta^4}{252}\left[(\Delta\varphi)^2-\frac{10}{3}\partial^i\partial_j\varphi\partial^j\partial_i\varphi\right]+\frac{5\eta^2}{18}\left(\frac{4}{3}\varphi\Delta\varphi+\partial^i\varphi\partial_i\varphi\right). \tag{109}$$

2. For the part of $\chi^{(2)}$, $\chi^{(2)}_i$ and $\chi^{(2)}_{ij}$, they can be solved as a whole, and in the following calculations, we will find that they only show up at third order and always appear together.

$$\begin{aligned}&D_{ij}\chi^{(2)}+\partial_i\chi^{(2)}_j+\partial_j\chi^{(2)}_i+\chi^{(2)}_{ij}\\ =\;&\frac{\eta^4}{126}\left\{\left[4(\Delta\varphi)^2-\frac{19}{3}\partial^k\partial_m\varphi\partial^m\partial_k\varphi\right]\delta_{ij}+19\partial^k\partial_i\varphi\partial_k\partial_j\varphi-12\partial_i\partial_j\varphi\Delta\varphi\right\}\\ &+\frac{5\eta^2}{9}\left[\left(\frac{4}{3}\varphi\Delta\varphi+2\partial^k\varphi\partial_k\varphi\right)\delta_{ij}-6\partial_i\varphi\partial_j\varphi-4\varphi\partial_i\partial_j\varphi\right]+\pi_{ij},\end{aligned} \tag{110}$$

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[58] For all these derivations, we follow [86], and for final results, we refer to [89] and the pioneering works [90].

[59] Components of the perturbed Einstein tensor are shown in App. C.5 (Eqs. (240) - (244)), and there we see that the quadratic terms of first order metric perturbations significantly contribute to second order perturbed Einstein tensor. This means that even there were no original second order metric perturbations, first order ones would also lead to vector and tensor perturbation modes at second order, *i.e.*, they would cause the rotation of cosmic medium and gravitational waves. Moreover, the products of two linear metric perturbations make the equations of motion no longer decoupled at second order.

where the transverse and traceless contribution π_{ij}, representing the second order tensor mode generated by scalar initial perturbations, is determined by the inhomogeneous wave equation,

$$\pi''_{ij} + \frac{4}{\eta}\pi'_{ij} - \Delta\pi_{ij} = -\frac{\eta^4}{21}\Delta\mathcal{S}_{ij}(\mathbf{x}),$$

with $\mathcal{S}_{ij} = \Delta\Psi_0\delta_{ij} + \partial_i\partial_j\Psi_0 + 2(\partial_i\partial_j\varphi\Delta\varphi - \partial_i\partial_k\varphi\partial^k\partial_j\varphi)$, and

$$\Delta\Psi_0 = \frac{1}{2}\left[\partial^i\partial_j\varphi\partial^j\partial_i\varphi - (\Delta\varphi)^2\right]. \tag{111}$$

This Poisson equation can be solved by the Green function method, [60] and

$$\pi_{ij}(\eta,\mathbf{x}) = \frac{\eta^4}{21}\mathcal{S}_{ij}(\mathbf{x}) + \frac{4\eta^2}{3}\mathcal{T}_{ij}(\mathbf{x}) + \tilde{\pi}_{ij}(\eta,\mathbf{x}),$$

where $\Delta\mathcal{T}_{ij} = \mathcal{S}_{ij}$ and the remaining piece $\tilde{\pi}_{ij}$, containing one constant and one oscillating term with damping amplitude, can be written as

$$\tilde{\pi}_{ij}(\eta,\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{40}{k^4}\left(\frac{1}{3} - \frac{j_1(k\eta)}{k\eta}\right)\mathcal{S}_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{112}$$

with $\mathcal{S}_{ij}(\mathbf{k}) = \int \mathrm{d}\mathbf{x}\mathcal{S}_{ij}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$ and $j_1(x) = (\sin x - x\cos x)/x^2$ the spherical Bessel function.

In Sec. 6, we will show that only the *leading* terms in the metric perturbations, *i.e.*, the terms with highest power of the conformal time η are of interest to us; other terms all decay so fast that they are irrelevant for the evolution of the perturbed Universe at late times. So, when we use Eqs. (109) and (110), it is sufficient to take into account the leading pieces only,

$$\begin{aligned}\Psi^{(2)} &= \frac{\eta^4}{252}\left[(\Delta\varphi)^2 - \frac{10}{3}\partial^i\partial_j\varphi\partial^j\partial_i\varphi\right], &(113)\\ D_{ij}\chi^{(2)} + \partial_i\chi^{(2)}_j + \partial_j\chi^{(2)}_i + \chi^{(2)}_{ij} &= \frac{\eta^4}{126}\Bigg[\left((\Delta\varphi)^2 - \frac{10}{3}\partial^k\partial_m\varphi\partial^m\partial_k\varphi\right)\delta_{ij} \\ &\qquad + 7\partial_i\partial_k\varphi\partial^k\partial_j\varphi + 6\partial_i\partial_j\Psi_0\Bigg]. &(114)\end{aligned}$$

We will see in Sec. 7 that these solutions show up in the calculation of the third order term Q_0 of the kinematical backreaction term $\langle Q\rangle_D$.

Till now, we have collected all the solutions for a, $\Psi^{(1)}$ and $\chi^{(1)}$, $\Psi^{(2)}$ and $D_{ij}\chi^{(2)} + \partial_i\chi^{(2)}_j + \partial_j\chi^{(2)}_i + \chi^{(2)}_{ij}$, which we will use in the next three sections to calculate the averaged physical observables from first to third order, respectively.

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[60] In Sec. 7.1, this equation will be "solved", and here we just keep Ψ_0 everywhere.

5 First order perturbative calculations of the averaged physical observables

From now on, we proceed to the perturbative calculations of the averaged physical observables. The purpose of this section is to gather all the necessary equations, get familiar with the perturbative calculations and make preparations for the next two sections. For the perturbative calculations, we make use of the conformal time η defined in Eq. (50). We calculate the averaged volume expansion rate $\langle\theta\rangle_D$, spatial curvature $\langle\mathcal{R}\rangle_D$ and energy density $\langle\rho\rangle_D$ to first order. The averaged kinematical backreaction term $\langle Q\rangle_D$ will be proven to start from second order in Sec. 6.1.1, so we do no discuss it here. To make the perturbative calculations more compact, we also postpone the investigations of the effective equation of state w_{eff} and square of the effective speed of sound c^2_{eff} to the next section, as they are related to $\langle Q\rangle_D$, although they do have first order terms.

For the start, we summarize all the mathematical preparations for the coming perturbative calculations, *i.e.*, rewrite the necessary equations in terms of the conformal time η. To get these equations, we only need to change $\frac{\partial}{\partial t}$ to $\frac{1}{a(\eta)}\frac{\partial}{\partial\eta}$. The corresponding ADM decompositions, commutation rule, Buchert equations and integrability condition consequently read

$$\begin{aligned} \frac{1}{2}\left(\mathcal{R}+\theta^2-\theta^i{}_j\theta^j{}_i\right) &= 8\pi G\rho, && (115)\\ \theta_{,i}-\theta^j{}_{i;j} &= 0, && (116)\\ \frac{1}{a}\theta^{i\prime}_{\;j}+\theta\theta^i{}_j+\mathcal{R}^i{}_j &= 4\pi G\rho\delta^i{}_j, && (117) \end{aligned}$$

and

$$\langle O\rangle_D'-\langle O'\rangle_D = a(\langle O\theta\rangle_D-\langle O\rangle_D\langle\theta\rangle_D), \tag{118}$$

and

$$\begin{aligned} \mathcal{H}_D^2 &= \frac{8\pi G}{3}\rho_{\mathrm{eff}}a^2, && (119)\\ \mathcal{H}_D'+\mathcal{H}_D^2-\mathcal{H}\mathcal{H}_D &= -\frac{4\pi G}{3}(\rho_{\mathrm{eff}}+3p_{\mathrm{eff}})a^2, && (120) \end{aligned}$$

with $\mathcal{H}\equiv a'/a$ and $\mathcal{H}_D\equiv a_D'/a_D$, and

$$(a_D^6\langle Q\rangle_D)'+a_D^4(a_D^2\langle R\rangle_D)'=0. \tag{121}$$

Eqs. (115) - (121) establish the bases for the forthcoming calculations.

5.1 Evolution of the background Universe

We firstly list the results for the evolution of the background Universe: the simple case of the FLRW model, in which space-time is homogeneous and isotropic, and thus there exists no backreaction effects at all, *i.e.*, both $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$ vanish, and the effective scale factor a_D defined in Eq. (28) reduces to the scale factor a trivially. The unperturbed dynamical equations for the background dust Universe read

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho^{(0)}a^2, \quad \mathcal{H}' = -\frac{4\pi G}{3}\rho^{(0)}a^2.$$

Solving these equations, we directly obtain the behaviors of the background dust Universe,

$$a = \left(\frac{\eta}{\eta_0}\right)^2, \quad \mathcal{H} = \frac{2}{\eta}, \quad \rho^{(0)} = \frac{3}{2\pi G}\left(\frac{\eta_0}{\eta}\right)^4 \frac{1}{\eta^2}. \tag{122}$$

Returning to the cosmic time t, we have

$$a = \left(\frac{t}{t_0}\right)^{2/3}, \quad H = \frac{2}{3t}, \quad \rho^{(0)} = \frac{1}{6\pi G t^2}. \tag{123}$$

So we see easily that for the dust Universe,

$$t = \frac{\eta^3}{3\eta_0^2} \quad \text{and} \quad t_0 = \frac{\eta_0}{3}. \tag{124}$$

The spatial dependence of physical quantities on the background makes no sense, since everything is homogeneous and isotropic, leaving its spatial dependence trivial.

5.2 Temporal dependence of the averaged physical observables to first order

Let us continue to the first order perturbative calculations of the averaged physical observables. In this subsection, we calculate the temporal dependence of $\langle\theta\rangle_D$, $\langle\mathcal{R}\rangle_D$ and $\langle\rho\rangle_D$ in the perturbed dust Universe, both with the conformal and cosmic times. These results are the first step to the derivation of the second and third order contributions. We do not calculate $\langle Q\rangle_D$, as it is a pure second order term, shown in Sec. 6.1.1.

For the first order perturbative calculations of the averaged quantities, the integration measure J must be expanded to first order as well, [61]

$$J = a^3\left(1 - 3\Psi^{(1)}\right) = a^3\left(1 - \frac{\eta^2}{6}\Delta\varphi\right),$$

In the following, let us denote

$$\langle O\rangle \equiv \frac{\int_D O\mathrm{d}\mathbf{x}}{\int_D \mathrm{d}\mathbf{x}}, \tag{125}$$

which is defined to be the average on the background, *i.e.*, $J = a^3$. Watch out that the average is still over a physically comoving domain, which might have a distorted geometry, even on the background. Thus, for the first order perturbative calculations, the averages of the zeroth and first order quantities are

$$\begin{aligned}\langle O^{(0)}\rangle_D &= \frac{\int_D O^{(0)}J\mathrm{d}\mathbf{x}}{\int_D J\mathrm{d}\mathbf{x}} = O^{(0)}, \\ \langle O^{(1)}\rangle_D &= \frac{\int_D O^{(1)}J\mathrm{d}\mathbf{x}}{\int_D J\mathrm{d}\mathbf{x}} = \frac{\int_D O^{(1)}\mathrm{d}\mathbf{x}}{\int_D \mathrm{d}\mathbf{x}} = \langle O^{(1)}\rangle. \end{aligned} \tag{126}$$

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[61] At late times, the constant and decaying modes of $\Psi^{(1)}$ are negligible.

Therefore, the perturbation in J *does not affect the first order calculations.*

5.2.1 Averaged volume expansion rate $\langle\theta\rangle_D$

From the perturbative connections in Eq. (78), we have [62]

$$\theta^i{}_j = u^i{}_{;j} = \frac{1}{a}\Gamma^i_{0j} = \frac{1}{a}\left[\frac{a'}{a}\delta^i{}_j - \Psi^{(1)\prime}\delta^i{}_j + \frac{1}{2}D^i{}_j\chi^{(1)\prime}\right], \tag{127}$$

and taking its trace, we find the perturbative volume expansion rate to first order, [63]

$$\theta = \frac{3}{a}\left(\frac{a'}{a} - \Psi^{(1)\prime}\right). \tag{128}$$

Using Eqs. (95) and (106), we obtain the averaged expansion rate $\langle\theta\rangle_D$ as a function of η and φ,

$$\langle\theta\rangle_D = \frac{3}{a}\left(\frac{a'}{a} - \langle\Psi^{(1)\prime}\rangle_D\right) = \left(\frac{\eta_0}{\eta}\right)^2\left(\frac{6}{\eta} - \frac{\eta}{3}\langle\Delta\varphi\rangle\right), \tag{129}$$

and directly

$$H_D = \left(\frac{\eta_0}{\eta}\right)^2\left(\frac{2}{\eta} - \frac{\eta}{9}\langle\Delta\varphi\rangle\right), \tag{130}$$

From Eq. (129), the first order perturbation decays as $\propto 1/\eta$, slower than that of the zeroth order term, which is $\propto 1/\eta^3$. Therefore, the perturbation becomes more and more important as the Universe evolves. However, this does not mean that cosmological perturbation dominates at late times, as in a perturbative approach, we must restrict our analysis to $|\langle\Psi^{(1)}\rangle_D| \ll 1$, and if it is already of $\mathcal{O}(1)$, the perturbative approach naturally breaks down.

5.2.2 Averaged spatial curvature $\langle\mathcal{R}\rangle_D$

From Eq. (115) and the trace of Eq. (117), we have

$$\mathcal{R} = 16\pi G\rho - \theta^2 + \theta^i{}_j\theta^j{}_i, \quad \mathcal{R} = 12\pi G\rho - \frac{1}{a}\theta' - \theta^2, \tag{131}$$

so

$$\mathcal{R} = -\theta^2 - \frac{4}{a}\theta' - 3\theta^i{}_j\theta^j{}_i. \tag{132}$$

By means of Eqs. (127), (128), (95) and (106), we find to first order

$$\langle\mathcal{R}\rangle_D = \frac{120\langle A_1\rangle}{a^2} = 120\left(\frac{\eta_0}{\eta}\right)^4\langle A_1\rangle.$$

Closer inspection of Eq. (133) shows that

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[62] Attention, this result is slightly different from that in Eq. (15), where we use the cosmic time t, and $u^0 = 1$. However, in our perturbative calculations, we make use of the conformal time η, and $u^0 = 1/a$.

[63] in which we have used the property that $D^i{}_j$ is trace free.

1. $\langle \mathcal{R} \rangle_D$ has only a first order term, with the zeroth order one vanishing, as the background metric is spatially flat.

2. $\langle \mathcal{R} \rangle_D$ decays as $1/\eta^4$. From Eq. (95), $a \propto \eta^2$, so to linear order $\langle \mathcal{R} \rangle_D \propto 1/a^2$ [91], also $\propto 1/a_D^2$, see later in Eq. (139).

3. We may, with the help of Eq. (100), rewrite $\langle \mathcal{R} \rangle_D$ as $\langle \mathcal{R} \rangle_D = \left(\frac{\eta_0}{\eta}\right)^4 \left(\frac{20}{3}\langle \Delta\varphi \rangle + 120K\right)$. So we see that the constant K in Eq. (100) contributes to the averaged spatial curvature a term $120K(\eta_0/\eta)^4 \propto 1/a^2$. We know that in the unperturbed $k \neq 0$ universe (see the metric in Eq. (2)), the Ricci scalar is $6k/a^2$. Thus, $120K(\eta_0/\eta)^4$ is nothing but the background spatial curvature. As we focus on the perturbations in the spatially flat Universe, it vanishes automatically. This is the reason that we set $A_1 = -\Delta C_1/6$ in Eq. (101).

Let us give the final first order perturbative result for the averaged spatial curvature here,

$$\langle \mathcal{R} \rangle_D = \frac{20}{3} \left(\frac{\eta_0}{\eta}\right)^4 \langle \Delta\varphi \rangle. \tag{133}$$

5.2.3 Averaged energy density $\langle \rho \rangle_D$

Similarly, from Eqs. (131), (127) and (128), we have

$$\langle \rho \rangle_D = -\frac{\frac{1}{a}\langle \theta' \rangle_D + \langle \theta^i{}_j \theta^j{}_i \rangle_D}{4\pi G} = \frac{3}{2\pi G} \left(\frac{\eta_0}{\eta}\right)^4 \left(\frac{1}{\eta^2} + \frac{\langle \Delta\varphi \rangle}{6}\right). \tag{134}$$

We find for a domain overdense in average that $\langle \Delta\varphi \rangle$ is positive. Simultaneously, from Eq. (133), we have a positive averaged spatial curvature, and from Eq. (129), the averaged expansion rate is surpressed. This is consistent with our intuitional understanding of the gravitational collapse, which decreases the expansion rate of the Universe.

Some cautious people may doubt that from the Friedmann equations, overdense regions would expand faster than the underdense ones. This doubt seems reasonable if we would just take into account the perturbation in the averaged energy density $\langle \rho \rangle_D$. However, we see from Eq. (119) that the effective expansion rate is not only influenced by the averaged energy density $\langle \rho \rangle_D$, but also the averaged spatial curvature $\langle \mathcal{R} \rangle_D$. Combing the first order perturbations of $\langle \rho \rangle_D$ and $\langle \mathcal{R} \rangle_D$ in Eqs. (133) and (134), we find that the first order perturbation in ρ_{eff} is $-\frac{1}{6\pi G}\left(\frac{\eta_0}{\eta}\right)^4 \langle \Delta\varphi \rangle < 0$ for the overdense regions. Thus, the averaged expansion rate is indeed reduced.

We find in the first order perturbative calculations that only $\Psi^{(1)}$ enters the expressions of $\langle \theta \rangle_D$, $\langle \mathcal{R} \rangle_D$ and $\langle \rho \rangle_D$, and the metric perturbation $\chi^{(1)}$ does not show up. We will show in the next section that $\sigma^2 = \frac{1}{8a^2} D^i{}_j \chi^{(1)\prime} D^j{}_i \chi^{(1)\prime}$, so $\chi^{(1)}$ is linked to the shear of the perturbed Universe. This means that only the expansion influences the evolution of the perturbed Universe at linear order.

5.2.4 Temporal dependence of the first order averaged physical observables on the cosmic time

Now we convert the temporal dependence of the first order averaged physical observables from the conformal time η to cosmic time t, which can directly be used to compare with cosmological observations. From Eq. (124), it is straightforward to perform this task, and we list the corresponding results as

$$\begin{aligned} \langle \mathcal{R} \rangle_D &= \frac{20}{3}\left(\frac{t_0}{t}\right)^{4/3} \langle \Delta\varphi \rangle, \\ \langle \rho \rangle_D &= \frac{1}{6\pi G t^2}\left(1 - \frac{3}{2} t^{2/3} t_0^{4/3} \langle \Delta\varphi \rangle\right), \\ H_D &= \frac{2}{3t}\left(1 - \frac{1}{2} t^{2/3} t_0^{4/3} \langle \Delta\varphi \rangle\right). \end{aligned} \tag{135}$$

5.3 Spatial dependence of the averaged physical observables to first order

For the spatial dependence of the first order perturbative results, we do not expand the averaged physical observables as the series of the conformal time η, but the effective scale factor a_D. This means that we firstly need to expand η as a Taylor series of a_D. From Eqs. (29) and (128), to linear order we have

$$\frac{a_D'}{a_D} = \frac{a}{3}\langle \theta \rangle_D = \frac{a'}{a} - \langle \Psi^{(1)\prime} \rangle = \frac{2}{\eta} - \frac{\eta}{9}\langle \Delta\varphi \rangle, \tag{136}$$

so at late times,

$$\frac{a_D}{a_{D_0}} = \left(\frac{\eta}{\eta_0}\right)^2 \left(1 - \frac{\eta^2}{18}\langle \Delta\varphi \rangle\right) = a\left(1 - \frac{\eta^2}{18}\langle \Delta\varphi \rangle\right). \tag{137}$$

Thus, if $\langle \Delta\varphi \rangle$ is negative, the effective scale factor a_D grows faster than the ordinary result η^2 in the unperturbed dust model. This is consistent with the analysis in Sec. 5.2.3 that the underdense regions expand faster than the overdense ones.

Solving this equation perturbatively to first order, we have

$$\left(\frac{\eta}{\eta_0}\right)^2 = \frac{a_D}{a_{D_0}} + \frac{\eta_0^2}{18}\langle \Delta\varphi \rangle \left(\frac{a_D}{a_{D_0}}\right)^2 = \frac{a_D}{a_{D_0}} + \frac{t_0^2}{2}\langle \Delta\varphi \rangle \left(\frac{a_D}{a_{D_0}}\right)^2. \tag{138}$$

Substituting Eq. (138) into Eq. (135), we obtain the spatial dependence of the averaged physical observables to first order, [64]

$$\langle \mathcal{R} \rangle_D = \frac{20}{3}\frac{a_{D_0}^2}{a_D^2}\langle \Delta\varphi \rangle, \tag{139}$$

$$\langle \rho \rangle_D = \frac{1}{6\pi G t_0^2}\frac{a_{D_0}^3}{a_D^3}, \tag{140}$$

$$H_D = \frac{2}{3t_0}\frac{a_{D_0}^{3/2}}{a_D^{3/2}}\left(1 - \frac{5}{4}\frac{a_D}{a_{D_0}} t_0^2 \langle \Delta\varphi \rangle\right). \tag{141}$$

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[64] To directly compare these results directly with the experimental data and simulations, we only list the results with the cosmic time t, as those with the conformal time η is not so relevant.

We see from Eqs. (139) and (141) that their first order terms are not proportional to their temporal dependence in Eqs. (130) and (134). This is not strange, because the first order terms pick up modifications from zeroth order when converting from temporal to spatial dependence (see Eq. (137) for details). We also discover from Eq. (140) that $\langle\rho\rangle_D \propto 1/a_D^3$, without any fluctuation. The reason is that we use the comoving coordinate system, and thus Eq. (140) must be a natural consequence. This provides also an independent check for the validity of our results.

Let us finally note that the first order contributions to $\langle\theta\rangle_D$, $\langle\mathcal{R}\rangle_D$ and $\langle\rho\rangle_D$ are all *surface terms*, as we may write them as integral of total derivatives

$$\langle\Delta\varphi\rangle = \frac{\int_D \partial^i(\partial_i\varphi)\mathrm{d}\mathbf{x}}{\int_D \mathrm{d}\mathbf{x}}.$$

This means all the information about the first order averaged physical observables is encoded on the boundaries of the perturbed domains that we are interested in.

More surface terms show up below, when we turn to the second order perturbative calculations.

6 Second order perturbative calculations of the averaged physical observables

We move on to the second order perturbative calculations of averaged physical observables. Second order cosmological perturbation theory has been discussed widely in the literature. However, in these previous papers, the metric perturbations of second order are always needed for calculations, and these calculations are always rather complicated and tedious. In this paper, we show how to obtain the leading terms of second order contributions to $\langle Q\rangle_D$, $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$ from the metric perturbations of first order only.

We first prove that the kinematical backreaction term $\langle Q\rangle_D$ is a second order term, and then using the integrability condition, which is a crucial new input, find the second order terms of $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$. In these calculations, the shear scalar σ^2 and thus $\chi^{(1)\prime}$ enter in the final results. The effective equation of state w_{eff} and square of speed of sound c_{eff}^2 are also given to second order.

Different from the first order cases, at second order we have to consider the perturbation of the measure of integral J. Therefore, the averaged physical observables of different orders now become

$$\begin{aligned}
\langle O^{(0)}\rangle_D &= O^{(0)},\\
\langle O^{(1)}\rangle_D &= \langle O^{(1)}\rangle + 3\langle O^{(1)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(1)}\Psi^{(1)}\rangle,\\
\langle O^{(2)}\rangle_D &= \langle O^{(2)}\rangle.
\end{aligned}\tag{142}$$

We see that at second order, the average of a first order quantity $\langle O^{(1)}\rangle_D$ *picks up two second order modifications* $3\langle O^{(1)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(1)}\Psi^{(1)}\rangle$. In the following, we will show that these modifications show up in the second order calculations, especially for that of $\langle\theta\rangle_D$.

6.1 Temporal dependence of the averaged physical observables to second order

In this subsection, we extend the temporal dependence of the averaged physical observables to second order. We calculate $\langle Q\rangle_D$, $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$ in order. Although dealing with the second order perturbation theory, we demand nothing more than linear perturbative metric. All these simplifications rely on the fact that the integrability condition is an exact relation valid to any order, and $\langle Q\rangle_D$ has no linear contribution, which is proven immediately.

6.1.1 Averaged kinematical backreaction term $\langle Q\rangle_D$

Let us recall the kinematical backreaction term $\langle Q\rangle_D$ defined in Eq. (36),

$$\langle Q\rangle_D \equiv \frac{2}{3}\left(\langle\theta^2\rangle_D - \langle\theta\rangle_D^2\right) - 2\langle\sigma^2\rangle_D.$$

Now we prove that $\langle Q\rangle_D$ is a pure second order term, by showing that both the first part $\langle\theta^2\rangle_D - \langle\theta\rangle_D^2$ and second one $\langle\sigma^2\rangle_D$ are of second order.

To calculate the variance $\langle\theta^2\rangle_D - \langle\theta\rangle_D^2$ to second order, we expand θ as

$$\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)}, \tag{143}$$

where $\theta^{(0)}$, $\theta^{(1)}$ and $\theta^{(2)}$ are the zeroth, first and second order contributions to θ, respectively. $\theta^{(0)}$ and $\theta^{(1)}$ have been calculated in Eq. (128). Using Eq. (142), to second order, we have

$$\begin{aligned}\langle\theta^2\rangle_D - \langle\theta\rangle_D^2 &= \langle(\theta^{(0)} + \theta^{(1)} + \theta^{(2)})^2\rangle_D - (\langle\theta^{(0)} + \theta^{(1)} + \theta^{(2)}\rangle_D)^2 \\ &= \langle(\theta^{(1)})^2\rangle_D - \langle\theta^{(1)}\rangle_D^2 = \langle(\theta^{(1)})^2\rangle - \langle\theta^{(1)}\rangle^2. \end{aligned} \tag{144}$$

This means that the first piece of $\langle Q\rangle_D$ is a second order term. However, to calculate it, the detailed form of $\theta^{(2)}$ is unnecessary. All we need is θ up to first order. Using Eq. (128), we directly have

$$\langle\theta^2\rangle_D - \langle\theta\rangle_D^2 = \frac{9}{a^2}\left[\langle(\Psi^{(1)\prime})^2\rangle - \langle\Psi^{(1)\prime}\rangle^2\right] = \frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left[\langle(\Delta\varphi)^2\rangle - \langle\Delta\varphi\rangle^2\right]. \tag{145}$$

Similarly, we calculate the average of the shear scalar $\langle\sigma^2\rangle_D$. From Eqs. (14) and (127), we find to first order that

$$\sigma^i{}_j = \theta^i{}_j - \frac{1}{3}\theta\delta^i{}_j = \theta^{(0)i}{}_j + \theta^{(1)i}{}_j - \frac{1}{3}(\theta^{(0)} + \theta^{(1)})\delta^i{}_j = \frac{1}{2a}D^i{}_j\chi^{(1)\prime}, \tag{146}$$

so $\sigma^i{}_j$ has no zeroth order contribution. Hence, using Eq. (107), we have

$$\sigma^2 = \frac{1}{2}\sigma^i{}_j\sigma^j{}_i = \frac{1}{8a^2}D^i{}_j\chi^{(1)\prime}D^j{}_i\chi^{(1)\prime} = \frac{\eta_0^2}{18}\left(\frac{\eta_0}{\eta}\right)^2\left[\partial^i\partial_j\varphi\partial^j\partial_i\varphi - \frac{1}{3}(\Delta\varphi)^2\right]. \tag{147}$$

Thus, $\langle\sigma^2\rangle_D$, the second piece of $\langle Q\rangle_D$, is also of second order, but can again be calculated by using the expression of $\chi^{(1)}$ at first order only.

So far, we have proved that both parts of $\langle Q\rangle_D$ are of second order, and consequently $\langle Q\rangle_D$ is a second order term, but nevertheless can be calculated from the first order contributions to $\Psi^{(1)}$ and $\chi^{(1)}$. Using Eqs. (145) and (147), we get $\langle Q\rangle_D$ to second order

$$\langle Q\rangle_D = \frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left[\langle(\Delta\varphi)^2\rangle - \langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle - \frac{2}{3}\langle\Delta\varphi\rangle^2\right] \equiv \frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2 B(\varphi), \tag{148}$$

where

$$B(\varphi) \equiv \langle(\Delta\varphi)^2\rangle - \langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle - \frac{2}{3}\langle\Delta\varphi\rangle^2 = \langle\partial^i(\partial_i\varphi\Delta\varphi)\rangle - \langle\partial^i(\partial_j\varphi\partial^j\partial_i\varphi)\rangle - \frac{2}{3}\langle\Delta\varphi\rangle^2.$$

$B(\varphi)$ has only second order terms and is a function of spatial coordinates only.

Some remarks on this result for $\langle Q\rangle_D$ are in order. From Eq. (148) we find that

1. $\langle Q\rangle_D$, written in the abbreviated form $B(\varphi)$, contains two second order terms, which are total derivatives and become surface terms when averaging. Meanwhile, the third term $\langle\Delta\varphi\rangle^2$ is the square of a first order surface term, and thus its second order modifications in Eq. (142) do not show up in $B(\varphi)$. Therefore, $\langle Q\rangle_D$ is a function of the total derivatives of φ on the boundaries of the averaged domains only.

2. Because $\Delta\varphi$ is a fluctuating term, and can be stochastically positive in some regions and negative in others, its average is expected to be negligible, if the averaged domains become large enough (but are still on subhorizon scales). However, $(\Delta\varphi)^2$ and $\partial^i\partial_j\varphi\partial^j\partial_i\varphi$ are positive definite, and therefore give nontrivial surface terms when averaging. Thus, $\langle Q\rangle_D$ consists of these two surface terms on large scales. If they cancel, we say that there is no kinematical backreaction at second order. In the Newtonian limit, this cancelation was discussed in [64] for periodic boundary conditions, and in [92] for spherically symmetric spaces. In relativistic cosmological perturbation theory, this problem was treated in [93]. However, in general case, there is no reason for this cancelation. A good review of this cancelation problem can be found in [91].

3. We see from Eq. (148) that $\langle Q\rangle_D$ decreases as $1/\eta^2$, indicating that $\langle Q\rangle_D \propto 1/a$. And we have already known that $\langle \mathcal{R}\rangle_D \propto 1/a^2$ and $\langle \rho^{(0)}\rangle_D \propto 1/a^3$ in Sec. 5.2.2 and 5.2.3. So $\langle Q\rangle_D$ decays slower than $\langle \mathcal{R}\rangle_D$ and $\langle \rho\rangle_D$. Therefore, in the course of the evolution of the perturbed Universe, the kinematical backreaction becomes more and more important in the effective energy density ρ_{eff} and pressure p_{eff}. Of course, we should note that $\langle Q\rangle_D$ is a pure second order term, but $\langle \mathcal{R}\rangle_D$ has got a first order term, and $\langle \rho\rangle_D$ even contains a zeroth order one, so we cannot indiscreetly conclude that $\langle Q\rangle_D$ dominates the late time behavior of the averaged Universe. The effect of $\langle Q\rangle_D$ is determined not only on its temporal dependence on η (or equivalently the scale factor a) in its denominator, but also by the value of the surface terms in its numerator.

6.1.2 Averaged spatial curvature $\langle \mathcal{R}\rangle_D$

In Sec. 5.2.2, we have calculated the averaged spatial curvature $\langle \mathcal{R}\rangle_D$ to first order in Eq. (133), and here, we use the integrability condition Eq. (121) to obtain its second order part. Since the integrability condition is an exact relation to any order, and we have already got $\langle Q\rangle_D$ to second order in Eq. (148), solving this differential equation $(a_D^6\langle Q\rangle_D)' + a_D^4(a_D^2\langle \mathcal{R}\rangle_D)' = 0$, it is possible to obtain $\langle \mathcal{R}\rangle_D$ to second order. Needless to say, the effective scale factor a_D should also be expanded to second order in terms of the perturbation φ. However, in the following, we show that we do not need that, but the first order result attained in Eq. (137) is sufficient for our purpose.

We rewrite the integrability condition as

$$6\frac{a_D'}{a_D}\langle Q\rangle_D + \langle Q\rangle_D' + 2\frac{a_D'}{a_D}\langle \mathcal{R}\rangle_D + \langle \mathcal{R}\rangle_D' = 0. \tag{149}$$

Because $\langle Q\rangle_D$ is already of second order, in the first term of Eq. (149), we only need the zeroth order term of a_D'/a_D. In the third one, since $\langle \mathcal{R}\rangle_D$ has no zeroth order term, we need the zeroth and first order terms of a_D'/a_D. Altogether, its second order piece is irrelevant. The relation between a_D and η to first order has already been shown in Eq. (137). Substituting Eqs. (137) and (148) into Eq. (149), to second order we have

$$\langle \mathcal{R}\rangle_D' + \left(\frac{4}{\eta} - \frac{2\eta}{9}\langle\Delta\varphi\rangle\right)\langle \mathcal{R}\rangle_D + \frac{10\eta_0^4}{9\eta^3}B(\varphi) = 0.$$

Solving this differential equation exactly, we find

$$\langle\mathcal{R}\rangle_D = D\left(\frac{\eta_0}{\eta}\right)^4 \exp\left(\frac{\langle\Delta\varphi\rangle}{9}(\eta^2-\eta_0^2)\right) + \frac{5B(\varphi)}{\langle\Delta\varphi\rangle}\left(\frac{\eta_0}{\eta}\right)^4\left[1-\exp\left(\frac{\langle\Delta\varphi\rangle}{9}(\eta^2-\eta_0^2)\right)\right],$$

where D is the constant of integration, which is a function of spatial coordinates. For consistency, we must expand this solution up to second order

$$\langle\mathcal{R}\rangle_D = \left(\frac{\eta_0}{\eta}\right)^4\left(D - \frac{D\eta_0^2}{9}\langle\Delta\varphi\rangle + \frac{5\eta_0^2}{9}B(\varphi)\right) + \eta_0^2\left(\frac{\eta_0}{\eta}\right)^2\left(\frac{D}{9}\langle\Delta\varphi\rangle - \frac{5}{9}B(\varphi)\right).$$

There is only one undetermined constant of integration D in the above expression. From Eq. (133), we know that $\langle\mathcal{R}\rangle_D$ has no zeroth order term, so D must only have the first and second order terms; otherwise, the terms in the first bracket would give rise to a zeroth order contribution. We expand $D = D^{(1)} + D^{(2)}$, with $D^{(1)}$ and $D^{(2)}$ being the first and second order terms of D. Because $\langle\Delta\varphi\rangle$ is a first order term and $B(\varphi)$ is a second order one, $\langle\mathcal{R}\rangle_D$ thus becomes

$$\begin{aligned}\langle\mathcal{R}\rangle_D &= \left(\frac{\eta_0}{\eta}\right)^4\left(D^{(1)} + D^{(2)} - \frac{D^{(1)}\eta_0^2}{9}\langle\Delta\varphi\rangle + \frac{5\eta_0^2}{9}B(\varphi)\right)\\ &\quad+\eta_0^2\left(\frac{\eta_0}{\eta}\right)^2\left(\frac{D^{(1)}}{9}\langle\Delta\varphi\rangle - \frac{5}{9}B(\varphi)\right),\end{aligned} \tag{150}$$

where the first term $D^{(1)}(\eta_0/\eta)^4$ represents the first order result of $\langle\mathcal{R}\rangle_D$. It is matched with Eq. (133) to fix

$$D^{(1)} = \frac{20}{3}\langle\Delta\varphi\rangle. \tag{151}$$

Substituting Eq. (151) into Eq. (150) and using Eq. (101), we find $\langle\mathcal{R}\rangle_D$ to second order,

$$\langle\mathcal{R}\rangle_D = \frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\langle\Delta\varphi\rangle + E^{(2)}\left(\frac{\eta_0}{\eta}\right)^4 - \frac{5\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left(B(\varphi) - \frac{4}{3}\langle\Delta\varphi\rangle^2\right), \tag{152}$$

where $E^{(2)} \equiv D^{(2)} - D^{(1)}\eta_0^2\langle\Delta\varphi\rangle/9 + 5\eta_0^2 B(\varphi)/9$. Neither $D^{(2)}$ nor $E^{(2)}$ can be fixed by matching to some known coefficients. However, the term $E^{(2)}(\eta_0/\eta)^4$ is unimportant at any time. Early on, $\frac{20}{3}\langle\Delta\varphi\rangle$ is a first order term, while $E^{(2)}$ is a second order one, so it is negligible compared to the first term in Eq. (152). Similarly, at late times, $E^{(2)}(\eta_0/\eta)^4$ decays faster than $-\frac{5\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left(B(\varphi) - \frac{4}{3}\langle\Delta\varphi\rangle^2\right)$, because both numerators are of second order, but the exponent of the denominator in $E^{(2)}(\eta_0/\eta)^4$ is a larger one. Thus, $\frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\langle\Delta\varphi\rangle$ is the first order term of $\langle\mathcal{R}\rangle_D$, which is the same as the result in Eq. (133), and $-\frac{5\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left(B(\varphi) - \frac{4}{3}\langle\Delta\varphi\rangle^2\right)$ is the *leading* second order part at late times. Therefore, in the following calculations, we write $\langle\mathcal{R}\rangle_D$ as

$$\langle\mathcal{R}\rangle_D = \frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\langle\Delta\varphi\rangle - \frac{5\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left(B(\varphi) - \frac{4}{3}\langle\Delta\varphi\rangle^2\right). \tag{153}$$

Thus, at second order, $\langle\mathcal{R}\rangle_D$ is again a function of surface terms. So with Eq. (133), we find that $\langle\mathcal{R}\rangle_D$ is a function of surface terms at both first and second orders.

Here, we extend the calculation of the averaged spatial curvature $\langle\mathcal{R}\rangle_D$ up to second order by using the integrability condition. Its advantage is that we can do the second order calculation, but without knowing the metric perturbations of second order. All these are reflected in the fact that the integrability condition is an exact result valid to any order, and we have got $\langle Q\rangle_D$ to second order with only the first order cosmological perturbation theory.

6.1.3 Averaged volume expansion rate $\langle\theta\rangle_D$

Below, the second order perturbation of the volume expansion rate $\langle\theta\rangle_D$ is calculated, but again using the metric perturbations of first order only. For doing so, some small tricks playing with the commutation rule Eq. (118) will be helpful. Then, we show finally that our simple calculation is consistent with the result using metric perturbations of second order directly.

From Eqs. (132) and (15), we have

$$\mathcal{R} = -2\theta^2 - \frac{4}{a}\theta' - 6\sigma^2. \tag{154}$$

Since $\langle\mathcal{R}\rangle_D$ has already been calculated to second order in Eq. (153), $\langle\sigma^2\rangle_D$ is a pure second order term (see Eq. (147)), and we know the zeroth and first order terms of $\langle\theta\rangle_D$ from Eq. (129), we can obtain the second order perturbation of $\langle\theta\rangle_D$ from Eq. (154).

Using Eq. (128), we expand θ as

$$\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)} = 3\left(\frac{\eta_0}{\eta}\right)^2\left(\frac{2}{\eta} - \frac{\eta}{9}\Delta\varphi\right) + \theta^{(2)}. \tag{155}$$

so to second order

$$\begin{aligned}\theta^2 &= 9\left(\frac{\eta_0}{\eta}\right)^4\left(\frac{4}{\eta^2} - \frac{4}{9}\Delta\varphi + \frac{\eta^2}{81}(\Delta\varphi)^2\right) + \frac{12}{a\eta}\theta^{(2)},\\ \theta' &= \left(\frac{\eta_0}{\eta}\right)^2\left(-\frac{18}{\eta^2} + \frac{1}{3}\Delta\varphi\right) + \theta^{(2)\prime}.\end{aligned} \tag{156}$$

Substituting Eq. (156) into Eq. (154), and using Eq. (147), we find

$$\mathcal{R} = \frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\Delta\varphi - \frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left[3\partial^i\partial_j\varphi\partial^j\partial_i\varphi + (\Delta\varphi)^2\right] - \left(\frac{\eta_0}{\eta}\right)^2\left(\frac{24}{\eta}\theta^{(2)} + 4\theta^{(2)\prime}\right). \tag{157}$$

We see from Eq. (153) that $\mathcal{R}$ has both first and second order terms, so at second order, the first order term $\frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\Delta\varphi$ gives two additional second order modifications when averaging, as shown in Eq. (142). Therefore, the average of $\mathcal{R}$ to second order is

$$\langle\mathcal{R}\rangle_D = \frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\langle\Delta\varphi\rangle - \left(\frac{\eta_0}{\eta}\right)^2\left(\frac{24}{\eta}\langle\theta^{(2)}\rangle + 4\langle\theta^{(2)\prime}\rangle\right)$$

$$
\begin{aligned}
&+\frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left[10\langle\Delta\varphi\rangle^2-11\langle(\Delta\varphi)^2\rangle-3\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\right]\\
=&\ \frac{20}{3}\left(\frac{\eta_0}{\eta}\right)^4\langle\Delta\varphi\rangle-\left(\frac{\eta_0}{\eta}\right)^2\left(\frac{24}{\eta}\langle\theta^{(2)}\rangle+4\langle\theta^{(2)}\rangle'\right)\\
&+\frac{\eta_0^2}{9}\left(\frac{\eta_0}{\eta}\right)^2\left[10\langle\Delta\varphi\rangle^2-11\langle(\Delta\varphi)^2\rangle-3\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\right]. \qquad (158)
\end{aligned}
$$

Above, the Lemma Eq. (31) allows us to write $\langle\theta^{(2)\prime}\rangle=\langle\theta^{(2)}\rangle'$ at second order. Matching Eq. (158) with Eq. (153) yields

$$
\langle\theta^{(2)}\rangle'+\frac{6}{\eta}\langle\theta^{(2)}\rangle+\frac{\eta_0^2}{18}\left[3\langle(\Delta\varphi)^2\rangle+4\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\right]=0.
$$

Solving this differential equation provides us with the second order contribution to $\langle\theta\rangle_D$ from $\theta^{(2)}$,

$$
\langle\theta^{(2)}\rangle=-\frac{\eta_0^2\eta}{126}\left[3\langle(\Delta\varphi)^2\rangle+4\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\right]+\frac{F^{(2)}}{\eta^6},
$$

where $F^{(2)}$ is the constant of integration of second order, and at late times the term $F^{(2)}/\eta^6$ is negligible without doubt. Therefore, we finally find the averaged expansion rate $\langle\theta\rangle_D$ to second order,

$$
\begin{aligned}
\langle\theta\rangle_D &= \langle\theta^{(0)}\rangle_D+\langle\theta^{(1)}\rangle_D+\langle\theta^{(2)}\rangle_D\\
&= \left(\frac{\eta_0}{\eta}\right)^2\left[\frac{6}{\eta}-\frac{\eta}{3}\langle\Delta\varphi\rangle+\frac{2\eta^3}{63}\left(B(\varphi)-\frac{13}{12}\langle\Delta\varphi\rangle^2\right)\right], \qquad (159)
\end{aligned}
$$

where the first order term $\theta^{(1)}$ *again* contributes via averaging to the second order result (see Eq. (142)), *i.e.*, the second order part of $\langle\theta\rangle_D$ is subtly not the *mere* $\langle\theta^{(2)}\rangle$ or $\langle\theta^{(2)}\rangle_D$. Straightforwardly, the effective Hubble expansion rate H_D is

$$
H_D=\left(\frac{\eta_0}{\eta}\right)^2\left[\frac{2}{\eta}-\frac{\eta}{9}\langle\Delta\varphi\rangle+\frac{2\eta^3}{189}\left(B(\varphi)-\frac{13}{12}\langle\Delta\varphi\rangle^2\right)\right]. \qquad (160)
$$

We find that $\langle\theta\rangle_D$ is also a function of surface terms at both first and second orders.

Before going on to the perturbative calculations of other averaged physical observables to second order, we pause for a moment to show that the result in Eq. (159) can also be obtained by using the second order cosmological perturbation theory, which was discussed heavily in Sec. 4.4. [65]

---

[65] We briefly list the perturbed metric to second order and solutions for metric perturbations here, which can also be found in Eqs. (108), (104), (105) and (109),

$$
\mathrm{d}s^2 \;=\; a^2(\eta)\left[-\mathrm{d}\eta^2+\left(\delta_{ij}+\gamma_{ij}^{(1)}+\gamma_{ij}^{(2)}\right)\mathrm{d}x^i\mathrm{d}x^j\right],
$$

From the perturbed metric to second order Eq. (108), we have the volume expansion rate θ as

$$\theta = \frac{1}{a}\Gamma^i_{0i} = \frac{1}{a}\left(3\frac{a'}{a} - 3\Psi^{(1)\prime} - \frac{3}{2}\Psi^{(2)\prime} - 6\Psi^{(1)}\Psi^{(1)\prime} - \frac{1}{2}D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)\prime}\right). \tag{161}$$

We see from Eq. (161) that only the scalar metric perturbations at first and second orders matter in the derivation of $\langle\theta\rangle_D$ up to second order, with the second order vector and tensor perturbations uninfluential. Therefore, we obtain the average of the volume expansion rate θ,

$$\begin{aligned}\langle\theta\rangle_D \;=\; & \left(\frac{\eta_0}{\eta}\right)^2\left[\frac{6}{\eta} - \frac{\eta}{3}\langle\Delta\varphi\rangle + \frac{2\eta^3}{63}\left(B(\varphi) - \frac{13}{12}\langle\Delta\varphi\rangle^2\right)\right. \\ & \left. - \frac{5\eta}{18}\Big(2\langle\varphi\Delta\varphi\rangle + 3\langle\partial^i\varphi\partial_i\varphi\rangle + 6\langle\varphi\rangle\langle\Delta\varphi\rangle\Big)\right]. \end{aligned} \tag{162}$$

Thus, we find that the leading second order term in Eq. (162), which we get by using the explicit metric perturbations of second order in Eqs. (104), (105) and (109), is the same as that in Eq. (159). One can see as already argued for the case of $\langle\mathcal{R}\rangle_D$ that the subleading second order contributions show the same temporal dependence as the first order term. Thus, it is justified to neglect the subleading terms as they can never (in the perturbative regime) overcome the first order ones.

6.1.4 Averaged energy density $\langle\rho\rangle_D$

Similarly, from Eqs. (29) and (32), we have

$$\left(\frac{1}{3}\langle\theta\rangle_D\right)^2 = \frac{8\pi G}{3}\left(\langle\rho\rangle_D - \frac{\langle Q\rangle_D + \langle\mathcal{R}\rangle_D}{16\pi G}\right). \tag{163}$$

Using Eqs. (148), (153) and (159), we get the averaged energy density $\langle\rho\rangle_D$ up to second order,

$$\langle\rho\rangle_D = \frac{3\eta_0^4}{2\pi G\eta^6}\left[1 + \frac{\eta^2}{6}\langle\Delta\varphi\rangle - \frac{\eta^4}{126}\left(B(\varphi) - \frac{17}{6}\langle\Delta\varphi\rangle^2\right)\right], \tag{164}$$

and $\langle\rho\rangle_D$ is a function of surface terms at both first and second orders too.

---

where

$$\gamma^{(1)}_{ij} = -2\Psi^{(1)}\delta_{ij} + D_{ij}\chi^{(1)}, \quad \gamma^{(2)}_{ij} = -\Psi^{(2)}\delta_{ij} + \frac{1}{2}\left(D_{ij}\chi^{(2)} + \partial_i\chi^{(2)}_j + \partial_j\chi^{(2)}_i + \chi^{(2)}_{ij}\right),$$

and

$$\begin{aligned}\Psi^{(1)} &= \frac{\eta^2}{18}\Delta\varphi + \frac{5}{3}\varphi, \quad \chi^{(1)} = -\frac{\eta^2}{3}\varphi, \\ \Psi^{(2)} &= \frac{\eta^4}{252}\left[(\Delta\varphi)^2 - \frac{10}{3}\partial^i\partial_j\varphi\partial^j\partial_i\varphi\right] + \frac{5\eta^2}{18}\left(\frac{4}{3}\varphi\Delta\varphi + \partial^i\varphi\partial_i\varphi\right).\end{aligned}$$

Up to this point, we have obtained all the averaged physical observables: $\langle Q\rangle_D$, $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$ up to second order, and we only have to consult the first order metric perturbations $\Psi^{(1)}$ and $\chi^{(1)}$ (without the necessity of knowing them to second order). These simplifications are essentially based on the integrability condition and the fact that $\langle Q\rangle_D$ is a pure second order term. As a concequence, we are able to extend this method to higher orders. For example, if we go to third order, we only require the metric perturbations to second order, perform the same procedure for $\langle Q\rangle_D$ and then follow the previous processes, every averaged physical observable will be obtained without too much difficulty.

6.1.5 Effective equation of state $w_{\rm eff}$

Now we turn to the last two physical quantities: the effective equation of state $w_{\rm eff}$ and the square of the speed of sound, which are functions of $\langle Q\rangle_D$, $\langle\mathcal{R}\rangle_D$, $\langle\theta\rangle_D$ and $\langle\rho\rangle_D$.

From Eq. (37), using Eqs. (148), (152) and (164), we obtain the effective equation of state to second order,

$$w_{\rm eff} = \frac{5\eta^2}{54}\langle\Delta\varphi\rangle - \frac{\eta^4}{81}\left(B(\varphi) - \frac{5}{3}\langle\Delta\varphi\rangle^2\right). \tag{165}$$

Therefore, $w_{\rm eff}$ vanishes at zeroth order. This is different from the cosmological constant, with $w_\Lambda = -1$. Consequently, in a perturbative framework, the backreaction mechanism cannot induce accelerated expansion of the Universe, as that would imply $w_{\rm eff} < -1/3$. Nevertheless, the cosmological perturbations allow us to investigate a possible change of the expansion rate of the averaged Universe that might during the later nonlinear stage, lead to an accelerated expansion of the Universe. We discuss this on both small and large scales.

1. First, on small scales and at early times, $\Delta\varphi$ may significantly deviate from 0, so the first order term dominates the value of $w_{\rm eff}$. Using Eq. (84), we rewrite $w_{\rm eff}$ as

   $$w_{\rm eff} = \frac{10\pi G\eta^6}{27\eta_0^4}\langle\rho^{(1)}\rangle. \tag{166}$$

   We see from Eq. (166) that if $\langle\rho^{(1)}\rangle < 0$, which means that the energy density is underdense locally, $w_{\rm eff}$ is negative, and since $w_{\rm eff} \propto \eta^6$, this effect will be more and more influential as time goes on, and might represent the onset of the accelerated expansion of the averaged Universe. Of course, with the above expression, we can trace the evolution only for small perturbations. Once they are in the nonlinear regime, our approach fails.

2. Second, for large averaged domains (still on subhorizon scales) and at late times, the average of $\Delta\varphi$ is expected to become negligible, since it is a fluctuating term, and the two surface terms give nontrivial contributions (see also the similar discussion on $B(\varphi)$). Therefore the value of $w_{\rm eff}$ is dominated by these two second order terms on large scales,

   $$w_{\rm eff} = \frac{\eta^4}{81}\left(\langle\partial^i(\partial_j\varphi\partial^j\partial_i\varphi)\rangle - \langle\partial^i(\partial_i\varphi\Delta\varphi)\rangle\right). \tag{167}$$

We see from Eq. (167) that the sign of $w_{\rm eff}$ depends on the difference between the two surface terms. It vanishes for certain boundary conditions, see [64, 91, 92, 93]. However, we think that these boundary conditions are not very natural, and that the generic case for a finite domain in the Universe is that the effective equation of state is given by a finite surface term, which might be positive or negative, depending on the details of the fluctuations on the boundaries.

6.1.6 Square of the effective speed of sound $c^2_{\rm eff}$

Similarly, for the square of the effective speed of sound, we have

$$c^2_{\rm eff} = \frac{5\eta^2}{81}\langle\Delta\varphi\rangle - \frac{\eta^4}{243}\left(B(\varphi) - \frac{35}{18}\langle\Delta\varphi\rangle^2\right). \tag{168}$$

1. On small scales,

$$c^2_{\rm eff} = \frac{20\pi G\eta^6}{81\eta_0^4}\langle\rho^{(1)}\rangle.$$

So if the cosmic medium is overdense locally, $c^2_{\rm eff} > 0$. But we also see that $c^2_{\rm eff}$ can be negative in underdense regions. Usually this suggests that some damping is going on, which is related to dissipative phenomena and the increase of entropy.

2. On large scales, the second order terms dominate, and we find

$$c^2_{\rm eff} \ = \ \frac{\eta^4}{243}\left(\langle\partial^i(\partial_j\varphi\partial^j\partial_i\varphi)\rangle - \langle\partial^i(\partial_i\varphi\Delta\varphi)\rangle\right).$$

Again, the sign of the square of the effective speed of sound depends on the contrast of the two surface terms.

To summarize this subsection, we find that all the studied physical quantities: $\langle Q\rangle_D$, $\langle R\rangle_D$, $\langle\theta\rangle_D$, H_D, $\langle\rho\rangle_D$, $w_{\rm eff}$ and $c^2_{\rm eff}$, *can be expressed as functions of surface terms at both first and second orders*. Thus, to know the values of these averaged physical observables, we do not need to know anything about the interior of the averaged domains. Only the physical information, *i.e.*, the peculiar gravitational potential φ, and its derivatives, encoded on the boundaries of the domains matter.

6.1.7 Temporal dependence of the averaged physical observables to second order on the cosmic time

As in Eq. (135), the temporal dependence of the averaged physical observables to second order on cosmic time can be obtained by simply converting η to t with Eq. (124). The results are

$$\begin{aligned}
\langle Q\rangle_D &= \frac{t_0^{8/3}}{t^{2/3}}B(\varphi),\\
\langle\mathcal{R}\rangle_D &= \frac{20}{3}\left(\frac{t_0}{t}\right)^{4/3}\langle\Delta\varphi\rangle - 5\frac{t_0^{8/3}}{t^{2/3}}\left(B(\varphi) - \frac{4}{3}\langle\Delta\varphi\rangle\right),
\end{aligned}$$

$$\begin{aligned}
H_D &= \frac{2}{3t}\left[1-\frac{1}{2}t^{2/3}t_0^{4/3}\langle\Delta\varphi\rangle+\frac{3}{7}t^{4/3}t_0^{8/3}\left(B(\varphi)-\frac{13}{12}\langle\Delta\varphi\rangle^2\right)\right],\\
\langle\rho\rangle_D &= \frac{1}{6\pi G t^2}\left[1+\frac{3}{2}t^{2/3}t_0^{4/3}\langle\Delta\varphi\rangle-\frac{9}{14}t^{4/3}t_0^{8/3}\left(B(\varphi)-\frac{17}{6}\langle\Delta\varphi\rangle^2\right)\right],\\
w_{\text{eff}} &= \frac{5}{6}t^{2/3}t_0^{4/3}\langle\Delta\varphi\rangle-t^{4/3}t_0^{8/3}\left(B(\varphi)-\frac{5}{3}\langle\Delta\varphi\rangle^2\right),\\
c_{\text{eff}}^2 &= \frac{5}{9}t^{2/3}t_0^{4/3}\langle\Delta\varphi\rangle-\frac{1}{3}t^{4/3}t_0^{8/3}\left(B(\varphi)-\frac{35}{18}\langle\Delta\varphi\rangle^2\right).
\end{aligned} \tag{169}$$

6.2 Spatial dependence of the averaged physical observables to second order

An important lesson that we have learned is that the averaged physical observables are *not only time dependent, but also scale dependent.*

6.2.1 Spatial dependence

To obtain their spatial dependence up to second order, we now need to know the perturbative relation between η and a_D to second order. Following Eq. (136) and using Eq. (159), to second order, we have

$$\frac{a_D'}{a_D}=\frac{2}{\eta}-\frac{\eta}{9}\langle\Delta\varphi\rangle+\frac{2\eta^3}{189}\left(B(\varphi)-\frac{13}{12}\langle\Delta\varphi\rangle^2\right),$$

so at late times,

$$\frac{a_D}{a_{D_0}}=\left(\frac{\eta}{\eta_0}\right)^2\left[1-\frac{\eta^2}{18}\langle\Delta\varphi\rangle+\frac{\eta^4}{378}\left(B(\varphi)-\frac{1}{2}\langle\Delta\varphi\rangle^2\right)\right].$$

Solving this equation perturbatively to second order, we find

$$\begin{aligned}
\left(\frac{\eta}{\eta_0}\right)^2 &= \frac{a_D}{a_{D_0}}+\frac{\eta_0^2}{18}\langle\Delta\varphi\rangle\left(\frac{a_D}{a_{D_0}}\right)^2-\frac{\eta_0^4}{378}\left(B(\varphi)-\frac{17}{6}\langle\Delta\varphi\rangle^2\right)\left(\frac{a_D}{a_{D_0}}\right)^3\\
\left(\frac{t}{t_0}\right)^{2/3} &= \frac{a_D}{a_{D_0}}+\frac{t_0^2}{2}\langle\Delta\varphi\rangle\left(\frac{a_D}{a_{D_0}}\right)^2-\frac{3t_0^4}{14}\left(B(\varphi)-\frac{17}{6}\langle\Delta\varphi\rangle^2\right)\left(\frac{a_D}{a_{D_0}}\right)^3.
\end{aligned} \tag{170}$$

Substituting Eq. (170) into Eq. (169), we obtain the spatial dependence of the averaged physical observables to second order,

$$\langle Q\rangle_D = \frac{a_{D_0}}{a_D}t_0^2B(\varphi), \tag{171}$$

$$\langle\mathcal{R}\rangle_D = \frac{20}{3}\left(\frac{a_{D_0}}{a_D}\right)^2\langle\Delta\varphi\rangle-5\frac{a_{D_0}}{a_D}t_0^2B(\varphi), \tag{172}$$

$$H_D = \frac{2}{3t_0}\left(\frac{a_{D_0}}{a_D}\right)^{3/2}\left[1-\frac{5}{4}\frac{a_D}{a_{D_0}}t_0^2\langle\Delta\varphi\rangle+\frac{3}{4}\left(\frac{a_D}{a_{D_0}}\right)^2t_0^4\left(B(\varphi)-\frac{25}{24}\langle\Delta\varphi\rangle^2\right)\right], \tag{173}$$

$$\langle\rho\rangle_D = \frac{1}{6\pi G t_0^2}\left(\frac{a_{D_0}}{a_D}\right)^3, \tag{174}$$

$$w_{\text{eff}} = \frac{5}{6}\frac{a_D}{a_{D_0}} t_0^2 \langle\Delta\varphi\rangle - \left(\frac{a_D}{a_{D_0}}\right)^2 t_0^4 \left(B(\varphi) - \frac{25}{12}\langle\Delta\varphi\rangle^2\right), \tag{175}$$

$$c_{\text{eff}}^2 = \frac{5}{9}\frac{a_D}{a_{D_0}} t_0^2 \langle\Delta\varphi\rangle - \frac{1}{3}\left(\frac{a_D}{a_{D_0}}\right)^2 t_0^4 \left(B(\varphi) - \frac{25}{9}\langle\Delta\varphi\rangle^2\right). \tag{176}$$

We see that again $\langle\rho\rangle_D \propto 1/a_D^3$ due to the comoving gauge that we are working with.

6.2.2 Laurent series for the cosmological backreaction terms

Our perturbative results suggest us to go beyond the power-law scaling solutions of the spatial dependence of the averaged physical observalbes in Eq. (49) and write them in the series of the effective scale factor a_D. We see from Eq. (174) that the zeroth order term is proportional to $1/a_D^3$, so these series must be *Laurent* series, not just Taylor series. Here, we focus on the two backreaction terms $\langle Q\rangle_D$ and $\langle R\rangle_D$. We see from Eqs. (171) and (172) that they start from different powers: $1/a_D$ and $1/a_D^2$, so

$$\langle Q\rangle_D = \sum_{n=-1} Q_n \left(\frac{a_D}{a_{D_0}}\right)^n, \quad \langle\mathcal{R}\rangle_D = \sum_{n=-2} \mathcal{R}_n \left(\frac{a_D}{a_{D_0}}\right)^n. \tag{177}$$

The integrability condition then connects the coefficients:

$$Q_n = \frac{n+2}{n+6} R_n. \tag{178}$$

Thus, we find $Q_0 = -R_0/3$ at third order ($n = 0$) in perturbation theory, which *pretty well* fits the condition for leading to a cosmological constant via the backreaction mechanism in Eq. (47). Therefore, cosmological backreaction can really mimic a cosmological constant $\Lambda = Q_0$, and this is the motivation why we continue our work to third order. But unfortunately, cosmological backreaction is also expected to cause extra terms at lower orders as well.

Furthermore, from Eqs. (148) and (153), we find that the temporal dependence of the second order entries of $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$ is not proportional to each other, but the spatial dependence are exactly $Q_{-1} = -R_{-1}/5$, which is a direct consequence from Eq. (178), as the first order terms contribute to second order terms when converting the dependence from η to a_D.

6.2.3 Value of w_{de} from cosmological backreaction

At the end of this section, we briefly reconsider the equation of state of the "dark energy" mimicked by cosmological backreaction in Eq. (46),

$$w_{\text{de}} = \frac{3\langle Q\rangle_D - \langle\mathcal{R}\rangle_D}{3(\langle Q\rangle_D + \langle\mathcal{R}\rangle_D)}. \tag{179}$$

Since both $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$ are perturbative quantities, it is impossible to make a Taylor expansion here any more. But we can use Eq. (178) to discuss the possible values of w_{de}, which strongly depend on whether there is a cutoff in the Laurent series in Eq. (177). Let us investigate the value of w_{de} at both early and late times.

1. At early times, both the backreaction terms $\langle Q \rangle_D$ and $\langle \mathcal{R} \rangle_D$ are tiny, so Eq. (179) becomes 0/0 type and can be calculated by means of the l'Hôpital's rule. We find that $w_{\rm de} \to -1/3$ when $a_D \to 0$. Thus, the effect of cosmological backreaction in the early Universe is undistinguishable from the effect of a homogeneous curvature k in the FLRW model. [66]

2. It is harder to predict $w_{\rm de}$ for the future, as it is not clear whether the Laurent series converge at late times ($a_D \gg a_{D_0}$). Assume there existed an $n_{\rm max} = 0$, $w_{\rm de} \to -1$, then we would arrive at a cosmological constant, and the Universe would approach a de Sitter phase. If $n_{\rm max} > 0$, $w_{\rm de} \to -1 - n_{\rm max}/3$, meaning that the effective fluid would evolve like phantom dark energy. However, we do not see any reason for the existence of an $n_{\rm max}$. Then $w_{\rm de}$ can take any value in the far future. [67]

---

[66] This result seems inconsistent with our intuition of a vanishing cosmological backreaction in the early Universe, suggesting that $w_{\rm de}$ should also vanish. However, if there is cosmological backreaction in the early Universe, we have *new* degrees of freedom in the dynamics of the perturbed Universe, and we know that the limitation of the theory with more degrees of freedom does not always trivially reduce to the theory with fewer degrees of freedom. One example is the difference between the massive and massless gravity theories, namely the so-called vDVZ-discontinuity [94].

[67] To really excavate out whether there is an $n_{\rm max}$ or not, we must advance to even higher cosmological perturbation theory. But from the experiences to third order perturbative calculations, we modestly doubt that there should be no $n_{\rm max}$. Even if it did exist, before $Q_{n_{\rm max}}$ and $\mathcal{R}_{n_{\rm max}}$ would play the main role, our perturbative approach had already been invalidated due to structure formation in the cosmic medium.

7 Third order perturbative calculations of averaged physical observables

Now we proceed to third order perturbative calculations. Its motivation is that we discover a cosmological constant Q_0 at third order in Sec. 6.2.2. So it is worthy to find its expression.

To third order, the averages of physical quantities at different orders are

$$\begin{aligned}
\langle O^{(0)}\rangle_D &= O^{(0)}, \\
\langle O^{(1)}\rangle_D &= \langle O^{(1)}\rangle + 3\langle O^{(1)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(1)}\Psi^{(1)}\rangle + \text{third order modifications}, \\
\langle O^{(2)}\rangle_D &= \langle O^{(2)}\rangle + 3\langle O^{(2)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(2)}\Psi^{(1)}\rangle, \\
\langle O^{(3)}\rangle_D &= \langle O^{(3)}\rangle. \qquad (180)
\end{aligned}$$

We see that again the zeroth and third order terms are not affected by the perturbations in the measure of integral; a second order quantity picks two modification terms $3\langle O^{(2)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(2)}\Psi^{(1)}\rangle$ at third order, the same as a first order quantity does at second order; a first order quantity now has not only the second order modifications $3\langle O^{(1)}\rangle\langle\Psi^{(1)}\rangle - 3\langle O^{(1)}\Psi^{(1)}\rangle$ as before, but also third order ones when averaging. However, these third order terms are irrelevant for our following calculations, so we do not show their explicit forms.

In this section, we only give the detailed calculation of $\langle Q\rangle_D$ to third order, *i.e.*, Q_0, because $R_0 = -3Q_0$ directly from Eq. (178), $\langle\rho\rangle_D$ has no perturbative contributions due to the comoving coordinate system that we use, and the third order terms of H_D, w_{eff} and c^2_{eff} follow the expressions of Q_0 and R_0 trivially, as they are all functions of $\langle Q\rangle_D$ and $\langle\mathcal{R}\rangle_D$, and their specific forms are of little interest to us.

7.1 Averaged kinematical backreaction $\langle Q\rangle_D$

To calculate the averaged kinematical backreaction $\langle Q\rangle_D = \frac{2}{3}\left(\langle\theta^2\rangle_D - \langle\theta\rangle^2_D\right) - 2\langle\sigma^2\rangle_D$ to third order, we once more expand the volume expansion rate θ to third order as $\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)} + \theta^{(3)}$, so for the first piece of $\langle Q\rangle_D$, we get

$$\begin{aligned}
\langle\theta^2\rangle_D - \langle\theta\rangle^2_D &= \langle(\theta^{(0)} + \theta^{(1)} + \theta^{(2)} + \theta^{(3)})^2\rangle_D - \langle\theta^{(0)} + \theta^{(1)} + \theta^{(2)} + \theta^{(3)}\rangle^2_D \\
&= \langle(\theta^{(1)})^2\rangle_D - \langle\theta^{(1)}\rangle^2_D + 2(\langle\theta^{(1)}\theta^{(2)}\rangle_D - \langle\theta^{(1)}\rangle_D\langle\theta^{(2)}\rangle_D) \\
&= \langle(\theta^{(1)})^2\rangle - \langle\theta^{(1)}\rangle^2 + 3(\langle(\theta^{(1)})^2\rangle\langle\Psi^{(1)}\rangle - \langle(\theta^{(1)})^2\Psi^{(1)}\rangle) \\
&\quad -6(\langle\theta^{(1)}\rangle^2\langle\Psi^{(1)}\rangle - \langle\theta^{(1)}\rangle\langle\theta^{(1)}\Psi^{(1)}\rangle) \\
&\quad +2(\langle\theta^{(1)}\theta^{(2)}\rangle - \langle\theta^{(1)}\rangle\langle\theta^{(2)}\rangle). \qquad (181)
\end{aligned}$$

We have already obtained the volume expansion rate θ to second order in Eq. (161),

$$\theta = \frac{1}{a}\left(3\frac{a'}{a} - 3\Psi^{(1)'} - \frac{3}{2}\Psi^{(2)'} - 6\Psi^{(1)}\Psi^{(1)'} - \frac{1}{2}D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)'}\right).$$

Thus, the first part of $\langle Q\rangle_D$ becomes

$$\frac{2}{3}\left(\langle\theta^2\rangle_D - \langle\theta\rangle^2_D\right) = \frac{1}{a^2}\Big[6\left(\langle(\Psi^{(1)'})^2\rangle - \langle\Psi^{(1)'}\rangle^2\right) + 6\left(\langle\Psi^{(1)'}\Psi^{(2)'}\rangle - \langle\Psi^{(1)'}\rangle\langle\Psi^{(2)'}\rangle\right)$$

$$
\begin{aligned}
&+ 2\left(\langle\Psi^{(1)'}D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)'}\rangle - \langle\Psi^{(1)'}\rangle\langle D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)'}\rangle\right)\\
&+ 18\langle(\Psi^{(1)'})^2\rangle\langle\Psi^{(1)}\rangle + 12\langle\Psi^{(1)'}\rangle\langle\Psi^{(1)}\Psi^{(1)'}\rangle\\
&- 36\langle\Psi^{(1)}\rangle^2\langle\Psi^{(1)}\rangle + 6\langle\Psi^{(1)}(\Psi^{(1)'})^2\rangle\Bigg]. \qquad (182)
\end{aligned}
$$

Similarly, for the second piece of $\langle Q\rangle_D$, *i.e.*, $\langle\sigma^2\rangle_D \equiv \frac{1}{2}\sigma^i{}_j\sigma^j{}_i$, as we have known from Eq. (146) that $\sigma^i{}_j$ has no zeroth order term, so for third order calculation of $\langle\sigma^2\rangle_D$, we only need the perturbative expression of $\sigma^i{}_j$ up to second order. Following the procedure in Sec. 6.1.1, we have

$$
\begin{aligned}
\sigma^i{}_j &= \theta^i{}_j - \frac{1}{3}\theta\delta^i{}_j = \frac{1}{a}\Bigg[\frac{1}{2}D^i{}_j\chi^{(1)'} + \frac{1}{4}\left(D^i{}_j\chi^{(2)} + \partial^i\chi^{(2)}_j + \partial_j\chi^{(2)i} + \chi^{(2)i}{}_j\right)' + \Psi^{(1)}D^i{}_j\chi^{(1)'}\\
&\qquad + \Psi^{(1)'}D^i{}_j\chi^{(1)} - \frac{1}{2}D^i{}_k\chi^{(1)}D^k{}_j\chi^{(1)'} - \frac{1}{6}D^k{}_l\chi^{(1)}D^l{}_k\chi^{(1)'}\delta^i{}_j\Bigg].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
-2\langle\sigma^2\rangle_D &= -\frac{1}{a^2}\Bigg[\frac{1}{4}\langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\rangle + \frac{3}{4}\left(\langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\rangle\langle\Psi^{(1)}\rangle - \langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\Psi^{(1)}\rangle\right)\\
&\quad + \frac{1}{4}\langle D^i{}_j\chi^{(1)'}\left(D^j{}_i\chi^{(2)} + \partial^j\chi^{(2)}_i + \partial_i\chi^{(2)j} + \chi^{(2)j}{}_i\right)'\rangle\\
&\quad + \langle D^i{}_j\chi^{(1)'}\left(\Psi^{(1)}D^j{}_i\chi^{(1)'} + \Psi^{(1)'}D^j{}_i\chi^{(1)}\right)'\rangle\\
&\quad - \frac{1}{2}\langle D^i{}_j\chi^{(1)'}D^j{}_k\chi^{(1)}D^k{}_i\chi^{(1)'}\rangle\Bigg]. \qquad (183)
\end{aligned}
$$

Combining Eqs. (182) and (183), we have $\langle Q\rangle_D$ to third order in terms of the metric perturbations,

$$
\begin{aligned}
\langle Q\rangle_D &= \frac{1}{a^2}\Bigg[6\left(\langle(\Psi^{(1)'})^2\rangle - \langle\Psi^{(1)'}\rangle^2\right) - \frac{1}{4}\langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\rangle\\
&\quad + 6\left(\langle\Psi^{(1)'}\Psi^{(2)'}\rangle - \langle\Psi^{(1)'}\rangle\langle\Psi^{(2)'}\rangle\right)\\
&\quad + 2\left(\langle\Psi^{(1)'}D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)'}\rangle - \langle\Psi^{(1)'}\rangle\langle D^i{}_j\chi^{(1)}D^j{}_i\chi^{(1)'}\rangle\right)\\
&\quad + 18\langle(\Psi^{(1)'})^2\rangle\langle\Psi^{(1)}\rangle + 12\langle\Psi^{(1)'}\rangle\langle\Psi^{(1)}\Psi^{(1)'}\rangle - 36\langle\Psi^{(1)}\rangle^2\langle\Psi^{(1)}\rangle + 6\langle\Psi^{(1)}(\Psi^{(1)'})^2\rangle\\
&\quad - \frac{3}{4}\left(\langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\rangle\langle\Psi^{(1)}\rangle - \langle D^i{}_j\chi^{(1)'}D^j{}_i\chi^{(1)'}\Psi^{(1)}\rangle\right)\\
&\quad - \frac{1}{4}\langle D^i{}_j\chi^{(1)'}\left(D^j{}_i\chi^{(2)} + \partial^j\chi^{(2)}_i + \partial_i\chi^{(2)j} + \chi^{(2)j}{}_i\right)'\rangle\\
&\quad - \langle D^i{}_j\chi^{(1)'}\left(\Psi^{(1)}D^j{}_i\chi^{(1)'} + \Psi^{(1)'}D^j{}_i\chi^{(1)}\right)'\rangle - \frac{1}{2}\langle D^i{}_j\chi^{(1)'}D^j{}_k\chi^{(1)}D^k{}_i\chi^{(1)'}\rangle\Bigg]. \qquad (184)
\end{aligned}
$$

The first line of Eq. (184) is the second order result we got in Sec. 6.1.1 and the following six lines are the third order contributions to $\langle Q\rangle_D$ due to cosmological perturbations. We see that $\langle Q\rangle_D$ is the function of $\Psi^{(1)}$, $\chi^{(1)}$, $\Psi^{(2)}$ and $D^i{}_j\chi^{(2)} + \partial^i\chi^{(2)}_j + \partial_j\chi^{(2)i} + \chi^{(2)i}{}_j$ as a whole. The solutions for these metric perturbations have been solved in Secs. 4.3 and 4.4. Here we list their leading terms again, since the non-leading terms only induce slowly growing modes and thus are negligible at late times,

$$\begin{aligned}
&\Psi^{(1)} = \frac{\eta^2}{18}\Delta\varphi, \quad \chi^{(1)} = -\frac{\eta^2}{3}\Delta\varphi, \quad \Psi^{(2)} = \frac{\eta^4}{252}\left[(\Delta\varphi)^2 - \frac{10}{3}\partial^i\partial_j\varphi\partial^j\partial_i\varphi\right], \\
&D_{ij}\chi^{(2)} + \partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + \chi_{ij}^{(2)} = \frac{\eta^4}{126}\Bigg[\left((\Delta\varphi)^2 - \frac{10}{3}\partial^k\partial_m\varphi\partial^m\partial_k\varphi\right)\delta^i{}_j \\
&\qquad\qquad\qquad\qquad + 7\partial^i\partial_k\varphi\partial^k\partial_j\varphi + 6\partial^i\partial_j\Psi_0\Bigg]. \qquad (185)
\end{aligned}$$

Before substituting Eq. (185) into Eq. (184), we should be aware that Eq. (184) shows only the temporal dependence of $\langle Q\rangle_D$. So if we hope to obtain Q_0, *i.e.*, the third order spatial dependence of $\langle Q\rangle_D$, we must take into account the modifications from converting second order temporal dependence to spatial dependence, *i.e.*, the first row of Eq. (184) also leads to third order terms for the calculation of Q_0. From the second order result of $\langle Q\rangle_D$ in Eq. (169) and (138), we have the spatial dependence of $\langle Q\rangle_D$ up to third order, [68]

$$\langle Q\rangle_D = \frac{a_{D_0}}{a_D}\frac{\eta_0^2}{9}B(\varphi) - \frac{\eta_0^4}{162}\langle\Delta\varphi\rangle B(\varphi) + \text{the last six rows in Eq. (184)}. \qquad (186)$$

Now we can substitute Eq. (185) into Eq. (186), and use the relation $t_0 = \eta_0/3$ to finally express $\langle Q\rangle_D$ as

$$\langle Q\rangle_D = \frac{a_{D_0}}{a_D}t_0^2 B(\varphi) + Q_0,$$

where

$$\begin{aligned}
Q_0 = t_0^4\Bigg[&-3\langle\partial^i\partial_j\varphi\partial^j\partial_k\varphi\partial^k\partial_i\varphi\rangle + \frac{45}{14}\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\Delta\varphi\rangle - \frac{3}{14}\langle(\Delta\varphi)^3\rangle \\
&+ \frac{18}{7}\langle\partial^i\partial_j\varphi\partial^j\partial_i\Psi_0\rangle - \frac{15}{7}\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\langle\Delta\varphi\rangle \\
&+ \frac{15}{7}\langle\Delta\varphi\rangle\langle(\Delta\varphi)^2\rangle - \frac{5}{3}\langle\Delta\varphi\rangle^3\Bigg]. \qquad (187)
\end{aligned}$$

We see from Eq. (187) that it contains all the third order possibilities that we can construct. The unique undetermined quantity is Ψ_0, which is the solution of the Poisson equation in Eq. (111),

$$\Delta\Psi_0 = \frac{1}{2}\left[\partial^i\partial_j\varphi\partial^j\partial_i\varphi - (\Delta\varphi)^2\right]. \qquad (188)$$

Equation (188) can certainly be solved by the standard Green function method. Whereas, because of the complexity of the source term at the right hand side, the solution of Ψ_0

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[68] For the third order terms, there is no danger to change from the temporal dependence to spatial dependence, as they are already the highest order entries we are considering here.

will not be quite simple. Moreover, from Eq. (187), we find that what we really need is only $\partial^j\partial_i\Psi_0$, not the explicit form of Ψ_0, so here we make an educated Ansatz to *guess* $\partial^j\partial_i\Psi_0$.

Since Ψ_0 is a second order term, and there are four derivatives in $\partial^j\partial_i\Psi_0$, we may rewrite $\partial^j\partial_i\Psi_0$ as

$$\begin{aligned}\partial^j\partial_i\Psi_0 &= a\varphi\partial^j\partial_i\Delta\varphi + b\partial^j\varphi\partial_i\Delta\varphi + c\partial^j\partial_i\varphi\Delta\varphi + d\partial^j\partial_k\varphi\partial^k\partial_i\varphi \\ &\quad +(e\varphi\Delta\Delta\varphi + f\Delta\varphi\Delta\varphi + g\partial^k\partial_l\varphi\partial^l\partial_k\varphi + h\partial^k\varphi\partial_k\Delta\varphi)\delta^j{}_i. \qquad (189)\end{aligned}$$

Equation (189) exhausts all the possible ways to construct $\partial^j\partial_i\Psi_0$. From the constraint in Eq. (188), it is easy to fix $a = b = e = h = 0$, $c + 3f = -1/2$ and $d + 3g = 1/2$. So

$$\begin{aligned}\partial^j\partial_i\Psi_0 &= -\left(\frac{1}{2}+3f\right)\partial^j\partial_i\varphi\Delta\varphi + \left(\frac{1}{2}-3g\right)\partial^j\partial_k\varphi\partial^k\partial_i\varphi \\ &\quad +(f\Delta\varphi\Delta\varphi + g\partial^k\partial_l\varphi\partial^l\partial_k\varphi)\delta^j{}_i, \qquad (190)\end{aligned}$$

and

$$\begin{aligned}Q_0 &= t_0^4\Bigg[-\left(\frac{12}{7}+\frac{54}{7}g\right)\langle\partial^i\partial_j\varphi\partial^j\partial_k\varphi\partial^k\partial_i\varphi\rangle + \left(\frac{27}{14}-\frac{54}{7}f+\frac{18}{7}g\right)\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\Delta\varphi\rangle \\ &\quad -\frac{15}{7}\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\langle\Delta\varphi\rangle + \left(\frac{18}{7}f-\frac{3}{14}\right)\langle(\Delta\varphi)^3\rangle \\ &\quad +\frac{15}{7}\langle\Delta\varphi\rangle\langle(\Delta\varphi)^2\rangle - \frac{5}{3}\langle\Delta\varphi\rangle^3\Bigg]. \qquad (191)\end{aligned}$$

We guess that the coefficients in Eqs. (190) and (191) must be very symmetric and have special ratio, *i.e.*, they must be the same or at most have opposite signs. This guess is not totally irrational, since the coefficients in Eq. (188) are ±1/2. The unique possibility for this "non-standard" guess is $f = -g = 7/24$, and final result for $\partial^j\partial_i\Psi_0$ and Q_0 is thus "determined" as

$$\partial^j\partial_i\Psi_0 = -\frac{11}{8}(\partial^j\partial_i\varphi\Delta\varphi - \partial^j\partial_k\varphi\partial^k\partial_i\varphi) + \frac{7}{24}(\Delta\varphi\Delta\varphi - \partial^k\partial_l\varphi\partial^l\partial_k\varphi)\delta^j{}_i, \qquad (192)$$

and

$$\begin{aligned}Q_0 &= t_0^4\Bigg[\frac{28}{15}(\langle\partial^i\partial_j\varphi\partial^j\partial_k\varphi\partial^k\partial_i\varphi\rangle - 2\langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\Delta\varphi\rangle + \langle(\Delta\varphi)^3\rangle) \\ &\quad +\frac{15}{7}\Big(\langle(\Delta\varphi)^2\rangle - \langle\partial^i\partial_j\varphi\partial^j\partial_i\varphi\rangle\Big)\langle\Delta\varphi\rangle - \frac{5}{3}\langle\Delta\varphi\rangle^3\Bigg]. \qquad (193)\end{aligned}$$

We want to stress here that although this result is guessed, we think there would be no other solution, as they all make the coefficients in Eqs. (190) and (191) ugly. So from the belief that the true solution must be simple and beautiful, we regard Eq. (193) as our final expression of Q_0. [69]

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[69] Unfortunately, this Q_0 can no longer be written as surface terms, as the ratio of the coefficients in the first bracket is $1 : -2 : 1$. But it can be proved only with the ratio $1 : -3/2 : 1$, can we rewrite them as surface terms by construction.

The temporal dependence of $\langle Q \rangle_D$ and the temporal and spatial dependence of other averaged physical observables: $\langle \mathcal{R} \rangle_D$, $\langle \rho \rangle_D$, H_D, w_{eff} and c^2_{eff} will not be shown any more, since they are irrelevant for our following discussions. Listing them here makes no sense other than lengthening the text.

7.2 Gauge dependence of the averaged physical observables

Before the end of our perturbative calculations at different order, let us finally give a thorough discussion on the gauge dependence problem of the averaged physical observables. In Sec. 4.2.2, the gauge invariance of physical observables at different orders are discussed in detail: a tensor T is gauge invariant to the order n if and only if $\tilde{T}^{(k)} = T^{(k)}$ for every $k \leq n$, and this condition infers that in perturbative approaches, a quantity is gauge dependent except its *leading* term, apart from the trivial cases that it is a constant scalar field, or a linear combination of products of Kronecker deltas with constant coefficients on the background. Thus, we know that $\langle Q \rangle_D$ is a gauge invariant quantity to second order, since it has no zeroth and first order terms; second, the first order terms of $\langle \mathcal{R} \rangle_D$, w_{eff} and c^2_{eff} are gauge invariant; $\langle \theta \rangle_D$, H_D and ρ_{eff}, which have the zeroth order terms, depend on the gauge choice at any order of perturbation theory. In summary, we conclude that all leading terms of the averaged physical observables are gauge invariant, while the higher order ones are not.

8 Signatures of cosmological backreactions

In this section, we compare our theoretical results with experimental data and simulations in Newtonian gravity. For doing so, we first estimate 10% effects from cosmological backreaction. Then, we go beyond these order of magnitude estimations and calculate the variances of the backreaction terms. Lastly, we focus on the variance of the local Hubble expansion rate and identify our calculations, *i.e.*, the signatures of cosmological backreaction with the HST Key Project experimental data and demonstrate that these variances can be reinterpreted as a curvature effect.

8.1 Estimation of 10% effects from cosmological backreaction

We now estimate the order of magnitude of cosmological backreaction as a function of the averaging scale $R \sim V_{D_0}^{1/3}$. We show that cosmological averaging produces important modifications to local physical observables and determine the averaging scale at which corrections show up at the 10% level.

8.1.1 10% effect from the kinematical backreaction $\langle Q \rangle_D$

From Eq. (33), we know that the effective acceleration of the averaged Universe occurs if $\rho_{\text{eff}} + 3p_{\text{eff}} < 0$, *i.e.*, $\langle Q \rangle_D > 4\pi G \langle \rho \rangle_D$. Thus, from Eqs. (171) and (174), we estimate

$$\left| \frac{\langle Q \rangle_D}{4\pi G \langle \rho \rangle_D} \right| = \frac{3}{2} \left(\frac{a_D}{a_{D_0}} \right)^2 B(\varphi) t_0^4 = \frac{8}{27} \frac{R_{\text{H}}^4}{(1+z)^2} B(\varphi) \sim \frac{8}{75} \frac{1}{(1+z)^2} \left(\frac{R_{\text{H}}}{R} \right)^4 \mathcal{P}_\zeta, \qquad (194)$$

where $R_{\text{H}} = 2.998 \times 10^3 h^{-1}$ Mpc is the present Hubble distance, and $\mathcal{P}_\zeta = 2.457 \times 10^{-9}$ is the dimensionless power spectrum [10]. To the leading (second) order, we may safely use the solutions for the background Universe: $a = 1/(1+z)$ and $t_0 = 2R_{\text{H}}/3$, since $B(\varphi)$ is already of second order. Each derivative in $B(\varphi)$ is estimated as a factor of $1/R$ in front of the power spectrum, when we go to the Fourier space of φ. As φ is constant in time, and $\zeta \approx -5\varphi/3$ on superhorizon scales, we identify today's $\mathcal{P}_\varphi$ with $9\mathcal{P}_\zeta/25$ at superhorizon scales. The kinematical backreaction induces 10% modifications if $|\langle Q \rangle_D / 4\pi G \langle \rho \rangle_D| \geq 0.1$. This happens on scales [70]

$$r_Q \leq \frac{21 h^{-1}}{\sqrt{1+z}} \text{ Mpc}. \qquad (195)$$

For observations at $z \ll 1$, $r_Q \leq 30$ Mpc ($h = 0.7$).

8.1.2 10% effect from the averaged spatial curvature $\langle \mathcal{R} \rangle_D$

The averaged spatial curvature $\langle \mathcal{R} \rangle_D$ is the most important correction to the energy density. Similarly, from Eqs. (172) and (174), the criterion for the scale at which its effect

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[70] We use r to denote the critical scale for 10% effect from $\langle Q \rangle_D$, *i.e.*, r_Q, instead of R_Q. So are $r_{\mathcal{R}}$, r_H and r_{Q_0} for $\langle \mathcal{R} \rangle_D$, H_D and Q_0. Otherwise, $R_{\mathcal{R}}$ looks not good, and R_H may be confused with the Hubble distance R_{H}.

emerges is determined by

$$\left|\frac{\rho_{\rm eff}}{\langle\rho\rangle_D}-1\right| \approx \left|\frac{\langle R\rangle_D}{16\pi G\langle\rho\rangle_D}\right| \sim \frac{2}{3}\frac{1}{1+z}\left(\frac{R_{\rm H}}{R}\right)^2\sqrt{\mathcal{P}_\zeta}. \tag{196}$$

We evaluate a 10% effect within

$$r_{\mathcal{R}} \leq \frac{54h^{-1}}{\sqrt{1+z}}\ {\rm Mpc}. \tag{197}$$

At small redshifts, $r_{\mathcal{R}} \leq 77$ Mpc. Furthermore, a 1% effect is expected up to scales out to 240 Mpc. Note that the curvature of the Universe has been measured at the few per cent accuracy in the CMB [10], we will be able to detect this small curvature in the future experiments. [71]

8.1.3 10% effect from the normalized Hubble expansion rate δ_H

Furthermore, we discuss the normalized fluctuation on the local measurement of the Hubble expansion rate. From Eq. (173), we have

$$|\delta_H| \equiv \left|\frac{H_D-H_0}{H_0}\right| \sim \frac{1}{3}\frac{1}{1+z}\left(\frac{R_{\rm H}}{R}\right)^2\sqrt{\mathcal{P}_\zeta}. \tag{198}$$

An effect larger than 10% shows up for

$$r_H \leq \frac{38h^{-1}}{\sqrt{1+z}}\ {\rm Mpc}, \tag{199}$$

which reads $r_H \leq 54$ Mpc at small redshifts.

8.1.4 10% effect from the third order kinematical backreaction Q_0

In Sec. 8.1.1, we first investigate the possibility of the effective acceleration of the averaged Universe. There, we only use the result of $\langle Q\rangle_D$ in Eq. (171), *i.e.*, the leading (second) order result, in order to grasp the main ingredient of this problem. In Secs. 8.1.2 and 8.1.3, we also only take the leading order contributions from $\langle\mathcal{R}\rangle_D$ and H_D. Now we generalize our analysis to third order and really have a look at the higher order influences, *i.e.*, the influence from Q_0, the "cosmological constant".

Following the derivation in Eq. (194), and from Eqs. (193) and (174), we get

$$\left|\frac{Q_0}{4\pi G\langle\rho\rangle_D}\right| \sim \frac{32}{1125}\frac{1}{(1+z)^3}\left(\frac{R_{\rm H}}{R}\right)^6\mathcal{P}_\zeta^{3/2}. \tag{200}$$

If there is a 10% effect from this third order term, the critical scale is

$$r_{Q_0} \leq \frac{17h^{-1}}{\sqrt{1+z}}\ {\rm Mpc}. \tag{201}$$

For small redshifts, $r_{Q_0} \leq 24$ Mpc.

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[71] It was shown in [95] that even small curvatures might affect the analysis of high redshift SNe significantly.

8.1.5 Hierarchy of cosmological backreactions

From Eqs. (195) – (201), we directly find a hierarchy of effects that influence the evolution of the averaged Universe. 10% effects are found up to 80 Mpc for the averaged spatial curvature $\langle \mathcal{R} \rangle_D$. Below 50 Mpc, the fluctuation of the Hubble expansion rate is larger than 10% due to cosmic structure. Up to 30 Mpc and 25 Mpc, the kinematical backreaction $\langle Q \rangle_D$, *i.e.*, the second and third order perturbations, caused by the very local inhomogeneities and anisotropies, enter the game. This might put some of the steps on the cosmological distance ladder in question, as they are deeply within the domain of large cosmological backreaction. We summarize our results in Tab. (2).

Scale	Observable	Physical meaning	Order
≤ 80 Mpc	$\langle \mathcal{R} \rangle_D$	spatial curvature	first
≤ 55 Mpc	H_D	Hubble expansion rate	first
≤ 30 Mpc	$\langle Q \rangle_D$	kinematical backreaction	second
≤ 25 Mpc	Q_0	"cosmological constant"	third

Table 2: Hierarchy of cosmological backreactions.

At scales from 25 to 80 Mpc, different orders of cosmological backreactions show 10% effects, respectively.

The scale dependence of cosmological backreactions is shown in Fig. (5). We see from Eqs. (194) – (200), backreaction effects $\propto 1/R^2$ for first order quantities, and $\propto 1/R^4$ and $1/R^6$ for second and third order terms. Therefore, although third order effects grow faster than first and second ones, when we go to small scales, its numerator prevents it from becoming dominating. It seems that 20 Mpc is an interesting scale, where different orders of cosmological backreaction effects almost have the same order of magnitude. But of course, if so, our perturbative approach has broken down already.

8.2 Ensemble averages and variances of physical observables

Now we go beyond the rough estimations and calculate the variances of cosmological backreaction terms, as these variances (especially the variance of the local Hubble expansion rate) can be compared directly with experimental data. So although we have only one Universe and only one particular volume in the Universe to observe, these calculations are still worthy, at least for theoretical interest.

We first define the *ensemble variance* of an averaged physical observable as

$$\mathrm{Var}(\cdots) \equiv \overline{[(\cdots) - \overline{(\cdots)}]^2} = \overline{(\cdots)^2} - \overline{(\cdots)}^2, \tag{202}$$

where $\overline{(\cdots)}$ denotes the *ensemble average*, *i.e.*, the average over different parallel universes. In the following, we will calculate $[\mathrm{Var}(\delta_H)]^{1/2}$, $[\mathrm{Var}(\langle \mathcal{R} \rangle_D)]^{1/2}$, $[\mathrm{Var}(w_{\mathrm{eff}})]^{1/2}$ and $[\mathrm{Var}(\langle Q \rangle_D)]^{1/2}$, respectively.

In this subsection, we only consider the effects from the leading orders of cosmological backreactions, because at large scales, where perturbation theory can be applied, the third order results obtained in the previous section are too small to be seen in observations.

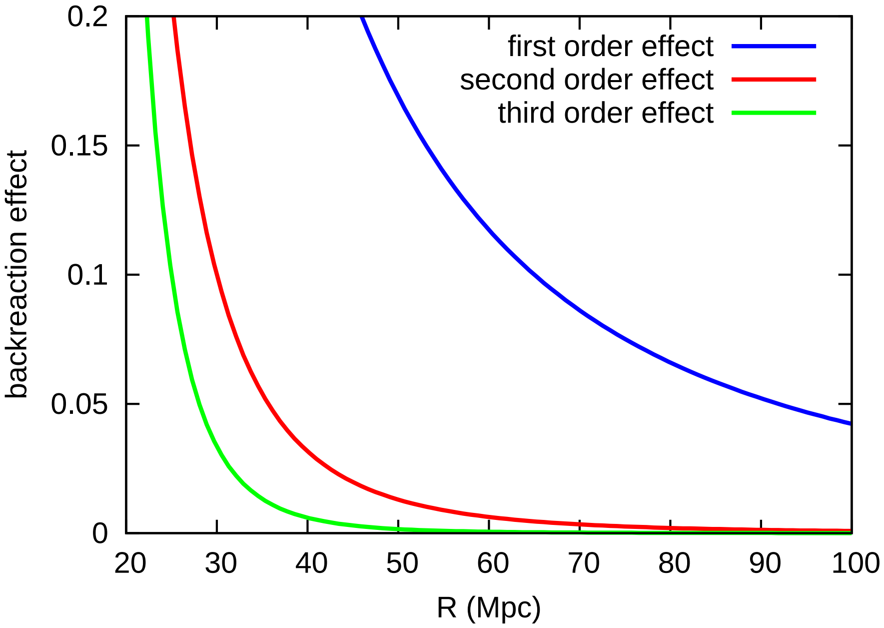

Figure 5: Scale dependence of cosmological backreaction at different orders.

Scale dependence of cosmological backreaction at first, second and third orders, which is proportional to $1/R^2$, $1/R^4$ and $1/R^6$, respectively. We illustrate this dependence by setting the these effects to be 0.1 at 65, 30 and 25 Mpc in order.

8.2.1 Variances of δ_H, $\langle\mathcal{R}\rangle_D$ and $w_{\rm eff}$

For physical quantities like $[\mathrm{Var}(\langle\delta_H\rangle_D)]^{1/2}$, $[\mathrm{Var}(\langle\mathcal{R}\rangle_D)]^{1/2}$ and $[\mathrm{Var}(w_{\rm eff})]^{1/2}$, which start from first order, we expand them as $O = O^{(1)} + O^{(2)} + O^{(3)} + \cdots$. Thus, from Eq. (202), to third order, we have

$$[\mathrm{Var}(O)]^{1/2} = \sqrt{\overline{(O^{(1)})^2}}\left(1 + \frac{\overline{(O^{(2)})^2} - (\overline{O^{(2)}})^2 + 2\overline{O^{(1)}O^{(3)}}}{2\overline{(O^{(1)})^2}}\right). \tag{203}$$

We find from Eq. (203) that for these terms, there are no second order variances, *i.e.*, the next order contributions are already at third order, so we can doubtlessly neglect them. [72]

From Eq. (203), the variance of the normalized effective Hubble expansion rate δ_H is

$$[\mathrm{Var}\,(\delta_H)]^{1/2} = \frac{5}{4}\frac{a_D}{a_{D_0}}t_0^2\left[\overline{\langle\Delta\varphi\rangle^2}\right]^{1/2} = \frac{5}{9}\frac{1}{1+z}R_{\rm H}^2\left[\overline{\langle\Delta\varphi\rangle^2}\right]^{1/2},$$

where [73]

$$\overline{\langle\Delta\varphi\rangle^2} \;=\; \frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^6}k_1^2k_2^2\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}}e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}$$

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[72] Note that this conclusion no longer holds true, when we rewrite $a_D/a_{D_0} = 1/(1+z)$, as this is the zeroth order result only. So strictly speaking, small second order variances also exist in Eq. (203), but since we constrain our attention to the leading order effects, these second order terms are negligible. We could define $1 + z_D \equiv \frac{a_{D_0}}{a_D}$, and then our results are exact to second order formally.

[73] Here we transform φ to the Fourier space as $\varphi(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\varphi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$.

$$= \frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \frac{\mathrm{d}\mathbf{k}}{32\pi^4} k\mathcal{P}_\varphi(k) e^{i\mathbf{k}\cdot(\mathbf{x}_1+\mathbf{x}_2)}. \tag{204}$$

In Eq. (204), we introduce the *two-point correlation function*, or equivalently, the *dimensionless power spectrum* ($k \equiv |\mathbf{k}|$) as [74]

$$\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}} \equiv 2\pi^2 \delta(\mathbf{k}_1 + \mathbf{k}_2)\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}. \tag{205}$$

For the Harrison-Zel'dovich spectrum [96], which is a scale free spectrum for the matter perturbations, *i.e.*, perturbations of all sizes behave in the same way, $\mathcal{P}_\varphi(k_1)$ is a constant, but thus the integral in Eq. (204) diverges. To regularize it, we must insert different *window functions* in Eq. (204).

1. The *top-hat window* $W_R^{\mathrm{T}}(r) = \theta(R - r)$, with

   $$V = \int \mathrm{d}\mathbf{x} W_R^{\mathrm{T}}(r) = \frac{4\pi}{3}R^3, \quad \frac{1}{V}\int \mathrm{d}\mathbf{x} W_R^{\mathrm{T}}(r) e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{3}{(kR)^3}[\sin(kR) - kR\cos(kR)].$$

   This window function is sometimes used in Newtonian simulations, and since we will compare our theoretical results with these simulations, we first utilize this window function to calculate the integral in Eq. (205). So

   $$\begin{aligned} \overline{\langle\Delta\varphi\rangle^2} &= \frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \frac{\mathrm{d}\mathbf{k}}{32\pi^4} k\mathcal{P}_\varphi(k)\left[W_R^{\mathrm{T}}(r)\right]^2 e^{i\mathbf{k}\cdot(\mathbf{x}_1+\mathbf{x}_2)} \\ &= \frac{9}{8\pi^3 R^3}\int_0^\infty \mathrm{d}k \frac{\mathcal{P}_\varphi(k)}{(kR)^3}[\sin(kR) - kR\cos(kR)]^2 \\ &= \frac{9}{16\pi^2 R^4}\int_0^\infty \mathrm{d}(kR)\mathcal{P}_\varphi(k) J_{3/2}^2(kR), \end{aligned} \tag{206}$$

   where $J_{3/2}(x) = \sqrt{\frac{2}{\pi x^3}}(\sin x - x\cos x)$ is the Bessel function of first kind. With the help of the top-hat window, we may regularize the spatial part of the integral in Eq. (204). However, the momentum integral is still divergent, if the dimensionless power spectrum $\mathcal{P}_\varphi(k)$ is scale invariant. To cure this problem, we again need different ways to cutoff the power spectrum. This is the disadvantage of the top-hat window. We will see immediately that with the Gaussian window, we have no trouble with the momentum integral, which means that the top-hat is useful, but still not very effective. We try two cutoffs here,

   (a) The *ultraviolet cutoff* $\mathcal{P}_\varphi(k) = \mathcal{P}_\varphi e^{-k/k_{\mathrm{c}}}$, with k_{c} a critical momentum, which is always taken to be 1/pc as a physical cutoff or 1/kpc in Newtonian simulations. Thus, we have

   $$\begin{aligned} \overline{\langle\Delta\varphi\rangle^2} &= \frac{9\mathcal{P}_\varphi}{16\pi^2 R^4}\int_0^\infty \mathrm{d}x J_{3/2}^2(x) e^{-x/k_{\mathrm{c}}R} \\ &= \frac{9\mathcal{P}_\varphi}{16\pi^2 R^4}\frac{1}{\pi}Q_1\left(1 + \frac{1}{2(k_{\mathrm{c}}R)^2}\right) \\ &\approx \frac{9\mathcal{P}_\varphi}{16\pi^2 R^4}\frac{1}{\pi}[\ln(2k_{\mathrm{c}}R) - 1], \quad (\text{if } k_{\mathrm{c}}R \gg 1), \end{aligned}$$

---

[74] We introduce the power spectrum in this way, *i.e.*, by setting k_1^3 in the denominator, in order to make $\mathcal{P}_\varphi(k_1)$ dimensionless.

where $Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1$ is the Legendre function of second kind. We have made an approximation from the second to the third line. Since R is about 50 to 100 Mpc and k_c is about 1/kpc to 1/pc, $k_\mathrm{c}R \sim 10^5$ to $10^8 \gg 1$, this approximation is very accurate. Therefore, we obtain

$$\mathrm{Var}\,(\delta_H) = \frac{5}{12\pi}\frac{1}{1+z}\left(\frac{R_\mathrm{H}}{R}\right)^2\sqrt{\mathcal{P}_\varphi}\left[\frac{1}{\pi}\ln(2k_\mathrm{c}R) - 1\right]^{1/2}. \tag{207}$$

(b) No cutoff is required for a *red-tilted spectrum* $\mathcal{P}_\varphi(k) = \mathcal{P}_\varphi(k/k_0)^{n_\mathrm{s}-1}$, with the *spectrum index* $n_\mathrm{s} < 1$. Similarly, we have

$$\begin{aligned}\overline{\langle\Delta\varphi\rangle^2} &= \frac{9\mathcal{P}_\varphi}{16\pi^2R^4}\frac{1}{(k_0R)^{n_\mathrm{s}-1}}\int_0^\infty \mathrm{d}xJ^2_{3/2}(x)x^{n_\mathrm{s}-1}\\ &= \frac{9\mathcal{P}_\varphi}{16\pi^2R^4}\frac{\Gamma(1-n_\mathrm{s})\Gamma\left(\frac{3+n_\mathrm{s}}{2}\right)}{\left(\frac{k_0R}{2}\right)^{n_\mathrm{s}-1}\Gamma\left(\frac{5-n_\mathrm{s}}{2}\right)\left(\Gamma\left(\frac{2-n_\mathrm{s}}{2}\right)\right)^2},\end{aligned}$$

where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$ is the Gamma function. Thus, we finally have

$$\mathrm{Var}\,(\delta_H) = \frac{5}{12\pi}\frac{1}{1+z}\left(\frac{R_\mathrm{H}}{R}\right)^2\sqrt{\mathcal{P}_\varphi}\left[\frac{\Gamma(1-n_\mathrm{s})\Gamma\left(\frac{3+n_\mathrm{s}}{2}\right)}{\left(\frac{k_0R}{2}\right)^{n_\mathrm{s}-1}\Gamma\left(\frac{5-n_\mathrm{s}}{2}\right)\left(\Gamma\left(\frac{2-n_\mathrm{s}}{2}\right)\right)^2}\right]^{1/2}. \tag{208}$$

2. Another frequently used window function is the *Gaussian window* $W_R^\mathrm{G}(r) = e^{-r^2/2R^2}$, with

$$V = \int \mathrm{d}\mathbf{x}W_R^\mathrm{G}(r) = (2\pi)^{3/2}R^3, \quad \frac{1}{V}\int \mathrm{d}\mathbf{x}W_R^\mathrm{G}(r)e^{i\mathbf{k}\cdot\mathbf{x}} = e^{-k^2R^2/2}.$$

The advantage of this window is its powerful ability to regularize integrals, *i.e.*, we usually do not need a cutoff for the power spectrum any more. For Eq. (205), now we have

$$\begin{aligned}\overline{\langle\Delta\varphi\rangle^2} &= \frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}}{32\pi^4}k\mathcal{P}_\varphi(k)\left[W_R^\mathrm{G}(r)\right]^2e^{i\mathbf{k}\cdot(\mathbf{x}_1+\mathbf{x}_2)}\\ &= \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}kk^3\mathcal{P}_\varphi(k)e^{-k^2R^2}.\end{aligned} \tag{209}$$

This integral is always convergent for the scale invariant dimensionless power spectrum $\mathcal{P}_\varphi(k) = \mathcal{P}_\varphi$. So we have

$$\overline{\langle\Delta\varphi\rangle^2} = \frac{1}{2}\frac{\mathcal{P}_\varphi}{(2\pi)^3}\frac{1}{R^4},$$

and

$$\mathrm{Var}\,(\delta_H) = \frac{5}{36\pi^{3/2}}\frac{1}{1+z}\left(\frac{R_\mathrm{H}}{R}\right)^2\sqrt{\mathcal{P}_\varphi}. \tag{210}$$

Similarly, for other quantities starting from first order terms, *e.g.*, the averaged spatial curvature $\langle\mathcal{R}\rangle_D$, its ensemble average and variance read

$$\begin{aligned}\overline{\langle\mathcal{R}\rangle_D} &= \frac{10}{3}\frac{a_{D_0}}{a_D}t_0^2\overline{\langle\Delta\varphi\rangle^2} = \frac{5}{54\pi^3}(1+z)\mathcal{P}_\varphi\frac{R_{\rm H}^2}{R^4},\\ [\mathrm{Var}(\langle\mathcal{R}\rangle_D)]^{1/2} &= \frac{20}{3}\left(\frac{a_{D_0}}{a_D}\right)^2\left[\overline{\langle\Delta\varphi\rangle^2}\right]^{1/2} = \frac{5}{3\pi^{3/2}}(1+z)^2\sqrt{\mathcal{P}_\varphi}\frac{1}{R^2}.\end{aligned}$$

For the effective equation of state, we have

$$\begin{aligned}\overline{w_{\rm eff}} &= \frac{11}{4}\left(\frac{a_D}{a_{D_0}}\right)^2 t_0^4\overline{\langle\Delta\varphi\rangle^2} = \frac{11}{324\pi^3}\frac{\mathcal{P}_\varphi}{(1+z)^2}\left(\frac{R_{\rm H}}{R}\right)^4,\\ [\mathrm{Var}(w_{\rm eff})]^{1/2} &= \frac{5}{6}\frac{a_D}{a_{D_0}}t_0^2\left[\overline{\langle\Delta\varphi\rangle^2}\right]^{1/2} = \frac{5}{54\pi^{3/2}}\frac{\sqrt{\mathcal{P}_\varphi}}{1+z}\left(\frac{R_{\rm H}}{R}\right)^2.\end{aligned}$$

8.2.2 Variances of $\langle Q\rangle_D$

We further calculate $[\mathrm{Var}(\langle Q\rangle_D)]^{1/2}$, which starts from second order term. We should firstly state that this calculation is rather complicated, and unfortunate, the final result is divergent, even if some window function is used.

First, the ensemble average of $\langle Q\rangle_D$ (to second order) is

$$\overline{\langle Q\rangle_D} = -\frac{2}{3}\frac{a_D}{a_{D_0}}t_0^2\overline{\langle\Delta\varphi\rangle^2} = -\frac{1}{54\pi^3}\frac{\mathcal{P}_\varphi}{1+z}\frac{R_{\rm H}^2}{R^4}.$$

We find that it is negative definite. Also, for the ensemble average of Q_0 (a third order term), if the fluctuation φ is gaussian, the third-point correlation function vanishes, its ensemble average $\overline{Q_0}$ is zero. This means that in a *typical* universe, to both second and third orders, *i.e.*, the leading order and the constant value of the kinematical backreaction, we *cannot* mimic a positive cosmological constant, leading to the acceleration. This naturally sounds a bad news for our purpose to achieve dark energy from structure formation. But we should be aware that this is just the ensemble average, and we actually have only one real Universe, so if the variance of one physical quantity is large enough, the possibility of accelerated expansion still cannot be ruled out.

Second, let us perform the variance of the kinematical backreaction $\langle Q\rangle_D$,

$$[\mathrm{Var}(\langle Q\rangle_D)]^{1/2} = \frac{a_{D_0}}{a_D}t_0^2[\mathrm{Var}(B(\varphi))]^{1/2},$$

where

$$\begin{aligned}&\mathrm{Var}(B(\varphi))\\ =\ &\overline{\langle\partial^i(\partial_i\varphi\Delta\varphi)-\partial^i(\partial_j\varphi\partial^j\partial_i\varphi)\rangle^2} - \frac{4}{3}\overline{\langle\partial^i(\partial_i\varphi\Delta\varphi)-\partial^i(\partial_j\varphi\partial^j\partial_i\varphi)\rangle\langle\Delta\varphi\rangle^2}\\ &+\frac{4}{9}\overline{\langle\Delta\varphi\rangle^4} - \frac{4}{9}(\overline{\langle\Delta\varphi\rangle^2})^2\\ =\ &\frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_3\mathrm{d}\mathbf{k}_4}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]\left[k_3^2k_4^2-(\mathbf{k}_3\cdot\mathbf{k}_4)^2\right]\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}}\times\end{aligned}$$

$$
\begin{aligned}
& e^{i[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}_1+(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{x}_2]} \\
& -\frac{4}{3}\frac{1}{V^3}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_3\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_3\mathrm{d}\mathbf{k}_4}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]k_3^2k_4^2\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}}\times \\
& e^{i[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}_1+\mathbf{k}_3\cdot\mathbf{x}_2+\mathbf{k}_4\cdot\mathbf{x}_3]} \\
& +\frac{4}{9}\frac{1}{V^4}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_3\mathrm{d}\mathbf{x}_4\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_3\mathrm{d}\mathbf{k}_4}{(2\pi)^{12}}k_1^2k_2^2k_3^2k_4^2\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}}e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2+\mathbf{k}_3\cdot\mathbf{x}_3+\mathbf{k}_4\cdot\mathbf{x}_4)} \\
& -\frac{4}{9}\left[\frac{1}{V^2}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^6}k_1^2k_2^2\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}}e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}\right]^2 \\
= \; & 2\frac{(2\pi^2)^2}{V^2}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]^2\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}\frac{\mathcal{P}_\varphi(k_2)}{k_2^3}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{x}_1+\mathbf{x}_2)} \\
& -\frac{8}{3}\frac{(2\pi^2)^2}{V^3}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_3\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]\frac{\mathcal{P}_\varphi(k_1)}{k_1}\frac{\mathcal{P}_\varphi(k_2)}{k_2}e^{i[\mathbf{k}_1\cdot(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{k}_2\cdot(\mathbf{x}_1+\mathbf{x}_3)]} \\
& +\frac{4}{3}\frac{(2\pi^2)^2}{V^4}\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_3\mathrm{d}\mathbf{x}_4\frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}k_1k_2\mathcal{P}_\varphi(k_1)\mathcal{P}_\varphi(k_2)e^{i[\mathbf{k}_1\cdot(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{k}_2\cdot(\mathbf{x}_3+\mathbf{x}_4)]} \\
& -\frac{4}{9}\frac{(2\pi^2)^2}{V^4}\left[\int \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\frac{\mathrm{d}\mathbf{k}}{(2\pi)^6}k\mathcal{P}_\varphi(k)e^{i\mathbf{k}\cdot(\mathbf{x}_1+\mathbf{x}_2)}\right]^2.
\end{aligned}
$$

Above, we have introduced the *four-point correlation function*, [75]

$$
\begin{aligned}
\overline{\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}} = \; & (2\pi^2)^2\Bigg[\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\mathbf{k}_3+\mathbf{k}_4)\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}\frac{\mathcal{P}_\varphi(k_3)}{k_3^3} \\
& + \delta(\mathbf{k}_1+\mathbf{k}_3)\delta(\mathbf{k}_2+\mathbf{k}_4)\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}\frac{\mathcal{P}_\varphi(k_2)}{k_2^3} \\
& + \delta(\mathbf{k}_1+\mathbf{k}_4)\delta(\mathbf{k}_2+\mathbf{k}_3)\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}\frac{\mathcal{P}_\varphi(k_2)}{k_2^3}\Bigg]. \qquad (211)
\end{aligned}
$$

Again some window functions are needed. We no longer consult the top-hat window, as this window function makes the integral nearly incalculable. Here, we only use the Gaussian window. Therefore,

$$
\begin{aligned}
\mathrm{Var}(B(\varphi)) = \; & 2(2\pi^2)^2\int \frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]^2\frac{\mathcal{P}_\varphi(k_1)}{k_1^3}\frac{\mathcal{P}_\varphi(k_2)}{k_2^3}e^{-|\mathbf{k}_1+\mathbf{k}_2|^2R^2} \\
& -\frac{8}{3}(2\pi^2)^2\int \frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}\left[k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right]\frac{\mathcal{P}_\varphi(k_1)}{k_1}\frac{\mathcal{P}_\varphi(k_2)}{k_2}e^{-(|\mathbf{k}_1+\mathbf{k}_2|^2+k_1^2+k_2^2)R^2/2} \\
& +\frac{4}{3}(2\pi^2)^2\int \frac{\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2}{(2\pi)^{12}}k_1k_2\mathcal{P}_\varphi(k_1)\mathcal{P}_\varphi(k_2)e^{-(k_1^2+k_2^2)R^2} \\
& -\frac{4}{9}(2\pi^2)^2\left[\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^6}k\mathcal{P}_\varphi(k)e^{-k^2R^2}\right]^2 \\
= \; & \frac{\mathcal{P}_\varphi^2}{(2\pi)^6}\int_0^\infty \mathrm{d}k_1\mathrm{d}k_2k_1^3k_2^3\left[J_R^{(0)}(k_1,k_2)-2J_R^{(2)}(k_1,k_2)+J_R^{(4)}(k_1,k_2)\right]
\end{aligned}
$$

---

[75] We point out that although there are three terms in $B(\varphi)$, we had better always put the first two together, or the calculation will be much more tedious, even not performable.

$$-\frac{64}{3}\frac{\mathcal{P}_\varphi^2}{(2\pi)^6}\int_0^\infty \mathrm{d}k_1\mathrm{d}k_2k_1^3k_2^3e^{-(k_1^2+k_2^2)R^2}\left[J_R^{(0)}(k_1,k_2)-J_R^{(2)}(k_1,k_2)\right]$$
$$+\frac{1}{3}\frac{\mathcal{P}_\varphi^2}{(2\pi)^6}\frac{1}{R^8}-\frac{1}{9}\frac{\mathcal{P}_\varphi^2}{(2\pi)^6}\frac{1}{R^8}. \tag{212}$$

In Eq. (212), we have made use of the *filter function*,

$$J_R^{(l)}(k_1,k_2) \equiv \int_{-1}^1 \mathrm{d}\mu\mu^l e^{-|\mathbf{k}_1+\mathbf{k}_2|^2R^2} = \int_{-1}^1 \mathrm{d}\mu\mu^l e^{-(k_1^2+k_2^2+2k_1k_2\mu)R^2},$$

with $\mu \equiv \cos\theta(\mathbf{k}_1,\mathbf{k}_2)$. But this time, for the scale invariant dimensionless power spectrum $\mathcal{P}_\varphi(k)=\mathcal{P}_\varphi$, the integral in Eq. (212) is *still divergent*.

We do not furthermore insert different cutoffs here any more, as they destroy the exact analyzability of Eq. (212) and thus lose the beauty of the calculations, although they are really able to regularize this integral. For instance, a ultraviolet cutoff would be physically well motivated, as there is no structure formation at small scales due to damping. We simply remark where this divergence comes. This divergence origins from the first integral in Eq. (212), where we meet an integral of $\int_0^\infty \mathrm{d}k_1\mathrm{d}k_2\exp(-(k_1-k_2)^2R^2)$ from the terms $J_R^{(0)}$ and $J_R^{(2)}$, where the singularities along the line $k_1=k_2$ cause the divergence of this integral. [76] The integral involving $J_R^{(4)}$ does not contain this divergence. [77] Similar results about this divergence can also be found in [67].

We now briefly summarize our results.

1. For quantities starting from first order terms, *e.g.*, δ_H, $\langle\mathcal{R}\rangle_D$ and w_{eff},

$$\overline{(\cdots)} \propto \frac{1}{R^4}, \quad [\mathrm{Var}(\cdots)]^{1/2} \propto \frac{1}{R^2}.$$

2. For quantities starting from second order terms, *e.g.*, $\langle Q\rangle_D$,

$$\overline{(\cdots)} \sim [\mathrm{Var}(\cdots)]^{1/2} \propto \frac{1}{R^4}.$$

So in the averaging problem in the perturbed Universe, variances of physical observables are always larger or at least as large as their ensemble averages. This means that to predict the sign of an averaged quantity in perturbed space-time is unfortunately impossible.

---

[76] This divergence appears similar with the co-linear divergence in quantum field theory.

[77] Some useful integrals,

$$\int_{-1}^1 \mathrm{d}\mu e^{-a\mu} = \frac{1}{a}e^a-\frac{1}{a}e^{-a},$$
$$\int_{-1}^1 \mathrm{d}\mu\mu^2 e^{-a\mu} = \left(\frac{1}{a}-\frac{2}{a^2}+\frac{2\cdot1}{a^3}\right)e^a-\left(\frac{1}{a}+\frac{2}{a^2}+\frac{2\cdot1}{a^3}\right)e^{-a},$$
$$\int_{-1}^1 \mathrm{d}\mu\mu^4 e^{-a\mu} = \left(\frac{1}{a}-\frac{4}{a^2}+\frac{4\cdot3}{a^3}-\frac{4\cdot3\cdot2}{a^4}+\frac{4\cdot3\cdot2\cdot1}{a^5}\right)e^a$$
$$-\left(\frac{1}{a}+\frac{4}{a^2}+\frac{4\cdot3}{a^3}+\frac{4\cdot3\cdot2}{a^4}+\frac{4\cdot3\cdot2\cdot1}{a^5}\right)e^{-a}.$$

8.3 Comparisons with experimental data and simulations

Based on the calculations above, our theoretical results can be compared with experimental data and simulations. Here, we concentrate on the normalized fluctuation of the local Hubble expansion rate δ_H and show the signatures of cosmological backreactions from it.

8.3.1 Comparison with the HST Key Project experimental data

We compare Eq. (208) with observations from the HST Key Project [53]. We use 64 individual measurements of H_0 in the CMB rest frame (corrected for the local flow) from SN Ia and the Tully-Fisher and fundamental plane relations in Tabs. (4), (5) and (6) in App. D (adopted from Tabs. (6), (7) and (9) in [53]). We restrict our analysis to objects between 31.3 to 467.0 Mpc, as Eq. (208) can be trusted only above 30 Mpc, *i.e.*, the critical scale of second order backreaction effects. Be r_i, H_i and σ_i the distance, Hubble expansion rate and 1σ error for the i'th datum, with distances increasing. We calculate the mean distance for the nearest k objects and the averaged Hubble expansion rate $\bar{H}_k$, *i.e.*, H_D for different subsets by

$$\bar{r}_k = \frac{\sum_{i=1}^{k} g_i r_i}{\sum_{i=1}^{k} g_i}, \quad \bar{H}_k = \frac{\sum_{i=1}^{k} g_i H_i}{\sum_{i=1}^{k} g_i}. \tag{213}$$

with weight $g_i = 1/\sigma_i^2$. The empirical variance of each subset is

$$\bar{\sigma}_k^2 = \frac{\sum_{i=1}^{k} g_i (H_i - \bar{H}_k)^2}{H_0^2 (k-1) \sum_{i=1}^{k} g_i}. \tag{214}$$

Let us stress that Eq. (208) is insensitive to global calibration issues.

The comparison of our result Eq. (208) with the HST Key Project experimental data is shown in Fig. (6). We normalize to the WMAP5 best-fit power-law spectrum, with the pivot $k_0 = 0.002$/Mpc and spectral index $n_\mathrm{s} = 0.960$ [10]. We see that the theoretical curve matches the experimental data well, without any fit parameter in the panel. Before we can claim that we observe the expected $1/R^2$ behavior in Eq. (208) and thus the evidence for cosmological backreaction, we have to make sure that the statistical noise cannot account for it. In the case of a perfectly homogenous coverage of the averaged domain with standard candles, we would expect a $1/R^{3/2}$ behavior. In Fig. (6), we show the statistical noise for the actual data set, which turns out to be well below our result in Eq. (208) and the data points. From Fig. (6), we find that at 45 (60) Mpc, the value of H_D differs from its global value 72 km/s/Mpc by about 10% (5%), consistent with our estimations in Eq. (199), whereas the expected variance for a perfectly homogeneous and isotropic Universe is only 5% (2%).

8.3.2 Comparison with the simulations in Newtonian gravity

At the end of this section, we compare our theoretical result in Eq. (207) to the N-body numerical simulations in Newtonian gravity.

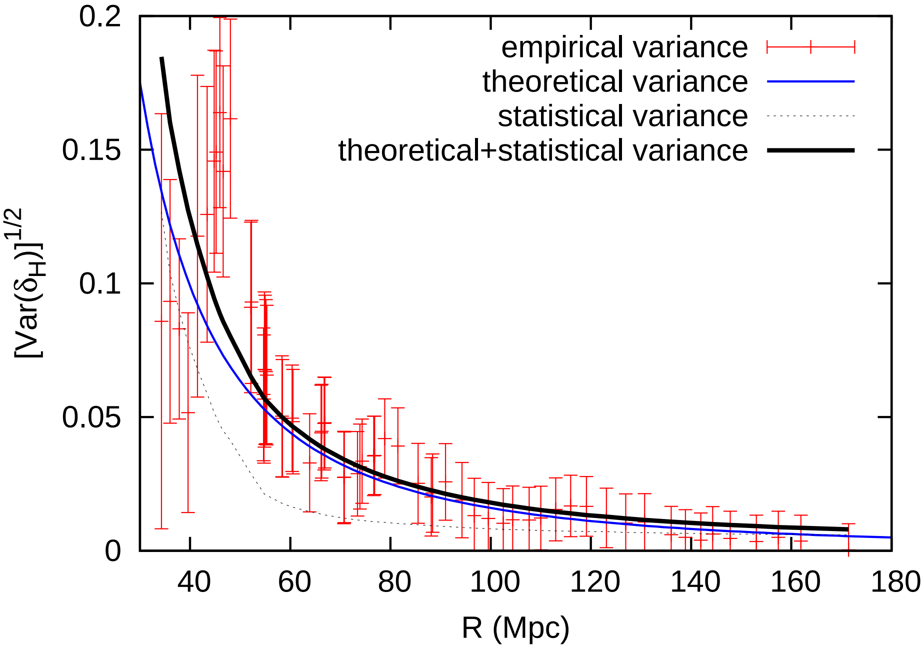

Figure 6: Comparison with the HST Key Project experimental data.

Cosmic variance of the normalized Hubble expansion rate (thick black line) from cosmological backreaction and sampling compared to the empirical variance of the HST Key Project experimental data [53]. The thin blue line indicates the effect of the inhomogeneities ($\propto 1/R^2$), and the dashed line shows the effect from sampling. The global Hubble constant is taken to be 72 km/s/Mpc [53].

The scale dependence of the cosmic variance of δ_H has previously been studied in the context of Newtonian cosmology [97, 98], largely based on the CDM simulations. In this setting, the variance of δ_H is due to peculiar motions (besides sampling variance and observational errors). While, in our *relativistic and comoving* approach, peculiar velocities vanish identically, and the cosmic variance of the Hubble expansion rate turns into a *curvature effect*; as Eqs. (172) and (173) give $\mathrm{Var}(\delta_H) \propto \overline{\langle R \rangle_D^2}$. In Fig. (7), we compare the relativistic result Eq. (207) to Newtonian "standard CDM" case in [98]. We find that our results for scale invariant power spectra (k_c =1/kpc corresponds to the physical cutoff in the primordial CDM spectrum and 1/pc to a typical cutoff in the CDM simulations) agree with Newtonian simulations. This agreement is not unexpected, as metric perturbations and peculiar velocities are small at the 100 Mpc scales.

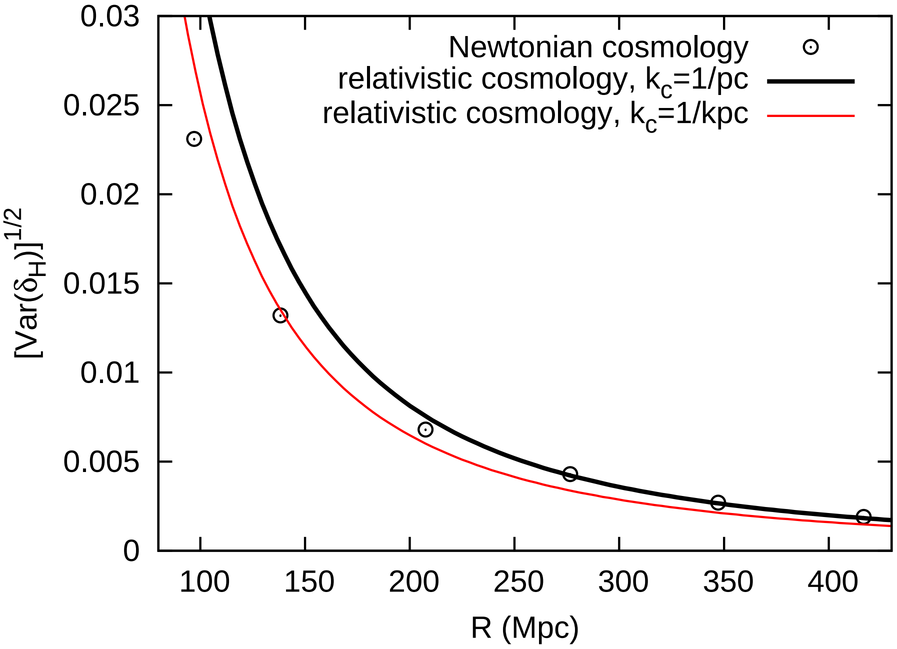

Figure 7: Comparison with the simulation in Newtonian gravity.

Scale dependence of the cosmic variance of the Hubble expansion rate. Data are from Newtonian CDM model in [98], with $h = 0.5$, $\Omega_m = 1$ and the COBE-normalized power spectrum. Thick and thin lines correspond to the relativistic result in Eq. (207) for a scale invariant power spectra with cutoffs at k_c=1/pc (physical) and 1/kpc (simulation), respectively.

9 Summary

This dissertation is motivated by the current debate on cosmological backreaction, *i.e.*, whether it is possible to achieve the effective acceleration of the averaged Universe in the MD era, without the necessity of dark energy. We point out that the evolution of the averaged Universe is intrinsically different from the oversimplified FLRW model, *i.e.*, even if the Universe decelerates locally everywhere, there is *still* the possibility that the global (averaged) Universe accelerates. This apparent contradiction is due to the inhomogeneities and anisotropies, *i.e.*, cosmic structures, in the Universe. Let us formulate it in detail.

First, for any irrotaional dust universe, using the Raychaudhuri equation Eq. (26), the deceleration parameter q can be expressed as

$$q = -\frac{3\dot{\theta} + \theta^2}{\theta^2} = \frac{6(\sigma^2 + 2\pi G\rho)}{\theta^2} > 0.$$

So we find that q is positive definite, and for the EdS model ($\Omega_{\mathrm{m}} = 1$) [99], we have $q = \Omega_{\mathrm{m}}/2 = 1/2$. Thus, it excludes the possibility of the accelerated expansion in the MD era. However, if we define an effective deceleration factor q_D, and from Eqs. (32), (33) and (42), we have

$$q_D \equiv -\frac{\ddot{a}_D}{a_D H_D^2} = \frac{1}{2}\Omega_{\mathrm{m}}^D + 2\Omega_Q^D.$$

Therefore, besides the ordinary result $q = \Omega_{\mathrm{m}}/2$, an extra term $2\Omega_Q^D$, caused by cosmological backreaction now enters the effective deceleration parameter. In [74], it was shown that the EdS model $(\Omega_{\mathrm{m}}, q) = (1, 1/2)$ appears as a *saddle point* for dynamics. So even a small amount of initial backreaction or averaged spatial curvature pushes the system far from it and might lead to a late time accelerated phase (see Fig. (8) for details. [78])

To understand this behavior quantitatively, we combine the exact Buchert equations and cosmological perturbation theory to study the evolution of the perturbed dust Universe in the comoving synchronous gauge. Some conclusions are listed in order.

We calculate the averaged kinematical backreaction term $\langle Q\rangle_D$ and the averaged spatial curvature $\langle \mathcal{R}\rangle_D$ and find that $\langle Q\rangle_D$ starts from second order and $\langle \mathcal{R}\rangle_D$ from first. As we use a perturbative approach, these terms can only affect the evolution of the Universe perturbatively, and thus we may only hope to find an *onset* of the cosmological backreaction mechanism in this work.

We conclude that cosmological backreaction is for real and it can both increase or decrease the expansion of the averaged Universe, depending on the averaged domain under consideration. Thus, we argue that the effective equation of state w_{eff} is both time and scale dependent. So are other averaged physical observables.

We find in our perturbative approach, that all physical quantities are surface terms or functions of them. This suggests the conjecture that a nonlinear treatment would also find only functions of surface terms.

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[78] The meanings of these five cases in Fig. (8) can be found in [74], and r in this diagram is just our r_D in this dissertation.

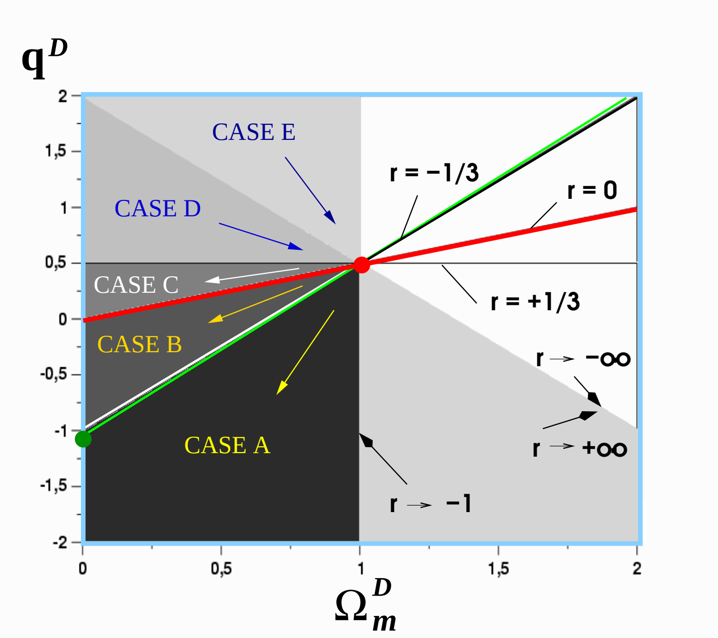

Figure 8: “Cosmic phase diagram”.

In this “cosmic phase diagram” [74], all the scaling solutions in Eq. (49) are represented by straight lines passing through the EdS model in the center $((\Omega_\mathrm{m}, q) = (1, 1/2))$. However, this central point is not stable (saddle point), and we can clearly see the behavior of different models (Cases A – E) around this saddle point (see the arrows). So any perturbation causes the system to move away from the EdS model, and effective acceleration may emerge even in the MD era.

Our pure relativistic treatment of the cosmological backreaction problem, *e.g.*, the variance of the normalized Hubble expansion rate, is consistent with experimental data and Newtonian simulations. This variance can be reinterpreted in our framework as a curvature effect, and the consistency with Newtonian simulations lies in the fact that the perturbative approach cannot be applied to both too small or too large scales, where non-perturbative effects dominate, or the light cone effects become crucial, but only in a *window* from about 50 to 200 Mpc, which just coincides with the scales, where Newtonian simulations work.

Cosmological backreaction may put some of the steps on the cosmological distance ladder in question, as they are deeply within the domain of large backreaction. Our findings call for revisiting local observations, like galaxy redshift surveys, in terms of possible backreaction signatures. The large scale physics of primordial CMB anisotropies is not affected. However, this statement cannot be made for secondary effects, like the late time integrated Sachs-Wolfe effect.

Another point of this dissertation is that we show in Sec. 6 how to calculate the leading growing modes of the averaged quantities to higher orders, but use only the perturbed metric to lower orders. This is a consequence of the integrability condition, which is valid to any order, and it greatly simplifies the perturbative calculations.

10 Outlook

At the end of this dissertation, we discuss several interesting aspects of the averaging and dark energy problems and talk about some relevant prospects in generalities.

10.1 Essence of the non-commutation of the temporal evolution and spatial averaging, entropy in the perturbed Universe

To begin with, we return to the non-commutation of the temporal evolution and spatial averaging in the inhomogeneous and anisotropic Universe and exploit its relation with the entropy in perturbed space-time. We have instigated this non-commutation in Sec. 2.4.3, which is the basis of the backreaction mechanism. A further question is naturally what its essence is. With the help of information theory, this non-commutation can be linked to the relative information entropy in the perturbed Universe.

10.1.1 Non-commutation and relative information entropy

In information theory, supposing the probability distribution is $\{q_i\}$, we would like to examine how close this distribution is to the actual one $\{p_i\}$ [100]. This distinguishability is characterized by the *relative information entropy* [101]

$$S\{p||q\} \equiv \sum_i p_i \ln \frac{p_i}{q_i}.$$

This definition of the relative entropy can be proven to be positive definite for $q_i \neq p_i$ and to vanish if q_i agrees with p_i, so it is a proper measure of entropy. Similarly, in cosmology we ask how matter distribution deviates from its spatial average and thus extend the relevant entropy to

$$\frac{S\{\rho||\langle\rho\rangle_D\}}{V_D} \equiv \left\langle \rho \ln \frac{\rho}{\langle\rho\rangle_D} \right\rangle_D .$$

If we merely generalize the relative information entropy from a discrete system to a continuum of cosmic medium, it is not surprising at all. The most wonderful is that this entropy can be proven (not conjectured axiomatically) to have a marvelous relation with the above non-commutation [100]. From the Lemma in Eq. (31), we find

$$\frac{\dot{S}\{\rho||\langle\rho\rangle_D\}}{V_D} = \langle\dot{\rho}\rangle_D - \langle\rho\rangle_D^{\bullet}. \tag{215}$$

Therefore, the non-commutation is described by the changing rate of the relative information entropy in the inhomogeneous and anisotropic Universe! Next, we demonstrate that this entropy increases in the process of structure formation. Again using the Lemma, we may rewrite Eq. (215) as

$$\frac{\dot{S}\{\rho||\langle\rho\rangle_D\}}{V_D} = -\langle(\theta - \langle\theta\rangle_D)(\rho - \langle\rho\rangle_D)\rangle_D.$$

So the changing rate of the information entropy is intrinsically a second order quantity as the product of two first order perturbative terms. Thus, we can calculate it with only the linear perturbative results in Eqs. (129) and (134). To second order, we have

$$\frac{\dot{S}\{\rho||\langle\rho\rangle_D\}}{V_D} \propto \eta \left(\frac{\eta_0}{\eta}\right)^6 (\Delta\varphi)^2 > 0.$$

Moreover, since $V_D/V_{D_0} = (a_D/a_{D_0})^3 = (\eta/\eta_0)^6$, we have $\dot{S} \propto \eta$, so $S \propto \eta^2 \propto a_D$, which means that the relative information entropy in the perturbed Universe increases *monotonically and linearly* with the effective scale factor, the same as the metric perturbations.

Actually, this conclusion can also be understood directly from the results in Sec. 5. In structure formation, for the process of cluster formation, *i.e.*, accumulation of matter, $\rho > \langle\rho\rangle_D$ and the expansion rate is decelerated $\theta < \langle\theta\rangle_D$, so $\dot{S} > 0$; similarly, for the process of void formation, *i.e.*, dilution of matter, $\rho < \langle\rho\rangle_D$ and $\theta > \langle\theta\rangle_D$, so again $\dot{S} > 0$.

This formulation of the non-commutation provides the link to arguments in favor of cosmological backreaction. In [102], it was argued that on the largest scales, we can view the Universe in the FLRW model with a single isotropic, but imperfect fluid, *i.e.*, we can understand structure formation as an dissipative process that creates entropy.

The further work of this aspect is in progress.

10.1.2 Entropy and Weyl tensor

Now we proceed to another exploration of the entropy in the Universe. Till now, the entropy of the gravitational field has not been well defined, whereas using the Weyl tensor [79] might shed light on this difficulty heuristically. The Weyl curvature vanishes in the early homogeneous and isotropic Universe owing to its conformal invariance and increases monotonically in the perturbed Universe at late times. Penrose [103] conjectured an analogue to the entropy increasing in thermodynamics. For a Schwarzschild black hole, $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} = 48G^2M^2/r^6 \propto$ its entropy S. In the process of structure formation, the cosmic medium little by little departs from the global expansion of the Universe and becomes self-gravitational systems. Generally speaking, these self-gravitational systems will finally end their evolution as black holes. Thus, both black holes and their corresponding Weyl tensors will emerge here and there in the perturbed Universe, and the average of the square of the Weyl tensor will increase monotonically. Stimulated by this, we are trying to formulate a general relation between the Weyl tensor and the entropy in the inhomogeneous and anisotropic Universe. As the Weyl tensor is zero for the background FLRW metric, we are again able to calculate the square of the Weyl tensor to second order, with only the first order perturbed metric.

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[79] The Weyl curvature is the part of the curvature tensor with all contractions vanishing. It is introduced to compensate the indetermination of algebraically independent components in the Riemann tensor, defined in n-dimensional space-time as

$$C_{\mu\nu\lambda\rho} \equiv R_{\mu\nu\lambda\rho} + \frac{2}{n-2}(g_{\mu\rho}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\rho} - g_{\mu\lambda}R_{\nu\rho} - g_{\nu\rho}R_{\mu\lambda}) + \frac{2}{(n-1)(n-2)}(g_{\mu\lambda}R_{\nu\rho} - g_{\mu\rho}R_{\nu\lambda}).$$

We work with the metric in Eq. (75). A straightforward but time-consuming calculation in the MD Universe shows

$$C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} = -8\left(\frac{\eta_0}{\eta}\right)^8\left[\partial^i(\partial_i\varphi\Delta\varphi) - \partial^i(\partial_j\varphi\partial^j\partial_i\varphi) - \frac{2}{3}(\Delta\varphi)^2\right].$$

Interestingly, $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$ looks very like the kinematical backreaction $\langle Q\rangle_D$ in Eq. (148), but *without* averaging. Moreover, $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$ can be proven to be proportional to the shear scalar σ^2. But we have not seen the reason. [80]

Are there other possibilities to build more invariants like the square of the Weyl tensor? Yes, we may also make use of the dual sector of the Weyl tensor $C^*_{\mu\nu\lambda\rho}$, and then there are three more choices: $C^*_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$, $C_{\mu\nu\lambda\rho}C^{\lambda\rho\sigma\tau}C_{\sigma\tau}{}^{\mu\nu}$ and $C^*_{\mu\nu\lambda\rho}C^{\lambda\rho\sigma\tau}C_{\sigma\tau}{}^{\mu\nu}$. Since we have the perturbative results to third order, we are investigating all these possibilities.

10.2 Fates of dark energy, cosmological constant and our Universe

Finally, let me freely talk about my perspectives on the fates of dark energy, cosmological constant and our Universe to terminate this dissertation.

10.2.1 Is dark energy really necessary?

After spending one hundred pages on the dark energy problem, we now return to the beginning and ask: is dark energy *really* necessary? Equivalently, to what degree can we *really* believe that we have observed dark energy, or have we got *direct* evidences for dark energy? Unfortunately, the answer is not so trivial. Did we *really* see it from the Supernova and CMB experiments? What we indeed observe in the SN experiment is that the distant SNe are *dimmer* than we have expected, and what we indeed see in the CMB experiment is that there is *something else* besides the ordinary matter. So, these evidences are *indirect*, and the existence of dark energy is *inferred* from them. [81]

Practically, it is quite unconvincing to conclude dark energy from one single experiment. In Fig. (9) [10], we show the joint two-dimensional constraint on the curvature parameter Ω_k and the constant equation of state for dark energy $w_{\rm de}$. If only consulting the data from the WMAP5 experiment, we find that the allowed region for the correlation of these two parameters is quite large, almost presenting nothing precise for us. This is what we mean by "indirect evidence". The improvement of this constraint must be from the combinations with other independent experiments, *e.g.*, the BAO, HST and SN observations. These joint constraints significantly reduce the uncertainty in the WMAP5-only experiment, and may help for finally establishing the existence of dark energy.

This indirectness puts forward another question: what is the *realistic* background of our Universe? In Einstein's static model, the geometry of space-time and also the cosmological constant are totally fixed by the cosmic medium in the Universe, and no

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[80] This proportionality has also been pointed out in [91].

[81] I once heard from Prof. S. Sarkar that except the late time integrated Sachs-Wolfe effect, dark energy is unnecessary for the explanations of all other experimental data.

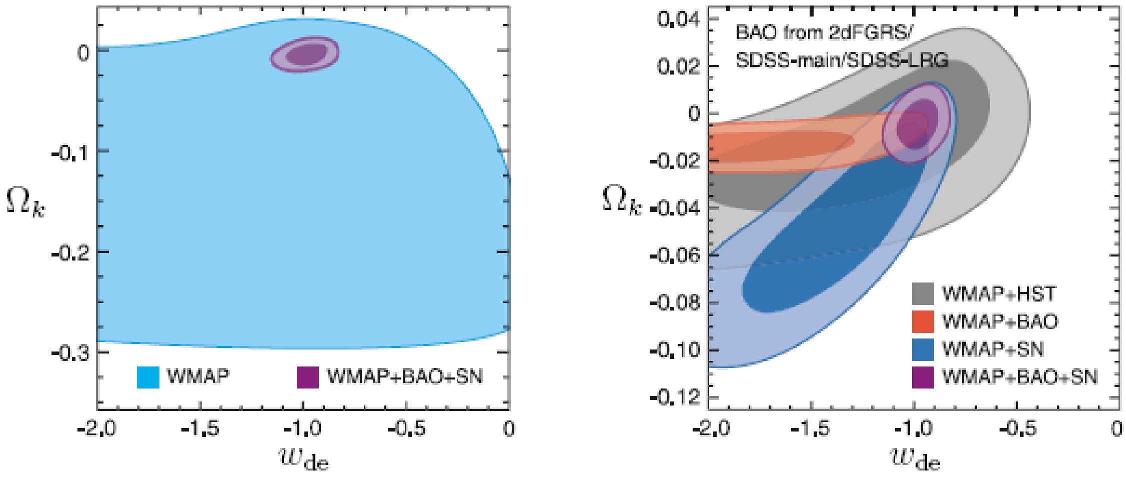

Figure 9: Constraint on the curvature parameter and the equation of state for dark energy.

> Joint two-dimensional constraint on the constant equation of state for dark energy $w_{\rm de}$ and the curvature parameter Ω_k [10]. The contours show the 68% and 95% CL. In the left panel, the WMAP5 constraint (light blue, 95% CL) is compared with the joint one from WMAP+BAO+SN (purple, 68% and 95% CL). This figure shows the power of extra information from BAO and SN for constraining Ω_k and $w_{\rm de}$ simultaneously. The right panel is a blow-up of the left one, showing the constraints from WMAP+HST (gray), WMAP+BAO (red), WMAP+SN (dark blue) and WMAP+BAO+SN (purple). This figure shows that we need both BAO and SN to constrain Ω_k and $w_{\rm de}$ simultaneously: WMAP+BAO fixes Ω_k, and WMAP+SN fixes $w_{\rm de}$.

room is left for free parameters. In the FLRW model, we have one free scale factor a. In the LTB model, the degrees of freedom for free parameters are even larger, *i.e.*, we have two parameters to play with. But strictly speaking, these improved models are still far from perfect. If we abandon all these toy models but only establish a model of the Universe entirely according to the experimental observations, we will find that our Universe is a void-dominating one, *i.e.*, most area in the inhomogeneous space is more underdense than the average level. This means the usual FLRW model as a background is actually not a suitable choice, and what we must perform should be to study the propagation of light in these voids. [82]

10.2.2 History and future of the cosmological constant

Lastly, let us not only limit ourselves to the present dark energy crisis, but widen our visual field to the history of the dark energy story, as a retrospection of history is the best guidance for future. Here we concentrate on the cosmological constant problem, and it is quite interesting to illustrate its history below. [83]

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[82] A recent discussion on this problem can be found in [104].

[83] I do not know the origin of this figure and thank Prof. J. Martin for presenting it to me.

The cosmological constant was led into physics at the same time with Einstein's initial static model of cosmology in 1917. Influenced by the works of Friedmann and Hubble, Einstein gave up his cosmological constant ten years later and declaimed it as "the greatest blunder of my life". The second round of this game was caused by the puzzle why the Earth or a typical star are older than the Universe, and the cosmological constant was restored in order to resolve this trouble. However, it was gone again by the reestimation of the Hubble constant in the 1950s: it should be about 100 km/s/Mpc, not 500 km/s/Mpc as first "measured" by Hubble. But the story did not stop here. With one of the four great discoveries in the 1960s by radio telescope: the extremely luminous and distant quasars, the cosmological constant was called back once more as a savior to help with the controversy why we only observe quasars in a small redshift interval at high z. This quasar excess was interpreted as a result of the dilution of the Universe at low redshifts due to the cosmological constant. However, the apparent controversy was resolved again about ten years later by the observation that the interval is not so small. Then, the forum of the cosmological constant was quiet for more than twenty years until the cosmology revolution from the SN experiments at the beginning of the new millennium. This time, it seems that the existence of the cosmological constant is much more stable than before and the cosmological constant would live there for ever. [84]

Now it is exactly the tenth year after this epochal event, and there still appears no smoking-gun to get rid of the cosmological constant once more. But at least for me, if dark energy were really confirmed by future experiments, it would actually not be a good news. To make my statement clear, let us reexperience the history of thermodynamics.

One and a half century ago, physicists were suffering the trouble due to the second law of thermodynamics, as according to which the entropy of the Universe would increase monotonically and everything would end in a mess. Thus, the fate of our Universe would be nothing but a hot soup, and the life of an individual would perish together with the cosmic medium in this hot soup. It sounds really horrible, but never comes true. Now we have understood that the Universe cannot be simply considered as an isolated system, and it is a negative heat capacity system once gravitation is taken into account. Therefore, we are happily retrieved from this tragedy.

Presently, the direction of the tragedy has a potential to turn around. If the Universe were to be dominated by dark energy, it would not be hotter and hotter as in thermodynamics, but colder and colder, since dark energy drives everything farther and farther, and the Universe would be cooling down eternally. So on the contrary to the heat death, we are now facing a cold death, unfortunately.

If science pointed nothing to the fates of the Universe and human being but two wretched finales, it would be very ironic that so many people are interested in it and take it as a labor of love. Hence, the analogy with the history of thermodynamics makes me intuitionally suspect of the dominance of dark energy and the cosmological constant.

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[84] Of course, this simple graph cannot exhaust every round in the history of the cosmological constant. Small ripples will show up on top of this wave, if we go into the fine structure of the history. For instance, just two or three years before the SN experiments, people once measured the Hubble constant to be $H_0 \approx 80$ km/s/Mpc but the ages of oldest stars to be about 15 Gyr. Thus, the cosmological constant replayed its role in the 1950s, but the Hipparcos experiment again removed this contradiction by more precise measurements.

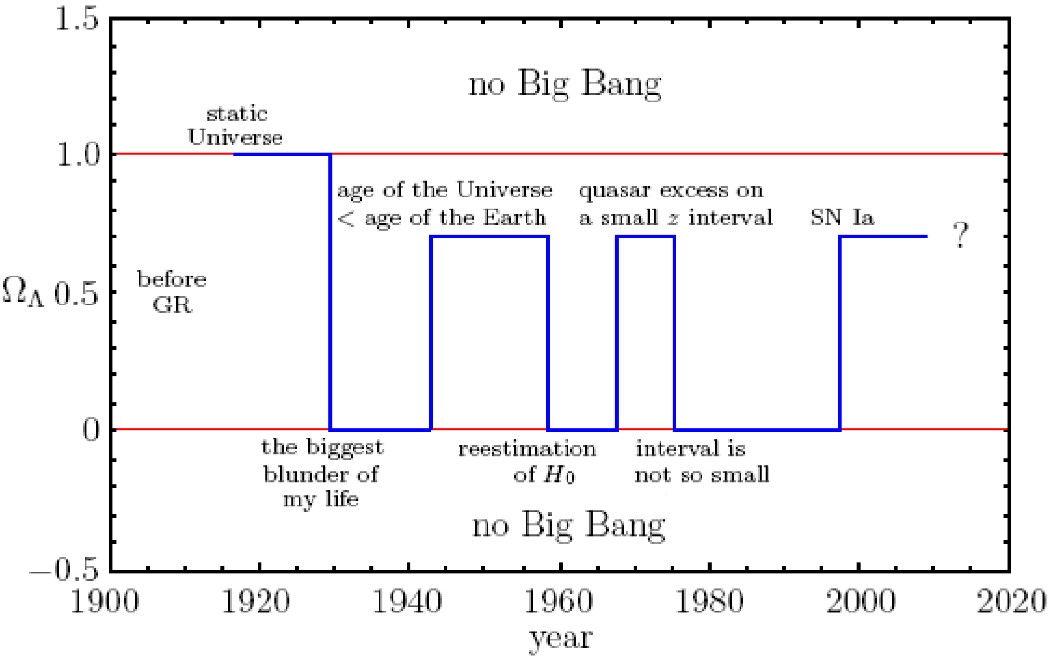

Figure 10: History of the cosmological constant.

We show the four rounds of the restorations and eliminations of the cosmological constant in the last ninety years. The cosmological constant was introduced into the Einstein equations in order to stabilize the Universe, lengthen the age of Universe, dilute the Universe at low redshifts or accelerate the Universe at late times. However, it was also weeded out correspondingly again and again when new and more accurate experimental data came. We are currently at the position of the question mark, and it seems that we would not be able to know where we will arrive in the recent years.

Doubtless to say, the only way out to understand them depends again on future experiments. But before waiting for these critical experiments, some preliminary theoretical attempts are still worthy, and this is the motivation that I accomplish this dissertation, since the backreaction mechanism is completely based on firm physical laws, so even if it turns out to be unimportant, we are anyhow far away from the danger to waste time on “wrong” directions, knowing that backreaction is not so unimportant in any case.

Our understanding of the Nature is like a spiral. It wanders, convolutes but ascends eventually. The knowledge of dark energy and the cosmological constant is exactly the same. Dark energy is not a mirage, and the cosmological backreaction is not a panacea. Today, an open mind is much better than an inflexible belief. What will happen? We shall see.

Appendices

A Basic notations

Latin indices run over 3-dimensional spatial coordinate labels from 1 to 3; while the Greek ones over space-time coordinate labels from 0 to 3, with x^0 the time coordinate (t for the cosmic time and η for the conformal time).

Time derivative is denoted by a dot over a physical quantity with respect to the cosmic time t and a superscript prime with respect to the conformal time η.

Spatial 3-vectors are indicated by letters in boldface, and their absolute values in italic.

The Minkowski metric $\eta_{\mu\nu}$ for flat space-time has the signature $(-,+,+,+)$.

Repeated indices are generally summed, understood as the Einstein convention.

Angle brackets $\langle\cdots\rangle_D$ with subscript D, define the average of physical observables on constant time hypersurfaces with the integral measure J; while only angle brackets $\langle\cdots\rangle$ define the average without it.

Bars over any physical quantity denotes its ensemble average.

Superscripts $^{(0)}$, $^{(1)}$, $^{(2)}$ and $^{(3)}$ refer to perturbed physical quantities at different order, respectively.

Speed of light is taken to be 1 throughout this dissertation.

The Christoffel connection is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} \right).$$

The Riemann tensor is defined as

$$R^{\lambda}{}_{\mu\nu\rho} \equiv \Gamma^{\lambda}_{\mu\rho,\nu} - \Gamma^{\lambda}_{\mu\nu,\rho} + \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - \Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu}.$$

The Ricci tensor is defined as one contraction of the Riemann tensor

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\lambda\nu}.$$

The Ricci scalar is given by contracting the Ricci tensor

$$R = R^{\mu}{}_{\mu}.$$

The Einstein tensor is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

The Einstein equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}.$$

Conventions for the Fourier and inverse Fourier transforms are

$$f(\mathbf{k}) \equiv \int \mathrm{d}\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{and} \quad f(\mathbf{x}) \equiv \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

B Notations of physical quantities

Below we list the some frequently used physical quantities and their definitions.

Symbol	Definition
$\Gamma^{\lambda}_{\mu\nu}$	Christoffel connection
Δ	Laplace operator
δ_H	normalized fluctuation of the Hubble expansion rate
ζ	hypersurface invariant variable
$\bar{\zeta}$	first integral of Eq. (92)
η	conformal time
θ	volume expansion rate
$\theta_{\mu\nu}$	expansion tensor
Λ	cosmological constant
$\xi^{(k)\mu}$	generator of the Lie derivative at k'th order
ρ	energy density
$\rho_{\rm c}$	critical energy density
$\rho_{\rm eff}$	energy density of the effective fluid
ρ_{ϕ_D}	energy density of the morphon field
σ^2	shear scalar
$\sigma_{\mu\nu}$	shear tensor
$\phi^{(1)}$	first order scalar perturbation
$\Phi^{(1)\rm inv}$	first order scalar gauge invariant variable
ϕ_D	morphon field
$\phi^{(1)}_{\rm l}$	first order scalar perturbation in the longitudinal gauge
φ	peculiar gravitational potential
$\chi^{(1)}$	first order scalar perturbation in the synchronous gauge
$\chi^{(2)}$	second order scalar perturbation in the synchronous gauge
$\chi^{(2)}_i$	second order vector perturbation in the synchronous gauge
$\chi^{(2)}_{ij}$	second order tensor perturbation in the synchronous gauge
$\psi^{(1)}$	first order scalar perturbation
$\Psi^{(1)}$	first order scalar perturbation in the synchronous gauge
$\Psi^{(1)\rm inv}$	first order scalar gauge invariant variable
$\Psi^{(2)}$	second order scalar perturbation in the synchronous gauge
$\psi^{(1)}_{\rm l}$	first order scalar perturbation in the longitudinal gauge
$\psi^{(1)}_{\rm s}$	first order scalar perturbation in the synchronous gauge
Ω	energy density parameter
$\Omega_{\rm b}$	energy density parameter for baryonic matter
$\Omega_{\rm CDM}$	energy density parameter for cold dark matter
$\Omega_{\rm de}$	energy density parameter for dark energy
Ω_k	energy density parameter for curvature
$\Omega_{\rm m}$	energy density parameter for matter
$\Omega_{\rm r}$	energy density parameter for radiation
Ω_Λ	energy density parameter for cosmological constant
Ω_ν	energy density parameter for neutrino
$\omega_{\mu\nu}$	rotation tensor
a	scale factor
a_D	effective scale factor
$B^{(1)}$	first order scalar perturbation
$c_{\rm eff}$	effective speed of sound

Symbol	Definition
D	domain of averaging
D_{ij}	traceless derivative
d_{L}	luminosity distance
$E^{(1)}$	first order scalar perturbation
$E_{\mathrm{s}}^{(1)}$	first order scalar perturbation in the synchronous gauge
$F_i^{(1)}$	first order vector perturbation
G	Newton's gravitational constant
$g_{\mu\nu}$	metric of space-time
g_{ij}	3-dimensional spatial metric
$G_{\mu\nu}$	Einstein tensor
h	dimensionless Hubble parameter
H	Hubble expansion rate in terms of the cosmic time t
$\mathcal{H}$	Hubble expansion rate in terms of the conformal time η
H_D	effective Hubble expansion rate
$h_{ij}^{(1)}$	first order tensor perturbation
$h_{ij}^{(1)\mathrm{inv}}$	first order tensor gauge invariant variable
$h_{\mu\nu}$	projection operator
J	measure of integral
k	curvature parameter
k_0	scale to measure the dimensionless power spectrum $\mathcal{P}_\varphi$
k_{c}	cutoff in the exponential power spectrum
$\mathcal{L}_{\xi^{(k)}}$	Lie derivative generated by $\xi^{(k)}$
n_{s}	index of the power spectrum
p	pressure
p_{eff}	pressure of the effective fluid
p_{ϕ_D}	pressure of the morphon field
$\mathcal{P}_\zeta$	dimensionless power spectrum of ζ
$\mathcal{P}_\varphi$	dimensionless power spectrum of φ
q	deceleration factor
q_D	effective deceleration factor
$\langle Q \rangle_D$	kinematical backreaction term
R	Ricci scalar
R	scale of observations
$\mathcal{R}$	3-dimensional spatial Ricci scalar
R_{H}	Hubble radius
$R_{\mu\nu}$	Ricci tensor
$\mathcal{R}_{ij}$	3-dimensional spatial Ricci tensor
$S_i^{(1)}$	first order vector perturbation
t	cosmic time
$T^{(k)}$	tensor at k'th order
t_{H}	Hubble time
$T_{\mu\nu}$	energy-momentum tensor
u_μ	4-velocity
V	volume
V_D	volume of the domain D for averaging
$V_i^{(1)\mathrm{inv}}$	first order vector gauge invariant variable
w_{de}	effective equation of state of dark energy
w_{eff}	equation of state of the effective fluid
W_R^{G}	Gaussian window function
W_R^{T}	top-hat window function
z	redshift

C Abbreviations

Here we list the abbreviations appearing in the context and their meanings.

Abbreviation	Physical meaning
ADM	Arnowitt-Deser-Misner
BAO	baryonic acoustic oscillation
BBN	Big Bang nucleosynthesis
BOOMERanG	Balloon Observations Of Millimetric Extragalactic Radiation and Geophysics
CDM	cold dark matter
CfA	Center for Astrophysics
CMB	cosmic microwave background
COBE	Cosmic Background Explorer
CL	confidence level
DGP	Dvali-Gabadadze-Porrati
EdS	Einstein-de Sitter
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	general relativity
Hipparcos	High Precision Parallax Collecting Satellite
HST	Hubble Space Telescope
KKLT	Kachru-Kallosh-Linde-Trivedi
LTB	Lemaître-Tolman-Bondi
MAXIMA	Millimeter Anisotropy eXperiment IMaging Array
MD	matter-dominated
QCD	chromodynamics
RD	radiation-dominated
SN	supernova
SN Ia	Supernova of Type Ia
vDVZ	van Dam-Veltman-Zakharov
WIMP	weakly interacting massive particle
WMAP	Wilkinson Microwave Anisotropy Probe
WMAP5	WMAP 5-year experiment

D Useful quantities

Below we enumerate the quantities useful for this dissertation. We first summarize some basic cosmological parameters, most of which are from the WMAP5 experiment [10]. Then, we list the Hubble constant from SN experiments, Tully-Fisher and fundamental plane relations, which are adopted from [53].

Parameter	Value
$\rho_{\rm c}^0$	$1.879 \times 10^{-26} h^2$ kg/m^3
	$8.098 \times 10^{-47} h^2$ Gev^4
k_0	0.002 Mpc^{-1}
Mpc	3.086×10^{22} m
$R_{\rm H}$	$9.778 h^{-1}$ Gyr
$t_{\rm H}$	$2.998 h^{-1} \times 10^3$ Mpc
σ_8	0.817 ± 0.026
τ	0.084 ± 0.016
$\Omega_{\rm b}^0 h^2$	0.02265 ± 0.00059
$\Omega_{\rm CDM}^0 h^2$	0.1143 ± 0.0034
$\Omega_{\rm de}^0$	0.721 ± 0.015
Ω_k^0	$(-0.0175, 0.0085)$
$\Omega_{\rm m}^0 h^2$	0.1369 ± 0.0037
$f_{\rm NL}^{\rm equil}$	$(-151, 253)$
$f_{\rm NL}^{\rm local}$	$(-9, 111)$
h	0.701 ± 0.013
$\sum m_\nu$	< 0.61 eV
$n_{\rm s}$	$0.960^{+0.014}_{-0.013}$
$\mathcal{P}_\zeta$	$(2.457^{+0.092}_{-0.093}) \times 10^{-9}$ at k_0
t_0	13.73 ± 0.12 Gyr
$1 + w_{\rm de}^0$	$(-0.11, 0.14)$

Table 3: Basic cosmological parameters.

We list some basic cosmological parameters in two sectors here. The upper sector is for the fundamental derived parameters, and the lower sector is for the parameters observed from the WMAP5 experiment [10].

Supernova	r	H_0	σ
SN 1990O	134.7	67.3	2.3
SN 1990T	158.9	75.6	3.1
SN 1990af	198.6	75.8	2.8
SN 1991S	238.9	69.8	2.8
SN 1991U	117.1	83.7	3.4
SN 1991ag	56.0	73.7	2.9
SN 1992J	183.9	74.5	3.1
SN 1992P	121.5	64.8	2.2
SN 1992ae	274.6	81.6	3.4
SN 1992ag	102.1	76.1	2.7
SN 1992al	58.0	72.8	2.4
SN 1992aq	467.0	64.7	2.4
SN 1992au	262.2	69.4	2.9
SN 1992bc	88.6	67.0	2.1
SN 1992bg	151.4	70.6	2.4
SN 1992bh	202.5	66.7	2.3
SN 1992bk	235.9	73.6	2.6
SN 1992bl	176.8	72.7	2.6
SN 1992bo	77.9	69.7	2.4
SN 1992bp	309.5	76.3	2.6
SN 1992br	391.5	67.2	3.1
SN 1992bs	280.1	67.8	2.8
SN 1993B	303.4	69.8	2.4
SN 1993O	236.1	65.9	2.1
SN 1993ag	215.4	69.6	2.4
SN 1993ah	119.7	71.9	2.9
SN 1993ac	202.3	72.9	2.7
SN 1993ae	71.8	75.6	3.1
SN 1994M	96.7	74.9	2.6
SN 1994Q	127.8	68.0	2.7
SN 1994S	66.8	72.5	2.5
SN 1994T	149.9	71.5	2.6
SN 1995ac	185.6	78.8	2.7
SN 1995ak	82.4	80.9	2.8
SN 1996C	136.0	66.3	2.5
SN 1996bl	132.7	78.7	2.7

Table 4: Hubble constant from SN data.

Cluster/Group	r	H_0	σ
Abell 1367	89.2	75.2	12.5
Abell 0262	66.7	70.9	11.8
Abell 2634	114.9	77.7	12.4
Abell 3574	62.2	76.2	12.2
Abell 0400	88.4	79.3	12.6
Antlia	45.1	68.8	11.3
Cancer	74.3	67.1	11.0
Cen 30	43.2	75.8	12.8
Cen 45	68.2	70.7	11.9
Coma	85.6	83.5	13.4
ESO 50	39.5	79.8	13.0
Hydra	58.3	69.6	11.1
MDL 59	31.3	73.6	11.8
NGC 3557	38.7	85.0	14.4
NGC 0383	66.6	73.9	11.9
NGC 0507	57.3	84.9	13.5
Pavo 2	50.9	86.3	14.2
Pegasus	53.3	66.4	10.7

Table 5: Hubble constant from the I-band Tully-Fisher relation.

Cluster/Group	r	H_0	σ
GRM 15	47.4	95.6	10.0
Hydra I	49.1	82.8	8.4
Abell S753	49.7	87.5	7.9
Abell 3574	51.6	92.0	10.0
Abell 194	55.9	91.3	7.5
Abell S639	59.6	109.7	9.9
Coma	85.8	83.2	6.0
Abell 539	102.0	86.2	6.5
DC 2345-28	102.1	83.2	6.4
Abell 3381	129.8	88.9	8.3

Table 6: Hubble constant from the fundamental plane relation.

E Second order perturbed gravity

The perturbative metric for spatially flat space-time up to second order in the synchronous gauge reads

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \left(\delta_{ij} + \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)}\right)\mathrm{d}x^i\mathrm{d}x^j\right], \tag{216}$$

where

$$\begin{aligned} \gamma_{ij}^{(1)} &= -2\Psi^{(1)}\delta_{ij} + D_{ij}\chi^{(1)}, \\ \gamma_{ij}^{(2)} &= -\Psi^{(2)}\delta_{ij} + \frac{1}{2}\left(D_{ij}\chi^{(2)} + \partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + \chi_{ij}^{(2)}\right). \end{aligned}$$

In the following, we enumerate all the non-vanishing perturbative quantities up to second order. We order these perturbative terms as: (1) we firstly assemble all the terms explicitly proportional to δ_{ij}; (2) the intrinsic second order terms precede those sourced from the quadratic combinations of first order perturbations; (3) the perturbative quantities are listed in the order as $\Psi^{(1)}$, $\chi^{(1)}$, $\Psi^{(2)}$, $\chi^{(2)}$, $\chi_i^{(2)}$ and $\chi_{ij}^{(2)}$, respectively.[85]

E.1 Perturbed metric

The perturbed metric to second order,

$$\begin{aligned} g_{00} &= -a^2, \\ g_{ij} &= a^2\left[\left(1 - 2\Psi^{(1)} - \Psi^{(2)}\right)\delta_{ij} + D_{ij}\left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right) + \frac{1}{2}\left(\partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + \chi_{ij}^{(2)}\right)\right]. \end{aligned}$$

The inverse perturbed metric to second order,

$$\begin{aligned} g^{00} &= -\frac{1}{a^2}, \\ g^{ij} &= \frac{1}{a^2}\left[\left(1 + 2\Psi^{(1)} + \Psi^{(2)} + 4\left(\Psi^{(1)}\right)^2\right)\delta^{ij} - D^{ij}\left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right)\right. \\ &\qquad \left. -\frac{1}{2}\left(\partial^i\chi^{(2)j} + \partial^j\chi^{(2)i} + \chi^{(2)ij}\right) - 4\Psi^{(1)}D^{ij}\chi^{(1)} + D^i{}_k\chi^{(1)}D^{kj}\chi^{(1)}\right]. \end{aligned}$$

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[85] These results were also given in, *e.g.*, [81] and [82]. However, there are several shortcomings in their works. First, though declaring that their results are suitable for any gauge, they used physical constraints to set $\omega_i^{(1)}$, $\chi_i^{(1)}$ and $\chi_{ij}^{(1)}$ to be zero. Albeit this handling is meaningful and reasonable, their perturbations in the metric Eq. (216), strictly speaking, are still not thoroughly complete as they claimed. Second, retaining the vector perturbations $\omega^{(1)}$, $\omega^{(2)}$ and $\omega_i^{(2)}$ is not useful, as in the frequently used gauges, *e.g.*, both longitudinal and synchronous gauges, these terms are 0 by definition. Furthermore, keeping these vector perturbations greatly lengthens the expressions in final results and thus increases unnecessary complexities. Third, keeping both $\phi^{(1)}$ and $\chi^{(1)}$ in the metric Eq. (216) induces terms like $\partial^i\partial_j\phi^{(1)}D^j{}_i\chi^{(1)}$. However, these two terms cannot be non-vanishing simultaneously in both longitudinal and synchronous gauges. So in practice, one never meets terms like that. All these inconveniences almost reduce their complicated results to a rather tedious mathematical exercise. Last, but not least, typos and missing terms can be found except the trivial zeroth order expressions, even in their long review article in *Physics Report*. For all the above reasons, it is a little bit unpractical and dangerous to directly follow their results, and therefore we do not give so “general” expressions as theirs, but fix our perturbed metric in the synchronous gauge in Eq. (216) and try to stop their errors from penetrating any more.

E.2 Christoffel connection

At zeroth order,

$$\Gamma^{0(0)}_{00} = \frac{a'}{a}, \quad \Gamma^{0(0)}_{ij} = \frac{a'}{a}\delta_{ij}, \quad \Gamma^{i(0)}_{0j} = \frac{a'}{a}\delta^i{}_j. \tag{217}$$

At first order,

$$\Gamma^{0(1)}_{ij} = -2\frac{a'}{a}\Psi^{(1)}\delta_{ij} - \Psi^{(1)\prime}\delta_{ij} + \frac{a'}{a}D_{ij}\chi^{(1)} + \frac{1}{2}D_{ij}\chi^{(1)\prime}, \tag{218}$$

$$\Gamma^{i(1)}_{0j} = -\Psi^{(1)\prime}\delta^i{}_j + \frac{1}{2}D^i{}_j\chi^{(1)\prime}, \tag{219}$$

$$\begin{aligned}\Gamma^{i(1)}_{jk} = {} & -\partial_j\Psi^{(1)}\delta^i{}_k - \partial_k\Psi^{(1)}\delta^i{}_j + \partial^i\Psi^{(1)}\delta_{jk} \\ & +\frac{1}{2}\partial_j D^i{}_k\chi^{(1)} + \frac{1}{2}\partial_k D^i{}_j\chi^{(1)} - \frac{1}{2}\partial^i D_{jk}\chi^{(1)}.\end{aligned} \tag{220}$$

At second order,

$$\begin{aligned}\Gamma^{0(2)}_{ij} = {} & -\left(\frac{a'}{a}\Psi^{(2)} + \frac{1}{2}\Psi^{(2)\prime}\right)\delta_{ij} + \frac{1}{2}\frac{a'}{a}\left(D_{ij}\chi^{(2)} + \partial_i\chi^{(2)}_j + \partial_j\chi^{(2)}_i + \chi^{(2)}_{ij}\right) \\ & +\frac{1}{4}\left(D_{ij}\chi^{(2)\prime} + \partial_i\chi^{(2)\prime}_j + \partial_j\chi^{(2)\prime}_i + \chi^{(2)\prime}_{ij}\right),\end{aligned} \tag{221}$$

$$\begin{aligned}\Gamma^{i(2)}_{0j} = {} & -\frac{1}{2}\Psi^{(2)\prime}\delta^i{}_j + \frac{1}{4}\left(D^i{}_j\chi^{(2)\prime} + \partial^i\chi^{(2)\prime}_j + \partial_j\chi^{(2)i\prime} + \chi^{(2)i}{}_j{}'\right) \\ & -2\Psi^{(1)}\Psi^{(1)\prime}\delta^i{}_j + \Psi^{(1)}D^i{}_j\chi^{(1)\prime} + \Psi^{(1)\prime}D^i{}_j\chi^{(1)} - \frac{1}{2}D^{ik}\chi^{(1)}D_{kj}\chi^{(1)\prime},\end{aligned} \tag{222}$$

$$\begin{aligned}\Gamma^{i(2)}_{jk} = {} & -\frac{1}{2}\left(\partial_j\Psi^{(2)}\delta^i{}_k + \partial_k\Psi^{(2)}\delta^i{}_j - \partial^i\Psi^{(2)}\delta_{jk}\right) \\ & +\frac{1}{4}\left(\partial_j D^i{}_k\chi^{(2)} + \partial_k D^i{}_j\chi^{(2)} - \partial^i D_{jk}\chi^{(2)}\right) \\ & +\frac{1}{2}\partial_j\partial_k\chi^{(2)i} + \frac{1}{4}\left(\partial_j\chi^{(2)i}{}_k + \partial_k\chi^{(2)i}{}_j - \partial^i\chi^{(2)}_{jk}\right) \\ & -2\Psi^{(1)}\left(\partial_j\Psi^{(1)}\delta^i{}_k + \partial_k\Psi^{(1)}\delta^i{}_j - \partial^i\Psi^{(1)}\delta_{jk}\right) \\ & +\Psi^{(1)}\left(\partial_j D^i{}_k\chi^{(1)} + \partial_k D^i{}_j\chi^{(1)} - \partial^i D_{jk}\chi^{(1)}\right) \\ & +\partial_j\Psi^{(1)}D^i{}_k\chi^{(1)} + \partial_k\Psi^{(1)}D^i{}_j\chi^{(1)} - \partial_m\Psi^{(1)}D^{im}\chi^{(1)}\delta_{jk} \\ & -\frac{1}{2}D^{im}\chi^{(1)}\partial_j D_{mk}\chi^{(1)} - \frac{1}{2}D^{im}\chi^{(1)}\partial_k D_{mj}\chi^{(1)} + \frac{1}{2}D^{im}\chi^{(1)}\partial_m D_{jk}\chi^{(1)}.\end{aligned} \tag{223}$$

E.3 Ricci tensor

At zeroth order,

$$R^{(0)}_{00} = 3\left[\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right], \quad R^{(0)}_{ij} = \left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right]\delta_{ij}. \tag{224}$$

At first order,

$$
\begin{aligned}
R_{00}^{(1)} &= 3\frac{a'}{a}\Psi^{(1)\prime} + 3\Psi^{(1)\prime\prime}, && (225)\\
R_{0i}^{(1)} &= 2\partial_i\Psi^{(1)\prime} + \frac{1}{2}\partial_j D^j{}_i\chi^{(1)\prime}, && (226)\\
R_{ij}^{(1)} &= \left[-2\left(\frac{a'}{a}\right)^2\Psi^{(1)} - 2\frac{a''}{a}\Psi^{(1)} - 5\frac{a'}{a}\Psi^{(1)\prime} - \Psi^{(1)\prime\prime} + \Delta\Psi^{(1)}\right]\delta_{ij} + \partial_i\partial_j\Psi^{(1)} \\
&\quad + \left(\frac{a'}{a}\right)^2 D_{ij}\chi^{(1)} + \frac{a''}{a}D_{ij}\chi^{(1)} + \frac{a'}{a}D_{ij}\chi^{(1)\prime} + \frac{1}{2}D_{ij}\chi^{(1)\prime\prime} \\
&\quad + \frac{1}{2}\partial_k\partial_i D^k{}_j\chi^{(1)} + \frac{1}{2}\partial_k\partial_j D^k{}_i\chi^{(1)} - \frac{1}{2}\Delta D_{ij}\chi^{(1)}. && (227)
\end{aligned}
$$

At second order,

$$
\begin{aligned}
R_{00}^{(2)} &= \frac{3}{2}\frac{a'}{a}\Psi^{(2)\prime} + \frac{3}{2}\Psi^{(2)\prime\prime} + 6\frac{a'}{a}\Psi^{(1)}\Psi^{(1)\prime} + 6\Psi^{(1)}\Psi^{(1)\prime\prime} + 3\left(\Psi^{(1)\prime}\right)^2 \\
&\quad + \frac{1}{2}\frac{a'}{a}D^{ij}\chi^{(1)}D_{ji}\chi^{(1)\prime} + \frac{1}{4}D^{ij}\chi^{(1)\prime}D_{ji}\chi^{(1)\prime} + \frac{1}{2}D^{ij}\chi^{(1)}D_{ji}\chi^{(1)\prime\prime}, && (228)\\
R_{0i}^{(2)} &= \partial_i\Psi^{(2)\prime} + \frac{1}{4}\partial_j D^j{}_i\chi^{(2)\prime} + \frac{1}{4}\Delta\chi_i^{(2)\prime} + 4\Psi^{(1)\prime}\partial_i\Psi^{(1)} + 4\Psi^{(1)}\partial_i\Psi^{(1)\prime} \\
&\quad + \Psi^{(1)}\partial_j D^j{}_i\chi^{(1)\prime} + \Psi^{(1)\prime}\partial_j D^j{}_i\chi^{(1)} - \frac{1}{2}\partial_j\Psi^{(1)}D^j{}_i\chi^{(1)\prime} + \partial_j\Psi^{(1)\prime}D^j{}_i\chi^{(1)} \\
&\quad - \frac{1}{2}\partial_k D^{km}\chi^{(1)}D_{mi}\chi^{(1)\prime} - \frac{1}{2}D^{km}\chi^{(1)}\partial_k D_{mi}\chi^{(1)\prime} \\
&\quad + \frac{1}{4}D^{km}\chi^{(1)\prime}\partial_i D_{mk}\chi^{(1)} + \frac{1}{2}D^{km}\chi^{(1)}\partial_i D_{mk}\chi^{(1)\prime}. && (229)
\end{aligned}
$$

For simplicity, we split $R_{ij}^{(2)}$ into two parts: the diagonal part $R_{ij}^{(2)\mathrm{d}}$, which is proportional to δ_{ij} explicitly, and the non-diagonal part $R_{ij}^{(2)\mathrm{nd}}$.[86]

$$
\begin{aligned}
R_{ij}^{(2)\mathrm{d}} &= \Bigg[-\left(\frac{a'}{a}\right)^2\Psi^{(2)} - \frac{a''}{a}\Psi^{(2)} - \frac{5}{2}\frac{a'}{a}\Psi^{(2)\prime} - \frac{1}{2}\Psi^{(2)\prime\prime} + \frac{1}{2}\Delta\Psi^{(2)} \\
&\quad + \left(\Psi^{(1)\prime}\right)^2 + \partial_k\Psi^{(1)}\partial^k\Psi^{(1)} + 2\Psi^{(1)}\Delta\Psi^{(1)} \\
&\quad - \partial_m\Psi^{(1)}\partial_k D^{km}\chi^{(1)} - \partial_k\partial_m\Psi^{(1)}D^{km}\chi^{(1)} - \frac{1}{2}\frac{a'}{a}D^{mk}\chi^{(1)}D_{km}\chi^{(1)\prime}\Bigg]\delta_{ij}, && (230)\\
R_{ij}^{(2)\mathrm{nd}} &= \frac{1}{2}\partial_i\partial_j\Psi^{(2)} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right]\left(D_{ij}\chi^{(2)} + \partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + \chi_{ij}^{(2)}\right) \\
&\quad + \frac{1}{4}\partial_j\partial_k D^k{}_i\chi^{(2)} + \frac{1}{4}\partial_i\partial_k D^k{}_j\chi^{(2)} - \frac{1}{4}\Delta D_{ij}\chi^{(2)} - \frac{1}{4}\Delta\chi_{ij}^{(2)} \\
&\quad + \frac{1}{2}\frac{a'}{a}\left(D_{ij}\chi^{(2)\prime} + \partial_i\chi_j^{(2)\prime} + \partial_j\chi_i^{(2)\prime} + \chi_{ij}^{(2)\prime}\right)
\end{aligned}
$$

---

[86] Strictly speaking, in $R_{ij}^{(2)\mathrm{nd}}$, there are also terms proportional to δ_{ij}, once we express D_{ij} as $\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta$, *e.g.*, in term $\frac{1}{2}\partial_k\partial_m D_{ij}\chi^{(1)}D^{km}\chi^{(1)}$. This is what we mean by "explicitly".

$$\begin{aligned}
&+\frac{1}{4}\left(D_{ij}\chi^{(2)''}+\partial_i\chi_j^{(2)''}+\partial_j\chi_i^{(2)''}+\chi_{ij}^{(2)''}\right)+3\partial_i\Psi^{(1)}\partial_j\Psi^{(1)}+2\Psi^{(1)}\partial_i\partial_j\Psi^{(1)}\\
&-3\frac{a'}{a}\Psi^{(1)'}D_{ij}\chi^{(1)}+\frac{1}{2}\Psi^{(1)'}D_{ij}\chi^{(1)'}+\frac{1}{2}\partial_k\Psi^{(1)}\partial_i D^k{}_j\chi^{(1)}+\frac{1}{2}\partial_k\Psi^{(1)}\partial_j D^k{}_i\chi^{(1)}\\
&-\frac{3}{2}\partial_k\Psi^{(1)}\partial^k D_{ij}\chi^{(1)}+\Psi^{(1)}\partial_k\partial_i D^k{}_j\chi^{(1)}+\Psi^{(1)}\partial_k\partial_j D^k{}_i\chi^{(1)}-\Psi^{(1)}\Delta D_{ij}\chi^{(1)}\\
&+\partial_i\Psi^{(1)}\partial_k D^k{}_j\chi^{(1)}+\partial_j\Psi^{(1)}\partial_k D^k{}_i\chi^{(1)}+\partial_k\partial_i\Psi^{(1)}D^k{}_j\chi^{(1)}+\partial_k\partial_j\Psi^{(1)}D^k{}_i\chi^{(1)}\\
&-\frac{1}{2}D^k{}_i\chi^{(1)'}D_{kj}\chi^{(1)'}-\frac{1}{2}\partial_i D_{mj}\chi^{(1)}\partial_k D^{km}\chi^{(1)}-\frac{1}{2}\partial_j D_{mi}\chi^{(1)}\partial_k D^{km}\chi^{(1)}\\
&+\frac{1}{2}\partial_m D_{ij}\chi^{(1)}\partial_k D^{km}\chi^{(1)}-\frac{1}{2}\partial_k\partial_i D_{mj}\chi^{(1)}D^{km}\chi^{(1)}-\frac{1}{2}\partial_k\partial_j D_{mi}\chi^{(1)}D^{km}\chi^{(1)}\\
&+\frac{1}{2}\partial_k\partial_m D_{ij}\chi^{(1)}D^{km}\chi^{(1)}+\frac{1}{2}D^{km}\chi^{(1)}\partial_i\partial_j D_{km}\chi^{(1)}+\frac{1}{4}\partial_i D^{mk}\chi^{(1)}\partial_j D_{mk}\chi^{(1)}\\
&+\frac{1}{2}\partial_k D^m{}_i\chi^{(1)}\partial^k D_{mj}\chi^{(1)}-\frac{1}{2}\partial_k D^m{}_i\chi^{(1)}\partial^m D_{kj}\chi^{(1)}. \qquad (231)
\end{aligned}$$

E.4 Ricci scalar

At zeroth order

$$R^{(0)}=\frac{6}{a^2}\frac{a''}{a}. \qquad (232)$$

At first order

$$R^{(1)} = \frac{1}{a^2}\left(-18\frac{a'}{a}\Psi^{(1)'}-6\Psi^{(1)''}+4\Delta\Psi^{(1)}+\partial_k\partial^i D^k{}_i\chi^{(1)}\right). \qquad (233)$$

At second order,

$$\begin{aligned}
R^{(2)} = \frac{1}{a^2}\Bigg(&-9\frac{a'}{a}\Psi^{(2)'}-3\Psi^{(2)''}+2\Delta\Psi^{(2)}+\frac{1}{2}\partial_k\partial_i D^{ki}\chi^{(2)}\\
&-12\Psi^{(1)}\Psi^{(1)''}-36\frac{a'}{a}\Psi^{(1)}\Psi^{(1)'}+6\partial_k\Psi^{(1)}\partial^k\Psi^{(1)}+16\Psi^{(1)}\Delta\Psi^{(1)}\\
&+4\Psi^{(1)}\partial_i\partial_k D^{ki}\chi^{(1)}-2\partial_i\partial_k\Psi^{(1)}D^{ki}\chi^{(1)}\\
&-3\frac{a'}{a}D^{ik}\chi^{(1)}D_{ki}\chi^{(1)'}-\frac{3}{4}D^{ik}\chi^{(1)'}D_{ki}\chi^{(1)'}-D^{ik}\chi^{(1)}D_{ik}\chi^{(1)''}\\
&-2\partial_k\partial^i D_{mi}\chi^{(1)}D^{km}\chi^{(1)}+\Delta D_{im}\chi^{(1)}D^{mi}\chi^{(1)}-\partial_k D^{km}\chi^{(1)}\partial^i D_{mi}\chi^{(1)}\\
&+\frac{3}{4}\partial^i D^{km}\chi^{(1)}\partial_i D_{mk}\chi^{(1)}-\frac{1}{2}\partial_k D^m{}_i\chi^{(1)}\partial_m D^{ki}\chi^{(1)}\Bigg). \qquad (234)
\end{aligned}$$

E.5 Einstein tensor

At zeroth order,

$$G^{0(0)}_{\ 0}=-\frac{3}{a^2}\left(\frac{a'}{a}\right)^2,\quad G^{i(0)}_{\ j}=\frac{1}{a^2}\left[\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\delta^i{}_j. \qquad (235)$$

At first order,

$$G^{0(1)}{}_{0} = \frac{1}{a^2}\left(6\frac{a'}{a}\Psi^{(1)\prime} - 2\Delta\Psi^{(1)} - \frac{1}{2}\partial_k\partial^i D^k{}_i\chi^{(1)}\right), \tag{236}$$

$$G^{0(1)}{}_{i} = \frac{1}{a^2}\left(-2\partial_i\Psi^{(1)\prime} - \frac{1}{2}\partial_k D^k{}_i\chi^{(1)\prime}\right), \tag{237}$$

$$G^{i(1)}{}_{0} = \frac{1}{a^2}\left(2\partial^i\Psi^{(1)\prime} + \frac{1}{2}\partial^k D^i{}_k\chi^{(1)\prime}\right), \tag{238}$$

$$\begin{aligned} G^{i(1)}{}_{j} = \frac{1}{a^2}\Bigg[&\left(4\frac{a'}{a}\Psi^{(1)\prime} + 2\Psi^{(1)\prime\prime} - \Delta\Psi^{(1)} - \frac{1}{2}\partial_k\partial^m D^k{}_m\chi^{(1)}\right)\delta^i{}_j \\ &+\partial^i\partial_j\Psi^{(1)} + \frac{a'}{a}D^i{}_j\chi^{(1)\prime} + \frac{1}{2}D^i{}_j\chi^{(1)\prime\prime} \\ &+\frac{1}{2}\partial_k\partial^i D^k{}_j\chi^{(1)} + \frac{1}{2}\partial_k\partial_j D^{ik}\chi^{(1)} - \frac{1}{2}\Delta D^i{}_j\chi^{(1)}\Bigg]. \end{aligned} \tag{239}$$

At second order,

$$\begin{aligned} G^{0(2)}{}_{0} = \frac{1}{a^2}\Bigg[&3\frac{a'}{a}\Psi^{(2)\prime} - \Delta\Psi^{(2)} - \frac{1}{4}\partial_k\partial_i D^{ki}\chi^{(2)} \\ &-3\partial_k\Psi^{(1)}\partial^k\Psi^{(1)} - 8\Psi^{(1)}\Delta\Psi^{(1)} + 12\frac{a'}{a}\Psi^{(1)}\Psi^{(1)\prime} - 3\left(\Psi^{(1)\prime}\right)^2 \\ &-2\Psi^{(1)}\partial_k\partial^i D^k{}_i\chi^{(1)} + \partial_k\partial_i\Psi^{(1)}D^{ki}\chi^{(1)} \\ &-\frac{1}{2}\Delta D_{mk}\chi^{(1)}D^{km}\chi^{(1)} + \partial_m\partial^k D_{ik}\chi^{(1)}D^{im}\chi^{(1)} + \frac{1}{2}\partial_k D^{km}\chi^{(1)}\partial^i D_{mi}\chi^{(1)} \\ &-\frac{3}{8}\partial^i D^{km}\chi^{(1)}\partial_i D_{km}\chi^{(1)} + \frac{1}{4}\partial_k D^m{}_i\chi^{(1)}\partial_m D^{ki}\chi^{(1)} \\ &+\frac{1}{8}D^{ik}\chi^{(1)\prime}D_{ki}\chi^{(1)\prime} + \frac{a'}{a}D^{ki}\chi^{(1)}D_{ik}\chi^{(1)\prime}\Bigg], \end{aligned} \tag{240}$$

$$\begin{aligned} G^{i(2)}{}_{0} = \frac{1}{a^2}\Bigg(&\partial^i\Psi^{(2)\prime} + \frac{1}{4}\partial_k D^{ki}\chi^{(2)\prime} + \frac{1}{4}\Delta\chi^{(2)i\prime} + 4\Psi^{(1)\prime}\partial^i\Psi^{(1)} + 8\Psi^{(1)}\partial^i\Psi^{(1)\prime} \\ &-\frac{1}{2}\partial_k\Psi^{(1)}D^{ki}\chi^{(1)\prime} + 2\Psi^{(1)}\partial_k D^{ki}\chi^{(1)\prime} + \Psi^{(1)\prime}\partial_k D^{ki}\chi^{(1)} - \partial_k\Psi^{(1)\prime}D^{ki}\chi^{(1)} \\ &-\frac{1}{2}\partial_k D^{km}\chi^{(1)}D^i{}_m\chi^{(1)\prime} - \frac{1}{2}\partial_k D^i{}_m\chi^{(1)\prime}D^{km}\chi^{(1)} + \frac{1}{4}\partial^i D_{mk}\chi^{(1)}D^{km}\chi^{(1)\prime} \\ &+\frac{1}{2}\partial^i D_{mk}\chi^{(1)\prime}D^{km}\chi^{(1)} - \frac{1}{2}D^{ik}\chi^{(1)}\partial_m D^m_k\chi^{(1)\prime}\Bigg), \end{aligned} \tag{241}$$

$$\begin{aligned} G^{0(2)}{}_{i} = \frac{1}{a^2}\Bigg(&-\partial_i\Psi^{(2)\prime} - \frac{1}{4}\partial_k D^k{}_i\chi^{(2)\prime} - \frac{1}{4}\Delta\chi_i^{(2)\prime} - 4\Psi^{(1)\prime}\partial_i\Psi^{(1)} - 4\Psi^{(1)}\partial_i\Psi^{(1)\prime} \\ &-\Psi^{(1)}\partial_k D^k{}_i\chi^{(1)\prime} + \frac{1}{2}\partial_k\Psi^{(1)}D^k{}_i\chi^{(1)\prime} - \Psi^{(1)\prime}\partial_k D^k{}_i\chi^{(1)} - \partial_k\Psi^{(1)\prime}D^k{}_i\chi^{(1)} \\ &+\frac{1}{2}\partial_k D^{km}\chi^{(1)}D_{im}\chi^{(1)\prime} + \frac{1}{2}\partial_k D_{im}\chi^{(1)\prime}D^{km}\chi^{(1)} \\ &-\frac{1}{4}\partial_i D_{mk}\chi^{(1)}D^{km}\chi^{(1)\prime} - \frac{1}{2}\partial_i D_{mk}\chi^{(1)\prime}D^{km}\chi^{(1)}\Bigg). \end{aligned} \tag{242}$$

As before, we also split $G^{(2)}_{ij}$ into the diagonal and non-diagonal parts: $G^{(2)\mathrm{d}}_{ij}$ and $G^{(2)\mathrm{nd}}_{ij}$.

$$\begin{aligned}
G^{i(2)\mathrm{d}}_{\ j} &= \frac{1}{a^2}\bigg[2\frac{a'}{a}\Psi^{(2)\prime} + \Psi^{(2)\prime\prime} - \frac{1}{2}\Delta\Psi^{(2)} - \frac{1}{4}\partial_k\partial_i D^{ki}\chi^{(2)} \\
&\quad -2\partial_k\Psi^{(1)}\partial^k\Psi^{(1)} - 4\Psi^{(1)}\Delta\Psi^{(1)} + \left(\Psi^{(1)\prime}\right)^2 + 8\frac{a'}{a}\Psi^{(1)}\Psi^{(1)\prime} + 4\Psi^{(1)}\Psi^{(1)\prime\prime} \\
&\quad -\partial_k\Psi^{(1)}\partial_m D^{mk}\chi^{(1)} - 2\Psi^{(1)}\partial_k\partial_i D^{ik}\chi^{(1)} + \partial_k\partial^l D_{ml}\chi^{(1)} D^{km}\chi^{(1)} - \frac{1}{2}\Delta D_{ml}\chi^{(1)} D^{ml}\chi^{(1)} \\
&\quad +\frac{1}{2}\partial^k D_{km}\chi^{(1)}\partial_l D^{ml}\chi^{(1)} - \frac{3}{8}\partial^l D_{km}\chi^{(1)}\partial_l D^{km}\chi^{(1)} + \frac{1}{4}\partial_k D^{ml}\chi^{(1)}\partial_m D_{kl}\chi^{(1)} \\
&\quad +\frac{a'}{a}D^{mk}\chi^{(1)} D_{km}\chi^{(1)\prime} + \frac{3}{8}D^{mk}\chi^{(1)\prime} D_{mk}\chi^{(1)\prime} + \frac{1}{2}D^{mk}\chi^{(1)} D_{mk}\chi^{(1)\prime\prime}\bigg]\delta^i_{\ j}, \qquad (243)
\end{aligned}$$

$$\begin{aligned}
G^{i(2)\mathrm{nd}}_{\ j} &= \frac{1}{a^2}\bigg[\frac{1}{2}\partial^i\partial_j\Psi^{(2)} + \frac{1}{2}\frac{a'}{a}\left(D^i_{\ j}\chi^{(2)\prime} + \partial^i\chi^{(2)\prime}_j + \partial_j\chi^{i(2)\prime} + \chi^{i(2)\prime}_{\ j}\right) \\
&\quad +\frac{1}{4}\partial_k\partial^i D^k_{\ j}\chi^{(2)} + \frac{1}{4}\partial_k\partial^j D^k_{\ i}\chi^{(2)} - \frac{1}{4}\Delta D^i_{\ j}\chi^{(2)} - \frac{1}{4}\Delta\chi^{i(2)}_{\ j} \\
&\quad +\frac{1}{4}\left(D^i_{\ j}\chi^{(2)\prime\prime} + \partial^i\chi^{(2)\prime\prime}_j + \partial_j\chi^{i(2)\prime\prime} + \chi^{i(2)\prime\prime}_{\ j}\right) + 3\partial^i\Psi^{(1)}\partial_j\Psi^{(1)} + 4\Psi^{(1)}\partial^i\partial_j\Psi^{(1)} \\
&\quad +\frac{1}{2}\partial_k\Psi^{(1)}\partial^i D^k_{\ j}\chi^{(1)} + \frac{1}{2}\partial_k\Psi^{(1)}\partial_j D^{ki}\chi^{(1)} - \frac{3}{2}\partial_k\Psi^{(1)}\partial^k D^i_{\ j}\chi^{(1)} \\
&\quad +2\Psi^{(1)}\partial_k\partial^i D^k_{\ j}\chi^{(1)} + 2\Psi^{(1)}\partial_k\partial_j D^{ki}\chi^{(1)} - 2\Psi^{(1)}\Delta D^i_{\ j}\chi^{(1)} - \Delta\Psi^{(1)} D^i_{\ j}\chi^{(1)} \\
&\quad +\partial^i\Psi^{(1)}\partial_k D^k_{\ j}\chi^{(1)} + \partial_j\Psi^{(1)}\partial_k D^{ki}\chi^{(1)} + \partial_k\partial^i\Psi^{(1)} D^k_{\ j}\chi^{(1)} \\
&\quad +2\frac{a'}{a}\Psi^{(1)\prime} D^i_{\ j}\chi^{(1)} + 2\frac{a'}{a}\Psi^{(1)} D^i_{\ j}\chi^{(1)\prime} + \frac{1}{2}\Psi^{(1)\prime} D^i_{\ j}\chi^{(1)\prime} + \Psi^{(1)} D^i_{\ j}\chi^{(1)\prime\prime} \\
&\quad -\frac{1}{2}\partial^i D_{mj}\chi^{(1)}\partial_k D^{km}\chi^{(1)} - \frac{1}{2}\partial_j D^i_{\ m}\chi^{(1)}\partial_k D^{km}\chi^{(1)} + \frac{1}{4}\partial^i D^{mk}\chi^{(1)}\partial_j D_{mk}\chi^{(1)} \\
&\quad +\frac{1}{2}\partial_m D^i_{\ j}\chi^{(1)}\partial_k D^{km}\chi^{(1)} + \frac{1}{2}\partial_k D^{mi}\chi^{(1)}\partial^k D_{mj}\chi^{(1)} - \frac{1}{2}\partial_k D^{mi}\chi^{(1)}\partial_m D^k_{\ j}\chi^{(1)} \\
&\quad -\frac{1}{2}\partial_k\partial^i D_{mj}\chi^{(1)} D^{km}\chi^{(1)} - \frac{1}{2}\partial_k\partial_j D^i_{\ m}\chi^{(1)} D^{km}\chi^{(1)} + \frac{1}{2}\partial_k\partial_m D^i_{\ j}\chi^{(1)} D^{km}\chi^{(1)} \\
&\quad +\frac{1}{2}\partial^i\partial_j D_{km}\chi^{(1)} D^{km}\chi^{(1)} - \partial_m\partial_k D^m_j\chi^{(1)} D^{ki}\chi^{(1)} + \frac{1}{2}\Delta D_{kj}\chi^{(1)} D^{ki}\chi^{(1)} \\
&\quad -\frac{a'}{a}D^{ki}\chi^{(1)} D_{kj}\chi^{(1)\prime} - \frac{1}{2}D^{ki}\chi^{(1)\prime} D_{kj}\chi^{(1)\prime} - \frac{1}{2}D^{ki}\chi^{(1)} D_{kj}\chi^{(1)\prime\prime}\bigg]. \qquad (244)
\end{aligned}$$

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