NTU prenucleoli

Inauguraldissertation zur Erlangung des Grades
eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.)
an der Fakultät für Wirtschaftswissenschaften
der Universität Bielefeld

vorgelegt von

Dipl.-Wirt.Math. Sven Klauke
Institut für Mathematische Wirtschaftsforschung
der Universität Bielefeld

Bielefeld, 2002

1. Gutachter: Prof.Dr. Joachim Rosenmüller

2. Gutachter: Prof.Dr. Peter Sudhölter

Tag der mündlichen Prüfung: 13. Januar 2003

Acknowledgements

This thesis was authored while I was a member of the Institute of Mathematical Economics (IMW) at the University of Bielefeld, at first also as a member of the graduate college ("Graduiertenkolleg") "Mathematical Economics" of the DFG ("Deutsche Forschungsgemeinschaft"), whose financial support is gratefully acknowledged, and thereafter as a Research Assistant.

My supervisor Prof. Dr. Joachim Rosenmüller deserves a special thank for reviving my interest in game theory and for encouraging me during the work on this thesis. To him and to Prof. Dr. Peter Sudhölter I am very grateful for many fruitful discussions about game theory in general and about my ideas. The very first of those ideas was the result of a seminar on NTU games that they gave at the University of Bielefeld.

A very special thank goes to my wife Diana for her patient attendances to those many private lectures and for her neverending confidence and encouragement. Also for her careful proofreading I am very grateful.

And last but not least at all my parents deserve my thank for their generous support and patience during the last years.

Contents

List of Figures

List of Tables

1 Introduction

Cooperative games or coalitional games (with or without transferable utility) deal with situations is which cooperation among some or all members of a group of players or agents is worthwile in the sense that it generates payoffs to cooperating groups. Apart from modelling the cooperation possibilities and the determination of the payoffs the main question thereby is to find allocation methods of payoffs to individuals that are fair is some yet to be specified sense.

In the case where utility is assumed to be freely transferable among individuals the coalitional game is exhaustively described by specifying a payoff for each group that can eventually form. If we denote the non-empty finite set of all players by N then possible groups are the subsets $\{S\,|S \subseteq N\}$, usually referred to as coalitions. A coalitional game with transferable utility (a TU game) can then be defined as a pair (N, v), where $v : \{S\,|S \subseteq N\} \to \mathbb{R}$, such that $v(\emptyset) = 0$, is the function that assigns to each coalition its payoff as a real number. If it so happens that, for example, all players agree to cooperate, thus N – the grand coalition – forms, the the value $v(N)$ is to be devided between them in a "fair" fashion. Depending on how the principle of fairness of this allocation is actually modelled various so-called solution concepts have been defined since the first introduction of coalitional games by John von Neumann and Oskar Morgenstern in [vNM44].

A more general model of such situations of cooperation is given by coalitional games with *non*-transferable utility (NTU games) as introduced by Robert

J. Aumann and Bezalel Peleg in [AP60]. In this framework we assume that coalitions are not able to freely transfer utility among their members and thus the specification of the overall (or sum) payoff to coalitions is no longer appropriate. We rather have to determine the set of all possible payoffs for every coalition, thus a coalitional game with non-transferable utility is given by a pair (N, V), where N is the grand coalition as before and $V : \{S \,|\, S \subseteq N\} \rightrightarrows \mathbb{R}^N$ is a correspondence that defines a set $V(S)$ of possible payoffs for every coalition $S \subseteq N$. Again the task is to develop solution concepts that yield fair allocations, i.e. a (possibly empty or single-valued) subset of $V(N)$, the possible payoffs for the grand coalition.

Many convincing solution concepts for TU games have been developed which all consider different types of fairness notions. As – technically – the class of all TU games is a subclass of the class of all NTU games[1] it seems natural to try to extend the definitions of known solution concepts from the TU case to the NTU case. This has been done for some solution concepts while other still withstand from a satisfactory extension.

The concept of the Core, that can be traced back to the work of Edgeworth ([Edg81]), has apparently the most canonical extension to the NTU case. Even the results concerning the conditions for the existence, i.e. non-emptiness, of it are quite similar in nature (see Bondareva ([Bon63]) and Shapley ([Sha67]) for the TU case and Scarf ([Sca67]) and Billera ([Bil70b], [Bil71]) for the NTU case). The Shapley value, introduced by Lloyd S. Shapley in [Sha53], has already two main extensions to the NTU case, one given by

[1] That means that every TU game can be as well formulated as an NTU game.

Shapley himself in [Sha69] and the other by Michael Maschler and Guillermo Owen in [MO89] and [MO92]. Various approaches to NTU bargaining sets (which concept has also many variants in the TU context, see, for example, [AM64], [DM67], and [Mas92] for an extensive survey and many references to the existing literature) were given by Peleg ([Pel63]), Billera ([Bil70a]), Asscher ([Ass76] and [Ass77]), and Yarom ([Yar85]). The kernel ([DM65]) and the prekernel ([MPS72]) for TU games were introduced as solution concepts that were originally meant to help to understand properties of the bargaining sets. Suggestions for an extension of these concepts to NTU games can be found in [OZ00] and [SS98].

The kernel and the prekernel for TU games use the auxiliary concept of the *excess* of a coalition. This is also the basis for that solution concept that will be the main subject of the present thesis, which is intended to serve as a contribution to the extension of the (pre-)nucleolus of David Schmeidler ([Sch69]) to NTU games. This task has previously been undertaken by Ehud Kalai in [Kal75] by extending the concept of the excess of a coalition, which is a function that for each coalition S maps payoffs to the "satisfaction" of the coalition with its share of the payoff. It is widely accepted that the canonical candidate for this function is[2] $e\,(S,x) = v(S) - x(S)$ for every coalition $S \subseteq N$ and every payoff vector $x \in \mathbb{R}^N$. Thus a negative excess means satisfaction of coalition S with the payoff x (S gets more than it could achieve by its own means) while a positive excess means dissatisfaction. In other words, the excess of a coalition is determined as the difference of the proposal ($x(S)$) to what the coalition can guarantee itself without the cooperation of other

[2] with the notational convention that $x(S) := \sum_{i \in S} x_i \quad \forall S \subseteq N, \forall x \in \mathbb{R}^N$.

players ($v(S)$). Roughly speaking, the nucleolus is that point that minimizes the excess of each coalition (thus maximizes its satisfaction) as much as possible whereat the least satisfied coalitions are always treated favoredly.

It is immediately clear from the definition that this simple form of an excess function cannot easily be reformulated within the NTU context where the possibilities for coalitions are described as sets rather than as real numbers. Kalai formulated properties that excess functions for NTU games should satisfy in order to measure satisfaction of coalitions appropriately. The result of this approach is a class of excess functions each of its members yielding an NTU nucleolus. These nucleoli share some properties with the TU nucleolus while fail to satisfy others although Kalai's properties for excess functions can be seen as extensions of the properties of the TU excess function.

In an attempt to define other excess functions for NTU games that yield NTU prenucleoli that share as many properties with the TU prenucleolus as possible we formulate different requirements for NTU excess functions to satisfy and thereby describe (actually axiomatize, i.e. uniquely characterize it) another class of NTU excess functions (called β-excess functions) yielding different NTU prenucleoli (called β-prenucleoli). By this approach we preserve a good deal of the properties of the TU prenucleolus like single-valuedness and validness of the Kohlberg criterion ([Koh71]) which is a quite elegant characterization of the nucleolus that reveals a further insight into the "minimization of dissatisfaction"-property.

In a recent paper, Chang and Chen ([CC02]) consider a class of so-called affine excess functions and its subclass of C-excess functions. The latter is a superclass of the β-excess functions and contains (like the class of Kalai) excess functions that do not necessarily coincide with the TU excess on TU games. They prove single-valuedness and validness of the Kohlberg criterion for the resulting prenucleoli.

Besides the results that are valid for this entire class of NTU prenucleoli we then focus an a special member of this class for which we are able to show some additional properties that increase the resemblance to the TU prenucleolus like covariance or the reduced game property. The first postulates that if the payoffs of the game are altered by a linear transformation of the utility scales then the solution concept should behave accordingly while the latter is a form of stability of solution concepts that covers situations in which (proper) subcoalitions look at the outcome of the solution on a reduced game with them as the grand coalition and thereby might find a reason to withdraw from N.

It is possible to define also the core of a TU game by using the excess functions for TU games. This is also true for the NTU excess functions of Kalai with respect to the NTU core. The new NTU excess functions that shall be introduced in the present thesis can not serve to define the NTU core in the same fashion but yield a different "core"-concept for NTU games. We are able to identify a condition that yields non-emptiness for this concept and also for the NTU core that is similar but different to the conditions for non-emptiness of the TU core and the NTU core.

This thesis is organized as follows. We provide the basic definitions concerning TU games and NTU games, included to prenucleolus concept, along with some notational agreements in Chapter 2. The approach of Kalai is introduced and discussed in Chapter 3 where we recall the main definitions and results. We also provide an example which serves to demonstrate some properties of Kalai's NTU prenucleoli that we regard as drawbacks and that motivated our new approach. This approach is introduced in Chapter 4. In that chapter we define the concept of β-excess functions and the according β-prenucleoli for a subclass of all NTU games. Also the basic results like single-valuedness, the Kohlberg criterion, and continuity of the β-prenucleoli are provided in Chapter 4, which finally contains the definition of the new set-valued solution concept called β-core, which we already mentioned, and an externsion of the β-prenucleoli to a more general class of NTU games. Chapter 5 is devoted to the analysis of the special member of the class of all β-prenucleoli for which further properties (reduced game property, monotonicity, core-inclusion, covariance) are shown. Also the β-core is reconsidered in connection with conditions that yield non-emptiness.

2 Definitions and Notations

Let us first agree on some notation. The set of all subsets of a set X is denoted by 2^X. Let U be the (finite or infinite) universe of all players and let $\emptyset \neq N \subseteq U$ be a finite subset. The subsets $S \subseteq N$ are called coalitions, N is called the grand coalition, 2^N is the set of all coalitions of N. $\mathbb{R}^N$ is the set of all functions from N to $\mathbb{R}$. Every $\mathbb{R}^N$ will be identified with $\mathbb{R}^{|N|}$; we will therefore write x_i instead of $x\left(i\right)$ for $x \in \mathbb{R}^N$ and $i \in N$. The set of all non-negative vectors in $\mathbb{R}^N$ is denoted by $\mathbb{R}^N_+$, while $\mathbb{R}^N_{++}$ is the set of all strictly positive vectors. If $S \in 2^N$ is a coalition and $x \in \mathbb{R}^N$, we denote by x_S or by $x|_S$ the projection of x on $\mathbb{R}^S := \left\{x \in \mathbb{R}^N \left| x_i = 0 \quad \forall i \notin S \right.\right\}$. The latter notation of a projection is used in cases where there are already indices attached to x to improve the readability.

If $x, y \in \mathbb{R}^N$, then $x \geq y$ means $x_i \geq y_i$ for every $i \in N$, while $x > y$ denotes the case where $x_i > y_i$ for every $i \in N$. The scalar product of x and y is denoted by $\langle x, y\rangle$, i.e. $\langle x, y\rangle := \sum_{i \in N} x_i y_i, x, y \in \mathbb{R}^N$. If $x \in \mathbb{R}^N$ is a vector and $r \in \mathbb{R}$ is some real number, then $rx := \left(rx_i\right)_{i \in N}$. The componentwise multiplication of two vectors $x, y \in \mathbb{R}^N$ is denoted by xy, i.e. $xy := \left(x_i y_i\right)_{i \in N}, x, y \in \mathbb{R}^N$. Of course $\dfrac{x}{y}$ means $\left(\dfrac{x_i}{y_i}\right)_{i \in N} \quad \forall x, y \in \mathbb{R}^N$, such that $y_i \neq 0 \quad \forall i \in N$. The vector in $\mathbb{R}^N$ that has components all equal to 1 is denoted by 1_N and its projection $1_N|_S$ by 1_S. Hence 1_S is the characteristic vector of coalition $S \in 2^N$.

For $\lambda \in \mathbb{R}^N$ and $A \subseteq \mathbb{R}^N$, λA is the set $\left\{\lambda a \left| a \in A\right.\right\}$, whereas rA for $r \in \mathbb{R}$ is the set $\left\{ra \left| a \in A\right.\right\}$. $\lambda + A$ denotes the set $\left\{\lambda + a \left| a \in A\right.\right\}$. The relative interior of A is denoted by $\texttt{int}(A)$ and its boundary by ∂A. If $x \in A$ and $y \leq x$ implies $y \in A$ then A is called comprehensive. We denote the set $A \cap \mathbb{R}^N_+$ by A^+.

The basic definitions concerning coalitional games are now recalled. Most of them are the standard definitions that are used in the literature.

Definition 2.1

A **coalitional game with transferable utility (TU game)** *is a pair (N, v), where $N \subseteq U$ is the set of players and $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$, is the coalitional function that assigns to each coalition $S \in 2^N$ its worth $v(S)$. Let Γ^{TU} be the class of all TU games.*

For every game $(N, v) \in \Gamma^{TU}$ let

$$I^*(N, v) = \left\{x \in \mathbb{R}^N \left| x(N) = v(N)\right.\right\} \text{ be the set of preimputations.}$$

Definition 2.2

Let $\Gamma \subseteq \Gamma^{TU}$ be a class of games. A **solution concept** *on Γ is a mapping*

$$\begin{aligned} \sigma : \Gamma &\to \bigcup_{(N,v)\in\Gamma} 2^{I^*(N,v)} \\ \sigma(N, v) &\subseteq I^*(N, v), \end{aligned}$$

that assigns to each game $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of the set of preimputations $I^*(N, v)$.

Definition 2.3

A **coalitional game with non-transferable utility (NTU game)** *is a pair* (N, V)*, where* $V : 2^N \to 2^{\mathbb{R}^N}, V(S) \subsetneq \mathbb{R}^S$*, is the coalitional function that assigns to each coalition* $S \in 2^N$ *a proper subset of* $\mathbb{R}^S$ *that is*

- *non-empty,*
- *closed,*
- *comprehensive, and that further satisfies*
- $V(S) \cap \mathbb{R}^S_+$ *is non-empty and compact.*

$V(S)$ *are those outcomes that are attainable to* S *through cooperation. Let* Γ^{NTU} *be the class of all NTU games. If* $x \in V(S)$ *we say that* S *is effective for* x*. We make the assumption that* $\max\{x_i \,|x \in V(\{i\})\} = 0$ *for every player* $i \in N$*.*

Remark 2.4

Another possible way to define an NTU game is to require that every $V(S)$ *is a subset of an* $|S|$*-dimensional space. We think that defining a game as a correspondence in this way suits better the purposes of this thesis. Our definition is also used in [Ros81] and [Kal75].*

A subclass of Γ^{NTU} is the class of all **hyperplane games**, denoted by Γ^H. In a hyperplane game, every $V(S)$ is a halfspace (relative to $\mathbb{R}^S$), given by

$$V(S) = \left\{x \in \mathbb{R}^S \,\middle|\, \langle x_S, p_V^S \rangle \leq c_V^S \right\},$$

where $p_V^S \in \mathbb{R}_{++}^S$ and $c_V^S \in \mathbb{R}_+$. $\left(p_V^S\right)_{S\in 2^N}$ and $\left(c_V^S\right)_{S\in 2^N}$ are then referred to as the representation of the game $(N,V) \in \Gamma^H$. Of course, the representation is not unique; in fact, if p_V^S and c_V^S represent $V(S)$, then so do $\frac{1}{r}p_V^S$ and rc_V^S for every $r \in \mathbb{R}_{++}$. When it does not cause confusion we will omit the index V if that simplifies the notation.

Those NTU games, for which $V(N)$ is a halfspace while every other $V(S), S \neq N$, is arbitrary (but satisfies of course the conditions of Definition 2.3), we will call **quasi hyperplane games** and denote by Γ^{qH} the class of all those games.

If $(N,V) \in \Gamma^{qH}$ is a quasi hyperplane game with $p_V^N = r1_N$ for some $r \in \mathbb{R}_{++}$, i.e. $\partial V(N)$ is parallel to the boundary of the unit simplex in $\mathbb{R}^N$, then (N,V) is called **simplex game**[3]. For every simplex game (N,V) we assume without loss of generality that $r = 1$, i.e. $p_V^N = 1_N$.

If $(N,V) \in \Gamma^{NTU}$ is an NTU game and $\lambda \in \mathbb{R}_{++}^N$, then we call (N,V) and the game $(N,\lambda V)$ **strategically equivalent under a linear transformation of utility**. Operations on games are always meant to be coalitionwise, thus the game $(N,\lambda V)$ is given by $(\lambda V)(S) = \lambda V(S)$ for every coalition $S \in 2^N$. When $(N,V) \in \Gamma^H$ is a hyperplane game then for every $\lambda \in \mathbb{R}_{++}^N$ the normal vectors $p_{\lambda V}^S$ $\left(S \in 2^N\right)$ that define the respective halfspaces are given by $p_{\lambda V,i}^S = \begin{cases} \dfrac{p_{V,i}^S}{\lambda_i} & \text{, if} \quad i \in S \\ 0 & \text{, if} \quad i \notin S \end{cases} \qquad \forall S \in 2^N$.

[3]The terms "quasi hyperplane game" and "simplex game" are apparently non-standard but are chosen in accordance with the geometrical contemplation.

The class Γ^{TU} of all TU games can be seen as a subclass of Γ^{NTU} as the following definition formalizes.

Definition 2.5

If $(N, v) \in \Gamma^{TU}$ is a TU game, then denote by $(N, V^v) \in \Gamma^H$ its derived NTU hyperplane game, i.e. (N, V^v) is represented by

$$p^S_{V^v} = 1_S,$$
$$c^S_{V^v} = v(S)$$

for every $S \in 2^N$. Of course, (N, V^v) is a simplex game.

If $(N, V) \in \Gamma^H$ is a hyperplane game, such that $p^S_V = r_S 1_S$ for some $r_S \in \mathbb{R}^S_{++}$ holds true for all $S \in 2^N$, then denote by $\left(N, v^V\right) \in \Gamma^{TU}$ its derived TU game, i.e.

$$v^V(S) = \frac{c^S}{r_S} \quad \forall S \in 2^N.$$

Definition 2.6 (monotonic NTU games)

Let $(N, V) \in \Gamma^{NTU}$ be an NTU game. V is **monotonic** *if for all coalitions $S, T \in 2^N$ with $\emptyset \neq S \subsetneq T$ and all $x \in V(S)$, there exists $y \in V(T)$ with $y_S \geq x$.*

An equivalent formulation of Definition 2.6 is to say that V is monotonic if the projection of $V(T)$ on $\mathbb{R}^S$ contains $V(S)$, which means that for every payoff vector that a coalition S is effective for it is possible to assign payoffs to the players in $T \setminus S$ such that the coalition T is effective for the resulting payoff vector. This definition of monotonicity disregards the players in $T \setminus S$ in the sense that it might be required to allocate payoffs to them that are not individual rational in order to find a $y \in V(T)$ with $y_S \geq x$. If we want to exclude such cases we get a stronger form of monotonicity which is called *individual superadditivity* and defined as follows.

Definition 2.7 (individual superadditive NTU games)
Let $(N, V) \in \Gamma^{NTU}$ be an NTU game. V is **individual superadditive** *if for every player $i \in N$, every coalition $\emptyset \neq S \in 2^{N \setminus \{i\}}$ and every $x \in V(S)$ there exists $y \in V(S \cup \{i\})$ with $y_S \geq x$ and $y_i \geq 0$.*

By the comprehensiveness assumption we can replace the last two inequalities by equations and hence individual superadditivity requires that $V(S) \subseteq V(S \cup \{i\})$.

Definition 2.8
A **solution concept** *on a class $\Gamma \subseteq \Gamma^{NTU}$ is a mapping*

$$\sigma : \Gamma \to \bigcup_{(N,V) \in \Gamma} 2^{V(N)}$$

$$\sigma(N, V) \subseteq V(N),$$

that assigns to each game $(N, V) \in \Gamma$ a subset $\sigma(N, V)$ of the set of outcomes $V(N)$ for which the grand coalition is effective.

The following properties are usually regarded as minimal requirements that solution concepts should satisfy.

Definition 2.9

A solution concept σ on Γ^{NTU} is called

- **efficient**, *if $\sigma(N,V) \subseteq \partial V(N)$ for every game $(N,V) \in \Gamma^{NTU}$,*
- **covariant**, *if for all $\lambda \in \mathbb{R}^N_{++}$ and all $x \in \sigma(N,V)$, $\lambda x \in \sigma(N, \lambda V)$ holds true for every game $(N,V) \in \Gamma^{NTU}$, and*
- **anonymous**, *if for every game $(N,V) \in \Gamma^{NTU}$ and for every injection $\pi : N \to U$ it follows that $\sigma(\pi N, \pi V) = \pi\sigma(N,V)$. The game $(\pi N, \pi V)$ is defined by $\pi V(\pi S) = V(S)$ for every $S \in 2^N$.*[4]

The main solution concept in this thesis is the (pre-)nucleolus. Therefore we will now introduce the concept of the general nucleolus. For this concept see [MPT92] and [Pel88]. Every nucleolus considered in this thesis, e.g. the (TU) prenucleolus, the Kalai (NTU) nucleoli or the (NTU) β-prenucleoli, are special cases of this general concept. Theorems about existence and uniqueness of these solution concepts are more or less simple corollaries of theorems that are valid for the general nucleolus.

Definition 2.10 (General Nucleolus)

Let X be a finite or infinite set (endowed with a topology), let D be a finite set and let $H := \{h_i\}_{i \in D}, h_i : X \to \mathbb{R} \quad \forall i \in D$, be a finite family of real-valued functions on X. Let $d := |D| < \infty$.

[4] This definition of anonymity is analog to the respective Definition in the TU case, see [Pel88]. Note that we identify both $\mathbb{R}^N$ and $\mathbb{R}^{\pi N}$ with $\mathbb{R}^{|N|}$.

Let $\Theta : X \to \mathbb{R}^d$ *be defined by*

$$\Theta_i(x) := \max\{\min\{h_j(x)\,|j \in S\}\,|S \subseteq D, |S| = i\} \quad (1 \le i \le d, x \in X).$$

Thus Θ *arranges the components of* $(h_i(x))_{i\in D}$ *non-increasingly.*

The set

$$\mathcal{N}(H,X) := \{x \in X\,|\Theta(x) \le_{lex} \Theta(y) \quad \forall y \in X\}$$

is the **general nucleolus** *of* X *w.r.t.* H*.*

Here $\le_{lex}$ denotes the lexicographic order of $\mathbb{R}^d$. That means that $x \le_{lex} y$ if either $x = y$ or there exists a number $k \in D$ with $x_i = y_i$ for all $1 \le i \le k-1$ and $x_k < y_k$.

The use of the lexicographic order on non-increasingly sorted vectors reflects the special treatment of those coalitions with highest excess resp. lowest satisfaction by the nucleolus.

The following results concerning the general nucleolus and the prenucleolus of TU games are taken from [Pel88].

Theorem 2.11

- *If* X *is non–empty and compact and* h_i *is continuous for every* $i \in D$*, then* $\mathcal{N}(H,X) \neq \emptyset$*.*
- *If* X *is convex and* h_i *is convex and continuous for every* $i \in D$*, then*
  1. $\mathcal{N}(H,X)$ *is convex and*

2. $h_i(x) = h_i(y) \quad \forall x, y \in \mathcal{N}(H, X), \forall i \in D.$

Proof:

See [Pel88], Theorem 5.1.3. and Theorem 5.1.5. ■

We now introduce the prenucleolus of TU games, a variation of Schmeidler's original concept, the nucleolus ([Sch69]), that is derived by waiving the requirement of individual rationality.

Definition 2.12 (Prenucleolus of TU games)

Let

$$\begin{aligned} \mathcal{P}N : \Gamma^{TU} &\rightarrow \bigcup_{(N,v)\in\Gamma^{TU}} 2^{I^*(N,v)} \\ \mathcal{P}N\,(N,v) &\subseteq I^*\,(N,v) \quad \forall\,(N,v) \in \Gamma^{TU} \end{aligned}$$

be defined by

$$\mathcal{P}N\,(N,v) := \mathcal{N}\left(\left(v(S) - \bullet(S)\right)_{S\in 2^N}, I^*\,(N,v)\right), (N,v) \in \Gamma^{TU}.$$

Then $\mathcal{P}N$ is called the **prenucleolus of TU games***. Let*

$$e\,(S,x,v) := v(S) - x(S) \quad \forall\,(N,v) \in \Gamma^{TU}, x \in \mathbb{R}^N, S \in 2^N,$$

denote the **excess of coalition S at x***.*

Theorem 2.13

$|\mathcal{P}N\,(N,v)| = 1 \quad \forall\,(N,v) \in \Gamma^{TU}.$

Proof:

Let $(N,v) \in \Gamma^{TU}$ be a game. It is easily checked that $I^*(N,v)$ is non-empty and convex and that the excess function $e(S,x,v) = v(S) - x(S), x \in I^*(N,v), S \in 2^N$, is continuous and convex (even affine linear).

Of course, $I^*(N,v)$ is not compact, thus the first part of Theorem 2.11 does not apply directly to show non-emptiness of $\mathcal{P}N(N,v)$. But let $x \in I^*(N,v)$ and define $t := \max\left\{e(S,x,v) \,\middle|\, S \in 2^N\right\}$. Let

$$X := \left\{y \in I^*(N,v) \,\middle|\, e(S,x,v) \leq t \quad \forall S \in 2^N\right\},$$

then X is non-empty ($x \in X$), convex and compact and

$$\mathcal{P}N(N,v) = \mathcal{N}\left((e(S,\bullet,v))_{S\in 2^N}, X\right) \neq \emptyset$$

.

The second part of Theorem 2.11 ensures $e(S,x,v) = e(S,y,v)$ for all $S \in 2^N$ and all $x,y \in \mathcal{P}N(N,v)$. From this $x = y$ follows, thus $|\mathcal{P}N(N,v)| = 1$. ■

Beside the definition of the prenucleolus of TU games this solution concept does also admit of an elegant description by characterizing the special state if which those coalitions are whose satisfaction at the prenucleolus point is below an arbitrarily chosen threshold. This characterization is due to Elon Kohlberg ([Koh71])[5] and can be informally described as follows. Suppose there is an imputation $x \in I^*(N,v)$ then take a look at the set $\mathcal{S}$ of those

[5]for the nucleolus, our formulation is the adoption of his results with respect to the prenucleolus

coalitions that attain maximal excess at this point. If it were possible to find a vector y sucht that $y(N) = 0$, hence y describes a reallocation, and such that $y(S) \geq 0$ for every coalition $S \in \mathcal{S}$ then none of these coalitions has a greater excess at $x + y$ then at x. If furthermore there is a coalition $\bar{S} \in \mathcal{S}$ with $y\left(\bar{S}\right) > 0$ then x can not be the prenucleolus because $\bar{S}$ can be made better off of deviating from x "in the direction given by y" without making other coalitions worse off then $\bar{S}$.

Hence the prenucleolus has the property that whenever there is $y \in \mathbb{R}^N$ sucht that $y(N) = 0$ and $y(S) \geq 0$ for every coalition S attaing maximal excess then $y(S) = 0$ follows for all those coalitions. Furthermore, this proposition is also true when we also consider the set of all coalitions that attain the second highest excess or above, third highest excess or above etc.

What we present here actually is an equivalent formulation that makes use of the concept of balancedness of coalitions and that was also described in [Koh71]. We do it this way because the concept of balancedness does also play a role in further parts of this thesis.

Let $(N, v) \in \Gamma$ be a TU game. Then

$$\mathcal{D}\left(\alpha, x, v\right) := \left\{S \in 2^N \left| e\left(S, x, v\right) \geq \alpha \right.\right\} \quad \forall \alpha \in \mathbb{R}, x \in \mathbb{R}^N,$$

denotes the set of all coalitions with excess greater than α.

A collection of coalitions $\mathcal{S} \subseteq 2^N$ is said to be **balanced**, if there exist balancing coefficients $\left(\delta_S\right)_{S \in \mathcal{S}}, \delta_S \in \mathbb{R} \quad \forall S \in \mathcal{S}$, such that

$$\sum_{S \in \mathcal{S}} \delta_S 1_S = 1_N.$$

The well-known Kohlberg characterization of the prenucleolus can now be stated as follows.

Theorem 2.14

Let $(N,v) \in \Gamma$ be a TU game and let $x \in I^*(N,v)$ be a preimputation. Then $x = \mathcal{P}N(N,v) \Leftrightarrow \mathcal{D}(\alpha,x,v)$ is balanced for all $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\alpha,x,v) \neq \emptyset$.

Proof:

See [Koh71]. Since this paper considers the nucleolus of a TU game, see also [Pel88] for the proof concerning the prenucleolus. ■

For shortage of notation we will say that Theorem 2.14 means that for $\mathcal{P}N$ the Kohlberg criterion holds.

An important concept for solution concepts in cooperative game theory is the reduced game property (RGP). Suppose a solution concept Φ on a class Γ of (TU or NTU) games is agreed upon by all players. Then in a game $(N,V) \in \Gamma$ a coalition $S \in 2^N$ not equal to $\emptyset$ or to N might want to analyse "its own game" (S,V^*), called the *reduced game*, where V^* is that coalitional function that reflects in some sense the possible gains of cooperation, when the "outside players" $N \backslash S$ are payed according to Φ. Whenever the outcomes according to Φ for players in S differ from game V to game V^* then some players might prefer forming coalition S (and "play" the game (S,V^*)) rather than joining in the grand coalition N.

The solution Φ is immune against such sort of instability, if $\Phi(S, V^*) = \Phi(N, V)|_S$, i.e. the payoffs to the players of the "split off" coalition S do not change. Thus there are actually no incentives to depart from the grand coalition. Of course, specifying the coalitional function V^* of the reduced game is crucial to this concept, but by no means canonical.

We present here the definition of a reduced (TU) game that was introduced by Davis and Maschler ([DM65])[6], because it plays an important role in the theory of the (TU) prenucleolus and we will later define an extension of this reduced game to the class of all NTU games which will be useful in the analysis of the (yet to be defined) NTU prenucleolus.

Definition 2.15

Let $(N, v) \in \Gamma^{TU}$ be a TU game, let $x \in \mathbb{R}^N$ be an arbitrary vector and let $S \in 2^N \setminus \{\emptyset, N\}$ be a coalition. The (TU) **reduced game** $\left(S, v_x^S\right)$ *of S w.r.t. x is defined by*

$$\begin{aligned} v_x^S(T) &:= \max_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\}, \forall T \in 2^S \setminus \{\emptyset, S\} \\ v_x^S(S) &:= v(N) - x(N \setminus S) \\ v_x^S(\emptyset) &:= 0. \end{aligned}$$

Definition 2.16

Let Φ be a solution concept on a class $\Gamma \subseteq \Gamma^{TU}$. Φ satisfies the **reduced**

[6] In fact, a slight variation of it that coincides with their original definition if $x(N) = v(N)$.

game property (RGP), *if for every* $x \in \Phi(N,v)$ *and every* $S \in 2^N \setminus \{\emptyset, N\}$ *the following is true:* $\left(S, v_x^S\right) \in \Gamma$ *and* $x_S \in \Phi\left(S, v_x^S\right)$.

Lemma 2.17

The prenucleolus $\mathcal{P}N$ *satisfies RGP on* Γ^{TU}.

For the proof [Sob75] and again [Pel88][7] are referred to.

The core is another set-valued solution concept that has been extensively studied, both in the TU and in the NTU case. The main idea behind the definition of the core is a form of "internal stability". Considering the TU case, an imputation x is in the core of a game (N,v) if for every coalition S the worth $x(S)$ is greater or equal to $v(S)$. This means that the coalition has no incentive to deviate from the grand coalition because it is served at least as good as it could by itself. Speaking in terms of excess functions (Definition 2.12) we could as well say that no coalition has a strictly positive excess at x, i.e. does not regard itself as dissatisfied. If some imputation y yields $y(S) < v(S)$ for some coalition S then this coalition is frequently said to be able to improve upon y by their own means.

The fact that the core of a (TU or NTU) game can be set-valued can be well regarded either as an advantage or as a disadvantage, depending on the purpose the solution concept is to be used. If for a game the core consists of more then one imputation the problem of choosing one of them might not be easily solvable. The contrary problem might also occur since the core can

[7] Lemma 5.2.1 (and Corollary 5.2.2) and Theorem 5.2.7 (and Corollary 5.2.8)

be empty. Thus it is interesting to know which (TU or NTU) games posses an actually non-empty core. We elaborate only on the TU case here and postpone the NTU case to section 5.4.

Definition 2.18 (TU and NTU core)

1. *Let $(N, v) \in \Gamma^{TU}$ be a TU game. The* **(TU) core** *of (N, v) is defined as*

$$\mathbf{Core}\,(N, v) := \left\{x \in I^*\,(N, v)\,\middle|\, x(S) \geq v(S) \quad \forall S \in 2^N \right\}.$$

2. *Let $(N, V) \in \Gamma^{NTU}$ be an NTU game. The* **(NTU) core** *of (N, V) is defined as*

$$\mathbf{Core}\,(N, V) := \left\{x \in V(N)\,\middle|\, \forall S \in 2^N : \nexists y \in V(S) \text{ s.t. } y_S > x_S \right\}.$$

If a TU game $(N, v) \in \Gamma^{TU}$ satisfies

$$v(N) \geq \sum_{S \in \mathcal{S}} \delta_S v(S)$$

for every balanced collection $\mathcal{S} \subseteq 2^N$ with balancing coefficients $(\delta_S)_{S \in \mathcal{S}}$, then (N, v) is called **balanced**. It was proved independently in [Bon63] and [Sha67] that balancedness is a sufficient and necessary condition for the core of a game to be non-empty.

3 The NTU nucleoli of Kalai

The considerations about (dis-)satisfaction of coalitions with respect to proposed imputations that seems best expressed via the excess functions has led to two widely accepted solution concepts for TU games, the kernel and the nucleolus[8]. It is therefore natural to look for similar concepts in the NTU context. It is furthermore obvious that – having in mind the general nucleolus and results about it – "only" the concept of an excess function has to be appropriately reformulated for non-transferable utility situations. But as we hope to demonstrate convincingly hereafter this reformulation is not at all canonical. Comparing two real numbers ($v(S)$ and $x(S)$) is most naturally done by their difference but what about – turning now to the NTU context – a point (x_S) and a subset ($V(S)$) both in $\mathbb{R}^N$? Of course it is possible to look at their distance[9]. But as we will show at the end of this chapter this can lead into trouble.

What we present in this thesis are two axiomativ approaches to the problem of modelling satisfaction or excess concepts. The first is due to Ehud Kalai ([Kal75]) and is the focus of this chapter while the other is our suggestion for a perhaps more convincing concept, see the chapters 4 and 5. We use the term "axiomatic" because both postulate a set of axioms that an NTU excess function should satisfy and analyze the class of NTU (pre-)nucleoli that are defined via those excess functions that fit into the axiom system. Let us now introduce the approach of Kalai.

[8] and, of course, to their respective "pre"-versions.

[9] See Definition 4.28 for a possible definiton of such a distance.

Let $\Gamma \subseteq \Gamma^{NTU}$ be the subclass of NTU games that satify

$$\forall S \in 2^N : \quad \exists a^S \in \mathbb{R}^S \text{ such that } a^S \geq x \quad \forall x \in V(S). \tag{1}$$

Definition 3.1 (K(alai)-excess function)

Let $(N, V) \in \Gamma$ be a game. The function

$$l^{(N,V)} : 2^N \times \mathbb{R}^N \to \mathbb{R}$$

is called K-excess function, if the following conditions hold for all coalitions $S \in 2^N$:

1. *Independence of other coalitions*

   $\left[x, y \in \mathbb{R}^N, x_S = y_S\right] \Rightarrow l^{(N,V)}(S, x) = l^{(N,V)}(S, y).$

2. *Monotonicity*

   $\left[x, y \in \mathbb{R}^N, x_S < y_S\right] \Rightarrow l^{(N,V)}(S, x) > l^{(N,V)}(S, y).$

3. *Normalization*

   $\left[x \in \mathbb{R}^N, x_S \in \partial V(S)\right] \Rightarrow l^{(N,V)}(S, x) = 0.$

4. *Continuity in x and, if we fix S and regard $l^{(N,V)}$ as a function on games, also continuity in (N, V).*

The topology on the game space that Kalai used in order to define continuity will be described in detail in section 4.6 where we will use the same topology for the purposes of this thesis.

As we already mentioned earlier, one can easily see that the conditions of Definition 3.1 are satisfied by the (TU) excess function $e(S, x, v) = v(S) - x(S)$ (see Definition 2.12), when they are properly reformulated within the TU environment. But notice that TU games as members of Γ^{NTU} do not belong to the class Γ considered by Kalai because they violate condition (1). This condition is only needed in order to define a metric on the game space to be able to speak of continuity of the excess functions (4. in Definition 3.1). If hyperplane games, which include the TU games and which all violate (1), are to be considered it is possible to use another metric on this game space and so condition (1) can be omitted in this case.

Definition 3.2

Let $(N, V) \in \Gamma$ be a game and let $l^{(N,V)}$ be a K-excess function. Define the **K(alai)-nucleolus** *of (N, V) w.r.t. $l^{(N,V)}$ as*

$$K\mathcal{N}^l (N, V) := \mathcal{N} \left(l^{(N,V)}, IR(N, V) \right),$$

where $IR(N, V) := \{x \in V(N) \,|\forall i \in N \; \nexists y \in V(\{i\}) \; \text{ with } y_i > x_i\}$ is the set of all individual rational *points of $V(N)$.*

Due to the restriction to individual rational points of $V(N)$, Kalai is indeed investigating a nucleolus concept rather than a prenucleolus concept.

Before we proceed, we will give some examples for K-excess functions. These examples are visualized in Figure 1.

Example 3.3

1. *Let* $\mu \in \mathbb{R}^N_{++}$ *be a vector and let* $(N, V) \in \Gamma$ *be a game. Define*

$$\begin{aligned} f_\mu(S,x) &:= f_\mu^{(N,V)}(S,x) \\ &:= \sup\{t \in \mathbb{R} \,|x_S + t\mu_S \in V(S)\} \end{aligned}$$

for every coalition $S \in 2^N$ *and for every* $x \in \mathbb{R}^N$*. When the vector* μ *is thought of as a direction in which coalitions are able to "move" from a starting point* $x \in \mathbb{R}^N$ *then* $f_\mu(S,x)$ *is the maximal distance the coalition* S *can "move in direction* μ*" without leaving* $V(S)$ *or the minimal distance* S *has to move when* x_S *is not a member of* $V(S)$ *and* S *is "forced back" to* $V(S)$*. See the end of this chapter for an interpretation and discussion of the notion that coalitions "move" in this sense from one imputation to another.*

It is easily checked that f_μ *is indeed a K-excess function, i.e. meets the conditions 1 to 4 of definition 3.1.*

2. *A special case of* $f^{(N,V)}$ *is given by*

$$g^{(N,V)}(S,x) := f^{(N,V)}_{\frac{1_N}{|S|}}(S,x).$$

Here the direction in which to move is the "egalitarian" one.

3. *The sum of "individual excesses" is another possibility.*

$$\begin{aligned} &h^{(N,V)} : 2^N \times \mathbb{R}^N \to \mathbb{R} \\ &h^{(N,V)}(S,x) = \sum_{i \in S} h_i^{(N,V)}(S,x), \end{aligned}$$

where

$$h_i^{(N,V)}(S,x) := \max\left\{t \in \mathbb{R} \,\middle|\, x_S + t1_{\{i\}} \in V(S)\right\}.$$

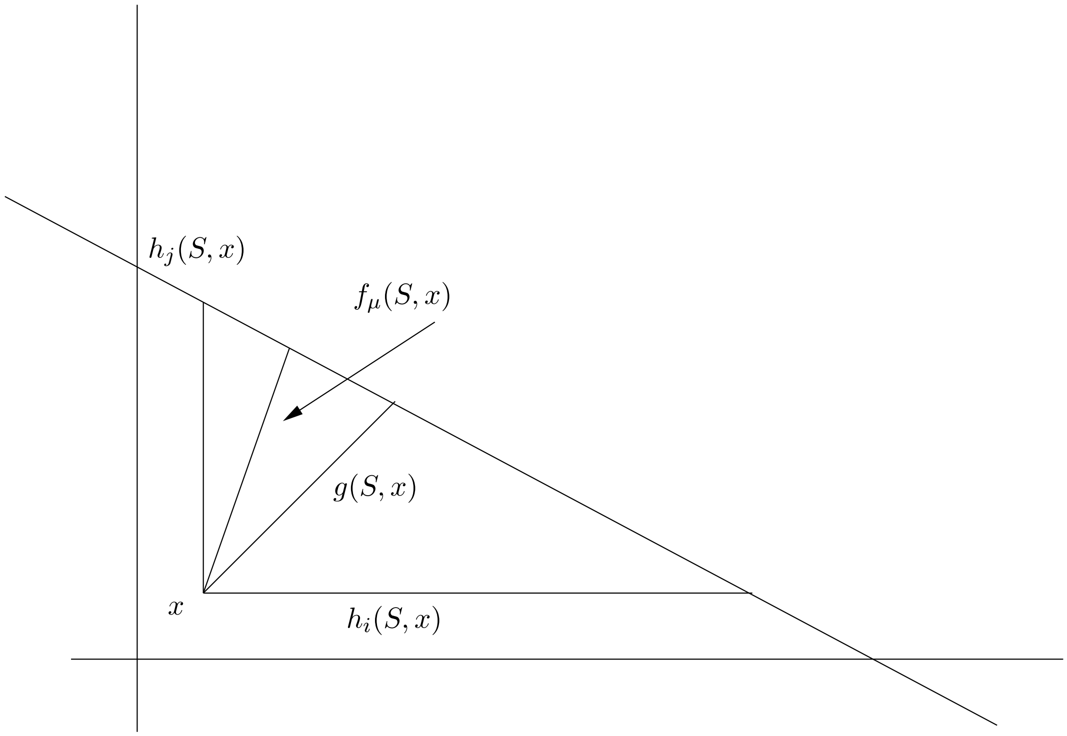

Figure 1: Three K-excess functions

Without going into further details, we briefly list some of the properties that K-nucleoli have or do not have in the next two remarks.

Remark 3.4

The following two important properties of the K-nucleoli are proven in [Kal75].

- $K\mathcal{N}^l(N,V) \neq \emptyset$ *for every game* $(N,V) \in \Gamma$ *and for every K-excess function* $l^{(N,V)}$.
- $K\mathcal{N}^l(N,V) \in \mathbf{Core}(N,V)$ *for every game* $(N,V) \in \Gamma$, *such that* $\mathbf{Core}(N,V) \neq \emptyset$, *and for every K-excess function* $l^{(N,V)}$.

Remark 3.5

1. *The results on single-valuedness of the general nucleolus can not be applied to state single-valuedness of $K\mathcal{N}^l$ for every choice of a K-excess function l, because in general K-excess functions are not convex. Look at $g^{(N,V)}$ of example 3.3, which might even be concave. There is, however, a "generic uniqueness" result in [Kal75], that holds under some additional restrictions to the K-excess functions. But also an example of a K-nucleoli that consists of three distinct points is given is that paper.*

2. *The Kohlberg criterion does not hold in general. A look at the proof of Theorem 2.14 reveals that the fact that the K-excess functions are in general not affine linear might be the reason for this. Also see example 3.6 below for a counterexample.*

3. *According to a theorem in [Yan97], there is no K-excess function l such such $K\mathcal{N}^l$ satisfies RGP. Of course, we did not yet specify any reduced (NTU) game. We postpone this until the discussion of the reduced game property of the (NTU) prenucleolus in section 5. We only mention that the (TU) reduced game (Defintion 2.15) has a direct analogon for NTU games, which is used in the stated theorem.*

4. *$K\mathcal{N}^l$ does not necessarily coincide with the nucleolus on the class of TU games considered as a subclasss of Γ^{NTU}. As mentioned earlier, this subclass does not belong to the class Γ used by Kalai. But we have also mentioned that his results can as well be formulated for hyperplane games such that this question of coincidence is valid.*

S	$h^{(N,V_0)}(S,\nu_0)$
$\{1,2\}$	$\frac{2}{3}$
$\{1,3\}$	$\frac{2}{3}$
$\{2,3\}$	$\frac{2}{3}$
N	0
$\{1\},\{2\},\{3\}$	$-\frac{1}{3}$

Table 1: K-excesses w.r.t. the K-nucleolus of V_0

Example 3.6

Let $N=\{1,2,3\}$ and let V_0 the hyperplane game where (omitting the subscript V_0) $p^S=(1,1,1)|_S \quad \forall S\in 2^N$ and

$$c^S=\begin{cases} 1 & ,\mathit{if} \quad |S|\geq 2 \\ 0 & ,\mathit{if} \quad |S|<2 \end{cases} \qquad \forall S\in 2^N.$$

Then $\nu_0 := K\mathcal{N}^h(N,V_0)=\frac{1}{3}(1,1,1)$ and the K-excesses are given in Table 1 (we use $h^{(N,V)}$ as K-excess function, see 3. in example 3.3).

When we change the game by decreasing $p_1^{\{1,2\}}$ and $p_1^{\{1,3\}}$, see Figure 2, then, as $p^{\{1,2\}}$ and $p^{\{1,3\}}$ approach $\left(\frac{1}{2},1,0\right)$ and $\left(\frac{1}{2},0,1\right)$, respectively, the K-nucleoli of the according games approach $x:=\left(\frac{3}{4},\frac{1}{8},\frac{1}{8}\right)$. So by a continuity argument this point is a candidate for the K-nucleolus of the game (N,V_1), in which $p^{\{1,2\}}=\left(\frac{1}{2},1,0\right)$ and $p^{\{1,3\}}=\left(\frac{1}{2},0,1\right)$. The respective K-excesses with respect

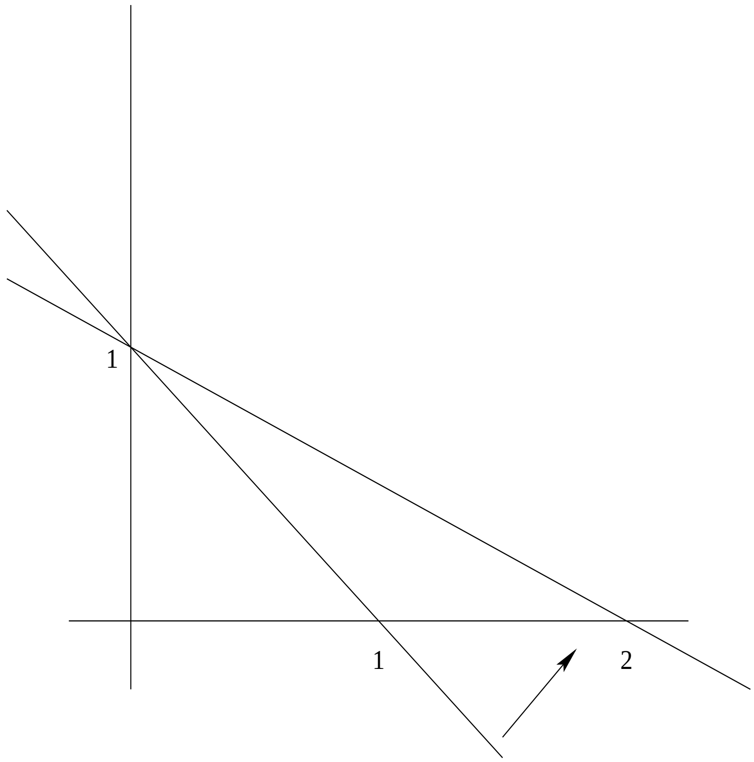

Figure 2: The game V_1

to x are given by Table 2.

From the view of the Kohlberg criterion this looks right, i.e. the collection of coalitions that attain maximal K-excess at x is balanced and so is every other collection attaining at least the second highest excess etc. But as we already mentioned, the Kohlberg criterion does not necessarily hold for the K-nucleoli and indeed we can show that x is not the K-nucleolus of the game

S	$h^{(N,V_1)}(S,x)$
$\{1,2\}$	$\frac{3}{2}$
$\{1,3\}$	$\frac{3}{2}$
$\{2,3\}$	$\frac{3}{2}$
N	0
$\{2\},\{3\}$	$-\frac{1}{8}$
$\{1\}$	$-\frac{3}{4}$

Table 2: K-excesses w.r.t. x in game V_1

(N, V_1). *Let therefore* $\epsilon > 0$ *and define* x^ϵ *by*

$$\begin{aligned} x_1^\epsilon &:= x_1 - \epsilon \\ x_2^\epsilon &:= x_2 + \frac{\epsilon}{2} \\ x_3^\epsilon &:= x_3 + \frac{\epsilon}{2}. \end{aligned}$$

Then we have $x^\epsilon \in \partial V(N)$. *Since* $x^\epsilon|_{\{1,2\}}$ *and* $x^\epsilon|_{\{1,3\}}$ *constitute a line that is parallel to* $\partial V(\{1,2\})$ *and* $\partial V(\{1,3\})$, *respectively, the K-excesses of the coalitions* $\{1,2\}$ *and* $\{1,3\}$ *are the same with respect to* x *and to* x^ϵ. *But we have*

$$h^{(N,V_1)}(\{2,3\}, x^\epsilon) < h^{(N,V_1)}(\{2,3\}, x)$$

from which it follows that x *is not the K-nucleolus of the game* (N, V_1). *Actually, it is* $y := \left(0, \frac{1}{2}, \frac{1}{2}\right) = x^{\frac{3}{4}}$. *The K-excesses with respect to* y *are given* *in Table 3.*

S	$h^{(N,V_1)}(S,y)$
$\{1,2\}$	$\frac{3}{2}$
$\{1,3\}$	$\frac{3}{2}$
$\{2,3\}$	0
$\{1\}, N$	0
$\{2\},\{3\}$	$-\frac{1}{2}$

Table 3: K-excesses w.r.t. y in game V_1

This example shows the non-validness of the Kohlberg criterion and some form of discontinuity of the K-nucleolus. This unwanted behavior can also be observed for other K-excess functions.

Now that we have introduced the approach of Kalai towards an extension of the (pre-)nucleolus concept to NTU games we end this chapter with a discussion of its main drawback (in the present auhtor's view, of course).

Have again a look at example 3.6. We do certainly not want to disqualify a solution concept by just looking at its result on one game, but we think that some general points can indeed be shown thereby.

The fact that the "limit point" $x = \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$ is not the K-nucleolus of the game V_1 is due to the fact that the coalitions $\{1,2\}$ and $\{1,3\}$ could (virtually) transfer utility according to the definiton of the points x^ϵ *without* changing their respective excesses[10] I.e., a payoff of $\left(\frac{3}{4}, \frac{1}{8}\right)$[11] yields the same excess to coalition $\{1, 2\}$ as $\left(0, \frac{1}{2}\right)$ and this, we think, is incompatible with the interpretation of the excess as a measure of (dis-)satisfaction.

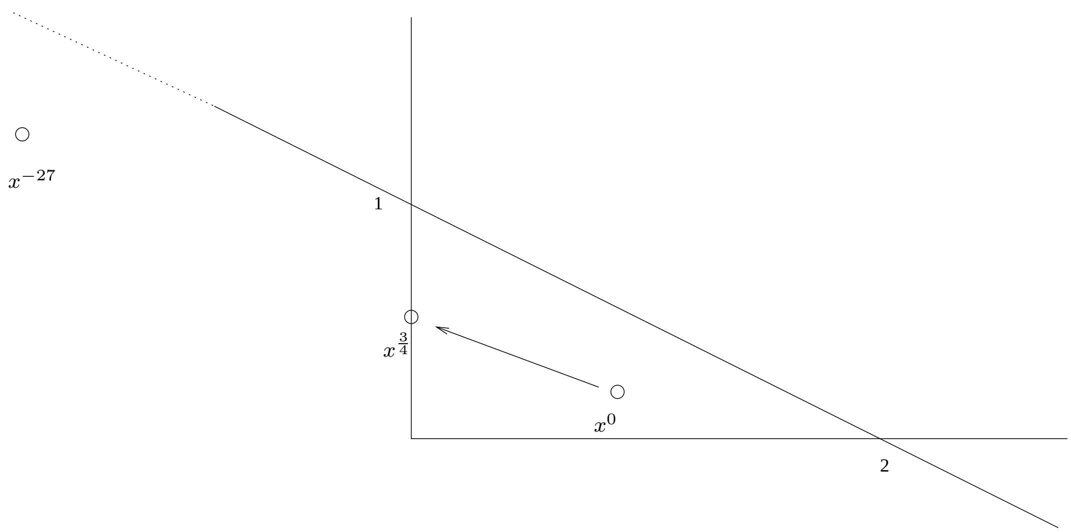

Figure 3: Three points with equal excess

Take a look at figure 3. Why would the coalition, say $\{1, 2\}$ be equally satisfied with $x^0 = \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$ and $x^{\frac{3}{4}} = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ resp. with their relevant part thereof? In the TU case the coalition compares what is proposed $(x(S))$ to what might be $(v(S))$ and expresses its excess via the difference between these values. The interpretation of this K-excess function is quite different. The coalition does not compare the proposed outcome to their own possibilities $(V(\{1, 2\}))$ but to what they could achieve *given* the proposal (for

[10]and the fact that the excess of coalition $\{2, 3\}$ is decreased thereby.

[11]We only consider the relevant part of the payoff vector, thus abusing the notation a little.

example $\left(0, \frac{1}{2}\right)$), i.e. they look at the distance of this point to the boundary of $V(\{1,2\})$. This is flawed in two ways. First the coalition ignores most of their possibilities by only considering the proposal and the nearby part of the boundary of $V(\{1,2\})$. On the contrary, the coalition should compare the proposal with *everything* else that is reachable. This becomes even more obvious when we notice that the point, say, $x^{-27} = \left(-26\frac{1}{4}, 13\frac{5}{8}, 13\frac{5}{8}\right)$ also yields the sames excess of $\frac{3}{2}$ to coalition $\{1,2\}$ (see figure 3 again).

The only explanation for this might be that the huge "gain" of player 2 exactly outweighs the huge "loss" of player 1 so that the coalition is again at the same satisfaction level (this is exactly what this specific K-excess function proposes). This is indeed valid in the TU case where utility can be freely transferred between players and thus only the sum of the individual outcomes matters. What we will propose in the next chapter uses this possibility of transfers between players but of course the "real" transfer rates as given by the shape of $V(N)$, the possible outcomes for the grand coalition, must be used for this. The K-excess functions completely ignore these transfer rates which is our second objection against them. Coalitions have to determine their (dis-)satisfaction with respect to proposals for the *grand* coalition so if there are transfer rates to be considered then surely the that rates are given by $V(N)$, resp. $\partial V(N)$ to be more exact, are the only valid ones.

However, we have based our discussion of the K-excess functions and the motivation for the need of another approach on one example game and one specific K-excess function. This procedure can itself be criticized since many known solution concepts for TU or NTU games produce strange or counterintuitive results on suitably constructed games. But to our knowledge there does not exist in the literature any K-excess function that is not very closed connected to the distance of the proposal to the boundary of the set of possible outcomes and hence to which our objectives do not apply. The examples we have chosen to demonstrate our objectives are therefore indeed representative for every K-excess function we know of.

See [CLT95] or [Pec98] for examples of papers utilizing Kalai's excess functions.

4 β-excess functions and β-prenucleoli

The previous section showed that Kalai's excess functions, although based on rather intuitive axioms, did not exhaustively establish a theory of nucleoli-like solution concepts for NTU games. In this section we will develope a new class of excess functions and investigate (in Chapter 5) in detail a member of this class with yields an (NTU) prenucleolus with some nice properties.

For the remainder of this section the class of NTU games under consideration is Γ^{qH}, the class of all quasi hyperplane games, thus for every game $(N, V) \in \Gamma^{qH}$ we have $p^N \in \mathbb{R}^N$ and $c^N \in \mathbb{R}$ such that $V(N) = \left\{ x \in \mathbb{R}^N \,\middle|\, \langle x, p^N \rangle \leq c^N \right\}$ and every $V(S), S \in 2^N \setminus \{N\}$, merely satisfies the conditions formulated in Definition 2.3. We will develop all the necessary theory for this class of NTU games in the first instance and propose an extension to general NTU games in section 4.7.

4.1 β-excess functions

We will now introduce the key concept of the new (NTU) excess functions.

Definition 4.1
Let $(N, V) \in \Gamma^{NTU}$ be a game. The set

$$X^N := \left\{ (x_S)_{S \in 2^N} \,\middle|\, x_S \in \mathbb{R}^S \quad \forall S \in 2^N \right\}$$

is the set of all **payoff configurations** *of the game (N, V).*

Definition 4.2

A function

$$\beta : \Gamma^{NTU} \to \bigcup_{(N,V)\in\Gamma^{NTU}} X^N,$$

s.t. $\beta(N,V) \in X^N \quad \forall (N,V) \in \Gamma^{NTU}$ *and* $\beta(N,V)_N \in \partial V(N)$*, that assigns to each NTU game a payoff configuration is called* **reference function***. For each coalition* $S \in 2^N$ *the point* $\beta(N,V)_S \in \mathbb{R}^S$ *is called* **reference point** *of* S*. Let* $\mathcal{B}$ *denote the set of all reference functions.*

Instead of $\beta(N,V)_S$ *we will use the notation* $\beta^{(N,V)}(S)$ *(or even* $\beta(S)$ *when there is no danger of confusion) to distinguish from the notation of projection.*

The purpose of introducing the concept of reference points and reference functions is to identify a "point of indifference" for every coalition, that means a point $x \in \mathbb{R}^S$ yielding an excess of zero to coalition S. Such an x is therefore a point at which the coalition is neither satisfied nor dissatisfied[12]. Although the concept of a reference function resembles somehow a solution concept itself[13], it is meant as a mere auxiliary concept. Note that so far we did not impose any conditions on β such as $\beta(S) \in V(S)$ or the like.

Once a point of indifference is chosen for a coalition S (we will later discuss the way this can or should be done), there are of course other points in $\mathbb{R}^S$ yielding equal excess to it. Another important feature of the excess function we are about to introduce is the way those points are characterized. This

[12] or, in other words, is indifferent between satisfaction and dissatisfaction.

[13] For example, the Harsanyi solution for NTU games ([Har63]) is defined as a function from games into payoff configurations.

characterization is based on the following considerations. Technically the domain of any excess function for coalition $S \in 2^N$ is $\mathbb{R}^N$, but the (dis-)-satisfaction of the coalition only depends on the outcome for this coalition and ignores the payments to the complementary coalition (compare Axiom 1 of Kalai in Definition 3.1). Thus no notion of envy is incorporated into the excess concept – like in the TU context. We might call this the principle of independence of the payoffs of other players.

Now suppose there is an imputation $x \in \partial V(N)$ such that x_S is a point of indifference for the coalition $S \in 2^N$. The coalition might consider a redistribution of its share x_S according to those transfer rates that are relevant to them in the grand coalition, namely p_S^N. Since the imputation x has been made possible through cooperation of all players and coalition S might well be not effective for x_S these transfer rates are surely the only possible basis for any such redistribution. Of course, these are only virtual redistributions: The imputation $x \in V(N)$ has not been allocated to the players yet. It is only a proposal which is to be checked whether or not it minimizes dissatisfaction. We are still within the process of determining the difference between the status quo x_S and what might be, i.e. we are "calculating dissatisfaction". Due to the principle of independence of the payoffs of other players such a redistribution should not effect the excess of any coalition outside of S. We argue that it should not change the excess of coalition S either.

Otherwise, i.e. when the coalition should be able to change its excess by redistributing x_S according to p_S^N, then the prenucleolus defined by such an excess function would not really be a lexicographical minimizer of dissatisfaction/excesses because it is in this sense not well defined what the dissatisfaction of a coalition actually is. This we want to avoid. Therefore we will impose another property on the new excess functions which might be informally described as "invariance under changes according to p_S^N". Since the motivation we gave for this property of course also holds for imputations $x \in \partial V(N)$ such that x_S is not a point of indifference for S, we might also say that the excess function for S should have contour sets that are hyperplanes with a normal vector proportional to p_S^N.

In the case that a TU game $(N, v) \in \Gamma^{TU}$ is under consideration, we already know which points $x \in \mathbb{R}^S$, $S \in 2^N$, are the only candidates for being a "point of indifference" by looking at the TU excess function $e\,(S, x) = v(S) - x(S)$. In other words, when considering (N, v) as a member of Γ^{NTU}, i.e. as (N, V^v), then the points of indifference of S lie on the boundary of $V^v(S)$[14]. This tells us that $\beta(S) \in \partial V^v(S)$ should be satisfied, or in other words $\sum_{i \in S} \beta_i(S) = v(S)$, if we want to make the new excess functions compatible to the TU excess. This motivates the next definition.

Definition 4.3
Let $\overline{\mathcal{B}} \subset \mathcal{B}$ be the set of all reference functions $\beta \in \mathcal{B}$ which satisfy

$$\sum_{i \in S} \beta_i^{(N,V^v)}(S) = v(S) \quad \forall S \in 2^N,$$

for every TU game $(N, v) \in \Gamma^{TU}$.

[14]thus satisfying $v(S) = x(S)$ and therefore $e(S, x) = 0$.

The following theorem states that if an excess function for NTU games should satisfy the two properties just discussed, i.e. vanishing on $\beta(S)$ for all $\beta \in \overline{\mathcal{B}}$ and for all $S \in 2^N$ and having contour sets which are hyperplanes with normal vectors proportional to p_S^N, and if it is furthermore an affine linear function that coincides with the TU excess function on Γ^{TU}, then it is uniquely defined for all reference functions $\beta \in \overline{\mathcal{B}}$.

Theorem 4.4

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. Let

$$e := e^{(N,V)} : 2^N \times \mathbb{R}^N \times \overline{\mathcal{B}} \to \mathbb{R}$$

be a function that satisfies

1. *invariance under changes according to $p_V^N|_S$:*

   $\forall x, y \in \mathbb{R}^N : \left\langle x_S, p_V^N|_S \right\rangle = \left\langle y_S, p_V^N|_S \right\rangle \Rightarrow e\left(S, x, \beta\right) = e\left(S, y, \beta\right) \ \forall S \in 2^N, \forall \beta \in \overline{\mathcal{B}}$,

2. *zero excess at points of indifference:*

   $e\left(S, \beta(S), \beta\right) = 0 \quad \forall S \in 2^N, \forall \beta \in \overline{\mathcal{B}}$,

3. *affine linearity:*

   $e(S, x, \beta) = \langle x_S, r_S \rangle + c_S$ *for some* $r_S \in \mathbb{R}^S$ *and* $c_S \in \mathbb{R}, \forall S \in 2^N, \forall x \in \mathbb{R}^N, \forall \beta \in \overline{\mathcal{B}}$, *and*

4. *coincidence with the TU excess function on Γ^{TU}:*

   If $(N, v) \in \Gamma^{TU}$, *then* $e^{(N,V^v)}(S, x, \beta) = v(S) - x(S) \quad \forall S \in 2^N, x \in \mathbb{R}^N, \beta \in \overline{\mathcal{B}}$.

Then

$$\begin{aligned} e(S,x,\beta) &= \sum_{i\in S} (\beta_i(S) - x_i)\, p^N_{V,i} \\ &= \left\langle \beta(S) - x_S, p^N_V|_S \right\rangle . \end{aligned}$$

Proof:

Let $S \in 2^N$ *be a coalition and let* $\beta \in \overline{\mathcal{B}}$ *be a reference function.*

Claim 1

Axioms 1 and 3 imply $r_S = \gamma_{r_S} p^N_V|_S$ *for some* $\gamma_{r_S} \in \mathbb{R}$.

The proof is straightforward and omitted.

Claim 2

The value γ_{r_S} *in Claim 1 is negative.*

Proof of Claim 2

Let $(N,v) \in \Gamma^{TU}$ *be a TU game. Then the Axioms 3 and 4 imply (together with Claim 1):*

$$\gamma_{r_S} \left\langle x_S, p^N_{V^v}|_S \right\rangle + c_S = \langle x_S, -1_S \rangle + v(S),$$

which at once yields $\gamma_{r_S} < 0$ *because of* $p^N_V|_S > 0$.

Claim 3

Claim 1 and Axiom 2 imply

$$e(S,x,\beta) = \alpha_{r_S} \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle,$$

with $\alpha_{r_S} \in \mathbb{R}_{++}$.

Proof of Claim 3

By Claim 1 we have $r_S = \gamma_{r_S} p_V^N|_S$ *for some* $\gamma_{r_S} \in \mathbb{R}$ *with* $\gamma_{r_S} < 0$ *by Claim 2. By axiom 2 we have*

$$\begin{aligned} e\left(S, \beta(S), \beta\right) &= \langle \beta(S), r_S \rangle + c_S \\ &= 0 \\ \Leftrightarrow \qquad c_S &= -\gamma_{r_S} \left\langle \beta(S), p_V^N|_S \right\rangle, \end{aligned}$$

which yields

$$\begin{aligned} e(S,x,\beta) &= \langle x_S, r_S \rangle + c_S \\ &= \gamma_{r_S} \left\langle x_S, p_V^N|_S \right\rangle - \gamma_{r_S} \left\langle \beta(S), p_V^N|_S \right\rangle \\ &= \gamma_{r_S} \left\langle x_S - \beta(S), p_V^N|_S \right\rangle \\ &= \alpha_{r_S} \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle \end{aligned}$$

with $\alpha_{r_S} := -\gamma_{r_S} > 0$ *and the proof of Claim 3 is complete.*

Now Claim 3 together with axiom 4 yield (for any TU game $(N,v) \in \Gamma^{TU}$*)*

$$e^{(N,V^v)}(S,x,\beta) = \alpha_{r_S} \left\langle \beta(S) - x_S, p_{V^v}^N|_S \right\rangle = v(S) - x(S), x \in \mathbb{R}^N,$$

which is equivalent to

$$\alpha_{r_S}\left(\sum_{i\in S}\beta_i(S)-x(S)\right)=v(S)-x(S),x\in\mathbb{R}^N.$$

Since $\beta\in\overline{\mathcal{B}}$, *i.e.* $\sum\limits_{i\in S}\beta_i^{(N,V^v)}(S)=v(S)$, *this in turn yields* $\alpha_{r_S}=1$. ■

Theorem 4.4 has anticipated the next definition, which is reformulated now for the sake of clarity.

Definition 4.5 ($\boldsymbol{\beta}$-excess function)
Let $(N,V)\in\Gamma^{qH}$ *be a (quasi hyperplane) game. The function*

$$\begin{aligned}e:=e^{(N,V)} &\quad:\quad 2^N\times\mathbb{R}^N\times\overline{\mathcal{B}}\to\mathbb{R}\\ e\left(S,x,\beta\right) &\quad:=\quad \sum_{i\in S}\left(\beta_i^{(N,V)}(S)-x_i\right)p_{V,i}^N\\ &\quad=\quad \left\langle\beta^{(N,V)}(S)-x_S,p_V^N|_S\right\rangle\end{aligned}$$

is called **$\boldsymbol{\beta}$-excess function**. *We sometimes also write* $e^\beta\left(S,x\right)$ *instead of* $e\left(S,x,\beta\right)$.

The previous theorem 4.4 proved that the axioms 1 – 4 uniquely determine an excess function for NTU quasi-hyperplane games. The next lemma will answer the question affirmatively if these axioms are logically independent, i.e. no axiom is an implication of the others.

Lemma 4.6
The axioms of Theorem 4.4 which characterize the β*-excess function are logically independent.*

Proof:

Let $\beta \in \overline{\mathcal{B}}$ be a reference function. We show the independence of the axioms by giving an example of an excess function for every axiom, respectively, that satisfies only the other axioms and is different from the β-excess function. For notational convenience we omit the superscript β in the definitions.

Let $S \in 2^N$ be a coalition.

1. *Let*
$$e^1(S,x) := \langle \beta(S) - x_S, 1_S \rangle .$$
Then e^1 satisfies the axioms 2, 3 and 4.

2. *Let*
$$e^2(S,x) := c_V^S - \left\langle x_S, p_V^N|_S \right\rangle .$$
Then e^2 satisfies the axioms 1, 3 and 4.

3. *Let*
$$e^3(S,x) := \left(\left\langle \beta(S) - x_S, p_V^N|_S \right\rangle\right)^{p_{V,i}^N} .$$
for any $i \in N$. Then e^3 satisfies the axioms 1, 2 and 4.

4. *Let*
$$e^4(S,x) := \alpha \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle$$
with $\alpha > 1$. Then e^4 satisfies the axioms 1, 2 and 3.

It is easily checked that all excess functions $e^i, i = 1, 2, 3, 4,$ do not coincide with the β-excess function. ■

By Theorem 4.4 and Lemma 4.6 we have shown that the axioms (1) – (4) indeed constitute an axiomatization of the β-excess function.

Remark 4.7

A look at the axioms 2 and 4 of Theorem 4.4 reveals that we can not relax the condition $\beta \in \overline{\mathcal{B}}$ to $\beta \in \mathcal{B}$ because these axioms would then be incompatible. We nevertheless allow also β-excess functions for $\beta \in \mathcal{B} \setminus \overline{\mathcal{B}}$ to be defined, since most of the propositions of this chapter apply to every excess functions $\beta \in \mathcal{B}$. Of course, we must be aware that we thereby possibly define β-prenucleoli for NTU games that do not coincide with the (TU) prenucleolus on TU games.

Apart from the axioms that axiomatize e^β, the β-excess functions also satisfy those properties as stated by the next lemma. These properties are in fact simple corollaries of Definition 4.5, thus the proofs are omitted.

Lemma 4.8

Let $(N, V) \in \Gamma^{qH}$ be a game. For every reference function $\beta := \beta^{(N,V)} \in \mathcal{B}$, the β-excess function e^β has the following properties (compare Definition 3.1):

1. *Independence of other players*

   $$\left[x, y \in \mathbb{R}^N, x_S = y_S\right] \Rightarrow e^\beta(S, x) = e^\beta(S, y).$$

2. *Monotonicity*

$$\left[x, y \in \mathbb{R}^N, x_S < y_S\right] \Rightarrow e^\beta\left(S, x\right) > e^\beta\left(S, y\right).$$

3. *Normalization*

$$x \in \mathbb{R}^N, \left\langle x_S, p_V^N|_S\right\rangle = \left\langle \beta(S), p_V^N|_S\right\rangle \Rightarrow e^\beta\left(S, x\right) = 0.$$

4. *Continuity in x.*

Note that the β-excess functions meet three of the four properties that define the Kalai-excess functions. Of course, the contour sets of β-excess functions generally differ from those of Kalai-excess functions.

4.2 The β-prenucleolus of quasi hyperplane games

With the definition of the β-excess functions at hand we can now apply the general nucleolus (Definition 2.10) to define a prenucleolus on the class of quasi hyperplane games.

Definition 4.9 ($\boldsymbol{\beta}$-prenucleolus of quasi hyperplane games)
Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game and let $\beta \in \mathcal{B}$ be a reference function. The set

$$\mathcal{P}N^\beta\left(N, V\right) := \mathcal{N}\left(\left(e^{(N,V)}\left(S, \bullet, \beta\right)\right)_{S \in 2^N}, V(N)\right)$$

is called **$\boldsymbol{\beta}$-prenucleolus of quasi hyperplane games** *or* **(NTU) $\boldsymbol{\beta}$-prenucleolus** *for short.*

We did not restrict the set of possible candidates for the β-prenucleolus to individual rational outcomes but to the entire set $V(N)$. Thus we indeed consider a *pre*-nucleolus concept. We will show later under which conditions the β-prenucleoli are individual rational.

Remark 4.10

It is straightforward to see that the β-prenucleoli satisfy efficiency and anonymity, two of the basic properties that solution concepts should always satisfy (Definition 2.9). We will later also consider the third property, i.e. covariance.

Example 4.11

Let us introduce two reference functions and have a look at the resulting (NTU) β-prenucleoli.

Let $(N, V) \in \Gamma^H$ be a hyperplane game and let the reference functions β_1 and β_2 be given by

$$\beta_{1,i}(S) := \begin{cases} \dfrac{c^S}{|S| p_i^S} & , \quad i \in S \\ 0 & , \quad i \notin S \end{cases} \quad \left(S \in 2^N, i \in N\right).$$

and

$$\beta_{2,i}(S) := \begin{cases} \dfrac{c^S}{p_i^S} & , \quad i \in S \\ 0 & , \quad i \notin S \end{cases} \quad \left(S \in 2^N, i \in N\right).$$

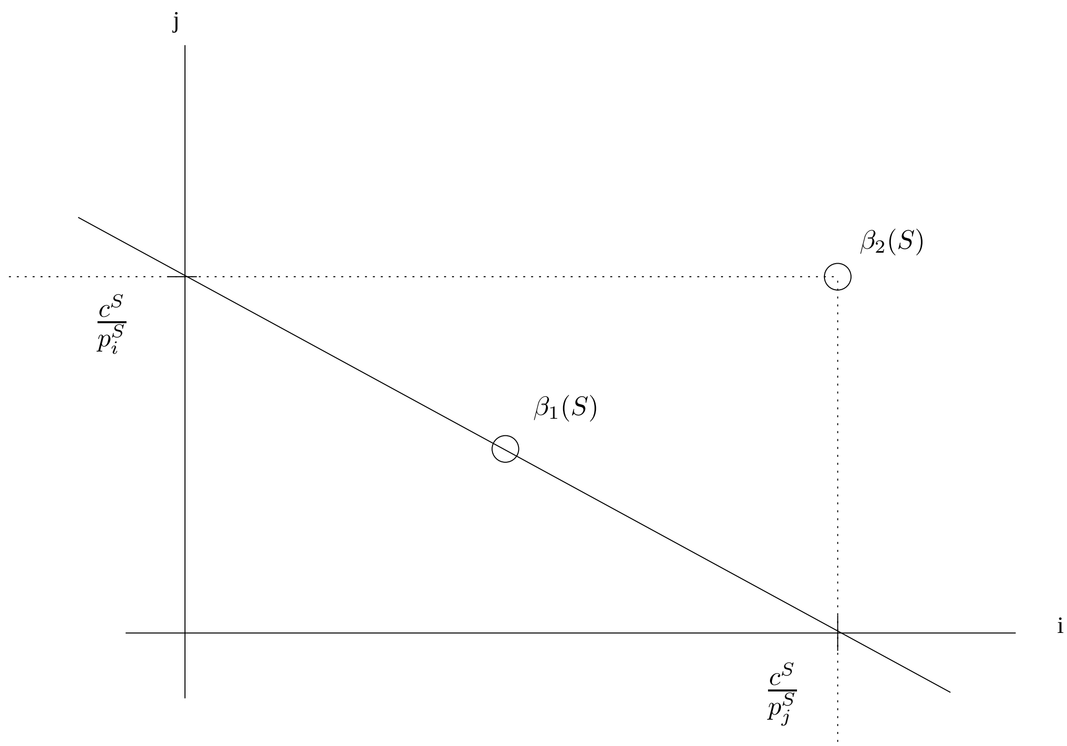

Figure 4: Two reference functions

The number $\dfrac{c^S}{p_k^S}$ $(k \in S)$ *is the maximal amount that player* $k \in S$ *can achieve under an imputation for coalition* S *which is individual rational for all players in* S*, i.e. is contained in* $V(S) \cap \mathbb{R}_+^S$*.* $\beta_1(S), S \in 2^N$*, is the mean value of these extreme points.* β_2 *reflects the situation where indifference of the coalition between satisfaction and dissatisfaction is not achieved until every player receives his maximal outcome that is possible under an individual rational imputation for which coalition* S *is effective. See Figure 4 for an illustration of these definitions.*

Let us compute the two β-prenucleoli of the game (N, V_1) of example 3.6, i.e. (N, V_1) consists of $N := \{1, 2, 3\}$ and the coalitional function V_1, which is given by Table 4.

S	$p^S_{V_1}$	$c^S_{V_1}$
$\{1\}$	$(1, 0, 0)$	0
$\{2\}$	$(0, 1, 0)$	0
$\{3\}$	$(0, 0, 1)$	0
$\{1, 2\}$	$\left(\frac{1}{2}, 1, 0\right)$	1
$\{1, 3\}$	$\left(\frac{1}{2}, 0, 1\right)$	1
$\{2, 3\}$	$(0, 1, 1)$	1
N	$(1, 1, 1)$	1

Table 4: The game V_1 of example 3.6

The computation yields $\mathcal{P}N^{\beta_1}(N, V_1) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$ and $\mathcal{P}N^{\beta_2}(N, V_1) = (1, 0, 0)$. The resulting β-excesses for both excess functions β_1 and β_2 are given in Table 5. The following remarks can be made concerning these two examples of β-excess functions and the resulting β-prenucleoli.

1. *For both β-prenucleoli the Kohlberg criterion holds, i.e. the respective collections of coalitions with maximal excess etc. are balanced. We will later on see that this is true in general.*

2. *The excess function β_2 does not coincide with the (TU) excess function, thus $\beta_2 \notin \overline{\mathcal{B}}$. This means that $\mathcal{P}N^{\beta_2}(N, V^v)$ for $(N, v) \in \Gamma^{TU}$ is in general not equal to $\mathcal{P}N(N, v)$.*

3. *For every hyperplane game $(N, V) \in \Gamma^H$ we have $\beta_1^{(N,V)}(S) \in \partial V(S)$, thus $\beta_1 \in \overline{\mathcal{B}}$ holds true.*

S	$e^{\beta_1}(S, x)$	$e^{\beta_2}(S, x)$
$\{1, 2\}$	$\frac{2}{3}$	2
$\{1, 3\}$	$\frac{2}{3}$	2
$\{2, 3\}$	$\frac{2}{3}$	2
N	0	0
$\{2\}, \{3\}$	$-\frac{1}{6}$	0
$\{1\}$	$-\frac{2}{3}$	-1

Table 5: β-excess for the β-prenucleoli of example 3.6

We now turn to the analysis of some basic properties of the (NTU) β-prenucleoli. The first result on existence and uniqueness of $\mathcal{P}N^{\beta}$ follows as directly as those about the TU prenucleolus from Theorem 2.11 about the general nucleolus.

Theorem 4.12

$\left|\mathcal{P}N^{\beta}(N, V)\right| = 1 \quad \forall (N, V) \in \Gamma^{qH}, \forall \beta \in \mathcal{B}.$

Proof:

The proof is a straightforward modification of the proof of Theorem 2.13 about the single-valuedness of the prenucleolus of TU games. The β-excess functions are affine linear and hence convex, $V(N)$ is convex and the fact that it is not compact must be treated similarly. ■

The simplicty of the proof of the single-valuedness of the β-prenucleolus is of course a consequence of the way in which we have designed the β-excess functions – mainly of the assumption of affine linearity. These resemble very much the TU excess and it is thus not surprising that we can directly adopt Theorems and proofs from the TU context. But remember that we still consider quasi hyperplane games. For those games we have already explained the reasons for which (NTU) excess functions should be affine linear.

The next property we want to consider in relation with $\mathcal{P}N^{\beta}$ is covariance (Definition 2.9). We will therefore further restrict the set of reference functions that we will consider in the sequel. It turns out that if we impose a covariance assumption on the reference functions then we can easily show the covariance of $\mathcal{P}N^{\beta}$.

Definition 4.13 (covariant reference functions)

Let $\mathcal{B}^c$ denote the set of all reference functions $\beta \in \mathcal{B}$ that furthermore satisfy $\beta^{(N,\lambda V)}(S) = \lambda\beta^{(N,V)}(S)\ \forall (N,V) \in \Gamma^{qH}, \forall S \in 2^N, \forall \lambda \in \mathbb{R}^N_{++}$.

Theorem 4.14

If $\beta \in \mathcal{B}^c$ then $\mathcal{P}N^\beta(N, \lambda V) = \lambda \mathcal{P}N^\beta(N, V)$ for every game $(N, V) \in \Gamma^{qH}$ and every $\lambda \in \mathbb{R}^N_{++}$, i.e. the β-prenucleolus satisfies covariance.

Proof:

Let $(N, V) \in \Gamma^{qH}$ be a game and let $\beta \in \mathcal{B}^c$ be a reference function. Then

$$
\begin{aligned}
e^{(N,\lambda V)}(S, \lambda x, \beta) &= \sum_{i \in S} \left(\beta_i^{(N,\lambda V)}(S) - \lambda_i x_i \right) \cdot p^N_{\lambda V, i} \\
&= \sum_{i \in S} \left(\lambda_i \beta_i^{(N,V)}(S) - \lambda_i x_i \right) \cdot \frac{p^N_{V,i}}{\lambda_i} \\
&= e^{(N,V)}(S, x, \beta) \\
&\forall \quad S \in 2^N, x \in \mathbb{R}^N.
\end{aligned}
$$

Hence $\mathcal{P}N^\beta(N, \lambda V) = \lambda \mathcal{P}N^\beta(N, V) \quad \forall \lambda \in \mathbb{R}^N_{++}$. ■

Remark 4.15

Both reference functions of example 4.11 are covariant.

From now on we will mostly consider reference functions in $\mathcal{B}^c$. Since we know now that $\mathcal{P}N^\beta$ is then a covariant solution concept it is no loss of generality to assume that every quasi hyperplane game under current consideration is a simplex game. This simplifies most of the proofs that will follow. Whenever appropriate we will give some hints how to prove results without the covariance assumption. But since we judge covariance as an essential property for solution concepts we formulate and prove most of the results with respect to covariant reference functions.

We have seen is chapter 3 that for the K-nucleoli the Kohlberg criterion (Theorem 2.14) does not hold in general contrary to the (TU) (pre-)nucleolus. Since this criterion has both an interpretational and a technical relevance (it is frequently used in proof of propositions about the TU (pre-)nucleolus) we consider the validness of this criterion for the β-prenucleoli – as stated by the following Theorem – as an important property of this solution concept.

We will give a detailed proof of the validness of the Kohlberg criterion for the β-prenucleoli. It is in fact an adoption of the proof for the TU case which we have omitted in Chapter 2.

Definition 4.16

Let N be a finite set and let $\mathcal{S} \subseteq 2^N$ be a collection of subsets of N. We say that $\mathcal{S}$ has **property I**[15] *if for every $y \in \mathbb{R}^N$ such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{S}$ it follows that $y(S) = 0$ for every $S \in \mathcal{S}$ holds true.*

This property is the formalization of the considerations we gave to motivate the introduction of the Kohlberg criterion in Chapter 2. The connection of the property I to the balancedness concept is explained by the following Lemma.

Lemma 4.17

Let N be a finite set. A collection $\mathcal{S} \subseteq 2^N$ of subsets of N has property I if and only if it is balanced.

[15] This expression is due to [Koh71].

Proof:

1. $\Longrightarrow$: *Let $\mathcal{S} = \{S_1, \ldots, S_p\}$ be a collection that satisfy property I. Consider the following linear program (P):*

$$\begin{array}{rcl} \sum_{S\in\mathcal{S}} y(S) & \rightarrow & \max! \\ & s.t. & \\ y(N) & = & 0 \\ -y(S) & \leq & 0 \quad \forall S \in \mathcal{S} \\ y & \in & \mathbb{R}^N \end{array}$$

and the dual program (D) of (P):

$$\begin{array}{rcl} (u_1, u_2) \cdot 0 & \rightarrow & \min! \\ & s.t. & \\ -u_1 1_N + u_2 \left(1_S\right)_{S\in\mathcal{S}} & = & -\sum_{S\in\mathcal{S}} 1_S \\ u_2 & \geq & 0 \\ u_1 & \in & \mathbb{R} \\ u_2 & \in & \mathbb{R}^p. \end{array}$$

Every feasible solution of (P) yields the value 0 *due to property I. Since* $0 \in \mathbb{R}^N$ *is indeed feasible for (P), the dual program (D) has feasible and optimal solutions as well. Hence there exist* $\bar{u}_1 \in \mathbb{R}$ *and* $\bar{u}_2 \in \mathbb{R}^p_+$ *such that*

$$-\bar{u}_1 1_N + \bar{u}_2 \left(1_S\right)_{S\in\mathcal{S}} = -\sum_{S\in\mathcal{S}} 1_S. \qquad (2)$$

If we define $(\lambda_S)_{S\in\mathcal{S}}$ via $\lambda_S := \bar{u}_{2,i}$ if $S = S_i$ for some $i \in \{1,\ldots,p\}$, for every $S \in \mathcal{S}$ we have

$$\bar{u}_1 1_N = \sum_{S\in\mathcal{S}} (1+\lambda_S)\, 1_S. \tag{3}$$

Now $1+\lambda_S > 0$ holds for every $S \in \mathcal{S}$ and thus by (2) also $\bar{u}_1 > 0$. Therefore (3) means that $\mathcal{S}$ is balanced.

2. *$\Longleftarrow$: Now let $\mathcal{S} \subseteq 2^N$ be balanced and let $(\lambda_S)_{S\in\mathcal{S}}$ be balancing coefficients for $\mathcal{S}$, i.e.*

$$\sum_{S\in\mathcal{S}} \lambda_S 1_S = 1_N. \tag{4}$$

Let $y \in \mathbb{R}^N$ such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{S}$. Multiplying (4) by y we get $\sum_{S\in\mathcal{S}} \lambda_S y(S) = y(N) = 0$. This yields $y(S) = 0$ because of $\lambda_S > 0$ for every $S \in \mathcal{S}$. Hence $\mathcal{S}$ has property I.

■

Now let $\beta \in \mathcal{B}$ be a reference function. For every quasi hyperplane game $(N,V) \in \Gamma^{qH}$ and every $x \in V(N)$ define as before the set of all coalitions whose excess at x is greater or equal than $\alpha \in \mathbb{R}$ by $\mathcal{D}(\alpha,x,V) := \left\{S \in 2^N \,\middle|\, e^{(N,V)}(S,x,\beta) \geq \alpha\right\}$.

Theorem 4.18

Let $(N,V) \in \Gamma^{qH}$ be a game and let $\beta \in \mathcal{B}^c$ be a reference function. Let $x \in V(N)$ be an imputation. Then $x = \mathcal{P}N^{\beta}(N,V)$ if and only if $\mathcal{D}(\alpha,x,V)$ is balanced for all $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\alpha,x,V) \neq \emptyset$.

Proof:

Let $(N,V) \in \Gamma^{qH}$ be a game and assume w.l.o.g. that (N,V) is simplex. Let $\beta \in \mathcal{B}^c$ be a reference function. In view of Lemma 4.17 we proof the Theorem by showing that $x = \mathcal{P}N^{\beta}(N,V)$ if and only if every such non-empty collection has property I.

1. $\Longrightarrow$: *Denote by $x := \mathcal{P}N^{\beta}(N,V)$ the β-prenucleolus of (N,V). Let $\alpha \in \mathbb{R}$ such that $\mathcal{D} := \mathcal{D}(\alpha, x, V) \neq \emptyset$.*

   Let $y \in \mathbb{R}^N$ such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{D}$. Define $z(t) := x + ty, t \in \mathbb{R}$. Then $z(t) \in \partial V(N)$ for every $t \in \mathbb{R}$ holds because of the simplex assumption on (N,V). Furthermore, again because of $p_V^N = 1_N$,

$$\begin{aligned} e^{\beta}(S, z(t)) &= \left\langle \beta(S) - z(t)_S, p_V^N|_S \right\rangle \\ &= \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle - \left\langle t y_S, p_V^N|_S \right\rangle \\ &= \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle - \underbrace{t y(S)}_{\geq 0} \\ &\leq e^{\beta}(S, x) \end{aligned}$$

   holds for every $t \in \mathbb{R}$ with $t \geq 0$. Since $\bar{t}$ can be chosen small enough such that $\mathcal{D}$ remains the set of minimal satisfied coalitions at $z(\bar{t})$, the existence of a coalition $\bar{S} \in \mathcal{D}$ with $y\left(\bar{S}\right) > 0$ implies that x is not the β-prenucleolus, contrary to our assumption. Thus $y(S) = 0$ for every $S \in \mathcal{D}$.

2. $\Longleftarrow$: *Now let $x \in \partial V(N)$ such that every non-empty $\mathcal{D}(\alpha, x, V), \alpha \in \mathbb{R}$, has property I. Let $z := \mathcal{P}N^{\beta}(N,V)$ and $y := z - x$. Let e_1 denote the*

maximal occuring β-excess at x and let $S \in \mathcal{D}(e_1, x, V)$ be a coalition that attains this β-excess. We have $e^\beta(S,z) \le e^\beta(S,x)$ which yields

$$\begin{aligned} & e^\beta(S,z) - e^\beta(S,x) \le 0 \\ \Leftrightarrow\quad & \left\langle \beta(S) - z_S, p_V^N|_S \right\rangle - \left\langle \beta(S) - x_S, p_V^N|_S \right\rangle \le 0 \\ \Leftrightarrow\quad & z(S) - x(S) \ge 0 \\ \Leftrightarrow\quad & y(S) \ge 0 \end{aligned}$$

Therefore we can imply $y(S) = 0$ for every $S \in \mathcal{D}(e_1, x, V)$ due to property I. An analog computation for every other occuring β-excess at x (from 2^{nd} highest excess to lowest successively) finally yields $y(S) = 0$ for every $S \in 2^N$. Hence $x = z = \mathcal{P}N^\beta(N,V)$. ■

This proof is an almost verbatim copy of the proof of the Kohlberg criterion for the TU prenucleolus in [Pel88].

Remark 4.19

A careful inspection of the use of the simplex assumption in the proof of Theorem 4.18 reveals what can be said about those cases in which $\beta \in \overline{\mathcal{B}}$ is not covariant and hence simplexity can not be assumed. If we say that $\mathcal{S} \subseteq 2^N$ has property $I(p)$ for $p \in \mathbb{R}^N_{++}$ whenever $\langle y, p\rangle = 0$ and $\langle y_S, p_S\rangle \ge 0$ for every $S \in \mathcal{S}$ imply $\langle y_S, p_S\rangle = 0$ for every $S \in \mathcal{S}$, then $x = \mathcal{P}N^\beta(N,V)$ if and only if every non-empty $\mathcal{D}(\alpha, x, V)$ has property $I\left(p_V^N\right)$. This was also noticed in [CC02].

So far no severe restrictions to the choice of the reference function β were made, i.e. Theorem 4.12 and Theorem 4.18 about single-valuedness and the

Kohlberg criterion are true for every choice of $\beta \in \mathcal{B}^c$. This fact establishes a quite comfortable basis for the following investigations, since no matter what subset of $\mathcal{B}^c$ is under current consideration, the β-prenucleolus exists, is even single-valued and covariant and the validness of the Kohlberg criterion makes computations of β-prenucleoli much more easier.

A question that arises in connection with every solution concept for NTU games is its behavior on the class of TU games Γ^{TU}. If a solution concept on Γ^{NTU} claims to be an extension of a solution concept on Γ^{TU}, it has to coincide on Γ^{TU} with the (TU) solution it stems from, otherwise it would not be an "extension". As we claimed to extend the TU prenucleolus to Γ^{NTU}, we have to examine the behavior of $\mathcal{P}N^{\beta}$ on Γ^{TU}.

As we pointed out in section 3 the coincidence of $K\mathcal{N}^l$ with the nucleolus on TU games is not independent of the choice of a K-excess function. For the β-prenucleolus this coincidence is valid for every β-excess function, for which the axiomatization (Theorem 4.4) is valid, i.e. for $\beta \in \overline{\mathcal{B}}$.

Lemma 4.20

Let $(N, v) \in \Gamma^{TU}$ be a TU game and let $(N, V^v) \in \Gamma^H$ be its associated NTU (simplex) game. Then $\mathcal{P}N\,(N, v) = \mathcal{P}N^{\beta}\,(N, V^v)$ holds true for every $\beta \in \bar{\mathcal{B}}$.

Proof:

Remember that $\beta \in \bar{\mathcal{B}}$ means $\sum\limits_{i \in S} \beta_i^{(N,V^v)}(S) = v(S)$ for every coalition $S \in 2^N$. Thus the β-excess of a coalition $S \in 2^N$ w.r.t. an imputation $x \in V(N)$

computes as

$$\begin{aligned} e^{(N,V^v)}(S,x) &= \left\langle \beta(S) - x_S, p^N_{V^v}|_S \right\rangle \\ &= \sum_{i \in S} \beta_i(S) - \sum_{i \in S} x_i \\ &= v(S) - x(S) \end{aligned}$$

and thus coincides with the TU excess function. From this the statement follows immediately. ■

4.3 A useful expression of $\mathcal{P}N^\beta$ via the TU prenucleolus

The proof of Lemma 4.20 motivates the following considerations. Let $(N,V) \in \Gamma^{qH}$ be a simplex game and let $\beta \in \mathcal{B}$ be a reference function. The excess of a coalition $S \in 2^N$ at $x \in \mathbb{R}^N$ computes as

$$\begin{aligned} e^{(N,V)}\left(S,x,\beta\right) &= \left\langle \beta(S) - x_S, p^N_V|_S \right\rangle \\ &= \sum_{i \in S} \beta_i^{(N,V)}(S) - x(S). \end{aligned}$$

Note the resemblance to the TU excess function. Thus the reference function β induces a TU game $\left(N, v^\beta_V\right) \in \Gamma^{TU}$ via

$$v^\beta_V(S) := \sum_{i \in S} \beta_i^{(N,V)}(S) \quad \forall S \in 2^N \tag{5}$$

with the property that for every coalition $S \in 2^N$ and every imputation $x \in \mathbb{R}^N$ the (TU) and (NTU) excesses are equal:

$$e^{(N,V)}\left(S,x,\beta\right) = e\left(S,x,v^\beta_V\right)$$

and thus the β-prenucleolus and the (TU) prenucleolus coincide:

$$\mathcal{P}N^{\beta}(N,V) = \mathcal{P}N\left(N, v_V^{\beta}\right).$$

This fact will not only help in computing $\mathcal{P}N^{\beta}$ (see section 5.3 for an elaboration on this) but also yields conclusions about $\mathcal{P}N^{\beta}$.

Definition 4.21

Let $(N,V) \in \Gamma^{qH}$ be a simplex game and let $\beta \in \mathcal{B}$ be a reference function. The TU game $\left(N, v_V^{\beta}\right)$ as defined by (5) is called the **(TU) $\boldsymbol{\beta}$-game** *of the NTU game (N,V).*

Note that from this definition and from the definition of the reference function $v_V^{\beta}(N) = c_V^N$ follows.

Lemma 4.22

Let $(N,V) \in \Gamma^{qH}$ be a game and let $\beta \in \mathcal{B}^c$ be a reference function. Let $(N,W) \in \Gamma^{qH}$ be a simplex game which is derived from (N,V) by a linear transformation of utility[16]. If v_W^{β} is individual superadditive, i.e. $v_W^{\beta}(S) + v_W^{\beta}(\{i\}) \leq v_W^{\beta}(S \cup \{i\}) \quad \forall S \in 2^{N\setminus\{i\}}, i \in N$, then $\mathcal{P}N^{\beta}(N,V)$ is individual rational.

Proof:

In view of the comments about the (TU) β-game of an NTU game above, we show that $\mathcal{P}N\left(v_W^{\beta}\right)$ is individual rational. To this end, let $x := \mathcal{P}N\left(v_W^{\beta}\right)$

[16] I.e., $(N,W) = \left(N, p_V^N V\right)$.

and suppose there exists a player $i \in N$ such that $x_i < v_W^\beta(\{i\})$. For every coalition $S \in 2^{N\setminus\{i\}}$ that does not contain i the following is true:

$$\begin{aligned} e\left(S, x, v_W^\beta\right) &= v_W^\beta(S) - x(S) \\ &< v_W^\beta(S) + v_W^\beta(\{i\}) - x(S) - x_i \\ &\leq v_W^\beta(S \cup \{i\}) - x(S \cup \{i\}) \\ &= e\left(S \cup \{i\}, x, v_W^\beta\right). \end{aligned}$$

Thus coalitions that attain maximal excess under the imputation x must all contain player i. Since $e\left(N, x, v_W^\beta\right) = 0$ and $e\left(\{i\}, x, v_W^\beta\right) > 0$, N is not a coalition with maximal excess. It follows that the collection of coalitions with maximal excess is not balanced, contrary to our assumption that x is the prenucleolus of v_W^β. ■

This proof is normally given within the TU context to show the following corollary.

Corollary 4.23

If a TU game (N, v) is individual superadditive, then $\mathcal{PN}(N, v)$ is individual rational.

If $(N, V) \in \Gamma^{qH}$ is not a simplex game but $\beta \in \mathcal{B}^c$ is covariant, then we can also describe the (NTU) β-prenucleolus of (N, V) by the (TU) prenucleolus of a suitably chosen TU game. Since the game $\left(N, p_V^N V\right)$ is a simplex game, we have

$$\mathcal{PN}^\beta\left(N, p_V^N V\right) = \mathcal{PN}\left(N, v^\beta_{p_V^N V}\right).$$

Now the covariance of β and of $\mathcal{P}N^{\beta}$ (Lemma 4.14) yield

$$\mathcal{P}N^{\beta}(N,V) = \frac{1}{p_V^N}\mathcal{P}N\left(N, v^{\beta}_{p_V^N V}\right).$$

4.4 $\mathcal{P}N^{\beta}$ and the core

The core is a very well established solution concept both for TU and for NTU games. Therefore it is considered a major advantage of the (TU) prenucleolus that it is always a member of the core whenever the latter is non-empty. This is important in situations where there is demand for (in the core-sense) stable but *single-valued* solutions for TU games. Sometimes the (TU) prenucleolus is therefore said to be a core-selector.

As noticed in section 3, the Kalai prenucleoli for NTU games are also contained in the NTU core, when it exists. We will now investigate this property in connection with the class of β-prenucleoli.

The simplicity of the proofs that both the (TU) prenucleolus and the Kalai (NTU) prenucleoli are core-selectors is due to the fact that the respective cores can be defined as those imputations yielding non-positive excesses. This is not true for the β-prenucleoli and, moreover, core-inclusion will turn out to be a property of some reference functions on some subclass of games. We begin with a counterexample.

Example 4.24

Let $N = \{1, 2, 3\}$ and consider the hyperplane game as given by Table 6.

S	p_V^S	c_V^S
$\{1\}$	(1,0,0)	0
$\{2\}$	(0,1,0)	0
$\{3\}$	(0,0,1)	0
$\{1,2\}$	(5,10,0)	7
$\{1,3\}$	(8,0,10)	17
$\{2,3\}$	(0,3,7)	11
$\{1,2,3\}$	(9,1,9)	16

Table 6: The game of example 4.24

The core of this game is a singleton:

$$\mathbf{Core}\,(N,V) = \left\{\left(0, \frac{7}{10}, \frac{17}{10}\right)\right\}.$$

Take, for example, the reference functions β_1 and β_2 as defined in example 4.11, then

$$\begin{aligned} \mathcal{P}N^{\beta_1}\,(N,V) &= \left(\frac{48221}{60480}, -\frac{97}{160}, \frac{63373}{60480}\right) \\ &\approx (0.7973; -0.60625; 1.0478)\,, \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}N^{\beta_2}\,(N,V) &= \left(\frac{23609}{22680}, -\frac{9131}{1260}, \frac{34973}{22680}\right) \\ &\approx (1.0410; -7.2478; 1.5420)\,, \end{aligned}$$

thus $\mathcal{P}N^{\beta_i}(N,V) \notin \mathbf{Core}(N,V)$ *for* $i = 1, 2$. *Furthermore, with Lemma 4.22 in mind we can imply that each respective* β*-game of* (N,V) *is not individual superadditive.*

Example 4.24 also serves to prove the next lemma, which states an impossibility.

Lemma 4.25

There exists a game $(N,V) \in \Gamma^H$ *with* $|N| = 3$, *such that* $\mathbf{Core}(N,V) \neq \emptyset$ *and* $\mathcal{P}N^{\beta}(N,V) \notin \mathbf{Core}(N,V)$ *for every* $\beta \in \bar{\mathcal{B}}$ *with* $\beta^{(N,V)}(S) \in \partial V(S) \cap \mathbb{R}^S_{++} \quad \forall S \in 2^N$.

Proof:

Consider the game of example 4.24 and denote by $x := \left(0, \frac{7}{10}, \frac{17}{10}\right)$ *its unique core-element. Let* $\beta \in \bar{\mathcal{B}}$ *be a reference function, such that* $\beta^{(N,V)}(S) \in \partial V(S) \cap \mathbb{R}^S_{++}$. *Then* $e^{\beta}(\{1,2\}, x) > 0$ *and* $e^{\beta}(\{1,3\}, x) > 0$. *Since the* β*-excess for the coalitions* $\{1\}$ *and* N *are zero and the* β*-excesses for the coalitions* $\{2\}$ *and* $\{3\}$ *are negative, the* β*-excess for the coalition* $\{2,3\}$ *must be positive - and furthermore equal to* $e^{\beta}(\{12\}, x)$ *and* $e^{\beta}(\{13\}, x)$ *- for the collection of coalitions with highest excesses to be balanced.*

For $0 < \lambda < 1$ *let* $\beta^{\lambda} := \lambda\left(0, 0, \frac{11}{7}\right) + (1-\lambda)\left(0, \frac{11}{3}, 0\right)$ *be any choice of*

$\beta(\{2,3\})$ on the open line segment $\partial V(\{2,3\}) \cap \mathbb{R}^{\{2,3\}}_{++}$. Then

$$\begin{aligned} e^{\beta^{\lambda}}(\{2,3\},x) =& \left((1-\lambda)\frac{11}{3} - \frac{7}{10}\right) + \left(\lambda\frac{11}{7} - \frac{17}{10}\right) \cdot 9 \\ =& \lambda\frac{220}{21} - \frac{37}{3} \end{aligned}$$

Thus $e^{\beta^{\lambda}}(\{2,3\},x) > 0$ if and only if $\lambda > \frac{259}{220} > 1$, a contradiction. ■

4.5 The β-core

The fact that the core of TU games consists of those imputations that yields non-positive (TU) excesses for all coalitions motivates the definition of the so-called (NTU) β-core which are those points that yield non-positive β-excesses for all coalitions. A brief discussion of this solution concept follows.

Definition 4.26
Let $(N,V) \in \Gamma^{qH}$ be a game and let $\beta \in \mathcal{B}$ be a reference function. The set

$$\mathbf{Core}^{\beta} := \left\{x \in V(N) \,\middle|\, e^{\beta}(S,x) \leq 0 \quad \forall S \in 2^N \right\}$$

*is called the **β-core** of the game (N,V).*

We will not undertake a detailed analysis of the β-core for all possible choices of a reference function $\beta \in \mathcal{B}$. We rather make the following more or less obvious remarks about this solution concept.

The β-core is efficient. As well as the (NTU) core of hyperplane games, the (NTU) β-core is always a convex compact polyhedron that might be empty.

Whenever it is not empty, it contains the β-prenucleolus, which is therefore a β-core-selector. Non-emptiness of the core does not imply non-emptiness of the β-core, as the game and the reference function used in example 4.24 show. For this game the β-core is empty while the core is not. For those reference functions that give rise to the definition of the NTU prenucleolus in chapter 5 we will provide a sufficient and necessary condition for the β-core to be non-empty (see section 5.7).

4.6 Continuity

A desirable property of solution concepts for coalitional games with or without transferable utility is robustness against small pertubations of the game. This property is best described by the concept of continuity, i.e. when there is a converging sequence of games, then the solutions of these games should converge to the solution of the limit game.

In order to speak of converging sequences of games we first have to specify a metric on the space of all games. For TU games this can be done quite canonically. Considering the set of players N as fixed, a TU game (N, v) is given by the values $v(S)$ for all coalitions $S \in 2^N$. Since $v(\emptyset) = 0$ is fixed, the class of all TU games with player set N can be identified with $\mathbb{R}^{2^{|N|}-1}$ and thus we can define metrics on games and indeed talk about convergence of games.

It is well known ([Sch69]) that the (pre-)nucleolus is continuous on Γ^{TU} for fixed player set N. We formulate this fact as a theorem because it will prove useful in the analysis of the continuity of the (NTU) β-prenucleoli.

For a fixed player set N let therefore $\Gamma_N^{TU} := \left\{(N,v) \,\middle|\, (N,v) \in \Gamma^{TU}\right\}$ be the set of all TU games with N as the grand coalition.

Theorem 4.27 ([Sch69])
The (TU) prenucleolus $\mathcal{P}N : \Gamma_N^{TU} \to \mathbb{R}^N$ is continuous.

Turning to the analysis of the continuity of the β-prenucleoli we need to define a metric on NTU games, resp. on the subclass of quasi hyperplane games which is the domain of the β-prenucleoli. Let therefore the player set N be fixed and let $\| \; \| : \mathbb{R}^N \to \mathbb{R}$ be any norm on $\mathbb{R}^N$.

Definition 4.28
Let $x \in \mathbb{R}^N$ and $A, B \subset \mathbb{R}^N$. Define

$$d_0(x, A) := \inf_{a \in A} \|x - a\|$$

and

$$d(A, B) := \max\left\{\sup_{b \in B} d_0(b, A), \sup_{a \in A} d_0(a, B)\right\}. \tag{6}$$

Remark 4.29
If for a set $A \subset \mathbb{R}^N$ and for $r \in \mathbb{R}, r > 0$, we define the open r-neighbourhood of A via

$$N_r(A) := \left\{y \in \mathbb{R}^N \,|\, \|x - y\| < r \text{ for some } x \in A\right\},$$

then an equivalent definition of (6) is

$$d\,(A,B) = \inf\left\{r \in \mathbb{R}_{++} \,|A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\right\}.$$

The function $d : 2^{\mathbb{R}^N} \times 2^{\mathbb{R}^N} \to \mathbb{R} \cup \{\infty\}$ is the Hausdorff distance of subsets of $\mathbb{R}^N$. See, for example, [Hau44] or [Edg90]. If it is defined only on the non-empty and compact subsets it is even a metric. We cannot use the Hausdorff distance like that because we want to find a metric on sets that are comprehensive and thus not compact. In this case the Hausdorff distance might be infinite.

We show that we can nevertheless use the distance d of Definition 4.28 to define a metric on those subsets of $\mathbb{R}^N$ that we are interested in.

For any $a \in \mathbb{R}^N$ define therefore $G(a) := \left\{x \in \mathbb{R}^N \,|x \leq a\right\}$ and let

$$\mathcal{K} := \left\{X \subseteq \mathbb{R}^N \,\middle|\, \exists a \in \mathbb{R}^N : X \subseteq G(a), X \text{ is comprehensive and closed}\right\}$$

be the set of all comprehensive closed subsets of $\mathbb{R}^N$ that are contained in $G(a)$ for some $a \in \mathbb{R}^N$. Within this setup already Kalai used the Hausdorff distance to define a metric on the game space, apparently the details are to be found in his Ph.D. Thesis ([Kal72]). However, we will perform a proof of this fact that is an almost verbatim copy of the proof of the metric property of d on compact sets, see for example [Edg90].

Lemma 4.30

The Hausdorff distance (Definition 4.28) is a metric on $\mathcal{K}$.

Proof:

Symmetry and positiveness are clear. Let $A, B, C \in \mathcal{K}$*. If* $A = B$ *then* $A \subseteq N_r(B)$ *and* $B \subseteq N_r(A)$ *for every* $r > 0$*. Thus* $d(A, B) = 0$*. If, conversely,* A *and* B *satisfy* $d(A, B) = 0$*, then for each* $a \in A$ *we have* $a \in N_r(B)$ *for every* $r > 0$ *thus* $d_0(a, B) = 0$*. Since* B *is closed it follows that* $x \in B$ *holds and hence* $A \subseteq B$*. The converse* $B \subseteq A$ *is clearly true by the same argument. Hence* $A = B$*.*

For the finiteness part notice that there exists $\bar{a} \in \mathbb{R}^N$ *such that* $A \subseteq G(\bar{a})$ *and* $B \subseteq G(\bar{a})$*. Notice further that the comprehensiveness of* A *and* B *guarantee* $d(A, G(\bar{a})) < \infty$ *and* $d(B, G(\bar{a})) < \infty$*. Then if* $G(\bar{a}) \subseteq N_r(A)$ *for some* $0 \leq r < \infty$ *then* $B \subseteq N_r(A)$ *because of* $B \subseteq G(\bar{a})$ *and vice versa. Hence* $d(A, B) < \infty$*.*

Finally, to show the triangle inequality, let $\epsilon > 0$ *and* $a \in A$*. Then there exists* $b \in B$ *such that* $\|a - b\| < d(A, B) + \epsilon$*. Then there is also* $c \in C$ *such that* $\|b - c\| < d(B, C) + \epsilon$*. This means that* A *is within the* $(d(A, B) + d(B, C) + 2\epsilon)$*-neighbourhood of* C *and vice versa. Therefore* $d(A, C) \leq d(A, B) + d(B, C) + 2\epsilon$ *holds. Since this is true for every* $\epsilon > 0$ *the proof is complete.* ■

Now we are able to define a metric on the following subclass of all quasi hyperplane games. Let $\Gamma^{\mathcal{K}} \subset \Gamma^{qH}$ be the subclass of the class of all quasi hyperplane games whose members additionally satisfy

$$\text{for every } S \in 2^N \setminus \{N\} \text{ there exists } a^S \in \mathbb{R}^S \text{such that } V(S) \subseteq G\left(a^S\right).$$

By our basic assumptions on quasi hyperplane games each $V(S)$ is also closed and comprehensive, i.e. $V(S) \in \mathcal{K}$ for every $S \in 2^N \setminus \{N\}$.

Then by

$$d\left(\left(N,V\right),\left(N,W\right)\right) := \max\left\{\left(d\left(V(S),W(S)\right)\right)_{S\in 2^N\setminus\{N\}}, \left\|\frac{p_V}{c_V}-\frac{p_W}{c_W}\right\|\right\}$$

for all games $\left(N,V\right),\left(N,W\right)\in\Gamma^{\mathcal{K}}$, a metric on $\Gamma^{\mathcal{K}}$ is defined. Thus we can say that

$$\lim_{k\to\infty}\left(N,V^k\right)=\left(N,V\right)$$

for all (N,V), $\left(N,V^k\right)\in\Gamma^{\mathcal{K}}, k\in\mathbb{N}$, when

$$\lim_{k\to\infty} d\left(\left(N,V^k\right),\left(N,V\right)\right)=0$$

holds true.

After these technical preliminaries we can now analyse the continuity property of the β-prenucleoli on $\Gamma^{\mathcal{K}}$. It seems obvious that continuity of $\mathcal{P}N^\beta$ can at most be achieved for continuous reference functions $\beta:\Gamma^{\mathcal{K}}\to X^N$, i.e. β must satisfy

$$\lim_{k\to\infty}\beta\left(N,V^k\right)=\beta(N,V)$$

for every sequence $\left(\left(N,V^k\right)\right)_{k\in\mathbb{N}}$ of games[17] with

$$\lim_{k\to\infty}\left(N,V^k\right)=\left(N,V\right),$$

where $(N,V)\in\Gamma^{\mathcal{K}}$ and $\left(N,V^k\right)\in\Gamma^{\mathcal{K}}\quad\forall k\in\mathbb{N}$. Let $\mathcal{B}^{cc}$ be the set of all continuous and covariant reference functions on $\Gamma^{\mathcal{K}}$. We discuss a possible weakening of this continuity assumption in section 5.6.

The next theorem states that this requirement is indeed sufficient to prove the continuity of $\mathcal{P}N^\beta$.

[17] With a standard metric on X^N which can be seen as $\prod_{S\in 2^N}\mathbb{R}^S$.

Theorem 4.31

Let $\beta \in \mathcal{B}^{cc}$ be a covariant and continuous reference function. Then the β-prenucleolus $\mathcal{P}N^{\beta}$ is continuous on $\Gamma^{\mathcal{K}}$.

Proof:

Let $\left(N, V^k\right) \in \Gamma^{\mathcal{K}} \quad \forall k \in \mathbb{N}$ and $(N,V) \in \Gamma^{\mathcal{K}}$ be given such that $\lim_{k\to\infty}\left(N, V^k\right) = (N,V)$ in the metric as defined above. Let $\beta \in \mathcal{B}^{cc}$ be a covariant and continuous reference function.

Let $\nu^k := \mathcal{P}N^{\beta}\left(N, V^k\right) \quad \forall k \in \mathbb{N}$ be the β-prenucleoli of the games $\left(N, V^k\right), k \in \mathbb{N}$, and let $\nu := \mathcal{P}N^{\beta}(N,V)$. We have to show that

$$\lim_{k\to\infty} \nu^k = \nu$$

holds.

From the definition of the (TU) β-games we can easily imply

$$\lim_{k\to\infty}\left(N, v^{\beta}_{V^k}\right) = \left(N, v^{\beta}_{V}\right)$$

and thus also

$$\lim_{k\to\infty}\left(N, v^{\beta}_{p^N_{V^k}V^k}\right) = \left(N, v^{\beta}_{p^N_V V}\right)$$

holds true. From the continuity of the TU prenucleolus we can imply

$$\lim_{k\to\infty}\mathcal{P}N\left(N, v^{\beta}_{p^N_{V^k}V^k}\right) = \mathcal{P}N\left(N, v^{\beta}_{p^N_V V}\right)$$

and thus we have

$$\lim_{k\to\infty}\frac{1}{p^N_{V^k}}\mathcal{P}N^{\beta}\left(N, v^{\beta}_{p^N_{V^k}V^k}\right) = \frac{1}{p^N_V}\mathcal{P}N^{\beta}\left(N, v^{\beta}_{p^N_V V}\right)$$

which means nothing else than[18]

$$\lim_{k\to\infty} \nu^k = \nu. \qquad \blacksquare$$

4.7 Extension to general NTU games

Now that we have established a prenucleolus on a subclass of all NTU games, namely the class of all quasi hyperplane games Γ^{qH}, we now propose an extension of this solution concept to a more general class of NTU games. The way this extension will be carried out is motivated by a Theorem of Aumann ([Aum85]), as we shall mention at the end of this section.

Suppose now that we consider an NTU game for which the set $V(N)$ is no longer a halfspace but an arbitrary closed, convex and comprehensive set with a smooth boundary. Smoothness means that the function $p : \partial V(N) \to \mathbb{R}^N$ that maps each point of the boundary of $V(N)$ to a normal vector of the hyperplane that weakly separates this point from $V(N)$ is continuous. We shall assume that $\sum_{i\in N} p_i(x) = 1$ for every $x \in \partial V(N)$.

If a point $x \in \partial V(N)$ on the boundary of $V(N)$ is given we could replace $V(N)$ by the halfspace the boundary of which is the hyperplane that weakly separates x from $V(N)$ (a normal of this hyperplane is thus given by $p(x)$) then the game so constructed is a quasi hyperplane game. It is therefore a member of the domain of the β-prenucleolus which can thus be calculated for this game. If it so happens that this β-prenucleolus coincides with x then

[18] Remember the representation of the β-prenucleoli by TU prenucleoli of suitably chosen TU games, section 4.3.

x is a canonical candidate for being a β-prenucleolus point of the original game. Of course the question of existence and uniqueness of such a point lies at hand.

Let us start with describing that class of NTU games for which we can hope to answer these questions. The informal description we gave above how we wish to find a β-prenucleolus of general NTU games suggests that we need some fixed point argumentation. For this the continuity of the function $p : \partial V(N) \to \mathbb{R}^N$ is of course essential. Further we must exclude games for which parts of the boundary of $V(N)$ admit of normal vectors with components equal to zero. At those points we can not define quasi hyperplane games in the way described above. By the same argument we can neither allow points on $\partial V(N)$ to have normal vectors with negative components. This we can easily achieve by requiring comprehensiveness of $V(N)$ which is indeed a standard assumption.

To summarize and to formalize this let $\Gamma^l \subset \Gamma^{NTU}$ be the class of NTU games whose members satisfy the following assumptions:

1. For every proper subcoalition $S \in 2^N, S \neq N$, $V(S)$ meets the assumptions of Definition 2.3.

2. $V(N)$ is a closed, convex and comprehensive subset of $\mathbb{R}^N$ such that $V(N) \cap \mathbb{R}^N_+$ is non-empty and compact and the function $p : \partial V(N) \to \mathbb{R}^N$ that maps each $x \in \partial V(N)$ to a normal vector of the hyperplane that weakly separates x from $V(N)$ satisfies

   (a) p is continuous,

(b) $\sum_{i \in N} p_i(x) = 1 \quad \forall x \in \partial V(N)$, and

(c) there exists $\epsilon > 0$ such that $p_i(x) \geq \epsilon$ for all $x \in \partial V(N)$ and for all $i \in N$.

Condition 2c can be seen as a form of a non-levelness condition which is frequently used for the analysis of NTU games. It is though a stronger assumption for it additionally prevents the normal vectors at the boundary of $V(N)$ to have components which are arbitrarily closed to zero.

Next we formalize the replacement of the set $V(N)$ by a halfspace. See also Figure 5.

Definition 4.32

Let $(N, V) \in \Gamma^l$ be a game. For every $x \in \partial V(N)$ define the quasi hyperplane game $(N, V^x) \in \Gamma^{qH}$ by

$$\begin{aligned} V^x(S) &:= V(S) \quad \forall S \in 2^N, S \neq N, \text{ and} \\ V^x(N) &:= \left\{ y \in \mathbb{R}^N \,|\, \langle y, p(x) \rangle \leq \langle x, p(x) \rangle \right\}. \end{aligned}$$

For every $x \in \partial V(N)$ the game (N, V^x) coincides with (N, V) on all proper subsets of N, while $V(N)$ is replaced by the halfspace that is given by x and $p(x)$, as described in the introduction of this section.

The definition of β-prenucleoli for games in Γ^l can now be formalized as follows.

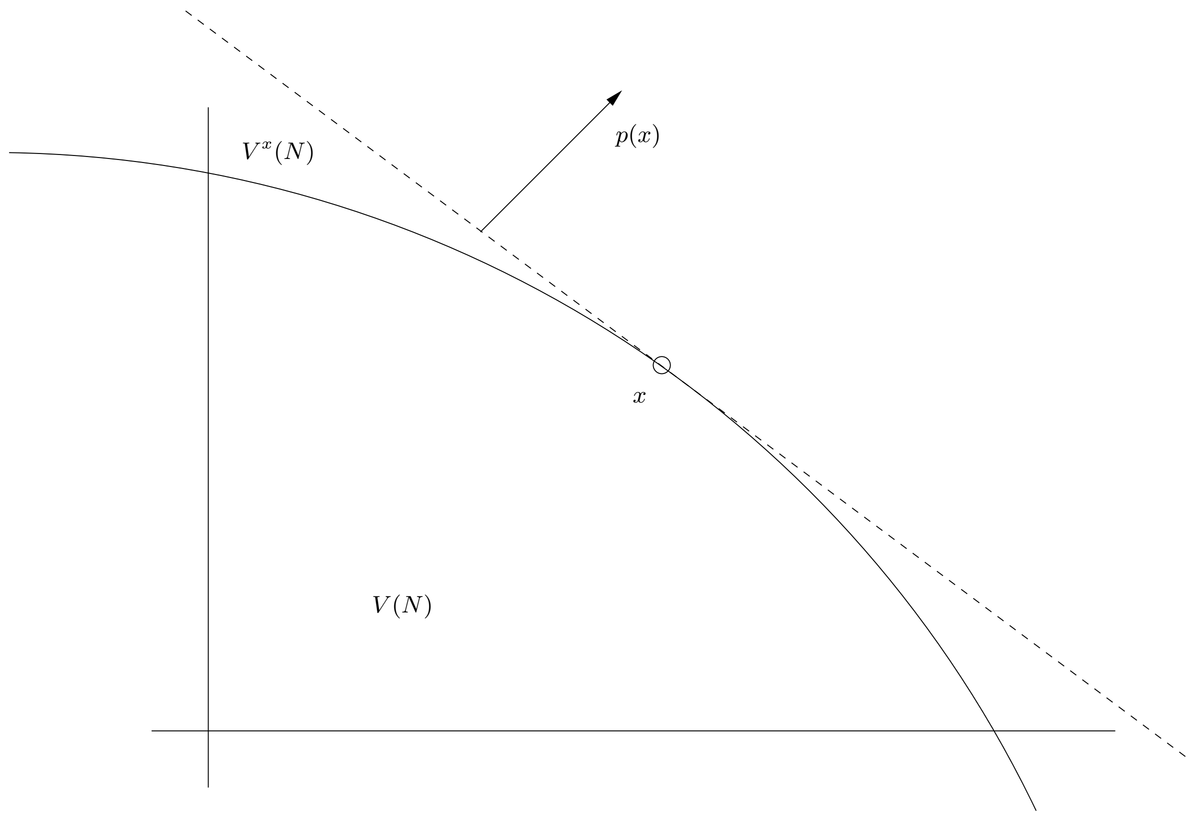

Figure 5: Construction of the game V^x

Definition 4.33

Let $(N, V) \in \Gamma^l$ be a game. Let $\beta \in \mathcal{B}^{cc}$ be a reference function. A point $x \in \partial V(N)$ is called **$\boldsymbol{\beta}$-prenucleolus** *of (N, V) if $x = \mathcal{P}N^\beta(N, V^x)$. Denote by $\mathcal{P}N^\beta(N, V)$ the set of all β-prenucleoli of (N, V).*

The question of non-emptiness of $\mathcal{P}N^\beta$ can be answered affirmatively as follows.

Theorem 4.34

Let $(N,V) \in \Gamma^l$ be a game and let $\beta \in \mathcal{B}^{cc}$ be a reference function. Then $\mathcal{P}N^\beta(N,V) \neq \emptyset$ holds true.

Proof:

Let $C := \left\{ \lambda \in \mathbb{R}^N \,\middle|\, \lambda(N) = 1, \sup_{x \in V(N)} \langle x, \lambda \rangle < \infty \right\}$ be the set of all (normalized) vectors that give rise to hyperplanes that are tangent to $V(N)$[19]. By our assumption on games in Γ^l, C is a compact and convex set of strictly positive vectors of $\mathbb{R}^N$.

Let $f^0 : C \to \partial V(N)$ be the function that maps each $c \in C$ to the point $x \in \partial V(N)$ such that $c = p(x)$. f^0 is a continuous function because of the continuity of p.

Let $f^1 : \partial V(N) \to \Gamma^{qH}$ be defined by $f^1(x) := (N, V^x)$ (see Definition 4.32). Again f^1 and hence $f^1 \circ f^0$ are continuous. The β-prenucleolus function $\mathcal{P}N^\beta : \Gamma^{qH} \to \mathbb{R}^N$ is continuous due to Theorem 4.31, so up to now we have constructed a continuous function from C to $\mathbb{R}^N$ by $\mathcal{P}N^\beta \circ f^1 \circ f^0$.

Further let $f^2 : \mathbb{R}^N \to \partial V(N)$ map each point $x \in \mathbb{R}^N$ to the point in $\partial V(N)$ that is nearest to x in some metric on $\mathbb{R}^N$. Since f^2 is continuous and also p is a continuous function from $\partial V(N)$ to C we can finally define the following function. Let $f : C \to C$ be defined by $f \equiv p \circ f^2 \circ \mathcal{P}N^\beta \circ f^1 \circ f^0$, then f is a continuous function from C into C itself. Therefore there exists $\bar{c} \in C$ such that $f(\bar{c}) = \bar{c}$ by Brouwer's fixed point theorem.

[19] C can also be seen as the closure of $p(\partial V(N))$.

It is now easily verified that $x := f^0(\bar{c})$ *satisfies* $x = \mathcal{P}N^\beta(N, V^x)$ *and therefore the proof is complete.* ■

This proof has been very much inspired by Robert J. Aumann's proof of the existence of the NTU Shapley value in [Aum85].

The question of uniqueness of the β-prenucleolus on Γ^l must at present be left as open.

Remark 4.35

The continuity property of the β*-prenucleolus on* Γ^{qH} *(Theorem 4.31) was used in the proof of Theorem 4.34 but in fact a weaker form of continuity would have been sufficient for this proof. What was actually used is "continuity in* p^N*", with fixed sets* $V(S)$, $S \in 2^N$ *and* $S \neq N$*. It is a straightforward corollary of Theorem 4.31 that the* β*-prenucleolus is also continuous in this sense.*

5 The NTU prenucleolus

In chapter 4 we have introduced the new class of β-excess functions for NTU games and the according class of NTU β-prenucleoli. Of course among the members of this class there exist prenucleoli which fail to satisfy some essential conditions such as covariance, coincidence with the TU prenucleolus on the class of all TU games and the like. Thus not much more than the definition of these solution concepts seem to justify the name *prenucleolus*, although we have already identified subsets of the set of all reference functions $\mathcal{B}$ which give rise to covariant β-prenucleoli and to β-prenucleoli that coincide with the TU prenucleolus.

In this chapter we will examine more detailed a subclass of some NTU β-excess functions which all yield the same NTU β-prenucleolus. This NTU β-prenucleolus (denoted by $\mathcal{P}N$) is covariant, symmetric, single-valued, continuous, efficient, monotonic[20] and satisfies the Kohlberg criterion. Moreover it also satisfies RGP with respect to a new reduced game and is contained in the core for a subclass of games. It should be noticed at this point that for the TU prenucleolus the properties single-valuedness, covariance, RGP and anonimity are enough requirements to characterize it uniquely. This axiomatization is due to Sobolev ([Sob75]). It is one of the most interesting open questions about $\mathcal{P}N$ whether or not some of its properties also constitute an axiomatization in the NTU case. In any case, we feel that they are certainly enough reason to justify the name 'NTU prenucleolus'.

[20]in a special sense, see section 5.2

5.1 Maximal feasible reference functions and the NTU prenucleolus

In the sequel we will have to deal with some form of monotonicity for hyperplane games, but we encounter a problem with the known concepts as described in section 2.

Lemma 5.1

1. *Hyperplane games are always monotonic (Definition 2.6).*

2. *If a hyperplane game $(N, V) \in \Gamma^H$ is individual superadditive (Definition 2.7), then (N, V) is strategically equivalent (under a linear transformation of utility) to a simplex game (N, V^v) belonging to some TU game (N, v).*

Proof:

1. *Since $p_V^S > 0 \quad \forall S \in 2^N$, it is immediately clear that the projection of any $V(T)$ on $\mathbb{R}^S, S \subset T$, is always $\mathbb{R}^S$ itself, thus it contains $V(S)$.*

2. *If (N, V) is individually superadditive, then $p^S = p^T|_S$ must hold for all $S \subset T, S, T \in 2^N$. Thus $p^S = p^N|_S \quad \forall S \in 2^N$.* ■

We will use yet another concept of monotonicity, which we call *weak individual superadditivity.*

Definition 5.2

Let $(N,V) \in \Gamma^{NTU}$ be a game. (N,V) is called **weak individual superadditive**, *if for every player $i \in N$, every coalition $\emptyset \neq S \in 2^{N\setminus\{i\}}$ and every $x \in V(S)^+$ there exists $y \in V(S \cup \{i\})$ with $y_S \geq x$ and $y_i \geq 0$.*

Definition 5.2 requires that at least the individual rational outcomes in $V(S), S \in 2^N \setminus \{\emptyset, N\}$, are also obtainable in $S \cup \{i\}$ by assigning zero payoff to player i.

We now design a special reference function that will yield the NTU prenucleolus. In example 4.11 we defined two reference functions β_1 and β_2 that were motivated as follows. In the case of the reference function β_1 the coalition was supposed to be indifferent between satisfaction and dissatisfaction when each player received $\frac{c^S}{|S|p_i^S}$, that means a fraction of his or her maximal outcome under an individual rational imputation for which coalition S is effective. For β_2 each player had to receive exactly this maximal outcome. Of course, by the way we constructed the β-excess functions, every redistribution of these outcomes according to $p_V^N|_S$ also yields points of indifference.

We now propose a slightly different story. We could allow the coalitions to maximize this outcome that can then be redistributed without changing its excess. Since players would not accept points that are not individual rational we restrict this maximization to the individual rational part of $V(S)$, that is to $V(S)^+$. For technical reasons we assume that the maximization is furthermore done on $p_V^N|_S V(S)^+$ thus the players at first consider a transformation into a simplex game.

Definition 5.3

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. For every coalition $S \in 2^N, S \neq N$, define

$$m_V^S := \max_{x \in p_V^N|_S V(S)^+} \left\langle x, p_V^N|_S \right\rangle$$

and denote the maximizers by M_V^S, i.e.

$$M_V^S := arg \max_{x \in p_V^N|_S V(S)^+} \left\langle x, p_V^N|_S \right\rangle .$$

Remark 5.4

1. *If $(N,V) \in \Gamma^H$, i.e. (N,V) is a hyperplane game, then an alternative formulation of Definition 5.3 is*

$$m_V^S := \frac{c_V^S}{\min\limits_{i \in S} \frac{p_{V,i}^S}{p_{V,i}^N}} \quad \left(S \in 2^N, S \neq N\right).$$

   If (N,V) is even simplex, then this reduces further to

$$m_V^S := \frac{c_V^S}{\min\limits_{i \in S} p_{V,i}^S} \quad \left(S \in 2^N, S \neq N\right).$$

   Thus for a TU game $(N,v) \in \Gamma^{TU}$ we have $m_{V^v}^S = c_{V^v}^S = v(S)$ for every $S \in 2^N, S \neq N$.

2. *The value m_V^S is invariant under positive linear transformations of utility, i.e. for every $\lambda \in \mathbb{R}_{++}^N$:*

$$m_{\lambda V}^S = m_V^S \quad \forall\, (N,V) \in \Gamma^{qH} \quad \forall S \in 2^N, S \neq N.$$

According to the motivation we gave for this definition, we now define special reference functions that declare those imputations to be points of indifference that are redistributions of m_V^S according to $p_V^N|_S$ for every coalition $S \neq N$.

Definition 5.5

Let $\beta \in \mathcal{B}$ be a reference function. β is called **maximal feasible**, *if*

$$\left\langle \beta^{(N,V)}(S), p_V^N|_S \right\rangle = m_V^S$$

holds true for every quasi hyperplane game $(N,V) \in \Gamma^{qH}$ and every coalition $S \in 2^N, S \neq N$.

The next lemma shows that maximal feasible reference functions belong to $\overline{\mathcal{B}}$, thus to a class of reference functions that we already analyzed is chapter 4. Every result we have proved for this class therefore still hold within this special setup.

Lemma 5.6

If $\beta \in \mathcal{B}$ is maximal feasible, then $\beta \in \overline{\mathcal{B}}$.

Proof:

Let $\beta \in \mathcal{B}$ be a maximal feasible reference function. Let $(N,v) \in \Gamma^{TU}$ be a

TU game and let $(N, V^v) \in \Gamma^H$ be its according NTU (simplex) game. Then

$$\begin{aligned}\sum_{i \in S} \beta_i^{(N,V^v)}(S) &= m_{V^v}^S \\ &= c_{V^v}^S \\ &= v(S)\end{aligned}$$

holds true, hence $\beta \in \overline{\mathcal{B}}$. ■

Lemma 5.6 tells us that if we consider maximal feasible reference functions, we are within the framework of the axiomatization of β-excess functions (Theorem 4.4). Figure 6 illustrates definitions 5.3 and 5.5. The dotted line denotes the set of all possible choices for a reference point that is given by a maximal feasible reference function.

Hence it is clear that maximal feasibility does not determine a reference function uniquely. But for every two maximal feasible reference functions β and β' the excess functions e^β and $e^{\beta'}$ coincide for every game $(N, V) \in \Gamma^{qH}$, every coalition $S \in 2^N, S \neq N$,[21] and every $x \in \mathbb{R}^N$:

$$\begin{aligned}e^\beta(S, x) &= \sum_{i \in S} \left(\beta_i^{(N,V)}(S) - x_i\right) p_{V,i}^N \\ &= m_V^S - \sum_{i \in S} x_i p_{V,i}^N \\ &= \sum_{i \in S} \left(\beta_i'^{(N,V)}(S) - x_i\right) p_{V,i}^N \\ &= e^{\beta'}(S, x),\end{aligned}$$

and hence also $\mathcal{P}N^\beta$ and $\mathcal{P}N^{\beta'}$ coincide. We will therefore shorten the notation by defining $\mathcal{P}N(N, V) := \mathcal{P}N^\beta(N, V)$ whenever β is maximal feasible

[21] The coincidence for N follows from the definition of the reference function.

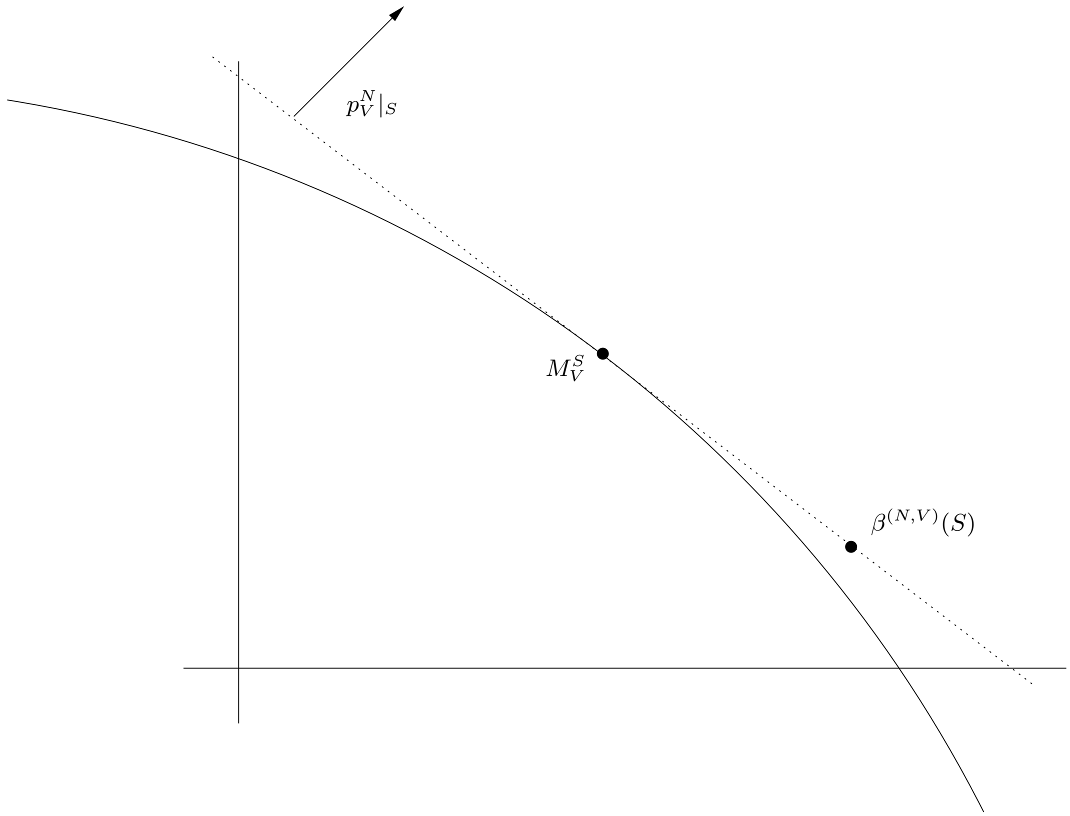

Figure 6: A maximal feasible reference function

and call $\mathcal{P}N$ **the (NTU) prenucleolus of quasi hyperplane games**.

Lemma 5.7

Let $\beta \in \overline{\mathcal{B}}$ be maximal feasible. Then for every game $(N,V) \in \Gamma^{qH}$ and every $\lambda \in \mathbb{R}^N_{++}$ we have $\beta^{(N,\lambda V)}(S) = \lambda\beta^{(N,V)}(S)$ for every coalition $S \in 2^N$.

Proof:

Only the case $S \neq N$ has to be considered. From 2. in Remark 5.4 we can *imply*

$$\begin{aligned}\left\langle \beta^{(N,\lambda V)}(S), p^N_{\lambda V}|_S \right\rangle &= m^S_{\lambda V} \\ &= m^S_V \\ &= \left\langle \beta^{(N,V)}(S), p^N_V|_S \right\rangle \\ &= \left\langle \lambda\beta^{(N,V)}(S), p^N_{\lambda V}|_S \right\rangle .\end{aligned}$$

The desired equation follows immediately. ■

In other words if β is maximal feasible then β is covariant, hence $\beta \in \mathcal{B}^c$. Lemma 5.7 therefore directly yields the covariance of the (NTU) prenucleolus.

Lemma 5.8

The NTU prenucleolus $\mathcal{P}N$ satisfies covariance.

The proof is an easy concequence of Theorem 4.14.

As the next result concerning the NTU prenucleolus we will present a Theorem similar to Lemma 4.22, which states that for weak individual superadditive quasi hyperplane games the NTU prenucleolus is individual rational.

Theorem 5.9

If $(N, V) \in \Gamma^{qH}$ is weak individual superadditive (see Definition 5.2), then the NTU prenucleolus $\mathcal{P}N\,(N, V)$ is individual rational.

Proof:

Let $\beta \in \mathcal{B}^c$ be a maximal feasible reference function. Let $(N, V) \in \Gamma^{qh}$ be weak individual superadditive and w.l.o.g. simplex. We know that $\mathcal{P}N(N,V) = \mathcal{P}N\left(N, v_V^\beta\right)$, thus all we have to show is that v_V^β is individual superadditive, i.e. v_V^β satisfies

$$v_V^\beta(S) + v_V^\beta(\{i\}) \leq v_V^\beta(S \cup \{i\}) \quad \forall S \in 2^{N\setminus\{i\}}, i \in N,$$

since for individual superadditive TU games the TU prenucleolus is individual rational (Corollary 4.23).

Now suppose that v_V^β is not individual superadditive, thus there is a player $i \in N$ and a coalition $S \in 2^{N\setminus\{i\}}$ such that

$$v_V^\beta(S) + v_V^\beta(\{i\}) > v_V^\beta(S \cup \{i\}).$$

Since $v_V^\beta(\{i\}) = 0$ and $v_V^\beta(S) = m_V^S$ this means

$$m_V^S > m_V^{S\cup\{i\}}.$$

Let $\bar{x}$ be "a maximizer for m^S", i.e.

$$\bar{x} \in M_V^S = \arg\max_{x \in V(S)^+} \left\langle x, p_V^N|_S \right\rangle.$$

Since $\bar{x} \in V(S)^+$ and the game is weak individual superadditive, $\bar{x} \in V(S \cup \{i\})$ must hold. But

$$\begin{aligned}
\left\langle \bar{x}, p_V^N|_{S\cup\{i\}} \right\rangle &= \left\langle \bar{x}, p_V^N|_S \right\rangle \\
&= m_V^S \\
&> m_V^{S\cup\{i\}}
\end{aligned}$$

holds, thus $\bar{x} \notin V(S \cup \{i\})^+$. *By* $\bar{x} \in \mathbb{R}_+^{S\cup\{i\}}$ *also* $\bar{x} \notin V(S \cup \{i\})$ *holds, a contradiction to the assumption of weak individual superadditivity of* (N, V).■

5.2 Monotonicity

For the NTU prenucleolus we have a form of ”independence of irrelevant alternatives” in the sense that if for two games (N, V) and (N, W) in Γ^{qH} such that for some $S \in 2^N, S \neq N$, the inclusion $V(S) \subset W(S)$ holds but still $m_S^V = m_S^W$ is true (and $V(T) = W(T)$ for all $S \neq T$), then $\mathcal{P}N(N, V) = \mathcal{P}N(N, W)$. We will show that a more general form of monotonicity holds for the NTU prenucleolus that is similar to a monotonicity result for the TU prenucleolus.

Young proved in [You85] that a solution concept ϕ for TU games that is a core-selector (like the TU prenucleolus) is in general not monotonic, i.e. from $v(S) < w(S)$ for some $S \in 2^N$ it does in general not follow that $\phi_i(N, v) < \phi_i(N, w) \quad \forall i \in S$. Zhou showed in [Zho91] that the TU prenucleolus is weakly coalitional monotonic in the following sense.

Definition 5.10
A solution concept ϕ *for TU games is weakly coalitional monotonic on a class* $\Gamma \subseteq \Gamma^{TU}$ *if for all games* $(N, v) \in \Gamma$ *and* $(N, w) \in \Gamma$ *such that* $v(S) \leq w(S)$ *for some* $S \in 2^N$ *and* $v(T) = w(T)$ *for all* $T \in 2^N, T \neq S$, *it follows that* $\sum_{i\in S} \phi_i(N, v) \leq \sum_{i\in S} \phi_i(N, w)$.

Theorem 5.11 ([Zho91])

The TU prenucleolus is weakly coalitional monotonic on Γ^{TU}.

So the (TU) prenucleolus ensures that at least the sum of the payoffs to a coalition S increases when its worth $v(S)$ increases. Now of course the question arises if we can show some monotonicity property for the NTU prenucleolus. We therefore extend the concept of weak coalitional monotonicity to the class of quasi hyperplane games as follows.

Definition 5.12

A solution concept ϕ *for quasi hyperplane games is weakly coalitional monotonic on a class* $\Gamma \subseteq \Gamma^{qH}$ *if for all games* $(N,V) \in \Gamma$ *and* $(N,W) \in \Gamma$ *such that* $V(S) \subseteq W(S)$ *for some* $S \in 2^N$ *and* $V(T) = W(T)$ *for all* $T \in 2^N, T \neq S$, *it follows that* $\left\langle \phi\left(N,V\right)|_S, p_V^N|_S \right\rangle \leq \left\langle \phi\left(N,W\right)|_S, p_W^N|_S \right\rangle$.

This definition coincides with Zhou's definition on the class of all TU games. We are now able to state and to prove the following result.

Theorem 5.13

The NTU prenucleolus is weakly coalitional monotonic on Γ^{qH}.

Proof:

Let $(N,V), (N,W) \in \Gamma^{qH}$ *be two quasi hyperplane games such that* $V(S) \subseteq W(S)$ *for some* $S \in 2^N, S \neq N$ *and* $V(T) = W(T)$ *for every* $T \in 2^N, T \neq S$.

Two cases can occur:

1. *case:* $m_V^S = m_W^S$.

   Then we have $\mathcal{P}N(N,V) = \mathcal{P}N(N,W)$ *and nothing more has to be shown.*

2. *case:* $m_V^S < m_W^S$.

   Since we have $V(N) = W(N)$, *we can define* $p := p_V^N = p_W^N$ *for abbreviation. Now* $m_V^S < m_W^S$ *implies* $v_{pV}^\beta(S) < v_{pW}^\beta(S)$ *which yields* $\sum_{i\in S} \mathcal{P}N\left(N, v_{pV}^\beta\right)_i \leq \sum_{i\in S} \mathcal{P}N\left(N, v_{pW}^\beta\right)_i$ *by Zhou's result.*
   Now the proof of the Theorem is completed by

$$\begin{aligned}
\langle \mathcal{P}N(N,V)|_S, p|_S\rangle &= \sum_{i\in S} \mathcal{P}N(N,V)_i \cdot p_i \\
&= \sum_{i\in S} \frac{1}{p_i}\mathcal{P}N\left(N, v_{pV}^\beta\right)_i \cdot p_i \\
&= \sum_{i\in S} \mathcal{P}N\left(N, v_{pV}^\beta\right)_i \\
&\leq \sum_{i\in S} \mathcal{P}N\left(N, v_{pW}^\beta\right)_i \\
&= \langle \mathcal{P}N(N,W)|_S, p|_S\rangle .
\end{aligned}$$

■

5.3 Computation

In chapter 4 we showed how general β-prenucleoli can be expressed via (TU) prenucleoli of suitably chosen TU games. In the case of maximal feasible reference functions, i.e. for the NTU prenucleolus, things are even simpler because of the invariance of the values $m_V^S, S \in 2^N$.

To be more precise, let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game, not necessarily simplex. The game $\left(N, p_V^N V\right)$ is simplex and, because of the maximal feasibility of β and with $p := p_V^N$ for abbreviation

$$\begin{aligned} v_{pV}^{\beta}(S) &= \sum_{i\in S} \beta_i^{pV}(S) \\ &= \sum_{i\in S} \beta_i^{pV}(S) \cdot p_{pV,i}^N \\ &= m_{pV}^S \\ &= m_V^S \quad \forall S \in 2^N, S \neq N, \end{aligned}$$

holds true.

Again by the covariance of the NTU prenucleolus we have

$$\mathcal{P}N\,(N,V) = \frac{1}{p}\mathcal{P}N\left(N, v_{pV}^{\beta}\right).$$

It follows that the computation of the NTU prenucleolus for a given quasi hyperplane game (N,V) consists of determining the values $m_V^S, S \in 2^N$, and computing the (TU) prenucleolus of the TU game $\left(N, v_{pV}^{\beta}\right)$.

Having this in mind, we are now able to apply the results of [Kla97] which will yield a set-valued dynamical system that converges to the (NTU) prenucleolus[22]. Hence the question to find a dynamical system approach concerning the NTU prenucleolus, which was declared an open question both in [Kal75] and in [CC02], can be answered affirmatively. As it was shown in [Kla97] these results can also be used to develop a computer program to approximate

[22] See also Justman ([Jus77]) for a different set-valued dynamical system and [MP76] for a general theory of set-valued dynamical systems.

the (NTU) prenucleolus. And of course all other algorithms that can compute the TU prenucleolus can be used to compute also the NTU prenucleolus.

5.4 Inclusion in the core

As mentioned earlier, the core of a cooperative game *with* transferable utility (TU game) can as well be defined to be those imputations yielding non-positive (TU) excesses for all coalitions. Therefore it is easily seen that the (TU) prenucleolus is a member of the (TU) core whenever the (TU) core is not empty. For the same reason this is also true for Kalai's nucleoli.

We have seen in example 4.24 that β-prenucleoli need not be members of the (NTU) core even when the latter is not empty.

We are about to show in this subsection that the NTU prenucleolus is a core-member for a certain subclass of quasi hyperplane games. As a corollary of this result it is shown that all games in this subclass, called "m-balanced games", posses a non-empty core. This corollary leads to interesting questions: Of what type is the relation between the class of all "m-balanced" games and the class of all "balanced" games — for which non-emptiness of the core of its members was proved by Scarf in [Sca67]. The same question applies to π-balanced games which also have a non-empty core due to a Theorem by Billera ([Bil70b],[Bil71]). We discuss this concepts and show some connections at the end of this section.

Let us now begin with the central definition of this section.

Definition 5.14

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game. If for all balanced collections $\mathcal{S} \subseteq 2^N$ with balancing coefficients $(\delta_S)_{S\in\mathcal{S}}$ we have

$$c^N \geq \sum_{S\in\mathcal{S}} \delta_S m_V^S, \tag{7}$$

then (N, V) is called **m-balanced**.

The resemblance of this definition to the balancedness condition for TU games is obvious. In fact, both definitions share the same intuition. For the core of a game to be non-empty the game must not allocate too much utility to subcoalitions relative to what the grand coalition gets. If subcoalitions receive too much they are more likely to be able to improve upon proposed imputations and the set of pssible candidates for the core might shrink to the empty set. Both balancedness conditions specify upper bounds for the worth of subcoalitions or lower bounds for the worth of the grand coalition, depending on how these inequalities are interpreted. We will see that the definition of m-balancedness is a sufficient condition for

- the NTU prenucleolus to be contained in the core (Theorem 5.16),
- the core to be non-empty (Corollary 5.17), and
- the β-core to be non-empty (Theorem 5.36).

For the latter m-balancedness is also necessary.

Let us continue with a remark concerning the TU β-game of a quasi hyperplane game with maximal feasible $\beta \in \mathcal{B}$. In Chapter 4 we have defined and frequently used the TU β-game of a simplex quasi hyperplane game (Definition 4.21). At this point it is useful to define a generalized version. For a reference function $\beta \in \mathcal{B}$ and a quasi hyperplane game $(N, V) \in \Gamma^{qH}$ (not necessarily simplex) let the *generalized* β-game $\left(N, v_V^\beta\right)$ be defined by $v_V^\beta(S) := \left\langle \beta^{(N,V)}(S), p_V^N|_S \right\rangle$ for every $S \in 2^N$. Hence the generalized version coincides with the original β-game if (N, V) happens to be simplex. Of course the generalized β-game of (N, V) does not help in determining the β-prenucleolus of (N, V) because in general the respective excess functions differ.[23] Nevertheless we can use the generalized β-game to characterize the m-balanced quasi hyperplane games by referring to the balancedness condition for TU games as follows.

Lemma 5.15

Let $\beta \in \mathcal{B}$ be a maximal feasible reference function. A game $(N, V) \in \Gamma^{qH}$ is m-balanced if and only if its generalized TU β-game $\left(N, v_V^\beta\right)$ is balanced.

Proof:

Since β is maximal feasible we have $v_V^\beta(S) = \left\langle \beta^{(N,V)}(S), p_V^N|_S \right\rangle = m_V^S$ for all $S \in 2^N, S \neq N$, and $v_V^\beta(N) = c_V^N$. Inserting this into the definition of m-balancedness (Definition 5.14) completes the proof at once. ■

[23] The TU excess function for $\left(N, v_V^\beta\right)$ is given by $e\left(S, x, v_V^\beta\right) = v_V^\beta(S) - x(S) = \left\langle \beta^{(N,V)}(S), p_V^N|_S \right\rangle - x(S)$, which is in general different from $e^\beta(S, x) = \left\langle \beta^{(N,V)}(S) - x_S, p_V^N|_S \right\rangle = \left\langle \beta^{(N,V)}(S), p_V^N|_S \right\rangle - \left\langle x_S, p_V^N|_S \right\rangle \left(S \in 2^N\right)$.

Since we about to analyze the connection between the NTU prenucleolus and the core which both are covariant solution concepts we need not use the generalized β-game any more because assuming all occuring games to be simplex is no loss of generality.

The main result of this section is that $\mathcal{P}N$ is an NTU core-selector for m-balanced games.

Theorem 5.16

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. If (N,V) is m-balanced, then $\mathcal{P}N\,(N,V) \in \mathbf{Core}\,(N,V)$.

Proof:

Without loss of generality, assume that (N,V) is a simplex game. Let β be a maximal feasible reference function. Let $\left(N, v_V^{\beta}\right)$ be the TU β-game of (N,V), i.e.

$$v_V^{\beta}(S) = m_V^S \quad \forall S \in 2^N.$$

Since the m-balancedness of (N,V) is equivalent to the balancedness of $\left(N, v_V^{\beta}\right)$ we have $\mathbf{Core}\left(N, v_V^{\beta}\right) \neq \emptyset$ by the result of Bondareva and Shapley.

Let $\nu := \mathcal{P}N\left(N, v_V^{\beta}\right)$ be the (TU) prenucleolus of $\left(N, v_V^{\beta}\right)$, then $\nu \in \mathbf{Core}\left(N, v_V^{\beta}\right)$, in other words:

$$\sum_{i \in S} \nu_i \geq v_V^{\beta}(S) \quad \forall S \in 2^N.$$

Thus ν satisfies

$$\sum_{i \in S} \nu_i \geq m_V^S \quad \forall S \in 2^N \tag{8}$$

*and, since $\nu \in$ **Core** $\left(N, v_V^\beta\right)$, ν is individual rational:*

$$\nu_i \geq 0 \quad \forall i \in N.$$

Thus we can imply from (8)

$$\nexists y_S \in V(S) \text{ such that } y_S > \nu_S \quad \forall S \in 2^N. \tag{9}$$

*This is true because m_V^S is the maximum of $\left\langle \bullet, p_V^N|_S \right\rangle$ over $V(S)^+$ and the game is simplex. Finally (9) implies $\nu \in$ **Core** (N, V). The observation $\nu = \mathcal{P}N(N, V)$ (see the considerations on page 58) now completes the proof.* ■

We immediately see that m-balancedness of a quasi hyperplane game is a sufficient condition for the core of this game to be non-empty, since it contains the NTU prenucleolus.

Corollary 5.17
If $(N, V) \in \Gamma^{qH}$ is m-balanced, then **Core** $(N, V) \neq \emptyset$.

In order to discuss this new condition of m-balancedness we recall the banlancedness concepts of Scarf ([Sca67]) and Billera ([Bil71]).

Definition 5.18 ([Sca67])
An NTU game $(N, V) \in \Gamma^{NTU}$ is balanced if for every balanced collection $\mathcal{S} \subseteq 2^N$ it follows that $x \in V(N)$ is true whenever $x_S \in V(S) \quad \forall S \in \mathcal{S}$ holds.

Theorem 5.19 ([Sca67])
If an NTU game $(N, V) \in \Gamma^{NTU}$ is balanced, then **Core** $(N, V) \neq \emptyset$.

It can easily be shown by examples that this condition is not necessary. Another sufficient condition for the non-emptiness of the core of NTU games was given by Billera ([Bil71]). Let therefore π be a $(2^n - 1) \times n$-matrix and consider its rows as indexed by the non-empty elements of 2^N (the coalitions). Let π^S be the row that corresponds to $S \in 2^N$. Assume that $\pi^N > 0$, $\pi^S \geq 0$ and $\pi_i^S = 0$ for all $S \in 2^N \setminus \{N\}$ and all $i \notin S$.

Definition 5.20 ([Bil71])
Let $(N, V) \in \Gamma^{NTU}$ be an NTU game and let π be a $(2^n - 1) \times n$-matrix as described above. Let $\mathcal{S} \subseteq 2^N$ be a collection of coalitions. $\mathcal{S}$ is π-balanced, if there exist real numbers $(\delta_S)_{S \in \mathcal{S}}$ such that $\sum\limits_{S \in \mathcal{S}} \delta_S \pi^S = \pi^N$.

Definition 5.21 ([Bil71])
An NTU game $(N, V) \in \Gamma^{NTU}$ is π-balanced if there exists a matrix π such that for every π-balanced collection $\mathcal{S} \subseteq 2^N$ it is true that $x \in V(N)$ whenever $x_S \in V(S) \quad \forall S \in \mathcal{S}$ holds.

Theorem 5.22 ([Bil71])
If an NTU game $(N, V) \in \Gamma^{NTU}$ *is* π*-balanced for some* π*, then* $\mathbf{Core}\,(N, V) \neq \emptyset$.

A similar result to the theorem of Bondareva and Shapley is the next theorem which states that π-balancedness is also a necessary condition for the core of hyperplane games to be non-empty.

Theorem 5.23 ([Bil71])
Let $(N, V) \in \Gamma^{H}$ *be a hyperplane game and let* p *be the matrix of all normal vectors* $p_V^S \quad \left(S \in 2^N\right)$[24]. *Then* $\mathbf{Core}\,(N, V) \neq \emptyset$ *if and only if* (N, V) *is* p*-balanced.*

Another interesting result of Billera states that for quasi hyperplane games π-balancedness is also necessary and sufficient when all sets $V(S), S \in 2^N$, are convex.

Theorem 5.24 ([Bil71])
Let $(N, V) \in \Gamma^{qH}$ *be a quasi hyperplane game such that* $V(S)$ *is convex for every* $S \in 2^N$*. Then* $\mathbf{Core}\,(N, V) \neq \emptyset$ *if and only if there exists a matrix* π *such that* (N, V) *is* π*-balanced.*

Corollary 5.17 now adds a third condition for the existence of the core of

[24] Thus p is of the form π.

quasi hyperplane games. There arise several questions about the relationship of m-balancedness to balancedness and π-balancedness.

Of course no two of those concepts are equivalent. Billera ([Bil70b]) provided an example of a game that is π-balanced but not balanced. See example 5.26 for a hyperplane game that is not m-balanced but has a non-empty core and is therefore π-balanced according to Theorem 5.23. The question which condition is sharper, if any, is open. Concering the connection between m-balanced and π-balanced games we can, however, make the following observation. For the class of convex valued quasi hyperplane games Theorem 5.23 together with Corollary 5.17 show that the class of all m-balanced games is a subclass of all π-balanced games.

Lemma 5.25
Let $(N, V) \in \Gamma^{qH}$ be a game such that $V(S)$ is convex for every $S \in 2^N$. If (N, V) is m-balanced then (N, V) is also π-balanced for some π.

Proof:
The core of (N, V) is non-empty due to Corollary 5.17. Therefore there exists π such that (N, V) is π-balanced due to Theorem 5.24. ■

Example 5.26 shows that this inclusion is strict by providing a hyperplane game that is not m-balanced but has a non-empty core and is therefore π-balanced. So the characterization of games with a non-empty core by m-balancedness only yields a smaller class of games. On the other hand, from a computational point of view, checking a given game for m-balancedness is a much easier task than to check for π-balancedness. Peleg ([Pel65]) provided a procedure to construct successively minimal balanced collections. It remains the computation of the values $m^S, S \in 2^N$, which is a simple maximization problem[25]. Finally it is to check whether or not the various inequalities (7) in Defintion 5.14 hold. Apparently there are no known methods to construct all π-balanced collections over a player set. Neither are the balancedness inclusions easy to verify. The concept of m-balancedness is a contribution to the problem of actually checking a game for non-emptiness of its core. In Section 5.7 m-balancedness is also used as a sufficient and necessary condition for the β-core, which was introduced in the present thesis (Definition 4.26), to be non-empty.

Example 5.26

Let $n = 4$ and let (N, V) be the hyperplane game as given by Table 7. The values m_V^S of this game are given in the same table in the right column. This game is not m-balanced. Take $\mathcal{S} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$. This collection is balanced with all balancing coefficients equal to $\frac{1}{3}$. But

$$\begin{aligned}\sum_{S \in \mathcal{S}} \delta_S m_V^S &= \frac{1}{3}(6 + 10 + 3 + 5) \\ &= 8 \\ &> 7 \\ &= c_V^N.\end{aligned}$$

[25] In the case of a hyperplane game these values are even given by equations, see Remark 5.4.

Nevertheless, the game has a non-empty core which is the convex hull of the following points: $\left(0, \frac{4}{3}, \frac{5}{3}, 4\right)$, $(2,0,2,3)$, $\left(2, \frac{7}{5}, \frac{3}{5}, 3\right)$, $\left(2, \frac{97}{25}, \frac{3}{5}, \frac{13}{25}\right)$, $\left(\frac{95}{54}, \frac{37}{9}, \frac{35}{54}, \frac{13}{27}\right)$, $(0,2,1,4)$, $\left(2, \frac{15}{4}, \frac{5}{4}, 0\right)$, $(2,0,3,2)$, $\left(\frac{5}{4}, \frac{1}{2}, \frac{15}{4}, \frac{3}{2}\right)$, $(2,2,3,0)$.

The NTU prenucleolus of this game is $\mathcal{PN}(N,V) = \left(4, \frac{19}{8}, -1, \frac{13}{8}\right)$.

5.5 Reduced game property

For the class Γ^{TU} of TU games two different versions of reduced games are used in axiomatizations of solution concepts. The reduced game defined in Chapter 2 is due to Davis and Maschler ([DM65]) and is used to axiomatize the (TU) prenucleolus and the (TU) prekernel. Another reduced game, due to Hart and Mas-Colell ([HMC89]), yields an axiomatization of the Shapley value via the same axioms used in the axiomatization of the prenucleolus just by exchanging the two reduced games in the definition of RGP.

These two (TU) reduced games can easily be generalized to the class of NTU games and the question arises wether or not the (NTU) prenucleolus satisfies RGP with respect to one of these (NTU) reduced games. But, as Maschler and Owen ([MO89]) have shown, there does not exist a solution concept for hyperplane games that is

- efficient,
- symmetric,

- covariant and
- that satisfies RGP with respect to the reduced games of Davis and Maschler or of Hart and Mas-Colell.

Since $\mathcal{P}N$ satisfies the first three properties, we can imply that $\mathcal{P}N$ does not satisfy RGP, although one should be aware that the covariance that [MO89] used contains also additive transformations of utility and they did not provide a definition of their notion of symmetry. But usually symmetry follows from anonimity. Also examples that show the non-validness of the reduced game property of $\mathcal{P}N$ with respect to either of the two reduced games are easily constructed. In their analysis of a possible extension of the prekernel to NTU games, also Orshan and Zarzuelo ([OZ00]) noticed that RGP might be a too strong requirement because it often causes the solution to be empty.

To overcome this problem, two different ways are possible. The first way, as undertaken in [MO89], is to keep the definition of the reduced game and to modify the definition of RGP. By this means they axiomatized their new solution concept, called the *consistent NTU Shapley value*, by efficiency, symmetry, covariance and the so-called bilateral consistency.

However, we will take the other possible way and keep the definition of RGP but use a new reduced game in order to show that the NTU prenucleolus $\mathcal{P}N$ satisfies a form of the reduced game property. This new reduced game coincides with that of Definition 2.15 on the class of all TU games. As yet we do not know if the properties of $\mathcal{P}N$ together with RGP constitute an axiomatization.

Definition 5.27

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game and let $S \in 2^N$ be a coalition. Let $\bar{M}_V^S \in p_V^S|_S V(S)^+$ be any maximizer in the definition of m_V^S.

Let $C_S := \left\{ i \in S \,\middle|\, \bar{M}_{V,i}^S > 0 \right\}$ be the set of those players of S whose outcome under $\bar{M}_V^S$ is strictly positive.

We now deviate from the standard definitions. We consider games (S,V) for a subcoalition $S \subsetneq N$ still as correspondences to $\mathbb{R}^N$ and not to an $|S|$-dimensional space as it would be required by our definition of an NTU game. Since $\mathbb{R}^{|S|}$ is isomorphic to $\mathbb{R}^S \subseteq \mathbb{R}^N$ this inconsistency should be tolerated.

Definition 5.28

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game, let $x \in \mathbb{R}^N$ be an imputation and let $S \in 2^N \setminus \{\emptyset, N\}$ be a coalition. The **(NTU) reduced game** $\left(S, V_x^S\right)$ *of S w.r.t. x is defined by*

$$
\begin{aligned}
V_x^S(T) &:= V(T) + s_T \quad (T \subset S) \\
V_x^S(S) &:= \left\{ x_S \in \mathbb{R}^S \,\middle|\, \left\langle x_S, p_V^N|_S \right\rangle \leq c^N - \left\langle x_{N\setminus S}, p_V^N|_{N\setminus S} \right\rangle \right\}.
\end{aligned}
$$

Here $s_T \in \mathbb{R}^T$ is defined by

$$
s_i := \begin{cases} \frac{\max\limits_{Q \subseteq N\setminus S} \left\{ m_V^{T\cup Q} - x(Q) \right\} - m_V^T}{|C_T| p_{V,i}^N} & : \quad i \in C_T \\ 0 & : \quad i \notin C_T \end{cases} \qquad (i \in N)
$$

Remark 5.29

In the case where (N, V) is a hyperplane game, then the definition of $V_x^S(T)$ can as well be formulated as

$$V_x^S(T) = \left\{ x_T \in \mathbb{R}^T \,\middle|\, \langle x_T, p_V^T \rangle \leq c_{V_x^S}^T \right\} \quad (T \subset S)$$

with

$$c_{V_x^S}^T := \max_{Q \subseteq N \setminus S} \left\{ m_V^{T \cup Q} - x(Q) \right\} \min_{i \in T} \frac{p_{V,i}^T}{p_{V,i}^N}.$$

Remark 5.30

1. *The (NTU) reduced game of a quasi hyperplane game is itself a quasi hyperplane game. If the original game was simplex then so is the reduced game.*

2. *Remark 5.29 shows that the (NTU) reduced game coincides with the (TU) reduced game of Davis and Maschler when the original game is a TU game. For this we only need to notice that in this case $p_V^S = 1_S$ and $m_V^S = v(S)$ holds true for all $S \in 2^N$.*

3. *The motivation for this reduced game is also quite similar. Every coalition $T \subset S$ considers an enlargement of their outcome set $V(T)$ by cooperating with players outside of S. If $Q \subseteq N \setminus S$ is such a group of players then $T \cup Q$ could "command" $m_V^{T \cup Q}$ reduced by what Q receives from x, that is $x(Q)$. The coalition T determines the – in this sense – most profitable coalition in $N \setminus S$. As in the TU case this maximazation is only virtual. Different subcoalitions of S might be dependend on intersecting coalitions to reach this maximum.*

Definition 5.31

Let Φ be a solution concept on Γ^{qH}. Φ satisfies the **reduced game property (RGP)**, *if the following is true for every game $(N,V) \in \Gamma^{qH}$:*

$$x \in \Phi(N,V) \Rightarrow x_S \in \Phi\left(S, V_x^S\right) \quad \forall S \in 2^N.$$

We do not need to require $\left(S, V_x^S\right) \in \Gamma^{qH}$ in the definition of the reduced game property because due to Remark 5.30 the property of being a quasi hyperplane game is preserved by reducing.

Theorem 5.32

The NTU prenucleolus $\mathcal{P}N$ satisfies RGP.

Proof:

Essential for this proof is the fact that the (TU) prenucleolus satisfies RGP (Lemma 2.17).

Let $\beta \in \mathcal{B}^c$ be a maximal feasible reference function and let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. Assume w.l.o.g. that (N,V) is a simplex game.

Denote by $\nu := \mathcal{P}N(N,V)$ the (NTU) prenucleolus of (N,V) and by $\left(N, v_V^\beta\right)$ the (TU) β-game of V, i.e. $v_V^\beta(S) = \sum_{i \in S} \beta_i^{(N,V)}(S) = m_V^S$, since β is maximal feasible.

Let $S \in 2^N, S \neq N$, be a coalition. We have to show that

$$\nu_S = \mathcal{P}N\left(S, V_\nu^S\right),$$

where $\left(S, V_\nu^S\right)$ *is the reduced game of* S *w.r.t.* ν*, or equivalently*

$$\nu_S = \mathcal{P}N\left(S, v^\beta_{V_\nu^S}\right). \tag{10}$$

Since $\nu = \mathcal{P}N\left(N, v_V^\beta\right)$ *and the (TU) prenucleolus satisfies RGP, it follows that*

$$\nu_S = \mathcal{P}N\left(S, \left(v_V^\beta\right)_\nu^S\right). \tag{11}$$

Thus in view of (10) and (11) all we have to show is

$$\left(v_V^\beta\right)_\nu^S = v^\beta_{V_\nu^S},$$

i.e., that the (TU) reduced game of v_V^β *w.r.t.* S *and* ν *is equal to the (TU)* β*-game of the (NTU) reduced game of* V *w.r.t.* S *and* ν*.*

Note that the definition of s_T *implies that if* M_V^T *is a maximizer for* m_V^T*, then* $M_V^T + s_T$ *is a maximizer for* $m^T_{V_x^S}$*. Thus we have for all* $T \subsetneq S$*:*

$$\begin{aligned}
m^T_{V_x^S} &= \left\langle M_V^T + s_T, p_V^N|_T \right\rangle \\
&= m_V^T + \left\langle s_T, p_V^N|_T \right\rangle \\
&= m_V^T + \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q} - x(Q) \right\} - m_V^T \\
&= \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q} - x(Q) \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
v^{\beta}_{V^S_\nu}(T) &= \sum_{i\in T}\beta_i^{\left(S,V^S_\nu\right)}(T) \\
&= m^T_{V^S_\nu} \\
&= \max_{Q\subseteq N\setminus S}\left\{m^{T\cup Q}-\nu(Q)\right\} \\
&= \max_{Q\subseteq N\setminus S}\left\{v^{\beta}_V\left(T\cup Q\right)-\nu(Q)\right\} \\
&= \left(v^{\beta}_V\right)^S_\nu(T).
\end{aligned}$$

It remains the case $T = S$. We have

$$\begin{aligned}
\left(v^{\beta}_V\right)^S_\nu(S) &= v^{\beta}_V(N)-\nu\left(N\setminus S\right) \\
&= c^N_V-\nu\left(N\setminus S\right) \\
&= \nu(S)
\end{aligned}$$

and

$$\begin{aligned}
v^{\beta}_{V^S_\nu}(S) &= \sum_{i\in S}\beta_i^{\left(S,V^S_\nu\right)}(S) \\
&= m^S_{V^S_\nu} \\
&= c^N_V-\nu\left(N\setminus S\right) \\
&= \nu(S) \qquad \blacksquare
\end{aligned}$$

5.6 Continuity and extension to general NTU games

In section 4.6 we have shown the continuity of $\mathcal{P}N^{\beta}$ for continuous reference functions β which was an essential ingredience for the proposed extension to general NTU games in section 4.7. To see if these results are still valid for the NTU prenucleolus only some thoughts about continuity of maximal feasible reference functions are to be made.

If we have a convergent sequence of games $\left(\left(N, V^k\right)\right)_{k\in\mathbb{N}}$, $\lim_{k\to\infty}\left(N, V^k\right) = (N, V)$, and a reference function that is maximal feasible then the payoff configurations $\left(\beta^{\left(N,V^k\right)}\right)_{k\in\mathbb{N}}$ need not necessatily converge because we only required for maximal feasibility that each $\beta^{\left(N,V^k\right)}(S)$ is *any* member of that hyperplane represented by m_V^S and p_V^N. But for the same reason it is therefore possible without any loss of generality to define each $\beta^{\left(N,V^k\right)}$, $k \in \mathbb{N}$, in a way that indeed $\lim_{k\to\infty}\beta^{\left(N,V^k\right)} = \beta^{(N,V)}$ holds.

As a matter of fact we could have weakened the requirement of continuity of the reference function in Theorem 4.31 in precisely this sense. But this would not have enlarged the class of continuous β-prenucleoli because all those reference functions that we could additionally consider would yield the same β-prenucleolus as we already explained in connection with maximal feasible reference functions in section 5.1.

Hence we can conclude for the purpose of this section that the following remarks hold true.

Remark 5.33

1. *The NTU prenucleolus $\mathcal{PN}$ is continuous.*

2. *$\mathcal{PN}$ can be extended to the class Γ^l as suggested by Definition 4.33 and then Theorem 4.34 ensures the existence of $\mathcal{PN}$ on Γ^l.*

5.7 The β-core for maximal feasible β

So far this chapter has been devoted to the analysis of the (NTU) β-prenucleolus for maximal feasible β which we called the (NTU) prenucleolus. We have, however, defined another new solution concept for quasi hyperplane games that uses β-excess functions: the (NTU) β-core (Definition 4.26). We now analyze the β-core for maximal feasible reference functions β. For this specific reference functions we can state some more propositions beside the remarks made in chapter 4. Note that analog to the reasoning about the NTU prenucleolus, every maximal feasible β yields the same β-core.

We show that the β-core for maximal feasible β is always a subset of the core, provided both are non-empty, and that the β-core of a game is non-empty if and only if the game is m-balanced (Definition 5.14). These results do not yield a new proof of Corollary 5.17 about non-emptiness of the (NTU) core for m-balanced games. In fact, they are themselves corollaries of the proof of Theorem 5.16.

Theorem 5.34
Let $(N,V) \in \Gamma^{qH}$ *be a game and let* $\beta \in \mathcal{B}^c$ *be maximal feasible. Then* $\mathbf{Core}^{\beta}(N,V) \subseteq \mathbf{Core}(N,V)$. *Especially, this means* $\mathbf{Core}(N,V) = \emptyset \Rightarrow \mathbf{Core}^{\beta}(N,V) = \emptyset$.

Proof:
Suppose w.l.o.g. that (N,V) *is simplex. We have used in the proof of Theo-*

rem 5.16 the fact that for every $S \in 2^N$ *and every* $x \in V(N)$ *the implication*

$$\left\langle x_S, p_V^N|_S \right\rangle \geq m_V^S \Rightarrow \nexists y_S \in V(S) \text{ such that } y_S > x_S$$

is true. Since $\left\langle x_S, p_V^N|_S \right\rangle \geq m_V^S$ *implies* $e^\beta(S,x) = \left\langle \beta(S), p_V^N|_S \right\rangle - \left\langle x_S, p_V^N|_S \right\rangle = m_V^S - \left\langle x_S, p_V^N|_S \right\rangle \leq 0$, *it follows that* $x \in \mathbf{Core}^\beta(N,V)$ *implies* $x \in \mathbf{Core}(N,V)$.$\blacksquare$

For every reference function $\beta \in \mathcal{B}^c$ the (NTU) β-excess function on a simplex quasi hyperplane game (N,V) coincides with the (TU) excess function on its (TU) β-game $\left(N, v_V^\beta\right)$. Thus also the respective cores coincide. We state this simple observation as a Lemma and omit the proof. Note that both solution concepts are covariant hence the simplex assumption can be dropped.

Lemma 5.35
Let $(N,V) \in \Gamma^{qH}$ *be a game and let* $\beta \in \overline{\mathcal{B}}$ *be maximal feasible. Then*

$$\mathbf{Core}^\beta(N,V) = \mathbf{Core}\left(N, v_V^\beta\right),$$

i.e., the β*-core of* (N,V) *coincides with the (TU) core of its* β*-game.*

We are now able to give a necessary and sufficient condition for the β-core of a game to be non-empty.

Theorem 5.36
Let $(N,V) \in \Gamma^{qH}$ *be a game and let* $\beta \in \overline{\mathcal{B}}$ *be maximal feasible. Then* $\mathbf{Core}^\beta(N,V) \neq \emptyset$ *if and only if* (N,V) *is m-balanced.*

Proof:

$\mathbf{Core}^{\beta}(N,V)$ *coincides with* $\mathbf{Core}\left(N, v_V^{\beta}\right)$ *(Lemma 5.35). Thus it is non-empty if and only if* $\left(N, v_V^{\beta}\right)$ *is balanced. But* $\left(N, v_V^{\beta}\right)$ *is balanced if and only if* (N,V) *is* m*-balanced (Lemma 5.15).* ■

These results reveal an interesting connection between the core and the β-core of m-balanced quasi hyperplane games. Both solution concepts are not empty and they have a non-empty intersection since the (NTU) prenucleolus is a member of both of them.

S	$p^S_{V_1}$	$c^S_{V_1}$	m^S_V
$\{1\}$	$(1,0,0,0)$	0	0
$\{2\}$	$(0,1,0,0)$	0	0
$\{3\}$	$(0,0,1,0)$	0	0
$\{4\}$	$(0,0,0,1)$	0	0
$\{1,2\}$	$(2,3,0,0)$	4	2
$\{1,3\}$	$(1,0,5,0)$	5	5
$\{1,4\}$	$(2,0,0,1)$	4	4
$\{2,3\}$	$(0,3,4,0)$	4	$\frac{4}{3}$
$\{2,4\}$	$(0,1,0,1)$	2	2
$\{3,4\}$	$(0,0,4,5)$	5	$\frac{5}{4}$
$\{1,2,3\}$	$(1,2,2,0)$	6	6
$\{1,2,4\}$	$\left(2,\frac{1}{2},0,7\right)$	5	10
$\{1,3,4\}$	$(3,0,2,2)$	6	3
$\{2,3,4\}$	$(0,1,1,1)$	5	5
N	$(1,1,1,1)$	7	

Table 7: The game of example 5.26

References

[AM64] R.J. Aumann and M. Maschler, *The bargaining set for cooperative games*, Advances in game theory (M. Dresher, L.S. Shapley, and A.W. Tucker, eds.), Princeton: Princeton University Press, 1964, pp. 443–476.

[AP60] R.J. Aumann and B. Peleg, *Von Neumann-Morgenstern solutions to cooperative games without sidepayments*, Bulletin of the American Mathematical Society **66** (1960), no. 173–179.

[Ass76] N. Asscher, *An ordinal bargaining set for games without sidepayments*, Mathematics of Operations Research **1** (1976), no. 4, 381–389.

[Ass77] ———, *A cardinal bargaining set for games without side payments*, International Journal of Game Theory **6** (1977), no. 2, 87–114.

[Aum85] R.J. Aumann, *An axiomatization of the non–transferable utility value*, Econometrica **53** (1985), no. 3, 599–612.

[Bil70a] L.J. Billera, *Existence of general bargaining sets for cooperative games without side payments*, Bulletin of the American Mathematical Society **76** (1970), 375–379.

[Bil70b] ———, *Some theorems on the core of an n–person game without side payments*, SIAM Journal of Appl. Math. **18** (1970), 567–579.

[Bil71] ———, *Some recent results in n–person game theory*, Mathematical Programming **1** (1971), 58–67.

[Bon63] O.N. Bondareva, *Some applications of linear programming methods to the theory of cooperative games*, Problemy Kybernitiki **10** (1963), 119–139.

[CC02] C. Chang and P.-A. Chen, *Notes on excess functions and the nucleolus on NTU games*, mimeo (2002).

[CLT95] F. Christensen, M. Lind, and J. Tind, *On the Nucleolus of NTU-games defined by multiple objective linear programs*, to appear in ZOR Mathematical Methods of Operations Research 43, 1996, 1995.

[DM65] M. Davis and M. Maschler, *The kernel of a cooperative game*, Naval Research Logistics Quarterly **12** (1965), 223–259.

[DM67] ———, *Existence of stable payoff configurations for cooperative games*, Essays in mathematical economics in honor of Oskar Morgenstern (M. Shubik, ed.), Princeton: Princeton University Press, 1967, pp. 39–52.

[Edg81] F.Y. Edgeworth, *Mathematical psychics*, London: Kegan Paul, 1881.

[Edg90] G.A. Edgar, *Measure, topology, and fractal geometry*, Undergraduate Texts in Mathematics, Springer-Verlag New York Inc., 1990.

[Har63] J.C. Harsanyi, *A simplified bargaining model for the n–person cooperative game*, International Economic Review **4** (1963), 194–220.

[Hau44] F. Hausdorff, *Mengenlehre*, Dover Publications, New York, 1944.

[HMC89] S. Hart and A. Mas-Colell, *Potential, value, and consistency*, Econometrica **57** (1989), no. 3, 589–614.

[Jus77] M. Justman, *Iterative processes with "nucleolar" restrictions*, International Journal of Game Theory **6** (1977), 189–212.

[Kal72] E. Kalai, *Cooperative non-sidepayment games: Extensions of sidepayment solutions, metrics, and representative functions*, Ph.D. thesis, Center for Applied Mathematics, Cornell University, Ithaca, N.Y., 1972.

[Kal75] ———, *Excess functions for cooperative games without sidepayments*, SIAM Journal of Appl. Math. **29** (1975), no. 1, 60–71.

[Kla97] S. Klauke, *Der Nukleolus als Endpunkt eines mengenwertigen dynamischen Systems*, Master's thesis, University of Bielefeld, 1997, in german.

[Koh71] E. Kohlberg, *On the Nucleolus of a characteristic function game*, SIAM Journal of Appl. Math. **20** (1971), 62–66.

[Mas92] M. Maschler, *The Bargaining Set, Kernel, and Nucleolus*, Handbook of Game Theory (Robert J. Aumann and Sergiu Hart, eds.), vol. 1, Elsevier Science Publisher B.V., 1992.

[MO89] M. Maschler and G. Owen, *The consistent Shapley value for hyperplane games*, International Journal of Game Theory **18** (1989), 389–407.

[MO92] ———, *The consistent Shapley value for games without side payments*, Rational Interaction (R. Selten, ed.), Springer Verlag, 1992, pp. 5–12.

[MP76] M. Maschler and B. Peleg, *Stable sets and stable points of set–valued dynamic systems with applications to game theory*, 985–995.

[MPS72] M. Maschler, B. Peleg, and L.S. Shapley, *The Kernel and Bargaining Set for Convex Games*, International Journal of Game Theory **1** (1972), 73–93.

[MPT92] M. Maschler, J.A.M. Potters, and S.H. Tijs, *The General Nucleolus and the reduced game property*, International Journal of Game Theory **21** (1992), 85–106.

[OZ00] G. Orshan and J.M. Zarzuelo, *The bilateral consistent prekernel for NTU games*, Games and Economic Behavior **32** (2000), no. 1, 67–84.

[Pec98] S. Pechersky, *On "gauge" excess functions for NTU games: axiomatic approach*, mimeo, 1998.

[Pel63] B. Peleg, *Bargaining sets of cooperative games without side payments*, Israel Journal of Mathematics **1** (1963), 197–200.

[Pel65] ———, *An inductive method for constructing minimal balanced collections of finite sets*, Naval Research Logistics Quarterly **12** (1965), 155–162.

[Pel88] ———, *Introduction to the theory of cooperative games*, Research Memorandum 81, Center for Research in Mathematical Economics and Game Theory, The Hebrew University of Jerusalem, 1988.

[Ros81] J. Rosenmüller, *Theory of games and markets*, North Holland, 1981.

[Sca67] H.E. Scarf, *The Core of an n-person game*, Econometrica **35** (1967), no. 1, 50–69.

[Sch69] D. Schmeidler, *The Nucleolus of a characteristic function game*, SIAM Journal of Appl. Math. **17** (1969), 1163–1170.

[Sha53] L.S. Shapley, *A value for n-person games*, Contributions to the Theory of Games (H.W. Kuhn and W. Tucker, eds.), vol. 2, Princeton University Press, 1953, pp. 307–317.

[Sha67] Lloyd S. Shapley, *On balanced sets and cores*, Naval Research Logistics Quarterly **14** (1967), 453–460.

[Sha69] ———, *Utility comparison and the theory of games*, La decision, Editions du CNRS, Paris, 1969, pp. 251–263.

[Sob75] A.I. Sobolev, *The characterization of optimality principles in cooperative games by functional equations*, Mathematical Methods in the Social Sciences (N.N. Vorobiev, ed.), vol. 6, Vilnius. Academy of Sciences of the Lithuanian SSR, 1975, pp. 95–151.

[SS98] R. Serrano and K.-I. Shimomura, *Beyond Nash bargaining theory: The Nash set*, Journal of Economic Theory **83** (1998), 286–307.

[vNM44] J. von Neumann and O. Morgenstern, *Theory of games and economics behavior*, Princeton University Press, 1944.

[Yan97] E. Yanovskaya, *Consistency properties of the nontransferable cooperative game solutions*, Game theoretical applications to economics and operations research (T. Parthasarathy, B. Dutta, J.A.M. Potters, T.E.S. Raghavan, and A. Sen, eds.), Kluwer Academic Publishers, 1997.

[Yar85] M. Yarom, *Dynamic systems of differential inclusions for the Bargaining Sets*, International Journal of Game Theory **14** (1985), no. 1, 51–61.

[You85] H. Peyton Young, *Monotonic solutions of cooperative games*, International Journal of Game Theory **14** (1985), no. 2, 65–72.

[Zho91] L. Zhou, *A weak monotonicity property of the nucleolus*, International Journal of Game Theory **19** (1991), 407–411.

Lebenslauf

09.02.1970 geboren in Bremerhaven

1976-1980 Besuch der Grundschule Bexhövede

1980-1982 Besuch der Orientierungsstufe Loxstedt

1982-1986 Besuch des Gymnasiums Loxstedt

1986-1989 Besuch des Gymnasiums Wesermünde

1989 Abitur

1989-1990 Grundwehrdienst

1990-1998 Studium der Wirtschaftsmathematik an der Universität Bielefeld

1998 Diplom in Wirtschaftsmathematik

1995-1996 Tutor an der Universität Bielefeld

1998-2000 Mitglied des Graduiertenkollegs "Mathematische Wirtschaftsforschung", Universität Bielefeld

2000-2003 Wissenschaftlicher Angestellter an der Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

2003 Abschluß des Promotionsverfahrens mit der mündlichen Prüfung