

# Consumption Selection on Incomplete Markets

Simon Krsnik

May 26, 2011

Gedruckt auf alterungsbeständigem Papier DIN-ISO 9706

Inaugural-Dissertation  
zur Erlangung des Grades eines Doktors  
der Wirtschaftswissenschaften  
durch die  
Fakultät für Wirtschaftswissenschaften  
der Universität Bielefeld

vorgelegt von  
Simon Krsnik

Bielefeld, 2011

Dekan: Prof. Dr. Fred G. Becker  
Erstreferent: Prof. Dr. Frank Riedel  
Zweitreferent: Jun.-Prof. Dr. Dr. Frederik Herzberg

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# Chapter 1

## Introduction

### Motivation

The question how to allocate capital best is as old as financial markets themselves. Maximizing expected gains only might be a good approach but cannot be the best answer because usually high expected gains are driven by highly speculative and risky investments. It can well be observed in the course of financial crises the demand for the less risky fixed income products, as government or sovereign bonds raise. Moreover capital guarantee products including portfolio insurance strategies became more important.

In this thesis we study economic agents who subordinate expected gains to other plans. The essential idea is justified and generally accepted in the common literature, since each portfolio insurance strategy subordinates expected gains to capital guarantees. E.g. the well known work of Black and Perold (1992) describes a *constant proportion portfolio insurance (CPPI)* strategy, where risky investment takes place up to a constant fraction of the capital guaranteed. This decision rule mirrors the plan to keep his *wealth* above a certain capital floor.

In contrast we consider agents whose plans are afflicted with future expenditures and mirror decisions based on *expenses*. Agents driven by those plans will be more careful withdrawing money today because high expenses narrow the capital stock and impede further enduring expenditures. Stating first ideas on how this plans may look like, we should interpret prior

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expenditures as a measure for standard of living. It can be widely observed that standard of living affects decisions on current expenses. E.g. if you need a new car and you are used to drive Porsche, a Škoda is no satisfactorily choice. Thus in some sense we might be addicted to our own (consumption) history.

Based upon this insight we may consider an agent who is frightened of an immoderate decline in standard of living. Holding up his current standard in the future the investor needs secure savings. Accordingly he may behave optimally only if risky investments will not endanger this savings. The first to cope with this kind of addicted (or *consumption ratcheting*) agents was Dybvig (1995). In a standard framework with time-additive utility evaluation, he found out that such agent behaves optimally only if risky investment takes place up to a certain fraction of the savings he need. This is a notable improvement for optimal strategies in a time-additive setting, since usually time-additivity generates optimal consumption pattern as random as the market itself. Until then all models based on time-additive utility evaluation led to results which were even harder to reconcile with the data observed on consumption behavior. Several other utility functionals has been proposed to mend this disadvantages. Most of them generate habit formation rather than narrowing the agents choice to rule out implausible consumption patterns.

Indeed, the problem of maximizing gains form utility obtained on a financial market is an essential problem in the areas of economics and finance. Starting with the economically meaningful assumption that the market does not allow arbitrage profits, an agent is faced with the problem to allocate an initial capital among several financial assets so as to maximize expected utility form consumption. To improve some weaknesses of time-additive utility maximization, we consider on optimization problem, where consumption selection is subject to individual likings. More precisely these individual likings are modeled via a closed convex cone in the space of consumption processes and might bare a rule on future expenditures.

Motivated by an optimal investment problem for a wealth-path dependent utility maximizer (Bouchard and Pham, 2004), we extend the classical the-



ory of maximizing utility from intertemporal (and terminal) consumption to more general distributions over time. This includes time-horizon uncertainty as well as a version of gaining utility from intertemporal consumption and terminal wealth (cf. Examples 1 and 2). A main outcome is the extension of the duality approach in the line of Kramkov and Schachermayer (1999) to our framework. Using techniques from convex duality we establish general existence and uniqueness results with and without exploiting asymptotic elasticity (Kramkov and Schachermayer, 1999).

This approach raises many questions and intersects with an ample source of related literature.

### **Maximizing Utility on Financial Markets**

First to consider the utility maximization problem in stochastic, time-continuous models were Samuelson and Merton (1969, 1971). He used the very strong assumption that asset prices are governed by Markovian dynamics with constant coefficients. Accordingly he could use the methods of stochastic programming and in particular the Hamilton-Jacobi-Bellman equation for dynamic programming. Merton found that optimal consumption is a constant fraction of the wealth process, moreover when relative risk aversion is constant this ratio is identified as the fraction of market price of risk to the relative risk aversion.

More recently, a martingale approach to the problem in complete Itô-process markets was introduced by Pilska (1986), Karatzas et al. (1987) and Cox and Huang (1989), exploiting more powerful techniques from convex duality. The key to this approach is to relate the marginal utility form a possibly optimal strategy to (the density of) the martingale measure. Difficulties with this approach arise in incomplete markets.

Originally the problem of employing these techniques in a time-continuous incomplete market model was treated by Xu - resp. (Shreve and Xu, 1992) - in his doctoral dissertation. He made use of the convexity of this problem to formulate and solve a dual variational problem. This approach turned out to be right one, also to study consumption-investment problems on more

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general *constraint markets* (Cvitanic and Karatzas, 1992). Under general convex constraints on the portfolio choice Cvitanic and Karatzas showed that the value of this constrained optimization problem corresponds to the value of an unconstrained optimization on an auxiliary market. Moreover they characterize the solution via stating several equivalent conditions.

As the most general under incomplete market conditions, the paper of Kramkov and Schachermayer (1999) must be mentioned. In this paper an agent gains utility from terminal wealth within a finite time horizon  $T < \infty$ . Kramkov and Schachermayer show that a necessary and sufficient condition for the existence of an optimal solution is the *asymptotic elasticity* of the utility function. An analytic condition on the behavior of the utility function at infinity, which excludes certain pathological situations. Furthermore, they show that the set of densities of equivalent local martingale measures may actually be too small to host the solution of the dual problem, see Kramkov and Schachermayer (1999, Proposition 5.1). In a later article (Kramkov and Schachermayer, 2003) they relaxed the assumptions on the utility function and imposed finiteness on the dual value function directly to show general existence of the primal solution. Moreover they ascertained that finiteness on the dual value function is the weakest assumption on the overall market structure to guarantee solvability of the primal problem in general. In the meantime one of the key theorems - The Bipolar Theorem of Brannath and Schachermayer - was further extended by Žitković (2002).

The insights in Kramkov and Schachermayer (1999) and (2003) mainly bases on techniques that rely on bidual properties. Since the space of measurable functions on  $(\Omega, \mathcal{F}, \mathbb{P})$  is not locally convex, which is an important property while proving the usual Bidual Theorem, those techniques have not been employed until then. Making use of the order structure of non-negative random variables, Brannath and Schachermayer (1999) were able to introduce a new polarity concept. Moreover they obtained an version of the Bipolar Theorem for sets of non-negative random variables. Žitković (2002) extended this Bidual Theorem for non-locally convex spaces to the space of processes, such that the treatment of nearly all standard investment and consumption problems was possible. To name a view list a selection of

some works, which will be important for our studies or might be interesting for further research as well.

Mnif and Pham (2001) studied the same optimization problem with utility from terminal wealth only, as Kramkov and Schachermayer (1999). On the basis of Föllmer and Kramkov (1997) they build a more general framework including the standards for incomplete markets. Thus they extend the results of Kramkov and Schachermayer (1999) to the case of general market constraints.

A problem of optimal consumption choice within this semimartingale model has been studied by Karatzas and Žitković (2003). Giving the investors a chance to rejoice in some random extra endowment the authors perturb the wealth process obtained on the market by adding a random cumulative endowment process. In this setting they used the Filtered Bipolar Theorem (Žitković, 2002). Moreover a definition of Asymptotic Elasticity with respect to the time-dependent intertemporal utility functions was needed to obtain a general existence result.

The most general model where utility is gained from wealth on incomplete markets is treated in Bouchard and Pham (2004). Based on the martingale approach Bouchard and Pham established results for investors gaining utility from the whole wealth process. They set up a very general model which may include many realistic assumptions as e.g. time-horizon uncertainty.

A non-standard model for intertemporal consumption choice within this semimartingale setting has been studied by Kauppila (2010) in her doctoral dissertation. In her model agents' preferences are not time-additive. More precisely preferences are based on the whole path of consumption up to the particular date. Those preferences were introduced by Hindy et al. (1992) as an economically more reasonable alternative to standard time-additive models.

Hindy et al. most important criticism is focused on the concept of local substitution; consumption on near by dates and slightly varying consumption rates should be good substitutes. In their seminal paper they show that preferences which are continuous in the Prohorov topology possess this property. In a following paper Hindy and Huang (1993) established the solution

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in a basic  $It\bar{o}$ -process model. In contrast to the solution found by Merton (1969), where optimal rate of consumption equals a constant fraction of current wealth, they found the following. An agent with *Hindy-Huang-Kreps*-preferences behaves optimally if his consumption policy is to keep the ratio of wealth to average of consumption history ( $\mathbb{P} - a.s.$ ) below a  $\mathbb{R}_{++}$ -valued *ratio barrier*. Following the ideas in Hindy et al. (1992) and Hindy and Huang (1993) various authors have worked on those preferences.

Bank and Riedel developed an approach on this optimization problem (in deterministic setting (Bank and Riedel, 2000) and in stochastic setting (Bank and Riedel, 2001)), based on an infinite-dimensional version of the Kuhn-Tucker conditions. More recently Kauppila (2010) embed the optimization problem for agents with those preference structure in the framework of Kramkov and Schachermayer and establish results corresponding to Kramkov and Schachermayer (1999). Evaluating cumulative consumption makes it necessary to employ the process based polarity definition achieved in Žitković (2002). In particular process polarity delivers a set of supermartingales for the dual variables. Kauppila narrows this set making use of a representation theorem for optional processes (cf. Bank and El Karoui (2004)). This representation theorem also delivers the solution in the complete market case. Moreover Kauppila has to develop a new Minimax-Theorem based on a weaker notion of compactness.

Žitković (2010) introduced the concept of *convex compactness*, which proper extends the comprehension of compactness for convex sets, and stated simple characterizations for convex compact sets. We should mention the two most important characterizations. Firstly for convex compact sets on  $L_+^{0,1}$  and secondly via a convergence property in the line of *Cesáro*-convergence. Moreover this convergence property heavily relates convex compact sets to the theorem of Komlòs (1967).

In spite of an entitled critique (Hindy et al., 1992), the use of agents gain-

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<sup>1</sup>Here  $L_+^0$  denotes the set of  $\mathbb{P} - a.s.$  non-negative random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

ing utility time-additively from intertemporal consumption rates is still very popular in economic theory (cf. Karatzas and Žitković (2003)). Moreover Hindy et al. have not been the first to criticize the time-additive utility functional claimed in Merton (1969). Various essential works as Ryder and Heal (1973), Constantinides (1990) or Ingersoll (1992) justify why loosening the assumption on preferences to be not time-additive is reasonable and join a huge literature on habit-formation models based on consumption in rates. In contrast to the introductory example (Dybvig, 1995), where the utility functional is unbiased on the whole set of permissible strategies, those models usually punish any deviation from the average consumption rate with increasing intensity.

For consumption choice influenced by time-additive expected utility a consumption policy similar to Hindy and Huang (1993) was established in Dybvig (1995). Since market dynamics are Markovian with constant coefficients, Dybvig the Hamilton-Jacobi-Bellman approach for dynamic programming. As denoted in the introductory example Dybvig studied the optimal behavior of an agent who is that frightened for a decline in standard of living that he only accepts non-decreasing processes for consumption. Those agents always reserve capital for holding up the current consumption level over their remaining lifetime. Obviously wealth can never lie below this reserves and as soon as wealth equals the reserves needed, the investor cannot invest in risky assets any longer. Moreover reserves must grow according to current consumption rates. For those investors the optimal consumption rule is to consume a constant fraction of the difference of wealth to reserve.

As can be obtained in Riedel (2009) the approach using Kuhn-Tucker like conditions (cf. Bank and Riedel (2001)) allows to derive the optimal consumption plan also for complete markets with more general Lévy process dynamics. Within a static infinite horizon setting Riedel verified that the optimal consumption policy is heavily related to the solution of the original problem (Merton, 1969). More precisely if a process is optimal for Merton's problem and its running maximum has a finite price then this running maximum process solves a consumption ratcheting problem.

A note how models for consumption ratcheting investors can be transferred

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into models for preferences with intertemporal substitution can be found in Schroder and Skiadas (2002).

More recently Schroder and Skiadas (2008) considered a utility maximization problem within a financial market driven by Brownian motion and point processes. They studied optimality conditions for agents with generalized recursive utility functionals, whose consumption selection is subjected to a closed convex cone in  $L^2$ . Unfortunately beside mathematical tractability they give no justification why consumption choice is restricted like that. For general consumption constraints - up to my knowledge - this model comes closest to the setting we choose.

### **Structure of the Thesis**

The entire expected utility maximization problem is embedded in a semimartingale model for incomplete markets in the line of Kramkov and Schachermayer (1999). More precisely, on the basis of the Filtered Bipolar Theorem (Žitković, 2002), we join the models introduced in Bouchard and Pham (2004) and Karatzas and Žitković (2003) to get a more general approach. Some of our proofs lean on corresponding proofs established in Bouchard and Pham (2004) or Kramkov and Schachermayer (1999). While utility is gained from a rate of consumption process as in Karatzas and Žitković, evaluation of the consumption process bases on a distribution function  $F$  (Bouchard and Pham, 2004), which weights the intertemporal utility function over time. This enables us to set up a general model on intertemporal consumption choice without causing exhausting calculations since the intertemporal utility function itself does not change over time. Moreover if we impute the investor to base his decision on a distribution function  $F$ , our model includes the setup with time-separable utility and time-horizon uncertainty as well. Our aim is to identify a suitable dual problem for both constrained and unconstrained consumption selection. Moreover we derive properties of primal and dual value functions as precise as possible. We go on as follows.

In the second section we introduce the semimartingale model for asset

prices, but instead of choosing the bond as numéraire right from the start we begin with an arbitrary non-decreasing process modeling bond price dynamics. By means of a short detour we repeat the properties needed to formulate the *process polar* for sets of non-negative semimartingales and the *Filtered Bipolar Theorem* (Žitković, 2002). We explain wealth dynamics under the usual No-Arbitrage assumption and derive a budget constraint employing process polarity. Although our results would hold as well, we abstain from general bond price dynamics for the analysis in Sections 3 - 6 and choose bond as numéraire. Most assertions stay unchanged, but since bond price dynamics influences wealth dynamics, they also affect the consumption choice. Moreover when bond prices raise, discounting of rate of consumption processes is necessary. In the Section 2 we give a taste how results must be carried over when the bond price is not constant.

In the third Section we analyze the problem of optimal consumption choice, when consumption choice is subject to the natural constraints only (Cf. Merton (1969)). Inter alia these natural constraints are necessary to rule out the possibility to select negative consumption rates on the one hand side and force the investor to decide with respect to the budget constraint on the other hand side. Since we want to apply techniques from convex analysis on this optimization problem, we first have to define a set of suitable dual variables.

Although we are maximizing over (progressively measurable) rate of consumption *processes* we will not employ the duality theory for processes as discussed in Žitković (2002). More precisely we take the consumption processes as non-negative *random variables* which are  $\mathcal{M}$ -measurable<sup>2</sup> and cope with duality as introduced originally (Cf. Brannath and Schachermayer (1999)). We choose the duality approach introduced in Bouchard and Pham (2004). This usually forces some difficulties since utility is maximized with respect to  $F$ , which influences some essential concepts like solidity. Solidity is absolutely essential for employing duality arguments and heav-

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<sup>2</sup>Here  $\mathcal{M}$  denotes the  $\sigma$ -field generated by all progressively measurable processes.

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ily depends on a measure defined on  $(\Omega \times [0, T], \mathcal{M})$ . Via the probability induced by the distribution function  $F$ , we make  $(\Omega \times [0, T], \mathcal{M})$  to a probability space. In this context we introduce polarity, we derive dual variables, and the dual problem. One should notice that for the pricing formula we still use the Lebesgue measure, reps.  $dt \otimes \mathbb{P}$ .

As the main results in this section we prove existence and uniqueness of primal and dual optimizers and list the main properties of primal and dual value function. Necessary and sufficient for solvability of both the primal and the dual problem will be introduced and discussed as well as main properties of primal and dual value function. Like in many other optimization problems on semimartingale models these assertions hold even if the intertemporal utility function  $u$  does not satisfy the usual condition on asymptotic elasticity (Kramkov and Schachermayer, 1999). Recall that asymptotic elasticity  $< 1$  is the weakest market independent condition, whereas finiteness of the dual value function is the weakest overall condition to guarantee these assertions (Kramkov and Schachermayer, 1999; Kramkov and Schachermayer, 2003).

Assuming that the distribution process  $F$  has a density with respect to the Lebesgue measure our model resembles the models in Karatzas and Žitković (2003) and Störmer (2010). When random endowment equals 0, we derive corresponding results as Karatzas and Žitković (2003). Karatzas and Žitković based their model on more general time-additive intertemporal utility, which intersect with our model. The price they had to pay for that, is to claim the existence of time independent minorant and majorant for the derivative of the intertemporal utility. Furthermore, since their intertemporal utility functions have a time variable, they had to impose an additional regularity condition on asymptotic elasticity over time. A similar regularity condition, valid in the time-separable case only, has also been used in Störmer (2010). In our model we do not have to look at changes in asymptotic elasticity over time. Thus, our model has obvious technical advantages.

In Section 4 we introduce the model for constrained consumption selection. We state reasonable axioms for sets of permissible consumption processes which sets limits for individual likings. The set of admissible con-



sumption processes will be narrowed according to this individual likings.

Ruling out admissible consumption rates like this, we should rethink the cause for market incompleteness.<sup>3</sup> If an investor stints himself to choose consumption strategies according to his likings, he could have similar partialities driving his portfolio selection. Thus, some source of risk which can be traded eventually is not be traded, because the investor dislikes those strategies (cf. Islamic Banking).

The influence of individual likings on the expected utility functional will be discussed as well. On the set of permissible consumption processes the investor acts as in the unconstrained case, on the remaining consumption processes we put the value  $-\infty$ . From a habit formation point of view one could say as long as consumption in permissible there is no punishment but as soon as consumption becomes non-permissible punishment is incredibly hard. The first comment on this kind of utility functions can be found in Dybvig (1995). There an agent evaluates a consumption process via the usual time-additive utility functional as long as it is non-decreasing.

When considering the maximization problem we look at the set of permissible consumption processes which are also admissible for a certain initial capital. This time we define the dual value function as the convex conjugate of the real valued value function directly. Having an evaluation functional (the unconstrained expected utility functional) which put a value on the whole underlying market structure becomes very important for the upcoming maximization problem. We will use the value of the market structure as a benchmark. Finite market value will play the same role as finiteness of the value function in the unconstrained setting. Making use of this benchmark we prove existence and uniqueness of an optimal primal strategy and state first properties of the value function. Unfortunately the dual problem cannot be set up as easy as before, where we had to minimize the convex conjugate over a set of dual variables. This time the set of primal variables is not solid, thus it is neither possible to apply the Bidual Theorem of Brannath

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<sup>3</sup>This becomes more clear if we model market incompleteness via portfolio constraints as in Karatzas and Shreve (1998, Examples 5.4.1).

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and Schachermayer directly nor it will be possible to derive properties as comprehensive as in the unconstrained case. At least it turns out that the set of dual variables introduced in Section 3 suffices to set up a suitable dual problem, which corresponds to the dual value function. Therefore we define a dual utility function on the set of dual variables.

A similar approach can be found in Kauppila (2010) for optimal consumption choice with intertemporal substitution. This concept slightly resembles the case of consumption ratcheting (cf. Section 6). As a main difference Kauppila let her agents choose *optional* processes, thus she is allowed to define a suitable dual function path wise.

In Section 5 and 6 we discuss some special cases and examples.

In the main parts of the fifth section we consider the case of unconstrained consumption choice and show some standard results. In particular we prove that the set of equivalent martingale measures suffices to set up a dual problem although the dual minimizer may not be contained within this set. Furthermore we derive a nice result on constrained consumption selection in the line of Cvitanić and Karatzas (1992, Theorem 10.1). In that paper they solved a utility maximization on constrained portfolio choice (cf. incomplete markets) and unconstrained consumption choice via auxiliary complete markets. Moreover they verified that the value of that (portfolio) constrained optimization problem corresponds to the value of that auxiliary market with minimal (dual) value. We found a version of this theorem on incomplete markets when consumption selection is constrained.

In the sixth section we study the case of consumption ratcheting in more detail. As usual consumption selection takes place on a set of progressively measurable processes. This time it turns out that it suffices to consider the smaller set of optional processes. Thus, we are able to apply the theory developed in Kauppila (2010). We will show how optimal consumption on complete markets looks like and state further properties of the (dual) value function. Finally we relate the optimization problem for consumption ratcheting agents to similar problems (resp. similar individual likings).

# Chapter 2

## The Model

Our aim is to analyze a consumption (and portfolio) choice problem in an incomplete market via convex duality methods. In this chapter we set up the model which builds the basis for the upcoming optimization problems.

To model incomplete market dynamics we choose a semimartingale approach based on the model of Kramkov and Schachermayer (1999) for optimal portfolio choice. More precisely we combine the techniques and ideas of Žitković (2002) and Bouchard and Pham (2004) and carry their results over to our problem where an investor gains utility from intertemporal consumption.

Within a finite time horizon  $T > 0$  the financial market consists of one riskless bond  $\mathcal{S}_0$  and  $n$  risky assets  $\mathcal{S} = (\mathcal{S}_i)_{1 \leq i \leq n}$ . The bond price process  $\mathcal{S}_0(t)$  is assumed to be an adapted, non-decreasing process with  $\mathcal{S}_0(0) = 1$ . We assume  $\mathcal{S}$  to be a  $\mathbb{R}^n$ -valued semimartingale on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{0 \leq t \leq T})$ . As we will see later this assumption allows for the existence of multiple equivalent martingale measures, thus the market may be incomplete. Further we assume  $\mathcal{F}_0$  as trivial.

By  $\hat{\mathcal{S}}$  we denote the price process of the assets discounted by the numéraire  $\mathcal{S}_0$ , i.e.

$$\hat{\mathcal{S}}(t) = \frac{\mathcal{S}(t)}{\mathcal{S}_0(t)}$$

The set of all  $\mathbb{R}^n$ -valued predictable and  $\hat{\mathcal{S}}$ -integrable processes  $\pi$  will be denoted by  $L(\hat{\mathcal{S}})$ .

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## 2.1 Utility Maximization

For the upcoming optimization problems consider an utility function  $u: \mathbb{R}_{++} \rightarrow \mathbb{R}$  which is continuously differentiable, strictly increasing, and strictly concave. We restrict our attention to utility functions  $u$  which satisfy the conditions  $u'(0) = \infty$  and  $u'(\infty) = 0$ .

For technical reasons we assume  $u(\infty) > 0$ , which can always be reached by adding a non-negative constant. From an economic point of view, this is not an additional assumption since affine transformations of a utility function do not effect the underlying preference structure of an economic agent. Concavity which implicitly is assumption on the derivative  $u'$  is a proper restriction, as concave functions correspond to risk averse investors.

For the upcoming optimization problem we change the measure space via a non-negative, non-decreasing  $\mathcal{F}$ -adapted process  $F$  called *distribution process*. Given some initial capital  $x > 0$  the problem is the following.

$$\text{maximize } \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad \text{s.t. } c \in \mathcal{C}(x) \quad (2.1.1)$$

Here  $\mathcal{C}(x)$  denotes the set of consumption patterns  $c$  admissible for  $x$ . Details will be given in the following section.

For the distribution process  $F$  we claim the following.

### Standing Assumption 1.

$$\mathbb{P}(F(T) > 0) > 0 \quad \text{SA 1.1}$$

$$\mathbb{E} \left[ \int_0^T \mathbf{1} dF(t) \right] = 1 \quad \text{SA 1.2}$$

In fact we only need to assume that  $F$  is bounded in expectations, then (SA 1.2) holds w.l.o.g.

Using a distribution process  $F$  has great advantages, since it includes the standard models and lots of well-established deviations.

On the one hand there are models where consumption takes place once, i.e.

at time  $T$  or at some random time  $\tau \leq T$ . Here the solutions should coincide with situations where the investor gains utility from its wealth process directly, see Kramkov and Schachermayer (1999) and Blanchet-Scalliet et al. (2002). Thanks to the assumption that intertemporal utility is strictly increasing the investor has incentives to spend his entire wealth for consumption at that predefined point in time. This point in time may be random as our setup includes time-horizon uncertainty. An assumption which might be closer to reality. Usually investors do not know with certainty the time they will exit the market. In practice, the time horizon can be affected by several factors, e.g. changes in investor's position like retirement or death.

**Example 1** (Uncertain Time Horizon). Consider the problem

$$\text{maximize } \mathbb{E} [u(c(\tau))]$$

where  $\tau \leq T$  is a random time, i.e. a non-negative random variable measurable with respect to  $\mathcal{F}$ . Typical cases are:

- (i)  $\tau$  *constant* ( $\tau = T$ ). This coincides with maximizing utility from terminal consumption.
- (ii)  $\tau$  *independent of*  $\mathcal{F}_T$ . In general the distribution of  $\tau$  is given via  $F(t) = \mathbb{P}(\tau \leq t)$ .
- (iii)  $\tau$  *stopping time*. In this special case,  $F(t) = \mathbf{1}_{\tau \leq t}$ .

On the other hand we include models of time continuous consumption. As a general formulation for maximizing utility from intertemporal and terminal consumption look at the following example with a *time-separable* intertemporal utility function.

**Example 2** (Intertemporal Consumption and Terminal “Wealth”). As at the terminal date the investor has no incentives to put capital aside, he should consume the remaining capital. From this point of view consider the problem

$$\text{maximize } \mathbb{E} \left[ \int_0^T f(t)u(c(t)) dt + \left( 1 - \int_0^T f(t) dt \right) u(c(T)) \right]$$

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Here  $f$  is a right-continuous, non-negative  $\mathcal{F}$ -adapted process with  $\int_0^T f(t) dt \leq 1$ . The process  $f$  can be interpreted as a density for  $F$ .

Expenditures on consumption will change investors wealth process enduringly. Thus when time continuous consumption is included the solution should deviate from the results for wealth-path dependent utility maximization (Bouchard and Pham, 2004). Wealth dynamics and the effects of withdrawing money for consumption will be introduced in the following.

We continue the studies of the general problem, analyzing how initial capital generates wealth processes. An investor on this market is an economic agent who acts as a price taker and who can decide at any time which amount  $\pi = (\pi_i)_{1 \leq i \leq d} \in L(\hat{\mathcal{S}})$  of each asset to hold in his portfolio and how much money to withdraw for consumption. The process  $\pi$  will be called the investor's *portfolio process* from now on.

Furthermore the investor chooses a *consumption rate process*  $c$ , a progressively measurable, non-negative process, which is related to a *cumulative consumption process*  $C$  via

$$C(t) = \int_0^t \frac{1}{\mathcal{S}_0}(s)c(s) ds \quad \text{for all } t \in [0, T]$$

Although we are only interested in consumption patterns that rely on rate of consumption processes we begin our analysis for all cumulative consumption processes  $C \in \mathcal{I}$ . Here  $\mathcal{I}$  denotes the set of all adapted, non-decreasing, càdlàg processes  $C$  with initial value  $C(0) = 0$  and  $C(t) \geq 0$   $\mathbb{P} - a.s.$  for all  $t \in [0, T]$ . Obviously  $C \equiv 0$  is a feasible consumption plan.

Given an initial capital  $x > 0$  we call the triple  $(x, \pi, C)$  a *consumption-investment strategy*. With these strategies we associate a *wealth process*  $X^{x, \pi, C}$  representing the investor's current holdings

$$X^{x, \pi, C}(t) = \mathcal{S}_0(t) \left( x + \int_0^t \pi(u) d\hat{\mathcal{S}}(u) - C(t) \right)$$

Thus, as usual in mathematical finance, discounted wealth equals earnings from a self financing portfolio minus cumulated consumption, i.e.

$$\frac{X^{x, \pi, C}(t)}{\mathcal{S}_0(t)} = x + \int_0^t \pi(u) d\hat{\mathcal{S}}(u) - C(t)$$

## 2.2. PROCESS DUALITY AND THE *FILTERED BIPOLAR THEOREM*

A consumption-investment strategy  $(x, \pi, C)$  is called *admissible for initial capital  $x$*  if its corresponding wealth process remains non-negative, i.e.

$$X^{x,\pi,C}(t) \geq 0 \quad \mathbb{P} - a.s. \text{ for all } t \in [0, T]$$

The set of all admissible consumption-investment strategies with an initial capital  $\bar{x} \leq x$  will be denoted by  $\mathcal{A}(x)$ . Obviously  $\mathcal{A}(x) = x\mathcal{A}(1)$  holds.

## 2.2 Concepts of Duality and the *Filtered Bipolar Theorem*

As a well known fact in functional analysis - the classical Bipolar Theorem states that - the bipolar of a subset  $D$  of a locally convex vector space is the smallest closed, balanced, and convex set containing  $D$ . The locally convex structure of the underlying space is of great importance since the proof relies heavily on the Hahn-Banach Theorem. Brannath and Schachermayer (1999) made use of the order structure of non-negative measurable functions on  $(\Omega, \mathcal{F}, \mathbb{P})$  to obtain an extension of the Bipolar Theorem to this space.<sup>1</sup> Furthermore Žitković (2002) extended this Bipolar Theorem of Kramkov and Schachermayer to the space of non-negative càdlàg supermartingales on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{0 \leq t \leq T})$ . Although we finally work with duality in the sense of Kramkov and Schachermayer, we discuss the most important properties and implications of the Filtered Bipolar Theorem (Žitković, 2002) in this section.

The set of  $\mathbb{R}$ -valued processes  $Y = Y(t, \omega)$  will be denoted by  $L^0$ . For two processes  $X, Y \in L$ , we write

$$X \geq Y \quad \text{if } X(t) \geq Y(t) \quad \mathbb{P} - a.s. \text{ for all } t \in [0, T]$$

The subcone of non-negative processes  $X \in L$  in the sense of  $\geq$  will be denoted with  $L_*$ .

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<sup>1</sup>Note that this space is not locally convex in general.

---

We will make use of two different concepts of duality. The first one in the notion of processes on  $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{0 \leq t \leq T})$  and later a second one in the notion of random variables on  $\Omega \times [0, T]$ . More precisely later we take certain non-negative processes as random variables to describe dual relations.

For a better understanding of the main properties of the process dual, we begin with some definitions.

For  $\mathcal{E} \subseteq L_*$  we define the *process polar*

$$\mathcal{E}^\times = \{F \in L_* \mid E(0)F(0) \leq 1; EF \text{ a supermartingale for all } E \in \mathcal{E}\}$$

We call a set  $\mathcal{E} \subseteq L_*$  *far-reaching* if there is an element  $E \in \mathcal{E}$  such that  $E(T) > 0$  holds  $\mathbb{P} - a.s.$

Further we denote

$$\mathcal{V} = \{V \in L_* \mid V(0) \leq 1 \text{ and } V \text{ is càdlàg and non-increasing}\}$$

A set  $\mathcal{E} \subseteq L_*$  is called *process solid* if for each  $E \in \mathcal{E}$  and  $B \in \mathcal{V}$  we have  $BE \in \mathcal{E}$ .

**Definition 1** (Fatou Convergence). *Let  $\{E^n\}_{n \geq 1} \subseteq L_*$  a sequence of processes. We say  $E^n$  Fatou-converges to a process  $E \in L_*$  if*

$$E(t) = \liminf_{s \searrow t} \liminf_{n \rightarrow \infty} E^n(s) = \limsup_{s \searrow t} \limsup_{n \rightarrow \infty} E^n(s)$$

*holds  $\mathbb{P} - a.s.$  for all  $t \in [0, T]$ .*

A set  $\mathcal{E} \subseteq L_*$  is called *Fatou-closed* if it is closed with respect to Fatou convergence.

For completeness we also state the definition of fork-convexity, which is well-established in the mathematical finance literature.

**Definition 2** (Fork-Convexity). *A set  $\mathcal{E}$  of non-negative supermartingales is called fork-convex, if for any  $u \in [0, T]$ , any  $S_1, S_2, S_3 \in \mathcal{E}$  and every  $\mathcal{F}_t$ -measurable random variable  $h$  with  $0 \leq h \leq 1$  ( $\mathbb{P} - a.s.$ ), the process  $S$  defined via*

$$S(t) := \begin{cases} S_1(t) & t < u \\ S_1(t) \left( h \frac{S_2(t)}{S_2(u)} + (1-h) \frac{S_3(t)}{S_3(u)} \right) & t \geq u \end{cases}$$



belongs to  $\mathcal{E}$ .

A famous (and for our forthcoming analysis also important) example of a fork-convex set is the set of equivalent local martingale measures for a semimartingale  $\mathcal{S}$ .

Finally we are able to write down the characterization of the *process-bipolar* (Žitković, 2002). Note that only supermartingales matter in the context of the process-polar.

**Theorem\* 1** (Filtered Bipolar Theorem (Žitković, 2002)). *Let  $\mathcal{E} \subseteq L_*$  denote a far-reaching set of supermartingales with initial value less or equal to 1. Then the process bipolar  $\mathcal{E}^{\times\times}$  is the smallest Fatou-closed, fork-convex and process solid set of supermartingales containing  $\mathcal{E}$ .*

## 2.3 Properties of the Market

To have a realistic model of a market, we assume a variant of the no-arbitrage property by postulating the existence of certain probability measures equivalent to  $\mathbb{P}$ . Therefore we introduce the set

$$\hat{\mathcal{X}}(x) = \left\{ X \geq 0 \mid X = \frac{1}{S_0} X^{\tilde{x}, \pi, 0} \text{ for } \tilde{x} \leq x \right\}$$

Obviously  $\hat{\mathcal{X}}(x)$  consists only of local martingales with  $X(0) \leq x$  and  $\hat{\mathcal{X}}(x) = x\hat{\mathcal{X}}(1)$ . We abbreviate  $\hat{\mathcal{X}} = \hat{\mathcal{X}}(1)$ .

We now introduce equivalent martingale measures in the sense of Kramkov and Schachermayer (1999).

A measure  $Q$  on  $(\Omega, \mathcal{F})$  is called an *equivalent martingale measure*, if  $Q \sim \mathbb{P}$  and each  $X \in \hat{\mathcal{X}}(1)$  is a local martingale under  $Q$ . The family of equivalent martingale measures will be denoted by  $\mathfrak{M}$ .

**Standing Assumption 2.**

$$\mathfrak{M} \neq \emptyset \tag{2.3.1}$$

---

This condition is strongly related to the absence of arbitrage opportunities. Furthermore under certain conditions  $\hat{\mathcal{S}}$  is a local martingale under  $Q$  if and only if  $Q \in \mathfrak{M}$ ; see Delbaen and Schachermayer (1994), (1995) or (1998).

For some duality considerations it is necessary to study the process-polar of  $\hat{\mathcal{X}}$

$$\hat{\mathcal{Y}} := (\hat{\mathcal{X}})^\times$$

Further we define the families

$$\mathcal{X} = \mathcal{S}_0 \hat{\mathcal{X}} = \{X \geq 0 \mid X = X^{\tilde{x}, \pi, 0}\} \quad \text{and} \quad \hat{\mathcal{A}} = \frac{1}{\mathcal{S}_0} \mathcal{A}$$

such that  $\mathcal{X}(x) \subseteq \mathcal{A}(x)$  (resp.  $\hat{\mathcal{X}}(x) \subseteq \hat{\mathcal{A}}(x)$ ) for all  $x > 0$ . Obviously the process-polar property of  $\hat{\mathcal{Y}}$  hands down to  $\mathcal{Y} = \mathcal{X}^\times$  such that

$$\mathcal{Y} = \frac{1}{\mathcal{S}_0} \hat{\mathcal{Y}}$$

In this setting Theorem 2.1 in Kramkov (1996) states that a non-negative càdlàg process  $X$  with  $X(0) \leq x$  is in  $\mathcal{A}(x)$  if and only if  $\frac{1}{\mathcal{S}_0} X$  is a supermartingale for each  $Q \in \mathfrak{M}$ . Similarly  $X$  is in  $\mathcal{X}$  if and only if  $\frac{1}{\mathcal{S}_0} X$  is a local martingale for each  $Q \in \mathfrak{M}$ .

Fix  $Q \in \mathfrak{M}$ . The càdlàg version of the process

$$H^Q(t) = \mathbb{E} \left[ \frac{dQ}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \quad \text{for all } t \in [0, T]$$

will be called a *local martingale density*. Let  $\hat{\mathcal{Y}}^e$  denote the set of all these processes.

**Remark 2.3.1.** Note that a non-negative càdlàg process  $X$  with  $X(0) \leq x$  is in  $\mathcal{A}(x)$  if and only if  $\frac{1}{\mathcal{S}_0} XH$  is a non-negative supermartingale for each  $H \in \hat{\mathcal{Y}}^e$ . In particular  $\hat{\mathcal{A}} = (\hat{\mathcal{Y}}^e)^\times$ , and further  $\hat{\mathcal{Y}}^e \subseteq \hat{\mathcal{Y}}$ .

*Proof.* Again this follows mainly from Kramkov (1996, Theorem 2.1).

Since  $\hat{\mathcal{A}} = \frac{1}{\mathcal{S}_0} \mathcal{A}$  is also the set of all non-negative processes  $X$ , such that  $X$  is a supermartingale under each  $Q \in \mathfrak{M}$ , we get  $XH^Q$  is a supermartingale under  $\mathbb{P}$  for each  $H^Q \in \mathcal{Y}^e$ . According to Bayes rule for stochastic processes<sup>2</sup>

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<sup>2</sup>e.g. see Karatzas and Shreve (1989, Lemma 3.5.3)

we get

$$\mathbb{E}_Q \left[ \int_0^T X(t) dt \right] = \mathbb{E} \left[ \int_0^T X(t) H^Q(t) dt \right]$$

Consequently  $\hat{\mathcal{A}} = (\hat{\mathcal{Y}}^e)^\times$  holds.

Contrarily  $XH^Q$  is a supermartingale under  $\mathbb{P}$  for each  $X \in \mathcal{X}$ , so

$$\hat{\mathcal{Y}}^e \subseteq \hat{\mathcal{Y}} = \{Y \in \mathcal{P} \mid Y(0) \leq 1, XY \text{ a supermartingale for all } X \in \hat{\mathcal{X}}\}$$

Finally  $\mathcal{A}(x)$  consists of all non-negative processes  $X$  such that  $\frac{1}{s_0}X$  is a supermartingale under each  $Q \in \mathfrak{M}$ . Thus we get  $X \in \mathcal{A}$  if and only if  $\frac{1}{s_0}XH^Q$  is a supermartingale under  $\mathbb{P}$  for each  $H^Q \in \mathcal{Y}^e$ .  $\square$

For later easy use and reference we quote some implications of the Filtered Bipolar Theorem (Žitković, 2002).

**Theorem\* 2** (Žitković (2002), Theorem 4).

(i) Let  $Y \in \hat{\mathcal{Y}}$ ,  $X \in \hat{\mathcal{X}}$  and  $C \in \mathcal{I}$ , then  $(X - C)Y$  is a supermartingale under  $\mathbb{P}$ .

(ii)  $\hat{\mathcal{Y}} = \hat{\mathcal{A}}^\times = (\hat{\mathcal{Y}}^e)^{\times\times}$  and  $\hat{\mathcal{X}}^{\times\times} = \hat{\mathcal{A}}$

(iii) For each  $Y \in \hat{\mathcal{Y}}$  there exists a sequence  $(Y^n)_{n \geq 0} \subseteq P\text{-solid}(\hat{\mathcal{Y}}^e)$ , such that  $Y^n$  Fatou-converges to  $Y \in \hat{\mathcal{Y}}$ .<sup>3</sup>

Employing the duality techniques we derived so far, we are able to easily characterize the cumulative consumption processes which are admissible for a given initial capital  $x > 0$ .

**Proposition 2.3.1.** *A process  $C \in \mathcal{I}$  is an  $x$ -admissible cumulative consumption process if and only if*

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t) dC(t) \right] \leq x \quad (2.3.2)$$

*Proof.* Note that according to Lemma 2.4.1 (see Section 2.4)

$$\mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right] = \mathbb{E}_Q [C(T)] \quad (2.3.3)$$

holds for all  $C \in \mathcal{I}$  and  $Q \in \mathfrak{M}$ .

<sup>3</sup>Here  $P$ -solid denotes the process-solid hull.

---

( $\Leftarrow$ ) Let  $C \in \mathcal{I}$  such that  $\mathbb{E} \left[ \int_0^T Y(t) dC(t) \right] \leq x (< \infty)$  holds for each  $Y \in \hat{\mathcal{Y}}$ . By assumption and Equation (2.3.3) we have

$$x \geq \mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right] = \mathbb{E}_Q [C(T)]$$

for all  $Q \in \mathfrak{M}$ .

Now let  $N(t) := \text{esssup}_{Q \in \mathfrak{M}} \mathbb{E}_Q [C(T) | \mathcal{F}_t]$ . Then obviously

$$N \geq 0 \quad \text{with } N(0) = x_0 \leq x$$

By El Karoui and Quenez (1995) we can assume that  $N$  is a càdlàg supermartingale under every  $Q \in \mathfrak{M}$ . The constrained version of the Optional Decomposition Theorem (Föllmer and Kramkov, 1997) guarantees the existence of a process  $X \in \hat{\mathcal{X}}(x_0)$  and process  $D \in \mathcal{I}$  such that  $N = X - D$ . Note that  $C \in \mathcal{I}$  and therefore

$$C(t) = \mathbb{E}_Q [C(t) | \mathcal{F}_t] \leq \mathbb{E}_Q [C(T) | \mathcal{F}_t] \leq N(t) \quad \mathbb{P} - a.s. \text{ for } t \in [0, T]$$

Finally we get

$$\underbrace{(x - x_0) + X}_{\in \hat{\mathcal{X}}(x)} - C \geq X - D - C = N - C \geq 0$$

Consequently  $\mathcal{S}_0((x - x_0) + X - C) \geq 0$  and  $C$  is an  $x$ -admissible consumption strategy.

( $\Rightarrow$ ) Conversely, suppose that  $C \in \mathcal{I}$  is an  $x$ -admissible consumption strategy, i.e. we find  $X \in \hat{\mathcal{X}}(x)$  such that

$$\mathcal{S}_0(t)(X(t) - C(t)) \geq 0 \quad \mathbb{P} - a.s. \text{ for all } t \in [0, T] \quad (2.3.4)$$

Moreover  $X - C$  is a  $Q$  supermartingale for every  $Q \in \mathfrak{M}$  and therefore

$$x \geq X(0) \geq \mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right]$$

holds for all  $Q \in \mathfrak{M}$ . Note that the second estimate requires to apply (2.3.4).

According to Theorem\* 2 for all  $Y \in \hat{\mathcal{Y}}$  we find  $H_n \in \hat{\mathcal{Y}}^e$  and  $D_n \in \mathcal{V}$  ( $n \geq 1$ ) such that the sequence  $H_n D_n$  Fatou-converges to  $Y$ . Recalling the definition of Fatou-convergence we get

$$Y(t) = \liminf_{s \searrow t} \liminf_{n \rightarrow \infty} H_n(s) D_n(s) \leq \liminf_{s \searrow t} \liminf_{n \rightarrow \infty} H_n(s)$$

$\mathbb{P} - a.s.$  for all  $t \in [0, T]$ . Note that the last inequality holds because  $D_n$  is dominated by the (constant) process  $\mathbf{1}$ . Furthermore taking conditional expectations and applying Fatou's Lemma yields

$$Y(t) = \mathbb{E}[Y(t) | \mathcal{F}_t] \leq \mathbb{E} \left[ \liminf_{s \searrow t} \liminf_{n \rightarrow \infty} H_n(s) \middle| \mathcal{F}_t \right] \leq \liminf_{n \rightarrow \infty} H_n(t)$$

$\mathbb{P} - a.s.$  for all  $t \in [0, T]$ . Note that the last inequality holds because  $H_n \in \hat{\mathcal{Y}}^e$  is a supermartingale. Applying Fatou's Lemma again, we get

$$\mathbb{E} \left[ \int_0^T Y(t) dC(t) \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T H_n(t) dC(t) \right] \leq x$$

which give us the second implication.  $\square$

We now concentrate on absolute continuous cumulative consumption processes, i.e. processes  $C \in \mathcal{I}$  such that

$$C(t) = \int_0^t \frac{1}{S_0}(s) c(s) ds \quad (0 \leq t \leq T)$$

for some rate of consumption process  $c$ . Therefore let  $L^0(\mathcal{M})$  denote the family of all  $\mathbb{R}$ -valued *progressively measurable* processes. By  $L_*^0(\mathcal{M})$  we denote the subcone of  $X \in L^0(\mathcal{M})$  with  $X(t) \geq 0$   $\mathbb{P} - a.s.$  for all  $t \in [0, T]$ . We define

$$\mathcal{C}(x) = \left\{ c \in L_*^0(\mathcal{M}) \mid \int_0^t \frac{1}{S_0}(s) c(s) ds = C(t) \text{ for some } X^{x, \pi, C} \in \mathcal{A}(x) \right\}$$

Note that  $\frac{1}{S_0}$  is non-increasing with  $\frac{1}{S_0}(0) = 1$ . Thus  $\frac{1}{S_0}c \in L_*^0(\mathcal{M})$  if  $c$  is progressively measurable.

---

**Corollary 2.3.2.** *A process  $c \in L_*^0(\mathcal{M})$  is the density process of an  $x$ -admissible absolute continuous cumulative consumption process  $C$  if and only if*

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T \frac{1}{S_0}(t)c(t)Y(t) dt \right] \leq x$$

*In particular*

$$\mathcal{C}(x) = \left\{ c \in L_*^0(\mathcal{M}) \mid \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T \frac{1}{S_0}(t)c(t)Y(t) dt \right] \leq x \right\} \quad (2.3.5)$$

*Proof.* This is a simple consequence of Proposition 2.3.1.  $\square$

From now on we will make use of identity (2.3.5) when we are dealing with  $\mathcal{C}(x)$ .

**Remark 2.3.2.** Note that the assertion of Corollary 2.3.2 can also be shown directly using Fubini's Theorem. Unfortunately Fubini fails in the setting of finitely-additive measures, see Yosida and Hewitt (1952, Theorem 3.3) for a counterexample. These circumstances force us to choose another approach in the proof of Proposition 2.3.1.

Moreover we may derive the following sharper estimate.

**Proposition 2.3.3.** *A rate of consumption process  $c \in L_*^0(\mathcal{M})$  is admissible for initial capital  $x > 0$  if and only if*

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q \left[ \int_0^T \frac{1}{S_0}(t)c(t) dt \right] \leq x$$

*Proof.* Notice that we have the identity

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q \left[ \int_0^T \frac{1}{S_0}(t)c(t) dt \right] = \sup_{H \in \hat{\mathcal{Y}}^e} \mathbb{E} \left[ \int_0^T H(t) \frac{1}{S_0}(t)c(t) dt \right]$$

by Bayes Rule for stochastic processes (Karatzas and Shreve, 1989, Lemma 3.5.3). Since  $\hat{\mathcal{Y}}^e \subseteq \hat{\mathcal{Y}}$  the estimate

$$\sup_{H \in \hat{\mathcal{Y}}^e} \mathbb{E} \left[ \int_0^T H(t) \frac{1}{S_0}(t)c(t) dt \right] \leq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t) \frac{1}{S_0}(t)c(t) dt \right] \leq x$$

is obvious.

Contrarily choose  $x > 0$  such that  $\mathbb{E} \left[ \int_0^T H(t) \frac{1}{\mathcal{S}_0}(t) c(t) dt \right] \leq x$  for all  $H \in \hat{\mathcal{Y}}^e$ .

According to Theorem\* 2 for all  $Y \in \hat{\mathcal{Y}}$  we find  $H_n \in \hat{\mathcal{Y}}^e$  and  $D_n \in \mathcal{V}$  ( $n \geq 1$ ) such that the sequence  $H_n D_n$  Fatou-converges to  $Y$ . As in the proof of Proposition 2.3.1 we derive

$$Y(t) = \mathbb{E} [Y(t) | \mathcal{F}_t] \leq \mathbb{E} \left[ \liminf_{s \searrow t} \liminf_{n \rightarrow \infty} H_n(s) \middle| \mathcal{F}_t \right] \leq \liminf_{n \rightarrow \infty} H_n(t)$$

$\mathbb{P}$  - *a.s.* for all  $t \in [0, T]$ . Thus

$$\mathbb{E} \left[ \int_0^T Y(t) \frac{1}{\mathcal{S}_0}(t) c(t) dt \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T H_n(t) \frac{1}{\mathcal{S}_0}(t) c(t) dt \right] \leq x$$

holds by applying Fatou's Lemma again. □

We finally state the optimization problem given in 2.1.1 more precisely. Given an initial capital  $x > 0$  we define the following value function.

**Problem 1** (Primal Problem).

$$V(x) = \max_{c \in \mathcal{C}_V(x)} \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad (2.3.6)$$

where  $\mathcal{C}_V(x) = \left\{ c \in \mathcal{C}(x) \mid \mathbb{E} \left[ \int_0^T u^-(c(t)) dF(t) \right] < \infty \right\}$ .

Notice that  $\mathcal{C}_V(x) = \mathcal{C}(x)$  if  $u$  is bounded below.

Up to now we do not know whether this problem is well defined or not. Given an initial capital  $x > 0$  the set of admissible consumption processes may be empty. We will discuss this problem in the following.

Recall that the bond price process is non-decreasing with  $\mathcal{S}_0(0) = 1$ . Thus  $\mathcal{S}_0$  may correspond to a non-negative interest rate via

$$\mathcal{S}_0(t) = \int_0^t e^{r(s)} ds$$

for a suitable non-negative interest rate process  $r$  with  $r(0) = 0$ .

As we will see now monotonicity of  $\mathcal{S}_0$  guarantees that the set of admissible consumption processes is always non-empty.

---

**Proposition 2.3.4.** *For all  $x > 0$  there exist  $\gamma_x > 0$  such that*

$$\gamma_x \mathbf{1} \in \mathcal{C}_V(x)$$

*Proof.* Recall that  $\mathcal{S}_0$  is non-decreasing with  $\mathcal{S}_0(0) = 1$ . Hence  $\frac{1}{\mathcal{S}_0}(t) \leq 1$   $\mathbb{P} - a.s.$  for all  $t \in [0, T]$ . Consequently for all  $Q \in \mathfrak{M}$  the integral  $\mathbb{E}_Q \left[ \int_0^T \frac{1}{\mathcal{S}_0}(t) dt \right]$  exists and is bounded from above by  $T < \infty$ .

Fubini's Theorem and Bayes rule for stochastic processes<sup>4</sup> now yield

$$\mathbb{E} \left[ \int_0^T H_Q(t) \frac{1}{\mathcal{S}_0}(t) dt \right] = \mathbb{E}_Q \left[ \int_0^T \frac{1}{\mathcal{S}_0}(t) dt \right] \quad \text{for all } H_Q \in \hat{\mathcal{Y}}^e \quad (2.3.7)$$

According to Proposition 2.3.3  $\gamma_1 := \left( \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q \left[ \int_0^T \frac{1}{\mathcal{S}_0}(t) dt \right] \right)^{-1}$  accomplishes the desired. Moreover multiplicativity of the set  $\mathcal{C}(x)$  give us  $\gamma_x := \gamma_1 \in \mathcal{C}(x)$ .

Thanks to the assumptions on  $F$ , namely (SA 1.2)

$$\mathbb{E} \left[ \int_0^T u(\gamma_x) dF(t) \right] = u(\gamma_x) \underbrace{\mathbb{E} \left[ \int_0^T \mathbf{1} dF(t) \right]}_{=1} = u(\gamma_x) > \infty$$

Thus  $\frac{x}{T} \mathbf{1} \in \mathcal{C}_V(x)$ . □

From now on to keep things simple we will assume that the bond price is constant. Unless denoted otherwise we take the trivial bond price, i.e.

**Standing Assumption 3.**

$$\mathcal{S}_0 \equiv 1$$

Under this assumption we have  $\hat{\mathcal{Y}} = \mathcal{Y}$ ,  $\hat{\mathcal{X}} = \mathcal{X}$ , etc. Moreover with  $\mathcal{S}_0 \equiv 1$  Equation (2.3.7) yields

$$\mathbb{E} \left[ \int_0^T H_Q(t) dt \right] = \mathbb{E}_Q \left[ \int_0^T \mathbf{1} dt \right] = T$$

Particularly  $\gamma_x = \frac{x}{T}$  holds for all  $x > 0$ .

**Remark 2.3.3.** At this point we should emphasize that Assumption 3 does not restrict the generality of the model since we always can choose the bond price as the numéraire. All further results can be extended easily to the general case  $\mathcal{S}_0 \not\equiv 1$ .

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<sup>4</sup>Compare Karatzas and Shreve (1989, Lemma 5.3)



## 2.4 Proofs of the Main Theorems in Section 2

We prove a Lemma which is fundamental to state the definition of admissible consumption processes.

**Lemma 2.4.1.** *For  $C \in \mathcal{I}$  and  $Q \in \mathfrak{M}$ , we have the following identity*

$$\mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right] = \mathbb{E}_Q [C(T)]$$

*Proof.* Fix  $C \in \mathcal{I}$  and choose  $H^Q \in \hat{\mathcal{Y}}^e$ . By virtue of left-continuity and existence of right limits of the process  $t \mapsto C(t-)$  the stochastic integral

$$M(t) := \int_0^t C(u-) dH^Q(u) \quad \text{for } t \in [0, T]$$

is also a local martingale (see Protter (1990, Theorem III. 17)). Thus we find a sequence of stopping times  $\{T_n\}_{n \geq 1}$  such that

$$\mathbb{P}(T_n = T) \xrightarrow[n \rightarrow \infty]{} 1 \quad (2.4.1)$$

and for all  $n \geq 1$  the processes  $M_{T_n \wedge \bullet}$  are uniformly integrable martingales.

The Monotone Convergence Theorem yields

$$\begin{aligned} \mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^{T_n} H_Q(t) dC(t) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^{T_n} H_Q(t-) dC(t) + \sum_{s \leq T_n} \Delta H^Q(s) \Delta C(s) \right] \end{aligned}$$

Using the integration by parts formula we go on with

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^{T_n} H_Q(t-) dC(t) + \sum_{s \leq T_n} \Delta H^Q(s) \Delta C(s) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ H_Q(T_n) C(T_n) + \int_0^{T_n} C(t-) dH_Q(t) \right] \end{aligned} \quad (2.4.2)$$

Since  $M$  is a martingale for the localizing sequence  $T_n$  with  $M(0) = 0$  the right addend in (2.4.2) vanishes and we continue with

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ H_Q(T_n) C(T_n) + \int_0^{T_n} C(t-) dH_Q(t) \right] = \lim_{n \rightarrow \infty} \mathbb{E}_Q [C(T_n)]$$

---

Finally we summarize

$$\mathbb{E} \left[ \int_0^T H_Q(t) dC(t) \right] = \lim_{n \rightarrow \infty} \mathbb{E}_Q [C(T_n)] \stackrel{(2.4.1)}{=} \mathbb{E}_Q [C(T)]$$

Thus the assertion holds. □

# Chapter 3

## Optimal Consumption Choice in Incomplete Markets

A basic problem in mathematical finance is the problem of an economic agent, who invests in a financial market so as to maximize expected utility from intertemporal consumption. In this chapter we utilize methods from convex duality to analyze and solve this problem in an incomplete financial market with a finite time-horizon. As usual in mathematical finance, we choose a market which is arbitrage free in the sense of

$$\mathfrak{M} \neq \emptyset$$

The investor gains utility from intertemporal consumption while his consumption choice is subjected to an initial capital  $x > 0$  and the natural constraints introduced in Merton (1969). This natural space of (rate of) consumption processes is given by the set of non-negative, progressively measurable processes. The budget constraint we use is the natural one in an incomplete financial market. A rate of consumption process  $c$  is admissible for the initial capital  $x$  if and only if it satisfies the constraint

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] \leq x$$

Here  $\hat{\mathcal{Y}}$  denotes the process dual of the non-negative wealth processes. The set of all rate of consumption processes  $c$  that satisfy this constraint has been

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denoted as  $\mathcal{C}(x)$ . Thus, the investor is faced with the problem to

$$\text{maximize } \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad \text{s.t. } c \in \mathcal{C}(x)$$

Here the distribution process  $F$  may reflect time-horizon uncertainty or if  $F$  admits of a density process some psychological discount factor. Our main assumptions are again

$$\begin{aligned} \mathbb{P}(F(T) > 0) &> 0 \\ \mathbb{E} \left[ \int_0^T \mathbf{1} dF(t) \right] &= 1 \end{aligned}$$

To guarantee solvability of the upcoming optimization problem we will find some sufficient conditions.

### 3.1 Duality and Existence of the Optimal Dual Strategy

In the following we develop a dual problem for the later utility maximization problem. Like in Bouchard and Pham (2004) we abstain from non-negative supermartingales and employ non-negative random variables instead. The measure space we use is  $(\Omega \times [0, T], \mathcal{M})$ .

More precisely we concentrate on non-negative progressively measurable processes on  $\Omega \times [0, T]$ . Since non-negativity heavily depends on the underlying measure, we first have to equip the space  $(\Omega \times [0, T], \mathcal{M})$  with an appropriate probability measure.

By  $p = F \otimes \mathbb{P}$  we denote the probability measure defined on  $(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}_{[0, T]})$  via

$$p(A \times B) = \mathbb{E} \left[ \int_0^T \mathbf{1}_{A \times B} dF(t) \right] \quad \text{for } A \in \mathcal{F}_T \text{ and } B \in \mathcal{B}_{[0, T]}$$

Notice that each progressively measurable process is  $\mathcal{F}_T \otimes \mathcal{B}_{[0, T]}$ -measurable. In particular  $(\Omega \times [0, T], \mathcal{M}, p)$  is a probability space.

### 3.1. THE OPTIMAL DUAL STRATEGY

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We already introduced  $L^0(\mathcal{M})$  as the set of real valued, progressively measurable processes. The subset of  $p$  integrable processes will be denoted as  $L^1(\mathcal{M})$ . Let  $Y_1, Y_2 \in L^0(\mathcal{M})$ . The order induced by  $p$  will be denoted by

$$Y_1 \succeq Y_2 \quad \text{if } Y_1 \geq Y_2 \text{ } p - a.s.$$

Moreover if the processes  $Y_1$  and  $Y_2$  are equal  $p - a.s.$  we write  $Y_1 \cong Y_2$ . The subcone of all non-negative processes in the sense of  $p$  (i.e.  $Y \succeq 0$   $p - a.s.$ ) will be denoted as  $L_+^0(\mathcal{M})$ . Furthermore we endow  $L_+^0(\mathcal{M})$  with the following  $[0, \infty]$ -valued bilinear form

$$\mathbb{E} \left[ \int_0^T X(t)Y(t) dF(t) \right] \quad \text{for } X, Y \in L_+^0(\mathcal{M})$$

For  $A \subseteq L_+^0(\mathcal{M})$  we define the  $p$ -polar

$$A^\circ = \left\{ b \in L_+^0(\mathcal{M}) \mid \mathbb{E} \left[ \int_0^T a(t)b(t) dF(t) \right] \leq 1, \text{ for all } a \in A \right\}$$

According to Brannath and Schachermayer (1999), the  $p$ -polar  $A^\circ$  of an arbitrary set  $A$  is always closed (with respect to convergence in probability  $p$ ), convex and solid. Furthermore the bipolar  $A^{\circ\circ}$  is the smallest closed, convex and solid set containing  $A$ . Recall that a subset  $\mathcal{E} \subseteq L_+^0(\mathcal{M})$  is called *solid* if

$$X \in L_+^0(\mathcal{M}), E \in \mathcal{E} \text{ with } X \preceq E \implies X \in \mathcal{E}$$

**Remark 3.1.1.** Obviously

$$\mathcal{C}(x) \subseteq L_*^0(\mathcal{M}) \subseteq L_+^0(\mathcal{M}) \quad \text{for all } x > 0 \quad (3.1.1)$$

Now we can define the dual problem. Therefore we introduce the *set of dual variables* corresponding to Problem 1. Fix  $z > 0$ .

$$\mathcal{Z}(z) := \left\{ Z \in L_+^0(\mathcal{M}) \mid \mathbb{E} \left[ \int_0^T c(t)Z(t) dF(t) \right] \leq z \text{ for all } c \in \mathcal{C} \right\}$$

Notice that by definition  $\mathcal{Z}(z) = z\mathcal{Z}(1)$  and  $\mathcal{C}^\circ = \mathcal{Z} (:= \mathcal{Z}(1))$  holds.

---

**Proposition 3.1.1.** *Let us abstain from Standing Assumption 3 during this Proposition.*

For all  $z > 0$  and  $Z \in \mathcal{Z}(z)$

$$\mathbb{E} \left[ \int_0^T Z(t) dF(t) \right] \leq \frac{z}{\gamma_1} \quad (= zT \text{ if } \mathcal{S}_0 \equiv 1) \quad (3.1.2)$$

holds. In particular the set  $\mathcal{Z}(z)$  is bounded in  $L^1(\mathcal{M})$ .

*Proof.*  $\gamma_1 > 0$  accomplishes the desired. Recall that  $\gamma_1 \mathbf{1} \in \mathcal{C}(1)$  by Proposition 2.3.4. By Lemma 3.3.2, we have  $\mathcal{Z}^\circ = \text{solid}(\mathcal{C})$ . Thus

$$\gamma_1 \mathbb{E} \left[ \int_0^T Z(t) dF(t) \right] \leq \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^T c(t) Z(t) dF(t) \right] \leq z$$

for all  $Z \in \mathcal{Z}(z)$ . □

Moreover

$$\mathcal{Z} = \text{solid}(\mathcal{C})^\circ \quad \text{but} \quad \text{solid}(\mathcal{C})^\circ \not\subseteq L_*^0(\mathcal{M}) \quad (3.1.3)$$

Here  $\text{solid}(\mathcal{C})$  denotes the solid hull of  $\mathcal{C}$ , cf. (3.3.1).

**Remark 3.1.2.** Fix  $x > 0$ . Since each  $d \in \text{solid}(\mathcal{C}(x))$  is  $p$ -a.s. dominated by a process  $c \in \mathcal{C}(x)$  we get  $\mathbb{E} \left[ \int_0^T u(d(t)) dF(t) \right] \leq V(x)$ . In particular

$$V(x) \leq \sup_{d \in \text{solid}(\mathcal{C}(x))} \mathbb{E} \left[ \int_0^T u(d(t)) dF(t) \right]$$

Further we define the *dual problem* and the corresponding *dual value function*.

**Problem 2 (Dual Problem).**

$$\tilde{V}(z) = \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \tilde{u}(Z(t)) dF(t) \right] \quad \text{for } z > 0$$

Here  $\tilde{u}$  denotes the conjugate function of  $u$ ,  $\tilde{u}(z) := \sup_{x>0} u(x) - xz$  for  $z > 0$ . Recall that  $\tilde{u}$  is continuous differentiable, strictly decreasing, strictly

convex, and satisfies  $\tilde{u}(0) = u(\infty)$ . We shall denote  $I := -\tilde{u}' = (u')^{-1}$ . Moreover note that

$$\tilde{u}(z) = u(I(z)) - zI(z) \quad (3.1.4)$$

for all  $z > 0$ .

Deriving a solution to a given optimization problem it is always easier if we know that problem has a finite value. Thus we sometimes directly ask for the following.

**Assumption 1.**

$$V(x_0) < \infty \quad \text{for some } x_0 > 0$$

**Remark 3.1.3.** Assumption 1 is equivalent to

$$V(x) < \infty \quad \text{for all } x > 0$$

*Proof.* Since concavity of  $V$  is obvious this assertion can be obtained using standard arguments from convex analysis.  $\square$

The following theorem is one of the most important results of this first part.

**Theorem 1.** *If Assumption 1 hold, then we have*

(i) *The value functions  $V$  and  $\tilde{V}$  are conjugate to each other*

$$V(x) = \inf_{z>0} \tilde{V}(z) + xz \quad x > 0 \quad (3.1.5)$$

$$\tilde{V}(z) = \sup_{x>0} V(x) - xz \quad z > 0 \quad (3.1.6)$$

(ii) *There exists  $z_0 > 0$  such that  $\tilde{V}(z) < \infty$  for all  $z > z_0$ .*

(iii) *If  $\tilde{V}(z) < \infty$  then the optimal solution  $\hat{Z}_z \in \mathcal{Z}(z)$  exists and is unique.*

(iv) *The function  $V$  is continuously differentiable on  $(0, \infty)$  and the function  $\tilde{V}$  is strictly convex on  $(z_0, \infty)$ . Here  $z_0$  denotes  $\inf\{z > 0 \mid \tilde{V}(z) < \infty\}$ .*

---

(v) The functions  $V'$  and  $\tilde{V}'$  satisfy

$$V'(0) = \infty \quad \text{and} \quad \tilde{V}'(\infty) = 0$$

Note that uniqueness is given in the sense of ' $\cong$ '.

Finally we take a closer look at the dual minimizer  $\hat{Z}_z$ . For  $z > 0$  we define the set of  $p - a.s.$  strictly positive dual variables

$$\mathcal{Z}^*(z) = \{Z \in \mathcal{Z}(z) \mid Z > 0 \text{ } p - a.s.\}$$

**Theorem 2.** *Let Assumption 1 hold, and fix  $z \geq z_0$ .*

*If  $\mathcal{Z}^*(z) \neq \emptyset$ , then the solution  $\hat{Z}_z$  to the dual problem  $\tilde{V}(z)$  lies in  $\mathcal{Z}^*(z)$ .*

## 3.2 Asymptotic Elasticity and Existence of the Optimal Primal Strategy

This section finally deals with existence and uniqueness of the optimal primal strategy. The following assumption has also been made in other works on optimization problems.

**Assumption 2.**

$$\tilde{V}(z) < \infty \quad \text{for all } z > 0$$

For many optimization problems within our semimartingale model this assumption has been identified as the weakest to guarantee existence of an optimal primal strategy. This has already been established in Kramkov and Schachermayer (2003) or Bouchard and Pham (2004) where an investor gains utility from his wealth process directly. In the course of this section it will turn out that it is the weakest assumption on the overall market structure to guarantee existence of the optimal consumption plan in our setting as well.

**Remark 3.2.1.** By definition of the dual set  $\mathcal{Z}(z)$  and the conjugate function  $\tilde{u}$  we always have

$$V(x) \leq \tilde{V}(y) + xy \quad \text{for all } x, y > 0$$



Thus Assumption 2 implies

$$V(x) < \infty \quad \text{for all } x > 0, \text{ i.e. Assumption 1} \quad (3.2.1)$$

$$\limsup_{x \rightarrow \infty} \frac{V(x)}{x} \leq 0 \quad (3.2.2)$$

**Remark 3.2.2.** Let  $u$  be an unbounded utility function, in the sense that  $u(\infty) = \infty$ . Note that in this case  $\tilde{u}(0) = \infty$ . Therefore Assumption 2 implies that  $\mathcal{Z}^*(z) \neq \emptyset$ .

We will now show that Assumption 2 implies the existence of an optimal consumption plan as well.

**Proposition 3.2.1.** *If Assumption 2 hold, then  $V(x)$  is strictly concave, strictly increasing, and continuous on  $(0, \infty)$ . Moreover an optimal consumption strategy  $\hat{c}_x \in \mathcal{C}_V(x)$  exists and is  $p$ -a.s. unique.*

*Proof.* According to (3.2.1) we know  $V(x) < \infty$  for all  $x > 0$ . Thus there exists a sequence  $\{c^n\}_{n \geq 1} \subseteq \mathcal{C}_V(x)$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c^n(t)) dF(t) \right] = V(x)$$

Thanks to Lemma 3.3.1 we also find a sequence of convex combinations  $\tilde{c}^n \in \text{conv}(c^k \mid k \geq n)$  and an element  $\tilde{c} \in \mathcal{C}(x)$  such that

$$\tilde{c}^n \xrightarrow[n \rightarrow \infty]{} \tilde{c} \quad p\text{-a.s.}$$

We claim that  $\tilde{c}$  is optimal to  $V(x)$ .

From concavity of  $u$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(\tilde{c}^n(t)) dF(t) \right] \geq \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c^n(t)) dF(t) \right] = V(x)$$

Making use of Fatou's Lemma we continue with

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u^-(\tilde{c}^n(t)) dF(t) \right] \geq \mathbb{E} \left[ \int_0^T u^-(\tilde{c}(t)) dF(t) \right] \quad (3.2.3)$$

---

which particularly shows that  $\tilde{c} \in \mathcal{C}_V(x)$ . Finally Lemma 3.3.8 gives us the  $p$ -uniform integrability of the set  $\{u^+(\tilde{c}^n) \mid n \geq 1\}$ , thus<sup>1</sup>

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u^+(\tilde{c}^n(t)) \, dF(t) \right] = \mathbb{E} \left[ \int_0^T u^+(\tilde{c}(t)) \, dF(t) \right] \quad (3.2.4)$$

holds. Equation (3.2.3) in addition to Equation (3.2.4) prove optimality of  $\tilde{c} \in \mathcal{C}_V(x)$ , while uniqueness in the sense of ‘ $\cong$ ’ follows straight forward from strict concavity of  $u$ .

For the remaining assertions let  $x_2 > x_1 > 0$ . The solutions of  $V(x_1)$  and  $V(x_2)$  will be denoted as  $c_1 \in \mathcal{C}_V(x_1)$  and  $c_2 \in \mathcal{C}_V(x_2)$ .

First we show strict concavity of  $V_K$ . Notice that for each  $\lambda \in (0, 1)$

$$\lambda x_1 + (1 - \lambda)x_2 \geq \mathbb{E} \left[ \int_0^T Y(t)(\lambda c_1(t) + (1 - \lambda)c_2(t)) \, dt \right]$$

holds for all  $Y \in \hat{\mathcal{Y}}$ , which in turn implies  $\lambda c_1 + (1 - \lambda)c_2 \in \mathcal{C}_V(\lambda x_1 + (1 - \lambda)x_2)$ .

By strict concavity of  $u$

$$\begin{aligned} & \lambda V(x_1) + (1 - \lambda)V(x_2) \\ &= \lambda \mathbb{E} \left[ \int_0^T u(c_1(t)) \, dF(t) \right] + (1 - \lambda) \mathbb{E} \left[ \int_0^T u(c_2(t)) \, dF(t) \right] \\ &< \mathbb{E} \left[ \int_0^T u(\lambda c_1(t) + (1 - \lambda)c_2(t)) \, dF(t) \right] \\ &\leq V(\lambda x_1 + (1 - \lambda)x_2) \end{aligned}$$

holds true and  $V$  is strictly concave. By definition  $V$  is non-decreasing. Hence,  $V$  must be strictly increasing, because otherwise this would contradict strict concavity of  $V$ . Since a concave function is always continuous on its domain, we also have continuity of  $V$  on  $(0, \infty)$ .  $\square$

Kramkov and Schachermayer (1999) developed a simple condition which was completely new for problems in convex optimization and very helpful finding an optimal primal strategy. They defined the *asymptotic elasticity* ( $AE$ ) of a utility function as

$$AE(u) = \limsup_{x \rightarrow \infty} \frac{x u'(x)}{u(x)}$$

---

<sup>1</sup>Note that Assumption 2 is necessary for this Lemma.

Obviously  $AE(u) < 1$  implies  $u'(\infty) = 0$ .

Furthermore they show that  $AE(u) < 1$  together with Assumption 1 implies Assumption 2. Note that many popular utility functions have an asymptotic elasticity strictly less than 1, e.g. *logarithmic-utility*:  $u(x) = \ln(x)$  or *power-utility*:  $u(x) = \frac{x^\alpha}{\alpha}$  for  $\alpha < 1$ .

We will see that their result also holds in our context.

**Assumption 3.**

$$AE(u) < 1$$

For later easy use and reference we first state an important result on asymptotic elasticity.<sup>2</sup>

**Lemma\* 3** (Kramkov and Schachermayer (1999, Lemma 6.3)). *Let  $u$  be a function with  $u(\infty) > 0$ , additionally satisfying the conditions*

$$u'(0) = \infty \quad \text{and} \quad u'(\infty) = 0$$

*In each of the following assertions the infimum of  $\gamma > 0$  for which these assertions hold true equals the asymptotic elasticity  $AE(u)$ .*

- (i) *There exists  $x_0 > 0$  such that  $u(\lambda x) < \lambda^\gamma u(x)$  for all  $\lambda > 1$ ;  $x \geq x_0$ .*
- (ii) *There exists  $x_0 > 0$  such that  $u'(x) < \gamma \frac{u(x)}{x}$  for all  $x \geq x_0$ .*
- (iii) *There exists  $z_0 > 0$  such that  $\tilde{u}(\mu z) < \mu^{-\frac{\gamma}{\gamma-1}} \tilde{u}(z)$  for all  $\mu \in (0, 1)$ ;  $0 < z \leq z_0$ .*
- (iv) *There exists  $z_0 > 0$  such that  $-\tilde{u}'(z) < \left(\frac{\gamma}{\gamma-1}\right) \frac{\tilde{u}(z)}{z}$  for all  $0 < z \leq z_0$ .*

We now come to the most important theorem of this section. Heuristically spoken it tells us that ‘the indirect utility function  $V$  is a utility function indeed’, i.e.  $V$  has the same properties as the underlying intertemporal utility function  $u$ .

---

<sup>2</sup>See Kramkov and Schachermayer (1999) for the proof.

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**Theorem 3.** *Let Assumptions 1 and 3 hold, then*

(i) *both functions  $V$  and  $-\tilde{V}$  are increasing, strictly concave, and continuously differentiable on  $(0, \infty)$ .*

(ii) *Assumption 2 holds and the optimal solution  $\hat{c}_x \in \mathcal{C}_V(x)$  exists and is unique  $p - a.s.$*

(iii) *the functions  $V'$  and  $-\tilde{V}'$  are strictly decreasing and satisfy*

$$V'(0) = \infty \quad \text{and} \quad \tilde{V}'(\infty) = 0 \quad (3.2.5)$$

$$V'(\infty) = 0 \quad \text{and} \quad -\tilde{V}'(0) = \infty \quad (3.2.6)$$

**Remark 3.2.3.** Contrarily the assertions of this theorem hold as well, if we claim Assumption 2 only (cf. Kramkov and Schachermayer (1999) and Kramkov and Schachermayer (2003)). While Assumption 2 is the weakest condition on the overall market structure to guarantee the assertions listed in the later theorem, Assumption 3 is the weakest (market independent) condition on the intertemporal utility function  $u$  to do so.

Kramkov and Schachermayer also found a relation between the asymptotic elasticity of the indirect utility function and the asymptotic elasticity of the intertemporal utility function. Under the assumptions of Theorem 3 and exerting the same arguments as in Kramkov and Schachermayer (1999, Lemma 3.12), we easily derive

$$\text{AE}(V)^+ \leq \text{AE}(u)^+ < 1 \quad (3.2.7)$$

Note that  $u(\infty) > 0$  is crucial for this consideration.

Closing this section we state a Theorem which emphasizes the relation between the solutions of the primal and the dual problem.

**Theorem 4.** *Let Assumptions 1 and 3 hold, and choose  $x, z > 0$  satisfying the equation  $x = -\tilde{u}'(z)$ . Then the solution to  $V(x)$  satisfies*

$$\hat{c}_x(t) = -\tilde{u}'(\hat{Z}_z(t)) \quad p - a.s. \quad (3.2.8)$$

where  $\hat{Z}_z$  solves  $\tilde{V}(z)$ . Furthermore

$$\mathbb{E} \left[ \int_0^T \hat{c}_x \hat{Z}_z(t) dF(t) \right] = xz \quad (3.2.9)$$

$$V'(x) = \frac{1}{x} \mathbb{E} \left[ \int_0^T \hat{c}_x(t) u'(\hat{c}_x(t)) dF(t) \right] \quad (3.2.10)$$

$$\tilde{V}'(z) = \frac{1}{z} \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) dF(t) \right] \quad (3.2.11)$$

---

## 3.3 Proofs of the Main Theorems in Section 3

### 3.3.1 Proofs of Section 3.1

Recall that a subset  $\mathcal{E} \subseteq L_+^0(\mathcal{M})$  is called *solid* if

$$X \in L_+^0(\mathcal{M}), E \in \mathcal{E} \text{ with } X \preceq E \implies X \in \mathcal{E}$$

For an arbitrary set  $\mathcal{E} \subseteq L_+^0(\mathcal{M})$  we define the *solid hull in  $L_+^0(\mathcal{M})$*

$$\text{solid}(\mathcal{E}) = \{X \in L_+^0(\mathcal{M}) \mid X \preceq E \text{ for some } E \in \mathcal{E}\} \quad (3.3.1)$$

Note that  $\text{solid}(\mathcal{E})$  is convex if  $\mathcal{E}$  is convex.

**Lemma 3.3.1.** *Let  $c_n$  a sequence in  $\mathcal{C}$ . Then there exists a sequence  $\bar{c}_n \in \text{conv}\{c_k \mid k \geq n\} \subseteq \mathcal{C}$  and an element  $\bar{c} \in \mathcal{C}$  such that simultaneously*

$$\bar{c}_n \longrightarrow \bar{c} \text{ } p\text{-a.s.} \quad \text{and} \quad \bar{c}_n \longrightarrow \bar{c} \text{ } dt \otimes \mathbb{P} \text{ - a.e.}$$

*Proof.* Let  $\mu$  the measure defined on  $(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}_{[0, T]})$  via

$$\mu(A \times B) = \mathbb{E} \left[ \int_0^T \mathbf{1}_{A \times B} dF(t) + \int_0^T \mathbf{1}_{A \times B} dt \right]$$

for all  $A \in \mathcal{F}$  and  $B \in \mathcal{B}_{[0, T]}$ .

Since  $c_n \in \mathcal{C}$  is a sequence of non-negative elements in  $L_*^0(\mathcal{M})$ , it follows from Delbaen and Schachermayer (1994, Lemma A1.1) that there exists a sequence  $\bar{c}_n \in \text{conv}\{c_k \mid n \geq k\}$  such that  $\bar{c}_n$  converges in measure  $\mu$  to some progressively measurable  $\bar{c}$  with values in  $[0, \infty]$ .

After passing to a subsequence, we may assume that the convergence holds  $p$ -a.s. and  $dt \otimes \mathbb{P}$ -a.e.

Let  $Y \in \mathcal{Y}$ . Using Fatou's Lemma we get

$$\mathbb{E} \left[ \int_0^T Y(t) \bar{c}(t) dt \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T Y(t) \bar{c}_n(t) dt \right] \leq 1$$

Finally  $1 \geq \sup_{Y \in \mathcal{Y}} \mathbb{E}[\int_0^T Y(t) \bar{c}(t) dt]$  holds, thus  $\bar{c} \in \mathcal{C}$ . □

**Lemma 3.3.2.**

$$\mathcal{Z}^\circ = \text{solid}(\mathcal{C})$$

*Proof.* We show that  $\text{solid}(\mathcal{C})$  is closed with respect to convergence in measure  $p$ . Let  $\{X_n\}_{n \geq 1} \subseteq \text{solid}(\mathcal{C})$  a sequence that converges in measure  $p$  to some  $X \in L_+^0$ . Further let  $c_n \in \mathcal{C}$  such that  $X_n \preceq c_n$ . According to Lemma 3.3.1, there exists a sequence  $\{\bar{c}_n\}_{n \geq 1} \subseteq \mathcal{C}$  converging  $p$ -a.s. to an element  $\bar{c} \in \mathcal{C}$ . This implies  $X \preceq \bar{c}$ , and so  $X \in \text{solid}(\mathcal{C})$ .

Now the set  $\text{solid}(\mathcal{C})$  is convex, solid and closed with respect to convergence in measure  $p$ . So we may apply the Bipolar Theorem of Brannath and Schachermayer (1999) in order to conclude

$$(\text{solid}(\mathcal{C}))^{\circ\circ} = \text{solid}(\mathcal{C})$$

Finally recall that  $\mathcal{Z} = \mathcal{C}^\circ = \text{solid}(\mathcal{C})^\circ$  holds by (3.1.3).  $\square$

**Remark 3.3.1.** Let  $\iota: (-\tilde{u}(0), -\tilde{u}(\infty)) \rightarrow (0, \infty)$  denote the inverse of  $-\tilde{u}$ . Recall that this function is strictly concave and strictly increasing.

Further the l'Hospital rule and (3.1.4) (resp. the condition on  $u$ ) yield

$$\lim_{x \rightarrow -\tilde{u}(\infty)} \frac{\iota(x)}{x} = \lim_{y \rightarrow \infty} \frac{y}{-\tilde{u}(y)} = \lim_{y \rightarrow \infty} \frac{1}{I(y)} = \infty \quad (3.3.2)$$

**Lemma 3.3.3.** *For any  $z > 0$  the family  $\{\tilde{u}^-(Z) \mid Z \in \mathcal{Z}(z)\}$  is  $p$ -uniformly integrable. Further if  $(Z^n)_{n \geq 1}$  is a sequence in  $\mathcal{Z}(z)$  which converges  $p$ -a.s. to a random variable  $Z \in L_+^0(\mathcal{M})$ , then  $Z \in \mathcal{Z}(z)$  and*

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(Z^n(t)) \, dF(t) \right] \geq \mathbb{E} \left[ \int_0^T \tilde{u}(Z(t)) \, dF(t) \right]$$

*Proof.* If  $\tilde{u}(\infty) \geq 0$ , then  $p$ -uniform integrability is trivial.

We assume  $\tilde{u}(\infty) < 0$ . Thanks to Remark 3.3.1,  $p$ -uniform integrability follows from the de la Vallée-Poussin Theorem, if we can verify

$$\sup_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \iota(\tilde{u}^-(Z(t))) \, dF(t) \right] < \infty$$

---

Since  $\tilde{u}(0) = u(\infty) > 0$ ,  $\iota(0)$  exists and is finite. For  $y > 0$  we obtain

$$\tilde{u}^-(y) = \begin{cases} -\tilde{u}(y) & \text{if } \tilde{u}(y) \leq 0 \\ 0 & \text{else} \end{cases} \quad (3.3.3)$$

Consequently the following estimate holds

$$\iota(\tilde{u}^-(y)) \leq \iota(-\tilde{u}(y)) + \iota(0) \quad \text{for all } y > 0$$

Now we get for every  $Z \in \mathcal{Z}(z)$

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \iota(\tilde{u}^-(Z(t))) \, dF(t) \right] &\leq \mathbb{E} \left[ \int_0^T \iota(-\tilde{u}(Z(t))) + \iota(0) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T Z(t) + \iota(0) \, dF(t) \right] \\ &\stackrel{(3.1.2)}{\leq} \frac{z}{\gamma_1} + \frac{\iota(0)}{\gamma_1} < \infty \end{aligned}$$

Since  $\frac{z+\iota(0)}{\gamma_1}$  is independent of the choice of  $Z$ , this gives us the desired uniform integrability of the family  $\{\tilde{u}^-(Z) \mid Z \in \mathcal{Z}(z)\}$ .

By definition and the Bipolar Theorem of Brannath and Schachermayer the set  $\mathcal{Z}(z)$  is closed in the topology of convergence in measure  $p$ . So, if a sequence  $Z^n \in \mathcal{Z}(z)$  converges in measure  $p$  to  $Z$ , then  $Z \in \mathcal{Z}(z)$ .

Further the uniform integrability of  $\{\tilde{u}^-(Z) \mid Z \in \mathcal{Z}(z)\}$  gives us  $L^1(\mathcal{M})$  convergence for the negative part of  $\tilde{u}$ , i.e.

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}^-(Z^n(t)) \, dF(t) \right] = \mathbb{E} \left[ \int_0^T \tilde{u}^-(Z(t)) \, dF(t) \right]$$

For the positive part we may apply Fatou's Lemma and get

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}^+(Z^n(t)) \, dF(t) \right] \geq \mathbb{E} \left[ \int_0^T \tilde{u}^+(Z(t)) \, dF(t) \right]$$

This completes our proof. □

**Lemma 3.3.4.** *Fix  $z > 0$ . For all  $Z \in \mathcal{Z}(z)$*

$$c \longmapsto \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] \quad (3.3.4)$$

*is upper semicontinuous on each set  $A \subseteq L_+^0(\mathcal{M})$ , on which  $\{u^+(c) \mid c \in A\}$  is  $p$ -uniformly integrable.*



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*Proof.* Notice that for all  $c \in \mathcal{K}$  and  $Z \in \mathcal{Z}$  the mapping

$$(c, Z) \mapsto \mathbb{E} \left[ \int_0^T c(t)Z(t) dF(t) \right]$$

and

$$c \mapsto \mathbb{E} \left[ \int_0^T u^-(c(t)) dF(t) \right]$$

are lower semicontinuous by Fatou's Lemma. In addition

$$c \mapsto \mathbb{E} \left[ \int_0^T u^+(c(t)) dF(t) \right]$$

is upper semicontinuous for  $c \in A$  by the Dominated Convergence Theorem. Thus the assertion holds.  $\square$

To prove the bidual relation between  $V$  and  $\tilde{V}$  we introduce some axillary objects.

Since we want to apply a Minimax Theorem for compact sets, but neither of the sets  $\mathcal{C}(x)$  and  $\mathcal{Z}(z)$  are compact, we define

$$\mathcal{C}_n = \{g \in L_+^0(\mathcal{M}) \mid 0 \leq g(\omega, t) \leq n \text{ for all } \omega \in \Omega, t \in [0, T]\} \quad \text{for } n \geq 1$$

These sets contain elements with radius  $n$  in  $L^\infty(\mathcal{M})$ , according to the Theorem of Banach-Alaoglu they are  $\sigma(L^\infty(\mathcal{M}), L^1(\mathcal{M}))$ -compact.

**Lemma 3.3.5.** *Let Assumption 1 hold, then*

$$\tilde{V}(z) = \sup_{x>0} V(x) - xz \quad \text{for all } z > 0$$

*Proof.* In the following steps, we will show

$$\begin{aligned} \tilde{V}(z) &= \lim_{n \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) dF(t) \right] \\ &= \sup_{x>0} V(x) - xz \end{aligned}$$

for all  $z > 0$ .

(1) First we verify the second identity. Applying the Kneser-Fan Minimax Theorem (see e.g. Terkelson (1972, Corollary 2)), we get

$$\begin{aligned} & \sup_{g \in \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \\ &= \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \end{aligned} \quad (3.3.5)$$

for all  $n \geq 1$ . As already mentioned compactness of  $\mathcal{C}_n$ , which is necessary for the Kneser-Fan Theorem, follows from the Banach-Alaoglu Theorem, and additionally

$$\mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right]$$

is upper-semicontinuous on  $\mathcal{C}_n$  by Lemma 3.3.4.

(2) We continue by proving

$$\begin{aligned} & \sup_{x > 0} \sup_{c \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \end{aligned} \quad (3.3.6)$$

The estimate " $\geq$ " is a straightforward consequence of Lemma 2.3.4. Since  $\frac{x}{T}\mathbf{1} \in \mathcal{C}(x)$  we know that  $\mathcal{C}_n \subset \mathcal{C}(Tn)$ . The converse estimate " $\leq$ " is true, because

$$\begin{aligned} & \sup_{c \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{C}(x) \cap \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \\ &\leq \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \end{aligned}$$

holds for fixed  $x > 0$ .

(3) Finally let

$$\partial \mathcal{C}(x) := \left\{ c \in \mathcal{C}(x) \mid \sup_{Z \in \mathcal{Z}(1)} \mathbb{E} \left[ \int_0^T c(t)Z(t) \, dF(t) \right] = x \right\} \quad (3.3.7)$$

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Obviously  $\partial\mathcal{C}(x) = x \partial\mathcal{C}(1)$  for all  $x > 0$ , and

$$\bigcup_{x>0} \mathcal{C}(x) = \left( \bigcup_{x>0} \partial\mathcal{C}(x) \right) \quad p - a.s. \quad (3.3.8)$$

(4) We summarize

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \inf_{g \in \mathcal{C}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \\ (3.3.6) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] \\ (3.3.7) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \partial\mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] \\ & = \sup_{x>0} \sup_{c \in \partial\mathcal{C}(x)} \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - xz \\ (3.3.8) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \mathcal{C}(x)} \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - xz \\ & = \sup_{x>0} (V(x) - xz) \end{aligned}$$

(5) It remains to verify

$$\tilde{V}(z) = \lim_{n \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \quad (3.3.9)$$

For  $z > 0$  let us define

$$\tilde{u}_n(z) := \sup_{0 < x \leq n} u(x) - xz$$

and

$$\tilde{V}_n(z) := \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \tilde{u}_n(Z(t)) \, dF(t) \right]$$

Obviously

$$\inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \tilde{u}_n(Z(t)) \, dF(t) \right] = \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{C}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right]$$

and  $\tilde{V}_n(z) \leq \tilde{V}(z)$  holds for all  $z > 0$ .

(6) In the following we show  $\lim_{n \rightarrow \infty} \tilde{V}_n(z) \geq \tilde{V}(z)$ , such that

$$\lim_{n \rightarrow \infty} \tilde{V}_n(z) = \lim_{n \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \tilde{u}_n(Z(t)) \, dF(t) \right] = \tilde{V}(z) \quad (3.3.10)$$

---

Let  $\{Z^n\}_{n \geq 1} \subseteq \mathcal{Z}(z)$  a sequence with

$$\lim_{n \rightarrow \infty} \tilde{V}_n(z) = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}_n(Z^n(t)) \, dF(t) \right]$$

According to Delbaen and Schachermayer (1994, Lemma A1.1) we find a sequence  $\bar{Z}^n \in \text{conv}\{Z^k | k \geq n\}$  that converges  $p - a.s.$  to some  $\bar{Z}$ . We know from Lemma 3.3.3 that  $\bar{Z} \in \mathcal{Z}(z)$ . Notice that  $\bar{Z}^n = \sum_{k=n}^N \lambda_k Z^k$  for suitable  $\lambda_k \in [0, 1]$  and

$$\tilde{u}_n(\bar{Z}^n) = \tilde{u}_n \left( \sum_{k=n}^N \lambda_k Z^k \right) \leq \sum_{k=n}^N \lambda_k \tilde{u}_n(Z^k) \leq \max_{n \leq k \leq N} \tilde{u}_n(Z^k)$$

holds by convexity of  $\tilde{u}_n$ . Since  $\tilde{u}_n$  increase in  $n$ , this particularly verifies

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}_n(Z^n(t)) \, dF(t) \right] \geq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}_n(\bar{Z}^n(t)) \, dF(t) \right] \quad (3.3.11)$$

Recall that for  $y > 0$  we may always write

$$\tilde{u}(y) = u(I(y)) - yI(y)$$

Furthermore it is easy to see, that

$$\tilde{u}_n(y) = \tilde{u}(y) \quad \text{for all } y > u'(n)$$

This in conjunction with Lemma 3.3.3 gives us  $p$ -uniform integrability of the sequence  $(\tilde{u}_n^-(\bar{Z}^n))_{n \geq 1}$ . Finally we apply Fatou's Lemma on  $(\tilde{u}_n^+(\bar{Z}^n))_{n \geq 1}$ , and we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}_n(\bar{Z}^n(t)) \, dF(t) \right] &\geq \mathbb{E} \left[ \int_0^T \liminf_{n \rightarrow \infty} \tilde{u}_n(\bar{Z}^n(t)) \, dF(t) \right] \\ &\geq \mathbb{E} \left[ \int_0^T \limsup_{n \rightarrow \infty, m \geq n} \tilde{u}_m(\bar{Z}^n(t)) \, dF(t) \right] \\ &\geq \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t)) \, dF(t) \right] \\ &\geq \tilde{V}(z) \end{aligned}$$

This in addition to (3.3.11) finally proves (6) and completes the proof of Lemma 3.3.5.  $\square$

**Lemma 3.3.6.** *Let Assumption 1 hold. For all  $z > 0$  with  $\tilde{V}(z) < \infty$  there exists a unique optimal solution  $\bar{Z} \in \mathcal{Z}(z)$  to  $\tilde{V}(z)$ . Moreover,  $\tilde{V}$  is strictly convex on the set  $\{\tilde{V} < \infty\}$ .*

*Proof.* Let  $z > 0$  such that  $\tilde{V}(z) < \infty$ . We choose a minimizing sequence  $Z_n \in \mathcal{Z}(z)$  for  $\tilde{V}(z)$ , i.e.

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(Z_n(t)) dF(t) \right] = \tilde{V}(z)$$

By Proposition 2.3.4 the sequence  $Z_n$  is bounded in  $L^1(\mathcal{M})$ . According to Komlòs (1967, Theorem 1), the sequence

$$\bar{Z}_n = \frac{1}{n} \sum_{k=1}^n Z_k$$

converges in measure  $p$  to some non-negative process  $\bar{Z} \in L^1(\mathcal{M})$ . We deduce from Lemma 3.3.3 that  $\bar{Z} \in \mathcal{Z}(z)$  and

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}_n(t)) dF(t) \right] \geq \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t)) dF(t) \right] \quad (3.3.12)$$

holds. Further, convexity of the function  $\tilde{u}$  yields that

$$\mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}_n) dF(t) \right] \leq \max_{n \geq k \geq 1} \mathbb{E} \left[ \int_0^T \tilde{u}(Z_k) dF(t) \right] \quad (3.3.13)$$

Thus we summarize

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t)) dF(t) \right] &\stackrel{(3.3.12)}{\leq} \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}_n) dF(t) \right] \\ &\stackrel{(3.3.13)}{\leq} \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(Z_k) dF(t) \right] \\ &= \tilde{U}(z) \end{aligned}$$

Moreover, since  $Z_n$  is a minimizing sequence, equality must hold. So  $\bar{Z} \in \mathcal{Z}(z)$  is a solution to  $\tilde{V}(z)$ .

To prove strict convexity of  $\tilde{V}$ , let  $z_1, z_2 \in \{\tilde{V} < \infty\}$  and  $\lambda \in (0, 1)$ . By  $Z_1$  (resp.  $Z_2$ ) we denote the solution to  $\tilde{V}(z_1)$  (resp.  $\tilde{V}(z_2)$ ). Note that

$$\lambda Z_1 + (1 - \lambda) Z_2 \in \mathcal{Z}(\lambda z_1 + (1 - \lambda) z_2)$$

---

Now, using strict convexity of  $\tilde{u}$ , we get

$$\begin{aligned}\tilde{V}(\lambda z_1 + (1 - \lambda)z_2) &\leq \mathbb{E} \left[ \int_0^T \tilde{u}(\lambda Z_1 + (1 - \lambda)Z_2) \, dF(t) \right] \\ &< \lambda \tilde{V}(z_1) + (1 - \lambda)\tilde{V}(z_2)\end{aligned}$$

Ultimately uniqueness of the optimal solution follows from the strict convexity of  $\tilde{u}$ .  $\square$

**Lemma 3.3.7.** *Let Assumption 1 hold, then*

$$\lim_{x \rightarrow 0} V'(x) = \infty \quad \text{and} \quad \lim_{z \rightarrow \infty} \tilde{V}'(z) = 0$$

*Proof.* Recall that the functions  $V$  and  $-\tilde{V}$  are concave and increasing. According to Rockafellar (1970, Theorem V.23.4) we have the following representations for derivatives of concave functions

$$\begin{aligned}V'(x) &= \inf_{-\tilde{V}(\bar{z}) < x} \bar{z} && \text{for all } x > 0 \\ -\tilde{V}'(z) &= \inf_{V(\bar{x}) < z} \bar{x} && \text{for all } z > 0\end{aligned}$$

So, obviously both assertions of this Lemma are equivalent and w.l.o.g. it is sufficient to prove the second one.

The function  $-\tilde{V}$  is concave and increasing. Hence there is a finite and non-negative limit

$$-\tilde{V}'(\infty) := \lim_{z \rightarrow \infty} -\tilde{V}'(z)$$

Since  $-\tilde{u}$  is increasing with  $-\tilde{u}(z) \rightarrow 0$  when  $z \rightarrow \infty$ , we have the following. For every  $\varepsilon > 0$  there exist  $a_\varepsilon \in \mathbb{R}$  such that

$$-\tilde{u}(z) \leq a_\varepsilon + \varepsilon z \quad \text{for all } z > 0$$

Notice that the l'Hospital rule yields

$$0 \leq -\tilde{V}'(\infty) = \lim_{z \rightarrow \infty} \frac{-\tilde{V}(z)}{z}$$

Thus we may deduce from  $L^1(\mathcal{M})$  boundedness of  $\mathcal{Z}(z)$  that

$$\begin{aligned}
 0 \leq \lim_{z \rightarrow \infty} \frac{-\tilde{V}(z)}{z} &= \lim_{z \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \frac{\tilde{u}(Z(t))}{z} dF(t) \right] \\
 &\leq \lim_{z \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T \frac{a_\varepsilon + \varepsilon Z(t)}{z} dF(t) \right] \\
 &\stackrel{(3.1.2)}{\leq} \lim_{z \rightarrow \infty} \mathbb{E} \left[ \int_0^T \frac{a_\varepsilon}{z} dF(t) \right] + \frac{\varepsilon}{\gamma_1} \\
 &= \frac{\varepsilon}{\gamma_1}
 \end{aligned}$$

for all  $\varepsilon > 0$ . Consequently  $\tilde{V}'(\infty) = 0$ .  $\square$

### Proof of Theorem 1

*Proof.* (i) Equation (3.1.6) has been shown in Lemma (3.3.5). Equation 3.1.5 follows from the general bidual property of Legendre-Fenchel transforms, cf. Rockafellar (1970, Theorem III.12.2).

(ii) According to Assumption 1 and Lemma 3.3.7 there exist  $z_0$  such that  $\tilde{V}(z_0) = \sup_{x>0} V(x) - xz_0$  is finite. Let  $z > z_0$ , then

$$\tilde{V}(z) = \sup_{x>0} V(x) - xz \leq \sup_{x>0} V(x) - xz_0 = \tilde{V}(z_0) \quad \text{for all } z > z_0$$

which induces finiteness of  $\tilde{V}$  on  $\{\tilde{V}(z) < \infty\} = (z_0, \infty)$ .

(iii) This has been shown in Lemma 3.3.6.

(iv) The strict convexity of  $\tilde{V}$  on the set  $\{\tilde{V}(z) < \infty\} = (z_0, \infty)$  has also been proved in Lemma 3.3.6. Continuous differentiability of  $V$  now follows from convex analysis, cf. Rockafellar (1970, Theorem V.26.3).

(v) Again this has already been verified in a previous Lemma, namely Lemma 3.3.7.  $\square$

### Proof of Theorem 2

*Proof.* We distinguish the two cases  $u(\infty) = \infty$  and  $u(\infty) < \infty$ .

First assume that  $u(\infty) = \infty$ . We know from convex analysis, that  $\tilde{u}(0) = \infty$ . Thus  $\tilde{V}(z) < \infty$  implies  $\bar{Z}_z > 0$   $p$ -a.s., i.e.  $\bar{Z}_z \in \mathcal{Z}^*(z)$ .

---

Now assume  $u(\infty) < \infty$ . The verification of this assertion will be divided into two parts.

Assuming uniform integrability of a certain set, we first prove that the minimizer  $\bar{Z}_z \in \mathcal{Z}^*(z)$  as well. In a second step we will verify uniform integrability.

1) We abbreviate  $\bar{Z} = \bar{Z}_z$  and perturb  $\bar{Z}$  with an arbitrary  $Z \in \mathcal{Z}(z)$  via

$$Z^\varepsilon = \varepsilon Z + (1 - \varepsilon)\bar{Z} \quad \text{for } \varepsilon \in (0, \tfrac{1}{2})$$

By optimality of  $\bar{Z}$  and convexity of  $\tilde{u}$ , we get

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon} \mathbb{E} \left[ \int_0^T (\tilde{u}(\bar{Z}(t)) - \tilde{u}(Z^\varepsilon(t))) \, dF(t) \right] \\ &\geq \mathbb{E} \left[ \int_0^T \tilde{u}'(Z^\varepsilon(t)) (\bar{Z}(t) - Z(t)) \, dF(t) \right] \end{aligned} \quad (3.3.14)$$

We will see in (2) that the family  $\{(\tilde{u}'(Z^\varepsilon)(\bar{Z} - Z))^- \mid \varepsilon \in (0, \frac{1}{2})\}$  is  $p$ -uniformly integrable. Combined with Fatou's Lemma, this  $p$ -uniform integrability gives us

$$0 \stackrel{(3.3.14)}{\geq} \lim_{\varepsilon \searrow 0} \mathbb{E} \left[ \int_0^T \tilde{u}'(Z^\varepsilon(t)) (\bar{Z}(t) - Z(t)) \, dF(t) \right] \quad (3.3.15)$$

$$\begin{aligned} &\geq \mathbb{E} \left[ \int_0^T \lim_{\varepsilon \searrow 0} \tilde{u}'(Z^\varepsilon(t)) (\bar{Z}(t) - Z(t)) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T \tilde{u}'(\bar{Z}(t)) (\bar{Z}(t) - Z(t)) \, dF(t) \right] \end{aligned} \quad (3.3.16)$$

We will prove this Lemma by exacting a contradiction. Therefore assume that  $\bar{Z} \notin \mathcal{Z}^*(z)$  and choose  $Z \in \mathcal{Z}^*(z)$ . By assumption  $\tilde{u}(0) = -\infty$  and  $\{\bar{Z} = 0\}$  is a non-null set under probability  $p$ . Moreover, on this set

$$\underbrace{\tilde{u}'(\bar{Z})}_{-\infty} \underbrace{(\bar{Z} - Z)}_{-Z < 0} \cdot \mathbf{1}_{\{\bar{Z}=0\}} = \infty \cdot \mathbf{1}_{\{\bar{Z}=0\}}$$

holds. Consequently the right hand side of (3.3.16) equals infinity, which actually contradicts (3.3.15).

2) Closing this proof we show  $p$ -uniform integrability of the family

$$\left\{ (\tilde{u}'(Z^\varepsilon)(\bar{Z} - Z))^- \mid \varepsilon \in (0, \tfrac{1}{2}) \right\} \quad (3.3.17)$$



### 3.3. PROOFS OF THE MAIN THEOREMS IN SECTION 3

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First note that for  $x, y \in \mathbb{R}_+$  we have

$$(x - y)^- = \max\{0, y - x\} \leq y$$

Since  $\tilde{u}'$  is increasing and non-positive, we have

$$(\tilde{u}'(Z^\varepsilon)(\bar{Z} - Z))^- = (\tilde{u}'(Z^\varepsilon)\bar{Z} - u'(Z^\varepsilon)Z)^- \leq -\tilde{u}'(Z^\varepsilon)\bar{Z} \leq -\tilde{u}'((1 - \varepsilon)\bar{Z})\bar{Z}$$

Note that up to now we have not used  $u(\infty) < \infty$ . According to Kramkov and Schachermayer (1999, Lemma 6.1)  $u(\infty) < \infty$  implies  $\text{AE}(u) < 1$ .<sup>3</sup> So we can apply Lemma\* 3 and find  $a, z_0 > 0$  such that

$$-\tilde{u}'(\hat{z})\hat{z} < \frac{a}{1-a}\tilde{u}(\hat{z}) \quad \text{for all } 0 < \hat{z} \leq z_0 \text{ and all } \varepsilon \in (0, \frac{1}{2})$$

Particularly we have  $\hat{a} := \frac{a}{1-a} > 0$ . Thus for all  $\varepsilon \in (0, \frac{1}{2})$  we get

$$-\tilde{u}'((1 - \varepsilon)\bar{Z}(t))\bar{Z}(t) \leq \hat{a}\tilde{u}(\bar{Z}(t))\mathbf{1}_{\{\bar{Z} \leq z_0\}} - \tilde{u}'((1 - \varepsilon)\bar{Z}(t))\bar{Z}(t)\mathbf{1}_{\{\bar{Z} > z_0\}}$$

Since  $-\tilde{u}'$  is bounded from above on the interval  $[\frac{z_0}{2}, \infty)$ , we can find  $b > 0$  such that

$$-\tilde{u}'((1 - \varepsilon)\bar{Z}(t))\bar{Z}(t)\mathbf{1}_{\{\bar{Z} > z_0\}} \leq b\bar{Z}(t)\mathbf{1}_{\{\bar{Z} > z_0\}} \leq b\bar{Z}(t)$$

We summarize

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T (\tilde{u}'(Z^\varepsilon)(\bar{Z} - Z))^- dF(t) \right] \\ & \leq \mathbb{E} \left[ \int_0^T \hat{a}\tilde{u}(\bar{Z}(t))\mathbf{1}_{\{\bar{Z} \leq z_0\}} + b\bar{Z}(t)\mathbf{1}_{\{\bar{Z} > z_0\}} dF(t) \right] \quad (3.3.18) \\ & \leq \hat{a} \underbrace{\mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t))\mathbf{1}_{\{\bar{Z} \leq z_0\}} dF(t) \right]}_{< \infty} + b \underbrace{\mathbb{E} \left[ \int_0^T \bar{Z}(t) dF(t) \right]}_{\substack{(3.1.2) \frac{z}{\gamma_1} \\ \leq}} \end{aligned}$$

Note that the left addend stays finite, because  $\tilde{V}(z) < \infty$  and  $\bar{Z}$  is the corresponding minimizer.

$$\mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t))\mathbf{1}_{\{\bar{Z} \leq z_0\}} dF(t) \right] \leq \mathbb{E} \left[ \int_0^T \tilde{u}(\bar{Z}(t)) dF(t) \right] = \tilde{V}(z)$$

---

<sup>3</sup>See Section 3.2 for the definition of asymptotic elasticity  $\text{AE}(u)$ .

Finally finiteness of (3.3.18) as well as independence of  $\varepsilon$  implies the desired  $p$ -uniform integrability of the family in (3.3.17).  $\square$

### 3.3.2 Proofs of Section 3.2

In this Section, we are aiming at the proof of Theorem 3. Nevertheless we first to derive a result, which is necessary for the proof of Proposition 3.2.1.

**Lemma 3.3.8.** *Let Assumption 1 hold and assume that*

$$\limsup_{x \rightarrow \infty} \frac{V(x)}{x} = 0$$

*For every countable set  $D \subseteq \mathcal{C}(x)$  the family  $\{u^+(c(t)) \mid c \in D\}$  is  $p$ -uniformly integrable.*

*Proof.* We will prove the assertion of this Lemma by forcing a contradiction. Therefore we set  $D = \{c^n\}_{n \geq 1}$ . Suppose that the family  $\{u^+(c^n(t)) \mid n \geq 1\}$  is not  $p$ -uniformly integrable. Now we are able to find  $\alpha > 0$ , a subsequence of  $\{c^n\}_{n \geq 1}$  (again denoted by  $c^n$ ), and a sequence  $\{A_n\}_{n \geq 1}$  of disjoint sets  $A_n$  of  $(\Omega \times [0, T], \mathcal{M})$  such that

$$\mathbb{E} \left[ \int_0^T u^+(c^n(t)) \mathbf{1}_{A_n} dF(t) \right] \geq \alpha \quad \text{for all } n \geq 1$$

We define a sequence of random variables  $\{h^m\}_{m \geq 1}$  via

$$h^m = x_0 + \sum_{k=1}^m c^k \mathbf{1}_{A_k}$$

where  $x_0 = \inf \{x > 0 \mid u(x) \geq 0\}$ . For arbitrary  $Z \in \mathcal{Z}$

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T h^m(t) Z(t) dF(t) \right] \\ & \leq x_0 \mathbb{E} \left[ \int_0^T Z(t) dF(t) \right] + \sum_{k=1}^m \mathbb{E} \left[ \int_0^T c^k(t) Z(t) dF(t) \right] \\ & \stackrel{(3.1.2)}{\leq} x_0 T + mx \end{aligned}$$

holds. So according to Lemma 3.3.2  $h^m \in \text{solid}(\mathcal{C}(Tx_0 + mx))$ . On the other hand

$$\mathbb{E} \left[ \int_0^T u(h^m(t)) \, dF(t) \right] \geq \sum_{k=1}^m \mathbb{E} \left[ \int_0^T u^+(c^k(t) \mathbf{1}_{A_k}) \, dF(t) \right] \geq \alpha m$$

Since  $h$  will be  $p$ -a.s. dominated by an element of  $\mathcal{C}(Tx_0 + mx)$ , the two inequalities we derived lately imply

$$\frac{V(x_0T + mx)}{x_0T + mx} \geq \frac{\mathbb{E} \left[ \int_0^T u(h^m(t)) \, dF(t) \right]}{x_0T + mx} \geq \frac{\alpha m}{x_0T + mx}$$

for all  $x > 0$ . Thus,

$$\limsup_{x \rightarrow \infty} \frac{V(x)}{x} = \limsup_{m \rightarrow \infty} \frac{V(x_0T + m)}{x_0T + m} \geq \lim_{m \rightarrow \infty} \frac{\alpha}{\frac{x_0T}{m} + 1} = \alpha > 0$$

holds, which contradicts the assumption  $\limsup_{x \rightarrow \infty} \frac{V(x)}{x} = 0$ .  $\square$

We now concentrate on the proof of Theorem 3. Deriving some auxiliary results we will make use of the following fact

$$\tilde{V} \text{ is convex and therefore continuous on the set } \{\tilde{V} < \infty\}. \quad (3.3.19)$$

The following Lemma is an important result for the later proof of Theorem 3. The solution to  $\tilde{V}(z)$  with  $z \in \{\tilde{V} < \infty\}$  will be denoted as  $\hat{Z}_z \in \mathcal{Z}(z)$ .

**Lemma 3.3.9.** *Let Assumption 1 and Assumption 3 hold. Let  $z_n > 0$  ( $n \geq 1$ ), with  $\tilde{V}(z_n) < \infty$ . If the sequence  $z_n$  converges to  $z > 0$ , with  $\tilde{V}(z) < \infty$ , then*

- (i)  $\hat{Z}_{z_n}$  converges to  $\hat{Z}_z$  in probability  $p$ .
- (ii)  $\tilde{u}(\hat{Z}_{z_n})$  converges to  $\tilde{u}(\hat{Z}_z)$  in  $L^1(\mathcal{M})$ .
- (iii)  $\hat{Z}_{z_n} \tilde{u}'(\hat{Z}_{z_n})$  converges to  $\hat{Z}_z \tilde{u}'(\hat{Z}_z)$  in  $L^1(\mathcal{M})$ .
- (iv)  $\hat{Z}_{z_n} \tilde{u}'(a_n \hat{Z}_{z_n})$  converges to  $\hat{Z}_z \tilde{u}'(\hat{Z}_z)$  in  $L^1(\mathcal{M})$ .  
Here  $\{a_n\}_{n \geq 1} \subseteq \mathbb{R}$  denotes a sequence with  $\lim_{n \rightarrow \infty} a_n = 1$ .

---

We will prove this Lemma step by step. Note that Assumption 3 is necessary only for Lemma 3.3.9 (iii) and (iv).

*Proof of 3.3.9 (i).* We will see: If  $\hat{Z}_{z_n}$  does not converge to  $\hat{Z}_z$  in measure  $p$ , this causes a contradiction.

Assume that  $\hat{Z}_{z_n}$  does not converge to  $\hat{Z}_z$  in measure  $p$ . Then there exist  $\tilde{\varepsilon} > 0$ , such that

$$\limsup_{n \rightarrow \infty} p \left( |\hat{Z}_{z_n} - \hat{Z}_z| > \tilde{\varepsilon} \right) > \tilde{\varepsilon}$$

According to Proposition 3.1.1 we have

$$\mathbb{E} \left[ \int_0^T \hat{Z}_{z_n}(t) dF(t) \right] \leq \frac{z_n}{\gamma_1} \quad \text{and} \quad \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) dF(t) \right] \leq \frac{z}{\gamma_1}$$

Hence passing to a possibly smaller  $\varepsilon > 0$  will give us

$$\limsup_{n \rightarrow \infty} p \left( |\hat{Z}_{z_n} - \hat{Z}_z| > \varepsilon, |\hat{Z}_{z_n} + \hat{Z}_z| > \frac{1}{\varepsilon} \right) > \varepsilon \quad (3.3.20)$$

Moreover we find a sequence  $\{A_n\}_{n \geq 1} \subseteq \Omega \times [0, T]$  with  $\limsup_{n \rightarrow \infty} p(A_n) > \varepsilon$  such that

$$|\hat{Z}_{z_n} - \hat{Z}_z| > \varepsilon \quad \text{as well as} \quad |\hat{Z}_{z_n} + \hat{Z}_z| > \frac{1}{\varepsilon} \quad \text{on } A_n$$

By convexity of  $\tilde{u}$  we have

$$\tilde{u} \left( \frac{1}{2}(\hat{Z}_{z_n} + \hat{Z}_z) \right) \leq \frac{1}{2} \left( \tilde{u}(\hat{Z}_{z_n}) + \tilde{u}(\hat{Z}_z) \right) \quad \text{for all } (\omega, t) \in \Omega \times [0, T]$$

Now Equation (3.3.20) and strict convexity of  $\tilde{u}$  ensures the existence of  $\eta > 0$  such that

$$\limsup_{n \rightarrow \infty} p \left( \tilde{u} \left( \frac{1}{2}(\hat{Z}_{z_n} + \hat{Z}_z) \right) \leq \frac{1}{2}(\tilde{u}(\hat{Z}_{z_n}) + \tilde{u}(\hat{Z}_z)) - \eta \right) > \eta \quad \text{on } A_n$$

Consequently

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T \tilde{u} \left( \frac{1}{2}\hat{Z}_{z_n}(t) + \frac{1}{2}\hat{Z}_z(t) \right) dF(t) \right] \\ & \leq \frac{1}{2} \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_{z_n}(t)) + \tilde{u}(\hat{Z}_z(t)) dF(t) \right] - \eta^2 \\ & = \frac{1}{2} \left( \tilde{V}(z_n) + \tilde{V}(z) \right) - \eta^2 \end{aligned}$$

holds. Making use of (3.3.19) we are able to conclude with

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u} \left( \frac{1}{2} \hat{Z}_{z_n}(t) + \frac{1}{2} \hat{Z}_z(t) \right) dF(t) \right] \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{2} (\tilde{V}(z_n) + \tilde{V}(z)) - \eta^2 \\
 &= \tilde{V}(z) - \eta^2
 \end{aligned} \tag{3.3.21}$$

We will now construct a dual process  $h \in \mathcal{Z}(z)$  with

$$\mathbb{E} \left[ \int_0^T \tilde{u}(h(t)) dF(t) \right] < \tilde{V}(z)$$

this obviously contradicts the definition of  $\tilde{V}(z)$ . Therefore we define

$$g_n := \frac{1}{2} (\hat{Z}_{z_n} + \hat{Z}_z) \quad \text{for all } n \geq 1$$

By Delbaen and Schachermayer (1994, Lemma A1.1) we can find a sequence  $h_n \in \text{conv}(g_k \mid k \geq n)$ , which converges *p-a.s.* to a random variable  $h$  with values in  $[0, \infty]$ . Note that  $h_n \in \mathcal{Z} \left( \frac{\sup_{m \geq n} (z_m + z)}{2} \right)$  by construction. In addition  $\lim_{n \rightarrow \infty} z_n = z$  yields  $h \in \mathcal{Z}(z)$ .

Using continuity of  $\tilde{u}$ , Fatou's Lemma, and convexity of  $\tilde{u}$ , we get

$$\begin{aligned}
 \mathbb{E} \left[ \int_0^T \tilde{u}(h(t)) dF(t) \right] &= \mathbb{E} \left[ \int_0^T \liminf_{n \rightarrow \infty} \tilde{u}(h_n(t)) dF(t) \right] \\
 &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(h_n(t)) dF(t) \right] \\
 &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(g_n(t)) dF(t) \right] \\
 &\stackrel{(3.3.21)}{\leq} \tilde{V}(z) - \eta^2
 \end{aligned}$$

Thus  $\mathbb{E} \left[ \int_0^T \tilde{u}(h(t)) dF(t) \right] < \tilde{V}(z)$  and convergence in probability  $p$  is necessary.  $\square$

*Proof of 3.3.9 (ii).* We already know about the convergence in probability  $p$ , cf. 3.3.9 (i). In particular  $\tilde{u}^-(\hat{Z}_{z_n})$  converges to  $\tilde{u}^-(\hat{Z}_z)$  in this probability. In additionally Lemma 3.3.3 resp.  $p$ -uniform integrability of  $\{\tilde{u}^-(\hat{Z}_{z_n})\}_{n \geq 1}$  implies convergence of  $\tilde{u}^-(\hat{Z}_{z_n})$  in  $L^1$ .

---

Since the random variables  $\tilde{u}^+(\hat{Z}_{z_n})$  are non-negative and integrable (recall that  $\tilde{V}(z) < \infty$ ), we get  $L^1$  convergence if

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}^+(\hat{Z}_{z_n}(t)) \, dF(t) \right] = \mathbb{E} \left[ \int_0^T \tilde{u}^+(\hat{Z}_z(t)) \, dF(t) \right]$$

Notice that  $\tilde{V}$  is continuous on  $\{\tilde{V} < \infty\}$ . Thus

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}^+(\hat{Z}_{z_n}(t)) \, dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_{z_n}(t)) + \tilde{u}^-(\hat{Z}_{z_n}(t)) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_z(t)) \, dF(t) \right] + \mathbb{E} \left[ \int_0^T \tilde{u}^-(\hat{Z}_z(t)) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T \tilde{u}^+(\hat{Z}_z(t)) \, dF(t) \right] \end{aligned}$$

This finally yields  $L^1(\mathcal{M})$  convergence of  $\tilde{u}(\hat{Z}_{z_n})$ .  $\square$

*Proof of 3.3.9 (iii).* Again one of the key observations is the convergence  $(\hat{Z}_{z_n})$  in measure  $p$ , cf. 3.3.9 (i). By continuity of  $\tilde{u}'$ , we conclude that  $\tilde{u}'(\hat{Z}_{z_n})\hat{Z}_{z_n}$  converges to  $\tilde{u}'(\hat{Z}_z)\hat{Z}_z$  in measure  $p$ . In order to obtain assertion 3.3.9 (iii) we have to show  $p$ -uniform integrability of the family

$$\left\{ \tilde{u}'(\hat{Z}_{z_n})\hat{Z}_{z_n} \mid n \geq 1 \right\}$$

Thanks to Assumption 3, asymptotic elasticity of  $u$  is less than 1. Recall Lemma\* 3, which guarantees the existence of  $z_0 > 0$  and a constant  $C < \infty$  such that

$$-\tilde{u}'(z) < C \frac{\tilde{u}(z)}{z} \quad \text{for all } 0 < z \leq z_0 \quad (3.3.22)$$

Thus  $-\tilde{u}'(\hat{Z}_{z_n})\hat{Z}_{z_n} \mathbf{1}_{\{\hat{Z}_{z_n} < z_0\}}$  is absolutely dominated by  $C \tilde{u}(\hat{Z}_{z_n}) \mathbf{1}_{\{\hat{Z}_{z_n} < z_0\}}$ . We already derived  $p$ -uniform integrability of  $\tilde{u}(\hat{Z}_{z_n})$  in 3.3.9 (ii). This in turn implies the  $p$ -uniform integrability of

$$\left\{ -\tilde{u}'(\hat{Z}_{z_n})\hat{Z}_{z_n} \mathbf{1}_{\{\hat{Z}_{z_n} < z_0\}} \mid n \geq 1 \right\}$$

It remains to show the  $p$ -uniform integrability of

$$\left\{ -\tilde{u}'(\hat{Z}_{z_n}) \hat{Z}_{z_n} \mathbf{1}_{\{\hat{Z}_{z_n} \geq z_0\}} \mid n \geq 1 \right\}$$

Since  $-\tilde{u}'$  is positive and strictly decreasing, we are able to conclude with

$$-\tilde{u}'(\hat{Z}_{z_n}) \mathbf{1}_{\{\hat{Z}_{z_n} \geq z_0\}} \leq -\tilde{u}'(z_0) \quad \text{for all } n \geq 1$$

Finally  $L^1(\mathcal{M})$  boundedness of  $\hat{Z}_{z_n}$  in combination with  $\tilde{u}'(\infty) = 0$ , implies the desired  $p$ -uniform integrability.  $\square$

*Proof of 3.3.9 (iv).* We will state an inequality similar to Equation (3.3.22). Lemma\* 3 (iii)&(iv) imply the following.

For fixed  $\mu \in [0, T]$  there exist a constant  $\hat{C} < \infty$  and  $z_1 > 0$  such that

$$-\tilde{u}(\mu z) < \hat{C} \frac{\tilde{u}(z)}{z} \quad \text{for all } 0 < z < z_1$$

Now the assertion of Lemma 3.3.9 (iv) can be obtained analogously to 3.3.9 (iii).  $\square$

**Lemma 3.3.10.** *Let Assumption 1 and Assumption 3 hold, then the dual value function  $\tilde{V}$  satisfies the following:*

(i)  $\tilde{V}: (0, \infty) \rightarrow (0, \infty)$  is continuous differentiable

(ii) For all  $z \in (0, \infty)$

$$z\tilde{V}'(z) = \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) dF(t) \right]$$

*Proof.* By  $\partial_R \tilde{V}$  we denote the right derivative of  $\tilde{V}$ .

In the first step we will prove the assertions (i) and (ii) simultaneously. Later will verify the estimates

$$\liminf_{\varepsilon \searrow 1} \frac{\tilde{V}(z) - \tilde{V}(\varepsilon z)}{\varepsilon - 1} \geq -\mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) dF(t) \right] \geq \limsup_{\varepsilon \searrow 1} \frac{\tilde{V}(z) - \tilde{V}(\varepsilon z)}{\varepsilon - 1}$$

for all  $z > 0$ .

1) Since  $\liminf \leq \limsup$  always holds, equality must hold in all estimations stated above. In particular we get

$$z \partial_R \tilde{V}(z) = -\mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) dF(t) \right] \quad \text{for all } z > 0$$

---

Note that, according to Lemma 3.3.9 (iii), the function  $\partial_R \tilde{V}(z)$  is continuous in  $z$ . Thanks to convexity of  $\tilde{V}$  we have  $\tilde{V}' = \partial_R \tilde{V}$ , which proves (ii). In particular  $\tilde{V}$  is continuous differentiable, which proves (i).

2) Note that  $\tilde{u}'(\varepsilon z)$  is increasing in  $\varepsilon > 0$ . The following estimate holds by  $\varepsilon \hat{Z}_z \in \mathcal{Z}(z)$ , convexity of  $\tilde{u}$ , and the Monotone Convergence Theorem.

$$\begin{aligned}
& \liminf_{\varepsilon \searrow 1} \frac{\tilde{V}(z) - \tilde{V}(\varepsilon z)}{\varepsilon - 1} \\
& \geq \liminf_{\varepsilon \searrow 1} \frac{1}{\varepsilon - 1} \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_z(t)) - \tilde{u}(\varepsilon \hat{Z}_z(t)) \, dF(t) \right] \\
& \geq \liminf_{\varepsilon \searrow 1} \frac{1}{\varepsilon - 1} \mathbb{E} \left[ \int_0^T (1 - \varepsilon) \hat{Z}_z(t) \tilde{u}'(\varepsilon \hat{Z}_z(t)) \, dF(t) \right] \\
& = - \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right]
\end{aligned}$$

We now prove the second inequality. For all  $\varepsilon > 0$  we consider the dual variables  $\hat{Z}_{\varepsilon z} \in \mathcal{Z}(\varepsilon z) = \varepsilon \mathcal{Z}(z)$ . Thus,  $\frac{1}{\varepsilon} \hat{Z}_{\varepsilon z} \in \mathcal{Z}(z)$  and in particular

$$\begin{aligned}
\tilde{V}(z) - \tilde{V}(\varepsilon z) &= \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_z(t)) - \tilde{u}(\hat{Z}_{\varepsilon z}(t)) \, dF(t) \right] \\
&\leq \mathbb{E} \left[ \int_0^T \tilde{u}\left(\frac{1}{\varepsilon} \hat{Z}_{\varepsilon z}(t)\right) - \tilde{u}(\hat{Z}_{\varepsilon z}(t)) \, dF(t) \right]
\end{aligned}$$

holds. By convexity of  $\tilde{u}$  we are able to conclude with

$$\begin{aligned}
& \limsup_{\varepsilon \searrow 1} \frac{\tilde{V}(z) - \tilde{V}(\varepsilon z)}{\varepsilon - 1} \\
& \leq \limsup_{\varepsilon \searrow 1} \frac{1}{\varepsilon - 1} \mathbb{E} \left[ \int_0^T \tilde{u}\left(\frac{1}{\varepsilon} \hat{Z}_{\varepsilon z}(t)\right) - \tilde{u}(\hat{Z}_{\varepsilon z}(t)) \, dF(t) \right] \\
& \leq \limsup_{\varepsilon \searrow 1} \frac{1}{\varepsilon - 1} \mathbb{E} \left[ \int_0^T \left(\frac{1}{\varepsilon} - 1\right) \hat{Z}_{\varepsilon z}(t) \tilde{u}'\left(\frac{1}{\varepsilon} \hat{Z}_{\varepsilon z}(t)\right) \, dF(t) \right] \\
& \stackrel{L \ 3.3.9(iv)}{=} - \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right]
\end{aligned}$$

which verifies the second estimate.  $\square$

**Lemma 3.3.11.** *Let Assumption 1 and Assumption 3 hold. Choose  $x, z > 0$  with  $x = -\tilde{V}'(z)$ , then  $-\tilde{u}'(\hat{Z}_z) \in \mathcal{C}(x)$  will satisfy the identity*

$$V(x) = \mathbb{E} \left[ \int_0^T u(-\tilde{u}'(\hat{Z}_z(t))) \, dF(t) \right]$$



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*Proof.* According to Remark 3.1.2 it suffices to prove that  $-\tilde{u}(\hat{Z}_z)$  is solves the problem

$$\sup_{d \in \text{solid}(\mathcal{C}(x))} \mathbb{E} \left[ \int_0^T u(d(t)) \, dF(t) \right]$$

Before we start proving optimality, we have to ensure that  $-\tilde{u}(\hat{Z}_z) \in \text{solid}(\mathcal{C}(x))$  indeed.

By the bipolar Theorem of Kramkov and Schachermayer we know,  $c \in \text{solid}(\mathcal{C}(x))$  is equivalent to

$$\sup_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T Z(t)c(t) \, dF(t) \right] \leq xz$$

According to Lemma 3.3.10, we have

$$xz = -z\tilde{V}'(z) = -\mathbb{E} \left[ \int_0^T \hat{Z}_z(t)\tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right]$$

Moreover since  $-\tilde{u}'$  is a non-negative function we get  $-\tilde{u}'(\hat{Z}_z(t)) \geq 0$  and further obtain  $-\tilde{u}'(\hat{Z}_z) \in \mathcal{C}(x)$ . Consequently we need to show that

$$\mathbb{E} \left[ \int_0^T Z(t)\tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right] \geq \mathbb{E} \left[ \int_0^T \hat{Z}_z(t)\tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right] \quad (3.3.23)$$

holds for all  $Z \in \mathcal{Z}(z)$ .

Let  $\varepsilon > 0$ . We will perturb  $\hat{Z}_z$  by an arbitrary  $Z \in \mathcal{Z}(z)$  and define

$$Z^\varepsilon = (1 - \varepsilon)\hat{Z}_z + \varepsilon Z$$

By convexity  $Z^\varepsilon \in \mathcal{Z}(z)$  for all  $(0, 1)$ . Optimality of  $\hat{Z}_z$  and by differentiability of  $\tilde{u}$  induce

$$0 \leq \mathbb{E} \left[ \int_0^T \tilde{u}(Z^\varepsilon(t)) - \tilde{u}(\hat{Z}_z(t)) \, dF(t) \right] \leq \mathbb{E} \left[ \int_0^T \tilde{u}'(Z^\varepsilon(t))(Z^\varepsilon(t) - \hat{Z}_z(t)) \, dF(t) \right]$$

Since  $\tilde{u}'$  is increasing and  $Z^\varepsilon \geq (1 - \varepsilon)\hat{Z}_z$  by definition we continue with

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \tilde{u}'(Z^\varepsilon(t))\hat{Z}_z(t) \, dF(t) \right] &\geq \mathbb{E} \left[ \int_0^T \tilde{u}'(Z^\varepsilon(t))Z(t) \, dF(t) \right] \quad (3.3.24) \\ &\geq \mathbb{E} \left[ \int_0^T \tilde{u}'((1 - \varepsilon)\hat{Z}_z(t))Z(t) \, dF(t) \right] \end{aligned}$$

---

Here the (non-positive) random variable  $\tilde{u}'((1 - \varepsilon)\hat{Z}_z)Z$  stays integrable if  $\varepsilon$  is sufficient small. Finally

$$\begin{aligned}
& - \mathbb{E} \left[ \int_0^T \hat{Z}_z(t) \tilde{u}(\hat{Z}_z(t)) \, dF(t) \right] \\
&= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \int_0^T -\tilde{u}'((1 - \varepsilon)\hat{Z}_z(t)) Z(t) \, dF(t) \right] \\
&\stackrel{(3.3.24)}{\geq} \liminf_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \int_0^T -\tilde{u}'(Z_\varepsilon(t)) \hat{Z}_z(t) \, dF(t) \right] \\
&\geq - \mathbb{E} \left[ \int_0^T \liminf_{\varepsilon \rightarrow 0} -\tilde{u}'(Z_\varepsilon(t)) \hat{Z}_z(t) \, dF(t) \right] \\
&= - \mathbb{E} \left[ \int_0^T Z(t) \tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right]
\end{aligned}$$

Notice that we plug in a minus sign to make the random variables non-negative. Thus we may use the Monotone Convergence Theorem in the second line in addition to Fatou's Lemma in the forth line.

This verifies Equation (3.3.23) and we derive  $-\tilde{u}(\hat{Z}_z) \in \text{solid}(\mathcal{C}(x))$ .

Finally we prove the optimality of  $-\tilde{u}(\hat{Z}_z)$ . Recall that by definition of  $\mathcal{Z}(z)$

$$\mathbb{E} \left[ \int_0^T c(t) \hat{Z}_z(t) \, dF(t) \right] \leq xz \quad \text{holds for all } c \in \mathcal{C}(x)$$

Further by biduality of  $u$  and  $\tilde{u}$

$$u(c) \leq \tilde{u}(\hat{Z}_z) + c\hat{Z}_z$$

holds  $p - a.s.$  These preliminary ideas lead us to

$$\begin{aligned}
\mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] &\leq \tilde{V}(z) + xz \\
&\stackrel{L \ 3.3.10(ii)}{=} \mathbb{E} \left[ \int_0^T \tilde{u}(\hat{Z}_z(t)) - \hat{Z}_z(t) \tilde{u}'(\hat{Z}_z(t)) \, dF(t) \right] \\
&= \mathbb{E} \left[ \int_0^T u(-\tilde{u}'(\hat{Z}_z(t))) \, dF(t) \right]
\end{aligned}$$

where the last equality holds by the general bidual properties of  $u$  and  $\tilde{u}$ . This finally proves the optimality of  $-\tilde{u}(\hat{Z}_z)$ .  $\square$

**Proof of Theorem 3**

*Proof.* (i) All assertions which are not known from Theorem 1, are straight forward implications of Lemma 3.3.10.

(ii) According to Lemma 3.3.10 (i),  $\tilde{V}(z) < \infty$  for all  $z > 0$ . Thus Assumption 2 holds and existence of the unique optimal strategy is just a consequence of Proposition 3.2.1.

(iii) Monotonicity follows from differentiability and concavity. Similarly as in the proof of Theorem 1, both equations  $V'(\infty) = 0$  and  $\tilde{V}'(0) = \infty$  are equivalent. Recall Equation (3.2.7), i.e.  $\text{AE}(V) < 1$ . This observation moreover implies  $V'(\infty) = 0$ .  $\square$

**Proof of Theorem 4**

*Proof.* According to Theorem 1 (iii) the dual solution  $\hat{Z}_z$  exist and Identity (3.2.8) hold by Lemma 3.3.11. Moreover Identity (3.2.11) has been derived in Lemma 3.3.10 (ii). Making use of the assumption  $x = -\tilde{V}(z)$  and Identity (3.2.8), all other identities can be derived easily.  $\square$



## Chapter 4

# Constrained Consumption Selection in Incomplete Markets

In this section we set up our model for expected utility maximization when consumption selection is constrained. As usual in mathematical finance, we choose a market which is arbitrage free in the sense of

$$\mathfrak{M} \neq \emptyset$$

Again, we consider an investor who gains utility from intertemporal consumption while his consumption choice is subject to an initial capital  $x > 0$ . But this time the natural constraints may not be enough to mirror the investor's interests. We now consider an investor who abstains from unconstrained consumption selection in order to realize individual likings or deep-seated higher values.

The natural space for (rate of) consumption processes is given by the set of non-negative, progressively measurable processes  $c \in L_*^0(\mathcal{M})$ . A process is admissible for the initial capital  $x$ , if it satisfies the constraint

$$\sup_{Y \in \mathcal{Y}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] \leq x \quad (4.0.1)$$

---

where  $\hat{\mathcal{Y}}$  denotes the process dual of the non-negative wealth processes. The set of all admissible processes is denoted  $\mathcal{C}(x)$ .

Although the investor will not be able to benefit from all consumption processes  $c \in L_*^0(\mathcal{M})$ , he will be able to evaluate them. His preferences will be heavily related to the following functional, which depends on intertemporal utility  $u$ .

$$\mathcal{E}(c) = \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad \text{for } c \in L_*^0(\mathcal{M}) \quad (4.0.2)$$

In the following  $\mathcal{E}$  is called *evaluation function*. Again we assume

$$\begin{aligned} \mathbb{P}(F(T) > 0) &> 0 \\ \mathbb{E} \left[ \int_0^T \mathbf{1} dF(t) \right] &= 1 \end{aligned}$$

The assumptions on intertemporal utility  $u$  are the same as in the last sections.

Processes which are permitted by the investor will lie in a smaller set  $\bar{\mathcal{K}} \subseteq L_*^0(\mathcal{M})$ . Thus the investor is accompanied by the following problem.

$$\text{maximize } \mathcal{E}(c) \quad \text{s.t. } c \in \bar{\mathcal{K}} \cap \mathcal{C}(x) \quad (4.0.3)$$

According to the given initial capital  $x$  the investor is able to evaluate the whole market structure via  $V(x)$  as defined in Section 3. To guarantee solvability of the former optimization problem we may ask for assumptions that imply finiteness of the unconstrained optimization problem  $V(x)$ . Several sufficient circumstances have been discussed in the last section.

## 4.1 Investors Consumption Choice

Our aim is to restrict the investor's attention to a predefined set of consumption patterns. As usual in economic theory we impute rationality to the investor, i.e. he acts as an expected utility maximizer. His preferences will be influenced (*but not rigorously defined*) by the evaluation function, resp. the utility functional as defined in Section 2.

Coming to a decision the investor takes not all admissible consumption plans  $c \in \mathcal{C}(x)$  into account. Depending on his *individual likings* (or *partiality*) the investor stints himself of choosing a consumption process  $c$  that lies in a predefined *acceptance* set  $\bar{\mathcal{K}}$ . Processes  $c \in \bar{\mathcal{K}}$  will be called *permissible*.

We now state some basic rules on the investors choice.

**Axiom 1.** *A permissible process  $c$  is admissible, i.e.  $\bar{\mathcal{K}} \subseteq L_*^0(\mathcal{M})$ .*

Obviously, negative consumption is counterfactual. Thus, we are dealing more with a question of practicability than of personal partialities.

**Axiom 2.** *If two processes  $c_1, c_2 \in L_*^0(\mathcal{M})$  are permissible, then  $c_1 + c_2$  as well as  $ac_1$  for all  $a \geq 0$  are permissible.*

**Axiom 3.** *If the sequence  $\{c_n\}_{n \geq 1} \subseteq L_*^0(\mathcal{M})$  of permissible consumption processes converging in probability  $p$  to a process  $c \in L_*^0(\mathcal{M})$ , then  $c$  is also permissible.*

To keep things simple and to exclude the trivial case as well, we further assume

**Axiom 4.** *The constant process  $\mathbf{1}$  is permissible.*

According to the Axioms, we see immediately

**Remark 4.1.1.** The *acceptance set*  $\bar{\mathcal{K}}$  is a non-empty, closed, convex subcone of  $L_*^0(\mathcal{M})$ <sup>1</sup>.

This definition includes two nongeneric cases.

**Example 3** (Merton's Investor). Since  $L_*^0(\mathcal{M})$  itself satisfies the axioms we stated lately, we are able to choose  $\bar{\mathcal{K}} = L_*^0(\mathcal{M})$ . Investors acting according to this acceptance set have been studied in the prevailing literature since Merton (1969).

---

<sup>1</sup>Remember: A convex cone in a vector space is a non-negative homogeneous set which is closed under addition.

---

**Example 4** (Inflexible Investors). Imagine an investor who is intolerant for every change in the rate of consumption. Those inflexible investors will accept constant rate of consumption processes only, i.e.  $c = a\mathbf{1}$  for  $\tilde{a} > 0$ . Obviously, investors like that will choose among all rate of consumption processes in  $\bar{\mathcal{K}} = \{a\mathbf{1} \mid a \in \mathbb{R}_+\}$ .

Likings which are not as generic could be some future plan, which forces the investor to be more careful withdrawing money for consumption today because he is addicted to past consumption and need a secured reserve. As a border case of addiction to past consumption we can handle the problem of an investor whose consumption decision is driven by ratcheting behavior, i.e. the rate of consumption will never decrease, cf. Dybvig (1995), Riedel (2009).

**Example 5** (Consumption Ratcheting).

$$\bar{\mathcal{K}} = \{c \in L_*^0(\mathcal{M}) \mid c(t) \geq c(s) \text{ for all } t \geq s\}$$

Consumption ratcheting is a non-standard example which allows for a good mathematical treatment. Moreover one can easily construct more behavior patterns related to this ratcheting partialities.

**Example 6** (Exponentially Weaning from Past Consumption). Fix a discount factor  $\delta \in \mathbb{R}_+$ , displaying investors' ability to wean from past consumption. This investor is addicted to previous standard of living as well, but overcomes addiction exponentially.

$$\bar{\mathcal{K}} = \{c \in L_*^0(\mathcal{M}) \mid c(t) \geq e^{-\delta(t-s)}c(s) \text{ for all } t \geq s\}$$

**Example 7** ("Consumption based CPPI"). Fix a factor  $\varrho \in [0, 1]$ , displaying the actual requirements of the investor. He gets accustomed to previous standard of living which influences his future requirements.

$$\bar{\mathcal{K}} = \left\{ c \in L_*^0(\mathcal{M}) \mid c(t) \geq \varrho \sup_{s \leq t} c(s) \text{ for all } t \geq s \right\}$$



Up to now we have not given a formal definition of investors preferences or the utility function. We will make up for that in the succeeding discussion.

As we have already mentioned, investors' preferences are strongly related the evaluation function. We define the utility function  $\mathcal{E}_{\mathcal{K}}$  as the restriction of evaluation function on the choice set  $\bar{\mathcal{K}}$ .

$$\begin{array}{ccc} \mathcal{C}_V & \xrightarrow{\mathbb{E}\left[\int_0^T u(c(t)) dF(t)\right]} & \mathbb{R} \cup \{\infty\} \\ \cup \uparrow & & \parallel \\ \mathcal{E}_{\mathcal{K}}: \mathcal{K}_V & \longrightarrow & \mathbb{R} \cup \{\infty\} \end{array}$$

Here  $\mathcal{C}_V = \left\{ c \in L_*^0(\mathcal{M}) \mid \mathbb{E}\left[\int_0^T u^-(c(t)) dF(t)\right] \right\}$ , accordingly  $\mathcal{K}_V$  denotes  $\bar{\mathcal{K}} \cap \mathcal{C}_V$  holds.

Actually this is not a misuse of notation. If we choose  $\bar{\mathcal{K}} = L_*^0(\mathcal{M})$ , we get the identity  $\bar{\mathcal{C}}_V = L_*^0(\mathcal{M}) \cap \bar{\mathcal{C}}_V$ . According to that notation  $\mathcal{K}_V(x) = \bar{\mathcal{K}} \cap \mathcal{C}_V(x)$ . If  $u$  is bounded below, we get  $\mathcal{K}_V = \bar{\mathcal{K}}$  (resp.  $\mathcal{K}_V(x) = \mathcal{K}(x)$  for all  $x > 0$ ).

From a technical point of view the results remain the same if we define the utility function for all potentially admissible consumption rates  $c \in \mathcal{C}_V$  as

$$\mathcal{E}_{\mathcal{K}}(c) := \begin{cases} \mathbb{E}\left[\int_0^T u(c(t)) dF(t)\right] & \text{if } c \in \mathcal{K}_V \\ -\infty & \text{else} \end{cases} \quad (4.1.1)$$

Although the technical treatment of both function will be identical we should think about these definitions. Compared with other models in habit formation or decision theory it is definitively questionable why one should punish any deviation from individual likings that hard. Restricting investors' choice to an acceptance set does not change the overall situation but might be better justifiable as the utility functional (4.1.1). We display some critical aspects in the following.

**Example 8.** Consider two lotteries  $A$  and  $B$ . The first lottery  $A$  pays 1 up to time  $\frac{T}{2}$  and 2 thereafter or the other way around, it pays 2 up to time  $\frac{T}{2}$  and 1 thereafter. Here each cash flow will be realized with the same probability (i.e. 50%). The second lottery  $B$  pays  $\frac{u(1)+u(2)}{2}$  constantly.

---

Now consider two different types of investors. One investor who acts according to Merton's setting without any further constraints, and one consumption ratcheting investor (see Example 5).

Obviously the Merton type investor will be indifferent between these two lotteries whereas the ratcheting investor strictly prefers lottery  $B$ . Beyond that the ratcheting investor will strictly prefer any lottery with a sure outcome above lottery  $A$ . The existence of a possible decrease frightens him that much that he evaluates lottery  $A$  with  $-\infty$ .

## 4.2 Two Benchmark Cases

In the following we will discuss two generic cases for constraint consumption selection.

The first case is already fully introduced in the Section 3 and known as the unconstrained situation. It is an improper example of constraint consumption selection where consumption choice is restricted only by natural assumptions. In this case the acceptance set is as maximal as possible.

The second case is the most simple example for proper constrained consumption selection. In that situation the investor stints himself to choose a constant consumption rate and represses savings to finance this standard for the rest of his life (see Example 4). Accordingly the investor's acceptance set is reduced to a minimum.

In both cases we will point out that the (dual) value function is close to the (dual) intertemporal utility function.

### The Unconstrained Case

The unconstrained setup has been elaborately discussed in the previous section. We pointed out that main properties of value and dual value function correspond to properties of the underlying intertemporal utility function. Nevertheless we recall the most important results in what follows.

**Theorem\* 4.** *Let Assumption 1 and Assumption 3 hold, then*

- (i) both functions  $V$  and  $-\tilde{V}$  are finite, increasing, strictly concave, and continuously differentiable on  $(0, \infty)$ .
- (ii) for all  $x, z > 0$  the optimal solutions to  $V(x)$  and  $\tilde{V}(z)$  exist and are unique  $p - a.s.$
- (iii) the functions  $V'$  and  $-\tilde{V}'$  are strictly decreasing and satisfy

$$\begin{aligned} V'(0) = \infty \quad \text{and} \quad \tilde{V}'(\infty) = 0 \\ V'(\infty) = 0 \quad \text{and} \quad -\tilde{V}'(0) = \infty \end{aligned}$$

As in many other (unconstrained) optimization problems formulated on incomplete semimartingale models, these result holds also if  $u$  does not satisfy Assumption 3. For investors gaining utility from (intertemporal) wealth, Kramkov and Schachermayer (2003) and Bouchard and Pham (2004) verified that Assumption 2 is the weakest known condition to guarantee this useful properties and relations.

### The Inflexible Case

In the situation where only constant consumption plans can attract the investor, our problem reduces to a very simple question. Since the evaluation function is increasing with respect to the order induced by  $\geq^2$ , we just have to find a constant consumption plan which has the price of our initial capital.

By  $\bar{\mathcal{K}}_0$  we denote the acceptance set of an inflexible investor, i.e.

$$\bar{\mathcal{K}}_0 = \{c\mathbf{1} \mid c \in \mathbb{R}_+\}$$

The value function of an inflexible investor will be denoted by  $V_0 := V_{\bar{\mathcal{K}}_0}$ . For  $x > 0$ ,  $V_0(x)$  corresponds to the problem

$$\sup_{c \in \mathbb{R}_+} u(c) \quad \text{s.t.} \quad \sup_{Y \in \bar{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T cY(t) dt \right] \leq x \quad (4.2.1)$$

---

<sup>2</sup>This ordering means that  $X \geq Y$  if and only if  $X(t) \geq Y(t)$   $p - a.s.$  for all  $t \in [0, T]$

---

Standing Assumption 1 implies the identity

$$\mathbb{E} \left[ \int_0^T u(c) \, dF(t) \right] = u(c) \underbrace{\mathbb{E} \left[ \int_0^T \mathbf{1} \, dF(t) \right]}_{=1}$$

We now establish the pricing formula for inflexible consumption processes, which in turn gives us the optimal consumption plan.

**Proposition 4.2.1.** *Let  $c \geq 0$ . The price of the constant consumption plan  $c = c\mathbf{1}$  is given via*

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T cY(t) \, dt \right] = cT \quad (4.2.2)$$

Thus, according to (4.2.1), the value function for the inflexible investor is given as

$$V_0(x) = u\left(\frac{x}{T}\right) \quad \text{for all } x > 0 \quad (4.2.3)$$

and the dual value function satisfies the identity

$$\tilde{V}_0(z) = \tilde{u}(zT) \quad \text{for all } z > 0 \quad (4.2.4)$$

*Proof.* Note that

$$\sup_{Y \in \hat{\mathcal{Y}}^e} \mathbb{E} \left[ \int_0^T H(t)c(t) \, dt \right] = \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) \, dt \right]$$

by Proposition 2.3.3. According to Fubini's Theorem and Bayes rule for stochastic processes we derive

$$\mathbb{E} \left[ \int_0^T cH(t) \, dt \right] = c \mathbb{E}_Q \left[ \int_0^T \mathbf{1} \, dt \right] = cT \quad \text{for all } H \in \hat{\mathcal{Y}}^e$$

Thus (4.2.2) holds.

The last Equation (4.2.4) is implied by Equation (4.2.3)

$$\begin{aligned} \tilde{V}_0(z) &= \sup_{x>0} u\left(\frac{x}{T}\right) - xz \\ &= \sup_{x>0} u(x) - x(zT) = \tilde{u}(zT) \end{aligned}$$

Equation (4.2.3) itself follows from strict monotonicity of  $u$  and Standing Assumption 1, resp. (SA 1.2).  $\square$

Obviously the value function and the dual value function of an inflexible investor have the same properties as in the unconstrained case.

**Corollary 4.2.2.** *Value function  $V_0$  and dual value function  $\tilde{V}_0$  of an inflexible investor have the following properties.*

- (i) *Both functions  $V_0$  and  $-\tilde{V}_0$  are finite, increasing, strictly concave, and continuously differentiable on  $(0, \infty)$ .*
- (ii) *The functions  $V'_0$  and  $-\tilde{V}'_0$  are strictly decreasing and satisfy*

$$\begin{aligned} V'_0(0) = \infty & \quad \text{and} \quad \tilde{V}'_0(\infty) = 0 \\ V'_0(\infty) = 0 & \quad \text{and} \quad -\tilde{V}'_0(0) = \infty \end{aligned}$$

*Proof.* Thanks to Proposition 4.2.1 these assertions are easy to check.  $\square$

### 4.3 Optimal Consumption Selection under Constraints

In this section we study the optimization problem of the investor. Of course taking individual likings into account affects his rationality in some sense. But although optimization is subordinated to some higher values, the situation itself is that of an (*unbounded*) rational decision maker. Thus introducing acceptance sets leads to a much richer class of rational investors.

Formally the consumption choice problem is

**Problem 3** (Constrained Primal Problem).

$$V_{\mathcal{K}}(x) = \max_{c \in \mathcal{K}_V(x)} \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] \quad (4.3.1)$$

Thanks to Axiom 4,  $V_{\mathcal{K}} > -\infty$  always holds. According to the previous section we generally denote  $V_{L^0_+(\mathcal{M})}$  as  $V$ . This coincides with the previous definition and moreover

$$V_{\mathcal{K}} \leq V \quad (4.3.2)$$

---

holds for all acceptance sets  $\mathcal{K}$ . We already mentioned that  $V$  mirrors the potential value of the whole underlying market structure.  $V(x)$  may be seen as the maximum gains form utility which can be reached on this market given the initial capital  $x$ . Finiteness of  $V$  will still be important benchmark for the upcoming analysis of  $V_{\mathcal{K}}$ .

**Remark 4.3.1.** Let Assumption 1 hold. For arbitrary acceptance sets  $\bar{\mathcal{K}}$  we get the following identity<sup>3</sup>

$$V_{\mathcal{K}}(\infty) = u(\infty)$$

*Proof.* Since  $V(x) \geq V_{\mathcal{K}}(x) \geq u(x)$  holds for all  $x > 0$  it suffices to verify the identity for  $V$  instead of  $V_{\mathcal{K}}$ .

We will verify that for each  $x > 0$  there exist  $\bar{x} > 0$  such that  $V(x) \leq u(\bar{x})$ . Since  $u$  is strictly increasing the inverse  $u^{-1}$  exist and  $\bar{x} = u^{-1}(V(x))$  satisfies the estimate. If  $u$  is unbounded above  $\bar{x}$  is obviously well defined. On the other hand when  $u$  is bounded above, we have

$$V(x) = \mathbb{E} \left[ \int_0^T u(\hat{c}_x(t)) dF(t) \right] < u(\infty) \quad (< \infty)$$

by (SA 1.2). Thus,  $\bar{x}$  is well defined. □

In the following we tie in with the duality considerations from Section 2. Instead of studying the dual of Problem 3 as we did lately, we first define the (real-valued) *dual value function* via

$$\tilde{V}_{\mathcal{K}}(z) := \sup_{x>0} V_{\mathcal{K}}(x) - xz$$

Note that according to Identity 4.3.2,  $V_{\mathcal{K}}(x) - xz \leq V_{L_+^0(\mathcal{M})}(x) - xz$  holds for all  $x, z > 0$ . Thus

$$\tilde{V}_{\mathcal{K}} \leq \tilde{V}_{L_+^0(\mathcal{M})} \tag{4.3.3}$$

holds for all acceptance sets  $\mathcal{K}$ .

---

<sup>3</sup>Notice that we abbreviate  $f(\infty) := \lim_{x \rightarrow \infty} f(x)$ .

**Remark 4.3.2.** According to standard arguments in convex analysis  $V_{\mathcal{K}}(x) = \inf_{z>0} \tilde{V}_{\mathcal{K}}(z) + xz$  holds for all  $x > 0$ , cf. Rockafellar (1970, V Theorem 23.5)

In the previous section we established the dual relation between  $V_{L_+^0(\mathcal{M})}$  and  $\tilde{V}_{L_+^0(\mathcal{M})}$ . Furthermore we have seen, that Assumption 1 is sufficient to imply this dual relation. Note that the corresponding abbreviation  $\tilde{V} = \tilde{V}_{L_+^0(\mathcal{M})}$  is allowed only if the dual relation between  $V$  and  $\tilde{V}$  holds.

As in the previous section we now show that Assumption 2 suffices to guarantee existence of an optimal primal strategy.

**Proposition 4.3.1.** *Let Assumption 2 hold. The optimal consumption strategy  $c_x^{\mathcal{K}} \in \mathcal{K}_V(x)$  exists and is  $p$ -a.s. unique. Furthermore the value function  $V_{\mathcal{K}}$  is finite, strictly increasing, strictly concave, and continuous on  $(0, \infty)$ .*

*Proof.* We first prove existence and uniqueness of an optimal primal strategy. Fix  $x > 0$ . Recall that Assumption 2 induces  $V < \infty$  (Assumption 1). Thus  $V_{\mathcal{K}}$  is finite as well and there exist a sequence  $\{c^n\}_{n \geq 1} \subseteq \mathcal{K}_V(x)$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c^n(t)) \, dF(t) \right] = V(x)$$

Recall Lemma 3.3.1. Thus, we find a sequence of convex combinations  $\tilde{c}^n \in \text{conv}(c^k \mid k \geq n)$  and an element  $\tilde{c} \in \mathcal{C}(x)$  such that  $\tilde{c}_n \rightarrow \tilde{c}$   $p$ -a.s.

According to the definition of  $\tilde{\mathcal{K}}$  (Axioms 2 and 3), we get  $\tilde{c} \in \tilde{\mathcal{K}}$ . Additionally  $\tilde{c}$  satisfies the budget constraint, thus  $\tilde{c} \in \mathcal{K}(x)$ . Using the same arguments as in the proof of Proposition 3.2.1, we show that  $c_x^{\mathcal{K}} := \tilde{c} \in \mathcal{K}_V(x)$  is optimal to  $V_{\mathcal{K}}(x)$  and unique in the sense of ‘ $\cong$ ’.

Thanks to Axiom 2 we may also derive strict concavity of  $V_{\mathcal{K}}$  as in Proposition 3.2.1.

For the remaining assertions let  $x_2 > x_1 > 0$ . The solutions of  $V_{\mathcal{K}}(x_1)$  and  $V_{\mathcal{K}}(x_2)$  will be denoted as  $c_1 \in \mathcal{K}_V(x_1)$  and  $c_2 \in \mathcal{K}_V(x_2)$ .

The remaining assertions follow from basic convex analysis. By definition  $V_{\mathcal{K}}$  is non-decreasing. Moreover  $V_{\mathcal{K}}$  must be strictly increasing, because otherwise this would contradict strict concavity of  $V_{\mathcal{K}}$ . Since a concave function is always continuous on its domain, we also have continuity of  $V_{\mathcal{K}}$  on  $(0, \infty)$ .  $\square$

---

We now discuss some further properties of the dual value function  $\tilde{V}_{\mathcal{K}}$ .

**Proposition 4.3.2.** *Let Assumption 2 hold, then  $\tilde{V}_{\mathcal{K}}$  is strictly decreasing, convex, continuous, and continuously differentiable on  $(0, \infty)$ . Furthermore for each  $z > 0$  there exist a  $x_{\mathcal{K}}^z > 0$  (namely  $x_{\mathcal{K}}^z = -\tilde{V}'_{\mathcal{K}}(z)$ ) such that*

$$\tilde{V}_{\mathcal{K}}(z) = V(x_{\mathcal{K}}^z) - zx_{\mathcal{K}}^z \quad (4.3.4)$$

*Proof.*  $V_{\mathcal{K}}$  is a concave function. According to Rockafellar (1970, III Theorem 12.2) the function  $\tilde{V}_{\mathcal{K}}$  is convex and finite by assumption  $\infty > \tilde{V} \geq \tilde{V}_{\mathcal{K}}$ . Thus  $\tilde{V}_{\mathcal{K}}$  is continuous on  $(0, \infty)$ .

Further  $V_{\mathcal{K}}$  is strictly concave. Thanks to Rockafellar (1970, V Theorem 26.3)  $\tilde{V}_{\mathcal{K}}$  is continuously differentiable on  $(0, \infty)$  and according to Rockafellar (1970, V Theorem 23.5)

$$\tilde{V}_{\mathcal{K}}(z) = V_{\mathcal{K}}(-\tilde{V}'_{\mathcal{K}}(z)) + z\tilde{V}'_{\mathcal{K}}(z) \quad \text{holds for all } z > 0$$

In particular  $x_{\mathcal{K}}^z := -\tilde{V}'_{\mathcal{K}}(z)$  fulfills (4.3.4). Furthermore  $\tilde{V}'_{\mathcal{K}} < 0$ , which in turn implies that  $\tilde{V}_{\mathcal{K}}$  is strictly decreasing.  $\square$

## 4.4 A Duality Approach for Constrained Consumption Selection

In the unconstrained setting the relation between the unconstrained value functions  $V$  and  $\tilde{V}$  is induced by a more general duality concept. For incomplete semimartingale models Kramkov and Schachermayer (1999) determined the dual value function to be the value function of a corresponding dual problem. This approach allowed them to discover properties of the value functions, we have not been able to derive. In this section we establish a very similar dual formulation for the constrained problem.

Recall that we implicitly defined the dual value function as

$$\tilde{V}_{\mathcal{K}}(z) = \sup_{x>0} \sup_{c \in \mathcal{C}_V(x)} \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] - zx$$



For the definition of the constrained dual problem, we need to define a  $(-\infty, \infty]$ -valued functional on the space  $L_+^1(\mathcal{M})$ .<sup>4</sup> We define a dual for the evaluation function via

$$\tilde{\mathcal{E}}_{\mathcal{K}} : h \longmapsto \sup_{c \in \mathcal{K}_V} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)h(t) \, dF(t) \right] \quad (4.4.1)$$

The function  $\tilde{\mathcal{E}}_{\mathcal{K}}$  will be called the  $\mathcal{K}$ -dual of  $\mathcal{E}$ .

Notice that this function slightly deviates from its analogon in the unconstrained setting where we had studied the functional

$$\mathbb{E} \left[ \int_0^T \underbrace{\left( \sup_{x(t) > 0} u(x(t)) - h(t)x(t) \right)}_{= \tilde{u}(h(t))} \, dF(t) \right] \quad (4.4.2)$$

**Remark 4.4.1.** The  $\mathcal{K}$ -dual  $\tilde{\mathcal{E}}_{\mathcal{K}}$  is a convex and decreasing functional.

Thus, we also consider consumption processes  $x > 0$  which may have an infinite price with respect to  $p$ , in the sense that

$$\mathbb{E} \left[ \int_0^T h(t)x(t) \, dF(t) \right] \quad (4.4.3)$$

may not be integrable.<sup>5</sup>

Making use of the  $\mathcal{K}$ -dual functional, we may introduce a suitable dual problem to  $V_{\mathcal{K}}(z)$  as follows.

**Problem 4** (Constrained Dual Problem).

$$\inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) \quad (4.4.4)$$

Unfortunately we will not be able to verify equality between (4.4.4) and  $\tilde{V}_{\mathcal{K}}(z)$  in general.

<sup>4</sup>Note that for each  $z > 0$  the set of dual variables  $\mathcal{Z}(z)$  is a subset of  $L_+^1(\mathcal{M})$ .

<sup>5</sup>Note that both processes  $x, h$  are non-negative. Thus, integrability reduces to finiteness of (4.4.3).

---

Since we aim for employing a Minimax Theorem for bounded sets, we will construct approximate dual functionals by cutting off the  $\mathcal{K}$ -dual functional. For  $n \geq 1$  we define *approximate dual functions*  $\tilde{\mathcal{E}}_n$ . We set

$$\mathcal{K}_n = \{g \in \bar{\mathcal{K}} \mid 0 \leq g(\omega, t) \leq n \text{ for all } \omega \in \Omega, t \in [0, T]\}$$

and

$$\begin{aligned} \tilde{\mathcal{E}}_n(Z) &:= \sup_{g \in \mathcal{K}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ &= \sup_{c \in \bar{\mathcal{K}}} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - Z(t)(c(t) \wedge n) \, dF(t) \right] \end{aligned}$$

Proving duality between Problems 3 and 4 a convergence behavior of the approximate duals is needed. For  $Z \in L_+^1(\mathcal{M})$  the following convergence property will be of great interest

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) = \tilde{\mathcal{E}}_{\mathcal{K}}(Z) \quad (4.4.5)$$

Note that this property is satisfied at least in the trivial cases, when  $\bar{\mathcal{K}} = \mathbb{R}_+$  and  $\bar{\mathcal{K}} = L_*^0(\mathcal{M})$ .

At this point we should advert to the special structure of  $\mathcal{K}_n$ . Therefore we introduce a special concept of compactness which originally has been introduced in Žitković (2010).

For an arbitrary set  $A$  we define  $\text{Fin}(A)$  as the set of all non-empty, finite subsets of  $A$ .

**Definition 3** (Žitković). *A convex subset  $E$  of a topological vector space is called convexly compact if for any non-empty set  $A$  and any family  $\{F_a\}_{a \in A}$  of closed, convex subsets of  $E$ , the condition*

$$\forall D \in \text{Fin}(A), \bigcap_{a \in D} F_a \neq \emptyset \quad \text{implies} \quad \bigcap_{a \in A} F_a \neq \emptyset.$$

Without the restriction that the sets  $\{F_a\}_{a \in A}$  must be convex, this definition would be equivalent to compactness in the original sense. Thus any

convex and compact set is compactly convex and Definition 3 really extends the concept of convexity (Cf. Žitković (2010, Example 2.2) for convex compactness without compactness).

By definition of convex compactness looks very unwieldy. Luckily Žitković derived an easy characterization on the space of non-negative, measurable functions.

**Theorem\* 5** (Žitković (2010, Theorem 3.1)). *A closed convex subset  $E \subseteq L_+^0(\mathcal{M})$  is compactly convex if and only if it is bounded in probability.*

This has immediate consequences on the truncated acceptance sets  $\mathcal{K}_n$ .

**Remark 4.4.2.** The sets  $\mathcal{K}_n$  consists of elements with radius  $n$  in  $L_+^\infty(\mathcal{M})$ , and in addition these sets are convexly compact.

For the most parts lower semicontinuity of  $\tilde{\mathcal{E}}_{\mathcal{K}}$  is of great importance. Note that in general  $c \in \bar{\mathcal{K}}$  is not  $p$  integrable. Thus we do not know wether the mapping

$$Z \mapsto \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

(resp.  $Z \mapsto \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)h(t) dF(t) \right]$ ) is lower semicontinuous on  $\mathcal{Z}(z)$  or not. Since the approximate dual functions  $\tilde{\mathcal{E}}_n$  increase in  $n$ , we get

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) = \sup_{n \geq 1} \tilde{\mathcal{E}}_n(Z)$$

Thus, if (4.4.5) holds, then  $\tilde{\mathcal{E}}_{\mathcal{K}}$  is the pointwise supremum of lower semicontinuous functions (cf. Remark 4.5.1). Particularly  $\tilde{\mathcal{E}}_{\mathcal{K}}$  is lower semicontinuous itself. This is a valuable observation proving existence of a minimizing dual variable.

**Theorem 5.** *Let Assumption 2 hold. If (4.4.5) holds for all  $Z \in \mathcal{Z}(z)$  with  $\tilde{\mathcal{E}}_{\mathcal{K}}(Z) < \infty$ , then*

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) \tag{4.4.6}$$

*Furthermore this infimum is attained by a process  $Z_z^{\mathcal{K}} \in \mathcal{Z}(z)$ .*

Finally we set up a situation which satisfies the claims of Theorem 5.

**Proposition 4.4.1.** *Let  $u$  be bounded below, then for all  $Z \in \mathcal{Z}(z)$  with  $\tilde{\mathcal{E}}_{\mathcal{K}}(Z) < \infty$  the  $K$ -dual  $\tilde{\mathcal{E}}_{\mathcal{K}}$  satisfies the convergence property claimed in Equation (4.4.5).*

*Proof.* By definition we know  $\tilde{\mathcal{E}}_n(Z) \leq \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$ , so

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) \leq \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

holds for all  $Z \in \mathcal{Z}(z)$ . Contrarily notice that

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) \geq \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - (c(t) \wedge n)Z(t) \, dF(t) \right]$$

holds for all  $c \in \mathcal{K}_V$  or equivalently

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) &\geq \sup_{c \in \mathcal{K}_V} \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - (c(t) \wedge n)Z(t) \, dF(t) \right] \\ &\geq \sup_{c \in \mathcal{K}_V} \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - c(t)Z(t) \, dF(t) \right] \end{aligned}$$

Thus, it remains to verify

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - c(t)Z(t) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - \mathbb{E} \left[ \int_0^T c(t)Z(t) \, dF(t) \right] \end{aligned}$$

for all  $c \in \mathcal{K}_V$  with  $\mathbb{E} \left[ \int_0^T c(t)Z(t) \, dF(t) \right] < \infty$ . We observe

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) - c(t)Z(t) \, dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T u(c(t) \wedge n) \, dF(t) \right] - \mathbb{E} \left[ \int_0^T c(t)Z(t) \, dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - \mathbb{E} \left[ \int_0^T c(t)Z(t) \, dF(t) \right] \end{aligned}$$

by applying the Monotone Convergence Theorem. Here boundedness of  $u$  is of great importance. This verifies

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z) \geq \sup_{c \in \mathcal{K}_V} \mathbb{E} \left[ \int_0^T u(c(t)) - c(t)Z(t) \, dF(t) \right] = \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

which in turn proves the assertion of this Lemma.  $\square$

**Example 9** (Non-negative Intertemporal Utility). Let Assumption 2 hold. If the intertemporal utility function  $u$  is bounded below (resp. non-negative), then  $\tilde{V}_{\mathcal{K}}$  satisfies the identity

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

---

## 4.5 Proofs of the Main Theorems in Section 4

Before we come to the poof of the main theorem, we list some general considerations.

**Remark 4.5.1.** Obviously

$$\sup_{Z \in \mathcal{Z}} \frac{\mathbb{E} \left[ \int_0^T g(t) Z(t) dF(t) \right]}{\mathbb{E} \left[ \int_0^T Z(t) dF(t) \right]} \leq n$$

holds for all  $g \in \mathcal{K}_n$ . In particular the mapping

$$Z \mapsto \mathbb{E} \left[ \int_0^T u(c(t)) - c(t) Z(t) dF(t) \right] \quad (4.5.1)$$

is lower semicontinuous on  $\mathcal{Z}(z)$  with respect to the  $L^1$  topology. Thus  $\tilde{\mathcal{E}}_n$  is as a point-wise supremum of lower semicontinuous functions lower semicontinuous as well.

Auxiliary we define

$$\tilde{V}_n(z) := \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_n(Z) \quad \text{resp.} \quad U(z) := \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

**Remark 4.5.2.** The functions  $\tilde{V}_n$  are decreasing and convex. Furthermore they increase in  $n$ .

**Lemma 4.5.1.** *Let (4.4.5) hold. For all  $z > 0$  the following identity holds*

$$\inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) = \lim_{n \rightarrow \infty} \tilde{V}_n(z)$$

*Proof.* Obviously

$$\tilde{\mathcal{E}}_n(Z) \leq \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

and  $\tilde{V}_n \leq U$  for all  $n \geq 1$ .

In the following we show  $\lim_{n \rightarrow \infty} \tilde{V}_n(z) \geq U(z)$ , such that

$$\lim_{n \rightarrow \infty} \tilde{V}_n(z) = \lim_{n \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_n(Z) = U(z) \quad (4.5.2)$$

Let  $\{Z^n\}_{n \geq 1} \subseteq \mathcal{Z}(z)$  be a sequence such that

$$\lim_{n \rightarrow \infty} \tilde{V}_n(z) = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z^n)$$

According to Delbaen and Schachermayer (1994, Lemma A1.1) we find a sequence  $\bar{Z}^n \in \text{conv}\{Z^k \mid k \geq n\}$  that converges  $dt \otimes \mathbb{P} - a.e.$  to some  $\bar{Z} \in L_+^0$ . Thanks to closeness of  $\mathcal{Z}(z)$  we get  $\bar{Z} \in \mathcal{Z}(z)$ .

Now convexity of  $\tilde{\mathcal{E}}_n$  induces

$$\tilde{\mathcal{E}}_n(\bar{Z}^n) = \tilde{\mathcal{E}}_n\left(\sum_{k=n}^N \lambda_k Z^k\right) \leq \sum_{k=n}^N \lambda_k \tilde{\mathcal{E}}_n(Z^k) \leq \max_{n \leq k \leq N} \tilde{\mathcal{E}}_n(Z^k)$$

where  $\sum_{k=n}^N \lambda_k Z^k$  denotes the convex combination of  $\bar{Z}^n$ . Since  $\tilde{\mathcal{E}}_n$  is increasing in  $n$  we further observe

$$\tilde{\mathcal{E}}_n(\bar{Z}^n) \leq \max_{n \leq k \leq N} \tilde{\mathcal{E}}_n(Z^k) \leq \tilde{\mathcal{E}}_{k^*}(Z^{k^*})$$

for some  $k^* \in \{n, \dots, N\}$ . Summarizing the last considerations, we end up with

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(Z^n) \geq \liminf_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(\bar{Z}^n) \geq \tilde{\mathcal{E}}_N\left(\liminf_{n \rightarrow \infty} \bar{Z}^n\right) = \tilde{\mathcal{E}}_N(\bar{Z})$$

for arbitrary fixed  $N \geq 1$ . Here we employed lower-semicontinuity of  $\tilde{\mathcal{E}}_n$  (cf. Remark 4.5.2). Since  $\bar{Z} \in \mathcal{Z}(z)$ , and (4.4.5) holds we finally observe that

$$\lim_{n \rightarrow \infty} \tilde{V}_n(z) \geq \lim_{N \rightarrow \infty} \tilde{\mathcal{E}}_N(\bar{Z}) = \tilde{\mathcal{E}}_{\mathcal{K}}(\bar{Z}) \geq U(z)$$

Notice that the last inequality holds by definition of  $U$ . □

As preparation for the upcoming proof we define

$$\partial\mathcal{K}(x) := \left\{ c \in \mathcal{K}(x) \mid \sup_{Z \in \mathcal{Z}} \mathbb{E} \left[ \int_0^T Z(t)c(t) dF(t) \right] = x \right\} \quad (4.5.3)$$

Obviously  $\partial\mathcal{K}(x) = x \partial\mathcal{K}(1)$  for all  $x > 0$  and

$$\bigcup_{x>0} \mathcal{K}(x) = \left( \bigcup_{x>0} \partial\mathcal{K}(x) \right) \quad p - a.s. \quad (4.5.4)$$

---

**Lemma 4.5.2.** For all  $z > 0$  we get

$$\begin{aligned} & \sup_{x>0} V_{\mathcal{K}}(x) - xz \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \end{aligned}$$

*Proof.* We start with proving the fact

$$\begin{aligned} & \sup_{x>0} \sup_{c \in \mathcal{K}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \quad (4.5.5) \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \end{aligned}$$

The first inequality " $\geq$ " is a consequence of Axiom 4, resp.  $\frac{x}{T}\mathbf{1} \in \mathcal{K}(x)$ . Note that this assumption implies  $\mathcal{K}_n \subseteq \mathcal{K}(Tn)$ .

The converse inequality " $\leq$ " is true, because for fixed  $x > 0$

$$\begin{aligned} & \sup_{c \in \mathcal{K}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ &= \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}(x) \cap \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ &\leq \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \end{aligned}$$

Now fix  $z > 0$  and we observe

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ (4.5.5) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \mathcal{K}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - Z(t)c(t) \, dF(t) \right] \\ (4.5.3) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \partial \mathcal{K}(x)} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(c(t)) - Z(t)c(t) \, dF(t) \right] \\ &= \sup_{x>0} \sup_{c \in \partial \mathcal{K}(x)} \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - xz \\ (4.5.4) \quad & \stackrel{=}{=} \sup_{x>0} \sup_{c \in \mathcal{K}(x)} \mathbb{E} \left[ \int_0^T u(c(t)) \, dF(t) \right] - xz \\ &= \sup_{x>0} V_{\mathcal{K}}(x) - xz \end{aligned}$$

which proves the assertion. □



### Proof of Theorem 5

Again a Minimax-Theorem is the key to prove the dual relation between the optimization problem in Equation (4.4.6) and Problem 3. This time we will employ the following Minimax-Theorem for convexly compact sets.

**Theorem\* 6** (Kauppila (2010, Theorem A.1)). *Let  $E$  be a non-empty subset of a topological vector space, and  $F$  a non-empty, closed, convex, and convexly compact subset of a topological vector space. Let  $h: E \times F \rightarrow \mathbb{R}$  be convex on  $E$ , and concave and upper-semicontinuous on  $F$ . Then*

$$\inf_{x \in E} \sup_{y \in F} h(x, y) = \sup_{y \in F} \inf_{x \in E} h(x, y)$$

This Minimax-Theorem will replace the Kneser-Fan Minimax-Theorem we applied in Section 3.3. Moreover it will become important later when we study ratcheting investors (cf. Example 5).

*Proof.* The proof of the first assertion is almost given employing the previous Lemmata. Recall that for arbitrary  $z > 0$

$$\begin{aligned} & \tilde{V}_{\mathcal{K}}(z) \\ \stackrel{L 4.5.1}{=} & \lim_{n \rightarrow \infty} \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{K}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ = & \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - Z(t)g(t) \, dF(t) \right] \\ \stackrel{L 4.5.2}{=} & \sup_{x > 0} V_{\mathcal{K}}(x) - xz \end{aligned}$$

As we already mentioned we can verify the second equality employing Theorem\* 6. According to Remark 4.4.2 and Lemma 3.3.4  $\mathcal{K}_n$ ,  $\mathcal{Z}$ , and the functional fulfill all requirements to apply this theorem. Thus

$$\begin{aligned} & \sup_{g \in \mathcal{K}_n} \inf_{Z \in \mathcal{Z}(z)} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \\ = & \inf_{Z \in \mathcal{Z}(z)} \sup_{g \in \mathcal{K}_n} \mathbb{E} \left[ \int_0^T u(g(t)) - g(t)Z(t) \, dF(t) \right] \end{aligned} \quad (4.5.6)$$

for all  $n \geq 1$ , which finally proves  $\tilde{V}_{\mathcal{K}}(z) = \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$ .

---

In the following we construct a variable  $Z^* \in \mathcal{Z}(z)$  minimizing the expression

$$\inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z)$$

Choose a sequence  $\{Z^n\}_{n \geq 1} \subseteq \mathcal{Z}(z)$  such that

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_{\mathcal{K}}(Z^n) = \tilde{V}_{\mathcal{K}}(z)$$

According to Kramkov and Schachermayer (1999, Lemma A.1), there exist a sequence of convex combinations  $\bar{Z}^n \in \text{conv}\{Z^k \mid k \geq n\}$  that converges to some process  $\bar{Z} \in L_+^0$ . Thanks to completeness  $\bar{Z} \in \mathcal{Z}(z)$  as well. We will show that  $\bar{Z}$  satisfies the desired.

Recall that  $\tilde{\mathcal{E}}_{\mathcal{K}}$  is convex, thus

$$\tilde{\mathcal{E}}_{\mathcal{K}}(\bar{Z}_n) = \tilde{\mathcal{E}}_{\mathcal{K}}\left(\sum_{k=n}^N \lambda_k Z^k\right) \leq \sum_{k=n}^N \lambda_k \tilde{\mathcal{E}}_{\mathcal{K}}(Z^k) \leq \max_{n \leq k \leq N} \tilde{\mathcal{E}}_{\mathcal{K}}(Z^k) \quad (4.5.7)$$

where  $\sum_{k=n}^N \lambda_k Z^k$  denotes the convex combination of  $\bar{Z}^n$ . Consequently

$$\tilde{\mathcal{E}}_{\mathcal{K}}(\bar{Z}) \leq \liminf_{n \rightarrow \infty} \tilde{\mathcal{E}}_{\mathcal{K}}(\bar{Z}^n) \stackrel{(4.5.7)}{\leq} \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_{\mathcal{K}}(Z^n) = \tilde{V}_{\mathcal{K}}(z)$$

holds. Recall that (4.4.5) induces the lower semicontinuity of  $\tilde{\mathcal{E}}_{\mathcal{K}}$ , which is an important property for the first inequality.  $\square$

## Chapter 5

# Applications I: *Complete Markets and Artificial Markets*

In this section we state stronger assumption on the distribution function  $F$ . Thus employing the techniques from convex analysis and the theory derived in the previous sections we are able to get some sharper assertions.

More precisely we consider the dual problem on complete markets, we characterize the set of dual variable using discounted density processes of equivalent martingales only, and we restate the dual problem in the line of Cvitanić and Karatzas (1992).

### 5.1 Unconstrained Consumption Selection when Intertemporal Utility is Time-Separable

In this section we will restore the dual problem. We show that the set of equivalent martingale measures suffices to set up an appropriate dual function, although the dual minimizer may not be contained within this set. Therefore we restrict our attention to distribution processes  $F$ , which have a density with respect to the Lebesgue measure  $dt$ .

---

**Assumption 4.**

$$F(t) = \int_0^t f(u) du$$

Here  $f$  denotes a right-continuous,  $\mathcal{F}$ -adapted process with  $f(t) \geq 0$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ .

This setup includes time-separable intertemporal utility, e.g.

**Example 10** (Discounting of Utility). Let  $\delta > 0$ . Consider the following distribution function

$$F(t) = \int_0^T \beta e^{-\delta t} dt \quad \text{for } t \in (0, T]$$

with  $\beta = \delta(1 - e^{-\delta T})^{-1} > 0$ . This distribution corresponds to an investor with an exponential and time-separable utility function, i.e.

$$\mathbb{E} \left[ \int_0^T e^{-\delta t} \beta u(c(t)) dt \right]$$

As mentioned in the introduction, under Assumption 4 our model resembles the models in Karatzas and Žitković (2003) and Störmer (2010). We are able to derive corresponding results although we do not have to look at changes in asymptotic elasticity over time. Note that Assumptions on asymptotic elasticity over time cause implicitly assumptions on the underlying preference structure. Particularly this influences the investors attitude to risk.

Claiming Assumption 4 in our model *absolute risk aversion* (resp. *relative risk aversion*) is time-independent and fully described by  $u$ . Moreover note that Standing Assumption 1 does not influence the attitude to risk.

For the forthcoming analysis it makes sense to define the following right-continuous,  $(0, \infty]$ -valued adapted process

$$f^*(\omega, t) = \begin{cases} \frac{1}{f(\omega, t)} & \text{if } f(\omega, t) > 0 \\ \infty & \text{else} \end{cases} \quad (5.1.1)$$

Stating the convention  $0 \cdot \infty = 0$ , we get

$$ff^* = f^*f = \mathbf{1}_{\{f>0\}}$$

This leads us to a special set of dual variables. These dual variables will play the same role as the set of equivalent martingale measures  $\hat{\mathcal{Y}}^e$  in the case of utility maximization from terminal wealth (Kramkov and Schachermayer, 1999). For  $z > 0$  we define

$$\mathcal{Z}^e(z) = \left\{ \mathbf{1}_{\{f>0\}} \bar{z} f^* H \mid \text{where } \bar{z} \leq z \text{ and } H \in \hat{\mathcal{Y}}^e \right\} \quad (5.1.2)$$

**Remark 5.1.1.** Under Assumptions 3 and 4 we then have

$$\mathcal{Z}^e(z) \subseteq \mathcal{Z}^*(z) \quad \text{for all } z > 0$$

*Proof.* Let  $c \in \mathcal{C}$ ,  $H \in \hat{\mathcal{Y}}^e$  and  $\bar{z} \leq z$ , then

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \bar{z} \mathbf{1}_{\{f>0\}} f^*(t) H(t) c(t) dF(t) \right] &= \mathbb{E} \left[ \int_0^T \bar{z} \mathbf{1}_{\{f>0\}} H(t) c(t) dt \right] \\ &\leq \bar{z} \underbrace{\mathbb{E} \left[ \int_0^T H(t) c(t) dt \right]}_{\leq 1} \leq z \end{aligned}$$

Thus  $\bar{z} \mathbf{1}_{\{f>0\}} f^*(t) H(t) \in \mathcal{Z}(z)$  and by definition  $\bar{z} \mathbf{1}_{\{f>0\}} f^*(t) H(t) \in \mathcal{Z}^*(z)$  as well.  $\square$

For the remaining part of this section we state the following stronger assumption. We claim that the density process  $F$  is equivalent to the Lebesgue measure in the following sense.

**Assumption 5.**

$$F(t) = \int_0^t f(u) du \quad (5.1.3)$$

Here  $f$  denotes a càdlàg process with  $f(t) > 0$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ .

---

The set of dual variables  $\mathcal{Z} (= \mathcal{C}^e)$  is given as

$$\mathcal{Z} = \left\{ Z \in L_+^0(\mathcal{M}) \mid \sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^T f(t) Z(t) c(t) dt \right] \leq 1 \right\}$$

while the set of discounted equivalent martingale measures is defined as

$$\mathcal{Z}^e = \left\{ \lambda \frac{1}{f} H^Q \mid H^Q \in \hat{\mathcal{Y}}^e \text{ and } 0 \leq \lambda \leq 1 \right\} \quad (\subseteq \mathcal{Z})$$

Moreover under Assumption 5 the processes  $Z \in \mathcal{Z}^*$  are strictly positive  $dt \otimes \mathbb{P} - a.e.$  as well. According to Delbaen and Schachermayer (1994) the set  $\mathcal{Z}^e$  (resp.  $\hat{\mathcal{Y}}^e$ ) is closed under countable convex combinations. The following result will point out the importance of the set of equivalent martingale measures  $\mathfrak{M}$ .

**Proposition 5.1.1.** *Let Assumption 5 hold. The set of discounted equivalent martingale measures  $\mathcal{Z}^e$  satisfies the following*

$$\sup_{H \in \mathcal{Z}^e} \mathbb{E} \left[ \int_0^T f(t) H(t) c(t) dt \right] = \sup_{Z \in \mathcal{Z}} \mathbb{E} \left[ \int_0^T f(t) Z(t) c(t) dt \right]$$

holds for all  $c \in L_*^0(\mathcal{M})$ .

*Proof.* Fix  $x > 0$ . Notice that

$$\begin{aligned} \mathbb{E}_Q \left[ \int_0^T c(t) dt \right] &= \mathbb{E} \left[ \int_0^T H^Q(t) c(t) dt \right] \\ &= \mathbb{E} \left[ \int_0^T f(t) \left( \frac{1}{f}(t) H^Q(t) \right) c(t) dt \right] \end{aligned}$$

and  $\frac{1}{f} H^Q \in \mathcal{Z}^e$ . Hence the assertion follows directly from Proposition 2.3.3.  $\square$

We now come to the new result on unconstrained consumption choice. We will see that the set of discounted equivalent martingale measures suffices to define the dual problem, although the dual minimizer may not be contained in  $\mathcal{Z}^e$ .

**Theorem 6.** *Let Assumptions 2 and 5 hold, then*

$$\tilde{V}(z) = \inf_{Q \in \mathcal{M}} \mathbb{E} \left[ \int_0^T f(t) \tilde{u} \left( z \frac{1}{f}(t) H^Q(t) \right) dt \right]$$

*In particular  $\inf_{H \in \mathcal{Z}^e} \mathbb{E} \left[ \int_0^T f(t) \tilde{u} \left( z H(t) \right) dt \right] < \infty$  for all  $z > 0$ .*

## 5.2 Unconstrained Consumption Selection on Selected Complete Markets

In this section we aim for restoring Theorem 1 on complete markets. We abstain from market claiming existence of *multiple* equivalent martingale measures and postulate the existence of a unique equivalent martingale measure.

**Assumption 6.**

$$\mathfrak{M} = \{\mathbb{P}_*\}$$

With  $\mathbb{E}_*$  we denote the expectation with respect to  $\mathbb{P}_*$ , and let  $H_*$  denote the corresponding density process. Notice that Assumption 6 induces  $\hat{\mathcal{Y}}^e = \{H_*\}$ . Moreover we may simplify the pricing formula and the set of admissible rate of consumption processes changes as follows.

**Proposition 5.2.1.** *Let Assumption 6 hold, then*

$$\mathcal{C}(x) = \left\{ c \in L_*^0(\mathcal{M}) \mid \mathbb{E}_* \left[ \int_0^T c(t) dt \right] \leq x \right\} \quad (5.2.1)$$

*Proof.* Recall that by Theorem\* 1 the identity  $\hat{\mathcal{Y}} = (\hat{\mathcal{Y}}^e)^{\times\times}$  holds. So, according to the Filtered Bipolar Theorem of Žitković (2002, Theorem 2),  $\hat{\mathcal{Y}}$  is the fork-convex and process-solid hull of  $\hat{\mathcal{Y}}^e$ , i.e.

$$\hat{\mathcal{Y}} = \{AH_* \mid \text{for } A \in \mathcal{V}\}$$

Note that  $H_* \geq AH_*$  for all processes  $A \in \mathcal{V}$ , which induces

$$\mathbb{E} \left[ \int_0^T A(t)H_*(t)c(t) dt \right] \leq \mathbb{E} \left[ \int_0^T H_*(t)c(t) dt \right]$$

holds for all  $A \in \mathcal{V}$ . Thus

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^T H_*(t)c(t) dt \right]$$

holds for all  $c \in \mathcal{C}$ . □

---

Obviously  $\mathcal{C}(x) = \left\{ c \in L_*^0(\mathcal{M}) \mid \mathbb{E}_* \left[ \int_0^T \frac{1}{S_0}(t)c(t) dt \right] \leq x \right\}$  if  $\mathcal{S}_0 \neq 1$ .

Let Assumption 4 hold and recall (5.1.2). Aiming for a nice formula for the dual problem  $\tilde{V}$  we define the following two processes

$$H_z^* := z \mathbf{1}_{\{f>0\}} f^* H_* \in \hat{\mathcal{Z}}^e(z) \quad \text{and} \quad I_z^* := I(z f^* H_*) \quad (5.2.2)$$

Recall that  $I(\infty) = 0$ , thus  $\mathbf{1}_{\{f>0\}} I_z^* = I_z^*$  for  $z > 0$ . Moreover

$$\mathbf{1}_{\{f>0\}} \tilde{u}(H_z^*) = \mathbf{1}_{\{f>0\}} (u(I_z^*) - H_z^* I_z^*) \quad (5.2.3)$$

holds  $p - a.s.$  Here  $f^*$  is as defined in (5.1.1).

**Remark 5.2.1.** For all  $z > 0$  the following holds

$$\mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) dF(t) \right] > -\infty$$

*Proof.* Recall that  $p$  defines a probability measure, thus we may apply Jensen's inequality with respect to  $p$ . Since  $\tilde{u}$  is concave we get

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) dF(t) \right] &\geq \tilde{u} \left( \mathbb{E} \left[ \int_0^T H_z^*(t) dF(t) \right] \right) \\ &= \tilde{u} \left( \mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} z H_*(t) dt \right] \right) \\ &\geq \tilde{u}(zT) \quad (> -\infty) \end{aligned}$$

Here the last inequality holds because  $\tilde{u}$  is strictly decreasing. Additionally

$$\mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} z H_*(t) dt \right] = z \mathbb{E}_* \left[ \int_0^T \mathbf{1}_{\{f>0\}} dt \right] \leq zT$$

holds, because of Fubini's Theorem, Bayes rule for stochastic processes, and Proposition 2.3.4.  $\square$

In the following we will show that the processes defined in (5.2.2) will solve the optimization problems  $V$  and  $\tilde{V}$ . In Theorem 6 we already verified the formula

$$\tilde{V}(z) = \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(z \frac{1}{f}(t) H_*(t)) dt \right] \quad (5.2.4)$$

We will see, when markets are complete, this identity even holds under less restrictive assumptions. In particular we will not claim finiteness of  $\tilde{V}$ .



**Proposition 5.2.2.** *Let Assumptions 4 and 6 hold, then*

$$\tilde{V}(z) = \mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) dF(t) \right] \quad \text{for all } z > 0 \quad (5.2.5)$$

Thus  $\tilde{V}$  is strictly convex and strictly decreasing on its domain.

Furthermore we may define the set of dual variables via

$$\mathcal{Z}(z) = \{Z \preceq z f^* H_*\} \quad (= \text{solid}\{z f^* H_*\})$$

Notice that all these  $[0, \infty]$ -valued random variables are finite  $p - a.s.$

*Proof.* Fix  $Z \in \mathcal{Z}$ . We define  $A := \{\mathbf{1}_{\{f>0\}} Z > H_1^*\}$  such that  $\mathbf{1}_{\{f>0\}} \mathbf{1}_A = \mathbf{1}_A$ , and we abbreviate  $a := \mathbb{E}_0 \left[ \int_0^T \mathbf{1}_A dt \right]$ .

We will prove the identity (5.2.5) by constructing a contradiction.

Therefor assume that  $A \neq \emptyset$ , which in turn implies  $a > 0$ . Note that

$$\mathbb{E}_0 \left[ \int_0^T a^{-1} \mathbf{1}_A dt \right] = 1$$

Thus,  $\mathbf{1}_A \geq 0$  in addition to Proposition 5.2.1 induces  $a^{-1} \mathbf{1}_A \in \mathcal{C}$ . Furthermore we get

$$\mathbb{E} \left[ \int_0^T Z(t) a^{-1} \mathbf{1}_A dF(t) \right] > a^{-1} \mathbb{E} \left[ \int_0^T H_1^*(t) \mathbf{1}_A dF(t) \right]$$

because  $Z \mathbf{1}_A \geq H_1^* \mathbf{1}_A$   $p - a.s.$  and  $Z \mathbf{1}_A > H_1^* \mathbf{1}_A$  on the set  $A$  which has strictly positive measure by assumption. Finally

$$a^{-1} \mathbb{E} \left[ \int_0^T H_1^*(t) \mathbf{1}_A dF(t) \right] = a^{-1} \mathbb{E} \left[ \int_0^T \underbrace{\mathbf{1}_{\{f>0\}} \mathbf{1}_A}_{=\mathbf{1}_A} H_*(t) dt \right] = 1$$

which contradicts the assumption  $Z \in \mathcal{Z}$ .

W.l.o.g.  $\tilde{V}(z) = \mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) dF(t) \right]$  holds for all  $z > 0$ . Thus  $\tilde{V}$  is strictly increasing and strictly convex by definition of  $H_z^*$  and  $\tilde{u}$ .  $\square$

The later Proposition enables us to extend the assertions of Theorem 1. We define

$$z_0 = \inf \left\{ z > 0 \mid \tilde{V}(z) < \infty \right\} \quad \text{and} \quad x_0 = \lim_{z \searrow z_0} -\tilde{V}'(z) \quad (5.2.6)$$

---

**Theorem 7.** *Let the Assumptions 1, 4 and 6 hold. Additional to the assertions of Theorem 1 we have*

(i) *The function  $V$  is continuously differentiable on  $(0, \infty)$  and strictly concave on  $(0, x_0)$  and the function  $\tilde{V}$  is continuously differentiable and strictly convex on  $(z_0, \infty)$ .*

(ii) *Let  $x \in (0, x_0)$  and  $z \in (z_0, \infty)$  with  $z = V'(x)$ , then  $I_z^* \in \mathcal{C}(x)$ . Moreover the solution to the primal problem  $V(x)$  satisfies*

$$\hat{c}_x = I_z^* \quad p - a.s. \text{ for all } t \in [0, T]$$

(iii) *For  $x \in (0, x_0)$  and  $z \in (z_0, \infty)$  the following identities hold*

$$V'(x) = \mathbb{E} \left[ \int_0^T \frac{\hat{c}_x(t) u'(\hat{c}_x(t))}{x} dF(t) \right] \quad (5.2.7)$$

$$\tilde{V}'(z) = \mathbb{E} \left[ \int_0^T \frac{H_z^*(t) \tilde{u}'(H_z^*(t))}{z} dF(t) \right] \quad (5.2.8)$$

### 5.3 Artificial Markets and Secondary Topics in Duality Theory For Constrained Consumption Selection Incomplete Markets

In this section we are aiming for a duality theorem in the spirit of Cvitanić and Karatzas (1992, Theorem 10.1). Therefore we will analyze the constrained consumption choice problem within an complete market setting. Furthermore we establish a helpful result for unconstrained consumption choice when markets are incomplete.

Unfortunately we cannot handle the general case. Whenever necessary we restrict our analysis to the situation, where the intertemporal utility function  $u$  is bounded from below. We claim Assumption 6 again, i.e.

$$\mathfrak{M} = \{\mathbb{P}_*\}$$

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As derived lately the set of consumption rates permissible for capital  $x > 0$  is

$$\mathcal{K}(x) = \left\{ c \in \bar{\mathcal{K}} \mid \mathbb{E}_* \left[ \int_0^T c(t) dt \right] \leq x \right\}$$

and the set of dual variables satisfies

$$\mathcal{Z}(z) = \left\{ Z \in L_+^0(\mathcal{M}) \mid Z \leq z \frac{1}{f} H_* \right\}$$

We also established the formula

$$\tilde{V}(z) = \mathbb{E} \left[ \int_0^T f(t) \tilde{u} \left( z \frac{1}{f}(t) H_*(t) \right) dt \right] \quad \text{for } z > 0$$

This identity holds even if  $\tilde{V}(z)$  is not finite. Furthermore if  $\tilde{V}$  is finite

$$\tilde{V}(z) = \tilde{\mathcal{E}} \left( z \frac{H_*}{f}(t) \right) \tag{5.3.1}$$

holds for all  $z > 0$ .<sup>1</sup> This raises the question if we can find a similar relation for arbitrary acceptance sets  $\bar{\mathcal{K}}$ .

Obviously

$$u(c(t)) - c(t)Z_1(t) \leq u(c(t)) - c(t)Z_2(t)$$

holds  $p - a.s.$  whenever  $Z_1 \geq Z_2$  holds  $p - a.s.$  Thus

$$\inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) = \tilde{\mathcal{E}}_{\mathcal{K}} \left( z \frac{1}{f} H_* \right)$$

and we only need to verify  $\tilde{V}_{\mathcal{K}}(z) = \tilde{\mathcal{E}}_{\mathcal{K}} \left( z \frac{1}{f} H_* \right)$ .

In the following we derive an analogon to Equation (5.3.1) for the more general  $\mathcal{K}$ -dual  $\tilde{\mathcal{E}}_{\mathcal{K}}$ .

**Proposition 5.3.1.** *Let Assumptions 5 and 6 hold. Additionally we assume that the intertemporal utility function  $u$  is bounded below. Then for all  $z > 0$*

$$\tilde{V}_{\mathcal{K}}(z) = \tilde{\mathcal{E}}_{\mathcal{K}} \left( z \frac{H_*}{f} \right)$$

*holds.*

---

<sup>1</sup>Here  $\tilde{\mathcal{E}}$  denotes the  $\mathcal{K}$ -dual for  $\bar{\mathcal{K}} = L_*^0(\mathcal{M})$ .

---

*Proof.* The proof of the upcoming Proposition 5.3.2 can be easily transferred to this situation.  $\square$

For the remaining part of this section we abstain from claiming existence of one unique equivalent martingale measure. Let  $H \in \mathcal{Z}(1)$  be a dual process with

$$\sup_{c \in \mathcal{C}} \mathbb{E} \left[ \int_0^T f(t)c(t)H(t) dt \right] = 1 \quad (5.3.2)$$

The set of all dual processes satisfying (5.3.2) will be denoted as  $\partial\mathcal{Z}(1)$ . We easily construct an *artificial (complete) market*  $\mathfrak{M}_H$  induced by  $H$  as we assume that  $H$  is the density process of a ‘unique equivalent martingale measure’ in the sense that  $\hat{\mathcal{Y}}^e = \{H\}$ .

Notice that in general  $H$  is not equivalent to the Lebesgue measure  $dt$ . Thus this artificial market corresponds to the markets we studied in Section 5.1. Moreover utility maximization takes place with respect to the pricing formula

$$\mathbb{E} \left[ \int_0^T f(t)H(t)c(t) dt \right] \leq x \quad \text{for } c \in \bar{\mathcal{K}}_V \quad (5.3.3)$$

The set of all processes  $c \in \mathcal{K}_V$  satisfying the budget constraint (5.3.3) will be denoted as  $\mathcal{K}_H(x)$ . That the value function corresponding to the problem

$$V_{\mathcal{K}}^H(x) = \sup_{c \in \mathcal{K}_H(x)} \mathbb{E} \left[ \int_0^T f(t)\tilde{u}(c(t)) dt \right]$$

may not be finite even if the original value function  $V_{\mathcal{K}}$  is. We easily construct such a situation when  $u$  is not bounded above and  $H \notin \mathcal{Z}^*$ .<sup>2</sup>

According to the general theory the set of dual variables is given as

$$\mathcal{Z}^H(1) = \left\{ Z \in L_+^0(\mathcal{M}) \mid Z \preceq \frac{1}{f}H \right\}$$

We define the dual value function as usual for constrained consumption selection via

$$\tilde{V}_{\mathcal{K}}^H(z) = \sup_{x>0} V_{\mathcal{K}}^H(x) - xz \quad \text{for all } z > 0$$

---

<sup>2</sup>In Section 3.1 we defined the set  $\mathcal{Z}^*$  of dual variables  $Z > 0$   $F \otimes dt - a.s.$

**Proposition 5.3.2.** *Let Assumptions 5 and 6 hold. Furthermore we assume that the intertemporal utility function  $u$  is bounded below. Now if  $H \in \partial\mathcal{Z}(1)$  with  $V_{\mathcal{K}}^H < \infty$ , then*

$$\tilde{V}_{\mathcal{K}}^H(z) = \tilde{\mathcal{E}}_{\mathcal{K}}\left(z\frac{1}{f}H\right) \quad \text{for all } z > 0$$

*Proof.* Although we need only standard arguments, we sketch a proof for this assertion.

$$\tilde{V}_{\mathcal{K}}^H(z) = \sup_{x>0} V_{\mathcal{K}}^H(x) - xz = \sup_{x>0} \sup_{c \in \mathcal{K}_H(x)} \mathbb{E} \left[ \int_0^T f(t)u(c(t)) dt \right] - xz$$

With the standard arguments we obtain

$$\begin{aligned} & \sup_{x>0} \sup_{c \in \mathcal{K}_H(x)} \mathbb{E} \left[ \int_0^T f(t)u(c(t)) dt \right] - xz \\ &= \sup_{x>0} \sup_{c \in \mathcal{K}_H(x)} \mathbb{E} \left[ \int_0^T f(t)u(c(t)) - f(t)zH(t)c(t) dt \right] \\ &= \lim_{n \rightarrow \infty} \underbrace{\sup_{g \in \mathcal{K}_n} \mathbb{E} \left[ \int_0^T f(t)u(g(t)) - zf(t)H(t)g(t) dt \right]}_{= \tilde{\mathcal{E}}_n(zH)} \end{aligned}$$

where the last equality holds for the usual reasons. Thanks to boundedness of  $u$  we may apply (4.4.5) and

$$\tilde{V}_{\mathcal{K}}^H(z) = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(zH) = \tilde{\mathcal{E}}_{\mathcal{K}}(zH)$$

holds, which proves the Proposition. □

Inspired by Cvitanic and Karatzas (1992), we will point out a relation, connecting the dual value function with the dual value functions of the artificial complete markets.

Employing artificial markets, Theorem 6 may be read as follows

$$\tilde{V}(z) = \inf_{H \in \partial\mathcal{Z}^e(1)} \tilde{V}^H(z)$$

For constrained consumption selection we finally derive a similar identity.

---

**Corollary 5.3.3.** *Let Assumptions 2 and 5 hold. Further we assume that the intertemporal utility function  $u$  is bounded below. For all  $z > 0$ , the dual value function is given as*

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{H \in \partial \mathcal{Z}(1)} \tilde{V}_{\mathcal{K}}^H(z)$$

This assertion is nice a reformulation of Theorem 5.

Notice that infimum is attained by a dual variable  $H \in \mathcal{H}_z$ , here  $\mathcal{H}_z$  denotes the set of  $H \in \partial \mathcal{Z}(1)$  with  $\tilde{\mathcal{E}}_{\mathcal{K}}(zH) < \infty$ .<sup>3</sup> Moreover we see, that  $\tilde{V}_{\mathcal{K}} \leq \tilde{V}_{\mathcal{K}}^H$  holds for all  $H \in \partial \mathcal{Z}(1)$  and

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{H \in \mathcal{H}} \tilde{V}_{\mathcal{K}}^H(z) \quad \text{holds for all } \mathcal{H} \supseteq \mathcal{H}_z$$

*Proof.* Making use of the  $L_+^0$ -monotonicity of  $\tilde{\mathcal{E}}_{\mathcal{K}}$  and  $\mathcal{Z}(z) = z\mathcal{Z}(1)$  for  $z > 0$ , we get

$$\inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) = \inf_{H \in \partial \mathcal{Z}(1)} \tilde{\mathcal{E}}_{\mathcal{K}}(zH)$$

According to Proposition 4.4.1,

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{K}}(Z) = \inf_{H \in \partial \mathcal{Z}(1)} \tilde{\mathcal{E}}_{\mathcal{K}}(z \frac{1}{f} H)$$

holds. Notice that

$$\begin{aligned} \tilde{\mathcal{E}}_{\mathcal{K}}\left(z \frac{1}{f} H\right) &= \sup_{c \in \mathcal{K}_V} \mathbb{E} \left[ \int_0^T u(c(t)) - z \frac{1}{f} H(t) c(t) \, dF(t) \right] \\ &\geq \sup_{c \in \mathcal{K}_H} \mathbb{E} \left[ \int_0^T u(c(t)) - z \frac{1}{f} H(t) c(t) \, dF(t) \right] \geq V_{\mathcal{K}}^H(x) - xz \end{aligned}$$

holds for all  $x > 0$ . Thus we are allowed to employ Proposition 5.3.2 on  $H \in \mathcal{H}_z$ , which in turn implies

$$\tilde{V}_{\mathcal{K}}(z) = \inf_{H \in \mathcal{H}_z} \tilde{\mathcal{E}}_{\mathcal{K}}\left(z \frac{1}{f} H\right) = \inf_{H \in \mathcal{H}_z} \tilde{V}_{\mathcal{K}}^H(z) = \inf_{H \in \partial \mathcal{Z}(1)} \tilde{V}_{\mathcal{K}}^H(z)$$

Thus the assertion holds. □

---

<sup>3</sup>Assumption 2 guarantees that  $\mathcal{H} \neq \emptyset$ .

## 5.4 Proofs of the Main Theorems in Section 5

### 5.4.1 Proofs of Section 5.1

**Lemma 5.4.1.**

$$(\mathcal{Z}^e)^{\circ\circ} = \text{solid}(\overline{\mathcal{Z}^e})$$

Here  $\overline{\mathcal{Z}^e}$  denotes the closure with respect to convergence in probability  $p$ .

*Proof.* By Kramkov and Schachermayer (1999)  $\mathcal{Z}^e$  is closed under countable convex combinations, consequently  $\text{solid}(\overline{\mathcal{Z}^e})$  is solid and convex, and in addition  $\mathcal{Z}^e \subseteq \text{solid}(\overline{\mathcal{Z}^e}) \subseteq (\mathcal{Z}^e)^{\circ\circ}$ . The Bidual Theorem of Brannath and Schachermayer yields that  $(\mathcal{Z}^e)^{\circ\circ}$  is the smallest closed, convex and solid set in  $L_+^0(\mathcal{M})$  containing  $\mathcal{Z}^e$ . Hence we only need to show closeness of  $\text{solid}(\overline{\mathcal{Z}^e})$ .

For each sequence  $\{Z_n\} \subseteq \text{solid}(\overline{\mathcal{Z}^e})$  there exist a sequence  $\{H_n\}_{n \geq 1}$  such that  $Z_n \leq H_n$   $p - a.s.$  for all  $n \geq 1$ . Transferring into duals yields  $(\mathcal{Z}^e)^\circ \supseteq \mathcal{Z}^\circ = \text{solid}(\mathcal{C})$ . Thus  $\gamma_1 \mathbf{1} \in (\mathcal{Z}^e)^\circ$  and  $(\mathcal{Z}^e)^{\circ\circ}$  is bounded in  $L_+^1$ , which induces that  $\{H_n\}_{n \geq 1}$  is bounded as well.

With the help of Komlós Theorem (Komlós, 1967) we deduce the existence of a subsequence, again denoted as  $\{H_n\}_{n \geq 1}$ , which Cesaro-converges to a process  $\bar{H} \in L_+^0(\mathcal{M})$ , i.e.

$$\bar{H}_n := \sum_{k=1}^n \frac{H_k}{n} \xrightarrow[n \rightarrow \infty]{} \bar{H}$$

Moreover since  $\overline{\mathcal{Z}^e}$  is closed  $\bar{H} \in \overline{\mathcal{Z}^e}$  as well.

Using the same convex combination to define  $\bar{Z}_n$  we obtain  $\bar{Z}_n \leq \bar{H}_n$   $p - a.s.$  for all  $n \geq 1$  (resp.  $\bar{Z} \leq \bar{H}$   $p - a.s.$ ).

If we choose a sequence  $\{Z_n\} \subseteq \text{solid}(\overline{\mathcal{Z}^e})$  converging to some  $Z \in L_+^0(\mathcal{M})$ , then  $Z = \bar{Z} \leq \bar{H}$  holds  $p - a.s.$  This proves that  $\text{solid}(\overline{\mathcal{Z}^e})$  is closed.  $\square$

**Remark 5.4.1.** For all  $Z \in \mathcal{Z}$ , there exists a sequence  $\{H_n\}_{n \geq 1} \subseteq \mathcal{Z}^e$  such that the limit  $H = \lim_{n \rightarrow \infty} H_n$  exists  $p - a.s.$  and  $H \geq Z$  holds  $p - a.s.$  Moreover  $\lim_{n \rightarrow \infty} H_n = \hat{Z}$  holds almost surely if  $\hat{Z}$  solves  $\tilde{V}(1)$ .

---

*Proof.* We deduce from Proposition 5.1.1 that  $(\mathcal{Z}^e)^\circ = \text{solid}(\mathcal{C})$ . Since  $\mathcal{Z} = \text{solid}(\mathcal{C})^\circ$  by definition, this shows  $\mathcal{Z} = (\mathcal{Z}^e)^\circ$ , and thanks to the last Lemma we get  $\mathcal{Z} = \text{solid}(\overline{\mathcal{Z}^e})$ . So, for all  $Z \in \mathcal{Z}$  there exists a sequence  $\{H_n\}_{n \geq 1} \subseteq \mathcal{Z}^e$  such that the limit  $H = \lim_{n \rightarrow \infty} H_n$  exists  $p - a.s.$  and  $H \geq Z$  holds  $p - a.s.$

If  $\hat{Z}$  solves  $\tilde{V}(1)$  we deduce  $\lim_{n \rightarrow \infty} H_n = \hat{Z}$  by  $p - a.s.$  uniqueness of the minimizer  $\hat{Z}$ .  $\square$

**Lemma 5.4.2.** *Let Assumption 2 hold, then for all  $z > 0$*

$$\inf_{H \in \mathcal{Z}^e} \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(zH(t)) dt \right] < \infty$$

*Proof.* Note that it suffices to prove the assertion above for  $z = 1$ . In the following we prove the existence of  $H \in \mathcal{Z}^e$  with

$$\mathbb{E} \left[ \int_0^T f(t) \tilde{u}(H(t)) dt \right] < \infty$$

Let  $\{\delta_n\}_{n \geq 1}$  a sequence with  $\delta_n > 0$  and  $\sum_{n=1}^{\infty} \delta_n = 1$ . With  $\hat{Z}_n$  we denote the solution to  $\tilde{V}(\delta_n)$  ( $\leq \infty$ ). Let a strictly decreasing sequence  $\{\varepsilon_n\}_{n \geq 2}$  with  $\varepsilon_n > 0$  and  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . More precicely we choose  $\{\varepsilon_n\}_{n \geq 2}$  such that

$$\sum_{n=1}^{\infty} \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(\hat{Z}_n) \mathbf{1}_{A_n} dt \right] < \infty \quad (5.4.1)$$

for each sequence of sequence  $\{A_n\}_{n \geq 1} \subseteq \mathcal{M}$  with  $\mathbb{P}(A_n) \leq \varepsilon_n$  for  $n \geq 2$ .

In the following we construct a sequence  $\{A_n\}_{n \geq 1}$  satisfying the requirements stated above. According to Remark 5.4.1 to find a sequence  $\{H_n\}_{n \geq 1} \subseteq \mathcal{Z}^e$  such that

$$dt \otimes \mathbb{P} \left( \tilde{u}(\delta_n H_n) > \tilde{u}(\hat{Z}_n) + 1 \right) \leq \varepsilon_{n+1} \quad \text{for } n \geq 1$$

Now we define

$$\begin{aligned} A_1 &= \{ \tilde{u}(\delta_1 H_1) \leq \tilde{u}(\hat{Z}_1) + 1 \} \\ &\vdots \\ A_n &= \{ \tilde{u}(\delta_n H_n) \leq \tilde{u}(\hat{Z}_n) + 1 \} \setminus \bigcup_{k=1}^{n-1} A_k \\ &\vdots \end{aligned}$$



Thus,  $\{A_n\}_{n \geq 1}$  satisfies

$$\begin{aligned} A_k \cap A_l &= \emptyset \quad \text{for } k \neq l \\ \mathbb{P}(A_n) &\leq \varepsilon_n \quad \text{for } n \geq 2 \\ \text{and } \mathbb{P}\left(\bigcup_{n \geq 1} A_n\right) &= 1 \end{aligned}$$

Recall that the set of equivalent martingale measures  $\hat{\mathcal{Y}}^e$ , hence  $\mathcal{Z}^e$ , is closed under countable convex combinations (Cf. Delbaen and Schachermayer (1994)). Thus,  $H := \sum_{n \geq 1} \delta_n H_n \in \mathcal{Z}^e$  and thanks to the construction of  $A_n$

$$\begin{aligned} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(H(t)) dt\right] &= \sum_{n \geq 1} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(H(t)) \mathbf{1}_{A_n} dt\right] \\ &\leq \sum_{n \geq 1} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(\delta_n H_n(t)) \mathbf{1}_{A_n} dt\right] \end{aligned}$$

holds. Notice that the estimate holds because  $\tilde{u}$  is decreasing. Furthermore

$$\sum_{n \geq 1} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(\delta_n H_n(t)) \mathbf{1}_{A_n} dt\right] \leq \sum_{n \geq 1} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(\hat{Z}_n(t)) \mathbf{1}_{A_n} dt\right] + 1$$

holds. By the construction of the sequence  $\{\delta_n\}_{n \geq 1}$  (cf. (5.4.1)) we finally observe

$$\sum_{n \geq 1} \mathbb{E}\left[\int_0^T f(t) \tilde{u}(\hat{Z}_n(t)) \mathbf{1}_{A_n} dt\right] + 1 < \infty$$

Thus we have found  $H \in \mathcal{Z}^e$  with  $\tilde{V}(1) \leq \mathbb{E}\left[\int_0^T f(t) \tilde{u}(H(t)) dt\right] < \infty$ .  $\square$

### Proof of Theorem 6

*Proof.* Notice that the second assertion has already been show in Lemma 5.4.2.

We will show that there is  $\hat{H} \in \mathcal{Z}^e$  such that

$$\mathbb{E}\left[\int_0^T f(t) \tilde{u}((z + \varepsilon)\hat{H}(t)) dt\right] \leq \tilde{V}(z) + \varepsilon \quad (5.4.2)$$

for fixed  $z > 0$  and arbitrary  $\varepsilon > 0$ .

---

Let  $\hat{Z} \in \mathcal{Z}(z)$  the optimal solution to  $\tilde{V}(z)$ . According to Lemma 5.4.2 we are able to find an element  $H \in \mathcal{Z}^e$  such that

$$\mathbb{E} \left[ \int_0^T f(t) \tilde{u}(\varepsilon H(t)) dt \right] < \infty$$

Now choose  $\delta > 0$  sufficiently small such that

$$\mathbb{E} \left[ \int_0^T f(t) \left( \left| \tilde{u}(\hat{Z}(t)) \right| + \left| \tilde{u}(\varepsilon \hat{H}(t)) \right| \right) \mathbf{1}_A dt \right] \leq \frac{\varepsilon}{2} \quad (5.4.3)$$

holds for all sets  $A \in \mathcal{M}$  with  $\mathbb{P}(A) \leq \delta$ .

Thanks to Remark 5.4.1 we are able to find  $H_0 \in \mathcal{Z}^e$  such that

$$\mathbb{P} \left( \tilde{u}(zH_0) > \tilde{u}(\hat{Z}) \right) \leq \delta$$

Acting on the assumption in (5.4.3) we construct a set  $A := \left\{ \tilde{u}(zH_0) > \tilde{u}(\hat{Z}) \right\}$ .

Proving the estimate (5.4.2) we further define  $\hat{H} := \frac{zH_0 + \varepsilon H}{z + \varepsilon}$ .

Now, since  $\tilde{u}$  is decreasing we observe

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T f(t) \tilde{u}((z + \varepsilon)\hat{H}(t)) dt \right] \\ &= \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(zH_0(t) + \varepsilon H(t)) dt \right] \\ &\leq \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(\varepsilon H(t)) \mathbf{1}_A dt \right] + \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(zH_0(t)) \mathbf{1}_{A^c} dt \right] \end{aligned}$$

Moreover

$$\mathbb{E} \left[ \int_0^T f(t) \tilde{u}(\varepsilon H(t)) \mathbf{1}_A dt \right] + \mathbb{E} \left[ \int_0^T f(t) \tilde{u}(zH_0(t)) \mathbf{1}_{A^c} dt \right] \leq \tilde{V}(z) + \varepsilon$$

holds by the construction of  $A$ . This verifies (5.4.2) □

## 5.4.2 Proofs of Section 5.2

In the proof of Theorem 7 we will make use of the following identities.

**Lemma 5.4.3.** For  $x, z > 0$  such that  $x = -\tilde{V}'(z)$  we have the following.

$$\tilde{V}'(z) = \mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} H_*(t) \tilde{u}'(z f^*(t) H_*(t)) dt \right] \quad \text{for } z > z_0 \quad (5.4.4)$$

$$\mathbb{E} \left[ \int_0^T I_z^*(t) u'(I_z^*(t)) dF(t) \right] = z \mathbb{E}_* \left[ \int_0^T I_z^*(t) dF(t) \right] \quad (5.4.5)$$

*Proof.* First we prove Equation (5.4.4). Fix  $z > z_0$  and  $h > 0$ .

Thanks to convexity of  $\tilde{u}$ , the following identity holds  $\mathbb{P} \times dt - a.e.$

$$f \tilde{u}((z+h)f^* H_*) - f \tilde{u}(z f^* H_*) = \mathbf{1}_{\{f>0\}} H_* \int_z^{z+h} \tilde{u}'(\bar{z} f^* H_*) d\bar{z}$$

Thus

$$\begin{aligned} \tilde{V}(z+h) - \tilde{V}(z) &= \mathbb{E} \left[ \int_0^T \tilde{u}((z+h)f^*(t)H_*(t)) - \tilde{u}(zf^*(t)H_*(t)) dF(t) \right] \\ &= \mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} H_*(t) \left( \int_z^{z+h} \tilde{u}'(\bar{z} f^*(t)H_*(t)) d\bar{z} \right) dt \right] \end{aligned}$$

Since  $\tilde{V}(z_0) < \infty$ ,  $z > z_0$ , and  $h > 0$ , this triple integral is finite we are allowed to use Fubini's Theorem. We obtain

$$\begin{aligned} &\mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} H_*(t) \left( \int_z^{z+h} \tilde{u}'(\bar{z} f^*(t)H_*(t)) d\bar{z} \right) dt \right] \\ &= \int_z^{z+h} \mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} H_*(t) \tilde{u}'(\bar{z} f^*(t)H_*(t)) dt \right] d\bar{z} \end{aligned}$$

which proves identity (5.4.4).

We go on with Equation (5.4.5). Recall  $I = (u')^{-1}$ . Now straightforward calculus give us

$$\begin{aligned} \mathbb{E} \left[ \int_0^T I_z^*(t) u'(I_z^*(t)) dF(t) \right] &= \mathbb{E} \left[ \int_0^T f(t) I_z^*(t) \mathbf{1}_{\{f>0\}} z f^*(t) H_*(t) dt \right] \\ &= z \mathbb{E} \left[ \int_0^T \underbrace{I_z^*(t) \mathbf{1}_{\{f>0\}}}_{=I_z^*(t)} H_*(t) dt \right] \end{aligned}$$

Employing Fubini's Theorem and Bayes rule for stochastic processes, we conclude with

$$\mathbb{E} \left[ \int_0^T I_z^*(t) u'(I_z^*(t)) dF(t) \right] = z \mathbb{E} \left[ \int_0^T I_z^*(t) H_*(t) dt \right] = z \mathbb{E}_* \left[ \int_0^T I_z^*(t) dt \right]$$

which verifies Equation (5.4.5).  $\square$

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## Proof of Theorem 7

*Proof.* (i) According to identity (5.4.4),  $\tilde{V}$  is continuous differentiable and strictly convex on  $(z_0, \infty)$ . By the general properties of the Legendre-Fenchel transform, we have that  $V$  is continuously differentiable and strictly convex on  $(0, x_0)$ , see Rockafellar (1970, Theorem III 12.2).

The assertions (ii) and (iii) will be shown simultaneously in the following three steps. Therefor choose  $x \in (0, x_0)$  and  $z \in (z_0, \infty)$  such that  $-\tilde{V}'(z) = x$ .

1) We first prove  $I_z^* \in \mathcal{C}(x)$ . Recall Equation (5.4.4), thus

$$-\tilde{V}'(z) = -\mathbb{E} \left[ \int_0^T \mathbf{1}_{\{f>0\}} H_*(t) \tilde{u}'(z f^*(t) H_*(t)) dt \right] = \mathbb{E} \left[ \int_0^T H_*(t) I_z^*(t) dt \right]$$

Note that the last identity hold, because  $\mathbf{1}_{\{f>0\}} I_z^* = I_z^*$  and  $-\tilde{u}' = I$ . According to Fubini's Theorem and Bayes rule for stochastic processes, we are able to conclude with

$$x = -\tilde{V}'(z) = \mathbb{E} \left[ \int_0^T H_*(t) I_z^*(t) dt \right] = \mathbb{E}_* \left[ \int_0^T I_z^*(t) dt \right] \quad (5.4.6)$$

Finally  $I_z^* = -\tilde{u}'(z f^* H_*) \in \mathcal{C}(x)$  follows because  $-\tilde{u}$  is non-negative.

2) We continue with Equation (5.2.7). According to (5.4.5) and (5.4.6) we derive

$$\mathbb{E} \left[ \int_0^T I_z^*(t) u'(I_z^*(t)) dF(t) \right] = z \mathbb{E}_* \left[ \int_0^T I_z^*(t) dF(t) \right] = zx$$

3) Since  $z > z_0$ , we know that  $\tilde{V}(z) < \infty$ . Further  $V(x) < \infty$  holds, since by definition  $V(x) = \tilde{V}(z) + xz > -\infty$ . The usual procedure in convex analysis give us

$$\begin{aligned} \tilde{V}(z) + xz = V(x) &\geq \mathbb{E} \left[ \int_0^T u(I_z^*(t)) dF(t) \right] \\ &\stackrel{(5.2.3)}{=} \mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) + H_z^*(t) I_z^*(t) dF(t) \right] \\ &\stackrel{(5.4.6)}{=} \mathbb{E} \left[ \int_0^T \tilde{u}(H_z^*(t)) dF(t) \right] + xz \end{aligned}$$

#### 5.4. PROOFS OF THE MAIN THEOREMS IN SECTION 5

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According to Proposition 5.3.1  $H_z^*$  solves  $\tilde{V}(z)$  and equality must hold. This in turn verifies (ii), resp. optimality of  $I_z^*$ .

Now only (iii) is left. Employing (ii) we get

$$\mathbb{E} \left[ \int_0^T \hat{c}_x(t) u'(\hat{c}_x(t)) \, dF(t) \right] = \mathbb{E} \left[ \int_0^T I_z^*(t) u'(I_z^*(t)) \, dF(t) \right]$$

Thus obviously we already verified (5.2.8) in step 2). Now (5.2.7) follows by straight forward calculus.  $\square$



## Chapter 6

# Applications II: Consumption Ratcheting or *Optimal Consumption Choice with Intolerance for Decline in Standard of Living*

In this final part we reconsider the constrained consumption choice problem of an expected utility maximizing investor. Our main interest is to study the behavior of an investor who does not tolerate any (resp. an immoderate) decline in his rate of consumption process. His partialities obviously describe a nontrivial situation of constrained consumption choice. Again we utilize the semimartingale model including incomplete market dynamics.

### 6.1 The Problem of Consumption Ratcheting

In this section we analyze the consumption choice problem of an investor, who is intolerant for any decrease in his consumption rate (Example 5). Imputing to the investor to have these ratcheting partialities for consumption

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induces further useful properties. We will see that it suffices to concentrate on *optional* consumption plans which are permissible for ratcheting behavior.

When the investor stints himself to choose a non-decreasing rate of consumption process, the budget constraint can be reformulated. Heuristically spoken: Choosing a non-decreasing consumption rate forces the investor to decide which amount to withdraw not only for today, but continuously for the rest of his lifetime.

From now on we fix an acceptance set as the set containing all non-decreasing rate of consumption processes, i.e.

$$\bar{\mathcal{R}} := \{c \in L_*^0(\mathcal{M}) \mid c(t) \geq c(s) \text{ for all } t \geq s\}$$

Given an initial capital  $x > 0$ , the set of all rate of consumption processes  $c$ , which are admissible and also permissible for a ratcheting investor is given by

$$\mathcal{R}(x) := \left\{ c \in \bar{\mathcal{R}} \mid \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] \leq x \right\}$$

As usual we abbreviate  $\mathcal{R} = \mathcal{R}(1)$ .

Thus a ratcheting investor faces the following problem.

**Problem 5.**

$$V_{\mathcal{R}}(x) = \sup_{c \in \mathcal{R}_V(x)} \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad (6.1.1)$$

here  $\mathcal{R}_V(x)$  denotes the set of all consumption processes  $c \in \mathcal{R}(x)$  with

$$\mathbb{E} \left[ \int_0^T u^-(c(t)) dF(t) \right] < \infty$$

Again we deduce  $V_{\mathcal{R}}(x) \leq V(x)$  from  $\mathcal{R}(x) \subseteq \mathcal{C}(x)$ . Thus the value function  $V_{\mathcal{R}}$  stays finite if  $V$  does.

Fix  $c \in \bar{\mathcal{R}}$ . With  $\bar{c}$  we denote the (upper) right-continuous modification of  $c$ , thus

$$\bar{c}(t) := \inf_{(s \wedge T) > t} \sup_{t < u \leq (s \wedge T)} c(u) = \inf_{(s \wedge T) > t} c(s) \text{ for all } t \in [0, T]$$



Notice that the right-continuous version of an progressively measurable process is progressively measurable itself, which implies that  $\vec{c} \in \bar{\mathcal{R}}$  as well.<sup>1</sup> Moreover thanks to monotonicity the process  $\vec{c}$  is càdlàg . Since progressively measurable processes are adapted, we finally deduce

$$\vec{c} \in \mathcal{I} \tag{6.1.2}$$

Again  $\mathcal{I}_V$  will denote the set of  $c \in \mathcal{I}$  with  $\mathbb{E} \left[ \int_0^T u^-(c(t)) dF(t) \right] < \infty$ .

We summarize the previous considerations in the following Proposition.

**Proposition 6.1.1.** *For all  $x > 0$*

$$V_{\mathcal{R}}(x) = \sup_{c \in \mathcal{I}_V(x)} \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right]$$

*holds. Here  $\mathcal{I}_V(x)$  denotes the set of all non-decreasing optional processes which are admissible for the initial capital  $x$ .*

*Proof.* Fix  $c \in \mathcal{R}(x)$  and recall that  $\vec{c} \in \mathcal{I}$ . Since

$$\vec{c}(t) \geq c(t) \quad \mathbb{P} - a.s. \text{ for all } t \in [0, T]$$

and  $u$  is strictly increasing, we only need to verify

$$\mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^T Y(t)\vec{c}(t) dt \right] \quad \text{for all } Y \in \hat{\mathcal{Y}} \tag{6.1.3}$$

By monotonicity of  $c$  we get that for fixed  $\omega \in \Omega$  the paths  $c(\omega)$  and  $\vec{c}(\omega)$  may differ only at countably many  $t \in [0, T]$ . Thus the set  $\{t \in [0, T] \mid \vec{c}(t; \omega) \neq c(t; \omega)\}$  is of Lebesgue measure zero and Equation (6.1.3) holds.  $\square$

In the previous section we discussed the importance (resp. sufficiency) of Assumption 2 to derive useful properties of both the value and the dual value function. With respect to Proposition 6.1.1, the observations we made so far can be extended easily to the following.

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<sup>1</sup>See e.g. Bain and Crisan (2007, Lemma A.27.)

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**Proposition 6.1.2.** *Let Assumption 2 hold, then the optimal primal strategy  $c_{\mathcal{R},x} \in \mathcal{I}_V(x)$  exists and is  $p - a.s.$  unique. Furthermore the value function  $V_{\mathcal{R}}$  is finite, strictly increasing, strictly concave and continuous on  $(0, \infty)$ .*

*Proof.* This is a direct consequence of Proposition 6.1.1.  $\square$

Recall also the dual value function  $\tilde{V}_{\mathcal{R}}$  defined as

$$\tilde{V}_{\mathcal{R}}(z) = \sup_{x>0} V_{\mathcal{R}}(x) - xz \quad \text{for } z > 0$$

We easily derive  $\tilde{V}_{\mathcal{R}} \leq \tilde{V}$  from  $V_{\mathcal{R}}(x) - xz \leq V(x) - xz$  for all  $x, z > 0$ .

The assertions we made so far have no obvious impact on the dual value function.

**Proposition\* 7.** *Let Assumption 2 hold, then  $\tilde{V}_{\mathcal{R}}$  is strictly decreasing, convex, continuous and differentiable on  $(0, \infty)$ . Furthermore, for each  $z > 0$ , there exist a  $x_{\mathcal{R},z} > 0$  such that*

$$\tilde{V}_{\mathcal{R}}(z) = V_{\mathcal{R}}(x_{\mathcal{R},z}) - zx_{\mathcal{R},z}$$

We already verified that under certain conditions  $\tilde{V}_{\mathcal{R}}$  solves a corresponding dual problem, i.e.

$$\tilde{V}_{\mathcal{R}}(z) = \inf_{Z \in \mathcal{Z}(z)} \tilde{\mathcal{E}}_{\mathcal{R}}(Z)$$

Here  $\tilde{\mathcal{E}}_{\mathcal{R}}$  denotes the  $\mathcal{R}$ -dual of  $\mathcal{E}$ . According to Proposition 6.1.1 the  $\mathcal{R}$ -dual of  $\mathcal{E}$  obviously satisfies

$$\tilde{\mathcal{E}}_{\mathcal{R}}(Z) = \sup_{c \in \mathcal{I}_V} \mathbb{E} \left[ \int_0^T u(c(t)) - Z(t)c(t) dF(t) \right]$$

for  $Z \in L_+^0(\mathcal{M})$ .

As we will see now, when consumption plans are non-decreasing, the budget constraint can be reformulated.

Recall that optional processes are progressively measurable, i.e.  $\mathcal{I} \subseteq \bar{\mathcal{R}}$ .<sup>2</sup> Since adapted right-continuous processes play an important role for ratcheting investors, we may introduce the following important variant of the budget constraint.

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<sup>2</sup>See e.g. Métivier (1982, Chapter 1.1)

**Theorem 8.** *A process  $c \in \mathcal{I}$  is admissible for some initial capital  $x > 0$  if and only if*

$$x \geq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T-t) dc(t) \right]$$

*In particular*

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T-t) d\bar{c}(t) \right] = \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T H(t)\bar{c}(t) dt \right] \quad (6.1.4)$$

*holds for all  $c \in \bar{\mathcal{R}}$ .*

Moreover all relevant non-decreasing consumption plans, which are admissible for initial capital  $x > 0$  lie within the set

$$\mathcal{I}(x) = \left\{ c \in \mathcal{I} \mid \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T-t) dc(t) \right] \leq x \right\}$$

Finally we recheck the dual problem. The set of (discounted) equivalent martingale measures fulfills an important task in this problem.

In the very beginning we introduced  $\mathcal{Z}$  as the set of dual variables because of a grave advantage. This approach enables us to study a wider class of expected utility maximization problems.

Later, in Section 5.2, we employed Assumption 5 and showed, although the sets  $\hat{\mathcal{Y}}^e$  and  $\mathcal{Z}^e$  may not contain the dual minimizer, they are sufficient to set up a suitable dual problem. Moreover both of this sets are heavily related to each other. Under Assumption 5

$$\hat{\mathcal{Y}}^e = f \mathcal{Z}^e \quad (6.1.5)$$

holds. After the transformation into the bidual sets  $\hat{\mathcal{Y}}$  and  $\mathcal{Z}$  differences may raise. One of the main deviations is that processes in  $\hat{\mathcal{Y}}$  are non-negative in each point in time, while non-negativity in  $\mathcal{Z}$  is only considered with respect to  $F$ .

**Remark 6.1.1.** If Assumption 5 holds, we get  $(\hat{\mathcal{Y}})^{\circ\circ} = f\mathcal{Z}$ .

---

*Proof.* First notice that

$$1 \geq \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^T \left( f(t)\frac{1}{f}(t) \right) Y(t)c(t) dt \right]$$

for all  $Y \in \hat{\mathcal{Y}}$ . Thus  $\hat{\mathcal{Y}} \subseteq f\mathcal{Z}$  holds. Contrarily

$$(\hat{\mathcal{Y}})^{\circ\circ} \supseteq \hat{\mathcal{Y}} \supseteq \hat{\mathcal{Y}}^e = f\mathcal{Z}^e$$

holds. Thus the bipolar theorem (Brannath and Schachermayer, 1999) implies

$$(\hat{\mathcal{Y}})^{\circ\circ} \supseteq f\mathcal{Z} = f(\mathcal{Z}^e)^{\circ\circ}$$

by minimizing property of converting into biduals. Notice that by Proposition 5.1.1  $\mathcal{Z} = (\mathcal{Z}^e)^{\circ\circ}$  holds.  $\square$

Moreover, under Assumption 5, we may consider a suitable dual functional (for ratcheting investors) to be given as

$$\begin{aligned} \tilde{\mathcal{E}}_*(h) &:= \sup_{c \in \mathcal{I}_V} \mathbb{E} \left[ \int_0^T f(t)u(c(t)) - h(t)c(t) dt \right] \\ &= \sup_{c \in \mathcal{I}_V} \mathbb{E} \left[ \int_0^T f(t)u(c(t)) dt - \int_0^T h(t)(T-t) dc(t) \right] \end{aligned} \quad (6.1.6)$$

for  $h \in L_+^1(\mathcal{M})$ .

Although this functional slightly deviates from the previous  $\mathcal{R}$ -dual, all important properties hold. E.g. this functional is decreasing, concave and since  $\mathbf{1} \in \bar{\mathcal{R}}$ , this functional is  $(-\infty, \infty]$ -valued as well.

According to Remark 6.1.1 in addition to (6.1.6) we are able to restate Theorem 5 as follows.

$$\tilde{V}_{\mathcal{R}}(z) = \inf_{Y \in (\hat{\mathcal{Y}}(z))^{\circ\circ}} \tilde{\mathcal{E}}(Y)$$

Moreover we can show the following.

**Proposition 6.1.3.** *Let Assumptions 2 and 5 hold. If the intertemporal utility function  $u$  is bounded below, then*

$$\tilde{V}_{\mathcal{R}}(z) = \inf_{Y \in \hat{\mathcal{Y}}(z)} \tilde{\mathcal{E}}_*(Y)$$

*Furthermore this infimum is attained by a process  $Z^* \in \mathcal{Z}(z)$ .*

*Proof.* This assertion can be shown using the same arguments as in the proof to Theorem 5 □

## 6.2 Duality Theory for Non-decreasing Consumption Selection

The problem of optimal non-decreasing consumption selection resembles a problem of optimal cumulative consumption selection based on a special preference structure.<sup>3</sup> Kauppila (2010) considered a utility maximization problem within an incomplete semimartingale model for Hindy-Huang-Kreps type investors, whose decision is subject to a similar budget constraint. Inter alia her observations are based upon the following representation theorem.<sup>4</sup>

**Theorem\* 8** (Kauppila (2010, Corollary 5.5)). *Let Assumption 5 hold. Suppose that  $g: \mathbb{R} \mapsto \mathbb{R}$  is strictly decreasing from  $+\infty$  to  $-\infty$ . If  $X$  is a non-negative, right-continuous supermartingale with  $X(T) = 0$ , then there exists an optional process  $L$  such that for every stopping time  $\tau$*

$$X(\tau) = \mathbb{E} \left[ \int_{\tau}^T f(t) g \left( \sup_{\tau \leq s \leq t} L(s) \right) dt \mid \mathcal{F}_{\tau} \right]$$

holds  $\mathbb{P} - a.s.$

Recall that a process  $X$  is called *optional* if it is measurable with respect to the  $\sigma$ -algebra generated by the real-valued càdlàg processes. The process  $X^o$  with

$$X^o(\tau) := \mathbb{E}[X(\tau) \mid \mathcal{F}_{\tau}]$$

---

<sup>3</sup>An optimal consumption choice problem for an economic agent whose decisions are driven by Hindy-Huang-Kreps preferences. See Hindy et al. (1992), Hindy and Huang (1992) and (1993); Bank and Riedel (2000) and (2001)

<sup>4</sup>This kind of representation theorems is originally introduced and verified in Bank and El Karoui (2004)

---

for all stopping times  $0 \leq \tau \leq T$  is called *optional projection* of  $X$ . Notice that if  $X$  is adapted and bounded, the optional projection  $X^o$  exists and is unique within the set of optional processes.

Since  $\mathbb{E}[(T-t)Y(t) | \mathcal{F}_s] \geq (T-s)Y(s)$  for  $Y \in \hat{\mathcal{Y}}$ , we may apply that theorem to get a suitable representation

$$(T - \tau)Y(\tau) = \mathbb{E} \left[ \int_{\tau}^T f(t) u' \left( \sup_{\tau \leq s \leq t} L(s) \right) dt \middle| \mathcal{F}_{\tau} \right] \quad (6.2.1)$$

for each process  $Y \in \hat{\mathcal{Y}}$ .

Notice that, by non-negativity of  $X$ , the representation theorem holds also when we replace the function  $g$  by the derivative  $u'$  (compare Kauppila (2010)).

To point out the importance of this representation theorem, we anticipate the succeeding analysis and give the solution for the complete market case explicitly.

**Example 11** (Consumption Ratcheting in Complete Markets). Let Assumptions 2, 5, and 6 hold and choose  $x, z > 0$  such that  $u'(x) = z$ . As usual we denote the state price density of the unique (local) equivalent martingale measure by  $H_*$ . Suppose that the optional process  $\hat{L}_y$  satisfies (6.2.1) with  $Y = yH_*$ , then the process

$$\hat{c}_{\mathcal{R},x}(t) = \sup_{0 \leq s \leq t} \hat{L}_y(s)$$

is optimal for the initial capital  $x$ .

Furthermore this representation theorem points out an opportunity to narrow the set of dual variables. If we consider a ratcheting investor the set of relevant dual variables can be reduced to the set

$$\nabla \hat{\mathcal{Y}} = \left\{ X(t) = \int_t^T F(s) u'(L(s)) ds \middle| \begin{array}{l} \text{with } L \in \mathcal{I} \text{ and for some } Y \in \hat{\mathcal{Y}} \\ X^o(t) \leq (T-t)Y(t) \end{array} \right\}$$

Moreover for  $z > 0$  we define the set  $\nabla\hat{\mathcal{Y}}(z)$  by interchanging  $\hat{\mathcal{Y}}$  with  $\hat{\mathcal{Y}}(z)$ .

**Theorem 9.** *Let Assumptions 2 and 5 hold and assume that  $u$  is bounded below, then*

$$\tilde{V}_{\mathcal{R}}(z) = \inf_{Y \in \nabla\hat{\mathcal{Y}}(z)} \tilde{\mathcal{E}}(Y)$$

*holds for all  $z > 0$ .*

*Furthermore the the minimizer  $\hat{Y}_{\mathcal{R},z} \in \nabla\hat{\mathcal{Y}}(z)$  exists and is unique  $p - a.s.$*

*Proof.* See Kauppila (2010, Theorem 6.1). □

Indeed the assumptions in Kauppila (2010) slightly deviate from our assumptions. Kauppila uses finiteness of  $V_{\mathcal{R}}$  in addition to a time-dependent version of asymptotic elasticity, but the proof (Kauppila, 2010, Theorem 6.1) is not affected by these subtleties and holds also in our context

This observation strengthens our ability to reveal the full properties of the value functions.

**Corollary 6.2.1.** *Let Assumptions 2 and 5 hold, then*

(i) *both functions  $V_{\mathcal{R}}$  and  $-\tilde{V}_{\mathcal{R}}$  are finite, increasing, strictly concave and continuously differentiable on  $(0, \infty)$ .*

(ii) *for all  $x, z > 0$  the optimal solutions to  $V_{\mathcal{R}}(x)$  and  $\tilde{V}_{\mathcal{R}}(z)$  exist and are unique  $p - a.s.$*

(iii) *the functions  $V'_{\mathcal{R}}$  and  $-\tilde{V}'_{\mathcal{R}}$  are strictly decreasing.*

*Proof.* We only need to verify continuous differentiability of  $V_{\mathcal{R}}$ . Employing methods from convex analysis (Rockafellar, 1970, Theorem V.26.3) this follows from strict convexity of  $\tilde{V}_{\mathcal{R}}$ . For strict convexity of  $\tilde{V}_{\mathcal{R}}$  see Kauppila (2010, Theorem 6.3). □

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## 6.3 Beyond Consumption Ratcheting

In this section we discuss several consumption choice problems which are related to consumption ratcheting behavior. Solving a given optimization problem, it sometimes may be a good approach to transform the problem and then solve the transformed one. We explain the ideas given in Dybvig (1995) and Schroder and Skiadas (2002) which relate consumption choice problems to the ratcheting case.

To display the effects of the transformation we consider a financial market whose bond price process bases on a constant interest rate  $r > 0$  such that the bond price follows

$$\mathcal{S}_0(t) = e^{rt}$$

Implicitly we obtain

$$\sup_{Y \in \mathcal{Y}} \mathbb{E} \left[ \int_0^T e^{-rt} Y(t) c(t) dt \right]$$

as pricing formula (Corollary 2.3.2).

### Addiction to Past Consumption under Exponentially Decreasing Memory

Up to now we only considered investors who were addicted to past consumption in the sense that they were intolerant for any declining standard of living. In contrast we now study an addicted investor who exponentially weans from past consumption levels. These investors hold the current consumption level above their exponentially discounted consumption history. Consequently they accept a moderate decrease in their consumption rate. For fixed  $\delta \in \mathbb{R}_+$ , they choose among all consumption patterns with

$$c(t) \geq e^{-\delta(t-s)} c(s) \quad \text{for all } 0 \leq s \leq t$$

(cf. Example 6). The acceptance set of these investors will be denoted as  $\bar{\mathcal{C}}_\delta$  (or  $\mathcal{C}_\delta(x)$  if we restrict our attention to consumption processes which are admissible for initial capital  $x > 0$ ).



Inspired by the ideas developed in Dybvig (1995) we may go on as follows. Instead of solving the problem

$$\max \mathbb{E} \left[ \int_0^T u(c(t)) dF(t) \right] \quad \text{s.t. } c \in \mathcal{C}_\delta(x) \quad (6.3.1)$$

we define  $\tilde{c}$  via

$$\tilde{c}(t) = e^{\delta t} c(t)$$

and solve an equivalent problem for  $\tilde{c}$  on a similar financial market. Notice that this process  $\tilde{c}$  is non-decreasing, thus the constraints within the optimization problem formulated on  $\tilde{c}$  will be the same as for an ratcheting investor.

This procedure might not work in general. Dybvig (1995) introduced it on complete markets (Assumption 6) driven by a Brownian motion for a CRRA<sup>5</sup> intertemporal utility. Perusing this procedure one key observation is that we need to change the bond price dynamics from  $\mathcal{S}_0(t)$  into  $e^{\delta t} \mathcal{S}_0(t)$  which implicitly changes the pricing functional into

$$\sup_{Y \in \mathcal{Y}} \mathbb{E} \left[ \int_0^T e^{-(r+\delta)t} Y(t) c(t) dt \right]$$

Consequently the budget constraint will change as well.

### Hindy-Huang-Kreps Type Investors

In this last section we slightly change the interpretation of consumption strategies. Suppose an investor can choose a rate as consumption process  $c \in \bar{\mathcal{R}}$ . But here we do not think of  $c(t)$  as the rate of consumption at time  $t$ . Moreover, since  $c$  is non-decreasing, we take  $c(t)$  as the cumulative consumption up to time  $t$ . Further the investor will not gain his utility directly from the chosen cumulative consumption pattern. Based on his cumulative consumption choice, he will evaluate a process of average past consumption.

$$\Gamma(c; t) = \int_0^t e^{-\gamma(t-s)} dc(s) \quad \text{for some } \gamma \in \mathbb{R}_+ \quad (6.3.2)$$

---

<sup>5</sup>Here CRRA stands for *constant relative risk aversion*, i.e.  $u(x) = \frac{1}{\alpha} x^\alpha$  with  $\alpha \in (0, 1)$

---

In the following we call  $\gamma$  the average weighting factor. According to this cumulative consumption pattern  $c \in \bar{\mathcal{R}}$ , the investor gains

$$\mathbb{E} \left[ \int_0^T u(t, Y(c; t)) dF(t) \right]$$

Thus, given an initial capital  $x > 0$ , the investor's utility maximization problem is

$$\max \mathbb{E} \left[ \int_0^T u(t, \Gamma(c; t)) dF(t) \right] \quad \text{s.t. } c \in \mathcal{R}(x) \quad (6.3.3)$$

Here  $\mathcal{R}(x)$  denotes the set of all rate of consumption processes which satisfy the budget constraint

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T e^{-rt} \Gamma(t) dc(t) \right] \leq x$$

This model allows for nice economic interpretations. Its advantages have been discussed in Hindy and Huang (1992), Hindy et al. (1992) and Hindy and Huang (1993). See also Bank and Riedel (2001) for the analysis of the corresponding optimization problem. One of the main differences to our preference structure is that this model embodies the idea of local substitutions in the sense that consumption at near by dates can be almost perfect substitutes.

Following Schroder and Skiadas (2002) we can write down an isomorphism between the optimization problem for an Hindy-Huang-Kreps investor with average weighting factor  $\gamma$  and weighting factor 0, cf. Equation (6.3.2). When  $\gamma$  equals zero we are almost in the situation of consumption ratcheting investment. We only need to employ Theorem 8 to interchange the corresponding pricing functionals.

A full description of duality theory for Hindy-Huang-Kreps investors can be found in Kauppila (2010). In her doctoral dissertation Kauppila studied the consumption choice problem for an investor with an average weighting factor  $\gamma = 0$ . Moreover she originally established a duality Theorem for these kind of investors. Later she showed how the results can be carried over to the general case  $\gamma \in \mathbb{R}_+$ .

## 6.4 Proofs of the Main Theorems in Section 6

We start proving the important modification of the budget constraint, namely Theorem 8.

**Lemma 6.4.1.** *Let  $c \in \mathcal{I}$  with  $\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] < \infty$ , then*

$$\mathbb{E} \left[ \int_0^T H(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^T H(t)(T-t) dc(t) \right] \quad \text{for all } H \in \hat{\mathcal{Y}}^e$$

*Proof.* Fix  $H \in \hat{\mathcal{Y}}^e$ . Since  $H$  is a local martingale under  $\mathbb{P}$ , we find a sequence of stopping times  $\{T_n\}_{n \geq 1}$  such that

$$\mathbb{P}(T_n = T) \xrightarrow[n \rightarrow \infty]{} 1$$

and the processes  $\{H(T_n \wedge \bullet) \mid n \geq 1\}$  are uniformly integrable martingales.

If we can verify

$$\mathbb{E} \left[ \int_0^{T_n} H(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^{T_n} H(t)(T_n - t) dc(t) \right] \quad (6.4.1)$$

for  $n \geq 1$ , the desired assertion can be obtained by letting  $n \rightarrow \infty$  and using monotone convergence.

We continue proving Equation (6.4.1). Fubini's Theorem shows that

$$\begin{aligned} \mathbb{E} \left[ \int_0^{T_n} H(t)c(t) dt \right] &= \mathbb{E} \left[ \int_0^{T_n} H(t) \int_0^{T_n} \mathbf{1}_{\{s \leq t\}} dc(s) dt \right] \\ &= \mathbb{E} \left[ \int_0^{T_n} \int_0^{T_n} H(t) \mathbf{1}_{\{s \leq t\}} dt dc(s) \right] \\ &= \mathbb{E} \left[ \int_0^{T_n} \int_s^{T_n} H(t) dt dc(s) \right] \end{aligned}$$

Since  $c$  is particularly adapted, we may employ Jacod and Shiryaev (1987, Lemma I.3.12) and continue

$$\mathbb{E} \left[ \int_0^{T_n} \int_s^{T_n} H(t) dt dc(s) \right] = \mathbb{E} \left[ \int_0^{T_n} \mathbb{E} \left[ \int_s^{T_n} H(t) dt \mid \mathcal{F}_s \right] dc(s) \right]$$

Using the martingale property we further observe

$$\mathbb{E} \left[ \int_s^{T_n} H(t) dt \mid \mathcal{F}_s \right] = \int_s^{T_n} \mathbb{E} [H(t) \mid \mathcal{F}_s] dt = H(s)(T_n - s)$$

---

Thus we are able to conclude with

$$\begin{aligned}\mathbb{E} \left[ \int_0^{T_n} H(t)c(t) dt \right] &= \mathbb{E} \left[ \int_0^{T_n} \mathbb{E} \left[ \int_s^{T_n} H(t) dt \mid \mathcal{F}_s \right] dc(s) \right] \\ &= \mathbb{E} \left[ \int_0^{T_n} H(s)(T_n - s) dc(s) \right]\end{aligned}$$

and Equation (6.4.1) holds.  $\square$

**Lemma 6.4.2.** *Let  $c \in \mathcal{I}$  and  $x > 0$  with  $\mathbb{E} \left[ \int_0^T H(t)(T - t) dc(t) \right] \leq x$  for all  $H \in \hat{\mathcal{Y}}^e$ , then*

$$\sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T - t) dc(t) \right] \leq x$$

*Proof.* This assertion follows by using the same arguments as in the proof of Proposition 2.3.3.  $\square$

### Proof of Theorem 8

*Proof.* First let  $c \in \mathcal{I}$  be admissible for capital  $x > 0$ , i.e.

$$x \geq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)c(t) dt \right] \geq \sup_{H \in \hat{\mathcal{Y}}^e} \mathbb{E} \left[ \int_0^T H(t)c(t) dt \right]$$

Applying Lemma 6.4.1 we immediately see

$$x \geq \mathbb{E} \left[ \int_0^T H(t)c(t) dt \right] = \mathbb{E} \left[ \int_0^T H(t)(T - t) dc(t) \right]$$

for all  $H \in \hat{\mathcal{Y}}^e$ . Thus we may employ Lemma 6.4.2 to conclude with

$$x \geq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T - t) dc(t) \right]$$

which proves the first implication.

Contrarily let  $x \geq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T - t) dc(t) \right]$ . Then obviously

$$x \geq \sup_{Y \in \hat{\mathcal{Y}}} \mathbb{E} \left[ \int_0^T Y(t)(T - t) dc(t) \right] \geq \sup_{H \in \hat{\mathcal{Y}}^e} \mathbb{E} \left[ \int_0^T H(t)(T - t) dc(t) \right]$$

holds. Applying Lemma 6.4.1 we immediately see

$$x \geq \mathbb{E} \left[ \int_0^T H(t)(T-t) dc(t) \right] = \mathbb{E} \left[ \int_0^T H(t)c(t) dt \right]$$

for all  $H \in \hat{\mathcal{Y}}^e$ . Thus  $c$  is admissible for  $x$  by Proposition 2.3.3. □



# Chapter 7

## Concluding Remarks

In this thesis we set up a model for expected utility maximization when consumption rates are selected according to individual rules. As in many models (once the consumption process is given) the optimal portfolio choice can be obtained with the help of a martingale representation theorem, we only considered the consumption side from the original consumption-investment choice problem introduced in Merton (1969). We state some trivial and non-trivial examples for constrained consumption selection and sketch how explicit solutions may be derived. Moreover the survey paper of Bank and Föllmer (2003) could be a good source when searching explicit solutions for constrained consumption selection, at least for consumption ratcheting investment.

Solutions to various other examples must be developed from the very beginning. For Example 7 we may refer to the huge literature on CPPI portfolio strategies (Black and Perold, 1992). Although those portfolio strategies do not have a theoretical basis comparable to our constrained consumption selection problem the techniques developed there might be still useful.

Moreover one could transfer the idea of individual likings to the set of wealth processes. Under those consideration wealth-path dependent utility maximization (Bouchard and Pham, 2004) may give a theoretical foundation for CPPI strategies in the sense that a CPPI strategy is the optimal strategy for an investor with certain individual likings.





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