Microeconomic Foundation of Investment Decisions for Electronic Security Trading Systems

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Dedicated to my grandfather,

the origin of my aspiration and philosophy

Introduction

The Master standing by a stream, said, 'It passes on just like this, not ceasing day or night!' — Chapter XVII, Book IX, Confucian Analects (English translation by James Legge)

In the last century, our society has been experiencing new tides of technological revolution triggered by explosive application of information and communication technology. The innovation and application of new technology invokes profound influences on the evolution of financial markets. Since the last decade, the advances of electronic security trading have been one of the dominant characteristics in the evolution of financial markets. The comprehensive application of electronic trading in financial markets forces stock exchanges to adopt a new generation of the security trading platform - Electronic Security Trading System, e.g. Xetra (Frankfurt Stock Exchange), SETS (London Stock Exchange), and Universal Trading Platform (NYSE Euronext). Security markets in some countries have been dominated by electronic security trading systems. For example, over 90% of security transactions in Germany are executed by Xetra System operated by Frankfurt Stock Exchange.

There are three main advantages of using electronic security trading platforms instead of trading on conventional floor markets. First, electronic platforms provide more real-time trading information in the trading process. Second, electronic trading platforms are more transparent than conventional floor markets. Security prices are stipulated according to well-specified trading rules while market makers in conventional floor markets have considerable influence on the price determination. This 'black-box' argument applies in particular for prices which are negotiated among a small number of dealers. Third, transaction costs of electronic security trading platforms are on average lower than those of conventional floor markets.

These advantages together with the competition among global stock exchanges further accelerate the process of establishing electronic security trading platforms in many countries. For example, with the reference to the electronic trading system Xetra in Germany, China has started to launch an electronic trading platform called New Generation Trading Systems (NGTS).

Despite the popularity of electronic security trading systems, little is known about the microeconomic foundation of trading mechanisms in these markets, e.g. see Harris (1990), Huang & Stoll (1991), and O'Hara (1995).

Due to the lack of a proper formalization of trading mechanisms in electronic security trading systems, each electronic trading platform has its own set of trading rules, thus *' ... it is neither easy nor useful to describe how each trading mechanism works. But it is possible to discuss the operations of a specific mechanism, and detail, at least in principle, how mechanism of that type actually work.'* (O'Hara 1995, p. 9). Following this line, we consider one specific electronic security trading system, i.e. Xetra system operated by Frankfurt Stock Exchange. The official brochure Gruppe Deutsche Börse (2003) published by Frankfurt Stock Exchange provides a comprehensive description of Xetra market model with an explicit stipulation of certain trading rules.

According to Gruppe Deutsche Börse (2003), Xetra is an order-driven system in which traders can trade securities either in the form of continuous trading or in the form of Xetra auction by submitting certain types of order specifications through a computer interface. A central computer system then collects order specifications into a central order book and determines the security trading price and the trading volume according to well-specified trading rules.

There are two order sizes in Xetra, round lots and odd lots. Round lots correspond to a specific size designated to each security by the system, called round lot size. A round lot has one or multiple of the round lot size. Any order sizes other than round lots are referred to as odd lots. For example, if Xetra designates a round lot size of 200 shares for a certain security. An order specification with order size of 250 contains one round lot which size is 200, and one odd lot which size is $250 - 200 = 50$. Only round lots are accepted in continuous trading while all order sizes are allowed in Xetra auction.

Xetra accepts three basic order types: limit orders, market orders, and market-to-limit orders. A limit order is an order specification to buy or sell a security at a specific price called limit (price) or better. A market order is an unlimited order specification to buy or sell a security at the next trading price determined. A market-to-limit order is an unlimited order specification to buy or sell a security.

When a market-to-limit order enters the order book, it is treated as a market order and is to be executed by Xetra at the best possible price immediately. The unexecuted part of the market-to-limit order will enter the order book as a limit order with the limit price equal to the trading price of the executed part.

In order to support trading strategies, Xetra introduces stop orders as an additional order type, see (Gruppe Deutsche Börse 2003, p. 13). When a stop order enters Xetra, a special price called stop price must be designated. The stop order will not enter the order book until the current trading price reaches the stop price to 'trigger' the stop order. Currently there are two types of stop order in Xetra, the stop market order and the stop limit order. The stop market order enters the order book as a market order when the current trading price reaches the stop price, goes over the stop price for stop buy orders, or goes below the stop price for stop sell orders. The stop limit order enters the order book as a limit order when the current trading price reaches the stop price, goes over the stop price for stop buy orders, or goes below the stop price for stop sell orders.

In order to support large volume transactions without tremendously effecting the market, Xetra enables traders to enter iceberg order in which a mandatory limit price, an overall volume, and a peak volume must be specified. The overall volume of the iceberg order is fully considered in Xetra auction. In continuous trading, only the peak volume participates in the order book as the visible part of the iceberg order. When the peak volume in the order book is fully executed, a new peak is entered into the order book if the overall volume of the iceberg order is not totally fulfilled.

In addition, qualified traders in Xetra can submit quotes which are always treated in the order book as two synchronous order specifications (a limit buy order and a limit sell order).

A trading day of Xetra begins with the pre-trading phase followed by the trading phase and post-trading phase. Pre-trading phase and post-trading phase are the same for all securities, while different trading models and trading time schedules are applied in the trading phase according to the segmentation of securities.

In the pre-trading phase, traders can submit orders and quotes, and modify or delete existing orders and quotes. The order book is closed during this phase. Only the last trading price or the best sell/buy limits of the last auction of the previous day are displayed. In the post-trading phase, again, traders can submit orders and quotes, and modify or delete existing orders and quotes. No transactions happen after the post-trading phase and before the pre-trading phase.

Xetra supports two trading models in the trading phase:

1. Continuous trading in connection with auctions, which starts with an opening auction and is followed by continuous trading. Continuous trading can be interrupted by one or several intraday auction(s) and ends with either a closing auction or an end-of-day auction which is after an intraday closing auction followed by another segment of continuous trading.

2. One or more auctions per day at a predefined time schedule.

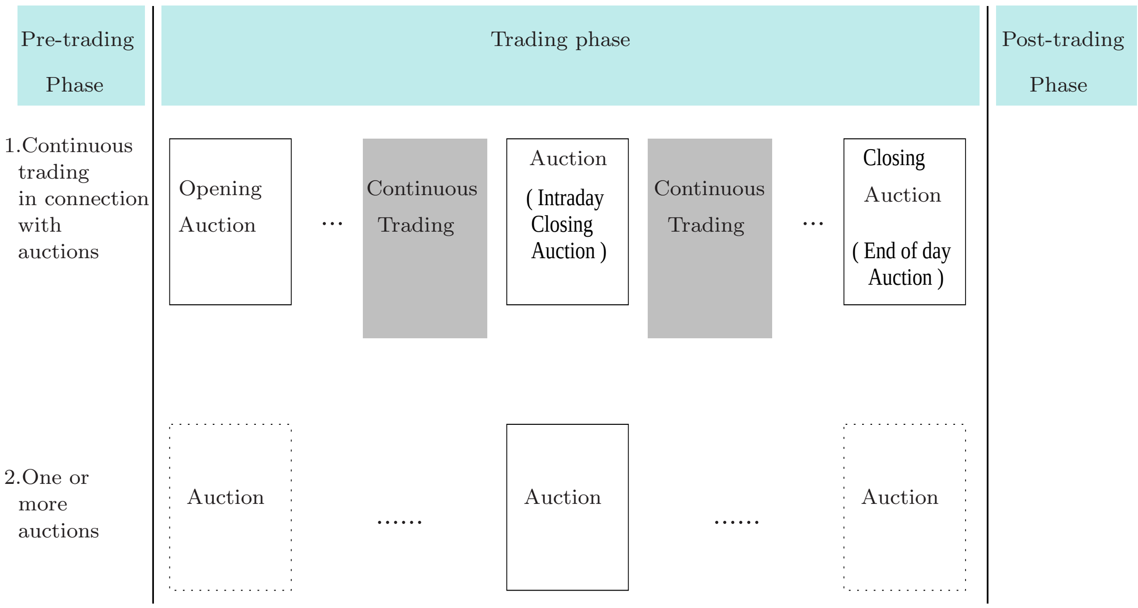


Figure 1: Two Xetra trading models.

As depicted above, these two Xetra trading models are essentially composed of the trading forms of continuous trading and Xetra auction.

Continuous trading accepts only round lot orders. Each new incoming order is immediately checked against the other side of the order book. The execution of orders is based on price/time priority. The order book is open during the course of continuous trading. Xetra discloses real-time trading data of the order book, e.g. limits, accumulated order volumes per limit, and the accumulated number of orders per limit.

Xetra auction accepts all order sizes and is composed of three phases: a call phase, a price determination phase, and an order book balancing phase. During the call phase, traders may submit order specifications. An order is tagged with a time-priority index and is collected in a central order book. The order book in the call phase remains partially closed and traders obtain the indicative price or the best sell/buy limits. In certain segmentation of securities, market imbalance information may be disseminated as well. The call phase has a random end after a fixed minimum time span in order to avoid price manipulation. It is followed by the price determination phase in which the auction price is determined according to Xetra auction trading rules. Once the auction price has been stipulated, orders are matched and transactions are carried out. For securities without market imbalance information, the surplus is offered again to the market in the order book balancing phase when not all orders in the order book can be fully executed. At the end of the auction process, all orders which were not or only partially executed are transferred to the next possible trading form or deleted according to their trading restrictions.

In essence, continuous trading and Xetra auction fall into the category of multi-unit double auction in which both sellers and buyers can submit order specifications in diversified trading quantity in order to trade on well-defined commodities or securities. The trading mechanism of continuous trading in Xetra is equivalent to a continuous double auction market known by experimentalists (known by practitioners as an open-out-cry market), and the trading mechanism of Xetra auction is equivalent to a clearinghouse (also known as a call market), cf. Friedman & Rust (1991). Xetra system is thus a hybrid market which contains the trading mechanism of continuous double auction and of clearinghouse.

Trading rules for Xetra auction and for continuous trading are explicitly depicted in Gruppe Deutsche Börse (2003). Despite the clarity of the corresponding trading rules in Xetra, literature in financial markets so far has provided little understanding on the nature of the formation of trading prices and final transactions in the system and on its implication for possible investment strategies. The price mechanism of electronic security markets has intuitively been described in Sharpe, Alexander & Bailey (1999), however, without formal rigor.

This work is intimately related to the theory of market microstructure which studies how the trading price and allocation are determined explicitly under the structure of specific trading mechanisms, e.g. see O'Hara (1995) and Stoll (2003). While a number of research in this strand has been engaged in the investigation of electronic trading platforms either from an empirical aspect, e.g. see Ahn, Bae & Chan (2001), Ellis, Michaely & O'Hara (2002), and Conrad, Johnson & Wahal (2003), or from an experimental aspect, e.g. see Sunder (1995) and Bloomfield, O'Hara & Saar (2005), only relatively little emphasis has been put on formalizing trading mechanisms of auctions in electronic security trading platforms so far. The lack of contribution in this topic is related to the fact that auctions in electronic security trading platforms belong to the type of multi-unit double auction since traders can submit an order specification with arbitrary quantity. Even in the scope of auction theory, the price determination and allocation mechanism in multi-unit double auction is still an open issue, see Friedman & Rust (1991) and Krishna (2002). Thus, the formalization of the trading mechanism in Xetra auction is essential both from the perspective of the theory of market microstructure and from that of auction theory.

Another concern in this work is the investment decision in Xetra auction market. Investment decision investigated in modern finance are generally based on the assumption of price taker, e.g. see Markowitz (1952), Merton (1992), and Horst & Wenzelburger (2008). This assumption considers that the investor's trading behavior has no impact on the market. On the other hand, Xetra auction determines the trading price according to the central order book which is an aggregation of orders submitted by traders. The trading price is dependent on the order specification that the trader submits. Thus the trader in Xetra auction market could potentially influence the trading price by its trading behavior, which implies that the trader is essentially not a price taker in Xetra auction market.

Xetra auction market provides traders with the knowledge on Xetra auction trading rules and the real-time trading information of the central order book which conventional floor markets are unlikely to provide. Given the knowledge on Xetra auction pricing rules, the price determination in Xetra auction market is no longer a 'black box' to traders. The trader can perceive the order book situation by combining its order to submit with the observation on the real-time order book. Then the trader can construct its forecast on the trading price by applying the knowledge on Xetra auction pricing rules to compute the trading price associated with its perceived order book. The trader's subjective forecast on the trading price has the control variable of the trader's order to submit. This implies that the trader is a price setter who can manipulate the trading price by its trading behavior.

To complete the investment decision model for the trader in Xetra auction market, it is necessary to incorporate the trader's forecast on the trading price with the portfolio selection model. We consider in this work the mean-variance model (M-V model in short), see Markowitz (1952) and Markowitz (1991). By integrating the forecast on the trading price with the M-V model, we construct an extended M-V model that depicts the investment decision of the price setter in Xetra auction market. As Xetra auction market only accepts integer shares of trading quantity, we develop a computational procedure to calculate the integer solution for the extended M-V model.

The last part of this work is to develop a financial market model for Xetra auction market where the interactions among traders and Xetra auction market mechanism generate the non market-clearing price dynamics. This requires an appropriate representation of complex interactions among traders and Xetra auction market mechanism in the dynamics of Xetra auction market.

The dynamic process of complex interactions among economic entities can be modelled by applying the methodology of Agent-based Computational Economics (ACE) that is a computational study on the dynamical economic system from 'bottom-up', see Tesfatsion (2006). Economists in this strand construct the ACE model for the economic system by modelling the interaction of agents that represent economic entities in the system.

ACE researchers have successfully handled complex financial market systems and have constructed agent-based financial market models with explicit forms of market mechanisms to determine market prices and trading volumes. For instance, Das (2003) introduced an agent-based financial market model which adopted a simple version of single-unit double auction as the explicit market mechanism to generate the market dynamics. Although the market mechanism considered in Das (2003) does not share the same type of auctions as Xetra auction market mechanism that belongs to the type of multi-unit double auction, the success of introducing explicit market mechanism in the agent-based financial market model suggests the possibility of applying the methodology of ACE modelling to construct the agent-based model with an explicit formulation of Xetra auction market mechanism.

The current difficulty of constructing agent-based models for financial market systems is the lack of general principles that economists can apply to construct agent-based models, see LeBaron (2006). In order to overcome this difficulty, we develop an integrative framework for ACE modelling that serves as general principles to investigate economic systems from 'bottom-up' and to construct the corresponding ACE models. Then we apply this integrative framework to construct the ACE model of Xetra auction market. With the implementation of the ACE model by employing the computer programming language, we conduct the computational simulation to generate the non market-clearing price dynamics of Xetra auction market.

The structure of this work is as follows. Chapter 1 presents the formal model of Xetra auction price mechanism and the allocation mechanism. Then it investigates the economic properties of Xetra auction allocation mechanism. Chapter 2 explores the economic properties of Xetra auction price mechanism. Then it develops a set of improved auction pricing rules that are regarded as an improvement on Xetra auction pricing rules from the perspective of market efficiency. Chapter 3 investigates the investment decision of the price setter in Xetra auction market. It starts with constructing the trader's subjective forecast on Xetra auction price. Then it presents the extended M-V model which integrates the conventional M-V model with the trader's subjective forecast on Xetra auction price. This chapter ends with the computational procedure for calculating the integer solution for the extended M-V model. Chapter 4 starts with the depiction on the integrative framework for ACE modelling. Then it applies this framework to construct the ACE model of Xetra auction market. With the implementation of the ACE model by employing the computer programming language Groovy/Java, this chapter presents simulation results of market dynamics. This work ends with concluding remarks presented in Chapter 5.

Chapter 1

Xetra Auction Market Mechanism

This chapter provides a formal model of Xetra auction market mechanism for limit orders and market orders. Xetra auction market mechanism handles several types of orders: limit orders, market orders, market-to-limit orders, iceberg orders, stop orders, and quotes. Market-to-limit orders are considered as market orders when entering the order book. Iceberg orders are considered as limit orders. Stop orders, depending on the imposed trading restrictions, are regarded as either market orders or limit orders when entering the order book. Quotes are handled as two simultaneous orders (a limit bid and a limit ask) in the order book. In essence, Xetra auction market mechanism handles two types of orders in the order book: limit orders and market orders. An order specification with a claim to sell is called an ask (limit/market) order, and an order with a claim to buy is called a bid (limit/market) order, see Gruppe Deutsche Börse (2003).

The market mechanism mentioned in the rest of this chapter refers to Xetra auction market mechanism, the price mechanism refers to Xetra auction price mechanism, and the allocation mechanism refers to Xetra auction allocation mechanism when no ambiguity happens. To model the market mechanism, we start from formalizing the concepts of limit orders and market orders in Section 1.1. Then we depict demand and supply schedule in Xetra auction market which are aggregation of limit orders and market orders. Section 1.2 describes Xetra auction pricing rules and proposes a formal presentation of the price mechanism. Section 1.3 formulates the allocation mechanism and investigates its properties.

1.1 Demand and Supply Schedule

During the call phase, Xetra auction collects all asks and bids submitted by traders in a central order book, tagged with a time-priority index. The call phase stops randomly after a fixed minimum time span. Assume that there are $I+1$ bids indexed by $i \in \{0, 1, \ldots, I\}$ and $J+1$ asks indexed by $j \in \{0, 1, \ldots, J\}$ when the call phase stops. In particular, assume that bid 0 and ask 0 are market orders while the rest are limit orders. Hence, $\{1, \ldots, I\}$ is the index set of limit bids and $\{1, \ldots, J\}$ the index set of limit asks.

Further assume that the order book situation is composed of bid 0 represented by a non-negative quantity $d_0 \geq 0$ and each bid $i \in \{1, \ldots, I\}$ by a price-quantity pair (a_i, d_i) stating the intention to buy $d_i \geq 0$ shares when the Xetra auction price is no higher than $a_i > 0$; ask 0 is represented by a non-negative quantity $s_0 \geq 0$ and ask $j \in \{1, \ldots, J\}$ by a price-quantity pair (b_j, s_j) stating the intention to sell $s_j \geq 0$ shares when the auction price is no lower than $b_j > 0$.

The price determination phase is invoked right after the call phase. The order book situation at the end of the call phase as well as a **reference price** P_{ref} referred to the last trading price in Xetra are regarded as information inputs in the market mechanism. Thus, the information inflows to the market mechanism is assumed as follows.

Assumption 1.1 (Assumption of Order Book)**.** *At the end of the call phase, Xetra auction market mechanism obtains an order book data set $\mathfrak{I}_0$ which contains all data of the order book and a reference price P_{ref} from Xetra:*

$$\mathfrak{I}_0 := \{d_0, (a_1, d_1), \ldots, (a_I, d_I); s_0, (b_1, s_1), \ldots, (b_J, s_J); P_{\text{ref}}\}. \tag{1.1}$$

Order specifications contained in $\mathfrak{I}_0$ constitute the demand and the supply schedule of the market when the trading price and the trading volume are determined. We start from the formal presentation of individual bids and asks, i.e. individual demand and supply schedules.

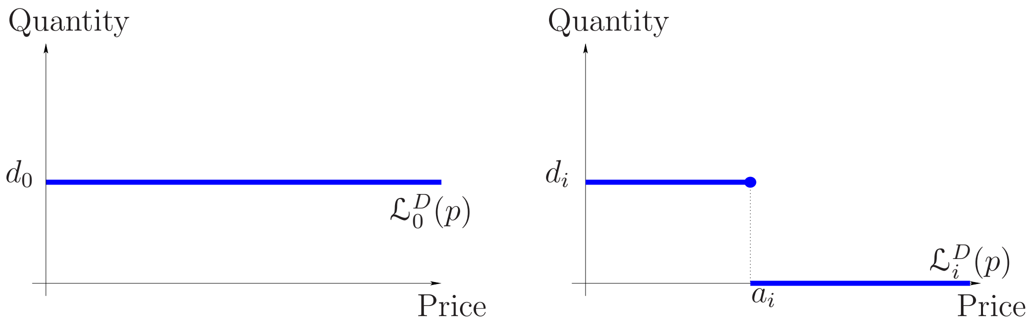
1.1.1 Demand-to-buy Schedule

For bid 0 which is a market order with a non-negative quantity d_0, the corresponding individual demand function is defined as:

$$\mathcal{L}_0^D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto d_0. \end{cases}$$

Each bid $i \in \{1, \ldots, I\}$ consists of a price-quantity pair (a_i, d_i). Let $1_{A_i^D}(p)$ denote a characteristic function of the compact interval $A_i^D = [0, a_i]$ such that

$$1_{A_i^D}(p) := \begin{cases} 1 & \text{when } \; p \in A_i^D, \\ 0 & \text{when } \; p \in \mathbb{R}_+ \setminus A_i^D. \end{cases}$$



(a) Market order: bid 0 (b) Limit order: bid i

Figure 1.1: Individual demand function.

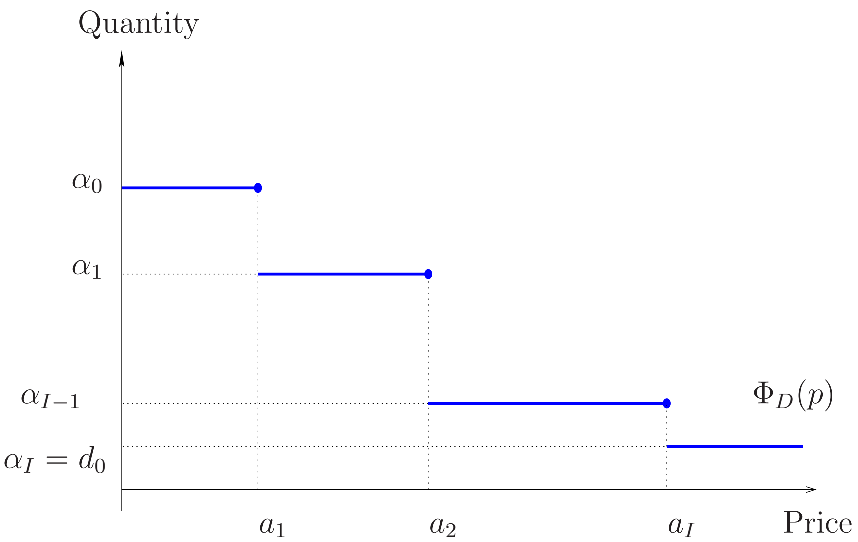


Figure 1.2: Aggregate demand function.

Then the individual demand function which represents bid i is the step function:

$$\mathcal{L}_i^D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto d_i 1_{A_i^D}(p). \end{cases} \tag{1.2}$$

The aggregate demand function is defined as the sum of individual demand functions:

$$\Phi_D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \sum_{i=0}^{I} \mathcal{L}_i^D(p). \end{cases} \tag{1.3}$$

Without loss of generality, assume $a_I > \ldots > a_2 > a_1 > 0$. Then we obtain the explicit form of aggregate demand function as follows:

Lemma 1.1. *Let $a_I > \ldots > a_2 > a_1 > 0$. The aggregate demand function $\Phi_D(p)$ is non-increasing and takes the form:*

$$\Phi_D(p) = \sum_{i=0}^{I} \alpha_i 1_{A_i}(p), \quad p \in \mathbb{R}_+, \tag{1.4}$$

where $\alpha_i := d_0 + \sum_{k=i+1}^{I} d_k$ for $i = 0, 1, \ldots, I-1$; $\alpha_I := d_0$; and $A_0 := [0, a_1]$; $A_i := (a_i, a_{i+1}]$ for $i = 1, \ldots, I-1$; $A_I := (a_I, +\infty)$.

Proof. The market order of bid 0 is executable for any $p \in \mathbb{R}_+$. $\{A_0, \ldots, A_I\}$ is by construction a partition of $\mathbb{R}_+$. Let $i_* \in \{0, \ldots, I-1\}$ be arbitrary but fixed. Then $p \in A_{i_*}$ implies that all bids $i = 0, i_*+1, \ldots, I$ are executable. The corresponding aggregate volume is $\alpha_{i_*} = d_0 + \sum_{k=i_*+1}^{I} d_k$. $p \in A_I$ implies that only bid 0 is executable. The corresponding aggregate volume is $\alpha_I = d_0$. This establishes the explicit form of aggregate demand function. Φ_D is non-increasing since $\alpha_0 > \alpha_1 > \ldots > \alpha_I$. □

1.1.2 Supply-to-sell Schedule

For ask 0 which is a market order with a non-negative quantity s_0, the corresponding individual supply function is defined as:

$$\mathcal{L}_0^S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto s_0. \end{cases}$$

Each ask $j \in \{1, \ldots, J\}$ consists of a price-quantity pair (b_j, s_j). Let $1_{B_j^S}(p)$ denote the characteristic function of the interval $B_j^S = [b_j, +\infty)$, the individual supply function which represents ask j is then the step function:

$$\mathcal{L}_j^S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto s_j 1_{B_j^S}(p). \end{cases} \tag{1.5}$$

The aggregate supply function is defined as the sum of individual supply functions

$$\Phi_S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \sum_{j=0}^{J} \mathcal{L}_j^S(p). \end{cases} \tag{1.6}$$

Without loss of generality, let $b_J > \ldots > b_2 > b_1 > 0$. Then we obtain the explicit form of the aggregate supply function as follows.

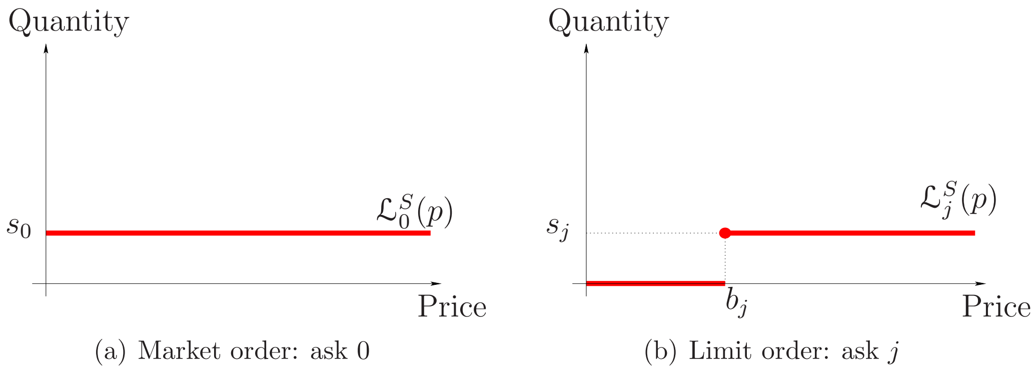


Figure 1.3: Individual supply function.

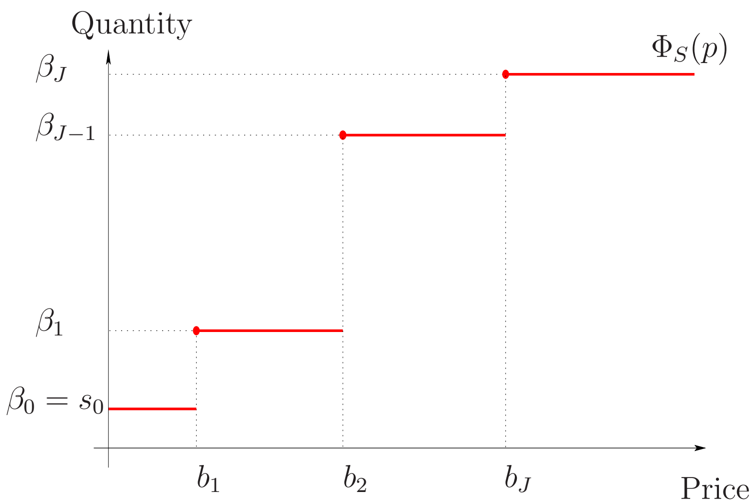


Figure 1.4: Aggregate supply function.

Lemma 1.2. *Let $b_J > \ldots > b_2 > b_1 > 0$. The aggregate supply function $\Phi_S(p)$ is non-decreasing and takes the form:*

$$\Phi_S(p) = \sum_{j=0}^{J} \beta_j 1_{B_j}(p), \quad p \in \mathbb{R}_+, \tag{1.7}$$

where $\beta_0 := s_0$; $\beta_j := s_0 + \sum_{k=1}^{j} s_k$ for $j = 1, \ldots, J$; and $B_0 := [0, b_1)$; $B_j := [b_j, b_{j+1})$ for $j = 1, \ldots, J-1$; $B_J := [b_J, +\infty)$.

Proof. The market order of ask 0 is executable for any $p \in \mathbb{R}_+$. $\{B_0, \ldots, B_J\}$ is by construction a partition of $\mathbb{R}_+$. $p \in B_0$ implies that only ask 0 is executable. The corresponding aggregate volume is $\beta_0 = s_0$. Let $j_* \in \{1, \ldots, J\}$ be arbitrary but fixed. Then $p \in B_{j_*}$ implies that all asks $j = 0, 1, \ldots, j_*$ are executable. The corresponding aggregate volume is $\beta_{j_*} = s_0 + \sum_{k=1}^{j_*} s_k$. This establishes the explicit form of aggregate supply function. Φ_S is non-decreasing since $\beta_J > \ldots > \beta_1 > \beta_0$. □

1.2 Xetra Auction Price Mechanism

Xetra auction price is determined according to a set of well-specified pricing rules described in (Gruppe Deutsche Börse 2003, p. 34). We denote **Xetra auction price** as P_{Xetra} and summarize Xetra auction pricing rules as follows:

We denote the limit price with the highest executable order volume and the lowest surplus as the **candidate price**.

Rule 1. The auction price is the candidate price if there is only one candidate price.

Rule 2. If there is more than one candidate price, then there are four cases:

Rule 2.1. If the surplus for all candidate prices is on the demand side, then the auction price is stipulated as the highest candidate price.

Rule 2.2. If the surplus for all candidate prices is on the supply side, then the auction price is stipulated as the lowest candidate price.

Rule 2.3. If there is no surplus for all candidate prices, the reference price P_{ref} is included as an additional criterion. The auction price is determined as follows:

Rule 2.3.1. The auction price is the highest candidate price if the reference price is higher than the highest candidate price.

Rule 2.3.2. The auction price is the lowest candidate price if the reference price is lower than the lowest candidate price.

Rule 2.3.3. The auction price is the reference price if the reference price lies between the highest candidate price and the lowest candidate price.

Rule 2.4. If there are some candidate prices with a surplus on the supply side and others with a surplus on the demand side, then the **upper bound price** is chosen as the lowest candidate price with a surplus on the supply side and the **lower bound price** is chosen as the highest candidate price with a surplus on the demand side. The lower bound price is always less than the upper bound price as we will show later. Xetra determines the auction price with these two prices and the reference price P_{ref}:

Rule 2.4.1. The auction price is the upper bound price if the reference price is higher than the upper bound price.

Rule 2.4.2. The auction price is the lower bound price if the reference price is lower than the lower bound price.

Rule 2.4.3. The auction price is the reference price if the reference price lies between the upper bound price and the lower bound price.

Rule 3. If there are only market orders on both sides of the order book, then the auction price is the reference price P_{ref}.

Rule 4. If Rule 1 to Rule 3 fails, there is no auction price.

1.2.1 Trading Volume and Surplus

Let $p \in \mathbb{R}_+$ be some arbitrary price such that the aggregate demand $\Phi_D(p)$ may be unequal to the aggregate supply $\Phi_S(p)$, then only the minimum of $\Phi_D(p)$ and $\Phi_S(p)$ could possibly be traded in Xetra auction. The quantity which is feasible to trade is called the **executable order volume** and is defined by:

$$\Phi_V : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \min\{\Phi_D(p),\ \Phi_S(p)\}. \end{cases} \tag{1.8}$$

The function (1.8) is also referred as the **trading volume function**. The **excess demand function** is given by:

$$\Phi_Z : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R} \\ p & \longmapsto \Phi_D(p) - \Phi_S(p). \end{cases} \tag{1.9}$$

Xetra auction refers the absolute value of excess demand $|\Phi_Z(p)|$ as the **surplus**.

1.2.2 Xetra Auction Price Model

Given the data set $\mathcal{I}_0$ of Assumption 1.1, Xetra auction determines a unique auction price P_{Xetra} by applying Xetra auction pricing rules. Only limit prices are taken into account by Xetra auction pricing rules. Let Ω_0 denote the set of all limit prices considered in $\mathcal{I}_0$. When there is no limit price in the order book, i.e.

$\Omega_0 = \emptyset$, Xetra auction checks if there are market orders on both sides (demand side and supply side) of the order book. Xetra auction then chooses the reference price P_{ref} as the auction price when market orders exist on both sides, i.e. $d_0 > 0$ and $s_0 > 0$. When market orders do not exist in both sides, there is no auction price in Xetra auction market.[1]

When there exists limit prices in the order book with $\Omega_0 = \{a_1, \ldots, a_I, b_1, \ldots, b_J\}$, Xetra auction computes the highest executable order volume $V_{\max}$ which is the maximum value of trading volume $\Phi_V(p)$ for all limit price $p \in \Omega_0$ such that

$$V_{\max} := \max \{\Phi_V(p) \mid p \in \Omega_0\}. \tag{1.10}$$

Notice that $V_{\max}$ exists and is finite since $\Phi_V(p)$ is finite and bounded in Ω_0. The set of **volume maximizing prices** is defined by

$$\Omega_V := \{p \in \Omega_0 \mid \Phi_V(p) = V_{\max}\},$$

which is non-empty and finite. When the order book is uncrossed with $V_{\max} = 0$, the executable order volume is zero. No transaction will be carried out in Xetra auction, and hence no Xetra auction price exists in the market.

When there exists executable order volume with $V_{\max} > 0$ and a non-empty set Ω_0, Xetra auction computes the lowest surplus $Z_{\min}$ defined by

$$Z_{\min} := \min \big\{ |\Phi_Z(p)| \bigm| p \in \Omega_V \big\}. \tag{1.11}$$

Candidate prices that correspond to the highest executable order volume and the lowest surplus are given by

$$\Omega_Z := \big\{ p \in \Omega_V \bigm| |\Phi_Z(p)| = Z_{\min} \big\}.$$

Since Ω_V is well defined and finite, $Z_{\min}$ exists and Ω_Z is also well defined and non-empty. Denote

$$\overline{P_Z} := \max \Omega_Z \quad \text{and} \quad \underline{P_Z} := \min \Omega_Z$$

as the highest candidate price and the lowest candidate price in Ω_Z respectively. When there is only one candidate price, the unique candidate price is chosen as Xetra auction price with $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$.

[1] In this case, Xetra auction will take a market order interruption: the call phase will be extended for a limited time span and market participants can submit new order specifications and quotes, modify or delete existing order specifications and quotes. The call phase terminates once all market orders in the order book can be fully executed or the time extension has expired, see (Gruppe Deutsche Börse 2003, p. 30).

Obviously, there could be more than one candidate price with $\sharp\Omega_Z > 1$. According to Rule 2.1, when all candidate prices are with a surplus on the demand side, i.e. $\Phi_Z(p) > 0$ for all $p \in \Omega_Z$, the highest candidate price $\overline{P_Z}$ is chosen as Xetra auction price with $P_{\text{Xetra}} = \overline{P_Z}$.

According to Rule 2.2, when all candidate prices are with a surplus on the supply side, i.e. $\Phi_Z(p) < 0$ for all $p \in \Omega_Z$, the lowest candidate price $\underline{P_Z}$ is chosen as Xetra auction price with $P_{\text{Xetra}} = \underline{P_Z}$.

Otherwise, there could be no surplus for all candidate prices or there could be a surplus on the demand side for some candidate prices while others with a surplus on the supply side. In these two cases, the reference price P_{ref} is introduced to determine Xetra auction price.

In the first case that corresponds to Rule 2.3, Xetra auction compares the reference price P_{ref} with the highest candidate price $\overline{P_Z}$ and the lowest candidate price $\underline{P_Z}$ to choose one of these three prices as Xetra auction price according to pricing rules described in Rule 2.3.1, Rule 2.3.2, and Rule 2.3.3.

In the second case that corresponds to Rule 2.4, Xetra auction chooses the upper bound price $P_{\max}$ that is the lowest candidate price with a surplus on the supply side and the lower bound price $P_{\min}$ that is the highest candidate price with a surplus on the demand side. The upper bound price and the lower bound price are formulated as:

$$\begin{aligned} P_{\max} &:= \min\{p \in \Omega_Z \mid \Phi_Z(p) = -Z_{\min}\}, \\ P_{\min} &:= \max\{p \in \Omega_Z \mid \Phi_Z(p) = Z_{\min}\}. \end{aligned}$$

The upper bound price $P_{\max}$ and the lower bound price $P_{\min}$ are well defined in this case. Notice that $P_{\max} > P_{\min}$ since $\Phi_Z(p)$ is non-increasing and $\Phi_Z(P_{\max}) < 0 < \Phi_Z(P_{\min})$. Xetra auction compares the reference price P_{ref} with the upper bound price $P_{\max}$ and the lower bound price $P_{\min}$ to choose one of these three prices as Xetra auction price according to pricing rules stated in Rule 2.4.1, Rule 2.4.2, and Rule 2.4.3. Xetra auction pricing rules are summarized in the following theorem.

Theorem 1.1. *Given the order book data set $\mathfrak{I}_0$ in Assumption 1.1, Xetra auction determines a unique auction price P_{Xetra} whenever it is possible.*

If $\Omega_0 \neq \emptyset$ and $V_{\max} > 0$, then P_{Xetra} exists and is determined as follows:

(i) If $\sharp\Omega_Z = 1$, then $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$;

(ii) If $\sharp\Omega_Z > 1$, then

$$P_{\text{Xetra}} = \begin{cases} \overline{P_Z}, & \textit{if} \;\; \Phi_Z(\overline{P_Z}) > 0, \\ \underline{P_Z}, & \textit{if} \;\; \Phi_Z(\underline{P_Z}) < 0, \\ \max\{\underline{P_Z}, \min\{P_{\text{ref}}, \overline{P_Z}\}\}, & \textit{if} \;\; \Phi_Z(\underline{P_Z}) = \Phi_Z(\overline{P_Z}) = 0, \\ \max\{P_{\min}, \min\{P_{\text{ref}}, P_{\max}\}\}, & \textit{if} \;\; \Phi_Z(\underline{P_Z}) > 0 \textit{ and } \;\; \Phi_Z(\overline{P_Z}) < 0; \end{cases} \tag{1.12}$$

where $\overline{P_Z} = \max \Omega_Z$, $\underline{P_Z} = \min \Omega_Z$, *and* $P_{\max} = \min\{p \in \Omega_Z \mid \Phi_Z(p) = -Z_{\min}\}$ *with* $P_{\min} = \max\{p \in \Omega_Z \mid \Phi_Z(p) = Z_{\min}\}$.

If only market orders exist such that $\Omega_0 = \emptyset$ *with* $d_0 > 0$ *and* $s_0 > 0$, *then the auction price is* $P_{\text{Xetra}} = P_{\text{ref}}$.

Otherwise, the auction price P_{Xetra} *remains unspecified.*

Proof. $\Omega_0 \neq \emptyset$ implies that there exists at least one limit order, and $V_{\max} > 0$ implies that $\Omega_Z \neq \emptyset$. Thus, Rule 1 and Rule 2 are considered when $V_{\max} > 0$ and $\Omega_0 \neq \emptyset$.

Rule 1 states that $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$ when $\sharp\Omega_Z = 1$, and Rule 2 describes four cases when $\sharp\Omega_Z > 1$, which corresponds to equation (1.12). Rule 2.1 corresponds to the case of all price $p \in \Omega_Z$ having the same surplus on the demand side for $\Phi_Z(p) > 0$. Xetra auction price under this rule is $P_{\text{Xetra}} = \overline{P_Z}$ when $\Phi_Z(\overline{P_Z}) > 0$. Analogously, Rule 2.2 corresponds to all price $p \in \Omega_Z$ having the same surplus on the supply side for $\Phi_Z(p) < 0$. Xetra auction price under this rule is $P_{\text{Xetra}} = \underline{P_Z}$ when $\Phi_Z(\underline{P_Z}) < 0$. According to Rule 2.3, Xetra auction price is $P_{\text{Xetra}} = \max\left\{\underline{P_Z}, \min\{P_{\text{ref}}, \overline{P_Z}\}\right\}$ when there is no surplus for all $p \in \Omega_Z$ with $\Phi_Z(\underline{P_Z}) = \Phi_Z(\overline{P_Z}) = 0$. Rule 2.4 states the case of $\Phi_Z(\overline{P_Z}) < 0$ and $\Phi_Z(\underline{P_Z}) > 0$. Rule 2.4.1, Rule 2.4.2, and Rule 2.4.3 stipulate Xetra auction price as $P_{\text{Xetra}} = \max\{P_{\min}, \min\{P_{\text{ref}}, P_{\max}\}\}$.

Rule 3 refers to the case that market orders exist on both market sides with no limit orders in the order book, i.e. $d_0 > 0$ and $s_0 > 0$ with $\Omega_0 = \emptyset$. Xetra auction price in this case is $P_{\text{Xetra}} = P_{\text{ref}}$. Rule 4 is applied when Rule 1 to Rule 3 fail. There is no Xetra auction price P_{Xetra} in this case. □

Theorem 1.1 is a formal characterization of Xetra auction pricing rules. Given any order book situation, Theorem 1.1 determines a unique Xetra auction price P_{Xetra} whenever it exists. The price determination process for Xetra auction is depicted in the form of flowchart in Appendix A. We demonstrate in the following example how to apply Theorem 1.1 to determine Xetra auction price P_{Xetra}.

Example 1.1. Consider an order book containing three bid orders with price-quantity pairs $(198, 100)$, $(200, 100)$, and $(201, 150)$. It also contains one ask order with the price-quantity pair $(199, 100)$. The order book situation is shown in Figure 1.5.

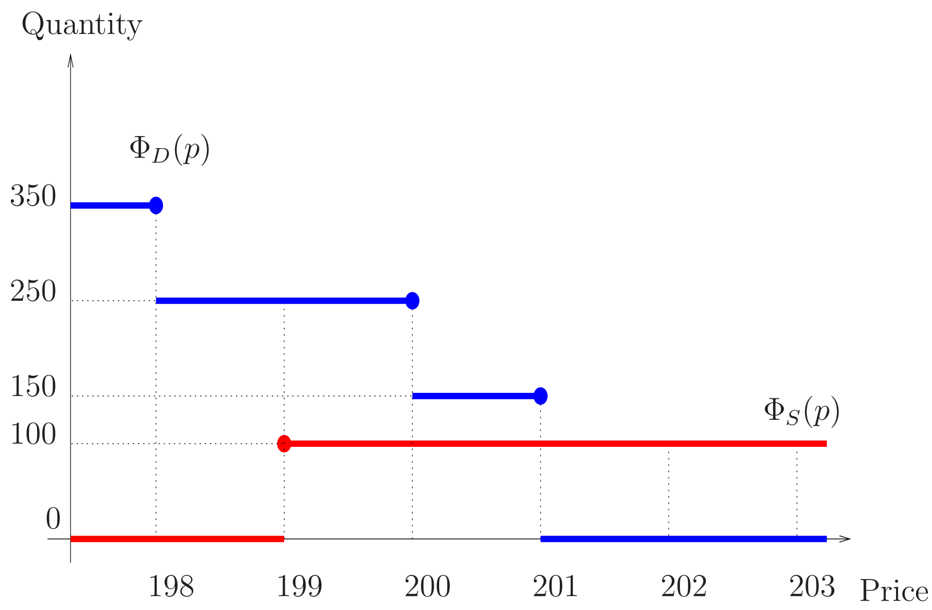


Figure 1.5: Crossed order book in Example 1.1.

Xetra auction first computes $V_{\max} = 100$ for $\Omega_0 = \{198, 199, 200, 201\} \neq \emptyset$. Since $V_{\max} = 100 > 0$, the set of all volume maximizing prices is $\Omega_V = \{199, 200, 201\}$. The lowest surplus is $Z_{\min} = 50$. The set of all candidate prices is $\Omega_Z = \{201\}$ where the highest candidate price is equal to the lowest candidate price with $\overline{P_Z} = \underline{P_Z} = 201$. Since $\sharp\Omega_Z = 1$, Xetra auction price is $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z} = 201$ according to case (i) in Theorem 1.1. □

1.3 Xetra Auction Allocation Mechanism

Given Xetra auction price P_{Xetra}, Xetra auction allocation mechanism computes the final transaction for each order specification.

1.3.1 Xetra Auction Allocation Model

The order execution in Xetra auction is based on the time priority. When an order is submitted to the order book, it is labelled with a time tag. The time tag attached to each order determines the time priority of the order and thus the ranking of its execution in the order book. There are two execution sequences corresponding to the demand side (bids) and the supply side (asks).

Denote the execution priority of bid i by $\iota_d(i) \in \{0, 1, \ldots, I\}$ and the execution priority of ask j by $\iota_s(j) \in \{0, 1, \ldots, J\}$ respectively. The position in the execution sequence of bid i is then $\iota_d(i)$, which implies that there are $\iota_d(i)$ bids which will be executed prior to bid i. Analogously, there are $\iota_s(j)$ asks which will be executed prior to ask j. Market orders always have higher execution priority than limit orders, thus bid 0 and ask 0 are positioned with the ranking of $\iota_d(0) = 0$ and $\iota_s(0) = 0$. The final transaction for each order is highly affected by its position in the execution sequence since Xetra auction applies the rule of **First Come First Serve (FCFS)** for the order execution.[2]

Given the fixed ranking of the execution sequence, bid i will not be executed until all higher ranked bids are executed. The maximum feasible quantity that bid i can get is therefore the quantity that higher ranked bids have left over, i.e. the positive difference between the highest executable order volume $\Phi_V(P_{\text{Xetra}}) = V_{\max}$ and the aggregation of the order volume realized prior to bid i. Thus, the maximum feasible quantity for bid $i \in \{0, 1, \ldots, I\}$ is given by:

$$\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}}) := \max\Big\{0, \Phi_V(P_{\text{Xetra}}) - \sum_{m=0}^{\iota_d(i)-1} \mathfrak{L}^D_{\iota_d^{-1}(m)}(P_{\text{Xetra}})\Big\}, \tag{1.13}$$

where $\iota_d^{-1}(m)$ denotes the index of the bid in position m of the execution sequence. If the individual demand $\mathfrak{L}_i^D(P_{\text{Xetra}})$ of bid i is no greater than $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$, then bid i is fully realized. The final transaction of bid i is thus $\mathfrak{L}_i^D(P_{\text{Xetra}})$. If $\mathfrak{L}_i^D(P_{\text{Xetra}})$ is greater than $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$, bid i can only be partially executed. The final transaction is thus $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$ and bid i is rationed. We denote the final transaction of bid i as:

$$X_i^D(P_{\text{Xetra}}) := \min\Big\{\mathfrak{L}_i^D(P_{\text{Xetra}}), \bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I. \tag{1.14}$$

In the supply side, the maximum feasible quantity for ask $j \in \{0, 1, \ldots, J\}$ is the positive difference between the highest executable order volume $\Phi_V(P_{\text{Xetra}})$

[2]FCFS is sometimes called First In First Out (FIFO).

$= V_{\max}$ and the aggregation of the order volume realized prior to ask j. The maximum feasible quantity for ask j is given by:

$$\bar{\mathcal{L}}_j^S(P_{\text{Xetra}}) = \max\Big\{0, \Phi_V(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathcal{L}_{\iota_s^{-1}(n)}^S(P_{\text{Xetra}})\Big\}, \tag{1.15}$$

where $\iota_s^{-1}(n)$ denotes the index of the ask in position n of the execution sequence. We denote the final transaction of ask j as:

$$X_j^S(P_{\text{Xetra}}) := \min\Big\{\mathcal{L}_j^S(P_{\text{Xetra}}), \bar{\mathcal{L}}_j^S(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J. \tag{1.16}$$

Xetra auction allocation mechanism is summarized as follows:

Theorem 1.2. *Given the order book data set $\mathcal{I}_0$ in Assumption 1.1, if P_{Xetra} exists, then the final transaction $X_i^D(P_{\text{Xetra}})$ for each bid $i = 0, 1, \ldots, I$ and $X_j^S(P_{\text{Xetra}})$ for each ask $j = 0, 1, \ldots, J$ are stated as:*

$$\begin{cases} X_i^D(P_{\text{Xetra}}) = & \min\Big\{\mathcal{L}_i^D(P_{\text{Xetra}}), \bar{\mathcal{L}}_i^D(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I; \\ X_j^S(P_{\text{Xetra}}) = & \min\Big\{\mathcal{L}_j^S(P_{\text{Xetra}}), \bar{\mathcal{L}}_j^S(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J. \end{cases} \tag{1.17}$$

The allocation mechanism (1.17) implies that the aggregate final transaction of bids is equal to the aggregate final transaction of asks in Xetra auction with

$$\sum_{i=0}^{I} X_i^D(P_{\text{Xetra}}) = \sum_{j=0}^{J} X_j^S(P_{\text{Xetra}}) = \Phi_V(P_{\text{Xetra}}) = V_{\max}.$$

The market-clearing situation is included in Xetra auction allocation mechanism as a special case where all orders are fully executed with

$$\begin{cases} X_i^D(P_{\text{Xetra}}) = & \mathcal{L}_i^D(P_{\text{Xetra}}), \quad i = 0, 1, \ldots, I; \\ X_j^S(P_{\text{Xetra}}) = & \mathcal{L}_j^S(P_{\text{Xetra}}), \quad j = 0, 1, \ldots, J. \end{cases} \tag{1.18}$$

1.3.2 Properties of Xetra Auction Allocation Mechanism

Xetra auction allocation mechanism satisfies some well-known properties of rationing mechanisms which are allocation mechanisms under the assumption of fixed market prices, see Benassy (1982) or Böhm (1989) for more details.

Voluntary exchange. The property of voluntary exchange states that no trader is forced to trade more than that she claims. Intuitively, this property holds for Xetra auction by the definition of Xetra order specifications. More formally, the allocation mechanism (1.17) satisfies such property because for all bids i and all asks j,

$$\begin{aligned} X_i^D(P_{\text{Xetra}}) &\leq \mathfrak{L}_i^D(P_{\text{Xetra}}), \\ X_j^S(P_{\text{Xetra}}) &\leq \mathfrak{L}_j^S(P_{\text{Xetra}}). \end{aligned}$$

The short-side rule. According to Benassy (1982), the 'short' side of a market is the market side where the aggregate trading volume is the smallest. It is thus the demand side if there is excess supply in the market and vice versa. The other side is called the 'long' side.

An allocation mechanism is called '**efficient**' or frictionless if no mutually advantageous trade can be carried out from the transaction attained. This implies that all traders on the short side will fully realize their claims.

Embedding with the property of voluntary exchange, we obtain the so-called '**short-side rule**' which states that all traders on the short side will fully realize their individual demand (supply). The allocation mechanism (1.17) satisfies the short-side rule if

$$\Phi_D(P_{\text{Xetra}}) \geq \Phi_S(P_{\text{Xetra}}) \;\Rightarrow\; X_j^S(P_{\text{Xetra}}) = \mathfrak{L}_j^S(P_{\text{Xetra}}), \quad \forall j; \qquad (1.19)$$

$$\Phi_D(P_{\text{Xetra}}) \leq \Phi_S(P_{\text{Xetra}}) \;\Rightarrow\; X_i^D(P_{\text{Xetra}}) = \mathfrak{L}_i^D(P_{\text{Xetra}}), \quad \forall i. \qquad (1.20)$$

Clearly, $\Phi_D(P_{\text{Xetra}}) \geq \Phi_S(P_{\text{Xetra}})$ implies $\Phi_V(P_{\text{Xetra}}) = \Phi_S(P_{\text{Xetra}})$ and hence

$$\bar{\mathfrak{L}}_j^S(P_{\text{Xetra}}) = \Phi_S(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathfrak{L}_{\iota_s^{-1}(n)}^S(P_{\text{Xetra}}) \geq \mathfrak{L}_j^S(P_{\text{Xetra}}), \quad j = 0, 1, \ldots, J,$$

which implies that (1.19) holds. Analogous argument holds for (1.20).

Manipulability. An allocation mechanism is called **non-manipulable** in quantity if the final transaction of a trader, when she is rationed, faces a bound which depends solely on quoted quantities of other traders that she can not manipulate. It is called manipulable in quantity if the trader can, when she is rationed, increase her final transaction by increasing the quoted quantity. Intuitively, non-manipulability implies that the quantity quoted by a trader has no impact on her maximum feasible quantity and vice versa.

Order specifications in Xetra auction face upper bounds $\bar{\mathcal{L}}_i^D(P_{\text{Xetra}})$ and $\bar{\mathcal{L}}_j^S(P_{\text{Xetra}})$ for their final transactions, should they be rationed. In the case of excess demand with $\Phi_D(P_{\text{Xetra}}) > \Phi_S(P_{\text{Xetra}})$, only bids are rationed. The maximum feasible quantity of bid i is

$$\bar{\mathcal{L}}_i^D(P_{\text{Xetra}}) = \max\Big\{0, \Phi_S(P_{\text{Xetra}}) - \sum_{m=0}^{\iota_d(i)-1} \mathcal{L}^D_{\iota_d^{-1}(m)}(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I;$$

which is independent of its individual quantity $\mathcal{L}_i^D(P_{\text{Xetra}})$.

Analogously, only asks are rationed in the case of excess supply with $\Phi_S(P_{\text{Xetra}}) > \Phi_D(P_{\text{Xetra}})$. The maximum feasible quantity of ask j is

$$\bar{\mathcal{L}}_j^S(P_{\text{Xetra}}) = \max\Big\{0, \Phi_D(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathcal{L}^S_{\iota_s^{-1}(n)}(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J;$$

which is independent of its individual quantity $\mathcal{L}_j^S(P_{\text{Xetra}})$.

It appears at first sight that this observation implies that the allocation mechanism is non-manipulable in the sense of classical rationing theory. However, traders do influence Xetra auction price through their order specifications. Hence, the situation in Xetra auction is more complicated than that in classical rationing theory in which prices are presumably fixed. To attain a profound understanding of the property of manipulability in Xetra auction, one has to further investigate the relationship between individual order specifications and Xetra auction price, which is carried out later in Chapter 3.

In this chapter we have established formal models for Xetra auction price mechanism and the allocation mechanism, which shed light on the future research on the market microstructure of multi-unit double auction and provide some hints on how to formalize market mechanisms in other auction markets of electronic security trading systems, e.g. Shanghai (2006) and Shenzhen (2006).

Chapter 2

Improvement on Auction Price Mechanism

This chapter analyzes the properties of Xetra auction price mechanism and presents an improvement on Xetra auction pricing rules from the perspective of market efficiency which concerns the trading price being market-clearing. Section 2.1 investigates the properties of Xetra auction price mechanism. Section 2.2 develops an improvement on Xetra auction pricing rules.

2.1 Properties of Xetra Auction Price Mechanism

In conventional microeconomic literature, e.g. Pindyck & Rubinfeld (2001), the aggregate demand and the aggregate supply schedule are represented by continuous curves in the price-quantity space. The market price and the trading volume are determined by the intersection point of the demand curve and the supply curve. The continuity of these two curves ensures that the market price has the property of the highest trading volume and zero surplus simultaneously. The market price of this type is called market-clearing in the sense that the quantity supplied is equal to the quantity demanded, i.e. zero surplus.

Compared with the conventional benchmark of equilibrium market system, the aggregate demand function and the aggregate supply function in Xetra auction market are step functions which are discontinuous. As stated in Gruppe Deutsche Börse (2003), Xetra auction price mechanism follows two principles to determine Xetra auction price:

1. The auction price is associated with the highest executable order volume;
2. The auction price is associated with the lowest surplus.

These two principles ensure Xetra auction to achieve an auction price close to market-clearing but cannot rule out the possibility of a non market-clearing trading price. We thus obtain the first property of Xetra auction price mechanism as follows:

Property 2.1 (Existence of non market-clearing price)**.** *Xetra auction price is generically non market-clearing.*

The property of non market-clearing price implies that the 'real' Xetra market may not be Pareto efficient all the time. This property exists not only in Xetra auction but also in another electronic security trading platform Euronext, see (Euronext 2006, p. 42). Trading rules stipulated in Shanghai (2006) and Shenzhen (2006) also imply that auction market prices in these two biggest security markets in China are generically non market-clearing. Thus, one should consider the potential effect of non market-clearing trading price when conducting economic investigation based on 'real' Xetra system or other financial market systems which share the same property; e.g. when empirically testing the efficient market hypothesis (EMH) in Xetra auction market under the assumption of market-clearing price.

Apart from the reference price P_{ref}, Xetra auction price mechanism only considers limit prices for possible auction price. This reflects that Xetra is an order-driven system such that limit prices submitted by traders are the main driving factors of Xetra auction price determination. On the other hand, the restriction on considering only limit prices in Xetra auction price mechanism excludes the possibility of a market-clearing trading price whenever there exists a market-clearing price but not a limit price. This can be seen from the following example which corresponds to Example 4 in Gruppe Deutsche Börse (2003).

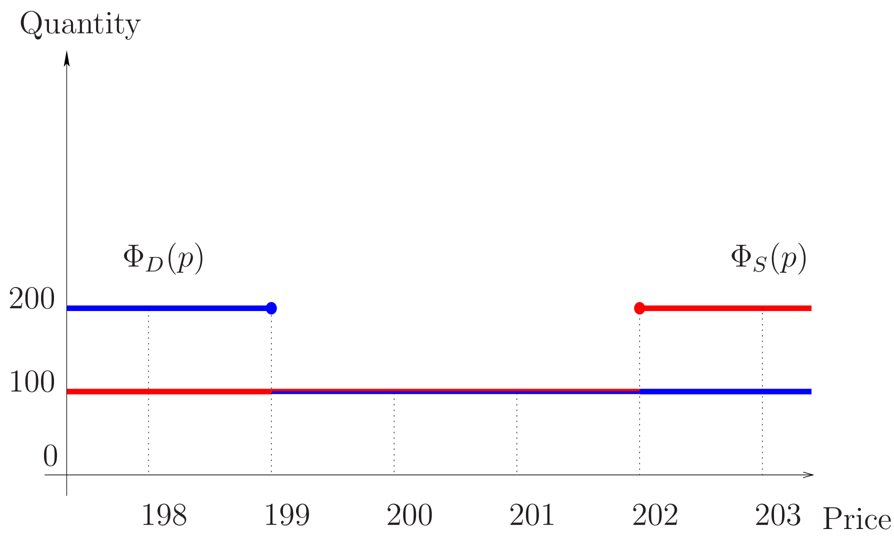


Figure 2.1: Crossed order book in Example 2.1.

Example 2.1. Consider an order book with one market order for buying 100 shares, another market order for selling 100 shares, one bid order with the price-quantity pair $(199, 100)$, and one ask order with the price-quantity pair $(202, 100)$. The order book situation is shown in Figure 2.1.

In this example we have $\Omega_0 = \{199, 202\} \neq \emptyset$. The highest executable order volume is $V_{\max} = 100$ and the set of volume maximizing prices is $\Omega_V = \{199, 202\}$. The lowest surplus is $Z_{\min} = 100$ and the set of candidate prices is $\Omega_Z = \{199, 202\}$. Since there is surplus on the demand side as well as on the supply side, Xetra auction applies Rule 2.4 which leads to $P_{\text{Xetra}} = 200$ when $P_{\text{ref}} = 200$, $P_{\text{Xetra}} = 202$ when $P_{\text{ref}} = 203$, and $P_{\text{Xetra}} = 199$ when $P_{\text{ref}} = 198$. □

As readily seen from Figure 2.1, any price $p \in (199, 202)$ is a market-clearing price but not a limit price. Only when the reference price $P_{\text{ref}} \in (199, 202)$ could Xetra auction price be market-clearing since then $P_{\text{Xetra}} = P_{\text{ref}}$. Market equilibrium could be obtained, for example, by taking the midpoint of the interval $(199, 202)$ as Xetra auction price with $P_{\text{Xetra}} = 201.5$. Xetra auction pricing rules exclude the possibility of market-clearing by not taking into account prices other than limit prices. This observation leads to the following property.

Property 2.2 (Limitation of limit prices)**.** *Xetra auction price mechanism excludes a market-clearing price but not a limit price from being Xetra auction price.*

We consider in this work the **market efficiency** of Xetra auction market as Xetra auction price being market-clearing. Property 2.2 implies that it is possible to select a market-clearing price but not a limit price as Xetra auction price by modifying the restriction of considering only limit prices in Xetra auction pricing rules. This modification on auction pricing rules, when it is possible, is an improvement on the market efficiency of Xetra auction market. We regard these modified auction pricing rules as **improved auction pricing rules** (improved rules in short).

2.2 Improvement on Auction Pricing Rules

We take a brief review on the price determination process of Xetra auction. Without loss of generality, we follow the order book data set $\mathcal{I}_0$ in Assumption 1.1 when the call phase ends. Xetra auction determines Xetra auction price P_{Xetra} according to the data set $\mathcal{I}_0$. When there exists at least one limit price for $\Omega_0 \neq \emptyset$

with a postive highest executable order volume for $V_{\max} > 0$, Xetra auction takes the following steps to compute Xetra auction price P_{Xetra}:

1. COMPUTE: the set of all volume maximizing prices Ω_V;
2. COMPUTE: the lowest surplus $Z_{\min}$ in Ω_V;
3. COMPUTE: the set of all candidate prices with the highest executable order volume and with the lowest surplus Ω_Z;
4. COMPUTE: the highest candidate price $\overline{P_Z}$;
5. COMPUTE: the lowest candidate price $\underline{P_Z}$;
6. SELECT: a unique auction price as P_{Xetra} according to Theorem 1.1.

The step functional form of trading volume function $\Phi_V(p)$ and the non-increasing excess demand function $\Phi_Z(p)$ suggest that we can start directly from Ω_V and apply a new set of auction pricing rules to choose an auction price which is equal to the auction price that Xetra auction pricing rules choose under the same order book data set $\mathfrak{I}_0$. We call this new set of auction pricing rules as **equivalent auction pricing rules** (equivalent rules in short).

2.2.1 Equivalent Auction Pricing Rules

Starting from the order book data set $\mathfrak{I}_0$ in Assumption 1.1, equivalent rules consider all limit prices in Ω_0. When there is no limit price in the order book with $\Omega_0 = \emptyset$, equivalent rules choose the reference price P_{ref} as the auction price when market orders exist in both sides with $d_0 > 0$ and $s_0 > 0$. There is no auction price when market orders do not exist on both sides.

When there exists limit prices in the order book with $\Omega_0 \neq \emptyset$, equivalent rules compute the highest executable order volume $V_{\max}$ for all limit prices. No auction price exists in equivalent rules when $V_{\max} = 0$.

When $V_{\max} > 0$, equivalent rules compute the set of all volume maximizing prices Ω_V. Define the highest volume maximizing price and the lowest volume maximizing price as

$$\overline{P_V} := \max \Omega_V \quad \text{and} \quad \underline{P_V} := \min \Omega_V.$$

Equivalent rules then choose a unique auction price P_{equate} from Ω_V as follows:

When there is only one volume maximizing price with $\sharp\Omega_V = 1$, the unique volume maximizing price is chosen as the auction price $P_{\text{equate}} = \overline{P_V} = \underline{P_V}$.

The excess demand function $\Phi_Z(p)$ is non-increasing. Thus, when there is more than one volume maximizing price with $\sharp\Omega_V > 1$, there are four possibilities by considering the surplus on the highest volume maximizing price $\overline{P_V}$ and the surplus on the lowest volume maximizing price $\underline{P_V}$.

CASE 1: $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) \geq 0$, i.e. all volume maximizing prices are either with a surplus on the demand side or with no surplus and there is at least one volume maximizing price with a surplus on the demand side. The highest volume maximizing price which corresponds to the lowest surplus in Ω_V is then chosen as the auction price $P_{\text{equate}} = \overline{P_V}$.

CASE 2: $\Phi_Z(\underline{P_V}) \leq 0$ and $\Phi_Z(\overline{P_V}) < 0$, i.e. all volume maximizing prices are either with a surplus on the supply side or with no surplus and there is at least one volume maximizing price with a surplus on the supply side. The lowest volume maximizing price which corresponds to the lowest surplus in Ω_V is then chosen as the auction price $P_{\text{equate}} = \underline{P_V}$.

CASE 3: $\Phi_Z(\overline{P_V}) = 0$ and $\Phi_Z(\underline{P_V}) = 0$, i.e. all volume maximizing prices are with no surplus. The reference price P_{ref} is introduced to compare with the highest volume maximizing price $\overline{P_V}$ and the lowest volume maximizing price $\underline{P_V}$. The auction price is the highest volume maximizing price when the reference price is higher than the highest volume maximizing price. The auction price is the lowest volume maximizing price when the reference price is lower than the lowest volume maximizing price. The auction price is the reference price when the reference price lies between the lowest volume maximizing price and the highest volume maximizing price. Thus the auction price is $P_{\text{equate}} = \max\{\underline{P_V}, \min\{P_{\text{ref}}, \overline{P_V}\}\}$.

CASE 4: $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, i.e. some volume maximizing prices are with a surplus on the demand side and others with a surplus on the supply side. Define the **upper bound volume maximizing price** $P'_{\max}$ as the lowest volume maximizing price with a surplus on the supply side and the **lower bound volume maximizing price** $P'_{\min}$ as the highest volume maximizing price with a surplus on the demand side. There are three possibilities when comparing the surplus of the upper bound volume maximizing price $P'_{\max}$ with the surplus of the lower bound volume maximizing price $P'_{\min}$:

(a) If the surplus of the lower bound volume maximizing price is less than the surplus of the upper bound volume maximizing price, the auction price is the lower bound volume maximizing price with $P_{\text{equate}} = P'_{\min}$;

(b) If the surplus of the lower bound volume maximizing price is greater than the surplus of the upper bound volume maximizing price, the auction price is the upper bound volume maximizing price with $P_{\text{equate}} = P'_{\max}$;

(c) If the surplus of the lower bound volume maximizing price is equal to the surplus of the upper bound volume maximizing price, the reference price P_{ref} is introduced and the auction price is $P_{\text{equate}} = \max\{P'_{\min}, \min\{P_{\text{ref}}, P'_{\max}\}\}$.

Formulate the upper bound volume maximizing price $P'_{\max}$ and the lower bound volume maximizing price $P'_{\min}$ as:

$$\begin{aligned} P'_{\max} &:= \min \ \{p \in \Omega_V \mid \Phi_Z(p) < 0\} \\ P'_{\min} &:= \max \ \{p \in \Omega_V \mid \Phi_Z(p) > 0\}. \end{aligned} \tag{2.1}$$

$P'_{\max}$ and $P'_{\min}$ are well defined in CASE 4. One property of $P'_{\min}$ and $P'_{\max}$ is stated in the following lemma.

Lemma 2.1. *When $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, $P'_{\max} > P'_{\min}$ is satisfied and no limit price $P^* \in (P'_{\min}, P'_{\max})$ exists.*

Proof. $P'_{\max} > P'_{\min}$ holds since $\Phi_Z(p)$ is non-increasing and $\Phi_Z(P'_{\max}) < 0 < \Phi_Z(P'_{\min})$. By contradiction, assume there exists a limit price $P^* \in (P'_{\min}, P'_{\max})$. Then it must be with $\Phi_Z(P^*) = 0$ and $\Phi_V(P^*) = V_{\max}$, which implies that $\Phi_D(P^*) = \Phi_S(P^*) = V_{\max}$. Since $\Phi_Z(P'_{\min}) > 0$ and $\Phi_Z(P'_{\max}) < 0$, we have $\Phi_S(P'_{\min}) = \Phi_D(P'_{\max}) = V_{\max}$. $\Phi_D(P^*) = \Phi_D(P'_{\max}) = V_{\max}$ implies that P^* is not a limit bid price and $\Phi_S(P'_{\min}) = \Phi_S(P^*) = V_{\max}$ implies that P^* is not a limit ask price. Thus, there exists no limit price $P^* \in (P'_{\min}, P'_{\max})$. □

The price determination process for equivalent rules is depicted in the form of flowchart in Appendix B. We formalize equivalent auction pricing rules in the following theorem.

Theorem 2.1. *Given the order book data set $\mathcal{I}_0$ in Assumption 1.1, equivalent auction pricing rules determine a unique auction price P_{equate} whenever it is possible.*

If $\Omega_0 \neq \emptyset$ and $V_{\max} > 0$, then P_{equate} exists and is determined as follows:

(I.) *If $\sharp\Omega_V = 1$, then $P_{\text{equate}} = \overline{P_V} = \underline{P_V}$.*

(II.) *If $\sharp\Omega_V > 1$, then*

(II.I) *if* $\Phi_Z(\underline{P_V}) > 0$ *and* $\Phi_Z(\overline{P_V}) \geq 0$*, then* $P_{\text{equate}} = \overline{P_V}$*;*

(II.II) *if* $\Phi_Z(\underline{P_V}) \leq 0$ *and* $\Phi_Z(\overline{P_V}) < 0$*, then* $P_{\text{equate}} = \underline{P_V}$*;*

(II.III) *if* $\Phi_Z(\underline{P_V}) = \Phi_Z(\overline{P_V}) = 0$*, then* $P_{\text{equate}} = \max\{\underline{P_V}, \min\{P_{\text{ref}}, \overline{P_V}\}\}$*;*

(II.IV) *otherwise,*

$$P_{\text{equate}} = \begin{cases} P'_{\min}, & \text{when } |\Phi_Z(P'_{\min})| < |\Phi_Z(P'_{\max})|; \\ P'_{\max}, & \text{when } |\Phi_Z(P'_{\min})| > |\Phi_Z(P'_{\max})|; \\ \max\{P'_{\min}, \min\{P_{\text{ref}}, P'_{\max}\}\}, & \text{when } |\Phi_Z(P'_{\min})| = |\Phi_Z(P'_{\max})|; \end{cases}$$

where $\overline{P_V} = \max \Omega_V$*,* $\underline{P_V} = \min \Omega_V$*, and* $P'_{\max} = \min\{p \in \Omega_V \mid \Phi_Z(p) < 0\}$ *with* $P'_{\min} = \max\{p \in \Omega_V \mid \Phi_Z(p) > 0\}$*.*

If only market orders exist such that $\Omega_0 = \emptyset$ *with* $d_0 > 0$ *and* $s_0 > 0$*, then the auction price is* $P_{\text{equate}} = P_{\text{ref}}$*.*

Otherwise, the auction price P_{equate} *remains unspecified.*

Proof. When there exists only one volume maximizing price with $\sharp\Omega_V = 1$, the auction price is the unique volume maximizing price with $P_{\text{equate}} = \overline{P_V} = \underline{P_V}$, which corresponds to (I) in this theorem.

The four cases in (II) correspond to the four cases when there is more than one volume maximizing price with $\sharp\Omega_V > 1$. CASE 1 with $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) \geq 0$ corresponds to (II.I) in the theorem. The auction price in this case is the highest volume maximizing price with $P_{\text{equate}} = \overline{P_V}$. CASE 2 with $\Phi_Z(\underline{P_V}) \leq 0$ and $\Phi_Z(\overline{P_V}) < 0$ corresponds to (II.II) in the theorem. The auction price in this case is $P_{\text{equate}} = \underline{P_V}$. CASE 3 corresponding to (II.III) in the theorem implies that the auction price $P_{\text{equate}} = \max\{\underline{P_V}, \min\{P_{\text{ref}}, \overline{P_V}\}\}$ when all volume maximizing prices have no surplus with $\Phi_Z(\underline{P_V}) = 0$ and $\Phi_Z(\overline{P_V}) = 0$. CASE 4 corresponds to (II.IV) in the theorem, referring to the case that some volume maximizing prices have a surplus on the demand side whereas others have a surplus on the supply side with $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$. In this case, equivalent rules introduce the upper bound volume maximizing price $P'_{\max}$ and the lower bound volume maximizing price $P'_{\min}$. When the surplus of $P'_{\min}$ is less than that of $P'_{\max}$, the auction price is $P_{\text{equate}} = P'_{\min}$ which is with lower surplus. When the surplus of $P'_{\min}$ is greater than that of $P'_{\max}$, the auction price is $P_{\text{equate}} = P'_{\max}$ which is with lower surplus. When the surplus of $P'_{\min}$ is equal to that of $P'_{\max}$, the reference price P_{ref} is included and the auction price is $P_{\text{equate}} = \max\{P'_{\min}, \min\{P_{\text{ref}}, P'_{\max}\}\}$.

Equivalent rules choose the reference price as the auction price $P_{\text{equate}} = P_{\text{ref}}$ when only market orders exist on both market sides for $d_0 > 0$ and $s_0 > 0$ with $\Omega_0 = \emptyset$. No auction price P_{equate} is specified when all other rules fail. □

The equivalence between the auction price P_{equate} chosen by equivalent rules and Xetra auction price P_{Xetra} chosen by Xetra auction pricing rules is stated formally in the following proposition.

Proposition 2.1. *Given the same order book data set $\mathcal{I}_0$ in Assumption 1.1, the auction price P_{equate} determined by Theorem 2.1 is equal to Xetra auction price P_{Xetra} determined by Theorem 1.1.*

The complete proof of Proposition 2.1 is presented in Appendix C. We demonstrate the equivalence of P_{equate} and P_{Xetra} in the following example.

Example 2.2. Consider the same order book as in Example 1.1 containing three bid orders with price-quantity pairs $(198, 100)$, $(200, 100)$, and $(201, 150)$. It also contains one ask order with the price-quantity pair $(199, 100)$. The order book situation is shown in Figure 2.2.

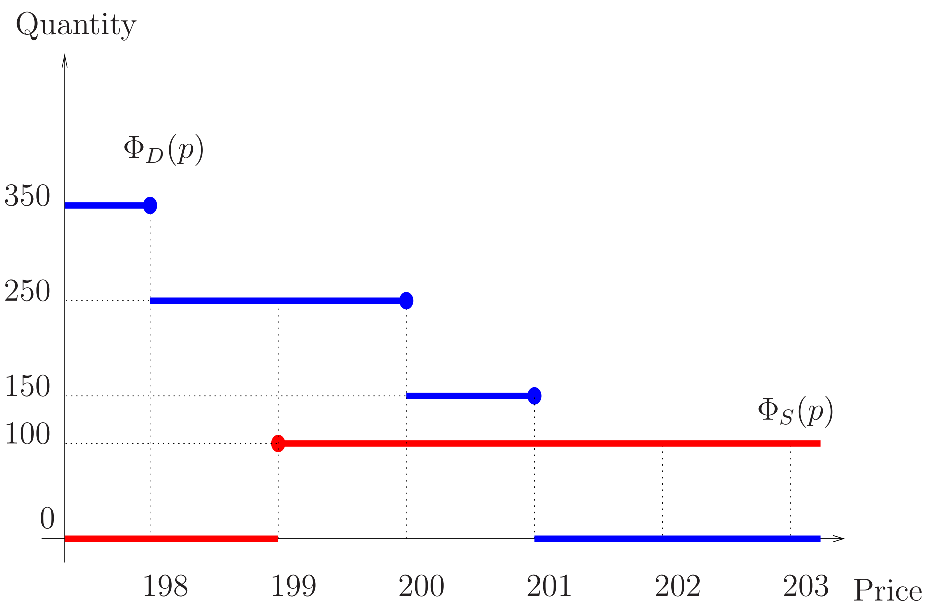


Figure 2.2: Crossed order book in Example 2.2.

In this example, equivalent rules determine P_{equate} by employing the following steps: Since $\Omega_0 = \{198, 199, 200, 201\} \neq \emptyset$, equivalent rules compute $V_{\max} = 100$.

Since $V_{\max} = 100 > 0$, equivalent rules compute the set of all volume maximizing prices $\Omega_V = \{199, 200, 201\}$. The highest volume maximizing price is $\overline{P_V} = 201$ and the lowest volume maximizing price is $\underline{P_V} = 199$. Since $\sharp\Omega_V > 1$, equivalent rules investigate $\Phi_Z(\overline{P_V}) = 50 > 0$ and $\Phi_Z(\underline{P_V}) = 150 > 0$. (II.I) in theorem 2.1 is thus applied and the auction price is $P_{\text{equate}} = \overline{P_V} = 201$, which is equal to Xetra auction price $P_{\text{Xetra}} = 201$ shown in Example 1.1. □

2.2.2 Improved Auction Pricing Rules

As stated in Property 2.2, Xetra auction pricing rules eliminate the possibility of choosing a market-clearing price but not a limit price as Xetra auction price. It leaves us space to attain an improvement from the perspective of market efficiency in Xetra auction market by relaxing the restriction of considering only limit prices. We achieve this improvement by modifying equivalent rules.

Considering each case in equivalent rules, we find out that a market-clearing price but not a limit price exists only when there is more than one volume maximizing price for some volume maximizing prices with a surplus on the demand side and others with a surplus on the supply side; i.e. CASE 4 of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$. We thus obtain improved rules by modifying the pricing rule in CASE 4 while inheriting all other pricing rules from equivalent rules.

For the case of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, consider the upper bound volume maximizing price $P^{'}_{\max}$ and the lower bound volume maximizing price $P^{'}_{\min}$ as formulated in equation (2.1). Any price lying between $P^{'}_{\max}$ and $P^{'}_{\min}$ is a market-clearing price.

Lemma 2.2. *When $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, any price $p \in (P^{'}_{\min}, P^{'}_{\max})$ is a market-clearing price.*

Proof. To prove any $p \in (P^{'}_{\min}, P^{'}_{\max})$ is a market-clearing price is to prove $\Phi_Z(p) = 0$ for any $p \in (P^{'}_{\min}, P^{'}_{\max})$. $\Phi_Z(p)$ is non-increasing, thus any price $p > P^{'}_{\min}$ is with $\Phi_Z(p) \le 0$ by definition of $P^{'}_{\min}$ and any price $p < P^{'}_{\max}$ is with $\Phi_Z(p) \ge 0$ by definition of $P^{'}_{\max}$. Therefore, $\Phi_Z(p) = 0$ for $p \in (P^{'}_{\min}, P^{'}_{\max})$. □

Lemma 2.2 together with Lemma 2.1 implies that any price $p \in (P^{'}_{\min}, P^{'}_{\max})$ is a market-clearing price but not a limit price when $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$.

Denote the auction price determined by improved rules as P_{impr}. We choose in improved rules the mid price of the upper bound volume maximizing price $P^{'}_{\max}$

and the lower bound volume maximizing price $P'_{\min}$ as the auction price with $P_{\text{impr}} = \frac{1}{2}(P'_{\min} + P'_{\max})$. The case of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$ in improved rules is stated as follows:

CASE 4′: $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, i.e. some volume maximizing prices are with a surplus on the demand side and others with a surplus on the supply side. The mid price of the upper bound volume maximizing price $P'_{\max}$ and the lower bound volume maximizing price $P'_{\min}$ is chosen as the auction price with $P_{\text{impr}} = \frac{1}{2}(P'_{\min} + P'_{\max})$.

The price determination process for improved rules is depicted in the form of flowchart in Appendix B. The formalization of improved auction pricing rules is presented as follows.

Theorem 2.2. *Given the order book data set $\mathcal{I}_0$ in Assumption 1.1, improved auction pricing rules determine a unique auction price P_{impr} whenever it is possible.*

If $\Omega_0 \neq \emptyset$ and $V_{\max} > 0$, then P_{impr} exists and is determined as follows:

(I.) *If $\sharp\Omega_V = 1$, then $P_{\text{impr}} = \overline{P_V} = \underline{P_V}$.*

(II.) *If $\sharp\Omega_V > 1$, then*

- (II.I) *if $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) \geq 0$, then $P_{\text{impr}} = \overline{P_V}$;*
- (II.II) *if $\Phi_Z(\underline{P_V}) \leq 0$ and $\Phi_Z(\overline{P_V}) < 0$, then $P_{\text{impr}} = \underline{P_V}$;*
- (II.III) *if $\Phi_Z(\underline{P_V}) = \Phi_Z(\overline{P_V}) = 0$, then $P_{\text{impr}} = \max\{\underline{P_V}, \min\{P_{\text{ref}}, \overline{P_V}\}\}$;*
- (II.IV) *otherwise, $P_{\text{impr}} = \frac{1}{2}(P'_{\min} + P'_{\max})$;*

where $\overline{P_V} = \max \Omega_V$, $\underline{P_V} = \min \Omega_V$, and $P'_{\max} = \min\{p \in \Omega_V \mid \Phi_Z(p) < 0\}$ with $P'_{\min} = \max\{p \in \Omega_V \mid \Phi_Z(p) > 0\}$.

If only market orders exist such that $\Omega_0 = \emptyset$ with $d_0 > 0$ and $s_0 > 0$, then the auction price is $P_{\text{impr}} = P_{\text{ref}}$.

Otherwise, the auction price P_{impr} remains unspecified.

Proof. Improved rules only differ from equivalent rules in the case of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, which corresponds to case (II.IV) in theorem 2.1. Thus we obtain this theorem by considering $P_{\text{impr}} = \frac{1}{2}(P'_{\min} + P'_{\max})$ in case (II.IV) while inheriting all other cases from theorem 2.1. □

The following example illustrates the improvement on market efficiency when applying improved rules to determine auction price in Xetra auction market.

Example 2.3. Consider the order book containing two bid orders with price-quantity pairs $(200, 100)$ and $(202, 150)$. It also contains two ask orders with price-quantity pairs $(199, 150)$ and $(201, 150)$. The order book situation is shown in Figure 2.3.

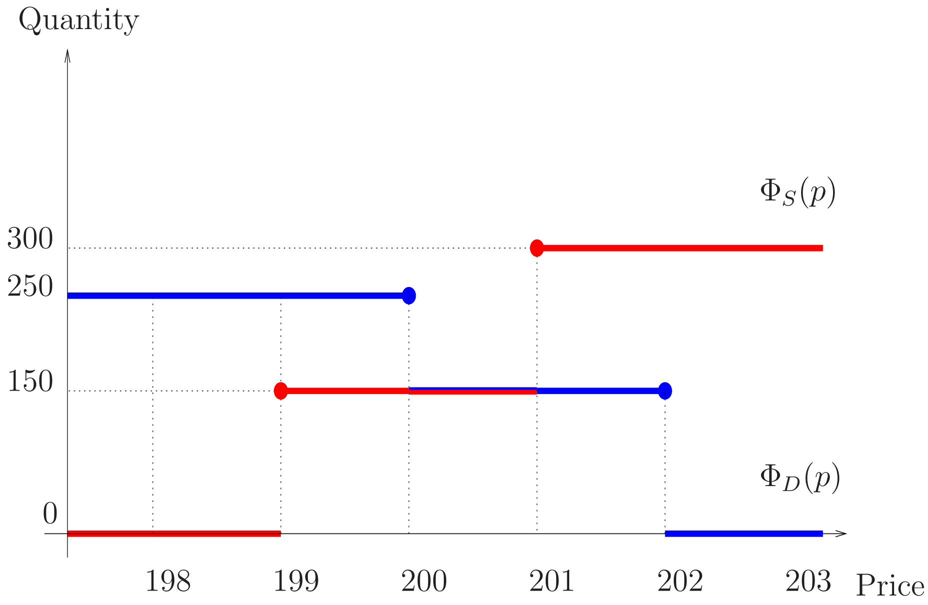


Figure 2.3: Crossed order book in Example 2.3.

In this example, improved rules determine P_{impr} by employing the following steps: Since $\Omega_0 = \{199, 200, 201, 202\} \neq \emptyset$, improved rules compute $V_{\max} = 150$. Since $V_{\max} = 150 > 0$, improved rules further compute the set of all volume maximizing prices $\Omega_V = \{199, 200, 201, 202\}$. The highest volume maximizing price is $\overline{P_V} = 202$ and the lowest volume maximizing price is $\underline{P_V} = 199$. Since $\sharp\Omega_V > 1$, improved rules compute $\Phi_Z(\underline{P_V}) = 100 > 0$ and $\Phi_Z(\overline{P_V}) = -150 < 0$. Thus, improved rules compute $P'_{\min} = 200$ and $P'_{\max} = 201$. Case (II.IV) in Theorem 2.2 is applied and the auction price is $P_{\text{impr}} = \frac{1}{2}(P'_{\min} + P'_{\max}) = 200.5$, which is a market-clearing price since $\Phi_Z(200.5) = 0$.

Equivalent rules compute $P'_{\min} = 200$ and $P'_{\max} = 201$ in this case. Since $|\Phi_Z(200)| < |\Phi_Z(201)|$, $P_{\text{equate}} = P'_{\min} = 200$ is chosen as the auction price by equivalent rules. The Xetra auction price is $P_{\text{Xetra}} = P_{\text{equate}} = 200$ in this example according to Proposition 2.1. $P_{\text{Xetra}} = P_{\text{equate}} = 200$ is not a market-clearing price since the excess demand $\Phi_Z(200) = 100 \neq 0$. □

This example verifies that Xetra auction market can be improved from the perspective of market efficiency when adopting improved rules to determine the auction price in the market. The following proposition formalizes this finding.

Proposition 2.2. *There exists improvement on market efficiency when Xetra auction market replace Xetra auction pricing rules depicted in Theorem 1.1 with improved pricing rules depicted in Theorem 2.2 to determine the auction price.*

Proof. Compare Theorem 2.2 with Theorem 2.1, the difference only lies in the case of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$, which is case (II.IV) for both theorems. Lemma 2.2 implies that Theorem 2.2 always selects a market-clearing price while Lemma 2.1 implies that Theorem 2.1 does not always choose a market-clearing price as the auction price in this case. Proposition 2.1 verifies the equivalence of auction price on Theorem 1.1 and Theorem 2.1. This equivalence implies that an improvement on market efficiency exists when Xetra auction market applies Theorem 2.2 instead of Theorem 1.1 to determine the auction price. □

In this chapter we have investigated the properties of Xetra auction price mechanism and have developed equivalent auction pricing rules and improved auction pricing rules which are derived from Xetra auction pricing rules. Equivalent auction pricing rules can be regarded as an equivalence of Xetra auction pricing rules in the sense that it attains the same auction price as Xetra auction pricing rules given the same order book data set. Improved auction pricing rules can be regarded as an improvement on Xetra auction pricing rules from the perspective of market efficiency. These two sets of auction pricing rules can be served as a reference when modifying Xetra auction pricing rules as well as auction pricing rules in other electronic security trading systems such as NGTS in Shanghai Stock Exchange of China.

Chapter 3

Investment Decision in Xetra auction market

Xetra auction market mechanism determines the trading price and the trading volume given the order book that is an aggregation of orders submitted by traders. This implies that the trading price and the final transaction volume are dependent on the order specification that the trader submits. The trader is thus not in the type of price takers who believe their trading behavior has no impact on the market. This triggers us to explicitly consider in the trader's investment decision the impact of the order submitted by the trader on the current Xetra auction price and the final transaction volume.

3.1 Security Trading Process

We consider an economy containing one risk-free asset market and one risky asset market with a consumption commodity regarded as the numeraire. The economy contains no transaction cost and no short-sale constraint.

The risk-free asset is divisible in any trading quantity with the trading price normalized to 1. It has a constant interest factor $R > 0$ which can be interpreted as $1 + r$ with r denoting the nominal interest rate. The risky asset market trades only integer shares, e.g. traders can trade 19 shares but not 19.81 shares in the risky asset market. The risky asset market is a Xetra auction market which is assumed to hold one Xetra auction for each trading period. Xetra auction market mechanism depicted in chapter 1 is applied to determine the trading price and the final transaction volume of the market.

We consider the trader's investment decision in one trading period. Assume that at the beginning of the trading period the trader obtains an initial endowment (y_0, Z_0) with y_0 shares of the risk-free asset and Z_0 shares of the risky asset. The trader observes trading information from markets and performs its investment decision to decide the order specification O_s to submit to Xetra auction market in the call phase. Xetra auction market mechanism collects order specifications submitted by traders into a central order book during the call phase and calculates in the price determination phase the Xetra auction price P_X and the final transaction volume Z_X for the trader given the central order book. Xetra auction price P_X is dependent on the order specification O_s that the trader submits and can be interpreted as **Xetra auction price function** $P_X(O_s)$. Correspondingly, the final transaction volume can be interpreted as **Xetra auction allocation function** $Z_X(O_s)$. After trading the risky asset, the trader participates in the risk-free asset market and attains the trading quantity y of the risk-free asset. The trader then obtains the portfolio holding $(y_0 + y, Z_0 + Z_X(O_s))$ which is the combination of the risk-free asset and the risky asset. Figure 3.1 illustrates the trader's security trading process.

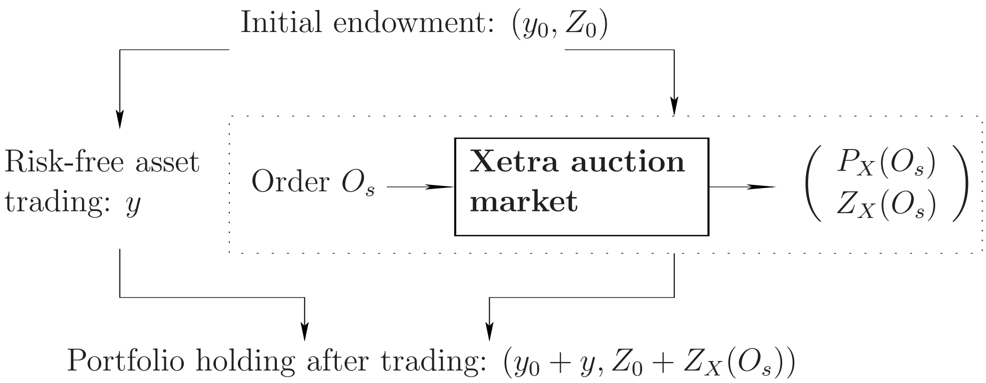


Figure 3.1: Trader's security trading process.

The trader in Xetra auction market has the knowledge on how Xetra auction market mechanism determines the trading price and the final transaction volume given the order book. It observes the real-time trading data in the market and considers in its investment decision the subjective forecast $P_X^e(O_s)$ on Xetra auction price function $P_X(O_s)$ and the forecast $Z_X^e(O_s)$ on Xetra auction allocation function $Z_X(O_s)$. We assume that the order book in Xetra auction market is transparent to the trader during the call phase.[1] We further assume that the order book is in the following form:

Assumption 3.1. *The trader obtains the real-time data of the complete order book in the form of:*

$$\mathcal{I}_0 = \{d_0, (a_1, d_1), \ldots, (a_I, d_I); s_0, (b_1, s_1), \ldots, (b_J, s_J); P_{\text{ref}}\} \tag{3.1}$$

[1] Currently the order book in the call phase of the auction is only partially open in Xetra platform operated in Deutsche Börse, but it is open with full order book depth in Xetra platform operated in Wiener Börse, see Wiener Börse (2006).

with d_0 denoting the market bid order, $(a_1, d_1), \ldots, (a_I, d_I)$ denoting I limit bid orders with $a_I > \ldots > a_2 > a_1$, s_0 denoting the market ask order, $(b_1, s_1), \ldots,$ (b_J, s_J) denoting J limit ask orders with $b_J > \ldots > b_2 > b_1$, and P_{ref} denoting the current reference price in Xetra auction market.

Xetra auction market allows two basic types of order specifications: limit orders and market orders. For simplicity, we assume that the trader submits only market orders. For convenience of the notation, we regard $Q_m > 0$ to denote a market bid order and $Q_m < 0$ to denote a market ask order. The following assumption is made to further simplify the analysis.

Assumption 3.2. *The trader is the last one who submits the order specification in the call phase.*

Under Assumptions 3.1 and 3.2, the trader constructs its subjective expectations on Xetra auction price and the final transaction volume which coincide with the 'real' Xetra auction price and the final transaction volume, i.e. $P^e_X(Q_m) = P_X(Q_m)$ and $Z^e_X(Q_m) = Z_X(Q_m)$. Our work start from deriving the explicit form of $P_X(Q_m)$ and $Z_X(Q_m)$.

3.2 Xetra Auction Price and Allocation

The formulation of Xetra auction price function and the allocation function requires the explicit form of excess demand function $\Phi_Z(p)$. We denote the aggregate demand function as $\Phi_D(p)$ and the aggregate supply function as $\Phi_S(p)$ for the order book data set $\mathcal{I}_0$ in Assumption 3.1. The excess demand function is defined as $\Phi_Z(p) := \Phi_D(p) - \Phi_S(p)$.

There are $I + J$ limit prices in $\mathcal{I}_0$. Let $\Omega_0 = \{a_1, \ldots, a_I, b_1, \ldots, b_J\}$ denote the set of all limit prices in $\mathcal{I}_0$ and rearrange all these limit prices in Ω_0 such that $P_1 \leq P_2 \leq \cdots \leq P_{I+J}$ for $P_i \in \Omega_0$ with $i = 1, \ldots, I + J$.

We introduce a new notation "$\rceil$" to represent a closed half or an open half of an interval. For example, $\rceil\, a, b \,\rceil = [a, b]\ ,\ [a, b)\ ,\ (a, b]\ ,$ or (a, b) with $a < b$. With the rearrangement of limit prices and the new notation "$\rceil$", the explicit form of $\Phi_Z(p)$ is stated as follows:

Lemma 3.1. *Given the order book data set $\mathcal{I}_0$ in Assumption 3.1, rearrange all the limit prices in $\mathcal{I}_0$ such that $P_1 \leq P_2 \leq \cdots \leq P_{I+J}$, then the excess demand* *function $\Phi_Z(p)$ is a non-increasing step function and takes the form:*

$$\Phi_Z(p) = \sum_{n=0}^{I+J} \phi_n^Z 1_{A_n^Z}(p), \tag{3.2}$$

where ϕ_n^Z is a constant for $n = 0, \ldots, I+J$ and $\{A_0^Z, A_1^Z, \ldots, A_{I+J}^Z\}$ is a partition of $\mathbb{R}_+$ with $A_0^Z := [0, P_1\ [$; $A_n^Z :=]\ P_n, P_{n+1}\ [$ for $n = 1, \ldots, I+J-1$; and $A_{I+J}^Z :=]\ P_{I+J}, +\infty)$.

We illustrate the explicit form of excess demand function $\Phi_Z(p)$ in the following example.

Example 3.1. Consider the order book containing three bid orders with price-quantity pairs $(198, 100)$, $(199, 100)$, and $(202, 150)$. It also contains two ask orders with price-quantity pairs $(200, 50)$ and $(201, 150)$. The reference price is $P_{\text{ref}} = 200$. The order book situation is shown in Figure 3.2.

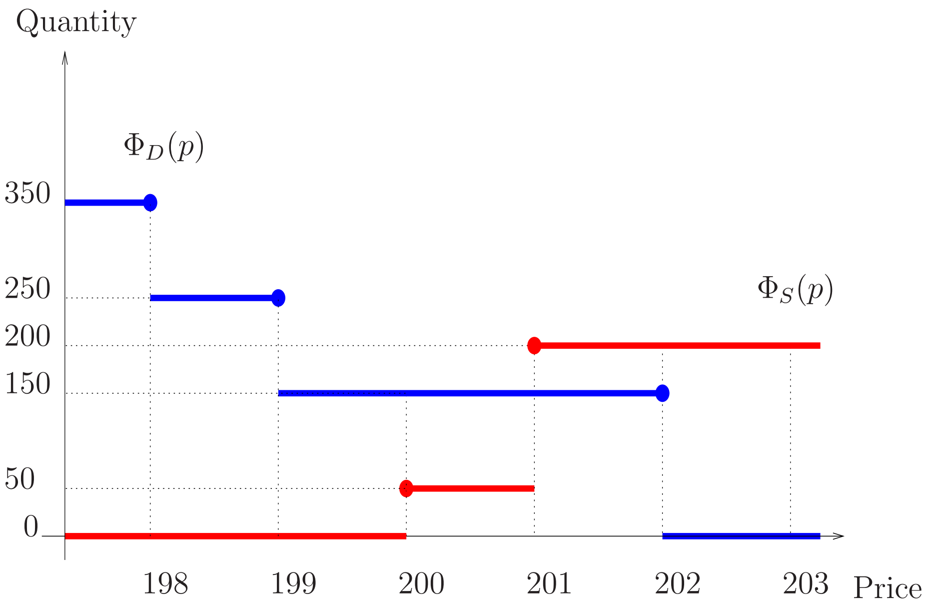


Figure 3.2: Crossed order book in Example 3.1.

Rearrange limit prices in the order book such that $P_1 = 198$, $P_2 = 199$, $P_3 = 200$, $P_4 = 201$, and $P_5 = 202$. Calculate the values of $\phi_0^Z = 350$, $\phi_1^Z = 250$, $\phi_2^Z = 150$, $\phi_3^Z = 100$, $\phi_4^Z = -50$, and $\phi_5^Z = -200$. The excess demand function is shown in

Figure 3.3, which takes the explicit form:

$$\Phi_Z(p) = \begin{cases} 350 & \text{for } p \in A_0^Z; \\ 250 & \text{for } p \in A_1^Z; \\ 150 & \text{for } p \in A_2^Z; \\ 100 & \text{for } p \in A_3^Z; \\ -50 & \text{for } p \in A_4^Z; \\ -200 & \text{for } p \in A_5^Z; \end{cases} \tag{3.3}$$

where $\{A_0^Z, A_1^Z, \ldots, A_5^Z\}$ is a partition of $\mathbb{R}_+$ with $A_0^Z = [0, 198]$; $A_1^Z = (198, 199]$; $A_2^Z = (199, 200)$; $A_3^Z = [200, 201)$; $A_4^Z = [201, 202]$; and $A_5^Z = (202, +\infty)$. $\Phi_Z(p)$ is non-increasing since $\phi_0^Z > \phi_1^Z > \phi_2^Z > \phi_3^Z > \phi_4^Z > \phi_5^Z$. □

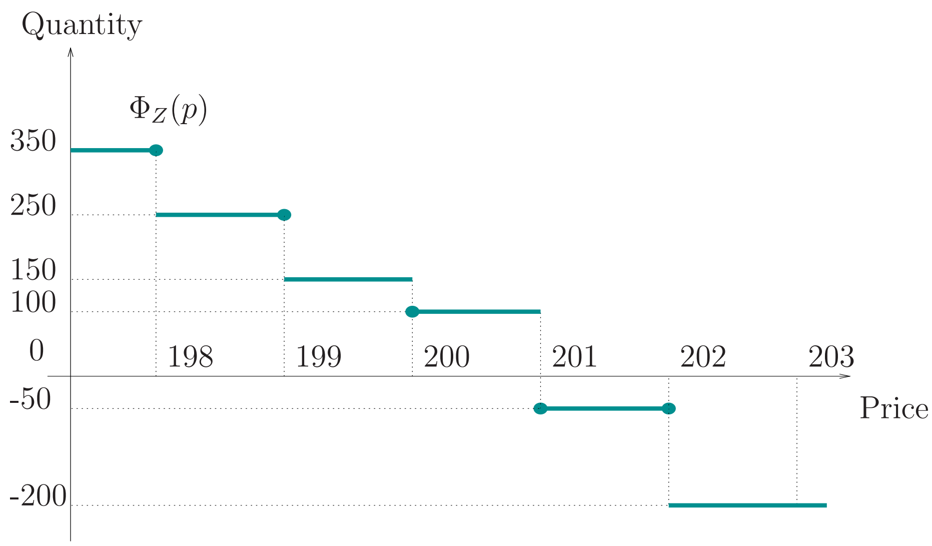


Figure 3.3: Excess demand function $\Phi_Z(p)$ for crossed order book in Example 3.1.

After the trader submits the market order Q_m, the updated order book data set is: $\mathcal{I}_0 \cup \{Q_m\}$. The new excess demand function denoted by $\Phi'_Z(Q_m, p)$ takes the form:

$$\begin{aligned} \Phi'_Z(Q_m, p) &= \Phi_Z(p) + Q_m \\ &= \sum_{n=0}^{I+J} \phi_n^Z 1_{A_n^Z}(p) + Q_m. \end{aligned} \tag{3.4}$$

Φ'_Z is increasing in Q_m and non-increasing in p. By applying Lemma 3.1, we further obtain a property for $\Phi'_Z(Q_m, p)$ as follows:

Property 3.1. *For any arbitrary but fixed price $p \in A_n^Z$, $\Phi'_Z(Q_m, p) = 0$ if and only if $Q_m = -\phi_n^Z$ for some $n = 0, 1, \ldots, I + J$.*

3.2.1 Xetra Auction Price Function

The order book situation is sophisticated in general. It is insightful to first investigate a simple case in order to get hints on how to derive Xetra auction price function $P_X(Q_m)$. Here we consider the case of uncrossed order book where the aggregate demand does not overlap with the aggregate supply. We derive the explicit form of Xetra auction price function $P_X(Q_m)$ in the following example.

Example 3.2. Consider an uncrossed order book containing two bid orders with price-quantity pairs $(198, 100)$ and $(199, 250)$. It also contains two ask orders with price-quantity pairs $(200, 50)$ and $(201, 150)$. The reference price is $P_{\text{ref}} = 202$. The order book situation is shown in Figure 3.4.

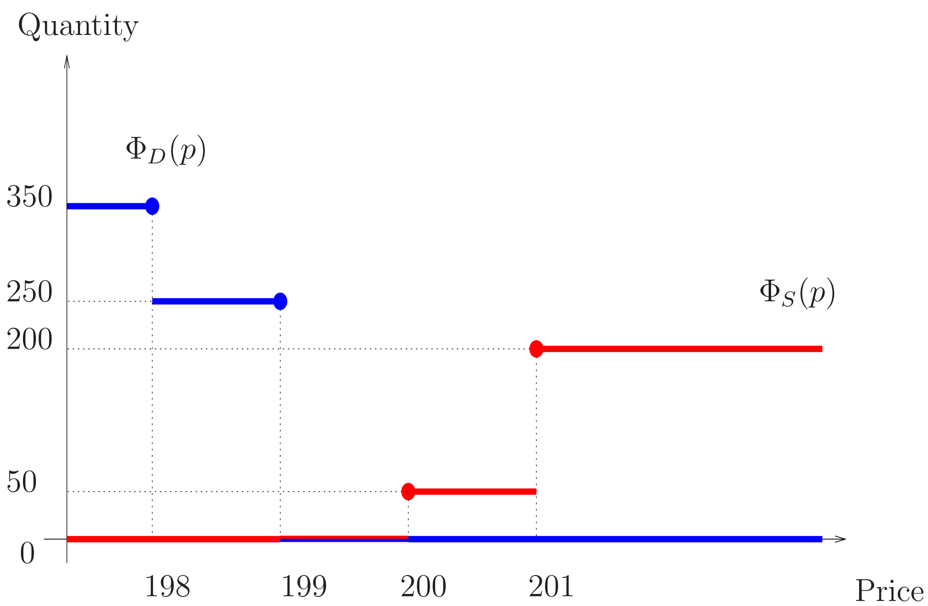


Figure 3.4: Uncrossed order book in Example 3.2.

Limit prices in this order book are $a_1 = 198$, $a_2 = 199$, $b_1 = 200$, and $b_2 = 201$ with $a_1 < a_2 < b_1 < b_2$. Calculate the values of $\phi_0^Z = \alpha_0 = 350$, $\phi_1^Z = \alpha_1 = 250$,

$\phi_2^Z = 0$, $\phi_3^Z = -\beta_1 = -50$, and $\phi_4^Z = -\beta_2 = -200$. The corresponding excess demand function that is shown in Figure 3.5 takes the explicit form:

$$\Phi_Z(p) = \begin{cases} 350 & \text{for } p \in A_0^Z; \\ 250 & \text{for } p \in A_1^Z; \\ 0 & \text{for } p \in A_2^Z; \\ -50 & \text{for } p \in A_3^Z; \\ -200 & \text{for } p \in A_4^Z; \end{cases} \tag{3.5}$$

where $A_0^Z = [0, 198]$, $A_1^Z = (198, 199]$, $A_2^Z = (199, 200)$, $A_3^Z = [200, 201)$, and $A_4^Z = [201, +\infty)$.

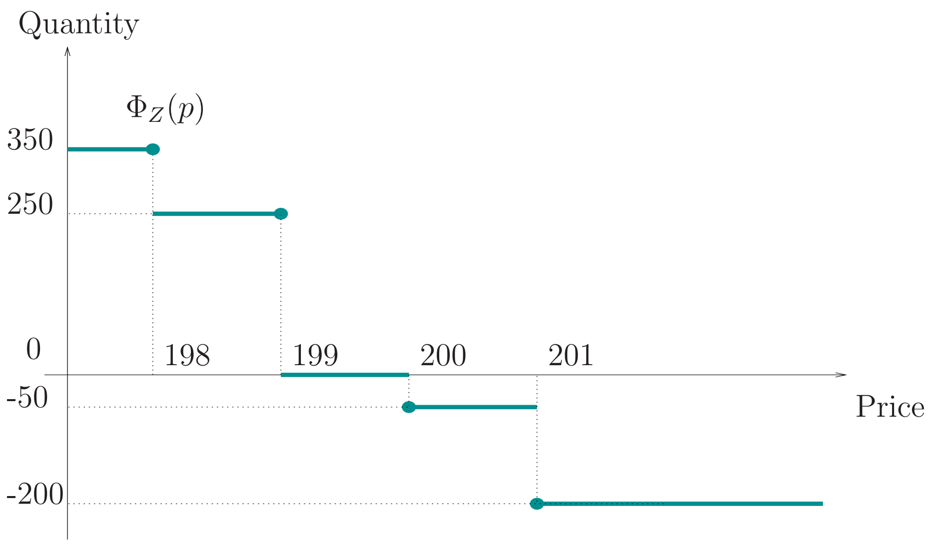


Figure 3.5: Excess demand function $\Phi_Z(p)$ for uncrossed order book in Example 3.2.

The quoted trading quantity Q_m that the trader can choose to trade against the excess demand $\Phi_Z(p)$ is with $Q_m \in (-\infty, +\infty)$. $\Phi_Z(p)$ in (3.5) has the values of $\{-350, -250, 0, 50, 200\}$, which divide the range of Q_m into:

$$\{\ (-\infty, -350),\ (-350, -250),\ (-250, 0),\ (0, 50),\ (50, 200),\ (200, +\infty)\ \}. \tag{3.6}$$

We consider the value of $P_X(Q_m)$ for Q_m in each interval of (3.6) and for $Q_m \in \{-350, -250, 0, 50, 200\}$. We discover that $\Phi_Z'(Q_m, p) \leq 0$ holds for any price $p \in \mathbb{R}_+$ when $Q_m \in (-\infty, -350)$. This implies that the lowest limit price $P_1 = 198$ is with the highest executable order volume and the lowest surplus. Thus,

$P_1 = 198$ is chosen as Xetra auction price when $Q_m \in (-\infty, -350)$. Analogously, $\Phi'_Z(Q_m, p) \geq 0$ holds for any price $p \in \mathbb{R}_+$ when $Q_m \in (200, +\infty)$. This implies that the highest limit price $P_{I+J} = 201$ is with the highest executable order volume and the lowest surplus. Thus, $P_{I+J} = 201$ is chosen as Xetra auction price when $Q_m \in (200, +\infty)$.

Another discovery is that $\Phi'_Z(Q_m, p) = 0$ for any price $p \in [0, 198]$ when $Q_m = -350$. Only the limit price $P_1 = 198$ is with the highest executable order volume and with no surplus. Thus, $P_1 = 198$ is chosen as Xetra auction price when $Q_m = -350$. Analogous results are obtained as Xetra auction price is $P_X = 199$ when $Q_m = -250$, $P_X = 200$ when $Q_m = 50$, and $P_X = 201$ when $Q_m = 200$. As the order book is uncrossed, there is no Xetra auction price P_X when the trader does not submit a market order with $Q_m = 0$.

Finally, we observe that $\Phi'_Z(Q_m, p) > 0$ holds for any price $p \in [0, 198]$ and that $\Phi'_Z(Q_m, p) < 0$ holds for any price $p \in (198, 199]$ when $Q_m \in (-350, -250)$. The limit price $P_1 = 198$ is with the highest executable order volume and the lowest surplus. Thus, $P_1 = 198$ is chosen as Xetra auction price when $Q_m \in (-350, -250)$. Analogous results are obtained as Xetra auction price is $P_X = 199$ when $Q_m \in (-250, 0)$, $P_X = 200$ when $Q_m \in (0, 50)$, and $P_X = 191$ when $Q_m \in (50, 200)$. In summary, Xetra auction price function $P_X(Q_m)$ takes the explicit form:

$$P_X(Q_m) = \begin{cases} 198 & \text{when } Q_m \in (-\infty, -250); \\ 199 & \text{when } Q_m \in [-250, 0); \\ 200 & \text{when } Q_m \in (0, 50]; \\ 201 & \text{when } Q_m \in (50, +\infty); \end{cases} \tag{3.7}$$

which is shown in Figure 3.6. □

These findings discovered in Example 3.2 can be generalized to construct Xetra auction price function $P_X(Q_m)$, which is shown in the following theorem.

Theorem 3.1. *Given the order book data set $\mathcal{I}_0$ in Assumption 3.1 and the explicit form of $\Phi_Z(p)$ from Lemma 3.1, Xetra auction price function $P_X(Q_m)$ which is a non-decreasing step function takes the form:*

$$P_X(Q_m) = \begin{cases} P_1 & \textit{when } Q_m \in (-\infty, -\phi^Z_1); \\ P^*_n & \textit{when } Q_m = -\phi^Z_n, \quad n \in \{1, 2, \ldots, I+J-1\}; \\ P_n & \textit{when } Q_m \in (-\phi^Z_{n-1}, -\phi^Z_n), \quad n \in \{2, 3, \ldots, I+J-1\}; \\ P_{I+J} & \textit{when } Q_m \in (-\phi^Z_{I+J-1}, +\infty); \end{cases}$$

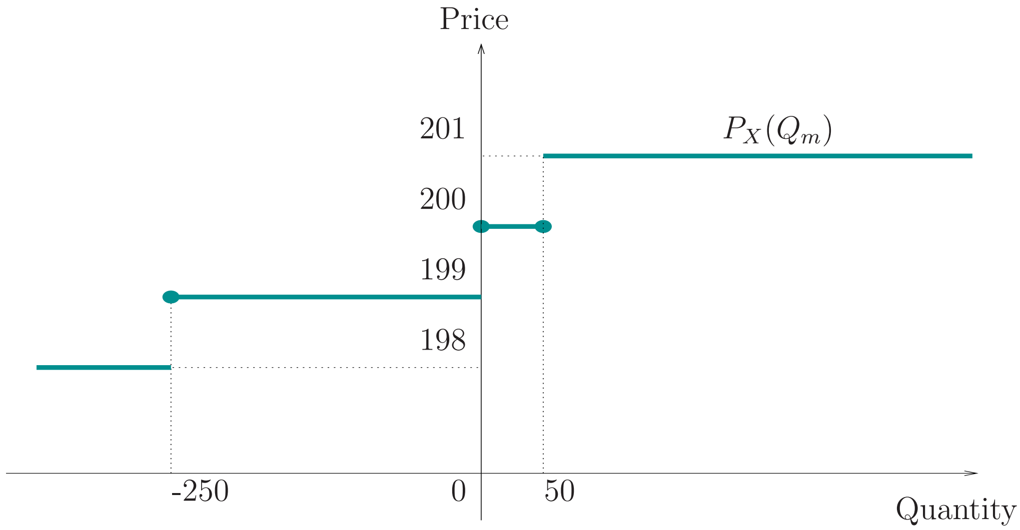


Figure 3.6: Xetra auction price function $P_X(Q_m)$ for uncrossed order book in Example 3.2.

where for any $n \in \{1, 2, \cdots, I + J - 1\}$,

$$P_n^* = \begin{cases} P_n & \textit{CASE 1;} \\ P_{n+1} & \textit{CASE 2;} \\ \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\} & \textit{CASE 3;} \end{cases} \tag{3.8}$$

for ***CASE 1:*** *either* $A_n^Z = [P_n, P_{n+1})$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z, P_n)| < |\Phi_Z'(-\phi_n^Z, P_{n+1})|$;

for ***CASE 2:*** *either* $A_n^Z = (P_n, P_{n+1}]$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z, P_n)| > |\Phi_Z'(-\phi_n^Z, P_{n+1})|$;

for ***CASE 3:*** *either* $A_n^Z = [P_n, P_{n+1}]$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z, P_n)| = |\Phi_Z'(-\phi_n^Z, P_{n+1})|$.

The proof of Theorem 3.1 is presented in Appendix D. Here we briefly interpret the construction of P_n^* in (3.8). According to Property 3.1, when $Q_m = -\phi_n^Z$ for $n \in \{1, 2, \ldots, I + J - 1\}$, the excess demand is with $\Phi_Z'(Q_m, p) = 0$ for $p \in A_n^Z$. A_n^Z could be $[P_n, P_{n+1})$, $(P_n, P_{n+1}]$, (P_n, P_{n+1}), or $[P_n, P_{n+1}]$.

When $A_n^Z = [P_n, P_{n+1})$, P_n is the only limit price associated with the highest executable order volume and zero surplus. This case is considered in CASE 1 of (3.8), and Xetra auction price function takes the value of $P_X(Q_m) = P_n$. Analogously, $A_n^Z = (P_n, P_{n+1}]$ is considered in CASE 2 of (3.8). Xetra auction price function takes the value of $P_X(Q_m) = P_{n+1}$, as P_{n+1} is the only limit price associated with the highest executable order volume and zero surplus in this case.

When $A_n^Z = (P_n, P_{n+1})$, we compare the surplus of P_n with the surplus of P_{n+1}. When P_n has lower surplus than P_{n+1} with $|\Phi'_Z(-\phi_n^Z, P_n)| < |\Phi'_Z(-\phi_n^Z, P_{n+1})|$, which is considered in CASE 1 of (3.8), Xetra auction price function takes the value of the limit price with the lower surplus as $P_X(Q_m) = P_n$. When P_n has higher surplus than P_{n+1} with $|\Phi'_Z(-\phi_n^Z, P_n)| > |\Phi'_Z(-\phi_n^Z, P_{n+1})|$, which is considered in CASE 2 of (3.8), Xetra auction price function takes the value of the limit price with the lower surplus as $P_X(Q_m) = P_{n+1}$. When P_n has the same surplus as P_{n+1} with $|\Phi'_Z(-\phi_n^Z, P_n)| = |\Phi'_Z(-\phi_n^Z, P_{n+1})|$, which is considered in CASE 3 of (3.8), the reference price P_{ref} is introduced and Xetra auction price function takes the value of $P_X(Q_m) = \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\}$.

$A_n^Z = [P_n, P_{n+1}]$ is considered in CASE 3 of (3.8). Both P_n and P_{n+1} are associated with the highest executable order volume and zero surplus in this case. The reference price P_{ref} is introduced and Xetra auction price function takes the value of $P_X(Q_m) = \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\}$.

We demonstrate how to apply Theorem 3.1 to compute Xetra auction price function $P_X(Q_m)$ in the following example.

Example 3.3. Consider the order book situation depicted in Example 3.1. The explicit form of excess demand function is described in (3.3) with $P_1 = 198$, $P_2 = 199$, $P_3 = 200$, $P_4 = 201$, and $P_5 = 202$; $\phi_0^Z = 350$, $\phi_1^Z = 250$, $\phi_2^Z = 150$, $\phi_3^Z = 100$, $\phi_4^Z = -50$, and $\phi_5^Z = -200$. The construction of Xetra auction price function $P_X(Q_m)$ is as follows:

For $Q_m \in (-\infty, -\phi_1^Z) = (-\infty, -250)$, the value of Xetra auction price function is $P_X(Q_m) = P_1 = 198$ according to Theorem 3.1.

For $Q_m = -\phi_1^Z = -250$, we obtain $P_X(Q_m) = P_2 = 199$ when CASE 2 of (3.8) is applied for $A_1^Z = (198, 199]$. For $Q_m \in (-\phi_1^Z, -\phi_2^Z) = (-250, -150)$, we have $P_X(Q_m) = P_2 = 199$.

For $Q_m = -\phi_2^Z = -150$, we obtain $P_X(Q_m) = P_3 = 200$ when CASE 2 of (3.8) is applied for $A_2^Z = (199, 200)$ with $|\Phi'_Z(-150, 199)| = 100 > 50 = |\Phi'_Z(-150, 200)|$. For $Q_m \in (-\phi_2^Z, -\phi_3^Z) = (-150, -100)$, we have $P_X(Q_m) = P_3 = 200$. For $Q_m = -\phi_3^Z = -100$, we obtain $P_X(Q_m) = P_3 = 200$ when CASE 1 of (3.8) is applied for $A_3^Z = [200, 201)$.

For $Q_m \in (-\phi_3^Z, -\phi_4^Z) = (-100, 50)$, we have $P_X(Q_m) = P_4 = 201$. For $Q_m = -\phi_4^Z = 50$, we obtain $P_X(Q_m) = \max\{201, \min\{P_{\text{ref}}, 202\}\} = 201$ when CASE 3 of (3.8) is applied for $A_4^Z = [201, 202]$ with the reference price $P_{\text{ref}} = 200$.

For $Q_m \in (-\phi_4^Z, +\infty) = (50, +\infty)$, we have $P_X(Q_m) = P_5 = 202$. Thus the explicit form of $P_X(Q_m)$ is:

$$P_X(Q_m) = \begin{cases} 198 & \text{when } Q_m \in (-\infty, -250); \\ 199 & \text{when } Q_m \in [-250, -150); \\ 200 & \text{when } Q_m \in [-150, -100]; \\ 201 & \text{when } Q_m \in (-100, 50]; \\ 202 & \text{when } Q_m \in (50, +\infty); \end{cases} \tag{3.9}$$

which is depicted in Figure 3.7. □

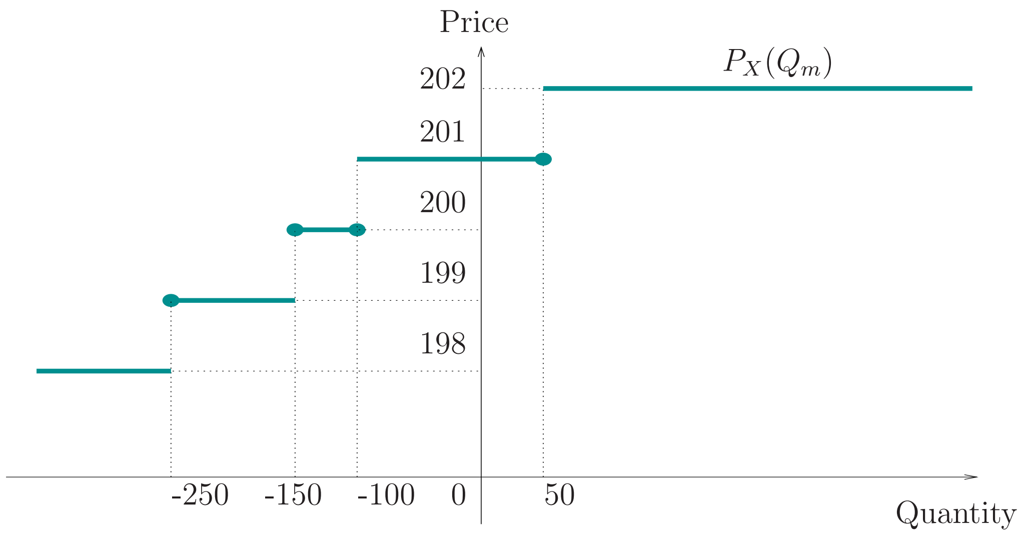


Figure 3.7: Xetra auction price function $P_X(Q_m)$ for crossed order book in Example 3.1.

3.2.2 Xetra Auction Allocation Function

To get hints on how to construct Xetra auction allocation function $Z_X(Q_m)$, we investigate the case of an uncrossed order book and derive Xetra auction allocation function in the following example.

Example 3.4. Consider the order book in Example 3.2. Observing from Figure 3.4, the maximum quantity that the trader would buy from the market is 200, and the maximum quantity that the trader would sell to the market is 350. The trader submits a market order Q_m that has higher execution priorities than limit orders in Xetra auction market. Xetra auction allocation function $Z_X(Q_m)$ thus takes the explicit form:

$$Z_X(Q_m) = \max\{-350, \min\{Q_m, 200\}\}; \tag{3.10}$$

which is shown in Figure 3.8. □

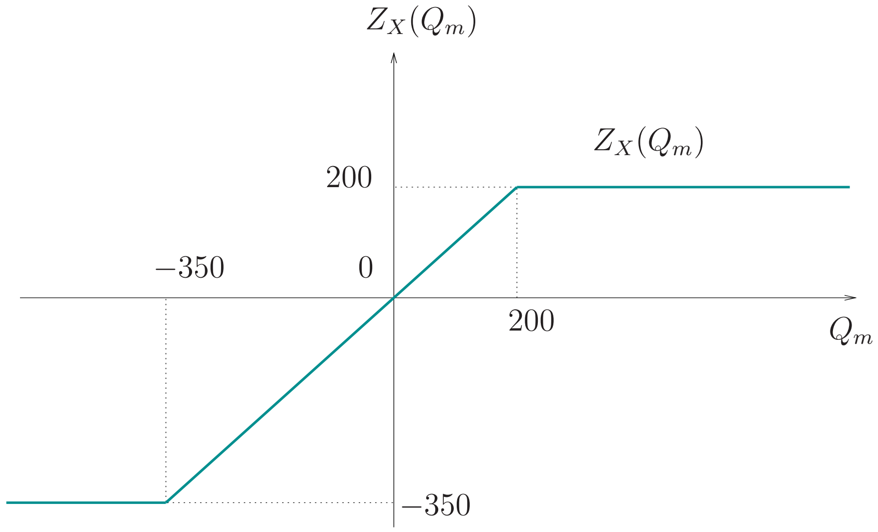


Figure 3.8: Xetra auction allocation function $Z_X(Q_m)$ for uncrossed order book in Example 3.2.

One can obtain Xetra auction allocation function $Z_X(Q_m)$ for general order book situation by reconsidering the maximum trading quantities shown in (3.10). Consider the trader trading against the order book depicted in Assumption 3.1 with the explicit form of $\Phi_Z(p)$ depicted in Lemma 3.1. The maximum quantity that the trader's market bid order $Q_m > 0$ could realize is $-\phi^Z_{I+J}$ when the order book has excess demand on the supply side with $\phi^Z_{I+J} < 0$. The trader can not realize its market bid order when all excess demands of the order book are on the demand side with $\phi^Z_{I+J} \geq 0$. Introduce the notation of $a^+ := \max\{0, a\}$ for $a \in \mathbb{R}$. The maximum trading quantity for the market bid order $Q_m > 0$ is thus $(-\phi^Z_{I+J})^+$. Similarly, the maximum trading quantity for the market ask order $Q_m < 0$ is $(-\phi^Z_0)^-$ with the notation of $a^- := \min\{0, a\}$. Xetra auction allocation function $Z_X(Q_m)$ thus takes the explicit form as follows:

Theorem 3.2. *Given the order book data set $\mathcal{I}_0$ in Assumption 3.1 with ϕ^Z_0 and ϕ^Z_{I+J} from the explicit form of $\Phi_Z(p)$ in Lemma 3.1, Xetra auction allocation* *function $Z_X(Q_m)$ takes the form:*

$$Z_X(Q_m) = \max\{(-\phi_0^Z)^-, \min\{Q_m, (-\phi_{I+J}^Z)^+\}\}. \tag{3.11}$$

By applying Theorem 3.2, we obtain Xetra auction allocation function $Z_X(Q_m)$ in Example 3.1 as:

$$\begin{aligned} Z_X(Q_m) &= \max\{(-\phi_0^Z)^-, \min\{Q_m, (-\phi_{I+J}^Z)^+\}\} \\ &= \max\{-350, \min\{Q_m, 200\}\}, \end{aligned}$$

which is depicted in Figure 3.9.

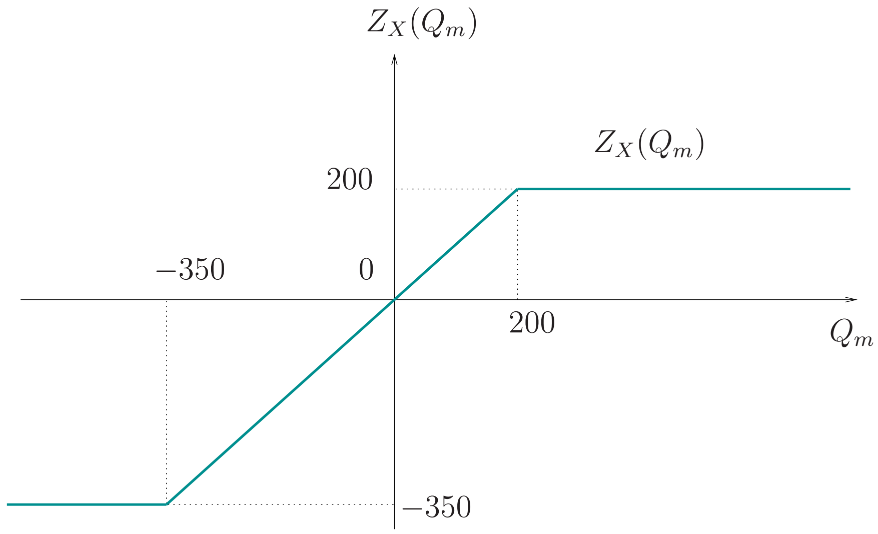


Figure 3.9: Xetra auction allocation function $Z_X(Q_m)$ for crossed order book in Example 3.1.

3.3 Expected Xetra Auction Price and Allocation

We relax Assumption 3.2 and consider that the trader is not the last who submits the order specification. Under this circumstances, the trader constructs its subjective expectations on Xetra auction price and the final transaction volume which are not consistent with the 'real' Xetra auction price and the final transaction volume in general. The trader has to consider the uncertainty that there might be orders entering the market after its market order Q_m. For simplicity, we assume:

Assumption 3.3. *The trader expects that other traders in the market submit only market orders with the aggregate amount of* $Q_D^e \geq 0$ *market bid orders and the aggregate amount of* $Q_S^e \leq 0$ *market ask orders after the trader submits its market order* Q_m*.*

3.3.1 Expected Xetra Auction Price Function

Since Q_m, Q_D^e, and Q_S^e are market orders, Xetra auction price mechanism treats all market orders as one aggregated market order $Q_m+Q_D^e+Q_S^e$ when determining Xetra auction price. Denote $\Delta Q^e := Q_D^e + Q_S^e$. Thus, the trader's subjective expectation $P_X^e(Q_m)$ on Xetra auction price function is equivalent to $P_X(Q_m + \Delta Q^e)$ where the explicit form of $P_X(\cdot)$ is depicted in Theorem 3.1.

Proposition 3.1. *Given Assumption 3.1 and Assumption 3.3, the trader's subjective expectation* $P_X^e(Q_m)$ *on Xetra auction price function is a non-decreasing step function and takes the form:*

$$P_X^e(Q_m) = \begin{cases} P_1 & \textit{when } Q_m \in (-\infty, -\phi_1^Z - \Delta Q^e); \\ P_n^* & \textit{when } Q_m = -\phi_n^Z - \Delta Q^e, \\ & \quad n \in \{1, 2, \ldots, I+J-1\}; \\ P_n & \textit{when } Q_m \in (-\phi_{n-1}^Z - \Delta Q^e, -\phi_n^Z - \Delta Q^e), \\ & \quad n \in \{2, 3, \ldots, I+J-1\}; \\ P_{I+J} & \textit{when } Q_m \in (-\phi_{I+J-1}^Z - \Delta Q^e, +\infty); \end{cases}$$

where for any $n \in \{1, 2, \cdots, I+J-1\}$

$$P_n^* = \begin{cases} P_n & \textit{CASE 1}; \\ P_{n+1} & \textit{CASE 2}; \\ \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\} & \textit{CASE 3}; \end{cases}$$

for ***CASE 1:*** *either* $A_n^Z = [P_n, P_{n+1})$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_n)| < |\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_{n+1})|$;

for ***CASE 2:*** *either* $A_n^Z = (P_n, P_{n+1}]$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_n)| > |\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_{n+1})|$;

for ***CASE 3:*** *either* $A_n^Z = [P_n, P_{n+1}]$ *or* $A_n^Z = (P_n, P_{n+1})$ *with* $|\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_n)| = |\Phi_Z'(-\phi_n^Z - \Delta Q^e, P_{n+1})|$.

3.3.2 Expected Xetra Auction Allocation Function

$Q_D^e > 0$ has no impact on the trader's final transaction volume when the trader submits a market bid order $Q_m > 0$, since the trader's market bid order has higher time priority and thus will be executed prior to Q_D^e. $Q_S^e < 0$ increases the maximum quantity that the trader can buy from the market. Analogously, $Q_S^e < 0$ has no impact on the trader's final transaction volume when the trader submits a market ask order $Q_m < 0$ and $Q_D^e > 0$ increases the maximum quantity that the trader can sell to the market. Thus, the trader's subjective expectation $Z_X^e(Q_m)$ on Xetra auction allocation function is obtained as follows:

Proposition 3.2. *Given Assumption 3.1 and Assumption 3.3, the trader's subjective expectation $Z_X^e(Q_m)$ on Xetra auction allocation function takes the form:*

$$Z_X^e(Q_m) = \max\{(-\phi_0^Z - Q_D^e)^-, \min\{Q_m, (-\phi_{I+J}^Z - Q_S^e)^+\}\}.$$

3.4 Portfolio Selection Problem

With its initial endowment (y_0, Z_0) at the beginning of the trading period, the trader conducts its forecast and expects the subjective probability distribution $\nu \in Prob(\mathbb{R})$ for the future cum-dividend risky asset price $\tilde{q}$ at the end of the trading period with the expected mean value $q^e \in \mathbb{R}$ and the associated variance $V^e \in \mathbb{R}_+$. With Assumption 3.1 and Assumption 3.3, the trader constructs its subjective expectation $P_X^e(Q_m)$ on Xetra auction price function and $Z_X^e(Q_m)$ on the allocation function which are derived from Proposition 3.1 and Proposition 3.2 respectively.

With the price of the risk-free asset normalized to 1, the trader has the budget constraint:

$$y^e + P_X^e(Q_m) \cdot Z_X^e(Q_m) = 0, \tag{3.12}$$

where y^e is the trader's expected trading quantity of the risk-free asset.

The trader expects to obtain the portfolio holding $(y_0 + y^e, Z_0 + Z_X^e(Q_m))$ after trading. Embedding with the budget constraint (3.12), the trader's future wealth at the end of the trading period is:

$$\begin{aligned} W_1 = &\quad R(y_0 + y^e) + \tilde{q} \cdot (Z_0 + Z_X^e(Q_m)) \\ = &\quad [\tilde{q} - RP_X^e(Q_m)] \cdot Z_X^e(Q_m) + \tilde{q}Z_0 + Ry_0. \end{aligned}$$

The expected value of the trader's future wealth is

$$\mathbb{E}_\nu[W_1] = [q^e - RP_X^e(Q_m)] \cdot Z_X^e(Q_m) + q^e Z_0 + Ry_0,$$

and the associated variance is

$$\mathbb{V}_\nu[W_1] = [Z_X^e(Q_m) + Z_0]^2 \cdot V^e.$$

We assume that the trader takes the preference on the mean μ and on the variance σ^2 of her future wealth. The utility function is represented as:

$$U : \begin{cases} \mathbb{R} \times \mathbb{R}_+ & \longrightarrow \mathbb{R} \\ (\mu, \sigma^2) & \longmapsto U(\mu, \sigma^2), \end{cases} \tag{3.13}$$

which is strictly quasi concave. It is increasing in μ and decreasing in σ^2.

The trader submits a market order to Xetra auction market with the integer trading quantity. The portfolio selection problem that the trader faces is to find the integer solution Q_m^* for:

$$\begin{aligned} & \max_{Q_m \in \mathbb{Z}} U(\mathbb{E}_\nu[W_1], \mathbb{V}_\nu[W_1]) && (3.14) \\ \Leftrightarrow \; & \max_{Q_m \in \mathbb{Z}} U([q^e - RP_X^e(Q_m)] \cdot Z_X^e(Q_m) + q^e Z_0 + Ry_0 \ , \ [Z_X^e(Q_m) + Z_0]^2 \cdot V^e) \\ \Leftrightarrow \; & \max_{Q_m \in \mathbb{Z}} \Upsilon(Q_m); \end{aligned}$$

where $\Upsilon(Q_m)$ is the short form of the trader's utility function.

The portfolio selection problem depicted in (3.14) considers that the trader is a price setter who can manipulate the current Xetra auction price through its market order with the quoted trading quantity Q_m, cf. the investment decision of the monopolist/monopsonist in Horst & Wenzelburger (2010). This portfolio selection problem is an extension of conventional M-V model and is called the **extended M-V model**.

3.4.1 Computational Process

$Z_X^e(Q_m)$ has both an upper bound $(-\phi_{I+J}^Z - Q_S^e)^+$ and a lower bound $(-\phi_0^Z - Q_D^e)^-$. We assume that the trader would not submit a market order Q_m out of these two bounds as in the 'real' market the trader would have to pay extra effort to handle the unexecuted part of market order after trading. Thus the trader considers $Q_m \in [(-\phi_0^Z - Q_D^e)^-, (-\phi_{I+J}^Z - Q_S^e)^+]$, which implies that $Z_X^e(Q_m) = Q_m$. Without loss of generality, we assume $-\phi_{I+J}^Z - Q_S^e \geq 0$ and $-\phi_0^Z - Q_D^e \leq 0$. The trader considers the integer quantity $Q_m \in [-\phi_0^Z - Q_D^e, -\phi_{I+J}^Z - Q_S^e]$ and aims at selecting the integer quantity ${Q_m}^*$ that solves portfolio selection problem (3.14).

The difficulty in solving portfolio selection problem (3.14) comes from the step functional form of $P_X^e(Q_m)$ which may not be tractable by employing conventional methods in mathematical analysis. To construct a computational process that solves portfolio selection problem (3.14), we investigate the structure of $P_X^e(Q_m)$.

Proposition 3.1 implies that one can derive a partition of $[-\phi_0^Z - Q_D^e, -\phi_{I+J}^Z - Q_S^e]$ with:

$$\Theta := \big\{[-\phi_0^Z - Q_D^e, -\phi_1^Z - \Delta Q^e \,[, \ldots,]\, -\phi_{I+J-1}^Z - \Delta Q^e, -\phi_{I+J}^Z - Q_S^e]\big\}.$$

The partition Θ contains $I+J$ intervals, denoted by Θ_i for $i = 1, \ldots, I+J$. The value of $P_X^e(Q_m)$ remains constant with $P_X^e(Q_m) = P_i$ for $Q_m \in \Theta_i$, $i = 1, \ldots, I+J$. When we consider portfolio selection problem (3.14) restricted in the interval Θ_i, we can ignore the step functional form of $P_X^e(Q_m)$ and take it as constant. Thus, the first step of solving portfolio selection problem (3.14) is to divide the portfolio selection problem into a series of subproblems:

$$\max_{Q_m \in \Theta_i \bigcap \mathbb{Z}} U([q^e - RP_i] \cdot Q_m + q^e Z_0 + Ry_0 \ , \ [Q_m + Z_0]^2 \cdot V^e), \tag{3.15}$$

for $i = 1, \ldots, I+J$.

The interval Θ_i is not necessarily a compact interval, thus it is difficult to apply conventional methods in mathematical analysis to solve portfolio selection subproblem (3.15). One possibility to resolve this difficulty is to make the compactification of partition Θ, i.e. to extend each interval $\Theta_i \in \Theta$ to a compact interval $\overline{\Theta_i}$ and allow $P_X^e(Q_m) = P_i$ for $Q_m \in \overline{\Theta_i}$ when $P_X^e(Q_m) = P_i$ for $Q_m \in \Theta_i$. One obtains the compactification $\overline{\Theta}$ of the partition Θ with

$$\overline{\Theta} := \big\{[-\phi_0^Z - Q_D^e, -\phi_1^Z - \Delta Q^e], \ldots, [-\phi_{I+J-1}^Z - \Delta Q^e, -\phi_{I+J}^Z - Q_S^e]\big\}.$$

Since $\overline{\Theta_i} \in \overline{\Theta}$ is a compact interval and $P_X^e(Q_m)$ remains constant with $P_X^e(Q_m) = P_i$ for $Q_m \in \overline{\Theta_i}$, one constructs the portfolio selection subproblem

$$\max_{Q_m \in \overline{\Theta_i}} U([q^e - RP_i] \cdot Q_m + q^e Z_0 + Ry_0 \ , \ [Q_m + Z_0]^2 \cdot V^e) \tag{3.16}$$

and computes the unique maximizer $Q_m^{i*} \in \overline{\Theta_i}$ for $i \in \{1, \ldots, I+J\}$.

For $i \in \{1, \ldots, I+J\}$, the unique maximizer $Q_m^{i*} \in \overline{\Theta_i}$ is generically a real number which is not the integer solution that the trader expects to obtain. We define the integer set $\mathrm{Int}[Q_m^{i*}]$ for each maximizer Q_m^{i*} of portfolio selection subproblem (3.16) as follows: If the maximizer Q_m^{i*} is on the boundary of $\overline{\Theta_i}$, then the integer set $\mathrm{Int}[Q_m^{i*}]$ collects the integer in Θ_i that is nearest to Q_m^{i*}. If the maximizer Q_m^{i*} is in the interior of $\overline{\Theta_i}$, then the integer set $\text{Int}[Q_m^{i*}]$ collects two consecutive integers in Θ_i between which Q_m^{i*} lies. The uniqueness of maximizer Q_m^{i*} and the well-behaveness of objective function infer that the integer maximizer for portfolio selection subproblem (3.15) lies in the integer set $\text{Int}[Q_m^{i*}]$ which contains the integer in Θ_i that is nearest to Q_m^{i*}.

Group together all integer sets $\text{Int}[Q_m^{i*}]$ for $i \in \{1, \ldots, I+J\}$ to construct a new integer set:

$$\text{Int}[Q_m] := \bigcup_{i \in \{1,\ldots,I+J\}} \text{Int}[Q_m^{i*}].$$

Since all integer maximizers of portfolio selection subproblem (3.15) for $i \in \{1, \ldots, I+J\}$ lie in the integer set $\text{Int}[Q_m]$, the integer solution of portfolio selection problem (3.14) also lies in the integer set $\text{Int}[Q_m]$. Calculate the maximum utility $U_{\max}$ of portfolio selection problem (3.14) for integer $Q_m \in \text{Int}[Q_m]$ as:

$$U_{\max} = \max_{Q_m \in \text{Int}[Q_m]} \Upsilon(Q_m).$$

Notice that the uniqueness of maximizer for $\max\limits_{Q_m \in \text{Int}[Q_m]} \Upsilon(Q_m)$ can not be ensured. Thus, one attains the set of all integer maximizers for portfolio selection problem (3.14):

$$\Theta^* = \{Q_m \in \text{Int}[Q_m] \mid \Upsilon(Q_m) = U_{\max}\}.$$

To achieve a unique maximizer ${Q_m}^*$, one has to apply additional criterion to choose one element from Θ^* when there is more than one maximizer, i.e. $\sharp\Theta^* > 1$.

The criterion of selecting a unique maximizer ${Q_m}^*$ from Θ^* is by and large subjective. For example, one criterion is to randomly choose ${Q_m}^*$ from Θ^*. The criterion applied in this work is to choose the minimum trading quantity from Θ^*, i.e. ${Q_m}^* = \min\limits_{Q_m \in \Theta^*} |Q_m|$. This criterion is based on the consideration that the trader intends to have the trading quantity as small as possible to attain the same utility and that the trader is inclined to buy the risky asset from the market and to hold it rather than to sell the risky asset to the market.

The computational process that achieves a unique integer solution for portfolio selection problem (3.14) is depicted in Procedure 3.1. Proposition 3.3 verifies that the solution calculated by Procedure 3.1 is the integer maximizer for portfolio selection problem (3.14).

Proposition 3.3. *The unique integer solution calculated by Procedure 3.1 is the integer maximizer of portfolio selection problem (3.14).*

Procedure 3.1 Computational process for portfolio selection problem

Derive

$$\Theta = \left\{ [-\phi_0^Z - Q_D^e, -\phi_1^Z - \Delta Q^e \,], \ldots,]\, -\phi_{I+J-1}^Z - \Delta Q^e, -\phi_{I+J}^Z - Q_S^e] \right\}$$

such that $P_X^e(Q_m) = P_i$ for $Q_m \in \Theta_i \in \Theta$, $i \in \{1, \ldots, I+J\}$.

Make the Compactification

$$\overline{\Theta} = \left\{ [-\phi_0^Z - Q_D^e, -\phi_1^Z - \Delta Q^e], \ldots, [-\phi_{I+J-1}^Z - \Delta Q^e, -\phi_{I+J}^Z - Q_S^e] \right\}$$

such that $P_X^e(Q_m) = P_i$ for $Q_m \in \overline{\Theta_i} \in \overline{\Theta}$ if $P_X^e(Q_m) = P_i$ for $Q_m \in \Theta_i \in \Theta$, $i \in \{1, \ldots, I+J\}$.

for $i = 1$ to $I + J$ **do**

  Compute the unique maximizer

$$Q_m^{i*} = \arg \max_{Q_m \in \overline{\Theta_i}} U([q^e - RP_i] \cdot Q_m + q^e Z_0 + Ry_0 \;,\; [Q_m + Z_0]^2 \cdot V^e).$$

  if Q_m^{i*} is on the boundary of $\overline{\Theta_i}$ **then**

    $\mathrm{Int}[Q_m^{i*}] = \{Q_m \in \Theta_i \mid Q_m$ is the nearest interger to $Q_m^{i*}\}$.

  else if Q_m^{i*} is in interior of $\overline{\Theta_i}$ **then**

    $\mathrm{Int}[Q_m^{i*}] = \{Q_m, Q_m + 1 \in \Theta_i \mid Q_m \leq Q_m^{i*} < Q_m + 1$ and $Q_m \in \mathbb{Z}\}$.

  end if

end for

Construct the integer set $\mathrm{Int}[Q_m] = \bigcup_{i \in \{1,\ldots,I+J\}} \mathrm{Int}[Q_m^{i*}]$.

Compute $U_{\max} = \max_{Q_m \in \mathrm{Int}[Q_m]} \upsilon(Q_m)$.

Compute the set of integer maximizer $\Theta^* = \{Q_m \in \mathrm{Int}[Q_m] \mid \upsilon(Q_m) = U_{\max}\}$.

if $\sharp\Theta^* = 1$ **then**

  return the unique maximizer ${Q_m}^* \in \Theta^*$.

else if $\sharp\Theta^* > 1$ **then**

  return ${Q_m}^* = \min_{Q_m \in \Theta^*} |Q_m|$.

end if

Proof. Step 1 in Procedure 3.1 derives the partition Θ from $P_X^e(Q_m)$ such that portfolio selection problem (3.14) can be divided into a series of subproblems (3.15) for $i \in \{1, \ldots, I+J\}$. Step 2 makes the compactification $\overline{\Theta}$ in order to transform subproblem (3.15) into subproblem (3.16) that can be solved by applying analytical tools. Step 4 calculates the unique maximizer $Q_m^{i*} \in \overline{\Theta_i}$ of portfolio selection subproblem (3.16) for $i \in \{1, \ldots, I+J\}$, given that the objective function is continuous strictly quasi concave in a compact interval.

Step 5 to Step 8 construct the integer set $\text{Int}[Q_m^{i*}]$ which contains the integer in Θ_i that is nearest to Q_m^{i*}. When Q_m^{i*} is on the boundary of $\overline{\Theta_i}$, $\text{Int}[Q_m^{i*}]$ contains the integer in Θ_i that is nearest to Q_m^{i*}. This is implemented in Step 6 with $\text{Int}[Q_m^{i*}] = \{Q_m \in \Theta_i \mid Q_m \text{ is the nearest interger to } Q_m^{i*}\}$. When Q_m^{i*} is in the interior of $\overline{\Theta_i}$, the integer set $\text{Int}[Q_m^{i*}]$ contains two consecutive integers in Θ_i between which Q_m^{i*} lies. This case is implemented in Step 8 with $\text{Int}[Q_m^{i*}] = \{Q_m, Q_m + 1 \in \Theta_i \mid Q_m \leq Q_m^{i*} < Q_m + 1 \text{ and } Q_m \in \mathbb{Z}\}$. Since Q_m^{i*} is the unique maximizer of subproblem (3.16) in the compact interval $\overline{\Theta_i}$ that differs from Θ_i only at boundary points, the integer maximizer of subproblem (3.15) lies in the integer set $\text{Int}[Q_m^{i*}]$ containing integers in Θ_i that are nearest to Q_m^{i*}.

Step 11 collects all elements in integer sets $\text{Int}[Q_m^{i*}]$ for $i \in \{1, \ldots, I+J\}$ into the integer set $\text{Int}[Q_m]$ which is a subset of $[-\phi_0^Z - Q_D^e, -\phi_{I+J}^Z - Q_S^e]$. Step 12 computes the maximum utility $U_{\max}$ for $Q_m \in \text{Int}[Q_m]$, and Step 13 collects all integer quantities corresponding to the maximum utility $U_{\max}$ into the set of integer maximizers Θ^*. The integer set $\text{Int}[Q_m]$ contains all integer solutions of subproblem (3.15) for $i \in \{1, \ldots, I+J\}$, which implies that the integer solution of portfolio selection problem (3.14) lies in $\text{Int}[Q_m]$. Hence, any quantity ${Q_m}^* \in \Theta^*$ is the integer maximizer of portfolio selection problem (3.14).

When there is only one element in Θ^*, i.e. $\sharp\Theta^* = 1$, Procedure 3.1 returns this unique quantity as the integer maximizer of portfolio selection problem (3.14). When $\sharp\Theta^* > 1$, Procedure 3.1 applies in Step 17 the criterion of minimum trading quantity to return a unique maximizer ${Q_m}^* = \min_{Q_m \in \Theta^*} |Q_m|$ as the integer maximizer of portfolio selection problem (3.14). Therefore, Procedure 3.1 calculates a unique integer maximizer for portfolio selection problem (3.14). □

We demonstrate in the following example how to apply Procedure 3.1 to solve portfolio selection problem for the trader in Xetra auction market.

Example 3.5. The market has the order book in Example 3.1 when the trader makes its investment decision. The order book is shown in Figure 3.2. The corresponding excess demand function is depicted in (3.3).

To simplify the calculation, we assume that the trader does not hold any risky asset or risk-free asset at the beginning of the trading period with $y_0 = 0$ and $Z_0 = 0$. The trader expects $Q_D^e = 50$ and $Q_S^e = -50$, which implies $\Delta Q^e = 0$. Thus, $P_X^e(Q_m)$ takes the form of (3.9). The trader's subjective expectation on Xetra auction allocation function takes the form:

$$\begin{aligned} Z_X^e(Q_m) &= \max\{(-\phi_0^Z - Q_D^e)^-, \min\{Q_m, (-\phi_{I+J}^Z - Q_S^e)^+\}\} \\ &= \max\{-400, \min\{Q_m, 250\}\}. \end{aligned} \tag{3.17}$$

The trader is assumed to take the linear mean-variance preference depicted with the utility function:

$$\begin{aligned} &U(\mathbb{E}_\nu[W_1], \mathbb{V}_\nu[W_1]) = \mathbb{E}_\nu[W_1] - \frac{\alpha}{2}\mathbb{V}_\nu[W_1] \qquad (3.18) \\ &= [q^e - RP_X^e(Q_m)] \cdot Z_X^e(Q_m) + q^e Z_0 + Ry_0 - \frac{\alpha}{2}[Z_X^e(Q_m) + Z_0]^2 \cdot V^e, \end{aligned}$$

where α is the measure of absolute risk aversion, see Wenzelburger (2004).

Assume that the interest factor of the risk-free asset is $R = 1.05$. The trader expects the mean value of the future cum-dividend price as $q^e = 250$ and its associated variance as $V^e = 25$. Consider $\alpha = 1.2$, the portfolio selection problem of the trader is then to compute the integer quantity Q_m^* that maximizes the trader's utility with its subjective expectation $P_X^e(Q_m)$ depicted in (3.9) and its subjective expectation $Z_X^e(Q_m)$ depicted in (3.17):

$$\max_{Q_m \in [-400,250] \bigcap \mathbb{Z}} [250 - 1.05 P_X^e(Q_m)] \cdot Q_m - 15 \cdot Q_m^2.$$

By applying Procedure 3.1, one derives the partition

$$\Theta = \big\{[-400, -250), [-250, -150), [-150, -100], (-100, 50], (50, 250]\big\}.$$

Make the Compactification

$$\overline{\Theta} = \big\{[-400, -250], [-250, -150], [-150, -100], [-100, 50], [50, 250]\big\}.$$

For each $i = 1, \ldots, 5$, compute the unique maximizer

$$Q_m^{i*} = \arg\max_{Q_m \in \overline{\Theta}_i} [250 - 1.05 P_X^e(Q_m)] \cdot Q_m - 15 \cdot Q_m^2$$

and the corresponding integer set $\mathrm{Int}[Q_m^{i*}]$. One obtains $Q_m^{1*} = -250$. Since Q_m^{1*} is on the boundary of $\overline{\Theta}_1$ and $Q_m^{1*} \notin \Theta_1$, the nearest integer to Q_m^{1*} in Θ_1 is -251. Thus, one obtains $\mathrm{Int}[Q_m^{1*}] = \{-251\}$. Similarly, one obtains $Q_m^{2*} = -150$

with $\mathrm{Int}[Q_m^{2*}] = \{-151\}$, $Q_m^{3*} = -100$ with $\mathrm{Int}[Q_m^{3*}] = \{-100\}$, $Q_m^{4*} = 1.30$ with $\mathrm{Int}[Q_m^{4*}] = \{1, 2\}$, and $Q_m^{5*} = 50$ with $\mathrm{Int}[Q_m^{5*}] = \{51\}$.

Construct the integer set $\mathrm{Int}[Q_m] = \{-251, -151, -100, 1, 2, 51\}$. Compute $U_{\max} = \max\{-955582.1, -348213.55, -154000, 23.95, 17.9, -37082.1\} = 23.95$ and the set of integer maximizer $\Theta^* = \{1\}$.[2] The unique maximizer is thus ${Q_m}^* = 1$, which implies that the trader submits a market bid order with the quoted trading quantity ${Q_m}^* = 1$. □

In this chapter we have investigated the investment decision of the trader in Xetra auction market. We have constructed the trader's subjective expectation $P_X^e(Q_m)$ on Xetra auction price and $Z_X^e(Q_m)$ on the final transaction volume which take the functional forms with the variable of the quoted trading quantity Q_m in its market order. Then we have developed the extended M-V model by incorporating the trader's subjective expectation $P_X^e(Q_m)$ and $Z_X^e(Q_m)$ with the mean-variance framework. We have constructed the computational process of Procedure 3.1 to calculate the integer solution for the trader's optimal trading quantity in the extended M-V model.

[2]The computation in Example 3.5 is conducted in the computer software *Mathematica*.

Chapter 4

Xetra Auction Market System

In this chapter we investigate the non market-clearing price dynamics in Xetra auction market by constructing the ACE model of Xetra auction market system (XAMS). Based on constructive aspects of the economic system, which are derived from systems theory and the methodology of Agent-Based Modeling (ABM), we propose an integrative framework for ACE modelling in section 4.1 that works as general guidance on constructing the ACE model from 'bottom-up'. We employ this integrative framework in section 4.2 to construct the ACE model of XAMS. With the implementation of the ACE model in a computer software system with the programming language Groovy/Java, we conduct in section 4.3 the computational market experiment and analyze the economic properties of market dynamics in simulation results.

4.1 Agent-based Modelling of Economic System

According to Tesfatsion (2006), Agent-based Computational Economics (ACE) is a computational study on the dynamical economic system from 'bottom-up'. Economists in this strand construct the ACE model of the economic system by modelling the interaction of the components that are termed as economic entities in the economic system. An **agent** in the ACE model is a bundle of computational processes that represent the functionality of the economic entity in the economic system. An **ACE model** is a collection of agents interacting with each other to represent the global behavior of the economic system.

In general, we follow the ACE modelling procedure that is depicted in Figure 4.1. First, by applying the ABM method which considers modelling the system as a collection of interacting agents, the ACE modeler analyzes the economic system and constructs the corresponding ACE model. Then the ACE modeler employs computer programming languages to implement the ACE model as the computer software system. After that, the ACE modeler initiates and executes the software system to observe in the computer environment the evolution of the software system which represents the dynamics of the economic system.

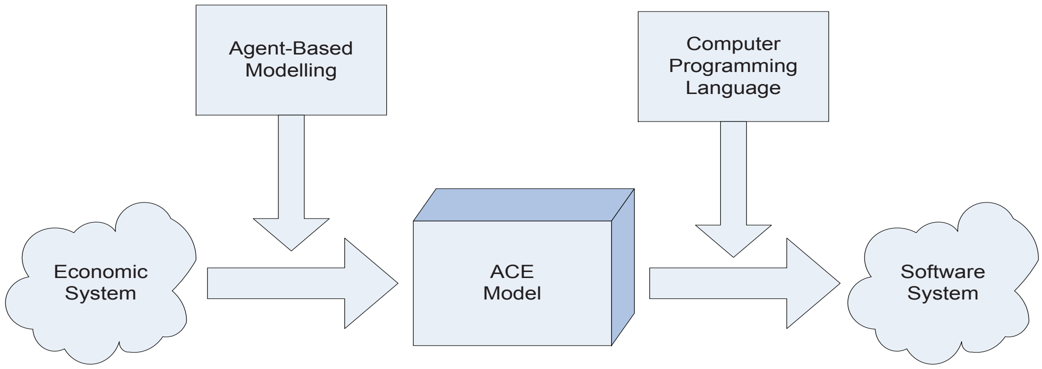


Figure 4.1: ACE modelling procedure.

An ACE model is an abstracted representation of the economic system. It is independent of the computer programming language and is not a computer software system. The current difficulty of constructing the ACE model is the lack of guidance and general principles which economists could follow when constructing ACE models, e.g. see LeBaron (2006). To overcome this difficulty, we propose in this work an integrative framework for ACE modelling which serves as general guidance to analyze the economic system from 'bottom-up' and to construct the corresponding ACE model. The foundation of this integrative framework is derived from systems theory and the methodology of ABM, which is depicted as constructive aspects of the economic system.

4.1.1 Constructive Aspects of Economic System

Constructive aspects of the economic system have the epistemic root in the systematic principle of economic phenomena and processes, i.e. economic phenomena and processes can be regarded as states and evolutionary processes of the corresponding economic systems.[1]

From systems theory and the methodology of ABM, an economic system can be regarded as a dynamical open system which interacts with its environment in

[1] The systematic principle of economic phenomena and processes is the specification of the systems theory in the field of economics, see Bertalanffy (1993).

society. An **economic system** is a collection of economic entities (consumers, firms, commodities, markets, etc.) interacting with each other such that the interactions of economic entities perform macroscopic behavior of the system given the influence from the environment. To explicitly analyze and model an economic system from 'bottom-up' is equivalent to stipulating the following **constructive aspects of the economic system**:

I. The scope of the economic system and its environment;

II. The interrelation between the economic system and its environment;

III. Elements of the economic system, which are economic entities considered in the economic system;

IV. The structure of the economic system, which is the interrelation among elements of the economic system.

To model an economic system from 'bottom-up', constructive aspects of the economic system suggest the integrative framework to formulate aspects I to IV of the economic system and their updating rules (also called state transition rules) which represent the dynamics of the economic system.

The economist normally specifies the scope of the economic system and its environment when initiating economic research. Consider the composite of the economic system and its environment as the economic world and regard the interrelation between the economic system and its environment as information flows.[2] The integrative framework formulates the ACE world which integrates the ACE model of the economic system and its environment with the information flows.

As observed in contemporary economic literature, economists tend to classify economic entities into different types in order to investigate the characteristics of each type. For example, Pindyck & Rubinfeld (2001) clusters microeconomic entities into consumers, producers (firms), commodities, markets, etc. The integrative framework follows this classification of economic entities to stipulate elements in the economic system and the corresponding agents in the ACE model.

The integrative framework represents the structure of the economic system as the information flows among agents in the ACE model. To explicitly depict in the ACE model the structure of the economic system, the integrative framework

[2]The concept of information can be termed as diversified meanings. Here the information is considered as quantitative representations of the economic world. Knowledge, methods, and actions are regarded as information in this sense once they can be quantified.

proposes the **diagram of the relationship** for the ACE model in which nodes represent agents in the ACE model and in which arcs represent information flows between agents.[3] The diagram of the relationship represents the interrelation between the economic system and its environment as the arcs which connect the corresponding ACE model with its environment.

Now the crucial point for the integrative framework is to model economic entities with the concept of agents. Most economic entities investigated in economic research are concerned with the functionality and behavior of individuals or a group of people in an economic world. We denote all these economic entities interpreting the functionality of human subject as **active economic entities** in the sense that they behave actively to fulfill their needs and objectives. Economic entities which are not directly involved with the functionality of human subject, e.g. commodities traded in the markets, are classified as **passive economic entities**. Correspondingly, we denote the agent representing the active economic entity as the **active economic agent** and the agent representing the passive economic entity as the **passive economic agent**.

An economic system can be treated as an economic entity that is used to construct another economic system, whereas an element in the economic system can be treated as an economic system. This property of system-element duality guarantees the hierarchy of economic systems, see Potts (2000). More importantly, this property infers that one can consider the economic entity as a collection of components whose interaction among each other provides the functionality of the economic entity and thus one can model the economic entity by formulating its constructive aspects.

4.1.2 Constructive Aspects of Active Economic Entity

In economic literature, active economic entities represent the functionality of decision makers in the economic world. The concept of the decision maker has been investigated in various fields in science, including psychology, sociology, computer science, etc. As we are looking for the general framework to construct the ACE model that is to be implemented as a computer software system, we employ the related concepts in computer science and combine with decision theory

[3] The diagram of the relationship for the ACE model serves the same purpose as the class diagram which describes the static structure of objects in a system and their relationships in Unified Modelling Language (UML), see Blaha & Rumbaugh (2004). Comparing with the class diagram in UML, the diagram of the relationship for the ACE model emphasizes on the connections of information flows among agents.

in economics to model the active economic entity.

The skeleton of the decision maker is stated in the concept of the agent in artificial intelligence (AI), a subfield of computer science. An agent in this field is *"anything that can be viewed as perceiving its environment through sensors and acting upon that environment through actuators"* (Russell & Norvig 2003, p. 32). Although it lacks of a unified model of the agent in AI, we take a pragmatic viewpoint and propose a general pattern by integrating behavioral rules of active economic entities with the current concept of the learning agent in AI, see (p. 53, ibid.). This general pattern, known as the **module of active economic agent (MAEA)**, is regarded as constructive aspects of the active economic entity and is composed of the submodule of information acquirement, the submodule of storage, the submodule of learning, the submodule of objectives, the submodule of forecasting, and the submodule of action transmission. One applies MAEA to construct the corresponding active economic agent from 'bottom-up' by specifying its submodules and the interrelation among submodules. We sketch the functionality of each submodule with the structure of MAEA illustrated in Figure 4.2.

The environment in Figure 4.2 represents those parts that are out of the scope of the active economic agent. The information flows between MAEA and the environment represent the interrelation of the agent with others in the economic system.

The submodule of information acquirement establishes the connections with its environment and collects information through the interrelations. The submodule of storage stores the information about the state of the agent. It collects the information transmitted from other submodules and sends out the information on request. The submodule of forecasting generates the forecast on uncertain factors that the agent considers. The submodule of objectives depicts the objectives that the agent intends to achieve, selects the action plan based on its designated objectives, and sends out the action plan to the submodule of action transmission to realize the action. The submodule of action transmission receives action plans from the submodule of objectives and realizes the action through its interrelations with the environment. The submodule of learning considers the learning process of the agent, which might not be constructed as it is not a compulsory piece when modelling the agent in economic literature.

The state of the active economic agent evolves when the agent acts to fulfill its objectives. The updating rules of the agent are thus the decision making process that is presented by interactions among submodules.

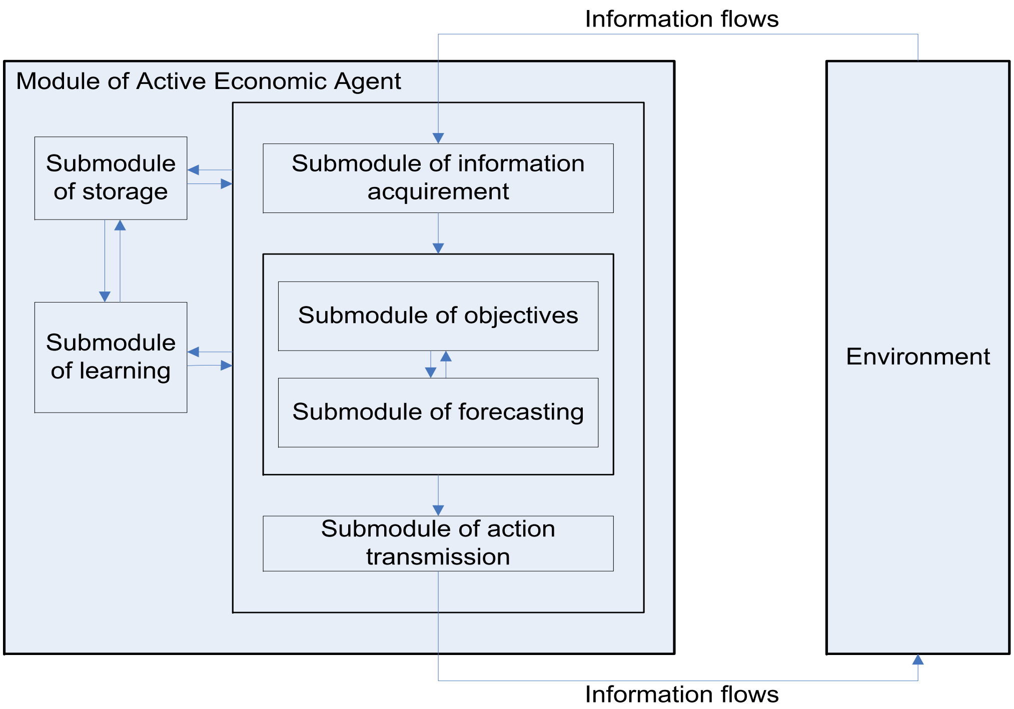


Figure 4.2: Structure of MAEA.

The decision making process generally starts when the agent initiates its state. The agent observes information via the submodule of information acquirement and keeps the information in its memory via the submodule of storage. Then it applies the submodule of learning to update itself, e.g. to update the forecasting methods currently applied in the submodule of forecasting in order to provide more accurate forecasts on uncertain factors that the agent considers. After that, the agent generates its subjective forecast via the submodule of forecasting, selects the action plan to fulfill its objectives via the submodule of objectives, and transmits the action to the economic system via the submodule of action transmission. Finally, the agent receives from the environment the feedback of its action. This general decision making process is illustrated in Figure 4.3 and is worked as a benchmark for depicting the updating rules of active economic agents.

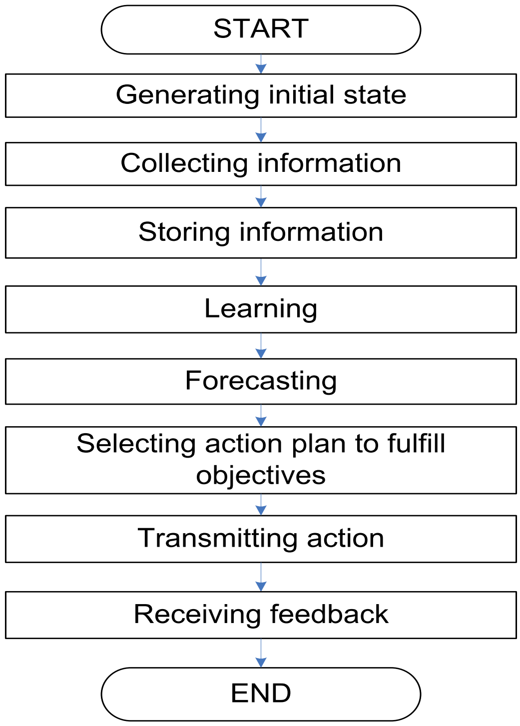


Figure 4.3: General decision making process of active economic agent.

4.1.3 Constructive Aspects of Passive Economic Entity

Passive economic entities do not behave actively to fulfill their objectives. They mainly act as information providers that disseminate information to active economic entities on request. We propose a general pattern named the **module of passive economic agent (MPEA)** to construct passive economic agents. MPEA consists of a set of **economic properties** that represent the information considered in the agent and associated **operations**, such as the operation of updating information that is regarded as the updating rules of the agent.

4.1.4 Integrative Framework for ACE Modelling

The integrative framework for ACE modelling is a modelling process that applies the constructive aspects of the economic system and of the economic entities to translate the economic system into the corresponding ACE model. Given the economic system in study, the integrative framework starts with specifying constructive aspects of the economic system. Then it applies MAEA and MPEA as templates to formulate the corresponding economic agents in the economic system. It models the updating rules of active economic agents with the decision making process and the updating rules of passive economic agents with the operation of updating information. The updating rules of economic agents constitute the updating rules of the ACE model. Given information flows between the ACE model and its environment, the interactions among agents generate the dynamics of the model. The integrative framework proposes in the form of the flowchart the **diagram of the interaction** for the ACE model that describes the sequence of workflows and activities among agents. This diagram explicitly depicts the interactions among agents in the dynamic process of the ACE model.[4] In summary, the integrative framework contains the modelling procedure as follows:

1. Specify constructive aspects of the economic system;
2. Construct corresponding agents in the ACE model by applying MAEA and MPEA respectively, model the decision making process of active economic agents and the operation of updating information in passive economic agents;
3. Present the diagram of the interaction to describe the sequence of workflows and activities among agents for the dynamics of the ACE model.

4.2 ACE Model of Xetra Auction Market System

We apply the integrative framework to construct the ACE model of XAMS. Consider an economic world with one risky asset market and one risk-free asset market in trading period $t \in \{1, \dots, T\}$. The economic world applies Euro (€) as the trading currency. The risky asset considered in the market has no dividend and is traded in integer shares, i.e. traders can trade 19 shares of the risky asset but not 19.81 shares. The risk-free asset is divisible in any trading quantity with the trading price normalized to 1. It has a constant interest factor R which can be interpreted as $1 + r$ with r denoting the nominal interest rate. N traders participate in the economic world to trade in the risky asset market and the risk-free asset market. The economic world has no transaction cost and no short sale constraint for traders.

[4]The diagram of the interaction can be regarded as the flowchart version of the activity diagram in Unified Modelling Language (UML), which is to represent the sequence of activities among components in the system, see Blaha & Rumbaugh (2004).

The risky asset market is a Xetra auction market that holds one Xetra auction for each trading period to determine the market price and the trading volume by XAMM. Xetra auction consists of the call phase and the price determination phase. During the call phase, XAMM disseminates the real-time trading data of the central order book in the market. Upon observing the real-time trading data from the market, traders perform their investment decisions and submit orders. XAMM then collects the submitted orders to the central order book and simultaneously updates the real-time trading data. The call phase stops randomly after a fixed time span and the price determination phase follows to determine Xetra auction price and the final transaction volumes. After that, XAMM cancels the unexecuted part of the orders and conducts the settlement to complete the payment for each transaction. After trading in Xetra auction market, traders interact with the risk-free asset market and trade for the risk-free asset.

We focus on Xetra auction market and regard the risk-free asset market as the environment of Xetra auction market. Given the information flows from the risk-free asset market, the interactions among traders and XAMM provide the functionality of Xetra auction market, i.e. determining the trading price and reallocating the risky asset among traders. We apply the integrative framework to construct the ACE model of XAMS. The first step is to consider constructive aspects of XAMS.

Example 4.1 (Constructive aspects of XAMS).

I. XAMS considers economic entities which operate in the market, i.e. N traders, XAMM, the numeraire employed in the market, and the risky asset traded in the market. The environment of XAMS is the risk-free asset market.

II. XAMS connects to the risk-free asset market to request the information of the interest factor R as well as to transmit trader's trading request on the risk-free asset. The risk-free asset market connects to XAMS to inform traders the interest factor R as well as their realized trading quantities of the risk-free asset. These information flows represent the interrelation between XAMS and its environment.

III. XAMS is regarded as a dynamical system which implicitly contains the concept of time. We propose the concept of the system clock to provide the information of the time considered in the ACE model. Thus, elements of XAMS as well as the corresponding ACE model are: N traders, XAMM, the numeraire, the risky asset, and the system clock.

IV. Consider a decentralized market. Traders connect with XAMM in order to perform the trading behavior, whereas there is no direct connection among traders. Traders and XAMM connect to the numeraire, the risky asset, and the system clock to have access to the associated information. Denote Xetra auction market center in the ACE model as a composite of XAMM, the numeraire, the risky asset, and the system clock. Then the structure of XAMS follows the type of the star network, see Figure 4.4. Xetra auction market center as the central node in the star network connects with all other nodes of traders in the ACE model. □

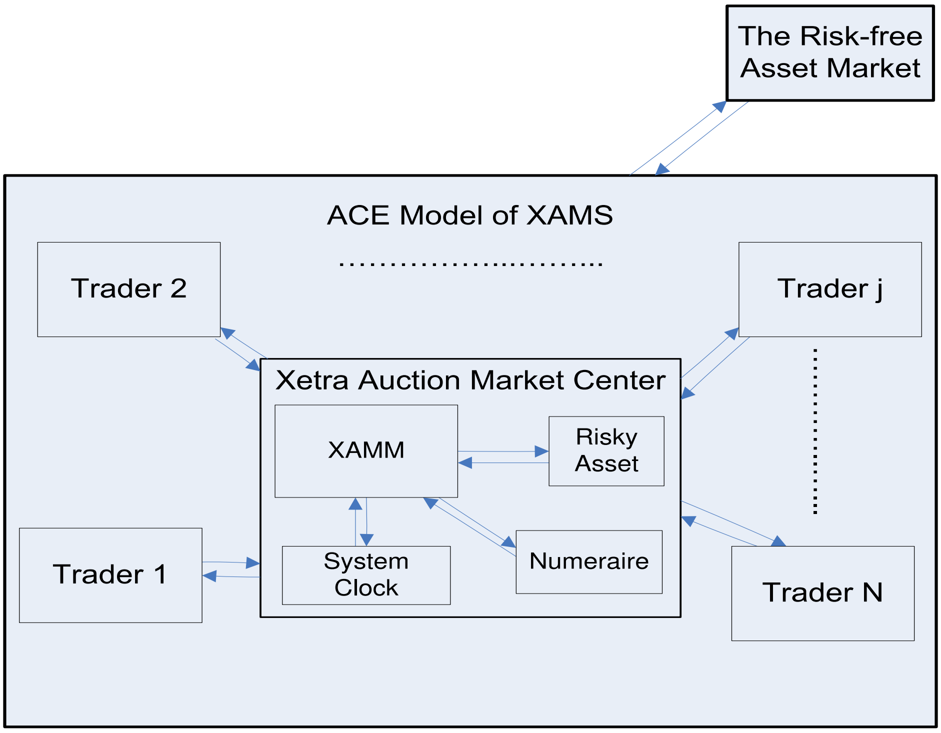


Figure 4.4: Diagram of the relationship for ACE model of XAMS.

Classify agents in XAMS as active economic agents of N traders and XAMM with passive economic agents of the numeraire, the risky asset, and the system clock. The second step of the integrative framework is to apply MAEA and MPEA to construct these agents respectively.

We consider three types of traders in XAMS. The first type is the price setter who assumes that, with the knowledge on XAMM and the real-time trading data, it could manipulate the current trading price as well as its transaction volume by its trading behavior. The second type is the price taker who believes that its trading behavior has no impact on the market. The last type is the noise trader who is assumed to act randomly in the market. Assume trader 1 in the ACE model as the price setter, trader $j \in \{2, \ldots, N-1\}$ as price takers, and trader N as the noise trader. For simplicity, further assume that each trader submits at most one order in each trading period with price setter and noise trader submitting the market order and price takers submitting the limit order. We apply MAEA to construct trader 1 in Example 4.2, trader $j \in \{2, \ldots, N-1\}$ in Example 4.3, and trader N in Example 4.4 respectively.

Example 4.2 (Trader 1). As stated in Figure 4.4, trader 1 connects with Xetra auction market center and the risk-free asset market. At the beginning of the trading period t, trader 1 is with the initial endowment $(y_0^{(1)}[t], Z_0^{(1)}[t])$ where $y_0^{(1)}[t]$ is the initial holding of the risk-free asset and $Z_0^{(1)}[t]$ is the initial holding of the risky asset. Trader 1 obtains through its submodule of information acquirement the interest factor R from the risk-free asset market and the real-time data set $\mathcal{I}_0[t]$ of the central order book from XAMM.

By applying the submodule of forecasting, trader 1 computes the expected mean value $q^{e(1)}[t]$ of the risky asset price for the next trading period $t+1$ and its associated variance $V^{e(1)}[t]$. As trader 1 can manipulate the current trading price as well as its transaction volume by submitting its market order, it performs its subjective forecast $P_X^{e(1)}[t](Q_m)$ on the current Xetra auction price and $Z_X^{e(1)}[t](Q_m)$ on the final transaction volume for period t which are functions with the control variable Q_m of the quoted trading quantity in its market order. Essentially, $P_X^{e(1)}[t](Q_m)$ presents the trader's subjective belief on the inverse demand function of the market with the control variable of the quoted trading quantity Q_m in its market order and $Z_X^{e(1)}[t](Q_m)$ depicts its subjective belief on the allocation function of the market.

With its forecast of $\{P_X^{e(1)}[t](Q_m), Z_X^{e(1)}[t](Q_m), q^{e(1)}[t], V^{e(1)}[t]\}$, the trader has the budget constraint:

$$P_X^{e(1)}[t](Q_m) \cdot Z_X^{e(1)}[t](Q_m) + y^{e(1)}[t] = 0, \tag{4.1}$$

where $y^{e(1)}[t]$ is the trader's expected trading quantity of the risk-free asset in period t. The trader expects the portfolio holding after trading in period t as $(y_0^{(1)}[t] + y^{e(1)}[t], Z_0^{(1)}[t] + Z_X^{e(1)}[t](Q_m))$. Complying with the budget constraint (4.1), the trader considers the mean value $mean^{(1)}[t]$ of its future wealth at the end of the trading period t as:

$$\begin{aligned} mean^{(1)}[t] &= \{q^{e(1)}[t] - R \cdot P_X^{e(1)}[t](Q_m)\} \cdot Z_X^{e(1)}[t](Q_m) \\ &\quad + q^{e(1)}[t] \cdot Z_0^{(1)}[t] + Ry_0^{(1)}[t]. \end{aligned} \tag{4.2}$$

The associated variance $var^{(1)}[t]$ is as:

$$var^{(1)}[t] = \{Z_0^{(1)}[t] + Z_X^{e(1)}[t](Q_m)\}^2 \cdot V^{e(1)}[t]. \tag{4.3}$$

We assume that trader 1 takes the linear mean-variance preference. The trader presents its objective in the submodule of objectives as the portfolio selection problem:

$$\begin{aligned} & \max_{Q_m \in \mathbb{Z}} \quad mean^{(1)}[t] - \frac{\alpha_1}{2} var^{(1)}[t] \qquad (4.4) \\ \Leftrightarrow\ & \max_{Q_m \in \mathbb{Z}} \quad \{q^{e(1)}[t] - R \cdot P_X^{e(1)}[t](Q_m)\} \cdot Z_X^{e(1)}[t](Q_m) \\ & \quad + q^{e(1)}[t] \cdot Z_0^{(1)}[t] + Ry_0^{(1)}[t] - \frac{\alpha_1}{2}\{Z_0^{(1)}[t] + Z_X^{e(1)}[t](Q_m)\}^2 \cdot V^{e(1)}[t], \end{aligned}$$

where α_1 is a constant measure of absolute risk aversion. Trader 1 solves this portfolio selection problem (4.4) and obtains the integer maximizer $Q_m^{(1)}[t]$ with $Q_m^{(1)}[t] > 0$ denoting the trading quantity on the demand side and $Q_m^{(1)}[t] < 0$ denoting the trading quantity on the supply side. Then the trader submits the market order with the quoted trading quantity $Q_m^{(1)}[t]$ to XAMM via the submodule of action transmission.

Trader 1 realizes the trading price $P_X[t]$ and its transaction volume $Z_X^{(1)}[t]$ after XAMM determines the trading price and the trading volume in the price determination phase. Then the trader completes with XAMM the payment for its transaction.

After trading the risky asset, trader 1 attains from the risk-free asset market the share $y^{(1)}[t] = -P_X[t] \cdot Z_X^{(1)}[t]$ of the risk-free asset. The portfolio holding that the trader acquires after trading in period t is $(y_0^{(1)}[t] + y^{(1)}[t], Z_0^{(1)}[t] + Z_X^{(1)}[t])$ and the trader's initial endowment of the next trading period $t+1$ is

$$\begin{cases} y_0^{(1)}[t+1] &= R(y_0^{(1)}[t] + y^{(1)}[t]), \\ Z_0^{(1)}[t+1] &= Z_0^{(1)}[t] + Z_X^{(1)}[t]. \end{cases} \tag{4.5}$$

The decision making process of trader 1 is illustrated in Figure 4.5. □

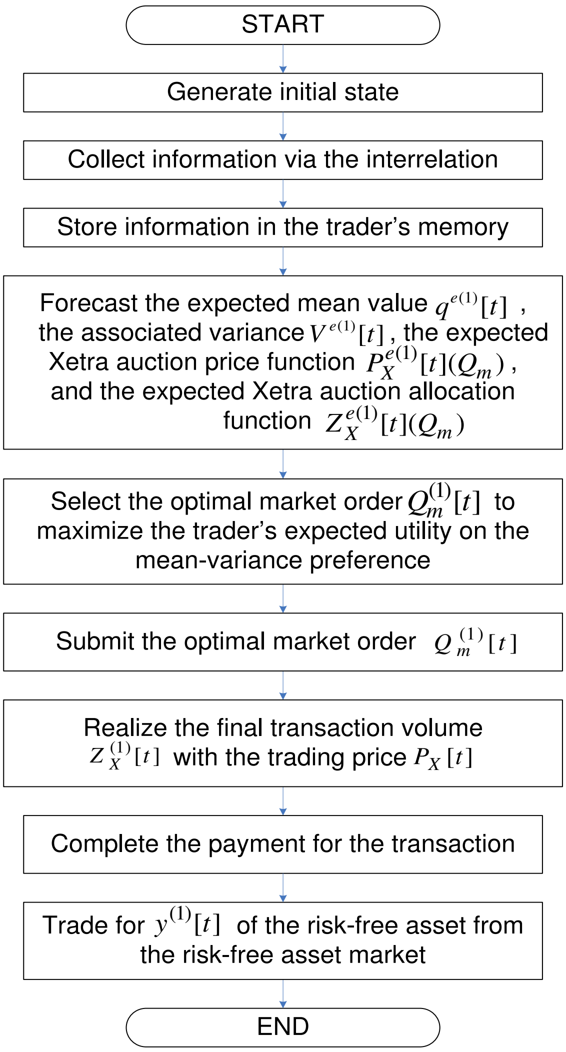


Figure 4.5: Decision making process of trader 1 in Example 4.2.

Example 4.3 (Trader $\mathbf{j = 2, \ldots, N-1}$). Analogous to trader 1, trader j connects with Xetra auction market center and the risk-free asset market. At the beginning of the trading period t, trader j is with the initial endowment $(y_0^{(j)}[t], Z_0^{(j)}[t])$. The trader obtains through its submodule of information acquirement the interest factor R and the real-time order book data set $\mathcal{I}_0[t]$.

By applying the submodule of forecasting, trader j computes the expected mean value $q^{e(j)}[t]$ of the risky asset price for the next trading period $t+1$ and its associated variance $V^{e(j)}[t]$. As the trader has to decide a limit price to quote in its limit order, the trader conducts its subjective forecast $P_X^{e(j)}[t]$ on the current

Xetra auction price and regards its forecast as the limit price. The trader expects it will realize from the market the quoted trading quantity Q_l in its limit order. With its forecast of $\{q^{e(j)}[t], V^{e(j)}[t], P_X^{e(j)}[t]\}$, the trader has the budget constraint:

$$P_X^{e(j)}[t] \cdot Q_l + y^{e(j)}[t] = 0, \tag{4.6}$$

where $y^{e(j)}[t]$ is the trader's expected trading quantity of the risk-free asset in period t. The trader expects the portfolio holding after trading in period t as $(y_0^{(j)}[t] + y^{e(j)}[t], Z_0^{(j)}[t] + Q_l)$. Complying with the budget constraint (4.6), the trader considers the mean value $mean^{(j)}[t]$ of its future wealth at the end of the trading period t as:

$$mean^{(j)}[t] = \{q^{e(j)}[t] - R \cdot P_X^{e(j)}[t]\} \cdot Q_l + q^{e(j)}[t] \cdot Z_0^{(j)}[t] + Ry_0^{(j)}[t]. \tag{4.7}$$

The associated variance $var^{(j)}[t]$ is as:

$$var^{(j)}[t] = \{Z_0^{(j)}[t] + Q_l\}^2 \cdot V^{e(j)}[t]. \tag{4.8}$$

Assume that trader j takes the linear mean-variance preference. The trader presents its objective in the submodule of objectives as the portfolio selection problem:

$$\begin{aligned} & \max_{Q_l \in \mathbb{Z}} \quad mean^{(j)}[t] - \frac{\alpha_j}{2}\, var^{(j)}[t] \\ \Leftrightarrow\ & \max_{Q_l \in \mathbb{Z}} \quad \{q^{e(j)}[t] - R \cdot P_X^{e(j)}[t]\} \cdot Q_l \\ & \qquad + q^{e(j)}[t] \cdot Z_0^{(j)}[t] + Ry_0^{(j)}[t] - \frac{\alpha_j}{2}\{Z_0^{(j)}[t] + Q_l\}^2 \cdot V^{e(j)}[t], \end{aligned} \tag{4.9}$$

where α_j is a constant measure of absolute risk aversion. Trader j solves this portfolio selection problem (4.9) and obtains the integer maximizer $Q_l^{(j)}[t]$. Then the trader submits its limit order with the price-quantity pair $(P_X^{e(j)}[t], Q_l^{(j)}[t])$ to XAMM via the submodule of action transmission.

Trader j realizes the trading price $P_X[t]$ and its transaction volume $Z_X^{(j)}[t]$ after XAMM determines the trading price and the trading volume in the price determination phase. Then the trader completes with XAMM the payment for its transaction.

After trading the risky asset, trader j attains from the risk-free asset market the share of risk-free asset $y^{(j)}[t] = -P_X[t] \cdot Z_X^{(j)}[t]$. The portfolio holding that the trader acquires after trading in period t is $(y_0^{(j)}[t] + y^{(j)}[t], Z_0^{(j)}[t] + Z_X^{(j)}[t])$ and the trader's initial endowment of the next trading period $t+1$ is

$$\begin{cases} y_0^{(j)}[t+1] & = R(y_0^{(j)}[t] + y^{(j)}[t]), \\ Z_0^{(j)}[t+1] & = Z_0^{(j)}[t] + Z_X^{(j)}[t]. \end{cases} \tag{4.10}$$

The decision making process of trader j is illustrated in Figure 4.6. □

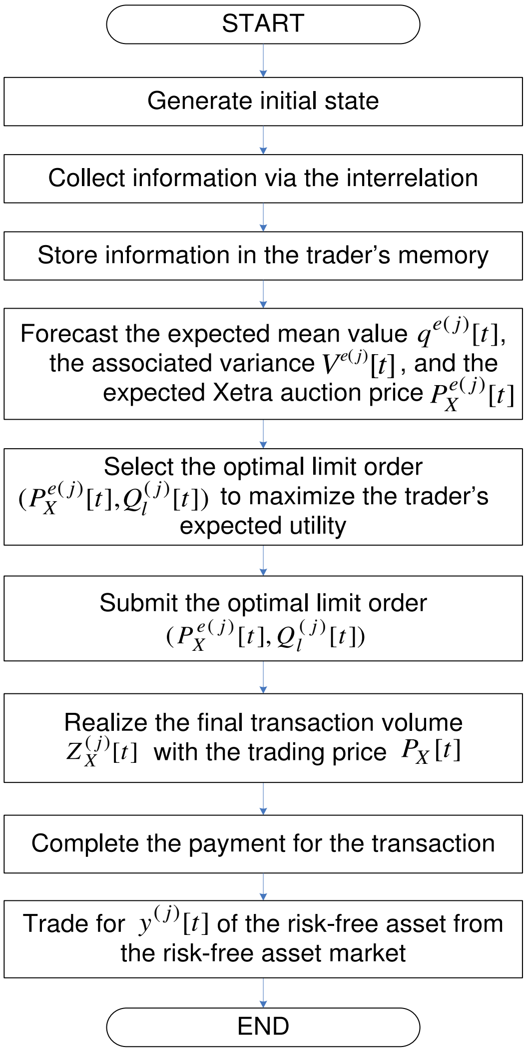


Figure 4.6: Decision making process of trader j in Example 4.3.

Example 4.4 (Trader N). Consider trader N is with the initial endowment of $(y_0^{(N)}[t], Z_0^{(N)}[t])$ at the beginning of the trading period t. Trader N obtains the interest factor R from the risk-free asset market.

The trader randomly selects $Q_m^{(N)}[t]$ from the set Q_{range} of all possible trading quantities considered by the noise trader. Then the trader constructs its market order with the quoted trading quantity $Q_m^{(N)}[t]$ and submits the order to XAMM.

Trader N realizes the trading price $P_X[t]$ and its transaction volume $Z_X^{(N)}[t]$ after XAMM determines the trading price and the trading volume in the price determination phase. Then the trader completes with XAMM the payment for its transaction.

After trading the risky asset, trader N attains from the risk-free asset market the shares of the risk-free asset $y^{(N)}[t] = -P_X[t] \cdot Z_X^{(N)}[t]$. The portfolio holding that the trader acquires after trading in period t is $(y_0^{(N)}[t] + y^{(N)}[t], Z_0^{(N)}[t] + Z_X^{(N)}[t])$ and the trader's initial endowment for the next trading period $t+1$ is

$$\begin{cases} y_0^{(N)}[t+1] & = R(y_0^{(N)}[t] + y^{(N)}[t]), \\ Z_0^{(N)}[t+1] & = Z_0^{(N)}[t] + Z_X^{(N)}[t]. \end{cases} \tag{4.11}$$

The decision making process of trader N is illustrated in Figure 4.7. □

XAMM is another type of active economic agent considered in the ACE model. Its objective is to determine Xetra auction price and the final transaction volume according to the central order book.

Example 4.5 (XAMM). At the beginning of the trading period t, XAMM has historical trading prices $P_X[i]$ for trading period $i \in \{-K_{\text{XAMM}}+1, \ldots, 0\}$ with the memory span $K_{\text{XAMM}} > 0$. It contains trading information $(\overline{\mathfrak{I}_0[i]}, P_X[i], Z_X[i])$ for trading period $i \in \{1, \ldots, t-1\}$ where $\overline{\mathfrak{I}_0[i]}$ is the order book data set at the end of the call phase, $P_X[i]$ is the Xetra auction price, and $Z_X[i] = \{Z_X^{(1)}[i], \ldots, Z_X^{(N)}[i]\}$ is the collection of the final transaction volume $Z_X^{(j)}[i]$ for each trader $j \in \{1, \ldots, N\}$. In summary, XAMM at the beginning of the trading period t is with the historical trading information set

$$\begin{aligned} Infor[t-1] \; = \{ \; & (\overline{\mathfrak{I}_0[t-1]}, P_X[t-1], Z_X[t-1]), \ldots, (\overline{\mathfrak{I}_0[1]}, P_X[1], Z_X[1]), \\ & P_X[0], \ldots, P_X[-K_{\text{XAMM}}+1] \; \}. \end{aligned} \tag{4.12}$$

During the call phase, XAMM works through the submodule of information acquirement to collect order specifications $\{Q_m^{(1)}[t], \ldots, (P_X^{e(j)}[t], Q_l^{(j)}[t]), \ldots, Q_m^{(N)}[t]\}$ submitted by traders and stores this information via the submodule of storage. It simultaneously disseminates the real-time order book data set $\mathfrak{I}_0[t]$.

The objective of XAMM is to determine in the price determination phase the Xetra auction price $P_X[t]$ and the final transaction volumes $Z_X[t] = \{Z_X^{(1)}[t], \ldots, Z_X^{(N)}[t]\}$ by applying Xetra auction trading rules stated in Gruppe Deutsche Börse (2003). We apply in our work the formulation of Xetra auction price mechanism

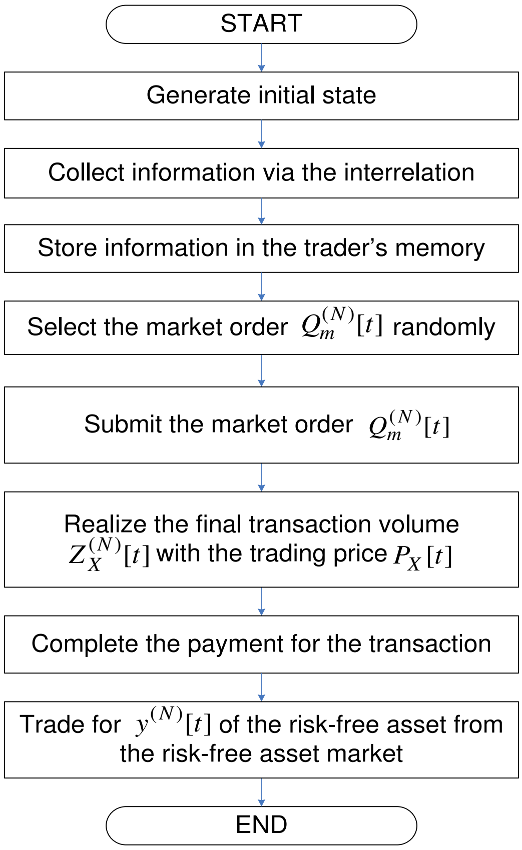


Figure 4.7: Decision making process of trader N in Example 4.4.

depicted in Theorem 1.1 and Xetra auction allocation mechanism depicted in Theorem 1.2 respectively. After determining Xetra auction price and the trading volume, XAMM cancels the unexecuted part of order specifications and conducts the settlement process to complete the payment for each transaction. Then XAMM closes the market until the next trading period $t+1$. The updating rules of XAMM are depicted in Figure 4.8. □

The numeraire, the risky asset, and the system clock are passive economic agents considered in the ACE model. They act as information providers to provide on request the information about the currency employed in the market, the security traded in the market, and the time considered in the model. As the environment of the ACE model, the risk-free asset market provides the trading on the risk-free asset.

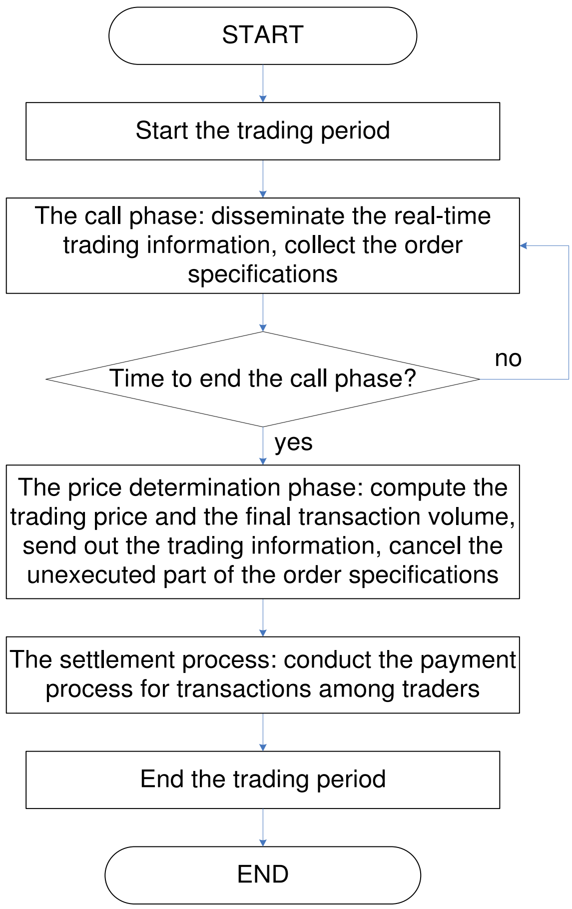


Figure 4.8: Updating rules of XAMM.

The last step in the integrative framework is to explicitly present the interactions among agents in the ACE model. We consider dynamics of XAMS with T trading periods. XAMM starts the call phase at the beginning of the trading period. It disseminates to traders the real-time trading information of the central order book and simultaneously collects order specifications submitted by traders. We assume that traders submit their order specifications in a random sequence during the call phase. To simplify our analysis, we further assume that price takers submit limit orders prior to the price setter and noise trader submit their market orders. The call phase stops randomly after a fixed time span and is followed by the price determination phase. XAMM determines in the price determination phase the Xetra auction price and the final transaction volume. Then it cancels the unexecuted part of the orders and conducts the settlement process to complete the payment for each transaction. After trading in Xetra auction market, traders obtain the risk-free asset holdings via the interaction with the risk-free asset market. XAMS iterates to the next trading period until it reaches the last trading period T. The diagram of the interaction in Figure 4.9 explicitly illustrates the workflows of activities among agents in the market dynamics.

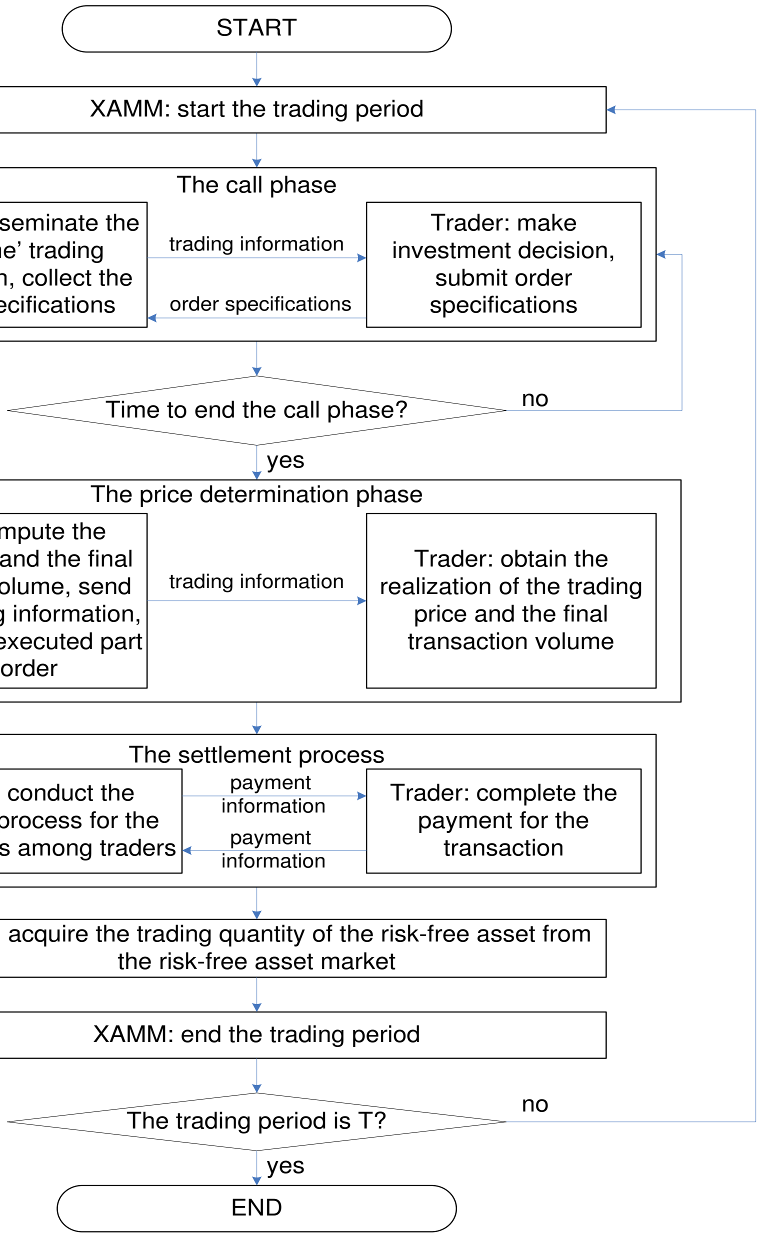


Figure 4.9: Diagram of the interaction for ACE model of XAMS.

4.3 Market Experiment

We implement the ACE model of XAMS by applying the computer language Groovy/Java with the database backend of Microsoft ACCESS/MySQL. Then we construct the market experiment and conduct the computer simulation of XAMS. The focus of the market experiment is on the generated dynamics of Xetra auction price. We are specifically concerned with:

1. whether the generated Xetra auction price is generically non market-clearing;

2. the impact of the price setter on the non market-clearing property and on the volatility of the Xetra auction price in the market.

4.3.1 Experimental Setup

We set up the simulation profile for Xetra auction market experiment by initializing the parameters and by specifying the forecasting methods employed by agents.

Model's Parameters to Initialize

- $T = 250 \ldots$ time horizon. The time horizon approximates the time span of one year when considering one auction for each trading day and 255 trading days for Frankfurt Stock Exchange in the year of 2009.

- $N = 22 \ldots$ number of traders. Three types of traders are considered in the model with 1 price setter, 20 price takers, and 1 noise trader.

- $r \ldots$ the interest rate of the risk-free asset. The interest rate is assumed to be constant in each profile. According to the Eurostat[5], the 3-months interest rate in the European Union (27 countries) for the period of October 2008 to September 2009 is in the range of $[1.04\%, 5.52\%]$. We choose r randomly from this range.

XAMM's Parameters to Initialize

[5] See http://epp.eurostat.ec.europa.eu.

- $\{P_X[0], \ldots, P_X[-K_{\text{XAMM}} + 1]\} \ldots$ the historical trading prices. We consider $P_X[0], \ldots, P_X[-K_{\text{XAMM}} + 1]$ as historical auction prices of the stock "Deutsche Börse AG" listed in Xetra for the period from August 27, 2009 to November 04, 2009, with the memory span $K_{\text{XAMM}} = 100$.[6]

- $\text{range}_{\text{p}} = 10\% \ldots$ the percentage of the price range. The Xetra platform requires transactions executed under certain price range from the last traded price P_{ref}. While it does not publicly provide the information of the percentage of the price range, another electronic trading platform Euronext requires the percentage of $\pm 10\%$. We employ the setting in Euronext and choose $\text{range}_{\text{p}} = 10\%$. Thus, XAMM considers Xetra auction price in the range of $[P_{\text{ref}}(1 - 10\%), P_{\text{ref}}(1 + 10\%)]$.

Trader's Parameters to Initialize

- $y_0^{(j)}[1] \ldots$ trader j's initial risk-free asset holding at the beginning of the trading period 1. We take $y_0^{(j)}[1]$ as a random positive number.

- $Z_0^{(j)}[1] \ldots$ trader j's initial risky asset holding at the beginning of the trading period 1. We take $Z_0^{(j)}[1]$ as a random integer number. To simplify the analysis, we assume that the aggregated volume in the market is constant with $\sum_{j=1}^{22} Z_0^{(j)}[1] = 1000$.

- $\alpha^{(j)} \ldots$ the measure of absolute risk aversion in the trader's utility function with the linear mean-variance preference. $\alpha^{(j)}$ is assumed to be constant in each profile and is selected randomly from the range of $(0, 2]$.

- $Q_{\text{range}} \ldots$ the set of the trading behavior for the noise trader $j = 22$ depicted in Example 4.4.

  We assume that the noise trader randomly chooses for each trading period the trading behavior from the set of $Q_{\text{range}} = \{$ selling 1 unit, buying 1 unit $\}$. We keep the noise trader's random choice of trading behavior for each period the same in the benchmark market experiment as in Xetra auction market experiment.

Trader's Forecasting Methods to Initialize

[6]This historical data of the stock trading price is provided online by Deutsche Börse, see http://deutsche-boerse.com.

1. Forecasting Methods in Common

   For each period t, trader $j = 1$ depicted in Example 4.2 and price takers $j \in \{2, \ldots, 21\}$ depicted in Example 4.3 compute the expected mean value $q^{e(j)}[t]$ of the risky asset price for the next trading period $t+1$ and its associated variance $V^{e(j)}[t]$. We assume two types of forecasting: the chartist and the trend trader. The forecasting type remains unchanged in each profile after trader j randomly chooses between these two types with equal probability.

   (a) Chartist

   - $q^{e(j)}[t] \ldots$ The chartist computes $q^{e(j)}[t]$ as the mean value of the historical trading prices $P_X[0], \ldots, P_X[-K_{\text{XAMM}} + 1]$ with

   $$q^{e(j)}[t] = \frac{1}{K_{\text{XAMM}}} \sum_{n=1}^{K_{\text{XAMM}}} P_X[1-n].$$

   - $V^{e(j)}[t] \ldots$ The chartist computes $V^{e(j)}[t]$ as the associated variance of the historical trading prices $P_X[0], \ldots, P_X[-K_{\text{XAMM}} + 1]$ with

   $$V^{e(j)}[t] = \frac{1}{K_{\text{XAMM}} - 1} \sum_{n=1}^{K_{\text{XAMM}}} (P_X[1-n] - q^{e(j)}[t])^2.$$

   Thus the chartist has constant forecast of $q^{e(j)}[t]$ and $V^{e(j)}[t]$ for each trading period.

   (b) Trend Trader

   - $q^{e(j)}[t] \ldots$ Trend trader expects the trend of the price movement based on the historical price movement. There are two types of trend traders in our simulation: the trend follower and the contrarian. The trend follower expects the trading price will increase (decrease) given the trading price increased (decreased) in the last trading period while the contrarian expects the opposite. Let the indicator $\text{id}_\text{c} = 1$ for trend follower and $\text{id}_\text{c} = -1$ for contrarian. With historical trading prices $P_X[t-1]$ and $P_X[t-2]$, the forecasting of the trend trader at period t is

   $$q^{e(j)}[t] = \begin{cases} P_X[t-1](1 + \text{id}_\text{c} \cdot \omega^{(j)}) & \text{if} \quad P_X[t-1] > P_X[t-2], \\ P_X[t-1] & \text{if} \quad P_X[t-1] = P_X[t-2], \\ P_X[t-1](1 - \text{id}_\text{c} \cdot \omega^{(j)}) & \text{if} \quad P_X[t-1] < P_X[t-2]; \end{cases}$$

where $\omega^{(j)}$ measures the aggressiveness of the price movement that the trader expects.
$\mathrm{id_c}$ and $\omega^{(j)}$ are assumed to be constant after $\mathrm{id_c}$ is chosen randomly from $\{-1, 1\}$ with equal probability and $\omega^{(j)}$ is chosen randomly from the range of $(0, \mathrm{range_p}]$.

- $V^{e(j)}[t] \ldots$ It is assumed that the trend trader keeps $V^{e(j)}[t]$ constant in the profile after it is chosen randomly from the range $(0, 5]$.

2. Forecasting Method for Price Setter $j = 1$ depicted in Example 4.2

   - $P_X^{e(1)}[t] \ldots$ the forecast on Xetra auction price in the current trading period t. By applying Proposition 3.1, the price setter computes the forecast $P_X^{e(1)}[t](Q_m^{(1)}[t])$ that is essentially its subjective belief on the inverse demand function of the market with the control variable of the quoted trading quantity $Q_m^{(1)}[t]$ in its market order.
   - $Z_X^{e(1)}[t] \ldots$ the forecast on the trader's final transaction volume in the current trading period t. By applying Proposition 3.2, the price setter computes $Z_X^{e(1)}[t](Q_m^{(1)}[t])$ that is essentially its subjective belief on the allocation function of the market with the control variable of the quoted trading quantity $Q_m^{(1)}[t]$ in its market order.

3. Forecasting Method for Price Taker $j = 2, \ldots, 21$ depicted in Example 4.3

   - $P_X^{e(j)}[t] \ldots$ the forecast on Xetra auction price in the current trading period t. It is assumed that $P_X^{e(j)}[t]$ is randomly chosen from the price range $[P_{\mathrm{ref}}(1 - 10\%), P_{\mathrm{ref}}(1 + 10\%)]$ stipulated in XAMM.

4.3.2 Experimental Procedure

To investigate the market dynamics and the impact of the price setter in Xetra auction market, we construct along with Xetra auction market experiment a benchmark market experiment. The benchmark market experiment has the same setup as Xetra auction market experiment except for the price setter $j = 1$ depicted in Example 4.2. The benchmark market experiment replaces the price setter $j = 1$ with the benchmark trader which follows the same setup as depicted in Example 4.2 except for adopting the forecast $Z_X^{e(1)}[t](Q_m) = Q_m$ and $P_X^{e(1)}[t](Q_m) = P_{\mathrm{ref}}$ where P_{ref} is the last traded price in Xetra auction market. Thus, the benchmark trader presents its objective in the submodule of objectives as the portfolio selection problem:

$$\max_{Q_m \in \mathbb{Z}} \quad \{q^{e(1)}[t] - R \cdot P_{\text{ref}}\} \cdot Q_m + q^{e(1)}[t] \cdot Z_0^{(1)}[t] + R y_0^{(1)}[t] - \frac{\alpha_1}{2}\{Z_0^{(1)}[t] + Q_m\}^2 \cdot V^{e(1)}[t], \tag{4.13}$$

where α_1 is a constant measure of absolute risk aversion. As observed in (4.13), the trader considers in its portfolio selection problem that the current trading price and the trader's final trading volume are independent of Q_m. The benchmark trader is thus regressed to a price taker who has no manipulation on the current trading price and on the trading volume by its market order.

We consider 50 rounds of market experiments. It starts with initiating 50 simulation profiles for each round of the market experiment with the index $s \in \{1, \ldots, 50\}$. Then for each profile s, we conduct the benchmark market experiment and Xetra auction market experiment with 250 trading periods of simulations.

4.3.3 Experimental Results

Price Dynamics. We consider the price dynamics for 250 trading periods generated in the market experiment. Figure 4.10 illustrates the price dynamics generated by all 50 simulation profiles. The blue color in this figure as well as in the following figures is for the benchmark market experiment and the red color is for Xetra auction market experiment.

Figure 4.10 demonstrates the divergence of the price dynamics in the market experiment. Consider the end-of-period trading price as the market price in the last trading period $t = 250$. It ranges $[0,\ 182.67]$ for all 50 benchmark market experiments and $[0,\ 205.39]$ for all 50 Xetra auction market experiments in the simulation. The histogram of the end-of-period trading price depicted in Figure 4.11 illustrates the divergence that emerges in both market experiments.

Performance of Price Setter. The performance of the trader can be measured by the ex post Sharpe ratio of the portfolio held by the trader, see Sharpe (1994). We apply the Wilcoxon signed ranks test to compare in each profile s the time series of the Sharpe ratio for the price setter with that for the benchmark trader who assumes its trading behavior has no impact on the market. Statistical results of the $\mathrm{p-value}$ are listed in Table E.1 and Table E.2 for the comparison on 250 trading periods in all 50 profiles. Table E.1 and Table E.2 show that 18

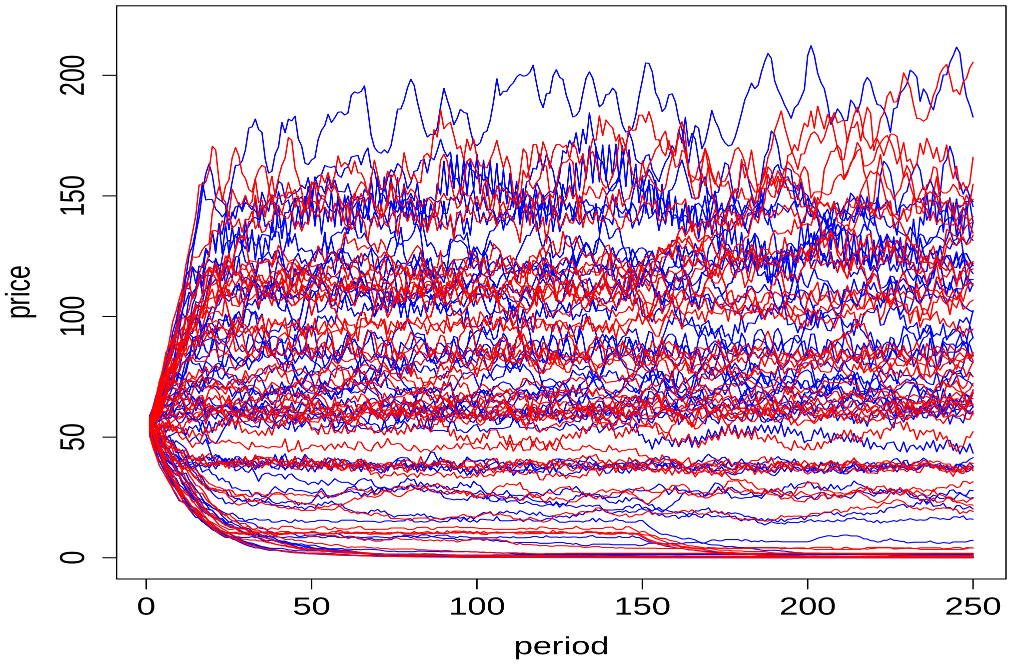


Figure 4.10: Price dynamics in 50 profiles.

profiles accept that the benchmark trader has lower Sharpe ratio than the price setter with 90% level of confidence; 31 profiles accept that the benchmark trader has higher Sharpe ratio than the price setter with 90% level of confidence; and 1 profile does not provide any inference from statistical tests. The price setter performs better than the benchmark trader in 18 of 50 profiles. Thus, it seems that the price setter does not perform significantly better than the benchmark trader.

We look into the 18 profiles where the price setter performs better than the benchmark trader and apply the the Wilcoxon signed ranks test to compare the percentage of forecast errors that the price setter commits on its expectation on the future trading price with that of the benchmark trader. The test results depicted in Table E.3 illustrate that the price setter has lower percentage of forecast errors than the benchmark trader in 12 profiles, a 67% of 18 profiles where the price setter performs better. If we look into the 31 profiles where the price setter does not have better performance, 48% of the 31 profiles have the price setter with lower percentage of forecast errors, see Table E.4. Thus, it seems that lower percentage of forecast errors triggers the price setter to have better

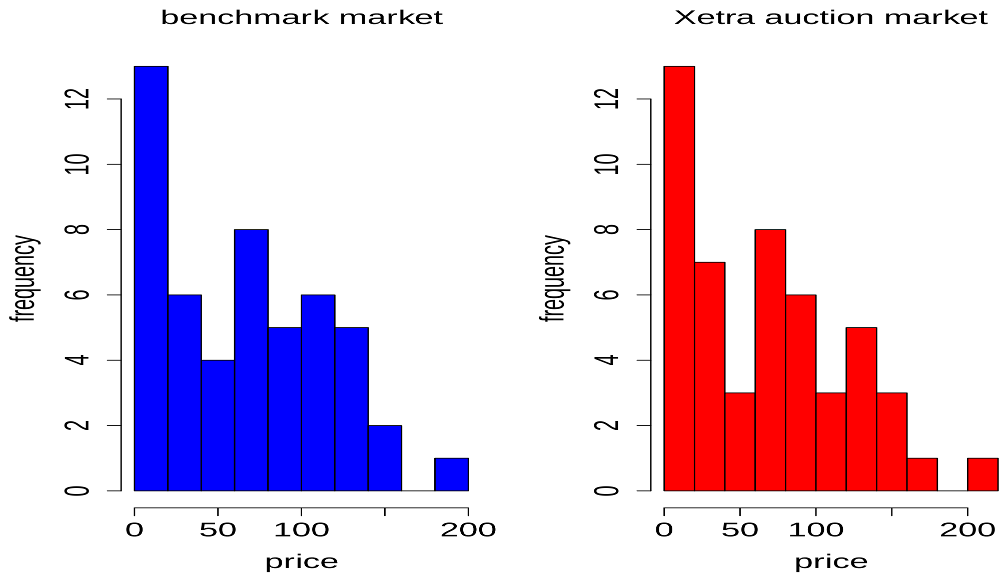


Figure 4.11: Histogram of end-of-period trading price in market experiments.

performance instead of the power of manipulating the current trading price.

The economical intuition behind this finding is that the price setter could only manipulate the current trading price while sharing the same forecast on the future price in the next trading period as the benchmark trader. The manipulation on the current trading price alone does not ensure higher performance.

Property of Non Market-Clearing Trading Price. We employ the percentage of the non market-clearing price $percent_{non}^{(s)}$ to investigate the non market-clearing property of the trading price for each profile $s \in \{1, \ldots, 50\}$ with

$$percent_{non}^{(s)} = \frac{\sharp\{\text{trading periods with non market-clearing trading price}\}}{\sharp\{\text{trading periods with trading price}\}}.$$

Figure 4.12 shows the percentage for the benchmark market experiment and for Xetra auction market experiment in each profile. The lowest percentage of non market-clearing price is 41.82% and the highest percentage is 96.58% in our simulation results, which implies that Xetra auction price generated in the market experiments is generically non market-clearing. This property can be easily observed in Figure 4.12 where the blue line and the red line are far above the x-axis.

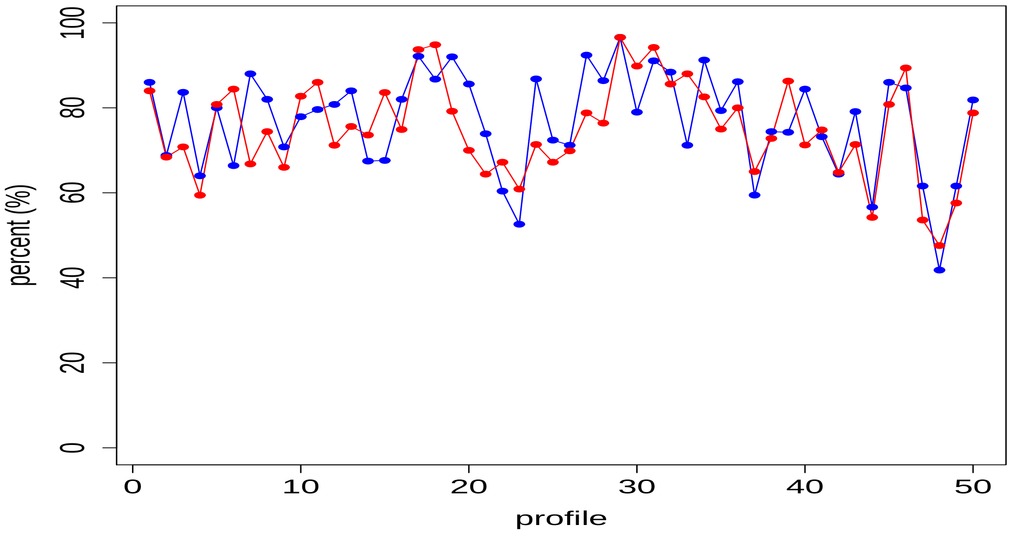


Figure 4.12: Percentage of non market-clearing price.

We conduct the Wilcoxon signed ranks test to investigate the impact of the price setter on the percentage of the non market-clearing price. We specifically test whether $percent^{(s)}_{non}$ associated with the benchmark market experiment is greater than that associated with Xetra auction market experiment. The null hypothesis is verbally presented as: $percent^{(s)}_{non}$ in the benchmark market experiment is no greater than that in Xetra auction market experiment. The test result has $\text{p} - \text{value} = 0.0659 < 0.1$. Thus, we reject the null hypothesis with 90% level of confidence and accept that $percent^{(s)}_{non}$ in the benchmark market experiment is greater than that in Xetra auction market experiment. Thus, Xetra auction market experiment where the price setter participates has a higher percentage of market equilibrium than the benchmark market experiment. This implies that the introduction of the price setter increases the possibility of market equilibrium and thus increases the market efficiency in Xetra auction market.

Intuitively, when there exists a surplus in Xetra auction market the price setter could submit a market order to accept the surplus without affecting the trading price in the market. When the price setter would submit a market order exceeding the surplus, the trading price would jump to an inferior position such that the trading price would fall down when the price setter would submit a market order in the sell side and vice versa. Thus the price setter has the incentive to meet but not outnumber the surplus in the market. When the price setter submits its market order to fully accept the surplus, Xetra auction market drives to equilibrium. The introduction of the price setter thus increases the market efficiency in Xetra auction market.

Price Volatility. We apply in this work the variance of the price dynamics $\{P_X^{(s)}[1], \ldots, P_X^{(s)}[250]\}$ for each profile s to measure price volatility in the market experiment. The variance of the price dynamics is formulated as:

$$Var^{(s)} = \frac{1}{249}\sum_{n=1}^{250}(P_X^{(s)}[n] - \overline{P_X^{(s)}})^2,$$

where $\overline{P_X^{(s)}}$ is the mean value of $\{P_X^{(s)}[1], \ldots, P_X^{(s)}[250]\}$. Figure 4.13 shows the variance for the benchmark market experiment and for Xetra auction market experiment.

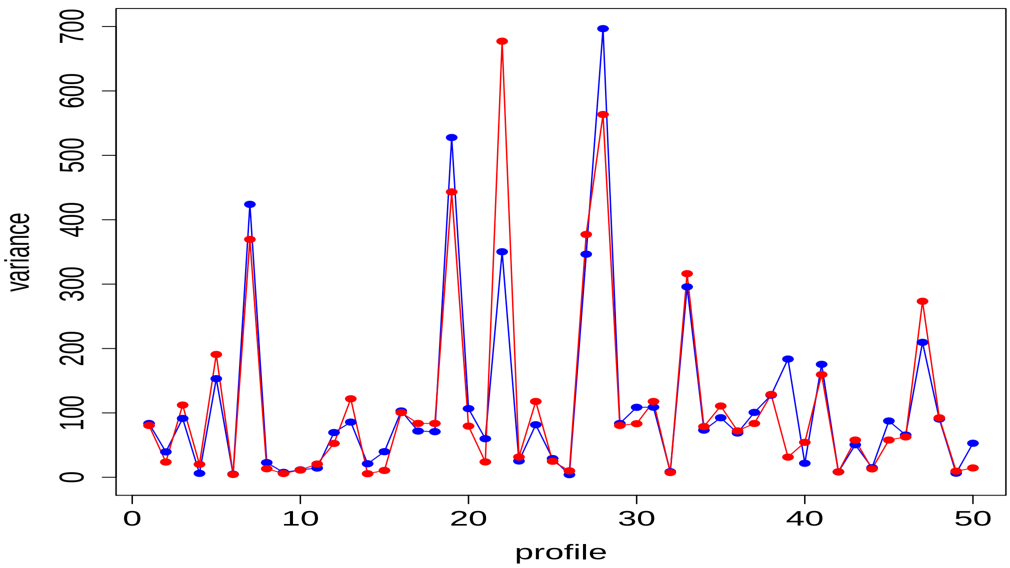


Figure 4.13: Variance of trading price dynamics.

We use the Wilcoxon signed ranks test to investigate the impact of the price setter on the price volatility. The null hypothesis that we test is verbally presented as: the variance on the price dynamics for the benchmark market experiment has the same measure as that for Xetra auction market experiment. The test result has $\mathrm{p-value} = 0.8545$. We cannot reject the null hypothesis to accept that the variance on the price dynamics for the benchmark market experiment is significantly different from that for Xetra auction market experiment. It seems that the introduction of the price setter does not significantly impact the price volatility in Xetra auction market.

Intuitively the price setter influences the price volatility of Xetra auction market in two different directions. The price setter exploits the market for a profit by its aggressive trading behavior, which would increase the price volatility in Xetra auction market. On the other hand, the introduction of the price setter increases the possibility of market equilibrium. The increase in market efficiency would allow traders to efficiently adjust their trading behavior to stabilize the market in equilibrium, which implies a decrease in the price volatility of the market. These two opposite impacts offset against each other so that the participation of the price setter would not significantly influence the price volatility in Xetra auction market.

Chapter 5

Concluding Remarks

In this work we focus on Xetra auction market. We have developed the formal model of Xetra auction market mechanism which is composed of the price mechanism and the allocation mechanism. We have investigated the economic properties of Xetra auction price mechanism, which provides hints on how to improve Xetra auction pricing rules from the perspective of market efficiency that is concerned with the auction market being market-clearing. We have constructed improved auction pricing rules that are regarded as an improvement on Xetra auction pricing rules from the perspective of market efficiency.

We have investigated the investment decision of the price setter in Xetra auction market. We consider that the trader submits a market order to Xetra auction market. Assuming that the trader obtains the real-time trading information of the order book and the knowledge on Xetra auction market mechanism, we have constructed the trader's forecast on Xetra auction price and on the final transaction in the step functional form with the control variable of the quoted trading quantity in its market order. Then we have constructed the extended M-V model, which combines the conventional M-V model with the trader's forecast on Xetra auction price and the final transaction. The computational procedure for calculating the optimal trading quantity, which solves the extended M-V model, has been constructed in this work as well.

We have derived in the last part of this work an integrative framework for ACE modelling from systems theory and the methodology of ABM. We have applied this integrative framework to develop the ACE model of Xetra auction market system with an explicit formulation of Xetra auction market mechanism and of the price setter. We have implemented the agent-based model into computer software system and conducted the computer simulation for the market experiment. The investigation of simulation results on market dynamics has validated the property of non market-clearing trading price in Xetra auction market.

Several extensions can be considered in the future. We have considered the market mechanism of Xetra auction in our work. There are two trading forms in Xetra, i.e. Xetra auction and continuous trading. The formalization of the market mechanism for continuous trading is still an open issue given the complexity of trading rules in continuous trading, see Gruppe Deutsche Börse (2003). Thus, to construct the formal model for the market mechanism of continuous trading and to investigate its economic properties can be considered in the future work.

Another extension is concerned with the trading behavior considered in the investment decision in Chapter 3 where the trader is assumed to submit a market order. Other than the market order, the trader can submit to Xetra auction market a limit order which also has an impact on Xetra auction price and the final transaction. One could investigate in the future the trader's forecast on Xetra auction price and the final transaction given the trading behavior of submitting a limit order to the market. Notice that the trader has to decide the limit price and the quoted trading quantity in the limit order. Thus, the trader's investment decision model is with two control variables of the limit price and the quoted trading quantity in the trader's limit order, which would make it more difficult in calculating the optimal solution for the investment decision model.

The final remark is concerned with the flexibility of the agent-based modelling. In the simulation results of the ACE model of Xetra auction market system, we have shown the property of non market-clearing trading price in Xetra auction market. This finding reveals the flexibility of the agent-based model in depicting non equilibrium market dynamics. Another flexibility of the agent-based modelling demonstrated in our work is on the aspect of modelling active economic agents. We have applied MAEA to model heterogenous traders in Xetra auction market with the mean-variance preference in the submodule of objectives. Essentially the submodule of objectives in MAEA presents the mechanism of selection that the agent employs to choose its action plan. By applying MAEA, one can easily extend the model of traders by replacing the mean-variance preference with new criteria of selection that allow more flavor of nonoptimizing and adaptive behavioral rules. One possibility is to introduce psychological patterns of decision making in the trader's submodule of objectives, which is open for the future work.

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Appendix A

Computational Process for Xetra Auction Pricing Rules

Based on the description in Gruppe Deutsche Börse (2003), the flowchart in Figure A.1 depicts the computational process for Xetra auction pricing rules. The subprocess Xetra-Auction-PDA($\mathcal{I}_0$) is depicted by the flowchart in Figure A.2.

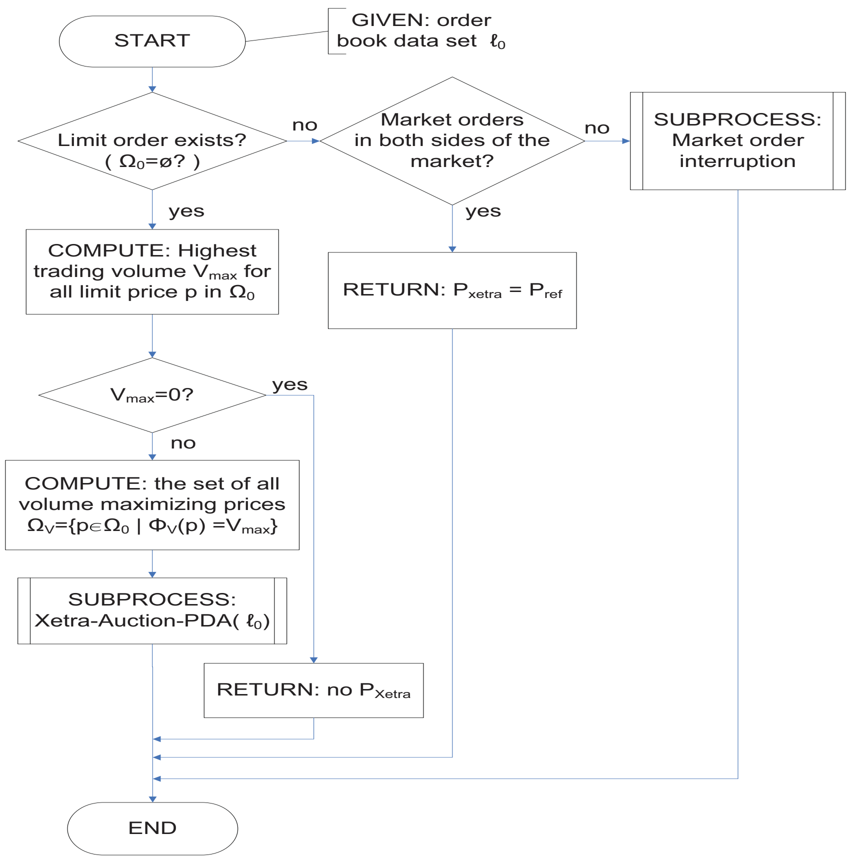

Figure A.1: Computational process for Xetra auction pricing rules.

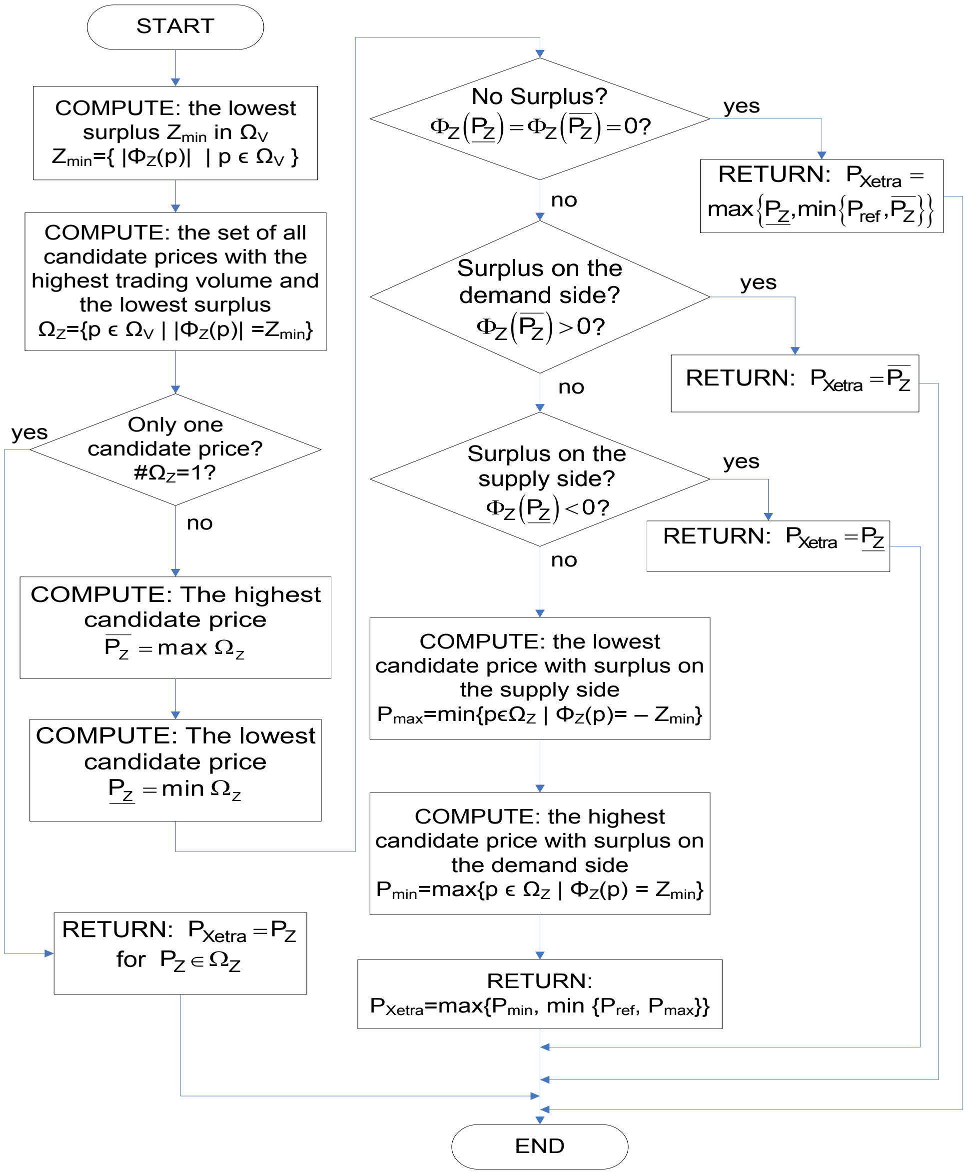

Figure A.2: Flowchart of subprocess: Xetra-Auction-PDA($\mathcal{I}_0$).

Appendix B

Computational Process for Improved Auction Pricing Rules

Computational Process for Equivalent Auction Pricing Rules. The flowchart in Figure B.1 illustrates the computational process for equivalent auction pricing rules to compute the auction price P_{equate}. The subprocess Xetra-Auction-Equivalent-PDA($\mathcal{I}_0$) is depicted by the flowchart in Figure B.2.

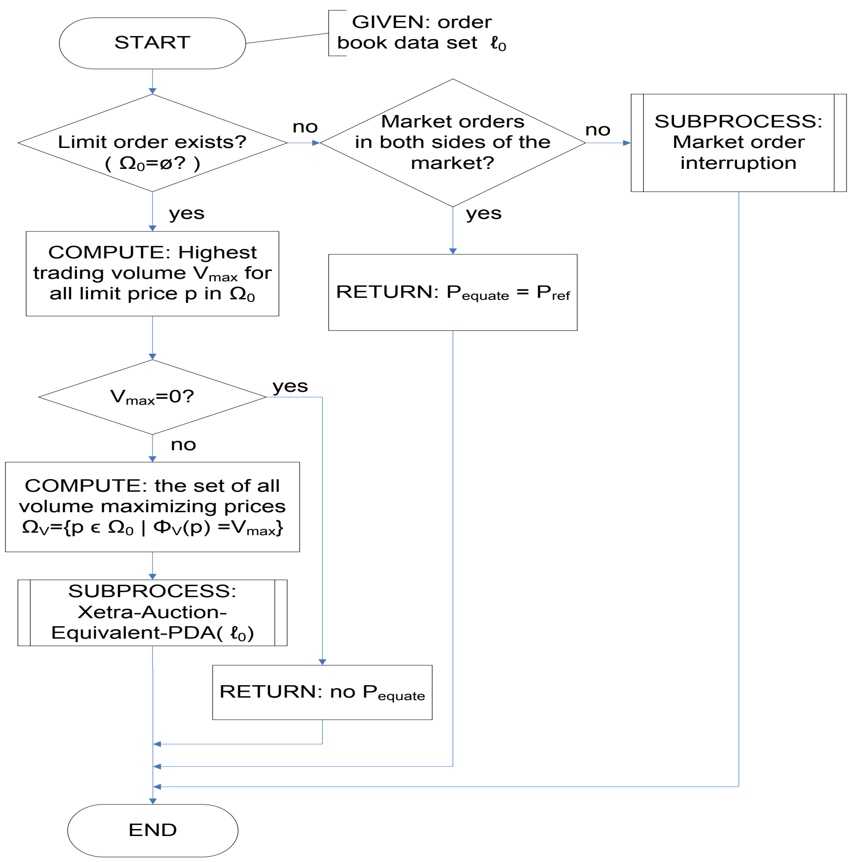

Figure B.1: Computational process for equivalent auction pricing rules.

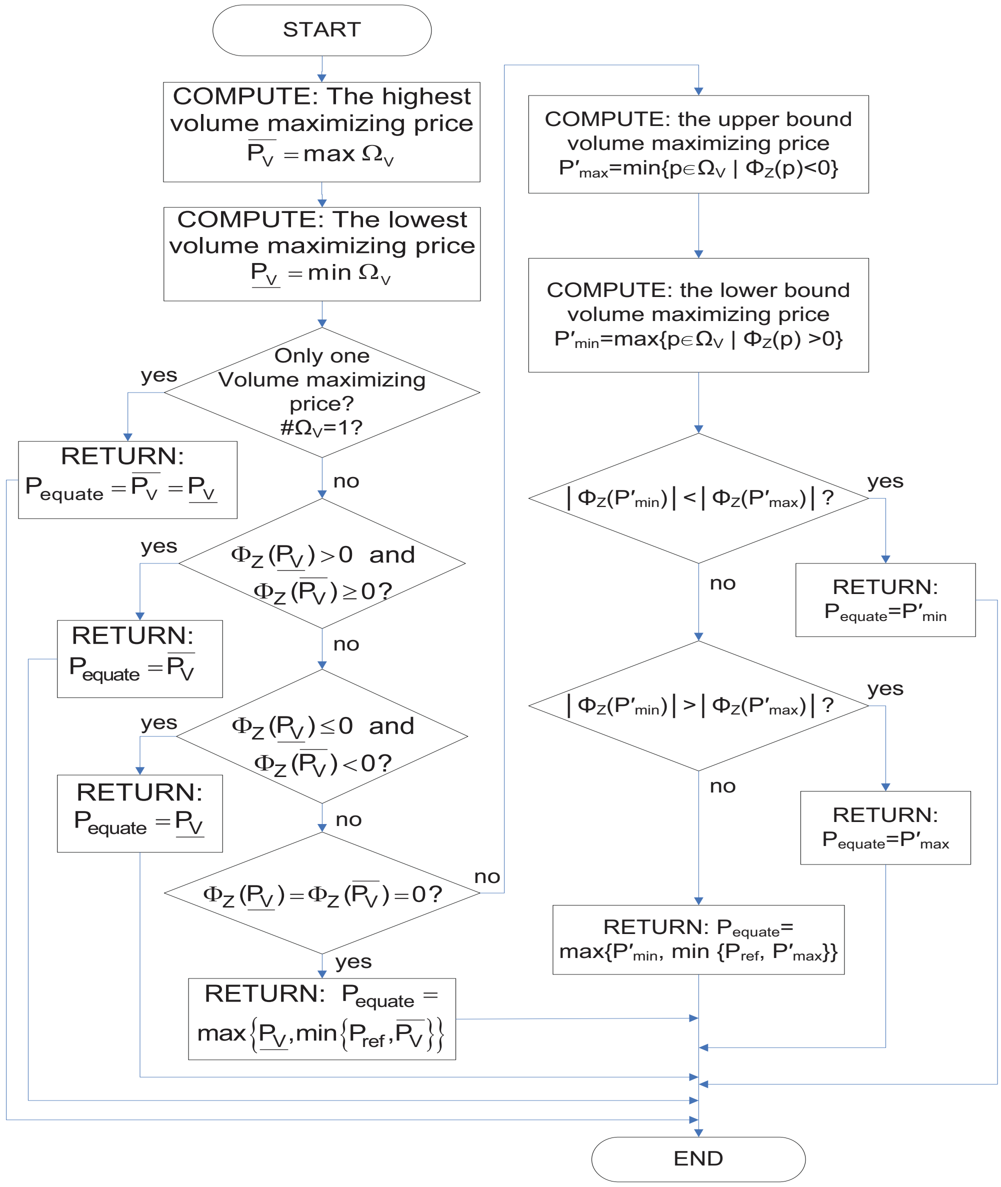

Figure B.2: Flowchart of subprocess: Xetra-Auction-Equivalent-PDA($\mathcal{I}_0$).

Computational Process for Improved Auction Pricing Rules. The flowchart in Figure B.3 illustrates the computational process for improved auction pricing rules to compute the auction price P_{impr}. The subprocess Xetra-Auction-Improved-PDA($\mathcal{I}_0$) is depicted by the flowchart in Figure B.4.

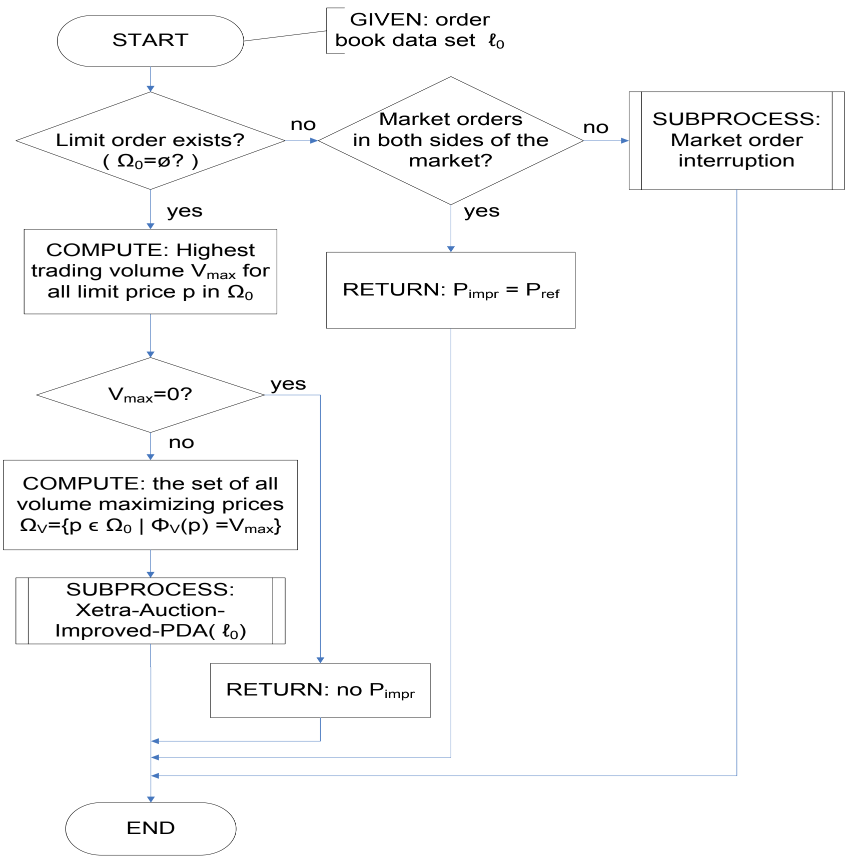

Figure B.3: Computational process for improved auction pricing rules.

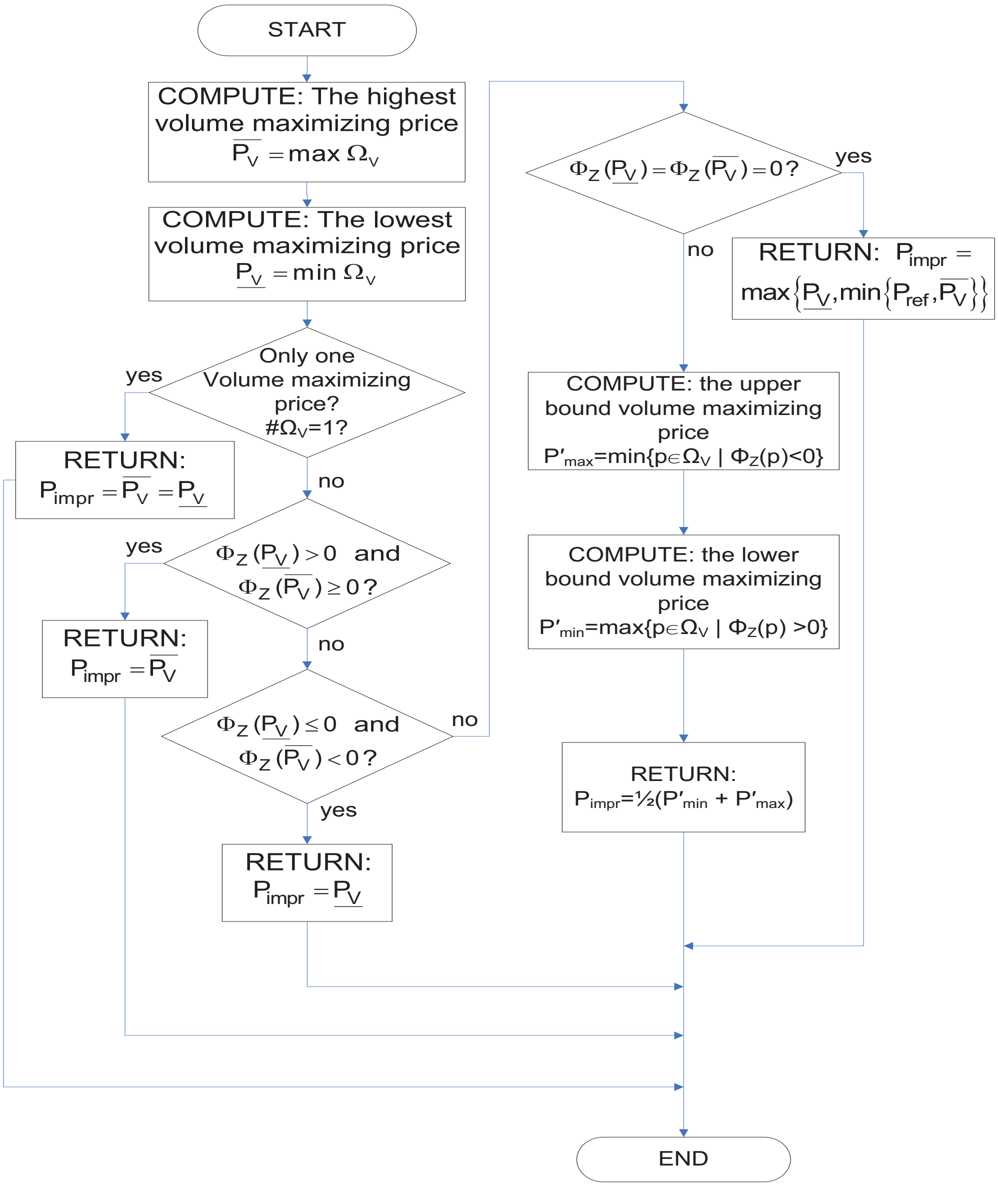

Figure B.4: Flowchart of subprocess: Xetra-Auction-Improved-PDA($\mathcal{I}_0$).

Appendix C

Proof of Proposition 2.1

Proof. To prove the equivalence of P_{equate} determined by Theorem 2.1 and P_{Xetra} determined by Theorem 1.1 given the same order book data set $\mathcal{I}_0$ in Assumption 1.1, we investigate each case depicted in Theorem 2.1 and show that P_{equate} determined by Theorem 2.1 is equal to P_{Xetra} determined by Theorem 1.1 in each case.

Case (I) in Theorem 2.1 selects the auction price $P_{\text{equate}} = \overline{P_V} = \underline{P_V}$ when there exists a unique volume maximizing price with $\sharp\Omega_V = 1$, which implies that case (i) of a unique candidate price in Theorem 1.1 fulfills with $\sharp\Omega_Z = 1$. Xetra auction price is thus $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$. Since $\sharp\Omega_V = \sharp\Omega_Z = 1$, we have $\overline{P_V} = \underline{P_V} = \overline{P_Z} = \underline{P_Z}$ in this case. Hence, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ for case (I) in Theorem 2.1.

Case (II.I) in Theorem 2.1 selects $P_{\text{equate}} = \overline{P_V}$ when $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) \geq 0$, which implies that candidate prices are with a surplus on the demand side or with no surplus. Either the situation of one candidate price or the situation of more than one candidate price with a surplus on the demand side happens in this case. For the first possibility of $\sharp\Omega_Z = 1$, Xetra auction price is $P_{\text{Xetra}} = \overline{P_Z}$ according to case (i) of Theorem 1.1. The second possibility corresponds to the case of $\Phi_Z(\overline{P_Z}) > 0$. Xetra auction price is $P_{\text{Xetra}} = \overline{P_Z}$ according to the first case in (1.12) of Theorem 1.1. Notice that $\overline{P_Z} = \overline{P_V}$ in this possibility since $\Phi_Z(p)$ is non-increasing and $\Phi_Z(p) > 0$ for all volume maximizing prices $p \in \Omega_V$. Thus, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ for case (II.I) in Theorem 2.1.

Case (II.II) in Theorem 2.1 selects $P_{\text{equate}} = \underline{P_V}$ when $\Phi_Z(\underline{P_V}) \leq 0$ and $\Phi_Z(\overline{P_V}) < 0$, which implies that candidate prices are with a surplus on the supply side or with no surplus. Either the situation of one candidate price or the situation of more than one candidate price with a surplus on the supply side happens in this case. For the first possibility of $\sharp\Omega_Z = 1$, Xetra auction price is $P_{\text{Xetra}} = \underline{P_Z}$ according to case (i) of Theorem 1.1. The second possibility corresponds to the case of $\Phi_Z(\overline{P_Z}) < 0$. Xetra auction price is $P_{\text{Xetra}} = \underline{P_Z}$ according to the second case in (1.12) of Theorem 1.1. Notice that $\underline{P_Z} = \underline{P_V}$ in this possibility since $\Phi_Z(p)$ is non-increasing and $\Phi_Z(p) < 0$ for all volume maximizing prices $p \in \Omega_V$. Thus, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ for case (II.II) in Theorem 2.1.

Case (II.III) in Theorem 2.1 selects $P_{\text{equate}} = \max\{\underline{P_V}, \min\{P_{\text{ref}}, \overline{P_V}\}\}$ when there exists $\Phi_Z(\underline{P_V}) = 0$ and $\Phi_Z(\overline{P_V}) = 0$, which implies that the third case in (1.12) of Theorem 1.1 fulfills with $\Phi_Z(\underline{P_Z}) = \Phi_Z(\overline{P_Z}) = 0$. Xetra auction price is thus $P_{\text{Xetra}} = \max\{\underline{P_Z}, \min\{P_{\text{ref}}, \overline{P_Z}\}\}$. Notice that $\underline{P_V} = \underline{P_Z}$ and $\overline{P_V} = \overline{P_Z}$ since $\Phi_Z(p) = 0$ for all $p \in \Omega_V$. Hence, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ for case (II.III) in Theorem 2.1.

Case (II.IV) in Theorem 2.1 states the situation of $\Phi_Z(\underline{P_V}) > 0$ and $\Phi_Z(\overline{P_V}) < 0$ for some volume maximizing prices with a surplus on the demand side while others with a surplus on the supply side, which contains three subcases on the surplus of the lower bound volume maximizing price $P^{'}_{\text{min}}$ and the surplus of the upper bound volume maximizing price $P^{'}_{\text{max}}$.

The first subcase is for the situation of $|\Phi_Z(P^{'}_{\text{min}})| < |\Phi_Z(P^{'}_{\text{max}})|$, i.e. the surplus of $P^{'}_{\text{min}}$ is less than the surplus of $P^{'}_{\text{max}}$. The auction price is equal to $P_{\text{equate}} = P^{'}_{\text{min}}$ according to the first case in (2.2) of Theorem 2.1. Either the situation of one candidate price or the situation of more than one candidate price with a surplus on the demand side happens in this subcase. $|\Phi_Z(P^{'}_{\text{min}})| < |\Phi_Z(P^{'}_{\text{max}})|$ implies that $|\Phi_Z(P^{'}_{\text{min}})|$ is the lowest surplus since $|\Phi_Z(P^{'}_{\text{min}})|$ is the lowest surplus on the demand side and $|\Phi_Z(P^{'}_{\text{max}})|$ is the lowest surplus on the supply side. In the situation of $\sharp\Omega_Z = 1$, Xetra auction price is thus $P_{\text{Xetra}} = P^{'}_{\text{min}}$ according to case (i) in Theorem 1.1 since $P^{'}_{\text{min}}$ is the unique candidate price. In the situation of more than one candidate price with a surplus on the demand side, $P^{'}_{\text{min}}$ is the highest volume maximizing price with the lowest surplus on the demand side, i.e. $\overline{P_Z} = P^{'}_{\text{min}}$. Xetra auction price is thus $P_{\text{Xetra}} = \overline{P_Z} = P^{'}_{\text{min}}$ according to the first case in (1.12) of Theorem 1.1. Hence, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ in this first subcase.

The second subcase is for the situation of $|\Phi_Z(P^{'}_{\text{min}})| > |\Phi_Z(P^{'}_{\text{max}})|$, i.e. the surplus of $P^{'}_{\text{min}}$ is greater than the surplus of $P^{'}_{\text{max}}$. The auction price is equal to $P_{\text{equate}} = P^{'}_{\text{max}}$ according to the second case in (2.2) of Theorem 2.1. Either the situation of one candidate price or the situation of more than one candidate price with a surplus on the supply side happens in this subcase. $|\Phi_Z(P^{'}_{\text{max}})|$ is the lowest surplus and $P^{'}_{\text{max}}$ is the lowest volume maximizing price with the lowest surplus on the supply side in this subcase, i.e. $\underline{P_Z} = P^{'}_{\text{max}}$. Thus, Xetra auction price is equal to $P_{\text{Xetra}} = \underline{P_Z} = P^{'}_{\text{max}}$ either according to case (i) in Theorem 1.1 in the situation of one candidate price or according to the second case in (1.12) of Theorem 1.1 in the situation of more than one candidate price with a surplus on the supply side. Hence, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ in this second subcase.

The third subcase is for the situation of $|\Phi_Z(P^{'}_{\text{min}})| = |\Phi_Z(P^{'}_{\text{max}})|$, i.e. the surplus of $P'_{\min}$ is equal to the surplus of $P'_{\max}$. The auction price is equal to $P_{\text{equate}} = \max\{P'_{\min}, \min\{P_{\text{ref}}, P'_{\max}\}\}$ according to the third case in (2.2) of Theorem 2.1. $|\Phi_Z(P'_{\min})| = |\Phi_Z(P'_{\max})|$ implies that $|\Phi_Z(P'_{\min})|$ and $|\Phi_Z(P'_{\max})|$ are both the lowest surplus on volume maximizing prices since $|\Phi_Z(P'_{\min})|$ is the lowest surplus on the demand side and $|\Phi_Z(P'_{\max})|$ is the lowest surplus on the supply side. The situation happens in this subcase for more than one candidate price and some candidate price with a surplus on the demand side while others with a surplus on the supply side. Xetra auction price is thus $P_{\text{Xetra}} = \max\{P_{\min}, \min\{P_{\text{ref}}, P_{\max}\}\}$ according to the fourth case in (1.12) of Theorem 1.1. Notice that $P'_{\min} = P_{\min}$ and $P'_{\max} = P_{\max}$ in this subcase according to the definitions of $P'_{\min}$ and $P'_{\max}$. Hence, the auction price is $P_{\text{equate}} = P_{\text{Xetra}}$ in this subcase.

Therefore, the auction price P_{equate} determined by Theorem 2.1 is equivalent to P_{Xetra} determined by Theorem 1.1 given the same order book data set $\mathcal{I}_0$ in Assumption 1.1. $\square$

Appendix D

Proof of Theorem 3.1

We start with presenting some notations that will be used in this proof. Denote the set of all limit prices in the order book data set $\mathcal{I}_0$ in Assumption 3.1 as Ω_0. After the trader submits the market order Q_m, the order book data set augments to $\mathcal{I}_0 \cup \{Q_m\}$ with the new set of all limit prices denoted by $\Omega_0^{'}$ which remains the same as $\Omega_0^{'} = \Omega_0$. The new aggregate demand function and the new aggregate supply function are denoted by $\Phi_D^{'}(Q_m, p)$ and $\Phi_S^{'}(Q_m, p)$ respectively. The corresponding trading volume function is denoted by $\Phi_V^{'}(Q_m, p)$. The highest executable order volume is $V_{\max}^{'}$ with the set of volume maximizing prices $\Omega_V^{'}$. The lowest surplus is $Z_{\min}^{'}$ and the set of candidate prices is $\Omega_Z^{'}$ with $\overline{P_Z^{'}} = \max \Omega_Z^{'}$ and $\underline{P_Z^{'}} = \min \Omega_Z^{'}$. We have the following lemmas to derive values of Xetra auction price function $P_X(Q_m)$ for different situations of Q_m.

Lemma D.1. *Given the order book data set $\mathcal{I}_0$ in Assumption 3.1 and the explicit form of $\Phi_Z(p)$ from Lemma 3.1, the value of Xetra auction price function is $P_X(Q_m) = P_1$ for $Q_m \in (-\infty, -\phi_0^Z]$ and $P_X(Q_m) = P_{I+J}$ for $Q_m \in [-\phi_{I+J}^Z, +\infty)$.*

Proof. Let $Q_m \in (-\infty, -\phi_0^Z)$. Then $\Phi_Z^{'}(Q_m, p) < 0$ holds for any price $p \in \mathbb{R}_+$, which implies that the lowest limit price P_1 is associated with the highest executable order volume and the lowest surplus. Thus P_1 is one of candidate prices with $P_1 = \underline{P_Z^{'}}$. $\Phi_Z^{'}(Q_m, p) < 0$ for all limit prices $p \in \Omega_0$ implies that Xetra auction price is $P_{\text{Xetra}} = \underline{P_Z^{'}} = P_1$ according to either case (i) in Theorem 1.1 or the second case in equation (1.12) of Theorem 1.1. When $Q_m = -\phi_0^Z$, we have $\Phi_Z^{'}(Q_m, p) = 0$ for any price $p \in A_0^Z$ and $\Phi_Z^{'}(Q_m, p) < 0$ for any price $p \in \mathbb{R}_+ \setminus A_0^Z$. This implies $\Phi_Z^{'}(Q_m, P_1) = 0$ or $\Phi_Z^{'}(Q_m, P_1) < 0$. When $\Phi_Z^{'}(Q_m, P_1) = 0$, P_1 is the only candidate price. Thus Xetra auction price is $P_{\text{Xetra}} = P_1$ according to case (i) in Theorem 1.1. When $\Phi_Z^{'}(Q_m, P_1) < 0$ for all limit prices $p \in \Omega_0$, P_1 is the lowest candidate price with $P_1 = \underline{P_Z^{'}}$. Thus Xetra auction price is $P_{\text{Xetra}} = \underline{P_Z^{'}} = P_1$ according to either the case (i) in Theorem 1.1 or the second case in equation (1.12) of Theorem 1.1. Hence, $P_X(Q_m) = P_1$ holds for $Q_m \in (-\infty, -\phi_0^Z]$.

Let $Q_m \in (-\phi_{I+J}^Z, +\infty)$. Then $\Phi_Z^{'}(Q_m, p) > 0$ holds for any price $p \in \mathbb{R}_+$, which implies that the highest limit price P_{I+J} is associated with the highest executable order volume and the lowest surplus. Thus P_{I+J} is one of candidate prices with $P_{I+J} = \overline{P'_Z}$. $\Phi'_Z(Q_m, p) > 0$ for all limit prices $p \in \Omega_0$ implies that Xetra auction price is $P_{\text{Xetra}} = \overline{P'_Z} = P_{I+J}$ according to either case (i) in Theorem 1.1 or the first case in equation (1.12) of Theorem 1.1. When $Q_m = -\phi^Z_{I+J}$, we have $\Phi'_Z(Q_m, p) = 0$ for any price $p \in A^Z_{I+J}$ and $\Phi'_Z(Q_m, p) > 0$ for any price $p \in \mathbb{R}_+ \setminus A^Z_{I+J}$. This implies that $\Phi'_Z(Q_m, P_{I+J}) = 0$ or $\Phi'_Z(Q_m, P_{I+J}) > 0$. When $\Phi'_Z(Q_m, P_{I+J}) = 0$, P_{I+J} is the only candidate price. Xetra auction price is thus $P_{\text{Xetra}} = P_{I+J}$ according to case (i) in Theorem 1.1. When $\Phi'_Z(Q_m, P_{I+J}) > 0$ for all limit prices $p \in \Omega_0$, P_{I+J} is the highest candidate price with $P_{I+J} = \overline{P'_Z}$. Xetra auction price is thus $P_{\text{Xetra}} = \overline{P'_Z} = P_{I+J}$ according to either case (i) in Theorem 1.1 or the first case in equation (1.12) of Theorem 1.1. Hence, $P_X(Q_m) = P_{I+J}$ holds for $Q_m \in [-\phi^Z_{I+J}, +\infty)$. □

Lemma D.2. *Given the order book data set $\mathfrak{I}_0$ in Assumption 3.1 and the explicit form of $\Phi_Z(p)$ from Lemma 3.1, the value of Xetra auction price function is $P_X(Q_m) = P_n$ when $Q_m \in (-\phi^Z_{n-1}, -\phi^Z_n)$ for $n \in \{1, 2, \ldots, I+J\}$.*

Proof. For any given $n \in \{1, 2, \ldots, I+J\}$, let $Q_m \in (-\phi^Z_{n-1}, -\phi^Z_n)$ be arbitrary but fixed. We have $\Phi'_Z(Q_m, p) > 0$ for any price $p \in A^Z_{n-1}$ and $\Phi'_Z(Q_m, p) < 0$ for any price $p \in A^Z_n$. The values of $\Phi'_D(Q_m, p)$, $\Phi'_S(Q_m, p)$, $\Phi'_V(Q_m, p)$, and $\Phi'_Z(Q_m, p)$ are constant for $p \in A^Z_n$ with $n = 0, \ldots, I+J$. With a little abuse of notations, denote all these constant values to be $\Phi'_D(A^Z_n)$, $\Phi'_S(A^Z_n)$, $\Phi'_V(A^Z_n)$, and $\Phi'_Z(A^Z_n)$ for $n = 0, \ldots, I+J$.

For any price $p \in A^Z_{n-1}$, $\Phi'_Z(Q_m, p) = \Phi'_Z(A^Z_{n-1}) > 0$ implies that $\Phi'_D(A^Z_{n-1}) > \Phi'_S(A^Z_{n-1})$ and $\Phi'_V(A^Z_{n-1}) = \Phi'_S(A^Z_{n-1})$. Similarly, $\Phi'_Z(Q_m, p) = \Phi'_Z(A^Z_n) < 0$ for any price $p \in A^Z_n$ implies that $\Phi'_D(A^Z_n) < \Phi'_S(A^Z_n)$ and $\Phi'_V(A^Z_n) = \Phi'_D(A^Z_n)$.

It is either $P_n \in A^Z_{n-1}$ or $P_n \in A^Z_n$. If $P_n \in A^Z_{n-1}$, then $A^Z_{n-1} = \llbracket P_{n-1}, P_n]$ which is right closed and $A^Z_n = (P_n, P_{n+1} \rrbracket$ which is left open. Thus P_n is a limit bid price since any limit price associated with a right-closed end point in $A^Z_0, \ldots, A^Z_{I+J}$ comes from a limit bid order. P_n is a limit bid price implies that $\Phi'_S(A^Z_{n-1}) = \Phi'_S(A^Z_n)$, which attains $\Phi'_V(A^Z_{n-1}) > \Phi'_V(A^Z_n)$ since $\Phi'_V(A^Z_{n-1}) = \Phi'_S(A^Z_{n-1})$ and $\Phi'_V(A^Z_n) = \Phi'_D(A^Z_n)$ with $\Phi'_D(A^Z_n) < \Phi'_S(A^Z_n)$. Hence, $V'_{\max} = \Phi'_V(A^Z_{n-1})$ since $\Phi'_V(A^Z_i) = \Phi'_S(A^Z_i) \leq \Phi'_S(A^Z_{n-1}) = \Phi'_V(A^Z_{n-1})$ for $i = 0, \ldots, n-1$, and $\Phi'_V(A^Z_j) = \Phi'_D(A^Z_j) \leq \Phi'_D(A^Z_n) = \Phi'_V(A^Z_n)$ for $j = n, \ldots, I+J$. Thus, P_n is the highest limit price associated with the highest executable order volume and the lowest surplus. This implies $P_n \in \Omega'_Z$ and $P_n = \overline{P'_Z}$. Notice that $\Phi'_Z(Q_m, P_n) > 0$ implies a surplus on the demand side, Xetra auction price is thus $P_{\text{Xetra}} = \overline{P'_Z} = P_n$ according to either case (i) in Theorem 1.1 or the first case in equation (1.12) of Theorem 1.1.

When $P_n \in A_n^Z$, one analogously obtains that Xetra auction price is $P_{\text{Xetra}} = \underline{P'_Z} = P_n$ according to either case (i) in Theorem 1.1 or the second case in equation (1.12) of Theorem 1.1. Therefore, one attains that the value of Xetra auction price function $P_X(Q_m) = P_n$ when $Q_m \in (-\phi_{n-1}^Z, -\phi_n^Z)$ for $n \in \{1, 2, \ldots, I + J\}$.

□

Lemma D.3. *Given the order book data set $\mathcal{I}_0$ in Assumption 3.1 and the explicit form of $\Phi_Z(p)$ from Lemma 3.1, the value of Xetra auction price function is $P_X(Q_m) = P_n^*$ when $Q_m = -\phi_n^Z$ for $n \in \{1, 2, \ldots, I + J - 1\}$, where*

$$P_n^* = \begin{cases} P_n & \textit{CASE 1}; \\ P_{n+1} & \textit{CASE 2}; \\ \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\} & \textit{CASE 3}; \end{cases}$$

*for **CASE 1:** either $A_n^Z = [P_n, P_{n+1})$ or $A_n^Z = (P_n, P_{n+1})$ with $|\Phi'_Z(-\phi_n^Z, P_n)| < |\Phi'_Z(-\phi_n^Z, P_{n+1})|$;*

*for **CASE 2:** either $A_n^Z = (P_n, P_{n+1}]$ or $A_n^Z = (P_n, P_{n+1})$ with $|\Phi'_Z(-\phi_n^Z, P_n)| > |\Phi'_Z(-\phi_n^Z, P_{n+1})|$;*

*for **CASE 3:** either $A_n^Z = [P_n, P_{n+1}]$ or $A_n^Z = (P_n, P_{n+1})$ with $|\Phi'_Z(-\phi_n^Z, P_n)| = |\Phi'_Z(-\phi_n^Z, P_{n+1})|$.*

Proof. For any given $n \in \{1, 2, \ldots, I + J - 1\}$, $Q_m = -\phi_n^Z$ implies that $\Phi'_Z(Q_m, p) = 0$ for any price $p \in A_n^Z$. A_n^Z could be $[P_n, P_{n+1})$, $(P_n, P_{n+1}]$, $[P_n, P_{n+1}]$, or (P_n, P_{n+1}).

If $A_n^Z = [P_n, P_{n+1})$, then $\Phi'_Z(Q_m, P_n) = 0$. P_n is the only limit price associated with a zero surplus. Xetra auction price is thus $P_{\text{Xetra}} = P_n$ according to case (i) in Theorem 1.1. Analogously, Xetra auction price is $P_{\text{Xetra}} = P_{n+1}$ if $A_n^Z = (P_n, P_{n+1}]$.

If $A_n^Z = [P_n, P_{n+1}]$, only P_n and P_{n+1} are associated with zero surplus. We have $\Omega'_Z = \{P_n, P_{n+1}\}$. According to the third case in equation (1.12) of Theorem 1.1, Xetra auction price is thus $P_{\text{Xetra}} = \max\{\ P_n, \min\{\ P_{\text{ref}}, P_{n+1}\ \}\ \}$.

If $A_n^Z = (P_n, P_{n+1})$, both P_n and P_{n+1} are associated with the highest executable order volume. P_n has a surplus on the demand side with $\Phi'_Z(-\phi_n^Z, P_n) > 0$ while P_{n+1} has a surplus on the supply side with $\Phi'_Z(-\phi_n^Z, P_{n+1}) < 0$. If the surplus of P_n is lower than the surplus of P_{n+1} with $|\ \Phi'_Z(-\phi_n^Z, P_n)\ | < |\ \Phi'_Z(-\phi_n^Z, P_{n+1})\ |$, Xetra auction price is $P_{\text{Xetra}} = \overline{P'_Z} = P_n$ according to either case (i) in Theorem 1.1 or the first case in equation (1.12) of Theorem 1.1. If the surplus of P_n is higher than the surplus of P_{n+1} with $\mid \Phi'_Z(-\phi_n^Z, P_n) \mid > \mid \Phi'_Z(-\phi_n^Z, P_{n+1}) \mid$, Xetra auction price is $P_{\text{Xetra}} = \underline{P'_Z} = P_{n+1}$ according to either case (i) in Theorem 1.1 or the second case in equation (1.12) of Theorem 1.1. If the surplus of P_n is equal to the surplus of P_{n+1} with $\mid \Phi'_Z(-\phi_n^Z, P_n) \mid = \mid \Phi'_Z(-\phi_n^Z, P_{n+1}) \mid$, Xetra auction price is $P_{\text{Xetra}} = \max\{P_n, \min\{P_{\text{ref}}, P_{n+1}\}\}$ according to the fourth case in equation (1.12) of Theorem 1.1. Hence, one attains the values of $P_X(Q_m)$ as stated when $Q_m = -\phi_n^Z$ for $n \in \{1, 2, \ldots, I + J - 1\}$. □

The explicit form of $P_X(Q_m)$ is attained by combining results in Lemma D.1, Lemma D.2, and Lemma D.3. Since $P_1 \leq P_2 \leq \cdots \leq P_{I+J}$, the structure of values of $P_X(Q_m)$ implies that it is a non-decreasing step function. Therefore, we verify Theorem 3.1. □

Appendix E

Results of Statistical Test

We use the Wilcoxon signed ranks test to compare the Sharpe ratio between the price setter and the benchmark trader for 250 trading periods. Table E.1 and Table E.2 illustrate test results of the $\mathrm{p-value}$ for all 50 profiles. These test results state that 18 profiles accept that the benchmark trader has lower Sharpe ratio than the price setter with 90% level of confidence; 31 profiles accept that the benchmark trader has higher Sharpe ratio than the price setter with 90% level of confidence; and 1 profile does not provide any inference from statistical tests.

Then we compare the percentage of forecast errors that the price setter commits on its expectation on the future trading price with that of the benchmark trader for the 18 profiles where the price setter performs better than the benchmark trader. The test results of the $\mathrm{p-value}$ are depicted in Table E.3. Table E.4 depicts the test results of the $\mathrm{p-value}$ for the comparison on the percentage of forecast errors for the price setter and the benchmark trader in the 31 profiles where the price setter does not have better performance.

These tables show three types of alternative hypothesis: H_A = ”l”, H_A = ”two sided”, and H_A = ”g”. H_A = ”l” refers to the alternative hypothesis that the benchmark trader has lower Sharpe ratio (percentage of forecast errors) than the price setter. H_A = ”two sided” refers to the alternative hypothesis that the Sharpe ratio (percentage of forecast errors) of the benchmark trader is not equal to that of the price setter. H_A = ”g” refers to that the benchmark trader has higher Sharpe ratio (percentage of forecast errors) than the price setter. The inference result ”$<$” denotes the acceptance of the alternative hypothesis H_A = ”l”; ”$>$” denotes the acceptance of the alternative hypothesis H_A = ”g”; and ”No Inference” denotes no significant inference derived from the tests.

Profile	H_A			inference
	"l"	"two sided"	"g"	
1	1	$1.83E-032$	$9.13E-033$	$>$
2	1	$1.36E-042$	$6.79E-043$	$>$
3	1	$1.36E-042$	$6.79E-043$	$>$
4	1	$2.43E-026$	$1.21E-026$	$>$
5	1	$1.98E-042$	$9.89E-043$	$>$
6	$9.84E-028$	$1.97E-027$	1	$<$
7	1	$7.42E-018$	$3.71E-018$	$>$
8	1	$6.72E-037$	$3.36E-037$	$>$
9	1	$2.95E-018$	$1.47E-018$	$>$
10	1	$1.36E-042$	$6.79E-043$	$>$
11	$1.12E-008$	$2.24E-008$	1	$<$
12	$1.33E-034$	$2.66E-034$	1	$<$
13	$6.87E-043$	$1.37E-042$	1	$<$
14	$1.43E-039$	$2.86E-039$	1	$<$
15	$1.20E-016$	$2.40E-016$	1	$<$
16	$2.08E-040$	$4.16E-040$	1	$<$
17	1	$1.36E-042$	$6.79E-043$	$>$
18	1	$2.09E-032$	$1.05E-032$	$>$
19	$2.88E-022$	$5.76E-022$	1	$<$
20	0.01	0.01	0.99	$<$
21	1	$1.43E-015$	$7.14E-016$	$>$
22	1	$3.24E-036$	$1.62E-036$	$>$
23	$1.31E-038$	$2.63E-038$	1	$<$
24	$5.26E-006$	$1.05E-005$	1	$<$
25	1	$5.92E-005$	$2.96E-005$	$>$

Table E.1: Test results on Sharpe ratio.

Profile	H_A			inference
	"l"	"two sided"	"g"	
26	1	$1.36E-042$	$6.79E-043$	>
27	1	$4.23E-010$	$2.12E-010$	>
28	$3.98E-032$	$7.96E-032$	1	<
29	1	$1.83E-032$	$9.13E-033$	>
30	1	$1.36E-042$	$6.79E-043$	>
31	$5.59E-005$	0	1	<
32	$1.31E-041$	$2.62E-041$	1	<
33	1	$1.48E-042$	$7.39E-043$	>
34	$1.58E-035$	$3.15E-035$	1	<
35	1	$1.61E-040$	$8.03E-041$	>
36	1	$1.36E-042$	$6.79E-043$	>
37	1	$1.36E-042$	$6.79E-043$	>
38	0.8	0.39	0.2	No Inference
39	1	$3.88E-019$	$1.94E-019$	>
40	1	$1.31E-022$	$6.53E-023$	>
41	1	$4.16E-042$	$2.08E-042$	>
42	1	$2.69E-013$	$1.35E-013$	>
43	$3.10E-019$	$6.20E-019$	1	<
44	$1.48E-040$	$2.96E-040$	1	<
45	1	0	$8.54E-005$	>
46	1	$1.36E-042$	$6.79E-043$	>
47	1	$1.46E-018$	$7.31E-019$	>
48	1	$1.35E-041$	$6.73E-042$	>
49	$6.79E-043$	$1.36E-042$	1	<
50	0.98	0.04	0.02	>

Table E.2: Test results on Sharpe ratio (continued).

Profile	H_A			inference
	"l"	"two sided"	"g"	
6	0.92	0.15	0.08	$>$
11	0.95	0.09	0.05	$>$
12	1	0	$7.90E-005$	$>$
13	0.82	0.36	0.18	No Inference
14	1	0.01	0	$>$
15	0	0	1	$<$
16	0.83	0.34	0.17	No Inference
19	1	0.01	0	$>$
20	1	0	0	$>$
23	1	0	0	$>$
24	0.99	0.01	0.01	$>$
28	0.99	0.02	0.01	$>$
31	$8.69E-028$	$1.74E-027$	1	$<$
32	0.77	0.47	0.23	No Inference
34	$9.24E-010$	$1.85E-009$	1	$<$
43	0.99	0.01	0.01	$>$
44	1	$1.38E-005$	$6.92E-006$	$>$
49	1	$4.12E-006$	$2.06E-006$	$>$

Table E.3: Test results on percentage of forecast errors in 18 profiles.

Profile	H_A			inference
	"l"	"two sided"	"g"	
1	0.74	0.53	0.26	No Inference
2	1	$8.67E-012$	$4.34E-012$	$>$
3	$2.28E-006$	$4.56E-006$	1	$<$
4	0.98	0.04	0.02	$>$
5	0.98	0.05	0.02	$>$
7	1	0	0	$>$
8	1	0	0	$>$
9	0.88	0.24	0.12	No Inference
10	0.92	0.16	0.08	$>$
17	$6.64E-020$	$1.33E-019$	1	$<$
18	$5.91E-029$	$1.18E-028$	1	$<$
21	0.73	0.55	0.27	No Inference
22	0.94	0.12	0.06	$>$
25	0.78	0.43	0.22	No Inference
26	0.92	0.15	0.08	$>$
27	0.93	0.13	0.07	$>$
29	0.74	0.53	0.26	No Inference
30	$5.11E-024$	$1.02E-023$	1	$<$
33	0.26	0.52	0.74	No Inference
35	$2.56E-017$	$5.13E-017$	1	$<$
36	0	0	1	$<$
37	$8.80E-019$	$1.76E-018$	1	$<$
39	$1.08E-008$	$2.17E-008$	1	$<$
40	1	0	0	$>$
41	1	$1.33E-007$	$6.63E-008$	$>$
42	0.97	0.06	0.03	$>$
45	0.99	0.02	0.01	$>$
46	$2.40E-023$	$4.81E-023$	1	$<$
47	1	0.01	0	$>$
48	0	0	1	$<$
50	1	$1.82E-008$	$9.11E-009$	$>$

Table E.4: Test results on percentage of forecast errors in 31 profiles.

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