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Pisot Substitutions and Beyond

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~~Mathematische Diffraktionstheorie geordneter und ungeordneter Systeme~~

~~Beyond Pisot Substitutions~~

Pisot Substitutions and Beyond!!

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A Note on this Thesis

This thesis deviates in two obvious aspects from an “average” mathematical PhD thesis – it is long and it is written in the style of a monograph. So let us remark on these observations and also stress in which points this book adds an original and new contribution to the mathematical theory.

The starting point of this thesis was the author’s master thesis [352] respectively the article [42]. There, we observed that the particular Pisot substitution we looked at, respectively the one-dimensional sequence it generates, has some very extraordinary properties which allow to prove pure point diffractivity respectively pure point dynamical spectrum for this sequence. A first check of the literature (compare the remark at the end of Section 3 in [42]) indicated, however, that these properties might not be that special for a Pisot substitution, and might be shared by a all Pisot substitutions. Moreover, we observed that although so-called non-unimodular Pisot substitutions exist, they were (and are) hardly ever explicitly discussed [39, 346]. The difficulty with this non-unimodular substitutions is that one also needs $\mathfrak{p}$-adic fields for their description, while in the unimodular case one can work in the “everyday” Euclidean world. So, the motivation and plan for this thesis was clear: Understand and work out the special properties of (unimodular) Pisot substitutions, and look at the non-unimodular case.

It soon became clear that the notation often used for Pisot substitution is not optimal – at least in the author’s eyes whose background is aperiodic order. Often, the sequences generated by Pisot substitutions are realized on $\mathbb{Z}$ and are consequently treated as discrete dynamical systems. More natural, in terms of cut and project sets, however, is the realization with natural intervals, wherefore the underlying dynamical system has $\mathbb{R}$-action. Especially, the so-called star-map has an obvious number-theoretic interpretation in the latter description, and even makes the generalisation to the unimodular case easier. Consequently, we have already the following conditions on this thesis:

- A coherent exposition of Pisot substitutions using the description of cut and project schemes.
- Self-contained introduction of model sets, so that (possible) results are also available to the community outside “quasicrystallinity”.
- Self-contained exposure to the “strange world” (at least, in the discrete dynamical systems community and the aperiodic order community) of $\mathfrak{p}$-adic fields, and thus to algebraic number theory.

However, here the story does not end, but expands in two additional directions. On the one hand, fractal geometry also comes into play as already the traditional name “Rauzy *fractal*” for the windows that appear for Pisot substitutions indicates. But objects like iterated function systems and Hausdorff measures/dimension are usually only defined on Euclidean spaces and not on products over local fields. On the other hand, also the fast developing theory on aperiodic order needed some adaptation to fit our needs, since we are looking at so-called multi-component Delone sets here. More importantly, the program to obtain a coherent picture also had its effects here, since iterated function systems and substitutions are treated similarly; this led to the notion "extended internal space" in this thesis.

With such heavy machinery, it is clear that also problems outside the main road of Pisot substitutions can be treated. One may argue that one loses the thread by discussing things like the visible lattice points in Chapter 5a, or reducible Pisot substitutions and substitutions with cobounds (in Chapter 6c). But, they also increase the awareness of the peculiarities of Pisot substitutions. However, one cannot treat everything, wherefore we have opted to only scratch diffraction theory and the theory of dynamical systems only and concentrate mainly on the "geometrical part" of the theory about Pisot substitutions.

Résumé: this thesis is written in the style of a monograph and has its length, since we try to present its topic in a self-contained coherent way and demonstrate connections to the "world" outside Pisot substitutions.

* * * * *

The main statement and ultimate goal of this thesis is Theorem 6.116, the list of equivalent conditions that identify a Pisot substitution as model set. This list summarises the (usually in the unimodular case) known conditions which are formulated and proved in full generality, *i.e.*, all statements apply to unimodular and non-unimodular Pisot substitutions. Moreover, there are also new equivalent conditions which appear for the first time here.

We now indicate which statements of this thesis are the author's original contributions to mathematical research:

- Chapter 3c about the visualisation of ultrametric spaces, in particular $\mathfrak{p}$-adic fields, brings together the approaches of [180] ("tree picture") and [312, 313] ("embedding into $\mathbb{R}^d$").

- The relationship of the Haar and the Hausdorff measure in a product space over local fields in Section 4.5 seems to be new. Consequently, also the term "metric dimension" appears, probably, for the first time here.

- The statements in Sections 4.8 – 4.10, if not indicated otherwise, are new. Here, the statements in Sections 4.9 & 4.10 are generalisations of results obtained by K.J. Falconer for (single component) self-affine iterated function systems in $\mathbb{R}^d$.

- The treatment of the multi-component case in this detail and, in particular, the introduction of the "height group" appear for the first time in Section 5.3.

- The theory of substitutions in Section 5.4 is formulated not only in the Euclidean setting. This has the effect that we have to assume property **(LT)** (or, later on, **(PLT)** & **(PLT+)**) to obtain a point sets/tiling with the desired properties (self-replication, representability *etc.*). In this general formulation, the statements in this section appear for the first time here. A weakness in this section, however, is that some statements towards the end (*e.g.*, Corollaries 5.89 & Corollary 5.90) are not proven explicitly, but we only point to the Euclidean case treated in [235].
  Let us remark that we need this general formulation for the statements about the substitution multi-component Delone set $\underline{\Upsilon}'$ in Theorem 6.116.

- The considerations, especially the construction of the "extended" internal space, in Section 5.7 are new, with the exception of Section 5.7.1. In particular, Theorem 5.154 is new. Some statements, with a proof different from ours, also appear in [230].
- The way visible lattice points are presented in Chapter 5a and the calculation of their autocorrelation coefficients (and the simplification of the used internal space) are due to the author.
- Most of the statements in Section 6.2 are well-known, but the statement that Pisot substitutions have trivial height group (Lemma 6.34) seems to appear for the first time here (but observe the footnote there).
- The presentations in Sections 6.3 & 6.4, which fit the needs of non-unimodular Pisot substitutions, are also new, if not otherwise indicated.
- Everything in Section 6.5 is due to the author.
- In Sections 6.6 – 6.9, the explicit generalisations to the non-unimodular case are new. The proof of Proposition 6.87 appears in this detail for the first time here. Proposition 6.106 is new.
- A fully worked out example of a non-unimodular Pisot substitution appears for the first time in Section 6.10. Especially, the corresponding figures, Figures 6.2 – 6.4, appear for the first time for a non-unimodular example. Similarly, the presentation of the non-unimodular examples in the next section is new.
- The interpretation of the examples discussed in Chapter 6a as possible model sets with mixed Euclidean and $\mathfrak{p}$-adic internal space is due to the author.
- Chapter 6b re-interprets the results of [235] (and [138]) in view of our findings about the (extended) internal space. Consequently, this also sheds new light on the term "height of a substitution of constant length" in [101].
- The definition "Euclidean connected" in Section 6c.1 is new.
- The examples in Section 6c.2 are new. Moreover, this also clarifies the term "cobound" and re-interprets it in terms of our height group.
- The observation that our methods (in particular, Theorem 5.154) can also be applied to reducible Pisot substitutions seems, in this generality, new.
- The indicated proof in Section 6c.4 that the "weak finiteness property" **(W)** of β-substitutions is equivalent to the tiling property, appears for the first time here (however, in the article [118] appears a reference to a preprint claiming that that result is also proved there).
- The definition of a hyperbolic toral automorphism in the "adelic case" in Definition 7.61 seems to be new.
- The explicit construction of the torus parametrisation and its relation to the Markov (-like) partition appear for the first time in Section 7.5.2.

1. Introduction

> Also, ich denke, dass mein Ausgangspunkt immer ist: ein Defizit in der gegenwärtigen intellektuellen Landschaft, was immer wieder dazu führt, dass einzelne Gesichtspunkte hochgezogen werden und dann für das Ganze verkauft werden.
>
> *Niklas Luhmann (im Radiogespräch mit Wolfgang Hagen)*

In the early eighties, the discovery of quasicrystals [189, 342] led to the – very practical – question, how the diffraction pattern of such aperiodic materials can be analysed and explained. We only remark that quasicrystals brought into question assumptions in crystallography about the relationship between long-range order and periodic order, compare the overview in [27]. Of course, since such quasicrystals exist "in the real world", the realm of long-range (or aperiodic) order has to be discovered – at least, if one happens to be a crystallographer. But in this territory of aperiodicity, there are also many fascinating questions to ask and to answer for a mathematician. Of course, mathematicians already looked at aperiodic structures and tilings before that time, but now research intensified and new questions arose. This thesis revolves around a conjecture that is around – in some way – for at least twenty years [71, 72, 306, 366], and, in fact, has its origin in the study of (one-dimensional) dynamical systems:

> Do all Pisot substitutions generate sequences that are pure point diffractive/have pure point dynamical spectrum/are model sets?

Unfortunately, the results of this thesis do not answer this question: No counterexample is known, *i.e.*, to date one does not know a Pisot substitution that is not a model set, but also no general proof that all Pisot substitutions are model sets is known. Fortunately, at least one has explicit methods to check a given Pisot substitution; these will appear here.

Often, the above question is only formulated for so-called "unimodular" Pisot substitutions, where one only has to deal with Archimedean local fields. Consequently, this work tries to treat all Pisot substitutions on equal footing. The effect of this is that we will not only give an overview of some results in algebraic number theory, but also have to review the construction of cut and project schemes. Thus, the main theme, namely "Pisot substitutions", appear for the first time only in Chapter 6, which is definitely the heart of this thesis. Up to that point, however, we have acquired much machinery that can be applied also to structures other than Pisot substitutions; this is done in the "additional" Chapters 5a, 6a–6c. But let us now give a detailed overview over the single chapters of this thesis.

Naturally, since we are talking about point sets in a space in the following, we need some background in topology. Thus, the corresponding notions in (set theoretic) topology are introduced in Chapter 2. The general setting we are working in are locally compact Abelian groups, usually equipped with a metric. This determines the choice of topics treated in this chapter. While completions (and metric spaces) also appear in algebraic number theory, the full theory reviewed in Chapter 2 will mainly be needed in the construction of cut and project schemes in Sections 5.3 & 5.7. The main references of this chapter are the books [75, 272, 301].

Algebraic number theory is the theme of Chapter 3. The goal of this chapter is to understand algebraic number fields and their completions, the local fields, in particular $\mathfrak{p}$-adic fields. A classical topic in number theory is the "geometry of numbers", wherefore we also discus lattices in algebraic number fields; in fact, the construction of the cut and project scheme in Section 6.4 might be regarded as a generalisation of these concepts, if one embeds orders respectively complete modules not only into the product over all Archimedean local fields, but even the product over all Archimedean and some non-Archimedean ($\mathfrak{p}$-adic) fields. Since we want to calculate diffraction patterns later on, we have to introduce the Fourier transformation at some stage. This is foreshadowed in the last section of this chapter, where we talk about characters and Pontryagin duality on local fields. The main references for this chapter are the books [211, 273, 341].

After Chapter 3, we have added three additional chapters with themes closely related to number theory. Chapter 3a introduces profinite groups, which give an alternative construction of p-adic integers. The goal, however, is a structure theorem for profinite groups. This will shed light on the internal space of lattice substitution systems (see Chapter 6b). Since the profinite groups are not needed elsewhere, the reader might postpone reading this chapter until lattice substitution systems are introduced. The main references for this chapter are the books [311, 393].

The adele ring, introduced in Chapter 3b, of an algebraic number field, is – besides the idele group – the most general and natural object that collects information about an algebraic number field in terms of its local fields (headword "local-global principle"). The point sets we will later consider, are "defined in" an algebraic number field or, more precisely, in one (special) local field (usually $\mathbb{R}$). Thus, it is natural to describe such a point set also by looking at the information provided by all other local fields. The most prominent example of this idea will be the visible lattice points in Chapter 5a, but it is also applied to Pisot substitutions in Section 6.5. The main reference here is [254].

Non-Archimedean fields are introduced in Chapter 3, but to get an intuitive feeling for $\mathfrak{p}$-adic fields, we introduce a method to "visualise" sets and points of $\mathfrak{p}$-adic fields. Moreover, the "strange" properties of such ultrametric spaces should to be caught by this visualisation. To this end, we introduce first a tree model for such spaces, and then provide an "embedding" into "everyday" Euclidean spaces. We hope that through this (pedagogical) chapter and the examples throughout Chapter 6, the reader gets a picture of $\mathfrak{p}$-fields in his mind. This chapter brings together ideas from [180] and [312, 313].

As the last two chapters, Chapters 2 & 3, Chapter 4 is written in a self-contained introductory style, to give readers not familiar with these branches of mathematics a chance to lay foundation stones there; they can be certainly be skipped by an expert. The first goal of Chapter 4 is the construction of Haar & Hausdorff measures and their relation to one another. Haar measures naturally appear here since we will do measure theory in locally compact Abelian groups. Hausdorff measures appear since the locally compact Abelian groups we will talk about, are usually metric spaces, namely, products over some local fields. Moreover, in these spaces, the Haar measure happens to be a special Hausdorff measure. This is essentially the contents of the first one third of this chapter, for which the main references are the books [59, 162, 270] and, in particular, [317]. The other part is concerned with "fractal geometry", *i.e.*, with iterated function systems, their attractors and the determination of the Hausdorff dimension for these attractors. Usually, iterated function systems are only treated on Euclidean spaces. We generalise this to product spaces of local fields, and show how (and that) the methods by K.J. Falconer [122, 124] (also see [123]) are applicable to this setting. This generalisation is needed to treat unimodular and non-unimodular Pisot substitutions equal in Chapter 6.

The central objects of interest in the study of aperiodic discrete point sets, are introduced in Chapter 5: Delone, Meyer and, in particular, model sets. In view of the above opening question about Pisot substitution, (regular) model sets are interesting, since they are pure point diffractive (and thus also have pure point dynamical spectrum). Associated to a model set is a cut and project scheme. Given a point set, the construction of a cut and project scheme associated to it (and using only information provided by this point set) has only been put on solid ground recently [36]; we will review this construction, with an special emphasis on the multi-component case, in Section 5.3. Afterwards, we turn our attention to a method how one can obtain aperiodic structures in Section 5.4, namely, by means of a substitution. Here, we stress the similarities with the somehow "dual" concept of an iterated function system, as also observed by [221]. But while an attractor of an iterated function system is unique, there is no such uniqueness for a substitution. Consequently, one has to demand additional properties like repetitivity, representability *etc.* for these substitution point sets (see [235]). In fact, these properties are closely related to properties of the dynamical system defined by such a point set, wherefore we introduce the basics of dynamical systems (Section 5.5) and their application to point sets (Section 5.6) afterwards. Section 5.7 brings together substitution point sets and model sets. To this end, unfortunately, quite some technicalities are necessary: We introduce the equivalent concepts algebraic and overlap coincidences (also compare [230, 364]) for substitution point sets – they will also play a prominent role for Pisot substitutions – which then allow the construction of an extended internal space for the cut and project scheme. Given such an extended internal space, one can define a substitution point set there. Consequently, one obtains a "duality" between the direct and the internal space of a cut and project scheme, and, moreover, criteria under which a given substitution point set is a model set.

An interesting example of a point set that is not a regular model set but nevertheless pure point diffractive, is discussed in Chapter 5a: The visible lattice points. Our goal, the calculation of the diffraction pattern of the visible lattice points, is already established in [37] (also see [36]). Our derivation, which depends on the considerations of Section 5.2, now shows how the "visualisation" of p-adic numbers (see Chapter 3c) leads to a direct way of calculating autocorrelation coefficients and thus the diffraction pattern. Furthermore, we simplify the description of the visible lattice points as weak model sets by "reducing" the internal space to its irredundant part.

Finally, Pisot substitutions appear in Chapter 6. In the first part of that chapter, we establish some properties of the point sets defined through these Pisot substitutions, *e.g.*, repetitivity or being a Meyer set. The proof of being a Meyer set, already establishes the connection to model sets. However, one first has to establish a suitable cut and project scheme to Pisot substitutions. Such a cut and project scheme, using our considerations about the local fields connected to an algebraic number field, is derived in Section 6.4. Unfortunately, one can not use the methods from Section 5.7 to construct a cut and project scheme – *e.g.*, showing that one has an algebraic coincidence seems only to be possible for a given example, but not for a whole family of examples like Pisot substitutions (in particular, since an algebraic coincidence would immediately imply that one has a model set). Recalling Chapter 3b, we review the construction of this cut and project scheme again and also establish the (Pontryagin) dual cut and project scheme, which is needed for the calculation of the diffraction pattern. In the following two sections, we can now apply our findings from Chapters 4 and Section 5.7, wherefore we obtain criteria how one can check that a given Pisot substitution is a model set. For Pisot substitutions, we can extended this list by looking at the so-called "stepped surface" in Section 6.8 and at certain iterated function systems (in Section 6.9) which yield the boundaries of the (possible) windows for the Pisot substitution in question – and which actually derives from the considerations about the overlap coincidence. Moreover, we also recover the coincidence conditions introduced in [50, 192]. Now, all concepts are introduced and we can look at examples. Especially, we explain the concepts on a non-unimodular example in Section 6.10, which seems to be the first – in this detail – worked out example (*e.g.*, explicit derivation and pictures of the aperiodic and periodic tiling as well as the stepped surface of the internal space seem to appear for the first time here). We end this chapter with the formulation of our main theorem, namely Theorem 6.116, about Pisot substitutions, which gives a list of equivalent statements for a Pisot substitution to generate a model set. We have to remark that this theorem, however, also includes statements that will only be proven in Chapter 7, wherefore they, at that stage, have to be taken for granted. Moreover, we also state conditions for the corresponding substitution point set in internal space; this is the reason why we have looked at substitutions on more general spaces than Euclidean spaces in Chapter 5.

There are again three additional chapters. They are intended to show how the methods and some of the equivalent statements, that enforce that a substitution point set is a model set, can be applied to more structures than just Pisot substitutions. Chapter 6a is such a first step towards (possibly) establishing that certain two-dimensional tilings [141, 386] are model sets with mixed Euclidean and $\mathfrak{p}$-adic internal space. However, it is really only an initial investigation in that it only establishes the possible cut and project scheme and remarks how to proceed from there. These examples are interesting, since to date only tilings with purely Euclidean (or the product of a Euclidean space and a finite group) internal space are known. Our considerations here are complementary to [141].

Chapter 6b shows how the well-established theory of lattice substitution systems (see [235]) fits into our setting. This chapter should be read in contrast to our considerations about Pisot substitutions. Especially, we indicate why one does not need the extended internal space there. Moreover, it should become clear why the calculation of an algebraic coincidence, which for lattice substitution systems is called modular coincidence, is considerably easier for lattice substitution systems than for Pisot substitutions. We also apply our "visualisation method" from Chapter 3c to a number of examples.

Chapter 6c treats additional topics which are closely related to Pisot substitutions. First, we review how connectivity is established for the windows of Pisot substitutions; this is due to [87], and we remark on a possible generalisation for non-unimodular Pisot substitutions. Section 6c.2 remarks on so-called "cobounds", which is a generalisation of the concept "height" that appears for constant length substitutions (see Chapter 6b). In fact, we show that our concept of the "height group" in Chapter 5 is closely related to "cobounds". Consequently, the considerations and examples in this section also shed light on our construction of cut and project schemes – especially why the height group appears in the first place – in Chapter 5. Moreover, Example 6c.13 seems to be the first explicit example outside the constant length case with "nontrivial conbounds" which is a model set. The following sections introduce reducible Pisot substitutions and β-substitutions (which are a special case of reducible Pisot substitutions). This is done because the methods established for Pisot substitutions can easily be applied to this examples. But while there are explicit examples of reducible Pisot substitutions which are not model sets (and not pure point diffractive, respectively which do not have pure point dynamical spectrum), a similar conjecture for β-substitutions as for Pisot substitutions seems to hold, namely, that all β-substitutions generate (regular) model sets.

The final chapter, Chapter 7, gives an overview of diffraction theory and the theory of dynamical systems for the point sets under consideration. Since there is no original contribution from the author to this fields, we only introduce and explain the concepts and give the main statements. Thus compared with the previous chapters, this might be quite harsh. The section about diffraction theory links the considerations in Section 5.2 to "real" diffraction patterns. The main reference here is [36]. The diffraction pattern is closely related to the spectrum of the underlying dynamical system, wherefore we consider the spectral theory of dynamical systems in the second part; this continues the considerations from Sections 5.5 & 5.6. In fact, one can show that if the diffraction spectrum is pure point so is the dynamical spectrum and *vice versa*. The main references for this section is [234]. Section 7.3 reviews a recent development: The characterisation of model sets in terms of dynamical systems, for which the so-called "torus parametrisation" plays an important role. The main reference here is [34]. The statements in this first three sections are straightforward applied to Pisot substitutions and are therefore already included in Theorem 6.116. But there are two explicit topics we still want to discus for Pisot substitutions, and which appear in the last section of that chapter: The explicit calculation of the eigenvalues of the underlying dynamical system, which already appeared in [216], and the torus parametrisation for Pisot substitutions. For the latter, we review subshifts, toral automorphisms and Markov partitions in Section 7.4. With these concepts at hand, one can establish the torus parametrisation.

One motivation for this work is to present the material about Pisot substitutions in a coherent way. Obviously, what "coherent" in this case means is highly subjective, wherefore we also point out the overviews over Pisot substitutions in [50, 298], which give a complementary picture on the material presented in Chapter 6. Moreover, the reader might first have a look at the slides of our PhD viva ("Disputation") which are presented in the chapter entitled "Extras" at the end of this book and serve as teaser for Pisot substitutions.

Furthermore, another motivation is to show and establish connections between the various concepts introduced in the theory of model sets and aperiodic order and in the theory of Pisot substitutions. Again, this depends on the limited abilities of the author to discover and understand such connections. So, we apologise to the reader for all trivialities, false claims and missed chances in the pages to come, and hope that, nevertheless, at least reading some of them is profitable.

2. Commutative Topological Groups

> Wir vergöttern die Vollkommenheit, weil wir sie nicht erreichen können; wir würden sie verwefen, wenn wir sie hätten. Das Vollkommene ist unmenschlich, denn das Menschliche ist unvollkommen.
>
> Das Buch der Unruhe – *Fernando Pessoa*

In a certain sense, this work is about understanding the theory of model sets in the setting of metrisable topological groups. Therefore, the intention of this chapter is to make the general results of the theory concrete in the case that the underlying spaces are metric spaces. This chapter is mainly intended to serve as a (self-contained) reference for statements needed later and might – after a look at our notation – be skipped by the expert.

2.1. Topology and Baire Spaces

We begin with some definitions and the general notation.

Definition 2.1. A *topological space* is a set X together with a *topology* $\mathcal{T}$ on X, *i.e.*, $\mathcal{T}$ is a family of subsets of X, called *open sets*, such that X and $\varnothing$ are open sets, the intersection of a pair of open sets is an open set and the union of any number of open sets is an open set. The *discrete topology* on the set X is the topology in which every subset of X is open; in this case, we say that X is a *discrete space*. A *refinement* of the topology $\mathcal{T}$ is a topology $\mathcal{T}'$ on the same set X such that each open set of $\mathcal{T}$ is also an open set of $\mathcal{T}'$; in this situation, we say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$ and $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

Remark 2.2. The *complement* $U^c = X \setminus U$ of an open set U is called a *closed set*. Therefore, one can also define the topology by the family of closed sets.

Definition 2.3. Let X and X' be topological spaces and $f : X \to X'$ be a function. Then f is *continuous* if the inverse image of each open set of X' is an open set of X or, equivalently, if the inverse image of each closed set of X' is a closed set of X. The function $f : X \to X'$ is called *open* if the image $f(U)$ is open in X' whenever U is open in X; it is called *closed* if the image $f(V)$ is closed in X' whenever V is closed in X. If f is a bijective function such that both f and f^{-1} are continuous, then f is called a *homeomorphism*; equivalently, a bijection f is a homeomorphism iff it is a continuous and open map or a continuous and closed map. X and X' are said to be *homeomorphic*, if there is a homeomorphism of X onto X'.

Lemma 2.4. [75, Prop. I.§5.9] *Let X, X' be two topological spaces. A mapping $f : X \to X'$ is continuous and closed iff $f(\operatorname{cl} A) = \operatorname{cl} f(A)$ for every subset A of X.* □

Remark 2.5. Let $(X_i)_{i\in I}$ be a family of topological spaces. The coarsest topology on the product set $X = \prod_{i\in I} X_i$ for which the projections $\pi_i : X \to X_i$ are continuous mappings, is called the *product* of the topologies of the X_i.

Definition 2.6. Let Y be a subset of the topological space X. The *closure* $\operatorname{cl} Y$ of Y is the intersection of the closed sets of X which contain Y. The *interior* $\operatorname{int} Y$ of Y is the union of the open sets of X which are contained in Y. The (set theoretic) difference of the closure and the interior of Y is called the *boundary* (or *frontier*) ∂Y of Y, *i.e.*, $\partial Y = \operatorname{cl} Y \setminus \operatorname{int} Y = \operatorname{cl} Y \cap (\operatorname{int} Y)^c = \operatorname{cl} Y \cap \operatorname{cl} Y^c$. Note that the closure and the boundary of a set Y are closed (*i.e.*, $\operatorname{cl}\operatorname{cl} Y = \operatorname{cl} Y$ and $\operatorname{cl} \partial Y = \partial Y$) and the interior is open (*i.e.*, $\operatorname{int}\operatorname{int} Y = \operatorname{int} Y$). If Y is both open and closed, we call it *clopen*; in this case its boundary is empty. We say that Y is *regularly open* if $Y = \operatorname{int}\operatorname{cl} Y$, and Y is *regularly closed* if $Y = \operatorname{cl}\operatorname{int} Y$. If Y is a subset of X and x is a point in X, x is said to be an *interior point* of Y if $x \in \operatorname{int} Y$ and an *adherence point* of Y if $x \in \operatorname{cl} Y$. The set Y is a *neighbourhood* of x if Y contains an open set which x belongs to; equivalently, Y is a neighbourhood of x if x is an interior point of Y. Therefore, an adherence point x of a set Y is a point such that each neighbourhood of x meets (*i.e.*, has nonempty intersection with) Y. Moreover, if a point $x \in Y$ has a neighbourhood which contains no point of Y except x, we say that x is an *isolated point* of Y. A set without isolated points is called *dense-in-itself*, a set which is both closed and dense-in-itself is called a *perfect set*.

Definition 2.7. A topological space X is called a *Hausdorff space* if any two distinct points of X have disjoint neighbourhoods.

Definition 2.8. A topological space X is *compact*[1] if every covering of X with open sets $\{U_i\}_{i\in I}$ (*i.e.*, $X = \bigcup_{i\in I} U_i$) admits a finite subcovering (*i.e.*, there is a finite set $I' \subset I$ such that $X = \bigcup_{i\in I'} U_i$). A subset Y of a topological space X is said to be *compact* if the subspace Y is compact (where the open sets of Y are the intersection with Y of the open sets of X). A set Y is called *relatively compact* if $\operatorname{cl} Y$ is compact. A topological space X is *locally compact* if each point in X has a neighbourhood which is compact. A topological space X is *σ-compact* (or *countable at infinity*) if it is the union of countably many compact sets; if it is locally compact and σ-compact, we say that X is *σ-locally compact*.

Lemma 2.9. [75, Prop. I.§9.15] and [301, Satz 8.22(b)] *If X is a σ-locally compact Hausdorff space, then there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of relatively compact open subsets of X which cover X (i.e., $X = \bigcup_{n=1}^{\infty} U_n$), such that $\operatorname{cl} U_n \subset U_{n+1}$ for each $n \in \mathbb{N}$.* □

We also note the following statement about continuous injections which is obtained from [75, Propositions I.§10.2 & I.§10.7].

Proposition 2.10. *Let f be a continuous injection of a Hausdorff space X into a locally compact space Y. Then, equivalent are:*

(i) *f is closed.*

(ii) *The inverse image under f of every compact subset of Y is compact.*

(iii) *f is a homeomorphism of X onto a closed subset of Y.*

Moreover, if any of these conditions is fulfilled, then X is locally compact. □

[1] Note that in [75] and [301] the term *quasi-compact* is used; there the word *compact* is used if X is a quasi-compact Hausdorff space.

Definition 2.11. A (nonempty) set $\mathcal{B}$ of nonempty subsets of a set X such that the intersection of any two sets from $\mathcal{B}$ contains a set of $\mathcal{B}$, is called a *filterbase*. A filterbase $\mathcal{B}$ with the additional property that any superset of a member of $\mathcal{B}$ is also in $\mathcal{B}$ is called a *filter*. Equivalently, a filter on a set X is a set $\mathcal{B}$ of subsets of X which has the following properties:

- Every subset of X which contains a set of $\mathcal{B}$ belongs to $\mathcal{B}$.
- Every finite intersection of sets of $\mathcal{B}$ belongs to $\mathcal{B}$.
- The empty set is not in $\mathcal{B}$.

Definition 2.12. In a topological space X, a *fundamental system of neighbourhoods* (or a *neighbourhood base*) of a point x is any set $\mathcal{N}$ of neighbourhoods of x such that for each neighbourhood V of x there is a neighbourhood $V' \in \mathcal{N}$ such that $V' \subset V$. This is a filterbase, and the set of all neighbourhoods of x is a filter, called the *neighbourhood filter* of x. A topological space X is said to satisfy the *first countability axiom* if there exists a countable fundamental system of neighbourhoods for each point $x \in X$. A *base of the topology* of a topological space X is any set $\mathcal{B}$ of open subsets of X such that every open subset of X is the union of sets belonging to $\mathcal{B}$. A topological space X is said to satisfy the *second countability axiom* if there exists a countable base of the topology. A set Y is called *dense* in a topological space X, if $\operatorname{cl} Y = X$. A topological space X is called *separable* if there is a countable dense subset of X. Every topological space that satisfies the second countability axiom is separable and satisfies the first countability axiom. A set Y is called *nowhere dense* (or *rare*) if $\operatorname{int}\operatorname{cl} Y = \varnothing$. A subset Y of a topological space X is called *meager* (or *of first category*) if it can be written as a countable union of nowhere dense sets. We note that a subset of a meager set is meager. If Y is not meager, we say that Y is *nonmeager* (or *of second category*). A nonempty topological space is a *Baire space* if each nonempty open set is nonmeager.

We note the following properties of Baire spaces.

Lemma 2.13. [272, Prop. 11.4.2], [76, Definition IX.§5.3] and [301, Satz 13.24] *The following conditions on a topological space X are equivalent:*

(i) *X is a Baire space.*

(ii) *Each nonempty open set is nonmeager.*

(iii) *The countable union of closed nowhere dense sets has no interior.*

(iv) *The intersection of countably many open dense sets is dense in X.*

(v) *The complement of a meager set is dense in X.* □

Important examples of Baire spaces are given in the following statement[2] proved by R.L. Moore.

Proposition 2.14. [272, Theorem 11.5.3] and [301, Satz 13.29(b)] *Every locally compact Hausdorff space is a Baire space.* □

[2]Sometimes Proposition 2.14 is also stated as *Baire category theorem*; but often the latter is only used for the statement that every complete pseudometric (or metrisable) space is a Baire space, see [272, Theorem 11.5.2] and [301, Satz 13.29(a)].

2.2. Commutative Topological Groups

Definition 2.15. A *commutative topological group* is an Abelian group G which carries a topology such that the group multiplication (*i.e.*, the addition) and the inversion is continuous; equivalently, it suffices that the map $(x\,,y) \mapsto x - y$ of $G \times G$ into G is continuous. A locally compact commutative topological group is called a *locally compact Abelian group* or *LCAG* for short (if it is σ-compact as well we denote this by $\sigma LCAG$). A ring R endowed with a topology compatible with the additive group structure and such that the multiplication map is continuous, is called a *topological ring.* A field K topologised in such a way as to be a topological ring and in which the (multiplicative) inverse on $K^\times = K \setminus \{0\}$ is continuous, is called a *topological field.*

Remark 2.16. A (commutative) topological group G is a *homogeneous* topological space, *i.e.*, given any two points x and y there is a homeomorphism of the space onto itself which maps x into y, namely $z \mapsto z - x + y$. Therefore, it suffices to investigate local properties of G at just one point. For example, we have the following properties:

- If A is open (respectively closed) and $x \in G$, then $x + A = \{x + y \mid y \in G\}$ and $-A = \{-y \mid y \in G\}$ are open (respectively closed).
- If U is open and A is any subset of G, then $U + A = \{x + y \mid x \in U,\, y \in A\}$ is open.
- If U is an (open) neighbourhood of $\{0\}$ and $x \in G$, then $x + U$ is an (open) neighbourhood of x. Consequently, if $\mathcal{N}$ is a neighbourhood base (filter) of 0, then $x + \mathcal{N} = \{x + N \mid N \in \mathcal{N}\}$ is a neighbourhood base (filter) of x.
- If $f : G \to G'$ is a homomorphism (*i.e.*, $f(x + y) = f(x) + f(y)$), where G and G' are (commutative) topological groups, then f is continuous iff it is continuous at one point of G.
- If $\{0\}$ is an open set, any one-point set in G is open; in other words, G is then a discrete group.

We also observe that if U is a neighbourhood of $\{0\}$, then $-U$ is also a neighbourhood of $\{0\}$; therefore $V = U \cap (-U)$ (as well as $V = U \cup (-U)$ and $V = U - U$) is a *symmetric neighbourhood* of $\{0\}$, *i.e.*, $V = -V$. Moreover, if U is a neighbourhood of 0, then there is also a neighbourhood V of 0 such that $V + V \subset U$ (by the continuity of the addition).

We are first interested in the Hausdorff property.

Lemma 2.17. [272, Theorem 3.1.7] and [75, Proposition III.§1.2] *For a (commutative) topological group G the following are equivalent:*

(i) *G is Hausdorff.*

(ii) *The set $\{0\}$ is closed in G.*

(iii) *The intersection of the neighbourhoods of* 0 *consists only of the point* 0. □

One also wants to compare two (commutative) topological groups. For this, we observe that a topological isomorphism is a bijective strict morphism.

Definition 2.18. A continuous homomorphism of a (commutative) topological group G into a (commutative) topological group G' is said to be a *strict morphism* of G into G' if the bijective homomorphism $\dot{f}$ of $G/f^{-1}(0')$ onto $f(G)$ (here, $0'$ denotes the identity element in G'), associated with f, is an isomorphism of topological groups.

Every continuous homomorphism of a topological group into a *discrete group* (*i.e.*, a topological group endowed with the discrete topology) is a strict homomorphism.

Lemma 2.19. [75, Prop. III.§2.29] *Let f be a continuous homomorphism of a topological group G into a topological group G'. Then the following three statements are equivalent:*

(i) *f is a strict morphism.*

(ii) *The image under f of every open set in G is an open set in $f(G)$.*

(iii) *The image under f of every neighbourhood of the identity element in G is a neighbourhood of the identity element in G'.* □

We are now interested in the relationship between filterbases and neighbourhood bases in a commutative topological group, see [272, Theorems 3.2.2 & 3.2.3], [159, Theorem II.§2.3.2] and [75, Prop. III.§1.1].

Proposition 2.20. *Let $\mathcal{B}$ be a family of subsets of a commutative group G. We consider the following properties:*

(i) *Finite intersections of sets from $\mathcal{B}$ are not empty.*

(ii) *$\mathcal{B}$ is a filterbase (obviously,* (i) *is satisfied in this case).*

(iii) *For each $U \in \mathcal{B}$ there is a $V \in \mathcal{B}$ such that $V + V \subset U$.*

(iv) *For each $U \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ such that $-V \subset U$.*

(v) *For each $U \in \mathcal{B}$ and any $x \in U$ there exists a $V \in \mathcal{B}$ such that $x + V \subset U$.*

If $\mathcal{B}$ satisfies (i), (iii) *and* (iv), *then $\mathcal{B}$ is a neighbourhood* subbase *of* 0 *for a group topology of G; it is a neighbourhood base of* 0 *if it satisfies* (ii), (iii) *and* (iv). *If in addition* (v) *is satisfied, the neighbourhood (sub)base of* 0 *consists of open sets in this group topology. If $\mathcal{B}$ satisfies properties* (iii), (iv) *and* (v), *$\mathcal{B}$ is a* prebase[3] *of neighbourhoods of* 0. □

We are interested in subgroups and quotient groups of topological groups.

Lemma 2.21. [75, Props. III.§2.1 & III.§2.2 & III.§2.5 & Corollary to Prop. III.§2.4] and [272, Theorems 3.3.1 & 3.3.2] *Let G be a topological group and H a subgroup of G.*

(i) *The closure* $\operatorname{cl} H$ *of H is a subgroup of G. If H is a normal subgroup of G, then so is* $\operatorname{cl} H$.

(ii) *If G is a Hausdorff topological group and H is a commutative subgroup of G, then* $\operatorname{cl} H$ *is a commutative subgroup of G.*

[3] A *prebase* of neighbourhoods is a system of neighbourhoods whose finite intersections (and – depending on the definition – also arbitrary unions) form a neighbourhood base.

(iii) *H is open iff it has an interior point. Every open subgroup is closed.*

(iv) *H is discrete iff H has an isolated point. Every discrete subgroup of a Hausdorff group is closed.* □

If H is an arbitrary subgroup of a topological group G, we introduce a topology on the quotient space G/H of left cosets in the following way: Let π be the canonical homomorphism of G to G/H (*i.e.*, $\pi : G \to G/H$, $\pi(x) = x + H$). The set $U \subset G/H$ is defined to be open (in G/H) if the set $\pi^{-1}(U)$ is open (in G). The topological space G/H will be called the *quotient space* of the group G by H. If H is a normal subgroup of G, then G/H is a topological group, the *quotient group* of G by H; in this case π is an open mapping.

Lemma 2.22. [75, Props. III.§2.17 & III.§2.18] *Let G be a topological group and H a normal subgroup of G. Let π be the canonical homomorphism of G to G/H.*

(i) *The quotient group G/H is Hausdorff iff H is closed in G.*

(ii) *The quotient group G/H is discrete iff H is open in G.*

(iii) *Let G be commutative. If $\mathcal{B}$ is a fundamental system of neighbourhoods of 0 in G, then $\pi(\mathcal{B})$ is a fundamental system of neighbourhoods of the neutral element $\pi(0)$ of G/H.* □

The following statement shows that the quotient of a factor group is topologically isomorphic to the factor group of the respective quotient groups.

Lemma 2.23. [75, Corollary to Prop. III.§2.26] *Let $(G_i)_{i\in I}$ be a family of topological groups, and for each $i \in I$ let H_i be a normal subgroup of G_i; let π_i denote the canonical mapping of G_i onto G_i/H_i. Let $G = \prod_{i\in I} G_i$ and $H = \prod_{i\in I} H_i$. Then the bijective homomorphism of G/H onto $\prod_{i\in I}(G_i/H_i)$ associated with the continuous homomorphism $(x_i)_{i\in I} \mapsto (\pi_i(x_i))_{i\in I}$ is an isomorphism of topological groups.* □

2.3. Metric Spaces

Definition 2.24. A space X is a *metric space* (X, d) if it is equipped with a *distance function* (or *metric*) $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ we have

- $d(x\,, y) \geq 0$ and $d(x\,, x) = 0$ (positivity).
- $d(x\,, y) = d(y\,, x)$ (symmetry).
- $d(x\,, y) \leq d(x\,, z) + d(z\,, y)$ (triangle inequality).
- $d(x\,, y) = 0$ iff $x = y$ (definiteness).

Without the definiteness, d is called a *pseudometric.* An *ultrametric distance* on a space X is a distance satisfying the *strong triangle inequality*

$$d(x\,, y) \leq \max\{d(x\,, z), d(z\,, y)\}$$

for all $x, y, z \in X$. An *ultrametric space* (X, d) is a metric space in which the distance satisfies this strong triangle inequality.

We define the closed ball of radius r and centre a as $B_{\leq r}(a) = \{x \in X \mid d(x,a) \leq r\}$ and the open ball of radius r and centre a as $B_{<r}(a) = \{x \in X \mid d(x,a) < r\}$ for $r \geq 0$ and $x \in X$. The open balls form a basis of a topology on X and are therefore called open (note that $B_{<0}(a) = \varnothing$, and that the closed balls are closed in this topology). Therefore, a metric generates a topology. Metric spaces are Hausdorff.

Moreover, we note that $\{B_{<1/n}(a) \mid n \in \mathbb{N}\}$ is a countable fundamental system of neighbourhoods for a point $a \in X$; therefore, a metric space (X,d) satisfies the first countability axiom. We also note that a metric space (X,d) satisfies the second countability axiom iff it is separable (or, equivalently, if for each $n \in \mathbb{N}$ the open covers $\{B_{<1/n}(x) \mid x \in X\}$ have countable subcovers, *i.e.*, if it is *Lindelöf*, see [367, p. 35]).

Definition 2.25. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X,d) is called a *Cauchy sequence* iff for every $\varepsilon > 0$ there exists an integer n_0 such that $d(x_m, x_n) < \varepsilon$ whenever $m,n > n_0$. Obviously, every convergent sequence is a Cauchy sequence.

We now give a list of elementary properties of ultrametric spaces, see [313, Lemmas 2.1.1.1–2.1.1.4 & Prop. 2.1.1] and [153, Corollary 2.3.4 & Prop. 2.3.6]. We denote by $S_r(a)$ the *sphere* of centre $a \in X$ and radius $r > 0$, *i.e.*, $S_r(a) = \{x \in X \mid d(x\,,a) = r\}$. The *diameter* of a nonempty set A is defined by $\operatorname{diam}(A) = \sup\{d(a\,,a') \mid a, a' \in A\}$.

Lemma 2.26. *Let (X,d) be an ultrametric space. It has the following properties:*

(i) *Any point of a ball is a centre of the ball, i.e., if $b \in B_{<r}(a)$ ($b \in B_{\leq r}(a)$), then $B_{<r}(a) = B_{<r}(b)$ ($B_{\leq r}(a) = B_{\leq r}(b)$).*

(ii) *If two balls have a common point, one ball is contained in the other.*

(iii) *The diameter of a ball is less than or equal to its radius.*

(iv) *If $d(x\,,y) > d(z\,,y)$, then $d(x\,,y) = d(x\,,z)$.*

(v) *If $d(x\,,z) \neq d(z\,,y)$, then $d(x\,,y) = \max\{d(x\,,z), d(z\,,y)\}$.*

(vi) *If $x \in S_r(a)$, then $B_{<r}(x) \subset S_r(a)$ and*

$$S_r(a) = \bigcup_{x\in S_r(a)} B_{<r}(x).$$

(vii) *The spheres $S_r(a)$ ($r > 0$) are* clopen *i.e., both open and closed.*

(viii) *The open balls $B_{<r}(a)$ are clopen; the closed balls $B_{\leq r}(a)$ are clopen for $r > 0$.*

(ix) *Let B and B' be two disjoint balls. Then, $d(B\,,B') = d(x\,,x')$ for any $x \in B$ and $x' \in B'$.*

(x) *A sequence $(x_n)_{n\in\mathbb{N}}$ with $d(x_n, x_{n+1}) \to 0$ ($n \to \infty$) is a Cauchy sequence.*

(xi) *If $x_n \to x \neq a$, then $d(x_n, a) = d(x\,,a)$ for all large indices n.*

(xii) *Let $W \subset X$ be compact. Then, for every $a \in X\backslash W$, the set of distances $\{d(x\,,a) | x \in W\}$ is finite.*

(xiii) *Let $W \subset X$ be compact. Then, for every $a \in W$, the set $\{d(x\,,a) \mid x \in W \setminus \{a\}\}$ of distances is discrete in $\mathbb{R}_{>0}$ (w.r.t. the topology on the multiplicative topological group $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$).* □

Remark 2.27. Properties (iv) and (v) are often formulated as

> In an ultrametric space, all triangles are *isosceles* (or *equilateral*), with at most one short side.

Remark 2.28. Obviously, an ultrametric space has a base of the topology consisting of clopen sets (and is then also called *zero-dimensional*, see Remark 4.60). Moreover, it is also totally disconnected, a property we recall next.

Definition 2.29. If two sets Y_1 and Y_2 have the property that $Y_1 \cap \operatorname{cl} Y_2 = \varnothing = \operatorname{cl} Y_1 \cap Y_2$, they are called *separated*. A set X is called *connected* if it cannot be written as union of two separated sets. The *connected component* of a point of a topological space X is the largest connected subset of X which contains this point. A space X is said to be *totally disconnected* if the connected component of each point of X consists of the point alone. Note that the connected component of a point is contained in the intersection of the sets which are clopen and which contain this point.

The following statement is well-known.

Lemma 2.30. [94, Theorem 3.1.14] *Suppose that Y_1 and Y_2 are disjoint subsets of a metric space $(X\,, d)$. If Y_1 is compact and Y_2 is closed, then they have positive distance $d(Y_1\,, Y_2) = \inf\{d(y_1\,, y_2) \mid y_1 \in Y_1,\, y_2 \in Y_2\} > 0$. Furthermore, if $d(Y_1\,, Y_2) = r$, then there is a $y \in Y_1$ such that $r = d(y\,, Y_2) = \inf\{d(y\,, z) \mid z \in Y_2\}$.* □

We also note the following statement about connectedness in a product space.

Lemma 2.31. [75, Prop. I.§11.10] *In a product space $X = \prod_{i\in I} X_i$, the connected component of $x = (x_i)$ in X is the product of the connected components of x_i in the factors X_i.* □

We know that every metric space is a topological space. We are now interested in the converse question.

Definition 2.32. A topological space X is called *(pseudo)metrisable* if there exists a (pseudo)metric d which yields the topology. Of course, the (pseudo)metric d is not uniquely determined by the topology. Two (pseudo)metrics on X are called *topologically equivalent* if they define the same topology.

Proposition 2.33. [301, Korollar 10.16] *Let X be a locally compact space. Then X is metrisable and σ-compact iff it satisfies the second countability axiom.* □

Proposition 2.34. [272, Theorem 3.6.1] *A commutative topological group is pseudometrisable iff it has a countable base at 0. If it is also Hausdorff, it is metrisable.* □

Remark 2.35. Note that the pseudometric on a commutative group can be chosen to be an *invariant* (or *translation-invariant*) pseudometric, *i.e.*, $d(x + a, y + a) = d(x, y)$ for all x, y and a in the group. Moreover, let $\{V_n\}_{n\in\mathbb{N}}$ be a countable base of neighbourhoods at 0. Then one can define a sequence $\{U_n\}_{n\in\mathbb{N}}$ of symmetric neighbourhoods of 0 such that $U_1 = V_1$ and

$U_{n+1} + U_{n+1} + U_{n+1} \subset U_n \cap V_n$, which also forms a countable base of neighbourhoods at 0. Define $\tilde{d}$ as follows:

$$\tilde{d}(x,y) = \begin{cases} 1, & \text{if } x - y \in U_1, \\ 0, & \text{if } x - y \in U_n \text{ for all } n, \\ 2^{-k}, & \text{if } x - y \notin U_{k+1} \text{ but } x - y \in U_k. \end{cases}$$

Then, $\tilde{d}$ is nonnegative, symmetric and invariant. The pseudometric d is obtained as $d(x,y) = \inf\{\sum_{i=0}^{p-1} \tilde{d}(z_i, z_{i+1}) \mid z_0 = x,\, z_p = y,\, p \in \mathbb{N}\}$ (so one takes a "path of shortest distance" from x to y). Here, a major step of the proof is to show that $\frac{1}{2}\tilde{d} \leq d \leq \tilde{d}$.

Lemma 2.36. [272, Theorem 3.6.2] *If G is pseudometrisable, so is any quotient group G/H. If G is metrisable, then every Hausdorff quotient group G/H is metrisable, too.* □

Remark 2.37. We also remark that the topology of any commutative topological group is determined by a family of pseudometrics, see [272, Theorem 3.6.3].

2.4. Completeness

Definition 2.38. A filterbase $\mathcal{B}$ in a topological space is said to *converge* to a point $x \in G$ if each neighbourhood V of x contains some $B \in \mathcal{B}$; in this case, x is called a *limit* of $\mathcal{B}$. We note that in a Hausdorff topological space each convergent filterbase has a unique limit. A filterbase $\mathcal{B}$ in a commutative topological group G is *Cauchy* if, for any neighbourhood V of 0, there exists an element B of $\mathcal{B}$ such that $B - B \subset V$; equivalently, $\mathcal{B}$ is a Cauchy filterbase iff for each neighbourhood V of 0, there exists $B \in \mathcal{B}$ and an element $x \in G$ such that $B \subset V + x$. Obviously, each convergent filterbase is Cauchy.

Definition 2.39. A set S together with a reflexive, transitive ordering relation $\leq$ such that finite subsets of S have upper bounds in S (*i.e.*, for $r, s \in S$ there is a $t \in S$ such that $r \leq t$ and $s \leq t$) is called a *directed set.* A *net* in a topological space X is a mapping $s \to x_s$ from a directed set S into X. We say that a net $(x_s)_{s \in S}$ *converges* to x (denoted by $x_s \to x$), if for any neighbourhood V of x there is an $s \in S$ such that $x_t \in V$ for $t \geq s$; in this case, we say that x is the *limit* of the net $(x_s)_{s \in S}$. A *Cauchy net* $(x_s)_{s \in S}$ is a net for which, given a neighbourhood U of 0 in a commutative topological group G, there exists an index r such that $x_s - x_t \in U$ for $s, t \geq r$.

Remark 2.40. Comparing the last two definitions of filterbases and nets (which both serve as generalisations of sequences, which are "inadequate" in general topological situations, see [272, Example 2.1.1]), we notice the following correspondence between them (see [272, Theorems 2.4.1 & 2.4.2]): For any filterbase in a topological space there is a net and vice versa. Given a net $(x_s)_{s \in S}$, let $B_s = \{x_t \mid t \geq s\}$, then $\mathcal{B} = \{B_s \mid s \in S\}$ is a filterbase. Conversely, a filterbase becomes a directed set by $B \leq B'$ if $B \supset B'$. Choosing one member $b_s \in B_s$ from each $B_s \in \mathcal{B}$, we obtain a net. Under this cited correspondence, a net is Cauchy iff the filterbase is Cauchy, see [272, Theorem 4.1.2].

Definition 2.41. A commutative topological group is *complete* if every Cauchy net (or, equivalently, every Cauchy filterbase) converges. A subset B of G is complete if each Cauchy net in B converges to a point in B. By a *completion* of a commutative topological group we mean a complete topological group $\hat{G}$ which contains a dense topologically isomorphic copy of G. Here, a *topological isomorphism* of a commutative topological group G into a commutative topological group G' is a *group isomorphism* (*i.e.*, a bijective group homomorphism) which is also a homeomorphism. We (often) denote this as $G \cong G'$.

Remark 2.42. We note that if d is an invariant pseudometric on the group G, then G is complete as topological group iff it is complete as pseudometric space, *i.e.*, if every net $(x_s)_{s\in S}$ such that $d(x_s, x_t)$ becomes arbitrarily small converges, see [272, Theorem 4.1.1]. Therefore, for a (pseudo)metrisable commutative group G, the completion may be obtained by the classical method of taking $\hat{G}$ to be the class of all Cauchy sequences from G. Furthermore, since the topology of any topological group is generated by a family of invariant pseudometrics (see Remark 2.37), one can "complete" the group with respect to each of them and form their product. The closure of the image of the group in this product is then its completion. So we can basically forget about nets and filterbases in our situation. We note that while a non-commutative topological group possesses a completion as uniform space, it may not be possible to extend the group operations to this completion.

In view of the last remark we note the following.

Proposition 2.43. [272, Theorem 4.7.1] *Any pseudometrisable commutative topological group possesses a completion.* □

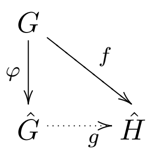
Proposition 2.44. [272, Theorem 4.7.2] *Any commutative topological group possesses a completion.* □

The main statement is now the uniqueness of the Hausdorff completion of a commutative topological group (compare the following definition), also see [75, Theorems III.§3.1 & III.§3.2].

Proposition 2.45. [272, Theorem 4.7.3] *A Hausdorff commutative topological group possesses a Hausdorff completion to which any Hausdorff completion must be topologically isomorphic.* □

Definition 2.46. If a commutative topological group G is not Hausdorff, the set $\{0\}$ is not closed in G by Lemma 2.17. But its closure $\operatorname{cl}\{0\}$ is a subgroup of G (by Lemma 2.21) and so the quotient group $G/\operatorname{cl}\{0\}$ is Hausdorff by Lemma 2.22 (note that $\operatorname{cl}\{0\}$ is the "smallest" subgroup of G such that the quotient group is Hausdorff). By the last proposition, the Hausdorff completion of $G/\operatorname{cl}\{0\}$ is unique up to topological isomorphism. This completion is also called the *Hausdorff completion* of G.

Proposition 2.47. [75, Prop. III.§3.8] *Let G be a (commutative) topological group which has a Hausdorff completion $\hat{G}$. Then every continuous homomorphism f of G into a complete Hausdorff group $\hat{H}$ can be uniquely factorised into $f = g \circ \varphi$, where g is a continuous homomorphism of $\hat{G} \to \hat{H}$ and φ is the canonical mapping of G into $\hat{G}$ (i.e., φ is the composition of the canonical injection of $G/\operatorname{cl}\{0\}$ into $\hat{G}$ and the canonical homomorphism π of G onto $G/\operatorname{cl}\{0\}$); i.e., to every continuous homomorphism f, there exists a continuous homomorphism g such that the following diagram is commutative:*



□

We also note the following properties of complete topological groups with respect to taking quotients.

Lemma 2.48. [272, Theorem 4.7.4] *If G is a complete pseudometrisable commutative topological group and H a closed subgroup, then G/H is complete.* □

Remark 2.49. Moreover, we also note that the Hausdorff completion $\hat{G}$ of a pseudometrisable commutative topological group G is metrisable: If G is pseudometrisable (with pseudometric ϱ), so is any quotient group G/H ([272, Theorem 3.6.2]), where the translation-invariant pseudometric is given by $\dot{\varrho}(x\,,y) = \dot{\varrho}(x-y\,,0) = \inf\{\varrho(x-y+h\,,0)\,|\,h \in H\}$. With the choice $H = \mathrm{cl}\{0\}$, $\dot{\varrho}$ is even a metric ($G/\,\mathrm{cl}\{0\}$ is Hausdorff), and the completion of a metric space is again a metric space (see [301, Satz 13.6]). The metric $\hat{\varrho}$ on $\hat{G}$, which can be shown to be a continuous homomorphism by Prop. 2.47, can constructively be obtained from $\dot{\varrho}$ by looking at the limit of Cauchy sequences from $G/\,\mathrm{cl}\{0\}$.

The next statement shows that LCAGs are complete, also see [313, Corollary 1.3.2.3].

Lemma 2.50. [272, Theorems 4.1.6 & 4.1.7] *If the commutative topological group G has a complete neighbourhood V of 0, then G is complete. Furthermore, a compact subset W of a topological group G is complete. Hence any LCAG is complete.* □

In connection with compactness, we also note the following definition and statement.

Definition 2.51. A subset W of a commutative topological group G is said to be *totally bounded* (or *precompact*) if for all neighbourhoods V of 0 there exists a finite number of elements $x_1, \dots, x_n \in G$ such that the sets $V + x_1, \dots, V + x_n$ cover W.

Lemma 2.52. [272, Theorem 4.4.1] *If W is a closed and totally bounded subset of a complete commutative topological group G, then W is compact.* □

We also note that the closure of a precompact set is precompact (see [75, Prop. II.§4.1]). Moreover, in a metric space every totally bounded set is bounded.

We also note that the sum $W_1 + W_2$ of two compact sets W_1, W_2 is again compact, while the sum $D + W$ of a closed set D and a compact set W in a topological group is closed (see [272, Theorem 3.1.10]).

We also introduce the notion of a compactly generated topological group, where we restrict ourselves to the case of a commutative group, see [169, Definition 5.12 & Theorem 5.14] for the non-Abelian case.

Definition 2.53. A commutative topological group G is said to be *compactly generated* if it contains a compact subset W for which the subgroup generated by W is G, *i.e.*, $G = \langle W \rangle_{\mathbb{Z}}$.

Lemma 2.54. *Let G be a LCAG and W be any compact subset of G. Then, there is an open and closed compactly generated subgroup of G containing W.* □

2.5. Uniform Spaces

Since results about topological groups and metric spaces can be regarded as special cases for results about uniform spaces (and, of course, since uniformities are used in the general theory), we give a short overview of uniform spaces here.

Definition 2.55. A *uniformity* (or *uniform structure*) on a set X is a filter $\mathcal{F}$ on the Cartesian product $X \times X$ consisting of members, called *entourages* (or *surroundings* or *relations*), such that

- each entourage U contains the diagonal $\Delta X = \{(x, x) \mid x \in X\}$.
- if U is an entourage, then there is a entourage V such that $V \circ V \subset U$, where $U \circ V = \{(x, z) \mid \exists y \in X : \ (x, y) \in V, \ (y, z) \in U\}$.
- if U is an entourage, then so is $U^{-1} = \{(y, x) \mid (x, y) \in U\}$.

If $\mathcal{F}$ is a filterbase satisfying these three conditions, we say that $\mathcal{F}$ is a *uniformity base.* The uniformity determined by the uniformity base consists of the supersets of the members of the uniformity base. A set X endowed with a uniformity is called a *uniform space.*

The topology of a uniform space is clarified in the following statement.

Lemma 2.56. [75, Prop. II.§1.1] and [301, Satz 11.8] *Let X be a set endowed with a uniformity $\mathcal{F}$, and for each $x \in X$ let $\mathcal{V}(x)$ be the set of subsets $V(x) = \{y \mid (x, y) \in V\}$ of X, where V runs through the set of entourages of $\mathcal{F}$. Then there is a unique topology on X such that, for each $x \in X$, $\mathcal{V}(x)$ is the neighbourhood filter of x in this topology.* □

Remark 2.57. A uniform space is called *Hausdorff* (*(locally) compact etc.*) if the induced topological space is *Hausdorff* (*(locally) compact etc.*).

Lemma 2.58. [75, Prop. II.§1.3] and [301, Satz 11.13(b)] *A uniform space is Hausdorff iff the intersection of all entourages of its uniformity is the diagonal ΔX.* □

Example 2.59. The *discrete uniformity* on the set X is the uniformity in which every superset of ΔX is an entourage. The *trivial uniformity* on the set X is the uniformity in which the full set $X \times X$ is the sole entourage. If X is a pseudometric space with pseudometric d, then the family $\mathcal{F}$ of all sets of the form $U_\varepsilon = \{(x, y) \mid d(x, y) < \varepsilon\}$ (where ε runs through all positive real numbers) is a uniformity base on X, which yields the same topology as the pseudometric d.

The last example prompts the question, in which case a uniform space is a pseudometric space. The answer is given in the following generalisation of Proposition 2.34.

Proposition 2.60. [301, Lemma 11.30 & Korollar 11.32] *Let X be a uniform space whose uniformity has a countable base. Then the uniformity of X can be defined by a pseudometric. Moreover, if it is a Hausdorff uniform space, then the uniformity can be defined by a metric.* □

While topological spaces can be compared by continuous functions, uniformly continuous functions play that role in uniform spaces.

Definition 2.61. A mapping f of a uniform space X into a uniform space X' is said to be *uniformly continuous* if, for each entourage V' of X', there is an entourage V of X such that $(x, y) \in V$ implies $(f(x), f(y)) \in V'$. If $\mathcal{F}$ and $\mathcal{F}'$ are two uniformities on the same set $X = X'$, $\mathcal{F}$ is said to be *finer* than $\mathcal{F}'$ (and $\mathcal{F}'$ *coarser* than $\mathcal{F}$) if the identity mapping $X \to X'$ is uniformly continuous.

Remark 2.62. While every uniform space is a topological space, the converse is not true, see [194, Beginning of Chapter 8 & Prop. (11.5)] and [367, p. 38]. Similarly, every uniformly continuous map is continuous but not vice versa, see [75, Prop. II.§2.1].

In view of the last remark we note.

Lemma 2.63. [75, Corollary 2 to Theorem II.§4.1] *Every locally compact space X is* uniformisable*, i.e., there exists a uniformity on X which is compatible with its topology.* □

Lemma 2.64. [272, Theorem 4.5.1] *A continuous homomorphism f mapping a subgroup H of a topological group G into the topological group G' is uniformly continuous.* □

Remark 2.65. Obviously, the discrete uniformity is the finest and the trivial uniformity the coarsest uniformity in the ordered set of uniformities on a set X. Also, the uniformity $\mathcal{F}$ is finer than $\mathcal{F}'$ iff every entourage of $\mathcal{F}'$ is an entourage of $\mathcal{F}$; moreover, in this case, the topology induced by $\mathcal{F}$ is finer than the topology induced by $\mathcal{F}'$.

We now look at completions in uniform spaces.

Definition 2.66. If X is a uniform space and if V is an entourage of X, a subset A of X is said to be *V-small* if $A \times A \subset V$. A filter $\mathcal{B}$ on a uniform space X is a *Cauchy filter* if for each entourage V of X there is a subset X which is V-small and belongs to $\mathcal{B}$. A *complete space* is a uniform space in which every Cauchy filter converges, where we say that a filter $\mathcal{B}$ *converges*[4] to x if $\mathcal{B}$ is *finer* than the neighbourhood filter $\mathcal{V}(x)$ of x, *i.e.*, if $\mathcal{B} \supset \mathcal{V}(x)$.

The corresponding generalisation of Proposition 2.45 is the following statement which states that every uniform space has a (up to isomorphism) unique Hausdorff completion (but also see Remark 2.42).

Proposition 2.67. [75, Theorem II.§3.3 & Propsition II.§3.12(i)] and [301, Satz 12.16 & Korollar 12.17] *Let X be a uniform space. Then there exists a complete Hausdorff uniform space $\hat{X}$ and a uniformly continuous mapping $i : X \to \hat{X}$ having the following property:*

(P) *Given any uniformly continuous mapping f of X into a complete Hausdorff uniform space $\hat{Y}$, there is a unique uniformly continuous mapping $g : \hat{X} \to \hat{Y}$ such that $f = g \circ i$.*

*If $(i', \hat{X}')$ is another pair of a complete Hausdorff uniform space $\hat{X}'$ and a uniformly continuous mapping $i' : X \to \hat{X}'$ having property **(P)**, then there is a unique isomorphism $\varphi : \hat{X} \to \hat{X}'$ such that $i' = \varphi \circ i$. Furthermore, the subspace $i(X)$ is dense in $\hat{X}$.* □

Note that the uniform space X is not assumed to be Hausdorff. We call $\hat{X}$, defined in the previous Proposition, the *Hausdorff completion* of X. Of course, if X is a commutative topological group G, then we are in the situation of the last section, *i.e.*, the Hausdorff completion of G as uniform space is the Hausdorff completion $\hat{G}$ (as topological group); in particular, in this case the (Hausdorff) completion is still a commutative topological group.

Also note that, if X is Hausdorff, the canonical mapping $i : X \to \hat{X}$ is an isomorphism of X onto the dense subspace $i(X)$ of $\hat{X}$, wherefore we may identify X and $i(X)$ in this case.

We also note the following statement, which is the generalisation of Proposition 2.47.

[4] This definition is consistent with the convergence of the associated filterbase defined in Definition 2.38.

Lemma 2.68. [75, Propsition II.§3.15] *Let X and X' be two uniform spaces. For each uniformly continuous mapping $f : X \to X'$, there is a unique uniformly continuous mapping $\hat{f} : \hat{X} \to \hat{X}'$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \hat{X} & \xrightarrow{\hat{f}} & \hat{X}' \end{array}$$

is commutative (i.e., $i' \circ f = \hat{f} \circ i$), where $i : X \to \hat{X}$ and $i' : X' \to \hat{X}'$ are the canonical mappings into the respective Hausdorff completions. □

As last point in this chapter, we are interested in completions of subspaces and of product spaces respectively product groups.

Lemma 2.69. [75, Corollary 1 to Propsition II.§3.18] *Let X be a uniform space and denote by i the canonical mapping of X into its Hausdorff completion $\hat{X}$. Let Y be a subspace of X and $j : Y \to X$ the canonical injection. Then, $\hat{\jmath} : \hat{Y} \to \hat{X}$ is an isomorphism of $\hat{Y}$ onto the closure of $i(Y)$ (more precisely, $i(j(Y))$) in $\hat{X}$.* □

Lemma 2.70. [75, Corollary 2 to Propsition II.§3.18] *Let $(X_i)_{i \in I}$ be a family of uniform spaces. Then the Hausdorff completion of the product space $\prod_{i \in I} X_i$ is canonically isomorphic to the product $\prod_{i \in I} \hat{X}_i$.* □

Corollary 2.71. *Let $(G_i)_{i \in I}$ be a family of commutative topological groups. Then the Hausdorff completion of the product group $G = \prod_{i \in I} G_i$ is topologically isomorphic to the product group $\prod_{i \in I} \hat{G}_i$.* □

3. Algebraic Number Theory

No sóc un home de lletres sino dóbres.

Antoni Gaudi

This is another chapter that serves as reference for statements later needed, and might again be skipped by the expert. Our primary goal is to understand local fields (in particular $\mathfrak{p}$-adic fields).

3.1. Algebraic Extensions

Definition 3.1. Let K and L be fields. If K is a subfield of L, we call L a *(field) extension* of K. In this case, L is called a *finite extension* of K if $[L : K]$ is finite, and a *simple extension* if $L = K(a)$ for some $a \in L$.

Definition 3.2. We say that $a \in L$ is *algebraic* over K if there is a nonzero polynomial $f(x) \in K[x]$ such that $f(a) = 0$. The extension L over K is called *algebraic extension* of K if each element of L is algebraic over K.

Finite and algebraic extensions are related as follows.

Lemma 3.3. [226, Prop. V.1.1], [253, Theorem I.2] and [73, Algebraische Ergänzung Satz §2.2] *If L is a finite extension of K, then L is an algebraic extension of K.* □

Definition 3.4. A field K is said to be *algebraically closed* if it has no proper algebraic extensions. A field L is called *algebraic closure* of a field K and denoted by $L = K^{\mathrm{a}}$, if L/K is algebraic and L is algebraically closed.

Definition 3.5. Let L_1 and L_2 be extensions of K. An isomorphism σ of L_1 into L_2 is called a *K-isomorphism* if $\sigma(a) = a$ for all $a \in K$. If there is a K-isomorphism of L_1 onto L_2 then L_1 and L_2 are said to be *K-isomorphic.*

We get the following existence and uniqueness statement for an algebraic closure of a field.

Lemma 3.6. [226, Theorem V.2.5 & Corollary V.2.9] and [253, Theorem I.25] *Every field K has an algebraic closure L, and any two algebraic closures L_1, L_2 of K are K-isomorphic.* □

Definition 3.7. If $f(x) \in K[x]$ and if a is an element of some extension of K, such that $f(a) = 0$, then a is called a *root* of $f(x)$. The unique monic irreducible polynomial $f(x)$ in $K[x]$ having a as a root, will be called the *minimal polynomial* (or *characteristic polynomial*) of a over K, denoted by $f(x) = \mathrm{Irr}(a, K, x)$.

Definition 3.8. An element $a \in L$ which is algebraic over K is said to be *separable* over K if it is a simple root of its minimal polynomial. The extension L is said to be a *separable extension* of K if it is algebraic over K and if each of its elements is separable over K.

The following statement is implied by [226, Prop. V.6.1], respectively [226, Corollary V.6.12].

Lemma 3.9. *If* char $K = 0$*, then every algebraic extension of* K *is separable.* □

Remark 3.10. We will mainly deal with extensions of fields of characteristic 0, *i.e.*, with separable extensions.

Definition 3.11. An *algebraic number field* L is a finite extension of $K = \mathbb{Q}$ lying in $\mathbb{C}$.

The next statement follows readily from the "Primitive Element Theorem", see [226, Theorem V.4.6] and [253, Theorem I.24].

Lemma 3.12. *If* L *is a finite separable extension of* K*, then there exists an element* $a \in L$ *such that* $L = K(a)$*, i.e.,* L *is a simple extension of* K*. In particular, an algebraic number field* L *is always of the form* $L = \mathbb{Q}(a)$*.* □

Definition 3.13. The degree $[L : K]$ of a finite separable extension L over K is called the *degree of* L *over* K. If L is an algebraic number field, *i.e.*, $K = \mathbb{Q}$, then the degree $[L : \mathbb{Q}]$ is simply called the *degree of* L.

We can identify a finite separable extension $L = K(a)$ with $K[x]/\operatorname{Irr}(a, K, x)$, *i.e.*, $K(a)$ is K-isomorphic to $K[x]/\operatorname{Irr}(a, K, x)$. Therefore, it is easy to see that the degree of $L = K(a)$ equals the degree of the minimal polynomial $\operatorname{Irr}(a, K, x)$ of a over K.

Definition 3.14. An extension L of K is called a *splitting field* of $f(x) \in K[x]$ over K if $f(x)$ splits in $K[x]$ into linear factors, *i.e.*, $f(x) = c \cdot (x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n)$ with $c \in K$ and $\alpha_i \in L$ $(1 \leq i \leq n)$, and such that $L = K(\alpha_1, \ldots, \alpha_n)$ is generated by the roots of f. An extension L of K is said to be a *normal extension* of K if it is algebraic over K and if every irreducible polynomial in $K[x]$, which has one root in L, splits into linear factors in K.

We obtain the following existence and uniqueness statement for splitting fields.

Lemma 3.15. [253, Theorem I.15] *Every nonconstant polynomial* $f(x) \in K[x]$ *has a splitting field over* K*, and any two splitting fields of* $f(x)$ *over* K *are* K*-isomorphic.* □

Furthermore, splitting fields and normal extensions are related by the following statements.

Lemma 3.16. [253, Theorem I.16] *Let* L *be a splitting field over* K *of a polynomial* $f(x) \in K[x]$*. Then* L *is normal.* □

Lemma 3.17. [253, Theorem I.17] *Let* L *be a finite normal extension of* K*. Then* L *is a splitting field over* K *of some polynomial in* $K[x]$*.* □

The following statement is implied in [253, Theorem I.20].

Lemma 3.18. *Let* L *be a finite separable extension of* K *and let* F *be a normal extension of* K *such that* $K \subset L \subset F$*. Then there are exactly* $[L : K]$ *distinct* K*-isomorphisms of* L *onto subfields of* F*.* □

Definition 3.19. An algebraic extension L of a field K is called a *Galois extension* if it is normal and separable (in the setting of Lemma 3.18, we have $F = L$). The group of K-isomorphisms of L is called the *Galois group* of L over K, and is denoted by $\operatorname{Gal}(L/K)$ (obviously, the Galois group always contains at least the identity). The elements of the Galois group, *i.e.*, the K-isomorphisms, are called *Galois automorphisms*. Furthermore, let $a, b \in L$. If $\operatorname{Irr}(a, K, x) = \operatorname{Irr}(b, K, x)$, then a and b are said to be *conjugate* (or *algebraic conjugate* or *Galois conjugate*) over K. In this case, there is a K-isomorphism $\sigma \in \operatorname{Gal}(L/K)$ such that $\sigma(a) = b$.

Remark 3.20. If $K = \mathbb{Q}$ and $L = \mathbb{Q}(a)$ is an algebraic number field, let F be the splitting field of $\mathrm{Irr}(a, \mathbb{Q}, x)$. Then, by Lemma 3.16, F is normal, and, by Lemma 3.18, there are exactly $n = [\mathbb{Q}(a) : \mathbb{Q}]$ different $\mathbb{Q}$-isomorphisms of $\mathbb{Q}(a)$ into F, namely, $\sigma_i \in \mathrm{Gal}(F/\mathbb{Q})$ $(1 \leq i \leq n)$. If $\mathrm{Irr}(a, \mathbb{Q}, x)$ splits as $(x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n)$ in F (where we can order the roots such that $\alpha_1 = a$), then (by an appropriate permutation) the Galois automorphism σ_i is given by sending a to α_i. By the definition of an algebraic number field, F is a subfield of the complete and algebraically closed field of complex numbers $\mathbb{C}$ (which is the completion of $\mathbb{Q}^a$ with respect to the usual absolute value). Consequently, there are n different embeddings of $\mathbb{Q}(a)$ into $\mathbb{C}$ which are equal to the identity on $\mathbb{Q}$. They are given by the embeddings of $\sigma_i(\mathbb{Q}(a))$ $(1 \leq i \leq n)$ into $\mathbb{C}$. By abuse of notation, we again denote these embeddings by σ_i $(1 \leq i \leq n)$. Moreover, if we speak of the "Galois automorphisms of an algebraic number field $\mathbb{Q}(a)$", we are actually referring to the Galois automorphisms of the associated splitting field F over $\mathbb{Q}$.

Definition 3.21. Let $\mathbb{Q}(a)$ be an algebraic number field of degree n and σ_i be its n different Galois automorphisms $(1 \leq i \leq n)$. For each $1 \leq i \leq n$, we say that $\sigma_i(\mathbb{Q}(a)) = \mathbb{Q}(\sigma_i(a))$ is a *field conjugated to the field* $\mathbb{Q}(a)$. Furthermore, we say that two elements $b, b' \in \mathbb{C}$ are *conjugate* if there exists a Galois automorphism σ (of $\mathbb{Q}(a)$, *i.e.*, of the splitting field associated to $\mathbb{Q}(a)$ over $\mathbb{Q}$) such that $\sigma(b) = b'$ (considered as embeddings into $\mathbb{C}$). We say that an embedding σ is *real* if $\sigma(\mathbb{Q}(a))$ is contained in the real field $\mathbb{R}$, otherwise it is called *complex*. Let $\sigma : \mathbb{Q}(a) \to \mathbb{C}$ be a complex embedding. Then we define the *complex conjugate embedding* $\overline{\sigma}$ *of* σ by $\overline{\sigma}(b) = \overline{\sigma(b)}$ for all $b \in \mathbb{Q}(a)$, where $\overline{z} = x - iy$ denotes the complex conjugate of $z = x + iy \in \mathbb{C}$. Obviously, $\overline{\sigma}$ is also a complex embedding of $\mathbb{Q}(a)$ different from σ. Therefore, the complex embeddings come in pairs of complex conjugate embeddings $\sigma, \overline{\sigma}$. We denote the numbers of real embeddings by r and half the number of complex embeddings by s. Then $r + 2s = n$, and the pair $[r, s]$ is called the *signature* of the algebraic number field $\mathbb{Q}(a)$. Usually, one fixes the order of the embeddings $\sigma_1, \ldots, \sigma_n$ in such a way that $\sigma_1, \ldots, \sigma_r$ are real (and, if a is a real number, such that σ_1 coincides with the identity on $\mathbb{Q}(a)$), the remaining embeddings are complex and, moreover, one uses the order $\sigma_{r+i} = \overline{\sigma}_{r+s+i}$ for $1 \leq i \leq s$.

We now turn our attention to rings and their quotient fields.

Definition 3.22. Let R be a commutative ring and a an element of some field containing R. We shall say that a is *integral* over R if there is a monic nonzero polynomial $f(x) \in R[x]$ with $f(a) = 0$.

Lemma 3.23. [224, Prop. I.1] and [226, Prop. VII.1.1] *Let R be an integral domain with* quotient field K, *i.e.*, $K = \{\frac{a}{b} \mid a \in R, b \in R \setminus \{0\}\}$. *Then for every algebraic element $a \in L$ over K, there exists a $c \in R$ such that ca is integral over R.* □

Definition 3.24. An integral domain (or entire ring) S is the *integral closure* of an integral domain R in a field L, if $S = \{a \in L \mid a \text{ is integral over } R\}$. An integral domain R is said to be *integrally closed in a field K* if every element of K which is integral over R in fact lies in R. It is said to be *integrally closed* if it is integrally closed in its quotient field.

Definition 3.25. The integral closure of $\mathbb{Z}$ in a number field K is called the ring of *algebraic integers* of K, and it is denoted by $\mathfrak{o}_K$, *i.e.*, $\mathfrak{o}_K = \{a \in K \mid a \text{ is integral over } \mathbb{Z}\}$. An invertible element ε of $\mathfrak{o}_K$ (*i.e.*, one also has $\varepsilon^{-1} \in \mathfrak{o}_K$) is called a *unit* of $\mathfrak{o}_K$.

We note that, for every algebraic number field K (say, of signature $[r\,,s]$), the set of units of $\mathfrak{o}_K$ forms a group (more precisely, it is a product of a finite cyclic group and a free group of rank $r+s-1$) by *Dirichlet's unit theorem* (see, *e.g.*, [211, Theorem 1.13] and [73, II.§4.5]).

The following statement about the ring of algebraic integers should serve as motivation for the next section about ideals.

Lemma 3.26. [273, Theorem 1.3] and [211, Theorem 1.34] *The ring $\mathfrak{o}_K$ of algebraic integers of a number field K is a* Dedekind ring, *i.e., integrally closed, Noetherian (i.e., every ideal of $\mathfrak{o}_K$ is finitely generated), and every prime ideal is a maximal ideal.* □

Remark 3.27. We will only deal with finite extensions (with the exception of algebraic closures), but not necessarily with algebraic number fields only.

3.2. Ideals

Definition 3.28. An ideal $\mathfrak{p}$ of a commutative ring R is called *prime* if the quotient $R/\mathfrak{p}$ is an integral domain. An ideal $\mathfrak{m}$ of a ring R is called *maximal* if the quotient $R/\mathfrak{m}$ is a field. A *fractional ideal* $\mathfrak{a}$ of a Dedekind ring R in K (where K is the quotient field of R) is a nonzero R-module $\mathfrak{a}$ contained in K such that there exists an element $0 \neq c \in R$ for which $c\mathfrak{a} \subset R$.

Remark 3.29. If an ideal $\mathfrak{a}$ of a commutative ring R is generated by elements $a_1, \ldots, a_n$, *i.e.*, if $\mathfrak{a} = a_1R + \ldots + a_nR$, we simply write $\mathfrak{a} = (a_1, \ldots, a_n)$ if the underlying ring R is clear.

Remark 3.30. Note that an element $\pi \in R$ is a *prime element*, if $\pi \neq 0$ and $(\pi) = \pi R$ is a prime ideal.

We first clarify the structure of ideals.

Lemma 3.31. [253, Prop. V.8] *Every ideal of a Dedekind ring R can be generated by two of its elements, i.e., if $\mathfrak{a}$ is an ideal of R then there are elements $a, b \in \mathfrak{a}$ such that $\mathfrak{a} = aR+bR = (a,b)$.* □

Definition 3.32. An ideal $\mathfrak{a}$ of a ring R is called *principal* if it is generated by one element a, *i.e.*, if $\mathfrak{a} = aR = (a)$ for some a. A commutative ring (with $1 \neq 0$) is called a *principal ring* if every ideal is principal. A principal integral domain is called a *principal ideal domain*.

Remark 3.33. If R is a Dedekind ring with finitely many prime ideals, then R is a principal ideal domain, see [211, Proposition 1.40].

Denote by $\mathbb{P}_R$ the set of all prime ideals $\mathfrak{p} \neq \{0\}$ of R, and by $\mathfrak{I}_R$ the set of all fractional ideals of R. Furthermore, define the multiplication of (fractional) ideals $\mathfrak{a}$ and $\mathfrak{b}$ by $\mathfrak{a} \cdot \mathfrak{b} = \{\text{ finite sums } \sum a_i b_i \,|\, a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\}$, and the inverse $\mathfrak{a}^{-1}$ of an ideal $\mathfrak{a}$ by $\mathfrak{a}^{-1} = \{a \in K \,|\, a \cdot \mathfrak{a} \subset R\}$ (where K is again the quotient field of R). Therefore, we can define arbitrary powers $\mathfrak{a}^n$ ($n \in \mathbb{Z}$), where $\mathfrak{a}^0 = R$. Then, we get in case of a Dedekind ring, the following alternative statements.

Lemma 3.34. [224, Theorem I.2] *Let R be a Dedekind ring. Then every ideal of R can be uniquely factored into prime ideals, and the non-zero fractional ideals form a group under multiplication.* □

Lemma 3.34'. [341, Prop. I.7] and [273, Theorem 1.4] *Every fractional ideal* $\mathfrak{a}$ *in a Dedekind ring* R *possesses a unique factorisation*

$$\mathfrak{a} = \prod_{\mathfrak{p} \in \mathbb{P}_R} \mathfrak{p}^{n_\mathfrak{p}}$$

where $n_\mathfrak{p} \in \mathbb{Z}$ *and all but a finite number of which are zero.* □

Lemma 3.34". [273, Corollary 1 of Theorem 1.4] *Let* R *be a Dedekind ring. Then the set of fractional ideals* $\mathfrak{I}_R$ *is the free Abelian group generated by* $\mathbb{P}_R$*.* □

The following statement shows how we can obtain new Dedekind rings.

Lemma 3.35. [211, Theorem I.29] and [341, Prop. I.9] *Let* R *be a Dedekind ring with quotient field* K*. Let* L *be a finite extension of* K *and* S *the integral closure of* R *in* L*. Then* S *is a Dedekind ring.* □

We recall the definition of norm, trace (of an element of a field extension) and discriminant (in $\operatorname{char} K = 0$).

Definition 3.36. Let L be a finite extension of K with $n = [L : K]$. Then there exists a finite normal extension F of K such that $K \subset L \subset F$, and there are exactly n K-isomorphisms $\sigma_1, \dots, \sigma_n$ of L onto subfields of F, see Lemma 3.18. For $a \in L$ we define the *norm of* a to be

$$N_{L/K}(a) = \prod_{i=1}^{n} \sigma_i(a),$$

and the *trace of* a to be

$$T_{L/K}(a) = \sum_{i=1}^{n} \sigma_i(a).$$

If $\operatorname{Irr}(a, K, x) = x^r + c_{r-1}x^{r-1} + \ldots + c_0$ is the minimal polynomial of a over K then $N_{L/K}(a) = (-1)^n \cdot c_0^{n/r}$ and $T_{L/K} = -\frac{n}{r} \cdot c_{r-1}$.

If K is an algebraic number field, then $a \in \mathfrak{o}_K$ is a unit iff $N_{K/\mathbb{Q}}(a) = \pm 1$, see [73, Satz II.§2.4].

Definition 3.37. Let R, S be Dedekind rings with quotient fields K, L respectively, where K is a finite extension of L. Let $\omega_1, \dots, \omega_n$ be a basis of S over K. Then,

$$d_{L/K} = \det(T_{L/K}(\omega_i \omega_j)_{ij})$$

is called the *discriminant of* L. For $K = \mathbb{Q}$, we simply write d_L (instead of $d_{L/\mathbb{Q}}$). In general, for a basis $\omega_1, \dots, \omega_n$ over K in L, $d_{L/K}(\omega_1, \dots, \omega_n) = \det(T_{L/K}(\omega_i \omega_j)_{ij})$ is the *discriminant of that basis.*

Remark 3.38. An extension L/K has nonzero discriminant iff L/K is separable, see [253, Theorem I.29].

We also recall some properties of norm and trace (see [253, Theorem I.27 & Corollary 1 to Theorem I.26] and [273, Prop. I.2.2]).

Lemma 3.39.
- *Both* $N_{L/K}(a)$ *and* $T_{L/K}(a)$ *are elements of* K*.*

- *For all $a, b \in L$ we have $N_{L/K}(a \cdot b) = N_{L/K}(a) \cdot N_{L/K}(b)$ and $T_{L/K}(a+b) = T_{L/K}(a) + T_{L/K}(b)$.*
- *If $a \in K$, then $N_{L/K}(a) = a^{[L:K]}$ and $T_{L/K}(a) = [L : K] \cdot a$.* □

We now turn our attention to ideals.

Definition 3.40. An ideal $\mathfrak{a}$ *divides* $\mathfrak{b}$, if there exists an ideal $\mathfrak{c}$ such that $\mathfrak{a} \cdot \mathfrak{c} = \mathfrak{b}$. We write $\mathfrak{a} \mid \mathfrak{b}$. This is equivalent to $\mathfrak{b} \subset \mathfrak{a}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are *relatively prime*, if there is no $\mathfrak{p} \in \mathbb{P}_R$ with $\mathfrak{p} \mid \mathfrak{a}$ and $\mathfrak{p} \mid \mathfrak{b}$. This is equivalent to $\gcd(\mathfrak{a}, \mathfrak{b}) = R$, where we define the *greatest common divisor (gcd) of two ideals* $\mathfrak{a}$ and $\mathfrak{b}$ to be $\gcd(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$.

Remark 3.41. For the Dedekind ring $\mathbb{Z}$, the ideals are given by $n \cdot \mathbb{Z}$ (where $n \in \mathbb{N}$) and the prime ideals by $p \cdot \mathbb{Z}$ with $p \in \mathbb{P}$. This correspondence of integer numbers and ideals justifies the use of the words "divides", "relatively prime" and "gcd" for ideals.

Definition 3.42. Let R be a Dedekind ring with quotient field K, and L a finite extension of K. We denote by S the integral closure of R in L. Let $\mathfrak{p}$ be a prime ideal in K with prime ideal factorisation $\mathfrak{p}\mathfrak{o}_L = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i}$ in L. Then, $e_i = e_{\mathfrak{P}_i} = e_{\mathfrak{P}_i|\mathfrak{p}}$ is called the *ramification index* of $\mathfrak{P}_i$ in the extension L/K. Furthermore, we define by $f_i = f_{\mathfrak{P}_i} = f_{\mathfrak{P}_i|\mathfrak{p}} = [S/\mathfrak{P}_i : R/\mathfrak{p}]$ the *residue degree* of $\mathfrak{P}_i$ in L/K.

Remark 3.43. One can consider $S/\mathfrak{P}_i$ as (finite) field extension of the (finite) field $R/\mathfrak{p}$. If not otherwise stated, we always assume that the Dedekind domain R is such that $R/\mathfrak{p}$ is a finite field for every prime ideal $\mathfrak{p}$.

We now define the ideal norm, respectively the (absolute) norm of an ideal.

Definition 3.44. For a prime ideal $\mathfrak{p}$ in a Dedekind domain R (with field of fractions K), we define the *(absolute) norm $N\mathfrak{p}$ of $\mathfrak{p}$* as the cardinality of the (by our convention) finite field $R/\mathfrak{p}$. We extend this definition multiplicatively to $\mathfrak{I}_K$. We define *ideal norm* $\mathfrak{N}_{L/K}$ in a finite extension L of K as follows: For every prime ideal $\mathfrak{P}$ in L set $\mathfrak{N}_{L/K}(\mathfrak{P}) = \mathfrak{p}^{f_{\mathfrak{P}|\mathfrak{p}}}$ and also extend this definition multiplicatively to $\mathfrak{I}_L$.

The following hold for the (absolute) norm of an ideal in an algebraic number field (which follow from statements about the index of a complete module, see Definition 3.88 and compare Remark 6.38), see [273, Theorem I.1.6 and Prop. II.2.7]

Lemma 3.45. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals in an algebraic number field L of degree n.*

- *$N\mathfrak{a} = [\mathfrak{o}_L : \mathfrak{a}]$ (this is well defined since both are free $\mathbb{Z}$-modules of rank n).*
- *$N\mathfrak{a} \in \mathfrak{a}$ and $\{0\} \neq \mathfrak{a} \cap \mathbb{Q} \subset \mathbb{Z}$.*
- *$N(\mathfrak{a} \cdot \mathfrak{b}) = N\mathfrak{a} \cdot N\mathfrak{b}$.*
- *If $\mathfrak{a}$ is generated by a, i.e., $\mathfrak{a} = a\mathfrak{o}_L = (a)$, then $N\mathfrak{a} = |N_{L/\mathbb{Q}}(a)|$.*
- *If $a_1, \ldots, a_n$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$, then $d(a_1, \ldots, a_n) = (N\mathfrak{a})^2 \cdot d_L$. In fact, if M is a free Abelian group (a free $\mathbb{Z}$-module) of rank n and a subgroup of $\mathfrak{o}_L$ generated by the $\mathbb{Z}$-basis $a_1, \ldots, a_n$, then one has $d_L(a_1, \ldots, a_n) = ([\mathfrak{o}_L : M])^2 \cdot d_L$.* □

The next statement gives us control on the behaviour of prime ideals in extensions, see [273, Prop. 4.2].

Lemma 3.46. *Let R be a Dedekind ring with quotient field K and L a finite extension of K with integral closure S. If $\mathfrak{P} \in \mathbb{P}_L$ is a prime ideal of S, then there exists exactly one prime ideal $\mathfrak{p} \in \mathbb{P}_K$ of R lying below $\mathfrak{P}$. Moreover, we have the following equivalent conditions:*

(i) $\mathfrak{p} = \mathfrak{P} \cap R$.

(ii) $\mathfrak{P} \mid \mathfrak{p}S$, *i.e.*, $\mathfrak{p}S \subset \mathfrak{P}$.

(iii) $\mathfrak{N}_{L/K}(\mathfrak{P}) = \mathfrak{p}^f$ *for some* $f \in \mathbb{N}$.

In case of an algebraic number field, i.e., $K = \mathbb{Q}$, $R = \mathbb{Z}$ and $S = \mathfrak{o}_L$, we have the equivalence of the properties $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$, $\mathfrak{P} \mid (p) = pS$, $p \in \mathfrak{P}$ and $N\mathfrak{P} = p^f$ for some $f \in \mathbb{N}$. □

The ramification indices and the residue degrees are connected by the following equality.

Lemma 3.47. [341, Chap. I, Prop. 10], [150, Theorem 5-1-3], [224, Chap. 1, Prop. 21] and [273, Theorem 4.1] *Let L be a finite extension of K. Then for every prime ideal $\mathfrak{p}$ in K and prime ideals $\mathfrak{P}$ of L we have*

$$[L : K] = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}|\mathfrak{p}} \cdot f_{\mathfrak{P}|\mathfrak{p}}. \qquad \square$$

An immediate consequence is the following Corollary.

Corollary 3.48. [273, Corollaries 1& 2 to Theorem 4.1] and [150, Corollary 5-1-4] *There are at most n primes $\mathfrak{P}$ above $\mathfrak{p}$, and $e_{\mathfrak{P}|\mathfrak{p}} \leq n$ and $f_{\mathfrak{P}|\mathfrak{p}} \leq n$.*

Remark 3.49. For primes $p \in \mathbb{P}$, the factorisation of (p) in a given quadratic extension L of $\mathbb{Q}$ ($[L : \mathbb{Q}] = 2$) gives rise to p being ramified ($2 = 2 \cdot 1$, *i.e.*, $p\mathfrak{o}_L = \mathfrak{P}^2$), splitting ($2 = 1 \cdot 1 + 1 \cdot 1$, *i.e.*, $p\mathfrak{o}_L = \mathfrak{P}_1 \cdot \mathfrak{P}_2$) or inert ($2 = 1 \cdot 2$, *i.e.*, $p\mathfrak{o}_L = \mathfrak{P}$).
We also remark that the ramification index of a prime $\mathfrak{P}$ above $\mathfrak{p}$ is also connected to the prime factorisation of the different.

Definition 3.50. Let R be a Dedekind domain with field of fractions K and L a finite separable extension of K with algebraic closure S. Let $A \subset L$ be a nonzero R-module. The set $A^\wedge = \{x \in L | T_{L/K}(xA) \subset R\}$ will be called the *codifferent of A over K*. It is an R-module (maybe even the zero-module for large A, *e.g.*, $A = L$). If $\mathfrak{a}$ is an arbitrary fractional ideal in L, then $(\mathfrak{a}^\wedge)^{-1}$ (which is also a fractional ideal, see the following lemma) will be called the *different of $\mathfrak{a}$ over K*. We denote it by $\mathfrak{D}_{L/K}(\mathfrak{a})$. The different of S will be called the *different of the extension L/K* and shall be denoted simply by $\mathfrak{D}_{L/K}$.

We collect some statements about the different and the codifferent, see [273, Propositions 4.6 & 4.8 & 4.11 & 4.12], [224, Propositions III.1 & III.2] and [341, Corollary 2 to Proposition III.11].

Lemma 3.51. *Let R, S, K, L be as before and $n = [L : K]$.*

(i) *If $\mathfrak{a}$ is a fractional ideal in L, then its codifferent $\mathfrak{a}^\wedge$ is also a fractional ideal in L and $\mathfrak{a} \cdot \mathfrak{a}^\wedge = S^\wedge$. Moreover, if $\mathfrak{a}$ is an ideal in S, then $(\mathfrak{a}^\wedge)^{-1}$ is also an ideal in S.*

(ii) *Assume that* $K = \mathbb{Q}$. *If* $\mathfrak{a}$ *is a fractional ideal in* L *with a* $\mathbb{Z}$*-basis* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *are defined by*

$$T_{L/K}(a_i \cdot b_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

then the $\mathbb{Z}$*-module generated by* $b_1, \ldots, b_n$ *equals* $\mathfrak{a}^\wedge$. *We call* $\{b_i\}$ *the* dual $\mathbb{Z}$-basis *relative to the trace of* $\mathfrak{a}$. *In fact, this statement already holds if we replace* $\mathfrak{a}$ *by a complete module* A *in the sense of Definition 3.88.*

(iii) *If* $K = \mathbb{Q}$ *and* $\mathfrak{a}$ *is a fractional ideal, then* $N\mathfrak{D}_{L/\mathbb{Q}}(\mathfrak{a}) = N\mathfrak{a} \cdot |d_L|$; *in particular* $N\mathfrak{D}_{L/\mathbb{Q}} = |d_L|$.

(iv) *Let* $L = K(a)$ *and* $f(x) = \mathrm{Irr}(a, K, x)$. *If*

$$\frac{f(x)}{x - a} = b_0 + b_1 \cdot x + \ldots + b_{n-1} \cdot x^{n-1},$$

then the dual $\mathbb{Z}$*-basis relative to the trace of the module generated by the* $\mathbb{Z}$*-basis* $1, a, \ldots, a^{n-1}$ *is given by* $b_0/f'(a), \ldots, b_{n-1}/f'(a)$, *where* f' *denotes the formal derivative of the polynomial* f.

(v) *If* $A = R[a]$ *and* $f(x) = \mathrm{Irr}(a, K, x)$, *then the codifferent* $A^\wedge$ *is generated as an* R*-module by the set* $a^j/f'(a)$, *where* $j \in \{0, \ldots, n-1\}$.

(vi) *If* $L = K(a)$, *then* $\mathfrak{D}_{L/K}$ *divides the principal ideal* $(f'(a))$. *We have* $\mathfrak{D}_{L/K} = (f'(a))$ *iff* $S = R[a]$. *More general, we have* $(f'(a)) = \mathfrak{D}_{L/K} \cdot \mathfrak{F}_{R[a]}$, *where*[1] $\mathfrak{F}_{R[a]} = \{x \in R[a] \,|\, xS \subset R[a]\}$. □

We finally make the connection between the ramification index of a prime ideal $\mathfrak{P}$ in S and the different.

Lemma 3.52. [224, Chap. III Prop. 8], [341, Proposition 13], [211, Theorem 1.44] and [273, Theorem 4.8] *Let* R, S, K, L *be as above. Let* $\mathfrak{P}$ *be a prime of* S *lying above* $\mathfrak{p}$, *and let* $e = e_{\mathfrak{P}|\mathfrak{p}}$ *be its ramification index. Then,* $\mathfrak{P}$ *is ramified (i.e.,* $e > 2$*) iff* $\mathfrak{P}$ *divides* $\mathfrak{D}_{L/K}$; *more precisely,* $\mathfrak{P}^{e-1}$ *divides* $\mathfrak{D}_{L/K}$. *Moreover,* $\mathfrak{P}^e$ *does not divide* $\mathfrak{D}_{L/K}$ *iff the number* e *is relatively prime to characteristic of the field* $R/\mathfrak{p}$ *(or, which means the same, to characteristic of* $S/\mathfrak{P}$, *or to* $N\mathfrak{p}$ *or to* $N\mathfrak{P}$*). Finally, there are only finitely many prime ideals* $\mathfrak{P}$ *in* S *which are* ramified *over* $\mathfrak{p} = \mathfrak{P} \cap R$, *i.e., prime ideals* $\mathfrak{P}$ *with* $e = e_{\mathfrak{P}|\mathfrak{p}} > 1$. □

3.3. Valuation Rings

We now define discrete valuation rings (see [341, Chap. I, §1]).

Definition 3.53. A ring R is called a discrete *valuation ring* if it is a principal ideal domain that has a unique non-zero prime ideal $\mathfrak{m}(R)$. The field $R/\mathfrak{m}(R)$ is called the *residue field* of R. The invertible elements of R are given by $R \setminus \mathfrak{m}(R) = R^\times$, $R^\times$ forms the (multiplicative)

[1] Let K, L, R, S be as above. Then for a ring A where $R \subset A \subset S$, $\mathfrak{F}_A$ denotes the *conductor* of A, *i.e.*, the greatest common divisor of ideals of S contained in A and one has $\mathfrak{F}_A = \{x \in L \mid xA^\wedge \subset S^\wedge\}$ (see [273, Prop. 4.10]).

group of *units* of R. By uniqueness of $\mathfrak{m}(R)$, we have $\mathfrak{m}(R) = \pi R$, where π is prime (unique up to multiplication of an element in $R^\times$). Such an element is called a *uniformising element* or *uniformiser*.

Definition 3.54. The non-zero ideals of a discrete valuation ring R are of the form $\pi^n R$. If $0 \neq x \in R$, one can write $x = \pi^n u$ with $n \in \mathbb{N}$ and $u \in R^\times$. The integer n is called the *valuation* (or the *exponential valuation* or the *order*) of x and denoted $v(x)$ (it does not depend on the choice of π). One can extend v to the field K of fractions of R by $v(\frac{a}{b}) = v(a) - v(b)$ and $v(0) = +\infty$.

Remark 3.55. We can also take the following axiomatic approach: Let K be a field, and let $v : K^\times \to \mathbb{Z}$ be a surjective homomorphism with $v(x + y) \geq \min(v(x), v(y))$ (with the convention $v(0) = +\infty$). Then the set $R = \{x \in K \mid v(x) \geq 0\}$ is a discrete valuation ring, having v as its associated valuation. Let π be an (any) element such that $v(\pi) = 1$, then π is a uniformiser (see [273, Theorem 1.10]).

The following statement describes a method how one can obtain discrete valuation rings from Dedekind rings, see [341, Prop. I.4].

Lemma 3.56. *For a Dedekind ring R set $R_\mathfrak{p} = \{\frac{a}{b} \mid a \in R, b \in R \setminus \mathfrak{p}\}$ for $\mathfrak{p} \in \mathbb{P}_R$. Then $R_\mathfrak{p}$ is a discrete valuation ring.* □

Explicitly, a valuation in the field of fractions of a Dedekind ring R is given by the exponent in the prime ideal factorisation (see [341, Prop. I.7], also compare to Lemma 3.34'):

$$v_\mathfrak{p}(x) = v_\mathfrak{p}(xR) = v_\mathfrak{p}((x)) \quad \text{where} \quad \mathfrak{a} = \prod_{\mathfrak{p} \in \mathbb{P}_R} \mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})}$$

for $x \neq 0$. Therefore, we have $\mathfrak{m}(R_\mathfrak{p}) = \{\frac{a}{b} \mid a \in \mathfrak{p},\, b \in R \setminus \mathfrak{p}\}$ and $R = \bigcap_{\mathfrak{p} \in \mathbb{P}_R} R_\mathfrak{p}$. Note that by $R_\mathfrak{p}/\mathfrak{m}(R_\mathfrak{p}) \cong R/\mathfrak{p}$ the residue field has $p^{f_{\mathfrak{p}|(p)}}$ elements in the case of algebraic number fields (where $\mathfrak{p}$ is a prime ideal above p) and therefore is isomorphic to $\mathbb{F}_{p^f}$ (where $f = f_{\mathfrak{p}|(p)}$), the finite field with p^f elements.
This shows, that every prime ideal $\mathfrak{p} \in \mathbb{P}_K$ yields a valuation of K, the $\mathfrak{p}$-*adic valuation* $v_\mathfrak{p}$. For an algebraic number field, these are all valuations by the following statement of Ostrowski, also see Lemma 3.57' on p. 31.

Lemma 3.57. [150, Theorem 3-1-5], [273, Theorem 3.2] and [211, Prop. 1.66] *Let L be an algebraic number field of degree n over $\mathbb{Q}$. If v is a (exponential) valuation of K, then there exists a prime ideal $\mathfrak{p}$ of the ring $S = \mathfrak{o}_L$ of integers of L such that $S_\mathfrak{p} = \{x \in L \mid v(x) \geq 0\}$. Conversely, every prime ideal of L defines in such a way an (exponential) valuation. Valuations defined by different prime ideals are non-equivalent.* □

We now compare valuations associated to prime ideals in extension fields.

Definition 3.58. Let R be a Dedekind ring with quotient field K and L a finite extension of K with integral closure S. For $\mathfrak{P} \in \mathbb{P}_L$, we have $\mathfrak{p} = \mathfrak{o}_K \cap \mathfrak{P}$. Therefore $v_\mathfrak{P}(x) = e_{\mathfrak{P}|\mathfrak{p}} \cdot v_\mathfrak{p}(x)$ if $x \in K$. We say, that the valuation $v_\mathfrak{P}$ *prolongs* the valuation $v_\mathfrak{p}$ with index $e_{\mathfrak{P}|\mathfrak{p}}$ (we denote this by $v_\mathfrak{P} \mid v_\mathfrak{p}$).

Conversely, we have the following.

Lemma 3.59. [341, Prop. I.11] *If w is a discrete valuation of L that prolongs $v_{\mathfrak{p}}$ with index e, then there exists a prime divisor $\mathfrak{P}$ of $\mathfrak{p}$ with $w = v_{\mathfrak{P}}$ and $e = e_{\mathfrak{P}|\mathfrak{p}}$.* □

We set $\mathfrak{o}_{v_{\mathfrak{p}}} = R_{\mathfrak{p}}$ and $\mathfrak{m}_{v_{\mathfrak{p}}} = \mathfrak{m}(R_{\mathfrak{p}})$. Obviously, we have $\mathfrak{o}_{v_{\mathfrak{p}}} = \mathfrak{o}_{v_{\mathfrak{P}}} \cap K$, $\mathfrak{m}_{v_{\mathfrak{p}}} = \mathfrak{m}_{v_{\mathfrak{P}}} \cap K$, $f_{v_{\mathfrak{P}}|v_{\mathfrak{p}}} = [\mathfrak{o}_{v_{\mathfrak{P}}}/\mathfrak{m}_{v_{\mathfrak{P}}} : \mathfrak{o}_{v_{\mathfrak{p}}}/\mathfrak{m}_{v_{\mathfrak{p}}}] \cong [S/\mathfrak{P} : R/\mathfrak{p}] = f_{\mathfrak{P}|\mathfrak{p}}$ and $e_{v_{\mathfrak{P}}|v_{\mathfrak{p}}} = [\mathbb{Z} : v_{\mathfrak{P}}(K^{\times})]$. So there is a 1-1-correspondence between the prime factors of $\mathfrak{p}$ in L and the prolongations of $v_{\mathfrak{p}}$ in L.

3.4. Completions

We continue with [341, Chap. II §1] and [224, Chap. II].

Definition 3.60. Let K be a field. An *absolute value*[2] on K is a real valued function $x \mapsto |x|_v$ on K satisfying the following three properties:

- We have $|x|_v \geq 0$ and $|x|_v = 0$ iff $x = 0$.
- For all $x, y \in K$ we have $|x \cdot y|_v = |x|_v \cdot |y|_v$.
- For all $x, y \in K$ we have $|x + y|_v \leq |x|_v + |y|_v$.

If instead of this last condition the absolute value satisfies the stronger condition

- $|x + y|_v \leq \max\{|x|_v, |y|_v\}$ for all $x, y \in K$,

then we shall say that it is a *non-Archimedean absolute value*[3] (see [224, Chap. II §1.]) or an *ultrametric absolute value* (see [341, Chap. II §1]) . The absolute value which is such that $|x|_v = 1$ for all $x \neq 0$ is called *trivial.*

Remark 3.61. We fix a prime number $p \in \mathbb{P}$. On the field of rational numbers $\mathbb{Q}$ (with $\mathfrak{o}_{\mathbb{Q}} = \mathbb{Z}$ and $\mathbb{P} = \mathbb{P}_{\mathbb{Z}}$), we define the *p-adic valuation* v_p of $x \in \mathbb{Q}^{\times}$ by $x = p^{v_p(x)}\frac{a}{b}$ where $p \not| (a \cdot b) \in \mathbb{Z} \setminus \{0\}$. Then, for any $x \in \mathbb{Q}$, we can also define the *p-adic absolute value* of x by $|x|_p = p^{-v_p(x)} = \left(\frac{1}{p}\right)^{v_p(x)}$ if $x \neq 0$ and $|0|_p = 0$ (the choice $\|x\|_p = \eta^{v_p(x)}$ with some $0 < \eta < 1$ gives an equivalent metric). This is a non-Archimedean absolute value (note that $\sup\{|x|_p \mid x \in \mathbb{Z}\} = 1$). The same construction to obtain absolute values works for all $\mathfrak{p}$-adic valuations, as we will see next.

Given a $\mathfrak{p}$-adic (exponential) valuation $v_{\mathfrak{p}}$, we easily obtain an non-Archimedian absolute value: If η is any real number between 0 and 1, put

$$\|x\|_{\mathfrak{p}} = \eta^{v_{\mathfrak{p}}(x)} \quad \text{for} \quad x \neq 0$$

and $\|0\|_{\mathfrak{p}} = 0$. For an algebraic number field K, the following choices of η are most often used, where $p \in \mathfrak{p}$:

[2]Sometimes, *e.g.*, in [211, 253, 273], the word "valuation" is used instead of "absolute value".

[3]An absolute value is called *Archimedean* if for all $x, y \in K^{\times}$ there is a natural number n such that $|n \cdot x|_v > |y|_v$. All other absolute values are called *non-Archimedean.* In this last case, an equivalent condition is $|x + y|_v \leq \max\{|x|_v, |y|_v\}$ for all $x, y \in K$. We also note that in [150, Definition 3-1-1] absolute values are defined with the third condition replaced by the statement: There exists a constant $C > 0$ such that $|x + y|_v \leq C \cdot \max\{|x|_v, |y|_v\}$ for all $x, y \in K$. Furthermore, we note that in [91, Definition 2.1.1] the third condition is replaced by: There exists a constant $C' > 0$ such that $|x|_v \leq 1$ implies $|x+1|_v \leq C'$. Then it is a non-Archimedean absolute value iff one can choose $C = 1$ (or $C' = 1$), otherwise it is called an Archimedean (note, however, that the triangle inequality, *i.e.*, $C = 2$ respectively $C' = 2$, does not necessarily hold with this definition, see remark in Definition 3.62). Consequently, we can equivalently call an absolute value $|\cdot|_v$ Archimedean if $|2|_v > 1$ and non-Archimedean if $|2|_v \leq 1$, see [326, Definition 8.1].

- $\eta = p^{-1/e}$ with $e = e_{\mathfrak{p}|(p)}$. This has the advantage that $\|x\|_p = \|x\|_{\mathfrak{p}}$ for $x \in \mathbb{Q}$, where $\|\cdot\|_p$ denotes the p-adic absolute value, compare Definition 3.58. This choice is used in Lemma 3.75.
- $\eta = p^{-f}$ with $f = f_{\mathfrak{p}|(p)}$. This is the so-called *normalised* absolute value. Normalised absolute values have some nice properties (see Lemma 3b.11), and are especially used if one considers the completion of K with respect to the metric induced by this absolute value (see Proposition 4.38).

Definition 3.62. Let K be a field. An absolute value $|\cdot|_v$ on K induces a metric $d(x,y) = |x-y|_v$ under which K becomes a topological field. Two absolute values in K are called *equivalent* if they define the same topology in K. Two absolute values $|\cdot|_{v_1}, |\cdot|_{v_2}$ of K are equivalent iff there is a positive number $c > 0$ such that $|x|_{v_1} = |x|_{v_2}^c$ for all $x \in K$. Note that for a non-Archimedean absolute value $|\cdot|_v$, $x \mapsto |x|_v^c$ with $c > 0$ still defines a non-Archimedean absolute value, while the same holds for $0 < c < 1$ in the Archimedean case (if $c > 1$, then the triangle inequality does not necessarily hold, but see Footnote 3 on p. 30).

We can now reformulate and give the complete statement of Ostrowski, which characterises all absolute values of an algebraic number field up to equivalence.

Lemma 3.57'. *Let L be an algebraic number field of degree n over $\mathbb{Q}$. If $|\cdot|_v$ is a non-Archimedean absolute value of L, then there exists a prime ideal $\mathfrak{p}$ of the ring $\mathfrak{o}_L$ of integers of L such that $|x|_v = \eta^{v_{\mathfrak{p}}(x)}$ for an η with $0 < \eta < 1$, where $v_{\mathfrak{p}}$ denotes the valuation induced by $\mathfrak{p}$. If $|\cdot|_v$ is an Archimedean absolute value of L, then we have $|x|_v = |\sigma(x)|$, where $\sigma \in \mathrm{Gal}(F/L)$ with Galois extension F of L and $|\cdot|$ denotes the usual complex absolute value. Conversely, every prime ideal of $\mathfrak{o}_L$ defines in such a way an absolute value on L which is non-Archimedean, and every $\sigma \in \mathrm{Gal}(F/L)$ (i.e., every embedding of L into $\mathbb{C}$) defines an Archimedean absolute value. Absolute values defined by different prime ideals are non-equivalent, and two absolute values defined by different embeddings σ_1, σ_2 of L into $\mathbb{C}$ are equivalent iff those embeddings are complex conjugate, i.e., if $\sigma_1(x) = \overline{\sigma_2(x)}$ for all $x \in L$ (where, as usual, $\overline{a+ib} = a - ib$ for $a, b \in \mathbb{R}$ denotes complex conjugation). Finally, these are (up to equivalence) all nontrivial absolute values of L.* □

Definition 3.63. The completions of an algebraic number field K are called *local fields*. A completion with respect to an Archimedean absolute value is either isomorphic to $\mathbb{R}$ (if K is embedded into $\mathbb{R}$) or to $\mathbb{C}$, both with the ordinary absolute value (see [211, Theorem 1.69]). We say that $\hat{K}$ is a $\mathfrak{p}$-*adic number field*[4] if it is the completion of K with respect to the (non-Archimedean) absolute value $\|\cdot\|_{\mathfrak{p}} = \eta^{v_{\mathfrak{p}}(\cdot)}$ which corresponds to the prime ideal $\mathfrak{p}$ of $\mathfrak{o}_K$. We denote this completion by $\hat{K}$, or $(\hat{K}, \hat{v})$ if we want to emphasise the dependency on the valuation, or by $K_{\mathfrak{p}}$ if we want to emphasise the dependency on the prime ideal. However, by an obvious abuse of notation, we will often denote the valuation and the absolute value on K and on its completion $\hat{K}$ in the same way. We set $\widehat{\mathfrak{o}_{\mathfrak{p}}} = \{x \in K_{\mathfrak{p}} \mid v_{\mathfrak{p}}(x) \geq 0\}$, the ring of $\mathfrak{p}$-*adic integers* (however note, that $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ is *not* the completion of $\mathfrak{o}_{\mathfrak{p}}$, but we have $\widehat{\mathfrak{o}_{\mathfrak{p}}} = \mathfrak{o}_{\hat{K}} = \mathfrak{o}_{K_{\mathfrak{p}}}$). We observe that $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ is a valuation ring, and that $\hat{K} = \widehat{\mathfrak{o}_{\mathfrak{p}}}[\frac{1}{\pi}]$, where π denotes a uniformiser. By abuse of notation, we denote its maximal ideal again by $\mathfrak{p}$, *i.e.*, $\mathfrak{p} = \mathfrak{m}(K_{\mathfrak{p}}) = \{x \in K_{\mathfrak{p}} \mid v_{\mathfrak{p}}(x) > 0\} = \pi\widehat{\mathfrak{o}_{\mathfrak{p}}}$. The residue field $\widehat{\mathfrak{o}_{\mathfrak{p}}}/\mathfrak{p}$ is isomorphic to $\mathbb{F}_{p^f}$, where $p \in \mathfrak{p}$ and $f = f_{\mathfrak{p}|(p)}$. For $K = \mathbb{Q}$ and $\mathfrak{p} = p\mathbb{Z}$, we get the *p-adic numbers* $\mathbb{Q}_p$, the ring of *p-adic integers* $\mathbb{Z}_p = \widehat{\mathfrak{o}_p}$ and residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$.

[4] Note that [150, Definition 4-1-1] uses the term "local field" for what we call "$\mathfrak{p}$-adic number field".

Lemma 3.64. [273, Theorem 5.4] *Let K be a field with a valuation v. The following properties are then equivalent:*

(i) *K is a $\mathfrak{p}$-adic field with the $\mathfrak{p}$-adic valuation.*

(ii) *K is a finite extension of a certain $\mathbb{Q}_p$ (where $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$).*

(iii) *K is complete with respect to v, $\operatorname{char} K = 0$ and the residue field of its valuation ring is finite.* □

Remark 3.65. Note that the set of values assumed by $\|\cdot\|_{\mathfrak{p}}$ on $K_{\mathfrak{p}}$ coincides with the set of values taken by $\|\cdot\|_{\mathfrak{p}}$ on K, see [150, Prop. 3-1-9].

We also note that $\mathfrak{p}$-adic fields are locally compact, where we view $\mathbb{Q}_{\mathfrak{p}}$ as metric space with the metric d defined by $d(x,y) = \|x-y\|_{\mathfrak{p}}$.

Lemma 3.66. [91, last Corollary in Section 4.1], [313, Theorem in Section 2.3.5] and [326, Corollary 12.2] *Let K be a field equipped with a nontrivial non-Archimedean absolute value $|\cdot|$ and consider the corresponding (ultra-)metric space. Then K is locally compact iff the following three conditions are satisfied:*

- *K is a complete metric space.*
- *$\{|x| \mid x \in K^\times\}$ is a discrete subgroup of $\mathbb{R}_{>0}$, hence of the form $\eta^{\mathbb{Z}}$ for some $0 < \eta < 1$.*
- *The residue field of its valuation ring is finite.* □

Since the metric $d(x,y) = \|x-y\|_{\mathfrak{p}}$ is an ultrametric, we refer to Lemma 2.26. Especially we have the following statement.

Lemma 3.67. [273, Theorem 5.2] *The additive and multiplicative groups of $K_{\mathfrak{p}}$ are both locally compact Abelian groups; the ring $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ and also the ideals $\mathfrak{p}^m$ ($m \in \mathbb{N}$) of it are compact. Those ideals, moreover, are clopen in $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ and form a basis of neighbourhoods of 0; thus $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ is a totally disconnected topological space.* □

Sometimes the p-adic numbers are introduced via formal power series (*e.g.*, in [313, Chapter 1], [326, Section 1.3] and [73, Section I.§3]). The justification is the following.

Lemma 3.68. [341, Chap. II Prop. 5], [211, Prop. 1.70], [326, Theorems 12.1 & 12.3], [91, Lemma 4.1.4 & following Corollary] and [273, Theorem 5.1] *Let R be a complete discrete valuation ring, with field of fractions K and residue field $k = R/\mathfrak{m}(R)$. Let S be a system of representatives of k, and π a uniformiser of R. Then every element $a \in R$ can be written uniquely as a convergent series*

$$a = \sum_{n=0}^{\infty} s_n \pi^n \quad \text{with } s_n \in S.$$

Similarly, every element $x \in K$ can be written as

$$x = \sum_{n=m}^{\infty} s_n \pi^n \quad \text{with } s_n \in S \text{ and } m \in \mathbb{Z}.$$ □

Remark 3.69. Note that it is preferable to take $0 \in S$. Also note that for an extension L of K with $f = 1$ one can choose the same set S for both L and K. Furthermore, if K is a $\mathfrak{p}$-adic number field with residue field of cardinality p^f, then the set S can be chosen such that $s_i = s_i^{p^f}$ for all $s_i \in S$ (*i.e.*, the s_i's are *Teichmüller digits*), see [210, Corollary in Section III.3].

Remark 3.70. The p-adic numbers $\mathbb{Q}_p$ can therefore be identified with the following set

$$\left\{ \sum_{n=m}^{\infty} s_n p^n \mid m \in \mathbb{Z},\ s_n \in \{0, \dots, p-1\} \right\}$$

of formal power series. The integers $\mathbb{Z} \subset \mathbb{Q}_p$ can then be identified with the subset of all *finite* power series (*i.e.*, there is an $N \in \mathbb{N}$ such that $s_n = 0$ for all $n \geq N$) with $m = 0$, while the rational numbers $\mathbb{Q} \subset \mathbb{Q}_p$ can be identified with the subset of all *eventually periodic* power series (*i.e.*, there are $k \in \mathbb{N}$ and $N \in \mathbb{Z}$ such that $s_n = s_{n+k}$ for $n \geq N$; the minimal such k is the *period length*, while for the minimal such N we call $(s_n)_{n=m}^{N-1}$ the *pre-period*), see [313, Prop. in Section 1.5.3].

Remark 3.71. If the valuation ring R and its uniformiser π is clear, then we use the notation $a = .s_0\, s_1\, s_2\, s_3 \dots$ for $a = \sum_{n=0}^{\infty} s_n\, \pi^n \in R$. Similarly, we use the notation $a = s_m \dots s_{-1}.s_0\, s_1\, s_2\, s_3 \dots$ for $a = \sum_{n=m}^{\infty} s_n\, \pi^n \in K$, so the dot marks (and precedes) the zeroth position. Furthermore, we indicate a period by overlining it, *i.e.*, $.s_0 \dots \overline{s_n \dots s_{n+k}} = .s_0 \dots s_n \dots s_{n+k}\, s_n \dots s_{n+k}\, s_n \dots s_{n+k} \dots$. Note that for $a = s_m \dots s_{-1}.s_0\, s_1\, s_2 \dots$, we have $v_{\mathfrak{m}(R)}(a) = \min\{k \mid s_k \neq 0\}$.

We now state the central result we will use in the following (see [341, Theorem II.1 & Corollaries 2 & 3], [211, Prop. 1.74] [91, Lemma 9.2.1 & Corollaries 1 & 2 in Section 9.2], also see [224, Chap. II §1, Corollaries 1–3] and [150, Theorem 5-1-5 & Corollaries 5-1-7 & 5-1-8 & 5-1-9]).

Theorem 3.72. *Let L/K be a finite separable extension of degree n, v a discrete valuation of K with ring A, and B the integral closure of A in L. Let w_i be the different prolongations of v to L, and let e_i, f_i be the corresponding numbers. Let $\hat{K}$ and $\hat{L}_i$ be the completions of K and L for v and the w_i.*

(i) *The field $\hat{L}_i$ is a separable extension of $\hat{K}$ of degree $n_i = e_i \cdot f_i$. Moreover, the global degree is the sum of the local degrees, i.e., $n = \sum_i n_i$.*

(ii) *The valuation $\hat{w}_i$ is the unique valuation of $\hat{L}_i$ prolonging $\hat{v}$, and $e_i = e_{\hat{w}_i|\hat{v}} = e_{\hat{L}_i/\hat{K}}$ and $f_i = f_{\hat{w}_i|\hat{v}} = f_{\hat{L}_i/\hat{K}}$.*

(iii) *If $a \in L$, the minimal polynomial p of a in L/K is equal to the product of the (pairwise distinct by separability) characteristic polynomials p_i of a in $\hat{L}_i/\hat{K}$. In particular,*

$$T_{L/K}(a) = \sum_i T_{\hat{L}_i/\hat{K}}(a), \qquad N_{L/K}(a) = \prod_i N_{\hat{L}_i/\hat{K}}(a).$$

(iv) *Let $\hat{K}^a$ be the algebraic closure of $\hat{K}$. If L is given by an irreducible polynomial $p \in K[x]$. i.e., $L \cong K[x]/p(x)$ (or $L \cong K(a)$, where $p(x) = \mathrm{Irr}(a, K, x)$), then the complete fields $\hat{L}_i$ correspond to the irreducible factors p_i of p over $\hat{K}$, i.e., $\hat{L}_i \cong \hat{K}[x]/p_i(x)$ (or $\hat{L}_i \cong \hat{K}(a_i)$, where $p_i(x) = \mathrm{Irr}(a_i, \hat{K}, x)$). In this case, $n_i = e_i \cdot f_i = [\hat{L}_i : \hat{K}] = \mathrm{grad}\, p_i$. Furthermore, there is exactly one K-monomorphism $\sigma_i : L \to \hat{L}_i$ with $\sigma(a) = a_i$.* □

Remark 3.73. We remark that $\sigma_i L$ is dense in $\hat{L}_i$, and that in (iii) we have used the short notation where a and (its embedding) a_i are identified, *i.e.*, in the setting of (iv) they read for $a \in L$:

$$T_{L/K}(a) = \sum_i T_{\hat{L}_i/\hat{K}}(\sigma_i a), \qquad N_{L/K}(a) = \prod_i N_{\hat{L}_i/\hat{K}}(\sigma_i a).$$

Remark 3.74. If $\hat{K}$ is complete under an valuation $\hat{v}$ and $\hat{L}$ is a finite (and complete) extension of $\hat{K}$ with valuation $\hat{w}$ and $\hat{w} \mid \hat{v}$, then two elements of $\hat{L}$ that are conjugate over $\hat{K}$ have the same valuation and therefore also the same non-Archimedean absolute value. Since the only nontrivial extension of $\mathbb{R}$ is $\mathbb{C}$, the same statement holds for Archimedean absolute values, where "conjugate" just means "complex conjugate".

In connection with the last theorem, we remark that the valuation of a finite extension of a $\mathfrak{p}$-adic field satisfies the following uniqueness statement, also see [153, Theorem 5.3.5].

Lemma 3.75. [273, Prop. 5.3], [91, Theorem 7.1.1], [326, Theorem 15.1] and [150, Theorem 4-1-8] *If $\hat{L}$ is a finite extension of degree n of a $\mathfrak{p}$-adic field $\hat{K}$, then there exists exactly one absolute value $|\cdot|_{\hat{w}}$ of $\hat{L}$ which coincides with $\|\cdot\|_{\mathfrak{p}}$ on $\hat{K}$. It is equal to*

$$|x|_{\hat{w}} = \left(\left\| N_{\hat{L}/\hat{K}}(x) \right\|_{\mathfrak{p}} \right)^{1/n}.$$

Moreover, $\hat{L}$ is also a $\mathfrak{P}$-adic field. □

Since we have $n = e \cdot f$ and considering the normalisations of the absolute value on p. 30, we obtain the following uniqueness statement for the normalised absolute value from the last Lemma.

Corollary 3.76. *If $\hat{L}$ is a finite extension of degree n of a $\mathfrak{p}$-adic field $\hat{K}$ with normalised absolute value $\|\cdot\|_{\hat{K}}$, then the (unique) normalised absolute value $\|\cdot\|_{\hat{L}}$ on $\hat{L}$ is given by*

$$\|x\|_{\hat{L}} = \left\| N_{\hat{L}/\hat{K}}(x) \right\|_{\hat{K}}.$$ □

We also give the following following specialisation to algebraic number fields of Theorem 3.72.

Proposition 3.77. [273, Theorem 5.5] *Let K be an algebraic number field, $\mathfrak{p}$ a (nonzero) prime ideal of $\mathfrak{o}_K$, and $K_{\mathfrak{p}}$ the completion of K corresponding to $\mathfrak{p}$. Moreover, let L/K be a finite extension, $\mathfrak{P}$ a prime ideal of $\mathfrak{o}_L$ lying above $\mathfrak{p}$, and $L_{\mathfrak{P}}$ the corresponding completion. Finally, let $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ and $\widehat{\mathfrak{o}_{\mathfrak{P}}}$ be the corresponding rings of integers in $K_{\mathfrak{p}}$ and $L_{\mathfrak{P}}$. Then we have:*

(i) $[L_{\mathfrak{P}} : K_{\mathfrak{p}}] = e_{\mathfrak{P}|\mathfrak{p}} \cdot f_{\mathfrak{P}|\mathfrak{p}}$.

(ii) *The field $L_{\mathfrak{P}}$ is generated by L and $K_{\mathfrak{p}}$, i.e., $L_{\mathfrak{P}} = K_{\mathfrak{p}} \cdot L$.*

(iii) *The ring $\widehat{\mathfrak{o}_{\mathfrak{P}}}$ is the integral closure of $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ in $L_{\mathfrak{P}}$.*

(iv) *If $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{P}}$ are the prime ideals in $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ and $\widehat{\mathfrak{o}_{\mathfrak{P}}}$, respectively, then $\overline{\mathfrak{P}}$ lies above $\overline{\mathfrak{p}}$ and $e_{\overline{\mathfrak{P}}|\overline{\mathfrak{p}}} = e_{\mathfrak{P}|\mathfrak{p}}$ and $f_{\overline{\mathfrak{P}}|\overline{\mathfrak{p}}} = f_{\mathfrak{P}|\mathfrak{p}}$. Furthermore, $\overline{\mathfrak{P}}$ is the only prime ideal in $\widehat{\mathfrak{o}_{\mathfrak{P}}}$ lying above $\overline{\mathfrak{p}}$.* □

Now, we know that each discrete valuation w_i that prolongs v is associated with a prime $\mathfrak{P}_i$ that lies above $\mathfrak{p}$. So in the case $L = K(a)$, Theorem 3.72 shows that we can associate to every $p_i \in \hat{K}[x]$ exactly one prime $\mathfrak{P}_i$. In many cases this connection is clarified by the following theorem of Kummer.

Remark 3.78. For a Dedekind ring R and fixed prime ideal $\mathfrak{p}$ in R, we denote for a polynomial $g \in R[x]$ by $\overline{g}$ the canonical map $R[x] \to R[x]/\mathfrak{p}[x] = k[x]$ resulting from the application of the residue map $R \to R/\mathfrak{p} = k$ to every coefficient ($\overline{g}$ is also called the reduction of g and $\mathfrak{p}$), *i.e.*, $g(x) \equiv \overline{g}(x) \bmod \mathfrak{p}$.

Proposition 3.79. [211, Theorem 1.42], [73, Satz IV.§2.3], [150, Theorem 5-1-11], [224, Prop. I.25] and [273, Theorem 4.12] *Let R be a Dedekind ring with quotient field K and $L = K(a)$ a finite separable extension of K with generating element $a \in S$, where S is the algebraic closure of R in L. Let $g \in R[x]$ be the minimal polynomial $g(x) = \mathrm{Irr}(a, K, x)$ of a over K. Moreover, let $\mathfrak{p}$ be a prime ideal of R such that $\mathfrak{p} \cdot S \cap R[a] = \mathfrak{p}[a]$. If $\overline{g} \in (R/\mathfrak{p})[x]$ has the decomposition $\overline{g} = \varphi_1^{e_1} \cdot \ldots \cdot \varphi_r^{e_r}$ in irreducible (pairwise distinct) monic polynomials $\varphi_i \in (R/\mathfrak{p})[x]$, then*

$$\mathfrak{p}\mathfrak{o}_L = \mathfrak{P}_1^{e_1} \cdot \ldots \mathfrak{P}_r^{e_r}$$

is the prime ideal decomposition of $\mathfrak{p}$ in S, $f_{\mathfrak{P}_i|\mathfrak{p}} = \mathrm{grad}\, \varphi_i$ and $\mathfrak{P}_i = \mathfrak{p}\mathfrak{o}_L + g_i(a)\mathfrak{o}_L$, where g_i is a polynomial in $R[x]$ such that $\overline{g_i} = \varphi_i$. □

Crucial is the condition $\mathfrak{p} \cdot S \cap R[a] = \mathfrak{p}[a]$. For $K = \mathbb{Q}$ this condition is fulfilled for all but a finite number of primes $p \in \mathbb{P}$. In this case we have $R[a] = \mathbb{Z}[a]$, $S = \mathfrak{o}_{\mathbb{Q}(a)}$ and $\mathfrak{p} = p\mathbb{Z}$. We set $r = [\mathfrak{o}_{\mathbb{Q}(a)} : \mathbb{Z}[a]]$. Suppose $p \not| r$, then we have $\mathbb{Z}[a] \subset \mathfrak{o}_{\mathbb{Q}(a)} \subset \frac{1}{r}\mathbb{Z}[a]$ and the above condition holds. Note that $d_{\mathbb{Q}(a)}(1, a, \ldots, a^{n-1}) = ([\mathfrak{o}_{\mathbb{Q}(a)} : \mathbb{Z}[a]])^2 \cdot d_{\mathbb{Q}(a)}$ by Lemma 3.45, so the discriminant can be used to determine for which algebraic number fields this proposition is applicable.
The principle application of Proposition 3.79 is the explicit factorisation of prime ideals. Maybe the best known result is the factorisation in quadratic fields. We first define the Legendre symbol and state some of its properties.

Definition 3.80. Let p be an odd prime. An integer $a \not\equiv 0 \bmod p$ is called a *quadratic residue* mod p if there is an integer b with $a \equiv b^2 \bmod p$. The *Legendre symbol* $\left(\frac{a}{p}\right)$ is equal to 1 if a is a quadratic residue mod p and it is equal to -1 if a is not a quadratic residue.

We state Euler's criterion and the quadratic reciprocal law due to Gauss.

Lemma 3.81. [211, Proposition A2.1 & Theorem A2.2] and [238, Satz 5.1 & Satz 5.3 & Satz 5.5] *Let p, q be distinct odd primes and $a, b \in \mathbb{Z}$.*

- *The Legendre symbol is multiplicative, i.e.,* $\left(\frac{a \cdot b}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$.
- *For the Legendre symbol the following congruence holds (Euler's criterion):*

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \bmod p.$$

- *In particular, one has* $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ *and* $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

- *The Legendre symbol satisfies the quadratic reciprocal law*

$$\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{(p-1)\cdot(q-1)/4}. \qquad \square$$

Now we are able to look at the prime ideal factorisation in quadratic extension fields.

Lemma 3.82. [273, Theorem 4.15], [73, Satz III.§8.1] and [211, Example I.30] *Let L be a quadratic extension of the rationals $\mathbb{Q}$ with discriminant $d = d_L$. We set $D = d/4$ if d is divisible by 4 and $D = d$ otherwise (then L is isomorphic to $\mathbb{Q}(\sqrt{D})$). Let p be a rational prime.*

- *If p is odd and $p \nmid d$, then*

$$(p) = p\mathfrak{o}_L = \begin{cases} \mathfrak{P}_1 \cdot \mathfrak{P}_2 & \text{iff } \left(\frac{d}{p}\right) = 1, \\ \mathfrak{P} & \text{iff } \left(\frac{d}{p}\right) = -1. \end{cases}$$

  For $\left(\frac{d}{p}\right) = 1$, we have $x^2 - D \equiv (x-a)\cdot(x+a) \bmod p$ with $a \in \mathbb{Z}$ and $a \not\equiv -a \bmod p$. Then we have $\mathfrak{P}_1 = (p, \sqrt{D} - a)$ and $\mathfrak{P}_2 = (p, \sqrt{D} + a)$.

- *If $p = 2$ and $2 \nmid d$, then*

$$(2) = 2\mathfrak{o}_L = \begin{cases} \mathfrak{P}_1 \cdot \mathfrak{P}_2 & \text{iff } d \equiv 1 \bmod 8, \\ \mathfrak{P} & \text{iff } d \equiv 5 \bmod 8. \end{cases}$$

  Here, in the splitting case, the prime ideals are given by $\mathfrak{P}_1 = (2, (\sqrt{D}+1)/2)$ and $\mathfrak{P}_2 = (2, (\sqrt{D}-1)/2)$.

- *If p divides d, then (irrespective of its parity) we have $(p) = \mathfrak{P}^2$. If p divides D, we have $\mathfrak{P} = (p, \sqrt{D})$. If $p = 2$ and $D \equiv 3 \bmod 4$ then $\mathfrak{P} = (2, \sqrt{D} + 1)$.* $\square$

We also note that, with the notation of the previous lemma, a basis of $\mathfrak{o}_{\mathbb{Q}(\sqrt{D})}$ is given by $\{1, \sqrt{D}\}$ if $D \equiv 2, 3 \bmod 4$ respectively $\{1, \frac{1+\sqrt{D}}{2}\}$ if $D \equiv 1 \bmod 4$.

3.5. Vector Spaces and Lattices

We can view a (finite) field extension L/K as K-vector space of dimension $\dim_K L = [L : K]$. If $L = K(a)$ is a finite extension of degree n, then the set $\{1, a, \ldots, a^{n-1}\}$ is a basis of the K-vector space L.

Definition 3.83. Let V be a vector space over a normed field K with norm $|\cdot|$. A *norm on the vector space V* is a real valued function $x \mapsto \|x\|_V$ on V satisfying the following three properties:

- We have $\|x\|_V \geq 0$ and $\|x\|_V = 0$ iff $x = 0$.
- For all $x \in V$ and $a \in K$ we have $\|a \cdot x\|_V = |a| \cdot \|x\|_V$.
- For all $x, y \in V$ we have $\|x + y\|_V \leq \|x\|_V + \|y\|_V$.

A K-vector space V which has a norm $\|\cdot\|_V$ is called a *normed vector space* over K. Again, a norm on V defines a metric d by putting $d(x, y) = \|x - y\|_V$ for any $x, y \in V$. Two norms on V are *equivalent* if they define the same topology on V.

Remark 3.84. If V is a finite dimensional K-vector space of dimension n, we can fix a basis $\{e_1, \ldots, e_n\}$. Then any vector $x \in V$ can be written (uniquely) in the form $x = a_1 \cdot e_1 + \ldots + a_n \cdot e_n$ with $a_i \in K$. The sup-norm on V with respect to this choice of the basis is given by

$$\|a_1 \cdot e_1 + \ldots + a_n \cdot e_n\|_V = \max_{1 \leq i \leq n} |a_i|.$$

If K is complete and V is finite dimensional, then any norm on V is equivalent to a sup-norm, or – alternatively – in the case of a $\mathfrak{p}$-adic field, by the norm defined in Lemma 3.75, as the following statement shows.

Lemma 3.85. [153, Theorem 5.2.1 & Prop. 5.2.4], [210, Theorem III.10], [326, Theorem 13.3] and [226, Prop. XII.2.2] *Let K be a complete field under a non-trivial absolute value, and let V be a finite dimensional K-vector space. Then any two norms on V are equivalent and V is complete with respect to the metric induced by any norm (i.e., V is a Banach space). Furthermore, if in addition K is locally compact, then V is also locally compact.* □

While $\mathbb{Z}_p$ is a compact (metric) ring, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$ are locally compact fields (see [153, Corollary 3.3.8]), and therefore also every $\mathfrak{p}$-adic field is locally compact. But these are all locally compact fields of characteristic 0, as the following classification theorem shows.

Lemma 3.86. [305, Theorem 4-12(i)] *Let K be any locally compact field where* $\operatorname{char} K = 0$ *and such that its topology is not the discrete topology. Then K is $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ (i.e., a $\mathfrak{p}$-adic field).* □

We also have the following converse of the last statement of Lemma 3.85.

Lemma 3.87. [305, Prop. 4-13(iv)] *If V is a locally compact vector space over a locally compact field K with nondiscrete topology, then V is finite dimensional over K.* □

We now turn our attention to lattices and their orders.

Definition 3.88. Let K be an algebraic number field of degree n and $\gamma_1, \ldots, \gamma_n$ be a $\mathbb{Q}$-basis of K. Then $M = \{c_1 \cdot \gamma_1 + \ldots + c_n \cdot \gamma_n \mid c_i \in \mathbb{Z}\}$ is called a *lattice in the number field K* (or a *complete module in the number field K*). It is a free Abelian group of rank n and a subgroup of K viewed as additive group.

Definition 3.89. Let K be an algebraic number field, M a lattice in K and define the set $\mathfrak{o}(M) = \{a \in K \mid aM \subset M\}$. We say that $\mathfrak{o}(M)$ is the *order of the lattice M*.

Lemma 3.90. [238, Satz 12.3] *The order $\mathfrak{o}(M)$ of a lattice M in an algebraic number field K is itself a lattice and a subring of K (with 1). Every element $a \in \mathfrak{o}(M)$ is integral over $\mathbb{Z}$ (in particular, we have $T_{K/\mathbb{Q}}(a), N_{K/\mathbb{Q}}(a) \in \mathbb{Z}$). Furthermore, if $b \in K$ is a root of a monic nonzero polynomial $f \in \mathbb{Z}[x]$, then $\mathfrak{o}(M)[b]$ is also an*[5] *order of K.* □

[5]In general, a ring $\mathfrak{O}$ in K with the following three properties is called an *order* of K: (i) K is the quotient field of $\mathfrak{O}$. (ii) $\mathfrak{O} \cap \mathbb{Q} = \mathbb{Z}$. (iii) The additive group of $\mathfrak{O}$ is finitely generated. Every order is an order of some lattice, and every order of a lattice is an order, see [211, Prop. 1.2].

We now make the connection to the ring of algebraic integers $\mathfrak{o}_K$ and the set of fractional ideals $\mathfrak{I}_K$ (for the latter, recall Lemma 3.34). Note that the product of two lattices M, M' is given by $M \cdot M' = \{a \cdot b \mid a \in M,\ b \in M'\}$.

Lemma 3.91. [238, Satz 12.4] and [73, Satz II.§2.6] *The ring of algebraic integers $\mathfrak{o}_K$ of the algebraic number field K is the maximal order in K with respect to inclusion.* □

Lemma 3.92. [238, Satz 13.1] *Let $\mathfrak{o}_K$ be the ring of integers of an algebraic number field K. The set of lattices in K with order $\mathfrak{o}_K$ is given by the set of fractional ideals $\mathfrak{I}_K$, and forms an Abelian group under multiplication of lattices.* □

Definition 3.93. We call the codifferent $M^\wedge$ of a lattice M over $\mathbb{Z}$ (see Definition 3.50) the *dual lattice* of M.

We now want to justify the use of the word "lattice" here by considering its embedding in a real vector space.
Let $\mathcal{L}^{r,s}$ be the set of sequences $x = (x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s})$ with $x_1, \ldots, x_r \in \mathbb{R}$ and $x_{r+1}, \ldots, x_{r+s} \in \mathbb{C}$. We define addition, multiplication and multiplication with a real number componentwise. Then $\mathcal{L}^{r,s}$ is a commutative ring with unit $(1, \ldots, 1; 1, \ldots, 1)$ and also a real vector space. Indeed, we have $\mathcal{L}^{r,s} \cong \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^{r+2s}$ by identifying x with the sequence

$$(x_1, \ldots, x_r, \operatorname{Re} x_{r+1}, \operatorname{Im} x_{r+1}, \ldots, , \operatorname{Re} x_{r+s}, \operatorname{Im} x_{r+s}).$$

Fix a point $x' \in \mathcal{L}^{r,s}$. The map $x \mapsto x \cdot x'$ ($x \in \mathcal{L}^{r,s}$), *i.e.*, the multiplication with x', is a linear map in the real vector space $\mathbb{R}^{r+2s}$, where the associated matrix $\boldsymbol{M}(x')$ is given by

$$\boldsymbol{M}(x') = \begin{pmatrix} x'_1 & & & & & & & \\ & \ddots & & & & & & \\ & & x'_r & & & & & \\ & & & \operatorname{Re} x'_{r+1} & \operatorname{Im} x'_{r+1} & & & \\ & & & -\operatorname{Im} x'_{r+1} & \operatorname{Re} x'_{r+1} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & \operatorname{Re} x'_{r+s} & \operatorname{Im} x'_{r+s} \\ & & & & & & -\operatorname{Im} x'_{r+s} & \operatorname{Re} x'_{r+s} \end{pmatrix}, \tag{3.1}$$

where components are 0 if they are not written down. The determinant $\det \boldsymbol{M}(x')$ equals $x'_1 \cdot \ldots \cdot x'_r \cdot |x'_{r+1}|^2 \cdot \ldots \cdot |x'_{r+1}|^2$ and the trace $\operatorname{Tr} \boldsymbol{M}(x')$ equals $x'_1 + \ldots + x'_r + 2 \cdot \operatorname{Re} x'_{r+1} + \ldots + 2 \cdot \operatorname{Re} x'_{r+1}$.

Definition 3.94. Let K be an algebraic number field of degree n and signature $[r, s]$. Furthermore, let $\sigma_1, \ldots, \sigma_r$ be the real embeddings of K into $\mathbb{C}$ and the remaining ones be the complex embeddings such that $\sigma_{r+i} = \overline{\sigma}_{r+s+i}$ ($1 \le i \le s$). To each $a \in K$, we associate the point

$$\tilde{\tilde{a}} = \big(\sigma_1(a), \ldots, \sigma_r(a); \sigma_{r+1}(a), \ldots, \sigma_{r+s}(a)\big)$$

in $\mathcal{L}^{r,s}$. We call $\tilde{\tilde{a}}$ the *geometric image* of $a \in K$ in $\mathcal{L}^{r,s}$. The map $\tilde{\tilde{}}: K \to \mathcal{L}^{r,s}$ is injective, *i.e.*, one-to-one (but, of course, not surjective, *i.e.*, "onto"). We note that $\widetilde{\widetilde{a + b}} = \tilde{\tilde{a}} + \tilde{\tilde{b}}$ and $\widetilde{\widetilde{a \cdot b}} = \tilde{\tilde{a}} \cdot \tilde{\tilde{b}}$ for $a, b \in K$. Moreover, $\widetilde{\widetilde{c \cdot a}} = c \cdot \tilde{\tilde{a}}$ for $c \in \mathbb{Q}$ and $N_{K/\mathbb{Q}}(a) = \det \boldsymbol{M}(\tilde{\tilde{a}})$ and $T_{K/\mathbb{Q}}(a) = \operatorname{Tr} \boldsymbol{M}(\tilde{\tilde{a}})$.

We give a topological definition of a lattice.

Definition 3.95. Let G be a locally compact Abelian topological group and L be a discrete subgroup of G (*i.e.*, the induced topology on L coincides with the discrete topology). We say that L is a *lattice* if the factor group G/L is compact.

We will use this general definition of a lattice later (see Sections 6.4 & 6.5), for now we are interested in the situation $G = \mathbb{R}^n$.

Definition 3.96. Let $e_1, \dots, e_n$ be a base of $\mathbb{R}^n$. Then the set $L = \{\sum_{i=1}^n a_i \cdot e_i \mid a_i \in \mathbb{Z}\}$ is a lattice in $\mathbb{R}^n$ and each lattice can be represented as such a set with an appropriate base. We call the set

$$\mathrm{FD}(L) = \{\sum_{i=1}^{n} \alpha_i \cdot e_i \mid \alpha_i \in [0,1[\}$$

the *fundamental domain* of the lattice L (with respect to the base $e_1, \dots, e_n$). The pairwise disjoint sets $\mathrm{FD}(L) + t$ where $t \in L$ cover the whole space $\mathbb{R}^n$ without overlaps. Moreover, representatives of the factor group $\mathbb{R}^n/L \cong \mathbb{T}^n$ are given by the points of $\mathrm{FD}(L)$.

We are finally able to connect to two notions of lattices.

Lemma 3.97. [73, Satz II.§4.2] *Let K be an algebraic number field of degree n with signature $[r,s]$. Let $M = \{\sum_{i=1}^n c_i \cdot \gamma_i \,|\, c_i \in \mathbb{Z}\}$ be a lattice in K where $\gamma_1, \dots, \gamma_n$ is a $\mathbb{Q}$-basis of K. Then the geometric image of M in $\mathcal{L}^{r,s} \cong \mathbb{R}^{r+2s} = \mathbb{R}^n$ is a lattice. Moreover, if $d = d_{K/\mathbb{Q}}(\gamma_1, \dots, \gamma_n)$, then the volume of any of its fundamental domains $\mathrm{FD}(\tilde{\tilde{M}})$ (i.e., the Lebesgue measure of $\mathrm{FD}(\tilde{\tilde{M}})$ in $\mathbb{R}^n$) is given by $2^{-s} \cdot \sqrt{|d|}$.* □

3.6. Roots of Polynomials in $\mathfrak{p}$-adic Fields

Hensel's lemma gives some knowledge about the factorisation of a polynomial $f(x) \in \widehat{\mathfrak{o}_\mathfrak{p}}[x]$ if we know something about the corresponding polynomial in $(\widehat{\mathfrak{o}_\mathfrak{p}}/\mathfrak{p})[x]$. We will state two versions of Hensel's lemma (stated for complete fields with a valuation, not just $\mathfrak{p}$-adic number fields; also see [153, Theorems 3.4.6 & 6.1.2]), the definition of the resultant is given afterwards. Note that as usual, π denotes the uniformiser/generator of the (unique) prime ideal $\mathfrak{p}$ and the formulations "mod π" and "mod $\mathfrak{p}$" are equivalent.

Proposition 3.98. [150, Theorem 4-1-6], [73, Satz IV.§3.2] and [211, Prop. 1.71] *Let $\hat{K}$ be complete with respect to a valuation $\hat{v}$. Let $f(x) \in \mathfrak{o}_{\hat{K}}[x]$ be a normed polynomial. Suppose that $\overline{g}(x), \overline{h}(x) \in (\mathfrak{o}_{\hat{K}}/\mathfrak{p})[x]$ are polynomials which are relatively prime and such that $f(x) \equiv \overline{g}(x) \cdot \overline{h}(x) \bmod \mathfrak{p}$. Then there exist polynomials $g(x), h(x) \in \mathfrak{o}_{\hat{K}}$ such that $f(x) = g(x) \cdot h(x)$, $g(x) \equiv \overline{g}(x) \bmod \mathfrak{p}$ and $h(x) \equiv \overline{h}(x) \bmod \mathfrak{p}$. Moreover, the degree of g equals the degree of $\overline{g}$.* □

Proposition 3.99. [73, Satz IV.§3.1] and [211, Prop. 1.72] *Let $\hat{K}$ be complete with respect to a valuation $\hat{v}$. Let $f(x)$, $g_0(x)$, $h_0(x)$ be polynomials in $\mathfrak{o}_{\hat{K}}[x]$ such that the following conditions are fulfilled:*

(i) *The highest coefficients of $f(x)$ and $g_0(x) \cdot h_0(x)$ coincide.*

(ii) *The exponent* $r = \hat{v}(R(g_0, f_0))$ *of the resultant of* g_0 *and* h_0 *is finite (i.e., the resultant is not zero).*

(iii) $f(x) \equiv g_0(x) \cdot h_0(x) \bmod \pi^{2r+1}$.

Then there are polynomials $g(x), h(x) \in \mathfrak{o}_{\hat{K}}[x]$ *such that* $f(x) = g(x) \cdot h(x)$, $g(x) \equiv g_0(x) \bmod \pi^{2r+1}$ *and* $h(x) \equiv h_0(x) \bmod \pi^{2r+1}$. □

Definition 3.100. The *resultant* of two polynomials $g, h \in R[x]$ with $g(x) = a_n x^n + \ldots + a_0$, $h(x) = b_m x^m + \ldots + b_0$ is defined as the determinant of the following matrix (where $R(f, g) = 1$ if $m = 0 = n$):

$$\begin{pmatrix} a_n & \cdots & a_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_n & \cdots & a_0 \\ b_m & \cdots & b_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_m & \cdots & b_0 \end{pmatrix} \quad \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n \text{ rows} \end{matrix}$$

We also state an easy Corollary of Proposition 3.98 about the existence of a root of a polynomial in a $\mathfrak{p}$-adic number field.

Corollary 3.101. [153, Theorem 5.4.8] *Let* $K_{\mathfrak{p}}$ *be a* $\mathfrak{p}$*-adic number field,* π *a uniformiser and* $f(x) \in \widehat{\mathfrak{o}_{\mathfrak{p}}}[x]$ *a polynomial. Suppose that there exists an* $\overline{a} \in \widehat{\mathfrak{o}_{\mathfrak{p}}}$ *such that* $f(\overline{a}) \equiv 0 \bmod \pi$ *and* $f'(\overline{a}) \not\equiv 0 \bmod \pi$*, where* $f'(x)$ *is the formal derivative of* $f(x)$*. Then there exists an* $a \in \widehat{\mathfrak{o}_{\mathfrak{p}}}$ *such that* $a \equiv \overline{a} \bmod \pi$ *and* $f(a) = 0$. □

We are also interested in criteria under which there is no factorisation of a polynomial. We begin by defining Eisenstein polynomials.

Definition 3.102. Let $\mathfrak{p}$ a prime ideal of R. A polynomial

$$f(x) = x^n + a_1 \cdot x^{n-1} + \ldots + a_n \ \in R[x]$$

is called *Eisenstein polynomial (for* $\mathfrak{p}$*)* if $a_1, \ldots, a_n \in \mathfrak{p}$ and $v_{\mathfrak{p}}(a_n) = 1$.

We are now ready to state Eisenstein's irreducibility criterion and a partial converse (also see [153, Prop. 5.3.11]).

Proposition 3.103. [211, Prop. 1.43]

(i) *Let* $f(x)$ *be an Eisenstein polynomial for* $\mathfrak{p}$ *and* a *a root of* $f(x)$ *in some extension of* R*. Then* $[K(a) : K] = n$*, i.e.,* $f(x)$ *is irreducible over* K*.*

(ii) *Let* S *be the integral closure of* R *in* $K(a)$ *and* $\mathfrak{P}$ *the ideal of* S *generated by* a *and* $\mathfrak{p}$*. Then* $\mathfrak{P}$ *is a prime ideal and* $\mathfrak{p} = \mathfrak{P}^n$*.*

(iii) *Let* L/K *be a finite extension of degree* n *and let* S *be the integral closure of* R *in* L*. Assume that* $\mathfrak{P}^n = \mathfrak{p}$ *for a certain prime ideal* $\mathfrak{P}$ *of* S*,* $\mathfrak{p} = \mathfrak{P} \cap R$ *and let* a *be an element of* S *with* $v_{\mathfrak{P}}(a) = 1$*. Then* $L = K(a)$ *and* a *is the root of an Eisenstein polynomial for* $\mathfrak{p}$*.* □

We now explore the extension $\hat{L}$ of a $\mathfrak{p}$-adic number field further.

Lemma 3.104. [273, Corollary to Proposition 5.5] *A $\mathfrak{p}$-adic field can have only finitely many extensions of a given degree.* □

Definition 3.105. Let $\hat{L}$ be a finite extension of a $\mathfrak{p}$-adic number field $\hat{K}$. We say that $\hat{L}/\hat{K}$ is *unramified* if $e_{\hat{L}/\hat{K}} = 1$, and *totally ramified* (or *fully ramified*) if $f_{\hat{L}/\hat{K}} = 1$. Furthermore, let $p \in \mathfrak{p}$ be the characteristic of the residue field $\widehat{\mathfrak{o}_{\mathfrak{p}}}/\mathfrak{p}$. Then we say that $\hat{L}/\hat{K}$ is *tamely ramified* if $e_{\hat{L}/\hat{K}} \not\equiv 0 \bmod p$, and *wildly ramified* (or *strongly ramified*) if $e_{\hat{L}/\hat{K}} \equiv 0 \bmod p$.

Eisenstein's irreducibility criterion in Proposition 3.103 (ii) assures that a root of an Eisenstein polynomial generates a totally ramified extension. Furthermore, we have the following characterisation, see [273, Lemma 5.4 & Corollary 1 to Theorem 5.8 & Theorems 5.9 & 5.10 & 5.11], [224, Props. II.7 & II.11 & II.12], [91, Theorem 7.7.1 & Corollary 1 in Section 8.2], [211, Props. 1.77 & 1.78] and [153, Props. 5.4.7 & 5.4.11].

Proposition 3.106. *Let $\hat{L}/\hat{K}$ be a finite extension of a $\mathfrak{p}$-adic field $\hat{K}$ of degree $n = e \cdot f$, where e is the ramification index of $\hat{L}/\hat{K}$ and f its residue degree. Let $k_{\hat{K}}$ and $k_{\hat{L}}$ be the residue field of $\hat{K}$ and $\hat{L}$, respectively, both of characteristic p, and $\mathfrak{o}_{\hat{K}}$ and $\mathfrak{o}_{\hat{L}}$ their rings of integers. Moreover, $\mathfrak{p}$ will be the prime ideal of $\mathfrak{o}_{\hat{K}}$ and $\mathfrak{P}$ the one of $\mathfrak{o}_{\hat{L}}$.*

(i) *The finite extension $\hat{L}/\hat{K}$ is unramified iff there exists an $a \in \mathfrak{o}_{\hat{L}}$ such that $\hat{L} = \hat{K}(a)$ and the image of a in $k_{\hat{L}}$ is a simple root of a polynomial $\varphi(x)$ over $k_{\hat{K}}$ which is obtained from $f(x) = \mathrm{Irr}(a, \hat{K}, x)$ by reduction* mod $\mathfrak{p}$ *of its coefficients.*

(ii) *The finite extension $\hat{L}/\hat{K}$ is unramified iff $\hat{L} = \hat{K}(\xi_m)$, where ξ_m is a primitive m-th root of unity and m is not divisible by p. More precisely, it is a primitive $(q-1)$-st root of unity and $\hat{L}$ is the splitting field of $x^{q-1} - 1$ in $\hat{K}$, where $q = p^f$.*

(iii) *There is one and only one intermediate field $\hat{F}$ of $\hat{L}/\hat{K}$ such that $\hat{F}/\hat{K}$ is unramified (with $[\hat{F} : \hat{K}] = f$) and $\hat{L}/\hat{F}$ is totally ramified.*

(iv) *If the finite extension $\hat{L}/\hat{K}$ is totally ramified, then there exists an Eisenstein polynomial $f(x)$ over $\hat{K}$ whose root a generates $\hat{L}/\hat{K}$, moreover, $\mathfrak{o}_{\hat{L}} = \mathfrak{o}_{\hat{K}}[a]$. Conversely, every extension of $\hat{K}$ which is generated by a root of an Eisenstein is totally ramified. Therefore, every $\mathfrak{p}$-adic field has totally ramified extensions of any prescribed degree.*

(v) *A finite extension $\hat{L}/\hat{K}$ is totally and tamely ramified iff $\hat{L} = \hat{K}(\Pi)$ where $\Pi \in \mathfrak{P} \setminus \mathfrak{P}^2$ (i.e., $v_{\mathfrak{P}}(\Pi) = 1$ and Π is a uniformiser in $\hat{L}$) is a root of $x^n - \pi = 0$ with $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and n is not divisible by p.* □

Remark 3.107. We can informally say that in the unramified case the systems of representatives S of $k_{\hat{L}}$ gets bigger since additional roots of unity are added, but the uniformiser stays the same. In the totally ramified case the situation is reversed, we can use the same S but have to choose a new uniformiser (basically – at least in the tamely ramified case – the n-th root of the former uniformiser).

As a last step, we look at quadratic extensions of p-adic fields. We begin with the following result about the existence of square roots in $\mathbb{Q}_p$.

Lemma 3.108. [273, Corollary 3 to Theorem 5.3] *If p is a rational odd prime and m is a rational integer not divisible by p, then the p-adic field $\mathbb{Q}_p$ contains a square root of m iff $\left(\frac{m}{p}\right) = 1$.* □

We note that for any field K (not necessarily $\mathfrak{p}$-adic) of characteristic not equal to 2, the quadratic extensions are in one-to-one correspondence with the elements of $K^\times/(K^\times)^2$, where we denote by $(K^\times)^2$ the group of all nonzero squares of the field K. Therefore, we obtain the following characterisation of quadratic extensions of $\mathbb{Q}_p$, see [273, Prop. 5.12], [383, Section I.4, Corollaries], [210, Props. in Section III.3], [91, Corollaries in Section 4.3 on p. 53], [238, Satz 10.1], [73, Section I.§6], [313, Section 1.6.6] and [153, Corollary 3.4.4 & Problem 116].

Proposition 3.109. (i) *Let $p \neq 2$. Then the numbers $\varepsilon_1 = p$, $\varepsilon_2 = \xi$ and $\varepsilon_3 = p \cdot \xi$, where ξ is the primitive $p-1$-th root of unity, are not squares of any p-adic numbers (alternatively, one can take $\varepsilon_2 = \eta$ and $\varepsilon_3 = p \cdot \eta$ where $\eta \in \mathbb{Z}_p^\times$ is any element such that $\eta \bmod p$ is not a quadratic residue, e.g., $\eta = -1$ if $p \equiv 3 \bmod 4$ or $\eta = 2$ if $p \equiv 3 \bmod 8$ or $p \equiv 5 \bmod 8$). Furthermore, $\mathbb{Q}_p$ has three non-isomorphic (or, more precisely, non-$\mathbb{Q}_p$-isomorphic) quadratic extensions, namely $\mathbb{Q}_p(\sqrt{\varepsilon_i})$, $i = 1, 2, 3$.*

(ii) *Let $p = 2$ Then the numbers $\tilde{\varepsilon}_1 = 3$, $\tilde{\varepsilon}_2 = 5$, $\tilde{\varepsilon}_3 = 7$, $\tilde{\varepsilon}_4 = 2$, $\tilde{\varepsilon}_5 = 6$, $\tilde{\varepsilon}_6 = 10$ and $\tilde{\varepsilon}_7 = 14$ (or, equivalently, $\tilde{\varepsilon}_1 = -5$, $\tilde{\varepsilon}_2 = -3$, $\tilde{\varepsilon}_3 = -1$, $\tilde{\varepsilon}_5 = -10$, $\tilde{\varepsilon}_6 = -6$ and $\tilde{\varepsilon}_7 = -2$) are not squares of any 2-adic number. There exist seven non-isomorphic quadratic extensions of the field $\mathbb{Q}_2$, namely, $\mathbb{Q}_2(\sqrt{\tilde{\varepsilon}_i})$, $i = 1, \ldots, 7$. Here, $\mathbb{Q}_2(\sqrt{5}) \cong \mathbb{Q}_2(\sqrt{-3})$ is the only unramified extension of $\mathbb{Q}_2$ which is quadratic, whereas the others are fully and wildly ramified.*

(iii) *Let $\varepsilon_0 = 1 = \tilde{\varepsilon}_0$. Then the quotient group $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ consists of four elements ε_i, $i = 0, 1, 2, 3$ for $p \neq 2$, and of eight elements $\tilde{\varepsilon}_i$, $i = 0, \ldots, 7$ for $p = 2$.* □

Remark 3.110. We have seen that contrary to $\mathbb{R}$, there are "many" extensions of $\mathbb{Q}_p$. In fact, the algebraic closure $\mathbb{Q}_p^a$ is an infinite extension of $\mathbb{Q}_p$ ([153, Corollary 5.3.10] and [326, Corollary 16.7]), but it is neither complete with respect to the (prolongated) p-adic valuation ([153, Theorem 5.7.4], [91, Lemma 8.3.1], [210, Theorem III.12], [326, Theorem 16.6] and [313, Corollary in Section 3.1.4]), nor locally compact ([313, Corollary in Section 3.1.4], compare to Lemma 3.66). However, its completion $\mathbb{C}_p = \widehat{\mathbb{Q}_p^a}$ is algebraically closed ([153, Prop. 5.7.8], [91, Lemma 8.3.1], [326, Corollary 17.2(i)] and [210, Theorem III.13]), but also not[6] locally compact ([153, Problem 269] and [326, Corollary 17.2(iii)]).

3.7. Harmonic Analysis in Local Fields

We first state a condition which ensures the relative compactness of a subset of a local field. It tells us that a subset of a local field is compact iff it is closed and bounded.

Lemma 3.111. [211, Prop. 1.94] *Let K be a local field with absolute value $|\cdot|$. A subset $W \subset K$ is* relatively compact*, i.e., its closure $\mathrm{cl}\, W$ is compact, iff there exists a number $b \in K$ such that $|a| \leq b$ for all $a \in W$, i.e., if it is bounded.* □

[6] $\mathbb{C}_p$ is also not *spherically complete* ([313, Prop. in Section 3.3.4] and [326, Corollary 20.6]), a property which is satisfied by another universal p-adic field $\Omega_p \supset \mathbb{C}_p$, see [313, Section 3.2]. However, in [210] and [91] the symbols "Ω" and "Ω_p" are used for "$\mathbb{C}_p$".

Definition 3.112. Let G be a (locally) compact Abelian group. A *character* χ of G is a continuous homomorphism of G into the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. The set of characters of G forms an Abelian group by defining the product $(\chi_1 \cdot \chi_2)(x) = \chi_1(x) \cdot \chi_2(x)$ (for all $x \in G$). This Abelian group is called the *character group* of G, and is denoted[7] by G^*. A generic element of G^* will be denoted x^*, and the value of x^* at y by either $x^*(y)$ or $\langle x^*, y\rangle$.

Definition 3.113. One can turn G^* into a topological group as follows: Let W be a compact subset of G and let V be an open neighbourhood of $1 \in \mathbb{T}$. Define subsets $N \subset G^*$ by

$$N(W,V) = \{x^* \in G^* \mid x^*(W) \subset V\}.$$

The sets $N(W,V)$ constitute a neighbourhood base for the trivial character and hence determine a topology on G^*, called the *compact-open topology*. It is also the topology of uniform convergence on compact subsets of G.

With this topology on G^*, we obtain the following statements.

Lemma 3.114. [305, Prop. 3-2(iii)–(v)], [169, Theorem 23.17] and [150, Theorem 7-1-2] *Let G be an Abelian topological group. Then the following assertions hold:*

(i) *If G is locally compact, then so is G^*.*

(ii) *If G is discrete, then G^* is compact.*

(iii) *If G^* is compact, then G is discrete.* □

For a locally compact group G, one can therefore consider the group $G^{**} = (G^*)^*$. This is done in the *Pontryagin Duality Theorem.*

Proposition 3.115. [150, Theorem 7-1-1] and [305, Theorem 3-20] *Let G be a locally compact Abelian group. Then the algebraic homomorphism of G into G^{**} given by $x \mapsto x^{**}$, $x^{**}(y^*) = y^*(x)$ ($x \in G$, $y^* \in G^*$) is an algebraic and topological isomorphism (or, an isomorphism of topological groups) of G onto G^{**}. Hence, G and G^* are mutually dual.* □

We now give explicit forms of the characters in local fields (viewed as additive group) and therefore compute their character groups, see [211, Section 1.§5.1], [273, Section 5.§3.1], [159, Section 2.§8.3], [169, Sections 23.27 & 25.1] and [150, Section 7-1 B].

Lemma 3.116. (i) *The field of real numbers $\mathbb{R}$ is self-dual. A character $a^* \in \mathbb{R}^* \cong \mathbb{R}$ is given by the map $a^*(x) = \exp(2\pi i \cdot x \cdot a)$ ($x \in \mathbb{R}$), where we identify a^* and a.*

(ii) *The field of complex numbers $\mathbb{C}$ is self-dual. A character $a^* \in \mathbb{C}^* \cong \mathbb{C}$ is given by the map*

$$a^*(x) = \exp(4\pi i \cdot \mathrm{Re}(x \cdot a)) = \exp(2\pi i \cdot (x \cdot a + \overline{x \cdot a})) = \exp(2\pi i \cdot T_{\mathbb{C}/\mathbb{R}}(x \cdot a)) \quad (x \in \mathbb{C}).$$

[7] In the literature, the character group of G is also often denoted by $\hat{G}$. However, we use the notation $\hat{\cdot}$ for completions, wherefore we use the notation G^* as, *e.g.*, [387]. We also note that the character group can also be viewed as *topological dual* (see p. 346) of G, see [387, Section II.§5].

(iii) *The field of p-adic numbers* $\mathbb{Q}_p$ *is self-dual. A character* $a^* \in \mathbb{Q}_p^* \cong \mathbb{Q}_p$*, where* $a = \sum_{n=m}^{\infty} s_n \cdot p^n$*, is given by the map (void sums are taken to be 0)*

$$\begin{aligned} a^*(x) &= \exp\left(-2\pi i \cdot \sum_{j=-\infty}^{\infty} t_j \cdot \sum_{n=j}^{\infty} s_{-n} \cdot p^{j-n}\right) \\ &= \exp\left(-2\pi i \cdot \sum_{j=k}^{-m} t_j \cdot p^j \cdot \sum_{n=j}^{-m} s_{-n} \cdot p^{-n}\right) = \exp\left(-2\pi i \cdot \vartheta_p(x \cdot a)\right), \end{aligned}$$

where $x = \sum_{n=k}^{\infty} t_n \cdot p^n \in \mathbb{Q}_p$ *and where we define*[8] $\vartheta_p(y) = \vartheta_p(\sum_{n=m}^{\infty} s_n p^n) = \sum_{n=m}^{-1} s_n p^n)$.

(iv) *The field of* $\mathfrak{p}$*-adic numbers* $\mathbb{Q}_\mathfrak{p}$ *is self-dual. Let* $p \in \mathfrak{p}$*, then a character* $a^* \in \mathbb{Q}_\mathfrak{p}^* \cong \mathbb{Q}_\mathfrak{p}$ *is given by*

$$a^*(x) = \exp\left(-2\pi i \cdot \vartheta_p(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}(x \cdot a))\right) \quad (x \in \mathbb{Q}_\mathfrak{p}).$$

(v) *The cyclic group* C_k *of order* $k \in \mathbb{N}$ *is self-dual. If we identify* C_k *with the set* $\{0, \ldots, k-1\}$ *and addition* mod k*, then a character* $a^* \in \mathsf{C}_k^* \cong \mathsf{C}_k$ *is given by th map* $a^*(x) = \exp\left(2\pi i \cdot \frac{x \cdot a}{k}\right)$ $(x \in \mathsf{C}_k)$. □

Remark 3.117. Of course, we have taken the freedom to make some choices in the representation of the characters by taking the factors of $\pm 2\pi i$ in the exponentials.

Definition 3.118. The *p-quasicyclic group* C_{p^∞} (or *Prüfer group*) is the group of all p^n-th complex roots of unity, with n running over all nonnegative integers, *i.e.*,

$$\mathsf{C}_{p^\infty} = \left\{\exp\left(2\pi i \cdot \frac{a}{p^n}\right) \mid n \in \mathbb{N},\ a \in \mathbb{Z}\right\}.$$

We remark that the p-quasicyclic group is the analogue[9] of the tours $\mathbb{T} = \{\exp(2\pi i \cdot x) \mid x \in [0,1[\} \cong \mathbb{R}/\mathbb{Z}$, see [169, Section 23.27 & 25.2], [311, Example 2.9.5(2)] and [159, Examples 2.§8.3(2)–(3)].

Lemma 3.119. (i) *The groups* $\mathbb{T}$ *and* $\mathbb{Z}$ *are a* dual pair*, i.e.,* $\mathbb{T}^* = \mathbb{Z}$ *and* $\mathbb{Z}^* = \mathbb{T}$.

(ii) *The groups* C_{p^∞} *and* $\mathbb{Z}_p$ *are a dual pair.* □

We are interested in the duals of a (finite) direct product of locally compact Abelian groups.

Proposition 3.120. [159, Example 2.§8.3(4)] and [169, Theorem 23.18] *Let* $G_1, \ldots, G_m$ *be LCAGs, with character groups* $G_1^*, \ldots, G_m^*$*, respectively. Let* $G = \prod_{i=1}^m G_i$ *be their finite direct product, then* $G^* \cong \prod_{i=1}^m G_i^*$*. Moreover, the topological isomorphism* $\prod_{i=1}^m G_i^* \to G^*$ *is given as follows: For every* $(\chi_1, \ldots, \chi_m) \in \prod_{i=1}^m G_i^*$*, let* $[\chi_1, \ldots, \chi_m] : G \to \mathbb{T}$ *denote the function* $(x_1, \ldots, x_m) \mapsto \chi_1(x_1) \cdot \ldots \cdot \chi_m(x_m)$*. Then the isomorphism is given by the mapping* $(\chi_1, \ldots, \chi_m) \mapsto [\chi_1, \ldots, \chi_m]$. □

[8] The function ϑ_p returns the "p-adic fractional part" of a p-adic number.

[9] We can even identify C_{p^∞} with the set of "p-adic fractional parts" of $\mathbb{Q}_p$, therefore justifying the notation $\mathsf{C}_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p$.

Remark 3.121. By the last proposition, the character group of an *elementary group*, *i.e.*, a group of the form $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^k \times F$ (where F is a finite group), is again an elementary group. Furthermore, a group of the form $\mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \ldots \times \mathbb{Q}_{\mathfrak{p}_1} \times \mathsf{C}_{m_1} \times \ldots \times \mathsf{C}_{m_\ell}$ is self-dual.

Definition 3.122. Let G and G^* be a dual pair of locally compact Abelian groups with $x \in G$ and $\chi \in G^*$. If $\chi(x) = \langle \chi\,, x \rangle = 1$, then one says that χ and x are *orthogonal* and denotes this by $\chi \perp x$. Furthermore, if A is a certain subgroup of the group G, then the set of all $\chi \in G^*$ that are orthogonal to A is called the *annihilator* of A and is denoted by $A^\perp$. Clearly, the annihilator of a subset $A \subset G$ is a closed subgroup of the group G^*. In particular, we have $A^{\perp\perp} = (A^\perp)^\perp = \operatorname{cl} A$. Therefore, if $A = \operatorname{cl} A$ is a closed subgroup of G, then the subgroups A and $A^\perp$ are said to be *orthogonal*.

Let us summarise the relationship between annihilators and their dual groups.

Lemma 3.123. [159, Theorem in Section 2.§9.1] and [150, Theorems 7-1-4 & 7-1-5] *Let H be a closed subgroup of a locally compact Abelian group G. Then*

(i) $(G/H)^* \cong H^\perp$.

(ii) $H^* \cong G^*/H^\perp$ □

Note that this last lemma together with Lemma 3.114 yields the following immediate statement.

Corollary 3.124. *Let H be a closed discrete subgroup of a locally compact Abelian group G with compact factor group G/H, i.e., H is a lattice in G. Then, its annihilator $H^\perp$ is a closed discrete subgroup of the LCAG G^* with compact factor group $G^*/H^\perp$, i.e., $H^\perp$ is a lattice in G^*.* □

Remark 3.125. For $G = \mathbb{R}$ and $A = \mathbb{Z}$ we obtain $\mathbb{Z}^\perp = \mathbb{Z}$. Moreover, if $G = \mathbb{R}^n$ and $L = \{\sum_{i=1}^n a_i \cdot e_i \mid a_i \in \mathbb{Z}\}$ is a lattice in $\mathbb{R}^n$ (where $e_1, \ldots, e_n$ is a basis, see Definition 3.96), then $L^\perp$ is the *dual lattice* of L defined by $L^\perp = \{\sum_{i=1}^n a_i \cdot e_i^\perp \mid a_i \in \mathbb{Z}\}$, where $e_1^\perp, \ldots, e_n^\perp$ denotes the *dual basis* to $e_1, \ldots, e_n$ defined by (compare Lemma 3.51)

$$e_i^\perp \cdot e_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

(thus, $(\mathbb{Z}^n)^\perp = \mathbb{Z}^n$). We also note that we can express any lattice $L \subset \mathbb{R}^n$ as $L = \boldsymbol{B}\,\mathbb{Z}^n = \{\boldsymbol{B}x \mid x \in \mathbb{Z}^n\}$ where $\boldsymbol{B} \in GL(n, \mathbb{R})$. Then, one easily checks that $L^\perp = (\boldsymbol{B}^t)^{-1}\,\mathbb{Z}^n$, where $\boldsymbol{B}^t$ denotes the transpose of $\boldsymbol{B}$.

We will calculate the dual lattices of lattices in more general spaces than $\mathbb{R}^n$ in Section 6.5.

Remark 3.126. In order to calculate Fourier transforms of functions in local fields, one now has to know the Haar measure on that field. Haar measures are introduced in the next chapter.

3a. Profinite Groups

Constantly talking isn't necessarily communicating.

Eternal Sunshine on a Spotless Mind
– *Charlie Kaufman*

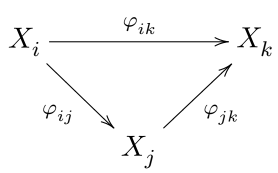
Profinite groups give an alternative description of p-adic integers. In the theory of substitution model set, profinite groups are needed to describe the internal space for lattice substitution system described in Chapter 6b. On first reading, however, this chapter can be skipped.

3a.1. Inverse Limits

Definition 3a.1. Let $I = (I, \preceq)$ denote a *directed partially ordered set* or *directed poset*, *i.e.*, I is a set with a binary relation $\preceq$ satisfying the following conditions:

- $i \preceq i$ for all $i \in I$;
- $i \preceq j$ and $j \preceq k$ imply $i \preceq k$ for $i, j, k \in I$;
- $i \preceq j$ and $j \preceq i$ imply $i = j$ for $i, j \in I$;
- if $i, j \in I$, then there exists some $k \in I$ such that $i \preceq k$ and $j \preceq k$.

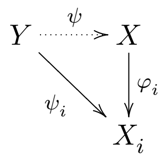
An *inverse* or *projective system* of topological spaces (respectively, topological groups) over I, consists of a collection $\{X_i \mid i \in I\}$ of topological spaces (respectively, topological groups) indexed by I, and a collection of continuous mappings (respectively, continuous group homomorphisms) $\varphi_{ij} : X_i \to X_j$, defined whenever $i \succeq j$, such that the diagrams of the form



commute whenever they are defined, *i.e.*, whenever $i, j, k \in I$ and $i \succeq j \succeq k$. In addition, we assume that φ_{ii} is the identity mapping id_{X_i} on X_i. We shall denote such a system by $\{X_i, \varphi_{ij}, I\}$, or by $\{X_i, \varphi_{ij}\}$ if the index set I is clearly understood.

Definition 3a.2. Let Y be a topological space (respectively, topological group), $\{X_i, \varphi_{ij}, I\}$ an inverse system of topological spaces (respectively, topological groups) over a directed poset I, and let $\psi_i : Y \to X_i$ be a continuous mapping (respectively, continuous group homomorphism) for each $i \in I$. These mappings are said to be *compatible* if $\varphi_{ij} \circ \psi_i = \psi_j$ whenever $j \preceq i$. One says that a topological space (respectively, topological group) X together with compatible continuous mappings (respectively, continuous homomorphisms) $\varphi_i : X \to X_i$

($i \in I$) is an *inverse limit* or *projective limit* of the inverse system $\{X_i, \varphi_{ij}, I\}$ and denote it by (X, φ_i), if the following universal property is satisfied:



whenever Y is a topological space (respectively, topological group) and $\psi_i : Y \to X_i$ ($i \in I$) is a set of compatible continuous mappings (respectively, continuous homomorhisms), then there is a unique continuous mapping (respectively, continuous homomorphism) $\psi : Y \to X$ such that $\varphi_i \circ \psi = \psi_i$ for all $i \in I$. We say that ψ is *induced* or *determined* by the compatible homomorphisms ψ_i. The maps $\varphi : X \to X_i$ are called *projections* (however, they are not necessarily surjections).

The inverse limit of an inverse system exists and is unique.

Lemma 3a.3. [311, Prop. 1.1.1] and [393, Prop. 1.1.4] *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of topological spaces (respectively, topological groups) over a directed poset I. Then:*

(i) *Let $\prod_{i \in I} X_i$ be the* direct product *(or* Cartesian product*) of topological spaces (respectively, topological groups), endowed with the product topology, and for each $i \in I$ write π_i for the projection map from $\prod_{i \in I} X_i$ to X_i. Define the subspace (respectively, subgroup)*

$$X = \{x = (x_i)_{i \in I} \mid \varphi_{ij} \circ \pi_i(x) = \pi_j(x) \text{ if } i \preceq j\}$$

and $\varphi_i = \pi_i|_X$ for each $i \in I$. Then (X, φ_i) is an inverse limit of $\{X_i, \varphi_{ij}, I\}$.

(ii) *This limit is unique in the following sense. If (X, φ_i) and (Y, ψ_i) are two limits of the inverse system $\{X_i, \varphi_{ij}, I\}$, then there is a unique homeomorphism (respectively, topological isomorphism) $\tilde{\varphi} : X \to Y$ such that $\varphi_i \circ \tilde{\varphi} = \varphi_i$ for each $i \in I$.* □

Remark 3a.4. If $\{X_i, \varphi_{ij}, I\}$ is an inverse system, we shall denote its inverse limit by $\varprojlim_{i \in I} X_i$ or $\varprojlim_i X_i$ or $\varprojlim_I X_i$ or $\varprojlim X_i$, depending on the context.

If all X_i share a certain property, we are interested if the inverse limit also has this property.

Lemma 3a.5. [393, Prop. 1.1.5] and [311, Lemma 1.1.2 & Props. 1.1.3 & 1.1.4] *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system indexed by I, and set $X = \varprojlim_{i \in I} X_i$.*

(i) *If each X_i is Hausdorff, so is X.*

(ii) *If each X_i is totally disconnected, so is X.*

(iii) *If each X_i is compact and Hausdorff, so is X.*

(iv) *If each X_i is Hausdorff, then X is closed in the direct product $\prod_{i \in I} X_i$.*

(v) *If each X_i is a nonempty compact Hausdorff space, then X is nonempty.* □

Definition 3a.6. If the topological space $X = \varprojlim_{i\in I} X_i$ arises as inverse limit of finite spaces X_i endowed with the discrete topology, we call X a *profinite space* (or *Boolean space*). Similarly, if a topological group $X = \varprojlim_{i\in I} X_i$ arises as inverse limit of finite groups X_i endowed with the discrete topology, we call X a *profinite group*. Also, if the X_i are finite cyclic groups (finite p-groups), then we call X a *procyclic group* (*pro-p group*).

We now characterise profinite spaces and groups.

Lemma 3a.7. [311, Thorem 1.1.12] *Let X be a topological space. Then the following conditions are equivalent:*

(i) *X is a profinite space.*

(ii) *X is compact Hausdorff and totally disconnected.*

(iii) *X is compact Hausdorff and admits a base of clopen sets for its topology.* □

Lemma 3a.8. [393, Corollary 1.2.4] *Let G be a topological group. The following are equivalent:*

(i) *G is profinite.*

(ii) *G is isomorphic (as a topological group) to a closed subgroup of a direct product of finite groups.*

(iii) *G is compact (and Hausdorff) and $\bigcap_{N \lhd_o G} N = 1$, where by $N \lhd_o G$ we mean that N is an open normal subgroup of G.*

(iv) *G is compact (and Hausdorff) and totally disconnected.* □

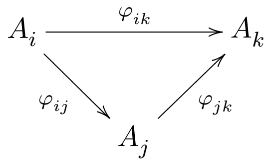
Lemma 3a.9. [393, Theorem 1.2.5(a)] *Let G be a profinite group. If $\mathcal{N}$ is a filterbase of closed normal subgroups of G such that $\bigcap_{N\in\mathcal{N}} N = 1$, then $G \cong \varprojlim_{N\in\mathcal{N}} G/N$. Moreover, $H \cong \varprojlim_{N\in\mathcal{N}} H/(H\cap N)$ for each closed subgroup H and $G/K \cong \varprojlim_{N\in\mathcal{N}} G/(KN)$ for each closed normal subgroup K.* □

Definition 3a.10. Let $\mathcal{C}$ be the class of nonempty finite groups (finite p-groups, or *etc.*), and let G be a group. Consider the collection $\mathcal{N} = \{N \lhd_f G \,|\, G/N \in \mathcal{C}\}$, where $N \lhd_f G$ indicates that N is a normal subgroup of finite index of G. $\mathcal{N}$ is a directed poset by defining $M \preceq N$ if N is a subgroup of M ($N, M \in \mathcal{N}$). For $M, N \in \mathcal{N}$ and $M \preceq N$, let $\varphi_{NM} : G/N \to G/M$ be the natural epimorphism. Then $\{G/N, \varphi_{NM}\}$ is an inverse system of groups in $\mathcal{C}$, and we say that the pro-$\mathcal{C}$ group $\varprojlim_{N\in\mathcal{N}} G/N$ is the *pro-$\mathcal{C}$ completion* of G. In particular, we use the terms *profinite completion* if $\mathcal{C}$ is the class of finite groups and *pro-p completion* if $\mathcal{C}$ is the class of finite p-groups.

Remark 3a.11. The name "completion" is justified. For this, let $\mathcal{N}$ be as above and assume that $\mathcal{N}$ is *filtered from below*, *i.e.*, whenever $N_1, N_2 \in \mathcal{N}$ then there exists an $N_3 \in \mathcal{N}$ such that N_3 is a subgroup of $N_1 \cap N_2$. Then one can make G into a topological group by considering $\mathcal{N}$ as a fundamental system of neighbourhoods of the identity element 1 of G. We refer to the corresponding topology on G as a *pro-$\mathcal{C}$ topology* (*profinite topology*, *pro-p topology etc.*). The completion of G with respect to this topology is the pro-$\mathcal{C}$ completion of G (which is unique up to isomorphisms).

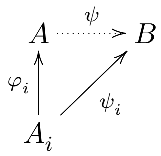
A dual concept to an inverse system is a direct or inductive system.

Definition 3a.12. Let $I = (I, \preceq)$ be a partially ordered set. A *direct* or *inductive system* of Abelian groups over I consists of a collection $\{A_i\}_{i\in I}$ of Abelian groups indexed by I and a collection of homomorphisms $\varphi_{ij} : A_i \to A_j$ defined whenever $i \preceq j$, such that the diagrams of the form



commute whenever $i \preceq j \preceq k$. In addition, we assume that φ_{ii} is the identity mapping id_{A_i} on A_i. We shall denote such a system by $\{A_i, \varphi_{ij}, I\}$, or by $\{A_i, \varphi_{ij}\}$ if the index set I is clearly understood.

Definition 3a.13. Let A be an Abelian group, $\{A_i, \varphi_{ij}, I\}$ a direct system of Abelian groups over a directed poset I and assume that $\psi_i : A_i \to A$ is a homomorphism for each $i \in I$. These mappings ψ_i are said to be *compatible* if $\psi_j \circ \varphi_{ij} = \psi_j$ whenever $i \preceq j$. One says that an Abelian group A together with compatible homomorphisms $\varphi_i : A_i \to A$ $(i \in I)$ is a *direct limit* or an *inductive limit* of the direct system $\{A_i, \varphi)ij, I\}$ if the following universal property is satisfied:



whenever B is an Abelian group and $\psi_i : A_i \to B$ $(i \in I)$ is a set of compatible homomorphisms, then there exists a unique homomorphism $\psi : A \to B$ such that $\psi \circ \varphi_i = \psi_i$ for all $i \in I$. We say that ψ is *induced* or *determined* by the compatible homomorphisms ψ_i.

As for inverse limits, the direct limit of an direct system exists and is unique.

Lemma 3a.14. [311, Prop. 1.2.1] and [393, Lemmas 6.3.1 & 6.3.2] *Let $\{A_i, \varphi_{ij}, I\}$ be a direct system of Abelian groups over a directed poset I. Then there exists a direct limit of the system. Moreover, this limit is unique in the following sense: If (A, φ_i) and (A', φ_i') are two direct limits, then there is a unique isomorphism $\tilde{\varphi} : A \to A'$ such that $\varphi_i' = \tilde{\varphi} \circ \varphi_i$ for each $i \in I$.* □

Remark 3a.15. If $\{A_i, \varphi_{ij}, I\}$ is a direct system, we denote its direct limit by $\varinjlim_{i\in I} A_i$ or $\varinjlim_i A_i$ or $\varinjlim_I A_i$ or $\varinjlim A_i$, depending on the context. The prototype of a direct limit is a union. In fact, if $A = \varinjlim A_i$ and $\varphi_i : A_i \to A$ are the canonical homomorphisms, then $A = \bigcup_{i\in I} \varphi_i(A_i)$. Also compare to the proof of the existence of a direct limit, where a suitable equivalence relation on the disjoint union of the A_i's yields the direct limit A as quotient set.

3a.2. Structure Theorems and Examples

We begin this Section with the pro-p and profinite completion of the integers, see [393, Section 1.5] and [311, Example 2.1.6(2)].

We note that $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$ of index n for every $n \in \mathbb{N}$ and that the quotient $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the cyclic group C_n with n elements. Therefore, the pro-p completion of $\mathbb{Z}$ is given by $\mathbb{Z}_p = \varprojlim_{i\in\mathbb{N}} \mathbb{Z}/p^i\mathbb{Z}$, the (ring of) p-adic integers. Here, $\varphi_i : \mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$ is given by $x \mapsto x + p^i\mathbb{Z}$.

For the profinite completion $\hat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ of $\mathbb{Z}$, observe that $\varphi_{nm} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is defined whenever $m \mid n$. Moreover, one can show that $\hat{\mathbb{Z}} = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$, a group we will meet again in the next section about adeles.

Conversely to $\mathbb{Z}_p$, we observe that the p-quasicyclic or Prüfer group C_{p^∞} (see Definition 3.118) can be defined as the direct limit $\mathsf{C}_{p^\infty} = \varinjlim_{i\in\mathbb{N}} \mathsf{C}_{p^i} = \varinjlim_{i\in\mathbb{N}} \mathbb{Z}/p^i\mathbb{Z}$.
In the following, we are interested in profinite groups which are finitely generated.

Definition 3a.16. Let G be a profinite group and let X be a subset of G. We say that X *generates* G (or, more precisely, *generates* G *as a profinite group or as a topological group*), if the abstract subgroup $\langle X\rangle$ of X generated by X is dense in G. In that case, we call X a *set of generators* (or, more precisely, a *set of topological generators*) of G and write $G = \overline{\langle X\rangle}$. A profinite group is called *finitely generated* if it can be generated by a finite set, and it is called a *d-generator group* (where d is a positive integer) if it can be generated by a set containing at most d elements. Clearly, a profinite group is a 1-generator group iff it is procyclic, *i.e.*, the inverse limit of finite cyclic groups.

For a finitely generated group G, denote the minimal number of generators by $d(G)$.

Lemma 3a.17. [311, Lemma 2.5.3] *Let* $\{G_i, \varphi_{ij}, I\}$ *be a* surjective inverse system *(i.e., all maps φ_{ij} are surjections) of finite groups. Define $G = \varprojlim_{i\in I} G_i$. Then G is finitely generated iff $\{d(G_i) \mid i \in I\}$ is a bounded set; in this case there exists some $i_0 \in I$ such that $d(G) = d(G_j)$ for each $j \succeq i_0$.* □

We will now classify procyclic groups, *i.e.*, profinite groups that can be generated by one element. But for this, we first define supernatural numbers.

Definition 3a.18. A *supernatural number* (or *Steinitz number*) is a formal infinite product $\prod_{p\in\mathbb{P}} p^{n(p)}$ over all primes p, in which each $n(p)$ is a non-negative integer or infinity.

Lemma 3a.19. [311, Prop. 2.7.1 & Theorem 2.7.2] *Let $n = \prod_{p\in\mathbb{P}} p^{n(p)}$ be a supernatural number, p be a prime and p^i also a supernatural number ($0 \le i \le \infty$).*

(i) *There exists a unique procyclic group C of order p^i up to isomorphism; namely, if $i < \infty$, $C \cong \mathbb{Z}/p^i\mathbb{Z}$, and if $i = \infty$, $C \cong \mathbb{Z}_p$.*

(ii) *There exists a unique procyclic group C of order n up to isomorphism.*

(iii) *The group $\mathbb{Z}_p$ has a unique closed subgroup H of index p^i. Moreover, $H = p^i\mathbb{Z}_p \cong \mathbb{Z}_p$ if i is finite, and $H = 1$ if i is infinite.*

(iv) *The group $\hat{\mathbb{Z}}$ has a unique closed subgroup H of index n. Moreover, $H \cong \prod_{p\in S} \mathbb{Z}_p$, where $S = \{p \mid n(p) < \infty\}$.*

(v) *Every procyclic group of order p^i appears as a quotient of $\mathbb{Z}_p$ in a unique way.*

(vi) *Every procyclic group of order n is a quotient of $\hat{\mathbb{Z}}$ in a unique way.*

(vii) *$\mathbb{Z}_p$ cannot be written as a direct product of nontrivial subgroups.* □

Finally, we can study the structure of finitely generated profinite Abelian groups.

Proposition 3a.20. [311, Theorem 4.3.5] *Let G be a finitely generated profinite Abelian group, with $d(G) = d$. Then, G is a direct sum of finitely many procyclic groups; more explicitly,*

$$G \cong \left[\bigoplus_{p\in\mathbb{P}} \left(\bigoplus_{m(p)} \mathbb{Z}_p\right)\right] \bigoplus \left[\bigoplus_{p\in\mathbb{P}} \left(\bigoplus_{i\in I_p} L_i(p)\right)\right],$$

where p ranges over all primes, each $L_i(p)$ is a finite cyclic p-group, each $m(p)$ is a natural number with $m(p) \leq d$, and each I_p is a finite set with $|I_p| \leq d$. □

As a last issue and aiming towards harmonic analysis (recall that $\mathbb{Z}_p^* \cong \mathsf{C}_{p^\infty}$), we are interested in the duals of profinite groups, also see [393, Lemmas 6.4.3 & 6.4.4].

Proposition 3a.21. [311, Lemmas 2.9.3 & 2.9.4]

(i) *Let $\{G_i, \varphi_{ij}, I\}$ be a surjective inverse system of profinite groups over a directed poset I and let $G = \varprojlim_{i\in I} G_i$ be its inverse limit. Then there exists an isomorphism $G^* \cong \varinjlim_{i\in I} G_i^*$.*

(ii) *Let $\{G_i \mid i \in I\}$ be a collection of profinite groups. Then $\left(\prod_{i\in I} G_i\right)^* \cong \bigoplus_{i\in I} G_i^*$.*

(iii) *Let $\{A_i, \varphi_{ij}, I\}$ be a directed system of discrete torsion Abelian groups over a directed poset I and let $A = \varinjlim_{i\in I} A_i$ be its direct limit. Assume that the canonical homomorphism $\varphi_i : A_i \to A$ are inclusion maps. Then there exists an isomorphism of profinite groups $A^* \cong \varprojlim_{i\in I} A_i^*$.*

(iv) *Let $\{A_i \mid i \in I\}$ be a collection of discrete torsion groups. Then $\left(\bigoplus_{i\in I} A_i\right)^* \cong \prod_{i\in I} A_i^*$.* □

3b. Adeles

The real trick lies in losing wars, in knowing which wars can be lost.

Catch-22 – *Joseph Heller*

The adele ring of an algebraic number field is one form in which the local-global principle manifests itself: The (global) number field is described by all corresponding local fields. Since we will look at point sets defined on number fields, the adele ring is the most natural and general object to be studied, see Section 6.5. We also work with adeles in Chapter 5a.

3b.1. Restricted Products and Adeles

Definition 3b.1. Let K be a field. An equivalence class ν of Archimedean absolute values is called an *infinite place*. An equivalence class ν of a non-Archimedean absolute value is called a *finite place*. If K is an algebraic number field of degree $n = [K : \mathbb{Q}]$ and signature $[r, s]$, then there are $r + s$ infinite places $\{\nu_1, \ldots, \nu_{r+s}\}$, where r of them are *real* and s are *complex*, *i.e.*, the completion of K with respect to an infinite place equals $\mathbb{R}$ or $\mathbb{C}$ (also see Definitions 3.21 & 3.63). Furthermore, each $\mathfrak{p} \in \mathbb{P}_K$ induces a finite place $\nu_\mathfrak{p}$. By Ostrowski's Theorem (see Lemmas 3.57 & 3.57'), these are all nontrivial (and non-equivalent) absolute values.

Remark 3b.2. Suppose that $S_\infty = \{\nu_1, \ldots, \nu_{r+s}\}$ with $\nu_1, \ldots, \nu_r$ real and $\nu_{r+1}, \ldots, \nu_{r+s}$ complex. Then we can order the $\mathbb{Q}$-isomorphisms $\sigma_1, \ldots, \sigma_n$ of K into $\mathbb{C}$ so that σ_i is associated with ν_i for $1 \le i \le r$ and $\sigma_{r+s+j} = \overline{\sigma}_{r+j}$ is associated with ν_{r+j} for $1 \le j \le s$. For each real ν_i, $K_{\nu_i} = (\hat{K}, \nu_i)$ is topologically isomorphic to $\mathbb{R}$, and for each complex ν_i, K_{ν_i} is topologically isomorphic to $\mathbb{C}$. Hence, if we define $K_\infty = \prod_{\nu \in S_\infty} K_\nu = K_{\nu_1} \times \ldots \times K_{\nu_{r+s}}$, we have a topological isomorphism $K_\infty \cong \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n$. Moreover, the field K is naturally embedded in K_∞ via the mapping $\tilde{\tilde{\cdot}}$ ("geometric image") defined in Definition 3.94. We will see that $\tilde{\tilde{K}}$ is dense in K_∞. Furthermore, for any positive integer d, we write $K_\infty^d = \prod_{\nu \in S_\infty} K_\nu^d$. Then we also have $K_\infty^d \cong \mathbb{R}^{d \cdot n}$.

Definition 3b.3. Let $J = \{\nu\}$ be a set of indices, and let J_∞ be a finite subset of J. Assume that for every index ν we are given a locally compact group G_ν (not necessarily Abelian), and we are further given a compact open (hence also closed) subgroup H_ν of G_ν for all $\nu \notin J_\infty$. We define the *restricted direct product* of the G_ν with respect to the H_ν as follows:

$$\prod_{\nu \in J}{}' G_\nu = \{(x_\nu)_{\nu \in J} \mid x_\nu \in G_\nu \text{ with } x_\nu \in H_\nu \text{ for all but finitely many } \nu\}.$$

Clearly, $\prod'_{\nu \in J} G_\nu$ is a subset of the direct product of the G_ν. In fact, it lies between the direct product and the direct sum of the component groups.

We now want to define a topology on a restricted product. For this, we note the following statement.

Lemma 3b.4. [75, Prop. I.§9.14]

(i) *Let $(X_i)_{i\in I}$ be a family of locally compact spaces such that X_i is compact for all but a finite number of indices. Then the product space $X = \prod_{i\in I} X_i$ is locally compact.*

(ii) *Conversely, if the product of a family $(X_i)_{i\in I}$ of nonempty topological spaces is locally compact, then the factors X_i are compact for all but a finite number of indices, and the factors which are not compact are locally compact.* □

We define a topology on $G = \prod'_{\nu\in J} G_\nu$ by specifying a neighbourhood base of the identity $(1)_{\nu\in J}$ consisting of sets of the form $\prod_{\nu\in J} N_\nu$, where N_ν is a neighbourhood of 1 in G_ν and $N_\nu = H_\nu$ for all but finitely many ν. This is *not* the product topology.

Let S be a finite set such that $J_\infty \subset S \subset J$, and consider the subgroup G_S of G defined by $G_S = \prod_{\nu\in S} G_\nu \times \prod_{\nu\in J\setminus S} H_\nu$. Then G_S is locally compact in the product topology. Moreover, the product topology on G_S is identical to that induced by the topology defined via the neighbourhood base of the identity above. Hence, each subgroup of the form G_S is locally compact with respect to the topology of the restricted direct product. Furthermore, since every $x \in G$ belongs to some subgroup of this form, we have established that G is locally compact (*i.e.*, by definition we have $G = \bigcup_S G_S$).

Lemma 3b.5. [305, Prop. 5-1] *Let G_ν and H_ν be as above, and let G be the restricted direct product of the G_ν with respect to the H_ν. Then*

(i) *G is a locally compact group.*

(ii) *A subset Y of G has compact closure iff $Y \subset \prod W_\nu$ for some family of compact subsets $W_\nu \subset G_\nu$ such that $W_\nu = H_\nu$ for all but finitely many indices ν.* □

Remark 3b.6. So, the restricted direct product of arbitrary many locally compact (but not necessarily compact) groups is again a locally compact group, while the direct product would only be locally compact if all but finitely many of them are compact, see Lemma 3b.4.

We now return to algebraic number fields.

Definition 3b.7. Let K be an algebraic number field of degree n and signature $[r, s]$. Then each completion K_ν (*i.e.*, $\mathbb{R}$, $\mathbb{C}$ or a $\mathfrak{p}$-adic field) is a locally compact additive group. For all finite places ν, K_ν admits $\widehat{\mathfrak{o}_\nu}$ as an open compact subgroup. The restricted product of the K_ν over all ν with respect to the subgroups $\widehat{\mathfrak{o}_\nu}$ (ν finite) is called the *adele group* of K and denoted $\mathbb{A}_K$. We note that $\mathbb{A}_K$ admits an obvious ring structure, wherefore we also speak of the *adele ring* $\mathbb{A}_K$ of K.

Let $x \in K$. Then $x \in K_\nu$ (where $x \in K_{\nu_i}$) is to be understood as $\sigma_i(x) \in K_{\nu_i}$ for $1 \le i \le r+s$ and σ_i is a $\mathbb{Q}$-isomorphisms of K into $\mathbb{C}$) for all ν, and $x \in \widehat{\mathfrak{o}_\mathfrak{p}}$ for almost all $\mathfrak{p}$. Thus, we may embed K in $\mathbb{A}_K$ via the diagonal map $x \mapsto (x, x, x, \dots) \in \mathbb{A}_K$ (therefore, this is – after suitable permutation – the geometric image $\tilde{\tilde{x}}$ in the first $r+s$ components followed by the embedding into all $\mathfrak{p}$-adic fields). With this we obtain the following (as reminder, we note that $K_{J_\infty} = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\mathfrak{p}\in\mathbb{P}_K} \widehat{\mathfrak{o}_\mathfrak{p}}$).

Lemma 3b.8. [150, Theorems 3-2-3 & 3-2-4], [224, Theorem VII.2], [305, Theorem 5-11] and [273, Props. 6.11 & 6.13] *K is a discrete subring of $\mathbb{A}_K$ and $\mathbb{A}_K/K$ is compact, i.e., K is a lattice in $\mathbb{A}_K$. Moreover, $K + K_{J_\infty} = \mathbb{A}_K$ and $K \cap K_{J_\infty} = \mathfrak{o}_K$.* □

Since K is a lattice in $\mathbb{A}_K$, we are interested in a fundamental domain, compare to Definition 3.96.

Lemma 3b.9. [224, Theorem VII.3] *Let K be an algebraic number field of degree $n = [K : \mathbb{Q}]$. Let $\gamma_1, \ldots, \gamma_n$ be a basis for the integers $\mathfrak{o}_K$ of K over $\mathbb{Z}$. We define* $\mathrm{FD}(K_\infty) = \{\sum_{i=1}^n \alpha_i \cdot \gamma_i \mid \alpha_i \in [0, 1[\}$. *Then*

$$\mathrm{FD}(K) = \mathrm{FD}(K_\infty) \times \prod_{\mathfrak{p} \in \mathbb{P}_K} \widehat{\mathfrak{o}_\mathfrak{p}}$$

is a fundamental domain of K in $\mathbb{A}_K$. □

Our next goal is *Artin's product formula.* For this we recall the definition of normalised absolute values, also see p. 30.

Definition 3b.10. If $\hat{K}$ is a local field, the *normalised absolute value* $|\cdot|_\nu$ on $\hat{K}$ (where we denote by ν associated place) is defined as follows:

- If $\hat{K} = \mathbb{R}$, then $|\cdot|_\nu$ is the usual absolute value.
- If $\hat{K} = \mathbb{C}$, then $|z|_\nu = z\overline{z}$ is the square of the usual absolute value.
- If $\hat{K} = \mathbb{Q}_\mathfrak{p}$ is non-Archimedean with uniformiser π, let $q = [\widehat{\mathfrak{o}_\mathfrak{p}} : \mathfrak{p}] = p^f$ (where $p \in \mathfrak{p}$ and $f = f_{\mathfrak{p}|(p)}$) be the order of the residue field. Then we use the normalisation such that $|\pi|_\nu = |\pi|_\mathfrak{p} = \frac{1}{q}$.

We will always use the notation $|\cdot|_\nu$ for a normalised absolute value.

Lemma 3b.11. [273, Theorem 3.3], [305, Theorem 5-14(i)], [91, Theorem 10.2.1] and [150, Theorem 3-2-7] *Let $x \in K^\times$ and J be the set of all places. Then* $\prod_{\nu \in J} |x|_\nu = 1$. □

We now state the *strong approximation theorem* which we may formulate in plain words as follows: If we choose any place ν_0, then K is dense in $\prod'_{\nu \in J \setminus \{\nu_0\}} K_\nu$ (while it is a lattice in $\mathbb{A}_K = \prod'_{\nu \in J} K_\nu$).

Proposition 3b.12. [211, Theorem 1.68] and [273, Theorem 6.7] *Let ν_0 be any place of the algebraic number field K and let $S' \subset J$ be a finite set of places of K distinct from ν_0. Furthermore, let $y_\nu \in K$ for all $\nu \in S'$. Then, for every $\varepsilon > 0$ there is an $x \in K$ such that $|x - y_\nu|_\nu < \varepsilon$ for all $\nu \in S'$ and $|x|_\nu \leq 1$ for all $\nu \notin S' \cup \{\nu_0\}$.* □

We also compile some statements from the harmonic analysis on $\mathbb{A}_K$, see [273, Prop. 6.17], [305, Lemmas 5-2 & 5-3 & Theorem 5-4], [211, Theorem 1.97] and [150, Theorem 7-1-16].

Lemma 3b.13. *Let K be an algebraic number field and $\mathbb{A}_K$ its adele ring.*

(i) *$\mathbb{A}_K$ is self-dual, i.e., $\mathbb{A}_K^* \cong \mathbb{A}_K$.*

(ii) *A character χ is given as product of the characters of its components, which is well-defined since they are trivial for all but finitely many places.*

(iii) *The annihilator $K^\perp$ of K is K itself, i.e., $K^\perp \cong K$.* □

3b.2. Higher Dimensions

Let K be an algebraic number field, with set J of places. Fix a place ν of K. For $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ belonging to the vector space K_ν^d put $|x|_\nu = \max_{1 \le i \le d} |x_i|_\nu$, and define a metric d_ν on K_ν^d by

- $d_\nu(x, y) = |x - y|_\nu^{1/2}$ if ν is a complex place,
- $d_\nu(x, y) = |x - y|_\nu$ if ν is not complex.

Definition 3b.14. Let $\mathbb{A}_K^d$ be the restricted direct product of the topological groups K_ν^d with respect to the subgroups $\widehat{\mathfrak{o}_\nu}^d$ (if ν is finite), so that $\mathbb{A}_K^d$ is a locally compact Hausdorff additive topological group. It is just the Cartesian product of d copies of the standard adele space $\mathbb{A}_K$ of K. We call $\mathbb{A}_K^d$ the *space of d-dimensional adeles of K*.

We now introduce a metric on $\mathbb{A}_K^d$. For each $x = (x_\nu)_{\nu \in J}, y = (y_\nu)_{\nu \in J} \in \mathbb{A}_K^d$, define

$$d(x, y) = \sup_{\nu \in J} \frac{d_\nu(x_\nu, y_\nu)}{N\nu},$$

where d_ν is the local metric defined above, and $N\nu = 1$ if $\nu \in J_\infty$ (*i.e.*, if ν is an infinite place) and $N\nu = N\mathfrak{p} = p^f$ if the finite place ν corresponds to the prime ideal $\mathfrak{p}$ with $p \in \mathfrak{p}$ and $f = f_{\mathfrak{p}|(p)}$.

Lemma 3b.15. [254, Lemma I.2] *The function $d : \mathbb{A}_K^d \times \mathbb{A}_K^d \to \mathbb{R}$ is a metric on $\mathbb{A}_K^d$. It is translational invariant, i.e., $d(x + a, y + a) = d(x, y) = d(a + x, a + y)$. The topology on $\mathbb{A}_K^d$ generated by this metric coincides with the restricted product topology. Moreover, a set $W \subset \mathbb{A}_K^d$ is compact iff W is closed and bounded (in terms of d).* □

We now define linear maps on K_ν^d and $\mathbb{A}_K^d$. In the later case, we again have a restricted product construction.

Let $\mathcal{L}_\nu^d$ be the ring of all $d \times d$ matrices over K_ν. The isomorphism of $K_\nu^{d^2}$ onto $\mathcal{L}_\nu^d$ (as additive group) induces a topology on $\mathcal{L}_\nu^d$ which makes it into a locally compact topological ring. From now, the symbol $\mathcal{L}_\nu^d$ denotes this topological ring.

If ν is finite, we denote by $\mathcal{I}_\nu^d$ the subring of $\mathcal{L}_\nu^d$ containing all those matrices with elements in $\widehat{\mathfrak{o}_\nu}^d$. Clearly, $\mathcal{I}_\nu^d$ is open and compact in $\mathcal{L}_\nu^d$.

Furthermore, we note that with the relative topology induced from $\mathcal{L}_\nu^d$, the group of units (*i.e.*, invertible matrices) of $\mathcal{L}_\nu^d$ is a multiplicative group; we denote this topological group by $\mathcal{G}_\nu^d$. Moreover, if ν is finite, we define

$$\mathcal{U}_\nu^d = \{\boldsymbol{M} \in \mathcal{I}_\nu^d \mid |\det \boldsymbol{M}|_\nu = 1\};$$

clearly, $\mathcal{U}_\nu^d$ is a subgroup of $\mathcal{G}_\nu^d$.

Definition 3b.16. Let $\mathcal{L}^d$ be the restricted direct product of the topological rings $\mathcal{L}_\nu^d$ with respect to the subrings $\mathcal{I}_\nu^d$. Then $\mathcal{L}^d$ is a locally compact topological ring. We call an element of $\mathcal{L}^d$ a *linear transformation of* $\mathbb{A}_K^d$ (since elements of $\mathcal{L}^d$ correspond to linear transformations of $\mathbb{R}^n$ in the classical theory). Each $\boldsymbol{M} \in \mathcal{L}^d$ is of the form $\boldsymbol{M} = (\boldsymbol{M}_\nu)_{\nu \in J}$, where $\boldsymbol{M}_\nu \in \mathcal{L}_\nu^d$ for all ν and $\boldsymbol{M}_\nu \in \mathcal{I}_\nu^d$ for almost all ν. Moreover, each such $\boldsymbol{M} \in \mathcal{L}^d$ can be regarded as a function on $\mathbb{A}_K^d$ by defining $\boldsymbol{M}x = (\boldsymbol{M}_\nu x_\nu)$. Let $\mathcal{G}^d$ be the restricted direct product of the groups $\mathcal{G}_\nu^d$ with respect to the subgroups $\mathcal{U}_\nu^d$. Then $\mathcal{G}^d$ is the group of units of $\mathcal{L}^d$. The elements of $\mathcal{G}^d$ are called the *lattice transformations of* $\mathbb{A}_K^d$ (each lattice transformation is a topological K-linear automorphism of $\mathbb{A}_K^d$, and thus an analogue of a non-singular linear transformation of $\mathbb{R}^n$ in the classical case). The group of non-singular $d \times d$ matrices over K is naturally embedded along the diagonal of $\mathcal{G}^d$ as a subgroup $\mathcal{P}^d$ of $\mathcal{G}^d$. We call an element $\boldsymbol{M} \in \mathcal{P}^d$ a *principal lattice transformation on* $\mathbb{A}_K^d$, and by abuse of notation, we use the same symbol $\boldsymbol{M}$ regardless of whether it is viewed as a $d \times d$ matrix over K or as an element of $\mathcal{G}^d$.

Remark 3b.17. The definition of $\mathcal{L}^d$ is analogous to the construction of the adele ring, while $\mathcal{G}^d$ parallels the construction of the group of ideles.

Remark 3b.18. For each $\boldsymbol{M} \in \mathcal{G}^d$, we define a (matrix) norm by $\|\boldsymbol{M}\|_{\mathcal{G}} = \prod_{\nu \in J} |\det \boldsymbol{M}_\nu|_\nu$. Then $\|\cdot\|_{\mathcal{G}}$ is a homomorphism of $\mathcal{G}^d$ into the multiplicative group of positive real numbers, and we obtain the analogue of Artin's product formula (see Lemma 3b.11), *i.e.*, $\|\boldsymbol{M}\|_{\mathcal{G}} = 1$ whenever $\boldsymbol{M} \in \mathcal{P}^d$.

Remark 3b.19. We also have the analogue of Lemma 3b.9: K^d is a lattice in $\mathbb{A}_K^d$ with fundamental domain $\mathrm{FD}(K^d) = \mathrm{FD}(K)^d$, see [254, Section 4.1].

3c. Visualisation of Ultrametric Spaces

A imaginaçao, o nosso último santuário.

NACHTZUG NACH LISSABON – *Pascal Mercier*

We are aiming at a description of certain geometrical objects, namely point sets or tilings. It is therefore only consequent to aid such a description in an intuitive geometrical way. Consequently, one needs a method "how to think about" (compact) sets in local fields. This is the goal of this chapter, and will then be applied at the examples considered in Chapters 5a & 6 – 6c.

3c.1. Topological Properties of $\mathfrak{p}$-adic Fields

We recall some topological properties of ultrametric spaces.

Lemma 2.26'. *Let (X, d) be an ultrametric space. It has the following properties:*

(i) *X is totally disconnected.*

(ii) *Any point of a ball is a centre of the ball, i.e., if $b \in B_{<r}(a)$ ($b \in B_{\leq r}(a)$), then $B_{<r}(a) = B_{<r}(b)$ ($B_{\leq r}(a) = B_{\leq r}(b)$).*

(iii) *If two balls have a common point, one ball is contained in the other.*

(iv) *The diameter of a ball is less than or equal to its radius.*

(v) *All triangles are isosceles with at most one short side.*

(vi) *If $x \in S_r(a)$, then $B_{<r}(x) \subset S_r(a)$ and*

$$S_r(a) = \bigcup_{x \in S_r(a)} B_{<r}(x).$$

(vii) *The spheres $S_r(a)$ ($r > 0$) are* clopen *i.e., both open and closed.*

(viii) *The open balls $B_{<r}(a)$ are clopen; the closed balls $B_{\leq r}(a)$ are clopen for $r > 0$.*

(ix) *Let B and B' be two disjoint balls. Then, $d(B, B') = d(x, x')$ for any $x \in B$ and $x' \in B'$.*

(x) *A sequence $(x_n)_{n \in \mathbb{N}}$ with $d(x_n, x_{n+1}) \to 0$ ($n \to \infty$) is a Cauchy sequence.*

(xi) *If $x_n \to x \neq a$, then $d(x_n, a) = d(x, a)$ for all large indices n.*

(xii) *Let $W \subset X$ be compact. Then, for every $a \in X \setminus W$, the set of distances $\{d(x, a) | x \in W\}$ is finite.*

(xiii) *Let* $W \subset X$ *be compact. Then, for every* $a \in W$*, the set* $\{d(x,a) \mid x \in W \setminus \{a\}\}$ *is discrete in* $\mathbb{R}_{>0}$ *of distances (w.r.t. the topology on the multiplicative topological group* $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$*).* □

It is now our goal to obtain a "realistic" model (*i.e.*, a "visualisation") of an ultrametric which reflects the above properties in some sense.

3c.2. Visualisation as Trees

In this section, we introduce the method used in [180] to visualise ultrametric spaces (also see [179, Section 2][1]).
In view of Remark 3.55, we make the following definition.

Definition 3c.1. A *valued field* is a field K with a valuation $v : K^\times \to \mathit{\Gamma}$ (extended to all of K by the convention $v(0) = +\infty$), where $\mathit{\Gamma}$ is a totally ordered Abelian group, such that v is a surjective homomorphism with $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$. The group $\mathit{\Gamma}$ is also called the *value group*.

Remark 3c.2. In Section 3.3 (see Remark 3.55), the choice $\mathit{\Gamma} = \mathbb{Z}$ for the value group is used. Here, we allow some more flexibility: the value group $\mathit{\Gamma}$ can be $q \cdot \mathbb{Z}$ (for some $q \in \mathbb{R}$; of course, in this case $\mathit{\Gamma} \cong \mathbb{Z}$) or even $\mathbb{Q}$.

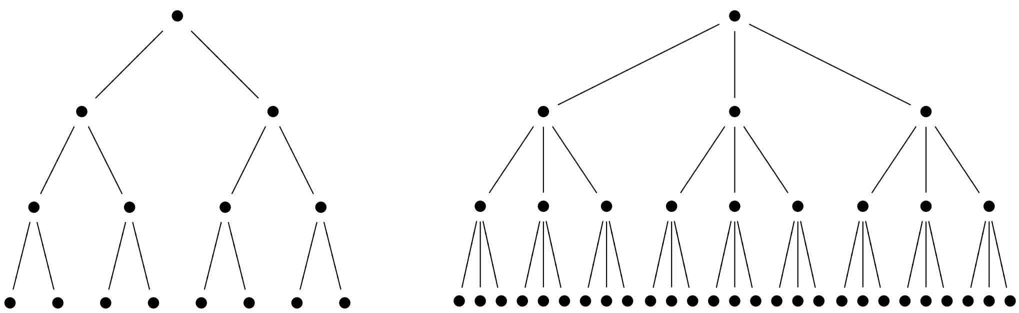
This flexibility is used as follows: For $\mathbb{Q}$ (or for $\mathbb{Q}_p$) we use the p-adic valuation v_p (where the value group is $\mathbb{Z}$) with normalisation $1/p$, *i.e.*, $\|x\|_p = p^{-v_p(x)}$, see p. 30. If we now look at an algebraic number field (or a finite extension of the p-adic field $\mathbb{Q}_p$), we keep the normalisation $1/p$ but change the value group to $\frac{1}{e}\,\mathbb{Z}$ or $f\,\mathbb{Z}$ (or whatever else is appropriate), depending on what properties of the absolute value we are interested in. The value group $\mathit{\Gamma}$ is then given by $\mathit{\Gamma} = \{-\log_p \|x\|_p \mid x \neq 0\}$.

We are now aiming at a graph-theoretic model for ultrametric spaces.

Definition 3c.3. A graph $G(V,E)$ (where V denotes the vertex set and $E \subset V \times V$ the edge set) is called a *tree* if it is connected and cycle-free. A *rooted tree* $(T(V,E), v)$ is a tree with a distinct vertex v called the *root* of $T(V,E)$. Since in a tree there is exactly one path between any two vertices, we have, in particular, exactly one path between the root and any vertex and we therefore can think of the edges in a rooted tree being *directed* away from the root. A vertex x is a *predecessor* of a vertex y (and y a *successor* of x) if x is contained in the path from v to y; we write $x < y$ in this case. If $x < y$ and there is no vertex z such that $x < z < y$, we say that x is a *direct predecessor* (or *parent*) of y (y is a *direct successor* or *child* of x). Similarly, we define (direct) predecessor and successor for edges. Obviously, the root v has no predecessor (all other vertices, of course, have at least one predecessor, namely v). A vertex with no successor is called a *leaf* or *end-vertex*. A vertex which is not a leaf, is called an *inner vertex* of the tree. A rooted tree $(T(V,E), v)$ is called a *m-ary tree* if all inner vertices have m direct successors.

[1] My favourite statement in that article is [179, Proposition 3.6]: "If S and T are Swiss cheeses in a valued field, such that $S \cap T \neq \varnothing$, then $S \cup T$ and $S \cap T$ are Swiss cheeses."

A binary ("2-ary") and a ternary ("3-ary") tree may look as follows:



We now recall Lemma 3.68, and then we want to visualise a complete discrete valuation ring as a m-ary tree.

Lemma 3.68 (First part). *Let R be a complete discrete valuation ring, with residue field $k = R/\mathfrak{m}(R)$ (and field of fractions K). Let S be a system of representatives of k, and π a uniformiser of R. Then every element $a \in R$ can be written uniquely as a convergent series*

$$a = \sum_{n=0}^{\infty} s_n \pi^n \quad \text{with } s_n \in S. \qquad \square$$

Given a complete discrete valuation ring R with residue field k, let $m = \operatorname{card} k = \operatorname{card} S$. Then we can identify each element $a \in R$ with an infinite path in the *infinite m-ary tree*, where by an *infinite rooted tree* we mean a rooted tree such that all paths starting at the root of length n can be extended to a path starting at the root of length $n+1$ for all $n \in \mathbb{N}$. For this, label the m direct successors of each (inner) vertex with the m elements of S. Then an infinite path w in this tree starting at the root can be uniquely be described by the labels of the edges it runs through, *e.g.*, $w = s_0 s_1 s_2 \ldots$. Of course, this path is identified with the number $\sum_{n=0}^{\infty} s_n \pi^n \in R$, and this identification is a bijective map between infinite paths in the tree starting at the root and R; we write $w(a)$ for the path corresponding to $a \in R$ (*i.e.*, the bijective map from R to the space of infinite paths is denoted by w). We also note that an infinite path is uniquely determined by the "leaf" (at infinity).

We now want to equip the paths in the tree with a metric $\tilde{d}$ such that, under the above identification of paths and elements of discrete valuation rings, $\tilde{d}$ corresponds to the ultrametric d on R.

Definition 3c.4. Let w and w' be two walks (*i.e.*, not necessarily of different edges but along directed edges) in a directed graph G, where w is finite, *i.e.*, w runs only through finitely many edges. We call the number of edges a path w runs through, its *length* $\#w$. By ww' we denote the path obtained by *concatenation* (or *juxtaposition*), provided it is also a path in G; *i.e.*, ww' is the path where we first run through w and then through w'. If w is a *prefix* (or *curtailment*) of w', *i.e.*, if w' starts with w ($w' = w \ldots$), we write $w \lhd w'$. By $w \barwedge w'$ we denote the maximal path such that both $w \barwedge w' \lhd w$ and $w \barwedge w' \lhd w'$, *i.e.*, the maximal common prefix.

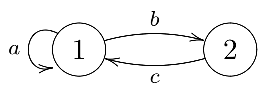
Definition 3c.5. Let $G(V, \vec{E})$ be a directed graph (*e.g.*, an infinite rooted tree), where we indicate by $\vec{E}$ that the edges are directed. We now construct a *path space* $\mathcal{E}_i^{\infty}$ for each vertex $i \in V$ as follows: We start at the vertex i and consider all infinite walks in the graph along directed edges. Each walk is indexed by the sequence (of labels) of the edges it runs through.

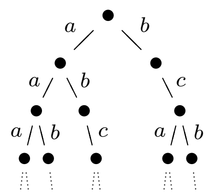
We define the following sets: $\mathcal{E}_i^0 = \varnothing$ and $\mathcal{E}_i^k$ is the set of all walks of length k with initial vertex i. The set of all finite walks with initial vertex i is denoted $\mathcal{E}_i = \bigcup_{k=0}^{\infty} \mathcal{E}_i^k$, and the set of all infinite walks with initial vertex i by $\mathcal{E}_i^\infty$. We topologise $\mathcal{E}_i^\infty$ in a natural way using the metric $\tilde{d} : \mathcal{E}_i^\infty \times \mathcal{E}_i^\infty \to \mathbb{R}$,

$$\tilde{d}(w, w') = \begin{cases} \eta^{\#(w \bar{\wedge} w')}, & \text{if } w \neq w', \\ 0, & \text{if } w = w', \end{cases} \tag{3c.1}$$

where $w, w' \in \mathcal{E}_i^\infty$ and $0 < \eta < 1$. This makes $\mathcal{E}_i^\infty$ into a compact ultrametric space (see Lemma 3c.8 below). If we define $\mathcal{N}_i(w) = \{w' \in \mathcal{E}_i^\infty \mid w \triangleleft w'\}$, the sets $\{\mathcal{N}_i(w) \mid w \in \mathcal{E}_i\}$ form a base of clopen sets for $\mathcal{E}_i^\infty$. We call a set of finite paths $A \subset \mathcal{E}_i$ a *covering set* for $\mathcal{E}_i^\infty$ if $\mathcal{E}_i^\infty \subset \bigcup_{w \in A} \mathcal{N}_i(w)$. Thus A is a covering set if for every $w' \in \mathcal{E}_i^\infty$ there exists a $w \in A$ with $w \triangleleft w'$.

Remark 3c.6. A path space $\mathcal{E}_i^\infty$ corresponding to a directed graph $G(V, \vec{E})$ can naturally be embedded onto an infinite rooted tree (and the path space of this tree is then homeomorphic to $\mathcal{E}_i^\infty$): Just connect all paths to a root such that two paths w, w' also share the edges of $w \bar{\wedge} w'$. For example, the path space $\mathcal{E}_1^\infty$ of the following directed graph and the one of the rooted tree are homeomorphic, and we speak of a path space[2] and not of a "walk space".





Conversely, we observe that the path space of the infinite m-ary tree is homeomorphic to the path space of the graph with only one vertex v and m directed loops (*i.e.*, m directed edges from v to v).

Remark 3c.7. These definitions can also be used for words and sequences (instead of paths). For now, however, we look at paths in infinite rooted trees, where we use the direction given by the root. But these definitions will be used again in Section 4.9 in a different context.

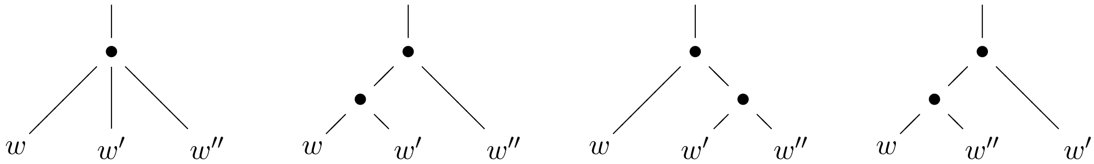
Lemma 3c.8. *Let R be a complete discrete valuation ring with residue field k and $m = \operatorname{card} k$. Then there is an $0 < \eta < 1$ such that the distance $d(x, x') = |x - x'|_{\mathfrak{m}(R)}$ between $x, x' \in R$ equals the distance $\tilde{d}(w(x), w(x'))$ between the corresponding infinite paths $w(x), w(x')$ in the infinite m-ary tree, where the metric $\tilde{d}$ for two paths w, w' is defined as in Equation* (3c.1).

Proof. We first show that $\tilde{d}$ is indeed an ultrametric in this case (this proof can easily be altered in case of a general directed graph, see Remark 3c.6). Obviously, $\tilde{d}$ is positive definite and symmetric. Therefore, we have to show the strong triangle inequality, *i.e.*, that all triangles are isosceles. We observe that two paths are close to each other if they have a long common prefix. Given three paths w, w' and w'' in a (rooted) tree, there are four possibilities[3] (we

[2]We have chosen the name "path space" as in [112, Section 2.5]. There, path spaces are introduced in connection with (graph-directed) iterated function systems. We will use and study path spaces associated to iterated function systems extensively in Section 4.9.

[3]In a binary tree the first depicted possibility does not occur.

name each leaf according to the path that leads to it) for their relationship (the following figure can also be found in [180, Fig. 4]).



One observes that two paths w, w' are close to each other (*i.e.*, they have a large common prefix $w \barwedge w'$), then their distance to a more distant path w'' (see the second case) must be the same (*i.e.*, $w \barwedge w'' = w' \barwedge w''$). But this is the strong triangle inequality. This applies analogously to the third and fourth case, while the first case even codes a regular triangle.

Given $a, b \in R$, observe that we obtain for the valuation $v_{\mathfrak{m}(R)}(a-b)$ of the difference $a-b$ (see Remark 3.71)

$$v_{\mathfrak{m}(R)}(a-b) = \min\{k \mid a_k - b_k = 0\} = \min\{k \mid a_k = b_k\} = \#(w(a) \barwedge w(b)),$$

and therefore

$$d(a,b) = \eta^{v_{\mathfrak{m}(R)}(a-b)} = \eta^{\#(w(a) \barwedge w(b))} = \tilde{d}(w(a), w(b)).$$

The η we are looking for is therefore the same η that appears in the definition of the norm on R as normalisation, see p. 30. □

Let v' be a vertex in the rooted tree $(T(E,V), v)$. We call the length of the path from the root v to the vertex v' the *depth* of v', denoted by $\mathrm{depth}(v')$. Two paths w, w' in the tree (starting at the root) which share a common prefix but separate at v', have distance $\tilde{d}(w, w') = \eta^{\mathrm{depth}(v')}$. We now visualise balls: Given a path w and a distance r, the open (respectively closed) ball $B_{<r}(w)$ ($B_{\leq r}(w)$) in the tree is just the (whole) subtree starting[4] at a vertex v' of the path w with appropriate depth. With this picture of a ball in mind, it is easy to see some of the surprising properties of balls in ultrametric spaces: Balls are subtrees of a tree. Therefore, two balls are either disjoint or one is contained in the other; the second case occurs iff they share a path. Also, any path of a ball is the centre of this ball because for a fixed radius r the ball is always the same subtree starting at a certain vertex (informally, we can also say that the ordering[5] of the (direct) successors in the pictures of the trees is arbitrary, therefore we can always order a tree such that any path is in the "centre", where as "path in the centre" we might define some path that runs "in the middle" of the tree in the picture). Also, for such discrete structures as trees, it is less surprising that balls are clopen. And if we only look at the leaves, we also might get a feeling for the total disconnectedness of ultrametric spaces (also see the next section).

Remark 3c.9. We note that one actually represents Cauchy sequences by the tree-picture, each limit of a Cauchy sequence corresponding to one infinite path. Since we look at complete rings

[4]By a *subtree starting at a vertex* v' we mean the induced subgraph of the tree consisting of the vertices v' and all successors of v' (and all the edges between them), *i.e.*, the graph spanned by v' and all its successors.

[5]We also note that there is no natural total ordering in a discrete valuation ring R, which can also be seen by the fact that the choice of the uniformiser π is arbitrary, also see Remark 3c.13.

(or fields), we have a bijection between paths and elements of that ring (or field). This point of view will play a role in Remark 3c.11.

We now look at an example, namely $\mathbb{Z}_3$.

Example 3c.10. We want to draw the tree that represents $\mathbb{Z}_3$. For this, we first recall some 3-adic representations[6] of some numbers[7] (for the notation, recall Remark 3.71):

$$0 = .\overline{0}, \quad 1 = .1\overline{0}, \quad 8 = .22\overline{0}, \quad 9 = .001\overline{0}, \quad 17 = .221\overline{0},$$
$$-1 = .\overline{2}, \quad -2 = .1\overline{2}, \quad -13 = .211\overline{2}, \quad -17 = .101\overline{2},$$
$$\frac{1}{2} = .2\overline{1}, \qquad -\frac{3}{2} = .0\overline{1}, \qquad \frac{3}{4} = .01\overline{20}, \qquad \frac{1}{17} = .\overline{2202110010201122 1},$$
$$\sqrt{7} = .2112022011020001201012 1012\ldots \quad \text{and/or} \quad .1110200021120222102121 0121\ldots.$$

For the computation of $\sqrt{7}$, we refer to [153, Section 1.2] (One solves the congruences $x^2 \equiv 7 \bmod 3^k$ for all $k \in \mathbb{N}$ to obtain the representation. Noting that $x^2 = 7$ has the solutions $\pm\sqrt{7}$, one can say that one of the above representations corresponds to $-\sqrt{7}$ and one to $+\sqrt{7}$ (although it is not a priori clear which one to which – of course, this is just the statement that algebraically one can not distinguish the different roots of an irreducible polynomial); indeed their sum as 3-adic numbers is 0. But we note that $x^2 - x - \frac{3}{2} = 0$ has the roots $\frac{1}{2} \pm \frac{1}{2}\sqrt{7}$, where one of them is greater than 1 in modulus and one of them is less than 1 in modulus. Also, as 3-adic roots of this last equation we obtain

$$\frac{1}{2} + \frac{1}{2}\sqrt{7} = .2\overline{1} + .2\overline{1}\,\sqrt{7} = .012212210212111010210\ldots \text{ and/or } .120010012010111212012\ldots.$$

The 3-adic norm of the first number is less than 1, wherefore (if we are interested in the solution of the polynomial $x^2 - x - \frac{3}{2} = 0$) we might justify the choice $-\sqrt{7}$ for the first representation above (so, metrically we can distinguish the different roots of an irreducible polynomial; we observe that the polynomial $x^2 - x - \frac{3}{2}$ is *not* irreducible over $\mathbb{Z}_3$). This procedure of distinguishing the roots of polynomials "metrically" will play an important role in Chapter 6, especially see the examples considered there.

The tree for $\mathbb{Z}_3$ is shown in Figure 3c.1 (compare to [180, Fig. 2]) on p. 66.

The next task is to extend the tree picture from the valuation ring R to its field of fractions K. For this, we recall the second part of Lemma 3.68.

Lemma 3.68 (Second part). *Let R be a complete discrete valuation ring, with residue field $k = R/\mathfrak{m}(R)$ and field of fractions K. Let S be a system of representatives of k, and π a uniformiser of R. Then every element $a \in K$ can be written uniquely as a convergent series*

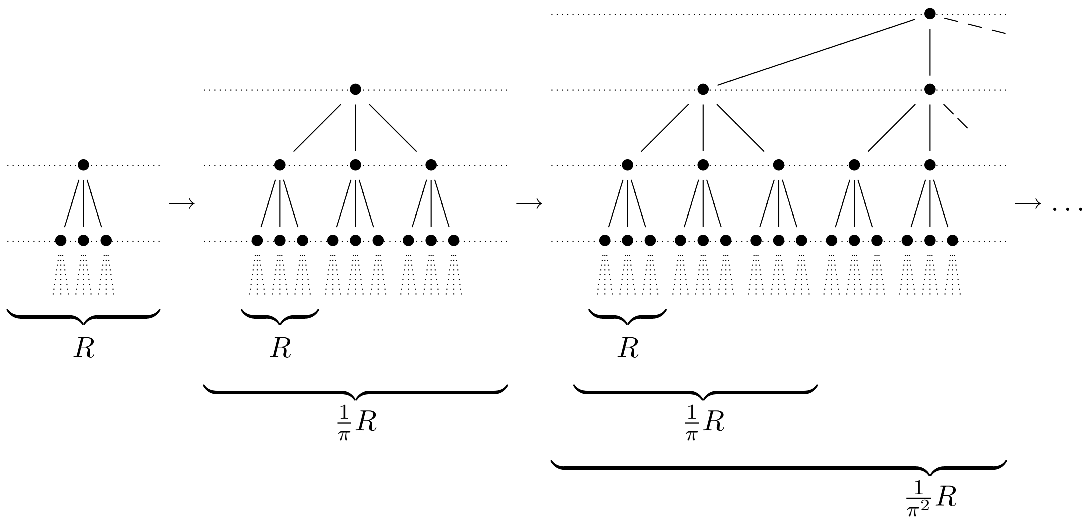
$$a = \sum_{n=m}^{\infty} s_n \pi^n \quad \text{with } s_n \in S \text{ and } m \in \mathbb{Z}. \qquad \square$$

While we have interpreted a valuation ring R as the set of all one-sided infinite sequences on the alphabet S (*i.e.*, one can identify R and $S^{\mathbb{N}}$), its field of fractions K can be interpreted as the set of all two-sided infinite sequences with a prefix consisting of infinitely many 0's,

[6] A nice elementary book about calculus with p-adic numbers is [289].

[7] We note that the period length of a rational number (in any number system) is a divisor of Euler's totient function of its denominator (recall that $\phi(17) = 16$).

i.e., we identify K and the set $\{(s_i)_{i\in\mathbb{Z}} \mid s_i \in S, \exists m : s_n = 0 \text{ for all } n \leq m\}$. By this, we can still use the tree-picture for K: R is a subtree in K (which shows that R is compact in K), where K is an infinite m-ary tree (where again $m = \operatorname{card} S$), for which the root is now located "at depth $-\infty$". We might think of obtaining the tree for K by starting with the tree for R and then successively interpreting this tree by a infinite m-ary tree with depth reduced by one, therefore giving meaning to the formula $K = R[\frac{1}{\pi}]$ where π denotes a uniformiser. Schematically, we have the following picture (here for a ternary tree) in obtaining K from R:



Here the depth basically denotes the member of the value group: If a path deviates from the 0-path $(0)_\mathbb{Z}$ at depth d, the (exponential) valuation of the corresponding number in K is d.

As an example, we show the corresponding tree representing $\mathbb{Q}_3$ in Figure 3c.2 (it also shown in [180, Fig. 3]).

A $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$ is a finite extension of the p-adic numbers $\mathbb{Q}_p$ of degree n. Its ramification index $e = e_{\mathfrak{p}|p}$ and its residue degree $f = f_{\mathfrak{p}|p}$ are related by $n = e \cdot f$. We can now also use our tree-picture to visualise what happens if we go from $\mathbb{Q}_p$ to $\mathbb{Q}_\mathfrak{p}$: If $f > 1$, then the tree changes from an infinite m-ary tree to an infinite (m^f)-ary tree. If $e > 1$, then the number of "depths-levels" increases by a factor of e, *i.e.*, if we use the uniquely extended absolute value of Lemma 3.75 and define the value group by $\mathit{\Gamma} = \{-\log_p(\|x\|_\mathfrak{p}) \mid x \in \mathbb{Q}_\mathfrak{p}^\times\}$ (therefore using $\eta = \frac{1}{p}$ for $\mathbb{Q}_p$ and $\mathbb{Q}_\mathfrak{p}$), then we get[8] $\mathit{\Gamma} = \frac{1}{e}\mathbb{Z}$ (where, of course, the value group of $\mathbb{Q}_p$ is $\mathbb{Z}$). We also recall Proposition 3.106: For an unramified extension ($e = 1$, $f > 1$), the new system of representatives for the residue field is given by 0 and the $(p^f - 1)$st roots of unity, while for a totally ramified extension ($f = 1$, $e > 1$), at least if it is also tamely ramified, the new uniformiser Π can be chosen such that $\Pi = \sqrt[e]{\pi}$. Moreover, we recall that every finite extension can be obtained by an unramified extension followed by a totally ramified extension. Therefore, schematically the situation is as follows (where we indicate by a double-line those

[8]In Remark 3c.11 we will use this value group for a $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$.

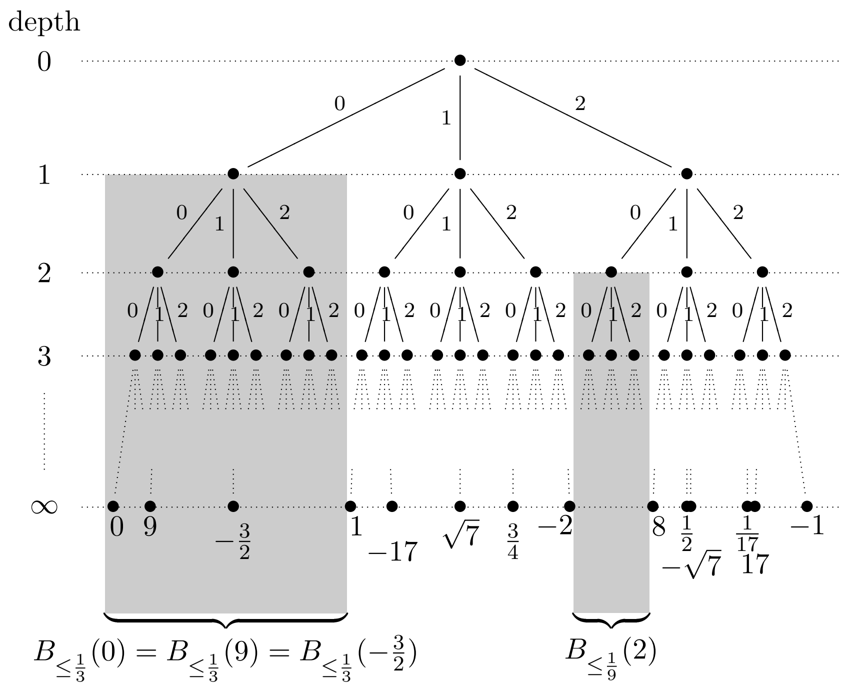
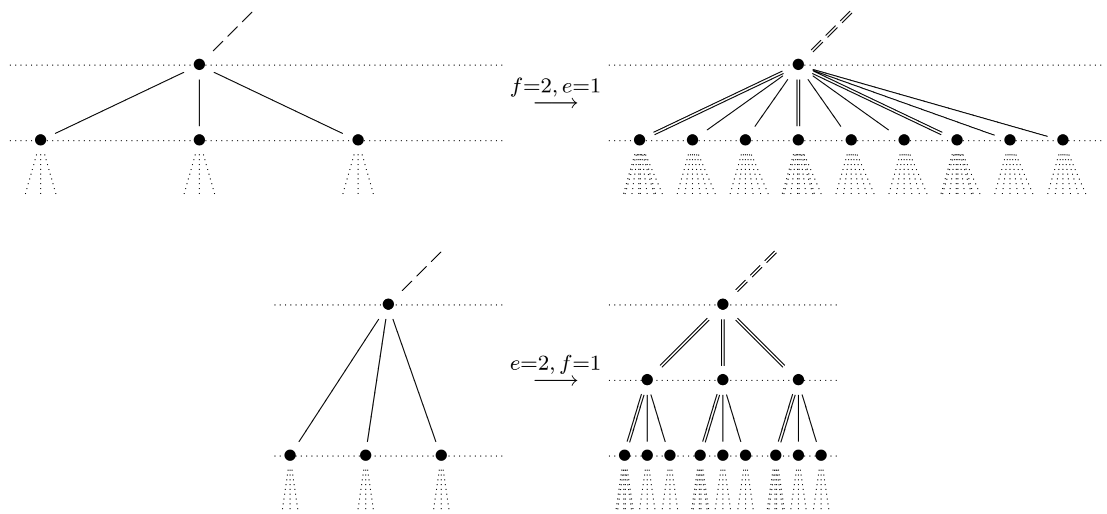


Figure 3c.1.: The tree representing the 3-adic integers $\mathbb{Z}_3$. We have also marked two balls, where we use the usual normalisation for the 3-adic ultrametric such that $\|3\|_3 = \frac{1}{3}$ (*i.e.*, $\eta = \frac{1}{3}$ and value group $\Gamma = \mathbb{Z}$).

paths in $\mathbb{Q}_\mathfrak{p}$ which correspond to numbers in $\mathbb{Q}_p$):



Instead of "depth", we also use the elements of the valuation group to label the "levels" in the tree.

Remark 3c.11. While every finite extension of $\mathbb{Q}_p$ is again a complete $\mathfrak{p}$-adic field, the algebraic closure $\mathbb{Q}_p^a$ is not complete. Its completion is denoted by $\mathbb{C}_p$; both $\mathbb{Q}_p^a$ and $\mathbb{C}_p$ are valuated

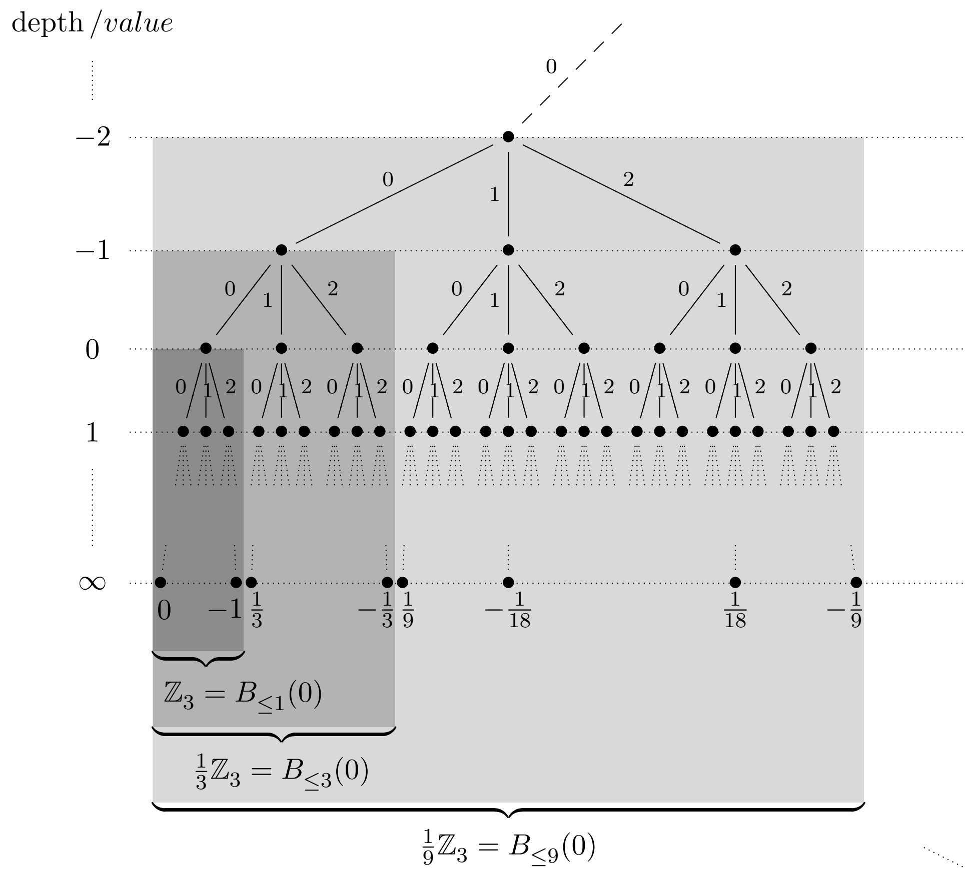


Figure 3c.2.: The tree representing $\mathbb{Q}_3$.

fields with value group $\mathbb{Q}$ (we denote the extended p-adic absolute value on $\mathbb{Q}_p^a$, respectively $\mathbb{C}_p$ by $\| \cdot \|_p'$). We also note that the countable set of algebraic numbers $\mathbb{Q}^a$ is dense in both $\mathbb{Q}_p^a$ and $\mathbb{C}_p$, therefore they are separable spaces. We now try to use our experience with the tree-picture to visualise $\mathbb{C}_p$. We first consider a special subfield of $\mathbb{Q}_p^a$: Starting from $\mathbb{Q}_p$, we adjoin all roots of unity having order coprime to p to obtain the maximal unramified extension $\mathbb{Q}_p^{unr}$. We note that for $\mathbb{Q}_p^{unr}$ the value group is still $\mathbb{Z}$, but the residue field is isomorphic to $\mathbb{F}_p^a$, the algebraic closure of the finite field $\mathbb{F}_p$ with p elements ($\mathbb{F}_p^a$ is also isomorphic to $\mathbb{F}_{p^\infty} = \mathsf{C}_{p^\infty}$, the p-quasicyclic group, see Definition 3.118 and Section 3a.2). We remark that $\mathbb{Q}_p^{unr}$ can be used to construct a Cauchy sequence which does not converge in $\mathbb{Q}_p^a$ by using higher and higher roots of unity in each step (see [153, Proof of Theorem 5.7.4]), showing that $\mathbb{Q}_p^a$ is not complete. We may think of the set of limits of Cauchy sequences in $\mathbb{Q}_p^{unr}$ as infinite p^∞-ary tree (where we understand p^∞ to mean $p^{\aleph_0}$).

Of course, $\mathbb{C}_p$ is obtained by adjoining roots of unity and roots of Eisenstein polynomials, see Proposition 3.106. The valuation ring of $\mathbb{C}_p$ is given by the closed unit ball $B_{\leq 1} = \{x \in \mathbb{C}_p \mid \|x\|_p' \leq 1\}$, its maximal ideal is given by the open unit ball $B_{<1} = \{x \in \mathbb{C}_p \mid \|x\|_p' < 1\}$, their quotient is the residue field which is isomorphic to $\mathbb{F}_p^a$. This also shows that one needs $p^{\aleph_0}$ many such disjoint open balls $B_{<1}$ to cover the valuation ring, indicating that $\mathbb{C}_p$ is not locally compact (for details, see [153, Hint 270 to Problem 269] and [326, Corollary 12.2]).

By adjoining roots of Eisenstein polynomials, we “increase” the value group from $\mathbb{Z}$ to $\mathbb{Q}$, *e.g.*, $\mathbb{Q}_p(\sqrt{p})$ is a totally ramified extension of $\mathbb{Q}_p$ for every $p \in \mathbb{P}$ with $e = 2$ and $f = 1$ (in fact, there are two such fully and wildly ramified quadratic extensions for $p \neq 2$ and six for $p = 2$, see Proposition 3.109), therefore $\|\sqrt{p}\|_p' = p^{-\frac{1}{2}}$. Furthermore, a root a of $x^c - p^d = 0$ has norm $\|a\|_p' = p^{-\frac{d}{c}}$ (see [153, Hint 268 to Problem 267]). But the denseness of the value group in $\mathbb{R}$ not only shows again that the valuation ring (the closed unit ball) of $\mathbb{C}_p$ is not compact (see Lemma 2.26'(xiii)) but it also destroys the *spherical completeness*[9] of $\mathbb{C}_p$, see [313, Proof in Section III.3.4], [326, Theorem 20.5] and [331, p. 4]. It is hard to "visualise" a tree which is not spherical complete (because one thinks of a tree as being a discrete structure), but an attempt is made in [313, Exercise III.7]. Still, since all triangles are isosceles and balls are "subtrees" in $\mathbb{C}_p$, it might still be helpful to visualise $\mathbb{C}_p$ on the basis of the previous considerations (see [180, Figs. 6 &7]).

We can justify and re-interpret the tree-picture for an ultrametric space X (with countable value group $\Gamma \subset \mathbb{R}$) as follows: For each $n \in \Gamma$, we denote by $r(n)$ the distance of two points $x, y \in X$ such that its value is n (*i.e.*, $r(n) = \eta^{-n} = \eta^{-v(x-y)}$), then the set $\{B_{\leq r(n)}(x) \,|\, n \in \Gamma\}$ is a fundamental system of neighbourhoods of closed balls of x. Of course, $\{B_{\leq r(n)}(y) \,|\, n \in \Gamma\}$ is a fundamental system of neighbourhoods of closed balls of y. Now, if two balls have a common point, one is contained in the other. We therefore look at the intersections $B_{\leq r(n)}(x) \cap B_{\leq r(n)}(y)$ for all $n \in \Gamma$. If such an intersection is nonempty, both balls are equal (since they are closed balls of equal radius), and, of course, the balls of bigger radius also have nonempty intersection. We can therefore construct a tree, now starting with the leaves: Identifying the leaves with the points in X, the paths for $x, y \in X$ (or "leading to" $x, y \in X$) separate at a vertex of value $\tilde{n}$ where $\tilde{n} = \max\{n \in \Gamma \mid B_{\leq r(n)}(x) \cap B_{\leq r(n)}(y) \neq \varnothing\}$.

We also note that the product of two ultrametric spaces can still be represented in one(!) tree. *E.g.*, consider the product $\mathbb{Z}_2 \times \mathbb{Z}_3$. Then $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ has the representation $x = .s_0\, s_1\, s_2 \ldots$ with $s_i \in \{0\,,1\}$ and $y = .s_0'\, s_1'\, s_2' \ldots$ with $s_i' \in \{0\,,1\,,2\}$. We use the obvious system of representatives for this product, namely $\tilde{S} = \{0\,,1\} \times \{0\,,1\,,2\} = \{(0\,,0), (0\,,1), (0\,,2), (1\,,0), (1\,,1), (1\,,2)\}$, therefore the "product tree" of a binary and a ternary tree is a 6-ary (sextary) tree, *i.e.*, we might represent the point (x, y) as $.\tilde{s}_0\, \tilde{s}_1\, \tilde{s}_2 \ldots$ where $\tilde{s}_i = (s_i\,, s_i') \in \tilde{S}$. Furthermore, we can think of the product of a (compact) ultrametric space X with a finite group G either as *forest* of $\operatorname{card} G$ disjoint rooted trees or as rooted tree (T, v), where the $\operatorname{card} G$ roots of the trees representing the $\operatorname{card} G$ copies of X are connected to the root v (which we may actually call the "super-root"). Especially, all finitely generated profinite Abelian groups can be visualised as trees, compare to Proposition 3a.20.

Remark 3c.12. In an infinite m-ary ($m < \infty$) tree one can obviously define a measure as follows: We give $B_{\leq 1}(0)$ measure 1. Now $B_{\leq 1}(0)$ is the union of m disjoint clopen balls with radius less than 1, *i.e.*, $B_{\leq 1}(0) = \bigcup_{s \in S} B_{<1}(s)$. We give each of them measure $1/m$, wherefore balls (or subtrees) starting at vertices of the same value (depth) have the same measure. Iterating this procedure (and extending to Borel sets) yields a measure on the tree, which is its Haar measure (unique up to a multiplicative constant, namely the choice of the measure of $B_{\leq 1}(0)$). This also indicates why the normalised absolute value, which is connected to the cardinality of the residue field (see p. 30 and Definition 3b.10) and therefore to the number of direct successors in the associated tree, is used in measure theory on $\mathfrak{p}$-adic fields, see Proposition 4.38.

[9] An ultrametric space is *spherically complete* if each nested sequence of closed balls has nonempty intersection. Spherical completeness is a concept one uses for example in functional analysis, see [326, Appendix A.8] and [331].

3c.3. “Embedding” into the Reals

Since we have already introduced the tree picture for an ultrametric space, it is now clear how to embed an ultrametric space into the reals (or – more general – into an Euclidean space $\mathbb{R}^n$): We already identified a path (representing an element of the space) with its leaf. Therefore, such an embedding is just a map from the leaves to (a subset of) $\mathbb{R}$, and pictures of trees for $\mathbb{Z}_3$, respectively $\mathbb{Q}_3$, in the last section already indicate how this can be done. The only task is to ensure that this map is bijective (or – at least – “essentially” bijective) and one would like to have a continuous map.

From now on we suppose that we are in the following situation: Either X is a $\mathfrak{p}$-adic field, a (finite) product of $\mathfrak{p}$-adic fields or some (compact) subset thereof. Then $x \in X$ is represented as a series $x = s_n\, s_{n+1}\, s_{n+2} \ldots$ for some $n \in \mathbb{Z}$ and $s_i \in S$ where S is some finite set of representatives. If we set $m = \operatorname{card} S$, then X can be represented as an infinite m-ary tree. If we fix a bijective map $\psi : S \to \{0, \ldots, m-1\}$ (with $\psi(0) = 0$), we can define a map $\varphi : X \to \mathbb{R}$ as follows:

- $\varphi(x) = \frac{m}{m+1} \sum_{j\in\mathbb{Z}} \psi(s_j) \cdot m^{-2j}$, which is injective and continuous and gives a sparse Cantor-like subset of the nonnegative reals. This choice follows [383, Equation I.6.1].
- In fact, every map $\varphi(x) = \frac{b-1}{m-1} \sum_{j\in\mathbb{Z}} \psi(s_j) \cdot b^{-j-1}$, where $b > m$, is a continuous and injective map, as shown in [313, Section 1.2.3]. In the case of $\mathbb{Q}_{\mathfrak{p}}$, the $\mathfrak{p}$-adic integers $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ are mapped into a Cantor-like (and therefore totally disconnected) subset of the interval $[0\,,1[$.
- $\varphi(x) = \frac{1}{m} \sum_{j\in\mathbb{Z}} \psi(s_j) \cdot m^{-j}$, which is continuous and “almost” injective, because the points $s_n \ldots s_{k-1}\, s_k \overline{m-1}$ and $s_n \ldots s_{k-1}\,(s_k + 1)\overline{0}$ (here, we identified s_i and its image $\psi(s_i)$) are mapped to the same real number. In the case of $\mathbb{Q}_{\mathfrak{p}}$, the $\mathfrak{p}$-adic integers $\widehat{\mathfrak{o}_{\mathfrak{p}}}$ are mapped onto the interval $[0\,,1]$, and the p^f (cl)open balls of radius less than 1 are mapped to the p^f intervals $[0\,,p^{-f}]$, $[p^{-f}\,,2\,p^{-f}]$, …, $[(p^f-1)\,p^{-f}\,,1]$.

We shall use the last map in the following although it is not bijective, bearing in mind that it is the limit $b \to p$ of the second possibility. It also resembles the well-known phenomenon $0.\overline{9} = 1$ *etc.* from the decimal expansion.

Remark 3c.13. Obviously, such an injective map φ induces an order on the ultrametric space X, see [383, Section I.4]. But this is no meaningful order, since the choice of the uniformiser as well as the map ψ bear some inherent arbitrariness (in fact, one could even let the bijective map ψ depend on the place j in the expansion (s_j), *e.g.*, use different maps for the odd and even indices). A different approach for an order-like structure based on balls is discussed in [326, Section 24].

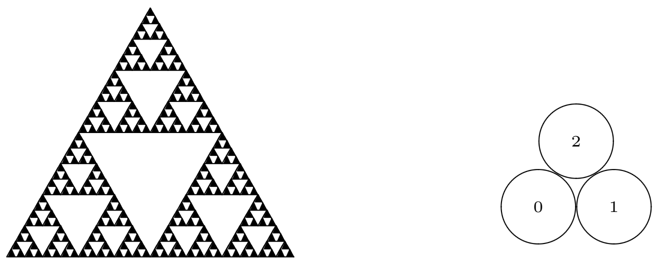
Remark 3c.14. We note that balls (subtrees) of the same radius are mapped to structures of the same size (or diameter) and shape. This is especially desirable in view of measure theory, since the Haar measure gives equal measure to balls of the same radius (by translation-invariance).

A Euclidean model of $\mathbb{Z}_7$ in the plane can be found – without any discussion – in books about algebraic number theory, see [326, Frontispiece] and [238, p. 326]. We believe that the discussion is absent because the tree-picture behind such models (and therefore the correct interpretation of such models in terms of topological properties of ultrametric spaces, *i.e.*, the properties stated in Lemma 2.26’) has not been analysed completely, in particular in view of the usefulness of such models.

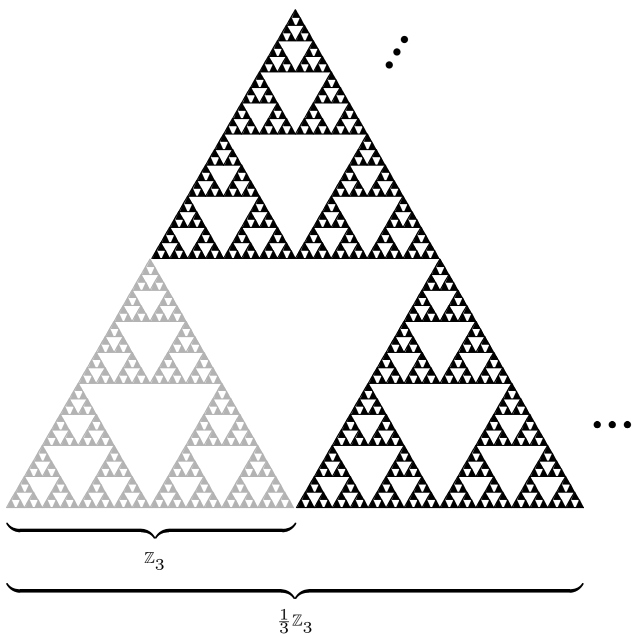
Nevertheless, a systematic approach (again, without reference to the tree-structure) of Euclidean models in $\mathbb{R}^2$ and $\mathbb{R}^3$ was carried out in [99], [312] and [313, Sections 1.2.5 & 1.5.6]. Basically, the only difference to models in $\mathbb{R}$ is that one uses a different bijective map ψ, *i.e.*, $\psi : S \to \mathbb{R}^d$. *E.g.*, if we consider $\mathbb{Z}_3$ (respectively $\mathbb{Q}_3$), we might choose the map $\psi : \{0,1,2\} \to \mathbb{R}^2$, $\psi(0) = (0\,,0)$, $\psi(1) = (\frac{1}{2},0)$ and $\psi(2) = (\frac{1}{4}\,,\frac{1}{4}\sqrt{3})$ and the parameter $b = 2$ in the formula

$$\varphi(x) = C \cdot \sum_{j\in\mathbb{Z}} \psi(s_j) \cdot b^{-j-1},$$

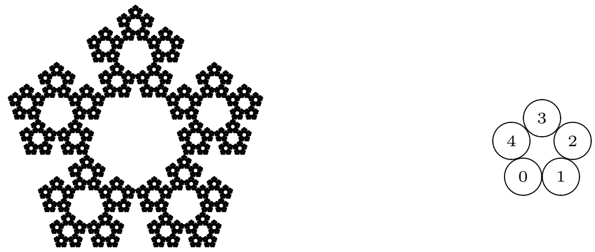
which yields (with normalisation constant $C = 1$) the Sierpinsky gasket/triangle of sidelength 1 as image $\psi(\mathbb{Z}_3)$.

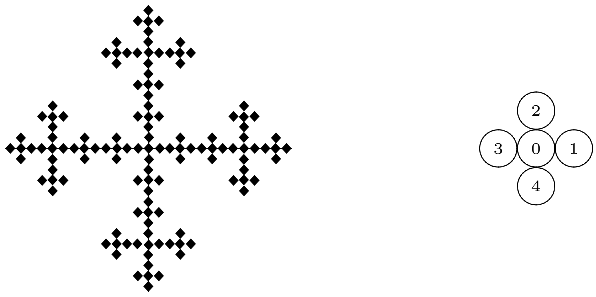


The circles on the right hand side indicate in which part of the figure the 3-adic integers go: The ones of the form .0 . . . go in the lower left, .1 . . . in the lower right and .2 . . . in the upper part of the Sierpinsky triangle (and – by iteration – the ones of form .00 . . . go in the lower left part of the .0 . . . part and so on). Parallel to extending the tree of the valuation ring R to its field of fractions K, we also obtain a model for $\mathbb{Q}_3$:



Obviously, there are many possibilities to visualise p-adic (or more generally, ultrametric) spaces. For example, two possibilities to visualise $\mathbb{Z}_5$ are the following:





Note that, in the latter possibility, the sphere $S_1(0) = \mathbb{Z}_5 \setminus 5 \cdot \mathbb{Z}_5$ (which consists of all 5-adic integers $.1\ldots$, $.2\ldots$, $.3\ldots$ and $.4\ldots$) manifests itself as a ring-like (or anulus-like) structure, *i.e.*, $S_1(0)$ is contained in a ring around the centre in this image.

Of course, it is no coincidence that these models are well-known objects in the theory of "fractals": In these objects, the "self-similarity" of infinite m-ary trees (or, in algebraic terms, of the p-adic expansion) manifests itself. We refer to [51] and Sections 4.8 & 4.10 for more on self-similarity, iterated functions systems *etc.*

Remark 3c.15. The (philosophical) reason, why we can visualise ultrametric spaces as Cantor-like subsets of $\mathbb{R}$, might be that every totally disconnected perfect compact metric space is homeomorphic to the Cantor set. Also, every totally disconnected space has (topological) dimension 0 and therefore can be embedded into a higher dimensional space like $\mathbb{R}$ (of dimension 1). A discussion of the term "dimension" follows in Remark 4.60.

4. Measure Theory: Haar & Hausdorff measures

Um ein Träumer sein zu können, fehlt mir das Geld.

Das Buch der Unruhe – *Fernando Pessoa*

After providing the basic facts in measure theory, we introduce Hausdorff measures *via* the method II construction and compare it with the Haar measure of the corresponding space. At the heart of this chapter, however, are Sections 4.8 – 4.10, where we consider iterated function system on mixed Euclidean and $\mathfrak{p}$-adic spaces and show how to derive bounds for the Hausdorff dimension of the attractor defined through an iterated function system. This extends results by K.J. Falconer obtained in the purely Euclidean (and one component) setting.

4.1. Measure Spaces

We first recall some basic definitions from measure theory.

Definition 4.1. Let X be any space and let $\mathfrak{C}$ be a class of subsets of X (*i.e.*, $\mathfrak{C} \subset \mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the power set of X). We say that $\mathfrak{C}$ is a *σ-algebra* (or a *completely additive class of sets* or *σ-field*) if it satisfies the postulates:

- If $A \in \mathfrak{C}$, then $A^c = X \setminus A \in \mathfrak{C}$.
- If $(A_n)_{n\in\mathbb{N}}$ is any sequence of sets from $\mathfrak{C}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{C}$.

A *σ-ring* is a nonempty class $\mathfrak{C}$ of subsets of X that satisfies the postulates:

- If $A, B \in \mathfrak{C}$, then $A \setminus B \in \mathfrak{C}$.
- If $(A_n)_{n\in\mathbb{N}}$ is any sequence of sets from $\mathfrak{C}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{C}$.

Observe that a σ-algebra is a σ-ring $\mathfrak{C}$ with $X \in \mathfrak{C}$.

Remark 4.2. With this definition, as $A\cap B = (A\cup B)\setminus(A \,\Delta\, B) = (A\cup B)\setminus((A \setminus B) \cup (B \setminus A))$, we also have: If $\mathfrak{C}$ is a σ-algebra and $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets from $\mathfrak{C}$, then $\varnothing \in \mathfrak{C}$, $X \in \mathfrak{C}$, $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{C}$, $\limsup_n A_n \in \mathfrak{C}$ and $\liminf_n A_n \in \mathfrak{C}$, where

$$\limsup_n B_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n \qquad \text{and} \qquad \liminf_n B_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n$$

for a sequence $(B_n)_{n\in\mathbb{N}}$ of sets. Thus, $\limsup_n B_n$ consists of those points in infinitely many B_n, and $\liminf_n B_n$ of those points in all but finitely many B_n.

Definition 4.3. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of sets. In view of $\liminf_n B_n \subset \limsup_n B_n$, we say that $(B_n)_{n\in\mathbb{N}}$ is *convergent* if $\limsup_n B_n = \liminf_n B_n$ and denote this by $\lim_n B_n$. We say that $(B_n)_{n\in\mathbb{N}}$ is *isotone* (or an *expanding sequence*) if $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$. It is *antitone* (or a *decreasing* or *contracting sequence*) if $B_n \supset B_{n+1}$. It is *monotone* if it is either isotone or antitone. Every monotone sequence of sets is convergent.

Definition 4.4. We shall say that a nonnegative extended real valued function μ (*i.e.*, μ takes values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$) is a *measure* provided it satisfies the following postulates:

- The domain of μ is a σ-algebra $\mathfrak{C}$ of X, *i.e.*, $\mu : \mathfrak{C} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets from $\mathfrak{C}$, then $\sum_{n=1}^{\infty} \mu(A_n)$ is defined in the extended real number system and
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$
- $\mu(\varnothing) = 0$.

A measure is *nondecreasing*, *i.e.*, we have $\mu(A) \geq \mu(B)$ whenever $A \supset B$. We also note that for $B \subset A$ and $\mu(B) < \infty$ we have $\mu(A \setminus B) = \mu(A) - \mu(B)$.

Proposition 4.5. [270, Theorem 10.7 & 10.8 & Corollaries 10.8.1 & 10.8.2] *Let μ be a measure on a σ-algebra $\mathfrak{C}$ on X and let $(A_n)_{n\in\mathbb{N}}$ be any sequence of sets from $\mathfrak{C}$. Then*

(i) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(ii) $\mu\left(\liminf_n A_n\right) \leq \liminf_n \mu(A_n)$.

(iii) $\mu\left(\limsup_n A_n\right) \geq \limsup_n \mu(A_n)$ *provided* $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < \infty$.

(iv) $\mu\left(\lim_n A_n\right) = \lim_n \mu(A_n)$ *provided* $(A_n)_n$ *is convergent and* $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < \infty$. □

We now show how one can obtain a measure from an outer measure by restricting the class of subsets of X.

Definition 4.6. We shall say that a nonnegative extended real valued function μ^* is an *outer measure* provided it satisfies the following postulates:

- The domain of μ^* is the class of all subsets of X, *i.e.*, $\mu^* : \mathcal{P}(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- μ^* is nondecreasing.
- If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$
- $\mu^*(\varnothing) = 0$.

Definition 4.7. Let μ^* be an outer measure. We say that a set B is *measurable* with respect to μ^* if for every $A \subset X$ we have $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B)$. Equivalently, B is measurable if for every $A \subset X$ for which $\mu^*(A) < \infty$ we have $\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \setminus B)$. Therefore, measurability of B has nothing to do with the outer measure of B itself; it depends on what B does to the outer measure of other sets.

The key theorem of outer measures is the following.

Proposition 4.8. [270, Theorem 11.2] *Let μ^* be an outer measure, and let $\mathfrak{M}$ be the class of μ^*-measurable sets. Then, $\mathfrak{M}$ is a σ-algebra, and the restriction of μ^* to $\mathfrak{M}$ is a measure.* □

Definition 4.9. Let X be a space with σ-algebra $\mathfrak{C}$ consisting of μ-measurable sets. Then the triple $(X, \mathfrak{C}, \mu)$ is called a *measure space*. Let $(X, \mathfrak{C}, \mu)$ and $(X', \mathfrak{C}', \mu')$ be two measure space, then the sets $A \times A'$ with $A \in \mathfrak{C}$ and $A' \in \mathfrak{C}'$ are called *measurable rectangles*. The σ-algebra on the (Cartesian product space) $X \times X'$ generated by all measurable rectangles is called the *product σ-algebra* and denoted by $\mathfrak{C} \otimes \mathfrak{C}'$. If B is a subset of $X \times X'$ and x is any point in X (x' is any point in X'), we define the *X-section* of B by $B_x = \{y \mid (x, y) \in B\}$ (*X'-section* of B by $B^{x'} = \{y \mid (y, x') \in B\}$). A measure μ defined on a class $\mathfrak{C}$ of measurable subset of X is called *σ-finite* on $\mathfrak{C}$, if there is an isotone sequence of sets $(A_n)_{n \in \mathbb{N}}$ from $\mathfrak{C}$ such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

We are interested in the measures on product spaces.

Lemma 4.10. [162, Theorem §35.B] *If $(X, \mathfrak{C}, \mu)$ and $(X', \mathfrak{C}', \mu')$ are σ-finite measure spaces, then the set function ν, defined for every set $B \in \mathfrak{C} \otimes \mathfrak{C}'$ by*

$$\nu(B) = \int \mu'(B_x)\, \mathrm{d}\mu(x) = \int \mu(B^{x'})\, \mathrm{d}\mu'(x'),$$

is a σ-finite measure with the property that, for every measurable rectangle $A \times A'$,

$$\nu(A \times A') = \mu(A) \cdot \mu'(A').$$

The latter condition determines ν uniquely. We call ν the product measure *of μ and μ' and write $\nu = \mu \otimes \mu'$. Moreover, $(X \times X', \mathfrak{C} \otimes \mathfrak{C}', \mu \otimes \mu')$ is called the* product measure space *of $(X, \mathfrak{C}, \mu)$ and $(X', \mathfrak{C}', \mu')$.* □

Remark 4.11. This lemma also holds for the product of a finite number of σ-finite measure spaces $(X_i, \mathfrak{C}_i, \mu_i)_{i=1}^n$, where one obtains, by noting that the product of measure spaces is associative, the product measure space $(\prod_{i=1}^n X_i, \bigotimes_{i=1}^n \mathfrak{C}_i, \bigotimes_{i=1}^n \mu_i)$, see [162, Section §37]. One can also consider the product of an infinite (countable) number of measure spaces $(X_i, \mathfrak{C}_i, \mu_i)_{i=1}^{\infty}$, provided the measure spaces $(X_i, \mathfrak{C}_i, \mu_i)$ are *probability spaces*, *i.e.*, measure spaces with $\mu_i(X_i) = 1$. Then we define a measurable rectangle as product $\prod_{i=1}^{\infty} A_i$ for which each $A_i \in \mathfrak{C}_i$ and for all but a finite number we have $A_i = X_i$. The σ-algebra generated by all measurable rectangles is again denoted by $\bigotimes_{i=1}^{\infty} \mathfrak{C}_i$. Let $A \in \bigotimes_{i=1}^n \mathfrak{C}_i$, then there exists a unique measure ν (denoted by $\bigotimes_{i=1}^n \mu_i$) on this product σ-algebra with the property that, for every measurable set B of the form $A \times X_{n+1} \times X_{n+2} \times \ldots$,

$$\nu(B) = (\mu_1 \otimes \ldots \otimes \mu_n)(A),$$

see [162, Theorem §38.B]. This construction should look familiar to the restricted product used in the definition of the ring of adeles, see Chapter 3b.

We also need some classes of sets in the following.

Definition 4.12. Let $\mathfrak{C}$ be any class of subsets of X. We define $\mathfrak{C}_\sigma$ as the class of all sets which are countable unions of sets from $\mathfrak{C}$. Similarly, $\mathfrak{C}_\delta$ is the class of all sets which are countable intersections of sets from $\mathfrak{C}$. Obviously, $\mathfrak{C} \subset \mathfrak{C}_\sigma \cap \mathfrak{C}_\delta$. The most widely used application of this construction is the following: Let $\mathfrak{G}$ be the class of all open sets of the topological space X and $\mathfrak{F}$ be the class of all closed sets of X (we have $\mathfrak{G} = \mathfrak{G}_\sigma$ and $\mathfrak{F} = \mathfrak{F}_\delta$). Then $(\mathfrak{G} \cup \mathfrak{F}) \subset (\mathfrak{G}_\delta \cap \mathfrak{F}_\sigma)$ holds. One can also iterate this process by forming $\mathfrak{G}_{\delta\sigma\delta\ldots\sigma\delta} = ((\ldots(((\mathfrak{G}_\delta)_\sigma)_\delta)\ldots)_\sigma)_\delta$ *etc.* Furthermore, we define the class $\mathfrak{B}$ of *Borel sets* as the minimal σ-algebra containing $\mathfrak{F}$ (or, equivalently $\mathfrak{G}$ or the class $\mathfrak{K}$ of all compact sets). We observe that $\mathfrak{G}_{\delta\sigma\delta\ldots\sigma\delta} \subset \mathfrak{B}$ (and similar for $\mathfrak{F}$) for all such iterates. Note that the construction denoted by δ and σ will only be used in connection with the symbols $\mathfrak{F}$ and $\mathfrak{G}$ in the following.

We are also interested in certain properties of measures. We begin with regular outer measures and afterwards turn to metric outer measures.

Definition 4.13. An outer measure μ^* is called *regular* if for every $A \subset X$ there is a measurable set $B \supset A$ such that $\mu^*(B) = \mu^*(A)$. We shall refer to B as a *measurable cover* for A. In these terms, an outer measure is regular if it determines measurable sets in such a way that every set has a measurable cover. If $\mathfrak{C}$ is a class of μ^*-measurable sets, an outer measure μ^* is said to be $\mathfrak{C}$*-regular*, if for each $A \subset X$ there is a $B \in \mathfrak{C}$ such that $A \subset B$ and $\mu^*(B) = \mu^*(A)$.

Lemma 4.14. [270, Theorem 12.1 & Corollary 12.1.1] and [127, Lemma 1.3] *Let μ^* be a regular outer measure.*

(i) *If $(A_n)_{n\in\mathbb{N}}$ is any sequence of sets, then $\mu^*(\liminf_n A_n) \leq \lim_n \mu^*(A_n)$.*

(ii) *If $(A_n)_{n\in\mathbb{N}}$ is any isotone sequence of sets, then $\mu^*(\lim_n A_n) = \lim_n \mu^*(A_n)$.* □

Lemma 4.15. [317, Theorem 10] and [270, Theorem 12.2] *Let μ^* be a regular outer measure. Let B be a μ^*-measurable subset of X with $\mu^*(B) < \infty$. A subset $A \subset B$ is μ^*-measurable, iff $\mu^*(B) = \mu^*(A) + \mu^*(B \setminus A)$.* □

Definition 4.16. Let X be a metric space with metric d. We define the distance of two nonempty sets A and B by $d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$ (and setting $d(A, B) = \infty$ if one of them is empty). We use the same symbol for the distance as for the metric. If A and B are disjoint nonempty sets, we say that A and B are said to be *positively separated* if the distance $d(A, B) > 0$. An outer measure μ^* defined on a metric space X is called a *metric outer measure*, if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ for every pair of disjoint nonempty sets A and B that are positively separated.

Lemma 4.17. [317, Theorems 17–19], [127, Lemmas 1.4 & 1.5] and [270, Theorem 13.2 & Corollary 13.2.1] *Let μ^* be a metric outer measure on a metric space X.*

(i) *Let $(A_n)_{n\in\mathbb{N}}$ be an isotone sequence of sets and let $A = \bigcup_{n=1}^\infty A_n$. If, for each n, the sets A_n and $A \setminus A_n$ are positively separated, then $\mu^*(A) = \sup_n \mu^*(A_n)$.*

(ii) *All closed subsets of X are μ^*-measurable.*

(iii) *All Borel sets in X are μ^*-measurable.* □

Lemma 4.18. [317, Theorem 22] *Let μ^* be a $\mathfrak{G}_\delta$-regular metric outer measure on a measure space X. If B is a μ^*-measurable set with finite μ^*-measure, there is an $\mathfrak{F}_\sigma$-set E with $E \subset B$, $\mu^*(E) = \mu^*(B)$, and, if $\varepsilon > 0$, there is a closed set C with $C \subset B$, $\mu^*(C) \geq \mu^*(B) - \varepsilon$.* □

4.2. Method I & II Constructions

Remark 4.19. The constructions called method I and method II can be found in [89], the terms are attributed to [270].

We start with the so-called method I construction of an outer measure on X.

Definition 4.20. Let $\mathfrak{C}$ be a class of subsets of X such that $\varnothing \in \mathfrak{C}$. We say that $\mathfrak{C}$ is a *sequential covering class* if for every $B \subset X$ there is a sequence $(A_n)_{n\in\mathbb{N}}$ of sets[1] from $\mathfrak{C}$ such that $B \subset \bigcup_{n=1}^{\infty} A_n$. We shall say that a nonnegative extended real valued function τ is a *pre-measure* provided it satisfies the following postulates:

- The domain of τ is a sequential covering class in X.
- $\tau(\varnothing) = 0$.

Proposition 4.21 (Method I Construction)**.** [270, Theorem 11.3] and [317, Theorem 4] *Let $\mathfrak{C}$ be a sequential covering class and τ be a pre-measure defined on $\mathfrak{C}$. Define*

$$\mu^*(B) = \inf\left\{\sum_{n=1}^{\infty}\tau(A_n) \mid A_n \in \mathfrak{C},\ \bigcup_{n=1}^{\infty} A_n \supset B\right\}$$

for each $B \subset X$. Then, μ^* is an outer measure on X. □

Lemma 4.22. [270, Corollary 12.3.1] *If μ^* is constructed by method I and if $\mathfrak{C}$ consists of μ^*-measurable sets, then μ^* is regular.* □

We now give the procedure of the method II construction, which is basically the method I construction followed by a limiting process.

Let $\mathfrak{C}$ be a sequential covering class in a metric space X with metric d. Let us define for each positive integer n the class

$$\mathfrak{C}_n = \{A \mid A \in \mathfrak{C},\ \operatorname{diam}(A) \le 1/n\},$$

where the diameter of a set A is defined by $\operatorname{diam}(A) = \sup\{d(a\,,a') \mid a, a' \in A\}$ and the convention $\operatorname{diam}(\varnothing) = 0$.

Proposition 4.23 (Method II Construction)**.** [317, Theorems 15 & 16 & 23] and [270, Theorem 13.3] *Let X be a metric space with metric d. Let $\mathfrak{C}$ be a sequential covering class such that for each n, $\mathfrak{C}_n$ is also a sequential covering class. Let τ be a pre-measure on $\mathfrak{C}$. For each n, use method I to construct an outer measure μ^* from $\mathfrak{C}_n$ and τ, i.e.,*

$$\mu_n^*(B) = \inf\left\{\sum_{i=1}^{\infty}\tau(A_i) \mid A_i \in \mathfrak{C}_n,\ \bigcup_{i=1}^{\infty} A_i \supset B\right\}$$

for each $B \subset X$. Then the set function μ^*, defined by

$$\mu^*(B) = \sup_n \mu_n^*(B) = \lim_{n\to\infty} \mu_n^*(B)$$

for every $B \subset X$, is a metric outer measure on X. Moreover, if each set of $\mathfrak{C}$ is open, then μ^* is a regular, $\mathfrak{G}_\delta$-regular metric outer measure, all Borel sets are μ^*-measurable, and each μ^*-measurable set of finite μ^*-measure contains an $\mathfrak{F}_\sigma$-set with the same measure. □

[1] If X is a σ-locally compact Hausdorff space, then the existence of such a sequential covering class (even of relatively compact sets) is assured by Lemma 2.9.

Remark 4.24. Equivalently, one can also define the class $\mathfrak{C}_\varepsilon = \{A \mid A \in \mathfrak{C}, \operatorname{diam}(A) \leq \varepsilon\}$, construct by method I the outer measure μ^*_ε for each ε and then obtain the above metric outer measure by $\lim_{\varepsilon\to 0} \mu^*_\varepsilon$.

There is also a criterion when method II reduces to method I, *i.e.*, under what circumstances method I yields a metric outer measure.

Lemma 4.25. [270, Theorem 13.8] *Let $\mathfrak{C}$ be a sequential covering class such that for each n, $\mathfrak{C}_n$ is a sequential covering class. Let τ be a pre-measure on $\mathfrak{C}$ such that, given any $A_0 \in \mathfrak{C}$, any $\varepsilon > 0$ and any positive integer n, there exists a sequence $(A_i)_{i\in\mathbb{N}}$ of sets from $\mathfrak{C}_n$ such that $\bigcup_{i=1}^\infty A_i \supset A_0$ and*

$$\sum_{i=1}^{\infty} \tau(A_i) \leq \tau(A_0) + \varepsilon.$$

Then, method I yields a metric outer measure. □

As a last point, we are again interested in measurability of sets with respect to outer measures constructed by method I or method II.

Lemma 4.26. [270, Theorems 13.5 & 13.7] *If μ^* is a metric outer measure constructed from $\mathfrak{C}$ and τ by either method I or method II, if $\mathfrak{C}$ consists of open sets, and if $X = \bigcup_{n=1}^\infty A_n$ where $\mu^*(A_n) < \infty$ for each n, then each of the following conditions is necessary and sufficient for measurability of a set B:*

(i) *There exists a set $C \in \mathfrak{G}_\delta$ such that $C \supset B$ and $\mu^*(C \setminus B) = 0$.*

(ii) *There exists a set $D \in \mathfrak{F}_\sigma$ such that $D \subset B$ and $\mu^*(B \setminus D) = 0$.*

Moreover, if the sets A_n are open sets, then each of the following conditions is necessary and sufficient for a set B to be measurable:

(i) *For every $\varepsilon > 0$, there exists an open set $C \supset B$ such that $\mu^*(C \setminus B) < \varepsilon$.*

(ii) *For every $\varepsilon > 0$, there exists a closed set $D \subset B$ such that $\mu^*(B \setminus D) < \varepsilon$.* □

4.3. Haar Measure

In connection with Lemma 4.26 we define regularity of a measure (not to be confused with regularity of an outer measure, compare Definition 4.13).

Definition 4.27. A measure μ defined on the Borel sets of a locally compact Hausdorff space X such that it is finite on compact sets, is called a *Borel measure*. Let μ be a Borel measure and let B be a Borel subset of X. We say that a measure μ is *outer regular on* B if $\mu(B) = \inf\{\mu(U) \mid U \supset B, U \text{ open }\}$. We say that μ is *inner regular on* B if $\mu(B) = \sup\{\mu(W) \mid W \subset B, W \text{ compact }\}$. The measure μ is said to be *regular* if it is both inner and outer regular on all Borel sets[2].

[2]A *Radon measure* is a measure defined on the Borel sets of a Hausdorff space X which is inner regular on all Borel sets and for which every point of X has an open neighbourhood of finite measure. If X is locally compact, the Radon measures on X are those Borel measures which are inner regular. Sometimes the term 'Radon measure' (with outer regularity on Borel sets) is used in the definition of the Haar measure ([305]).

Definition 4.28. A measure μ in a locally compact topological group G (not necessarily Abelian) is a *left Haar measure* if it satisfies the following conditions:

- μ is a regular Borel measure.
- μ is not identically zero.
- μ is invariant under left translations, *i.e.*, $\mu(xA) = \mu(A)$ for all $x \in G$ and all $A \in \mathfrak{B}$.

Here, xA denotes the *left translation* of A by x, *i.e.*, $xA = \{xz \,|\, z \in A\}$. A *right Haar measure* is defined similarly. A left Haar measure which is also a right Haar measure (*e.g.*, if G is Abelian) is simply called a *Haar measure*. Note that a (left/right) Haar measure is positive on every nonempty open set.

Remark 4.29. We remark that if μ is a Borel measure on G which is invariant under left translations for all Borel sets, then μ is automatically regular (and therefore by Proposition 4.30, proportional to the (left) Haar measure on G), see [59, Exercise 79.1]. Moreover, invariance under left translations for all compact $\mathfrak{G}_\delta$-sets suffices for this statement ([59, Exercise 90.1]). Furthermore, if μ is a Borel measure on a locally compact space X, then μ is regular iff $\mu(B) = \sup\{\mu(W) \mid W \supset B, W \text{ compact }\}$ for every Borel set B of finite measure ([59, Exercise 62.6]).

A fundamental theorem in the theory of Haar measures[3] (and also analysis, where they are used) reads as follows.

Proposition 4.30. [305, Theorem 1-8], [59, Theorems 78.1 & 78.2], [162, Theorems §58.B & §60.C] and [270, Theorem 17.5] *Let G be a locally compact group. Then G admits a left Haar measure. Moreover, this measure is unique up to scalar multiplication.* □

Remark 4.31. A left Haar measure is obtained by Proposition 4.8 from a certain outer measure μ^*. This outer measure μ^* is obtained by method I. Here, the sequential covering class used is the class that contains X together with the interiors of all compact sets. The pre-measure τ is defined in several steps (including an application of Tychonoff's theorem) from the set functions on the class of compact sets $\mathfrak{K}$ defined by $\tau_N(W) = (W : N)/(W_0 : N)$, where $W \in \mathfrak{K}$, W_0 is compact with nonempty interior, N is a neighbourhood of 1 and $(W : N)$ denotes the minimal number of left-translates of N that will cover W. For details see [270, Section 17], [305, Section 1.2] and [162, Chapter XI].

One is often interested whether two regular Borel measures are equal. Therefore, we introduce the following definition.

Definition 4.32. The σ-ring generated by the class of all compact $\mathfrak{G}_\delta$-sets in X is called the class of *Baire sets* in X. We observe the following special topological property of Baire sets (*Baire-sandwich theorem*, see [59, Thorem 56.1]): If $W \subset U$, where W is compact and U is open, then there exist Baire sets V and D, where V is open and the union of a sequence of compact $\mathfrak{G}_\delta$-sets and D is a compact $\mathfrak{G}_\delta$-set, such that $W \subset V \subset D \subset U$. Note that every Baire set is a Borel set and that on a metric space, the concept of Baire set and Borel set coincide ([59, Exercise 57.1]). A measure μ on X defined on the class of all Baire sets is called a *Baire measure*, if $\mu(W) < \infty$ for every compact $\mathfrak{G}_\delta$-set W.

[3] Note that the Haar measure is sometimes defined without the condition of regularity, *e.g.*, see [270] and [162]. Similarly, a Borel measure is sometimes defined without the finiteness on compact sets, *e.g.*, see [305].

Lemma 4.33. [59, Theorem 62.1] *If μ and μ' are regular Borel measures on the locally compact space X, the following conditions are equivalent:*

(i) $\mu = \mu'$, *i.e.*, $\mu(B) = \mu'(B)$ *for every Borel set* $B \in \mathfrak{B}$.

(ii) $\mu(W) = \mu'(W)$ *for every compact set* $W \in \mathfrak{K}$.

(iii) $\mu(U) = \mu'(U)$ *for every bounded open set* U.

(iv) $\mu(F) = \mu'(F)$ *for every Baire set* F.

(v) $\mu(D) = \mu'(D)$ *for every compact* $\mathfrak{G}_\delta$*-set* D.

(vi) $\mu(V) = \mu'(V)$ *for every bounded open Baire set* V. □

For Baire measure we have the following properties, which we would also like to have for Borel measures.

Lemma 4.34. [59, Lemma 56.2 & Theorem 52.3] *If X (X') is a locally compact space with Baire measure μ (μ'), then*

(i) *the σ-ring of Baire sets of the product topological space $X \times X'$ is equal to the Cartesian product of the σ-ring of the Baire sets of X with the σ-ring of Baire sets of X'.*

(ii) $\mu \otimes \mu'$ *is a Baire measure on the product topological space* $X \times X'$. □

We are interested in the Haar measure on product spaces. Unfortunately, Borel measures do not have the same nice properties as Baire measures. Fortunately, however, we are often in the situation that the product space is a metric space, where a Borel measure is also a Baire measure, see remark in Definition 4.32 and Remark 4.36. In general, we observe the following properties (we write $\mathfrak{B}(X)$ for the class of Borel sets in X):

- Let X, X' be locally compact spaces (or even compact groups). Then $\mathfrak{B}(X \times X') \supset \mathfrak{B}(X) \otimes \mathfrak{B}(X')$ and this inclusion can be proper ([59, Exercises 58.15 & 58.16 & 78.7]).
- Let μ (respectively μ') be a regular Borel measure on the locally compact space X (respectively X'). If $\mathfrak{B}(X \times X') = \mathfrak{B}(X) \otimes \mathfrak{B}(X')$, then $\mu \otimes \mu'$ is a regular Borel measure ([59, Exercise 62.10]).
- Let μ (respectively μ') be a regular Borel measure on the locally compact space X (respectively X'). Then there exists a unique regular Borel measure ν on $X \times X'$ which extends $\mu \otimes \mu'$. Equivalently, there exists a unique regular Borel measure ν on $X \times X'$ such that $\nu(B \times B') = \mu(B) \cdot \mu'(B')$ for all Borel sets $B \in \mathfrak{B}(X)$ ($B' \in \mathfrak{B}(X')$) ([59, Exercise 62.13]).
- If G and G' are two locally compact groups with Haar measures μ and μ' respectively, and if ν is a Haar measure in $G \times G'$, then, on the class of all Baire sets in $G \times G'$, ν is a constant multiple of $\mu \otimes \mu'$ ([162, Exercise §60(2)] and [59, Exercise 78.5]).

For later use and completeness, we state the following Lemma.

Lemma 4.35. [53, Theorem 29.12] *If the locally compact space X has a countable base for its topology (i.e., is metrisable), then every Borel measure on X is regular.* □

Remark 4.36. In addition to our remark in Definition 4.32 we have: The class of Borel sets in a LCAG G coincides with the class of Baire sets iff the topological group G is metrisable ([59, Exercise 71.1]).

Example 4.37. We now give examples of Haar measures on LCAGs G:

- If G is a finite group, the counting measure is a Haar measure μ. We will always normalise μ such that $\mu(G) = 1$, i.e.,
$$\mu(A) = \frac{1}{\operatorname{card} G} \cdot \operatorname{card} A, \qquad \text{for } A \subset G.$$
- The Lebesgue measure on $\mathbb{R}^d$ is a Haar measure. We use the normalisation such that the d-dimensional unit cube has Haar measure 1.
- The Lebesgue measure on $\mathbb{C}^d$ is a Haar measure which – *via* $\mathbb{C} \cong \mathbb{R}^2$ – coincides with the Haar measure on $\mathbb{R}^{2d}$. We will again normalise it such that the cube of unit sidelength has unit Haar measure, where we define the cube of unit sidelength by
$$\{x = (x_1, \ldots, x_d) \mid |\operatorname{Re} x_i| \leq \frac{1}{2},\ |\operatorname{Im} x_i| \leq \frac{1}{2} \text{ for all } 1 \leq i \leq d\}$$
(here $|\cdot|$ denotes the usual absolute value on $\mathbb{R}$).
- We also know that there is a Haar measure μ on the $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$. In this case, we use the normalisation[4]
$$\mu(\widehat{\mathfrak{o}_\mathfrak{p}}) = 1,$$
e.g., we have $\mu(\mathbb{Z}_2) = 1$ on $\mathbb{Q}_2$.

We recall from Definition 3b.10 that the normalised absolute value on a local field $\hat{K}$ is chosen as follows:

- If $\hat{K} = \mathbb{R}$, then $|\cdot|_\nu$ is the usual absolute value.
- If $\hat{K} = \mathbb{C}$, then $|z|_\nu = z\overline{z}$ is the square of the usual absolute value.
- If $\hat{K} = \mathbb{Q}_\mathfrak{p}$ is non-Archimedean with uniformiser π, let $q = [\widehat{\mathfrak{o}_\mathfrak{p}} : \mathfrak{p}] = p^f$ (where $p \in \mathfrak{p}$ and $f = f_{\mathfrak{p}|(p)}$) be the order of the residue field. Then we use the normalisation such that $|\pi|_\nu = |\pi|_\mathfrak{p} = \frac{1}{q}$.

We now give an "analytical" characterisation of the normalised absolute value in terms of the Haar measure, also see [273, Lemma 5.7]. Note that we write LCAGs additively.

[4]We mention that, if one wants to calculate Fourier transforms in $\mathfrak{p}$-adic fields, often the normalisation
$$\mu(\widehat{\mathfrak{o}_\mathfrak{p}}) = \frac{1}{\sqrt{N\mathfrak{D}_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}}} = \frac{1}{\sqrt{\|d_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}\|_\mathfrak{p}}}$$
is used (where $p = \mathfrak{p} \cap \mathbb{P}$, $\|\cdot\|_\mathfrak{p}$ denotes the normalised absolute value, $\mathfrak{D}_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}$ denotes the different (see Definition 3.50) and $d_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}$ the discriminant (see Definition 3.37)), since in this case the formulas for the Fourier transform and the corresponding inversion formula are symmetric, see [211, Example I.34] and [273, Prop. 5.17].

Proposition 4.38. [341, Prop. I.2] *Let K be a local field with Haar measure μ. Then, for every measurable subset A of K and every $x \in K$, one has*

$$\mu(x \cdot A) = |x|_\nu \cdot \mu(A),$$

where $|\cdot|_\nu$ denotes the normalised absolute value on K. □

This, of course, enables us to calculate the Haar measure of every (clopen) ball in a $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$, which just yields the value we obtain from our naive visualisation of such fields, see Chapter 3c. Moreover, in view of Lemma 4.34 and Remark 4.36, we note that the product measure of the Haar measure on local fields is the Haar measure on the product of the local fields. We state two special cases of this remark related to profinite groups (noting their characterisation in Proposition 3a.20) and adele rings (noting their construction on p. 54).

Lemma 4.39. [139, Prop. 16.10] *Let $G \times G'$ be the direct product of two profinite groups G and G' (which is again a profinite group). Denote the Haar measures on G, G' and $G \times G'$ by μ, μ' and λ, respectively. Then the Haar measure λ on $G \times G'$ coincides with $\mu \otimes \mu'$.* □

Proposition 4.40. [305, Prop. 5-5] *Let $G = \prod'_{\nu \in J} G_\nu$ be the restricted product of locally compact groups G_ν with respect to the family of compact subgroups $H_\nu \subset G_\nu$ (for $\nu \notin J_\infty$). Let μ_ν denote the corresponding (left) Haar measure on G_ν normalised such that $\mu_\nu(H_\nu) = 1$ for almost all $\nu \notin J_\infty$. Then there is a unique Haar measure μ on G such that for each finite set of indices S containing J_∞, the restriction μ_S of μ to $G_S = \prod_{\nu \in S} G_\nu \times \prod_{\nu \in J \setminus S} H_\nu$ is precisely the product measure.* □

We may also say that the product measure on G_S induces the Haar measure μ on G. Note that with the above choices of the normalisations, we get for an adele ring $\mathbb{A}_K$ of an algebraic number field K that $\mu(\mathrm{FD}(K)) = \mu'(\mathrm{FD}(K_\infty))$, where μ' denotes the Lebesgue measure on $\mathbb{R}^n$ with $n = [K : \mathbb{Q}]$. Moreover, we can now justify the introduction of the matrix norm $\|\!| \cdot |\!\|_\mathcal{G}$ in Remark 3b.18 (note that it is an infinite product of the normalised absolute values of the determinant of the linear map) for lattice transformations on $\mathbb{A}_K^d$, see Definition 3b.16.

Lemma 4.41. [254, Lemma 3] *Let $\boldsymbol{M} \in \mathcal{G}^d$ and suppose $A \subset \mathbb{A}_K^d$ is measurable. Then $\boldsymbol{M}A$ is measurable and $\mu(\boldsymbol{M}A) = \|\!|\boldsymbol{M}|\!\|_\mathcal{G} \cdot \mu(A)$.* □

4.4. Hausdorff Measure

We now want to define Hausdorff measures (see [317, Chapter 2, Section 1.1] and [270, Section 16]). For this, let X be any separable metric space. Then, the class $\mathfrak{G}$ of all open sets is a sequential covering class; furthermore, for each n, the class $\mathfrak{G}_n$ of all open sets with diameter less than or equal to $1/n$ is also a sequential covering class (here, separability enters).

We denote by $\mathfrak{H}$ the class of functions h defined for all $t \geq 0$, but perhaps having the value $+\infty$ for some values of t, monotonic increasing for $t \geq 0$, positive for $t > 0$ and continuous on the right for all $t \geq 0$. Furthermore $\mathfrak{H}_0$ denotes the subset of all h of $\mathfrak{H}$ with $h(0) = 0$.

Definition 4.42. Let $h \in \mathfrak{H}$ and let X be a separable metric space. Define $h(U)$ for $U \in \mathfrak{G}$ by $h(U) = h(\mathrm{diam}(U))$ if $U \neq \varnothing$ and $h(\varnothing) = 0$. Then the outer measure constructed from the pre-measure h on the sequential covering classes $\mathfrak{G}$ and $\mathfrak{G}_n$ ($n \in \mathbb{N}$), by method II is called the *outer Hausdorff measure* corresponding to the function h, or simply *outer h-measure*, and is denoted by $(\mu^*)^h$. The Hausdorff measure corresponding to the function $h(t) = t^r$ $(r > 0)$ is called the *outer* (r)*-measure* and denoted by $(\mu^*)^{(r)}$. Furthermore, the restriction of an outer Hausdorff measure to (the σ-algebra of) its measurable sets is called a *Hausdorff measure* or simply *h-measure* (and if $h(t) = t^r$, (r)*-measure*) and denoted by μ^h $(\mu^{(r)})$.

We note the following properties of an outer Hausdorff measure $(\mu^*)^h$, also see [127, Theorem 1.6] for the case $X = \mathbb{R}^n$.

Lemma 4.43. [317, Theorem 27] *An outer Hausdorff measure $(\mu^*)^h$ is a regular, $\mathfrak{G}_\delta$-regular metric outer measure, all Borel sets are $(\mu^*)^h$-measurable, and each $(\mu^*)^h$-measurable set of finite $(\mu^*)^h$-measure contains an $\mathfrak{F}_\sigma$-set with the same measure. In particular, each $(\mu^*)^h$-measurable set of finite $(\mu^*)^h$-measure contains a closed set differing from it by arbitrarily small measure.* □

An outer Hausdorff measure $(\mu^*)^h$ can be obtained by a number of different processes.

Lemma 4.44. [317, Theorem 28] *Let $h \in \mathfrak{H}$ and let A be a subset of a separable metric space X. For each $n \in \mathbb{N}$, let*

$$\begin{aligned}(\mu^*)^h_n(B) &= \inf\{\textstyle\sum_{i=1}^\infty h(U_i) \mid U_i \in \mathfrak{G},\ \operatorname{diam}(U_i) \le 1/n,\ \bigcup U_i \supset B\}\\ (\nu^*)^h_n(B) &= \inf\{\textstyle\sum_{i=1}^\infty h(F_i) \mid F_i \in \mathfrak{F},\ \operatorname{diam}(F_i) \le 1/n,\ \bigcup F_i \supset B\}\\ (\sigma^*)^h_n(B) &= \inf\{\textstyle\sum_{i=1}^\infty h(S_i) \mid \operatorname{diam}(S_i) \le 1/n,\ \bigcup S_i \supset B\}\\ (\tau^*)^h_n(B) &= \inf\{\textstyle\sum_{i=1}^\infty h(S_i) \mid \operatorname{diam}(S_i) \le 1/n,\ \bigcup S_i = B\}\end{aligned}$$

the sets $\{S_i\}_{i=1}^\infty$ in the definition of $(\sigma^*)^h_n$ and $(\tau^*)^h_n$ being arbitrary subsets of X. Then, for any m with $m < n$, we have

$$(\mu^*)^h_m(B) \le (\nu^*)^h_n(B) = (\sigma^*)^h_n(B) = (\tau^*)^h_n(B) \le (\mu^*)^h_n(B).$$

Further

$$(\mu^*)^h(B) = \sup_{n\in\mathbb{N}}(\mu^*)^h_n(B) = \sup_{n\in\mathbb{N}}(\nu^*)^h_{n\in\mathbb{N}}(B) = \sup_{n\in\mathbb{N}}(\sigma^*)^h_{n\in\mathbb{N}}(B) = \sup_{n\in\mathbb{N}}(\tau^*)^h_n(E).$$ □

For any B, it is clear that $(\mu^*)^{(r)}(B)$ is non-increasing as r increases from 0 to ∞. More precisely, we have the following.

Lemma 4.45. [270, Theorem 16.1] *If $(\mu^*)^{(r)}(B) < \infty$ for a subset B of a separable metric space, and if $r < s$, then $(\mu^*)^{(s)}(B) = 0$.* □

Definition 4.46. For a measurable subset B of a separable metric space we call the unique value

$$\sup\{r \in [0,\infty] \mid (\mu^*)^{(r)}(B) = \infty\} = \inf\{r \in [0,\infty] \mid (\mu^*)^{(r)}(B) = 0\}$$

the *Hausdorff dimension* (or *Hausdorff-Besicovich dimension*) of B, and denote it by $\dim_{\mathrm{Hd}} B$ (also see [127, Section 1.2]).

Remark 4.47. The Hausdorff dimension measures the metric size of any (measurable) subset B of a metric space. But often the value of the Hausdorff measure $\mu^{(\dim_{\mathrm{Hd}} B)}$ does not give much extra information, since it may not be positive and finite. So there are many cases where some function h other than $h(t) = t^r$ is more useful and natural. *E.g.*, the trajectories of the Brownian motion in $\mathbb{R}^n$ have positive and σ-finite Hausdorff measure almost surely with (for small t): $h(t) = t^2 \log\log t^{-1}$ in the case $n \ge 3$, and $h(t) = t^2 \log t^{-1} \log\log\log t^{-1}$ in the case $n = 2$. In particular, their dimension is 2 almost surely, see [251, Section 4.9] and references therein. See Remark 4.60 on the use of the word 'dimension' here.

In connection with the Hausdorff dimension, we define the following property.

Definition 4.48. A subset B of a separable metric space is called *Hausdorff rectifiable* if there is an $h \in \mathfrak{H}_0$ such that $0 < \mu^h(B) < \infty$. If the set B is Hausdorff rectifiable, then the Hausdorff dimension of B of the Hausdorff measure μ^h is defined (if it exists) by the the number γ such that

$$\lim_{r \to 0} \frac{h(a \cdot r)}{h(r)} = a^\gamma \text{ for all } a > 0,$$

also see [283] and references therein. We will say that a set B is *Hausdorff rectifiable in its dimension* if $0 < \mu^{(\dim_{\mathrm{Hd}} B)}(B) < \infty$, *i.e.*, if $h(t) = t^{\dim_{\mathrm{Hd}} B}$. We also note that it is "unusual" for a set B to be Hausdorff rectifiable, see the discussion in [317, Section 2.§3.5] and Remark 4.136. Nevertheless, there is always at least one compact set C in every uncountable complete separable metric space which is Hausdorff rectifiable, see [317, Theorem 35].

We state the following mapping theorem.

Lemma 4.49. [317, Theorem 29] *Let f be a function defined on a set B of a metric space X' with metric d'. Let f take values in a metric space X with metric d. Let g be a continuous strictly increasing function defined for $t \geq 0$ and with $g(0) = 0$. Suppose that f satisfies the Lipschitz condition $d(f(x), f(y)) \leq g(d'(x, y))$, for all $x, y \in B$. Then, for all $h \in \mathfrak{H}$ and all $\delta > 0$,*

$$(\mu^*)^{hg^{-1}}_{g(\delta)}(f(B)) \leq (\mu^*)^h_\delta(B), \qquad \text{and} \qquad (\mu^*)^{hg^{-1}}(f(B)) \leq (\mu^*)^h(B). \qquad \square$$

Corollary 4.50. *If $g(x) = c \cdot x$ (and therefore $d(f(x), f(y)) \leq c \cdot d'(x, y)$), then one has $(\mu^*)^{(r)}(f(B)) \leq c^r \cdot (\mu^*)^{(r)}(B)$.* □

An upper bound on the Hausdorff dimension is usually obtained by choosing a suitable covering by small sets. Obtaining a lower bound, however, is far more delicate, but possible by the *mass distribution principle.*

Definition 4.51. A *mass distribution* ν on a subset B of a measure space X is a Borel measure with support $\operatorname{supp} \nu \subset B$ such that $0 < \nu(B) < \infty$. The *support* of a Borel measure ν is defined as $\operatorname{supp} \nu = U^c$ where U is the union of all open Borel sets of ν-measure zero.

Proposition 4.52. [123, Theorem 4.2] *Let ν be a mass distribution on a subset B of a separable metric space. Suppose that for some $h \in \mathfrak{H}$ there are numbers $c > 0$ and $n \in \mathbb{N}$ such that $\nu(U) \leq c \cdot h(U)$ for all sets $U \in \mathfrak{G}_n$. Then $(\mu^*)^h(B) \geq \nu(B)/c$. In particular, if we can choose $h(t) = t^\gamma$, then $\dim_{\mathrm{Hd}} B \geq \gamma$.*

Proof. If $\{A_i\}_{i=1}^\infty$ is any cover of B with $A_i \in \mathfrak{G}_n$, then

$$0 < \nu(B) \leq \nu\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \nu(A_i) \leq c \cdot \sum_{i=1}^\infty h(A_i).$$

Taking the infimum over all such coverings yields $(\mu^*)^h_n(B) \geq \nu(B)/c$. Letting $n \to \infty$ (respectively, taking the supremum over n) establishes the claim. The statement about the Hausdorff dimension is immediate. □

4.5. Haar and Hausdorff Measures

Remark 4.53. Let G be a metrisable σ-locally compact Abelian group. Then there is an invariant metric d which generates a compatible topology, and the family of open balls $\{B_{<1/n}\}_{n\in\mathbb{N}} = \{x \in G \mid d(x,0) < 1/n\}$ with centre 0 is a (countable) base at 0. Since we assume that G is σ-compact, it is easy to see that the Hausdorff dimension of the open unit ball $\dim_{\mathrm{Hd}} B_{<1}$ equals the Hausdorff dimension of the whole space $\dim_{\mathrm{Hd}} G$.

Lemma 4.54. *Let G be a metrisable separable σ-locally compact Abelian group with invariant metric d and let $r = \dim_{\mathrm{Hd}} G$. Then the r-dimensional Hausdorff measure $\mu^{(r)}$ is a Haar measure iff $0 < \mu^{(r)}(B_{<1}) < \infty$, i.e., if the open unit ball is Hausdorff rectifiable in the Hausdorff dimension of G.*

Proof. Each Hausdorff measure on G is invariant under translations by the invariance of the metric d. Furthermore, all Borel sets are measurable.

We first check that $\mu^{(r)}$ is a nonzero Borel measure. We begin with the necessity of the condition: If $\mu^{(r)}(B_{<1}) = \infty$, then the compact unit ball $B_{\leq 1} \supset B_{<1}$ also has infinite measure, therefore $\mu^{(r)}$ is not a Borel measure. On the other hand, if $\mu^{(r)}(B_{<1}) = 0$, then all $B_{<1/n}$ ($n \in \mathbb{N}$) have zero measure, and consequently all compact sets (which can be covered by finitely many translates of $B_{1/n}$ for each n) have zero measure. Since G is σ-compact, even G has zero measure, therefore $\mu^{(r)}$ is identically zero in this case.

Conversely, if $0 < \mu^{(r)}(B_{<1}) < \infty$, then $\mu^{(r)}$ is nonzero and all compact sets have finite measure (they can be covered by finitely many translates of $B_{<1}$). Therefore, the condition is necessary and sufficient to have a translation invariant nonzero Borel measure.

Since X is metric space, $\mu^{(r)}$ is a regular Borel measure by Lemma 4.35. □

We characterise the locally compact Abelian groups we want to concentrate on. Let $\mathbb{M}$ be a locally compact Abelian group, given by

$$\mathbb{M} = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \ldots \times \mathbb{Q}_{\mathfrak{p}_k}. \tag{4.1}$$

As a finite product of separable Hausdorff spaces, $\mathbb{M}$ is a separable Hausdorff space. The product topology is generated by the maximum metric, which we denote by $d_{\mathbb{M}}$, *i.e.*, for $x, y \in \mathbb{M}$ with

$$x = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}, x_{r+s+1}, \ldots, x_{r+s+k})$$

(where $x_1, \ldots, x_r \in \mathbb{R}$, $x_{r+1}, \ldots, x_{r+s} \in \mathbb{C}$ and $x_{r+s+j} \in \mathbb{Q}_{\mathfrak{p}_j}$ for all $1 \leq j \leq k$) we have

$$\begin{aligned} d_{\mathbb{M}}(x,y) \quad = \quad & \max\{ |x_1 - y_1|, \ldots, |x_r - y_r|, |\operatorname{Re}(x_{r+1} - y_{r+1})|, |\operatorname{Im}(x_{r+1} - y_{r+1})|, \ldots, \\ & |\operatorname{Re}(x_{r+s} - y_{r+s})|, |\operatorname{Im}(x_{r+s} - y_{r+s})|, \|x_{r+s+1} - y_{r+s+1}\|_{\mathfrak{p}_1}, \ldots, \\ & \|x_{r+s+k} - y_{r+s+k}\|_{\mathfrak{p}_k} \}. \end{aligned}$$

Definition 4.55. Let $\mathbb{M}$ be given as in (4.1). Then we call the number $r + 2s + k$ the *metric dimension* of $\mathbb{M}$, and denote it by $\dim_{\mathrm{metr}} \mathbb{M}$ (on the use of the word 'dimension' here, see Remark 4.60).

Theorem 4.56. *Let $\mathbb{M}$ be given as in* (4.1). *Then* $\dim_{\mathrm{metr}} \mathbb{M} = \dim_{\mathrm{Hd}} \mathbb{M}$. *Moreover, $B_{<1}$ has finite nonzero* $\dim_{\mathrm{metr}} \mathbb{M}$*-dimensional Hausdorff measure. Therefore, the Haar measure on* $\mathbb{M}$ *coincides (up to a multiplicative constant) with the the* $\dim_{\mathrm{metr}} \mathbb{M}$*-dimensional Hausdorff measure.*

Proof. Obviously, we are done if the $\dim_{\mathrm{metr}} \mathbb{M}$-dimensional Hausdorff measure of $B_{<1}$ is nonzero and finite.

We first state the explicit form of $B_{<1}$:

$$]-1,1[\times \cdots \times]-1,1[\times (]-1,1[\times]-1,1[) \times \cdots \times (]-1,1[\times]-1,1[) \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k.$$

Therefore, the Haar measure is easily calculated as $\mu(B_{<1}) = 2^{r+2s}/(p_1^{f_1} \cdots p_k^{f_k})$ where $p_i = \mathfrak{p}_i \cap \mathbb{P}$ and f_i denotes the residue degree $f_i = f_{\mathfrak{p}_i|(p_i)}$.

We define a *hypercube* of sidelength (and diameter) δ as a translate of

$$[0,\delta] \times \cdots \times [0,\delta] \times ([0,\delta] \times [0,\delta]) \times \cdots \times ([0,\delta] \times [0,\delta]) \times \\ \{x \in \mathbb{Q}_{\mathfrak{p}_1} \mid \|x\|_{\mathfrak{p}_1} \leq \delta\} \times \cdots \times \{x \in \mathbb{Q}_{\mathfrak{p}_k} \mid \|x\|_{\mathfrak{p}_k} \leq \delta\}.$$

Obviously, the Haar measure of such a hypercube is less than or equal to δ^{r+2s+k}.

With these definitions, we obtain a lower bound for $r+2s+k$-dimensional Hausdorff measure of $B_{<1}$: Let $\{A_i\}_{i=1}^{\infty}$ be any cover of $B_{<1}$ and let $\varepsilon > 0$. Then we can cover each set A_i by a hypercube S_i^{ε} of diameter $(1+\varepsilon) \cdot \operatorname{diam} A_i$. By the property of the Haar measure μ, we obtain

$$\frac{2^{r+2s}}{p_1^{f_1} \cdots p_k^{f_k}} = \mu(B_{<1}) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} S_i^{\varepsilon}\right) \leq \sum_{i=1}^{\infty} \mu(S_i^{\varepsilon}) \\ \leq \sum_{i=1}^{\infty} ((1+\varepsilon) \cdot \operatorname{diam} A_i)^{r+2s+k} = (1+\varepsilon)^{r+2s+k} \cdot \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^{r+2s+k}$$

As this inequality holds for each cover $\{A_i\}_{i=1}^{\infty}$ of $B_{<1}$ and any $\varepsilon > 0$, we deduce that

$$\mu^{(r+2s+k)}(B_{<1}) \geq \frac{2^{r+2s}}{p_1^{f_1} \cdots p_k^{f_k}} \geq \frac{2^{r+2s}}{p_1^{f_1} \cdots p_k^{f_k}} \cdot \frac{1}{(1+\varepsilon)^{r+2s+k}}.$$

Note that this is basically the mass distribution principle of Proposition 4.52, where the mass distribution is given by the Haar measure.

For an upper bound, let $n \geq 1$ and $N > n$ and count the hypercubes of diameter $\frac{1}{N}$ needed to cover $B_{<1}$ efficiently:

- In the real and complex coordinates, these hypercubes are defined by the inequalities

$$\frac{m_i - N - 1}{N} \leq x_i \leq \frac{m_i - N}{N} \quad \text{where } 1 \leq m_i \leq 2 \cdot N \qquad (1 \leq i \leq r)$$

  and

$$\frac{m_i - N - 1}{N} \leq \operatorname{Re} x_i \leq \frac{m_i - N}{N} \quad \text{where } 1 \leq m_i \leq 2 \cdot N \qquad (r+1 \leq i \leq r+s)$$
$$\frac{m_i' - N - 1}{N} \leq \operatorname{Im} x_i \leq \frac{m_i' - N}{N} \quad \text{where } 1 \leq m_i' \leq 2 \cdot N \qquad (r+1 \leq i \leq r+s).$$

- In a $\mathfrak{p}$-adic component, we have to cover $\mathfrak{p} = \pi\,\widehat{\mathfrak{o}_{\mathfrak{p}}}$ (π denotes the uniformiser). Let $f = f_{\mathfrak{p}|(p)}$ denote the residue degree (where $p = \mathfrak{p} \cap \mathbb{P}$), then $\mathfrak{p}$ is a clopen ball of diameter (and also radius) p^f, it is covered by p^f disjoint clopen balls of diameter p^{2f}

etc. We note that in ultrametric spaces the diameter is less than or equal to the radius of a ball (see Lemma 2.26, respectively Lemma 2.26'). Let R be a positive real number and denote by $\lfloor R \rfloor_{\mathfrak{p}}$ the power of p^f such that $p^{f\cdot v} \le R < p^{f\cdot(v+1)}$ (then $R/p^f < \lfloor R \rfloor_{\mathfrak{p}} \le R$). Then $\mathfrak{p}$ is covered by $1/(\lfloor \frac{1}{N} \rfloor_{\mathfrak{p}} \cdot p^f)$ disjoint clopen balls of diameter/radius $\lfloor \frac{1}{N} \rfloor_{\mathfrak{p}}$, or, at most N disjoint clopen balls of radius $\frac{1}{N}$.

Counting everything, we need at most $2^{r+2s} \cdot N^{r+2s+k}$ hypercubes of diameter $\frac{1}{N}$ to cover $B_{<1}$, so that

$$(\mu^*)_n^{(r+2s+k)}(B_{<1}) \le 2^{r+2s} \cdot N^{r+2s+k} \cdot \left(\frac{1}{N}\right)^{r+2s+k} = 2^{r+2s}.$$

Hence we have $\mu^{(r+2s+k)}(B_{<1}) \le 2^{r+2s}$, which by Lemma 4.54 proves the claim. □

Remark 4.57. Of course, the covering for the upper bound of the Hausdorff measure can be made more efficient if one chooses the N with more care. One gets very efficient coverings if there are positive integers v_i such that $N \leqq \left(p_i^{f_i}\right)^{v_i}$ for $1 \le i \le k$, where by $a \leqq b$ we mean that $a \le b$ but both numbers are close. In fact, it is better to consider this condition in the form

$$v_i \leqq \log(N)/(f_i \cdot \log(p_i)).$$

We are looking for N, but for this we have to find numbers v_i. Obviously, this is no problem, if all p_i are the same (then we take a power of $p^{f_1 \cdots f_k}$ for N). So we suppose that we have two prime numbers $p_1 \ne p_2$ (we do not take residue degrees into account now). Suppose we have found integers v_1 and v_2 such that $p_1^{v_1} \approx p_2^{v_2}$, then we would take $N = \min\{p_1^{v_1}, p_2^{v_2}\}$. Suppose $N = p_1^{v_1}$ (so that $v_1 \cdot \log(p_1) = \log(N)$). Then we also have

$$v_2 > \frac{\log(N)}{\log(p_2)} = v_1 \cdot \frac{\log(p_1)}{\log(p_2)},$$

and by assumption, $v_2 - v_1 \cdot \frac{\log(p_1)}{\log(p_2)} \geqq 0$. The question is whether it is possible to find "good" values for v_1 and v_2.

Definition. A rational fraction a/b (where $b > 0$) is a *best approximation of the second kind* of a number $\lambda \in \mathbb{R}$ if the inequalities $c/d \ne a/b$ and $0 < d \le b$ imply $|d \cdot \lambda - c| > |b \cdot \lambda - a|$. For a number $\lambda \in \mathbb{R}$, we denote by $[\lambda_0 ; \lambda_1 , \lambda_2 , \lambda_3 \ldots]$ its *continued fraction expansion*, *i.e.*,

$$\lambda = \lambda_0 + \cfrac{1}{\lambda_1 + \cfrac{1}{\lambda_2 + \ldots}}$$

(which for rational numbers is finite and for irrational numbers infinite). We call the rational number $[\lambda_0 ; \lambda_1 , \lambda_2 , \lambda_3, \ldots , \lambda_k]$ the *k-th order convergent* (or *approximant*).

We note that the continued fraction expansion of $\log(p_1)/\log(p_2)$ (always irrational) yields the values for v_1 and v_2.

Proposition. [209, Theorem 16] *Every best approximation of the second kind is a convergent.* □

Also, the absolute value in the definition of the best approximation of the second kind plays no role, since we have the following statement.

Lemma. [209, Theorem 4] *Even-order convergents form an increasing and odd-order convergents a decreasing sequence. Also, every odd-order convergent is greater than any even-order convergent.* □

Furthermore, we note that there are explicit bounds in terms of the n-th order convergent.

Proposition. [209, Theorems 9 & 13] *Let a_n/b_n be the n-th order convergent of λ. Then we have*

$$\frac{1}{b_n + b_{n+1}} \leq |b_n \cdot \lambda - a_n| \leq \frac{1}{b_{n+1}}.$$ □

For more than two primes, this procedure has to be iterated in an appropriate way. Doing this carefully, one finally obtains $\mu^{(r+2s+k)}(B_{<1}) \leq \frac{2^{r+2s}}{p_1^{f_1}\cdots p_k^{f_k}}$, and therefore $\mu^{(r+2s+k)} = \mu$ (where μ, as above, denotes the Haar measure on $\mathbb{M}$).

We note that the upper estimate on the Hausdorff measure in the above proof of Theorem 4.56 can be seen as an estimate of the measure associated to the upper box-counting dimension, while the one here is connected to the measure associated to the lower box-counting dimension, see Definition 4.131 and [114, Chapter 1] and [123, Chapter 3].

We also want to add some remarks about the following statement, which compares the (d)-measure in Euclidean space $\mathbb{R}^d$ and the Lebesgue measure in $\mathbb{R}^d$.

Lemma 4.58. [317, Theorem 30] *For each natural number d, there is a finite positive constant κ_d so that, for every (measurable) set A in Euclidean space $\mathbb{R}^d$, we have*

$$\mu^{(d)}(A) = \kappa_d \cdot \lambda(A),$$

λ denoting the Lebesgue measure in $\mathbb{R}^d$. □

Explicitly, the constants are given by (where Γ denotes the Gamma-function):

$$\kappa_d = \left(\frac{4}{\pi}\right)^{d/2} \cdot \Gamma\left(1 + \frac{1}{2}d\right).$$

While λ assigns unit measure to the unit cube, $\mu^{(d)}$ assigns unit measure to the sphere of diameter 1.

We compare this statement with the result we have obtained: We use the metric $d_{\mathbb{M}}$ where balls are actually cubes; so the Hausdorff measure we obtain assigns unit measure to the cubes (or hypercubes) of unit side, and therefore, in $\mathbb{R}^d$, equals the Lebesgue measure. Usually, however, one uses the metric defined by the norm $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_d^2}$, where balls are "filled spheres", to define the Hausdorff measure; consequently, in that case, it assigns unit measure to the filled sphere of unit diameter (and the above constants κ_d arise).

Remark 4.59. We also stress that with the chosen metric $d_{\mathbb{M}}$, respectively the normalisations of the Haar measures in Example 4.37, the $(r + 2s + k)$-measure $\mu^{(r+2s+k)}$ equals the Haar measure μ on $\mathbb{M}$, which can be constructed as product measure of the Haar measures of its components. Moreover, on these components, we assign unit measure to the unit interval in $\mathbb{R}$, respectively to each set of $\mathfrak{p}_i$-adic integers $\widehat{\mathfrak{o}_{\mathfrak{p}_i}}$ $(1 \leq i \leq k)$.

Remark 4.60. In dimension theory, the *dimension* (or – better – the *strong inductive dimension*) of a topological space X is inductively defined as follows, see [112, Chapter 3], [271, Definition I.4] and [186].

Definition. A topological space X has dimension -1 if $X = \varnothing$. If for any disjoint closed sets A and B of a topological space X there exists an open set U such that $A \subset U \subset B^c$ and such that ∂U has dimension smaller than or equal to $n-1$, then X has dimension smaller than or equal to n. Moreover, if it is true that the dimension of X is smaller than or equal to n but not smaller than or equal to $n-1$, then we say that X has dimension n. If there is no n such that the dimension of X is bounded by n, then x has dimension $+\infty$.

It follows (immediately) from this definition that nonempty spaces in which there is a base of the topology consisting of clopen sets, is zero-dimensional, see [186, Remark B) to Definition II.1]. By Remark 2.28, ultrametric spaces, such as $\mathfrak{p}$-adic spaces, are zero-dimensional (*e.g.*, [271, Corollary on p. 113]). On the other hand, an Euclidean space $\mathbb{R}^d$ has dimension d (see [186, Theorem IV.1]). Therefore, topologically, the space $\mathbb{M}$ has dimension $r + 2s$.

But we are not primarily interested in $\mathbb{M}$ as a topological space, but as a metric space. Indeed, Hausdorff dimension is a metric (and, of course, measure theoretical) concept. Since each metric space is a topological space, it should be no surprise that a space of (topological) dimension d has Hausdorff dimension at least d ([186, Theorem VII.2]). More interestingly, the topological dimension of X is invariant under homeomorphisms (*i.e.*, if X' is homeomorphic to X, both have the same dimension), while the same need not be true for the Hausdorff dimension (*e.g.*, we note that $\mathbb{Q}_2(\sqrt{2}) \cong \mathbb{Q}_2^2$); but taking the infimum over the Hausdorff dimension of all spaces homeomorphic to X equals the (topological) dimension of X, see [186, Section VII.4].

Since we have chosen a space $\mathbb{M}$ with a fixed metric $d_{\mathbb{M}}$, and moreover, we are interested in this space and *not* in any homeomorphic image of it, we have chosen to call the Hausdorff dimension of $(\mathbb{M}\,, d_{\mathbb{M}})$ its metric dimension. We will later consider maps acting on this space which ("naturally") line up with this choice of $(\mathbb{M}\,, d_{\mathbb{M}})$, see Sections 4.8 & 4.9.

4.6. Hausdorff Metric and the Space $\mathcal{K}X$

Definition 4.61. Let X be a metric space with metric d (and therefore a Hausdorff space). We denote by $\mathcal{K}X$ the set of nonempty compact subsets of X. For a nonempty subset $A \subset X$ and $\delta \in [0,\infty[$, the *closed δ-fringe* of A is $A^\delta = \{x \in X \mid d(x\,,A) \leq \delta\}$ where $d(x\,,A) = \inf_{a\in A} d(x\,,a)$. If $A \in \mathcal{K}X$, we have $d(x\,,A) = \min_{a\in A} d(x\,,a)$ by the continuity of the map $d(x\,,\cdot)$. Note that $A^0 = \operatorname{cl} A$ and $\{y\}^r = B_{\leq r}(y)$. We say that $A, B \in \mathcal{K}X$ are *δ-indistinguishable* if $A \subset B^\delta$ and $B \subset A^\delta$. Noting that, for all $A, B \in \mathcal{K}X$, there exists a least $\delta \in [0,\infty[$ such that $A \subset B^\delta$, namely $\delta = \sup_{a\in A} d(a\,,B) = \max_{a\in A} d(a\,,B)$, we define the *Hausdorff metric* h_d on $\mathcal{K}X$ as follows: For $A, B \in \mathcal{K}X$, let $h_d(A\,,B) = \max\{\sup_{a\in A} d(a\,,B)\,, \sup_{b\in B} d(b\,,A)\}$. The space $\mathcal{K}X$ together with the Hausdorff metric h_d becomes a metric space; moreover, $(X\,,d)$ is embedded in $(\mathcal{K}X\,,h_d)$ by $x \mapsto \{x\}$.

Remark 4.62. The topology on $\mathcal{K}X$ induced by h_d is the so-called *Vietoris topology*, see [391, Section 1.2 & Prop. 2.1.5].

We now give a list of properties that $\mathcal{K}X$ inherits from the space X, where the last property will be the most important one in Section 4.8.

Proposition 4.63. [391, Props. 1.2.2 & 1.3.2 & 1.7.2 & 2.3.2] and [51, Theorem 2.3] *Let X be a metric (and therefore Hausdorff) space.*

(i) *$\mathcal{K}X$ is a metric (and therefore Hausdorff) space.*

(ii) *$\mathcal{K}X$ is compact iff X is compact.*

(iii) *$\mathcal{K}X$ is locally compact iff X is locally compact.*

(iv) *$\mathcal{K}X$ is complete iff X is complete.* □

We are also interested in sequences and their limits in $\mathcal{K}X$.

Lemma 4.64. [391, Prop. 2.4.4] *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}X$.*

(i) *If $(A_n)_{n\in\mathbb{N}}$ is antitone, then $(A_n)_{n\in\mathbb{N}} \to \bigcap_{n\in\mathbb{N}} A_n$.*

(ii) *If $(A_n)_{n\in\mathbb{N}}$ is isotone and bounded above in $\mathcal{K}X$, then $(A_n)_{n\in\mathbb{N}} \to \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.* □

We also note the following property of the union map.

Lemma 4.65. [391, Corollary 1.5.3] *The n-ary union map $\bigcup : (\mathcal{K}X)^n \to \mathcal{K}X$ is continuous, when $(\mathcal{K}X)^n$ is given the product topology.* □

Remark 4.66. The reader should note that in [391] a nonstandard analysis approach is chosen to obtain the results. But of course, they also hold in the "normal" context. Unfortunately, most books about fractals are only concerned with the case $X = \mathbb{R}^d$.

4.7. A Detour in Matrix Theory and Submultiplicative Sequences

Definition 4.67. For a matrix $\boldsymbol{M}$ we denote by $\rho(\boldsymbol{M})$ its *spectral radius*, *i.e.*, the maximum of the moduli of its eigenvalues (therefore, $\rho(\boldsymbol{M}) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \boldsymbol{M}\}$). A matrix $\boldsymbol{M}$ with real elements m_{ij} is called *nonnegative* (*positive*) if all elements of $\boldsymbol{M}$ are nonnegative (positive). A nonnegative square matrix (of order n) $\boldsymbol{M} = [m_{ij}]_{1\le i,j\le n}$ is called *reducible* if there is a permutation τ that puts it into the form

$$[m_{\tau(i)\tau(j)}]_{1\le i,j\le n} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix},$$

where $\boldsymbol{B}$ and $\boldsymbol{D}$ are square matrices (here, $\boldsymbol{0}$ denotes a rectangular matrix with zeroes only). Otherwise, $\boldsymbol{M}$ is called *irreducible*. A nonnegative matrix $\boldsymbol{M}$ is called *primitive* if some power of $\boldsymbol{M}$ is positive. We note that a nonnegative matrix $\boldsymbol{M} = [m_{ij}]_{1\le i,j\le n}$ is irreducible iff for every pair i, j there is a power $\boldsymbol{M}^p = \left[m_{ij}^{(p)}\right]_{1\le i,j\le n}$ of $\boldsymbol{M}$ such that its $(i\,,j)$-th element $m_{ij}^{(p)}$ is positive.

Remark 4.68. For an irreducible matrix $\boldsymbol{M} = [m_{ij}]_{1\le i,j\le n}$, the power p such that $m_{ij}^{(p)} > 0$ can always be chosen within the bounds

$$p \le \begin{cases} k-1, & \text{if } i \ne j, \\ k, & \text{if } i = j, \end{cases}$$

where k is the degree of the *minimal polynomial* of $\boldsymbol{M}$, see [144, Corollary in Section XIII.§1.2]. Here, the minimal polynomial is the polynomial $f(x)$ of least degree with highest coefficient 1 such that $f(\boldsymbol{M}) = \boldsymbol{0}$ (where $\boldsymbol{0}$ denotes the zero matrix); it divides each polynomial $g(x)$ with the property $g(\boldsymbol{M}) = \mathbf{0}$, *e.g.*, the *characteristic polynomial* of $\boldsymbol{M}$ defined by $\det(x \cdot \boldsymbol{E} - \boldsymbol{M})$ (where $\boldsymbol{E}$ denotes the square unit matrix, *i.e.*, the diagonal matrix with ones only on the diagonal).

Similarly, for a primitive matrix $\boldsymbol{M}$ one can find bounds for the power p such that $\boldsymbol{M}^p$ is positive, namely $p \leq n^2 - 2n + 2$ where n is the order of $\boldsymbol{M}$, see [392] and [61, Theorem II.4.14]. If some diagonal elements are positive or some other conditions are fulfilled, this bound can be improved considerably, see [61, Section II.4]. We note that if $\boldsymbol{M}^p$ is positive, so is every higher power of a nonnegative matrix $\boldsymbol{M}$.

These properties of nonnegative matrices have consequences on the "maximal" eigenvalue and its corresponding eigenvector. This is stated in the following proposition which (parts of it) is also known as *Perron-Frobenius theorem*, where part (ii) is attributed to Frobenius and (iii) to Perron.

Proposition 4.69. [144, Theorems XIII.1 – XII.3 & Section XIII.§5]

(i) *A nonnegative square matrix* $\boldsymbol{A}$ *always has a nonnegative eigenvalue* α *such that the moduli of all eigenvalues of* $\boldsymbol{A}$ *do not exceed* α, *i.e.,* $\alpha = \rho(\boldsymbol{A})$. *To this "maximal" eigenvalue* α *there corresponds a nonnegative eigenvector* y, *i.e.,* $\boldsymbol{A}y = \alpha\, y$ *where* $y_i \geq 0$ *(*$1 \leq i \leq n$*).*

(ii) *An irreducible nonnegative matrix* $\boldsymbol{A}$ *always has a positive eigenvalue* $\alpha = \rho(\boldsymbol{A})$ *that is simple. To this "maximal" eigenvalue* α *there corresponds an eigenvector* y *with only positive coordinates, i.e.,* $\boldsymbol{A}y = \alpha\, y$ *where* $y_i > 0$ *(*$1 \leq i \leq n$*). Moreover, if* $\boldsymbol{A}$ *has* h *eigenvalues of modulus* α, *then the whole spectrum of* $\boldsymbol{A}$, *regarded as a system of points in the complex plane, goes over into itself under a rotation of the plane by the angle* $2\,\pi/h$.

(iii) *A positive square matrix* $\boldsymbol{A}$ *always has a real and positive eigenvalue* $\alpha = \rho(\boldsymbol{A})$ *which is simple. Moreover, the moduli of all eigenvalues of* $\boldsymbol{A}$ *different from* α *are strictly smaller than* α. *To this "maximal" eigenvalue* α *there corresponds an eigenvector with only positive coordinates. The same holds, if* $\boldsymbol{A}$ *is a primitive matrix.* □

In the situation (ii) (and/or (iii)), we say that the maximal eigenvalue (*i.e.*, the one which equals the spectral radius) is the *Perron-Frobenius eigenvalue* or *PF-eigenvalue* and its corresponding eigenvector (with only positive entries) the *Perron-Frobenius eigenvector* or *PF-eigenvector* of the matrix.

Lemma 4.70. [144, Remark 2 in Section XIII.§2.5] *Let* $s_i = \sum_{k=1}^{n} a_{ik}$ *be the* i*-th row sum of an irreducible matrix* $\boldsymbol{A} = [a_{ik}]_{1 \leq i,k \leq n}$ *with nonnegative elements and PF-eigenvalue* α. *Then* $\min_i s_i \leq \alpha \leq \max_i s_i$, *and the equality sign on the left and on the right of* α *holds only if all row sums are equal.* □

Remark 4.71. Improved bounds can be found in [278].

The following statement is attributed to Wielandt.

Lemma 4.72. [144, Lemma XIII.2] *Let* $\boldsymbol{A}$, $\boldsymbol{C}$ *be two square matrices of the same order* n, *where* $\boldsymbol{A}$ *is non-negative (i.e.,* $a_{ik} \geq 0$*) and irreducible. Suppose* $|c_{ik}| \leq a_{ik}$ *for all components, then for every eigenvalue* γ *of* $\boldsymbol{C}$ *and the PF-eigenvalue* α *of* $\boldsymbol{A}$ *we have the inequality* $|\gamma| \leq \alpha$. *Furthermore, equality holds iff* $\boldsymbol{C} = \exp(i\phi) \cdot \boldsymbol{D}\boldsymbol{A}\boldsymbol{D}^{-1}$, *where* $\exp(i\phi) = \gamma/\alpha$ *and* $\boldsymbol{D}$ *is a diagonal matrix whose diagonal elements are of unit modulus.* □

The proof of this lemma shows that we also have the following immediate consequence.

Corollary 4.73. *Let $\boldsymbol{A}$, $\boldsymbol{C}$ be two square matrices of the same order n, both non-negative, with $c_{ik} \leq a_{ik}$. Then the spectral radius of $\boldsymbol{C}$ is strictly less than or equal to the spectral radius of $\boldsymbol{A}$. Moreover, if in addition $\boldsymbol{A}$ is irreducible and there is at least one strict inequality, then the spectral radius of $\boldsymbol{C}$ is strictly less than the PF-eigenvalue of $\boldsymbol{A}$.* □

To simplify notations, we write $\boldsymbol{C} \leq \boldsymbol{A}$ ($\boldsymbol{C} < \boldsymbol{A}$) if $\boldsymbol{A}$, $\boldsymbol{C}$ are two square matrices of the same order n, where both are nonnegative and $c_{ik} \leq a_{ik}$ ($c_{ik} \leq a_{ik}$ with at least one strict inequality). Similarly, we write $v \leq w$ if v, w are two nonnegative vectors in $\mathbb{R}^n$ and $v_i \leq w_i$ holds in each coordinate $1 \leq i \leq n$.

Lemma 4.74. [144, Remark 5 in Section XIII.§2.5] *Let $\boldsymbol{A}$ be a nonnegative irreducible matrix with PF-eigenvalue α and let v be a vector with nonnegative components. Then the inequalities $\alpha\, v \leq \boldsymbol{A}v$ or $\alpha\, v \geq \boldsymbol{A}v$ either implies that v is the corresponding PF-eigenvector (with only positive coordinates) or that $v = 0$. In either case, we have $\boldsymbol{A}v = \alpha\, v$.* □

Now we come to the second point of this section: Submultiplicative sequences and their convergence properties.

Definition 4.75. A sequence of positive real numbers $(a_k)_{k\in\mathbb{N}}$ is *submultiplicative* if it satisfies the inequality $a_{k+m} \leq a_k \cdot a_m$ for all $k, m \in \mathbb{N}$. A sequence of positive real numbers $(a_k)_{k\in\mathbb{N}}$ is *supermultiplicative* if it satisfies the inequality $a_{k+m} \geq a_k \cdot a_m$ for all $k, m \in \mathbb{N}$. If $(a_k)_{k\in\mathbb{N}}$ is supermultiplicative, then $(a_k^{-1})_{k\in\mathbb{N}}$ is submultiplicative.

Lemma 4.76. [126, Corollary 1.3] *If $(a_k)_{k\in\mathbb{N}}$ is a submultiplicative sequence, the sequence $\big((a_k)^{1/k}\big)_{k\in\mathbb{N}}$ converges, with $\lim_{k\to\infty}(a_k)^{1/k} = \inf_{k\in\mathbb{N}}(a_k)^{1/k}$.* □

Definition 4.77. A sequence of nonnegative square matrices $(\boldsymbol{A}_k)_{k\in\mathbb{N}}$ with $\boldsymbol{A}_k > \boldsymbol{0}$ for all $k \in \mathbb{N}$ is *submultiplicative* if it satisfies the inequality $\boldsymbol{A}_{k+m} \leq \boldsymbol{A}_k \cdot \boldsymbol{A}_m$ for all $k, m \in \mathbb{N}$. Similarly, we say that it is *supermultiplicative* if it satisfies the inequality $\boldsymbol{A}_{k+m} \geq \boldsymbol{A}_k \cdot \boldsymbol{A}_m$ for all $k, m \in \mathbb{N}$.

Our goal is a statement as Lemma 4.76 for the spectral radii of a sequence of submultiplicative matrices. For this, we first introduces matrix norms and state some of their properties.

Definition 4.78. A *matrix norm* $\|\!|\boldsymbol{A}|\!\|$ of an $n \times n$ square matrix $\boldsymbol{A}$ (over $\mathbb{C}$) is a nonnegative number associated with $\boldsymbol{A}$ having the following properties:

- $\|\!|\boldsymbol{A}|\!\| > 0$ if $\boldsymbol{A} \neq \boldsymbol{0}$ and $\|\!|\boldsymbol{A}|\!\| = 0$ iff $\boldsymbol{A} = \boldsymbol{0}$.
- $\|\!|r\boldsymbol{A}|\!\| = |r| \cdot \|\!|\boldsymbol{A}|\!\|$ for $r \in \mathbb{C}$.
- $\|\!|\boldsymbol{A} + \boldsymbol{B}|\!\| \leq \|\!|\boldsymbol{A}|\!\| + \|\!|\boldsymbol{B}|\!\|$ for square matrices $\boldsymbol{A}$, $\boldsymbol{B}$.
- $\|\boldsymbol{A}x\| \leq \|\!|\boldsymbol{A}|\!\| \cdot \|x\|$ where $x \in \mathbb{C}^n$ is vector with vector norm $\|x\|$; in this case, we say that the matrix norm and the vector norm are *compatible*.

Given a vector norm $\|\cdot\|$, we say that the matrix norm $\|\!|\cdot|\!\|$ is the *natural matrix norm induced* by this vector norm if $\|\!|\boldsymbol{A}|\!\| = \max\{\|\boldsymbol{A}x\| \mid \|x\| = 1\}$.

Remark 4.79. If $\boldsymbol{E}$ is the unit matrix, then for any natural matrix norm we have $\|\boldsymbol{E}\| = 1$. We also note that a matrix norm is submultiplicative, *i.e.*, $\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\| \cdot \|\boldsymbol{B}\|$.

Let $\boldsymbol{A} = [a_{ik}]_{1\leq i,k\leq n}$, then the most frequent matrix norms are:

- The *maximum absolute column sum norm* $\|\cdot\|_1$ is defined by $\|\boldsymbol{A}\|_1 = \max_k \sum_{i=1}^n |a_{ik}|$, which is the natural matrix norm induced by $\|x\|_1 = \sum_{i=1}^n |x_i|$.
- The *maximum absolute row sum norm* $\|\cdot\|_\infty$ is defined by $\|\boldsymbol{A}\|_\infty = \max_i \sum_{k=1}^n |a_{ik}|$, which is the natural matrix norm induced by $\|x\|_\infty = \max_i |x_i|$.
- The *spectral norm* $\|\cdot\|_2$ is defined by

$$\|\boldsymbol{A}\|_2 = \sqrt{\rho(\boldsymbol{A}^*\boldsymbol{A})} = \max\{\|\boldsymbol{A}x\|_2 \mid \|x\|_2 = 1\} = \sup\left\{\frac{\|\boldsymbol{A}x\|_2}{\|x\|_2} \mid \|x\|_2 \neq 0\right\},$$

which is the natural matrix norm induced by Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. Here, $\boldsymbol{A}^* = [\overline{a}_{ki}]_{1\leq i,k\leq n}$ denotes the *adjoint* or *Hermitian transpose* of $\boldsymbol{A}$ (also see the lemma in Remark 4.110).

We now state some properties of matrix norms.

Lemma 4.80. [154, Section 15.511] and [197, Section I.§4.2]

(i) *If $\boldsymbol{A}$ is any nonsingular $n \times n$ matrix (over $\mathbb{C}$) with eigenvalues λ_i, then $1/\|\boldsymbol{A}^{-1}\| \leq |\lambda_i| \leq \|\boldsymbol{A}\|$ for all i (and any natural matrix norm). Especially, we have $\rho(\boldsymbol{A}) \leq \|\boldsymbol{A}\|$.*

(ii) *For any natural matrix norm we have*

$$\lim_{n\to\infty} \|\boldsymbol{A}^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|\boldsymbol{A}^n\|^{1/n} = \rho(\boldsymbol{A}).$$ □

We now arrive at the statement we are looking for.

Lemma 4.81. *Let $(\boldsymbol{A}_k)_{k\in\mathbb{N}}$ be a submultiplicative sequence of nonnegative matrices. Then $(\|\boldsymbol{A}_k\|)_{k\in\mathbb{N}}$ is a submultiplicative sequence of positive numbers for any natural matrix norm. Moreover, we have*

$$\inf_{n\in\mathbb{N}} \|\boldsymbol{A}_n\|^{1/n} = \lim_{n\to\infty} \|\boldsymbol{A}_n\|^{1/n} = \lim_{n\to\infty} (\rho(\boldsymbol{A}_n))^{1/n} = \inf_{n\in\mathbb{N}} (\rho(\boldsymbol{A}_n))^{1/n}.$$

Proof. Since $(\boldsymbol{A}_k)_{k\in\mathbb{N}}$ is submultiplicative, we have $\boldsymbol{A}_{k+m} \leq \boldsymbol{A}_k \cdot \boldsymbol{A}_m$ for all $k, m \in \mathbb{N}$. For a nonnegative matrix $\boldsymbol{A}$, the maximum $\max\{\|\boldsymbol{A}x\| \mid \|x\| = 1\}$ is attained for a nonnegative vector x. So we can, by the definition of a natural matrix norm, just take the norm of this last inequality and use the submultiplicativity of the matrix norm to establish that $(\|\boldsymbol{A}_k\|)_{k\in\mathbb{N}}$ is a submultiplicative sequence of positive real numbers (note that $\|\boldsymbol{A}\| = 0$ iff $\boldsymbol{A} = \boldsymbol{0}$). By Lemma 4.76, we then have $\inf_{n\in\mathbb{N}} \|\boldsymbol{A}_n\|^{1/n} = \lim_{n\to\infty} \|\boldsymbol{A}_n\|^{1/n}$. We denote this limit by a.

By $\|\boldsymbol{A}_{k\cdot n}\| \leq \|\boldsymbol{A}_n^k\| \leq \|\boldsymbol{A}_n\|^k$ we have $\lim_{k\to\infty} \|\boldsymbol{A}_{k\cdot n}\|^{1/(k\cdot n)} \leq \lim_{k\to\infty} \|\boldsymbol{A}_n^k\|^{1/(k\cdot n)} = (\rho(\boldsymbol{A}_n))^{1/n}$, which establishes $a \leq (\rho(\boldsymbol{A}_n))^{1/n}$. On the other hand, we have $\rho(\boldsymbol{A}_n) \leq \|\boldsymbol{A}_n\|$, which then establishes the claim. □

Remark 4.82. If $(\boldsymbol{A}_k)_{k\in\mathbb{N}}$ is a supermultiplicative sequence of nonnegative nonsingular matrices, then $(\boldsymbol{A}_k^{-1})_{k\in\mathbb{N}}$ is a submultiplicative sequence of (not necessarily nonnegative) matrices. Therefore, we have not been able to establish a counterpart of this last lemma for a supermultiplicative sequence of matrices.

4.8. Iterated Function Systems

Definition 4.83. Let $(X\,,d)$ be a metric space. A map $f : X \to X$ is called *Lipschitz* if there is a constant $q < \infty$, called a *Lipschitz constant* of f, such that

$$d\,(f(x)\,,f(y)) \leq q \cdot d(x\,,y) \text{ for all } x, y \in X.$$

It is called a *contraction*, if one can choose $q < 1$; in this case, we call such a q a *contraction constant* associated to the contraction f. Similarly, $f : X \to X$ is called an *expansion* if there is a $q > 1$ such that $d\,(f(x)\,,f(y)) \geq q \cdot d(x\,,y)$ for all $x, y \in X$. Moreover, $f : X \to X$ is called a *similarity* if there is a $q > 0$ such that $d\,(f(x)\,,f(y)) = q \cdot d(x\,,y)$ for all $x, y \in X$. A similarity with constant $q = 1$ is also called an *isometry*.

We now define what we mean by an "iterated function system".

Definition 4.84. An *iterated function system*, or *IFS* for short, on a metric space $(X\,,d)$ is an $n \times n$ matrix $\boldsymbol{\Theta} = \left[\Theta_{ij}\right]_{1\leq i,j\leq n}$, where each $\Theta_{ij} = \{f_1, \ldots, f_k\}$ is a finite (possibly empty) set of contractions (with respect to d). An IFS is called *irreducible* (*primitive*) if its *substitution matrix* $\boldsymbol{S\Theta} = \left[\operatorname{card}\Theta_{ij}\right]_{1\leq i,j\leq n}$ is irreducible (primitive), where we set $\operatorname{card}\varnothing = 0$. More generally, an $n \times n$ matrix $\boldsymbol{\Theta}$, where each $\Theta_{ij} = \{f_1, \ldots, f_k\}$ is a finite (possibly empty) set of functions, is called a *matrix function system*, or *MFS* for short. Therefore, an IFS is an MFS on a metric space $(X\,,d)$ such that all maps are contractions.

Remark 4.85. Note that each entry Θ_{ij} of an IFS (respectively MFS) $\boldsymbol{\Theta}$ is a (maybe empty) set of maps. Moreover, we can define $\boldsymbol{\Theta}^k$ (for $k \in \mathbb{N}$) inductively by the composition of maps, *i.e.*,

$$\boldsymbol{\Theta}^{k+1} = \boldsymbol{\Theta}^k \circ \boldsymbol{\Theta} = \left[\bigcup_{\ell=1}^{n} \left\{\Theta_{i\ell}^{(k)} \circ \Theta_{\ell j}\right\}\right]_{1\leq i,j\leq n}$$

where

$$\Theta_{i\ell}^{(k)} \circ \Theta_{\ell j} = \begin{cases} \varnothing, & \text{if } \Theta_{i\ell}^{(k)} = \varnothing \text{ or } \Theta_{\ell j} = \varnothing, \\ \{g \circ f \mid g \in \Theta_{i\ell}^{(k)},\ f \in \Theta_{\ell j}\}, & \text{otherwise.} \end{cases}$$

Clearly, we have $\boldsymbol{S\Theta}^k \leq (\boldsymbol{S\Theta})^k$.

Since $(X\,,d)$ is a metric space, we know by Proposition 4.63 that $\mathcal{K}X$ is a metric space with the Hausdorff metric h_d. Moreover, the space $(\mathcal{K}X)^n$ is equipped with the product topology (see Lemma 4.65), *i.e.*, the topology of $(\mathcal{K}X)^n$ is generated by the corresponding maximum metric $h_d^{\max}$ (maximum taken over the Hausdorff metrics on its n components). On $\underline{A} = (A_i)_{i=1}^n \in (\mathcal{K}X)^n$ the action of an IFS $\boldsymbol{\Theta}$ is given by

$$\boldsymbol{\Theta}(\underline{A}) = \left(\bigcup_{j=1}^{n} \bigcup_{f\in\Theta_{ij}} f(A_j)\right)_{i=1}^{n}.$$

Note that the continuous image of a compact set is again compact, see [75, Corollary 1 to Theorem I.§9.2].

Lemma 4.86. *The IFS* $\boldsymbol{\Theta}$ *defines a contraction on* $(\mathcal{K}X)^n$*, provided no row sum of* $\boldsymbol{S\Theta}$ *is* 0.

Proof. For a contraction $f : X \to X$, we denote by $q(f)$ its contraction constant. For $A, B \in (\mathcal{K}X)^n$, *i.e.*, $\underline{A} = (A_i)_{i=1}^n$ and $\underline{B} = (B_i)_{i=1}^n$ where $A_i, B_i \in \mathcal{K}X$ for all $1 \leq i \leq n$, we obtain the following estimates:

$$\begin{aligned}
h_d^{\max}(\boldsymbol{\Theta}(\underline{A}), \boldsymbol{\Theta}(\underline{B})) &= \max_{1\leq i\leq n} h_d\left(\bigcup_{1\leq j\leq n} \Theta_{ij}(A_j), \bigcup_{1\leq j\leq n} \Theta_{ij}(B_j)\right) \\
&= \max_{1\leq i\leq n} h_d\left(\bigcup_{1\leq j\leq n} \bigcup_{f\in\Theta_{ij}} f(A_j), \bigcup_{1\leq j\leq n} \bigcup_{f\in\Theta_{ij}} f(B_j)\right) \\
&\leq \max_{1\leq i\leq n} \max_{1\leq j\leq n} {\max_{f\in\Theta_{ij}}}' h_d\left(f(A_j), f(B_j)\right) \\
&\leq \max_{1\leq i\leq n} \max_{1\leq j\leq n} {\max_{f\in\Theta_{ij}}}' \left(q(f) \cdot h_d\left(A_j, B_j\right)\right) \\
&\leq \left(\max_{1\leq i\leq n} \max_{1\leq j\leq n} {\max_{f\in\Theta_{ij}}}' q(f)\right) \cdot h_d^{\max}\left(\underline{A}, \underline{B}\right).
\end{aligned}$$

Since we take the maximum over finitely many contraction constants, we have

$$\left(\max_{1\leq i\leq n} \max_{1\leq j\leq n} {\max_{f\in\Theta_{ij}}}' q(f)\right) < 1$$

and the IFS $\boldsymbol{\Theta}$ is a contraction on $(\mathcal{K}X)^n$. The condition on the row sums ensures that the maximum over i in the above estimate is well-defined, the maximum over Θ_{ij} is only taken if the set Θ_{ij} is nonempty (wherefore we denoted this maximum by a prime); however, for every i, there is at least one j such that Θ_{ij} is nonempty by the row sum condition. □

Remark 4.87. This last estimate can also be found in [35, Proposition 4.1], also see [187, Section 3.2] and [126, Theorem 2.6]. It is also possible to extend the definition of an IFS to handle infinitely many contractions if their contraction constants have an upper bound less than 1 and if they form a (nonempty) compact set on the space of continuous maps $C(X, X)$ endowed with the compact-open topology, see [391, Chapter 3].

The importance of contractions lies in *Banach's fixed point theorem*, also called the *contraction principle* or *contraction mapping theorem*.

Theorem 4.88. [51, Theorem 3.2] and [112, Theorem 2.1.36] *Let $f : X \to X$ be a contraction on a complete metric space (X, d). Then f possesses a unique fixed point $x_f \in X$. Moreover, the sequence $\{f^k(x) \mid k \in \mathbb{N}\}$ converges for every $x \in X$ to x_f, i.e., $\lim_{k\to\infty} f^k(x) = x_f$ for every $x \in X$.* □

Proposition 4.89. *Let $\boldsymbol{\Theta}$ be an IFS on a complete metric space X (or, more precisely, on $(\mathcal{K}X)^n$), such that no row sum of $\boldsymbol{S\Theta}$ is 0. Then there exists a unique set $\underline{\Omega} \subset (\mathcal{K}X)^n$ (i.e., $\underline{\Omega} = (\Omega_i)_{i=1}^n$ where all Ω_i are nonempty compact subsets of X) that satisfies $\underline{\Omega} = \boldsymbol{\Theta}(\underline{\Omega})$. Moreover, for all $\underline{A} \subset (\mathcal{K}X)^n$ we have $\boldsymbol{\Theta}^k(\underline{A}) \to \underline{\Omega}$ in the metric $h_d^{\max}$. Furthermore, if $\boldsymbol{\Theta}(\underline{A}) \subset \underline{A}$ then $\underline{\Omega} = \bigcap_{k\in\mathbb{N}} \boldsymbol{\Theta}^k(\underline{A})$, and if $\underline{A} \subset \boldsymbol{\Theta}(\underline{A})$ and $\boldsymbol{\Theta}^k(\underline{A})$ is bounded above, then $\underline{\Omega} = \operatorname{cl}\left(\bigcup_{k\in\mathbb{N}} \boldsymbol{\Theta}^k(\underline{A})\right)$.*

Proof. Banach's fixed point theorem (Theorem 4.88) is applicable by Lemma 4.86 together with Proposition 4.63. The condition on $\boldsymbol{\Theta}(\underline{A})$ defines an antitone, respectively isotone sequence in $(\mathcal{K}X)^n$, see Lemma 4.64. □

Remark 4.90. We use the notation $\underline{A}$ to indicate that we are talking about a finite family of sets $\underline{A} = (A_i)_{i=1}^n$ (which is a set in the product space $(\mathcal{K}X)^n$). Moreover, we write $\underline{A} \subset \underline{B}$ to mean that $A_i \subset B_i$ for each component $1 \leq i \leq n$.

Remark 4.91. This proposition is proved in [187] (*e.g.*, [187, Theorems 1(1) & 3.1(3)]) for a metric space X and $n = 1$. Iterated functions systems on $\mathbb{R}^d$ (with all maps similarities, the main focus of [187]) have been popularised by the book [51].

Remark 4.92. Here, it becomes obvious that the space $\mathcal{K}X$, respectively $(\mathcal{K}X)^n$ is the appropriate space to consider an IFS. Obviously, $\varnothing$ and, if X is not a compact space (*e.g.*, $X = \mathbb{R}^d$), X, respectively $(\varnothing)_{i=1}^n$ and $(X)_{i=1}^n$ would otherwise always be solutions of the IFS.

Remark 4.93. The unique set $\underline{\Omega} = \boldsymbol{\Theta}(\underline{\Omega})$ satisfying the IFS is called the *attractor* or *invariant set* of the IFS $\boldsymbol{\Theta}$. By uniqueness, the IFS defines the attractor $\underline{\Omega}$. Moreover, we note that the iteration towards the attractor converges (in the Hausdorff metric) at a geometric rate for every $\underline{A} \in (\mathcal{K}X)^n$ since

$$h_d^{\max}(\boldsymbol{\Theta}^k(\underline{A}), \underline{\Omega}) \leq \left(\max_{1\leq i\leq n} \max_{1\leq j\leq n} \max_{f\in\Theta_{ij}}{}' q(f) \right)^k \cdot h_d^{\max}(\underline{A}, \underline{\Omega}).$$

To simplify the notation, we use the abbreviation

$$q(\boldsymbol{\Theta}) = \left(\max_{1\leq i\leq n} \max_{1\leq j\leq n} \max_{f\in\Theta_{ij}}{}' q(f) \right).$$

Remark 4.94. In the following, we write $f \in \boldsymbol{\Theta}$ if we mean $f \in \bigcup_{i=1}^n \bigcup_{j=1}^n \Theta_{ij}$, *i.e.*, if the map f occurs in (at least) one Θ_{ij}.

We now specify types of iterated function systems.

Definition 4.95. Let $\boldsymbol{\Theta}$ be an IFS on a metric space (X, d) with attractor $\underline{\Omega}$, where X is also a commutative group or, more generally, a module (wherefore we have linear maps, namely the module-homomorphisms of X into itself, and translations by any element of X). If all maps in $\boldsymbol{\Theta}$ are similarities, the attractor $\underline{\Omega}$ is called *self-similar.* If all maps in $\boldsymbol{\Theta}$ are affine transformations, *i.e.*, each map $f = t \circ g$ is a composition of a linear map g and a translation t on X, the attractor $\underline{\Omega}$ is called *self-affine*[5]. We say that the IFS $\boldsymbol{\Theta}$ is *nonsingular* if there is a $\varrho = \varrho(\boldsymbol{\Theta}) > 0$ such that

$$d(f(x), f(y)) \geq \varrho \cdot d(x, y) \text{ for all } x, y \in X \text{ and all maps } f \in \boldsymbol{\Theta}.$$

Basically, a nonsingular IFS maps balls to (proper) ellipsoids. All maps $f \in \boldsymbol{\Theta}$ of a nonsingular IFS $\boldsymbol{\Theta}$ are *bi-Lipschitz functions*, *i.e.*, there are constants $0 < C' \leq C < \infty$ such that $C' \cdot d(x, y) \leq d(f(x), f(y)) \leq C \cdot d(x, y)$ for all $x, y \in X$ (here, we can take $C' = \varrho(\boldsymbol{\Theta})$ and $C = q(\boldsymbol{\Theta})$). Note that every bi-Lipschitz function is injective, and that the inverse function of a bi-Lipschitz bijection is also bi-Lipschitz.

[5]Furthermore, if all maps f in $\boldsymbol{\Theta}$ are conformal maps, *i.e.*, their derivatives f' are similarities, then $\underline{\Omega}$ is called *self-conformal.* We will not consider self-conformal attractors here, but such iterated function systems are also frequently considered in the literature.

Remark 4.96. If we write the maps f in $\boldsymbol{\Theta}$ in the form $f(x) = g(x) + t$ where g is a linear (contractive) map on X and t is a translation, then the situation is often as follows: For all maps in $\boldsymbol{\Theta}$ the linear part is the same, they only differ in the translational part.

Proposition 4.97. *Let $\boldsymbol{\Theta}$ be a nonsingular irreducible IFS on a separable complete metric space X with attractor $\underline{\Omega} = (\Omega_i)_{i=1}^n$.*

(i) *The Hausdorff dimension of all Ω_i is the same.*

(ii) *If one set Ω_{i_0} is Hausdorff rectifiable, then all Ω_i are Hausdorff rectifiable.*

(iii) *If there is an i such that Ω_i has interior points, then all Ω_i have interior points.*

Proof. (i): Since the IFS is nonsingular, all maps occuring in $\boldsymbol{\Theta}$ are injective. Moreover, the inverse $f^{-1} : f(X) \to X$ of each such map f satisfies the Lipschitz condition $d\left(f^{-1}(x), f^{-1}(y)\right) \leq \frac{1}{\varrho} \cdot d(x, y)$. By the irreducibility of the IFS, there are for each pair $1 \leq i, j \leq n$ powers $p, p' \in \mathbb{N}$ such that $\Theta_{ij}^{(p)} \neq \varnothing$ and $\Theta_{ji}^{(p')} \neq \varnothing$. If $g \in \Theta_{ij}^{(p)}$, then $g(\Omega_j) \subset \Omega_i$; moreover, Corollary 4.50 (respectively Lemma 4.49) tells us that if Ω_j has Hausdorff measure 0, so has $g(\Omega_j)$ by the inequality $(\mu^*)^{(r)}\left(g(\Omega_j)\right) \leq q^r \cdot (\mu^*)^{(r)}(\Omega_j)$. Conversely, applying this inequality to g^{-1} yields $(\mu^*)^{(r)}(\Omega_j) \leq \varrho^{-r} \cdot (\mu^*)^{(r)}\left(g(\Omega_j)\right)$ and thus, if Ω_j has nonzero Hausdorff measure, so has $\Omega_i \supset g(\Omega_j)$. So by irreducibility, if one Ω_i has nonzero Hausdorff measure all have. By the definition of the Hausdorff dimension, their Hausdorff dimension is then the same.
(ii): We have already shown that if one set Ω_{i_0} has nonzero Hausdorff measure then (by irreducibility and nonsingularity) all sets Ω_i have nonzero Hausdorff measure. Similarly, if one set has infinite measure, all have.
(iii): Since the IFS is nonsingular, each map f of $\boldsymbol{\Theta}$ maps an open ball $B_{<r}(x)$ of radius r on some ellipsoid, in which we can inscribe an open ball $B_{<(\varrho \cdot r)}(f(x))$ of radius $\varrho \cdot r$. Since Ω_i has interior points if we find an open ball $B_{<r}(x) \subset \Omega_i$, the claim follows by irreducibility. □

Remark 4.98. In the situation of the last proposition, we also write $\dim_{\mathrm{Hd}} \underline{\Omega}$ to mean $\dim_{\mathrm{Hd}} \Omega_1 = \ldots = \dim_{\mathrm{Hd}} \Omega_n$.

The following statement will play an important role below, see Corollary 5.62 and Lemma 5.140.

Proposition 4.99. *Let (G, d) be a metrisable locally compact Abelian group with Haar measure μ. Let $\boldsymbol{\Theta}$ be a nonsingular irreducible IFS on G with attractor $\underline{\Omega} = (\Omega_i)_{i=1}^n$ with the following properties: there is an i_0 ($1 \leq i_0 \leq n$) such that Ω_{i_0} has interior points and $\mu(W) = \alpha \cdot \mu(f(W))$ for all compact sets $W \subset G$ and all maps f occurring in the IFS $\boldsymbol{\Theta}$, where α denotes the PF-eigenvalue of $\boldsymbol{S\Theta}$. Then the following hold:*

(i) *All Ω_i have nonzero Haar measure.*

(ii) *The unions in the IFS are* measure-disjoint*, i.e., for all i and all functions $g \in \Theta_{ij}$ and $h \in \Theta_{ik}$ with $g \neq h$ (but possibly $j = k$), we have*

$$\mu\left(g(\Omega_j) \cap h(\Omega_k)\right) = 0$$

(we note that $g(\Omega_j) \cup h(\Omega_k) \subset \Omega_i$).

(iii) *If all maps $f : G \to G$ in $\boldsymbol{\Theta}$ are homeomorphisms, the boundaries $\partial\Omega_i$ have zero Haar measure for all i. Moreover, in this case all Ω_i are perfect sets and regularly closed.*

Proof. The proof of Lemma 4.54 shows that the Haar measure of a compact set on G with interior points is positive, therefore $0 < \mu(\Omega_{i_0}) < \infty$. Furthermore, we note that the sets Ω_i and its boundaries are compact.
Using the IFS, we get the following inequality (in each component) for the Haar measure of the sets Ω_i:

$$\begin{aligned}\begin{pmatrix}\mu(\Omega_1)\\ \vdots \\ \mu(\Omega_n)\end{pmatrix} &= \begin{pmatrix}\mu\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{1j}} f(\Omega_j)\right)\\ \vdots \\ \mu\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{nj}} f(\Omega_j)\right)\end{pmatrix} \le \begin{pmatrix}\sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{1j}} \mu(f(\Omega_j))\\ \vdots \\ \sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{nj}} \mu(f(\Omega_j))\end{pmatrix}\\ &= \begin{pmatrix}\sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{1j}} \frac{1}{\alpha}\cdot\mu(\Omega_j)\\ \vdots \\ \sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{nj}} \frac{1}{\alpha}\cdot\mu(\Omega_j)\end{pmatrix} = \frac{1}{\alpha}\cdot\begin{pmatrix}\sum\limits_{1\le j\le n}(\operatorname{card}\Theta_{1j})\cdot\mu(\Omega_j)\\ \vdots \\ \sum\limits_{1\le j\le n}(\operatorname{card}\Theta_{nj})\cdot\mu(\Omega_j)\end{pmatrix} \qquad (4.2)\\ &= \frac{1}{\alpha}\cdot\boldsymbol{S\Theta}\begin{pmatrix}\mu(\Omega_1)\\ \vdots \\ \mu(\Omega_n)\end{pmatrix}.\end{aligned}$$

By Lemma 4.74, this inequality forces $(\mu(\Omega_1), \dots, \mu(\Omega_n))^T$ to be either the PF-eigenvector with positive coordinates or the vector with only zeroes. We already know that at least one coordinate is nonzero, therefore it must be the PF-eigenvector and all Ω_i have positive Haar measure. Moreover, the equality sign holds in the above inequality, which means (in the above notation) that

$$\mu\left(g(\Omega_j) \cup h(\Omega_k)\right) = \mu\left(g(\Omega_j)\right) + \mu\left(h(\Omega_k)\right)$$

and therefore $\mu\left(g(\Omega_j) \cap h(\Omega_k)\right) = 0$ for all g, h as above.

For the last claims, we note that, that since all maps $f : G \to G$ in the IFS are bijections, we have $f(A^c) = f(A)^c$ for all sets $A \subset G$. Moreover, since they are all closed and continuous, we have $f(\operatorname{cl} A) = \operatorname{cl} f(A)$ by Lemma 2.4 for all subsets $A \subset G$. Therefore we obtain $\partial f(A) = f(\partial A)$ and $\operatorname{int} f(A) = f(\operatorname{int} A)$ for all subsets A of G.

Observing that $\partial(A_1 \cup A_2) \subset \partial A_1 \cup \partial A_2$, it follows that

$$\begin{pmatrix} \mu(\partial\Omega_1) \\ \vdots \\ \mu(\partial\Omega_n) \end{pmatrix} = \begin{pmatrix} \mu\left(\partial\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{1j}} f(\Omega_j)\right)\right) \\ \vdots \\ \mu\left(\partial\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{nj}} f(\Omega_j)\right)\right) \end{pmatrix} \le \begin{pmatrix} \mu\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{1j}} f(\partial\Omega_j)\right) \\ \vdots \\ \mu\left(\bigcup\limits_{1\le j\le n}\bigcup\limits_{f\in\Theta_{nj}} f(\partial\Omega_j)\right) \end{pmatrix}$$

$$\le \begin{pmatrix} \sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{1j}} \mu(f(\partial\Omega_j)) \\ \vdots \\ \sum\limits_{1\le j\le n}\sum\limits_{f\in\Theta_{nj}} \mu(f(\partial\Omega_j)) \end{pmatrix} = \frac{1}{\alpha}\cdot \boldsymbol{S\Theta} \begin{pmatrix} \mu(\partial\Omega_1) \\ \vdots \\ \mu(\partial\Omega_n) \end{pmatrix}. \tag{4.3}$$

Again, either we have $\mu(\partial\Omega_i) = 0$ for all i or $\mu(\partial\Omega_i) > 0$ for all i. In both cases, the equality sign holds throughout the above inequality, and we want to exclude that the latter case.

Suppose to the contrary that $\mu(\partial\Omega_i) > 0$, then – with the same reasoning as above – all unions in the IFS are measure-disjoint. Moreover, since $\mu(g(\partial\Omega_j)) = \alpha\cdot\mu(\partial\Omega_j) > 0$ for all maps $g \in \Theta_{ij}^{(p)}$, all the measures occuring in the IFS are nonzero. Obviously, this also holds for higher iterations, where we then obtain

$$\frac{1}{\alpha^k}\cdot(\boldsymbol{S\Theta})^k \begin{pmatrix} \mu(\partial\Omega_1) \\ \vdots \\ \mu(\partial\Omega_n) \end{pmatrix} = \begin{pmatrix} \mu(\partial\Omega_1) \\ \vdots \\ \mu(\partial\Omega_n) \end{pmatrix} \le \frac{1}{\alpha^k}\cdot \boldsymbol{S\Theta}^k \begin{pmatrix} \mu(\partial\Omega_1) \\ \vdots \\ \mu(\partial\Omega_n) \end{pmatrix}. \tag{4.4}$$

Here, the (first) equality sign holds simply by iteration of the above equation. The inequality is obtained in a similar manner than the above calculations. Since we also have $\boldsymbol{S\Theta}^k \le (\boldsymbol{S\Theta})^k$, this last inequality is again an equality (here, of course, nonnegativeness/positivity of all entries and coordinates enters). This shows that the measure-disjointness also holds for higher iterates.

We define $D = \max_{1\le i\le n}\operatorname{diam}(\Omega_i)$. Then we have $\operatorname{diam}(f(\Omega_j)) \le q(\boldsymbol{\Theta})^k\cdot D$ for all $1\le i\le n$ and every $f\in\Theta_{ij}^{(k)}$. By assumption, Ω_{i_0} has interior points, therefore we find an $x\in\Omega_{i_0}$ and an $r>0$ such that $B_{<r}(x)\subset \operatorname{int}\Omega_{i_0}$. Choose k_0 such that $q(\boldsymbol{\Theta})^{k_0}\cdot D < 2\cdot r/3$, then there is a j such that $f(\Omega_j)\subset B_{<r}(x)$ for one $f\in\Theta_{i_0 j}^{(k_0)}$. Consequently, we have $\partial f(\Omega_j)\subset\operatorname{int}\Omega_{i_0}$, and this boundary (*i.e.*, $\partial f(\Omega_j)$) is not part of the boundary $\partial\Omega_{i_0}$. On the other hand, the equality sign in Equation (4.3), respectively Equation (4.4) implies that $\mu(\partial f(\Omega_j)) > 0$ contributes to $\mu(\partial\Omega_{i_0})$, a contradiction. Therefore, we must have $\mu(\partial\Omega_i) = 0$ for all i.

Now we show that all Ω_i are perfect and the regular closedness: Since Ω_{i_0} has interior points, so have all Ω_i by primitivity and $\operatorname{int} f(A) = f(\operatorname{int} A)$. Let $x\in\Omega_i$. We show that x is not an isolated point, *i.e.*, that for every neighbourhood U of x, there are points in $U\cap\Omega_i$ other than x. For every $\varepsilon>0$, $B_\varepsilon(x)$ is a neighbourhood of x. As before, we choose a k_0 such that $q(\boldsymbol{\Theta})^{k_0} \cdot D < 2 \cdot \varepsilon/3$, then there is a j such that $f(\Omega_j) \subset B_{<\varepsilon}(x)$ for one $f \in \Theta_{ij}^{(k_0)}$. But Ω_j and also $f(\Omega_j)$ have interior points, so every neighbourhood of x contains interior points. Therefore, $x \in \Omega_i$ is a limit point of a sequence of inner points of Ω_i and therefore, in particular, no isolated point. Consequently, since we have chosen $x \in \Omega_i$ arbitrary, Ω_i contains no isolated points and, as a compact (and therefore closed) set, it is also a perfect set. Moreover, Ω_i is also regularly closed, since the closure of a set is the set of its limit points. □

Remark 4.100. For self-similar or self-affine attractors of an IFS in $\mathbb{R}^d$, the statement of the last proposition can already be found in the literature. *E.g.*, the measure-disjointness can be found in [381, Proposition 10.1], while the statement about the boundaries appears in [221, Proof of Theorem 5.5 (vi)⇒(vii)] which itself is a generalisation of an argument used in [201] (we might call it the “Keesling-argument”: If the sets Ω_i of an attractor have interior points and are measure-disjoint, then (under suitable conditions) their boundaries have zero measure because some image of the boundary is mapped into the interior points under the IFS and can therefore not be part of any original boundary.).

Remark 4.101. In the situation of Proposition 4.99(iii), one can often show that the boundary has Hausdorff dimension strictly less than the metric dimension of the underlying space, see Lemma 6.70. For this, one needs the notion of “affinity dimension” and its relation to the Hausdorff dimension, which will be established in the next two sections.

Remark 4.102. Note that a regularly open set A (*i.e.*, $A = \operatorname{int} \operatorname{cl} A$) can have a boundary ∂A of positive Haar measure, see [74]. Obviously, the same holds for its closure $\operatorname{cl} A$, which is a regularly closed set.

We also note that there exist examples of self-similar attractors in $\mathbb{R}^2$ with positive Lebesgue measure but empty interior, see [98]. Note that in $\mathbb{R}$ the existence of such a set is an open problem.

We also remark that the Cantor set is a nowhere dense compact perfect subset of $\mathbb{R}$ with no interior points.

The proof of the above proposition also yields the following corollary.

Corollary 4.103. *In the situation of Proposition* 4.99*, we have* $\boldsymbol{S\Theta}^k = (\boldsymbol{S\Theta})^k$ *for all* $k \in \mathbb{N}$. □

Remark 4.104. An IFS $\boldsymbol{\Theta}$ can be visualised by a directed (multi-)graph $G(\boldsymbol{\Theta}) = G(V, \vec{E})$, where the vertex-set V is given by $V = \{\Omega_1, \dots, \Omega_n\}$. We draw $\operatorname{card} \Theta_{ij}$ directed edges from Ω_i to Ω_j, where we might label each one with exactly one of the maps in Θ_{ij}. Note that, by this labelling of the edges, a walk in $G(\boldsymbol{\Theta})$ is (uniquely) represented by the edges it runs through. The direction of the edges might seem counterintuitive, since we have an arrow from Ω_i to Ω_j labelled f if $f(\Omega_j)$ is a subset of Ω_i. This is chosen to match with the construction of the path spaces in the next section (see p. 106). However, this is only a convention and the other direction might – in different circumstances – be more appropriate (*e.g.*, see Section 7.5.2). See also the short remark in [44, Section 3] on the direction of the edges.

The graph $G(\boldsymbol{\Theta})$ is *strongly connected*, *i.e.*, for every (ordered) pair of vertices i, j (we use the notation i instead of Ω_i to denote the vertices from now on) there is a path along directed edges from i to j, iff $\boldsymbol{S\Theta}$ is irreducibel. In fact, by construction the matrix $\boldsymbol{S\Theta}$ is the *adjacency matrix* of $G(\boldsymbol{\Theta})$ (the element m_{ij} of the adjacency matrix is the number of (directed) edges from vertex i to j). If $G(\boldsymbol{\Theta})$ is strongly connected, we denote by h the greatest common divisor of the set of all lengths of *circles* in $G(\boldsymbol{\Theta})$. Then this h is the same as the number h mentioned in Proposition 4.69 (ii), and $\boldsymbol{S\Theta}$ is primitive iff $h = 1$ (see [340, Section 1.2]). Here, a circle is a closed path $\{(ij_1), (j_1j_2), \dots, (j_{\ell-1}i)\}$ (so $i, j_1, \dots, j_{\ell-1}$ are distinct), its length being ℓ.

Because of this representation as a graph, some people speak of a *(geometric) graph directed construction* or *graph-directed iterated function system (GIFS)* (in case of self-similarity, the term *mixed self-similar sets* is used in [44]). This parlance was introduced in [252], because before that time mainly (self-similar) iterated function systems of the form $\Omega = f_1(\Omega) \cup \dots \cup f_k(\Omega)$ were considered, wherefore the "G" in GIFS indicates that an IFS on $(\mathcal{K}X)^n$ with (possibly) $n > 1$ is considered (we cite [126, p. 48]: "Graph-directed sets, which generalise self-similar sets, provide our next example.")

4.9. Net Measures

We now return to the σLCAG $\mathbb{M}$ with metric $d_{\mathbb{M}}$ given in (4.1) as

$$\mathbb{M} = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \dots \times \mathbb{Q}_{\mathfrak{p}_k}.$$

Definition 4.105. We consider the following class of affine mappings in $\mathbb{M}$:
We denote by $\mathcal{D}$ the space of diagonal linear mappings from $\mathbb{M}$ to $\mathbb{M}$. Then, for each $T \in \mathcal{D}$, there are numbers $a_1, \dots, a_{r+s+k}$ such that

$$T(x) = T((x_1, \dots, x_{r+s+k})) = (a_1 \cdot x_1, \dots, a_{r+s+k} \cdot x_{r+s+k}), \tag{4.5}$$

where $x_1, \dots, x_r, a_1, \dots, a_r \in \mathbb{R}$, while $x_{r+1}, \dots, x_{r+s}, a_{r+1}, \dots, a_{r+s} \in \mathbb{C}$ and $x_{r+s+j}, a_{r+s+j} \in \mathbb{Q}_{\mathfrak{p}_j}$ $(1 \le j \le k)$.

We now look at the family of the $r + 2s + k$ numbers (the normalised absolute values of the numbers $a_1, \dots, a_r, a_{r+s+1}, \dots, a_{r+s+k}$ and the usual absolute value of the numbers $a_{r+1}, \dots, a_{r+s}$, but taken twice[6])

$$\begin{aligned}(\,&|a_1|, \dots, |a_r|, |a_{r+1}|, |a_{r+1}|, |a_{r+2}|, \dots, |a_{r+s-1}|, |a_{r+s}|, |a_{r+s}|,\\ &\qquad \|a_{r+s+1}\|_{\mathfrak{p}_1}, \dots, \|a_{r+2s+k}\|_{\mathfrak{p}_k}\,),\end{aligned} \tag{4.6}$$

called the *singular values* of T. We order them in descending order $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_{r+2s+k}$, where $(\alpha_1, \dots, \alpha_{r+2s+k})$ is a permutation of (4.6). We are only interested in maps $T \in \mathcal{D}$ which are *contracting* $(\alpha_1 < 1)$ and *nonsingular* $(\alpha_{r+2s+k} > 0)$. We denote the subset of nonsingular and contracting maps of $\mathcal{D}$ by $\mathcal{D}'$. A map $f(x) = T(x) + t$ with $T \in \mathcal{D}$, $t \in \mathbb{M}$ for all $x \in \mathbb{M}$ is an affine map; if we have $T \in \mathcal{D}'$, then f is an affine bi-Lipschitz function. We note that if $T \in \mathcal{D}$ is nonsingular, then T is a homeomorphism (so Proposition 4.99 applies in this case). We also note that the definition of "contracting" and "nonsingular" here lines up with the corresponding definition for an IFS, *i.e.*, if all maps $f \in \boldsymbol{\Theta}$ are of the form $f(x) = T(x) + t$ with $T \in \mathcal{D}'$, then $\boldsymbol{\Theta}$ is a nonsingular (self-affine) IFS.

[6] The reason for taking the usual absolute value of each factor in a complex coordinate twice is the following: Each factor of $\mathbb{C}$ contributes 2 to the metric dimension count and we want to incorporate the normalised absolute value (which happens to be the square, thus the 2, of the usual absolute value), because normalised absolute values are closely related to (Haar) measure, see Proposition 4.38 and Remark 4.107. It also matches with the definition of the metric $d_{\mathbb{M}}$, where real and imaginary coordinates are treated separately.

Remark 4.106. If all singular values are nonzero, the inverse map of $T \in \mathcal{D}$ (see Equation (4.5)) is given by

$$T^{-1}(x) = T^{-1}((x_1, \dots, x_{r+2s+k})) = (a_1^{-1} \cdot x_1, \dots, a_{r+2s+k}^{-1} \cdot x_{r+2s+k}).$$

With this, one can easily calculate the inverse functions of the affine mappings considered here.

Remark 4.107. We recall from Proposition 4.38 that, on a local field K, we have $\mu(x \cdot A) = |x|_\nu \cdot \mu(A)$ for every Haar measurable set $A \subset K$ and every $x \in K$, where $|\cdot|_\nu$ denotes the normalised absolute value on K. Let μ now denote the Haar measure on $\mathbb{M}$ (which, by the discussion following Proposition 4.38, is the product measure of the Haar measures of its components). If $A \subset \mathbb{M}$ is measurable, $T(A)$ is measurable for any $T \in \mathcal{D}$ and we have

$$\mu(T(A)) = \Big(|a_1| \cdots |a_r| \cdot |a_{r+1}|^2 \cdots |a_{r+s}|^2 \cdot \|a_{r+s+1}\|_{\mathfrak{p}_1} \cdots \|a_{r+s+k}\|_{\mathfrak{p}_k} \Big) \cdot \mu(A).$$

Since the Haar measure and the $(r+2s+k)$-measure on $\mathbb{M}$ coincide by Theorem 4.56 (and our goal is to determine the Hausdorff dimension and measure of subsets of $\mathbb{M}$), it is – at least – sensible to use the singular values in the following.

We first add some remarks on the space $\mathcal{D}$.

Remark 4.108. We recall the definition of linear maps on a d-dimensional vector space K_ν^d (where K_ν is a local field), see p. 56: By $\mathcal{L}_\nu^d$ we denote the ring of $d \times d$ matrices over K_ν. Then, a matrix $\boldsymbol{M} \in \mathcal{L}_\nu^d$ defines a linear transformation on the d-dimensional vector space K_ν^d. Also, the determinant of such a matrix is calculated as usual ("expansion according to the i-th row" *etc.*, see [226, Section XIII.4]), and the matrix $\boldsymbol{M}$ is invertible iff $\det \boldsymbol{M} \neq 0$, see [226, Proposition XIII.4.16]. Of course, one would like to have something like the Jordan decomposition/normal (or canonical) form of such a matrix, but the situation is much more complicated in general local fields than in $\mathbb{C}^d$ or even $\mathbb{R}^d$. Let us briefly indicate why.

We first note that the eigenvalues of a matrix $\boldsymbol{M}$ are precisely the roots of the characteristic polynomial of $\boldsymbol{M}$, *i.e.*, of $\det(x \cdot \boldsymbol{E} - \boldsymbol{M})$, see [226, Theorem XIV.3.2]. The following statement shows that we get a diagonal matrix, if the eigenvalues are distinct:

Lemma. [226, Corollary XIV.3.4] *If $\boldsymbol{M}$ has d distinct eigenvalues $\lambda_1, \dots, \lambda_d$ belonging to the eigenvectors $v_1, \dots, v_d$, and the underlying vector space V is d-dimensional, then $\{v_1, \dots, v_d\}$ is a basis for V. The matrix $\boldsymbol{M}$ with respect to this basis is the diagonal matrix*

$$\operatorname{diag}(\lambda_1, \dots, \lambda_d) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_d \end{pmatrix}$$
□

Of course, the characteristic polynomial might not have roots in the underlying field K_ν, because the (complete) local field K_ν is not necessarily algebraically closed (indeed, only $\mathbb{C}$ is, and this is the reason why there are additional Jordan normal forms for matrices acting on $\mathbb{R}^d$ compared to those on $\mathbb{C}^d$). But let us suppose that the characteristic polynomial has d distinct roots $\lambda_1, \dots, \lambda_d$ in the algebraic closure K_ν^a, then some of them might be algebraically conjugate over K_ν, *i.e.*, their minimal polynomials are equal. We can associate conjugate roots, *i.e.*, we build classes of algebraically conjugate roots of the characteristic polynomial over K_ν. Obviously, if such a class contains d' members, then their minimal polynomial has degree d'. Now we take one member from each class, *e.g.*, $\check{\lambda}$ with minimal polynomial of degree $\check{d}$. Then the field extension $K_\nu(\check{\lambda})$ is a finite extension of degree $\check{d}$ and in fact a $\check{d}$-dimensional K_ν-vector space with basis $\{1, \check{\lambda}, \dots, \check{\lambda}^{\check{d}-1}\}$. Moreover, Lemma 3.75 tells us that the absolute value of algebraically conjugate elements over local fields is the same (this also holds in the (only nontrivial) extension of $\mathbb{R}$, namely $\mathbb{C}$, since a complex number and its complex conjugate number have the same absolute value).

Now, these considerations lead to the following identification: Let the characteristic polynomial have d distinct roots $\lambda_1, \dots, \lambda_d$ in the algebraic closure K_ν^a, which, with respect to algebraic conjugation, form k classes in K_ν. We take one member $\check{\lambda}_i$ $(1 \le i \le k)$ from each class and denote by $\check{d}_i$ the degree of its minimal polynomial. Then we have $d = \check{d}_1 + \ldots + \check{d}_k$. Furthermore, we now identify K_ν^d and $K_\nu(\check{\lambda}_1) \times \cdots \times K_\nu(\check{\lambda}_k)$ and $\boldsymbol{M}$ with the $k \times k$ diagonal matrix

$$\operatorname{diag}(\check{\lambda}_1, \dots, \check{\lambda}_d).$$

An example for this procedure is the identification of the rotation matrix in $\mathbb{R}^2$ by the angle $\pi/2$ and the multiplication by i on $\mathbb{C}$. For a nontrivial example in $\mathbb{Q}_2$, see Example 6.62.

We also note that the product of the minimal polynomials equals the characteristic polynomial of the matrix $\boldsymbol{M}$, *i.e.*,

$$\prod_{i=1}^{k} \operatorname{Irr}(\check{\lambda}_i, K_\nu, x) = \det(x \cdot \boldsymbol{E} - \boldsymbol{M}),$$

wherefore we also have

$$\prod_{i=1}^{k} N_{K_\nu(\check{\lambda}_i)/K_\nu}(\check{\lambda}_i) = \det(\boldsymbol{M}).$$

Considering normalised[7] absolute values (see Corollary 3.76), this last equation yields

$$\prod_{i=1}^{k} \|\check{\lambda}_i\|_{K_\nu(\check{\lambda}_i)} = \|\det(\boldsymbol{M})\|_\nu. \tag{4.7}$$

Remark 4.109. We now take the adelic viewpoint of linear transformations, or more exactly of principal lattice transformations as considered in Definition 3b.16, since this viewpoint will apply in the following chapters.

Let $\boldsymbol{M}$ be a nonsingular $d \times d$ matrix over $\mathbb{Q}$ (so we have $\det \boldsymbol{M} \neq 0$). Moreover, we assume that the characteristic polynomial of $\boldsymbol{M}$ is irreducible over $\mathbb{Q}$, wherefore the characteristic polynomial is the minimal polynomial for any of its d roots. The set of places of $\mathbb{Q}$ is given by $J = \mathbb{P} \cup \{\infty\}$, the local fields of $\mathbb{Q}$ being $\mathbb{R}$ and $\mathbb{Q}_p$ for all $p \in \mathbb{P}$. Then, $\boldsymbol{M} = (\boldsymbol{M})_{\nu \in J}$ is a principle lattice transformation on $\mathbb{A}_\mathbb{Q}^d$ (note that we use the same symbol $\boldsymbol{M}$ with different meanings here, on the one hand as $d \times d$ matrix with coefficients over $\mathbb{Q}$ which acts on each local field, on the other hand as element of $\mathcal{P}^d$).

Let $K = \mathbb{Q}(\lambda)$ be the algebraic number field generated by a root of the characteristic polynomial of $\boldsymbol{M}$. We denote the signature of K by $[r\,, s]$ where $r + 2s = d = [K : \mathbb{Q}]$. Therefore, the characteristic polynomial has r real roots $\lambda_1, \dots, \lambda_r$ and s pairs of complex conjugate roots $\lambda_{r+1}, \overline{\lambda}_{r+1}, \dots, \lambda_{r+s}, \overline{\lambda}_{r+s}$. The considerations of the last remark now yield

[7] Here is one place where the choice to take the square of the usual absolute value on $\mathbb{C}$ enters.

that, with respect to the basis of its eigenvectors, the action of $\boldsymbol{M}$ on $\mathbb{R}^d$ can be identified with the mapping

$$T_\infty : \mathbb{R}^r \times \mathbb{C}^s \to \mathbb{R}^r \times \mathbb{C}^s, \quad (x_1, \ldots, x_{r+s}) \mapsto (\lambda_1 \cdot x_1, \ldots, \lambda_{r+s} \cdot x_{r+s}).$$

Of course, we have $\mathbb{R}^d \cong \mathbb{R}^r \times \mathbb{C}^s$.

On $\mathbb{Q}_p^d$, the situation is similar: We denote by $\mathbb{Q}_{\mathfrak{p}_i}$ ($1 \le i \le k$) the local fields of $K = \mathbb{Q}(\lambda)$ extending $\mathbb{Q}_p$ (therefore we have $p \in \mathfrak{p}_i$ and $\mathfrak{p}_i$ is a prime ideal in K). Then, by Theorem 3.72, we have $\mathbb{Q}_p^d \cong \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}$ and we can again identify the action of $\boldsymbol{M}$ on $\mathbb{Q}_p^d$ with the mapping

$$T_p : \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k} \to \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}, \quad (x_1, \ldots, x_k) \mapsto (\lambda \cdot x_1, \ldots, \lambda \cdot x_k).$$

Here, we have identified the algebraic number λ with its different embeddings in the $\mathfrak{p}$-fields.

As result, we have identified the principle lattice transformation $\boldsymbol{M} = (\boldsymbol{M})_{\nu \in J}$ on $\mathbb{A}_\mathbb{Q}^d$ with a "diagonal" mapping on $\mathbb{A}_K$, namely by an multiplication with λ (respectively its embeddings in the corresponding local fields). Moreover, we note that we have $1 = \|\boldsymbol{M}\|_\mathcal{G} = \prod_{\nu \in J} |\det \boldsymbol{M}|_\nu$, where $|\det \boldsymbol{M}|_\infty = |\lambda_1| \cdots |\lambda_r| \cdot |\lambda_{r+1}| \cdots |\lambda_{r+s}|$ (again noting that in the s complex cases we use the square of the usual absolute value) and $|\det \boldsymbol{M}|_p = \|\lambda\|_{\mathfrak{p}_1} \cdots \|\lambda\|_{\mathfrak{p}_k}$ (by the last remark and Theorem 3.72).

So we have diagonalised $\boldsymbol{M}$ on $\mathbb{A}_\mathbb{Q}^d$ and we might now consider its expansive, contractive and its indifferent part. This viewpoint will be taken in Section 6.5.

This last two remarks indicate that with some care it might be possible to consider a more general class of mappings than $\mathcal{D}$. The next remark, however, shows that $\mathbb{R}^d$ might not be a good template.

Remark 4.110. Let $\boldsymbol{T}$ be a (nonsingular) linear mapping on $\mathbb{R}^d$, *i.e.*, $\boldsymbol{T}$ can be represented as $d \times d$ matrix with coefficients in $\mathbb{R}$. Then the singular values of $\boldsymbol{T}$ can be defined as the lengths of the mutually perpendicular principle semiaxes of $\boldsymbol{T}(B)$, where $B = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2} \le 1\}$ denotes the unit ball in $\mathbb{R}^d$. We have

$$\begin{aligned}\boldsymbol{T}(B) &= \{y \in \mathbb{R}^d \mid y = \boldsymbol{T}x,\ \|x\|_2 \le 1\} = \{y \in \mathbb{R}^d \mid \|\boldsymbol{T}^{-1}y\|_2 \le 1\} \\ &= \{y \in \mathbb{R}^d \mid \sqrt{\langle \boldsymbol{T}^{-1}y\,, \boldsymbol{T}^{-1}y\rangle} \le 1\} = \{y \in \mathbb{R}^d \mid \sqrt{\langle \big(\boldsymbol{T}^{-1}\big)^* \boldsymbol{T}^{-1}y\,, y\rangle} \le 1\} \\ &= \{y \in \mathbb{R}^d \mid \sqrt{\langle (\boldsymbol{T}\boldsymbol{T}^*)^{-1}\, y\,, y\rangle} \le 1\}.\end{aligned}$$

Here, $\boldsymbol{T}^*$ denotes the *adjoint* of $\boldsymbol{T}$, which on $\mathbb{R}^d$ is just the transposed matrix (transposed and complex conjugated on $\mathbb{C}^d$). Now $\boldsymbol{T}\boldsymbol{T}^*$ (respectively its inverse) is a *symmetric* linear map, *i.e.*, it is equal to its transpose. Therefore we have the following spectral theorem.

Lemma. [226, Theorem XV.7.1] *Let V be a finite-dimensional vector space over $\mathbb{R}$ and $\boldsymbol{A} : V \to V$ be a symmetric linear map. Then V has an orthogonal basis consisting of eigenvectors of $\boldsymbol{A}$ (and $\boldsymbol{A}$ is a diagonal matrix over $\mathbb{R}$ with respect to this basis).* □

So we might equivalently define the singular value as the positive square roots (with multiplicity) of the eigenvalues of $\boldsymbol{T}\boldsymbol{T}^*$ (or, by similarity[8], of $\boldsymbol{T}^*\boldsymbol{T}$). Of course, this definition[9] of

[8]By [143, Eq. (82) in Section IX.§12.], we have $\boldsymbol{T}^*\boldsymbol{T} = \boldsymbol{U}^{-1}\boldsymbol{T}\boldsymbol{T}^*\boldsymbol{U}$, where $\boldsymbol{U}$ is an unitary matrix (which here can even be chosen to be orthogonal by [143, Footnote 17 in Chapter III.]).

[9]The singular values are originally defined as the lengths of the principle semiaxes of $\boldsymbol{T}(B)$ or, equivalently, as the positive square roots of the eigenvalues of $\boldsymbol{T}^*\boldsymbol{T}$ in [122] and [123, Section 9.4]. Therefore, we put some effort here in showing that these different definitions of the singular values match.

the singular values is consistent with the one in Definition 4.105 and used in the last remark, if we note the following: If $\boldsymbol{T}$ has a complex eigenvalue λ, then $\overline{\lambda}$ is also an eigenvalue and as singular values we would get $\sqrt{\lambda\overline{\lambda}}$ twice.

Clearly, if we consider a finite-dimensional vector space $\mathbb{R}^d$ over $\mathbb{R}$, we can consider all contractive nonsingular linear maps, see [122] and [123, Section 9.4]. But we do not know if it is possible to treat linear maps on $\mathbb{Q}_p^d$ (or $\mathbb{Q}_\mathfrak{p}^d$) similarly, an analogue (if possible?) of the above spectral theorem might be quite complicated, since one implicitly uses that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$ (and $[\mathbb{C}:\mathbb{R}]=2$).

As a last point in this remark, we give an example of a non-diagonalisable matrix, which also shows that the spectral radius is not (sub)multiplicative.

$$\begin{array}{lccccc} & \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix} & \begin{pmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{pmatrix} & = & \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \\ \text{spectrum:} & \{\sqrt{2}\} & \{\sqrt{2}\} & & \{1,4\} \end{array}$$

After these remarks, we now return to our situation with maps from $\mathcal{D}'$ and their singular values.

Definition 4.111. Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{r+2s+k}$ be the singular values of a map $T \in \mathcal{D}'$. The *singular value function* $\Phi^\gamma(T)$ of $T \in \mathcal{D}'$ is defined for $\gamma \geq 0$ as follows:

$$\Phi^\gamma(T) = \begin{cases} 1, & \text{if } \gamma = 0, \\ \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{j-1} \cdot \alpha_j^{\gamma-j+1}, & \text{if } j-1 < \gamma \leq j, \\ (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{r+2s+k})^{\gamma/(r+2s+k)}, & \text{if } \gamma > r+2s+k. \end{cases}$$

Then, $\Phi^\gamma(T)$ is continuous and strictly decreasing in γ.

We also define the *second singular value function* $\Psi^\gamma(T)$ of $T \in \mathcal{D}'$ for $\gamma \geq 0$ as follows:

$$\Psi^\gamma(T) = \begin{cases} 1, & \text{if } \gamma = 0, \\ \alpha_{r+2s+k} \cdot \ldots \cdot \alpha_{r+2s+k-j+2} \cdot \alpha_{r+2s+k-j+1}^{\gamma-j+1}, & \text{if } j-1 < \gamma \leq j, \\ (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{r+2s+k})^{\gamma/(r+2s+k)}, & \text{if } \gamma > r+2s+k. \end{cases}$$

Again, $\Psi^\gamma(T) = [\Phi^\gamma(T^{-1})]^{-1}$ is continuous and strictly decreasing in γ.

It is almost immediate from this definition that we have the following property.

Corollary 4.112. *For fixed γ, the singular value function is submultiplicative, i.e., one has $\Phi^\gamma(T \circ U) \leq \Phi^\gamma(T) \cdot \Phi^\gamma(U)$ for $T, U \in \mathcal{D}'$, while the second singular value function is supermultiplicative, i.e., $\Psi^\gamma(T \circ U) \geq \Psi^\gamma(T) \cdot \Psi^\gamma(U)$. Moreover, we have $\Phi^\gamma(T^n) = [\Phi^\gamma(T)]^n$ and $\Psi^\gamma(T^n) = [\Psi^\gamma(T)]^n$.*

Proof. If $0 \leq \gamma \leq r+2s+k$ is an integer, $\Phi^\gamma(T)$ (respectively $\Phi^\gamma(U)$) is the product of the γ biggest singular values of T (respectively U). By the multiplicativity of absolute values and since we look at diagonal linear maps, the singular values of $T \circ U$ are given by $\alpha_i \cdot \beta_{\pi(i)}$ ($1 \leq i \leq r+2s+k$), where we denote by α_i (respectively β_i) the singular values of T (respectively U) and π denotes an appropriate permutation. Obviously, the product of the γ biggest such products $\alpha_i \cdot \beta_{\pi(i)}$ is less than or equal to $\alpha_1 \cdots \alpha_\gamma \cdot \beta_1 \cdots \beta_\gamma$.

If $j-1 < \gamma < j$ where $1 \le j \le r+2s+k$ is an integer, then

$$\begin{aligned}\Phi^\gamma(T) &= \alpha_1\alpha_2\dots\alpha_{j-1}\alpha_j^{\gamma-j+1} = (\alpha_1\alpha_2\dots\alpha_{j-1})^{\gamma-j+1}\cdot(\alpha_1\alpha_2\dots\alpha_{j-1}\alpha_j)^{j-\gamma}\\ &= \left(\Phi^j(T)\right)^{\gamma-j+1}\cdot\left(\Phi^{j-1}(T)\right)^{j-\gamma},\end{aligned}$$

so this follows from the integral case.

For $\gamma > r+2s+k$, the multiplicativity of the absolute values implies that Φ^γ is multiplicative. Similarly, for T^n we get that Φ^γ is multiplicative.

The statements for Ψ^γ are obtained in the same manner, noting that $\Psi^\gamma(T)$ is the product of the γ smallest singular values of T for integral γ. □

Remark 4.113. For (arbitrary) linear mappings on $\mathbb{R}^d$ the last statement was proven in [122, Lemma 2.1]. Note that in that case one has the additional complication that the maps are not necessarily diagonal, wherefore one associates $\Psi^\gamma(T)$ with the γ-dimensional Lebesgue measure of γ-dimensional ellipsoids in $\mathbb{R}^d$ if γ is an integer. This then yields the required inequality. The other cases follow as above.

Remark 4.114. In what follows, the singular value functions will be used to estimate the Hausdorff measure of sets. A justification for the definition of the singular value functions is the following observation (this is implicit in the proof of Proposition 4.122 below): For the γ-dimensional Hausdorff measures $\mu^{(\gamma)}(W)$ respectively $\mu^{(\gamma)}(T(W))$ of a set W and its image $T(W)$ we have the following estimates:

$$\Psi^\gamma(T)\cdot\mu^{(\gamma)}(W) \le \mu^{(\gamma)}(T(W)) \le \Phi^\gamma(T)\cdot\mu^{(\gamma)}(W)$$

i.e., $\Phi^\gamma(T)$ ($\Psi^\gamma(T)$) is the factor the γ-dimensional Hausdorff measure of W is reduced at least (at most) under the map T. In other words, $\Phi^\gamma(T)$ ($\Psi^\gamma(T)$) is the least upper (the greatest lower) bound on the factor the γ-dimensional Hausdorff measure of a set is reduced under T. The results in this and especially the next section on iterated function systems should be interpreted with this observation in mind.

We now return to iterated function systems. In Remark 4.104, we have observed that an IFS $\boldsymbol{\Theta}$ can be visualised as directed multi-graph $G(\boldsymbol{\Theta}) = G(V, \vec{E})$. We can therefore construct a path space $\mathcal{E}^\infty$ as in Definition 3c.5: We start at some vertex i and consider all infinite walks in the graph along directed edges, where each walk and its initial vertex is indexed (uniquely) by the sequence of labels of the edges it runs through. We call this set the *path space* of $G(\boldsymbol{\Theta})$ (see [112, Section 2.5]). Since $\boldsymbol{\Theta}$ is a self-affine IFS here, the labels are given by the affine maps in $\boldsymbol{\Theta}$. If w is a walk, we denote by $\mathrm{ini}(w)$ its initial vertex and, if w is a finite walk, by $\mathrm{ter}(w)$ the terminal vertex of the walk w.

We also define the set of walks of length k and denote it by $\mathcal{E}^k$ (where we define $\mathcal{E}^0 = \varnothing$). Moreover, $\mathcal{E}^k_{ij}$ denotes the set of walks of length k with initial vertex i and terminal vertex j. Similarly, $\mathcal{E}^k_{\bullet j}$ (respectively $\mathcal{E}^k_{i\bullet}$) denotes the set of walks of length k with terminal vertex j (respectively, initial vertex i). We also define the set of all finite walks by $\mathcal{E}^{\mathrm{fin}} = \bigcup_{k\ge0}\mathcal{E}^k$ and set $\mathcal{E}^* = \mathcal{E}^{\mathrm{fin}}\cup\mathcal{E}^\infty$. The set of all finite walks with initial vertex i and terminal vertex j is defined by $\mathcal{E}^{\mathrm{fin}}_{ij} = \bigcup_{k\ge0}\mathcal{E}^k_{ij}$. We denote the set of all finite walks on $G(\boldsymbol{\Theta})$ of length greater than or equal to n by $\mathcal{E}^{[n]} = \{w\in\mathcal{E}^{\mathrm{fin}} \mid \#w \ge n\} = \bigcup_{k\ge n}\mathcal{E}^k$.

Similarly[10] as in Equation (3c.1) on p. 62, we can define a metric $d : \mathcal{E}^\infty \times \mathcal{E}^\infty \to \mathbb{R}$ by

$$d\left(w\,, w'\right) = \begin{cases} \eta^{-1}, & \text{if } \operatorname{ini}(w) \neq \operatorname{ini}(w'), \\ \eta^{\#(w \barwedge w')}, & \text{if } \operatorname{ini}(w) = \operatorname{ini}(w'),\ w \neq w', \\ 0, & \text{if } w = w', \end{cases}$$

where $0 < \eta < 1$. This makes $\mathcal{E}^\infty$ into a compact ultrametric space (as in Lemma 3c.8). Moreover, we define $\mathcal{N}(w) = \{w' \in \mathcal{E}^\infty \mid w \triangleleft w'\}$ for $w \in \mathcal{E}^{\text{fin}}$. Then the sets $\{\mathcal{N}(w) \mid w \in \mathcal{E}^{\text{fin}}\}$ form a base of clopen sets for $\mathcal{E}^\infty$. Again, we call a finite set $A \subset \mathcal{E}^{\text{fin}}$ a covering set for $B \subset \mathcal{E}^\infty$ if $B \subset \bigcup_{w \in A} \mathcal{N}(w)$.

Let ω be an edge label in $G(\boldsymbol{\Theta})$ (where $\boldsymbol{\Theta}$ is a self-affine IFS), *i.e.*, ω denotes an affine mapping $f(x) = T(x) + t$. Then, we denote the linear part of that affine map by T_ω, *i.e.*, $T_\omega = T$. For $w = \omega_1\,\omega_2\,\ldots\,\omega_N \in \mathcal{E}^{\text{fin}}$ we define $T_w = T_{\omega_1} \circ T_{\omega_2} \circ \cdots \circ T_{\omega_N}$, with $T_\varnothing = \text{id}$ the identity mapping, *i.e.*, $T_\varnothing(x) = x$ (and therefore $\Phi^\gamma(T_\varnothing) \equiv 1 \equiv \Psi^\gamma(T_\varnothing)$).
We now define a net, compare [317, Section 2.§6] and [127, Chapter 5].

Definition 4.115. A collection $\mathcal{N}$ of nonempty subsets of a metric space X will be called a *net* if it satisfies the following conditions:

- If N_1 and N_2 belong to $\mathcal{N}$, then $N_1 \subset N_2$ or $N_2 \subset N_1$ or $N_1 \cap N_2 = \varnothing$.
- Each point of X either belongs to an element of $\mathcal{N}$ of zero diameter or belongs to elements of $\mathcal{N}$ with arbitrarily small diameters.
- $\mathcal{N}$ is countable.
- Any element of $\mathcal{N}$ is contained in at most finitely many other sets of $\mathcal{N}$.
- Each element of $\mathcal{N}$ is an $\mathfrak{F}_\sigma$-set.

Since $\mathcal{E}^\infty$ is an ultrametric space, $\mathcal{E}^{\text{fin}}$ is countable, $\mathcal{N}(w)$ is a clopen set (namely, a clopen ball of radius $\eta^{\#w}$) for every $w \in \mathcal{E}^{\text{fin}}$ and $\operatorname{card} \boldsymbol{\Theta} = \sum_{i=1}^n \sum_{j=1}^n \operatorname{card} \Theta_{ij}$ is finite, the collection of sets $\{\mathcal{N}(w) \mid w \in \mathcal{E}^{\text{fin}}\}$ is a net on $(\mathcal{E}^\infty, d)$. Moreover, the definition of a "net" is consistent with the use of the word "net" in Definition 2.39, the directed set is given by $\mathcal{E}^{\text{fin}}$ itself together with the order relation $\triangleleft$, where $w \leq w'$ if $w' \triangleleft w$.

Let $w = \omega_1\,\ldots\,\omega_i\,\omega_{i+1}\,\ldots\,\omega_N = w_1\,w_2 \in \mathcal{E}^{\text{fin}}$ (where $w_1 = \omega_1\,\ldots\,\omega_i$ and $w_2 = \omega_{i+1}\,\ldots\,\omega_N$). Then, from Corollary 4.112 we have $\Phi^\gamma(T_{w_1 w_2}) \leq \Phi^\gamma(T_{w_1}) \cdot \Phi^\gamma(T_{w_2})$ and $\Psi^\gamma(T_{w_1 w_2}) \geq \Psi^\gamma(T_{w_1}) \cdot \Psi^\gamma(T_{w_2})$. We obtain the following so-called *net measures* on the space $\mathcal{E}^\infty$ by the method II construction (Proposition 4.23), see [122, p. 344] and [124, Proof of Prop. 2].

Lemma 4.116. *Given the metric space $(\mathcal{E}^\infty, d)$, the base of clopen sets given by $\{\mathcal{N}(w) \mid w \in \mathcal{E}^{\text{fin}}\}$ is a sequential covering class such that its members of diameter less than or equal to $1/n$ (i.e., the members where the lengths of the defining walks w fulfil $\#w \geq -\log n / \log \eta$) is also a sequential covering class.*

[10] The paths starting at a fixed vertex i yield an (infinite rooted) tree, wherefore all paths can be interpreted as *forest*, *i.e.*, as disjoint union of trees. Introducing a new vertex (a "super-root") which is joined to all roots, we can "embed" such a forest (of rooted directed trees) in a single (rooted directed) tree.

(i) *The set functions* $\overline{\tau}^{\gamma}$ *defined by* $\overline{\tau}^{\gamma}(\varnothing) = 0$ *and* $\overline{\tau}^{\gamma}(\mathcal{N}(w)) = \Phi^{\gamma}(T_w)$ *are pre-measures on the above base of clopen sets. Hence, the set functions* $(\overline{m}^*)^{(\gamma)}$*, defined by*

$$(\overline{m}^*)^{(\gamma)}(B) = \lim_{n\to\infty}\left(\inf\left\{\sum_{w\in A}\Phi^{\gamma}(T_w) \mid A\subset\mathcal{E}^{[n]},\ \bigcup_{w\in A}\mathcal{N}(w)\supset B\right\}\right)$$

for every $B\subset\mathcal{E}^{\infty}$*, are regular metric outer measures on* $\mathcal{E}^{\infty}$*.*

(ii) *The set functions* $\underline{\tau}^{\gamma}$ *defined by* $\underline{\tau}^{\gamma}(\varnothing) = 0$ *and* $\underline{\tau}^{\gamma}(\mathcal{N}(w)) = \Psi^{\gamma}(T_w)$ *are pre-measures on the above base of clopen sets. Hence, the set functions* $(\underline{m}^*)^{(\gamma)}$*, defined by*

$$(\underline{m}^*)^{(\gamma)}(B) = \lim_{n\to\infty}\left(\inf\left\{\sum_{w\in A}\Psi^{\gamma}(T_w) \mid A\subset\mathcal{E}^{[n]},\ \bigcup_{w\in A}\mathcal{N}(w)\supset B\right\}\right)$$

for every $B\subset\mathcal{E}^{\infty}$*, are regular metric outer measures on* $\mathcal{E}^{\infty}$*.*

We call $(\overline{m}^*)^{(\gamma)}$ *the* upper net measure of dimension γ *on* $\mathcal{E}^{\infty}$*; similarly, we call* $(\underline{m}^*)^{(\gamma)}$ *the* lower net measure of dimension γ *on* $\mathcal{E}^{\infty}$*.* □

Now, we are prepared to generalise [122, Proposition 4.1] and justify the use of the word "dimension" for these net measures by its similarity to the definition of the Hausdorff dimension. The term "Falconer dimension" was used in [185].

Lemma 4.117. *For a nonsingular self-affine IFS with irreducible substitution matrix* $\boldsymbol{S\Theta}$*, the following numbers exist and are equal:*

(i) $\inf\{\gamma \mid \sum_{w\in\mathcal{E}^{\text{fin}}}\Phi^{\gamma}(T_w) < \infty\} = \sup\{\gamma \mid \sum_{w\in\mathcal{E}^{\text{fin}}}\Phi^{\gamma}(T_w) = \infty\}$,

(ii) $\inf\{\gamma \mid (\overline{m}^*)^{(\gamma)}(\mathcal{E}^{\infty}) = 0\} = \sup\{\gamma \mid (\overline{m}^*)^{(\gamma)}(\mathcal{E}^{\infty}) = \infty\}$,

(iii) *the unique* $\gamma\geq 0$ *such that*

$$\lim_{\ell\to\infty}\left(\rho\left(\left[\sum_{w\in\mathcal{E}^{\ell}_{ij}}\Phi^{\gamma}(T_w)\right]_{1\leq i,j\leq n}\right)\right)^{1/\ell} = 1$$

(where $\sum_{w\in\varnothing}\Phi^{s}(T_w) = 0$*).*

We denote the common value by $\overline{\dim}_{\text{aff}}\boldsymbol{\Theta}$ *and call it the* (upper) affinity dimension *(or the* Falconer dimension*) of the IFS* $\boldsymbol{\Theta}$*.*

Proof. We denote by $(\overline{m}^*)^{(\gamma)}_n$ the (method I) outer measure defined by

$$(\overline{m}^*)^{(\gamma)}_n(B) = \inf\left\{\sum_{w\in A}\Phi^{\gamma}(T_w) \mid A\subset\mathcal{E}^{[n]},\ \bigcup_{w\in A}\mathcal{N}(w)\supset B\right\} \tag{4.8}$$

for every $B\subset\mathcal{E}^{\infty}$. Then $(\overline{m}^*)^{(\gamma)} = \lim_{n\to\infty}(\overline{m}^*)^{(\gamma)}_n$.

Existence. (i): Since Φ^γ is decreasing with γ, the numbers $\inf\{\gamma \mid \sum_{w\in\mathcal{E}^{\text{fin}}} \Phi^\gamma(T_w) < \infty\}$ and $\sup\{\gamma \mid \sum_{w\in\mathcal{E}^{\text{fin}}} \Phi^\gamma(T_w) = \infty\}$ are the same.

(ii): Let $\alpha_1(f) \geq \alpha_2(f) \geq \ldots \geq \alpha_{r+2s+k}(f)$ be the singular values of the linear part of a map $f \in \boldsymbol{\Theta}$, then we have the following inequality (for the numbers $q(\boldsymbol{\Theta})$ and $\varrho(\boldsymbol{\Theta})$, recall Remark 4.93 and Definition 4.95)

$$1 > q(\boldsymbol{\Theta}) \geq \alpha_1(f) \geq \alpha_2(f) \geq \ldots \geq \alpha_{r+2s+k}(f) \geq \varrho(\boldsymbol{\Theta}) > 0$$

for all $f \in \boldsymbol{\Theta}$. Therefore, we have $\varrho(\boldsymbol{\Theta})^{\gamma\cdot\#w} \leq \Phi^\gamma(T_w) \leq q(\boldsymbol{\Theta})^{\gamma\cdot\#w}$ for a finite walk $w \in \mathcal{E}^{\text{fin}}$. Let $\gamma' < \gamma$, then we obtain the following estimate:

$$\Phi^{\gamma'}(T_w)\cdot\varrho(\boldsymbol{\Theta})^{(\gamma-\gamma')\cdot\#w} \leq \Phi^\gamma(T_w) \leq \Phi^{\gamma'}(T_w)\cdot q(\boldsymbol{\Theta})^{(\gamma-\gamma')\cdot\#w}. \tag{4.9}$$

So, if $0 < (\overline{m}^*)^{(\gamma')}(\mathcal{E}^\infty) < \infty$, and therefore also $(\overline{m}^*)^{(\gamma')}_n(\mathcal{E}^\infty) > 0$ for large n, then we have $(\overline{m}^*)^{(\gamma')}_n(\mathcal{E}^\infty) \geq q(\boldsymbol{\Theta})^{(\gamma'-\gamma)\cdot n}\cdot(\overline{m}^*)^{(\gamma)}_n(\mathcal{E}^\infty)$ and (since $\gamma'-\gamma<0$ and $0<q(\boldsymbol{\Theta})<1$) $(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = 0$. This establishes that $\inf\{\gamma|(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty)=0\}$ and $\sup\{\gamma|(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty)=\infty\}$ are equal.

(iii): One has the following estimates:

$$\begin{aligned}\mathbf{0} \leq \left[\sum_{ww'\in\mathcal{E}^{\ell+\ell'}_{ij}} \Phi^\gamma(T_{ww'})\right]_{1\leq i,j\leq n} &= \left[\sum_{w\in\mathcal{E}^\ell_{ik}}\sum_{w'\in\mathcal{E}^{\ell'}_{kj}} \Phi^\gamma(T_w\circ T_{w'})\right]_{1\leq i,j\leq n}\\ \overset{\text{Corollary 4.112}}{\leq} \left[\sum_{w\in\mathcal{E}^\ell_{ik}}\sum_{w'\in\mathcal{E}^{\ell'}_{kj}} \Phi^\gamma(T_w)\Phi^\gamma(T_{w'})\right]_{ij} &= \left[\sum_{w\in\mathcal{E}^\ell_{ik}} \Phi^\gamma(T_w)\right]_{ik}\cdot\left[\sum_{w'\in\mathcal{E}^{\ell'}_{kj}} \Phi^\gamma(T_{w'})\right]_{kj}\end{aligned}$$

This inequality shows that $([\sum_{w\in\mathcal{E}^\ell_{ij}}\Phi^\gamma(T_w)]_{1\leq i,j\leq n})_{\ell\in\mathbb{N}}$ is a submultiplicative sequence of nonnegative matrices. Let us denote by $\rho_\ell(\gamma) = \rho([\sum_{w\in\mathcal{E}^\ell_{ij}}\Phi^\gamma(T_w)]_{1\leq i,j\leq n})$ the spectral radii of these matrices (with $\rho_0(\gamma)=1$, *i.e.*, the spectrum of the identity matrix $\boldsymbol{E}$). Then by Lemma 4.81 we have that $\lim_{\ell\to\infty}(\rho_\ell(\gamma))^{1/\ell}$ exists for each γ.

Using the estimate (4.9), we have for $\varepsilon > 0$

$$\varrho(\boldsymbol{\Theta})^{\varepsilon\cdot k}\left[\sum_{w\in\mathcal{E}^k_{ij}}\Phi^\gamma(T_w)\right]_{1\leq i,j\leq n} \leq \left[\sum_{w\in\mathcal{E}^k_{ij}}\Phi^{\gamma+\varepsilon}(T_w)\right]_{1\leq i,j\leq n} \leq q(\boldsymbol{\Theta})^{\varepsilon\cdot k}\left[\sum_{w\in\mathcal{E}^k_{ij}}\Phi^\gamma(T_w)\right]_{1\leq i,j\leq n}.$$

This yields

$$\varrho(\boldsymbol{\Theta})^\varepsilon\cdot\lim_{\ell\to\infty}(\rho_\ell(\gamma))^{1/\ell} \leq \lim_{\ell\to\infty}(\rho_\ell(\gamma+\varepsilon))^{1/\ell} \leq q(\boldsymbol{\Theta})^\varepsilon\cdot\lim_{\ell\to\infty}(\rho_\ell(\gamma))^{1/\ell}.$$

Thus, $\lim_{\ell\to\infty}(\rho_\ell(\gamma))^{1/\ell}$ is continuous and strictly decreasing in γ. In particular, for $\gamma=0$, we have $\left[\sum_{w\in\mathcal{E}^\ell_{ij}}\Phi^0(T_w)\right]_{1\leq i,j\leq n} = (\boldsymbol{S\Theta})^\ell$, which is a nonnegative integer matrix and a power of an irreducible matrix. Therefore we have $\rho_\ell(0) \geq 1$ by Lemma 4.70, since the row sums of any power of an irreducible matrix is always greater than 0. We even have $\rho_\ell(0) = (\rho_1(0))^\ell$ (this can be deduced from Prop. 4.69), wherefore $\lim_{\ell\to\infty}(\rho_r(0))^{1/\ell} = \rho_1(0) \geq 1$. Since $\lim_{\ell\to\infty}(\rho_\ell(\gamma))^{1/\ell}$ is strictly decreasing with γ and less than 1 for large γ, there is a unique γ for which the limit equals 1.

Equality. (i)"$\geq$"(ii): If $\sum_{w\in\mathcal{E}^{\text{fin}}}\Phi^\gamma(T_w) < \infty$, then there is, for every $\varepsilon > 0$, an $N \in \mathbb{N}$ such that $\sum_{w\in\mathcal{E}^{[\ell]}}\Phi^\gamma(T_w) < \varepsilon$ for all $\ell \geq N$. So by the definition of $(\overline{m}^*)^{(\gamma)}$ (see Lemma 4.116) we have $(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = 0$. Therefore, the number defined in (i) is greater than or equal to the number defined in (ii).

(i)"$\geq$"(iii): If $\sum_{w\in\mathcal{E}^{\text{fin}}}\Phi^\gamma(T_w) < \infty$, we have

$$\infty > \sum_{w\in\mathcal{E}^{\text{fin}}}\Phi^\gamma(T_w) = \sum_{\ell=0}^{\infty}\left(\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\sum_{w\in\mathcal{E}_{ij}^{\ell}}\Phi^\gamma(T_w)\right\}\right)$$
$$\geq \sum_{\ell=0}^{\infty}\left\|\left[\sum_{w\in\mathcal{E}_{ij}^{\ell}}\Phi^\gamma(T_w)\right]_{1\leq i,j\leq n}\right\|_\infty \geq \sum_{\ell=0}^{\infty}\rho_\ell(\gamma)$$

by the definition of the matrix norm $\|\cdot\|_\infty$ and Lemma 4.80. The numbers $\rho_\ell(\gamma)$ are nonnegative real numbers, wherefore the sum $\sum_{\ell=0}^{\infty}\rho_\ell(\gamma)$ converges iff $\lim_{\ell\to\infty}\sqrt[\ell]{\rho_\ell(\gamma)} < 1$. Therefore, the number defined in (i) is greater than or equal to the number defined in (iii).

(ii)"$\geq$"(iii): Suppose that we have $(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) < 1$ for some γ. Then there is a covering set A of $\mathcal{E}^\infty$ such that $\sum_{w\in A}\Phi^\gamma(T_w) \leq 1$. Let $\hat{\ell} = \max\{\#w \mid w \in A\}$. Define further covering sets A_ℓ $(\ell \geq \hat{\ell})$ by

$$A_\ell = \{w_1 w_2 \dots w_N \in \mathcal{E}^{\text{fin}} \mid w_i \in A,\ \#(w_1 w_2 \dots w_N) \geq \ell \text{ and } \#(w_1 w_2 \dots w_{N-1}) < \ell\}.$$

(That these are again covering sets can be seen by the tree-picture of $\mathcal{E}_\infty$, also see Remark 3c.6). The submultiplicativity of Φ^γ implies

$$\sum_{\substack{w'\in A \text{ s.t.}\\ ww'\mathcal{E}^{\text{fin}}}}\Phi^\gamma(T_{ww'}) \leq \Phi^\gamma(T_w)\cdot\sum_{\substack{w'\in A \text{ s.t.}\\ ww'\mathcal{E}^{\text{fin}}}}\Phi^\gamma(T_{w'}) \leq \Phi^\gamma(T_w)\cdot\sum_{w'\in A}\Phi^\gamma(T_{w'}) \leq \Phi^\gamma(T_w)$$

Using this inductively, we find $\sum_{w\in A_\ell}\Phi^\gamma(T_w) \leq 1$.

If $w' \in \mathcal{E}^{\ell+\hat{\ell}}$, then we can decompose $w' = w\hat{w}$ for some $w \in A_\ell$ and $\#\hat{w} \leq \hat{\ell}$. Moreover, for each such w, there are at most $\hat{r}^{\hat{\ell}}$ such $\hat{w}$, where we define $\hat{r} = \max\{\operatorname{card}\Theta_{ij} \mid 1 \leq i,j \leq n\}$. Since $\Phi^\gamma(T_{w'}) \leq \Phi^\gamma(T_w)$ we get

$$\sum_{w'\in\mathcal{E}^{\ell+\hat{\ell}}}\Phi^\gamma(T_{w'}) \leq \hat{r}^{\hat{\ell}}\sum_{w\in A_\ell}\Phi^\gamma(T_w) \leq \hat{r}^{\hat{\ell}}.$$

This being true for all $\ell \geq \hat{\ell}$, we can estimate

$$\sum_{w'\in\mathcal{E}_{ij}^{\ell+\hat{\ell}}}\Phi^\gamma(T_{w'}) \leq \sum_{w'\in\mathcal{E}^{\ell+\hat{\ell}}}\Phi^\gamma(T_{w'}) \leq \hat{r}^{\hat{\ell}},$$

i.e., $\left[\sum_{w\in\mathcal{E}_{ij}^m}\Phi^\gamma(T_w)\right]_{1\le i,j\le n} \le \left[\hat{r}^{\hat{\ell}}\right]_{1\le i,j\le n}$ for $m\ge\hat{\ell}$. By Corollary 4.73, $\rho_m(\gamma)$ is bounded by $n\cdot\hat{r}^{\hat{\ell}}$ and we have $\lim_{m\to\infty}\sqrt[m]{\rho_m(\gamma)}\le\lim_{m\to\infty}\sqrt[m]{n\cdot\hat{r}^{\hat{\ell}}}=1$. Therefore, the number defined in (ii) is greater than or equal to the number defined in (iii).

(iii)"$\ge$"(i): Suppose $\lim_{\ell\to\infty}\sqrt[\ell]{\rho_\ell(\gamma)}\le 1-2\cdot\varepsilon$, where $0<\varepsilon<\frac{1}{2}$, then there is an $\hat{\ell}\in\mathbb{N}$ such that

$$\left|\!\left|\!\left|\left[\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right]_{1\le i,j\le n}\right|\!\right|\!\right|_\infty^{1/\ell}<1-\varepsilon$$

for all $\ell\ge\hat{\ell}$ by Lemma 4.81. But then we have

$$\begin{aligned}\sum_{w\in\mathcal{E}^{\mathrm{fin}}}\Phi^\gamma(T_w)&=\sum_{\ell=0}^\infty\left(\sum_{i=1}^n\left\{\sum_{j=1}^n\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right\}\right)\le n\cdot\sum_{\ell=0}^\infty\left|\!\left|\!\left|\left[\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right]_{1\le i,j\le n}\right|\!\right|\!\right|_\infty\\&=n\cdot\sum_{\ell=0}^{\hat{\ell}-1}\left|\!\left|\!\left|\left[\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right]_{1\le i,j\le n}\right|\!\right|\!\right|_\infty+n\cdot\sum_{\ell=\hat{\ell}}^\infty\left|\!\left|\!\left|\left[\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right]_{1\le i,j\le n}\right|\!\right|\!\right|_\infty\\&\le n\cdot\sum_{\ell=0}^{\hat{\ell}-1}\left|\!\left|\!\left|\left[\sum_{w\in\mathcal{E}_{ij}^\ell}\Phi^\gamma(T_w)\right]_{1\le i,j\le n}\right|\!\right|\!\right|_\infty+n\cdot\frac{(1-\varepsilon)^{\hat{\ell}}}{\varepsilon}<\infty,\end{aligned}$$

where we used that the first sum is a finite sum while the second one is bounded by a geometric series. Since ε was chosen arbitrary, the number defined in (iii) is greater than or equal to the number defined in (i). □

Corollary 4.118. *Assume the setting of Lemma 4.117. If the linear part $T=T_f$ of all affine maps $f\in\boldsymbol{\Theta}$ is the same, the affinity dimension $\overline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta}$ is the unique value $\gamma\ge 0$ such that $\Phi^\gamma(T)\cdot\rho(\boldsymbol{S\Theta})=1$.*

Proof. In the case considered here, Φ^γ is multiplicative instead of submultiplicative by Corollary 4.112. Therefore, we also have $\rho_\ell(\gamma)=(\rho_1(\gamma))^\ell$, so that $\lim_{\ell\to\infty}\sqrt[\ell]{\rho_\ell(\gamma)}=\rho_1(\gamma)=\Phi^\gamma(T)\cdot\rho(\boldsymbol{S\Theta})$. □

Remark 4.119. Note that in the previous corollary it is *not* enough that all maps T_f have the same singular values. This can be seen from the example in [283, Section 7], where the IFS $\Theta=\{f_1,f_2,f_3,f_4\}$ with

$$f_1(x)=\begin{pmatrix}c&0\\0&1-c\end{pmatrix}x+\begin{pmatrix}0\\0\end{pmatrix},\qquad f_2(x)=\begin{pmatrix}c&0\\0&1-c\end{pmatrix}x+\begin{pmatrix}1-c\\c\end{pmatrix},$$

$$f_3(x)=\begin{pmatrix}1-c&0\\0&c\end{pmatrix}x+\begin{pmatrix}c\\0\end{pmatrix}\qquad\text{and}\qquad f_4(x)=\begin{pmatrix}1-c&0\\0&c\end{pmatrix}x+\begin{pmatrix}0\\1-c\end{pmatrix}$$

is considered. Here, $0\le c\le\frac{1}{2}$ is a free parameter (choosing $c=0$, one obtains the outline of the unit square, choosing $c=\frac{1}{2}$ yields the filled unit square; otherwise, one gets a "self-affine carpet", see [283, Fig. 1]). While for an IFS where all maps have the same linear part, say $\mathrm{diag}(c, 1-c)$, one obtains the singular values c^m and $(1-c)^m$ for the maps of its m-th power, the singular values of the maps in Θ^m are given by the set $\{c^{m-\ell} \cdot (1-c)^\ell \mid 0 \le \ell \le m\}$. Consequently, if one chooses $c = \frac{1}{3}$, one obtains for the affinity dimension $\overline{\dim}_{\mathrm{aff}}\Theta \approx 1.845$, see [283, p. 931] (while the affinity dimension of an IFS Θ' with $\mathrm{card}\,\Theta' = 4$ such that all linear parts are given by $\mathrm{diag}(\frac{1}{3}, \frac{2}{3})$ yields $\overline{\dim}_{\mathrm{aff}}\Theta' = 3 \cdot \frac{\log 2}{\log 3} \approx 1.893$; note that this number is an upper bound for the affinity dimension of Θ).

Because of Remark 4.82, the statements for Ψ^γ are weaker.

Lemma 4.120. *For a nonsingular self-affine IFS with irreducible substitution matrix $\boldsymbol{S\Theta}$, we call the number*

$$\inf\{\gamma \mid (\underline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = 0\} = \sup\{\gamma \mid (\underline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = \infty\}$$

the lower affinity dimension *of the IFS $\boldsymbol{\Theta}$ and denote it by $\underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta}$.*

If the linear part $T = T_f$ of all affine maps $f \in \boldsymbol{\Theta}$ is the same (or if $n = 1$), the lower affinity dimension is equal to the following numbers:

(i) $\inf\{\gamma \mid \sum_{w \in \mathcal{E}^{\mathrm{fin}}} \Psi^\gamma(T_w) < \infty\} = \sup\{\gamma \mid \sum_{w \in \mathcal{E}^{\mathrm{fin}}} \Psi^\gamma(T_w) = \infty\}$,

(ii) *the unique $\gamma \ge 0$ such that $\Psi^\gamma(T) \cdot \rho(\boldsymbol{S\Theta}) = 1$*
(respectively, $\lim_{\ell\to\infty} \left(\sum_{w \in \mathcal{E}^\ell} \Psi^\gamma(T_w)\right)^{1/\ell} = 1$).

Proof. The existence of the numbers can be proved as in Lemma 4.117. If all linear parts are the same, Ψ^γ is multiplicative instead of supermultiplicative by Corollary 4.112 and the proofs of Lemma 4.117 and Corollary 4.118 apply. If $n = 1$, the parallel statement of Lemma 4.76 can be applied to the supermultiplicative sequence $\left(\sum_{w \in \mathcal{E}^\ell} \Psi^\gamma(T_w)\right)_{\ell \in \mathbb{N}}$. □

In connection with the net measures we have constructed, we also note the following interesting statement, which can be found (without explicit proof) as [122, Lemma 4.2], [125, Prop. 3.1] and in [124, Proof of Proposition 2]. It should be compared to similar statements about net measures, *e.g.*, [317, Theorem 54], [251, Theorem 8.8] and [127, Theorem 5.4], which are also sometimes called *Frostman's lemma*. It will only be used in the proof of Proposition 4.129.

Lemma 4.121. *Suppose that we have $(\overline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = \infty$. Then there exists a set $E \subset \mathcal{E}^\infty$ such that $0 < (\overline{m}^*)^{(\gamma)}(E) < \infty$ and $(\overline{m}^*)^{(\gamma)}(E \cap \mathcal{N}(w)) \le \Phi^\gamma(T_w)$ for all $w \in \mathcal{E}^{\mathrm{fin}}$. Similarly, if we have $(\underline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = \infty$, then there exists a set $F \subset \mathcal{E}^\infty$ such that $0 < (\underline{m}^*)^{(\gamma)}(F) < \infty$ and $(\underline{m}^*)^{(\gamma)}(F \cap \mathcal{N}(w)) \le \Psi^\gamma(T_w)$ for all $w \in \mathcal{E}^{\mathrm{fin}}$.*

Proof. We define $(\overline{m}^*)^{(\gamma)}_n$ as in Equation (4.8) on p. 108. Then we have (for every $n \in \mathbb{N}$)

$$(\overline{m}^*)^{(\gamma)}_n(\mathcal{E}^\infty) = \inf\left\{\sum_{w \in A} \Phi^\gamma(T_w) \mid A \subset \mathcal{E}^{[n]},\ \bigcup_{w \in A} \mathcal{N}(w) \supset \mathcal{E}^\infty\right\} \le \sum_{w \in \mathcal{E}^n} \Phi^\gamma(T_w) < \infty,$$

where the first inequality follows because $\mathcal{E}^n$ is a special covering of $\mathcal{E}^\infty$, and the second inequality follows because it is a sum of finitely many positive numbers. Set $c = (\overline{m}^*)^{(\gamma)}_1(\mathcal{E}^\infty)$.

We now want to construct an antitone sequence $\{E_j\}_{j=1}^\infty$ such that $(\overline{m}^*)^{(\gamma)}_j(E_j) = c$. Set $E_1 = \mathcal{E}^\infty$ and define E_{j+1} inductively by specifying its intersection with each $\mathcal{N}(w)$ where

$w \in \mathcal{E}^j$ (as a reminder: $\mathcal{E}^\infty$ is an ultrametric space, and the sets $\{\mathcal{N}(w) \mid w \in \mathcal{E}^j\}$ are the pairwise disjoint clopen balls of radius η^{-j}). For this, we use the following procedure. Let $w \in \mathcal{E}^j$.

- If $(\overline{m}^*)^{(\gamma)}_{j+1}(E_j \cap \mathcal{N}(w)) \leq \Phi^\gamma(T_w)$, then $\mathcal{N}(w)$ is not the (only) cover of $E_j \cap \mathcal{N}(w)$ in the definition of the measure, *i.e.*, there are (also) sets of smaller diameter which cover it. We set $E_{j+1} \cap \mathcal{N}(w) = E_j \cap \mathcal{N}(w)$. Consequently, we have $(\overline{m}^*)^{(\gamma)}_{j+1}(E_{j+1} \cap \mathcal{N}(w)) = (\overline{m}^*)^{(\gamma)}_{j}(E_j \cap \mathcal{N}(w)) \leq \Phi^\gamma(T_w)$.

- If $(\overline{m}^*)^{(\gamma)}_{j+1}(E_j \cap \mathcal{N}(w)) > \Phi^\gamma(T_w)$, we have $(\overline{m}^*)^{(\gamma)}_{j}(E_j \cap \mathcal{N}(w)) = \Phi^\gamma(T_w)$ (since $\mathcal{N}(w)$ is a cover with $w \in \mathcal{E}^j$, while covers with sets of smaller radius yield a bigger measure). We claim that it is possible to construct an E_{j+1} such that $(\overline{m}^*)^{(\gamma)}_{j+1}(E_{j+1} \cap \mathcal{N}(w)) = \Phi^\gamma(T_w)$. Set $\ell_1 = j+1$ and define $S^{(1)} = \mathcal{N}(w) \cap \mathcal{E}^{\ell_1}$. This is a finite set, say with $N^{(1)} = \operatorname{card} S^{(1)}$ elements, and we choose some enumeration of its elements from 1 to $N^{(1)}$. Denote by $S^{(1)}_n$ the set of the first n elements of $S^{(1)}$. We note that $\Phi^\gamma(T_{w'}) \leq (q(\boldsymbol{\Theta}))^\gamma \cdot \Phi^\gamma(T_w) < \Phi^\gamma(T_w)$ for all $w' \in S^{(1)}$, so we have $(\overline{m}^*)^{(\gamma)}_{j+1}(E_j \cap \mathcal{N}(w')) < \Phi^\gamma(T_w)$ for every $w' \in S^{(1)}$, while $(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \bigcup_{w' \in S^{(1)}} \mathcal{N}(w')\right) = (\overline{m}^*)^{(\gamma)}_{j+1}(E_j \cap \mathcal{N}(w)) > \Phi^\gamma(T_w)$. As an outer measure, $(\overline{m}^*)^{(\gamma)}_{j+1}$ is nondecreasing, so there is an $1 \leq n_1 < N^{(1)}$ such that

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \bigcup_{w' \in S^{(1)}_{n_1}} \mathcal{N}(w')\right) \leq \Phi^\gamma(T_w),$$

but

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \bigcup_{w' \in S^{(1)}_{n_1+1}} \mathcal{N}(w')\right) > \Phi^\gamma(T_w).$$

We set $R^{(1)}_w = \bigcup_{w' \in S^{(1)}_{n_1}} \mathcal{N}(w')$. If $(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(1)}_w\right) = \Phi^\gamma(T_w)$, we set $E_{j+1} \cap \mathcal{N}(w) = E_j \cap R^{(1)}_w = (E_j \cap R^{(1)}_w) \cap \mathcal{N}(w)$ and we are done. Otherwise, we iterate this procedure. So, let $(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(1)}_w\right) < \Phi^\gamma(T_w)$ and let w'_{n_1+1} be the (n_1+1)-st element of $S^{(1)}$. Obviously, we have $\Phi^\gamma\left(T_{w'_{n_1+1}}\right) > \Phi^\gamma(T_w) - (\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(1)}_w\right)$. Define

$$\ell_2 = \left\lceil \log\left(\Phi^\gamma(T_w) - (\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(1)}_w\right)\right) / \log(q(\boldsymbol{\Theta})) \right\rceil.$$

We set $S^{(2)} = \mathcal{N}\left(w'_{n_1}\right) \cap \mathcal{E}^{\ell_2}$. We have $\ell_2 > \ell_1$, $S^{(2)}$ is finite and $\Phi^\gamma(T_{w'}) \leq \Phi^\gamma(T_w) - (\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(1)}_w\right)$ for every $w' \in S^{(2)}$. Enumeration of the $N^{(2)} = \operatorname{card} S^{(2)}$ elements and using the notation $S^{(2)}_n$ as above, there is an $1 \leq n_2 < N^{(2)}$ such that

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \left(R_1 \cup \bigcup_{w' \in S^{(2)}_{n_2}} \mathcal{N}(w')\right)\right) \leq \Phi^\gamma(T_w),$$

but

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \left(R_1 \cup \bigcup_{w' \in S^{(2)}_{n_2+1}} \mathcal{N}(w')\right)\right) > \Phi^\gamma(T_w).$$

We set $R^{(2)}_w = R^{(1)}_w \cup \bigcup_{w' \in S^{(2)}_{n_2}} \mathcal{N}(w')$ and proceed as before. By this we get a (finite or infinite) isotone (and therefore convergent) sequence $\{R^{(i)}_w\}$. Moreover, we have $j+1 = \ell_1 < \ell_2 < \ldots$ and

$$(q(\boldsymbol{\Theta}))^{\ell_i} > \Phi^\gamma(T_w) - (\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(i)}_w\right) \geq 0,$$

i.e., $\{(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap R^{(i)}_w\right)\}$ is a monotone increasing bounded sequence of positive numbers. We set $E_{j+1} \cap \mathcal{N}(w) = E_j \cap \lim_i R^{(i)}_w = (E_j \cap \lim_i R^{(i)}_w) \cap \mathcal{N}(w)$ and have $(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_j \cap \lim_i R^{(i)}_w\right) = \Phi^\gamma(T_w)$ (note that $(\overline{m}^*)^{(\gamma)}_{j+1}$ is not necessarily regular), where we use the obvious convention if the sequence of sets is finite.

We therefore have

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_{j+1} \cap \mathcal{N}(w)\right) = (\overline{m}^*)^{(\gamma)}_{j}\left(E_{j} \cap \mathcal{N}(w)\right) \leq \Phi^\gamma(T_w) \tag{4.10}$$

for all $w \in \mathcal{E}^j$. We sum over all $w \in \mathcal{E}^j$ and observe that any covering sets for calculating $(\overline{m}^*)^{(\gamma)}_{j+1}$ and $(\overline{m}^*)^{(\gamma)}_{j}$ must be contained in some such $\mathcal{N}(w)$, wherefore we obtain $(\overline{m}^*)^{(\gamma)}_{j+1}(E_{j+1}) = (\overline{m}^*)^{(\gamma)}_{j}(E_{j})$, and therefore by induction $(\overline{m}^*)^{(\gamma)}_{j}(E_{j}) = (\overline{m}^*)^{(\gamma)}_{1}(E_{1}) = c$ for all $j \in \mathbb{N}$.

Moreover, let $1 \leq j < m$, then $E_m \subset E_{j+1}$ and for $w \in \mathcal{E}^j$ we have

$$(\overline{m}^*)^{(\gamma)}_{j+1}\left(E_{m} \cap \mathcal{N}(w)\right) \leq (\overline{m}^*)^{(\gamma)}_{j+1}\left(E_{j+1} \cap \mathcal{N}(w)\right) \leq \Phi^\gamma(T_w).$$

In general, we have $(\overline{m}^*)^{(\gamma)}_{j+1}(E_m \cap \mathcal{N}(w)) \geq (\overline{m}^*)^{(\gamma)}_{j}(E_m \cap \mathcal{N}(w))$, but this last estimate shows that we, in fact, have the equality sign: If $(\overline{m}^*)^{(\gamma)}_{j+1}(E_m \cap \mathcal{N}(w)) < \Phi^\gamma(T_w)$, we use the same cover for $(\overline{m}^*)^{(\gamma)}_{j}(E_m \cap \mathcal{N}(w))$ (we only have the set $\mathcal{N}(w)$ as additional possibility for a cover), while in the case $(\overline{m}^*)^{(\gamma)}_{j+1}(E_m \cap \mathcal{N}(w)) = \Phi^\gamma(T_w)$ we know that there is no cover of sets with radius smaller than η^{-j} which yields a smaller measure as the cover with $\mathcal{N}(w)$. Summing over all $w \in \mathcal{E}^j$, then iterating and using Equation (4.10), we obtain for $1 \leq j < m$:

$$(\overline{m}^*)^{(\gamma)}_{j}(E_m) = (\overline{m}^*)^{(\gamma)}_{j+1}(E_m) = (\overline{m}^*)^{(\gamma)}_{m}(E_m) = c. \tag{4.11}$$

By construction, $\{E_j\}_{j=1}^\infty$ is an antitone sequence of sets, so we define and have (see Definition 4.3)

$$E = \liminf_j E_j = \limsup_j E_j = \lim_j E_j.$$

By construction, E and E_j ($j \in \mathbb{N}$) are Borel sets and therefore $(\overline{m}^*)^{(\gamma)}$-measurable by Proposition 4.23. We therefore obtain the following estimates for the measure of E:

$$\begin{aligned}(\overline{m}^*)^{(\gamma)}(E) &= \lim_{j\to\infty} (\overline{m}^*)^{(\gamma)}_{j}(E) = \lim_{j\to\infty} (\overline{m}^*)^{(\gamma)}_{j}\left(\liminf_m E_m\right) \\ &\overset{\text{Lemma 4.14}}{\leq} \lim_{j\to\infty}\lim_{m\to\infty} (\overline{m}^*)^{(\gamma)}_{j}(E_m) \overset{\text{Equation (4.11)}}{=} \lim_{j\to\infty} c = c\end{aligned}$$

and

$$\left(\overline{m}^*\right)^{(\gamma)}(E) = \left(\overline{m}^*\right)^{(\gamma)}(\lim_m E_m) \overset{\text{Proposition 4.5}}{=} \lim_{m\to\infty} \left(\overline{m}^*\right)^{(\gamma)}(E_m)$$
$$= \lim_{m\to\infty}\lim_{j\to\infty} \left(\overline{m}^*\right)^{(\gamma)}_j(E_m) \geq \lim_{m\to\infty} \left(\overline{m}^*\right)^{(\gamma)}_m(E_m) = c.$$

So, we have $0 < \left(\overline{m}^*\right)^{(\gamma)}(E) = c < \infty$ and using the first of these last two estimates on $\left(\overline{m}^*\right)^{(\gamma)}(E \cap \mathcal{N}(w))$ yields the other claim (of course, we have constructed the sequence $\{E_j\}_{j=1}^{\infty}$ in an appropriate way).

With the replacements $\overline{m}^* \leftrightarrow \underline{m}^*$, $\Phi^\gamma \leftrightarrow \Psi^\gamma$ and $E \leftrightarrow F$, the second part of the Lemma follows. □

4.10. Self-Affine Attractors

In this section, we show that the affinity dimension of an IFS is an upper estimate for the Hausdorff dimension of the invariant sets $\Omega_i \subset \mathbb{M}$ implicitly defined by the IFS (compare with [122, Proposition 5.1] and [123, Theorem 9.12]).

Proposition 4.122. *Let $\underline{\Omega} = (\Omega_i)_{i=1}^n$ be the invariant set of an irreducible nonsingular self-affine IFS $\boldsymbol{\Theta}$ in $\mathbb{M}$. If $\left(\overline{m}^*\right)^{(\gamma)}(\mathcal{E}^\infty) < \infty$ then $\mu^{(\gamma)}(\Omega_i) < \infty$ for all $1 \leq i \leq n$. In particular, we have $\dim_{\mathrm{Hd}} \underline{\Omega} \leq \min\{\overline{\dim}_{\mathrm{aff}} \boldsymbol{\Theta}, \dim_{\mathrm{metr}} \mathbb{M}\}$.*

Proof. The result is obtained by a covering argument. We also note that the Hausdorff dimension of all sets Ω_i is the same by Proposition 4.97.

Let $\underline{B} = (B_i)_{i=1}^n$ be a family of (closed and therefore compact) balls $B_i \subset \mathbb{M}$ (with centre at the origin) of diameter at least 1 and large enough to ensure that $f(B_j) \subset B_i$ for all $f \in \Theta_{ij}$ (we may, in fact, take the same ball B for all i). This is possible by the following reason: Denote by $t_{\max}$ the maximal translational part of all affine maps $f(x) = T(x) + t \in \boldsymbol{\Theta}$, *i.e.*, $t_{\max} = \max\{d_{\mathbb{M}}(f(0), 0) \mid f \in \boldsymbol{\Theta}\}$. Furthermore, since $\operatorname{diam} f(B) \leq q(\boldsymbol{\Theta}) \cdot \operatorname{diam} B$, we have the estimate $\sup\{d_{\mathbb{M}}(f(x), 0) \mid x \in B\} \leq q(\boldsymbol{\Theta}) \cdot \operatorname{diam} B + t_{\max}$ by the triangle inequality. Therefore, ball(s) of diameter greater than $t_{\max}/(1 - q(\boldsymbol{\Theta}))$ will do.

Given $\delta > 0$, there exists an integer $\hat{\ell} \in \mathbb{N}$ such that $\operatorname{diam} f_w(B) = \operatorname{diam} T_w(B) < \delta$ if $\#w \geq \hat{\ell}$ (all maps are contractions by at least $q(\boldsymbol{\Theta}) < 1$). Let A be any covering set of $\mathcal{E}^\infty$ such that $\#w \geq \hat{\ell}$ for each $w \in A$. Define $A_{ij} = A \cap \mathcal{E}_{ij}^{\mathrm{fin}}$ (the walks of the covering set A with initial vertex i and terminal vertex j). Then $\Omega_i \subset \bigcup_{j=1}^n \bigcup_{w \in A_{ij}} f_w(B_j)$. But $f_w(B_j) = f_w(B)$ is a translate of $T_w(B)$ which is given by

$$\begin{aligned} T_w(B) = \{(x_1, \ldots, x_{r+s+k}) \in \mathbb{M} \mid\, & |x_i| \leq |a_i^{(w)}| \cdot \frac{d}{2} \text{ for } 1 \leq i \leq r, \\ & \left|\operatorname{Re} x_i \cdot \mathrm{e}^{\left(-i \cdot \operatorname{Arg}(a_i^{(w)})\right)}\right| \leq |a_i^{(w)}| \cdot \frac{d}{2} \quad \text{and} \\ & \left|\operatorname{Im} x_i \cdot \mathrm{e}^{\left(-i \cdot \operatorname{Arg}(a_i^{(w)})\right)}\right| \leq |a_i^{(w)}| \cdot \frac{d}{2} \text{ for } r \leq i \leq r+s, \\ & \|x_{r+s+i}\|_{\mathfrak{p}_i} \leq \|a_{r+s+i}^{(w)}\|_{\mathfrak{p}_i} \cdot \lfloor d \rfloor_{\mathfrak{p}_i} \text{ for } 1 \leq i \leq k\}, \end{aligned} \tag{4.12}$$

where $T(x) = (a_1^{(w)} \cdot x_1, \ldots, a_{r+s+k}^{(w)} \cdot x_{r+s+k})$ and we use the notation $d = \operatorname{diam} B$, $\operatorname{Arg}(z)$ denotes the *argument* (or *phase*) of a complex number $z = |z| \cdot \exp(i \cdot \operatorname{Arg}(z))$ and $\lfloor d \rfloor_{\mathfrak{p}_i} = p_i^{f_i \cdot v}$ if $p_i^{f_i \cdot v} \leq d < p_i^{f_i \cdot (v+1)}$ (where $v \in \mathbb{Z}$, $p_i \in \mathfrak{p}_i \cap \mathbb{P}$ and f_i is the residue degree $f_i = f_{\mathfrak{p}_i | (p_i)}$). Thus $f_w(B)$ is contained in a translate of the *hypercuboid*[11]

$$\begin{aligned} &|a_1^{(w)}| \cdot [0\,, d] \times \cdots \times |a_r^{(w)}| \cdot [0\,, d] \times \\ &|a_{r+1}^{(w)}| \cdot \mathrm{e}^{\left(i \cdot \operatorname{Arg}(a_{r+1}^{(w)})\right)} \cdot ([0\,, d] \times [0\,, d]) \times \cdots \times |a_{r+s}^{(w)}| \cdot \mathrm{e}^{\left(i \cdot \operatorname{Arg}(a_{r+s}^{(w)})\right)} \cdot ([0\,, d] \times [0\,, d]) \times \\ &|a_{r+s+1}^{(w)}|_{\mathfrak{p}_1} \cdot \{x \in \mathbb{Q}_{\mathfrak{p}_1} \mid \|x\|_{\mathfrak{p}_1} \leq d\} \times \cdots \times |a_{r+s+k}^{(w)}|_{\mathfrak{p}_k} \cdot \{x \in \mathbb{Q}_{\mathfrak{p}_k} \mid \|x\|_{\mathfrak{p}_k} \leq d\}. \end{aligned} \tag{4.13}$$

We call these factors the *sides of the hypercuboid* and their diameters (w.r.t. to their corresponding metric) the *sidelengths*.

If $m = \lceil \dim_{\text{aff}} \boldsymbol{\Theta} \rceil$ and using the ordered version of the singular values, we can divide such a hypercuboid into at most

$$\begin{aligned} &p_1^{f_1} \cdots p_k^{f_k} \cdot \left(\frac{\alpha_1^{(w)}}{\alpha_m^{(w)}} + 1 \right) \cdots \left(\frac{\alpha_{m-1}^{(w)}}{\alpha_m^{(w)}} + 1 \right) \leq p_1^{f_1} \cdots p_k^{f_k} \cdot \left(2 \cdot \frac{\alpha_1^{(w)}}{\alpha_m^{(w)}} \right) \cdots \left(2 \cdot \frac{\alpha_{m-1}^{(w)}}{\alpha_m^{(w)}} \right) \leq \\ &p_1^{f_1} \cdots p_k^{f_k} \cdot 2^{m-1} \cdot \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \cdot \left(\alpha_m^{(w)}\right)^{1-m} \leq p_1^{f_1} \cdots p_k^{f_k} \cdot 2^{r+2s+k} \cdot \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \cdot \left(\alpha_m^{(w)}\right)^{1-m} \end{aligned}$$

hypercuboids R_i (which are translates of each other) of sidelength $\alpha_m^{(w)} \cdot \operatorname{diam} B < \delta$. Here, we have taken the effect in the discrete valuation rings into account that $\alpha_m^{(w)} / p_i^{f_i} < \lfloor \alpha_m^{(w)} \rfloor_{\mathfrak{p}_i} \leq \alpha_m^{(w)}$, wherefore we obtain the prefactor of powers of prime numbers $p_i^{f_i} = \mathfrak{p}_i \cap \mathbb{P}$ (*e.g.*, a 3-adic "interval" of diameter $\frac{1}{3}$ is divided into 3 "intervals" of diameter at most .332). Now, the sides in the complex components of R_i may not be parallel to the real and imaginary axes, therefore we need at most

$$p_1^{f_1} \cdots p_k^{f_k} \cdot 4^s \cdot 2^{r+2s+k} \cdot \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \cdot \left(\alpha_m^{(w)}\right)^{1-m}$$

hypercubes of sidelength (and also diameter) $\alpha_m^{(w)} \cdot \operatorname{diam} B$ to cover the hypercuboid of Equation (4.12), where we define a hypercube of sidelength δ by

$$\begin{aligned} &[0\,, \delta] \times \cdots \times [0\,, \delta] \times ([0\,, \delta] \times [0\,, \delta]) \times \cdots \times ([0\,, \delta] \times [0\,, \delta]) \times \\ &\qquad \{x \in \mathbb{Q}_{\mathfrak{p}_1} \mid \|x\|_{\mathfrak{p}_1} \leq \delta\} \times \cdots \times \{x \in \mathbb{Q}_{\mathfrak{p}_k} \mid \|x\|_{\mathfrak{p}_k} \leq \delta\}. \end{aligned}$$

Taking such a cover of each hypercuboid of Equation (4.12) with $w \in A$, it follows that (by the definition of the Hausdorff measure)

$$\begin{aligned} \bigcup_{i=1}^{n} (\mu^*)_\delta^{(\gamma)} (\Omega_i) &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{w \in A_{ij}} p_1^{f_1} \cdots p_k^{f_k}\, 2^{r+4s+k} \cdot \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \left(\alpha_m^{(w)}\right)^{1-m} \left(\alpha_m^{(w)} \cdot \operatorname{diam} B\right)^{\gamma} \\ &= p_1^{f_1} \cdots p_k^{f_k} \cdot 2^{r+4s+k} \cdot (\operatorname{diam} B)^{\gamma} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{w \in A_{ij}} \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \cdot \left(\alpha_m^{(w)}\right)^{1-m+\gamma} \\ &\leq p_1^{f_1} \cdots p_k^{f_k} \cdot 2^{r+4s+k} \cdot (\operatorname{diam} B)^{r+2s+k} \cdot \sum_{w \in A} \Phi^{\gamma}(T_w). \end{aligned}$$

[11] There is an additional but unnecessary factor of 2 in [122] and [128], which was then omitted in [123]. In these articles, the case $\mathbb{R}^d$ is treated.

This holds for any covering set A (where $n \geq \log\delta/\log\eta$, see Lemma 4.116; also see the beginning of the proof of Lemma 4.117), so

$$\bigcup_{i=1}^{n} (\mu^*)_\delta^{(\gamma)}(\Omega_i) \leq p_1^{f_1}\cdots p_k^{f_k}\cdot 2^{r+4s+k}\cdot(\operatorname{diam} B)^{r+2s+k}\cdot(\overline{m}^*)_n^{(\gamma)}(\mathcal{E}^\infty).$$

Letting $\delta \to 0$ (and therefore $n \to \infty$) establishes the result on the measures. The statement about the dimensions is an immediate consequence, using the definition in Lemma 4.117 and Definition 4.46. We also note that, since the sets Ω_i are compact subsets of $\mathbb{M}$, the Hausdorff dimension is bounded by $r+2s+k = \dim_{\text{metr}} \mathbb{M}$, see Theorem 4.56. □

Remark 4.123. In view of this proposition, we also use the notation $\overline{\dim}_{\text{aff}}\underline{\Omega} = \overline{\dim}_{\text{aff}}\boldsymbol{\Theta}$.

Remark 4.124. In view of the proof of this proposition, we define for $\mathbb{M} = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}$ the number

$$C_{\mathbb{M}} = p_1^{f_1}\cdots p_k^{f_k}\cdot 2^{r+4s+k},$$

where $p_i = \mathfrak{p}_i \cap \mathbb{P}$ and f_i denotes the residue degree $f_i = f_{\mathfrak{p}_i|(p_i)}$.

The methods used in the last proof also yield the following statement, see [124, Lemma 1].

Corollary 4.125. *Let $f(x) = T(x)+t$ be a nonsingular affine map on $\mathbb{M}$ and A be any subset of $\mathbb{M}$. Then there is a constant $C > 0$, depending only on the (metric dimension of the) space $\mathbb{M}$ ($C_{\mathbb{M}}$ will certainly do), such that*

$$\frac{1}{C}\cdot\Psi^\gamma(T)\cdot(\mu^*)^{(\gamma)}(A) \leq (\mu^*)^{(\gamma)}(f(A)) \leq C\cdot\Phi^\gamma(T)\cdot(\mu^*)^{(\gamma)}(A).$$

Proof. Let $A \subset \bigcup_{i=1}^\infty U_i$ where $\operatorname{diam} U_i \leq d$. Then $f(A) \subset \bigcup_{i=1}^\infty f(U_i)$. For each i, the set U_i is contained in a hypercube of sidelength d, $f(U_i)$ is contained in a translate of the hypercuboid in Equation (4.13) on p. 116. As before, we need at most

$$C_{\mathbb{M}}\cdot\alpha_1\cdots\alpha_{m-1}\cdot(\alpha_m)^{1-m}$$

hypercubes of sidelength $\alpha_m\cdot\operatorname{diam} U_i$ to cover each $f(U_i)$. We obtain the estimate

$$\begin{aligned}(\mu^*)_\delta^{(\gamma)}(f(A)) &\leq \sum_{i=1}^\infty C_{\mathbb{M}}\cdot\alpha_1\cdots\alpha_{m-1}\cdot(\alpha_m)^{1-m}(\alpha_m\cdot\operatorname{diam} U_i)^\gamma\\ &\leq C_{\mathbb{M}}\cdot\Phi^\gamma(T)\sum_{i=1}^\infty(\operatorname{diam} U_i)^\gamma.\end{aligned}$$

This holds for any cover of A, so

$$(\mu^*)_\delta^{(\gamma)}(f(A)) \leq C_{\mathbb{M}}\cdot\Phi^\gamma(T)\cdot(\mu^*)_\delta^{(\gamma)}(A).$$

Letting $\delta \to 0$ gives the right hand side of the above inequality.

Applying this inequality to the mapping T^{-1} and the set $f(A)$ yields

$$(\mu^*)^{(\gamma)}(A) \leq C_{\mathbb{M}}\cdot\Phi^\gamma(T^{-1})\cdot(\mu^*)^{(\gamma)}(f(A)),$$

giving the left hand side of the above inequality by $\Psi^\gamma(T) = \left(\Phi^\gamma(T^{-1})\right)^{-1}$. □

We use this result to obtain lower bounds on the Hausdorff dimension. The proof of the next lemma follows the discussion in [124, Section 2], while Proposition 4.127 refers to [283, Theorem 1.9].

Lemma 4.126. *Let $\boldsymbol{\Theta}$ be a nonsingular irreducible self-affine IFS in $\mathbb{M}$ with attractor $\underline{\Omega} = (\Omega)_{i=1}^n$. Assume that the sets Ω_i are pairwise disjoint and all unions of the IFS $\boldsymbol{\Theta}$ are disjoint. If one set Ω_{i_0} is Hausdorff rectifiable (and therefore all sets Ω_i are Hausdorff rectifiable by Proposition 4.97), then we have $\underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta} \le \dim_{\mathrm{Hd}} \underline{\Omega}$.*

Proof. By assumption, the sets Ω_i and the unions in $\underline{\Omega} = \boldsymbol{\Theta}(\underline{\Omega})$ are disjoint. Then by iteration, the unions in $\underline{\Omega} = \boldsymbol{\Theta}^\ell(\underline{\Omega})$ are disjoint for each $\ell \in \mathbb{N}$, *i.e.*, the unions in

$$\Omega_i = \bigcup_{j=1}^n \bigcup_{f \in \Theta_{ij}^{(\ell)}} f(\Omega_j) = \bigcup_{j=1}^n \bigcup_{w \in \mathcal{E}_{ij}^\ell} f_w(\Omega_j)$$

are disjoint.

Since all Ω_i are Hausdorff rectifiable, say with Hausdorff dimension γ, we define the nonzero constant $\tilde{C}_{\min} = \min\left\{\mu^{(\gamma)}(\Omega_i) \mid 1 \le i \le n\right\}$. Then we obtain by Corollary 4.125

$$\begin{aligned}
\infty > \mu^{(\gamma)}\left(\bigcup_{i=1}^n \Omega_i\right) &= \sum_{i=1}^n \mu^{(\gamma)}(\Omega_i) = \sum_{i=1}^n \mu^{(\gamma)}\left(\bigcup_{j=1}^n \bigcup_{w \in \mathcal{E}_{ij}^\ell} f_w(\Omega_j)\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \sum_{w \in \mathcal{E}_{ij}^\ell} \mu^{(\gamma)}\left(f_w(\Omega_j)\right) \ge \sum_{i=1}^n \sum_{j=1}^n \sum_{w \in \mathcal{E}_{ij}^\ell} \frac{1}{C} \cdot \Psi^\gamma(T_w) \cdot \mu^{(\gamma)}(\Omega_j) \\
&= \frac{1}{C} \cdot \sum_{j=1}^n \mu^{(\gamma)}(\Omega_j) \cdot \sum_{i=1}^n \sum_{w \in \mathcal{E}_{ij}^\ell} \Psi^\gamma(T_w) \ge \frac{\tilde{C}_{\min}}{C} \cdot \sum_{i=1}^n \sum_{j=1}^n \sum_{w \in \mathcal{E}_{ij}^\ell} \Psi^\gamma(T_w) \\
&= \frac{\tilde{C}_{\min}}{C} \sum_{w \in \mathcal{E}^\ell} \Psi^\gamma(T_w).
\end{aligned}$$

We note that $\mathcal{E}^\ell$ is a covering set for $\mathcal{E}^\infty$, wherefore Lemma 4.116 implies

$$\left(\underline{m}^*\right)^{(\gamma)}(\mathcal{E}^\infty) \le \frac{C}{\tilde{C}_{\min}} \cdot \sum_{i=1}^n \mu^{(\gamma)}(\Omega_i) < \infty.$$

By the definition of the lower affinity dimension in Lemma 4.120, we obtain the required statement. □

Proposition 4.127. *Let $\boldsymbol{\Theta}$ be a nonsingular irreducible self-affine IFS in $\mathbb{M}$ with attractor $\underline{\Omega} = (\Omega)_{i=1}^n$. Let γ be the Hausdorff dimension of the sets Ω_i. Assume that the sets Ω_i are pairwise (γ)-measure-disjoint, i.e., $\mu^{(\gamma)}\left(\Omega_i \cap \Omega_j\right) = 0$ for all $i \neq j$, and all unions of the IFS $\boldsymbol{\Theta}$ are (γ)-measure-disjoint. If one set Ω_{i_0} is Hausdorff rectifiable, then we have $\underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta} \le \dim_{\mathrm{Hd}} \underline{\Omega}$.*

Proof. We first show that if the sets Ω_i and the unions in $\underline{\Omega} = \boldsymbol{\Theta}(\underline{\Omega})$ are (γ)-measure-disjoint, so are the unions in $\underline{\Omega} = \boldsymbol{\Theta}^\ell(\underline{\Omega})$ for all $\ell \in \mathbb{N}$. This is proven inductively and to keep the notation simple, we will only explicitly show that the unions in $\underline{\Omega} = \boldsymbol{\Theta}^2(\underline{\Omega})$ are (γ)-measure-disjoint[12]. We have

$$\Omega_i = \bigcup_{k=1}^{n} \bigcup_{j=1}^{n} \bigcup_{f \in \Theta_{ik}} \bigcup_{g \in \Theta_{kj}} f(g(\Omega_j)),$$

and we want to show that $f(g(\Omega_j))$ and $\hat{f}(\hat{g}(\Omega_{\hat{j}}))$ are (γ)-measure-disjoint if $(f, g, j) \neq (\hat{f}, \hat{g}, \hat{j})$. Let $f \in \Theta_{ik}$, $g \in \Theta_{kj}$, $\hat{f} \in \Theta_{i\hat{k}}$ and $\hat{g} \in \Theta_{\hat{k}\hat{j}}$. We have $g(\Omega_j) \subset \Omega_k$ and $\hat{g}(\Omega_{\hat{j}}) \subset \Omega_{\hat{k}}$, and therefore $f(g(\Omega_j)) \subset f(\Omega_k)$ and $\hat{f}(\hat{g}(\Omega_{\hat{j}})) \subset \hat{f}(\Omega_{\hat{k}})$.

Suppose $f \neq \hat{f}$. By assumption, $f(\Omega_k)$ and $\hat{f}(\Omega_{\hat{k}})$ are (γ)-measure-disjoint, so this also holds for their subsets. Suppose $f = \hat{f}$, then $k = \hat{k}$ and $g(\Omega_j), \hat{g}(\Omega_{\hat{j}}) \subset \Omega_k$. By assumption, $g(\Omega_j)$ and $\hat{g}(\Omega_{\hat{j}})$ are (γ)-measure-disjoint. Noting that a nonsingular affine map on $\mathbb{M}$ is actually a homeomorphism, we have $f(g(\Omega_j)) \cap f(\hat{g}(\Omega_{\hat{j}})) = f\left(g(\Omega_j) \cap \hat{g}(\Omega_{\hat{j}})\right)$, and by Corollary 4.125 we have (T denotes the linear part of f)

$$\mu^{(\gamma)}\left(f\left(g(\Omega_j) \cap \hat{g}(\Omega_{\hat{j}})\right)\right) \leq C \cdot \Phi^\gamma(T) \cdot \mu^{(\gamma)}\left(g(\Omega_j) \cap \hat{g}(\Omega_{\hat{j}})\right) = 0,$$

which consequently proves that the unions in $\underline{\Omega} = \boldsymbol{\Theta}^2(\underline{\Omega})$ are (γ)-measure-disjoint.

By the (γ)-measure-disjointness, we have $\mu^{(\gamma)}\left(\bigcup_{i=1}^n \Omega_i\right) = \sum_{i=1}^n \mu^{(\gamma)}(\Omega_i)$ and

$$\sum_{i=1}^{n} \mu^{(\gamma)}\left(\bigcup_{j=1}^{n} \bigcup_{w \in \mathcal{E}_{ij}^\ell} f_w(\Omega_j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{w \in \mathcal{E}_{ij}^\ell} \mu^{(\gamma)}\left(f_w(\Omega_j)\right).$$

Therefore, a calculation as in the Proof of Lemma 4.126 shows that $(\underline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) < \infty$ and the claim follows. □

We also note the following corollary.

Corollary 4.128. *Let $\boldsymbol{\Theta}$ be a nonsingular irreducible self-affine IFS in $\mathbb{M}$ with attractor $\underline{\Omega} = (\Omega)_{i=1}^n$. Let γ be the Hausdorff dimension of the sets Ω_i. Assume that the sets Ω_i are pairwise (γ)-measure-disjoint and all unions of the IFS $\boldsymbol{\Theta}$ are (γ)-measure-disjoint. Then, for each $\ell \in \mathbb{N}$, all unions in $\underline{\Omega} = \boldsymbol{\Theta}^\ell(\underline{\Omega})$ are (γ)-measure-disjoint.* □

Unfortunately, the Hausdorff rectifiablility of the sets Ω_i is in general hard to justify (at least if the intersection $\Omega_i \cap \Omega_j$ for $i \neq j$ consists of more than countably many points). We thus give the following variant of Lemma 4.126 which avoids this condition, also see [124, Proposition 2].

Proposition 4.129. *Let $\boldsymbol{\Theta}$ be a nonsingular irreducible self-affine IFS in $\mathbb{M}$ with attractor $\underline{\Omega} = (\Omega)_{i=1}^n$. Assume that the sets Ω_i are pairwise disjoint and all unions of the IFS $\boldsymbol{\Theta}$ are disjoint. Then we have $\underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta} \leq \dim_{\mathrm{Hd}} \underline{\Omega}$.*

[12] The general case $\ell \in \mathbb{N}$ is obtained by setting $g \in \Theta_{kj}^{(\ell-1)}$ in the next equation.

Proof. We show: If $\gamma < \underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta}$, then $\mu^{(\gamma)}\left(\bigcup_{i=1}^n \Omega_i\right) > 0$. By the definition of the dimensions involved, this proves the claim.

If $\gamma < \underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta}$, we have $(\underline{m}^*)^{(\gamma)}(\mathcal{E}^\infty) = \infty$ by Lemma 4.120. By Lemma 4.121, there exists a set $F \subset \mathcal{E}^\infty$ such that $0 < (\underline{m}^*)^{(\gamma)}(F) < \infty$ and $(\underline{m}^*)^{(\gamma)}(F \cap \mathcal{N}(w)) \leq \Psi^\gamma(T_w)$ for all $w \in \mathcal{E}^{\mathrm{fin}}$. We now construct a mass distribution on $\bigcup_{i=1}^n \Omega_i$.

For $w = \omega_1\omega_2\omega_3\ldots \in \mathcal{E}^\infty$, let x_w be the point $\bigcap_{\ell=1}^\infty f_{\omega_1\ldots\omega_\ell}(\Omega_{\mathrm{ter}(\omega_\ell)})$, where $f_{\omega_1\ldots\omega_\ell} = f_{\omega_1} \circ \ldots \circ f_{\omega_\ell}$ and we observe that $f_\omega(\Omega_{\mathrm{ter}(\omega)}) \subset \Omega_{\mathrm{ini}(\omega)}$. Then $x_w \in \Omega_{\mathrm{ini}(w)}$, and by the assumed disjointness, the mapping $w \mapsto x_w$ is a continuous bijection. We define a finite measure ν by

$$\nu(A) = (\underline{m}^*)^{(\gamma)}\left(\{w \in F \mid x_w \in A\}\right).$$

Then, ν is a Borel measure on $\mathbb{M}$ with $\operatorname{supp}\nu \subset \bigcup_{i=1}^n \Omega_i$ and, moreover,

$$\nu\left(f_{w'}(\Omega_{\mathrm{ter}(w')})\right) = (\underline{m}^*)^{(\gamma)}\left(F \cap \mathcal{N}(w')\right) \leq \Psi^\gamma(T_{w'})$$

for all $w' \in \mathcal{E}^{\mathrm{fin}}$. We now want to show that this property ensures that ν satisfies $\nu(U) \leq c \cdot (\operatorname{diam} U)^\gamma$ for all sets $U \subset \mathbb{M}$ with $\operatorname{diam} U < 1$ and that we can therefore apply Proposition 4.52.

Let $m = \lceil \gamma \rceil$ and define

$$Q_\varepsilon^{[m]} = \{w = \omega_1 \ldots \omega_\ell \in \mathcal{E}^{\mathrm{fin}} \mid \alpha_{r+2s+k-m+1}^{(\omega_1\ldots\omega_\ell)} \leq \varepsilon,\ \text{but } \alpha_{r+2s+k-m+1}^{(\omega_1\ldots\omega_{\ell-1})} > \varepsilon\}$$

for $0 < \varepsilon < 1$, where we set $\alpha_{r+2s+k-m+1}^{\varnothing} = 1$. Note that $Q_\varepsilon^{[m]}$ is finite (since $(\varrho(\boldsymbol{\Theta}))^{\#w} \leq \alpha_{r+2s+k-m+1}^{(w)} \leq (q(\boldsymbol{\Theta}))^{\#w}$), that we have

$$\varrho(\boldsymbol{\Theta}) \cdot \varepsilon < \alpha_{r+2s+k-m+1}^{(w)} \leq \varepsilon$$

for all $w \in Q_\varepsilon^{[m]}$, and that $Q_\varepsilon^{[m]}$ is a covering set of $\mathcal{E}^\infty$ for every $0 < \varepsilon < 1$.

Let $U \subset \mathbb{M}$ be a set with $\delta = \operatorname{diam} U < 1$. Consequently, U can be inscribed in a hypercube B_δ of sidelength δ. We define

$$\hat{Q} = \{w \in Q_\delta^{[m]} \mid f_w(\Omega_{\mathrm{ter}(w)}) \cap B_\delta \neq \varnothing\}.$$

Then we have

$$\nu(U) \leq \nu(B_\delta) \leq \sum_{w \in \hat{Q}} \Psi^\gamma(T_w),$$

and we need an estimate for this last sum.

Since the compact sets Ω_i and Ω_j are disjoint for $i \neq j$, they are separated by a positive distance $d_{\mathbb{M}}(\Omega_i, \Omega_j) > 0$. Similarly, the compact sets $f(\Omega_j)$ and $g(\Omega_{\hat{j}})$ with $f \in \Theta_{ij}$ and $g \in \Theta_{j\hat{j}}$ with $f \neq g$ are separated by a positive distance. We denote the minimum of this finitely many distances by $\hat{\eta}$. Set $\eta = \min\{\hat{\eta}/3, 1\}$, then the closed η-fridges of the sets Ω_i, defined by

$$\Omega_i^\eta = \{x \in \mathbb{M} \mid d_{\mathbb{M}}(x, \Omega_i) \leq \eta\},$$

satisfies

$$\Omega_i^\eta \supset \bigcup_{j=1}^n \bigcup_{f \in \Theta_{ij}} f(\Omega_j^\eta)$$

with these unions and the union $\bigcup_{i=1}^{n} \Omega_i^{\eta}$ being disjoint (observe that $f(\Omega_i^{\eta}) \subset (f(\Omega_i))^{q(\boldsymbol{\Theta})\cdot\eta}$). For $w \in \mathcal{E}_{\bullet i}^{\mathrm{fin}}$, the set $f_w(\Omega_i^{\eta})$ contains a hypercuboid R_w centred at each point of $f_w(\Omega_i)$, where R_w is given by

$$
\begin{aligned}
&|a_1^{(w)}| \cdot [-\eta\,,\eta] \ \times \cdots \times \ |a_r^{(w)}| \cdot [-\eta\,,\eta] \ \times \\
&|a_{r+1}^{(w)}| \cdot \mathrm{e}^{\left(i\cdot \mathrm{Arg}(a_{r+1}^{(w)})\right)} \cdot ([-\eta\,,\eta] \times [-\eta\,,\eta]) \ \times \cdots \times \ |a_{r+s}^{(w)}| \cdot \mathrm{e}^{\left(i\cdot \mathrm{Arg}(a_{r+s}^{(w)})\right)} \cdot ([-\eta\,,\eta] \times [-\eta\,,\eta]) \ \times \\
&|a_{r+s+1}^{(w)}|_{\mathfrak{p}_1} \cdot \{x \in \mathbb{Q}_{\mathfrak{p}_1} \mid \|x\|_{\mathfrak{p}_1} \le \eta\} \ \times \cdots \times \ |a_{r+s+k}^{(w)}|_{\mathfrak{p}_k} \cdot \{x \in \mathbb{Q}_{\mathfrak{p}_k} \mid \|x\|_{\mathfrak{p}_k} \le \eta\}.
\end{aligned}
$$

Let $w \in \hat{Q}$. Then we might choose $t_w \in B_\delta \cap f_w(\Omega_{\mathrm{ter}(w)}^{\eta})$. By the disjointness, we have $R_w + t_w \cap R_{w'} + t_{w'} = \varnothing$ if $w, w' \in \hat{Q}$ and $w \neq w'$. We now estimate the Haar measure $\mu(B_\delta \cap (R_w + t_w))$. For this, we note the following:

- Since $w \in \hat{Q}$, we have $\alpha_{r+2s+k}^{(w)} \le \cdots \le \alpha_{r+2s+k-m+1}^{(w)} \le \delta$.

- The sides of $R_w + t_w$ in a complex coordinate (which is a (two-dimensional) square) might not be parallel to the real and imaginary axis. But we can always inscribe a square with sidelength reduced by a factor of $\frac{1}{\sqrt{2}}$, having therefore half of its measure. Since we have s complex coordinates, we obtain a factor of $\left(\frac{1}{2}\right)^s$.

- Looking at the $\mathfrak{p}$-coordinates, η is not necessarily a power of p^f, where $p \in \mathfrak{p} \cap \mathbb{P}$ and f is the residue degree $f = f_{\mathfrak{p}|(p)}$. But there is exactly one power $v \in \mathbb{Z}$ of p, such that $\eta/p^f < p^{f\cdot v} \le \eta$ (we have used the notation $\lfloor \eta \rfloor_{\mathfrak{p}}$ in the proof of Proposition 4.122). We can replace η in the $\mathfrak{p}$-adic coordinate by this power $p^{f\cdot v} = \lfloor \eta \rfloor_{\mathfrak{p}}$ without changing the hypercuboid. Doing this for all $\mathfrak{p}$-adic coordinates, we obtain a factor of $1/(p_1^{f_1} \cdots p_k^{f_k})$, where $p_i \in \mathfrak{p}_i \cap \mathbb{P}$ and $f_i = f_{\mathfrak{p}_i|(p_i)}$.

- Suppose $\alpha_1 \cdot \eta \le \delta$. Then in the real coordinates, the side $|a_i^{(w)}| \cdot [-\eta\,,\eta] + (t_w)_i$ might only be half inside B_δ. The same holds in the complex coordinates, if they are parallel to the real and imaginary axes.

- We do not know whether $\alpha_i \cdot \eta \le \delta$ for all i, *i.e.*, if the (half-)sidelength is always smaller than δ and therefore (at least "half" of) $R_w + t_w$ is inside B_δ; but we do know that $\alpha_{r+2s+k-m+1}^{(w)} \cdot \eta \le \alpha_{r+2s+k-m+1}^{(w)} \le \delta$. So we might replace $\alpha_i^{(w)}$ for $1 \le i \le r+2s+k-m$ by $\alpha_{r+2s+k-m+1}^{(w)}$.

With these remarks, the following lower bound is obtained:

$$
\begin{aligned}
\mu(B_\delta \cap (R_w + t_w)) &\ge \frac{\eta^{r+2s+k}}{2^s \cdot p_1^{f_1} \cdots p_k^{f_k}} \cdot \alpha_{r+2s+k}^{(w)} \cdots \alpha_{r+2s+k-m+2}^{(w)} \cdot \left(\alpha_{r+2s+k-m+1}^{(w)}\right)^{r+2s+k-m+1} \\
&= \frac{2^s \cdot (2\cdot\eta)^{r+2s+k}}{C_{\mathbb{M}}} \cdot \Psi^{\gamma}(T_w) \cdot \left(\alpha_{r+2s+k-m+1}^{(w)}\right)^{r+2s+k-\gamma}.
\end{aligned}
$$

We now sum over all $w \in \hat{Q}$ and obtain:

$$\frac{2^s \cdot (2 \cdot \eta)^{r+2s+k}}{C_{\mathbb{M}}} \cdot \sum_{w \in \hat{Q}} \Psi^\gamma(T_w) \cdot \left(\alpha^{(w)}_{r+2s+k-m+1}\right)^{r+2s+k-\gamma} \le \sum_{w \in \hat{Q}} \mu(B_\delta \cap R_w + t_w)$$
$$= \mu\left(B_\delta \cap \bigcup_{w \in \hat{Q}} R_w + t_w\right) \le \mu(B_\delta) \le \delta^{r+2s+k}.$$

With the estimate $\alpha^{(w)}_{r+2s+k-m+1} \ge \varrho(\boldsymbol{\Theta}) \cdot \delta$ for all $w \in \hat{Q}$, one finally obtains

$$\nu(U) \le \sum_{w \in \hat{Q}} \Psi^\gamma(T_w) \le \frac{C_{\mathbb{M}}}{2^s \cdot (2 \cdot \eta)^{r+2s+k} \cdot (\varrho(\boldsymbol{\Theta}))^{r+2s+k-\gamma}} \cdot \delta^\gamma,$$

where $\delta = \operatorname{diam} U < 1$, and therefore by the mass distribution principle (Proposition 4.52) that

$$\mu^{(\gamma)}\left(\bigcup_{i=1}^n \Omega_i\right) \ge \frac{2^s \cdot (2 \cdot \eta)^{r+2s+k} \cdot (\varrho(\boldsymbol{\Theta}))^{r+2s+k-\gamma}}{C_{\mathbb{M}}} \cdot (\underline{m}^*)^{(\gamma)}(F) > 0,$$

which consequently proves the claim. □

Remark 4.130. If the conditions in Propositions 4.127 or 4.129 are fulfilled, we also write $\underline{\dim}_{\mathrm{aff}}\Omega = \underline{\dim}_{\mathrm{aff}}\boldsymbol{\Theta}$.

Definition 4.131. For a nonempty bounded subset $A \subset X$ of a separable metric space (X, d), we denote by $N_\delta(A)$ the smallest number of sets of diameter δ (respectively, of closed balls of diameter δ) that can cover A. The *lower* and *upper box-counting dimensions* of A are defined to be

$$\underline{\dim}_{\mathrm{box}} A = \liminf_{\delta \to 0}\left(-\frac{\log(N_\delta(A))}{\log(\delta)}\right) \quad \text{and} \quad \overline{\dim}_{\mathrm{box}} A = \limsup_{\delta \to 0}\left(-\frac{\log(N_\delta(A))}{\log(\delta)}\right),$$

respectively. If these two numbers are equal, we call the common value the *box-counting dimension* $\dim_{\mathrm{box}} A$ of A. By definition, it is clear that $N_\delta(A) \cdot \delta^\gamma$ is an upper bound for the (γ)-Hausdorff measure of A, which yields

$$\dim_{\mathrm{Hd}} A \;\le\; \underline{\dim}_{\mathrm{box}} A \;\le\; \overline{\dim}_{\mathrm{box}} A \;\le\; \dim_{\mathrm{metr}} X,$$

see [51, Theorem 5.8] and [123, Proposition 4.1].

Remark 4.132. The box-counting dimension is easier to calculate than the Hausdorff dimension and "often" they coincide, which makes the box-counting dimension quite popular (see [51, Chapter 5]). We note that sometimes other names for the lower/upper box-counting dimension are used, *e.g.*, " fractal dimension", "lower/upper entropy index", "lower/upper capacity", "Minkowski dimension" or "Bouligand-Minkowski dimension". A good overview of various "measures" and "dimensions" used for fractals and their relationships to one another, can be found in [114, Chapter 1] and [123, Chapter 3].

As for the Hausdorff dimension (see Proposition 4.97), it is clear that for a nonsingular irreducible IFS with attractor $\underline{\Omega}$ the (lower/upper) box-counting dimension of all sets Ω_i ($1 \le i \le n$) is the same[13], wherefore we write $\underline{\dim}_{\text{box}}\underline{\Omega}$ and $\overline{\dim}_{\text{box}}\underline{\Omega}$, respectively $\dim_{\text{box}}\underline{\Omega}$. We now compare these numbers with $\overline{\dim}_{\text{aff}}\underline{\Omega}$, see[14] [122, Theorem 5.4].

Lemma 4.133. *Let $\underline{\Omega} = (\Omega_i)_{i=1}^n$ be the invariant set of an irreducible nonsingular self-affine IFS $\boldsymbol{\Theta}$ in $\mathbb{M}$. Then we have*

$$\dim_{\text{Hd}}\underline{\Omega} \;\le\; \underline{\dim}_{\text{box}}\underline{\Omega} \;\le\; \overline{\dim}_{\text{box}}\underline{\Omega} \;\le\; \min\{\overline{\dim}_{\text{aff}}\underline{\Omega}\,, \dim_{\text{metr}}\mathbb{M}\}.$$

Proof. We only have to show that $\overline{\dim}_{\text{box}}\underline{\Omega} \le \overline{\dim}_{\text{aff}}\underline{\Omega}$. For this, we actually show that for every $\gamma > \overline{\dim}_{\text{aff}}\underline{\Omega}$, we have $\overline{\dim}_{\text{box}}\underline{\Omega} \le \gamma$. This proves the claim.

Let $\gamma > \overline{\dim}_{\text{aff}}\underline{\Omega}$ and set $m = \lceil \gamma \rceil$. Then, we know from Lemma 4.117 (i) that the sum $\sum_{w\in\mathcal{E}^{\text{fin}}} \Phi^\gamma(T_w) < \infty$. For $0 < \varepsilon < 1$ we define the following sets:

$$Q_\varepsilon^{(m)} = \{w = \omega_1 \ldots \omega_\ell \in \mathcal{E}^{\text{fin}} \mid \alpha_m^{(\omega_1\ldots\omega_\ell)} \le \varepsilon,\ \text{but}\ \alpha_m^{(\omega_1\ldots\omega_{\ell-1})} > \varepsilon\},$$

i.e., we collect all finite walks, such that the m-th singular value is just smaller than or equal to ε. We note that $Q_\varepsilon^{(m)}$ is finite (since $(\varrho(\boldsymbol{\Theta}))^{\#w} \le \alpha_m^{(w)} \le (q(\boldsymbol{\Theta}))^{\#w}$) and, moreover, it is a covering set for $\mathcal{E}^\infty$ for all $0 < \varepsilon < 1$ (and also all $1 \le m \le r + 2s + k$). We also note that for $w \in Q_\varepsilon^{(m)}$ we have $\varrho(\boldsymbol{\Theta}) \cdot \varepsilon < \alpha_m^{(w)} \le \varepsilon$, and therefore

$$Q_\varepsilon^{(m)} \cap Q_{\varepsilon\cdot\varrho(\boldsymbol{\Theta})}^{(m)} = \varnothing.$$

So we have

$$\infty > \sum_{w\in\mathcal{E}^{\text{fin}}} \Phi^\gamma(T_w) \ge \sum_{\ell=1}^{\infty} \sum_{w\in Q^{(m)}_{(\varrho(\boldsymbol{\Theta}))^\ell}} \Phi^\gamma(T_w) > 0,$$

wherefore there is an $\hat{\ell} \in \mathbb{N}$ such that $\sum_{w\in Q^{(m)}_{(\varrho(\boldsymbol{\Theta}))^\ell}} \Phi^\gamma(T_w) \le 1$ for all $\ell \ge \hat{\ell}$. For small enough $\delta > 0$, we choose for ℓ' the least integer such that $(\varrho(\boldsymbol{\Theta}))^{\ell'} \cdot \operatorname{diam} B < \delta \le (\varrho(\boldsymbol{\Theta}))^{\ell'-1} \cdot \operatorname{diam} B$ (here, δ is small enough if $\ell' \ge \hat{\ell}$), where B is as in the proof of Proposition 4.122.

We also recall from the proof of Proposition 4.122 that we need at most

$$C_{\mathbb{M}} \cdot \alpha_1^{(w)} \cdots \alpha_{m-1}^{(w)} \cdot \left(\alpha_m^{(w)}\right)^{1-m} = C_{\mathbb{M}} \cdot \Phi^\gamma(T_w) \cdot \left(\alpha_m^{(w)}\right)^{-\gamma}$$

hypercubes of sidelength (and also diameter) $\alpha_m^{(w)} \cdot \operatorname{diam} B$ to cover the hypercuboid of Equation (4.12) on p. 115.

[13] Basically, we have a "mapping statement" for the box-counting dimension (compare the proof of Lemma 4.49): If Ω_i is covered by N_δ balls of radius δ, then $f(\Omega_i) \subset \Omega_j$ is covered by at most N_δ balls of radius $q(\boldsymbol{\Theta}) \cdot \delta$. This yields an upper bound on the box-counting dimension of $f(\Omega_i)$. Applying the same argument to f^{-1} gives the required statement.

[14] Note that there is a mistake in the proof of [122, Theorem 5.4] on p. 348 in line 7, where "$\epsilon \ge \alpha_m > b^r\epsilon$" must read "$\epsilon \ge \alpha_m > b^{qr}\epsilon$". But then, however, the estimates in [122, p. 348, Lines 15–17] break down. This is the reason why we use the sets $Q_\varepsilon^{(m)}$ in the proof of Lemma 4.133.

We can now estimate the number of hypercubes needed to cover $\bigcup_{i=1}^{n} \Omega_i$, namely, we need at most

$$C_{\mathbb{M}} \cdot \sum_{w \in Q^{(m)}_{(\varrho(\boldsymbol{\Theta}))^{\ell'}}} \Phi^{\gamma}(T_w) \cdot \left(\alpha_m^{(w)}\right)^{-\gamma} < C_{\mathbb{M}} \cdot (\varrho(\boldsymbol{\Theta}))^{-\gamma \cdot (\ell'+1)} \cdot \sum_{w \in Q^{(m)}_{(\varrho(\boldsymbol{\Theta}))^{\ell'}}} \Phi^{\gamma}(T_w)$$

$$\leq C_{\mathbb{M}} \cdot (\varrho(\boldsymbol{\Theta}))^{-\gamma \cdot (\ell'+1)} \leq C_{\mathbb{M}} \cdot \left(\frac{\operatorname{diam} B}{(\varrho(\boldsymbol{\Theta}))^2 \cdot \delta} \right)^{\gamma}$$

hypercubes of diameter δ. Hence

$$\frac{\log\left(N_\delta\left(\bigcup_{i=1}^{n} \Omega_i\right)\right)}{-\log(\delta)} < \frac{\log(C_{\mathbb{M}}) + \gamma \cdot (\log(\operatorname{diam} B) - 2\log(q(\boldsymbol{\Theta}))) - \gamma \cdot \log(\delta)}{-\log(\delta)},$$

so that

$$\limsup_{\delta \to 0} \left(\frac{\log\left(N_\delta\left(\bigcup_{i=1}^{n} \Omega_i\right)\right)}{-\log(\delta)} \right) \leq \gamma.$$

Consequently, this proves the claim. □

Remark 4.134. We remark that dimension calculations for self-affine sets are, in general, quite laborious and therefore there are not many worked examples; see [56, 62, 145, 185, 206, 222, 255, 283, 285, 286], [374, Section I.4], [251, Sections 4.9 & 4.15] and [114, Sections 3.4 & 3.6] and references therein for a collection of articles about self-affine sets.

Remark 4.135. In the case of a self-similar IFS (*i.e.*, where $\alpha = \alpha_1 = \ldots = \alpha_{r+2s+k}$ and therefore $\Phi^{\gamma}(T) = \alpha^{\gamma}$ (if $\gamma \leq r + 2s + k$) for a map T with $(r + 2s + k)$ singular values α), the name *similarity dimension* is commonly used for what we call the affinity dimension, see [44, Section 9], [127, Section 8.3], [126, Chapters 2 & 3], [112, Section 6.4] *etc.* So, we can view the affinity dimension as the correct[15] generalisation of the similarity dimension to a self-affine IFS.

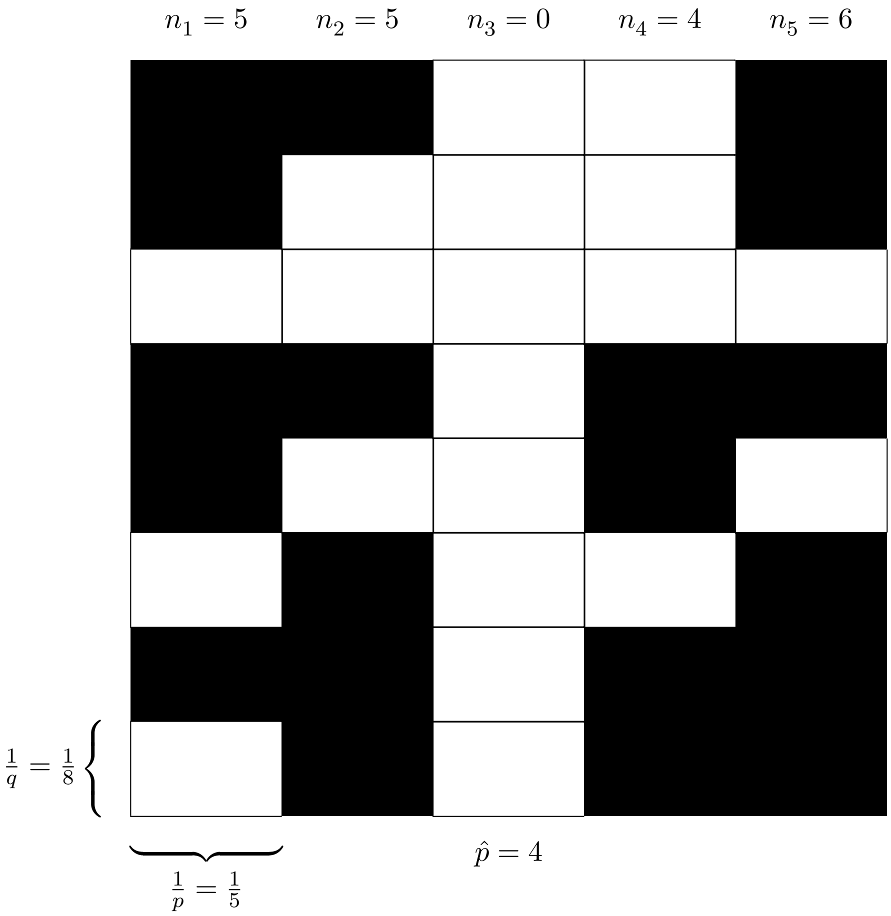
We also note that the similarity dimension equals the Hausdorff dimension of the self-similar attractor, if the IFS $\boldsymbol{\Theta}$ satisfies the *open set condition*, or *OSC* for short, see [264], [187], [248], [51, Theorem 5.9], [127, Theorem 8.6], [125, Theorem 9.3], [126, Theorem 2.7], [112, Section 6.3 & 6.4] *etc.* Moreover, in this case the attractor is also Hausdorff rectifiable, *i.e.*, has finite nonzero (γ)-measure (where γ denotes the Hausdorff dimension). The OSC informally states that there is not too much overlap: An IFS $\boldsymbol{\Theta}$ satisfies the open set condition, if there exists a family of nonempty bounded open sets $\underline{U} = (U_i)_{i=1}^{n}$ such that $U_i \supset \bigcup_{j=1}^{n} \bigcup_{f \in \Theta_{ij}} f(U_j)$ with all unions disjoint. If for the attractor $\underline{\Omega}$ all unions in the (self-similar) IFS are disjoint, the IFS is also called *totally disconnected*, while an IFS which is not totally disconnected but (still) satisfies the OSC is called *just touching*, see [51, Section 4.2]. Note that $\boldsymbol{\Theta}^k(\operatorname{cl} \underline{U})$ defines an antitone sequence, wherefore we have for the attractor $\underline{\Omega} = \bigcap_{k \in \mathbb{N}} \boldsymbol{\Theta}^k(\operatorname{cl} \underline{U})$ by Proposition 4.89. If the IFS is neither totally disconnected nor just touching, then it is called *overlapping* and, in fact, there is consequently "substantial" overlap for the unions. Frequently, the similarity dimension equals the Hausdorff dimension even if the OSC fails to hold, see [121]. We note that there is also a *strong open set condition* or *SOSC*, where additional one has $\underline{\Omega} \cap \underline{U} \neq \varnothing$. However, often the OSC and the SOSC are equivalent, see [325].

[15] An earlier attempt is the so-called *gap dimension*, which is essentially defined through the singular value function $\tilde{\Phi}^{\gamma}(T) = (\alpha_1 \ldots \alpha_{r+2s+k})^{(\gamma/(r+2++k))}$. It has the drawback that it is in general *not* an upper bound for the Hausdorff dimension, see [246, Section 4] (also see Remark 4.114). It is also called *affine dimension* in [62] and [374].

Remark 4.136. In view of the last remark, one might wonder if the affinity dimension equals the Hausdorff dimension if the OSC holds, or at least if in the IFS all unions are disjoint and the sets Ω_i are pairwise disjoint. We first quote from [122, p. 340]:

> Secondly, even if these components [*i.e.*, the sets $f(U)$ in the OSC, here denoted $S_i(U)$] are all disjoint, it can happen that the components align with each other in such a way that unusually efficient coverings are possible and the Hausdorff dimension is reduced. (This second possibility cannot occur if the S_i are similarities.)

We now give an example[16] for this phenomenon, see [57, Chapter 4], [255, 285, 286] and [123, Example 9.11]: Let the unit cube $[0\,,1]\times[0\,,1]\subset\mathbb{R}^2$ be divided into a $p\times q$ array of rectangles (of sidelengths $\frac{1}{p}$ and $\frac{1}{q}$), where $p,q\in\mathbb{N}$ with $1<p<q$. We say that we have q rows and p columns. The linear part of the affine maps considered maps the unit cube in such a rectangle, *i.e.*, it is given by the diagonal matrix $\operatorname{diag}(\frac{1}{p}\,,\frac{1}{q})$. We now select a subcollection S of these rectangles, and let n_j denote the number of rectangles selected from the j-th column for $1\le j\le p$ (then $\sum_{j=1}^{p} n_j=\operatorname{card} S$). Let $\hat{p}$ denote the number of columns containing at least one rectangle of S. Now $\boldsymbol{\Theta}$ maps the unit cube to S, and by iteration we obtain an self-affine attractor Ω (only one set!).



[16]We note that [124, Example 2] is *not* such an example which eludes this phenomenon of self-affine attractors. On the one hand, it is self-similar (and satisfies the OSC, as correctly noted there), on the other hand, the attractor is *incorrectly* stated as the single point $\{(0\,,0)\}$. But [113, Example 1] is the correct version of what [124, Example 2] had in mind.

We get the following dimensions[17] (see the references cited for the Hausdorff and box-counting dimension):

$$\underline{\dim}_{\mathrm{aff}}\Theta = \begin{cases} \log\left(\sum\limits_{j=1}^{p} n_j\right) \cdot \frac{1}{\log(q)}, & \text{if } \sum\limits_{j=1}^{p} n_j \le q, \\ \log(q) \cdot \left(\frac{1}{\log(q)} - \frac{1}{\log(p)}\right) + \log\left(\sum\limits_{j=1}^{p} n_j\right) \cdot \frac{1}{\log(p)}, & \text{otherwise,} \end{cases}$$

$$\dim_{\mathrm{Hd}} \Omega = \log\left(\sum_{j=1}^{p} n_j^{\log(p)/\log(q)}\right) \cdot \frac{1}{\log(p)}$$

$$\dim_{\mathrm{box}} \Omega = \log(\hat{p}) \cdot \left(\frac{1}{\log(p)} - \frac{1}{\log(q)}\right) + \log\left(\sum_{j=1}^{p} n_j\right) \cdot \frac{1}{\log(q)}$$

$$\overline{\dim}_{\mathrm{aff}}\Omega = \begin{cases} \log\left(\sum\limits_{j=1}^{p} n_j\right) \cdot \frac{1}{\log(p)}, & \text{if } \sum\limits_{j=1}^{p} n_j \le p, \\ \log(p) \cdot \left(\frac{1}{\log(p)} - \frac{1}{\log(q)}\right) + \log\left(\sum\limits_{j=1}^{p} n_j\right) \cdot \frac{1}{\log(q)}, & \text{otherwise,} \end{cases}$$

and one can check that $\underline{\dim}_{\mathrm{aff}}\Theta \le \dim_{\mathrm{Hd}} \Omega \le \dim_{\mathrm{box}} \Omega \le \overline{\dim}_{\mathrm{aff}}\Omega$ (in the example shown above, we have $\dim_{\mathrm{Hd}} \Omega \approx 1.6342$, $\dim_{\mathrm{box}} \Omega \approx 1.6353$, $\overline{\dim}_{\mathrm{aff}}\Omega = 1\frac{2}{3}$ and $\underline{\dim}_{\mathrm{aff}}\Theta \approx 1.5693$). Note that one can even find examples for the IFS, such that all unions in the IFS are disjoint (the example above only satisfies the OSC), but nevertheless, all inequalities for the dimensions are strict!

It is shown in [286] that these sets have infinite Hausdorff measure in their dimension, except in the cases where the Hausdorff dimension and the box-counting dimension coincide (this also holds for the family of self-affine fractals considered in [222]), which is the case iff all n_i are the same and we consequently have $\underline{\dim}_{\mathrm{aff}}\Theta \le \dim_{\mathrm{Hd}} \Omega = \dim_{\mathrm{box}} \Omega = \overline{\dim}_{\mathrm{aff}}\Omega$ (where $\underline{\dim}_{\mathrm{aff}}\Theta = \dim_{\mathrm{Hd}} \Omega$ iff $n_i = q$ for all i, *i.e.*, in the extremal case where the whole unit hypercube is the attractor). Also, the h-measure with respect to $h(t) = t^\gamma \cdot \exp(-c \cdot |\log(t)|/(\log|\log(t)|)^2)$ (with $c > 0$ small and where γ denotes the Hausdorff dimension) is considered and shown to be infinite, while it is zero with respect to $h(t) = t^\gamma \cdot \exp(-c \cdot |\log(t)|/(\log|\log(t)|)^{2-\delta})$ for any $\delta > 0$. These findings should be compared with Remark 4.47.

Remark 4.137. It is, however, "generic" for a self-affine attractor that the Hausdorff dimension and the affinity dimension coincide. This follows (in $\mathbb{R}^d$) from the following statement.

Proposition. [122, Theorem 5.3] *Let* $\boldsymbol{\Theta} : \mathcal{K}(\mathbb{R}^d) \to \mathcal{K}(\mathbb{R}^d)$ *be an IFS given by the set* $\boldsymbol{\Theta} = \{f_1, \dots, f_n\}$ *where* $f_i(x) = T_i(x) + t_i$ *are affine maps for* $1 \le i \le n$ *(with linear part* T_i *and translation* t_i*). Assume that* $\|T_i\|_2 < \frac{1}{3}$ *for* $1 \le i \le n$*, and denote the self-affine attractor by* Ω*. Then for almost all* $(t_1, \dots, t_n) \in \mathbb{R}^{n \cdot d}$ *(in the sense of* $(n \cdot d)$*-dimensional Lebesgue measure) one has* $\dim_{\mathrm{Hd}} \Omega = \min\{d, \overline{\dim}_{\mathrm{aff}}\Omega\}$. □

This result is obtained by a potential-theoretic characterisation of the Hausdorff dimension; it might not be easy to generalise it to sets in $\mathbb{M}$, since one has to evaluate integrals of

[17] Note that neither Proposition 4.127 (see ahead, the set Ω is "usually" not Hausdorff rectifiable) nor Proposition 4.129 (as the above example shows) apply, wherefore we have to write $\underline{\dim}_{\mathrm{aff}}\Theta$ (instead of $\underline{\dim}_{\mathrm{aff}}\Omega$) in the following.

(continuous) functions. We note that the requirement $\|T_i\|_2 < \frac{1}{3}$ is a little more than "an inconvenient technical requirement" (as it is called in [122, Remark (3) in Section 6]), see [113, Example 2] (which relies on [297, Theorem 8]); however, this requirement is always met by using an appropriate power of $\boldsymbol{\Theta}$. We also note that the Hausdorff dimension is not a continuous function on the translational parts, see [124, Example 1] (also given as [123, Example 9.10]) which has a jump in the dimension as the translational parts are continuously varied.

However, a statement as in this proposition does not help at all in any given example – it can be in the null set. But as noted in the previous remark, a checkable criterion (like the OSC in the self-similar case), which excludes this null set, seems out of reach in the self-affine case.

5. Model Sets & Substitutions

> In the words of the ancients, one should make his decision within the space of seven breaths. It is a matter of being determined and having the spirit to break through to the other side.
>
> Ghost Dog – *Jim Jarmusch*

In this chapter, the notions "Delone", "Meyer" and "model" sets are introduced. Special emphasis is put on the multi-component case, see Section 5.3 where a cut and project scheme in the multi-component case is established. There is a big interest in point sets and their ergodic properties which arise from a substitution rule (see Sections 5.4 & 5.6), wherefore we give a short introduction to dynamical systems in Section 5.5. Substitutions are brought together with model sets in Section 5.7, where, unfortunately, it gets quite technical.

5.1. Model Sets and the Density Formula

Definition 5.1. Let G be an LCAG. A subset Λ of G is *relatively dense* if there is a compact subset W of G so that $G = \Lambda + W$. A set Λ is called *uniformly discrete* if there is an open neighbourhood V of 0 in G such that for all $x \in \Lambda$ one has $(V + x) \cap \Lambda = \{x\}$. A subset Λ is a *Delone set* (or a *Delaunay set*) if it is both uniformly discrete and relatively dense.

Remark 5.2. If G is metrisable and the metric is such that boundedness implies total boundedness, we can also express these properties in terms of balls: A set Λ is relatively dense if there is a value R such that each closed ball of radius R contains at least one point of Λ. The minimal value of R is called the *covering radius* of Λ. A subset Λ is uniformly discrete if there is a value r such that each open ball of radius r contains at most one point of Λ. The maximal such r is called the *packing radius* for Λ. Sometimes, a Delone set is also called an *(r, R)-set*.

Remark 5.3. If G is also σ-compact, a uniformly discrete set Λ is countable.

Definition 5.4. A *cut and project scheme*, or *CPS* for short, is a triple $(G, H, \tilde{L})$ consisting of a σLCAG G (called the *direct space* or *physical space*), an LCAG H (called the *internal space*) and a lattice $\tilde{L}$ in $G \times H$ (*i.e.*, $\tilde{L}$ is discrete and the (Abelian) factor group $(G \times H)/\tilde{L}$ is compact), such that the two natural projections $\pi_1 : G \times H \to G$, $(x, y) \mapsto x$ and $\pi_2 : G \times H \to H$, $(x, y) \mapsto y$ satisfy the following properties:

- The restriction $\pi_1|_{\tilde{L}}$ of π_1 to $\tilde{L}$ is injective.
- The image $\pi_2(\tilde{L})$ is dense in H.

We set $L = \pi_1(\tilde{L})$ and define the so-called *star-map* $(\cdot)^\star = \pi_2 \circ (\pi_1|_{\tilde{L}})^{-1} : L \to H$, which is well-defined on L by the injectivity of $\pi_1|_{\tilde{L}}$ (we shall later also extend the star-map to larger subgroups of G, see Section 5.7.3). It is therefore justified to use the notation $L^\star = \pi_2(\tilde{L})$ (and we also have $\tilde{L} = \{(x\,,x^\star) \mid x \in L\}$). The situation is summarised in the following diagram:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \text{ dense} \\ L & \overset{\text{bijective}}{\longleftrightarrow} & \tilde{L} & \longrightarrow & L^\star \end{array}$$

Remark 5.5. Let $(G\,,H\,,\tilde{L})$ be a CPS. Then there are unique (up to a multiplicative constant) Haar measures μ_G and μ_H on G and H, respectively (see Proposition 4.30). We recall that $\mu_G \otimes \mu_H$ is a Baire measure but not necessarily a Borel measure (and therefore not necessarily a Haar measure) on $G \times H$ (see Lemma 4.34 and the discussion following it). We will always choose the normalisation of the Haar measure $\mu_{G\times H}$ on $G \times H$ such that on the class of all Baire sets it coincides with the measure $\mu_G \otimes \mu_H$.

Furthermore, we are often only interested in the case that both G and H are metrisable (and therefore also $G \times H$, *e.g.*, by the maximum metric). With this assumption, the Haar measure $\mu_{G\times H}$ is in fact equal to $\mu_G \otimes \mu_H$, see Lemma 4.35 and Remark 4.36. The justification for this assumption is that our main interest lies in substitutions, see Section 5.4.

We also note that we may always assume that H is metrisable, see Proposition 5.48.

Definition 5.6. Given a CPS $(G\,,H\,,\tilde{L})$ and a subset $S \subset H$, we define

$$\Lambda(S) = \{x \in L \mid x^\star \in S\}.$$

A *model set* (or *cut and project set*), associated with the CPS $(G\,,H\,,\tilde{L})$, is a nonempty subset $\varLambda$ of G of the form

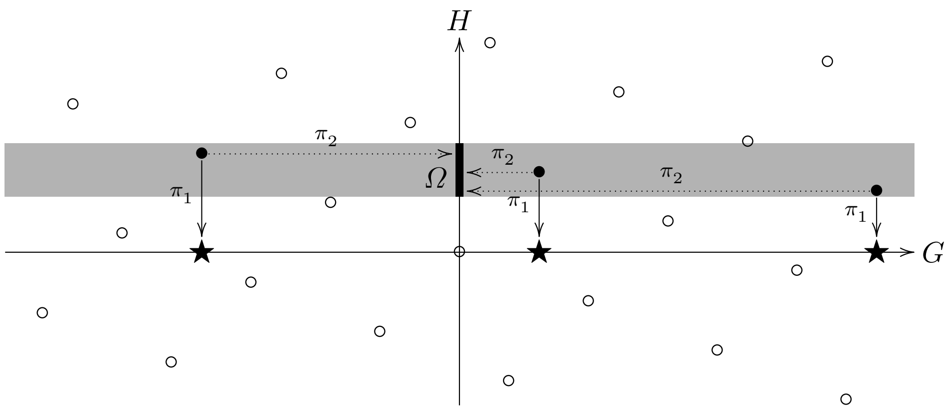
$$\varLambda = x + \Lambda(\varOmega + y), \tag{5.1}$$

where $x \in G$, $y \in H$, and $\varOmega \subset H$ is compact and regularly closed (*i.e.*, $\varOmega = \operatorname{cl}\operatorname{int}\varOmega$). The set[1] $\varOmega + y$ is called the *window* (or *acceptance domain*) of the model set. Often the boundary $\partial\varOmega$, or points in $\varLambda$ arising from it, needs some additional care, wherefore we call a set $\varLambda'$ such that (where x, y and $\varOmega$ are as before)

$$x + \Lambda(\operatorname{int}\varOmega + y) \subset \varLambda' \subset x + \Lambda(\varOmega + y) \tag{5.2}$$

an *inter model set*, or *IMS* for short (following [34, Section 5.1]).

Schematically, we have the following situation for a model set:



[1] Of course, the "y" in the definition of a model set is superfluous, setting $\varOmega' = \varOmega + y$ we have $\varLambda = x + \Lambda(\varOmega')$ and $\varOmega'$ is also compact and regularly closed. Here, the "y" is used to treat direct space (the "x") and internal space similarly.

Here, "$\circ$" denotes the lattice points, "$\bullet$" the lattice points which lie in the (grey) "cut strip" $G \times \Omega$ (*i.e.*, their image under π_2 lies in the (bold) window Ω), and "$\bigstar$" the points of the model set $\Lambda(\Omega)$.

Remark 5.7. We can alternatively define $\Lambda(S)$ by

$$\Lambda(S) = \{x \in L \mid 0 \in x^\star - S\}.$$

Note that although we get the same set $\Lambda(S)$, the interpretation is slightly different: In the original definition, one projects all lattice points which lie in the "cut strip" $G \times \Omega$. Here, one forms the set $\tilde{L} - S$ and then looks at its intersection with $G \times \{0\}$.

Lemma 5.8. *If S has nonempty interior, then $\Lambda(S)$ is relatively dense. If S is relatively compact, then $\Lambda(S)$ is uniformly discrete. Therefore, (inter) model sets are Delone sets.* □

Proof. We refer to [259, Lemma 2.5 & Prop. 2.6] and [256, Lemma II.10] (for the case where both G and H are metrisable). □

Definition 5.9. An (inter) model set (defined as in Equation (5.1) respectively (5.2) on p. 130) is called *regular* if $\mu_H(\partial\Omega) = 0$, where μ_H denotes the Haar measure on H. An (inter) model set is called *generic* if $(\partial\Omega + y) \cap L^\star = \varnothing$; in this case we have equality signs in Equation (5.2) (instead of "$\subset$"). Let S be a subset of (the Abelian group) H. Then $\operatorname{Stab} S = \{h \in H \mid S + h = S\}$ is called the *stabiliser* of S. We say that S has a *trivial stabiliser* or that it is *irredundant* if $\operatorname{Stab} S = \{0\}$. Consequently, we say that an (inter) model set is *irredundant* if the window $\Omega + y$ is irredundant (has a trivial stabiliser).

Remark 5.10. We give a first indication, why the term "generic" is used: By Proposition 2.14, the internal space H is a Baire space. Furthermore, $\partial\Omega$ and $-\partial\Omega$ (and their translates) are nowhere dense sets (because Ω is regularly closed), while $\tilde{L}$ and therefore $L^\star$ is countable (if H is also σ-compact). But then the set $\{y \in H \mid (y + \partial\Omega) \cap L^\star \neq \varnothing\}$ is meager: If $(y + \partial\Omega) \cap L^\star \neq \varnothing$ then there is at least one $t_y^\star \in L^\star$ such that $t_y^\star \in y + \partial\Omega$ and therefore $y \in t_y^\star - \partial\Omega \subset L^\star - \partial\Omega$. So we have established that

$$\{y \in H \mid (y + \partial\Omega) \cap L^\star \neq \varnothing\} \subset -\partial\Omega + L^\star,$$

and this last set (and any of its subsets) is – as countable union of (closed) nowhere dense sets – meager and therefore has no interior by Lemma 2.13, respectively the complement $E = \{y \in H \mid (y + \partial\Omega) \cap L^\star = \varnothing\}$ is dense (and nonmeager) in H. This argument involving Baire's category theorem appeared before in [327, Section 2.2.2]. We note that by Baire's category theorem, we can think of meager sets being "small" while nonmeager sets are "big" (or *fat*), wherefore the term "generic" is used (because then the y of $y + \partial\Omega$ is chosen from the "big" set); however, we note the following, see [279, Theorem 1.6]: $\mathbb{R}$ can be decomposed into two complementary sets A and B such that A is meager and B is a set of Lebesgue measure zero. We will see (Lemma 7.53) that this last situation does not occur here; in fact, the meager set here is also a set of (Haar) measure zero, which then consequently justifies the use of the word "generic" in all (*i.e.*, topological and measure theoretical) aspects. As a last point, we observe that, by [279, Theorem 9.2], the set E contains a dense $\mathfrak{G}_\delta$ subset of H.

Definition 5.11. A set $\Lambda \subset G$ is called a *Meyer set* if it is a Delone set and $\Lambda - \Lambda = \{x - y \mid x, y \in \Lambda\}$ is contained in a finite number of translates of Λ, *i.e.*, $(\Lambda - \Lambda) \subset (\Lambda + F)$ where $\operatorname{card} F < \infty$.

Remark 5.12. If $\varLambda \subset G$ is a Meyer set, then $\varLambda + F$ and therefore $\varLambda - \varLambda$ is a Delone set. A proof of this can be found in [217, Section 3] (if $G = \mathbb{R}^d$, respectively if G is metrisable) and in [34, Prop. 14]. In fact, if $G = \mathbb{R}^d$ or – more general – if G is compactly generated, then a Delone set $\varLambda$ is a Meyer set iff $\varLambda - \varLambda$ is also a Delone set, see [217, Theorem 1.1] and [34, Theorem 11 & Corollary 2]. Moreover, $\varLambda$ is a Meyer set iff $\varLambda$ and $\varLambda - \varLambda - \varLambda$ are Delone sets, see [256, Section II.14.2] and [257, Lemma 1]. Further equivalent characterisations of Meyer sets, especially in $\mathbb{R}^d$, appear in [217, Theorem 1.1], [218, Theorem 3.1] and [259, Theorem 9.1].

We also note that if $\varLambda$ is a Meyer set, then any finite combination of the form $\pm\varLambda\pm\varLambda\pm\ldots\pm\varLambda$ is also a Meyer set (subsequent use of the definition yields that it is contained in $\varLambda+F+\ldots+F$ and the set-theoretic sum of finite sets is again a finite set).

Y. Meyer proved the following statements, which relate Meyers sets and model sets. Note that although the statement is formulated for an open set $\varOmega$, it also holds for a regularly closed subset of H.

Proposition 5.13. *Let* $(G\,, H\,, \tilde{L})$ *be a CPS.*

(i) *If* $\varOmega$ *is a relatively compact open subset of* H*, then* $\Lambda(\varOmega)$ *is a Meyer set.*

(ii) *Let* $\varLambda$ *be a subset of* G*. If* $\varLambda$ *is a Meyer set, then there is a relatively compact open subset* $\varOmega$ *of* H *such that* $\varLambda$ *is contained in* $\Lambda(\varOmega)$*.*

Proof. These are [256, Prop. II.3], respectively [256, Prop. II.3] together with [256, Theorem II.X]. □

The following is the inverse statement to (ii) of the previous proposition, see [259, Corollary 5.6] for the case $G = \mathbb{R}^d$.

Corollary 5.14. *Any relatively dense subset* $\varLambda$ *of a model set* $\Lambda(\varOmega)$ *is a Meyer set.*

Proof. As relatively dense subset of a model set $\Lambda(\varOmega)$, $\varLambda$ is a Delone set. Moreover, we note that $\varOmega - \varOmega - \varOmega$ is also a relatively compact set, wherefore $\varLambda - \varLambda - \varLambda \subset \Lambda(\varOmega - \varOmega - \varOmega)$ is a relatively dense subset of a model set and therefore a Delone set. Consequently, [257, Lemma 1] (compare [256, Section II.14.2], see Remark 5.12) establishes the claim. □

We first define a property which is related to uniform discreteness and relative denseness.

Definition 5.15. We say that a set $\varLambda \subset G$ is *uniformly locally finite* (or *weakly uniformly discrete*) if, for all compact $W \subset G$, we have $\sup_{t\in G} \operatorname{card}\left((t+W)\cap \varLambda\right) \le C_W < \infty$ for some constant C_W which only depends on W.

Remark 5.16. Every uniformly discrete set is uniformly locally finite. There are sets which are uniformly locally finite but not uniformly discrete, *e.g.*, $\mathbb{Z} \cup \{n + \frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$. If G is σ-compact, then every uniformly locally finite set is countable. We note that a set $\varLambda$ is relatively dense if there is a compact set W such that $\inf_{t\in G} \operatorname{card}\left((t+W)\cap\varLambda\right) \ge 1$.

Let G be a σLCAG with Haar measures μ_G. Since G is a σLCAG, there also exists a sequence $\{A'_n\}_{n\in\mathbb{N}}$ of relatively compact open subsets of G such that $X = \bigcup_{i=1}^{\infty} A'_n$ and $\operatorname{cl} A'_n \subset A'_{n+1}$ for each $n \in \mathbb{N}$ by Lemma 2.9. In particular, we also have $0 < \mu_G(A'_n) < \infty$ for all $n \in \mathbb{N}$ (noting that the Haar measure $\mu_G(A'_n)$ is always finite since $\operatorname{cl} A'_n$ is compact for all $n \in \mathbb{N}$ and it is positive since A'_n is nonempty and open for all $n \in \mathbb{N}$). Note that for $G = \mathbb{R}^d$ we may take $A'_n = B_{<n}(0)$ for all $n \in \mathbb{N}$.

We now want to take averages, wherefore we define (appropriate) van Hove sequences in G.

Definition 5.17. Let G be a σLCAG. If A and W are compact subsets of G, the *W-boundary* of A is

$$\partial^W A = ((W + A) \setminus \operatorname{int} A) \ \cup \ ((-W + \operatorname{cl} A^c) \cap A)\,.$$

Therefore, the W-boundary of A consists of the points that have fallen out of A by adding points of W and the points of A that come into A from outside by adding points of W (observing that $\partial^{\{0\}} A = \partial A$, it can be understood as some sort of W-thickening of ∂A). Moreover, $\partial^W A$ is compact. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of relatively compact subsets of G is called a *van Hove sequence* if $\mu_G(A_n) > 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\mu_G(\partial^W \operatorname{cl} A_n)}{\mu_G(A_n)} = 0$$

for every compact $W \subset G$. Moreover, if we have $\operatorname{cl} A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, then we say that the van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$ is *nested*. Furthermore, we will assume that (for every G) we have fixed a (nested) van Hove sequence with the additional property that there is a constant C such that

$$\mu_G(A_n - A_n) \le C \cdot \mu_G(A_n) \qquad \textbf{(vH)}$$

for all $n \in \mathbb{N}$.

Remark 5.18. Of course, one uses the σ-compactness of G to prove that such a nested van Hove sequence with Property **(vH)** exists. Ideally, one would like to use the sequence $\{A'_n\}_{n\in\mathbb{N}}$, but in fact one needs the structure theorem for compactly generated Abelian groups (see Definition 2.53), see [329, Page 145] and [169, Section 9]: Every set $\operatorname{cl} A'_n$ generates a compactly generated open subgroup $G_n \subset G$. Therefore, every such G_n is of the form $\mathbb{Z}^d \times \mathbb{R}^{d'} \times K$, where d, d' are integers (depending on n) and K is a compact group (depending on n), see [169, Theorem 9.6]. But then there is an $m = m(n) \in \mathbb{N}$ such that the compact set

$$A_n = \{(x\,, y\,, z) \in \mathbb{Z}^d \times \mathbb{R}^{d'} \times K \mid |x_i| \le m,\ |y_j| \le m\}$$

fulfils $\mu_G(A_n - A_n) \le 2^{d+d'} \cdot \mu_G(A_n)$ and $\mu_G(\partial^{(\operatorname{cl} A'_n)} A_n)/\mu_G(A_n) \le \frac{1}{n}$. So, $\{A_n\}_{n\in\mathbb{N}}$ is a nested van Hove sequence with Property **(vH)** (since for every compact set W there is an $N \in \mathbb{N}$ such that $W \subset \operatorname{cl} A'_n$ for all $n \ge N$).

We are now able to take averages in uniformly locally finite sets.

Definition 5.19. Let $\varLambda$ be a uniformly locally finite subset of a σLCAG G. Let $\{A_n\}_{n\in\mathbb{N}}$ be a nested van Hove sequence as before. We set

$$\eta_\varLambda^{(n)}(z) = \frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,, y) \in G \times G \mid x, y \in \varLambda \cap A_n,\ x - y = z\}.$$

If the limit $\eta_\varLambda(z) = \lim_{n\to\infty} \eta_\varLambda^{(n)}(z)$ exists for all $z \in G$, we call $\eta_\varLambda(z)$ the *autocorrelation coefficient* of $\varLambda$ at z.

Remark 5.20. We use the observation of [36, Remark to Axiom 2 (p. 65)]: By properly thinning out the sequence $\{A_n\}_{n\in\mathbb{N}}$ (which then is still a nested van Hove sequence with Property **(vH)**), one can always achieve that the autocorrelation coefficients exist for all $z \in G$. In the following, we will therefore always use a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that the autocorrelation coefficients exist for the given set $\varLambda$.

Definition 5.21. Let $(G\,,H\,,\tilde{L})$ be a CPS. Let $\varOmega \subset H$, $(t\,,t') \in G \times H$ and $n \in \mathbb{N}$, then we define

$$d^n(\varOmega) = \frac{\operatorname{card}\left(\tilde{L} \cap (A_n \times \varOmega)\right)}{\mu_G(A_n)} = \frac{\operatorname{card}\left(\Lambda(\varOmega) \cap A_n\right)}{\mu_G(A_n)}$$

and

$$d^n_{(t\,,t')}(\varOmega) = \frac{\operatorname{card}\left(\tilde{L} \cap ((A_n + t) \times (\varOmega + t'))\right)}{\mu_G(A_n)} = \frac{\operatorname{card}\left(\Lambda(\varOmega + t') \cap (A_n + t)\right)}{\mu_G(A_n)}.$$

We say that $\varOmega$ has the *uniform density property*, or is a *UDP set*, if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\left| d^n_{(t\,,t')}(\varOmega) - \frac{\mu_H(\varOmega)}{\mu_{G\times H}(\operatorname{FD}(\tilde{L}))} \right| < \varepsilon$$

for all $n \geq N$ and all $(t\,,t') \in G \times H$.

In plain words, $\lim_{n\to\infty} d^n(\varOmega)$ is the average number of points of $\Lambda(\varOmega)$ "per unit area" (the density of $\Lambda(\varOmega)$). Thus, although we have defined UDP for a set $\varOmega$ in H, it is actually a property of a model set $\Lambda(\varOmega)$ within a CPS.

In a CPS, we can relate the autocorrelation coefficients and the numbers defined here, also see [261, Prop. 3].

Lemma 5.22. *Let $(G\,,H\,,\tilde{L})$ be a CPS. Let $S \subset H$ and assume that $\varLambda = \Lambda(S)$ is uniformly locally finite. Then*

$$\begin{aligned}\eta_\varLambda^{(n)}(z) &\leq d^n(S \cap (S + z^\star)) \leq \eta_\varLambda^{(n)}(z) + \\ &\frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,,y) \in G \times G \mid x \in \Lambda(S) \cap A_n,\ y \in \Lambda(S) \cap ((A_n - z) \setminus A_n),\ z = x - y\}.\end{aligned}$$

Moreover, we have

$$\left|\eta_\varLambda^{(n)}(z) - d^n(S \cap (S + z^\star))\right| \leq \frac{\operatorname{card}\left(\tilde{L} \cap (\partial^{\{-z\}} A_n \times S)\right)}{\mu_G(A_n)}.$$

So, if S is relatively compact, we have $\eta_\varLambda(z) = \lim_{n\to\infty} d^n(S \cap (S + z^\star))$.

Proof. We have (we note that $\tilde{L}$ is an additive subgroup of $G \times H$, so if $\tilde{x}, \tilde{y} \in \tilde{L}$ then also $\tilde{x} - \tilde{y} \in \tilde{L}$)

$$\begin{aligned}\eta_\varLambda^{(n)}(z) &= \frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,,y) \in G \times G \mid x, y \in \Lambda(S) \cap A_n,\ x - y = z\} \\ &= \frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,,y) \in G \times G \mid \tilde{x}, \tilde{y} \in \tilde{L} \cap (A_n \times S),\ \tilde{x} - \tilde{y} = \tilde{z}\} \\ &= \frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,,y) \in G \times G \mid \tilde{x} \in \tilde{L} \cap (A_n \times S),\ \tilde{x} \in \tilde{L} \cap ((A_n + z) \times (S + z^\star)), \\ &\qquad\qquad \tilde{y} = \tilde{x} - \tilde{z}\} \\ &\leq \frac{1}{\mu_G(A_n)} \operatorname{card}\{(x\,,y) \in G \times G \mid \tilde{x} \in \tilde{L} \cap (A_n \times (S \cap (S + z^\star))),\ \tilde{y} = \tilde{x} - \tilde{z}\} \\ &\leq \frac{1}{\mu_G(A_n)} \operatorname{card}\{x \in G \mid \tilde{x} \in \tilde{L} \cap (A_n \times (S \cap (S + z^\star)))\} = d^n(S \cap (S + z^\star)).\end{aligned}$$

We also observe, that to every $z^\star = x^\star - y^\star$ there exists (at least) one z such that $(z\,,z^\star) \in \tilde{L}$. We fix such a z in the following and obtain

$$\begin{aligned}
d^n(S\cap(S+z^\star)) &= \frac{1}{\mu_G(A_n)}\operatorname{card}\{x\in G \mid \tilde{x}\in\tilde{L}\cap(A_n\times(S\cap(S+z^\star)))\}\\
&= \frac{1}{\mu_G(A_n)}\operatorname{card}\{x\in G \mid \tilde{x}\in\tilde{L}\cap(A_n\times S),\ \tilde{x}\in\tilde{L}\cap(A_n\times(S+z^\star))\}\\
&\le \frac{1}{\mu_G(A_n)}\operatorname{card}\{x\in G \mid \tilde{x}\in\tilde{L}\cap(A_n\times S),\\
&\qquad\qquad \tilde{x}\in\tilde{L}\cap((A_n+(A_n+z))\times(S+z^\star))\}\\
&= \frac{1}{\mu_G(A_n)}\operatorname{card}\{(x\,,y)\in G\times G \mid \tilde{x}\in\tilde{L}\cap(A_n\times S),\\
&\qquad\qquad \tilde{y}\in\tilde{L}\cap(((A_n-z)+A_n)\times S),\ \tilde{y}=\tilde{x}-\tilde{z}\}\\
&\le \frac{1}{\mu_G(A_n)}\operatorname{card}\{(x\,,y)\in G\times G \mid \tilde{x},\,\tilde{y}\in\tilde{L}\cap(A_n\times S),\ \tilde{y}=\tilde{x}-\tilde{z}\}\\
&\quad+\frac{1}{\mu_G(A_n)}\operatorname{card}\{(x\,,y)\in G\times G \mid \tilde{x}\in\tilde{L}\cap(A_n\times S),\\
&\qquad\qquad \tilde{y}\in\tilde{L}\cap(((A_n-z)\setminus A_n)\times S),\ \tilde{y}=\tilde{x}-\tilde{z}\},
\end{aligned}$$

which proves the first inequality.

We have $y\in(A_n-z)\setminus A_n\subset(-z+A_n)\setminus\operatorname{int}A_n$, and therefore also $x=z+y\in A_n\setminus(A_n+z)=A_n\cap(A_n^c+z)\subset(z+\operatorname{cl}A_n^c)\cap A_n$. This yields

$$\begin{aligned}
\left|\eta_\Lambda^{(n)}(z)-d^n(S\cap(S+z^\star))\right| &\le \frac{\operatorname{card}\{(x\,,y)\in G\times G \mid x,y\in\Lambda(S)\cap\partial^{\{-z\}}A_n,\ z=x-y\}}{\mu_G(A_n)}\\
&\le \frac{\operatorname{card}\{x\in G \mid x\in\Lambda(S)\cap\partial^{\{-z\}}A_n\}}{\mu_G(A_n)}\\
&= \frac{\operatorname{card}\left(\tilde{L}\cap(\partial^{\{-z\}}A_n\times S)\right)}{\mu_G(A_n)}.
\end{aligned}$$

We now want to estimate the number $\operatorname{card}\left(\tilde{L}\cap(\partial^{\{-z\}}A_n\times S)\right)$. As in the proof of Lemma 5.8, there is a compact set $W\subset G\times H$ such that $\tilde{L}+W=G\times H$. Setting $W_1=\pi_1(W)$ and $W_2=\pi_2(W)$, we have $W\subset W_1\times W_2$. W.l.o.g., we may assume that $0\in W$ and that $W_1=-W_1$. Moreover, we can choose a fundamental domain $\mathrm{FD}(\tilde{L})$ such that $\mathrm{FD}(\tilde{L})\subset W$. If S is relatively compact, so is $S+W_2$ and we have $S\subset S+W_2$. Similarly, we have $\partial^{\{-z\}}A_n\subset\partial^{W_1-z}A_n$. We have now achieved that if $x\in\tilde{L}\cap(\partial^{\{-z\}}A_n\times S)$ then $x+(W_1\times W_2)\subset\left(\partial^{W_1-z}A_n\times(S+W_2)\right)$. Therefore, we can bound the above number by the Haar measures of the involved sets as

$$\frac{1}{\mu_G(A_n)}\operatorname{card}\left(\tilde{L}\cap(\partial^{\{-z\}}A_n\times S)\right)\le\frac{\mu_H(S+W_2)}{\mu_{G\times H}(\mathrm{FD}(\tilde{L}))}\cdot\frac{\mu_G(\partial^{W_1-z}A_n)}{\mu_G(A_n)}.$$

The first factor is a constant (independent of n), while the second factor vanishes for $n\to\infty$ by the definition of the van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$. This proves the claim. □

In particular, this last lemma shows that if $S \cap (S + z^\star)$ is a UDP set, then the autocorrelation coefficient $\eta_\Lambda(z)$ is given by $\mu_H(S \cap (S + z^\star))/\mu_{G\times H}(\mathrm{FD}(\tilde{L}))$. Actually, the "uniformness" is not even needed (see Chapter 5a), but for now we are interested in knowing which sets are UDP sets. The following statement is a consequence of the Birkhoff ergodic theorem (where one uses Property **(vH)**), and can be found in [261, Theorem 1].

Lemma 5.23. *Let $(G\,, H\,, \tilde{L})$ be a CPS. If $\varOmega \subset H$ is a measurable relatively compact set whose boundary $\partial\varOmega$ has Haar measure 0, then $\varOmega$ is a UDP set.* □

Remark 5.24. We note that a compact set is Riemann measurable iff its boundary has Haar measure 0. Here, a set is Riemann measurable if its characteristic function can be approximated appropriately by continuous functions from below and above. Therefore, sometimes the formulation "Riemann measurable window" is used in the situation of the last statement, *e.g.*, compare [175].

Remark 5.25. The proof of this Lemma in [261] is held in the spirit of [175, Section 4], where it is proven if both G and H are Euclidean spaces. However, the generalisation to general LCAGs G and H is already indicated in [175, Section 5]. Independently, the proof of the last lemma can also be found in [328, Sections 2–4] if G is Euclidean and H is a general LCAG. We borrowed the name "UDP set" from this last article.

As a consequence, we have now obtained the *density formula*, compare to [328, Theorem 1], [175, Prop. 4.2] and [261, Theorem 1].

Theorem 5.26. *Let $(G\,, H\,, \tilde{L})$ be a CPS. If $\varOmega$ is a relatively compact subset of H with almost no boundary, i.e., $\mu_H(\partial\varOmega) = 0$, then*

$$\lim_{n\to\infty} d^n_{(t,t')}(\varOmega) = \lim_{n\to\infty} \frac{\mathrm{card}\left(\tilde{L} \cap ((t + A_n) \times (t' + \varOmega))\right)}{\mu_G(A_n)} = \frac{\mu_H(\varOmega)}{\mu_{G\times H}(\mathrm{FD}(\tilde{L}))}$$

uniformly in $(t\,, t') \in G \times H$. □

Corollary 5.27. *Assume the setting of Theorem 5.26. Then the autocorrelation coefficient at $z \in L$ is given by*

$$\eta_\Lambda(z) = \frac{\mu_H(\varOmega \cap (\varOmega - z^\star))}{\mu_{G\times H}(\mathrm{FD}(\tilde{L}))}.$$

If $z \notin L$, then we have $\eta_\Lambda(z) = 0$.

Proof. If $\varOmega$ is relatively compact with almost no boundary, so is $\varOmega \cap (\varOmega + t)$ for every $t \in H$ (and thus also $((\varOmega - t) \cap \varOmega) + t$). So the density formula applies. If $z \notin L$, then we also have $z \notin \Lambda - \Lambda$, *i.e.*, there are no $x, y \in G$ such that $x, y \in \Lambda$ with $x - y = z$. □

Note that one may also write $\eta_\Lambda(z) = \mu_H(\varOmega \cap (\varOmega + z^\star))/\mu_{G\times H}(\mathrm{FD}(\tilde{L}))$. However, with the minus sign the interpretation is clearer: The autocorrelation coefficient at z arises from the (projected) lattice points that fall inside the cut strip and stay inside the cut strip after translation by $\tilde{z} = (z\,, z^\star)$.

For the use in the following sections, we now define multi-component versions of the sets we have considered so far.

Definition 5.28. We say that a finite family $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ of Delone sets Λ_i $(1 \le i \le n)$ is a *multi-component Delone set* if $\operatorname{supp} \underline{\Lambda} = \bigcup_{i=1}^n \Lambda_i$ is a Delone set. We say that a multi-component Delone set $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ is a *multi-component Meyer set* if each set Λ_i $(1 \le i \le n)$ and[2] $\operatorname{supp} \underline{\Lambda}$ is a Meyer set. Moreover, we say that a finite family $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ of model sets $\Lambda_i = x + \Lambda(\Omega_i)$ $(1 \le i \le n)$ with respect to the same CPS is a *multi-component model set*. In this case, we also write $\underline{\Lambda} = x + \Lambda(\underline{\Omega})$. Moreover, if $\underline{\Lambda}$ is a multi-component (inter) model set such that each component Λ_i $(1 \le i \le n)$ is a regular (inter) model set, then we say that $\underline{\Lambda}$ is a *regular multi-component (inter) model set*.

5.2. Pure Point Diffractive Sets

Definition 5.29. We say that a subset Λ of an LCAG G is *locally finite*, if for any compact set W in G we have $\operatorname{card}(W \cap \Lambda) < \infty$. We note that every uniformly locally finite set (see Definition 5.15) is also locally finite. Let Λ be a locally finite subset of an LCAG G. We say that Λ has *finite local complexity*, or is an *FLC set*[3], if its *difference set* $\Delta = \Lambda - \Lambda$ is closed and discrete. Equivalently, Λ is an FLC set if its difference set Δ is locally finite.

If Λ is uniformly locally finite and G is a σLCAG, we also define the *essential difference set* $\Delta^{\text{ess}} = \{z \in G \mid \eta_\Lambda(z) \neq 0\}$. Obviously, we have $\Delta^{\text{ess}} \subset \Delta$. Moreover, we denote the group generated by Δ^{ess} by $\mathcal{L}$, *i.e.*, $\mathcal{L} = \langle \Delta^{\text{ess}} \rangle_{\mathbb{Z}}$, which we may interpret as a subgroup of G.

Example 5.30. Define for every $n \in \mathbb{N}$ the set

$$\Lambda_n = \{(n-1) + \frac{r}{n} \mid 0 \le r < n\} \cup \{-\left((n-1) + \frac{r}{n}\right) \mid 0 \le r < n\}.$$

Then $\Lambda = \bigcup_{n \in \mathbb{N}} \Lambda_n$ is locally finite, but not uniformly locally finite.

The next statement shows that FLC already implies uniform local finiteness.

Proposition 5.31. *Any relatively dense FLC set Λ is uniformly locally finite (and therefore, in particular, locally finite).*

Proof. Since Λ is relatively dense, there exists a compact set W' such that $\Lambda + W' = G$. We want to show that, for all compact sets $W \subset G$, we have $\sup_{t \in G} \operatorname{card}(\Lambda \cap (t + W)) \le C_W < \infty$ for some constant C_W. By $t \in G = \Lambda + W'$, we can choose for any $t \in G$ elements $\lambda_t \in \Lambda$ and $w_t \in W'$ such that $t = \lambda_t + w_t$. Thus, we have

$$\Lambda \cap (W + t) = \Lambda \cap (W + \lambda_t + w_t) = ((\Lambda - \lambda_t) \cap (W + w_t)) + \lambda_t.$$

Consequently, we have the following estimate

$$\operatorname{card}(\Lambda \cap (W + t)) \le \operatorname{card}\left(((\Lambda - \Lambda) \cap (W + W')) + \lambda_t\right).$$

As sum of two compact sets, $W + W'$ is also compact, wherefore by FLC one has

$$\operatorname{card}(\Lambda \cap (W + t)) \le \operatorname{card}\left(((\Lambda - \Lambda) \cap (W + W')) + \lambda_t\right) = \operatorname{card}\left((\Lambda - \Lambda) \cap (W + W')\right) < \infty,$$

and this estimate is independent of t. This establishes the claim. □

[2] In [233], multi-component Delone (respectively Meyer) sets are defined without the condition that the support is also Delone (respectively Meyer), see [233, Definition 2.1]. However, in that case it is not clear if statements like Proposition 5.13 generalise to the multi-component case.

[3] A Delone set Λ which is an FLC set is also called a *Delone set of finite type*, see [218, Def. 2.1(ii)].

Definition 5.32. An FLC set $\varLambda$ in G is called *repetitive* if, for every compact set $W \subset G$, the set of repetitions $\varLambda \cap W$, *i.e.*,

$$\{t \in G \mid (\varLambda - t) \cap W = \varLambda \cap W\}$$

is relatively dense in G.

We also formulate the properties "FLC set" and "repetitivity" in the multi-component case. But first we note the following equivalent characterisation of an FLC set, which also justifies the term "finite local complexity", see [329, Prop. 2.3].

Lemma 5.33. *If $\varLambda$ is a locally finite set in G, the following properties are equivalent:*

- *$\varLambda$ is an FLC set.*
- *For every compact $W \subset G$, there is a compact $W' \subset G$ such that, for every $t \in G$, there is some $t' \in W'$ with $(\varLambda - t) \cap W = (\varLambda - t') \cap W$.*

Moreover, this last property can also be stated as:

- *For every compact $W \subset G$, there is a finite set $Y \subset \varLambda$ such that, for every $t \in \varLambda$, there is some $t' \in Y$ with $(\varLambda - t) \cap W = (\varLambda - t') \cap W$.* □

In plain words, this equivalent statement says that, up to translations, there are only finitely many different patterns in $\varLambda$ of any given "finite size" W, *i.e.*, $\operatorname{card}\{(\varLambda - t) \cap W \mid t \in \varLambda\} < \infty$.

Definition 5.34. Let $\underline{\varLambda}$ be a *multi-component locally finite set*, *i.e.*, each $\varLambda_i$ is a locally finite set (obviously, this is the case iff $\operatorname{supp} \underline{\varLambda}$ is locally finite); in particular, this is the case if $\underline{\varLambda}$ is a multi-component Delone set. We say that $\underline{\varLambda}$ is an *FLC multi-component set* if, for every compact $W \subset G$, there is a finite set $Y \subset \operatorname{supp} \underline{\varLambda}$ such that, for every $t \in \operatorname{supp} \underline{\varLambda}$, there is some $t' \in Y$ with

$$(\underline{\varLambda} - t) \cap W = ((\varLambda_i - t) \cap W)_{i=1}^{n} = \big((\varLambda_i - t') \cap W\big)_{i=1}^{n} = (\underline{\varLambda} - t') \cap W.$$

Obviously, in this case, each $\varLambda_i$ itself as well as $\operatorname{supp} \underline{\varLambda}$ are FLC sets. Furthermore, we say that an FLC multi-component set $\underline{\varLambda}$ is *repetitive* if, for every compact set $W \subset G$, the set of repetitions of $\underline{\varLambda} \cap W$, *i.e.*,

$$\{t \in G \mid (\underline{\varLambda} - t) \cap W = \underline{\varLambda} \cap W\} = \{t \in G \mid ((\varLambda_i - t) \cap W)_{i=1}^{n} = (\varLambda_i \cap W)_{i=1}^{n}\}$$

is relatively dense in G. We note that the set of repetitions of $\underline{\varLambda} \cap W$ is the intersection over $1 \leq i \leq n$ of the set of repetitions of $\varLambda_i \cap W$, wherefore, if $\underline{\varLambda}$ is repetitive, each $\varLambda_i$ and $\operatorname{supp} \underline{\varLambda}$ are repetitive.

Remark 5.35. If each $\varLambda_i$ (and even $\operatorname{supp} \underline{\varLambda}$) is repetitive, $\underline{\varLambda}$ need not[4] be repetitive. Similarly, if each $\varLambda_i$ is FLC, then $\operatorname{supp} \underline{\varLambda}$ (and therefore also $\underline{\varLambda}$) is not necessarily FLC. We suspect that this is the case for a multi-component set in G whose components are model sets with different ("not related") internal spaces and lattices.

[4] We refer to a variant of the chair tiling in Example 6b.20 for such an example [229]: Using the substitution on $\mathbb{Z}^2$ with the alphabet $\mathcal{A} = \{a, b, c, d\}$ (see Example 6b.20), the diagonal in the first quadrant (*i.e.*, the points $\{(m\,, m) \in \mathbb{Z}^2 \mid m \in \mathbb{N}\}$) belongs to $\varLambda_a$. Similarly, the diagonal in the second quadrant belongs to $\varLambda_b$, the diagonal in the third quadrant to $\varLambda_c$ and the one in the fourth to $\varLambda_d$ (note that the origin belongs to $\varLambda_a$). Now, we change $\varLambda_a$ and $\varLambda_c$, such that all points of the diagonal $\{(m\,, m) \in \mathbb{Z}^2 \mid m \in \mathbb{Z}\}$ belong to $\varLambda_a$, but none to $\varLambda_c$. One now checks that these changed sets are still repetitive (they belong to the hulls $\mathbb{X}(\varLambda_a)$ respectively $\mathbb{X}(\varLambda_c)$, compare Lemma 5.108), but $\underline{\varLambda}$ is not repetitive (here, the point $\{-1\,, -1\}$ destroys the repetitivity; moreover, this changed multi-component set does not belong to the the hull $\mathbb{X}(\underline{\varLambda})$). This can be seen by recovering the associated "chair" tiling (modulo the tile at $\{-1\,, -1\}$) again.

However, we note that we have the following equivalent characterisation of FLC in the multi-component case.

Lemma 5.36. *Let* $\underline{\Lambda}$ *be a multi-component locally finite set. Then* $\underline{\Lambda}$ *is an FLC multi--component set iff* $\operatorname{supp}\underline{\Lambda}$ *is a locally finite FLC set.*

Proof. Obviously, we only have to show the "only if" part. To this end, let $\operatorname{supp}\underline{\Lambda}$ be a locally finite FLC set. Then, for every compact set $W \subset G$, there is a finite set $Y_W \subset \operatorname{supp}\underline{\Lambda}$ (depending only on W) such that for all $s \in \operatorname{supp}\underline{\Lambda}$ there is an $s' \in Y_W$ with $(\operatorname{supp}\underline{\Lambda}-s)\cap W = (\operatorname{supp}\underline{\Lambda}-s')\cap W$. By local finiteness, each such pattern has only finite cardinality, wherefore there is a constant C_W (depending only on W) such that $\operatorname{card}\{(\operatorname{supp}\underline{\Lambda}-t)\cap W \,|\, t \in G\} \leq C_W$ for $t \in \operatorname{supp}\underline{\Lambda}$ (this does not imply uniform local finiteness). Obviously, every Λ_i has this property, *i.e.*, $\operatorname{card}\{(\Lambda_i - t)\cap W \mid t \in G\} \leq C_W$ for $t \in \operatorname{supp}\underline{\Lambda}$. Moreover, every Λ_i is also an FLC set: Given $s \in \operatorname{supp}\underline{\Lambda}$, there are at most C_W points in $(\operatorname{supp}\underline{\Lambda} - s)\cap W$, of which at most C_W are also points in Λ_i. If $s \in \Lambda_i$, there are at most $\sum_{j=0}^{C_W-1}\binom{C_W-1}{j} = 2^{C_W-1}$ possible different patterns in Λ_i which are consistent with a single pattern of $\operatorname{supp}\underline{\Lambda}$ (always of "size" W). But there are only $\operatorname{card}Y_W$ patterns of "size" W in $\operatorname{supp}\underline{\Lambda}$, wherefore we have the estimate $\operatorname{card}\{(\Lambda_i - s)\cap W \,|\, s \in \Lambda_i\} \leq (\operatorname{card}Y_W)\cdot 2^{C_W-1}$ and (since this is a finite number for every compact set W) consequently each Λ_i is FLC.

Similarly, $\underline{\Lambda}$ is FLC: Given $s \in \operatorname{supp}\underline{\Lambda}$, there are at most C_W points in $(\operatorname{supp}\underline{\Lambda} - s)\cap W$ and they may all belong to each set Λ_i (and every point to at least one Λ_i), wherefore we can bound the number of patterns in $(\underline{\Lambda} - s)\cap W$ by $n \cdot 2^{C_W}$, and consequently the total number of patterns of "size" W by $(\operatorname{card}Y_W)\cdot n \cdot 2^{C_W}$. This proves the claim. □

We now return to the "single-component" case.

Remark 5.37. By the choice of the van Hove sequence used, the autocorrelation coefficients $\eta_\Lambda(z)$ exist for all $z \in G$ for a uniformly locally finite subset Λ of a σLCAG G. If Λ is also repetitive, then $\eta_\Lambda(z) > 0$ for all $z \in \Lambda - \Lambda$. So, in this case we have $\Delta^{\text{ess}} = \Delta$.

We remark that the definition of an IMS (see Equation (5.2) on p. 130) is basically chosen in such a way to ensure that to a compact and regularly closed window Ω there exists a repetitive IMS Λ'; for details compare Lemma 5.118, Proposition 7.54 and the references there.

Definition 5.38. Let Λ be an FLC set such that its density $\operatorname{dens}\Lambda = \lim_{n\to\infty}\operatorname{card}(\Lambda \cap A_n)/\mu_G(A_n) = \eta_\Lambda(0) > 0$ (then its essential difference set Δ^{ess} is nonempty). We define a nonnegative translation-invariant symmetric function ϱ_Λ on $G \times G$ by

$$\varrho_\Lambda(x\,,y) = 1 - \frac{\eta_\Lambda(x-y)}{\eta_\Lambda(0)}.$$

In fact, it even satisfies the triangle inequality[5], wherefore it is a translation-invariant pseudometric on G, respectively on $\mathcal{L}$. Following [250, Theorem 1], we call ϱ_Λ the *variogram* of Λ.

[5]The function ϱ_Λ satisfies the triangle inequality $\varrho_\Lambda(x\,,z) \leq \varrho_\Lambda(x\,,y) + \varrho_\Lambda(y\,,z)$: If we can show that

$$\begin{aligned}\operatorname{card}\{(x'\,,y') \in G\times G | x',y' \in \Lambda\cap A_n,\ x'-y' = x-y\} + \operatorname{card}\{(y''\,,z'') \in G\times G | y'',z'' \in \Lambda\cap A_n,\ y''-z'' = y-z\}\\ \leq \operatorname{card}\{(x'''\,,z''') \in G\times G \mid x''',z''' \in \Lambda\cap A_n,\ x'''-z''' = x-z\} + \operatorname{card}(\Lambda\cap A_n)\end{aligned}$$

for all $n \in \mathbb{N}$, then taking the limit $n \to \infty$ and by the definition of η_Λ and ϱ_Λ establishes the triangle inequality. Let us denote by χ_S the *indicator function* or *characteristic function* of a set S, *i.e.*, $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. Note that for a finite set $S \subset G$ we have $\operatorname{card}S = \sum_{x\in G}\chi_S(x)$. Let us fix

As in Example 2.59, the family $\mathcal{F}$ of all sets the form $\{(x\,,y) \in G \times G \mid \varrho_\Lambda(x\,,y) \le \varepsilon\}$, where ε runs through all positive real numbers, is a uniformity base on G (similarly, the family of sets $\{(x\,,y) \in \mathcal{L} \times \mathcal{L} \mid \varrho_\Lambda(x\,,y) \le \varepsilon\}$ is a uniformity base on $\mathcal{L}$). We call the topology defined by this uniformity the *autocorrelation topology* on G (respectively $\mathcal{L}$), or *AC topology* for short. Since this topology coincides with the topology defined by the pseudometric itself, we define, for $\varepsilon > 0$, the set $P_\varepsilon = P_\varepsilon(\Lambda)$ of *ε-almost periods* of Λ on G (respectively $\mathcal{L}$) through

$$P_\varepsilon = \{t \in G \mid \varrho_\Lambda(t\,,0) < \varepsilon\}.$$

We note that for $\varepsilon > 1$, we have $P_\varepsilon = G$, respectively $P_\varepsilon = \mathcal{L}$ depending on which space we use for ϱ, while for $0 < \varepsilon \le 1$ we have (in either case) $P_\varepsilon \subset \Delta^{\mathrm{ess}}$. Obviously, the sets P_ε are symmetric and (by the triangle inequality) satisfy $P_\varepsilon + P_{\varepsilon'} \subset P_{\varepsilon+\varepsilon'}$, see [36, Fact 2]. In fact, Proposition 2.20 establishes the following.

Lemma 5.39. *The family $\mathcal{B} = \{P_\varepsilon \mid \varepsilon > 0\}$ is a neighbourhood base of 0 consisting of open sets in the AC topology on G respectively on $\mathcal{L}$.* □

We will later continue from this last observation. Now, we define the following property of a set.

Definition 5.40. Let Λ be a (uniformly locally finite) FLC set in a σLCAG G. If for all $\varepsilon > 0$ the sets $P_\varepsilon = P_\varepsilon(\Lambda)$ are Delone sets, then we say that Λ is *strictly pure point diffractive* or a *SPPD set* for short. Note that the uniform local finiteness follows automatically from Proposition 5.31.

Remark 5.41. The name “pure point diffractive” in this case is suggested by [36, Theorem 1], see Theorem 7.10. For the justification of “strictly”, see Remark 7.38.

A cornerstone in the theory of model sets is the following statement.

Proposition 5.42. [329, Theorem 4.5(2)] *Let $(G\,,H\,,\tilde{L})$ be a CPS and Λ a regular (inter) model set. Then Λ is an SPPD set.* □

Sketch of Proof. Originally, the previous statement is proven in terms of dynamical systems. We are now able to sketch a topological proof, also compare [261, Corollary 1]: Since the window Ω is compact, any IMS Λ is (a subset of and therefore also a) uniformly discrete and thus uniformly locally finite. The difference set Δ of Λ fulfils

$$\Lambda(\operatorname{int}\Omega - \operatorname{int}\Omega) \subset \Delta \subset \Lambda(\Omega - \Omega),$$

wherefore it is a Delone set (noting that $\Omega - \Omega$ is totally bounded and therefore relatively compact); this establishes that Λ is an FLC set. Since $P_\varepsilon \subset \Delta$, the ε-almost periods P_ε are also uniformly discrete for all $\varepsilon > 0$. So it remains to check that the ε-almost periods are relatively dense for all $\varepsilon > 0$.

We first note that $\operatorname{int}\Omega$ and Ω are (by assumption) relatively compact sets with almost no boundary, wherefore we have $\mu_H(\operatorname{int}\Omega) = \mu_H(\Omega)$, but also $\mu_H(\operatorname{int}\Omega \cap (t' + \operatorname{int}\Omega)) =$

$x, y, z \in G$. Then this last inequality is established by summing the following inequality over all $\check{y} \in G$:

$$\chi_{\Lambda\cap A_n}(\check{y}-y+x)\cdot\chi_{\Lambda\cap A_n}(\check{y})+\chi_{\Lambda\cap A_n}(\check{y})\cdot\chi_{\Lambda\cap A_n}(\check{y}-y+z) \le \chi_{\Lambda\cap A_n}(\check{y}-y+x)\cdot\chi_{\Lambda\cap A_n}(\check{y}-y+z)+\chi_{\Lambda\cap A_n}(\check{y})$$

This last inequality, however, is proven by considering the different cases. Consequently, the triangle inequality for ϱ_Λ holds.

$\mu_H(\varOmega \cap (t' + \varOmega))$ for all $t' \in H$. So, for a given (regular) window $\varOmega$, the autocorrelation coefficient (and the sets P_ε) of all its possible inter model sets are the same.

We are done, if we can show the following: Given $\varepsilon > 0$, there exists an open neighbourhood U_ε of 0 in H such that $\mu_H(\varOmega \cap (t' + \varOmega)) > (1 - \varepsilon) \cdot \mu_H(\varOmega)$ for all $t' \in U_\varepsilon$. Because in that case we have $\Lambda(U_\varepsilon) \subset P_\varepsilon$ and, since U_ε has nonempty interior, $\Lambda(U_\varepsilon)$ and therefore P_ε is relatively dense.

But if H is metrisable, Lemma 4.26 yields such an neighbourhood: We first note that $\operatorname{int} \varOmega$ is measurable and bounded, wherefore $(\operatorname{int} \varOmega)^c$ is closed. Then, for any $\varepsilon > 0$, there exists a compact (*i.e.*, closed and bounded), and therefore measurable, set $W \subset \operatorname{int} \varOmega$ such that $0 \le \mu_H(\varOmega) - \mu_H(W) = \mu_H(\operatorname{int} \varOmega) - \mu_H(W) = \mu_H(\operatorname{int} \varOmega \setminus W) < \varepsilon$. But then the compact set W has positive distance from the closed set $(\operatorname{int} \varOmega)^c$; we denote this distance by δ. It follows that $W \subset \operatorname{int} \varOmega \cap (t + \operatorname{int} \varOmega)$ for all $t \in B_{<\delta}(0)$, and therefore

$$\mu_H(\varOmega) - \varepsilon < \mu_H(W) \le \mu_H(\operatorname{int} \varOmega \cap (t + \operatorname{int} \varOmega) = \mu_H(\varOmega \cap (t + \varOmega)) \le \mu_H(\varOmega)$$

for all $t \in B_{<\delta}(0)$. This establishes that $B_{<\delta}(0)$ is such an open neighbourhood U_ε we are looking for. □

In the following, we will often deal with multi-component (inter) model sets and multi-component Meyer sets. Note that these are, in particular, FLC multi-component uniformly locally finite sets *i.e.*, each component is uniformly locally finite and, by Lemma 5.36, its support is an FLC set. Let $\underline{\varLambda} = (\varLambda_i)_{i=1}^n$ be such an FLC multi-component uniformly locally finite set in a σLCAG G. If for all $\varepsilon > 0$ the sets $P_\varepsilon^{(i)} = P_\varepsilon(\varLambda_i)$ $(1 \le i \le n)$ are Delone sets, then we say that each component of $\underline{\varLambda}$ is an SPPD set.
From the previous Proposition 5.42, the following statement follows immediately.

Corollary 5.43. *Let $(G, H, \tilde{L})$ be a CPS and $\underline{\varLambda}$ a regular multi-component (inter) model set. Then, each component of $\underline{\varLambda}$ is an SPPD set. In particular, for every $\varepsilon > 0$, there exists a common relatively dense (and also uniformly discrete) set P'_ε in G, such that $P'_\varepsilon \subset P_\varepsilon(\varLambda_i)$ for all $1 \le i \le n$. Moreover, $\operatorname{supp} \underline{\varLambda}$ is also an SPPD set and, for every $\varepsilon > 0$, the relatively dense sets P'_ε can be chosen such that $P'_\varepsilon \subset P_\varepsilon(\operatorname{supp} \underline{\varLambda})$.*

Proof. The last statement follows from the proof of Proposition 5.42: Given $\varepsilon > 0$, denote by $U_\varepsilon^{(i)}$ an open (and bounded) neighbourhood of 0 for the component i $(1 \le i \le n)$ such that $\mu_H(\varOmega_i \cap (t' + \varOmega_i)) > (1 - \varepsilon) \cdot \mu_H(\varOmega_i)$ for all $t \in U_\varepsilon^{(i)}$. Then $U_\varepsilon = \bigcap_{i=1}^n U_\varepsilon^{(i)}$ is also a nonempty open neighbourhood of 0 in H, and $P'_\varepsilon = \Lambda(U_\varepsilon)$ is such a relatively dense set (which is also uniformly discrete since U_ε is bounded).

For the statements about $\operatorname{supp} \underline{\varLambda}$, we note that if $\underline{\varLambda} = x + \Lambda(\underline{\varOmega})$ then $\operatorname{supp} \underline{\varLambda} = x + \Lambda(\bigcup_{i=1}^n \varOmega_i) = x + \Lambda(\operatorname{supp} \underline{\varOmega})$, wherefore it is also a regular model set. The statement about the ε-almost periods follows as before. □

In the next section we are interested in the converse task: Given an FLC multi-component set $\underline{\varLambda}$ such that for every $\varepsilon > 0$ there is a relatively dense set P'_ε with $P'_\varepsilon \subset P_\varepsilon(\varLambda_i)$ for all $1 \le i \le n$, we construct a CPS for which $\underline{\varLambda}$ might be a multi-component model set.

5.3. Construction of a Cut and Project Scheme

In this section, we review the following construction from [36, Section 3]: Let $\varLambda$ be a uniformly locally finite FLC set in a σLCAG G such that the sets $P_\varepsilon = P_\varepsilon(\varLambda)$ are relatively dense, then

we can associate a CPS $(G\,, H\,, \tilde{\mathcal{L}})$ with it, *i.e.*, we obtain an LCAG H and a lattice $\tilde{\mathcal{L}} \subset G \times H$. In particular, if Λ is pure point diffractive, we can associate a CPS with it.
Here, we will review the construction in the multi-component case. We first note the following:

- Let $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be an FLC multi-component uniformly locally finite set. On the one hand, as finite union of uniformly locally finite sets, $\underline{\Lambda}$ is also uniformly locally finite. In fact, it is enough if $\operatorname{supp} \underline{\Lambda}$ is uniformly locally finite. On the other hand, by Lemma 5.36, $\underline{\Lambda}$ is an FLC set iff $\operatorname{supp} \underline{\Lambda}$ is an FLC set. We also note that a multi-component IMS is an FLC multi-component uniformly locally finite set. Here we are interested in the converse: Which conditions ensure that $\underline{\Lambda}$ is a multi-component IMS?

- As in Definition 5.29, the difference sets $\Delta_i = \Lambda_i - \Lambda_i$ are closed and discrete. Moreover, we also have the essential difference sets $\Delta_i^{\text{ess}} = \{z \in G \mid \eta_{\Lambda_i}(z) \neq 0\}$ and the groups $\mathcal{L}_i = \langle \Delta_i^{\text{ess}} \rangle_{\mathbb{Z}}$ generated by them (always $1 \le i \le n$). Here, of course, $\eta_{\Lambda_i}(z)$ denotes the autocorrelation coefficient at z of the set Λ_i. As finite union of discrete and closed subsets of G, the set $\Delta' = \bigcup_{i=1}^{n} \Delta_i$ is also closed and discrete. Moreover, we define $(\Delta^{\text{ess}})' = \bigcup_{i=1}^{n} \Delta_i^{\text{ess}}$ and $\mathcal{L}' = \langle (\Delta^{\text{ess}})' \rangle_{\mathbb{Z}}$. We note that $\mathcal{L}' = \mathcal{L}_1 + \ldots + \mathcal{L}_n$. Furthermore, each $\mathcal{L}_i$ is a subgroup of $\mathcal{L}'$ which is a subgroup of G.

- As in Definition 5.38, we can define for each Λ_i a variogram ϱ_{Λ_i} and, for each $\varepsilon > 0$, the set of ε-almost periods $P_\varepsilon(\Lambda_i)$ on $\mathcal{L}_i$, respectively $\mathcal{L}'$, respectively G. We define[6] the *maximum variogram* $\varrho_{\underline{\Lambda}}$ (or the *variogram of* $\underline{\Lambda}$) on $G \times G$ (respectively on $\mathcal{L}' \times \mathcal{L}'$) by $\varrho_{\underline{\Lambda}}(x\,, y) = \max_{1 \le i \le n} \varrho_{\Lambda_i}(x\,, y)$. Consequently, for each $\varepsilon > 0$, we define and establish the set $P'_\varepsilon = P_\varepsilon(\underline{\Lambda})$ of ε-almost periods on $\mathcal{L}' \subset G$ by

$$P'_\varepsilon = \{t \in G \mid \varrho_{\underline{\Lambda}}(t\,, 0) < \varepsilon\} = \bigcap_{i=1}^{n} P_\varepsilon(\Lambda_i). \tag{5.3}$$

- To simplify the notation, we use $\Lambda = \operatorname{supp} \underline{\Lambda} = \bigcup_{i=1}^{n} \Lambda_i$ in the following. We may – without loss of generality – assume that $0 \in \Lambda$, and therefore that $0 \in \Lambda_i$ for at least one $i \in \{1, \ldots, n\}$. Otherwise, we choose a $t \in \Lambda$ and consider the multi-component set $\underline{\Lambda} - t = (\Lambda_i - t)_{i=1}^{n}$ (obviously, then we have $0 \in \operatorname{supp}(\underline{\Lambda} - t)$).

- We also define the sets $\Delta = \Lambda - \Lambda$ and $\Delta^{\text{ess}} = \{z \in G \mid \eta_\Lambda(z) \neq 0\} \subset \Delta$ and the group $\mathcal{L} = \langle \Delta^{\text{ess}} \rangle_{\mathbb{Z}}$. We observe that $\mathcal{L}'$ is a subgroup of $\mathcal{L}$ (since $\Delta_i^{\text{ess}} \subset \Delta^{\text{ess}}$ for all $1 \le i \le n$). Therefore, we equip $\mathcal{L}$ with the topology defined by the maximum variogram $\varrho_{\underline{\Lambda}}$. We call this the *AC topology* on $\mathcal{L}$.

We make the following additional assumption about the FLC multi-component uniformly locally finite set $\underline{\Lambda}$:

[6] In [233, Section 3], [231, Section 4] and [230, Section 5], a G-invariant pseudometric ϱ on the space of all locally finite multi-component sets is defined by

$$d_{\text{AC}}(\underline{\Lambda}'\,, \underline{\Lambda}'') = \limsup_{m \to \infty} \frac{\sum_{i=1}^{n} \operatorname{card}\left((\Lambda'_i \,\Delta\, \Lambda''_i) \cap A_m\right)}{\mu_G(A_m)},$$

where $\underline{\Lambda}', \underline{\Lambda}''$ denote such locally finite multi-component sets and $\{A_m\}$ a van Hove sequence on G. Consequently, the set of ε-almost periods of $\underline{\Lambda}$ is there defined as $\{t \in G \mid \varrho(t + \underline{\Lambda}, \underline{\Lambda}) \le \varepsilon\}$, which – up to equivalence in the definition of the used pseudometric – corresponds to our definition in Equation (5.3).

(As) For all $\varepsilon > 0$, the set P'_ε is relatively dense.

Assumption **(As)** is suggested by Corollary 5.43 for the multi-component case and will enable us to construct a CPS. We also note that **(As)** also implies that Λ_i and $\operatorname{supp} \underline{\Lambda}$ are relatively dense, wherefore Proposition 5.31 applies. We also make the following observations:

- If the sets Λ_i are repetitive, then we have $\Delta_i = \Delta_i^{\text{ess}}$ for all $1 \le i \le n$ and therefore $\Delta' = (\Delta^{\text{ess}})'$. Moreover, if $\Delta_i = \Delta_i^{\text{ess}}$ for all $1 \le i \le n$, then for each $t_i \in \Lambda_i$ we have $\Lambda_i \subset t_i + \mathcal{L}'$. If, in addition, $\underline{\Lambda}$ is a repetitive[7] FLC multi-component set (so we also have $\Delta^{\text{ess}} = \Delta$) then we may interpret such a $t_i + \mathcal{L}'$ as coset of the factor group $\mathcal{L}/\mathcal{L}'$ (of course, one gets the same coset for every $t_i \in \Lambda_i$). If there is a coset $t + \mathcal{L}'$ of the factor group $\mathcal{L}/\mathcal{L}'$ such that $\Lambda_i \cap t + \mathcal{L}' \neq \varnothing$, we say that Λ_i *belongs to the coset* $t + \mathcal{L}'$. Obviously, there might be more than one Λ_i that belong to a certain coset and, by assuming repetitivity and FLC of $\underline{\Lambda}$, each Λ_i belongs to only one coset.
- We can also define, for every $\varepsilon > 0$, the set $P_\varepsilon(\Lambda)$ of ε-almost periods of Λ. Observing that $0 < \eta_{\Lambda_i}(0) \le \eta_\Lambda(0) = \operatorname{dens} \Lambda \le \operatorname{dens} \Lambda_1 + \ldots + \operatorname{dens} \Lambda_n = \eta_{\Lambda_1}(0) + \ldots + \eta_{\Lambda_n}(0)$ and $\eta_\Lambda(z) \ge \max_{1 \le i \le n} \eta_{\Lambda_i}(z)$ shows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $P'_\delta \subset P_\varepsilon(\Lambda)$, wherefore, for all $\varepsilon > 0$, the set $P_\varepsilon(\Lambda)$ is also relatively dense by **(As)**. Note that the topology defined by the sets P'_ε is finer than the one defined by the sets $P_\varepsilon(\Lambda)$.

In view of these remarks, we note that $\mathcal{L}$ is the minimal subgroup of G such that (up to an overall translation) it catches all "relevant elements" of $\underline{\Lambda}$ (meaning all except possibly the ones which have vanishing autocorrelation with the rest of the structure, and therefore occur with zero density). Moreover, the AC topology on $\mathcal{L}$ is the coarsest topology which is finer than any AC topology defined by the components.

Remark 5.44. If $\underline{\Lambda}$ is a multi-component (inter) model set, so is Λ. The window of Λ is then given by the union of the windows of the components of $\underline{\Lambda}$. So in this case, Λ is a Meyer set, and therefore also an FLC set. Basically, we are interested in establishing which multi-component Meyer sets are multi-component model sets. Moreover, the sets we will consider satisfy $\Delta_i = \Delta_i^{\text{ess}}$ for all $1 \le i \le n$, because usually they are repetitive (or, at least, have the same autocorrelation coefficients as a repetitive set).

We now want to continue with Lemma 5.39, which now reads as follows.

Lemma 5.45. *The family* $\mathcal{B} = \{P'_\varepsilon \mid \varepsilon > 0\}$ *is a neighbourhood base of* 0 *consisting of open sets in the AC topology (defined by the maximum variogram as in Equation* (5.3) *on p. 142) on* G *respectively on* $\mathcal{L}$ *(respectively on* $\mathcal{L}'$*).* □

One also has $\bigcap_{\varepsilon > 0} P'_\varepsilon = \operatorname{cl}\{0\}$ (where closure is taken w.r.t. the AC topology). Moreover, we observe that $\operatorname{cl}\{0\}$ is a subgroup of G, (even of $\mathcal{L}'$) (since we have a translation-invariant pseudometric that defines the ε-almost periods). We call $\operatorname{cl}\{0\}$ the *group/set of almost periods* of $\underline{\Lambda}$ and each $t \in \operatorname{cl}\{0\}$ an *almost period*. We obtain from Lemma 2.22 the following.

Lemma 5.46. *Let* π *be the canonical homomorphism of* G *to* $G/\operatorname{cl}\{0\}$ *(respectively of* $\mathcal{L}$ *to* $\mathcal{L}/\operatorname{cl}\{0\}$*). Then* $\pi(\mathcal{B}) = \{\pi(P'_\varepsilon) \mid \varepsilon > 0\}$ *is a fundamental system of neighbourhoods of the neutral element* $\pi(0)$ *consisting of open sets in the quotient topology* $G/\operatorname{cl}\{0\}$ *(respectively* $\mathcal{L}/\operatorname{cl}\{0\}$*) of the AC topology.* □

[7] If $\underline{\Lambda}$ is repetitive (has FLC), so are (have) Λ_i for $i \in \{1, \ldots, n\}$ and $\operatorname{supp} \underline{\Lambda}$.

We now take the completion of this Hausdorff space $\mathcal{L}/\operatorname{cl}\{0\}$, which is unique, see Proposition 2.45. By Definition 2.46, we call this completion the Hausdorff completion of $\mathcal{L}$ (with respect to the AC topology). Note that $\mathcal{L}/\operatorname{cl}\{0\}$, and therefore also the Hausdorff completion, is a metric spaces, see Remark 2.49.

Of course, it would be interesting to know the conditions for $\{0\}$ itself to be closed (or, equivalently, under which condition the (maximum) variogram is a metric). At the moment, we note that group of *periods* of $\underline{\varLambda}$ is contained in $\operatorname{cl}\{0\}$, the group of almost periods. Here, we say that $t \in G$ is a period of $\underline{\varLambda}$ if $t + \underline{\varLambda} = \underline{\varLambda}$, *i.e.*, if $t + \varLambda_i = \varLambda_i$ for all $1 \le i \le n$ (we stress the "for all" in this definition, it is not enough if t is a period for some $\varLambda_i$). We may call $\underline{\varLambda}$ *aperiodic* if the only period is 0. An aperiodic multi-component set may still contain almost periods, *e.g.*, $\mathbb{Z} \setminus \{0\}$ is aperiodic and all $t \in \mathbb{Z}$ are almost periods.
Now, the next step is to repeat the construction of [36, Section 3] for $\mathcal{L}$. We will omit the proofs here, and only state the results together with some remarks. We begin with the following statement, cf. [36, Lemma 2].

Lemma 5.47. *For all* $0 < \varepsilon < 1$*, the set* P'_ε *is totally bounded in the AC topology.* □

We note that this last lemma does in general not include $P_1 = (\varDelta^{\text{ess}})' = \varDelta'$. The proof makes use of the relative denseness of the sets P_ε.
As already indicated, applying Proposition 2.45 to the commutative topological group $\mathcal{L}$ yields the following statement, cf. [36, Prop. 1 & Lemma 4].

Proposition 5.48. *There is a (unique up to topological isomorphism) Hausdorff completion* H *of* $\mathcal{L}$*. This completion* H *is an LCAG and the maximum variogram* $\varrho_{\underline{\varLambda}}$ *on* $\mathcal{L}$ *induces a metric on* H*, which generates its topology. Moreover, for all* $0 < \varepsilon < 1$*, the sets* $\operatorname{cl}_H \varphi(P'_\varepsilon)$ *are compact and have nonempty interior (where* φ *denotes the canonical mapping of* $\mathcal{L} \to H$*).* □

Let $x \in \mathcal{L}$ and denote the image of x in H under φ by $x^\star$ (instead of $\varphi(x)$, since this is the star-map, as we will see in a moment), and set $\widetilde{\mathcal{L}} = \{(x\,, x^\star) \mid x \in \mathcal{L}\}$. Then one obtains the following statement, cf. [36, Lemmas 3 & 4 & Corollary 1].

Lemma 5.49. *The set* $\widetilde{\mathcal{L}}$ *is a Delone set in* $G \times H$*. Moreover,* $\widetilde{\mathcal{L}}$ *is a subgroup of* $G \times H$*, hence the factor group* $(G \times H)/\widetilde{\mathcal{L}}$ *is compact, i.e.,* $\widetilde{\mathcal{L}}$ *is a lattice in* $G \times H$*.* □

In order to prove that $\widetilde{\mathcal{L}}$ is uniformly discrete in $G \times H$ one uses that $\varLambda$ is an FLC set, while $\widetilde{\mathcal{L}}$ is relatively dense because the sets P'_ε are. This establishes that $\widetilde{\mathcal{L}}$ is a uniformly discrete and relatively dense subgroup of $G \times H$ and therefore a lattice.

We have therefore obtained the following CPS:

$$
\begin{array}{ccccc}
G & \xleftarrow{\;\pi_1\;} & G \times H & \xrightarrow{\;\pi_2\;} & H \\
\cup & & \cup & & \cup \text{ dense} \\
\mathcal{L} & \xleftrightarrow{\text{bijective}} & \widetilde{\mathcal{L}} & \longrightarrow & \mathcal{L}^\star = \pi_2(\mathcal{L})
\end{array}
\tag{5.4}
$$

We note that if $\{0\}$ is closed in the AC topology, then we can identify $\mathcal{L}$ and its image $\varphi(\mathcal{L})$ in H; so in this case, also π_2 is bijective on $\widetilde{\mathcal{L}}$. Moreover, we may interpret $\widetilde{\mathcal{L}}$ as (some sort of) diagonal embedding of $\mathcal{L}$ in $G \times H$.
Before we think about the relevance of this CPS, we first introduce another topology and relate its Hausdorff completion to the factor group in the last lemma, cf. [36, Prop. 2].

Definition 5.50. Consider $G \times \mathcal{L}$ with the product topology, where G has the standard topology and $\mathcal{L}$ the AC topology. Then, $(G \times \mathcal{L})/\alpha(\mathcal{L})$ is a topological group, where $\alpha(t) = (t\,,t)$ denotes the diagonal map[8]. As groups, G and $(G \times \mathcal{L})/\alpha(\mathcal{L})$ are isomorphic via $x \mapsto (x\,,0) \bmod \mathcal{L}$. We call this product topology on $G \times \mathcal{L}$ the *mixed topology on G*.

Lemma 5.51. *The (compact) factor group $(G \times H)/\widetilde{\mathcal{L}}$ is the Hausdorff completion of G with respect to the mixed topology.* □

Having established the existence of a CPS, we are now interested in the specific input of the multi-component set, *i.e.*, the relation between the CPS in Equation (5.4) and the CPS one may establish for each component.

We begin with the following observation:

- The topological group $(\mathcal{L}\,,\varrho_{\underline{\Lambda}})$ (*i.e.*, $\mathcal{L}$ with the AC topology, which is defined through the maximum variogram) is topological isomorphic to $(\mathcal{L}'\,,\varrho_{\underline{\Lambda}}) \times \mathcal{L}/\mathcal{L}'$, where the factor group $\mathcal{L}/\mathcal{L}'$ is endowed with the discrete topology: Obviously, as (commutative) groups, $\mathcal{L}$ and $\mathcal{L}' \times \mathcal{L}/\mathcal{L}'$ are isomorphic. Moreover, if $x, y \in \mathcal{L}$ such that $x + \mathcal{L}' \neq y + \mathcal{L}'$, then $\varrho_{\underline{\Lambda}}(x\,,y) = 1$, which establishes the claim.
- The Hausdorff completion of a discrete group is this discrete group itself, and the Hausdorff completion of a product group is (topologically isomorphic to) the product group of the completions of its factors, see Corollary 2.71. So, if we denote the Hausdorff completion of $(\mathcal{L}'\,,\varrho_{\underline{\Lambda}})$ by H', then the Hausdorff completion of $(\mathcal{L}'\,,\varrho_{\underline{\Lambda}}) \times \mathcal{L}/\mathcal{L}'$ is $H' \times \mathcal{L}/\mathcal{L}'$. Since the Hausdorff completion is unique up to topological isomorphism, we have $H \cong H' \times \mathcal{L}/\mathcal{L}'$.
- Therefore, if we are interested in the internal space H, it is actually enough to determine the Hausdorff completion H' of $(\mathcal{L}'\,,\varrho_{\underline{\Lambda}})$. We also note that the above construction also yields a lattice $\widetilde{\mathcal{L}}'$ in $G \times H'$, and we observe the following: Let $\alpha(\mathcal{L}/\mathcal{L}')$ denote the diagonal embedding of $\mathcal{L}/\mathcal{L}'$ into $(\mathcal{L}/\mathcal{L}')_G \times \mathcal{L}/\mathcal{L}'$. Then, there is a bijective map between $(\widetilde{\mathcal{L}}' \times \{0\}) + \alpha(\mathcal{L}/\mathcal{L}')$ and $\widetilde{\mathcal{L}}$. Thus, in crystallographic[9] terms, it might be convenient to think of the lattice $\widetilde{\mathcal{L}}$ as lattice $\widetilde{\mathcal{L}}'$ with "basis" $\alpha(\mathcal{L}/\mathcal{L}')$.

  However, note that the structure of $(\mathcal{L}/\mathcal{L}')_G$, *i.e.*, the factor group $\mathcal{L}/\mathcal{L}'$ with the quotient topology induced by G, might in general be complicated, wherefore it might be hard to argue that the sum of $\alpha(\mathcal{L}/\mathcal{L}')$ with $\widetilde{\mathcal{L}}' = (\widetilde{\mathcal{L}}' \times \{0\})$ is a lattice.

Example 5.52. Let β be an irrational number with $0 < \beta < \frac{1}{3}$. Set $\Lambda_1 = \mathbb{Z}$ and $\Lambda_2 = \mathbb{Z} + \beta$. Then $\underline{\Lambda} = (\Lambda_1\,,\Lambda_2)$ is a periodic (every $t \in \mathbb{Z}$ is a period) multi-component uniformly locally finite set, which is an FLC set: For every compact set W there are at most two patterns, namely $\underline{\Lambda} \cap W$ and $(\underline{\Lambda} - \beta) \cap W$. Alternatively, the sets $\Delta_1 = \mathbb{Z} = \Delta_2$ and $\Delta = \mathbb{Z} + \{-\beta, 0, \beta\}$ are all locally finite. We note that[10] $\varrho_{\underline{\Lambda}}(z\,,0) = 0$ if $z \in \mathbb{Z}$ and $\varrho_{\underline{\Lambda}}(z\,,0) = 1$ otherwise. So

[8] If $\{0\} = \mathrm{cl}\{0\}$, then $\alpha(\mathcal{L}) = \widetilde{\mathcal{L}}$. Moreover, if we denote by $\pi : \mathcal{L} \to \mathcal{L}/\,\mathrm{cl}\{0\}$ the canonical homomorphism, then $(G \times \mathcal{L})/\alpha(\mathcal{L})$ and $(G \times \pi(\mathcal{L}))/\widetilde{\mathcal{L}}$ are topologically isomorphic.

[9] In crystallography, the body centred cubic lattice $\mathbb{Z}^3 + \{(0,0,0)\,,(\frac{1}{2},\frac{1}{2},\frac{1}{2})\}$ can also be interpreted as simple cubic lattice $\mathbb{Z}^3$ with "basis" $\{(0,0,0)\,,(\frac{1}{2},\frac{1}{2},\frac{1}{2})\}$.

[10] The variogram on Λ is given by

$$\varrho_{\Lambda}(z\,,0) = \begin{cases} 0 & \text{if } z \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } z \in (\mathbb{Z} + \beta) \cup (\mathbb{Z} - \beta) \\ 1 & \text{otherwise.} \end{cases}$$

we obtain as Hausdorff completion H' of $\mathcal{L}' = \mathbb{Z}$ simply $H' = \{0\}$ (since $\mathrm{cl}\{0\} = \mathbb{Z}$). Since $\mathcal{L}$ is the free Abelian group generated by 1 and β, wherefore we may write $\mathcal{L} = \mathbb{Z} \oplus \beta\mathbb{Z}$, the internal space H is given by

$$H \cong H' \times \mathcal{L}/\mathcal{L}' = \{0\} \times (\mathbb{Z} \oplus \beta\mathbb{Z})/\mathbb{Z} \cong \{m \cdot \beta \bmod 1 \mid m \in \mathbb{Z}\}.$$

Indeed, $\underline{\Lambda}$ is a multi-component model set, where the windows are given by $\{0\}$ and $\{\beta\}$. Moreover, the above construction also establishes that $\tilde{\mathcal{L}} = \{(k+m\cdot\beta\,, m\cdot\beta \bmod 1) \,|\, k, m \in \mathbb{Z}\}$ is a lattice in $\mathbb{R} \times (\mathbb{Z} \oplus \beta\mathbb{Z})/\mathbb{Z}$.

Remark 5.53. If $\underline{\Lambda}$ is a multi-component uniformly locally finite set such that each component is an FLC set (but not $\underline{\Lambda}$), and if property **(As)** holds, then one still can establish the CPS $(G\,, H'\,, \widetilde{\mathcal{L}}')$. This is relevant in the following case: Suppose $\underline{\Lambda} = x + \Lambda(\underline{\Omega})$ is a multi-component regular model set. For every component choose[†] a $t_i \in G$ and form the multi-component set $\underline{\Lambda}^{\bowtie} = (\Lambda_i + t_i)_{i=1}^{m}$. Then $\underline{\Lambda}^{\bowtie}$ is still a multi-component uniformly locally finite set such that each component is an FLC set, but it might not be an FLC multi-component set anymore. By construction, we obtain the same H' and $\widetilde{\mathcal{L}}'$ for $\underline{\Lambda}$ and $\underline{\Lambda}^{\bowtie}$, but the sets $\mathcal{L}$ and $\mathcal{L}^{\bowtie} = \langle \mathrm{supp}\, \underline{\Lambda}^{\bowtie} \rangle_{\mathbb{Z}}$ may differ. So, while $\widetilde{\mathcal{L}}$ is a lattice in $G \times H' \times \mathcal{L}/\mathcal{L}'$, the subgroup $\widetilde{\mathcal{L}^{\bowtie}}$ might only be relatively dense in $G \times H' \times \mathcal{L}^{\bowtie}/\mathcal{L}'$. We emphasise that each component is always a model set in the CPS $(G\,, H'\,, \widetilde{\mathcal{L}}')$.

In view of these remarks and examples, we can alternatively write the CPS in Equation (5.4) on p. 144 as follows:

$$\begin{array}{ccccc}
G & \overset{\pi_1}{\longleftarrow} & G \times H' \times \mathcal{L}/\mathcal{L}' & \overset{\pi_2}{\longrightarrow} & H' \times \mathcal{L}/\mathcal{L}' \\
\cup & & \cup & & \cup \text{ dense} \\
\mathcal{L} & \overset{\text{bijective}}{\longleftrightarrow} & \widetilde{\mathcal{L}} & \longrightarrow & \mathcal{L}^{\star}
\end{array}$$

We also make the following definition.

Definition 5.54. We call the group $\mathcal{L}/\mathcal{L}'$ the *height group* of $\underline{\Lambda}$. We always endow the height group with the discrete topology. The origin of the name will become clear in Chapter 6b (see Definition 6b.12).

As observed on p. 143, each component Λ_i of a repetitive FLC multi-component set belongs to one coset $t + \mathcal{L}'$ of the height group $\mathcal{L}/\mathcal{L}'$. Therefore, we may simply interpret an element of the height group as a (first and rough) statement about the position of a component Λ_i inside $\mathcal{L}$. Often, we are in the situation of the following statement, wherefore the factor group $\mathcal{L}/\mathcal{L}'$ is finite.

Lemma 5.55. *Let $\underline{\Lambda}$ be a repetitive FLC multi-component uniformly locally finite set. If, for every $1 \leq i \leq n$, there is a $t_i \in \Lambda_i$ and an $a_i \in \mathbb{N}$ such that $a_i \cdot t_i \in \mathcal{L}'$, then the factor group $\mathcal{L}/\mathcal{L}'$ is a finite group and its cardinality is bounded by $\prod_{i=1}^{n} a_i$.*

Proof. As observed on p. 143, we may assume that $\Lambda_i \subset t_i + \mathcal{L}'$. Then, we also have $\Lambda \subset \{t_1, \ldots, t_n\} + \mathcal{L}'$ and $\Lambda - \Lambda \subset \bigcup_{1 \leq i,j \leq n} \mathcal{L}' + t_i - t_j$. But then $\mathcal{L} \subset \langle t_1, \ldots, t_n \rangle_{\mathbb{Z}} + \mathcal{L}'$, and the cardinality of the factor group

$$\left(\langle t_1, \ldots, t_n \rangle_{\mathbb{Z}} + \mathcal{L}' \right) / \mathcal{L}'$$

This might indicate that in general there is no $\delta > 0$ such that $P_{\delta}(\Lambda) \subset P'_{\varepsilon}$ for all $\varepsilon > 0$ (of course, in the periodic case considered here, it is the case).

[†]The construction of "control points" (see Remark 5.125) for substitution tilings seems to ensure a coherent choice for such t_i.

is bounded by the product $a_1 \cdots a_n$. This establishes the claim. □

If $\mathcal{L}$ and $\mathcal{L}'$ are complete modules in an algebraic number field, finiteness of the height group follows by the "elementary divisors theorem", see p. 221.

Remark 5.56. Of course, one can also repeat the above construction for each Λ_i, respectively each $\mathcal{L}_i$ separately. We note the following points:

- Let us denote the Hausdorff completion of $\mathcal{L}_i$ (endowed with the AC topology given by the variogram ϱ_{Λ_i}) by H_i. Noting that condition **(As)** in particular implies that, for every $\varepsilon > 0$, the sets $P_\varepsilon(\Lambda_i)$ are relatively dense, we have the following cut and project schemes (noting that the star-map $(\cdot)^\star$ here may in general be different for each i and from the star in Equation (5.4) on p. 144):

$$
\begin{array}{ccccc}
G & \xleftarrow{\pi_1} & G \times H_i & \xrightarrow{\pi_2} & H_i \\
\cup & & \cup & & \cup \text{ dense} \\
\mathcal{L}_i & \xleftrightarrow{\text{bijective}} & \widetilde{\mathcal{L}_i} & \longrightarrow & (\mathcal{L}_i)^\star
\end{array}
$$

- Each ϱ_{Λ_i} is a translation-invariant pseudometric on $\mathcal{L}_i$ and also on $\mathcal{L}'$ (respectively $\mathcal{L}$). By definition, the AC topology on $\mathcal{L}'$, defined by the maximum variogram, is generated by the family $(\varrho_{\Lambda_i})_{i=1}^n$ of pseudometrics (variograms). Therefore, the AC topology $(\mathcal{L}', \varrho_{\underline{\Lambda}})$ on $\mathcal{L}'$ is finer than any topology $(\mathcal{L}', \varrho_{\Lambda_i})$ generated by a single pseudometric and the identity map $\mathrm{id}_i : (\mathcal{L}', \varrho_{\underline{\Lambda}}) \to (\mathcal{L}', \varrho_{\Lambda_i})$ is continuous for all $1 \le i \le n$.

- If $x, y \in \mathcal{L}'$ are such that $x + \mathcal{L}_i \neq y + \mathcal{L}_i$ (*i.e.*, x, y are representatives of different cosets of the factor group $\mathcal{L}'/\mathcal{L}_i$), we have $\eta_{\Lambda_i}(x - y) = 0$ and therefore $\varrho_{\Lambda_i}(x, y) = 1$. But this shows that the topological group $(\mathcal{L}', \varrho_{\Lambda_i})$ is topologically isomorphic to the product topology of $(\mathcal{L}_i, \varrho_{\Lambda_i})$ and $\mathcal{L}'/\mathcal{L}_i$, the latter factor group endowed with the discrete topology; *i.e.*, $(\mathcal{L}', \varrho_{\Lambda_i}) \cong (\mathcal{L}_i, \varrho_{\Lambda_i}) \times \mathcal{L}'/\mathcal{L}_i$.

- Similarly as before, the Hausdorff completion of $(\mathcal{L}', \varrho_{\Lambda_i})$ is given by $H_i \times \mathcal{L}'/\mathcal{L}_i$ (where $1 \le i \le n$). Moreover, by Proposition 2.47 (respectively Lemma 2.68), there is a (uniformly) continuous homomorphism $\hat{\mathrm{id}}_i$ such that the following diagram commutes for every $1 \le i \le n$:

$$
\begin{array}{ccc}
\mathcal{L}' & \xrightarrow{\mathrm{id}_i} & \mathcal{L}_i \times \mathcal{L}'/\mathcal{L}_i \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_i'} \\
H' & \xrightarrow{\hat{\mathrm{id}}_i} & H_i \times \mathcal{L}'/\mathcal{L}_i
\end{array}
$$

  where φ and φ_i' denote the canonical uniformly continuous homomorphisms from a commutative topological group into its Hausdorff completion.

- Let $\pi_2^{(i)} : H_i \times \mathcal{L}'/\mathcal{L}_i \to \mathcal{L}'/\mathcal{L}_i$ denote the canonical projection. Then the uniformly continuous homomorphism $\pi_2^{(i)} \circ \hat{\mathrm{id}}_i : H' \to \mathcal{L}'/\mathcal{L}_i$ is a strict morphism, recall Definition 2.18 and Lemma 2.19. Therefore, $\mathcal{L}'/\mathcal{L}_i$ and $H'/\ker(\pi_2^{(i)} \circ \hat{\mathrm{id}}_i)$ are topologically isomorphic. By Lemma 2.22, $\ker(\pi_2^{(i)} \circ \hat{\mathrm{id}}_i)$ is an open and therefore (by Lemma 2.21) also closed subgroup of H'. We may think of this clopen subgroup as "contribution" of H_i to H', see Example 5.158 which clarifies this remark.

- As a last point, we mention that the situation according to Remark 2.42 is as follows: Let σ denote the map

$$\sigma : \mathcal{L}' \to \prod_{i=1}^{n} (H_i \times \mathcal{L}'/\mathcal{L}_i),\ x \mapsto (\varphi_1' \circ \mathrm{id}_1(x), \ldots, \varphi_n' \circ \mathrm{id}_n(x)).$$

  Then the closure of $\sigma(\mathcal{L}')$ in the product topology $\prod_{i=1}^{n}(H_i \times \mathcal{L}'/\mathcal{L}_i)$ is exactly H'. Note that one may view the AC topology on $\mathcal{L}'$ defined by the maximum variogram $\varrho_{\underline{\Lambda}}$ as the coarsest topology such that all maps id_i are continuous.

Consequently, these remarks connect the CPS in Equation (5.4) on p. 144 constructed from $\mathcal{L}$ with the CPS constructed from each $\mathcal{L}_i$.

Of course, one would like to have criteria under which condition one not only obtains a CPS but one can prove that a given, say, multi-component Meyer set is a multi-component model set. A first step is indicated in Proposition 5.48: the sets $\mathrm{cl}_H\, \varphi(P_\varepsilon')$ are compact with nonempty interior, and therefore have the property one would like to have for a window.

5.4. Substitutions I

We begin with a dual definition to an IFS, see Definition 4.84.

Definition 5.57. An *expansive matrix function system*, or *EMFS* for short, on a metric space (X, d) is an $(n \times n)$-matrix $\boldsymbol{\Theta} = \left[\Theta_{ij}\right]_{1 \le i,j \le n}$ (*i.e.* an MFS), where each $\Theta_{ij} = \{f_1, \ldots, f_k\}$ is a finite (possibly empty) set of expansions[11] (with respect to d), *i.e.*, for every $\boldsymbol{\Theta} \ni f : X \to X$ there is a $q > 1$ such that $d(f(x), f(y)) \ge q \cdot d(x, y)$ for all $x, y \in X$. As in Definition 4.84, we say that an EMFS $\boldsymbol{\Theta}$ is *primitive* (respectively *irreducible*) if its substitution matrix $\boldsymbol{S\Theta}$ is primitive (respectively irreducible). Similarly, Remark 4.85 also applies to an EMFS.

Remark 5.58. For a metrisable commutative topological group (G, d), we always assume that the metric d is translation-invariant, compare Remark 2.35. Therefore, we note that on a (nontrivial) metrisable compact commutative topological group (G, d), one cannot have an expansion: Since G is covered by finitely many open balls $B_{<\varepsilon}(x)$ of radius ε, an application of the triangle inequality shows that there is a constant $C > 0$ such that $d(x, y) < C$ for all $x, y \in G$. But by the definition of an expansion f, we have $d\left(f^k(x), f^k(y)\right) \ge q^k \cdot d(x, y)$ for all $k \in \mathbb{N}$, which, for sufficiently large k, contradicts the previous statement. In view of this, we assume that G is not compact (and in particular not finite).

For similar reasons, we assume that the metric d has the property that every bounded set is also totally bounded, wherefore a set is compact iff it is bounded and closed. Otherwise, let S be a bounded but not totally bounded set (therefore $\operatorname{diam} S < \infty$). Then its preimage $f^{-1}(S)$ is also bounded, in fact we have $\operatorname{diam} f^{-1}(S) \le (\operatorname{diam} S)/q$. Assuming that f is bi-Lipschitz, $f^{-1}(S)$ is not totally bounded (otherwise S would also be totally bounded). Iterating this procedure, we get sets of smaller and smaller diameter which are not totally bounded. But this also implies that, for every $x \in X$, the only compact ball (*i.e.*, the only closed ball $B_{\le\varepsilon}(x)$ which is compact) around x is the set $\{x\}$. From this, we can conclude that only finite subsets

[11] In fact, we assume that the maps $f \in \boldsymbol{\Theta}$ are bi-Lipschitz, *i.e.*, there is also a constant $C < \infty$ such that $d(f(x), f(y)) \le C \cdot d(x, y)$ for all $x, y \in X$. Basically, this condition ensures that, for every map $f \in \boldsymbol{\Theta}$, the image of a bounded set is bounded.

of G are compact. If G is a σLCAG, we can further conclude (from Lemma 2.9) that G has to be equipped with the discrete topology. Moreover, we are interested in Delone sets in G, but the existence of a Delone set then implies that G is a countable discrete group, *i.e.*, $G \cong \mathbb{Z}$. So, the assumption about (total) boundedness basically only excludes the case $G \cong \mathbb{Z}$ in what follows.

Definition 5.59. A multi-component Delone set $\underline{\varLambda}$ on a metrisable LCAG $(G\,,d)$ (where d denotes the metric) is called a *substitution multi-component Delone set* if there is an EMFS $\boldsymbol{\Theta}$ on $(G\,,d)$ such that $\underline{\varLambda} = \boldsymbol{\Theta}(\underline{\varLambda})$, *i.e.*, we have

$$\varLambda_i = \bigcup_{j=1}^{n} \bigcup_{f \in \Theta_{ij}} f(\varLambda_j) \tag{5.5}$$

for all $1 \le i \le n$, and all unions on the right hand side of Equation (5.5) are disjoint. We call the maps $f \in \boldsymbol{\Theta}$ the corresponding *substitutions* of $\underline{\varLambda}$. A substitution multi-component Delone set is called *primitive* respectively *irreducible* if the EMFS is primitive respectively irreducible.

We now make some additional assumption about the EMFS $\boldsymbol{\Theta}$. These are sufficient conditions which will enable us to define a so-called “adjoint IFS” to the EMFS and connect the EMFS and this IFS. We will comment on these assumptions at the end of this section.

From now on, we assume the following “linear and translation property” of the maps in a MFS $\boldsymbol{\Theta}$:

(LT) The MFS $\boldsymbol{\Theta}$ is an irreducible MFS on a metrisable (noncompact) LCAG $(G\,,d)$ (with Haar measure μ) and either an EMFS or an IFS. Let α denote the PF-eigenvalue of $\boldsymbol{S\Theta}$. Then all maps $\boldsymbol{\Theta} \ni f_i : G \to G$ (with $i \in I$ for some finite set I) satisfy the following four properties:

(i) All substitutions are bi-Lipschitz bijections.

(ii) Every substitution is an affine map, *i.e.*, for every i one has $f_i(x+y) = f_i(x) + f_i(y) - f_i(0)$ for all $x, y \in G$.

(iii) If $\boldsymbol{\Theta}$ is an EMFS, then $\mu(f_i(W)) = \alpha \cdot \mu(W)$ for all compact subsets $W \subset G$. If $\boldsymbol{\Theta}$ is an IFS, then $\mu(f_i(W)) = \frac{1}{\alpha} \cdot \mu(W)$ for all compact subsets $W \subset G$.

(iv) For every pair f_i, f_j of substitutions there is a $t_{ij} \in G$ such that $f_i(x) + t_{ij} = f_j(x)$, *i.e.*, the substitutions only differ from each other by a translation.

For every $a \in G$, denote by $t_{(a)} : G \to G$ the translation $t_{(a)}(x) = x + a$ by a on G. Then – without loss of generality – we can write every map f_i as composition $f_i = t_{(f_i(0))} \circ f_0$ where $f_0(0) = 0$. Indeed, if f_i is a map with the above properties, set $f_0 = t_{(-f_i(0))} \circ f_i$ (then f_0 is independent of the choice $i \in I$). For a given MFS $\boldsymbol{\Theta}$ satisfying **(LT)**, we will always denote by f_0 (respectively, by $f_0^{\boldsymbol{\Theta}}$ if there is any possibility of confusion) this last map. Therefore, the components of such an MFS $\boldsymbol{\Theta}$ are given by

$$\Theta_{ij} = \{t_{(a_{ij1})} \circ f_0, \dots, t_{(a_{ijk})} \circ f_0\} = \{f_0 \circ t_{(f_0^{-1}(a_{ij1}))}, \dots, f_0 \circ t_{(f_0^{-1}(a_{ijk}))}\}, \tag{5.6}$$

where $k = \operatorname{card} \Theta_{ij}$ and $a_{ij1}, \dots, a_{ijk} \in G$.

We note that for a bi-Lipschitz bijection $f : G \to G$ (with respect to d) one has: $f : G \to G$ is a bi-continuous map (*i.e.*, f and f^{-1} are continuous maps) and there are constants $0 < C' \leq C < \infty$ such that $C' \cdot d(x\,,y) \leq d(f(x)\,,f(y)) \leq C \cdot d(x\,,y)$ for all $x, y \in G$; consequently, we also have $d(x\,,y)/C \leq d(f^{-1}(x)\,,f^{-1}(y)) \leq d(x\,,y)/C'$ for all $x, y \in G$. One can therefore make the following connection between the notions EMFS and IFS.

Definition 5.60. Let $\boldsymbol{\Theta} = \left[\Theta_{ij}\right]_{1\leq i,j\leq n}$ be an MFS on a metrisable LCAG $(G\,,d)$ satisfying **(LT)** (in particular, it is either an EMFS or an IFS). Let the component Θ_{ij} of $\boldsymbol{\Theta}$ be given as in Equation (5.6). Then we define the *adjoint MFS* $\boldsymbol{\Theta}^{\#} = \left[\Theta^{\#}_{ij}\right]_{1\leq i,j\leq n}$ of $\boldsymbol{\Theta}$ by[12]

$$\Theta^{\#}_{ij} = \bigcup_{t_{(a)}\circ f_0 \in \Theta_{ji}} \{t_{(f_0^{-1}(a))} \circ f_0^{-1}\} = \bigcup_{t_{(a)}\circ f_0 \in \Theta_{ji}} \{f_0^{-1} \circ t_{(a)}\}.$$

Obviously, we have $(\boldsymbol{\Theta}^{\#})^{\#} = \boldsymbol{\Theta}$, *i.e.*, the adjoint of the adjoint MFS is the original MFS, and $\boldsymbol{S\Theta}^{\#} = (\boldsymbol{S\Theta})^t$, *i.e.*, the substitution matrix of the adjoint MFS is the transposed of the substitution matrix of the MFS. Moreover, the adjoint of an EMFS is an IFS satisfying property **(LT)** and vice versa.

Corollary 5.61. *The adjoint MFS of an EMFS with* **(LT)** *is an IFS satisfying* **(LT)**. *Similarly, the adjoint MFS of an IFS with* **(LT)** *is an EMFS satisfying* **(LT)**.

Proof. Obviously, the adjoint MFS of an EMFS is an IFS and vice versa. Moreover, only property **(LT)**(iii) is not immediate for the adjoint MFS. But by the assumptions on f_0, if W is compact (*i.e.*, totally bounded and closed), so is its image $f_0(W)$ respectively its preimage $f_0^{-1}(W)$. □

For each substitution multi-component Delone set, where the associated EMFS satisfies **(LT)**, one can therefore set up an IFS. We therefore note the following easy consequences of statements from Section 4.8.

Corollary 5.62. *Let* $\boldsymbol{\Theta}$ *be an EMFS with* **(LT)**. *Then, since the adjoint MFS* $\boldsymbol{\Theta}^{\#}$ *is an IFS, there is a unique set* $\underline{A} \subset (\mathcal{K}G)^n$ *that satisfies* $\underline{A} = \boldsymbol{\Theta}^{\#}(\underline{A})$.

Proof. See Proposition 4.89. □

Corollary 5.63. *Let* $\boldsymbol{\Theta}$ *with* **(LT)**. *Denote by* $\underline{A}$ *the unique attractor of the adjoint IFS* $\boldsymbol{\Theta}^{\#}$. *Suppose that there is an* $1 \leq i_0 \leq n$ *such that* A_{i_0} *has interior points. Then the following hold:*

(i) *All* A_i *have non-zero Haar measure.*

(ii) *The unions in the IFS* $\boldsymbol{\Theta}^{\#}$ *are measure-disjoint.*

(iii) *The boundaries* ∂A_i *have zero Haar measure for all* i.

(iv) *All* A_i *are perfect sets and regularly closed.*

Proof. This is Proposition 4.99, if we note that bi-Lipschitz bijections are homeomorphisms and since we also have $\mu(W) = \alpha \cdot \mu(f_0^{-1}(W))$ for all compact sets W. □

[12]Informally, the adjoint MFS is obtained from the MFS by replacing f_0 by its inverse f_0^{-1}, applying f_0^{-1} to all translations and transposing "the matrix".

Example 5.64. We consider the following two expansive matrix function systems

$$\boldsymbol{\Theta}^{(1)} = \begin{pmatrix} \{f_0\} & \{f_1\} \\ \{f_1\} & \{f_0\} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Theta}^{(2)} = \begin{pmatrix} \{f_0\} & \{f_0, f_1\} \\ \{f_1\} & \varnothing \end{pmatrix}$$

in $\mathbb{R}$, where $f_0(x) = 2x$ and $f_1(x) = t_{(1)} \circ f_0(x) = 2x + 1$.

- For any $x \in \mathbb{Z}_{\geq 0}$, there is a finite (unique) binary expansion $x = \sum_{k\geq 0} a_k 2^k$ with $a_i \in \{0\,, 1\}$. Thus, we can define the map $S_2 : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, $x = \sum_{k\geq 0} a_k \cdot 2^k \mapsto S_2(x) = \sum_{k\geq 0} a_k$. We obtain the sets

  $$\varLambda'_{\text{odd}} = \{x \in \mathbb{Z}_{\geq 0} \mid S_2(x) \text{ is odd}\} \qquad \text{and} \qquad \varLambda'_{\text{even}} = \{x \in \mathbb{Z}_{\geq 0} \mid S_2(x) \text{ is even}\}.$$

  Then, one easily establishes that any pair

  $$\varLambda_1^{(1)} = \varLambda'_i \cup -(\varLambda'_j + 1) \qquad \text{and} \qquad \varLambda_2^{(1)} = \mathbb{Z} \setminus \varLambda_1^{(1)}$$

  with $i, j \in \{\text{odd}\,, \text{even}\}$ is a substitution multi-component Delone set $\underline{\varLambda}^{(1)} = (\varLambda_1^{(1)}\,, \varLambda_2^{(1)})$ with $\left(\boldsymbol{\Theta}^{(1)}\right)^2 (\underline{\varLambda}^{(1)}) = \underline{\varLambda}^{(1)}$, compare [298, Section 2.1.1] and [160, p. 2]. Such a fixed point on $\mathbb{Z}$ is called a *Thue-Morse sequence* (see [265, 371, 372]). Symbolically, we can also write this substitution as $a \mapsto ab$ and $b \mapsto ba$, compare Example 5.158.

- Similarly, we define the sets

  $$\begin{array}{rclcl} \varLambda'_1 & = & 2\mathbb{Z} \,\cup\, 8\mathbb{Z}+3 \,\cup\, 32\mathbb{Z}+15 \,\cup\, \ldots & = & \bigcup_{k\geq 0} 2^{2k+1}\mathbb{Z} + 2^{2k} - 1, \\ \varLambda'_2 & = & 4\mathbb{Z}+1 \,\cup\, 16\mathbb{Z}+7 \,\cup\, 64\mathbb{Z}+31 \,\cup\, \ldots & = & \bigcup_{k\geq 0} 2^{2k+2}\mathbb{Z} + 2^{2k+1} - 1, \end{array}$$

  and observe that $\varLambda'_1 \cup \varLambda'_2 \cup \{-1\} = \mathbb{Z}$ (where this union is disjoint). One establishes, that $\varLambda_1^{(2)} = \varLambda'_1 \cup \{-1\}$ and $\varLambda_2^{(2)} = \varLambda'_2$ (respectively, $\varLambda_1^{(2)} = \varLambda'_1$ and $\varLambda_2^{(2)} = \varLambda'_2 \cup \{-1\}$) are the components of the substitution multi-component Delone set $\underline{\varLambda}^{(2)}$ with $\left(\boldsymbol{\Theta}^{(2)}\right)^2 (\underline{\varLambda}^{(2)}) = \underline{\varLambda}^{(2)}$. Such a fixed point on $\mathbb{Z}$ is called a *period-doubling sequence* (see [11, Example I.4.4]). Symbolically, we can also write this substitution as $a \mapsto ab$ and $b \mapsto aa$.

Now, the adjoint iterated function systems are given by

$$\left(\boldsymbol{\Theta}^{(1)}\right)^{\#} = \begin{pmatrix} \{g_0\} & \{g_1\} \\ \{g_0\} & \{g_1\} \end{pmatrix} \qquad \text{and} \qquad \left(\boldsymbol{\Theta}^{(2)}\right)^{\#} = \begin{pmatrix} \{g_0\} & \{g_1\} \\ \{g_0, g_1\} & \varnothing \end{pmatrix}$$

in $\mathbb{R}$, where $g_0 = f_0^{-1}(x) = \frac{1}{2}x$ and $g_1(x) = t_{(\frac{1}{2})} \circ g_0(x) = \frac{1}{2}x + \frac{1}{2}$. We note that in both cases a (and thus, by uniqueness, the) attractor of these iterated function systems is given by $\underline{A} = ([0\,, 1]\,, [0\,, 1])$, wherefore the properties mentioned in Corollary 5.63 are obvious.

Definition 5.65. A *tile* T in an LCAG G is a compact set, which is regularly closed and[13] has a boundary of Haar measure 0. A *covering* $\mathcal{T}$ of G is a collection of tiles $(T_i)_{i\in I}$ which

[13] Often a tile is defined without the condition on the Haar measure of its boundary. However, in view of Remark 4.102 and because it is often implicitly used, we use the above definition. Note that for the Haar measure of a tile T we have $0 < \mu(T) = \mu(\operatorname{int} T) < \infty$ (see Definition 4.27).

covers G, *i.e.*, $G = \bigcup_{i\in I} T_i$. A *tiling* $\mathcal{T}$ of G is a covering, which contains no overlapping tiles, *i.e.*, $\operatorname{int} T_i \cap \operatorname{int} T_j = \varnothing$ for all distinct pairs $T_i, T_j \in \mathcal{T}$. For a covering $\mathcal{T}$ – or, more generally, for any collection of compact sets respectively tiles – and any $x \in G$ we define the *covering degree* $\deg^{cov}_{\mathcal{T}}(x)$ of $\mathcal{T}$ at x by $\deg^{cov}_{\mathcal{T}}(x) = \operatorname{card}\{T_i \in \mathcal{T} \,|\, x \in T_i\}$. A collection of tiles $\mathcal{T}$ which contains no overlapping tiles is called a *packing* (note that this also explains the names covering/packing radius in Remark 5.2). Therefore, if $\mathcal{T}$ is a covering and a packing, it is a tiling. If the tiles of a tiling (respectively a covering) $\mathcal{T}$ belong to finitely many translation classes $[T_1], [T_2], \ldots, [T_n]$, *i.e.*, $T_i, T_j \in \mathcal{T}$ belong to the same translation class iff there is a $t \in G$ such that $T_i = T_j + t$, then the class representatives $T_1, \ldots, T_n$ are called *prototiles*. Sometimes, the tiles have another feature – usually called *type* or *colour* – wherefore two tiles T_1 and T_2 belong to the same translation class $[T]$ if there is a $t \in G$ such that $T_1 = T_2 + t$ and if they are of the same type (*e.g.*, with this one can, for $i \in \{1, 2\}$, discriminate the tiles of $\mathcal{T} = \underline{\varLambda}^{(i)} + \underline{A} = [0\,,1] + \mathbb{Z}$ in Example 5.64 that belong to $[0\,,1] + \varLambda_1^{(i)}$ (which are of "type 1") from the ("type 2") tiles belonging to $[0\,,1] + \varLambda_2^{(i)}$).

We get the following immediate consequence from Corollary 5.62.

Corollary 5.66. *Assume the setting of Corollary* 5.62. *If there is an* $1 \leq i_0 \leq n$ *such that* A_{i_0} *has interior points, then the sets* A_i *are tiles* $(1 \leq i \leq n)$. □

If $\underline{\varLambda} = (\varLambda_i)_{i=1}^n$ is a multi-component set and $\underline{T} = (T_i)_{i=1}^n$ is a set of prototiles (or, more generally, a set of n compact sets), we use the notation $\underline{\varLambda} + \underline{T}$ to mean the collection $(T_i + t_i \,|\, t_i \in \varLambda_i,\ 1 \leq i \leq n)$ of tiles. Note that $\operatorname{supp}(\underline{\varLambda} + \underline{T}) = \bigcup_{i=1}^n \varLambda_i + T_i = \bigcup_{i=1}^n \bigcup_{t_i \in \varLambda_i} T_i + t_i$.

We note that in the setting considered here, there is a natural choice for the prototiles, namely the components of the attractor of the adjoint IFS (and not some translated copies of these prototiles). We will always use this choice of the prototiles. We are interested in conditions which guarantee that $\underline{\varLambda} + \underline{A}$ is a tiling.

Lemma 5.67. *Let* $\underline{\varLambda}$ *be a multi-component Delone set and* $\underline{A}$ *be a set of* n *compact sets in a metrisable LCAG* G. *If* $G = \operatorname{supp}(\underline{\varLambda} + \underline{A})$ *(i.e., if* $\underline{\varLambda} + \underline{A}$ *covers* G*), then at least one* A_{i_0} $(1 \leq i_0 \leq n)$ *has interior points.*

Proof. Since G is metrisable, it is also σ-compact. So each $\varLambda_i$ (which are uniformly discrete) and therefore also $\underline{\varLambda}$ are countable by Remark 5.3. Suppose none of the compact (and therefore closed) sets A_i has interior points. Then $\operatorname{supp}(\underline{\varLambda} + \underline{A})$ is a countable union of nowhere dense sets, wherefore G would be meager. But this contradicts Proposition 2.14. □

We will later use the previous lemma to prove that for the iterated function systems we are interested in, we are in the situation of Corollary 5.63, see Corollary 5.81.

Definition 5.59 implies that we can put the structure of a directed graph on a substitution multi-component Delone set. We will clarify this now.

Definition 5.68. Let $\boldsymbol{\Theta}$ be an $n \times n$-EMFS and let us denote by $\omega_i(x)$ the *elementary multi-component point set* in the i-th component at x given by

$$\omega_i(x) = (\varnothing, \ldots, \varnothing, \underset{\substack{\uparrow \\ i\text{-th component}}}{\{x\}}, \varnothing, \ldots, \varnothing).$$

Then the EMFS $\boldsymbol{\Theta}$ acts on such an $\omega_i(x)$ as

$$\boldsymbol{\Theta}\left(\omega_i(x)\right) = \bigcup_{j=1}^{n} \bigcup_{f \in \Theta_{ji}} \omega_j(f(x)),$$

and we say that the EMFS $\boldsymbol{\Theta}$ defines a substitution on the point sets of G. Note that $\operatorname{supp} \omega_i(x) = \{x\}$.

Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a substitution multi-component Delone set. Then, by σ-compactness of G, we may think of G as countable union of elementary multi-component point sets. We put the structure of a directed graph $G(\underline{\Lambda})$ on $\underline{\Lambda}$ (also compare Definition 3c.5 and p. 106): The vertices are the elementary multi-component point sets $\omega_i(x) \subset \underline{\Lambda}$. We put a directed edge from $\omega_i(x)$ to $\omega_j(y)$ if $\omega_j(y) \subset \boldsymbol{\Theta}\left(\omega_i(x)\right)$.

Since, by definition, the unions in each component (*i.e.*, the unions in Equation 5.5) of a substitution multi-component Delone set are disjoint, each vertex $\omega_i(x)$ has at most one direct predecessor (parent). Moreover, since $\underline{\Lambda}$ is a fixed point under $\boldsymbol{\Theta}$, each vertex $\omega_i(x)$ has at least one parent. Therefore, we obtain the following corollary.

Corollary 5.69. *Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a substitution multi-component Delone set. Then, in the directed graph $G(\underline{\Lambda})$, each vertex $\omega_i(x)$ has exactly one parent and $\sum_{j=1}^{n} \operatorname{card} \Theta_{ji}$ children.* □

Using the graph $G(\underline{\Lambda})$, we now want to show that one obtains the substitution multi-component Delone set $\underline{\Lambda}$ from a "finite seed", compare [221, Section 3]. Since we want to define (converging) sequences of (subsets of) multi-component Delone sets, we actually need a topology. We will introduce the topology used here, the "local topology", in Definition 5.102 and only use the following intuitive notion of closeness with respect to the local topology here: Two multi-component uniformly discrete subsets $\underline{\Lambda}, \underline{\Lambda}'$ are close to each other, if they agree (possibly after a small translation) on a large compact set. In the following, we can neglect the small translation, wherefore the larger the compact set is on which $\underline{\Lambda}, \underline{\Lambda}'$ agree, the smaller is the distance between them.

Definition 5.70. Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a substitution multi-component Delone set. Let $\underline{\mathcal{P}} = \bigcup_{k=1}^{m} \omega_{i_k}(x_k)$ be a finite disjoint union of elementary multi-component point sets such that[14] $\underline{\mathcal{P}} \subset \underline{\Lambda}$. We say that $\underline{\mathcal{P}}$ satisfies the *inclusion condition* if $\underline{\mathcal{P}} \subset \boldsymbol{\Theta}(\underline{\mathcal{P}})$. By iteration, if $\underline{\mathcal{P}}$ satisfies the inclusion condition, then $\boldsymbol{\Theta}^{\ell-1}(\underline{\mathcal{P}}) \subset \boldsymbol{\Theta}^{\ell}(\underline{\mathcal{P}})$ for all $\ell \in \mathbb{N}$. We note that by the definition of the substitution multi-component Delone set $\underline{\Lambda}$, all unions in $\boldsymbol{\Theta}^{\ell}(\underline{\mathcal{P}})$ are disjoint. We say that $\underline{\mathcal{P}}$ is a *finite seed* if it satisfies the inclusion property and $\underline{\Lambda} = \lim_{\ell \to \infty} \boldsymbol{\Theta}^{\ell}(\underline{\mathcal{P}})$.

Lemma 5.71. [221, Lemma 3.2] *Every substitution multi-component Delone set $\underline{\Lambda}$ possesses a finite seed $\mathcal{P}$. Also, the graph $G(\underline{\Lambda})$ contains a directed cycle.*

Sketch of Proof. Let $\boldsymbol{\Theta}$ be the associated EMFS. We define $\rho = \max\{d(0\,, t_{(a)}) \mid t_{(a)} \circ f_0 \in \Theta_{ij},\ 1 \leq i,j \leq n\}$, *i.e.*, ρ is the maximal distance of the translational parts in the EMFS $\boldsymbol{\Theta}$. Let $x \in G$ and $1 \leq i \leq n$. Then, for any child $\omega_j(y) \subset \boldsymbol{\Theta}(\omega_i(x))$ of $\omega_i(x)$, we have the following estimate:

$$q \cdot d(x\,, 0) \leq d(f_0(x)\,, f_0(0)) = d(f_0(x)\,, 0) \leq d(f_0(x)\,, y) + d(y\,, 0) \leq \rho + d(y\,, 0).$$

[14] In Definition 5.83, we will call such a set $\mathcal{P}$ a "cluster".

Noting that $q > 1$, the inequality $\frac{R+\rho}{q} < \frac{2}{q+1} \cdot R$ holds for all $R > \rho \cdot \frac{q+1}{q-1}$, wherefore if a child $\omega_j(y)$ has distance $d(y\,,0)$ greater than $\rho \cdot \frac{q+1}{q-1}$ from the origin, then its parent is closer (by a factor of at least $2/(q+1) < 1$) to the origin. So, we can conclude that every vertex $\omega_j(y)$ in $G(\underline{\Lambda})$ with distance $d(y\,,0) > \rho \cdot \frac{q+1}{q-1}$ from the origin is a successor of a vertex in $\underline{\Lambda} \cap B_{\leq \rho \cdot \frac{q+1}{q-1}}(0)$ (compare this bound with the bound on the radius of the balls used in the proof of Proposition 4.122). But this is a finite set by the compactness of the closed bounded ball, so we may take this as candidate for the finite seed. But the conditions on the finite seed are easily checked, since no vertex inside this ball has a predecessor from outside this ball and all vertices outside are successors of vertices inside the ball. The finiteness of this seed and the fixed point property $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ consequently yield the existence of a directed cycle of vertices of this seed. □

Remark 5.72. In the proof of [221, Lemma 3.2] (which applies to $\mathbb{R}^d$), first an equivalent metric is defined. Our proof shows that this is not necessary.

We want to understand the cycle structure of $G(\underline{\Lambda})$ better. We note that all directed cycles are inside the finite seed constructed in the proof of Lemma 5.71.

Definition 5.73. Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a substitution multi-component Delone set in an LCAG G such that the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)**. We say that $\underline{\Lambda}$ is *indecomposable* if it cannot be partitioned into two nonempty multi-component sets $\underline{\Lambda}^{(1)}$ and $\underline{\Lambda}^{(2)}$ (*i.e.*, $\underline{\Lambda} = \underline{\Lambda}^{(1)} \cup \underline{\Lambda}^{(2)} = (\Lambda_i^{(1)} \cup \Lambda_i^{(2)})_{i=1}^n$ and $\underline{\Lambda}^{(1)} \cap \underline{\Lambda}^{(2)} = (\Lambda_i^{(1)} \cap \Lambda_i^{(2)})_{i=1}^n = (\varnothing)_{i=1}^n$, but at least one component of each multi-component set is nonempty) such that $\underline{\Lambda}^{(i)} = \boldsymbol{\Theta}(\underline{\Lambda}^{(i)})$ for $1 \leq i \leq 2$. Otherwise, it is called *decomposable*. If $\underline{\Lambda}$ is decomposable, say $\underline{\Lambda} = \underline{\Lambda}^{(1)} \cup \underline{\Lambda}^{(2)}$, we say that $\underline{\Lambda}^{(1)}$ is an *indecomposable component* of $\underline{\Lambda}$ if there is no nonempty multi-component set $\underline{\Lambda}' \subset \underline{\Lambda}^{(1)}$ with $\underline{\Lambda}' = \boldsymbol{\Theta}(\underline{\Lambda}')$. Note that the complement $(\underline{\Lambda}')^c = (\Lambda_i \setminus \Lambda_i')_{i=1}^n$ of a multi-component set $\underline{\Lambda}' \subset \underline{\Lambda}$ which satisfies $\underline{\Lambda}' = \boldsymbol{\Theta}(\underline{\Lambda}')$ also is a fixed point $(\underline{\Lambda}')^c = \boldsymbol{\Theta}((\underline{\Lambda}')^c)$ by the disjointness of the unions in Equation (5.5) on p. 149.

Note that each indecomposable component defines a subgraph of $G(\underline{\Lambda})$.

Corollary 5.74. *Let $\underline{\Lambda}$ be a substitution multi-component Delone set. Then the number of directed cycles in $G(\underline{\Lambda})$ equals the number of its indecomposable components.*

Proof. The proof of Lemma 5.71 also implies that each indecomposable component contains at least one directed cycle. Suppose such an indecomposable component contains more than one cycle. Since it is indecomposable, there is at least one vertex in $G(\underline{\Lambda})$ for which the vertices of at least two cycles are predecessors (obviously, if one vertex of a directed cycle is a predecessor then all vertices of this cycle are predecessors), otherwise it would not be indecomposable. But this contradicts the disjointness of the unions in Equation (5.5) on p. 149 and establishes the claim. □

Remark 5.75. Obviously, each directed cycle is a finite seed for the corresponding indecomposable component of a substitution multi-component Delone set. Therefore, the (with respect to set inclusion) minimal finite seed of a substitution multi-component Delone set consists of the union of the vertices of all directed cycles of $G(\underline{\Lambda})$.

We have already noted that the EMFS $\boldsymbol{\Theta}$ defines a substitution on (multi-component) point sets of G via

$$\boldsymbol{\Theta}\left(\omega_i(x)\right) = \bigcup_{j=1}^{n} \bigcup_{t_a \circ f_0 \in \Theta_{ji}} \omega_j(f_0(x) + a)$$

(noting that a MFS $\boldsymbol{\Theta}$ is linear on the point set, wherefore it suffices to consider the action on its elementary parts). But we may also consider the *associated tile substitution* to $\boldsymbol{\Theta}$ of the tiling/covering/packing $\mathcal{T} = \underline{\Lambda} + \underline{A}$ as follows:

- There are two steps. Either we subdivide and inflate, *i.e.*,
  - (i) Replace the (proto)tiles $\underline{A}$ by applying the adjoint IFS. Then, from $\mathcal{T} = \underline{\Lambda} + \underline{A}$ we obtain $\mathcal{T}' = \underline{\Lambda} + \boldsymbol{\Theta}^{\#}(\underline{A})$.
  - (ii) The prototiles in $\mathcal{T}'$ are given by $(f_0^{-1}(A_i))_{i=1}^{n}$. To obtain again a tiling with the prototiles $(A_i)_{i=1}^{n}$, we apply the map f_0 to $\mathcal{T}'$.
- Or we inflate and then subdivide, *i.e.*,
  - (i) Apply the map f_0 to $\mathcal{T}$ to obtain $\mathcal{T}'' = f_0(\underline{\Lambda}) + f_0(\underline{A})$, where we have $f_0(\omega_i(x)) = \omega_i(f_0(x))$. Therefore, $\mathcal{T}''$ has prototiles $(f_0(A_i))_{i=1}^{n}$.
  - (ii) To obtain a tiling with the prototiles $(A_i)_{i=1}^{n}$, we again use the adjoint IFS $\underline{A} = \boldsymbol{\Theta}^{\#}(\underline{A})$, but in its "inflated version" $f_0(\underline{A}) = f_0(\boldsymbol{\Theta}^{\#}(\underline{A}))$, *i.e.*,

    $$f_0(A_i) = \bigcup_{j=1}^{n} \bigcup_{t_{(a)} \circ f_0 \in \Theta_{ji}} A_j + a,$$

    to replace the "inflated prototiles" $f_0(A_i)$ by translated copies of the original prototiles A_i.

  Obviously, both methods yield the same result (by the linearity of f_0).

The combined action of inflation and subdividing (respectively subdividing and inflation) on a single (proto)tile is called a *tile substitution*, *i.e.*, since a single tile in $\mathcal{T}$ is given by $\omega_i(x) + \underline{A} = (\varnothing, \ldots, \varnothing, A_i + x, \varnothing, \ldots, \varnothing) = A_i + x$, the tile substitution is given by

$$A_i + x \quad \mapsto \quad \Big(A_j + t_{(a)} \circ f_0(x) \mid 1 \leq j \leq n, t_{(a)} \circ f_0 \in \Theta_{ji}\Big). \tag{5.7}$$

Moreover, by Corollary 5.63, the unions that occur in Equation (5.7) are measure-disjoint, and this also holds for higher iterates of the tile substitution. So, starting with a single tile and applying the tile substitution over and over again (maybe with some overall translations), one might obtain a tiling in the limit. We already note that, as a consequence of this measure-disjointness, Definition 5.84 will be sensible.

We are now interested how the result of the tile substitution relates to the collection $\underline{\Lambda} + \underline{A}$ we started with. In the following statement, we do not assume that the sets A_i are tiles for now (so they may have empty interior), they are just the solution of the IFS.

Lemma 5.76. *Let $\underline{\Lambda}$ be a substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)***. Denote by $\underline{A}$ the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$. Then, $\underline{\Lambda} + \underline{A}$ has the following* self-replicating property *with respect to f_0^{-1} and f_0:*

$$
\begin{aligned}
f_0^{-1}(\underline{\Lambda} + \underline{A}) &\overset{\substack{\text{linearity}\\ \text{of } f_0^{-1}}}{=} \big(f_0^{-1}(\Lambda_i) + f_0^{-1}(A_i) \mid 1 \le i \le n\big) \\
&\overset{\substack{\text{using the}\\ \text{EMFS } \boldsymbol{\Theta}\\ \text{and the}\\ \text{disjointness}}}{=} \left(f_0^{-1}\left(\bigcup_{j=1}^{n} \bigcup_{t_{(a)} \circ f_0 \in \Theta_{ij}} t_{(a)} \circ f_0(\Lambda_j) \right) + f_0^{-1}(A_i) \mid 1 \le i \le n \right) \\
&\overset{\substack{\text{linearity}\\ \text{of } f_0^{-1}}}{=} \Big(t_{(f_0^{-1}(a))}(\Lambda_j) + f_0^{-1}(A_i) \mid t_{(a)} \circ f_0 \in \Theta_{ij},\ 1 \le i,j \le n\Big) \\
&\overset{\substack{\text{linearity of}\\ \text{translations}}}{=} \Big(\Lambda_j + t_{(f_0^{-1}(a))}(0) + f_0^{-1}(A_i) \mid t_{(a)} \circ f_0 \in \Theta_{ij},\ 1 \le i,j \le n\Big) \\
&\overset{\substack{\text{linearity}\\ \text{of } f_0^{-1}}}{=} \Big(\Lambda_j + f_0^{-1} \circ t_{(a)}(A_i) \mid t_{(a)} \circ f_0 \in \Theta_{ij},\ 1 \le i,j \le n\Big) \\
&\overset{\substack{\text{using the}\\ \text{IFS } \boldsymbol{\Theta}^{\#}}}{=} \big(\Lambda_j + A_j \mid 1 \le j \le n\big) = \underline{\Lambda} + \underline{A}
\end{aligned}
$$

respectively

$$
\begin{aligned}
f_0(\underline{\Lambda} + \underline{A}) &\overset{\substack{\text{linearity}\\ \text{of } f_0}}{=} (f_0(\Lambda_i) + f_0(A_i) \mid 1 \le i \le n) \\
&\overset{\substack{\text{using the}\\ \text{IFS } \boldsymbol{\Theta}^{\#}}}{=} \left(f_0(\Lambda_i) + f_0\left(\bigcup_{j=1}^{n} \bigcup_{t_{(a)} \circ f_0 \in \Theta_{ji}} f_0^{-1} \circ t_{(a)}(A_j) \right) \mid 1 \le i \le n \right) \\
&\overset{\substack{\text{linearity of}\\ f_0\ \&\ t_{(\cdot)}}}{=} \Big(f_0(\Lambda_i) + \big(A_j + t_{(a)}(0)\big) \mid t_{(a)} \circ f_0 \in \Theta_{ji},\ 1 \le i,j \le n\Big) \\
&\overset{\substack{\text{linearity of}\\ \text{translations}}}{=} \Big(t_{(a)} \circ f_0(\Lambda_i) + A_j \mid t_{(a)} \circ f_0 \in \Theta_{ji},\ 1 \le i,j \le n\Big) \\
&\overset{\substack{\text{using the}\\ \text{EMFS } \boldsymbol{\Theta}\\ \text{and the}\\ \text{disjointness}}}{=} \big(\Lambda_j + A_j \mid 1 \le j \le n\big) = \underline{\Lambda} + \underline{A}.
\end{aligned}
\tag{5.8}
$$

In the step "disjointness", we make use of the disjointness in the definition of a substitution multi-component Delone set. □

Definition 5.77. Let $(X, \mathfrak{C}, \mu)$ be a topological measure space. We say that a function $f : X \to \mathbb{R}$ is *locally constant μ-a.e.* if for every $x \in X \setminus N$, where N is a set of μ-measure 0 (possibly empty), there is a neighbourhood V of x such that $f(y) = f(x)$ for all $y \in V$. We say that f is *constant μ-a.e.* if there is a constant C such that $\mu(\{x \in X \mid f(x) \neq C\}) = 0$.

Obviously, a collection $\mathcal{T}$ of tiles is a tiling iff its covering degree is 1 μ-a.e.

Lemma 5.78. *Let $\underline{\Lambda}$ be a multi-component Delone set and $\underline{A}$ be a set of n compact sets in a metrisable LCAG G. Assume that the boundary of each A_i has Haar measure 0. Then the covering degree of $\underline{\Lambda} + \underline{A}$ is bounded and locally constant μ-a.e.*

Proof. The covering degree at x is given by

$$\deg^{cov}_{\underline{\Lambda}+\underline{A}}(x) = \sum_{i=1}^{n} \mathrm{card}\{t \in \Lambda_i \mid x \in t + A_i\} = \sum_{i=1}^{n} \mathrm{card}(\Lambda_i \cap (x - A_i)). \tag{5.9}$$

As finite union of compact sets, $\mathrm{supp}\,\underline{A} = \bigcup_{i=1}^{n} A_i$ is compact. Since $\mathrm{supp}\,\underline{\Lambda}$ is uniformly discrete, we have the estimate

$$\deg^{cov}_{\underline{\Lambda}+\underline{A}}(x) \le n \cdot \mathrm{card}\left((\mathrm{supp}\,\underline{\Lambda}) \cap (x - \mathrm{supp}\,\underline{A})\right) < \infty.$$

So, the covering degree is bounded.

Since $\underline{\Lambda}$ is countable, we have $\mu(\mathrm{supp}(\underline{\Lambda} + \partial\underline{A})) = 0$ by Proposition 4.5. Let $x \in \mathrm{supp}((\underline{\Lambda} + \partial\underline{A}))^c$, wherefore, for every $1 \le i \le n$ and every $t \in \Lambda_i$, the point x is either an interior point of $t + A_i$ or in the complement of the interior of $t + A_i$. So the covering degree at such an x is given by

$$\deg^{cov}_{\underline{\Lambda}+\underline{A}}(x) = \sum_{i=1}^{n} \mathrm{card}\{t \in \Lambda_i \mid x \in t + \mathrm{int}\,A_i\} = \sum_{i=1}^{n} \mathrm{card}(\Lambda_i \cap (x - \mathrm{int}\,A_i)).$$

We already know that the covering degree is bounded, therefore, for every $1 \le i \le n$, we have $\Lambda_i \cap (x - \mathrm{int}\,A_i) = \{t_{i1}, \ldots, t_{ik_i}\}$ with $k_i \in \mathbb{N}$ (or the empty set[15]). To each such t_{ij}, there is an open neighbourhood V_{ij} such that $x + V_{ij} \subset t_{ij} + \mathrm{int}\,A_i$ (in plain words, we may translate x by an element of V_{ij} and still remain inside A_i). Moreover, the compact set $x - \mathrm{supp}\,\underline{A}$ has positive distance, say ε, from the (since $\mathrm{supp}\,\underline{\Lambda}$ is uniformly discrete) closed set $(\mathrm{supp}\,\underline{\Lambda}) \setminus (x - \mathrm{supp}\,\underline{A})$ (so we may translate x by an element of $B_{<\varepsilon}(0)$ without getting any "new" element from any Λ_i in the above intersection in Equation (5.9)). Taking the intersection of all the (finitely many) V_{ij} and $B_\varepsilon(x)$, yields an open neighbourhood of x of constant covering degree. This establishes the claim. □

Lemma 5.79. *Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a repetitive substitution multi-component Delone set in an LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**. *Denote by $\underline{A}$ the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$. If the sets A_i are tiles, the covering degree of $\mathcal{T} = \underline{\Lambda} + \underline{A}$ is constant μ-a.e and greater than or equal to* 1.

Proof. By Lemma 5.78, the covering degree is bounded and locally constant μ-a.e. The proof of Lemma 5.78 shows that, for every $x \in \mathrm{supp}((\underline{\Lambda}+\partial\underline{A}))^c$, there is a neighbourhood V_x such that $\deg^{cov}_{\mathcal{T}}(x) = \deg^{cov}_{\mathcal{T}}(x')$ for all $x' \in V_x$. We have to show that for any pair $x, y \in \mathrm{supp}((\underline{\Lambda}+\partial\underline{A}))^c$ one has $\deg^{cov}_{\mathcal{T}}(x) = \deg^{cov}_{\mathcal{T}}(y)$.

Since the sets A_i are tiles, we can choose an $x \in \mathrm{supp}((\underline{\Lambda} + \partial\underline{A}))^c$ with $\deg^{cov}_{\mathcal{T}}(x) \ge 1$. Consequently, there is a radius $r > 0$ such that at all $x' \in B_{<r}(x)$ one has the same covering degree. We now use the self-replicating property to obtain bigger and bigger open balls which have the same μ-a.e. constant covering degree. To this end, we observe the following:

- Since f_0 is an expansion, there is a $q > 1$ such that $d(f_0(x'), f_0(x'')) \ge q \cdot d(x', x'')$ for all $x', x'' \in G$. Therefore, we also have $B_{<q \cdot r}(f_0(x)) \subset f_0\left(B_{<r}(x)\right)$.

[15] If $\Lambda_i \cap (x - \mathrm{int}\,A_i) = \varnothing$, the compact set $\{x\}$ has positive distance, say ε', from the closed set $\Lambda_i + A_i$. We then use the open neighbourhood $V_{i1} = B_{<\varepsilon'}(x)$ of x in the following construction.

- By Corollary 5.63, the unions in the adjoint IFS $\boldsymbol{\Theta}^{\#}$ are measure-disjoint and the boundaries have Haar measure 0. Trivially, we have $\deg^{cov}_{f_0(A_i)}(x') = 1$ for all $x' \in \operatorname{int} f_0(A_i)$. Applying the tile substitution, we have

$$f_0(A_i) \quad \mapsto \quad \Big(A_j + t_{(a)}(0) \mid 1 \le j \le n, t_{(a)} \circ f_0 \in \Theta_{ji}\Big),$$

  and the covering degree of the collection of compact sets on the right hand side equals 1 for μ-almost every $x' \in \operatorname{int} f_0(A_i)$ (since we may only get overlaps on the boundaries of the tiles but not on interior points).

- By the definition of the substitution multi-component Delone set $\underline{\Lambda}$, the unions in Equation (5.5) on p. 149 are disjoint (so here the disjointness enters). Consequently, every vertex $\omega_i(x)$ in $G(\underline{\Lambda})$ has exactly one parent.

Now, the covering degree of $(\underline{\Lambda} + \underline{A}) \cap B_{<r}(x)$ inside $B_{<r}(x)$ equals the covering degree of the expanded version $f_0(\underline{\Lambda}+\underline{A}) \cap f_0(B_{<r}(x))$ (where we here mean by $f_0(\underline{\Lambda}+\underline{A})$ the associated tiling of the multi-component Delone set $(f_0(\Lambda_i))_{i=1}^n$ with the (expanded) prototiles $(f_0(A_i))_{i=1}^n$). Looking at the series of equations in Equation (5.8) on p. 156, the covering degree of $f_0(\underline{\Lambda}+\underline{A})$ and $\underline{\Lambda} + \underline{A}$ are the same at μ-almost every $x \in f_0(B_{<r}(x))$. So the covering degree of $\mathcal{T}$ inside $B_{<r}(x)$ and inside $B_{<q\cdot r}(f_0(x))$ are μ-a.e. constant and the same. We can now iterate this procedure, wherefore the covering degree inside $B_{<q^k\cdot r}(f_0^k(x))$ is the same for every $k \in \mathbb{N}$.

Let $y \in \operatorname{supp}((\underline{\Lambda}+\underline{\partial A}))^c$. Then the covering degree inside the neighbourhood V_y is constant. By repetitivity[16], we have the following: For every relatively compact neighbourhood W of y, there is a $k \in \mathbb{N}$ such that a translate of W is found inside $B_{<q^k\cdot r}(f_0^k(x))$. Consequently, $\deg^{cov}_{\mathcal{T}}(x) = \deg^{cov}_{\mathcal{T}}(y)$ for every $y \in \operatorname{supp}((\underline{\Lambda} + \underline{\partial A}))^c$, which proves the claim. □

Remark 5.80. The argument in the previous proof can basically be found in [192, Proof of Theorem 1.1].

From the Lemmas 5.67 & 5.76 & 5.79, we now obtain the following statement.

Corollary 5.81. *Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a repetitive substitution multi-component Delone set in an LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**. *Denote by $\underline{A}$ the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$. Then the sets A_i are tiles, and $\mathcal{T} = \underline{\Lambda}+\underline{A}$ is a self-replicating covering of μ-a.e. constant covering degree.* □

In view of Lemmas 5.78 & 5.79 respectively Corollary 5.81, we make the following definition.

Definition 5.82. Let $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$ be a substitution multi-component Delone set in an LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)**. Denote by $\underline{A}$ the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$. We say that $\underline{\Lambda}$ is *representable* if $\underline{\Lambda} + \underline{A}$ is a tiling of G. In this case, $\underline{\Lambda} + \underline{A}$ is said to be the *associated tiling of* $\underline{\Lambda}$ (with prototiles $\{A_i\}_{i=1}^n$).

We note that a given EMFS does not have a unique solution (while its associated adjoint IFS has a unique solution). Consequently, there can be solutions of the EMFS which are representable and solutions which are not representable. This is shown in Example 6b.2 for

[16] Actually, we use the repetitivity of the associated tiling, *i.e.*, repetitivity of a tiling and not of an FLC multi-component set. But both definitions of repetitivity are – *mutatis mutandis* – parallel, and the associated tiling is repetitive iff the FLC multi-component set is.

the Thue-Morse substitution (see Example 5.64). Note that these findings are closely related to the notions "(in)decomposable" (see Definition 5.73 and [221]).

Obviously, if $\underline{\Lambda} + \underline{A}$ is a tiling of G where $\underline{\Lambda}$ is a multi-component Delone set, then the sets A_i are tiles by Lemma 5.67. Therefore, representability implies that each set A_i is a (proto)tile (as does repetitivity!).

We are now interested in two points: We want to justify why it is in principle enough to look at primitive (instead of only irreducible) substitutions, and we want to have a criterion for representability and/or repetitivity. We begin with the following observation, which also sheds more light on the calculations in Lemma 5.76.

Let $\underline{\Lambda}$ be a substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)**, and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. Then, by Lemma 5.79, the covering $\mathcal{T} = \underline{\Lambda} + \underline{A}$ has μ-a.e. constant covering degree, wherefore $\underline{\Lambda}$ is representable iff the covering degree is 1 μ-a.e.

Definition 5.83. We say that a finite set $\mathcal{P}$ of tiles is a *patch* if the tiles of $\mathcal{P}$ have mutually disjoint interiors, *i.e.*, for every pair $T_i, T_j \in \mathcal{P}$ with $T_i \neq T_j$ we have $\operatorname{supp}(\operatorname{int} T_i) \cap \operatorname{supp}(\operatorname{int} T_j) = \varnothing$. By a translate $\mathcal{P} + t$ of a patch $\mathcal{P}$ (with $t \in G$) we mean that all tiles of $\mathcal{P}$ are translated by t (of course, this is also a patch). A subpatch of a patch $\mathcal{P}$ is a subset of the tiles of $\mathcal{P}$ and also a patch.

Similarly, for a multi-component set $\underline{\Lambda}$, we say that $\underline{\mathcal{P}} = (\mathcal{P}_i)_{i=1}^n$ is a *cluster* if $\mathcal{P}_i \subset \Lambda_i$ is finite for all $1 \leq i \leq n$. Subcluster and translate of a cluster are defined accordingly.

Definition 5.84. Let $\underline{\Lambda}$ be a substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)**, and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. Then a patch $\mathcal{P}$ of the covering $\mathcal{T} = \underline{\Lambda} + \underline{A}$ will be called *legal* if, for a $k \in \mathbb{N}$ and some $1 \leq i \leq n$, it is a translate of a subpatch of the k-th iterate of the tile substitution of A_i.

Similarly, a cluster $\underline{\mathcal{P}}$ of $\underline{\Lambda}$ is called *legal* if, for a $k \in \mathbb{N}$ and some $1 \leq i \leq n$, it is a translate of a subcluster of $\boldsymbol{\Theta}^k(\omega_i(0))$.

Legality of all patches of $\mathcal{T}$ respectively all clusters of $\underline{\Lambda}$ has several consequences. Obviously, if $\underline{\Lambda}$ is representable, the two notions "patch" and "cluster" (and therefore the two uses of the word "legal") coincide.
First, we begin with a statement about the structure of irreducible matrices, see [144, Theorem XIII.2 & Definition XIII.3] and compare Proposition 4.69(ii).

Lemma 5.85. *An irreducible nonnegative matrix $\boldsymbol{A}$ always has a positive eigenvalue α that is simple. Moreover, if $\boldsymbol{A}$ has h eigenvalues of modulus α (the number h is called the* index of imprimitivity *of $\boldsymbol{A}$), then these eigenvalues are all distinct. If $h > 1$, then $\boldsymbol{A}$ can be put by means of a permutation into the following "cyclic" form:*

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{0} & \boldsymbol{A}_{1,2} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_{2,3} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \boldsymbol{A}_{h-1,h} \\ \boldsymbol{A}_{h,1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix},$$

where there are square blocks along the main diagonal. The matrix $\boldsymbol{A}$ is primitive iff $h = 1$. □

We observe that the h-th matrix power of an irreducible nonnegative matrix $\boldsymbol{A}$ has the following form:

$$\boldsymbol{A}^h = \begin{pmatrix} \boldsymbol{A}_{1,2}\boldsymbol{A}_{2,3}\cdots\boldsymbol{A}_{h-1,h}\boldsymbol{A}_{h,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_{2,3}\cdots\boldsymbol{A}_{h-1,h}\boldsymbol{A}_{h,1}\boldsymbol{A}_{1,2} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots \\ \mathbf{0} & \mathbf{0} & \ddots & \boldsymbol{A}_{h,1}\boldsymbol{A}_{1,2}\boldsymbol{A}_{2,3}\cdots\boldsymbol{A}_{h-1,h} \end{pmatrix},$$

where the square blocks along the main diagonal are primitive matrices. Moreover, if we take the n^2-2n+2-th power of $\boldsymbol{A}^h$, then the square blocks along the main diagonal are surely positive matrices, see Remark 4.68.

For the EMFS $\boldsymbol{\Theta}$ respectively the adjoint IFS $\boldsymbol{\Theta}^{\#}$, this last lemma has the following consequence: Let $\underline{\Lambda}$ be a substitution multi-component Delone set with **(LT)**. In particular, $\boldsymbol{S\Theta}$ is irreducible, and we denote its index of imprimitivity by h. Then we can partition the family $\underline{\Lambda} = (\Lambda_i)_{i=1}^n$ of Delone sets into exactly h equivalence classes $[\Lambda_i]$ as follows: Λ_i is equivalent to Λ_j if for one (or, equivalently, every) pair $x \in \Lambda_i$, $y \in \Lambda_j$ the same components in $\boldsymbol{\Theta}^{h\cdot(n^2-2n+2)}(\omega_i(x))$ and $\boldsymbol{\Theta}^{h\cdot(n^2-2n+2)}(\omega_j(y))$ are (non)empty. Consequently, if $\omega_i(x) \subset \Lambda_i$, then, for every $k \in \mathbb{N}$, the nonempty components of $\boldsymbol{\Theta}^k(\omega_i(x))$ belong to one such equivalence class. Moreover, the i-th component of $\boldsymbol{\Theta}(\omega_i(x))$ is empty if $h > 1$. The same statement holds (by matrix transposition) for the IFS $\boldsymbol{\Theta}^{\#}$. Note that equivalence classes for the IFS and the EMFS are in direct correspondence: $(\Lambda_{i_1}, \dots, \Lambda_{i_\ell})$ forms an equivalence class iff $(A_{i_1}, \dots, A_{i_\ell})$ forms an equivalence class. We call the equivalence classes defined here, the *imprimitivity equivalence classes*.

Corollary 5.86. *Let $\underline{\Lambda}$ be a substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**, *and let $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. If every patch of the covering $\mathcal{T} = \underline{\Lambda} + \underline{A}$ is legal, then the EMFS $\boldsymbol{\Theta}$ is primitive and $\mathcal{T}$ is a tiling.*

Proof. If a patch is legal, all its tiles must belong to the same imprimitivity equivalence class. Since every patch is legal, no patch of $\mathcal{T}$ contains tiles in different imprimitivity equivalence classes. Also, $\underline{\Lambda}$ (and therefore also $\mathcal{T}$) is a fixed point of the substitution, wherefore there is only one imprimitivity equivalence class. So, primitivity follows.

By legality, every patch is a subpatch of some iterate of the tile substitution of some single tile. But the covering degree of such an iterate is, by the measure-disjointness of the tile substitution (see Equation (5.7) on p. 155), at most 1 μ-a.e. Therefore, every patch of $\mathcal{T}$ has covering degree at most 1 μ-a.e. and $\mathcal{T}$ is a packing. Suppose that $\mathcal{T}$ is not a tiling, then there exists an open set V such that $\deg_{\mathcal{T}}^{cov}(x) = 0$ for all $x \in V$. As in the proof of Lemma 5.79, we now obtain bigger and bigger balls of zero covering degree. This contradicts the assumption that $\underline{\Lambda}$ is Delone. Therefore, $\mathcal{T}$ is a tiling. □

Lemma 5.87. *Let $\underline{\Lambda}$ be a substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**, *and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. If every cluster of $\underline{\Lambda}$ is legal, then the EMFS $\boldsymbol{\Theta}$ is primitive and $\underline{\Lambda}$ is representable.*

Proof. Primitivity follows as in the previous proof.

Also, the previous proof shows that we have $\deg^{cov}_{\underline{\Lambda}+\underline{A}} \geq 1$ μ-a.e. Moreover, since by assumption the sets A_i are tiles, the tile substitution and the substitution on the point set line up, *i.e.*, a cluster $\underline{\mathcal{P}}$ is legal iff the patch $\underline{\mathcal{P}} + \underline{A}$ is legal. Therefore representability also follows from the previous proof. □

Remark 5.88. The previous lemma in a Euclidean space can be found as [235, Theorem 3.7].

Corollary 5.89. *Let $\underline{\Lambda}$ be a primitive substitution FLC multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**, *and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. Suppose that $\mathcal{T} = \underline{\Lambda} + \underline{A}$ is a tiling, i.e., $\underline{\Lambda}$ is representable. Then, every patch of $\mathcal{T}$ is legal iff $\mathcal{T}$ is repetitive.*

Proof. The proof of [235, Lemma 3.3] (although given for $\mathbb{R}^d$) also applies here. □

Corollary 5.90. *Let $\underline{\Lambda}$ be a repetitive primitive substitution multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**, *and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. Then every cluster of $\underline{\Lambda}$ is legal iff $\underline{\Lambda}$ is representable.*

Proof. This is [235, Corollary 3.8] in the Euclidean setting. Basically, the only thing to note is that under representability a cluster $\underline{\mathcal{P}}$ is legal iff the patch $\underline{\mathcal{P}} + \underline{A}$ is legal. □

Remark 5.91. A consequence of Lemma 5.71 respectively Remark 5.75 is the following, see [235, Remark 3.9]: In order to check that every cluster of a substitution multi-component Delone set is legal, one only needs to check if some cluster containing a finite seed is legal.

We now want to give a justification of the property **(LT)** and why we only look at primitive substitutions from now on.

Remark 5.92. Obviously, we want all maps $f \in \boldsymbol{\Theta}$ to be bi-Lipschitz bijections to ensure the existence of an adjoint IFS. Irreducibility ensures the existence of an attractor of this IFS. Moreover, we also used that the substitutions are affine and that they have the same linear part f_0 in the definition of the adjoint IFS. Actually, these conditions (and, of course, the definition of the adjoint IFS) are also chosen as to have the self-replicating property in Lemma 5.76. For a repetitive self-replicating substitution multi-component set, we can then prove that the covering degree is constant almost everywhere (in Lemma 5.79), therefore we might have a chance that the substitution multi-component set under consideration is representable. Now, for representable multi-component Delone set, we have the above interplay between legality, representability and repetitivity. Here, however, primitivity enters automatically.

So far, we have not justified property **(LT)**(iii). On the one hand, it is needed to have Proposition 4.99 available, *i.e.*, to have that the components of the attractor of the adjoint IFS are proper tiles (given interior points, see Corollary 5.63). On the other hand, it is suggested by the following result in $\mathbb{R}^d$ for substitution multi-component Delone sets, which we conjecture to hold also in the general situation.

In $\mathbb{R}^d$, the map f_0 is given by $x \mapsto \boldsymbol{Q}x$ where $\boldsymbol{Q}$ is some matrix over $\mathbb{R}$. We note that the application of this map to any (Lebesgue measurable) subset of $\mathbb{R}^d$ increases the measure by a factor of $|\det \boldsymbol{Q}|$. The following is stated as *Perron eigenvalue condition* in [221].

Lemma 5.93. *Let $\boldsymbol{\Theta}$ be a primitive EMFS on $\mathbb{R}^d$, where all substitutions f are affine maps of the form $f(x) = \boldsymbol{Q}x + t_f$ (i.e., they only differ in the translation $t_f \in \mathbb{R}^d$) with invertible matrix $\boldsymbol{Q}$. Let λ be the PF-eigenvalue (and therefore the spectral radius) of $\boldsymbol{S\Theta}$. Suppose $\underline{\Lambda}$ is a substitution multi-component set such that $\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})$. Then the following hold:*

(i) $\operatorname{supp} \underline{\varLambda}$ *is uniformly discrete only if* $|\det \boldsymbol{Q}| \leq \lambda$.

(ii) *The components of the attractor* $\underline{A}$ *of the adjoint IFS* $\boldsymbol{\varTheta}^{\#}$ *have interior points only if* $|\det \boldsymbol{Q}| \geq \lambda$. □

In plain words, the first condition says that in order to have uniform discreteness, not too many points can be added in each substitution step. For the second condition, we observe the following: $|\det \boldsymbol{Q}|$ is the absolute value of the product of the eigenvalues of $\boldsymbol{Q}$; moreover, since an invertible matrix $\boldsymbol{Q}$ can be written in a unique way as a product of a unitary matrix and the matrix $(\boldsymbol{Q}^*\boldsymbol{Q})^{1/2}$ (see [226, Theorem XV.6.9]), one also has $|\det \boldsymbol{Q}| = |\det((\boldsymbol{Q}^*\boldsymbol{Q})^{1/2})|$, wherefore the alternative definition of the singular values in Remark 4.110 (in $\mathbb{R}^d$) and Corollary 4.118 establishes that $\overline{\dim}_{\mathrm{aff}} \boldsymbol{\varTheta} \geq d$. So, only under this condition it is possible that $\dim_{\mathrm{Hd}} A_i = d$ by Proposition 4.122 (otherwise, the Hausdorff dimension is strictly smaller than d and therefore A_i has no interior points).

5.5. An Interlude on Dynamical Systems

Geometrical properties of an FLC multi-component Delone set are often reflected in properties of its associated point set dynamical system. Therefore, we introduce dynamical systems here.

Definition 5.94. Let G be a group[17] and X be a nonempty set. The pair (X, G) is called a *dynamical system* if there is a G-action φ on X, *i.e.*, if we have a map $\varphi : G \times X \to X$ with the following property: For any $g \in G$, define the map $T_g : X \to X$ by the relation $T_g(x) = \varphi(g, x)$. Then the transformations T_g determine a representation of the group G, *i.e.*, $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$ for all $g_1, g_2 \in G$ and $T_e = \mathrm{id}$ for the unit element $e \in G$. The group (semigroup) of transformations $\{T_g \,|\, g \in G\}$ is called a *G-flow* (a *G-semiflow*) on X. If (X, G) is a dynamical system, then $\mathcal{O}(x) = \{T_g(x) \,|\, g \in G\}$ is called the *orbit* of $x \in X$. We say that $x \in X$ is a *fixed point* if $\operatorname{card} \mathcal{O}(x) = 1$.

Often, there is more structure on G and X.

Definition 5.95. Let G be a measurable group (or semigroup) with σ-algebra $\mathfrak{B}$ and measure μ. For any measure space $(X, \mathfrak{C}, \nu)$, we can form the product measure space $(G \times X, \mathfrak{B} \otimes \mathfrak{C}, \mu \otimes \nu)$. Then, the quadruple $(X, \mathfrak{C}, \nu, G)$ (respectively, the triple (X, ν, G) if the σ-algebra is clear) is called a *measure theoretical dynamical system* if

- (X, G) is a dynamical system.
- For any $g \in G$, the map $T_g : X \to X$ is an *automorphism*[18] of the (measure) space X, *i.e.*, the maps T_g are bijective maps of X onto itself such that for any $A \in \mathfrak{C}$ we have $T_g(A), T_g^{-1}(A) \in \mathfrak{C}$ and $\nu(A) = \nu(T_g^{-1}(A)) = \nu(T_g(A))$, where $T_g^{-1}(A)$ is the inverse image of a set A. We also say that the maps T_g are *measure-preserving.*

If T is an automorphism, the measure ν is said to be an *invariant measure* for that automorphism or simply a *T-invariant measure.* An automorphism T is called *ergodic* if the only members $A \in \mathfrak{C}$ with $T^{-1}(A) = A$ satisfy $\nu(A) = 0$ or $\nu(A^c) = 0$. In this case, we also say that the measure ν is an *ergodic* T-invariant measure. The measure theoretical dynamical system $(X, \mathfrak{C}, \nu, G)$ is called *ergodic* if all automorphisms T_g are ergodic.

[17] One can also consider the case, where G is only a semigroup with unit element.

[18] If G is a semigroup, the map T_g is an *endomorphism* of the (measure) space X, *i.e.*, the map T_g is a surjective map of the space X onto itself such that for any $A \in \mathfrak{C}$ we have $T_g^{-1}(A) \in \mathfrak{C}$ and $\nu(A) = \nu(T_g^{-1}(A))$.

Moreover, often the space X is a compact topological space, wherefore the measure ν can be normalised such that it is a probability measure (*i.e.*, $\nu(X) = 1$). Furthermore, we can also define dynamical systems on topological spaces.

Definition 5.96. Let G be a topological group (or semigroup). For any topological space X, we can form the product space $G \times X$ (equipped with the product topology). Then, the pair (X, G) is called a *topological dynamical system* if

- (X, G) is a dynamical system.
- For any $g \in G$, the map $T_g : X \to X$ is a *homeomorphism*[19].

A subset $A \subset X$ is called *G-invariant* if $T_g(A) = A$ for all $g \in G$. A nonempty closed G-invariant subset $A \subset X$ is called *minimal* if there exists no proper closed G-invariant subset of A. If X itself is minimal, one says that the topological dynamical system (X, G) is *minimal*. We also say that a homeomorphism T is *minimal* if the only nonempty closed T-invariant subset of X is X itself.

Definition 5.97. The *hull* of an element $x \in X$ is the closure of its orbit $\mathcal{O}(x)$ with respect to the topology of X. We denote the hull of x by $\mathbb{X}(x)$, *i.e.*, $\mathbb{X}(x) = \operatorname{cl} \mathcal{O}(x)$.

If (X, G) is minimal, the hull of any element $x \in X$ equals X, *i.e.*, $\mathbb{X}(x) = X$ for all $x \in X$.

One is interested on the interplay between these two notions of dynamical systems. We make the following assumptions on G and X, which will enable us to compare topological and measure theoretical dynamical systems in our setting:

- G is an LCAG. By Proposition 4.30, G admits a Haar measure μ, where the σ-algebra is given by the Borel sets (see Definition 4.12).
- The space X is a compact[20] topological Hausdorff space. We let the σ-algebra $\mathfrak{C}$ of X be given by the Borel sets of X and the measure ν be a Borel probability measure.
- Often, the G-action is given by the translations $T_g(x) = x - g$. This is the *canonical action* of G on X. Obviously, the topology respectively the σ-algebra structure should line up with this G-action, *i.e.*, the maps T_g should be homeomorphisms and automorphisms (measure-preserving).

Sometimes, the topological group G is given by $\mathbb{Z}$ equipped with the discrete topology. Consequently, the maps T_g are given by $f^n = f \circ f \circ \ldots \circ f$, $(f^n)^{-1} = (f^{-1})^n = f^{-n}$ and $f^0 = \mathrm{id}$ for all $n \in \mathbb{N}$, where f an automorphism. In this case, the notation (X, f) (respectively (X, ν, f)) instead of $(X, \mathbb{Z})$ (respectively $(X, \nu, \mathbb{Z})$) is used, and one says that the dynamical system is *discrete*.

Let X be a compact metric space (with metric d) and $f : X \to X$ be a homeomorphism. We denote by $\mathfrak{M}(X)$ the set of all Borel measures of X, and by $\mathfrak{M}_f(X)$ the set of all f-invariant Borel measures. We note the following properties:

[19] If G is a semigroup, the map T_g is a continuous map on X.
[20] Compactness ensures that there exists a minimal subset of X, see [108, Proposition 7]

- $\mathfrak{M}(X)$ is a compact metrisable space[21], *e.g.*, see [109, Prop. 2.8] and [384, Theorems 6.4 & 6.5].

- $\mathfrak{M}(X)$ is convex, *i.e.*, $t \cdot \mu + (1-t) \cdot \nu \in \mathfrak{M}(X)$ for any $t \in [0\,,1]$ and $\mu, \nu \in \mathfrak{M}(X)$. We note that the *extreme* points of $\mathfrak{M}(X)$ (where a point is called extreme if it cannot be represented as a nontrivial convex combination of two other points) are the probability measures supported on points and are called *Dirac measures*.

- $\mathfrak{M}_f(X)$ is a compact and convex subset of $\mathfrak{M}(X)$. The extreme points of $\mathfrak{M}_f(X)$ are the ergodic f-invariant measures, see [109, Prop. 3.5], [80, Prop. 4.6.2] and [384, Theorem 6.10].

- If $f \neq \mathrm{id}$, then $\mathfrak{M}_f(X)$ is nowhere dense in $\mathfrak{M}(X)$, see [109, Prop. 3.5].

The Theorem of Krylov-Bogoliubov (alternative spellings Kryloff, Boglioboff, Bogolioubov and Bogolubov) asserts that the set $\mathfrak{M}_f(X)$ is nonempty.

Lemma 5.98. [109, Theorem 3.6],[96, Theorem I.§8.1], [384, Corollary 6.9.1] and [108, Satz 90] *For any continuous map f of the compact metric space X into itself, there exists an f-invariant Borel probability measure ν, i.e., $\mathfrak{M}_f(X)$ is nonempty .* □

The analogue for a G-flow follows from [148, Theorems 3.1 & 3.2].

Proposition 5.99. *Let G be a commutative topological group and X be a compact Hausdorff space, such that $(X\,,G)$ is a topological dynamical system. Then there exists a G-invariant Borel probability measure on X.* □

Definition 5.100. We say that the measure theoretical dynamical system (X, ν, G) is *supported* by the topological dynamical system $(X\,,G)$ if ν is a G-invariant Borel measure. Thus, the Theorem of Krylov-Bogoliubov, respectively its analogue for G-flows, asserts that every topological dynamical system (where G is a commutative topological group and X a compact Hausdorff space) supports at least one measure theoretical dynamical system. We say that a topological dynamical system $(X\,,G)$ is *uniquely ergodic* if it supports exactly one measure theoretical dynamical system. If $(X\,,G)$ is uniquely ergodic and minimal, we say that the topological dynamical system $(X\,,G)$ is *strictly ergodic*.

We also want to compare dynamical systems.

Definition 5.101. Let $(X\,,G)$ and $(X'\,,G)$ be two topological dynamical systems. Then, $(X'\,,G)$ is said to be a *factor* of $(X\,,G)$, if there exists a continuous surjective map $\phi : X \to X'$ such that $\phi \circ T_g = T'_g \circ \phi$ (note that we might have different representations of the group G),

[21] The *Prohorov metric* (alternative spelling Prohoroff) $\overline{d}$ defined by

$$\overline{d}(\mu\,,\nu) = \inf\{\varepsilon > 0 \mid \mu(B^\varepsilon) \leq \nu(B) + \varepsilon \text{ and } \nu(B^\varepsilon) \leq \mu(B) + \varepsilon \text{ for all } B \in \mathfrak{C}\}$$

where B^ε denotes the ε-fringe of B (see Definition 4.61), yields the so-called *weak-* topology* on $\mathfrak{M}(X)$ (see p. 346). The weak-* topology is also defined via the Riesz representation theorem by identifying the Borel measures with the positive linear functionals on the Banach space of all continuous complex values functions on X equipped with the supremum norm, *e.g.*, see [384, Section §6.1] and [244, Section I.8].

i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T_g} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X' & \xrightarrow{T'_g} & X' \end{array}$$

The map ϕ is called a *topological homomorphism*. If ϕ is a homeomorphism (*i.e.*, $(X\,, G)$ is also a factor of $(X'\,, G)$), then $(X\,, G)$ and $(X'\,, G)$ are said to be *topologically conjugate* and ϕ is said to be an *isomorphism* between the two topological dynamical systems.

Similarly, let $(X\,, \mathfrak{C}\,, \nu\,, G)$ and $(X'\,, \mathfrak{C}'\,, \nu'\,, G)$ be two measure theoretical dynamical systems. Then, $(X'\,, \mathfrak{C}'\,, \nu'\,, G)$ is said to be a *factor* of $(X\,, \mathfrak{C}\,, \nu\,, G)$, if there exist sets $B \in \mathfrak{C}$ and $B' \in \mathfrak{C}'$ and a map $\phi : B \to B'$ such that the following hold:

- B and B' have full measure, *i.e.*, $\nu(B) = 1$ and $\nu'(B') = 1$.
- ϕ is a measure-preserving map from[22] $(B\,, \mathfrak{C} \cap B\,, \nu|_B)$ onto $(B'\,, \mathfrak{C}' \cap B'\,, \nu'|_{B'})$.
- $\phi \circ T_g = T'_g \circ \phi$.

In this case, ϕ is called a *measure theoretic homomorphism*. If ϕ is an *isomorphism*, *i.e.*, ϕ is invertible and both ϕ and ϕ^{-1} are measure-preserving, between $(B\,, \mathfrak{C} \cap B\,, \nu|_B)$ and $(B'\,, \mathfrak{C}' \cap B'\,, \nu'|_{B'})$ (so, $(X\,, \mathfrak{C}\,, \nu\,, G)$ is also a factor of $(X'\,, \mathfrak{C}'\,, \nu'\,, G)$), then we say that the two measure theoretical dynamical systems are *measure theoretically conjugate* or *isomorphic* mod 0.

If two topological dynamical system are topologically conjugate and one of them is minimal/uniquely ergodic/strictly ergodic, so is the other. This follows from the observation that if $(X\,, G)$ and $(X'\,, G)$ are topologically conjugate (with isomorphism ϕ), then the measure theoretic dynamical systems[23] $(X\,, \mathfrak{C}\,, \nu\,, G)$ and $(X'\,, \mathfrak{C}'\,, \phi\nu\,, G)$ are measure theoretic conjugate for any G-invariant Borel probability measure ν on X, see [109, p. 19]. On the other hand, we note that measure theoretic conjugacy does not say anything about topological conjugacy.

5.6. Delone Sets and Dynamical Systems

We define a topology on the space of all uniformly discrete subsets of an LCAG G.

Definition 5.102. We denote by $\mathcal{D}_m$ the set of all multi-component uniformly discrete subsets[24] of an LCAG G with m components. We equip $\mathcal{D}_m$ with a topology, called the *local topology*, defined via the uniform structure given by the entourages

$$U_{\mathrm{LT}}(W, V) = \{(\underline{\Lambda}\,, \underline{\Lambda}') \in \mathcal{D}_m \times \mathcal{D}_m \mid (v + \underline{\Lambda}) \cap W = \underline{\Lambda} \cap W \text{ for some } v \in V\}$$

for $W \subset G$ compact and V a neighbourhood of $0 \in G$. Thus, two multi-component uniformly discrete sets are close if they agree on a "large" compact set up to a "small" (global) translation.

[22] We denote by $\mathfrak{C} \cap B$ the *trace σ-algebra* $\{A \cap B \mid A \in \mathfrak{C}\}$ and by $\nu|_B$ the *conditional measure* defined by $\nu|_B(A) = \nu(A \cap B)/\nu(B)$.

[23] Let X, X' be compact metric spaces and ϕ a continuous map from X into x'. Then, ϕ transports every measure $\nu \in \mathfrak{M}(X)$ into a measure $\phi\nu \in \mathfrak{M}(X')$. This map is again denoted by ϕ and called the *extension of* ϕ, see [109, Def. 3.1].

[24] So, each component is uniformly discrete and therefore also uniformly locally finite.

If $(G\,,d)$ is a metrisable LCAG such that with respect to the metric d every bounded set is totally bounded, the local topology can alternatively be defined via a metric d_{LT} as follows: For $\underline{\Lambda}, \underline{\Lambda}' \in \mathcal{D}_m$, let

$$d_{\mathrm{LT}}(\underline{\Lambda}\,, \underline{\Lambda}') = \min\left\{ d'_{\mathrm{LT}}(\underline{\Lambda}\,, \underline{\Lambda}')\,, \frac{1}{\sqrt{2}} \right\},$$

where

$$d'_{\mathrm{LT}}(\underline{\Lambda}\,, \underline{\Lambda}') = \inf\{\varepsilon > 0 \mid (v + \underline{\Lambda}) \cap B_{\leq 1/\varepsilon}(0) = \underline{\Lambda} \cap B_{\leq 1/\varepsilon}(0) \text{ for some } v \in B_{\leq \varepsilon}(0)\}.$$

The fact that d_{LT} is a metric (only the triangle inequality has to be checked), is proven basically in [234, p. 3–4] (although only the case $G = \mathbb{R}^d$ is considered there, the proof only makes use of the triangle inequality for d and therefore also works in the more generally setting here).

The following statement is proven in [329, Prop. 2.1] for $\mathcal{D}_1$, *i.e.*, for uniformly discrete subsets of an LCAG G, but the proof also applies to $\mathcal{D}_m$.

Lemma 5.103. *The topological space $\mathcal{D}_m$ equipped with the local topology is a complete Hausdorff space.* □

As is immediate from the definition of the local topology, the canonical action of G on $\mathcal{D}_m$ given by

$$G \times \mathcal{D}_m \to \mathcal{D}_m, \quad (t, \underline{\Lambda}) \mapsto -t + \underline{\Lambda},$$

is continuous. We are especially interested in the case, where we have a compact subset of $\mathcal{D}_m$ which is invariant under this action, *i.e.*, we are interested in minimal topological dynamical systems defined on subsets of $\mathcal{D}_m$. The following statement "constructs" such G-invariant compact subsets of $\mathcal{D}_m$. It is proven in [329, Prop. 2.3] for $\mathcal{D}_1$, but – as before – the proof extends to $\mathcal{D}_m$.

Lemma 5.104. *If $\underline{\Lambda}$ is a multi-component Delone set, the hull $\mathbb{X}(\underline{\Lambda})$ is compact (in $\mathcal{D}_m$ with respect to the local topology) iff $\underline{\Lambda}$ is an FLC multi-component Delone set.* □

Corollary 5.105. *If $\underline{\Lambda}$ is an FLC multi-component Delone set in an LCAG G, the pair $(\mathbb{X}(\underline{\Lambda})\,, G)$ is a topological dynamical system. We call such a dynamical system a* point set dynamical system. □

Remark 5.106. Analogously, we can also define a *tiling dynamical system* $(\mathbb{X}(\mathcal{T})\,, G)$: Let $\mathcal{T} = \underline{\Lambda} + \underline{A}$ be a tiling of G, where $\underline{\Lambda}$ is a multi-component Delone set and $\underline{A}$ a finite family of prototiles. Then the definitions of FLC, repetitivity *etc.* of $\underline{\Lambda}$ are consistent with the corresponding notions for $\mathcal{T}$. Moreover, one can also define a corresponding local topology on the set of all tilings with m prototiles, wherefore all statements about $(\mathbb{X}(\underline{\Lambda})\,, G)$ in this section hold analogously for $(\mathbb{X}(\mathcal{T})\,, G)$.

Geometrical properties of an FLC multi-component Delone set are often reflected in properties of its associated point set dynamical system.

Definition 5.107. Two multi-component Delone sets $\underline{\Lambda}, \underline{\Lambda}'$ are *locally indistinguishable*, or *LI* for short, if each cluster of $\underline{\Lambda}$ is, up to a translation, a subset of $\underline{\Lambda}'$ and vice versa. This is obviously an equivalence relation, and we can define the *LI-class* of a multi-component Delone set $\underline{\Lambda}$.

Minimality and this last notion are closely related to repetitivity, as the following statement shows. As before, it is proven in [329, Prop. 3.1] for $\mathcal{D}_1$, but the proof extends to $\mathcal{D}_m$. Note that it is a variant of Gottschalk's theorem [151] (compare [287, Theorem 4.1.2] and [316, Section 5.1]).

Lemma 5.108. *Let G be an LCAG and $\underline{\Lambda}$ an FLC multi-component Delone set. Then the following properties are equivalent:*

(i) *The multi-component set $\underline{\Lambda}$ is repetitive.*

(ii) *The hull $\mathbb{X}(\underline{\Lambda})$ coincides with the LI-class of $\underline{\Lambda}$.*

(iii) *The point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ is minimal.* □

We are now looking for the geometric counterpart of unique ergodicity.

Definition 5.109. Let $\{A_n\}_{n\in\mathbb{N}}$ a van Hove sequence. A FLC multi-component Delone set has *uniform cluster frequencies* (relative to $\{A_n\}_{n\in\mathbb{N}}$), or is a *UCF set* for short, if, for all $t \in G$ and any cluster $\underline{\mathcal{P}}$, the *frequency* of $\underline{\mathcal{P}}$ defined by

$$\lim_{n\to\infty} \frac{\operatorname{card}\{x \in G \mid (\underline{\mathcal{P}} + x) \subset ((\underline{\Lambda} - t) \cap A_n)\}}{\mu_G(A_n)}$$

exists, with the convergence being uniform in t. If we consider a tiling $\mathcal{T}$ instead of a $\underline{\Lambda}$, we may speak of *uniform patch frequencies* (*UPF* for short).

In plain language, $\operatorname{card}\{x \in G \mid (\underline{\mathcal{P}} + x) \subset ((\underline{\Lambda} - t) \cap A_n)$ is the number of translates of $\underline{\mathcal{P}}$ contained in (the relatively compact set) $(\underline{\Lambda} - t) \cap A_n$, which is finite by the σ-compactness of G.

The connection between UCF and unique ergodicity is established in [329, Theorem 3.2 & Corollary 3.3] for $\mathcal{D}_1$, but the proof again extends to $\mathcal{D}_m$.

Lemma 5.110. *Let $\underline{\Lambda}$ be an FLC multi-component Delone set. Then, the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ is uniquely ergodic iff $\underline{\Lambda}$ is a UCF set.* □

Remark 5.111. Let $\underline{\Lambda}$ be a repetitive Delone set in $\mathbb{R}^d$. Then for every radius $r > 0$ there is a radius $R > 0$, such that every closed ball of of radius R contains the centre of a translate of every possible patch $\{B_{\leq r} \cap \underline{\Lambda} - t \,|\, t \in \operatorname{supp} \underline{\Lambda}\}$. We call the least such radius R for each r the *repetitivity function* $R_{\underline{\Lambda}}(r)$. A repetitive Delone set is *linearly repetitive* (or *strongly repetitive*) if $R_{\underline{\Lambda}}(r) = \mathcal{O}(r)$ as $r \to \infty$ (*i.e.*, there are constants r_0 and $C > 0$ such that $R_{\underline{\Lambda}}(r) \leq C \cdot r$ for $r > r_0$). Therefore, a tiling $\mathcal{T}$ is linearly repetitive if there exists a constant $C > 0$ such that for every patch $\mathcal{P}$, any ball of radius $C \cdot \operatorname{diam}(\operatorname{supp} \mathcal{P})$ contains a translate of $\mathcal{P}$. By [219, Theorem 6.1], linear repetitivity implies UCF. Moreover, self-similar tilings are linearly repetitive by [365, Lemma 2.3] (where we mean by a self-similar tiling, a tiling for which the tiles are given as attractor of a self-similar IFS). However, as noted in [365, p. 272], self-affine tilings are in general not linearly repetitive.

Remark 5.112. The unique invariant probability measure of a uniquely ergodic point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ can explicitly be given as follows, see [234, Corollary 2.8]: Let V be the open neighbourhood of 0 in the definition of uniform discreteness of $\operatorname{supp} \underline{\Lambda}$, and let

$\underline{\mathcal{P}}$ be any nonempty cluster of $\underline{\Lambda}$ or some translate of $\underline{\Lambda}$. Then, for any Borel set $U \subset V$, we define the following *cylinder sets*

$$\{\underline{\Lambda}' \in \mathbb{X}(\underline{\Lambda}) \mid \underline{\mathcal{P}} - t \subset \underline{\Lambda}' \text{ for some } t \in U\}.$$

The unique invariant probability measure on these cylinder sets is then given by the product of $\mu_G(U)$ and the corresponding cluster frequency. This determines the measure completely.

Corollary 5.113. *Let $\underline{\Lambda}$ be an FLC multi-component Delone set. Then, the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ is strictly ergodic iff $\underline{\Lambda}$ is a repetitive UCF set.* □

Before we look for a condition which ensures UCF, we first note that for a representable FLC multi-component Delone sets the point set dynamical system and the associated tiling dynamical system are topologically conjugate. For this we make the following definition. Note that we assume that closed bounded balls are compact.

Definition 5.114. Let A, B be two multi-component Delone sets, or two tilings, or one multi-component Delone set and one tiling. We denote by $\underline{\mathcal{P}}_r(x, A)$ respectively $\mathcal{P}_r(x, A)$ the finite portion of the structure A in a ball of radius r around x, the *r-cluster* respectively the *r-patch* at x. So, if $A = \mathcal{T}$ is a tiling, then $\mathcal{P}_r(x, \mathcal{T}) = \{T \in \mathcal{T} \mid T \cap B_{\leq r}(x) \neq \varnothing\}$. If $A = \underline{\Lambda}$ is a multi-component Delone set, then $\underline{\mathcal{P}}_r(x, \underline{\Lambda}) = \underline{\Lambda} \cap B_{\leq r}(x)$.

We say that B is *locally derivable* (with radius r) from A, if, for all $x, y \in G$, one has:

$$\underline{\mathcal{P}}_r(x, A) = \underline{\mathcal{P}}_r(y, A) + (x - y) \quad \Longrightarrow \quad \underline{\mathcal{P}}_0(x, B) = \underline{\mathcal{P}}_0(y, B) + (x - y).$$

If B is locally derivable from A and A is locally derivable from B, then A and B are *mutually locally derivable*, or *MLD* for short.

MLD was introduced in [40] (also see [25] and references therein). For dynamical systems it has the following consequence, since it provides a (topological) isomorphism, see [288, 303, 304] (also compare [235, Lemma 3.10] and [316, Lemma 5.10]).

Lemma 5.115. *Let A, B be two FLC multi-component Delone sets, or two FLC tilings, or one FLC multi-component Delone set and one FLC tiling in an LCAG G. If A and B are MLD, then the topological dynamical systems $(\mathbb{X}(A), G)$ and $(\mathbb{X}(B), G)$ are topologically conjugate.* □

Obviously, if a substitution FLC multi-component Delone set $\underline{\Lambda}$ is representable, it is MLD to its associated tiling $\mathcal{T} = \underline{\Lambda} + \underline{A}$, wherefore consequently $(\mathbb{X}(\underline{\Lambda}), G)$ and $(\mathbb{X}(\mathcal{T}), G)$ are topologically conjugate in this case, also see [235, Lemma 3.10].

We now state under which condition the structures we are interested in have uniform cluster/patch frequencies and therefore yield uniquely ergodic topological dynamical systems (also see [316, Theorem 6.1]).

Proposition 5.116. *Let $\underline{\Lambda}$ be a primitive substitution FLC multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)***, and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. If $\mathcal{T} = \underline{\Lambda} + \underline{A}$ is a tiling, then it has uniform patch frequencies.*

Proof. The proof of [235, Lemma A.6] (and therefore [235, Theorem 4.1]) in $\mathbb{R}^d$ carries over to our more general situation. □

We note that repetitivity and UCF (and therefore strict ergodicity of the associated point set dynamical system) implies that all cluster frequencies are positive.

Corollary 5.117. *Let $\underline{\Lambda}$ be a substitution FLC multi-component Delone set, where the EMFS $\boldsymbol{\Theta}$ satisfies* **(LT)**, *and $\underline{A}$ be the attractor of the adjoint IFS such that the sets A_i are tiles. If every cluster of $\underline{\Lambda}$ is legal, then $\underline{\Lambda}$ is a repetitive UCF set and the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ is strictly ergodic.*

Proof. If every cluster is legal, then the substitution is primitive and $\underline{\Lambda}$ is representable by Lemma 5.87. But then the associated tiling is repetitive by Corollary 5.89. By the previous lemma, the associated tiling also has uniform patch frequencies. So the tiling dynamical system of the associated tiling is strictly ergodic. Consequently, by MLD, the point set dynamical system is also strictly ergodic, see Lemma 5.115. □

For multi-component model sets, sufficient condition for UCF and repetitivity are the following.

Lemma 5.118. (i) *Let $\underline{\Lambda}$ be a regular multi-component model set. Then it has UCF.*

(ii) *Let $\underline{\Lambda}$ be a multi-component model set, such that each component is a generic model set. Then it is repetitive.*

Proof. The first statement is [329, Theorem 4.5(1)], and the second statement is [327, Lemma 2.1], both extended to the multi-component case. □

Let $(A_i)_{i=1}^n$ be a set of prototiles such that there exists a tile substitution as in Equation (5.7) on p. 155 (where the unions on the right hand side are measure-disjoint). Let us denote such a tile substitution by $\boldsymbol{\Theta}_{\underline{A}}$ (again, we can write it as an MFS). Furthermore, let $G_{\underline{A}}$ denote the set of all tilings of G with translates of the prototiles $(A_i)_{i=1}^n$. Then, we can extend the tile substitution to a map $\boldsymbol{\Theta}_{\underline{A}} : G_{\underline{A}} \to G_{\underline{A}}$ (we again denote this map by $\boldsymbol{\Theta}_{\underline{A}}$). Moreover, we may look at the following subspace of $G_{\underline{A}}$: Let $G_{\underline{A}}^{\boldsymbol{\Theta}_{\underline{A}}}$ be the set of all tilings of G such that every patch of each tiling is legal (with respect to the tile substitution $\boldsymbol{\Theta}_{\underline{A}}$). Note that this implies that the tile substitution is primitive (by Corollary 5.86). Moreover, under FLC, it also implies that all tilings in $G_{\underline{A}}^{\boldsymbol{\Theta}_{\underline{A}}}$ are repetitive (by Corollary 5.89), wherefore $G_{\underline{A}}^{\boldsymbol{\Theta}_{\underline{A}}}$ forms a single LI class (by Lemma 5.108 respectively its counterpart for tilings) and therefore coincides with $\mathbb{X}(\mathcal{T})$ for any tiling $\mathcal{T} \in G_{\underline{A}}^{\boldsymbol{\Theta}_{\underline{A}}}$.

Definition 5.119. Let $\mathcal{T}$ be a primitive repetitive substitution tiling. The tilings from $\mathbb{X}(\mathcal{T})$, respectively simply the tiling $\mathcal{T}$, are said to have the *unique composition property*, or is *UCP* for short, if the tile substitution $\boldsymbol{\Theta}_{\underline{A}} : \mathbb{X}(\mathcal{T}) \to \mathbb{X}(\mathcal{T})$ is bijective. Alternatively, $\mathcal{T}$ is UCP iff $f_0(\mathcal{T})$ (*i.e.*, the inflated tiling) is locally derivable from $\mathcal{T}$.

In plain words, a tiling $\mathcal{T}$ is UCP if there is a unique way to compose (some of) its tiles into a legal "super"-patch belonging to a tiling with "super"-prototiles $(f_0(A_i)_{i=1}^n$. The equivalence of the two definitions of UCP can be found in [365, Lemma 2.5] (although in the setting $G = \mathbb{R}^d$, but the proof generalises easily to our setting).

Proposition 5.120. *Let $\mathcal{T}$ be a primitive repetitive substitution tiling. Then $\mathcal{T}$ has the unique composition property iff it is aperiodic.*

Proof. The technical laborious proof of [365, Theorem 1.1] for self-affine (self-similar) tilings in $\mathbb{R}^d$ carries over to our setting. □

The previous statement is a higher dimensional generalisation of a statement for one-dimensional (symbolic) sequences (see [266, 267], also see [298, Section 7.2.1] and [50, Section 20]), where UCP is called *recognisability.*

5.7. Substitutions II

In view of the results at the end of Section 5.4 and their connection to the theory of dynamical systems, we define the following "primitive **(LT)** property" of a substitution multi-component Delone set $\underline{\Lambda}$:

(PLT) $\underline{\Lambda}$ is a representable primitive substitution multi-component Delone set in a LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)** and the components A_i of the attractor $\underline{A}$ of the adjoint IFS $\boldsymbol{\Theta}^{\#}$ are the prototiles of the associated tiling $\mathcal{T} = \underline{\Lambda} + \underline{A}$.

If a FLC multi-component set $\underline{\Lambda}$ satisfies **(PLT)**, then it is repetitive iff every cluster of $\underline{\Lambda}$ is legal (by Corollary 5.89). Moreover, in this case the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$ is strictly ergodic by Corollary 5.117. In particular, it is also a UCF set.

Let $\underline{\Lambda}$ be a repetitive substitution multi-component Delone set in an LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)** and the components A_i of the attractor $\underline{A}$ of the adjoint IFS $\boldsymbol{\Theta}^{\#}$ are tiles (we do not assume representability). Then, by Lemma 5.79, the covering degree of $\mathcal{T} = \underline{\Lambda} + \underline{A}$ is constant μ-a.e. Moreover, we note that the density of the point sets Λ_i is given by the autocorrelation coefficient at 0, *i.e.*, $\operatorname{dens}\Lambda_i = \eta_{\Lambda_i}(0)$. So, we easily establish the following equality:

$$\deg_{\mathcal{T}}^{cov}(x) = \sum_{i=1}^{n} \eta_{\Lambda_i}(0) \cdot \mu(A_i) \qquad \mu\text{-a.e.}$$

Suppose that every cluster of $\underline{\Lambda}$ is legal (or, equivalently, that $\underline{\Lambda}$ is representable), *i.e.*, that $\underline{\Lambda}$ satisfies **(PLT)** and is repetitive. We define the following function

$$\operatorname{dens}_{\mathcal{T}}^{ovlap}(x) = \sum_{i=1}^{n} \eta_{\Lambda_i}(x) \cdot \mu(A_i).$$

We have $\operatorname{dens}_{\mathcal{T}}^{ovlap}(0) = 1$ and $0 \le \operatorname{dens}_{\mathcal{T}}^{ovlap}(x) \le 1$. Moreover, one can check that $\operatorname{dens}_{\mathcal{T}}^{ovlap}(x) = \operatorname{dens}(\mathcal{T} \sqcap (\mathcal{T} + x))$, where we understand that the intersection "$\sqcap$" of a tile $T \in \mathcal{T}$ and a tile $T' \in \mathcal{T} + x$ equals T if $T = T'$ and is the empty set otherwise (so, $\mathcal{T} \sqcap (\mathcal{T} + x)$ yields a (maybe empty) packing of G with the same prototiles A_i).

To establish a cut and project scheme for $\underline{\Lambda}$, we have made the following assumption **(As)**: For all $\varepsilon > 0$, the set $P'_\varepsilon = \bigcap P_\varepsilon(\Lambda_i)$ is relatively dense in G. Now, if **(As)** holds, we have the following: For every $\varepsilon > 0$, there is a relatively dense set $S \subset G$ (namely P'_ε) such that

$$\operatorname{dens}_{\mathcal{T}}^{ovlap}(x) > 1 - \varepsilon \tag{5.10}$$

for all $x \in S$. This is because for all $x \in P'_\varepsilon$, we have the inequality $1 - \eta_{\Lambda_i}(x)/\operatorname{dens}\Lambda_i < \varepsilon$ and therefore $\eta_{\Lambda_i}(x) > (1 - \varepsilon) \cdot \operatorname{dens}\Lambda_i$ for all $1 \le i \le n$. Multiplying by $\mu(A_i)$ and summing over i establishes Equation (5.10).

Corollary 5.121. *Let $\underline{\mathit{\Lambda}}$ be a substitution multi-component Delone set that satisfies* **(PLT)**. *Then, for every $\varepsilon > 0$ we have $f_0(P'_\varepsilon) \subset P'_\varepsilon$. Consequently, we also have* $\operatorname{dens}^{\mathit{ovlap}}_{\mathcal{T}}(x) \leq \operatorname{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0(x))$.

Proof. Let $x \in P'_\varepsilon$. Then, $\eta_{\Lambda_i}(x) > (1-\varepsilon) \cdot \operatorname{dens} \Lambda_i$ for all $1 \leq i \leq n$. We take a pair $\omega_i(t), \omega_i(t+x) \subset \underline{\Lambda}$. By the assumption on the (primitive!) EMFS $\boldsymbol{\Theta}$ there are "on average" α many different pairs $\omega_j(t'), \omega_j(t'+f_0(x)) \subset \underline{\Lambda}$ such that $\omega_j(t') \subset \boldsymbol{\Theta}(\omega_i(t))$ and $\omega_j(t'+f_0(x)) \subset \boldsymbol{\Theta}(\omega_i(t+x))$. We recall that $\underline{\Lambda}$ is a fixed point of $\boldsymbol{\Theta}$ and all unions in $\boldsymbol{\Theta}(\underline{\Lambda})$ are disjoint.

Now, one way to establish the claim is the following: For A_0 take the support of a patch $\mathcal{P}$ of $\mathcal{T}$, which contains a neighbourhood of 0 (*e.g.*, the patch associated to a finite seed of $\underline{\Lambda}$). Set $A_n = f_0^n(A_0)$. Then, this defines a van Hove sequence (which is nested, if we start with a finite seed). But the previous considerations imply that $\eta^{(n)}_{\Lambda_i}(x) \lessapprox \eta^{(n+1)}_{\Lambda_i}(f_0(x))$, where by "$\lessapprox$" we indicate that the inequality may not hold for an individual n but "on average" in such a way that in the limit we have the inequality

$$\eta_{\Lambda_i}(x) = \lim_{n\to\infty} \eta^{(n)}_{\Lambda_i}(x) \leq \lim_{n\to\infty} \eta^{(n+1)}_{\Lambda_i}(f_0(x)) = \eta^{(n)}_{\Lambda_i}(f_0(x)).$$

This, of course, establishes the claim. □

We will continue from these observations about $\operatorname{dens}^{\mathit{ovlap}}_{\mathcal{T}}$ in Section 5.7.1. It is our aim in the remaining parts of this section, to see if a (representable primitive) substitution multi-component Delone set $\underline{\Lambda}$ (therefore, with **(PLT)**) can also be described as (regular) multi-component (inter) model set. Suppose this is the case and the internal space $H = H' \times \mathcal{L}/\mathcal{L}'$ is obtained as in Section 5.3. Then, we have $\Lambda(\underline{\operatorname{int} \Omega}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Omega})$ with – in the case of regularity – $\mu_H(\partial \Omega_i) = 0$ for all $1 \leq i \leq n$ (comparing this with Equation (5.2) on p. 130, recall that we assume that 0 is contained in at least one Λ_i wherefore we may take $x = 0$). We now like to understand how the EMFS $\boldsymbol{\Theta}$ acts in the internal space.

- Let $t_{(a)} \circ f_0 \in \Theta_{ij}$. Then, by linearity, one has $f_0(\Lambda_i - \Lambda_i) \subset \Lambda_j - \Lambda_j$, *i.e.*, we can restrict f_0 to $\mathcal{L}'$. Consequently, $f_0 : \mathcal{L}' \to \mathcal{L}'$ is a group homomorphism (as is $f_0 : \mathcal{L} \to \mathcal{L}$). For every translational part $t_{(a)}$, one has $a \in \mathcal{L}$.
- We now equip $\mathcal{L}$ respectively $\mathcal{L}'$ with the AC topology. Obviously, since $(\mathcal{L}, \varrho_{\underline{\Lambda}})$ is a commutative topological group, the translations $t_{(a)} : (\mathcal{L}, \varrho_{\underline{\Lambda}}) \to (\mathcal{L}, \varrho_{\underline{\Lambda}})$ are continuous. Corollary 5.121 shows, that f_0 is also continuous on $(\mathcal{L}, \varrho_{\underline{\Lambda}})$ and $(\mathcal{L}', \varrho_{\underline{\Lambda}})$.
- Although nontrivial translations are no group homomorphisms, they are uniformly continuous on every commutative topological group. So every map $f \in \boldsymbol{\Theta}$ extends to a uniformly continuous (affine) map $\hat{f} : H \to H$ on the completion of $(\mathcal{L}, \varrho_{\underline{\Lambda}})$ by Lemma 2.68. Moreover, we note that $\hat{f}_0 : H' \to H'$ (which we may interpret as restriction of $\hat{f}_0 : H \to H$ on H') is also uniformly continuous. Note that, for every $a \in \mathcal{L}$, we have $\hat{t}_{(a)} = t_{(a^\star)}$.

We denote the MFS on the Hausdorff completion H, obtained from the EMFS $\boldsymbol{\Theta}$ by applying Lemma 2.68 to all its maps, by $\boldsymbol{\Theta}^\star$.

We would like to have that $\boldsymbol{\Theta}^\star$ is actually an IFS on H, which is equivalent to saying that $\hat{f}_0$ is a contraction on H. But this is not true in general, by the special role played by the (discrete) height group $\mathcal{L}/\mathcal{L}'$: Let $x = (y, z), x' = (y', z') \in H \cong H' \times \mathcal{L}/\mathcal{L}'$ with $z \neq z'$.

Then, $\hat{f}_0(x) = (\breve{f}_0(y)\,, \dot{f}_0(z))$ and $\hat{f}_0(x') = (\breve{f}_0(y')\,, \dot{f}_0(z'))$, where we denote the action of (the linear and continuous map on H and H') $\hat{f}_0$ on the discrete height group by $\dot{f}_0$. Since in general $\dot{f}_0(z) \neq \dot{f}_0(z')$, $\hat{f}_0$ is not an contraction on H, see the next example.

Example 5.122. We look at an periodic sequence on $\mathbb{Z}$: Let $\Lambda_a = 3\mathbb{Z}$, $\Lambda_b = 3\mathbb{Z}+1$ and $\Lambda_c = 3\mathbb{Z}+2$, *i.e.*, we have the sequence $\dots cabcabc\dot{a}bcabcab \dots$ realised on $\mathbb{Z}$ (where "$\cdot$" denotes the zeroth position). Then, similar as in Example 5.52, we have $H' = \{0\}$ and $\mathcal{L}/\mathcal{L}' \cong \mathsf{C}_3$ and therefore $H = \{0\} \times \mathsf{C}_3$, where C_3 denotes the cyclic group $\mathsf{C}_3 = \mathbb{Z}/3\mathbb{Z}$ of order 3.

One can also obtain this sequence as fixed point of the following substitution: Let $f_0(x) = 2x$ and $f_1(x) = t_{(1)} \circ f_0(x) = 2x+1$. Let the EMFS $\boldsymbol{\Theta}$ be given by

$$\begin{array}{ccc} \Theta_{aa} = \{f_0\} & \Theta_{ba} = \{f_1\} & \Theta_{ab} = \{f_1\} \\ \Theta_{cb} = \{f_0\} & \Theta_{bc} = \{f_0\} & \Theta_{cc} = \{f_1\} \end{array}$$

and all other components are empty. The shorthand notation is $a \mapsto ab$, $b \mapsto ca$ and $c \mapsto bc$ (note that a finite seed is given by $c\dot{a}$, *i.e.*, $\omega_a(0) \cup \omega_c(-1)$). For example, we have

$$3\mathbb{Z} = \Lambda_a = \Theta_{aa}(\Lambda_a) \cup \Theta_{ab}(\Lambda_b) = (6\mathbb{Z}) \cup (6\mathbb{Z}+3)$$

and similarly for the sets Λ_b and Λ_c (so it is a primitive substitution multi-component Delone set on $G = \mathbb{R}$).
We observe that $\dot{f}_0$ acts on C_3 as multiplication by 2 modulo 3. So, we have for $(0\,,0), (0\,,1) \in H$: $\hat{f}_0((0\,,0)) = (0\,,0)$ and $\hat{f}_0((0\,,1)) = (0\,,2)$ and $\hat{f}_0$ is not an contraction on H (since we equip the discrete height group with the discrete metric).

In view of this, we make the following "re-interpretation" for model sets obtained by a substitution: By assumption, we have a finite family $\underline{\Lambda} = (\Lambda_i)_{i=1}^n$ of Delone sets respectively a finite family $\underline{\Omega} = (\Omega_i)_{i=1}^n$ of corresponding windows. Now, each such window Ω_i is contained in exactly one coset $H' + t \subset H$ with $\Lambda_i \subset t + \mathcal{L}' \subset \mathcal{L}$ (compare the discussion on p. 143). So, we may also say that each window Ω_i is a subset of H' (instead of H). Thus, it is enough if $\boldsymbol{\Theta}^\star$ defines a contraction (and therefore is an IFS) on $(\mathcal{K}H')^n$, *i.e.*, we actually define the MFS $\breve{\boldsymbol{\Theta}}^\star$ by

$$\breve{\Theta}^\star_{ij} = \bigcup_{\hat{f}=(\breve{f},\dot{f})\in\Theta^\star_{ij}} \{\breve{f}\} = \bigcup_{f\in\Theta_{ij}} \{\breve{f}\},$$

where, for a continuous map $\hat{f} : H \to H$ with $H \cong H' \times \mathcal{L}/\mathcal{L}'$, we denote the action of $\hat{f}$ restricted on H' by $\breve{f}$ and restricted on $\mathcal{L}/\mathcal{L}'$ by $\dot{f}$ (*i.e.*, using the projections $\pi : H \to H'$ and $\pi' : H \to \mathcal{L}/\mathcal{L}'$, we have $\breve{f} = \pi \circ \hat{f}$ and $\dot{f} = \pi' \circ \hat{f}$). Then, $\breve{\boldsymbol{\Theta}}^\star$ is an IFS if $\breve{f}_0$ is a contraction on H'.

We observe that for all maps $\hat{f} \in \boldsymbol{\Theta}^\star$, the restriction $\dot{f}$ on $\mathcal{L}/\mathcal{L}'$ can simply be interpreted as a constant which yields the correct coset. More precisely, if $a_i \in \mathcal{L}/\mathcal{L}'$ where $\Lambda_i \subset a_i + \mathcal{L}'$, then we have $\hat{f} = (\breve{f}, a_i)$ for all $1 \leq j \leq n$ and all $\hat{f} \in \Theta^\star_{ij}$. Thus, if $\breve{\boldsymbol{\Theta}}^\star$ is an IFS on H' with attractor $\underline{\breve{\Pi}}$, then $\underline{\Pi}$ with $\Pi_i = \breve{\Pi}_i \times \{a_i\}$ is a fixed point of $\boldsymbol{\Theta}^\star$. In this case and by abuse of notation, we say that this $\underline{\Pi}$ is the "attractor" of the "IFS" $\boldsymbol{\Theta}^\star$. So by $\boldsymbol{\Theta}^\star$ we always mean an IFS on $(\mathcal{K}H')^n$.

We now have a closer look at this IFS. We suppose that the substitution multi-component Delone set $\underline{\Lambda}$ satisfies **(PLT)**, and that $\underline{\Lambda}$ can also be described as multi-component (inter) model set where the internal space $H = H' \times \mathcal{L}/\mathcal{L}'$ is obtained as in Section 5.3. Then, we observe the following:

- All maps $\hat{f} \in \boldsymbol{\Theta}^\star$ are affine continuous maps $\hat{f} : H \to H$. The same holds for its restriction $\breve{f} : H' \to H'$. Since $\Lambda(\operatorname{int} \Omega_i) \subset \Lambda_i \subset \Lambda(\Omega_i)$, we also have

$$\Lambda\left(\hat{f}(\operatorname{int} \Omega_i)\right) = f\left(\Lambda(\operatorname{int} \Omega_i)\right) \subset f(\Lambda_i) \subset f\left(\Lambda(\Omega_i)\right) = \Lambda\left(\hat{f}(\Omega_i)\right) \tag{5.11}$$

for every $f \in \Theta_{ij}$. The unions in Equation (5.5) on p. 149 are disjoint. But here, that equation reads

$$\Lambda\left(\bigcup_{j=1}^{n} \bigcup_{\hat{f} \in \Theta_{ij}^\star} \hat{f}(\operatorname{int} \Omega_j)\right) \subset \Lambda_i \subset \Lambda\left(\bigcup_{j=1}^{n} \bigcup_{\hat{f} \in \Theta_{ij}^\star} \hat{f}(\Omega_j)\right),$$

wherefore we also have that

  - $\hat{f}(\operatorname{int} \Omega_j) \cap \hat{g}(\operatorname{int} \Omega_k) \cap (\mathcal{L}')^\star = \varnothing$, where $f \neq g$ with $f \in \Theta_{ij}$ and $g \in \Theta_{ik}$.
  - Consequently, the sets $\hat{f}(\operatorname{int} \Omega_j)$ and $\hat{g}(\operatorname{int} \Omega_k)$ do not have a common interior point.

- These considerations make it plausible think of $\breve{f}_0 : H' \to H'$ as contraction. If $\breve{f}_0$ *is* a contraction, then $\breve{\boldsymbol{\Theta}}^\star$ is an IFS on H' (respectively $(\mathcal{K}H')^n$). So, we may apply the findings of Section 4.8: By Proposition 4.89, there exists a unique attractor $\underline{\breve{\Pi}} \in (\mathcal{K}H')^n$. We want to establish that $\underline{\Omega} = \underline{\Pi}$ where $\Pi_i = \breve{\Pi}_i \times \{a_i\}$ (*i.e.*, we want to establish that the attractor of the IFS *is* the window for the IMS):
Denote by $\underline{\mathcal{P}} \subset \underline{\Lambda}$ a finite seed of the substitution multi-component Delone set $\underline{\Lambda}$. Then, we have $\underline{\Lambda} = \lim_{\ell \to \infty} \boldsymbol{\Theta}^\ell(\underline{\mathcal{P}})$. Moreover, by the inclusion condition, we have that $\underline{\mathcal{P}}^\star \subset \boldsymbol{\Theta}^\star(\underline{\mathcal{P}}^\star)$ in[25] H, and therefore

$$\underline{\Pi} = \operatorname{cl}_H\left(\bigcup_{\ell \in \mathbb{N}} (\boldsymbol{\Theta}^\star)^\ell(\underline{\mathcal{P}}^\star)\right) = \operatorname{cl}_H(\underline{\Lambda}^\star). \tag{5.12}$$

So, by construction, each Π_i is the smallest(!) closed set in $H' \times \{a_i\} \subset H$ such that $\Lambda_i \subset \Lambda(\Pi_i)$, wherefore $\Pi_i \subset \Omega_i$. But $\Omega_i \setminus \Pi_i$ does not contain interior points, therefore $\operatorname{int} \Omega_i = \operatorname{int} \Pi_i$ and the regular closedness of Ω_i also establishes $\underline{\Pi} = \underline{\Omega}$.
We have argued that all maps $\hat{f} \in \boldsymbol{\Theta}^\star$ of the "IFS" $\boldsymbol{\Theta}^\star$ can simply be interpreted as $\hat{f} = (\breve{f}, a_i)$ with constant $a_i \in \mathcal{L}/\mathcal{L}'$ (see p. 172). Note that Equation (5.12) suggests this interpretation: We have $\Pi_i = \operatorname{cl}_H \Lambda_i^\star$ and $\Lambda_i^\star \subset H' \times \{a_i\}$, so it is "sensible" to use $\hat{f} = (\breve{f}, a_i)$. In a sense, using a constant in the $\mathcal{L}/\mathcal{L}'$-component counterbalances that $\boldsymbol{\Theta}^\star$ is not really an IFS on $(\mathcal{K}H)^n$, but we only have an IFS $\breve{\boldsymbol{\Theta}}^\star$ on $(\mathcal{K}H')^n$.

- Furthermore, the density formula (Theorem 5.26) together with Equation (5.11) suggests that $\hat{f}_0$ not only is a contraction, but one also has $\mu_H(W) = \alpha \cdot \mu_H(\hat{f}_0(W))$ for all compact sets $W \subset H' \times \{0\}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$. In general, however, we only have that $\boldsymbol{S\Theta} \geq \boldsymbol{S\Theta}^\star$ (and not equality), since different maps $f, g \in \Theta_{ij}$ might yield identical maps $\hat{f}, \hat{g}$: Consider the periodic (one-component) Delone set $\Lambda_a = \mathbb{Z}$. One can obtain Λ_a as substitution Delone set with $\boldsymbol{\Theta} = \{f_0, g = t_{(1)} \circ f_0\}$ where $f_0(x) = 2 \cdot x$, *i.e.*, $\mathbb{Z} = (2\mathbb{Z}) \cup (2\mathbb{Z}+1)$. Then, Λ_a is a model set (with $H = H'$) in the CPS $(\mathbb{R}, \{0\}, \mathbb{Z} \times \{0\})$ and $\hat{f}_0, \hat{g} : H \to H$, $0 \mapsto 0$ coincide.

[25] We use the following obvious notation: If $\omega_i(x) = (\varnothing, \ldots, \varnothing, x, \varnothing, \ldots, \varnothing) \in G^n$, then $(\omega_i(x))^\star = (\varnothing, \ldots, \varnothing, x^\star, \varnothing, \ldots, \varnothing) = \omega_i(x^\star) \in H^n$.

- Suppose, $\boldsymbol{S\Theta} = \boldsymbol{S\Theta}^\star$, which, for example, is the case if $\varrho_{\underline{\Lambda}}$ is a metric on $\mathcal{L}$. Also suppose that $\mu_H(W) = \alpha \cdot \mu_H(\hat{f}_0(W))$ for all compact sets $W \subset H$ of the form $W = \breve{W} \times \{0\}$. Then we can apply Proposition 4.99: The unions in the IFS $\boldsymbol{\Theta}^\star$ (more precisely, in the IFS $\breve{\boldsymbol{\Theta}}^\star$) are measure-disjoint (by Remark 4.102, this is more than having no common interior point), and the boundaries $\partial \Pi_i$ have Haar measure μ_H zero (if $\hat{f}_0$ is a homeomorphism). Thus, under these conditions, $\underline{\Lambda}$ is (automatically) a regular(!) multi-component (inter) model set.

These observations should serve as motivation and guideline for the results later in this section.

5.7.1. Algebraic and Overlap Coincidences

We now continue with the discussion from the beginning of this section. For a substitution tiling $\mathcal{T}$, we now introduce a (different from P'_ε) candidate for a relatively dense set S such that Equation (5.10) on p. 170 holds for all $x \in S$. This is also the start of the discussion of "coincidence conditions".

Definition 5.123. Let $\underline{\Lambda}$ be a repetitive substitution multi-component Delone set that satisfies **(PLT)**. As before, we denote the union of the difference sets by Δ', *i.e.*, $\Delta' = \bigcup_{i=1}^n \Delta_i = \bigcup_{i=1}^n (\Lambda_i - \Lambda_i)$ (note that $\Delta_i = \Delta_i^{\mathrm{ess}}$). We say that $\underline{\Lambda}$ admits an *algebraic coincidence* if Δ' is a Meyer set and there exists an $1 \le i \le n$, an $m \in \mathbb{Z}_{\ge 0}$ and a $t \in \Lambda_i$ such that $t + f_0^m(\Delta') \subset \Lambda_i$ (where $f_0^0(\Delta') = \Delta'$, *i.e.*, $f_0^0 = \mathrm{id}$).

Remark 5.124. If $\underline{\Lambda}$ is a multi-component (inter) model set, then by $\Delta_i = \Lambda(\Omega_i - \Omega_i)$ we have $\Delta' = \Lambda(\bigcup_{i=1}^n (\Omega_i - \Omega_i))$ and Δ' is a Meyer set.

The name "algebraic coincidence" is due to [230, 231]. One can also formulate this condition in terms of "overlaps" of the tiles of a substitution tiling (here, the connection between $\mathrm{dens}_{\mathcal{T}}^{\mathit{ovlap}}(x) = \mathrm{dens}\,(\mathcal{T} \sqcap (\mathcal{T} + x))$, *i.e.*, the density of the "$\underline{\text{over}}$lap coincidences", and the algebraic coincidence consequently comes into play). Therefore, for tilings this condition is known as "overlap coincidence" and was introduced earlier by B. Solomyak, see *e.g.* [235, 364]. Moreover, the condition "overlap coincidence" is checkable/decidable for a given example by constructing a finite graph, while the importance of the condition "algebraic coincidence" lies in the theoretical description, namely it provides (as we will show) a convenient base at 0 for the AC topology. The same phenomenon, of course, arises in so-called lattice substitution systems, which we will review in Chapter 6b: The "modular coincidence" (which is a special case of "algebraic coincidence") is often inconvenient to check but has a more direct interpretation for the (possible) underlying multi-component (inter) model set, while the converse is true for the generalised "Dekking coincidence" (which is a special case of "overlap coincidence").

In view of this, we now introduce the notion "overlap coincidence" and establish to which condition of the function $\mathrm{dens}_{\mathcal{T}}^{\mathit{ovlap}}$ these coincidence conditions are equivalent. But first, we start with the following remark and an algorithm how to check for coincidences.

Remark 5.125. We have already implicitly used the following construction: Any (repetitive) tiling $\mathcal{T}$ can be converted into a multi-component Delone set $\underline{\Lambda}$ by the following procedure: To each prototile T_i, assign a *control point* $x_{T_i} \in G$ (often, one uses[26] a point $x_{T_i} \in \operatorname{int} T_i$, but

[26]The construction of an appropriate control point $x_{T_i} \in \operatorname{int} T_i$ for a substitution tiling $\mathcal{T}$ is described in [295, Prop. 1.3] and [294, Prop. 1.7] (also compare [373, p. 37] and [202]). The

a more convenient choice might be $x_{T_i} = 0$ for all $1 \leq i \leq n$). Then, to each tile $T_i + t \in \mathcal{T}$, we can associate the *control point* $x_{T_i} + t$, wherefore the control points of tiles in the same translation class $[T_i]$ are in the same "relative position" to its associated tile. The set of all control points for the tiles of a given translation class $[T_i]$ yield a Delone set Λ_i (it is relatively dense by repetitivity and uniformly discrete since $\mathcal{T}$ is a tiling and therefore, in particular, T_i is a regularly closed compact set which consequently has interior points, wherefore the positions of two tiles in $[T_i]$ must have – since G is a metric space and, in particular, Hausdorff and locally compact – a minimal distance). Now, the choice $x_{T_i} \in \operatorname{int} T_i$ ensures that $\underline{\Lambda}$ is a multi-component Delone set (but we will see that in the examples we will consider, also the choice $x_{T_i} = 0$ is possible). Clearly, $\mathcal{T}$ can be reconstructed from $\underline{\Lambda}$ as $\mathcal{T} = \underline{\Lambda} + (T_i - x_{T_i})_{i=1}^n$, which consequently establishes that $\mathcal{T}$ and $\underline{\Lambda}$ are MLD. We also note that the difference set of tilings in the same translation class $[T_i]$ is given by $\Delta_i = \Lambda_i - \Lambda_i$, and we can define $\Delta' = \bigcup_{i=1}^n \Delta_i$ also for a tiling.

In view of this remark, we also note that the notation $\mathcal{T} = \underline{\Lambda} + \underline{A}$ implies that the choice $x_{A_i} = 0$ for the control points of the prototiles is used (for all $1 \leq i \leq n$).

Definition 5.126. Let $\underline{\Lambda}$ be a substitution multi-component Delone set that satisfies **(PLT)**. Then, as in Equation (5.7) on p. 155, we have the following tile substitution

$$A_i + x \quad \mapsto \quad \Big(A_j + t_{(a)} \circ f_0(x) \mid 1 \leq j \leq n, t_{(a)} \circ f_0 \in \Theta_{ji}\Big).$$

We introduce the notation "$\Cap$" for the *ordered intersection*, *i.e.*, we discriminate the left and right hand side of the intersection. Then, we use the following notation for all $x \in G$:

$$\Xi(i,j,x) = A_i \Cap (A_j + x) = \big((A_i - x) \Cap A_j\big) + x.$$

For simplicity, we may identify $\Xi(i,i,0) = A_i \Cap A_i$ with A_i. We let the tile substitution act on $\Xi(\cdot,\cdot,\cdot)$ as follows: Apply the tile substitution to each side of the ordered intersection separately, then form all possible ordered intersection, *i.e.*,

$$\begin{aligned}
\Xi(i,k,x) &= A_i \Cap (A_k + x) \\
&\mapsto \Big(A_j + a \mid 1 \leq j \leq n, t_{(a)} \circ f_0 \in \Theta_{ji}\Big) \Cap \\
&\qquad \Big(A_\ell + b + f_0(x) \mid 1 \leq \ell \leq n, t_{(b)} \circ f_0 \in \Theta_{\ell k}\Big) \\
&= \Big(\big(A_j + a\big) \Cap \big(A_\ell + b + f_0(x)\big) \mid 1 \leq j, \ell \leq n, t_{(a)} \circ f_0 \in \Theta_{ji}, t_{(b)} \circ f_0 \in \Theta_{\ell k}\Big) \\
&= \Big(\Xi(j,\ell,b + f_0(x) - a) + a \mid 1 \leq j, \ell \leq n, t_{(a)} \circ f_0 \in \Theta_{ji}, t_{(b)} \circ f_0 \in \Theta_{\ell k}\Big)
\end{aligned} \tag{5.13}$$

We now restrict the possible values of x: Let $A_i + t_i, A_k + t_k \in \mathcal{T}$ be two tiles in the tiling and $y \in \Delta'$. We call the triple $(A_i + t_i, A_k + t_k, y) \in \mathcal{T} \times \mathcal{T} \times \Delta'$ an *overlap* if $\Xi(i,k,t_k + y - t_i) \neq \varnothing$

idea is that, by primitivity (and nonempty interior as well as repetitivity), there is a power $m = m(T_i)$ such that a translate $T_i + b$ can be found inside the support of the m-th power of the tile substitution of T_i. Then, defining $x_{T_i} = -\sum_{k=1}^{\infty} f_0^{-m\cdot k}(-b)$ (this is well-defined since f_0^{-1} is a contraction) establishes the control point with the following two properties (the second is a direct consequence of the first): (*i*) $\omega_i(-x_{T_i})$ satisfies the inclusion property $\omega_i(-x_{T_i}) \subset \boldsymbol{\Theta}^m(\omega_i(-x_{T_i}))$. (*ii*) $\operatorname{int}(-x_{T_i} + T_i)$ is a relatively compact open neighbourhood of 0.

and a *strong overlap* if $\operatorname{int} \Xi(i,k,t_k+y-t_i) \neq \varnothing$ (this is equivalent to $\operatorname{int}(A_i+t_i) \cap \operatorname{int}(A_k+t_k+y) \neq \varnothing$). In these cases, we simply call $\Xi(i,k,t_k+y-t_i)$ a *(strong) overlap*. We call an overlap, which is not a strong overlap, a *weak overlap*. An overlap of the form $\Xi(i,i,0) = A_i \Cap A_i \overset{\text{iden.}}{=} A_i = A_i \sqcap A_i$ is called a *coincidence* (we identify $\Xi(i,i,0)$ with A_i).

Using the substitution in Equation (5.13), we may define the following graphs:

- The vertices of the *strong overlap graph* $G^{ovlap}_{str}(\mathcal{T})$ are the strong overlaps $\Xi(i,k,x)$ (nonempty interior) where $A_i+t_i, A_k+t_k \in \mathcal{T}$ and $x = t_k+y-t_i$ with $y \in \Delta'$. We put a directed edge (labelled by a) from $\Xi(i,k,x)$ to $\Xi(j,\ell,b+f_0(x)-a)$ if $\Xi(j,\ell,b+f_0(x)-a)$ is a strong overlap in the substitute of $\Xi(i,k,x)$ (*i.e.*, it appears in the right hand side of Equation (5.13)).

- The vertices of the *weak overlap graph* $G^{ovlap}_{weak}(\mathcal{T})$ are the overlaps $\Xi(i,k,x)$ (nonempty) where $A_i+t_i, A_k+t_k \in \mathcal{T}$ and $x = t_k+y-t_i$ with $y \in \Delta'$. We put a directed edge (labelled by a) from $\Xi(i,k,x)$ to $\Xi(j,\ell,b+f_0(x)-a)$ if $\Xi(j,\ell,b+f_0(x)-a)$ is an overlap in the substitute of $\Xi(i,k,x)$ (*i.e.*, it appears in the right hand side of Equation (5.13)).

Obviously, $G^{ovlap}_{str}(\mathcal{T})$ is a subgraph of $G^{ovlap}_{weak}(\mathcal{T})$.

We say that $\mathcal{T}$ admits an *overlap coincidence* if Δ' is a Meyer set and, for every strong overlap, there is a directed path to a coincidence in $G^{ovlap}_{str}(\mathcal{T})$.

We note the following:

- Since the union (respectively the support) on the right hand side in the tile substitution of A_i+x yields $f_0(A_i)+f_0(x)$, from every (strong) overlap there is a directed edge to at least one (strong) overlap, *i.e.*, every vertex has at least one child. Moreover, if $\Xi(i,k,x)$ is empty, then there are no overlaps in its substitute.

- We also observe that $b+f_0(x)-a = b+f_0(t_k)+f_0(y)-(a+f_0(t_i))$ in Equation (5.13). But $b+f_0(t_k) \in \Lambda_\ell$, $a+f_0(t_i) \in \Lambda_j$ and $f_0(x) \in f_0(\Delta') \subset \Delta'$, and the above construction is really well-defined (in the sense that the intersections on the right hand side of Equation (5.13) are either overlaps or empty sets).

- Since $\Delta' = -\Delta'$ and we use the ordered intersection, $\Xi(i,k,x)$ is a (strong) overlap iff $\Xi(k,i,-x)$ is a (strong) overlap. Moreover, there is an edge from $\Xi(i,k,x)$ to $\Xi(j,\ell,b+f_0(x)-a)$ (labelled a) iff there is an edge from $\Xi(k,i,-x)$ to $\Xi(\ell,j,a-f_0(x)-b)$ (labelled b). Therefore, if we do not care about the order in $\Cap$, we may obtain new directed graphs by identifying the vertices $\Xi(i,k,x)$ and $\Xi(k,i,-x)$, forgetting the edge labels and replacing multiple directed edges by a single one. The graph obtained from $G^{ovlap}_{str}(\mathcal{T})$ by this process is called *subdivision graph for overlaps* in [364] and [235, Appendix A.2]. Note that this graph is sufficient to decide if there is an overlap coincidence.

- Assuming that Δ' is a Meyer set implies that $G^{ovlap}_{weak}(\mathcal{T})$ is a finite graph, see [235, Lemma A.8]: By construction $t_i \in \Lambda_i \subset \Delta'$ and $t_k \in \Lambda_k \subset \Delta'$. By the Meyer set property, $x = t_k+y-t_i \in \Delta'+\Delta'-\Delta' \subset \Delta'+F+F$, where F is a finite set, wherefore the allowed x for an overlap $\Xi(i,k,x)$ form a discrete set in G. The tiles A_i are compact, and we set $\breve{R} = \max\{\operatorname{diam}(A_i \cup \{0\}) \mid 1 \le i \le n\}$. Then $\Xi(i,k,x)$ is nonempty only if $x \in B_{\le 2\breve{R}}(0)$ (otherwise, one easily establishes that the tiles A_i and A_k+x have positive distance, since the maximal distance of any of their points from the origin is bounded by $\check{R}$, wherefore $A_i \subset B_{\leq \check{R}}(0)$ and $A_k + x \subset B_{\leq \check{R}}(x)$). Together with the discreteness, the finiteness follows.

- A weak overlap $\Xi(i,k,x)$ is a subset of the boundary ∂A_i.

Since every (strong) overlap has at least one child and by the finiteness, there is at least one directed cycle in $G^{ovlap}_{str}(\mathcal{T})$. In fact, we already know some of its structure.

Remark 5.127. As noted in Remark 4.104, we can represent the (adjoint) IFS $\boldsymbol{\Theta}^{\#}$ as directed multi-graph $G(\boldsymbol{\Theta}^{\#})$. Moreover, the vertices of this graph $G(\boldsymbol{\Theta}^{\#})$ are the sets A_i which we may identify with the coincidences $\Xi(i,i,0)$. An edge in $G(\boldsymbol{\Theta}^{\#})$ is labelled by some map $f_0^{-1} \circ t_{(a)}$, but by **(LT)** it is enough to simply label it by the translation a.

Then, by construction (or, more precisely, since the tile substitution is derived from the IFS), $G(\boldsymbol{\Theta}^{\#})$ is a subgraph of $G^{ovlap}_{weak}(\mathcal{T})$. Since the IFS $\boldsymbol{\Theta}^{\#}$ is irreducible (even primitive), $G(\boldsymbol{\Theta}^{\#})$ is strongly connected and therefore contains at least one directed cycle. Moreover, this also shows that if there is a directed path from a strong overlap to one coincidence, then there is a directed path to all coincidences in $G^{ovlap}_{str}(\mathcal{T})$.

Remark 5.128. In a given example, it might be hard to judge if a given set $\Xi(i,k,x)$ is empty or not or if it even has interior points. We note the following:

- As noted before, $\Xi(i,k,x)$ is nonempty only if $x \in B_{\leq 2\check{R}}(0)$.

- The previous remark shows that the coincidences $\Xi(i,i,0)$ are given as solution of an IFS. In fact, using the same interpretation as for $G(\boldsymbol{\Theta}^{\#})$, one sees that $G^{ovlap}_{weak}(\mathcal{T})$ and $G^{ovlap}_{str}(\mathcal{T})$ define iterated function systems for the sets $\Xi(i,k,x)$, although these iterated function systems might not be irreducible. But the connection to the theory of iterated function systems enables us to characterise these sets.

- By the measure-disjointness of the unions in the tile substitution (respectively, the unions in the adjoint IFS $\boldsymbol{\Theta}^{\#}$), there are no directed edges from a coincidence $\Xi(i,i,0)$ to a non-coincidence $\Xi(i,j,x)$ (with $i \neq j$ if $x = 0$) in $G^{ovlap}_{str}(\mathcal{T})$. Moreover, if there is an edge from a coincidence to a non-coincidence $\Xi(i,j,x)$ in $G^{ovlap}_{weak}(\mathcal{T})$, then $\Xi(i,j,x)$ is actually a weak overlap and therefore part of the boundary ∂A_i (namely, the part given by $A_i \cap (A_j + x)$).

- If there is an directed edge from $\Xi(i,k,x)$ to $\Xi(j,\ell,y)$ in $G^{ovlap}_{str}(\mathcal{T})$, then some translate of $f_0^{-1}(\Xi(j,\ell,y))$ can be found inside $\Xi(i,k,x)$.

- The boundary has Haar measure 0. So by the mapping theorem (see Corollary 4.50), if $\Xi(i,j,x)$ is a weak overlap (*i.e.*, part of the boundary ∂A_i) and therefore a vertex in $G^{ovlap}_{weak}(\mathcal{T})$ (but not in $G^{ovlap}_{str}(\mathcal{T})$), then there is no path leading from $\Xi(i,j,x)$ to an coincidence (otherwise, the boundary would have interior points!). So any path from a weak overlap $\Xi(i,j,x)$ can only lead to other weak overlaps in $G^{ovlap}_{weak}(\mathcal{T})$; consequently, there is at least one circle consisting only of weak overlaps in $G^{ovlap}_{weak}(\mathcal{T})$.

We can now often determine the graphs $G^{ovlap}_{weak}(\mathcal{T})$ and $G^{ovlap}_{str}(\mathcal{T})$, such that all their vertices $\Xi(i,k,x)$ are implicitly given by an attractor of some iterated function systems:

(i) Start with all vertices $\Xi(i,k,x)$ such that $x \in B_{\leq 2\check{R}}(0)$ where $\check{R} = \max\{\mathrm{diam}(A_i \cup \{0\}) \,|\, 1 \leq i \leq n\}$. Add all possible directed edges according to Equation (5.13) on p. 175

between them. So, we first do not care if a vertex with $x \in B_{\le 2\check{R}}(0)$ is actually an empty set or not. We call this the *starting graph*.

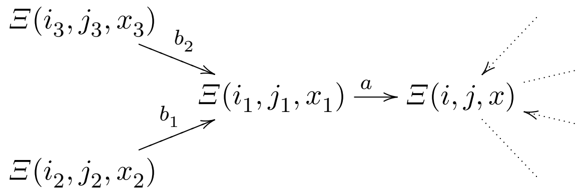
(ii) Remove iteratively all vertices which have no outgoing directed edge (since, *e.g.*, all its children would be outside $B_{\le 2\check{R}}(0)$). These are obviously empty sets.

(iii) We know the solution for the coincidences $A_i = \varXi(i,i,0)$, namely, it is the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$. From now on, we only care about vertices different from these coincidences A_i.

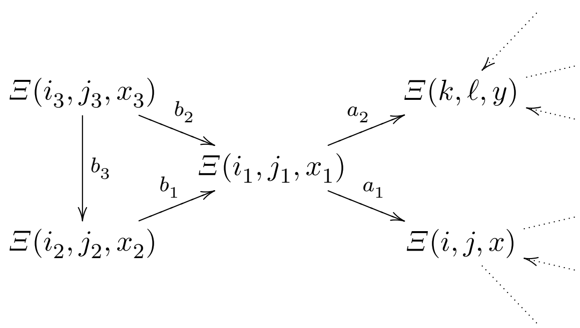
(iv) If there is an edge from a coincidence to a non-coincidence $\varXi(i,k,x)$, then this vertex does not belong to $G^{\mathit{ovlap}}_{\mathit{str}}(\mathcal{T})$, but may be a weak overlap and therefore belong to $G^{\mathit{ovlap}}_{\mathit{weak}}(\mathcal{T})$. Remove all edges leading from coincidences to non-coincidences. Then mark all vertices to which a path from a possible weak overlap leads to. These are the new possible weak overlaps.

(v) All vertices from which a path leads to coincidences are strong overlaps and belong to $G^{\mathit{ovlap}}_{\mathit{str}}(\mathcal{T})$. Note that there might be more strong overlaps (iff $\mathcal{T}$ admits no(!) overlap coincidence). Remove all edges between the strong overlaps we already have and the possible weak overlaps.

(vi) If the induced subgraph given by all successors of a vertex $\varXi(i,k,x)$ (more precisely, we only need the successors of one directed edge leading to $\varXi(i,k,x)$) is a directed tree, then we can easily determine these set as in the following example:



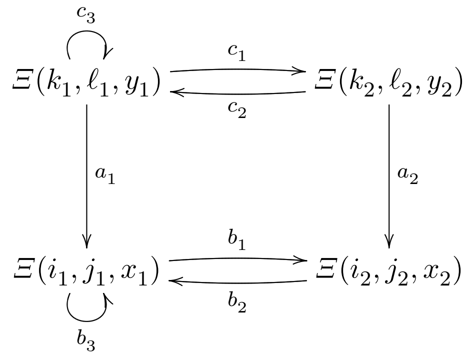
One has $\varXi(i_1,j_1,x_1) = f_0^{-1}(\varXi(i,j,x)+a)$, $\varXi(i_2,j_2,x_2) = f_0^{-2}(\varXi(i,j,x)+a) + f_0^{-1}(b_1)$ and $\varXi(i_3,j_3,x_3) = f_0^{-2}(\varXi(i,j,x)+a) + f_0^{-1}(b_2)$. A similar situation is the following:



where one has $\varXi(i_1,j_1,x_1) = f_0^{-1}(\varXi(i,j,x)+a_1) \cup f_0^{-1}(\varXi(k,\ell,y)+a_2)$, $\varXi(i_2,j_2,x_2) = f_0^{-1}(\varXi(i_1,j_1,x_1)+b_1)$ and $\varXi(i_3,j_3,x_3) = f_0^{-1}(\varXi(i_1,j_1,x_1)+b_2) \cup f_0^{-1}(\varXi(i_2,j_2,x_2)+b_3)$.

(vii) Therefore, we may, in fact, remove successively all *stranded vertices*. Here, we say that a vertex in a directed graph is stranded, if it either has no outgoing or no incoming edges. We say that a graph is *essential* if no vertex of the graph is stranded (see [239, Definition 2.2.9]). So we determine an essential subgraph here; in fact, one can show that every (finite) graph has a unique essential subgraph (see [239, Proposition 2.2.10]).

(viii) This essential subgraph may consist of several connected (but not necessarily strongly connected) components[27], so we might have a situation as the following for such a component:



(5.14)

The strongly connected part (consisting of induced subgraph by the vertices $\Xi(i_1, j_1, x_1)$ and $\Xi(i_2, j_2, x_2)$) then defines an (irreducible) IFS, which has a unique attractor.

(ix) Implicitly, one knows the solution of all iterated function systems (in particular, of the coincidences $\Xi(i, i, 0)$). So, the only problem are sets like $\Xi(k_1, \ell_1, y_1)$ and $\Xi(k_2, \ell_2, y_2)$ in Equation (5.14): One has

$$\begin{aligned} \Xi(k_1, \ell_1, y_1) &= f_0^{-1}(\Xi(i_1, j_1, x_1) + a_1) \cup f_0^{-1}(\Xi(k_2, \ell_2, y_2) + c_1) \cup f_0^{-1}(\Xi(k_1, \ell_1, y_1) + c_3) \\ \Xi(k_2, \ell_2, y_2) &= f_0^{-1}(\Xi(i_2, j_2, x_1) + a_2) \cup f_0^{-1}(\Xi(k_1, \ell_1, y_1) + c_2), \end{aligned} \tag{5.15}$$

which consequently does not define an IFS (one can not use an estimate like in the proof of Lemma 4.86 to show that one has a contraction in $\mathcal{K}G$). One can only say that if the sets $\Xi(i_1, j_1, x_1)$ and $\Xi(i_2, j_2, x_2)$ have interior points, so have $\Xi(k_1, \ell_1, y_1)$ and $\Xi(k_2, \ell_2, y_2)$. Moreover, one might hope to prove further statements for a given system as in Equation (5.15) by recursive substitution on the right hand side (in this case, this yields $\Xi(k_1, \ell_1, y_1) = f_0^{-1}(\Xi(i_1, j_1, x_1) + a_1) \cup f_0^{-1}(f_0^{-1}(\Xi(i_2, j_2, x_1) + a_2) + +c_1) \cup \ldots)$, *e.g.*, that if the sets $\Xi(i_1, j_1, x_1)$ and $\Xi(i_2, j_2, x_2)$ have no interior points, then neither have $\Xi(k_1, \ell_1, y_1)$ and $\Xi(k_2, \ell_2, y_2)$. It is this case, where we do not have a general statement about!

(x) If we know for each set $\Xi(i, j, x)$ if it is empty, nonempty but without interior points or has interior points, one can easily obtain the graphs $G^{\mathit{ovlap}}_{\mathit{str}}(\mathcal{T})$ and $G^{\mathit{ovlap}}_{\mathit{weak}}(\mathcal{T})$ as induced subgraphs (with the appropriate vertices) of the starting graph.

Examples of overlap graphs will appear in the next chapters (see Section 6.10). After these comments on the computability of overlap coincidences, we now establish the connection between the function $\mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}$ and the overlap and the algebraic coincidence.

[27] We say that a directed graph is *connected* if its associated undirected graph (*i.e.*, if we forget the direction of the edges) is a connected graph. A directed graph is *strongly connected* if, for any (ordered) pair v_i, v_j of vertices, there is path along directed edges from v_i to v_j.

Lemma 5.129. *Let $\underline{\varLambda}$ be a repetitive substitution multi-component Delone set that satisfies* **(PLT)**. *Suppose that $\varDelta'$ is a Meyer set. Then the following are equivalent:*

(i) $\lim_{m\to\infty} \mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^m(x)) = 1$ *for every* $x \in \varDelta'$.

(ii) $1-\mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^m(x)) \le C\cdot r^m$ *for an* $m \in \mathbb{N}$, *every* $x \in \varDelta'$ *and some constants (independent of x)* $C > 0$ *and* $r \in]0\,,1[$.

(iii) $\mathcal{T}$ *admits an overlap coincidence.*

(iv) $\underline{\varLambda}$ *admits an algebraic coincidence.*

Sketch of Proof. (i)⇔(ii)⇔(iii): For $G = \mathbb{R}^d$, this can be found in [235, Lemma A.9] (respectively [364, Prop. 6.7]). We indicate the most important steps:

- If (iii) holds, then there is an $\ell \in \mathbb{N}$ such that from every overlap $\varXi(i,k,x)$ there is a path of at most length ℓ leading to a coincidence in $G^{\mathit{ovlap}}_{\mathit{str}}(\mathcal{T})$. But this means that the "non-overlap coincidence density" $1 - \mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^m(x))$ increases as follows:
$$1 - \mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^{m+\ell}(x)) \le \left(1 - \frac{\min\{\mu(A_i) \mid 1 \le i \le n\}}{\alpha^\ell \cdot \max\{\mu(A_i) \mid 1 \le i \le n\}}\right) \cdot \left(1 - \mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^m(x))\right),$$
since after ℓ substitutions every tile (of measure at most $\max\{\mu(A_i) \mid 1 \le i \le n\}$) yields at least one coincidence (of measure at least $\min\{\mu(A_i) \mid 1 \le i \le n\}$). Consequently, writing $m = k\cdot\ell + s$ where $0 \le s < \ell$ for any $m \in \mathbb{N}$, one can establish (ii).

- Suppose to the contrary that (iii) does not hold. Then there is an overlap $\varXi(i,k,x) = \mathrm{int}\, A_i \cap (\mathrm{int}\, A_k + x)$ which does not lead to an coincidence and, by UPF and repetitivity, the patch $A_i \cup (A_k + x)$ has positive frequency. Moreover, inflating this cluster by f_0^m does not yield coincidences, wherefore the "non-overlap coincidence density" is bounded away from 0 for all m.

(iii)⇔(iv): For $G = \mathbb{R}^d$, this can be found in [235, Propositions 3.7 & 3.9 & Theorem 3.10]. One establishes (iii)⇒(iv) as follows: Since $G^{\mathit{ovlap}}_{\mathit{str}}(\mathcal{T})$ is a finite graph, there is an $M \in \mathbb{N}$ such that for any $x \in \varDelta'$ there is a $t_x \in \varLambda_i$ with $t_x + f_0^M(x) \in \varLambda_i$. One then has to show that there is a t independent of x; for this, one actually uses that one has an FLC set together with (ii). By FLC, there are finitely many configurations $\{(\mathcal{T} \sqcap (\mathcal{T}+x)) \cap W \mid x \in \varDelta'\}$ for any compact set W. Say, there are ℓ many such configurations and choose representatives $\mathcal{T} \sqcap (\mathcal{T}+x_i) \cap W$ with $1 \le i \le \ell$. Then, the "multiple non-overlap coincidence density" can be estimated by
$$1 - \frac{\mu\left((\mathcal{T} \sqcap (\mathcal{T} + f_0^m(x_1)) \sqcap \ldots \sqcap (\mathcal{T} + f_0^m(x_\ell)))\ \cap f_0^m(W)\right)}{\mu(f_0^m(W))} < C\cdot r^m \cdot \ell$$
using (ii). Consequently, for some (big) m, there is at least one common tile in the multiple "intersection" of the numerator. This yields the (i and the) $t \in \varLambda_i$ in the definition of the algebraic coincidence.
For the converse direction, we assume $\xi + f^M(\varDelta') \subset \varLambda_j$ and one observes $f_0(\varDelta') \subset \varDelta'$ and $f_0^M(\varDelta') - f_0^M(\varDelta') \subset \varDelta'$ (see Equation (5.17) on p. 184). One then argues that for any strong overlap $\varXi(i,k,t_k+y-t_i)$ there is an $M' \ge 2\,M$ such that $f_0^{M'}(\mathrm{supp}\,\varXi(i,k,t_k+y-t_i))$ contains

a ball of radius bigger than the covering radius of (the relatively dense set) $f_0^{2M}(\varLambda_j - \varLambda_j)$. Thus, there is a $z \in \varLambda_j - \varLambda_j$ such that

$$\Big(\boldsymbol{\Theta}^{M'}(\omega_i(t_i)) + \underline{A}\Big) \;\ni\; \xi + f_0^{2M}(z) + A_j \;\in\; \Big(\boldsymbol{\Theta}^{M'}(\omega_k(t_k)) + \underline{A}\Big) + f_0^{M'}(y),$$

where on the left and on the right[28] are patches of $\mathcal{T}$. But this shows that there is a path (of length M') in $G_{str}^{ovlap}(\mathcal{T})$ from the overlap $\varXi(i,k,t_k + y - t_i)$ to the coincidence $\varXi(j,j,0)$. □

5.7.2. Coincidences and the CPS

The following is immediate from Equation (5.10) on p. 170 and (ii) in the previous lemma, also see [230, Prop. 5.1]. Note that if $\varDelta'$ is a Meyer set, so is $f_0^m(\varDelta')$.

Lemma 5.130. *Let $\underline{\varLambda}$ be a representable repetitive substitution multi-component Delone set in an LCAG G as above, with associated tiling $\mathcal{T} = \underline{\varLambda} + \underline{A}$. Suppose that $\varDelta'$ is a Meyer set. If $\underline{\varLambda}$ admits an algebraic coincidence, then, for every $\varepsilon > 0$, there is an $m = m(\varepsilon) \in \mathbb{Z}_{\geq 0}$ such that $f_0^m(\varDelta') \subset P'_\varepsilon$.* □

An immediate consequence of the previous lemma and the (proof of) Corollary 5.43 is the following statement, where we show that if $\underline{\varLambda}$ is a primitive repetitive substitution multi-component Delone set and a multi-component regular IMS, then it admits an algebraic coincidence, also see [230, Theorem 6.6] for a different proof. This justifies the introduction of "algebraic coincidence" in Definition 5.123.

Proposition 5.131. *Let $(G\,,H\,,\tilde{L})$ be a CPS. Let $\underline{\varLambda}$ be a regular multi-component (inter) model set which is also a representable repetitive substitution multi-component Delone set, where the EMFS satisfies* **(LT)**. *Suppose $\hat{f}_0 : H' \to H'$ is a contraction. Then $\underline{\varLambda}$ admits an algebraic coincidence.*

Proof. Since all $\varOmega_i \in H$ have nonempty interior, there is an open ball $B_{<r}(0) \subset H' \times \{0\} = H$ such that for every $1 \leq i \leq n$ there exists an $s_i \in H$ with $B_{<r}(s_i) \subset \operatorname{int} \varOmega_i$. Moreover, by the denseness of $\mathcal{L}^\star \subset H$ and the definition of the operator $\Lambda(\cdot)$, we can even choose elements $s_i = t_i^\star$ with $t_i \in \varLambda_i$.

We have $\varDelta' \subset \Lambda(\bigcup_{i=1}^n(\varOmega_i - \varOmega_i))$, and $\bigcup_{i=1}^n(\varOmega_i - \varOmega_i)$ is (as finite union of compact sets) compact; in particular, $\operatorname{diam} \bigcup_{i=1}^n(\varOmega_i - \varOmega_i) < \infty$. Since $\hat{f}_0$ is a contraction, there is an $m \in \mathbb{Z}_{\geq 0}$ such that $\hat{f}_0^m(\bigcup_{i=1}^n(\varOmega_i - \varOmega_i)) \subset B_{<r}(0)$. But then $t_i^\star + \hat{f}_0^m(\bigcup_{i=1}^n(\varOmega_i - \varOmega_i)) \subset \operatorname{int} \varOmega_i$, and consequently $t_i + f_0^m(\varDelta') \subset \varLambda_i$. So, we have an algebraic coincidence. □

Remark 5.132. Under the assumptions of Lemma 5.130 and similar to our considerations in Section 5.3, we observe the following:

- Since $f_0^{m+1}(\varDelta') \subset f_0^m(\varDelta')$ for all $m \in \mathbb{Z}_{\geq 0}$, the set $\{f_0^m(\varDelta') \mid m \in \mathbb{Z}_{\geq 0}\}$ is a filterbase and is a countable base at 0 for the commutative groups G, $\mathcal{L}$ and $\mathcal{L}'$. Therefore, we have commutative topological groups $(\mathcal{L}\,, \{f_0^m(\varDelta')\})$ and $(\mathcal{L}'\,, \{f_0^m(\varDelta')\})$ (here, we define the topology by the base at 0). Moreover, since $\varDelta' \subset \mathcal{L}'$, one has $(\mathcal{L}\,, \{f_0^m(\varDelta')\}) \cong (\mathcal{L}'\,, \{f_0^m(\varDelta')\}) \times \mathcal{L}/\mathcal{L}'$ (compare p. 145; we recall that $\mathcal{L}/\mathcal{L}'$ is the height group equipped with the discrete topology, see Definition 5.54).

[28] The "2" in $M' \geq 2M$ is needed to establish that $\xi + f_0^{2M}(z) - f_0^{M'}(y) \in \varLambda_j$.

- Since f_0 is an expansion, these topological groups $(\mathcal{L}, \{f_0^m(\Delta')\})$ and $(\mathcal{L}', \{f_0^m(\Delta')\})$ are Hausdorff.
- Denote the Hausdorff completion of $(\mathcal{L}', \{f_0^m(\Delta')\})$ by H'_{sub} and the Hausdorff completion of $(\mathcal{L}, \{f_0^m(\Delta')\})$ by H_{sub}. Then, H_{sub} is topologically isomorphic to $H'_{\text{sub}} \times \mathcal{L}/\mathcal{L}'$ (again compare p. 145).
- Lemma 5.130 shows that the topology defined by the filterbase $, \{f_0^m(\Delta')\}$ is finer than the topology given through the pseudometric $\varrho_{\underline{\Lambda}}$. Therefore, by Proposition 2.47 (respectively Lemma 2.68), there is a (uniformly) continuous homomorphism $\widehat{\text{id}}$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{L}, \{f_0^m(\Delta')\}) & \xrightarrow{\text{id}} & (\mathcal{L}, \varrho_{\underline{\Lambda}}) \\ \downarrow \varphi & & \downarrow \varphi' \\ H_{\text{sub}} \cong H'_{\text{sub}} \times \mathcal{L}/\mathcal{L}' & \xrightarrow{\widehat{\text{id}}} & H \cong H' \times \mathcal{L}/\mathcal{L}' \end{array} \tag{5.16}$$

  where φ and φ' denote the canonical uniformly continuous homomorphisms from a commutative topological group to its Hausdorff completion.

Actually, the entire construction of a CPS in Section 5.3 goes through and one obtains the following CPS, see Proposition 5.137 and [230, Section 4]:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H_{\text{sub}} & \xrightarrow{\pi_2} & H_{\text{sub}} \\ \cup & & \cup & & \cup \text{ dense} \\ \mathcal{L} & \xleftrightarrow{\text{bijective}} & \widetilde{\mathcal{L}} = \{(t, t) \mid t \in \mathcal{L}\} & \xleftrightarrow{\text{bijective}} & \mathcal{L} \end{array}$$

While Equation (5.16) indicates that, in general, H'_{sub} is not "optimal" since it is "bigger than" H' (see Example 5.156 on this phenomenon), one might wonder under which condition(s) H'_{sub} and H' are topologically isomorphic.

The converse inclusion $P'_\varepsilon \subset f_0^m(\Delta')$ in Lemma 5.130 cannot hold in general: Suppose $\underline{\Lambda}$ is periodic, *e.g.*, $\underline{\Lambda} = \underline{\Lambda} + t$ for some $t \in G \setminus \{0\}$. Then $\mathbb{Z}t = \langle t \rangle_{\mathbb{Z}} \subset P'_\varepsilon$ for all $\varepsilon > 0$, but – since f_0 is an expansion – one certainly does not have $\mathbb{Z}t \subset f_0^m(\Delta')$ for all m. If $\underline{\Lambda}$ is a repetitive UCF multi-component set, this is the only case one has to exclude.

Lemma 5.133. *Let $\underline{\Lambda}$ be a repetitive substitution multi-component Delone set that satisfies* **(PLT)**. *Then, every almost period of $\underline{\Lambda}$ is a period. Consequently, $\varrho_{\underline{\Lambda}}$ is a metric on $\mathcal{L}$ if $\underline{\Lambda}$ is aperiodic.*

Proof. Let $t \in \Delta'$ be an almost period, *i.e.*, $t \in \text{cl}\{0\}$ in $(\mathcal{L}, \varrho_{\underline{\Lambda}})$. This means that $\varrho_{\Lambda_i}(t, 0) = 0$ and therefore $\eta_{\Lambda_i}(t) = \text{dens}\,\Lambda_i$ for all $1 \le i \le m$.

Since $\mathcal{T}$ is a repetitive UCP tiling ($\underline{\Lambda}$ is a repetitive UCF multi-component set), $\mathcal{T}$ has UPF and, in particular, the patch frequencies exists (uniformly) and are positive for any patch $\mathcal{P}$ occuring in $\mathcal{T}$. Let us denote the patch frequency of a patch $\mathcal{P}$ by $\text{freq}_{\mathcal{T}}(\mathcal{P})$.

For $1 \le i \le n$, we set $\mathcal{P}_i = \{A_i, A_i + t\}$. Then, one has $\eta_{\Lambda_i}(t) = \text{freq}_{\mathcal{T}}(\mathcal{P}_i)$, wherefore – since $\eta_{\Lambda_i}(t)$ is positive – the patch $\mathcal{P}_i$ occurs in $\mathcal{T}$. Now, we consider patches (enumerated by j) of the following form: $\mathcal{P}_j^{(i)} = \{A_i\} \cup \breve{\mathcal{P}}_j^{(i)}$ where $\text{supp}\,\breve{\mathcal{P}}_j^{(i)} \supset \text{int}\,A_i + t$, $T \cap (\text{int}\,A_i + t) \neq \varnothing$ for any tile $T \in \breve{\mathcal{P}}_j^{(i)}$ and $\mathcal{P}_j^{(i)}$ is a patch of $\mathcal{T}$. Obviously, $\mathcal{P}_i$ itself is such a patch. Moreover, one has $\operatorname{dens} \varLambda_i = \sum_j \operatorname{freq}_{\mathcal{T}}(\mathcal{P}_j^{(i)})$. From $\eta_{\varLambda_i}(t) = \operatorname{dens} \varLambda_i$ and the positivity of the patch frequencies, we deduce that there is only one such patch $\mathcal{P}_j^{(i)}$, namely $\mathcal{P}_i$. Consequently, t is a period of $\varLambda_i$. But this holds for any $1 \le i \le n$, wherefore t is a period of $\underline{\varLambda}$. □

We now arrive at the statement we are looking for. This also appears as [230, Prop. 5.2], but with a different proof.

Lemma 5.134. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT)**. *Suppose that $\varDelta'$ is a Meyer set. If $\underline{\varLambda}$ admits an algebraic coincidence, then, for every $m \in \mathbb{N}$, there is an $\varepsilon = \varepsilon(m) > 0$ such that $P'_\varepsilon \subset f_0^m(\varDelta')$.*

Proof. We prove this lemma by contradiction. We first observe the following properties: $P'_\varepsilon \subset P'_{\varepsilon'}$ for $\varepsilon < \varepsilon'$, $f_0^n(\varDelta') \subset f_0^{n'}(\varDelta')$ for $n > n'$, $P'_1 = \varDelta' = f_0^0(\varDelta')$ and $0 \in P'_\varepsilon$ and $0 \in f_0^n(\varDelta')$ for all $\varepsilon > 0$ and all $n \in \mathbb{Z}_{\ge 0}$.

Since $\underline{\varLambda}$ is aperiodic, there are no almost periods by the previous lemma, *i.e.*, $\operatorname{cl}\{0\} = \{0\}$ in $(\mathcal{L}, \varrho_{\underline{\varLambda}})$.

Suppose there is an $n \in \mathbb{Z}_{\ge 0}$ such that $P'_\varepsilon \not\subset f_0^n(\varDelta')$ for any $\varepsilon > 0$. But this means that there is an $x \neq 0$ such that, for every $\varepsilon > 0$, $x \in P'_\varepsilon$ (but $x \notin f_0^n(\varDelta')$). Consequently, such an x would be an almost period. This contradiction proves the claim. □

Putting everything together, one obtains the following statement.

Corollary 5.135. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT)**. *Suppose that $\varDelta'$ is a Meyer set. If $\underline{\varLambda}$ admits an algebraic coincidence, then $\varrho_{\underline{\varLambda}}$ is a metric, $\{f_0^m(\varDelta') \mid m \in \mathbb{Z}_{\ge 0}\}$ is (alternatively to $\{P'_{1/k} \mid k \in \mathbb{N}\}$) a countable base at 0 for $(\mathcal{L}, \varrho_{\underline{\varLambda}})$ and the continuous homomorphism $\hat{\mathrm{id}}$ in Equation (5.16) on p. 182 is an isomorphism. Moreover, for every $m \in \mathbb{Z}_{\ge 0}$ there exist $\varepsilon > 0$ and $m' \in \mathbb{Z}_{\ge 0}$ such that $f_0^{m'}(\varDelta') \subset P'_\varepsilon \subset f_0^m(\varDelta')$; similarly, for every $\varepsilon > 0$ there are $m \in \mathbb{Z}_{\ge 0}$ and $\varepsilon' > 0$ such that $P'_{\varepsilon'} \subset f_0^m(\varDelta') \subset P'_\varepsilon$.* □

Remark 5.136. For a commutative topological group with countable base at 0, Remark 2.35 shows how to define a translation-invariant pseudometric which defines the topology (of course, $\varrho_{\underline{\varLambda}}$ is such a pseudometric in our case). The previous corollary indicates that, assuming aperiodicity and algebraic coincidence, f_0 indeed is a contraction (on $\mathcal{L}'$ equipped with the AC topology) with respect to the metric defined through the countable base $\{P'_{1/k} \mid k \in \mathbb{N}\}$ (and therefore even $\varrho_{\underline{\varLambda}}$ itself) respectively $\{f_0^m(\varDelta') | m \in \mathbb{Z}_{\ge 0}\}$. At the moment, unfortunately, we do not know how to prove this rigorously and what happens if one replaces such a metric with an equivalent (translation-invariant) metric.

5.7.3. Extending the Internal Space

For a given example, it is often hard to check for an algebraic coincidence, since the set $\varDelta'$ is usually hard to control. Therefore we would like to have statements which are easier to check. As a first step in this direction, we would like to derive statements as in Lemmas 5.78 & 5.79 for the sets $\breve{\varPi}_i$ (respectively $\varPi_i = \breve{\varPi}_i \times \{a_i\}$), *i.e.*, we would like to define something like the adjoint EMFS to the IFS $\breve{\boldsymbol{\varTheta}}^\star$ (respectively $\boldsymbol{\varTheta}^\star$). This is not possible in general, since $\hat{f}_0^{-1}$ may not be defined on H' respectively H. Thus, one has to "enlarge" or "extend" H' respectively H appropriately. For this, we note the following points:

- Let $\underline{\varLambda}$ be a substitution FLC multi-component Delone set in an LCAG G having property **(LT)**, with associated tiling $\mathcal{T} = \underline{\varLambda} + \underline{A}$ and such that every cluster is legal. Then, it is repetitive, wherefore $\varDelta_i = \varDelta_i^{\mathrm{ess}}$. As before, we set $\varDelta' = \bigcup_{i=1}^{n} \varDelta_i$ and $\varDelta = \operatorname{supp} \underline{\varLambda} - \operatorname{supp} \underline{\varLambda}$. Since $f_0 : G \to G$ is a bi-Lipschitz bijection, we can define $f_0^m(\varDelta')$ and $f_0^m(\varDelta)$ for any $m \in \mathbb{Z}$. Note that we always have $f_0^{k+1}(\varDelta') \subset f_0^k(\varDelta')$ for all $k \in \mathbb{Z}$, since $f_0(\varLambda_i - \varLambda_i) = f_0(\varLambda_i) - f_0(\varLambda_i) = f(\varLambda_i) - f(\varLambda_i) \subset \varLambda_j - \varLambda_j$, where $f \in \varTheta_{ji} \neq \varnothing$.
  We also note that $\varDelta \subset \varDelta' + F$ where F is a finite set given by $F = \{t_i - t_j \,|\, 1 \leq i, j \leq n\}$ with fixed $t_i \in \varLambda_i$ (also compare Lemma 5.55).

- We define

$$(\mathcal{L}'_{\mathrm{ext}})^{(m)} = \langle \bigcup_{k=0}^{m} f_0^{-k}(\varDelta') \rangle_{\mathbb{Z}} = \bigcup_{k=0}^{m} f_0^{-k}(\langle \varDelta' \rangle_{\mathbb{Z}}) = \bigcup_{k=0}^{m} f_0^{-k}(\mathcal{L}')$$

  (where the second equality follows by linearity) and $(\mathcal{L}_{\mathrm{ext}})^{(m)} = \langle \bigcup_{k=0}^{m} f_0^{-k}(\varDelta) \rangle_{\mathbb{Z}} = \bigcup_{k=0}^{m} f_0^{-k}(\mathcal{L})$ (note that $(\mathcal{L}'_{\mathrm{ext}})^{(0)} = \mathcal{L}'$ and $(\mathcal{L}_{\mathrm{ext}})^{(0)} = \mathcal{L}$). By construction, each $(\mathcal{L}'_{\mathrm{ext}})^{(m)}$ and $(\mathcal{L}_{\mathrm{ext}})^{(m)}$ is a commutative group.

- For the commutative groups $(\mathcal{L}'_{\mathrm{ext}})^{(m)}$ and $(\mathcal{L}_{\mathrm{ext}})^{(m)}$, we note that the family of sets $\mathcal{B}^{(m)} = \{f_0^k(\varDelta') \mid k \in \mathbb{Z}, k \geq -m\}$ satisfies properties (ii) and (iv) of Proposition 2.20. We prove property (iii) under the assumption of an algebraic coincidence, see [230, Lemma 4.1]: If $\underline{\varLambda}$ admits an algebraic coincidence, then there is an $x \in \varLambda_i$ and an $M \in \mathbb{Z}_{\geq 0}$ such that $x + f_0^M(\varDelta') \subset \varLambda_i$. Noting that $\varDelta' = -\varDelta'$ and f_0 is linear, we have

$$f_0^M(\varDelta') + f_0^M(\varDelta') = f_0^M(\varDelta') - f_0^M(\varDelta') \subset \varLambda_i - \varLambda_i \subset \varDelta', \tag{5.17}$$

  and consequently $f_0^{M+k}(\varDelta') + f_0^{M+k}(\varDelta') \subset f_0^k(\varDelta')$ for all $k \in \mathbb{Z}$. By Proposition 2.20, $\mathcal{B}^{(m)}$ is a neighbourhood base of 0 of the commutative groups $(\mathcal{L}'_{\mathrm{ext}})^{(m)}$ and $(\mathcal{L}_{\mathrm{ext}})^{(m)}$. We note that (since f_0 is an expansion on G) these groups are then Hausdorff commutative topological groups.

- By construction, $f_0^{m-k} : (\mathcal{L}'_{\mathrm{ext}})^{(m)} \to (\mathcal{L}'_{\mathrm{ext}})^{(k)}$ is a continuous group homomorphism. Moreover, for $m \leq k$, the embedding $(\mathcal{L}'_{\mathrm{ext}})^{(m)} \hookrightarrow (\mathcal{L}'_{\mathrm{ext}})^{(k)}$ is a group homomorphism. Therefore, we may say that $\{(\mathcal{L}'_{\mathrm{ext}})^{(m)}, \hookrightarrow, m \in \mathbb{Z}_{\geq 0}\}$ is a direct system of commutative topological groups, see Definition 3a.12. We denote the (existing and unique) direct limit (see Definition 3a.13 & Lemma 3a.14) by

$$\mathcal{L}'_{\mathrm{ext}} = \varinjlim_{m \in \mathbb{Z}_{\geq 0}} (\mathcal{L}'_{\mathrm{ext}})^{(m)} = \bigcup_{m \in \mathbb{Z}_{\geq 0}} (\mathcal{L}'_{\mathrm{ext}})^{(m)} = \bigcup_{m \in \mathbb{Z}_{\geq 0}} f_0^{-m}(\mathcal{L}'),$$

  where the second equality follows by the construction of the direct limit (compare Remark 3a.15, respectively [311, Proof of Prop. 1.2.1 & Exercise 1.2.3]). *Mutatis mutandis*, the same holds for the direct limit $\mathcal{L}_{\mathrm{ext}} = \varinjlim_{m \in \mathbb{Z}_{\geq 0}} (\mathcal{L}_{\mathrm{ext}})^{(m)}$. We also observe that, although we have

$$\begin{aligned} \left((\mathcal{L}_{\mathrm{ext}})^{(m)}, \mathcal{B}^{(m)}\right) &\cong \left((\mathcal{L}'_{\mathrm{ext}})^{(m)}, \mathcal{B}^{(m)}\right) \times (\mathcal{L}_{\mathrm{ext}})^{(m)} / (\mathcal{L}'_{\mathrm{ext}})^{(m)} \\ &\cong \left((\mathcal{L}'_{\mathrm{ext}})^{(m)}, \mathcal{B}^{(m)}\right) \times \mathcal{L}/\mathcal{L}' \end{aligned}$$

for every $m \in \mathbb{Z}_{\geq 0}$, this does *not* imply $\mathcal{L}_{\text{ext}} \cong \mathcal{L}'_{\text{ext}} \times \mathcal{L}/\mathcal{L}'$ (since it might happen that $(\mathcal{L}_{\text{ext}})^{(m)} \subset (\mathcal{L}'_{\text{ext}})^{(m+1)}$ for all m, wherefore $\mathcal{L}_{\text{ext}} \cong \mathcal{L}'_{\text{ext}}$; see Example 6c.13 in Section 6c.2 on this phenomenon). Moreover, we note that $\mathcal{B} = \{f_0^k(\varDelta') \mid k \in \mathbb{Z}\}$ is a neighbourhood base of 0 of these direct limits by construction. This yields

$$(\mathcal{L}_{\text{ext}}\,, \mathcal{B}) \cong \big(\mathcal{L}'_{\text{ext}}\,, \mathcal{B}\big) \times \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}},$$

where the factor group $\mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$ is equipped with the discrete topology. Note that the latter factor group is (isomorphic to) a subgroup of $\mathcal{L}/\mathcal{L}'$.

- We therefore have Hausdorff commutative topological groups $(\mathcal{L}'_{\text{ext}}\,, \mathcal{B})$ and $(\mathcal{L}_{\text{ext}}\,, \mathcal{B})$. Note that by construction we have that f_0 and f_0^{-1} are continuous. Since we assume that $\underline{\varLambda}$ admits an algebraic coincidence, we have (by definition) that $\varDelta'$ is a Meyer set and therefore, in particular, it is relatively dense in G. So there is a compact set $W \subset G$ such that $W + \varDelta' = G$. Moreover, we have $f_0^m(W) + f_0^m(\varDelta') = G$, and $f_0^m(W)$ is, as continuous image of a compact set, also compact. So, $f_0^m(\varDelta')$ is relatively dense for all $m \in \mathbb{Z}$. But then, for every $m \in \mathbb{Z}$, $f_0^m(\varDelta')$ is totally bounded, since it can be covered by finitely many translates of $f_0^{m'}(\varDelta')$ for any $m' \in \mathbb{Z}$, compare [36, Lemma 2] and [230, Proof of Theorem 4.2]: There is nothing to show for $m' \leq m$. Otherwise, let $x \in f_0^m(\varDelta')$ and write $x = y + z$ with $y \in f_0^{m'}(\varDelta')$ and $z \in f_0^{m'}(W)$. Then we get $z = x - y \in f_0^m(\varDelta') - f_0^{m'}(\varDelta') \subset f_0^m(\varDelta') + f_0^m(\varDelta') \subset f_0^{m-M}(\varDelta')$. Thus, $z \in f_0^{m'}(W) \cap f_0^{m-M}(\varDelta')$ and $F = f_0^{m'}(W) \cap f_0^{m-M}(\varDelta')$ is a finite set as intersection of a compact and uniformly discrete set. But then $f_0^m(\varDelta') \subset F + f_0^{m'}(\varDelta')$, which proves the claim.
  Note that, since f_0^{-1} is a contraction on G, the covering radius and the packing radius in $f_0^m(\varDelta')$ go to 0 for $m \to -\infty$. Therefore, $\mathcal{L}_{\text{ext}}$ (and $\mathcal{L}'_{\text{ext}}$) is dense in G.

- We denote the Hausdorff completion of $(\mathcal{L}'_{\text{ext}}\,, \mathcal{B})$ by H'_{ext} and the Hausdorff completion of $(\mathcal{L}_{\text{ext}}\,, \mathcal{B})$ by $H_{\text{ext}} \cong H'_{\text{ext}} \times \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$. We denote the canonical homomorphism from $(\mathcal{L}_{\text{ext}}\,, \mathcal{B})$ to its completion by φ. Observe that the sets $\text{cl}_{H_{\text{ext}}}\, \varphi(f_0^m(\varDelta')) \subset H'_{\text{ext}} \times \{0\}$ are compact in H_{ext}, wherefore H_{ext} is an LCAG, compare [36, Prop. 1]. Moreover, they have nonempty interior, compare [36, Lemma 4] respectively [230, Proof of Theorem 4.2].

- We may also denote the Hausdorff completions of the (Hausdorff commutative) topological groups $(\mathcal{L}'_{\text{ext}})^{(m)}$ by $(H'_{\text{ext}})^{(m)}$. By the uniqueness of the direct limit, the uniqueness of the Hausdorff completion and since the image of a group in its completion is dense, we also have
$$H'_{\text{ext}} = \varinjlim_{m \in \mathbb{Z}_{\geq 0}} \big(H'_{\text{ext}}\big)^{(m)}\,,$$
  and a similar statement also holds for H_{ext}.

- Similarly to the construction in Section 5.3, we define $\widetilde{\mathcal{L}}_{\text{ext}} = \{(t\,, \varphi(t)) \in G \times H_{\text{ext}} \mid t \in \mathcal{L}_{\text{ext}}\}$. Then, $\widetilde{\mathcal{L}}_{\text{ext}}$ is a lattice, where both uniform discreteness and relative denseness follow since $\varDelta'$ is a Meyer set (noting that $\varDelta \subset \varDelta' + F$ where F is a finite set), compare [36, Lemmas 3 & 5] and [230, Proof of Theorem 4.2].
  Note that we can identify $\varphi(\mathcal{L}_{\text{ext}})$ and $\mathcal{L}_{\text{ext}}$ here.

- We now make the connection to the LCAG H_{sub} of Remark 5.132: Obviously, H_{sub} is a subspace of H_{ext}. Moreover, $\varphi(\varDelta')$ is relatively compact in both H_{sub} and H_{ext}, respectively H'_{sub} and H'_{ext}, and

$$\mathrm{cl}_{H_{\mathrm{ext}}}\,\varphi(\varDelta') \cap H'_{\mathrm{sub}} = \mathrm{cl}_{H_{\mathrm{sub}}}\,\varphi(\varDelta').$$

  So, H'_{sub} is the clopen compactly generated subgroup of H'_{ext} containing the compact set $\mathrm{cl}_{H'_{\mathrm{ext}}}\,\varphi(\varDelta')$, see Definition 2.53 and Lemma 2.54.

- By Remark 5.132 and Lemma 5.134 we have: If $\underline{\varLambda}$ is aperiodic, then H_{sub} and H are topologically isomorphic. Obviously, in this case we have an embedding $H' \hookrightarrow H'_{\mathrm{ext}}$, respectively $H \hookrightarrow H_{\mathrm{ext}}$; moreover, by this embedding, we can naturally identify $\widetilde{\mathcal{L}}$ with the discrete subgroup $\widetilde{\mathcal{L}}_{\mathrm{ext}} \cap (G \times H)$.

- By Proposition 2.47 (respectively Lemma 2.68), there are (uniformly) continuous homomorphism $\hat{f}_0$ and $\hat{f}_0^{-1}$ such that the following diagrams commute:

$$\begin{array}{ccc} (\mathcal{L}'_{\mathrm{ext}}, \mathcal{B}) & \xrightarrow{f_0, f_0^{-1}} & (\mathcal{L}'_{\mathrm{ext}}, \mathcal{B}) \\ \varphi \downarrow & & \downarrow \varphi \\ H'_{\mathrm{ext}} & \xrightarrow{\check{f}_0, \check{f}_0^{-1}} & H'_{\mathrm{ext}} \end{array}$$

We summarise our findings, compare to [230, Theorem 4.2] for the case $G = \mathbb{R}^d$ and H_{sub} (the construction of H_{ext} is new).

Proposition 5.137. *Let $\underline{\varLambda}$ be a substitution FLC multi-component Delone set in an LCAG G having property* **(LT)***, with associated tiling $\mathcal{T} = \underline{\varLambda} + \underline{A}$ and such that every cluster is legal. Assume that $\underline{\varLambda}$ admits an algebraic coincidence. Then, we have the following CPS:*

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H_{\mathrm{ext}} & \xrightarrow{\pi_2} & H_{\mathrm{ext}} \cong H'_{\mathrm{ext}} \times \mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}} \\ \text{dense} \cup & & \cup & & \cup \text{ dense} \\ \mathcal{L}_{\mathrm{ext}} & \overset{\text{bijective}}{\longleftrightarrow} & \widetilde{\mathcal{L}}_{\mathrm{ext}} & \overset{\text{bijective}}{\longleftrightarrow} & \mathcal{L}_{\mathrm{ext}} \end{array} \tag{5.18}$$

where $\mathcal{L}_{\mathrm{ext}} = \bigcup_m f_0^{-m}(\mathcal{L})$ (and, similarly, $\mathcal{L}'_{\mathrm{ext}} = \bigcup_m f_0^{-m}(\mathcal{L}')$, where both arise as direct limits) and H_{ext} denotes its Hausdorff completion.
Moreover, if $\underline{\varLambda}$ is aperiodic, we can identify the LCAG H' in $H = H' \times \mathcal{L}/\mathcal{L}'$ of Equation (5.4) on p. 144 with a clopen compactly generated subgroup of H_{ext}, which also identifies the lattice $\widetilde{\mathcal{L}}$ in a natural way with a subgroup of the lattice $\widetilde{\mathcal{L}}_{\mathrm{ext}}$. □

Corollary 5.138. *Assume the setting of the previous proposition. Then the continuous homomorphisms $f_0, f_0^{-1} : G \to G$ are also continuous homomorphisms on $(\mathcal{L}'_{\mathrm{ext}}, \mathcal{B})$ and extend to continuous homomorphisms $\check{f}_0, \check{f}_0^{-1} : H'_{\mathrm{ext}} \to H'_{\mathrm{ext}}$.* □

These statements show that it is possible to extend the internal space H in such a way that we can also define the inverse of $\hat{f}_0$.

5.7.4. Tiles and Tilings in the Extended Internal Space

From now on, we assume that for a given aperiodic substitution multi-component Delone set which admits a CPS as in Section 5.3, we have an "extended" internal space H_{ext} and the following property:

(PLT+) $\underline{\varLambda}$ is a representable aperiodic primitive substitution multi-component Delone set in a LCAG G, where the EMFS $\boldsymbol{\Theta}$ satisfies **(LT)** and the components A_i of the attractor $\underline{A}$ of the adjoint IFS $\boldsymbol{\Theta}^{\#}$ are the prototiles of the associated tiling $\mathcal{T} = \underline{\varLambda} + \underline{A}$ (*i.e.*, $\underline{\varLambda}$ is aperiodic and satisfies **(PLT)**). Moreover, $\hat{f}_0$ is a bijection (and therefore a homeomorphism) in the ("extended") internal space H_{ext}, and $\breve{\boldsymbol{\Theta}}^{\star}$ (see pp. 171–172) is an IFS with property **(LT)** on H'_{ext}.

We now have a closer look at the IFS $\breve{\boldsymbol{\Theta}}^{\star}$, and afterwards also at its adjoint.

Lemma 5.139. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT+)**. *Let $\underline{\mathcal{P}}$ be a finite seed of $\underline{\varLambda}$ and suppose that $\breve{f}_0 : H'_{\text{ext}} \to H'_{\text{ext}}$ is a contraction. Then, there is a family $\underline{\breve{\varPi}} \subset (\mathcal{K}H'_{\text{ext}})^n$ of nonempty compact subsets of H'_{ext} such that $\underline{\varLambda} \subset \Lambda(\underline{\varPi})$ where $\varPi_i = \breve{\varPi}_i \times \{a_i\}$ for some $a_i \in \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$.*

Proof. We use the notation $\boldsymbol{\Theta}^{\star}$ and $\breve{\boldsymbol{\Theta}}^{\star}$ as on pp. 171–172. Let $\pi : H_{\text{ext}} \to H'_{\text{ext}}$ and $\pi' : H_{\text{ext}} \to \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$ be the canonical projections, and denote by a_i the element in $\pi'(\varLambda_i) = \{a_i\}$.

Define $\underline{\breve{\varPi}}^{(0)} = \pi(\underline{\mathcal{P}}^{\star})$, *i.e.*,

$$\breve{\varPi}_i^{(0)} = \bigcup_{\omega_i(x) \subset \underline{\mathcal{P}}} \{\pi(x^{\star})\}.$$

Note that the sets $\breve{\varPi}_i^{(0)}$ are either compact or empty, at least one being nonempty. Moreover, we note that for $x \in \varLambda_i$ one has $x^{\star} = (\pi(x^{\star}), a_i)$. We now set $\underline{\breve{\varPi}}^{(k)} = (\breve{\varPi}_i^{(k)})_{i=1}^n$ and

$$\underline{\breve{\varPi}}^{(k+1)} = \breve{\boldsymbol{\Theta}}^{\star}(\underline{\breve{\varPi}}^{(k)}) = \left(\breve{\boldsymbol{\Theta}}^{\star}\right)^{k+1} (\underline{\breve{\varPi}}^{(0)}).$$

By the primitivity of $\breve{\boldsymbol{\Theta}}^{\star}$, all components of $\underline{\breve{\varPi}}^{(n^2-2n+2)}$ are nonempty (and compact) by Remark 4.68.

By construction, we have $\bigcup_{i=1}^n \breve{\varPi}_i^{(k)} \times \{a_i\} \subset \mathcal{L}^{\star}$ and $\left(\Lambda(\breve{\varPi}_i^{(k)} \times \{a_i\})\right)_{i=1}^n = \boldsymbol{\Theta}^k(\underline{\mathcal{P}})$ (note that the star-map is bijective). By the inclusion property of the finite seed, we have that the (unique) attractor $\underline{\breve{\varPi}}$ of the IFS $\breve{\boldsymbol{\Theta}}^{\star}$ is given by

$$\underline{\breve{\varPi}} = \text{cl}_{H_{\text{ext}}} \left(\bigcup_{k \in \mathbb{Z}_{\geq 0}} \left(\breve{\boldsymbol{\Theta}}^{\star}\right)^k (\underline{\breve{\varPi}}^{(0)}) \right),$$

see Proposition 4.89. Consequently, each $\varLambda_i$ is a subset of $\Lambda(\breve{\varPi}_i \times \{a_i\})$. □

By the uniqueness of the attractor (irrespective of the compact "starting value" $\underline{\breve{\varPi}}^{(0)}$), we actually do not need the finite seed. Note that the sets $\varPi_i$ may have empty interior. To exclude this case, we once again assume the setting of Proposition 4.99 in the previous lemma, which consequently yields the following statement. Note that by $\mu_{H'_{\text{ext}}}$ we denote the Haar measure obtained from the Haar measure $\mu_{H_{\text{ext}}}$ restricted to H'_{ext}.

Lemma 5.140. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT+)**. *Suppose that $\hat{f}_0 : H'_{\mathrm{ext}} \to H'_{\mathrm{ext}}$ is a contraction such that $\mu_{H'_{\mathrm{ext}}}(W) = \alpha \cdot \mu_{H'_{\mathrm{ext}}}(\hat{f}_0(W))$ for every compact set $W \subset H'_{\mathrm{ext}}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$ respectively $\boldsymbol{S\breve{\Theta}}^\star$. Let $\underline{\breve{\varPi}}$ be the attractor of $\breve{\boldsymbol{\Theta}}^\star$ in $(\mathcal{K}H'_{\mathrm{ext}})^n$ and suppose that at least one component $\breve{\varPi}_{i_0}$ has nonempty interior. Then, one has:*

(i) *All $\breve{\varPi}_i$ have nonzero Haar measure $\mu_{H'_{\mathrm{ext}}}(\breve{\varPi}_i) > 0$.*

(ii) *The unions in $\breve{\boldsymbol{\Theta}}^\star$ are measure-disjoint.*

(iii) *The boundaries $\partial\breve{\varPi}_i$ have zero Haar measure for all i.*

(iv) *All $\breve{\varPi}_i$ are perfect sets and regularly closed.*

(v) *$\underline{\varLambda}$ is a subset of a (regular) multi-component model set.*

(vi) *$\underline{\varLambda}$ is a multi-component Meyer set.*

Proof. This is just Proposition 4.99 (together with the statement of the last lemma) applied to our situation here. The last statement follows directly from Corollary 5.14. □

Note that by $\varPi_i = \breve{\varPi}_i \times \{a_i\}$, the same statements about nonzero Haar measure, boundaries of zero Haar measure *etc.* also hold for these sets.

Corollary 5.141. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT+)**. *Assume that $\underline{\varLambda}$ admits an algebraic coincidence and suppose that $\hat{f}_0 : H'_{\mathrm{ext}} \to H'_{\mathrm{ext}}$ is a contraction such that $\mu_{H'_{\mathrm{ext}}}(W) = \alpha \cdot \mu_{H'_{\mathrm{ext}}}(\hat{f}_0(W))$ for every compact set $W \subset H'_{\mathrm{ext}}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$ respectively $\boldsymbol{S\breve{\Theta}}^\star$. Let $\underline{\breve{\varPi}}$ be the attractor of $\breve{\boldsymbol{\Theta}}^\star$ in $(\mathcal{K}H'_{\mathrm{ext}})^n$. Then the sets $\breve{\varPi}_i$ have nonempty interior.*

Proof. By Lemma 5.140, it is enough to prove that one set $\breve{\varPi}_i$ has interior points. Since there is an algebraic coincidence, there is an $1 \le i \le n$, an $m \in \mathbb{Z}_{\ge 0}$ and a $t \in \varLambda_i$ such that $t + f_0^m(\varDelta') \subset \varLambda_i$. By Corollary 5.135, $\mathrm{cl}_{H_{\mathrm{ext}}}(t + f_0^m(\varDelta'))^\star$ is a compact neighbourhood of $t^\star$ (with nonempty interior) contained in $\varPi_i = \breve{\varPi}_i \times \{a_i\}$ (compare proof of Lemma 5.139). Consequently, $\breve{\varPi}_i$ has interior points (*e.g.*, $t^\star$). □

For an MFS $\boldsymbol{\Theta}^\star$ on H_{ext} satisfying (essentially – see the following discussion) **(LT)**, we can now define the adjoint MFS $\boldsymbol{\Theta}^{\star\#}$ as before. Note that by construction, we have $\boldsymbol{\Theta}^{\star\#} = \boldsymbol{\Theta}^{\#\star}$.

However, we note the following: Let $\hat{f} : H_{\mathrm{ext}} \to H_{\mathrm{ext}}$ be a continuous map and $H_{\mathrm{ext}} \cong H'_{\mathrm{ext}} \times \mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}$. Similarly as on p. 172, we denote the action of $\hat{f}$ restricted on H'_{ext} by $\breve{f}$ and restricted on $\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}$ by $\dot{f}$. If we restrict all maps of an MFS $\boldsymbol{\Theta}^\star$ on H'_{ext}, then we denote this by $\breve{\boldsymbol{\Theta}}^\star$. We have argued that $\breve{\boldsymbol{\Theta}}^\star$ is an IFS on H'_{ext}, while – because of the discrete group $\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}$ – the MFS $\boldsymbol{\Theta}^\star$ is not an IFS on H_{ext}. Similarly, the adjoint $\boldsymbol{\Theta}^{\star\#}$ is not an EMFS on H_{ext} (again, because of $\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}$), but $\breve{\boldsymbol{\Theta}}^{\star\#}$ is an EMFS on H'_{ext}. So, we say that an MFS $\boldsymbol{\Theta}^\star$ (respectively $\boldsymbol{\Theta}^{\star\#}$) on H_{ext} satisfies **(LT)**, if the associated MFS $\breve{\boldsymbol{\Theta}}^\star$ (respectively $\breve{\boldsymbol{\Theta}}^{\star\#}$) on H'_{ext} satisfies **(LT)** (and therefore is either an IFS or an EMFS).

We have also argued that all maps $\hat{f} \in \boldsymbol{\Theta}^{\star}$ of the "IFS" $\boldsymbol{\Theta}^{\star}$ can simply be interpreted as $\hat{f} = (\check{f}, a_i)$ with some constant a_i (see p. 172). However, such an interpretation for the maps in the "EMFS" $\boldsymbol{\Theta}^{\star\#}$ is not possible; in fact, $\boldsymbol{\Theta}^{\star\#}$ should yield a multi-component Delone set in (all of) H_{ext}. We now establish that there is natural candidate for such a multi-component Delone set $\underline{\Upsilon}$. Here, we make use of the *symmetric CPS* in Equation (5.18), *i.e.*, we actually have two cut and project schemes, namely $(G\,, H_{\text{ext}}\,, \widetilde{L}_{\text{ext}})$ and $(H_{\text{ext}}\,, G\,, \widetilde{L}_{\text{ext}})$.

Lemma 5.142. *Let* $\underline{A}$ *a repetitive aperiodic substitution multi-component Delone set that satisfies* **(PLT+)**. *Define the following regular multi-component inter model sets:*

$$\underline{\Upsilon}^1 = \Lambda(\operatorname{int} \underline{A}) \qquad \textit{and} \qquad \underline{\Upsilon}^2 = \Lambda(\underline{A})$$

where $\Lambda(\cdot)$ *is defined with respect to the CPS* $(\check{H}'_{\text{ext}}\,, G\,, \widetilde{\mathcal{L}}'_{\text{ext}})$. *Then*

(i) $\underline{\Upsilon}^1 \supset \boldsymbol{\Theta}^{\star\#}(\underline{\Upsilon}^1)$, *and the inclusion can be proper.*

(ii) $\underline{\Upsilon}^2 = \boldsymbol{\Theta}^{\star\#}(\underline{\Upsilon}^2)$, *and the unions on the right hand side may not be disjoint.*

Proof. (i): We have to confirm the above inclusion:

$$\begin{aligned}
\Upsilon^1_i &\overset{\text{def}}{=} \Lambda(\operatorname{int} A_i) \overset{\text{IFS}}{=} \Lambda\left(\operatorname{int} \bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} f(A_j)\right) \\
&\supset \Lambda\left(\bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} \operatorname{int} f(A_j)\right) = \Lambda\left(\bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} f(\operatorname{int} A_j)\right) \\
&= \bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} \Lambda\left(f(\operatorname{int} A_j)\right) = \bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} \hat{f}\left(\Lambda(\operatorname{int} A_j)\right) \\
&\overset{\text{def}}{=} \bigcup_{j=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} \hat{f}(\Upsilon^1_j) \overset{\text{def}}{=} \bigcup_{j=1}^{n} \bigcup_{\hat{f} \in \Theta^{\#\star}_{ij}} \hat{f}(\Upsilon^1_j),
\end{aligned} \tag{5.19}$$

where $\operatorname{int} f(A) = f(\operatorname{int} A)$ follows since all maps are homeomorphisms, and we get $\hat{f}$ by definition from f. For the inclusion sign, we note that the unions in the IFS $\boldsymbol{\Theta}^{\#}(\underline{A})$ are measure-disjoint, wherefore the unions in $\boldsymbol{\Theta}^{\#}(\operatorname{int} \underline{A})$ are disjoint.

Obviously, the inclusion is proper if

$$\left(\bigcup_{i=1}^{n} \bigcup_{f \in \Theta^{\#}_{ij}} f(\partial A_j)\right) \setminus \left(\bigcup_{i=1}^{n} \partial A_i\right) \cap \mathcal{L}'_{\text{ext}} \neq \varnothing,$$

i.e., if there are "internal boundary points" which are projected lattice points.

(ii): A calculation similar to Equation (5.19) establishes that $\Upsilon^2 = \hat{\boldsymbol{\Theta}}^{\star\#}(\Upsilon^2)$. Since the unions in $\boldsymbol{\Theta}^{\#}(\underline{A})$ are only measure-disjoint but not disjoint, the unions in $\boldsymbol{\Theta}^{\star\#}(\Upsilon^2)$ may not be disjoint. The latter are disjoint iff, for every distinct pair $f \in \Theta^{\#}_{ij}$ and $g \in \Theta^{\#}_{ik}$, one has

$$f(\partial A_j) \cap g(\partial A_k) \cap \mathcal{L}'_{\text{ext}} = \varnothing.$$

□

Remark 5.143. This last statement suggests that there is a multi-component inter model set $\underline{\Upsilon}^1 \subset \underline{\Upsilon} \subset \underline{\Upsilon}^2$ which is also a primitive (and repetitive) substitution multi-component set. Basically, the idea is that one starts with $\underline{\Upsilon}^2$ and then removes boundary points in some "coherent" way. Unfortunately, we are – at least in the general case – not able to establish such a claim.

Note that under the assumptions of the previous proposition, we may form the collection $\underline{\Upsilon} + \underline{\Pi}$ for any multi-component inter model set $\underline{\Upsilon}^1 \subset \underline{\Upsilon} \subset \underline{\Upsilon}^2$. Furthermore, we have Lemma 5.78 for $\underline{\Upsilon} + \underline{\Pi}$, *i.e.*, its covering degree is bounded and locally constant $\mu_{H_{\text{ext}}}$-a.e. We now formulate a consequence of Lemma 5.79 in the case we consider here.

Proposition 5.144. *Let $\underline{\Lambda}$ be a repetitive aperiodic substitution multi-component Delone set that satisfies* **(PLT+)**. *Denote the attractor of the IFS $\breve{\Theta}^\star$ by $\underline{\breve{\Pi}}$ and define the tiles $\Pi_i = \breve{\Pi}_i \times \{a_i\}$. Suppose $\underline{\Upsilon}$ is a repetitive substitution multi-component Delone set (with respect to the "EMFS" $\Theta^{\star\#}$) such that it is also given as a multi-component inter model set $\Lambda(\operatorname{int} \underline{A}) \subset \underline{\Upsilon} \subset \Lambda(\underline{A})$. Then, $\underline{\Lambda}$ is a multi-component inter model set with $\Lambda(\operatorname{int} \underline{\Pi}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Pi})$ iff $\underline{\Upsilon} + \underline{\Pi}$ is a tiling of H_{ext}.*

Proof. We actually prove the following: Let $\underline{\Lambda}'$ be a repetitive substitution multi-component inter model set with $\Lambda(\operatorname{int} \underline{\Pi}) \subset \underline{\Lambda}' \subset \Lambda(\underline{\Pi})$. Then, the covering degree of $\underline{\Lambda}' + \underline{A}$ equals k μ_G-a.e. iff the covering degree of $\underline{\Upsilon} + \underline{\Pi}$ equals k $\mu_{H_{\text{ext}}}$-a.e.

Both collections $\underline{\Lambda}' + \underline{A}$ and $\underline{\Upsilon} + \underline{\Pi}$ meet the requirements of Lemma 5.79 (in particular, they are self-replicating). So we calculate the covering degree as:

$$\deg^{cov}_{\underline{\Lambda}' + \underline{A}}(x) = \sum_{i=1}^{n} \mu_G(A_i) \cdot \operatorname{dens}(\Lambda'_i) \qquad \mu_G\text{-a.e.}$$
$$\deg^{cov}_{\underline{\Upsilon} + \underline{\Pi}}(y) = \sum_{i=1}^{n} \mu_G(\Pi_i) \cdot \operatorname{dens}(\Upsilon_i) \qquad \mu_{H_{\text{ext}}}\text{-a.e.}$$

We establish the equality of these sums by the density formula, see Corollary 5.27 using $\eta_X(0) = \operatorname{dens} X$ for a uniformly discrete point set X in an LCAG. The density formula yields

$$\operatorname{dens}(\Lambda'_i) = \frac{\mu_{H_{\text{ext}}}(\Pi_i)}{\mu_{G\times H_{\text{ext}}}(\operatorname{FD}(\widetilde{\mathcal{L}}_{\text{ext}}))} \qquad \text{and} \qquad \operatorname{dens}(\Upsilon_i) = \frac{\mu_G(A_i)}{\mu_{G\times H_{\text{ext}}}(\operatorname{FD}(\widetilde{\mathcal{L}}_{\text{ext}}))},$$

which consequently establishes the equality of the covering degrees. □

Corollary 5.145. *Assume the setting of Proposition* 5.144. *Then, $\underline{\Lambda}$ is a multi-component inter model set with $\Lambda(\operatorname{int} \underline{\Pi}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Pi})$ iff the only strong overlaps $\Xi(i,k,x)$ with $x \in \Lambda(\operatorname{int} \Pi_k - \operatorname{int} \Pi_i)$ are coincidences.*

Proof. The discussion around Equation (5.12) on p. 173 shows that $\underline{\Lambda} \subset \Lambda(\underline{\Pi})$. Therefore, if $A_i + t_i$ and $A_k + t_k$ are two different tiles from $\mathcal{T}$ (so, $t_i \neq t_k$ if $i = k$), we have $t_i \in \Lambda(\Pi_i)$, $t_k \in \Lambda(\Pi_k)$ and $\operatorname{int} A_i + t_i \cap \operatorname{int} A_k + t_k = \varnothing$.

So, if $\underline{\Lambda}$ is a multi-component IMS $\Lambda(\operatorname{int} \underline{\Pi}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Pi})$, then the previous equality must especially hold for all $t_i \in \Lambda(\operatorname{int} \Pi_i)$ and all $t_k \in \Lambda(\operatorname{int} \Pi_k)$, which establishes one direction.

Conversely, suppose that, for all pairs $t_i \in \Lambda(\operatorname{int} \Pi_i)$ and $t_k \in \Lambda(\operatorname{int} \Pi_k)$, one has $\operatorname{int} A_i + t_i \cap \operatorname{int} A_k + t_k = \varnothing$. By $\mu_{H_{\text{ext}}}(\operatorname{int} \Pi_i) = \mu_{H_{\text{ext}}}(\Pi_i)$, we can conclude that the covering degree of $\mathcal{T}$ is 1 μ_G-a.e. This establishes the claim. □

This last result is a variant of the overlap coincidence we explored in Section 5.7.1. In fact, we will show in the next section (see Theorem 5.154) that this last criterion is equivalent to the overlap coincidence. We also note that it might be possible that the iteration of $\Xi(i,k,x)$ with $x \in \Lambda(\operatorname{int} \mathit{\Pi}_k - \operatorname{int} \mathit{\Pi}_i)$ (see Equation (5.13) on p. 175) may lead to sets $\Xi(j,m,y)$ where $y \in \Lambda(\mathit{\Pi}_m - \mathit{\Pi}_j) \setminus \Lambda(\operatorname{int} \mathit{\Pi}_m - \operatorname{int} \mathit{\Pi}_j)$.

We note that up to intersections at the boundary $(\partial A_i \times (-\mathit{\Pi}_i)) \cup (A_i \times (-\partial \mathit{\Pi}_i))$ (respectively $((-\partial A_i) \times \mathit{\Pi}_i) \cup ((-A_i) \times \partial \mathit{\Pi}_i))$ and with $\Lambda(\operatorname{int} \underline{\mathit{\Pi}}) \subset \underline{\mathit{\Lambda}}' \subset \Lambda(\underline{\mathit{\Pi}})$, we have the following *klotz construction*[29]:

$$\begin{aligned} (G \times \{0\}) \cap \left(\widetilde{\mathcal{L}}_{\text{ext}} + \left(\bigcup_{i=1}^{n} A_i \times (-\mathit{\Pi}_i) \right) \right) &= \left((\underline{\mathit{\Lambda}}' + \underline{A}) \times \{0\} \right) \\ (\{0\} \times H_{\text{ext}}) \cap \left(\widetilde{\mathcal{L}}_{\text{ext}} + \left(\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i \right) \right) &= (\{0\} \times (\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}})) \end{aligned} \tag{5.20}$$

These equalities follow by Remark 5.7. The following statement is now no surprise.

Corollary 5.146. *Assume the setting of Proposition* 5.144. *Then,* $\underline{\mathit{\Lambda}}$ *is a multi-component inter model set with* $\Lambda(\operatorname{int} \underline{\mathit{\Pi}}) \subset \underline{\mathit{\Lambda}} \subset \Lambda(\underline{\mathit{\Pi}})$ *iff* $\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i$ *is a measure-disjoint union and a fundamental domain of* $\widetilde{\mathcal{L}}_{\text{ext}}$.

Proof. We note that the union $\bigcup_{i=1}^{n} A_i \times \mathit{\Pi}_i$ is measure-disjoint iff $\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i$ (respectively $\bigcup_{i=1}^{n} A_i \times (-\mathit{\Pi}_i)$) is measure-disjoint. In general, we have the following estimate:

$$\sum_{i=1}^{n} \frac{\mu_G(A_i) \cdot \mu_{H_{\text{ext}}}(\mathit{\Pi}_i)}{\mu_{G \times H_{\text{ext}}}(\mathrm{FD}(\widetilde{\mathcal{L}}_{\text{ext}})} = \sum_{i=1}^{n} \frac{\mu_G \otimes \mu_{H_{\text{ext}}}(A_i \times \mathit{\Pi}_i)}{\mu_{G \times H_{\text{ext}}}(\mathrm{FD}(\widetilde{\mathcal{L}}_{\text{ext}})} \geq \frac{\mu_G \otimes \mu_{H_{\text{ext}}}(\bigcup_{i=1}^{n} A_i \times \mathit{\Pi}_i)}{\mu_{G \times H_{\text{ext}}}(\mathrm{FD}(\widetilde{\mathcal{L}}_{\text{ext}})}.$$

If the union is measure-disjoint, then we have equality, and if it is a fundamental domain, then this equation equals 1. The proof of Proposition 5.144 shows that in this case $\underline{\mathit{\Lambda}} + \underline{A}$ is then a tiling.

In the converse direction, we first note that this last inequality is also an equality, since the union is measure-disjoint: otherwise, if the sets $A_i \times \mathit{\Pi}_i$ would not be pairwise measure-disjoint, then, since their boundaries $(\partial A_i \times \mathit{\Pi}_i) \cup (A_i \times \partial \mathit{\Pi}_i)$ have Haar measure 0, there would be at least one pair $A_i \times \mathit{\Pi}_i$ and $A_j \times \mathit{\Pi}_j$ with a common interior point. Consequently, A_i and A_j as well as $\mathit{\Pi}_i$ and $\mathit{\Pi}_j$ would have a common interior point x respectively y. If $B_{<r}(x) \subset A_i \cap A_j$ and $B'_{<r'}(y) \subset \mathit{\Pi}_i \cap \mathit{\Pi}_j$ are corresponding open neighbourhoods of these interior points, let $z \in \Lambda(B'_{<r'}(y))$ and $z' \in \Lambda(B_{<r}(x))$. Then, the covering degree of $\underline{\mathit{\Lambda}} + \underline{A}$ inside $B_{<r}(x+z)$ (of $\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$ inside $B'_{<r'}(y+z')$) would be at least 2, contradicting the assumption that they are tilings (since $\underline{\mathit{\Lambda}}$ is representable).

We have to establish that $\widetilde{\mathcal{L}}_{\text{ext}} + (\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i)$ is a tiling of $G \times H_{\text{ext}}$. As product of compact sets, each $(-A_i) \times \mathit{\Pi}_i$ is compact, and as finite union $\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i$ is compact. Since $\mu_G \otimes \mu_{H_{\text{ext}}} (\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i) = \mu_{G \times H_{\text{ext}}}(\mathrm{FD}(\widetilde{\mathcal{L}}_{\text{ext}})$, we either have $\widetilde{\mathcal{L}}_{\text{ext}} + (\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i) = G \times H_{\text{ext}}$ or there is a nonempty open set U such that

$$\left(\widetilde{\mathcal{L}}_{\text{ext}} + \left(\bigcup_{i=1}^{n} (-A_i) \times \mathit{\Pi}_i \right) \right) \setminus \left(\widetilde{\mathcal{L}}_{\text{ext}} + U \right) = \varnothing,$$

[29] This is a generalisation of the klotz construction (or the construction with "oblique tilings") as used in [212, 213, 215, 277].

i.e., there are "holes". We first establish that there is no third possibility: Since the set $(\bigcup_{i=1}^{n}(-A_i) \times \mathit{\Pi}_i)$ is compact and $\widetilde{\mathcal{L}}_{\text{ext}}$ is closed, $\widetilde{\mathcal{L}}_{\text{ext}} + (\bigcup_{i=1}^{n}(-A_i) \times \mathit{\Pi}_i)$ is also closed (see [272, Theorem 3.1.10]). So, any point $x \in (G \times H_{\text{ext}}) \setminus (\widetilde{\mathcal{L}}_{\text{ext}} + (\bigcup_{i=1}^{n}(-A_i) \times \mathit{\Pi}_i))$ has positive distance from $\widetilde{\mathcal{L}}_{\text{ext}} + (\bigcup_{i=1}^{n}(-A_i) \times \mathit{\Pi}_i)$, which establishes the existence of a set U if such an x exists.

We now prove the claim by contraposition: Suppose we have such an open set $U \subset G \times H_{\text{ext}}$. Then, by the denseness of the image $\pi_2(\widetilde{\mathcal{L}}_{\text{ext}})$, we would get an open set of covering degree 0 in $\underline{\mathit{\Lambda}} + \underline{A}$ by the first equation of Equation (5.20). Then, $\underline{\mathit{\Lambda}} + \underline{A}$ is not a tiling. □

Putting it all together, we have established the following equivalent statements.

Theorem 5.147. *Let $\underline{\mathit{\Lambda}}$ be a repetitive aperiodic substitution multi-component Delone set that satisfies* **(PLT+)**. *Denote the attractor of the IFS $\breve{\boldsymbol{\Theta}}^{\star}$ by $\underline{\breve{\mathit{\Pi}}}$ and define the tiles $\mathit{\Pi}_i = \breve{\mathit{\Pi}}_i \times \{a_i\}$. Suppose $\underline{\mathit{\Upsilon}}$ is a repetitive substitution multi-component Delone set (with respect to the "EMFS" $\boldsymbol{\Theta}^{\star\#}$) such that it is also given as a multi-component inter model set $\Lambda(\operatorname{int}\underline{A}) \subset \underline{\mathit{\Upsilon}} \subset \Lambda(\underline{A})$. Then, the following properties are equivalent:*

(i) *$\underline{\mathit{\Lambda}}$ is a multi-component inter model set with $\Lambda(\operatorname{int}\underline{\mathit{\Pi}}) \subset \underline{\mathit{\Lambda}} \subset \Lambda(\underline{\mathit{\Pi}})$.*

(ii) *$\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$ is a tiling of H_{ext}.*

(iii) *$\bigcup_{i=1}^{n}(-A_i) \times \mathit{\Pi}_i$ is a measure-disjoint union and a fundamental domain of $\widetilde{\mathcal{L}}_{\text{ext}}$.*

(iv) *The only strong overlaps $\mathit{\Xi}(i,k,x)$ with $x \in \Lambda(\operatorname{int}\mathit{\Pi}_k - \operatorname{int}\mathit{\Pi}_i)$ are coincidences.* □

Remark 5.148. The condition that $\underline{\mathit{\Upsilon}}+\underline{\mathit{\Pi}}$ is a tiling of H_{ext}, and therefore has covering degree 1 $\mu_{H_{\text{ext}}}$-a.e., is sometimes also stated by saying that it has an "exclusive inner point", *i.e.*, a point which lies in the interior of one tile but does not belong to any other tile. Since an algebraic coincidence establishes that $\underline{\mathit{\Lambda}}$ is a multi-component inter model set, we can say that algebraic coincidence and the existence of an exclusive inner point are equivalent. Unfortunately, there seems to be no direct way to establish this connection (by proving that algebraic coincidence is equivalent to any of the above statements directly): Although one wants to show that $\Lambda(\operatorname{int}\underline{\mathit{\Pi}}) \subset \underline{\mathit{\Lambda}} \subset \Lambda(\underline{\mathit{\Pi}})$ and the sets $\mathit{\Pi}_i$ are the prototiles in $\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$, the patch $(\mathit{\Pi}_i \mid 1 \le i \le n)$ may in general not appear in $\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$; *e.g.*, consider the CPS $(H_{\text{ext}}, \mathbb{R}, \widetilde{\mathcal{L}}_{\text{ext}})$ in Section 6.7 (where the internal space is $\mathbb{R}$): There, the prototiles in the "internal space" $\mathbb{R}$ are given by $[0, \ell_i]$ with some $\ell_i > 0$ for all $1 \le i \le n$, but the family of tiles $([0, \ell_i] \mid 1 \le i \le n)$ – which, for some $\varepsilon > 0$, covers the interval $[0, \varepsilon]$ n times – does not occur in the corresponding tiling of the internal space $\mathbb{R}$.

We add two remarks to this section.

Remark 5.149. One would like to call $\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$ the "dual tiling" of $\underline{\mathit{\Lambda}}' + \underline{A}$, for example, because of Equation (5.20) on p. 191. However, we think that the name "dual tiling" is only justified if the CPS construction starting with $\underline{\mathit{\Upsilon}} + \underline{\mathit{\Pi}}$ yields the internal space G and the same lattice $\widetilde{\mathcal{L}}$. However, it is not clear that this will happen in the general case.

However, note that our findings in Chapter 6 suggest that in the case where $\mathcal{L}$ is associated to an algebraic number field, one really has this duality.

Remark 5.150. Classically, most results about substitution multi-component Delone sets (respectively substitution tilings) are derived in $\mathbb{R}^d$. There, the results seem to be more general in the sense that one does not assume **(LT)** for $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}^{\star}$. However, we observe the following statement which can be found in [203] (also see [236, Lemma 3.2]): We note that in $\mathbb{R}^d$ the map f_0 is simply a multiplication by some matrix $\boldsymbol{Q}$, *i.e.*, $f_0(x) = \boldsymbol{Q}x$. If $\mathcal{L}$ is a finitely generated free Abelian group in $\mathbb{R}^d$ such that $\mathcal{L}$ spans $\mathbb{R}^d$ and $\boldsymbol{Q}\mathcal{L} \subset \mathcal{L}$, then all eigenvalues of $\boldsymbol{Q}$ are algebraic integers. This suggests that substitutions in $\mathbb{R}^d$ are essentially defined on algebraic number fields.

But on algebraic number fields, the condition **(LT)**(iii) for $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}^\star$ is easily established by the Artin's product formula for normalised absolute values (Lemma 3b.11), see Section 6.4 for details. So, in fact, our findings might be more general.

5.7.5. Coincidences and Model Sets

We now want to establish the converse to Proposition 5.131, also compare [230, Theorem 6.5] which is obtained by following a different path to such a statement.

Before we attack the general case, we first establish a statement which is a side-effect of our considerations on the finite seed. We assume that we have very special algebraic coincidences, namely that every vertex of the finite seed $\mathcal{P}$ admits an algebraic coincidence (wherefore their image under the star-map consists of interior points of the sets $\varPi_i$ only, see Corollary 5.141). Then, we can easily establish the following result. We only note that this condition may also be used to establish that $\underline{\varLambda}$ is even a generic multi-component model set, see [42, Section 4] for details.

Lemma 5.151. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT+)**. *Denote its finite seed by $\mathcal{P}$. Suppose that $\hat{f}_0 : H'_{\text{ext}} \to H'_{\text{ext}}$ is a contraction such that $\mu_{H'_{\text{ext}}}(W) = \alpha \cdot \mu_{H'_{\text{ext}}}(\hat{f}_0(W))$ for every compact set $W \subset H'_{\text{ext}}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$ respectively $\boldsymbol{S\breve{\Theta}}^\star$. Let $\underline{\breve{\varPi}}$ be the attractor of $\breve{\boldsymbol{\Theta}}^\star$ in $(\mathcal{K}H'_{\text{ext}})^n$. If for every vertex $\omega_i(x) \in \mathcal{P}$ we have an algebraic coincidence, i.e., there is an $m = m(x,i) \in \mathbb{N}$ such that $x + f_0^m(\varDelta') \subset \varLambda_i$ for every $\omega_i(x) \in \mathcal{P}$, then one has $\underline{\varLambda} = \Lambda(\operatorname{int}\underline{\varPi})$.*

Proof. By Corollary 5.141, all points corresponding to a vertex in the finite seed are interior points of some $\varPi_i$. Moreover, by iteration this actually applies to all points of $\underline{\varLambda}$, which proves the claim. □

We now assume the setting of Lemma 5.140 and note the following immediate consequences:

- Since the unions in the IFS $\breve{\boldsymbol{\Theta}}^\star$ (as well as in the IFS $\boldsymbol{\Theta}^\#$) are measure-disjoint and by the definition of the corresponding tile substitution, one has that

$$\left(\boldsymbol{\Theta}^{\star\#}\right)^N (\omega_i(0)) + \underline{\varPi}, \quad \text{respectively} \quad \boldsymbol{\Theta}^N (\omega_i(0)) + \underline{A}$$

  is a translate of a patch of $\underline{\varUpsilon} + \underline{\varPi}$ respectively $\underline{\varLambda} + \underline{A}$.

- If

$$\omega_j(v), \omega_k(z) \in \left(\boldsymbol{\Theta}^{\star\#}\right)^N (\omega_i(0)), \quad \text{respectively} \quad \omega_j(x), \omega_k(y) \in \boldsymbol{\Theta}^N (\omega_i(0)),$$

  then $\varPi_j + v$ and $\varPi_k + z$, respectively $A_j + x$ and $A_k + y$, are measure-disjoint (here, $v, z \in H_{\text{ext}}$ and $x, y \in G$).

- We also have

$$\operatorname{supp}\left[\left(\breve{\boldsymbol{\Theta}}^{\star\#}\right)^{N}(\omega_i(0)) + \underline{\breve{\varPi}}\right] = \hat{f}_0^{-N}(\breve{\varPi}_i), \quad \text{respectively}$$
$$\operatorname{supp}\left[\boldsymbol{\Theta}^{N}(\omega_i(0)) + \underline{A}\right] = \hat{f}_0^{-N}(A_i).$$

  Consequently, the radius of the largest ball contained in such a patch tends to infinity for $N \to \infty$.

- By the construction of the dual MFS, we also have that

$$\omega_i(z) \in \left(\boldsymbol{\Theta}^{\star\#}\right)^{N}\left(\omega_j(0)\right) \qquad \Longleftrightarrow \qquad \omega_j\left(\hat{f}_0^{N}(z)\right) \in \left(\boldsymbol{\Theta}^{\star}\right)^{N}\left(\omega_i(0)\right),$$

  respectively

$$\omega_i(x) \in \left(\boldsymbol{\Theta}^{\#}\right)^{N}\left(\omega_j(0)\right) \qquad \Longleftrightarrow \qquad \omega_j\left(f_0^{N}(x)\right) \in \boldsymbol{\Theta}^{N}\left(\omega_i(0)\right). \tag{5.21}$$

We now use these findings to prove the following statement. This statement connects the tiling $\underline{\varUpsilon} + \underline{\varPi}$ with the notion of an overlap coincidence. We note that tiles belonging to different cosets $H'_{\text{ext}} + t$ of H_{ext} are automatically disjoint.

Lemma 5.152. *Let $\underline{\varLambda}$ be an aperiodic repetitive substitution multi-component Delone set that satisfies* **(PLT+)**. *Suppose that $\hat{f}_0 : H'_{\text{ext}} \to H'_{\text{ext}}$ is a contraction such that $\mu_{H'_{\text{ext}}}(W) = \alpha \cdot \mu_{H'_{\text{ext}}}(\hat{f}_0(W))$ for every compact set $W \subset H'_{\text{ext}}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$ respectively $\boldsymbol{S\breve{\Theta}}^{\star}$. Let $\underline{\breve{\varPi}}$ be the attractor of $\breve{\boldsymbol{\Theta}}^{\star}$ in $(\mathcal{K}H'_{\text{ext}})^n$, and define $\varPi_i = \breve{\varPi}_i \times \{a_i\}$. Suppose $\underline{\varUpsilon}$ is a repetitive substitution multi-component Delone set such that it is also given as a multi-component inter model set $\Lambda(\operatorname{int}\underline{A}) \subset \underline{\varUpsilon} \subset \Lambda(\underline{A})$. Then, equivalent are*

(i) *$\underline{\varUpsilon} + \underline{\varPi}$ is a tiling.*

(ii) *If $x^\star \in \varUpsilon_i$ and $y^\star \in \varUpsilon_j$ (where $x^\star \neq y^\star$ if $i = j$) and $x^\star - y^\star \in H'_{\text{ext}} \times \{a_j - a_i\}$ (i.e., $\varPi_i + x^\star$ and $\varPi_j + y^\star$ belong to the some coset in $H'_{\text{ext}} \times \mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$), then $\varXi(i, j, -y + x)$ leads to an coincidence.*

Proof. (ii)⇒(i): If $\varXi(i, j, -y + x)$ leads to a coincidence, then there is a $z' \in G$, an $N \in \mathbb{N}$ and a $1 \leq k \leq n$ such that

$$\begin{aligned} A_k + z' &\in \left[\boldsymbol{\Theta}^{N}(\omega_i(-x)) + \underline{A}\right] \sqcap \left[\boldsymbol{\Theta}^{N}\left(\omega_j(-y)\right) + \underline{A}\right] \\ &= \left[\boldsymbol{\Theta}^{N}(\omega_i(0)) + \underline{A}\right] - f_0^{N}(x) \sqcap \left[\boldsymbol{\Theta}^{N}\left(\omega_j(0)\right) + \underline{A}\right] - f_0^{N}(y). \end{aligned}$$

Setting $z' = f_0^N(z)$, one obtains

$$\omega_k\left(f_0^{N}(x+z)\right) \in \boldsymbol{\Theta}^{N}(\omega_i(0)) \text{ and } \omega_k\left(f_0^{N}(y+z)\right) \in \boldsymbol{\Theta}^{N}\left(\omega_j(0)\right).$$

Using Equation (5.21), we obtain the statement

$$\omega_i(x+z), \omega_i(y+z) \in \left(\boldsymbol{\Theta}^{\#}\right)^{N}(\omega_k(0)).$$

We apply the (bijective) star-map, which yields

$$\omega_i(x^\star + z^\star),\, \omega_i(y^\star + z^\star) \in \left(\boldsymbol{\Theta}^{\star\#}\right)^N (\omega_k(0)),$$

and therefore the measure-disjointness of $\mathit{\Pi}_i + x^\star + z^\star$ and $\mathit{\Pi}_j + y^\star + z^\star$. Translation by $-z^\star$ establishes the claim.

(i)⇒(ii): By Corollary 5.89, for every $x^\star \in \mathit{\Upsilon}_i$ and $y^\star \in \mathit{\Upsilon}_j$, the patch consisting of $\mathit{\Pi}_i + x^\star$ and $\mathit{\Pi}_j + y^\star$ is legal, wherefore there is a $z^\star \in H_{\text{ext}}$, an $N \in \mathbb{N}$ and a $1 \le k \le n$ such that

$$\omega_i(x^\star + z^\star),\, \omega_i(y^\star + z^\star) \in \left(\boldsymbol{\Theta}^{\star\#}\right)^N (\omega_k(0)).$$

Backtracing the above steps (applying the inverse of the star-map and using Equation (5.21)), one derives that $\mathit{\Xi}(i, j, -y + x)$ leads to a coincidence. This proves the claim. □

Together with Proposition 5.144, we therefore derive the statement that $\underline{\mathit{\Lambda}}$ is a multi-component IMS $\Lambda(\operatorname{int} \underline{\mathit{\Pi}}) \subset \underline{\mathit{\Lambda}} \subset \Lambda(\underline{\mathit{\Pi}})$ iff $\mathit{\Xi}(i, j, z)$ leads to a coincidences for all $z \in \left(A_i - A_j\right) \cap (\mathcal{L}_{\text{ext}} + a_j - a_i)$ (since only strong overlaps are in question for coincidences, one can restrict to $z \in \left(\operatorname{int} A_i - \operatorname{int} A_j\right) \cap (\mathcal{L}_{\text{ext}} + a_j - a_i)$). Moreover, we may state this last condition also as: $\mathit{\Xi}(i, j, z)$ leads to a coincidences for all $z \in \left(\operatorname{int} A_i - \operatorname{int} A_j\right) \cap (\mathit{\Lambda}_j + \mathcal{L}_{\text{ext}} - \mathit{\Lambda}_i)$. This is quite close to the definition of an overlap coincidence in Definition 5.126, which can be stated as: $\underline{\mathit{\Lambda}} + \underline{\mathit{\Lambda}}$ admits an overlap coincidence iff $\mathit{\Xi}(i, j, z)$ leads to a coincidences for all $z \in \left(\operatorname{int} A_i - \operatorname{int} A_j\right) \cap (\mathit{\Lambda}_j + \mathit{\Delta}' - \mathit{\Lambda}_i)$ (since for strong overlaps, only the values $z \in \left(\operatorname{int} A_i - \operatorname{int} A_j\right)$ are in question). In fact, the two conditions are equivalent.

Corollary 5.153. *Let $\mathcal{L}'$ be finitely generated by $\mathit{\Delta}'$. Then, $\underline{\mathit{\Lambda}} + \underline{\mathit{\Lambda}}$ admits an overlap coincidence iff $\mathit{\Xi}(i, j, z)$ leads to a coincidence for all $z \in \left(\operatorname{int} A_i - \operatorname{int} A_j\right) \cap (\mathit{\Lambda}_j + \mathcal{L}'_{\text{ext}} - \mathit{\Lambda}_i)$.*

Proof. Since $\mathit{\Delta}' \subset \mathcal{L}' = \langle \mathit{\Delta}' \rangle_{\mathbb{Z}} \subset \mathcal{L}'_{\text{ext}}$, the case that we do not have an overlap coincidence is obvious. For the other direction, we assume that $\underline{\mathit{\Lambda}} + \underline{\mathit{\Lambda}}$ admits an overlap coincidence and prove that for every $x \in \mathcal{L}'_{\text{ext}}$ there is an $N \in \mathbb{N}$ such that $f^N + 0(x) \in \mathit{\Delta}'$.

To this end, we observe that for every $x \in \mathcal{L}'_{\text{ext}} = \bigcup_{k \ge 0} f_0^{-k}(\mathcal{L}')$ there is an $m \in \mathbb{N}$ (depending on x) such that $f_0^m(x) \in \mathcal{L}'$. Moreover, we note that $\underline{\mathit{\Lambda}} + \underline{\mathit{\Lambda}}$ admits an algebraic coincidence by Lemma 5.129. But an algebraic coincidence (see Definition 5.123) can be used to show that for any $z' \in \mathit{\Delta}' \pm \mathit{\Delta}' \pm \ldots \pm \mathit{\Delta}'$ (with a finite number of sets $\mathit{\Delta}' = -\mathit{\Delta}'$), there is a power $M \in \mathbb{N}$ such that $f_0^M(z') \in \mathit{\Lambda}_{\hat{\imath}} - \mathit{\Lambda}_{\hat{\imath}} \subset \mathit{\Delta}'$ for some $1 \le \hat{\imath} \le n$ (since there is an m' and a $t \in \mathit{\Delta}_{\hat{\imath}}$ such that $f_0^{m'}(\mathit{\Delta}') + f_0^{m'}(\mathit{\Delta}') = t + f_0^{m'}(\mathit{\Delta}') - (t + f_0^{m'}(\mathit{\Delta}')) \subset \mathit{\Delta}_{\hat{\imath}} - \mathit{\Delta}_{\hat{\imath}}$). Consequently, since we assume that $\mathcal{L}'$ is finitely generated by $\mathit{\Delta}'$, this establishes the claim. □

We note that we already used the algebraic coincidence (or, equivalently, the overlap coincidence) to establish the extended CPS. Moreover, putting everything together, we have now established the following statement.

Theorem 5.154. *Let $\underline{\mathit{\Lambda}}$ be a repetitive aperiodic substitution multi-component Delone set that satisfies* **(PLT+)**. *Suppose that $\hat{f}_0 : H'_{\text{ext}} \to H'_{\text{ext}}$ is a contraction such that $\mu_{H'_{\text{ext}}}(W) =$*

$\alpha \cdot \mu_{H'_{\mathrm{ext}}}(\hat{f}_0(W))$ for every compact set $W \subset H'_{\mathrm{ext}}$, where α is the PF-eigenvalue of $\boldsymbol{S\Theta}$ respectively $\boldsymbol{S\breve{\Theta}^{\star}}$. Let $\underline{\breve{\Pi}}$ be the attractor of $\boldsymbol{\breve{\Theta}^{\star}}$ in $(\mathcal{K}H'_{\mathrm{ext}})^n$, and define $\Pi_i = \breve{\Pi}_i \times \{a_i\}$. Suppose $\underline{\Upsilon}$ is a repetitive substitution multi-component Delone set such that it is also given as a multi-component inter model set $\Lambda(\mathrm{int}\,\underline{A}) \subset \underline{\Upsilon} \subset \Lambda(\underline{A})$. Assume that $\mathcal{L}'$ is finitely generated by Δ'. Then, equivalent are:

(i) *$\underline{\Lambda}$ is a multi-component inter model set with $\Lambda(\mathrm{int}\,\underline{\Pi}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Pi})$.*

(ii) *Any of the properties in Theorem 5.147, i.e.,*

- *$\underline{\Upsilon} + \underline{\Pi}$ is a tiling of H_{ext}.*
- *$\bigcup_{i=1}^{n}(-A_i) \times \Pi_i$ is a measure-disjoint union and a fundamental domain of $\widetilde{\mathcal{L}}_{\mathrm{ext}}$.*
- *The only strong overlaps $\Xi(i,k,x)$ with $x \in \Lambda(\mathrm{int}\,\Pi_k - \mathrm{int}\,\Pi_i)$ are coincidences.*

(iii) *Any of the properties in Lemma 5.129, i.e.,*

- *$\lim_{m\to\infty} \mathrm{dens}^{\mathit{ovlap}}_{\underline{\Lambda}+\underline{A}}(f_0^m(x)) = 1$ for every $x \in \Delta'$.*
- *$1 - \mathrm{dens}^{\mathit{ovlap}}_{\underline{\Lambda}+\underline{A}}(f_0^m(x)) \leq C \cdot r^m$ for an $m \in \mathbb{N}$, every $x \in \Delta'$ and some constants (independent of x) $C > 0$ and $r \in]0\,,1[$.*
- *$\underline{\Lambda} + \underline{A}$ admits an overlap coincidence.*
- *$\underline{\Lambda}$ admits an algebraic coincidence.* □

Remark 5.155. The last statement parallels [230, Theorem 6.7] in the case $\mathbb{R}^d$ (where a lot of the technical conditions are hidden since one works in an algebraic number field, see Remark 5.150). However, the statement there is obtained *via* results about the corresponding dynamical systems (which we will only review in Section 7.3), see [230, Theorems 6.4 & 6.5], and therefore avoids the construction of $\mathcal{L}_{\mathrm{ext}}$, H_{ext} and the corresponding EMFS and IFS on them. Our derivation of the above result is inspired by the statements in [192, Section 4.2].

5.8. Examples of Internal Spaces

We begin this section with examples of a simple periodic sequence, which explains the difference between H, H_{sub} and H_{ext}

Example 5.156. The lattice $\Lambda_a = \mathbb{Z}$ in $\mathbb{R}$ is a (FLC one-component) Delone set. We observe that $\Delta_a = \mathbb{Z} - \mathbb{Z} = \mathbb{Z}$, $\mathcal{L} = \mathbb{Z} = \mathcal{L}'$ and that, by periodicity with every $t \in \mathbb{Z}$, we have $\varrho_{\Lambda_a}(x) = 0$ for $x \in \mathbb{Z}$ and 1 otherwise. Consequently, the closure of 0 in the AC topology is again $\mathbb{Z}$ and the Hausdorff completion of Δ_a with respect to the AC topology is $H = \{0\}$. So, the lattice $\mathbb{Z}$ is a model set relative to the CPS $(\mathbb{R}\,, \{0\}\,, \mathbb{Z} \times \{0\})$.

One may also obtain Λ_a as substitution Delone set: Take the EMFS $\Theta = \{f_0\,, t_{(1)} \circ f_0\}$ where $f_0(x) = 2 \cdot x$ (we may also use the shorthand notation $a \mapsto aa$). The equation $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z}+1) = \Theta(\mathbb{Z})$ establishes that it is a substitution (one-component) model set. Now, we equip $\mathbb{Z}$ with the topology defined by the base $\{2^m\mathbb{Z} \mid m \in \mathbb{Z}_{\geq 0}\}$ at 0. In the parlance of algebraic number theory, this base at 0 consists of the (nonnegative) powers of the prime ideal $(2) = 2\mathbb{Z}$ of the Dedekind ring $\mathbb{Z}$. Comparing with the remarks in Definition 3.63, we also obtain the completion $H_{\mathrm{sub}} = \mathbb{Z}_2$, the 2-adic integers. Therefore, we obtain the CPS $(\mathbb{R}\,, \mathbb{Z}_2\,, \widetilde{\mathcal{L}} = \{(t\,,t) \mid t \in \mathbb{Z}\})$. Moreover, $\mathbb{Z}_2$ is a compact space, and $\Lambda_a = \mathbb{Z} = \Lambda(\mathbb{Z}_2)$.

We note that $\mathcal{L}$ in the AC topology is just $\mathbb{Z}$ equipped with the 2-adic topology. Moreover, using the notation of Section 5.7.3, we have $(\mathcal{L}_{\text{ext}})^{(m)} = \frac{1}{2^m}\mathbb{Z}$, whose Hausdorff completion is $\frac{1}{2^m}\mathbb{Z}_2$. Consequently, one obtains $H_{\text{ext}} = \mathbb{Z}_2[\frac{1}{2}] = \mathbb{Q}_2$, the 2-adic numbers.

The next example is again a periodic sequence, which now illuminates the construction of a CPS in the multi-component case.

Example 5.157. Let $\varLambda_a = 2\mathbb{Z}$, $\varLambda_b = 6\mathbb{Z}+1$, $\varLambda_c = 6\mathbb{Z}+3$ and $\varLambda_d = 6\mathbb{Z}+5$. This is the periodic (with period 6) sequence $\ldots adabacad\dot{a}bacadaba \ldots$ (where $\dot{}$ denotes the zeroth position). As indicated in Example 5.156, each of these lattices is a model set with trivial internal space: The CPS for $\varLambda_a$ is $(\mathbb{R}\,, \{0\}\,, (2\mathbb{Z}) \times \{0\})$ (and we obviously have $\varLambda_a = \Lambda(0)$), while the CPS for $\varLambda_b, \varLambda_c$ and $\varLambda_d$ is $(\mathbb{R}\,, \{0\}\,, (6\mathbb{Z}) \times \{0\})$ (and we have $\varLambda_b = \Lambda(0) + 1$ *etc.*). We now want to establish that $\underline{\varLambda} = (\varLambda_a, \varLambda_b, \varLambda_c, \varLambda_d)$ is a multi-component model set.

For this, we note that $\mathcal{L}' = \bigcup_i \langle \varLambda_i - \varLambda_i \rangle_{\mathbb{Z}} = 2\mathbb{Z} \cup 6\mathbb{Z} \cup 6\mathbb{Z} \cup 6\mathbb{Z} = 2\mathbb{Z}$. Obviously, we have $\mathcal{L} = \mathbb{Z}$. Moreover, the maximum variogram is given by $\varrho_{\underline{\varLambda}}(t) = \varrho_{\varLambda_j}(t) = 1$ if $t \in 6\mathbb{Z}$ and 0 otherwise, where $j \in \{b, c, d\}$. Consequently, the height group is $\mathcal{L}/\mathcal{L}' = \mathbb{Z}/2\mathbb{Z} = \mathsf{C}_2$ and we have $H' = \mathsf{C}_3$. Therefore, we have established the CPS $(\mathbb{R}\,, \mathsf{C}_3 \times \mathsf{C}_2\,, \widetilde{\mathcal{L}} = \{(t, t \bmod 3, t \bmod 2) \mid t \in \mathbb{Z}\})$. One now easily obtains the windows $\varOmega_i \subset H = \mathsf{C}_3 \times \mathsf{C}_2$ for each $\varLambda_i$: $\varOmega_a = \mathsf{C}_3 \times \{0\}$, $\varOmega_b = \{0\} \times \{1\}$, $\varOmega_c = \{1\} \times \{1\}$ and $\varOmega_d = \{2\} \times \{1\}$.

We note that $\mathcal{L}'/\mathcal{L}_a = \{0\}$, and, with the notation of p. 147, $\ker(\pi_2^{(a)} \circ \hat{\text{id}}_a) = \mathsf{C}_3 = H'$. For $j \in \{b, c, d\}$, we have $\mathcal{L}'/\mathcal{L}_j \cong \mathsf{C}_3$ and $\ker(\pi_2^{(j)} \circ \hat{\text{id}}_j) = \{0\}$. Note that in each case $\ker(\pi_2^{(i)} \circ \hat{\text{id}}_i) = \{0\}$ is a clopen subgroup of H'. We note that the "finer" the maximum variogram $\varrho_{\underline{\varLambda}}$ is compared to $\varrho_{\varLambda_i}$, the "bigger" $\ker(\pi_2^{(i)} \circ \hat{\text{id}}_i)$ is. So we may interpret $\ker(\pi_2^{(i)} \circ \hat{\text{id}}_i)$ as the contribution to H' that counterbalances the change from the individual variogram to the maximum variogram.

The so-called "paperfolding sequence" (see [105] and [11, Sections I.3 & II.]) is an example where some of the components of a substitution multi-component Delone set are periodic and others are aperiodic. Again, we illuminate the construction of a CPS with this example

Example 5.158. The paperfolding sequence is obtained by the substitution rule $a \mapsto ab$, $b \mapsto cb$, $c \mapsto ad$ and $d \mapsto cd$ on $\mathbb{Z}$, *i.e.*, with $f(x) = 2x$, $g(x) = 2x+1 = t_{(1)} \circ f(x)$, $f'(x) = f^{-1}(x) = \frac{1}{2}x$ and $g'(x) = f' \circ t_{(1)}(x) = \frac{1}{2}x + \frac{1}{2}$ we have the following EMFS and adjoint IFS:

$$\boldsymbol{\varTheta} = \begin{pmatrix} \{f\} & \varnothing & \{f\} & \varnothing \\ \{g\} & \{g\} & \varnothing & \varnothing \\ \varnothing & \{f\} & \varnothing & \{f\} \\ \varnothing & \varnothing & \{g\} & \{g\} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varTheta}^{\#} = \begin{pmatrix} \{f'\} & \{g'\} & \varnothing & \varnothing \\ \varnothing & \{g'\} & \{f'\} & \varnothing \\ \{f'\} & \varnothing & \varnothing & \{g'\} \\ \varnothing & \varnothing & \{f'\} & \{g'\} \end{pmatrix}$$

(So, the notation $i \mapsto jk$ indicates that $f \in \varTheta_{ji}$ and $g \in \varTheta_{ki}$.) One easily establishes that all components of the attractor of the adjoint IFS are given by $[0\,, 1]$, so the paperfolding sequence is indeed representable. In fact, the paperfolding sequence reads $\ldots abcdadc\bullet\dot{a}bcbadcba \ldots$ on $\mathbb{Z}$, where $\bullet$ is either b or d (using $\boldsymbol{\varTheta}^2$ yields a unique fixed point). It is easy to see that $\varLambda_a = 4\mathbb{Z}$ and $\varLambda_c = 4\mathbb{Z} + 2$ are periodic, and one can then establish that

$$\varLambda_b = \bigcup_{m \geq 1} (2^{m+2}\mathbb{Z} + 2^m - 1) \quad \text{and} \quad \varLambda_d = \bigcup_{m \geq 1} (2^{m+2}\mathbb{Z} + 3 \cdot 2^m - 1), \tag{5.22}$$

and the singleton $\{-1\}$ added to either of them, see [38, Section 2.8.5].

One therefore obtains for the multi-component Delone set $\underline{\varLambda} = (\varLambda_a, \varLambda_b, \varLambda_c, \varLambda_d)$: $\mathcal{L}_a = 4\mathbb{Z} = \mathcal{L}_c$, $\mathcal{L}_b = 2\mathbb{Z} = \mathcal{L}_d$ and therefore $\mathcal{L}' = 2\mathbb{Z}$ and the height group $\mathcal{L}/\mathcal{L}' = \mathsf{C}_2$. Moreover, $\varLambda_b$ and $\varLambda_d$ are aperiodic, so we can make use of the topological isomorphism $H \cong H_{\text{sub}}$. Alternatively, we can calculate the $\varrho_{\varLambda_b}$ respectively $\varrho_{\varLambda_d}$ (which equal the maximum variogram $\varrho_{\underline{\varLambda}}$) from the above explicit from in Equation (5.22) as

$$\varrho_{\varLambda_b}(t) = \varrho_{\varLambda_d}(t) = \varrho_{\underline{\varLambda}}(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t \in \mathbb{Z} \setminus 2\,\mathbb{Z}, \\ \frac{1}{2} & \text{if } t \in 2\,\mathbb{Z} \setminus 4\,\mathbb{Z}, \\ \frac{3}{2^m} & \text{if } t \in 2^m\,\mathbb{Z} \setminus 2^{m+1}\,\mathbb{Z} \text{ where } m > 1. \end{cases}$$

This is basically the 2-adic metric on $2\mathbb{Z}$ (respectively $\mathbb{Z}$), wherefore $H' = 2\mathbb{Z}_2$ (note that $2\mathbb{Z}_2$ is a clopen subgroup of $\mathbb{Z}_2$, namely just the unique prime ideal of this valuation ring). We note that $H \cong H' \times \mathcal{L}/\mathcal{L}' = 2\mathbb{Z}_2 \times \mathsf{C}_2 \cong \mathbb{Z}_2$. So, we have established the CPS $(\mathbb{R}\,, \mathbb{Z}_2\,, \widetilde{\mathcal{L}} = \{(t\,,t) \mid t \in \mathbb{Z}\})$ respectively $(\mathbb{R}\,, 2\mathbb{Z}_2 \times \mathsf{C}_2\,, \widetilde{\mathcal{L}} = \{(t\,, t-(t \bmod 2)\,, t \bmod 2) \mid t \in \mathbb{Z}\})$.

In fact, noting that $\mathrm{cl}_{\mathbb{Z}_2}(2^k\mathbb{Z} + \ell) = 2^k\mathbb{Z}_2 + \ell$, one establishes that $\underline{\varLambda}$ is a multi-component model set with the following windows $\varOmega_i$ for the component $\varLambda_i$ (the first form is given as subset of H, the second as subset of $H' \times \mathsf{C}_2$):

$$\begin{array}{lclcl} \varOmega_a & = & 4\mathbb{Z}_2 & = & 4\mathbb{Z}_2 \times \{0\} \\ \varOmega_b & = & \bigcup_{m\geq 1}(2^{m+2}\mathbb{Z}_2 + 2^m - 1) & = & \Big(\bigcup_{m\geq 1}(2^{m+2}\mathbb{Z}_2 + 2^m - 2)\Big) \times \{1\} \\ \varOmega_c & = & 4\mathbb{Z}_2 + 2 & = & 4\mathbb{Z}_2 + 2 \times \{0\} \\ \varOmega_d & = & \bigcup_{m\geq 1}(2^{m+2}\mathbb{Z}_2 + 3\cdot 2^m - 1) & = & \Big(\bigcup_{m\geq 1}(2^{m+2}\mathbb{Z}_2 + 3\cdot 2^m - 2)\Big) \times \{1\} \end{array}$$

and the singleton $\{-1\} = \{.\bar{1}\} = \{\{.0\bar{1}\} \times \{1\}\}$ added to either $\varOmega_b$ or $\varOmega_d$ (we note that $\partial\varOmega_b = \{-1\} = \partial\varOmega_d$). Moreover, similarly as in the previous example, we note that $\ker(\pi_2^{(i)} \circ \hat{\mathrm{id}}_i) \cong 4\mathbb{Z}_2$ for $i \in \{a, c\}$ and $\ker(\pi_2^{(i)} \circ \hat{\mathrm{id}}_i) = \{0\}$ for $i \in \{b, d\}$.

Since each $\varLambda_i$ is also a model set relative to a CPS $(G\,, H_i\,, \widetilde{\mathcal{L}}_i)$ (*e.g.*, $\varLambda_a = 4\mathbb{Z}$ is also a model set relative to the CPS $(\mathbb{R}\,, \{0\}\,, \widetilde{\mathcal{L}}_a = \{(t\,,t) \mid t \in 4\mathbb{Z}\})$, compare with the previous example), we may also say that each point in H_i corresponds to a translate of the set $\ker(\pi_2^{(i)} \circ \hat{\mathrm{id}}_i)$ in H' (*e.g.*, relative to $(\mathbb{R}\,, \{0\}\,, \widetilde{\mathcal{L}}_a)$, one has $\varLambda_a = \Lambda(\{0\})$, while relative to the $(\mathbb{R}\,, \mathbb{Z}_2\,, \widetilde{\mathcal{L}})$ one has $\varLambda_a = \Lambda(4\mathbb{Z}_2)$).

5a. Weak Model Sets

> I don't want to move to a city where the only cultural advantage is being able to make a right turn on a red light.
>
> ANNIE HALL – *Woody Allen & Marshall Brickman*

This chapter introduce a class of point sets that can be described by a weak version of model sets. Still, some properties – especially diffractive properties – also hold for this generalisation.

5a.1. Visible Lattice Points on the Square Lattice

The visible lattice points are a subset of the square lattice $\mathbb{Z}^2$.

Definition 5a.1. The *visible lattice points* V of $\mathbb{Z}^2$ are given by

$$V = \{(x\,, y) \in \mathbb{Z}^2 \mid \gcd(x\,, y) = 1\} = \mathbb{Z}^2 \setminus \left(\bigcup_{p\in\mathbb{P}} p \cdot \mathbb{Z}^2 \right).$$

Obviously, one can extend this definition to any dimension $d \geq 2$ (*i.e.*, one can define the visible lattice points of $\mathbb{Z}^d$) as well as any lattice, see [37, Equation (9)].

We first show a picture of the set V of visible lattice points of $\mathbb{Z}^2$ in Figure 5a.1. The visible lattice points are precisely the lattice points (of $\mathbb{Z}^2$) that are visible from the origin (here, marked as $\circ$), in the sense that the line segment joining them from the origin contains no other lattice point. Figure 5a.1 can also be found on [15, front cover], also compare [16, Section 8] and [28].

We would like to establish that V is a strictly pure point diffractive set, see Definition 5.40. For this we observe:

- Any subset of a lattice L is uniformly locally finite since the lattice itself is (by definition) uniformly locally finite as Delone subgroup.
- Any subset V of a lattice L is an FLC, since $\Delta = V - V \subset L - L = L$, where the difference set is also locally finite.

So, all we are left with is the determination of the sets of ε-almost periods $P_\varepsilon(V)$ and establishing that they are Delone sets. Again, since $V - V \subset \mathbb{Z}^2$ (respectively, in general $V - V \subset L$), all we have to prove is that the sets $P_\varepsilon(V)$ are relatively dense.

We calculate the autocorrelation coefficients $\eta_V(z)$ by showing that V can be obtained as a particular weak model set.

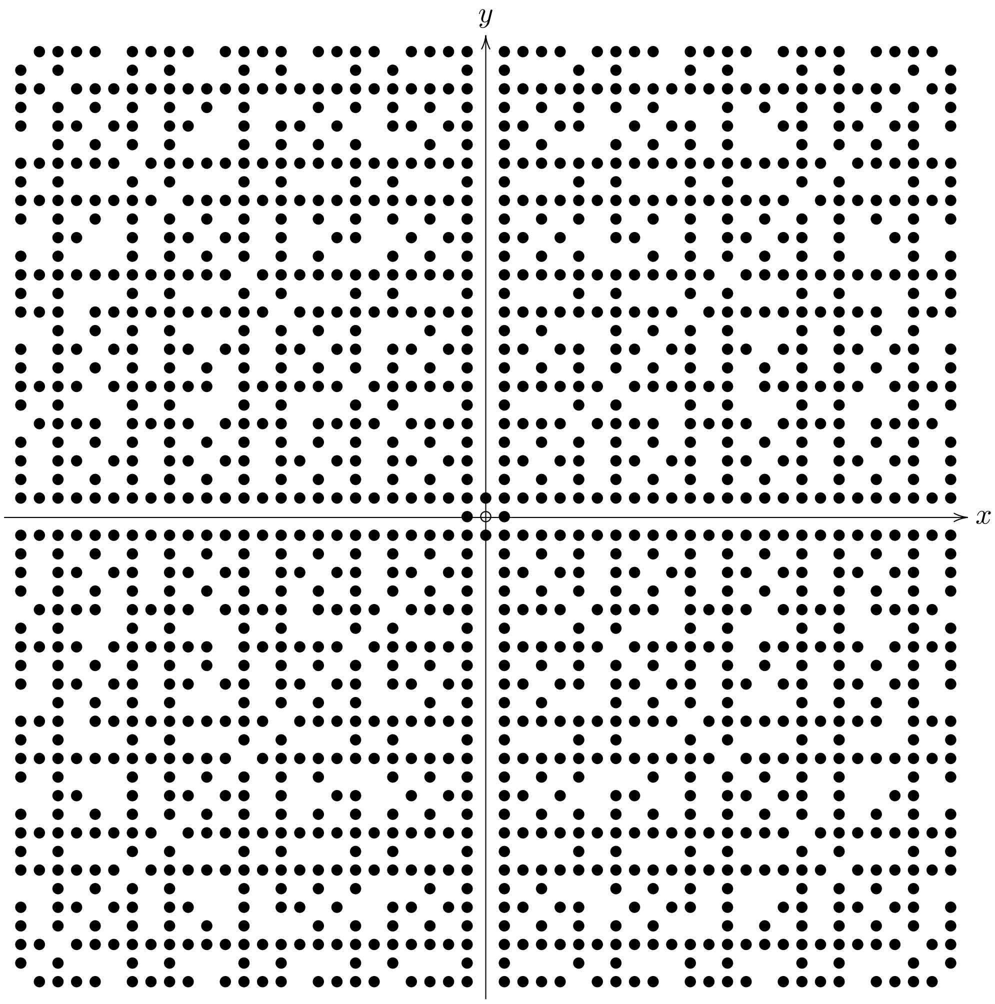


Figure 5a.1.: A patch of the visible lattice points.

Definition 5a.2. A *weak model set*, associated with a CPS $(G\,, H\,, \tilde{L})$, is a nonempty subset $\varLambda = x + \Lambda(\varOmega + y) \subset G$, where $x \in G$, $y \in H$ and $\varOmega \subset H$ is compact, but not necessarily regularly closed. Therefore, it might happen that $\operatorname{int} \varOmega = \varnothing$ and $\varLambda$ is not relatively dense (compare Lemma 5.8).

Moreover, to have the chance to establish that $\varLambda$ is a SPPD set, we might also demand in the definition of a weak model set that the difference set $\varLambda - \varLambda$ is a Delone set.

In fact, the set of visible lattice points is not relatively dense; there are arbitrarily big holes as an application of the *Chinese Remainder Theorem* (*e.g.*, [226, Theorem II.2.1]) shows, see [37, Prop. 4] (also compare [15, Theorem 5.29]).

We now establish a CPS for V by first recalling some facts from the theory of adele rings and writing the lattice $\mathbb{Z}^2$ as a particular model set within the framework of adele rings.

Example 5a.3. We give some examples of adele rings, see Chapter 3b, in particular, Definition 3b.7 and Lemmas 3b.8 & 3b.9:

- The adele ring of $\mathbb{Q}$ is given by the restricted product $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_{p\in\mathbb{P}} \mathbb{Q}_p$. Then, $\mathbb{Q}$ is a lattice in $\mathbb{A}_{\mathbb{Q}}$ with fundamental domain $\mathrm{FD}(\mathbb{Q}) = [0\,,1[\times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$. Moreover, we have $\mathbb{Q} \cap (\mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p) = \mathbb{Z}$.

- Similarly, the d-dimensional adele $\mathbb{A}_{\mathbb{Q}}^d$ of $\mathbb{Q}$ is given by $\mathbb{A}_{\mathbb{Q}}^d = \mathbb{R}^d \times \prod'_{p\in\mathbb{P}} \mathbb{Q}_p^d$. Here, the fundamental domain is given by $\mathrm{FD}(\mathbb{Q}^d) = ([0\,,1[)^d \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^d$, and we can think of $\mathbb{Q}^d$ as product of d lattice $\mathbb{Q}$. Also, we have $\mathbb{Q}^d \cap (\mathbb{R}^d \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^d) = \mathbb{Z}^d$.

- For the adele ring of $\mathbb{Q}(i)$ we observe that the prime 2 is ramified ($(2) = (1-i)^2$), the primes $p \equiv 3 \bmod 4$ are inert, and the primes $p \equiv 1 \bmod 4$ are splitting ($(p) = \mathfrak{p}^{(1)} \cdot \mathfrak{p}^{(2)}$), see Lemma 3.82. Therefore, we use the notation

$$\mathbb{A}_{\mathbb{Q}(i)} = \mathbb{C} \times \mathbb{Q}_{(1-i)} \times \prod\nolimits'_{p\equiv 3 \bmod 4} \mathbb{Q}_{(p)} \times \prod\nolimits'_{p\equiv 1 \bmod 4} \mathbb{Q}_{\mathfrak{p}_p^{(1)}} \times \mathbb{Q}_{\mathfrak{p}_p^{(2)}}.$$

As before, we have

$$\mathbb{Q}(i) \cap (\mathbb{C} \times \widehat{\mathfrak{o}_{(1-i)}} \times \prod\nolimits'_{p\equiv 3 \bmod 4} \widehat{\mathfrak{o}_{(p)}} \times \prod\nolimits'_{p\equiv 1 \bmod 4} \widehat{\mathfrak{o}_{\mathfrak{p}_p^{(1)}}} \times \widehat{\mathfrak{o}_{\mathfrak{p}_p^{(2)}}}) = \mathbb{Z}[i].$$

- Similarly, for the adele ring of $\mathbb{Q}(\sqrt{-3}) \cong \mathbb{Q}(\xi_6)$ we observe that 3 is ramified, the primes $p \equiv 2 \bmod 3$ are inert, and the primes $p \equiv 1 \bmod 3$ are splitting. Now, the situation is as in the previous example $\mathbb{Q}(i)$.

By Proposition 4.40, there is a unique Haar measure μ on $\mathbb{A}_K$ such that its restriction to any finite product coincides with the product measure of the Haar measure of its factors, where the Haar measures are chosen as in Example 4.37. So we may think of μ as the (infinite) product measure $\bigotimes_{i\in J} \mu_i$ of the Haar measures of its components (compare Remark 4.11).

Of course, we identify $x \in K$ with its diagonal embedding $(x, x, x, \ldots) \in \mathbb{A}_K$. With this convention, we note that the condition

$$\mathbb{Z}^2 = \mathbb{Q}^2 \cap (\mathbb{R}^2 \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^2) = \{z \in \mathbb{Q}^2 \mid z \in \prod_{p\in\mathbb{P}} \mathbb{Z}_p^2\}$$

almost establishes that $\mathbb{Z}^2$ is a model set; all we have to check is whether the image of the lattice $\mathbb{Q}^2$ in $\prod'_{p\in\mathbb{P}} \mathbb{Q}_p^2$ is dense. But this is an easy consequence of the strong approximation theorem (see Proposition 3b.12) and the topology of an adele space which is a metric space with metric given in Lemma 3b.15: Given any open ball of radius $\varepsilon > 0$, let the finite set S' of places in Proposition 3b.12 be the set of places ν with $1/N\nu \geq \varepsilon$ (of course, ν_0 is/are the infinite place(s)). Then, Proposition 3b.12 states that there is a "projected" lattice point inside this ball, wherefore the image of the lattice is dense.

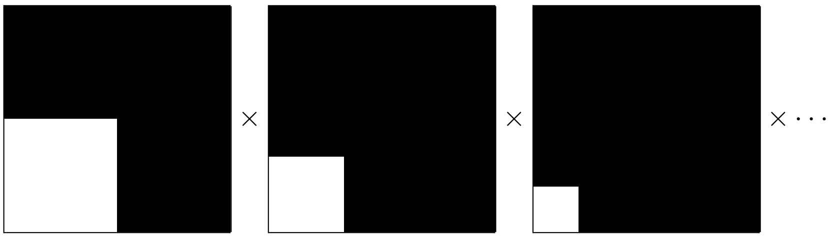
So, we have established the CPS $(\mathbb{R}^d\,, \prod'_{p\in\mathbb{P}} \mathbb{Q}_p^d\,, \mathbb{Q}^d)$ and that $\mathbb{Z}^d = \Lambda(\prod_{p\in\mathbb{P}} \mathbb{Z}_p^d)$. Indeed, the set $\prod_{p\in\mathbb{P}} \mathbb{Z}_p^d$ is a regularly closed compact subset of $\prod'_{p\in\mathbb{P}} \mathbb{Q}_p^d$, wherefore $\mathbb{Z}^d$ is a model set within this CPS (this also "confirms" that $\mathbb{Z}^d$ is a Delone set). Moreover, since for $x, y \in \mathbb{Z}_p$ one has $x + y, x - y \in \mathbb{Z}_p$ ("all triangles are isosceles"), one can also "confirm" the group law for elements of $\mathbb{Z}^d$.

Now, we can describe the visible lattice points V of $\mathbb{Z}^2$ as a weak model set: If $\gcd(m\,,n) \neq 1$, then there is (at least one prime) p such that $p \mid m$ and $p \mid n$, *i.e.*, if $\gcd(m\,,n) \neq 1$ then there is a prime p such that $(m\,,n) \in \mathbb{Z}^2 \setminus p\mathbb{Z}^2$. Therefore, we obtain the following description of V as weak model set:

$$V = \left\{ z \in \mathbb{Q}^2 \mid z \in \prod_{p\in\mathbb{P}} (\mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2) \right\} = \Lambda\left(\prod_{p\in\mathbb{P}} (\mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2) \right).$$

We set $\varOmega_V = \prod_{p\in\mathbb{P}} (\mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2)$. We note that relative to the metric of Lemma 3b.15, $\varOmega_V$ is bounded and closed, and therefore a compact set in the internal space $\prod'_{p\in\mathbb{P}}(\mathbb{Q}_p^2)$. But $\varOmega_V$ has empty interior, by the definition of the topology of a restricted product (even no component $\varOmega_V$ contains a full subgroup of the form $\mathbb{Z}_p$, wherefore it is not contained in any neighbourhood of some point).

With the methods from Chapter 3c, we can visualise the window $\varOmega_V$ as follows:



$(\mathbb{Z}_2^2 \setminus 2\cdot\mathbb{Z}_2^2) \subset \mathbb{Z}_2^2$ $\quad(\mathbb{Z}_3^2 \setminus 3\cdot\mathbb{Z}_3^2) \subset \mathbb{Z}_3^2$ $\quad(\mathbb{Z}_5^2 \setminus 5\cdot\mathbb{Z}_5^2) \subset \mathbb{Z}_5^2$

One can also calculate the Haar measure of $\varOmega_V$:

$$\mu(\varOmega_V) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^2}\right) = \prod_{p\in\mathbb{P}} \frac{p^2-1}{p^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.607927, \tag{5a.1}$$

where ζ is Riemann's zeta function. This calculation now starts our determinations of the autocorrelation coefficients.

Remark 5a.4. Since $\varOmega_V$ is a compact set with no interior, we have $\partial\varOmega_V = \varOmega_V$ (and our previous calculation shows $\mu(\varOmega_V) = \mu(\partial\varOmega_V) > 0$). So, we cannot apply the density formula (see Corollary 5.27) to determine the autocorrelation coefficients. The reason for this is that V has arbitrarily large holes, so one might choose a van Hove sequence $\{A_n\}$ in the definition of the autocorrelation coefficients, such that all its members A_n "lie in a hole"; this would establish that all correlation coefficients are 0 relative to this van Hove sequence, while the above formula suggests that one would expect $\eta_V(0) = \operatorname{dens} V = \mu(\varOmega_V)$. Indeed, the density of V is given by $6/\pi^2$, see [15, Theorem 3.9] and [184, Theorem 6.6.3] (compare [37, Prop. 6]).

So, the problem is that V is not a UDP set. But we will now establish that there is a special van Hove sequence $\{A_n\}$, which is nested around the origin, such that for every $\varepsilon > 0$ there is an $N = N(z) \in \mathbb{N}$ such that

$$\left| \eta_V^n(z) - \frac{\mu(\varOmega_V \cap z + \varOmega_V)}{\mu_{\mathbb{R}^2} \otimes \mu(\mathrm{FD}(\mathbb{Q}^2))} \right| = |\eta_V^n(z) - \mu(\varOmega_V \cap z + \varOmega_V)| < \varepsilon \tag{5a.2}$$

for all $n \geq N$. Consequently, this establishes the autocorrelation coefficients.
We first calculate $\mu\,(\varOmega_V \cap (z + \varOmega_V))$. To this end, we observe that for $z = (x\,,y) \in \mathbb{Z}^2$, the translation $(x\,,y) + \varOmega$ acts as a rotation, *e.g.*, see Figure 5a.2 for $\mathbb{Z}_2^2 \setminus 2\mathbb{Z}_2^2$.

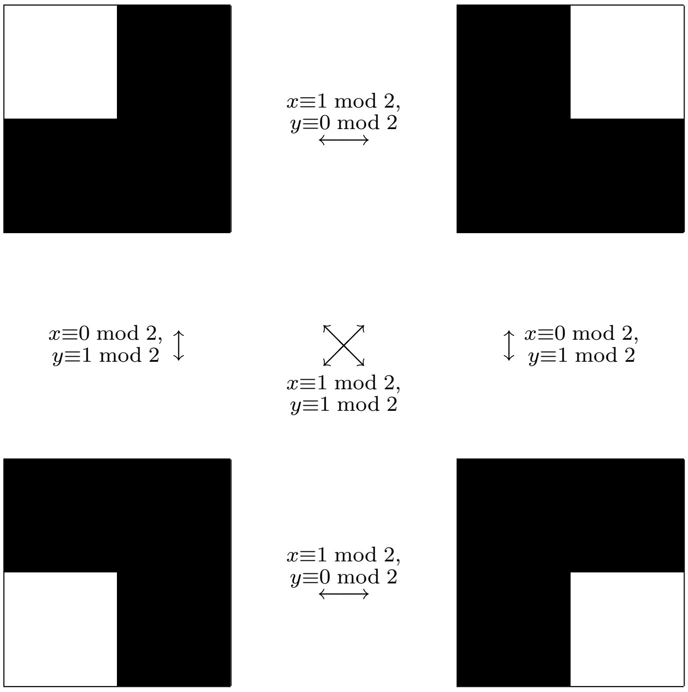


Figure 5a.2.: The translation $(x\,,y)+\varOmega$ with $(x\,,y)\in\mathbb{Z}^2$ acts as a rotation, here shown for the $\mathbb{Z}_2^2$-component of the internal space (the corresponding $\mathbb{Z}_2^2$-component of $\varOmega$ is given by $\mathbb{Z}_2^2\setminus 2\mathbb{Z}_2^2$).

So, for the Haar measure μ_p of the component $\mathbb{Q}_p^2$ (which is the restriction of the Haar measure μ on the adele ring to this component) we have:

$$\mu_p\big(\big(\varOmega_{V,p}\cap((x\,,y)+\varOmega_{V,p})\big)=\begin{cases}1-\frac{1}{p^2}, & \text{if } x\equiv y\equiv 0 \bmod p,\\ \\ 1-\frac{2}{p^2}, & \text{otherwise.}\end{cases}$$

where we use the notation $\varOmega_{V,p}=\mathbb{Z}_p^2\setminus p\mathbb{Z}_p^2$. Defining $g_p(x\,,y)=1$ if $x\equiv y\equiv 0 \bmod p$ and $g_p(x\,,y)=2$ otherwise, we obtain:

$$\mu\left(\varOmega_V\cap((x\,,y)+\varOmega_V)\right)=\prod_{p\in\mathbb{P}}\left(1-\frac{g_p(x\,,y)}{p^2}\right)=\prod_{p\in\mathbb{P}}\left(\frac{p^2-g_p(x\,,y)}{p^2}\right). \tag{5a.3}$$

We also note that, since $\mu\left(\varOmega_V\cap((x\,,y)+\varOmega_V)\right)>0$ for all $(x\,,y)\in\mathbb{Z}^2$, this also establishes that $V-V=\mathbb{Z}^2$.

Moreover, we have the following immediate statement (recalling our findings for $\mu(\Omega_V)$ in Equation (5a.1)).

Lemma 5a.5. *Suppose the autocorrelation coefficients for $z\in\mathbb{Z}^2$ of V are given by $\eta_V(z)=$*

$\mu\left(\Omega_V \cap (z+\Omega_V)\right)$ and are 0 otherwise. Then, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\prod_{\substack{p\in\mathbb{P},\\ p>N}} \left(\frac{p^2-2}{p^2-1}\right) > 1-\varepsilon, \text{ wherefore } \left(\prod_{\substack{p\in\mathbb{P},\\ p\le N}} p\right)\mathbb{Z}^2 \subset P_\varepsilon(V),$$

which establishes the strict pure point diffractivity of V. □

Remark 5a.6. Basically, this is the proof of [36, Prop. 5]. However, there the autocorrelation coefficients are taken from [37, Theorem 1], which are derived using a different method than the one we will use next.

The rigorous approach in establishing the diffraction properties in the article [37] is a response to the history of the problem as described in [28] and differing numerical calculations in [268] and [338] (also see [339, Fig. 4.9]).

We now calculate the autocorrelation coefficients. For this, we first calculate the autocorrelation coefficients of the sets

$$V_N = \mathbb{Z}^2 \setminus \left(\bigcup_{\substack{p\in\mathbb{P},\\ p\le N}} p\mathbb{Z}^2\right);$$

we call V_N the set of *N-visible lattice points*. We note that V_N is a periodic set with group of periods $(\prod_{p\in\mathbb{P},p\le N} p)\mathbb{Z}^2$. Moreover, one obviously has $V \subset V_N$ for all $N \in \mathbb{N}$. The autocorrelation coefficients $\eta_{V_N}(z)$ are easily established by observing that V_N are model sets

$$V_N = \Lambda\left(\prod_{\substack{p\in\mathbb{P},\\ p\le N}} \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2\right), \text{ relative to the CPS } \left(\mathbb{R}^2, \prod_{\substack{p\in\mathbb{P},\\ p\le N}} \mathbb{Q}_p^2, \mathbb{Q}^2\right),$$

wherefore

$$\eta_{V_N}(z) = \prod_{\substack{p\in\mathbb{P},\\ p\le N}} \left(1-\frac{g_p(z)}{p^2}\right) = \prod_{\substack{p\in\mathbb{P},\\ p\le N}} \left(\frac{p^2-g_p(z)}{p^2}\right).$$

For fixed z, this defines a bounded decreasing sequence in N with limit $\lim_{N\to\infty} \eta_{V_N}(z) = \mu\left(\Omega_V \cap (z+\Omega_V)\right)$. Define $C_N = \prod_{p\in\mathbb{P},p\le N} p$. Then, for the van Hove sequence $\{A_N\}_{N\in\mathbb{N}}$ with $A_N = [-\frac{1}{2}C_N+\frac{1}{2}, \frac{1}{2}C_N+\frac{1}{2}]\times[-\frac{1}{2}C_N+\frac{1}{2}, \frac{1}{2}C_N+\frac{1}{2}]$ one has the estimate $\eta_V^{(N)}(z) \le \eta_{V_N}(z)$.

Moreover, we note that, up to boundary points whose contribution can be estimated by Lemma 5.22 as

$$0 \le \eta_{V_{C_N}}(z) - \eta_V^{(N)}(z) \le \frac{\operatorname{card}(\mathbb{Q}^2 \cap (\partial^{\{-z\}} A_N \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2)}{\mu_{\mathbb{R}^2}(A_N)} \le \frac{\operatorname{card}(\mathbb{Z}^2 \cap \partial^{\{-z\}} A_N)}{\mu_{\mathbb{R}^2}(A_N)},$$

one has $\eta_V^{(N)}(z) \approxeq \eta_{V_{C_N}}(z)$ (note that we look at V_{C_N} now, because V_{C_N} coincides inside A_N with V). Consequently, we have establish the autocorrelation coefficients for the visible lattice points, and they are obtained by a formula as in Corollary 5.27 respectively Equation (5a.2). Thus, the condition in Lemma 5a.5 holds.

Remark 5a.7. Here, we give some alternative descriptions of the visible lattice points.

- We can also interpret V as subset of $\mathbb{Z}[i]$, since $\mathbb{Z}^2 \cong \mathbb{Z}[i]$. We observe that $\mathbb{Q}(i)$ is a lattice in the adele

$$\mathbb{A}_{\mathbb{Q}(i)} = \mathbb{C} \times \mathbb{Q}_{(1-i)} \times {\prod}'_{p\equiv 3 \bmod 4} \mathbb{Q}_{(p)} \times {\prod}'_{p\equiv 1 \bmod 4} \mathbb{Q}_{\mathfrak{p}_p^{(1)}} \times \mathbb{Q}_{\mathfrak{p}_p^{(2)}},$$

  wherefore the direct space is here given by $\mathbb{C} \cong \mathbb{R}^2$. Similarly to the above derivation, one obtains the window

$$\varOmega_V = \left(\widehat{\mathfrak{o}_{(1-i)}} \setminus (1-i)^2 \cdot \widehat{\mathfrak{o}_{(1-i)}}\right) \times \prod_{p\equiv 3 \bmod 4} \left(\widehat{\mathfrak{o}_{(p)}} \setminus p\widehat{\mathfrak{o}_{(p)}}\right)$$
$$\times \prod_{p\equiv 1 \bmod 4} \left((\widehat{\mathfrak{o}_{\mathfrak{p}_p^{(1)}}} \times \widehat{\mathfrak{o}_{\mathfrak{p}_p^{(2)}}}) \setminus (\mathfrak{p}_p^{(1)} \cap \mathfrak{p}_p^{(2)}) \right).$$

  Let us explore the factors that appear in the window. For this, we note that we have the following situation:

  - $\mathbb{Q}_{(1-i)}$ is a totally ramified (quadratic) extension of $\mathbb{Q}_2$ (*i.e.*, we have for the ramification index $e = 2$ and for the residue degree $f = 1$). The uniformiser is given by $(1 - i)$ and its normalised absolute value is $\|1 - i\|_{(1-i)} = \frac{1}{2}$. Note that in $\mathbb{Z}[i]$, the ideal $(1 - i)$ is the "checkerboard sublattice", *i.e.*, the sublattice of $\mathbb{Z}[i]$ with basis vectors $1 \pm i$ (and $(1 - i)^2$ is "the checkerboard of the checkerboard" and therefore $2\mathbb{Z}[i]$).
  - For every inert prime $p \equiv 3 \bmod 4$, $\mathbb{Q}_{(p)}$ is an unramified (quadratic) extension of $\mathbb{Q}_p$ (*i.e.*, $e = 1$ and $f = 2$). The uniformiser is given by p, and one may choose the following system of representatives S in Lemma 3.68:

$$\begin{aligned} S =& \{0, 1, \dots, p-1,\ i, i+1, \dots, i+p-1,\ 2i, 2i+1, \dots, 2i+p-1, \dots, \\ & (p-1)i, (p-1)i+1, \dots, (p-1)(i-1)\} \end{aligned}$$

    We also have $\|p\|_{(p)} = \frac{1}{p^2}$. Moreover, this system of representatives shows that $\mathbb{Q}_{(p)} \cong \mathbb{Q}_p^2$.
  - For every splitting prime $p \equiv 1 \bmod 4$, we have $(p) = \mathfrak{p}_p^{(1)} \cdot \mathfrak{p}_p^{(2)}$ and the $\mathfrak{p}$-adic fields $\mathbb{Q}_{\mathfrak{p}_p^{(1)}}$ and $\mathbb{Q}_{\mathfrak{p}_p^{(2)}}$ are actually just $\mathbb{Q}_p$, *i.e.*, we have $\mathbb{Q}_{\mathfrak{p}_p^{(1)}} \times \mathbb{Q}_{\mathfrak{p}_p^{(2)}} = \mathbb{Q}_p^2$, $\widehat{\mathfrak{o}_{\mathfrak{p}_p^{(1)}}} \times \widehat{\mathfrak{o}_{\mathfrak{p}_p^{(2)}}} = \mathbb{Z}_p^2$ and $\mathfrak{p}_p^{(1)} \cap \mathfrak{p}_p^{(2)} = p\mathbb{Z}_p^2$. For the last claim, we also look at the following example $p = 5$: Then, a possible prime ideal factorisation is $(5) = (2+i) \cdot (2-i)$. Note that in $\mathbb{Z}[i]$, the ideal $(2 \pm i)$ is a sublattice of $\mathbb{Z}[i]$ with "basis vectors" $\{2 \pm i, \mp 1 + 2i\}$ (whose fundamental domain has measure 5, after identification $\mathbb{Z}[i] \cong \mathbb{Z}^2$). Note that the "coincidence site lattice" of $(2 + i)$ and $(2 - i)$ (*i.e.*, the lattice obtained by the intersection of the two lattices $(2 + i) \cdot \mathbb{Z}[i]$ and $(2 - i) \cdot \mathbb{Z}[i]$) is $5 \cdot \mathbb{Z}[i]$.

  The considerations establish the equivalence of this description with the previous one.

- Actually, we can forget everything about adele rings respectively restricted products, and simply take the (infinite) product of compact spaces $\prod_{p\in\mathbb{P}} \mathbb{Z}_p^2$ (which is again a compact space by Tychonoff's theorem) as internal space. Obviously, we then have to take $\mathbb{Z}^2$ as the lattice in the CPS.

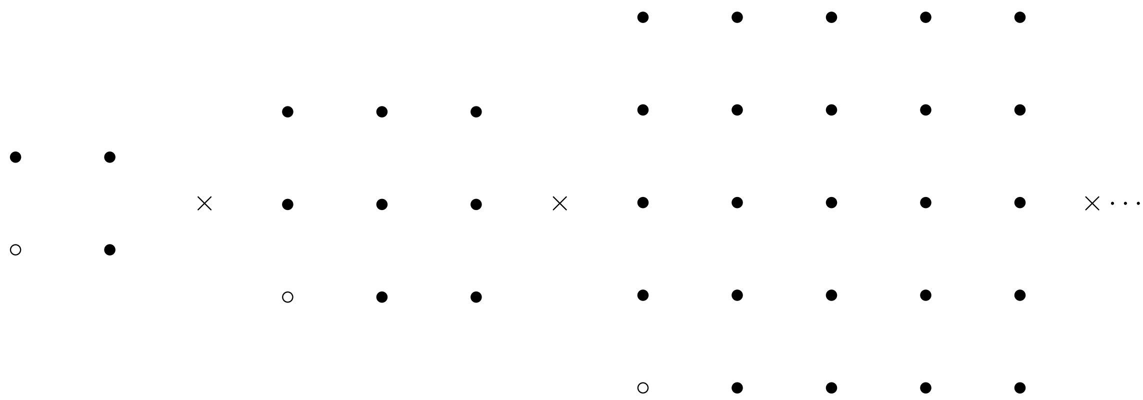
- A further considerable simplification of the description of the visible lattice points as model set by the following observation: The stabiliser of the window $\varOmega_V$ is given by

$$\operatorname{Stab}\varOmega_V = \{z \in \prod_{p\in\mathbb{P}} \mathbb{Z}_p^2 \mid z + \varOmega_V = \varOmega_V\} = \prod_{p\in\mathbb{P}} p\mathbb{Z}_p^2 .$$

  We have basically seen this in Figure 5a.2, since vectors $z = (x\,, y)$ with $x \equiv y \equiv 0 \bmod p$ do not change the component $\Omega_{V,p}$ of the window. So, as (weak) model set, the visible lattice points V are not irredundant. We can achieve irredundancy by factoring the internal space $\prod_{p\in\mathbb{P}} \mathbb{Z}_p^2$ (a commutative group) through this stabiliser (see Lemma 2.23), which yields $\prod_{p\in\mathbb{P}}(\mathbb{Z}/p\mathbb{Z})^2 = \prod_{p\in\mathbb{P}} \mathsf{C}_p^2$, *i.e.*, an infinite product of cyclic groups. So, we ultimately arrived at the CPS $(\mathbb{R}^2\,, \prod_{p\in\mathbb{P}} \mathsf{C}_p^2\,, \tilde{L})$ where the lattice is given as image of $\mathbb{Z}^2$ under the map

$$(x\,, y) \mapsto ((x\,, y)\,, (x \bmod 2\,, y \bmod 2)\,, (x \bmod 3\,, y \bmod 3)\,, (x \bmod 5\,, y \bmod 5)\,, \ldots).$$

  Moreover, the window of V relative to this CPS is given by $\prod_{p\in\mathbb{P}}(\mathsf{C}_p^2 \setminus (0\,, 0))$, which we may depict as follows:



5a.2. Diffraction Pattern of the Visible Lattice Points

Although we have not established the connection between strictly pure point diffractive sets and the actual "physical" diffraction pattern so far (see Theorem 7.10), we will here indicate how the diffraction pattern for (weak) model sets can be calculated. Note that this section is informal in nature, because we do not state rigorous proofs here but indicate that the result [37, Theorem 3] can be understood by our knowledge of the window $\varOmega_V$ for the visible lattice points. For this we need define the Fourier transformation of integrable functions.

Definition 5a.8. Let G be an LCAG and denote by G^* its character group (which is also an LCAG by Lemma 3.114), see 3.7. Let[1] $f \in L^1(G)$. Then we define $f^* : G^* \to \mathbb{C}$, the Fourier transform of f, by the formula

$$f^*(x^*) = \int_G f(y)\overline{\langle x^*\,, y\rangle}\,\mathrm{d}\mu(y)$$

[1]Here, $f : G \to \mathbb{C}$ and the L^1-norm is defined by $\|f\|_1 = \int_G |f(y)|\,\mathrm{d}\mu(y)$, where μ is the unique Haar measure on G.

for $x^* \in G^*$. Note that this formula makes sense, since for all $y \in G$, $\overline{\langle x^* , y \rangle}$ has absolute value 1.

Remark 5a.9. As in $\mathbb{R}^d$, one can prove *Plancherel's theorem*, which states that the Fourier transform is an isometry of $L^1(G) \cap L^2(G)$ onto a dense subspace of $L^2(G^*)$ and can therefore be extended to an isometry of $L^2(G)$ onto $L^2(G^*)$, *e.g.*, see [199, Section VII.4] and [305, Theorem 3-26]. Moreover, one also has an inversion formula, convolution *etc.* available. We will fix a normalisation in the Fourier transform (which we may think of as the normalisation of the Haar measure) such that the inversion formula $(f^*)^*(-x) = f(x)$ holds for all $f \in L^2(G)$. For a $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$ this is achieved as indicated in Footnote 4 on p. 81.

In general, the diffraction pattern of a structure is obtained as Fourier transform of the autocorrelation measure associated to the structure in question. However, for regular model sets, one can calculate the diffraction pattern from the knowledge of the CPS and the window. This result is well-established in the purely Euclidean setting (*i.e.*, where both $G = \mathbb{R}^d$ and $H = \mathbb{R}^{d'}$ are Euclidean spaces), but also extends naturally to more general LCAGs. Unfortunately, it only seems to be folklore in the more general setting. It then reads as follows, compare [171, Section 5], [257, Theorem 7], [260, Theorems 12 & 13], [329, Theorem 4.5], [63, Theorem 2.6] and [33, Section 7] (for some early accounts see [84, 120]).

Folklore Theorem 5a.10. *Let $\Lambda = \Lambda(\Omega)$ be a regular model set in the CPS $(G , H , \tilde{L})$. Denote by $\tilde{L}^\perp$ the annihilator of $\tilde{L}$ (see Definition 3.122), which is a lattice in $G^* \times H^*$ by Corollary 3.124. Denote by $\pi_1^* : G^* \times H^* \to G^*$ and $\pi_2^* : G^* \times H^* \to H^*$ the canonical projections. Then, the diffraction pattern/spectrum exists and is a pure point measure (i.e., it can be written as a countable union of Dirac measures, see Definition 7.7), which is concentrated on $\pi_1^*(\tilde{L}^\perp)$. Assuming that the projection π_1^* is bijective on the lattice $\tilde{L}^\perp$, the intensity at a point $k \in \pi_1^*(\tilde{L}^\perp)$ is given by the square of the* Fourier-Bohr coefficient $a(k)$ *which is given by*

$$a(k) = \frac{\operatorname{dens} \Lambda}{\mu_H(\Omega)} \int_\Omega \langle \pi_2^* \circ (\pi_1^*)^{-1}(k) , y \rangle \, \mathrm{d}\mu_H(y)$$

We note that $\operatorname{dens} \Lambda / \mu_H(\Omega) = 1/\mu_{G \times H}(\mathrm{FD}(\tilde{L}))$ *by the density formula (Corollary 5.27).* □

Of course, there are several things to note. For example, there seems to be a "dual" CPS $(G^* , H^* , \tilde{L}^\perp)$ (or, actually, more precisely $(H^* , G^* , \tilde{L}^\perp)$) involved in the determination of the diffraction pattern, see Definition 7.17. Moreover, the Fourer-Bohr coefficients are obtained by the inverse Fourier transform of the characteristic function of the window, which we will use below. Note that more on diffraction will appear in Section 7.1, for now, we state a general result for the diffraction of lattice subsets, see [24, Theorem 2 & Corollary 4].

Proposition 5a.11. *Let L be a lattice in a σLCAG G, and let $S \subset L$ be a subset with existing autocorrelation coefficients $\eta_S(z)$. Then the diffraction pattern/spectrum is periodic with the annihilator $L^\perp$ as lattice of periods.* □

In view of this statements, we note that we have the following situation for the visible lattice points:

- By Lemma 3b.13, we have $\mathbb{A}_K^* \cong \mathbb{A}_K$ (and therefore also $(\mathbb{A}_K^d)^* \cong \mathbb{A}_K^d$) and $K^\perp \cong K$ (and therefore also $(K^d)^\perp \cong K^d$).
- Lemma 3b.13 also shows, that the Fourier transform in an adelic spaces can be calculated as product of the Fourier transforms in its components (as one expects).

- From Remark 3.125, we have $(\mathbb{Z}^d)^\perp \cong \mathbb{Z}^d$.

We now calculate the Fourier transform of the window Ω_V in the CPS $(\mathbb{R}^2, \prod'_{p\in\mathbb{P}} \mathbb{Q}_p^2, \mathbb{Q}^2)$. The reason for taking this ("complicate") CPS is that, by the observed self-duality in adele rings respectively $\mathbb{R}^2$, we can identify the direct (respectively internal) space with its character group. Moreover, the annihilator of $\mathbb{Q}^2$ is $\mathbb{Q}^2$ itself, which (if the Folklore Theorem 5a.10 holds for V) establishes that the diffraction pattern is concentrated on the (double) rational coordinates. To calculate the Fourier transform of the window Ω_V, we establish the Fourier transform of clopen balls in $\mathbb{Q}_\mathfrak{p}$. On the self-dual field $\mathbb{Q}_\mathfrak{p}$ with Haar measure normalised such that $\mu(\widehat{\mathfrak{o}_\mathfrak{p}}) = 1$, we obtain (with the appropriate normalisation) the following formula for the Fourier transform of a function $f \in L^2(\mathbb{Q}_\mathfrak{p})$:

$$f^*(x) = \frac{1}{\sqrt{\|d_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}\|_\mathfrak{p}}} \int_{\mathbb{Q}_\mathfrak{p}} f(y) \cdot \exp\left(2\pi i \cdot \vartheta_p(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}(x \cdot y))\right) \mathrm{d}\mu(y)$$

where $p = \mathfrak{p} \cap \mathbb{P}$, $\|\cdot\|_\mathfrak{p}$ denotes the normalised absolute value, $d_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}$ the discriminant (see Definition 3.37) and $\vartheta_p(y) = \vartheta_p(\sum_{n=m}^{\infty} s_n p^n) = \sum_{n=m}^{-1} s_n p^n$, see Lemma 3.116.

We denote the characteristic function of a set S by χ_S. Then we obtain the following formula for the Fourier transform of clopen balls in $\mathbb{Q}_p$.

Lemma 5a.12. [383, Example IV.6] and [382, Example 1] *The Fourier transform of the clopen ball $B_{\leq p^\gamma}(0) \subset \mathbb{Q}_p$ (with $\gamma \in \mathbb{Z}$) around the origin is given by*

$$\chi^*_{B_{\leq p^\gamma}(0)}(x) = \begin{cases} p^\gamma, & \text{if } x \in B_{\leq p^{-\gamma}}(0), \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the Fourier transform of an arbitrary ball is given by

$$\chi^*_{B_{\leq p^\gamma}(z)}(x) = \begin{cases} p^\gamma \cdot \exp\left(2\pi i \cdot \vartheta_p(x \cdot z)\right), & \text{if } x \in B_{\leq p^{-\gamma}}(0), \\ 0, & \text{otherwise.} \end{cases}$$ □

For a general $\mathfrak{p}$-adic field $\mathbb{Q}_\mathfrak{p}$, has to know the prime ideal factorisation of the different $\mathfrak{D}_{K/\mathbb{Q}}$ (compare Lemma 3.52) to calculate the Fourier transform of clopen balls. Using normalised absolute values, we note that $B_{p^{f\cdot\gamma}}(0) = \mathfrak{p}^\gamma$ where $f = f_{\mathfrak{p}|(p)}$ is the residue degree and $\gamma \in \mathbb{Z}$. Then, the following is implicit from [387, Prop. II.12] and [305, pp. 253 ff].

Lemma 5a.13. *Let $\mathfrak{p}$ a prime ideal in an algebraic number field K (i.e., $\mathfrak{p} \in \mathbb{P}_K$). Let $\mathfrak{D}_{K/\mathbb{Q}} = \prod_{\mathfrak{p}\in\mathbb{P}_K} \mathfrak{p}^{n_\mathfrak{p}}$ the prime ideal factorisation of the different of K. Then, the Fourier transform of the clopen ball $\mathfrak{p}^\gamma$ (with $\gamma \in \mathbb{Z}$) of radius $f_{\mathfrak{p}|(p)} \cdot \gamma$ around the origin is given by*

$$\chi^*_{\mathfrak{p}^\gamma}(x) = \begin{cases} p^{\gamma - \frac{n_\mathfrak{p}}{2}}, & \text{if } x \in \mathfrak{p}^{n_\mathfrak{p} - \gamma}, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the Fourier transform of an arbitrary ball is given by

$$\chi^*_{z+\mathfrak{p}^\gamma}(x) = \begin{cases} p^{\gamma - \frac{n_\mathfrak{p}}{2}} \cdot \exp\left(2\pi i \cdot \vartheta_p(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}(x \cdot z))\right), & \text{if } x \in \mathfrak{p}^{n_\mathfrak{p} - \gamma}, \\ 0, & \text{otherwise.} \end{cases}$$ □

We note that for the part $\Omega_{V,p}$ of the window Ω_V one has

$$\Omega_{V,p} = \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2 = (B_{\leq 1}(0) \times B_{\leq 1}(0)) \setminus (B_{\leq \frac{1}{p}}(0) \times B_{\leq \frac{1}{p}}(0)).$$

By the linearity of the Fourier transform, we therefore obtain:

$$\chi^*_{\Omega_{V,p}}(x,y) = \begin{cases} 0, & \text{if } \|x\|_p > p \text{ or } \|y\|_p > p, \\ 1 - \frac{1}{p^2}, & \text{if } \|x\|_p \leq 1 \text{ and } \|y\|_p \leq 1, \\ -\frac{1}{p^2}, & \text{otherwise.} \end{cases}$$

In particular, we obtain 0 if the denominators of x or y contains a square of p. We also observe that $\chi^*_{\Omega_{V,p}}(x,y) = \chi^*_{\Omega_{V,p}}(x+t_1, y+t_2)$ for all $t_1, t_2 \in \mathbb{Z}_p$. Taking the product over all $p \in \mathbb{P}$, we obtain the following Fourier-Bohr coefficients (of course, provided that the Folklore Theorem 5a.10 holds for V) for the visible lattice points:

$$a(z) = \begin{cases} \left(\prod_{p\in\mathbb{P}} 1 - \frac{1}{p^2}\right) \cdot \left(\prod_{p|q} \frac{-1}{p^2-1}\right), & \text{if } z = (\frac{k_1}{m_1}, \frac{k_2}{m_2}) \in \mathbb{Q}^2 \text{ and} \\ & q = \operatorname{lcm}(m_1, m_2) \text{ is square-free} \\ & (\text{in this case, } z \in \frac{1}{q}\mathbb{Z}^2), \\ 0, & \text{otherwise.} \end{cases} \tag{5a.4}$$

Here, $\operatorname{lcm}(\cdot,\cdot)$ denotes the *least common multiple* of two numbers and we assume that k_1 and m_1 respectively k_2 and m_2 are coprime.

We now note that these findings are rigorously established (using a different method) in [37, Theorem 3], which reads for $V \subset \mathbb{Z}^2$ as follows.

Lemma 5a.14. *The diffraction spectrum of the set of visible points V exists and is a pure point measure which is concentrated on the set of points in $\mathbb{Q}^2$ with square-free denominators and whose intensity at a point z is given by the square of the Fourier-Bohr coefficient $a(z)$ of Equation* (5a.4). □

Remark 5a.15. If one uses $\mathbb{A}_{\mathbb{Q}(i)}$ instead of $\mathbb{A}_{\mathbb{Q}}^2$ to describe the visible lattice points, one has to observe that $\mathfrak{o}_{\mathbb{Q}(i)} = \mathbb{Z}[i]$ and thus $\mathfrak{o}_{\mathbb{Q}(i)}^\wedge = \mathbb{Z}[i]^\wedge = \frac{1}{2}\,\mathbb{Z}[i] = \frac{1}{2}\,\mathfrak{o}_{\mathbb{Q}(i)} = (\sqrt{2})^{-2}$ by Lemma 3.51. Therefore, $\mathfrak{D}_{\mathbb{Q}(i)/\mathbb{Q}} = (\sqrt{2})^2$ (also observe Lemma 3.52), and one has $\chi^*_{(1-i)^\gamma} = 2^{\gamma-1}\,\chi_{(1-i)^{2-\gamma}}$. One may now check that both points of view (*i.e.*, regarding the visible lattice points as subset of $\mathbb{Z}^2$ or as subset of $\mathbb{Z}[i]$) yield the same diffraction pattern.

A (numerically calculated) picture of the diffraction pattern is shown in Figure 5a.3.

5a.3. Additional Topics and Further Examples

Remark 5a.16. Our findings can easily be generalised to higher dimensions and more general lattices, where the results stay virtually the same, see [37]. We show the visible lattice points of the hexagonal lattice $\mathbb{Z}[\xi_6]$.

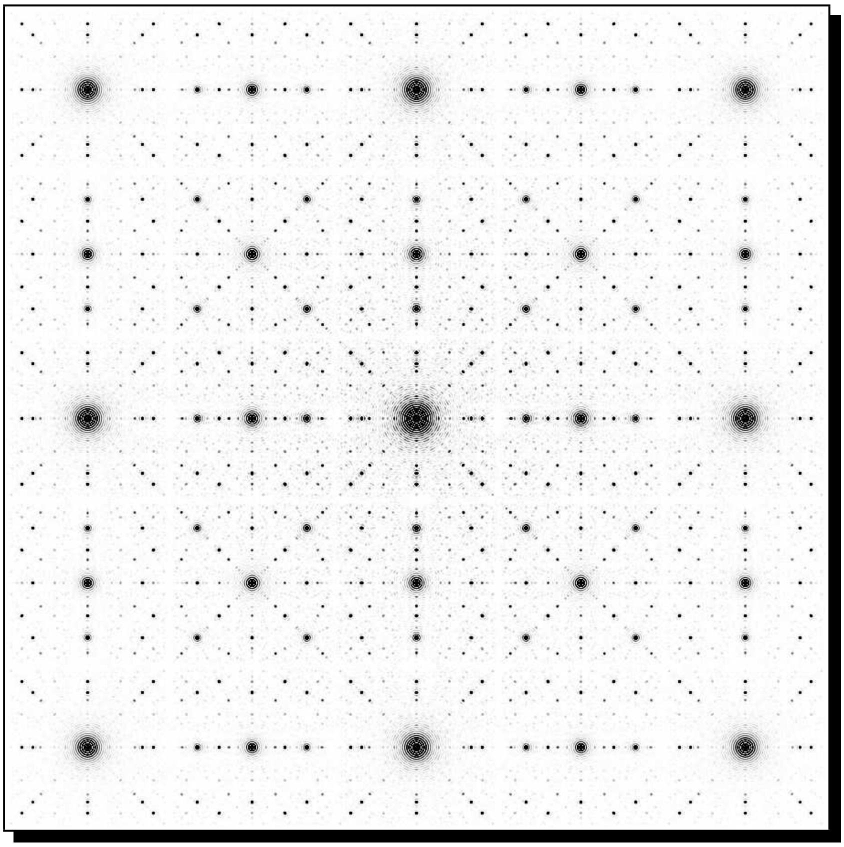
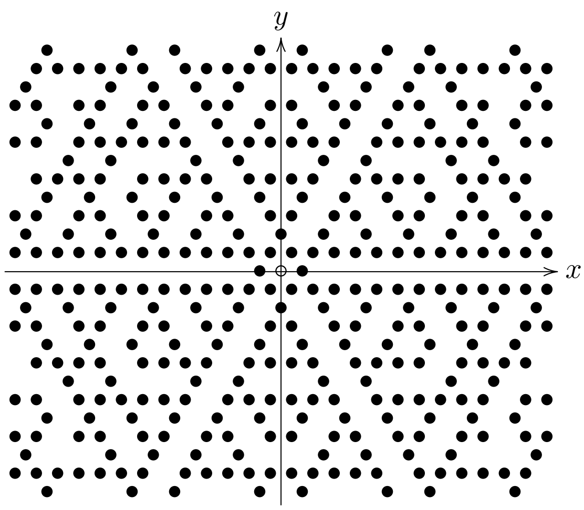


Figure 5a.3.: The part $[-1.25\,,1.25] \times [-1.25\,,1.25]$ of the diffraction pattern of the visible lattice points of $\mathbb{Z}^2$. Numerically calculated from a patch of radius 80 on a 901×901-grid using DISCUS [296]. Observe that there is (up to numerics) no intensity on the lines corresponding to rational numbers without square-free denominator, *e.g.*, on $x, y = \pm\frac{1}{4}, \pm\frac{3}{4}$.



Remark 5a.17. We also note that the set $\varOmega_V \cap ((x\,,y)+\varOmega)$, which appears in the determination of the autocorrelation coefficient at $(x\,,y)$ (see Equations (5a.2) & (5a.3) on p. 202), may also be interpreted as window for the set of lattice points which are visible from the origin *and* $(x\,,y)$. Consequently, the measure in Equation (5a.3) gives the density of visible lattice points which are visible from the origin and $(x\,,y)$. Obviously, one can easily generalise this to obtain statements about the density of lattice points which are visible from a finite set of points, compare [16, Section 8] and [308, 318]. Moreover, one can also calculate autocorrelation coefficients (and thus the diffraction pattern) with the methods established above.

Remark 5a.18. Inspired by the pictures in [388] and [375, pp. 25–26], we note the following: The points $\{(x\,,y) \in \mathbb{Z}^2 \mid \gcd(x,y) = p_1^{b_1} \cdots p_\ell^{b_\ell}\}$ can be obtained as weak model set relative to the CPS $(\mathbb{R}^2\,, \prod_{p\in\mathbb{P}} \mathbb{Z}_p^2\,, \mathbb{Q}^2)$ with window

$$(p_1^{b_1}\mathbb{Z}_{p_1}^2 \setminus p_1^{b_1+1}\mathbb{Z}_{p_1}^2) \cdots (p_\ell^{b_\ell}\mathbb{Z}_{p_\ell}^2 \setminus p_\ell^{b_\ell+1}\mathbb{Z}_{p_\ell}^2) \cdot \prod_{\substack{p\in\mathbb{P},\\ p\neq p_i (1\leq i\leq \ell)}} \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2.$$

Moreover, one easily establishes that the difference set is exactly given by $p_1^{b_1} \cdots p_\ell^{b_\ell}\mathbb{Z}^2$, that the set of ε-periods is relatively dense (wherefore it is a SPPD set) and one can calculate the Fourier-Bohr coefficients.

Therefore, we can partition $\mathbb{Z}^2$ in sets with points of the same gcd, which we may interpret as a "∞-component model set". It would be interesting if one can prove or disprove that this "∞-component model set" is pure point diffractive, where we give (the atoms of) different components different scattering strengths (obviously, the methods for multi-component model sets do not apply here, since there is no common relatively dense P'_ε-set). Note that one can (easily) calculate the Fourier-Bohr coefficients (namely, as in the last section) if Folklore Theorem 5a.10 can be extended to this case. See Section 7.1 on "scattering strengths".

Remark 5a.19. In the article [37], also the diffraction pattern of the set of *k-th power-free numbers* in $\mathbb{Z}$ is calculated, *i.e.*, of the set $\{x \in \mathbb{Z} \mid p^k \nmid x \text{ for all } p \in \mathbb{P}\}$ (also see [184, §6.6]). Obviously, this is a model set in the CPS $(\mathbb{R}\,, \prod'_{p\in\mathbb{P}} \mathbb{Q}_p\,, \mathbb{Q})$ with window $\prod \mathbb{Z}_p \setminus p^k\mathbb{Z}_p$. Moreover, the simplified CPS is given by $(\mathbb{R}\,, \prod_{p\in\mathbb{P}} \mathsf{C}_{p^k}\,, \tilde{L})$, where the lattice $\tilde{L}$ is given as as image of $\mathbb{Z}$ under the map $x \mapsto (x, x \bmod 2^k, x \bmod 3^k, x \bmod 5^k, \ldots)$, with window $\prod_{p\in\mathbb{P}} \mathsf{C}_{p^k} \setminus \{0\}$. Diffractive properties are essentially established as before, see [37, Theorems 4 & 5]

Remark 5a.20. We think that our treatment of the visible lattice points by visualising the window, should also be useful if one wants to calculate higher correlations (*i.e.*, correlations between three or more lattice points), cluster frequencies and entropies. For the latter, more precisely, for the topological entropy, see [291].

Remark 5a.21. The visible lattice points are obtained by (an infinite number of) number theoretic sieves, wherefore they become aperiodic. This manifests in the infinite ("card $\mathbb{P}$ many") components of the internal space $\prod'_{p\in\mathbb{P}} \mathbb{Q}_p^2$. We may think of the aperiodic visible lattice points as limit of the periodic N-visible lattice points, obtained by adding more and more components to the internal space (respectively, removing part of the window in higher and higher components).

For substitutions, especially the ones with compact internal space H respectively H_{sub} like lattice substitution systems (see Chapter 6b), the situation is in a certain sense analogous and yet different: Here, the internal space is fixed, but (if we start with the whole compact space in the iteration, wherefore we have an antitone sequence that converges to the window) the IFS $\boldsymbol{\Theta}^{\star}$ adds smaller and smaller parts to the window, which consequently (may) force aperiodicity.

6. Pisot Substitutions

The analytic trickery is over; it is time to harvest the corollaries.

LECTURES ON ERGODIC THEORY – *Paul R. Halmos*

After the last few technical chapters, it is no time to apply our established theory to "real-world" examples. We look at a certain class of one-dimensional point sets, namely, the ones which arise from Pisot substitutions. In all considerations, we also include point sets with mixed Euclidean and $\mathfrak{p}$-adic internal spaces (and not only restrict ourselves to the purely Euclidean case). Explicit examples (especially see Section 6.10) show how the concepts introduced are applied. The central statement of this thesis is Theorem 6.116.

6.1. One-Dimensional Sequences

We begin our discussion with symbolic sequences and substitutions that generate them.

Definition 6.1. Let $\mathcal{A}$ be a (nonempty) finite set that we call the *alphabet*. The elements of $\mathcal{A}$ are called *letters* (or *symbols*) and are either denoted as digits (*i.e.*, $\mathcal{A} = \{1, \dots, n\}$ where $n = \operatorname{card} \mathcal{A}$) or as letters (*i.e.*, $\mathcal{A} = \{a_1, \dots, a_n\}$). A *word* or *block* is a finite string of elements in $\mathcal{A}$. Similarly, as in Definitions 3c.4 & 3c.5 respectively on p. 106, we define the length of a word as the number of letters it consists of, the set $\mathcal{A}^k$ of all words of length k, the set $\mathcal{A}^{\text{fin}}$ of all words of finite length and the set $\mathcal{A}^*$ of all words. Moreover, ϵ denotes the empty word, and we have the *concatenation*[1] of two words $v = v_1 \dots v_\ell$ and $w = w_1 \dots w_m$ defined by $vw = v_1 \dots v_\ell w_1 \dots w_m$. A *one-sided sequence* in $\mathcal{A}$ (or a *(right) infinite word* on $\mathcal{A}$) is an element $u = (u_n)_{n \in \mathbb{N}} = u_1 u_2 u_3 \dots \in \mathcal{A}^*$. We denote the set of one-sided sequences by $\mathcal{A}^{\mathbb{N}}$. A *(two-sided) sequence* in $\mathcal{A}$ (or a *bi-infinite word* on $\mathcal{A}$) is an element $u = (u_n)_{n \in \mathbb{N}} = \dots u_{-2} u_{-1} \dot{u}_0 u_1 u_2 \dots \in \mathcal{A}^*$ (here, "$\dot{}$" denotes the zeroth position). We denote the set of sequences by $\mathcal{A}^{\mathbb{Z}}$.

Remark 6.2. The set of all finite words that occur in a sequence u, is called the *language* of the sequence u. Knowing the language, one can also consider the combinatorics/complexity of the sequence u, see [298, Section 1.1.2]. Moreover, we note that we can also topologise $\mathcal{A}^{\mathbb{N}}$ with the canonical product metric as in Equation (3c.1) on p. 62. Consequently, one also has sets $\mathcal{N}(w)$ (which form a base of clopen sets for $\mathcal{A}^{\mathbb{N}}$) which are called *cylinders* here. All these notions extend in a natural way to $\mathcal{A}^{\mathbb{Z}}$ by defining a metric $d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ by

$$d(w, w') = \begin{cases} \eta^{\min\{\#((w_0 w_1 w_2 \dots) \barwedge (w'_0 w'_1 w'_2 \dots)),\, \#((w_0 w_{-1} w_{-2} \dots) \barwedge (w'_0 w'_{-1} w'_{-2} \dots))\}}, & \text{if } w \neq w', \\ 0, & \text{if } w = w', \end{cases}$$

[1] Actually, the set $\mathcal{A} \setminus \{\epsilon\}$ is endowed with the structure of a free semi-group. This can be extended to a free group with $\operatorname{card} \mathcal{A}$ generators.

for any $0 < \eta < 1$ (so one "divides" each sequence in its left-sided and right-sided infinite part). Note that for u interpreted as multi-component subset (in $\mathbb{R}$) with points on $\mathbb{Z}$, this is simply the local topology of Definition 5.102. Since any multi-component set on $\mathbb{Z}$ is an FLC set, one can consequently define *symbolic dynamical systems* and consider questions about minimality *etc.*, see the books [298, 299] and references therein. Here, the corresponding ($\mathbb{Z}$-)action is the shift $S((u_n)_{n\in\mathbb{Z}}) = S(\ldots u_{-2}u_{-1}\dot{u}_0u_1u_2\ldots) = \ldots u_{-2}u_{-1}u_0\dot{u}_1u_2\ldots = (u_{n+1})_{n\in\mathbb{Z}}$.

From now on, we will only look at (two-sided) sequences (and not one-sided sequences). They can be produced by substitutions.

Definition 6.3. A *substitution* σ on $\mathcal{A}$ is a map from $\mathcal{A}$ into the set $\mathcal{A}^{\text{fin}} \setminus \{\epsilon\}$ of nonempty finite words on $\mathcal{A}$; we use the notation $i \mapsto \sigma(i) = i_1 \ldots i_\ell$ (where $\ell = \#\sigma(i)$). It extends to a morphism of $\mathcal{A}^*$ by concatenation (*i.e.*, $\sigma(vw) = \sigma(v)\sigma(w)$) and $\sigma(\epsilon) = \epsilon$, and therefore also to a map over $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$. A *fixed point* u of σ is a sequence $u \in \mathcal{A}^{\mathbb{Z}}$ with $u = \sigma(u)$. If one has $u = \sigma^m(u)$ for some $m \in \mathbb{N}$, the sequence u is called a *periodic point* (of *period* m). See Remark 6.13 on how one calculate the number of periodic points.

Remark 6.4. (Primitive) substitutions are used to produce sequences: By the finiteness of the alphabet $\mathcal{A}$, there is a letter $a \in \mathcal{A}$ such that $\sigma^k(a)$ begins with a for some power $k \in \mathbb{N}$. Moreover, we assume that $\#(\sigma^{k\cdot m}(a)) \geq 2$ for some $m \in \mathbb{N}$, *i.e.*, we have some "expansion". Moreover, primitivity implies that all letters occur in $\sigma^N(d)$ for (at least) all $N \geq n^2 - 2n + 2$ (by Remark 4.68) and all letters $d \in \mathcal{A}$.

Take any *legal* pair ba (*i.e.*, ba occurs in some $\sigma^{k'}(a)$ with $k' \in \mathbb{N}$). Look at the substitutes of $b\dot{a}$ under σ^k. Again, by finiteness of $\mathcal{A}$, here is a letter $c \in \mathcal{A}$ and $m', \ell' \in \mathbb{N}$ such that $\sigma^{k\cdot m'}(b)$ and $\sigma^{k\cdot \ell'}(c)$ end in c. Thus, there exists a unique fixed point $u \in \mathcal{A}^{\mathbb{Z}}$ of $\sigma^{k\cdot\ell'}$ with $\ldots c\dot{a}\ldots$ and which is obtained as limit $n' \to \infty$ in the local topology of the sequence of finite words $\sigma^{k\cdot\ell'\cdot n'}(c\dot{a})$. Alternatively, one may simply look at the substitutes of $b\dot{a}$ under σ^k, say $\sigma^k(b\dot{a}) = \ldots b_k\dot{a}_k \ldots$. By the legality of ba, all pairs b_ka_k are legal, and by the finiteness of the alphabet $\mathcal{A}$, there are only finitely (at most $(\operatorname{card}\mathcal{A})^2$) many distinct such pairs. Therefore, there is a cycle (of, say, period $\check{k}$) of pairs $b_k\dot{a}_k$ that we obtain under the action of σ. Let $b_i\dot{a}_i$ such a pair in this cycle; then, $\sigma^{\check{k}}(a_i) = a_i \ldots$, $\sigma^{\check{k}}(b_i) = \ldots b_i$ and one obtains a unique fixed point $\lim_{n'\to\infty} \sigma^{n'\cdot\check{k}}(b_i\dot{a}_i)$. In the one-sided case, the previous considerations may be found in [299, Prop. V.1].

We are interested in minimal dynamical systems, wherefore by Lemma 5.108 and Corollary 5.89 we are actually interested in primitive substitutions.

Definition 6.5. Let σ be a substitution over $\mathcal{A}$ with $n = \operatorname{card}\mathcal{A}$. The $(n \times n)$ *substitution matrix* $\boldsymbol{S}\sigma$ of the substitution σ is given by

$$(\boldsymbol{S}\sigma)_{ij} = \operatorname{card}\{k \mid a_k = i, \text{ where } \sigma(j) = a_1 \ldots a_k \ldots a_\ell\},$$

i.e., $(\boldsymbol{S}\sigma)_{ij}$ is the number of occurrences of i in $\sigma(j)$. We say that σ is *primitive* if the matrix $\boldsymbol{S}\sigma$ is primitive.

Definition 6.6. Let $\mathcal{A}$ be an alphabet with $n = \operatorname{card}\mathcal{A}$. Let $l : \mathcal{A}^{\text{fin}} \to \mathbb{Z}^n$ denote the *canonical homomorphism*, also called the *homomorphism of Abelianisation*, defined as

$$l(w) = (\operatorname{card}\{k \mid w_k = i \text{ for } w = w_1 \ldots w_\ell\})_{1\leq i\leq n}.$$

Note that one has $\boldsymbol{S}\sigma = (l(\sigma(i)))_{1\leq i\leq n}$ (where $l(\sigma(i))$ is the i-th column). Moreover, one immediately derives $l(\sigma(w)) = (\boldsymbol{S}\sigma)\, l(w)$ for all $w \in \mathcal{A}^{\text{fin}}$.

To apply the theory established in the last chapter, we want to associate a tiling in $\mathbb{R}$ to a fixed point u of σ. For this, we observe the following:

- For a primitive substitution σ, denote by λ the PF-eigenvalue of $\boldsymbol{S}\sigma$ and by $\ell = (\ell_1, \dots \ell_n)$ its left PF-eigenvector, see the Perron-Frobenius theorem in Proposition 4.69. Then, the length $\sigma(w)$ increases on average by a factor of λ compared to w.
- With the Abelianisation l, we define a map $l' : \mathcal{A}^{\text{fin}} \cup \{\epsilon\} \to \mathbb{R}$ by $l'(w) = \langle \ell \, , l(w) \rangle$ and $l'(\epsilon) = 0$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^d$.
- We define the following tile substitution in $\mathbb{R}$: We denote by $u \triangleleft v$ that u is a prefix of v, *i.e.*, $v = u \dots$. The prototiles are given by $A_i = [0 \, , \ell_i]$, and we have

$$[0 \, , \ell_j] + x \mapsto \big([0 \, , \ell_i] + l'(w) + \lambda x \mid wi \triangleleft \sigma(j) \big) \, . \tag{6.1}$$

  Note that this corresponds naturally to the steps "inflation" and "subdivision": We represent each letter j by a compact interval $[0 \, , \ell_j]$, whose inflated version $[0 \, , \lambda \cdot \ell_j]$ can be subdivided into the corresponding intervals given by the substitution σ.
- Consequently, we can associate a primitive EMFS $\boldsymbol{\Theta}$ to each primitive substitution σ given by

$$\Theta_{ij} = \bigcup_{\{w \mid wi \triangleleft \sigma(j)\}} \{ t_{(l'(w))} \circ f_0 \} \tag{6.2}$$

  where $f_0(x) = \lambda x$ and Then, we have $\boldsymbol{S}\sigma = \boldsymbol{S\Theta}$.
- By construction, the attractor of the IFS $\boldsymbol{\Theta}^{\#}$ is given by $\underline{A} = ([0 \, , \ell_i])_{1 \le i \le n}$ since

$$[0 \, , \ell_i] = \bigcup_{j=1}^{n} \Theta^{\#}_{ij}([0 \, , \ell_j]) = \bigcup_{\{w \mid wj \triangleleft \sigma(i)\}} \frac{1}{\lambda}([0 \, , \ell_j] + l'(w)) = \frac{1}{\lambda}[0 \, , \lambda \cdot \ell_i].$$

- By Remark 6.4, there is a pair $c\dot{a}$ such that one obtains a fixed point for some power σ^N of the the substitution σ on $\mathcal{A}$. But this corresponds now to the following: Applying successively the (N-th power of the) tile substitution in Equation (6.1) to $[0 \, , \ell_c] - \ell_c \cup [0 \, , \ell_a]$ (with support $[-\ell_c \, , \ell_a]$) yields – as limit in the local topology – a tiling $\mathcal{T} = \underline{\Lambda} + \underline{A}$ of $\mathbb{R}$, where Λ_i is the set of left endpoints of the tiles belonging to the equivalence class $[A_i]$. Note that one might to introduce further labels (*i.e.*, colour all tiles $[A_i]$ with colour i if $\ell_i = \ell_j$ for some $i \neq j$) to be able to discriminate the tiles A_i according to their substitution rule. Moreover, this procedure also shows that the sets Δ_i are primitive substitution multi-component Delone sets with finite seed $\omega_c(-\ell_c) \cup \omega_a(0)$. Here, the relative denseness of the sets Λ_i is established by primitivity and the finiteness of the numbers ℓ_i, while the disjointness of the unions in Equation 5.5 follows automatically since we define $\underline{\Lambda}$ *via* the tile substitution. Note that every cluster in $\underline{\Lambda}$ is legal and $\underline{\Lambda}$ is representable.
- We also note condition **(LT)** is satisfied by $\boldsymbol{\Theta}$ respectively $\boldsymbol{\Theta}^{\#}$.
- Since $0 \in \operatorname{supp} \underline{\Lambda}$, we also obtain $\mathcal{L} = \langle \ell_1, \dots \ell_n \rangle_{\mathbb{Z}}$. We call $A_i = [0 \, , \ell_i]$ the *natural intervals* for a one-dimensional substitution and $\mathcal{T} = \underline{\Lambda} + \underline{A}$ the *representation with natural intervals* of a fixed point u of σ.

Remark 6.7. Obviously, the left eigenvector is only unique up to a multiplicative constant. But one can always choose $\ell_i \in \mathbb{Q}(\lambda)$ (since the eigenvector equation $\ell\, \boldsymbol{S}\sigma = \lambda\,\ell$ is a linear equation with coefficients in the field $\mathbb{Q}(\lambda)$) and positive (by primitivity). Moreover, one might always choose one component of ℓ to be 1, which, of course, also implies that all components are in $\mathbb{Q}(\lambda)$, or – by multiplying with an appropriate number – that all components are in $\mathbb{Z}[\lambda]$ or $\mathfrak{o}_{\mathbb{Q}(\lambda)}$.

Remark 6.8. We say that σ is a *substitution of constant length* if there is an $N \in \mathbb{N}$ such that $\#\sigma(i) = N$ for all $i \in \mathcal{A}$. In this case, the eigenvector to $\lambda = N$ is (up to multiplicative constant) given by $\ell = (1, \dots, 1)$ (also, the IFS $\Theta^{\#}$ reads $[0\,,1] = \frac{1}{N}[0\,,1] \cup \frac{1}{N}[0\,,1] + \frac{1}{N} \cup \ldots \cup \frac{1}{N}[0\,,1] + \frac{N-1}{N}$ for all components). Consequently, each $\varLambda_i$ is a subset of $\mathbb{Z}$. We will look at substitutions of constant length in Chapter 6b.

Remark 6.9. We can also define prototiles $A_i' = [-\ell_i\,,0]$, and define the tile substitution by

$$[-\ell_j\,,0] + x \mapsto \big([-\ell_i\,,0] - l'(w) + \lambda x \mid iw \triangleright \sigma(j)\big)\,,$$

where $u \triangleright v$ denotes that u is a suffix of v, *i.e.*, $v = \ldots u$.

One then obtains a representable substitution multi-component Delone set $\underline{\varLambda}'$. In fact, the tiling $\underline{\varLambda}' + \underline{A}'$ coincides with $\mathcal{T} = \underline{\varLambda} + \underline{A}$, the only difference is that the sets $\varLambda_i$ now are the positions of the right endpoints of the intervals in the equivalence class $[A_i] = [A_i']$.

So far, the only thing we are missing to apply the results at the end of Section 5.4 about the interplay between repetitivity, legality and representability is FLC. This will be the issue of the next section. Here, we note the following consequence of Corollary 5.117.

Corollary 6.10. *Let $\underline{\varLambda}$ be the primitive substitution multi-component Delone set in $\mathbb{R}$ associated to a sequence u, which is a fixed point of a primitive substitution σ, as above. If $\underline{\varLambda}$ is an FLC set, then $\underline{\varLambda}$ is a repetitive UCF set and the point set dynamical system $(\mathbb{X}(\underline{\varLambda})\,,\mathbb{R})$ is strictly ergodic.* □.

This statement should be compared to the following.

Corollary 6.11. [299, Theorems V.2 & V.13, Props. V.9 & V.10 and Corollary V.14], [300, Results 5.1.3.3 & 5.1.3.5] and [298, Prop. 1.2.3 & Theorem 1.2.7 & Propositions 5.1.13 & 5.4.4] *If σ is primitive, any of its periodic points u is a* repetitive *sequence, i.e., every word occuring in u occurs in an infinite number of positions with bounded gaps. Consequently, the symbolic dynamical system $(\mathbb{X}(u)\,,S)$ is minimal (and independent of the chosen periodic point u). In fact, it is even strictly ergodic, wherefore the frequency of any word in u is positive. In particular, the frequencies of the letters are given by the coordinates of the (right) PF-eigenvector of $\boldsymbol{S}\sigma$, normalised in such a way that the sum of its coordinates equals* 1. □

Remark 6.12. One can think of Corollary 6.11 as Corollary 6.10 where we "represent" $\underline{\varLambda}$ not with the intervals $[0\,,\ell_i]$ but always with $[0\,,1]$. However, the situation is somehow more delicate, since in Corollary 6.10 we have a continuous dynamical system, while in Corollary 6.11 we have a discrete dynamical system. We refer to Remark 7.40 for a further discussion.

Remark 6.13. The number of fixed points of σ^m is computable with the help of the *dynamical zeta-function*, see [13, Sections 9 & 10] and [239, Section 6.4]. For each $m = 1, 2, \ldots$ let N_m denote the number of fixed points of σ^m. We set

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} z^m\right),$$

at least formally. We recover the number N_m by the first m derivatives of $\log(\zeta(z))$ at $z = 0$.

Note that the number of fixed points of σ^m which are not fixed points for σ^n for any $n < m$ is given by $M_m = N_m - \sum_{n<m,\, n|m} M_n$. Also, the numbers M_m are multiples of m, since substitutions by σ^n for $1 < n < m$ yield different fixed points, *i.e.*, we have an *inflation orbit* of length m. So the number of different inflation orbits is given by M_m/m, cf. [29, 168]. Here, an inflation orbit is the orbit under σ in $\mathcal{A}^{\mathbb{Z}}$. So, we actually define a symbolic dynamical system for substitution sequences with unique composition property (*i.e.*, for aperiodic sequences by Proposition 5.120), where the action is now given by the substitution σ. Thus, we look at the topological dynamical system $(\mathbb{X}(u)\,, \sigma)$ (respectively $(\mathbb{X}(\underline{\Lambda})\,, \sigma)$ where the action is given by the associated tile substitution), also see the books [298, 299] and references therein.

The dynamical zeta-function is calculated by two matrices corresponding to two "types" of periodic points of σ. For a substitution σ, let $\sigma(i) = i_1 \dots i_k$ with $k = \#\sigma(i)$. In connection with Remark 6.4, we use the notation $\sigma^{(+)}(i) = i_1$ and $\sigma^{(-)}(i) = i_k$ for the first respectively last symbol of the substitute $\sigma(i)$. Then, with the notation of Remark 6.4, one has $\left(\sigma^{\check{k}}\right)^{(-)}(b_i)\left(\sigma^{\check{k}}\right)^{(+)}(a_i) = b_i\, a_i$. We might calculate this number $\check{k}$ as follows: Consider all two-letter words in the fixed point u (respectively, in the "language" defined *via* σ); we denote the set of these two-letter words by $\mathcal{W}_2$. We note that one has $\operatorname{card}\mathcal{W}_2 \le (\operatorname{card}\mathcal{A})^2$. Let $\boldsymbol{A}_0$ be the $\operatorname{card}\mathcal{W}_2 \times \operatorname{card}\mathcal{W}_2$-matrix obtained as follows: the component $(\boldsymbol{A}_0)_{ab,cd}$ equals 1 if $\sigma^{(-)}(a)\sigma^{(+)}(b) = cd$ and 0 otherwise. Then, the statement in Remark 6.4 can be reformulated as: there exists a $\check{k} \in \mathbb{N}$ such that $\operatorname{tr}(\boldsymbol{A}_0^{\check{k}}) > 0$, *i.e.*, $\operatorname{tr}(\boldsymbol{A}_0^m)$ gives the number of fixed points of the form $\dots a\dot{b} \dots$ of σ^m.

Using the representation with natural lengths, $\boldsymbol{A}_0$ can be used to calculate periodic points under the tile substitution where the origin 0 is a boundary point of the tiles associated with the corresponding letters. But there are also periodic points of $(\mathbb{X}(\underline{\Lambda})\,, \sigma)$ where the origin is an interior point of some tile (note that the origin is either an interior point or a boundary point for *all* tilings in the corresponding inflation orbit). The number of these periodic points is calculated as follows: We denote the set of all three-letter words of u by $\mathcal{W}_3$. Then, we define the $\operatorname{card}\mathcal{W}_3 \times \operatorname{card}\mathcal{W}_3$-matrix $\boldsymbol{A}_1$ as

$$\boldsymbol{A}_1 = \Big(\operatorname{card}\{k \mid xyz = i_k i_{k+1} i_{k+2} \text{ where } \sigma^{(-)}(a)\sigma(b)\sigma^{(+)}(c) = i_1 \dots i_k i_{k+1} i_{k+2} \dots i_m\}\Big)_{abc,xyz},$$

i.e, $(\boldsymbol{A}_1)_{abc,xyz}$ is the number of occurrences of xyz in $\sigma^{(-)}(a)\sigma(b)\sigma^{(+)}(c)$. However, $\operatorname{tr}(\boldsymbol{A}_1^m)$ not only gives the number of fixed points of σ^m such that the origin is in each case an interior point of some tile, but also includes the above fixed points (given by $\operatorname{tr}(\boldsymbol{A}_0^m)$), which are here counted twice.

Thus the number of periodic points for σ is given by $N_m = \operatorname{tr}(\boldsymbol{A}_1^m) - \operatorname{tr}(\boldsymbol{A}_0^m)$. Elementary manipulation in the zeta-function yields

$$\zeta(z) = \frac{\det(\boldsymbol{E} - z\boldsymbol{A}_0)}{\det(\boldsymbol{E} - z\boldsymbol{A}_1)} \tag{6.3}$$

(also see [13, Sections 9 & 10], where graphs are used to calculate the matrices $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$).

In [29, 167, 168], the dynamical zeta-function is used to determine so-called Wyckoff positions of the lattice $\tilde{L}$ of a model set with CPS $(G\,, H\,, \tilde{L})$; it is therefore connected to the so-called "torus parameterisation", see Section 7.3. For more on the dynamical zeta-function, also see [100, 282][2]. Note the zeta-function in Equation (6.3) is a rational function. This indicates that $(\mathbb{X}(u)\,,\sigma)$ is related to a sofic shift (see [239, Theorem 6.4.8]). We will have a closer look at this relationship in Section 7.5.2.

To clarify the concepts in this remark, we give an example: the Fibonacci substitution $\sigma = \sigma_{\text{Fib}}$ given by $a \mapsto ab$, $b \mapsto a$, also see Section 6.10.1. One obtains $\mathcal{W}_2 = \{aa, ab, ba\}$ and $\mathcal{W}_3 = \{aab, baa, bab, aba\}$. This yields

$$\begin{array}{rcl} \sigma^{(-)}(a)\sigma^{(+)}(a) & = & ba \\ \sigma^{(-)}(a)\sigma^{(+)}(b) & = & ba \\ \sigma^{(-)}(b)\sigma^{(+)}(a) & = & aa \end{array} \qquad \boldsymbol{A}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\begin{array}{rcl} \sigma^{(-)}(a)\sigma(a)\sigma^{(+)}(b) & = & baba \\ \sigma^{(-)}(b)\sigma(a)\sigma^{(+)}(a) & = & aaba \\ \sigma^{(-)}(b)\sigma(a)\sigma^{(+)}(b) & = & aaba \\ \sigma^{(-)}(a)\sigma(b)\sigma^{(+)}(a) & = & baa \end{array} \qquad \boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We therefore obtain the zeta-function $\zeta(z) = \frac{1-z}{1-z-z^2}$. We note that $\operatorname{tr}(\boldsymbol{A}_0^{2m}) = 2$ while $\operatorname{tr}(\boldsymbol{A}_0^{2m+1}) = 0$, which corresponds to the following two fixed points of σ^2 (respectively, the following inflation orbit of period 2 of σ):

$$\ldots baabaababaabaab\dot{a}baababaabaab \ldots$$
$$\ldots baabaababaababa\dot{a}baababaabaab \ldots$$

We note that

$$\log(\zeta(z)) = \frac{2}{2}z^2 + \frac{3}{3}z^3 + \frac{6}{4}z^4 + \frac{10}{5}z^5 + \frac{17}{6}z^6 + \ldots$$

where the three fixed points of σ^3 (one inflation orbit) are

$$\ldots baababaababaab\ddot{a}ababaababaabaa \ldots$$
$$\ldots aababaabaababa\ddot{a}babaabaababaab \ldots$$
$$\ldots abaababaabaaba\ddot{b}aabaababaababa \ldots$$

Here "∵" indicates that the origin is an interior point of the corresponding tile. We note that this also shows that $(\mathbb{X}(u)\,,\sigma)$ is not minimal (and also not uniquely ergodic).

6.2. Pisot-Vijayaraghavan Numbers and Pisot Substitutions

Definition 6.14. An algebraic integer $\lambda > 1$ is a *Pisot-Vijayaraghavan number*, or *PV-number* for short, if $|\lambda'| < 1$ holds for every algebraic conjugate λ' of λ with $\lambda' \neq \lambda$. An algebraic integer $\lambda > 1$ is a *Salem number*, if $|\lambda'| \leq 1$ holds for every algebraic conjugate λ' of λ with $\lambda' \neq \lambda$ and λ is not a PV-number (*i.e.*, we have $|\lambda'| = 1$ for at least one[3] algebraic conjugate of λ).

[2] Note the radius of convergence of the $\zeta(z)$ is $\exp(-h)$ where h is is the topological entropy of the substitution [282, Proposition 5.1]. One can also prove a "prime orbit theorem" (see [282, Theorem 6.9]), *i.e.*, the number of periodic orbits of period less than or equal to x is asymptotic to $\exp(hx)/hx$ as $x \to \infty$. Also note that by the theorem of Bowen-Sinai (see [100, Theorem 2.2] and [244, Theorem IV.9.1]) these periodic orbits are "typical" if we look at the integration of continuous maps $\mathbb{X}(u) \to \mathbb{R}$ with respect to the corresponding invariant measure of the measure theoretic dynamical system.

[3] One can show that equality holds for all but one algebraic conjugate, wherefore Salem numbers have even degree and at least degree 4.

Remark 6.15. Pisot-Vijayaraghavan numbers first appeared in the study of distributions modulo 1, see [290, 378] (also see Lemma 7.67). For a recent reference on this subject, see [68].

The reason why we look at Pisot and Salem numbers is the following.

Proposition 6.16. [218, Theorem 4.1(iii)] and [257, Theorem 6] *Let Λ be a Delone set in $\mathbb{R}^d$ such that $\eta\Lambda \subset \Lambda$ for a real number $\eta > 1$. If Λ is a Meyer set, then η is a Pisot-Vijayaraghavan number or a Salem number.* □

Remark 6.17. It is noted in [218, Remarks to Section 4] (without proof) that for a repetitive Meyer set, Salem numbers can be ruled out, also compare Proposition 6.42 and Example 6.43.

Obviously, the Delone sets in $\mathbb{R}$ we considered in the last section admit this property of the previous proposition. Moreover, we are interested in substitutions σ which can be described as model sets, and therefore, in particular, as Meyer sets.

Definition 6.18. We say that σ is a *Pisot substitution* (or a *substitution of Pisot type*), if the substitution matrix $\boldsymbol{S}\sigma$ has dominant (simple) eigenvalue $\lambda > 1$ and all other eigenvalues λ' satisfy $0 < |\lambda'| < 1$. Furthermore, we call[4] the substitution of Pisot type *unimodular* if $|\det \boldsymbol{S}\sigma| = 1$.

Remark 6.19. Trivially, all natural numbers $n \geq 2$ are Pisot-Vijayaraghavan numbers with minimal polynomial $x - n$. The only possible associated substitutions σ for these PV-numbers are the periodic ones over one symbol a given by $\sigma(a) = a \ldots a = a^n$ (therefore $\boldsymbol{S}\sigma = (n)$). Trivially, the sequences defined by such a substitution are model sets, see Example 5.156.

We first note some consequences of this definition.

Lemma 6.20. [88] and [298, Proposition 1.2.8] *The characteristic polynomial $p(x) = \det(x \cdot \boldsymbol{E} - \boldsymbol{S}\sigma)$ is irreducible over $\mathbb{Q}$ (irreducible and normed over $\mathbb{Z}$; so $p(x)$ is also the minimal polynomial of λ, i.e., $p(x) = \mathrm{Irr}(\lambda, \mathbb{Q}, x)$).*

Proof. We have $p(x) \in \mathbb{Z}[x]$ and p is irreducible over $\mathbb{Q}$ iff it is irreducible over $\mathbb{Z}$. Suppose $p(x)$ is reducible, *i.e.*, $p(x) = q(x) \cdot r(x)$ with $q(x), r(x) \in \mathbb{Z}[X]$ and $q(x), r(x) \notin \mathbb{Z}$. Since $\lambda > 1$ is a simple root of $p(x)$, it is either a root of $q(x)$ or of $r(x)$. Suppose λ is a root of $q(x)$. Then all roots of $r(x)$ must have modulus smaller than 1, wherefore the modulus of their product is smaller than 1, so its constant term is 0 since $r(x) \in \mathbb{Z}[X]$. But then also $p(x)$ has root 0, contradicting the assumption that 0 is not an eigenvalue of $\boldsymbol{S}\sigma$. □

Since $p(x) = \det(x \cdot \boldsymbol{E} - \boldsymbol{S}\sigma)$ is irreducible and equals the minimal polynomial $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$, the eigenvalues of $\boldsymbol{S}\sigma$ are given by the algebraic conjugates of λ and are all simple (and algebraic integers of degree n).

Lemma 6.21. [88] and [298, Prop. 1.2.8] *A substitution σ of Pisot type with* $\operatorname{card} \mathcal{A} > 1$ *is not of* constant length*, i.e., there is a pair $i, j \in \mathcal{A}$ such that $\#\sigma(i) \neq \#\sigma(j)$.*

Proof. If the substitution is of constant length $\bar{\ell}$, then the image of each letter under the substitution contains exactly $\bar{\ell}$ letters and $\bar{\ell}$ is an eigenvalue for the left eigenvector $(1, \ldots, 1)$. In this case, the characteristic polynomial $p(x)$ is reducible over $\mathbb{Q}$ (hence $\mathbb{Z}$). □

[4] In this case, the square of $\boldsymbol{S}\sigma$ belongs to the *unimodular linear group* $SL(n, \mathbb{Z})$, hence the term "unimodular", see [20, Definition 3].

Lemma 6.22. [88] and [298, Theorem 1.2.9] *A substitution σ of Pisot type is primitive.*

Proof. Since the characteristic polynomial $p(x)$ is irreducible over $\mathbb{Q}$, the substitution matrix $\boldsymbol{S}\sigma$ is also irreducible (by contraposition). Now, the statement follows from the Perron-Frobenius theorem (see Proposition 4.69, compare to Lemma 5.85): A matrix $\boldsymbol{M}$ is primitive iff it is irreducible and has exactly one root with maximal modulus. □

We get the lengths ℓ_i of the intervals $A_i = [0\,, \ell_i]$ as components of the left eigenvector of $\boldsymbol{S}\sigma$ for the eigenvalue λ, *i.e.*, $\ell\,(\boldsymbol{S}\sigma) = \lambda \cdot \ell$ where $\ell = (\ell_1, \dots, \ell_n)$. Since $\boldsymbol{S}\sigma$ is an integer matrix, this eigenvector equation has solutions for $\ell_i \in \mathbb{Q}(\lambda)$, wherefore (by suitable multiplication) we may assume that $\ell_i \in \mathbb{Z}[\lambda]$. We note that $\mathbb{Z}[\lambda] = \{c_1\lambda^{n-1} + \ldots + c_n\lambda^0 \mid c_i \in \mathbb{Z}\}$ is an order in $\mathbb{Q}(\lambda)$ but not necessarily the maximal order $\mathfrak{o}_{\mathbb{Q}(\lambda)}$, see Lemma 3.91 and observe $\mathfrak{o}_{\mathbb{Q}(\sqrt{5})} \neq \mathbb{Z}[\sqrt{5}]$. Nevertheless, $\mathbb{Z}[\lambda]$ is a $\mathbb{Z}$-module of rank n: the set $\{1, \lambda, \lambda^2, \dots, \lambda^{n-1}\}$ is a basis. And we may assume

$$\mathcal{L} = \langle \ell_1, \dots, \ell_n \rangle_{\mathbb{Z}} \subset \mathbb{Z}[\alpha]$$

and $\ell_i = g_i(\lambda)$, where $g_i(x) \in \mathbb{Z}[x]$ and $\deg(g_i) \leq n-1$.

We recall Definition 3.88 and observe the following.

Lemma 6.23. *$\mathcal{L}$ is a complete module in $\mathbb{Q}(\lambda)$, i.e., $\mathcal{L}$ has rank n over $\mathbb{Z}$. Consequently, the numbers ℓ_i are rationally independent.*

Proof. We have $0 \neq \ell_1 \in \mathcal{L} \subset \mathbb{Z}[\lambda]$, in particular, $\mathcal{L} \neq \{0\}$. For all $x \in \mathcal{L}$ we also have $\lambda x \in \mathcal{L}$ as a consequence of the eigenvector equation $\ell\,(\boldsymbol{S}\sigma) = \lambda \cdot \ell$. Then, $\ell_1, \lambda\ell_1, \lambda^2\ell_1, \dots, \lambda^{n-1}\ell_1$ are all in $\mathcal{L}$, and are obviously linearly independent over $\mathbb{Q}$ (by the irreducibility). So, $\langle \ell_1, \lambda\ell_1, \dots, \lambda^{n-1}\ell_1 \rangle_{\mathbb{Z}} \subset \langle \ell_1, \dots, \ell_n \rangle_{\mathbb{Z}} = \mathcal{L} \subset \mathbb{Z}[\alpha]$ and $\mathcal{L}$ has rank n. □

We note that one cannot always achieve that $\mathcal{L} = \mathbb{Z}[\lambda]$, see Remark 6.36 at the end of this section. The following corollary should be compared with the statement that for any unit ϵ belonging to the order $\mathfrak{o}(M)$ of a complete module M, one has $\epsilon\, M = M = \epsilon^{-1}\, M$, see [73, p. 106].

Corollary 6.24. *If the PV-number λ is a unit, we have $\lambda\,\mathcal{L} = \mathcal{L} = \frac{1}{\lambda}\mathcal{L}$.*

Proof. The proof of the previous lemma shows that $\lambda\,\mathcal{L} \subset \mathcal{L}$. If λ is a unit, one has $\frac{1}{\lambda} \in \mathbb{Z}[\lambda]$, wherefore $\frac{1}{\lambda}\,\mathcal{L} \subset \mathcal{L}$. But this proves the claim. □

Since $\mathcal{L}$ is a group and since $\lambda\,\mathcal{L} \subset \mathcal{L}$, one also the following statement.

Corollary 6.25. *For a Pisot substitution with PV-number λ, one has $\mathbb{Z}[\lambda] \cdot \mathcal{L} = \mathcal{L}$.* □

Lemma 6.26. *Any fixed point of a Pisot substitution σ (with* $\operatorname{card} \mathcal{A} \geq 2$*) is aperiodic.*

Proof. By Corollary 6.11, the frequency of the letters are given by the coordinates of the left eigenvector ϱ of $\boldsymbol{S}\sigma$. The eigenvector equation $(\boldsymbol{S}\sigma)\,\varrho = \lambda \cdot \varrho$ establishes that (since λ is irrational) not all components of ϱ are rational.

But on the other hand, if a sequence u is periodic then the frequencies of the letters are rational (in fact, the denominator of a frequency divides the period length in this case). □

We now look at $\mathcal{L}' = \bigcup_{i=1}^{n} \varLambda_i - \varLambda_i$, where $\underline{\varLambda}$ is a representation with natural intervals of a fixed point u of a Pisot substitution. As for $\mathcal{L}$, we obtain the following statement.

Lemma 6.27. *$\mathcal{L}'$ is a complete module in $\mathbb{Q}(\lambda)$.*

Proof. We first note that $\mathcal{L}'$ is a subgroup of the free commutative group $\mathcal{L}$ and therefore also free; moreover, the cardinality of basis for $\mathcal{L}'$ is less than or equal to the cardinality of the basis for $\mathcal{L}$, see [226, Theorem I.7.3]. If $0 \neq x \in \varDelta'$ (say, $x \in \varLambda_i - \varLambda_i$), then by the substitution process we also have $\lambda x \in \mathcal{L}'$ (namely, $\lambda x \in \varLambda_j - \varLambda_j$ for every j with $\varTheta_{ji} \neq \varnothing$) and the higher iterates $\lambda^k x \in \mathcal{L}'$. Thus, as in the proof of Lemma 6.23, we conclude that $\mathcal{L}'$ has rank n. □

Thus, $\mathcal{L}$ and $\mathcal{L}'$ are free $\mathbb{Z}$ modules and both of rank n. Therefore, they are isomorphic (see [226, Corollary III.4.3]) and, given a basis of $\mathcal{L}$, there is a (nonsingular) $n \times n$-matrix $\boldsymbol{A}$ over $\mathbb{Z}$ that maps this basis of $\mathcal{L}$ to a basis of $\mathcal{L}'$ (as consequence of the *elementary divisors theorem*, see [226, Theorem III.7.8]). In particular, we have the following corollary.

Corollary 6.28. *The height group $\mathcal{L}/\mathcal{L}'$ is a finite group with* $\operatorname{card} \mathcal{L}/\mathcal{L}' = |\det \boldsymbol{A}|$. □

Remark 6.29. We can also prove similar statements for each $\mathcal{L}_i$:

- Each $\mathcal{L}_i$ is a complete module in $\mathbb{Q}(\lambda)$: Again, $\mathcal{L}_i$ is a subgroup of the free group $\mathcal{L}$ and therefore free. If $x \in \varDelta_i$, then by the substitution process and primitivity, we also have $\lambda^k \cdot x \in \varDelta_i$ for all $k \geq n^2 - 2n + 2$ (compare Remark 4.68), and the claim follows.
- The factor groups $\mathcal{L}'/\mathcal{L}_i$ (respectively, $\mathcal{L}/\mathcal{L}_i$) are finite groups.

We emphasise that so far we have not made use of all properties of a Pisot substitution; in fact, one only needs that the one-dimensional substitution σ is primitive, also compare to the example in Section 6c.2.

We have now derived the following setting: $\mathcal{T} = \underline{\varLambda} + \underline{A}$ is a tiling in $\mathbb{R}$ with n different prototiles $A_i = [0\,, \ell_i]$ where the corresponding interval lengths ℓ_i are linearly independent over $\mathbb{Q}$. Moreover, we have $0 \in \operatorname{supp} \underline{\varLambda}$ and $\mathcal{L} = \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}} = \{ \sum_{i=1}^{n} m_i \cdot \ell_i \mid m_i \in \mathbb{Z} \}$, and since $\ell_1, \ldots, \ell_n$ are linearly independent over $\mathbb{Q}$, the set $\{\ell_1, \ldots, \ell_n\}$ is a basis of $\mathcal{L}$, wherefore $\operatorname{rank}(\mathcal{L}) = n$ and $\mathcal{L}$ is a free Abelian group (or a free module over $\mathbb{Z}$) of rank n. We define two maps.

Definition 6.30. We define the *Parikh map* (see [87]) $\vartheta : \mathcal{L} \to \mathbb{Z}^n$ by

$$x = \sum_{i=1}^{n} m_i \cdot \ell_i \quad \mapsto \quad \vartheta(x) = (m_i)_{i=1}^{n}$$

Its values are the *Parikh vectors* $\vartheta(x)$.

Secondly, we define a map $\chi : \mathcal{L} \to \mathbb{Z}$ by

$$x = \sum_{i=1}^{n} m_i \cdot \ell_i \quad \mapsto \quad \chi(x) = \sum_{i=1}^{n} m_i = \langle \vartheta(x)\,, (1, \ldots, 1) \rangle.$$

Multiplication of ℓ_i with λ yields $\lambda \cdot \ell_i = \sum_{j=1}^{n} [\boldsymbol{S}\sigma]_{ji} \cdot \ell_j$, wherefore – using the Parikh map – one obtains $\vartheta(\lambda \cdot x) = (\boldsymbol{S}\sigma)\vartheta(x)$.

The map χ is a linear surjection, and $\mathcal{M} = \ker(\chi)$ is a $\mathbb{Z}$-module of rank $n - 1$ (again, because of the linear independence of the ℓ_i over $\mathbb{Q}$, or, alternatively, because $\chi(\mathcal{L}) = \mathbb{Z}$ and $\operatorname{rank}(\mathbb{Z}) = 1$):

$$\begin{aligned} \mathcal{M} &= \{x \in \mathcal{L} \mid \chi(x) = 0\} = \left\{ \sum_{i=1}^{n} m_i \cdot \ell_i \;\middle|\; \sum_{i=1}^{n} m_i = 0 \right\} \\ &= \langle \ell_1 - \ell_2, \ell_2 - \ell_3, \ldots, \ell_{n-1} - \ell_n \rangle_{\mathbb{Z}} = \langle \ell_2 - \ell_1, \ell_3 - \ell_1, \ldots, \ell_n - \ell_1 \rangle_{\mathbb{Z}} \end{aligned}$$

This $\mathcal{M}$ is a free Abelian group of rank $n-1$ and one obtains the (short) exact sequence:

$$0 \longrightarrow \mathcal{M} \xrightarrow{\text{id}} \mathcal{L} \xrightarrow{\chi} \mathbb{Z} \longrightarrow 0$$

Obviously, since there are only free modules involved (in particular, $\mathbb{Z}$ is free), this exact sequence splits, compare [170, Section I.4] and [226, Section III.§4]. Therefore, the Abelian groups $\mathcal{L}$ and $\mathcal{M} \oplus \mathbb{Z}$ are isomorphic.
The following lemma is obvious. Here, we indicate by $\dot{\bigcup}$ that the union is disjoint.

Lemma 6.31. *We have* $\mathcal{L} = \dot{\bigcup}_{n\in\mathbb{Z}} \chi^{-1}(n)$*, with* $\chi^{-1}(n) = \{x \in \mathcal{L} \mid \chi(x) = n\}$*. Also, we get* $\mathcal{M} = \chi^{-1}(0)$. □

Lemma 6.32. *For all* $n \in \mathbb{Z}$*, we have* $\operatorname{card}(\operatorname{supp} \underline{\Lambda} \cap \chi^{-1}(n)) = 1$.

Proof. We note that the union $\operatorname{supp} \underline{\Lambda} = \bigcup_{i=1}^{n} \Lambda_i$ is disjoint (since the interval lengths ℓ_i are positive). If $x \in \operatorname{supp} \underline{\Lambda} \subset \mathcal{L}$, then $x = \sum_{i=1}^{n} m_i \cdot \ell_i$, where $m_i \in \mathbb{Z}$. Starting at 0 and moving to the right (left) tile by tile, one obtains a bi-infinite sequence $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ with[5] $x_n \in \mathcal{L}$ and $\chi(x_n) = n$ for all $n \in \mathbb{Z}$. □

Proposition 6.33. *If* $\underline{\Lambda}$*,* $\mathcal{L}$ *and* $\mathcal{M}$ *are given as above,* $\mathcal{L}$ *admits the partitions*

$$\mathcal{L} = \dot{\bigcup_{x \in \operatorname{supp} \underline{\Lambda}}} (x + \mathcal{M})$$

and

$$\mathcal{L} = \dot{\bigcup_{y \in \mathcal{M}}} (y + \operatorname{supp} \underline{\Lambda}).$$

We also use the notation $\mathcal{L} = (\operatorname{supp} \underline{\Lambda}) \dot{+} \mathcal{M}$.

Proof. The first follows from Lemma 6.32, the linearity of χ and Equation (6.4) in Footnote 5.

The second follows from the observation that $\mathcal{L} = (\operatorname{supp} \underline{\Lambda}) + \mathcal{M} = \{x + y \mid x \in \operatorname{supp} \underline{\Lambda},\ y \in \mathcal{M}\}$ gives a unique representation, *i.e.*, if $x + y = x' + y'$ for $x, x' \in \operatorname{supp} \underline{\Lambda}$ and $y, y' \in \mathcal{M}$ then $x = x'$ and $y = y'$. So we have $\mathcal{L} = (\operatorname{supp} \underline{\Lambda}) \dot{+} \mathcal{M}$.

Alternatively, both follow from $\mathcal{L} \cong \mathcal{M} \oplus \mathbb{Z}$ by identifying $\mathbb{Z}$ with $\operatorname{supp} \underline{\Lambda}$ *via* Lemma 6.32 (and therefore we equip $\operatorname{supp} \underline{\Lambda}$ with the following group structure: For x_i, $x_j \in \operatorname{supp} \underline{\Lambda}$ set $x_i \boxplus x_j = x_{i+j}$. This group structure also appears in [119, Section 3.5]). □

For the previous statements, we actually only used that $\operatorname{card}(\operatorname{supp} \underline{\Lambda} \cap \chi^{-1}(n)) = 1$ for all $n \in \mathbb{Z}$. Especially, the rational independence of the ℓ_i's ensures that the map χ is well-defined, since the number of different tiles $A_i = [0, \ell_i]$ equals the rank of $\mathcal{L}$.

We now infer further conclusions from this linear independence. For a Pisot substitution σ, not only the ℓ_i's – as components of the left PF-eigenvector of $\boldsymbol{S}\sigma$ – are rationally independent, but also the frequencies ϱ_i of the letters, *i.e.*, the (normalised) components of the right

[5] So, representing $\underline{\Lambda}$ that way, we have

$$\operatorname{supp} \underline{\Lambda} \cap \chi^{-1}(n) = \{x_n\}. \tag{6.4}$$

Note that for $x = \sum_{i=1}^{n} m_i \cdot \ell_i \in \operatorname{supp} \underline{\Lambda}$ either all m_i's are nonnegative or nonpositive.

PF-eigenvector of $\boldsymbol{S}\sigma$, are linearly independent. Using this, which is a consequence of the irreducibility of the characteristic polynomial $\det(x\,\boldsymbol{E} - \boldsymbol{S}\sigma)$, we now show[6] that $\mathcal{L}/\mathcal{L}'$ is trivial (we observe that we actually only assume primitivity of the substitution and irreducibility of the characteristic polynomial).

Lemma 6.34. *Let σ be a Pisot substitution. Let $\underline{\Lambda}$ be the representation with natural lengths of any fixed point u of σ. Assume that $\underline{\Lambda}$ is FLC. Then, $\mathcal{L}/\mathcal{L}'$ is trivial, i.e., $\mathcal{L} = \mathcal{L}'$.*

Proof. To prove this statement, we calculate the frequency that a point in $\underline{\Lambda}$ belongs to a certain coset of $\mathcal{L}/\mathcal{L}'$ in two different ways. For this, we define the *coset frequency*

$$\begin{aligned} d_x(\underline{\Lambda}) &= \lim_{m\to\infty} \frac{1}{2\,m+1} \operatorname{card}\{\omega_i(y) \subset \underline{\Lambda} \mid\ 1 \le i \le n,\ y \in x + \mathcal{L}',\ -2\,m \le \chi(y) \le 2\,m\} \\ &= \lim_{m\to\infty} \frac{1}{2m+1} \operatorname{card}\{k \mid -m \le k \le m,\ \chi^{-1}(k) \in x + \mathcal{L}'\}. \end{aligned}$$

(In plain words, the right hand side counts the number of symbols in the fixed point u of σ between position $-m$ and m, such that in their representation with natural lengths these symbols belong to the coset $x + \mathcal{L}'$; then one takes the limit $m \to \infty$).

Since one has $\Lambda_i \subset \mathcal{L}' + t_i$ for any $t_i \in \Lambda_i$ (see p. 143, each Λ_i belongs to one coset), we can easily calculate this coset frequency as

$$d_x(\underline{\Lambda}) = \sum_{i\in\mathcal{A}:\ \Lambda_i \subset x+\mathcal{L}'} \varrho_i,$$

where ϱ_i is the frequency of the letter i.

We now observe the following: Since we use the representation with natural intervals, and we know that $\omega_i(x) \subset \Lambda_i$ (*i.e.*, that there is a point in Λ_i at position x), we also know that there is a point $x + \ell_i$ in some (but exactly one) Λ_j. In other words, given an $\omega_i(x) \subset \underline{\Lambda}$ with $\chi(x) = k$ (*i.e.*, the k-th symbol of the fixed point u is an i and $\chi^{-1}(k) = x$), we know that $\chi^{-1}(k+1) = x + \ell_i$, but we do not know what letter the $k+1$-th symbol is. But for the cosets, this means that if $\Lambda_i \subset x + \mathcal{L}'$ and if ij is a legal two letter word of u, then $\Lambda_j \subset x + \ell_i + \mathcal{L}'$. Thus, all successors j of the symbol i in u belong to the same coset. Consequently, we can determine the coset frequency using predecessors:

$$d_x(\underline{\Lambda}) = \sum_{kj\in u:\ \Lambda_k \subset x-\ell_k+\mathcal{L}'} \varrho_k.$$

Thus, for each x (more precisely, for each representative $x + \mathcal{L}/\mathcal{L}'$) we have two equations for $d_x(\underline{\Lambda})$. Obviously, if all Λ_i belong to the same coset $x + \mathcal{L}'$, we have $d_x(\underline{\Lambda}) = 1$ and $\operatorname{supp}\underline{\Lambda} = \bigcup_i \Lambda_i \subset x + \mathcal{L}' = \mathcal{L}' = \mathcal{L}$ (since we assume that $0 \in \operatorname{supp}\underline{\Lambda}$). Otherwise, there is at least one two letter word $kj \in u$ such that Λ_k and Λ_i belong to different cosets, say $x_k + \mathcal{L}'$ and $x_j + \mathcal{L}'$. Since all numbers ϱ_i are positive, the equations for $d_{x_j}(\underline{\Lambda})$ establish a (non-trivial) linear dependence amongst the ϱ_i. Consequently, this case cannot occur for Pisot substitutions. □

[6] It seems that the following statement (with, maybe, a similar proof, compare [50, Lemma 12.4]) is basically the statement of [50, Theorem 12.1]. However, the notation and language used in that article is highly elusive, at least to us.

Remark 6.35. By the linear independence of the natural lengths, one can identify $\mathcal{L}$ and $\mathbb{Z}^n$ (as free groups). With this identification, the $\mathcal{L}_i$'s (and, in general, also $\mathcal{L}'$) are sublattices of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

It seems that often this identification between $\mathcal{L}$ and $\mathbb{Z}^n$ is used to derive statements about Pisot substitutions (especially see [298, Chapters 7 & 8] and references therein). However, it is our task in the next section to show that, by the construction of Section 5.3, one may in a natural way associate $\mathcal{L}$ with a (different) lattice, and then proceed from there. Also, compare Section 6.5.

We end this section with a remark about the relationship between $\boldsymbol{S}\sigma$ and $\mathcal{L}$.

Remark 6.36. Usually, we do not get $\mathcal{L} = \mathbb{Z}[\lambda]$ and λ even does not fix $\mathcal{L}$ as the following[7] matrices show:

$$\boldsymbol{M}^{(1)} = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{M}^{(2)} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

For both matrices one has (with $\det(\boldsymbol{M}^{(i)}) = 1$ and $\operatorname{tr}(\boldsymbol{M}^{(i)}) = 6$ for $i = 1, 2$) $\lambda = 3 + \sqrt{8} \approx 5.83$ and $\lambda' = 3 - \sqrt{8} \approx 0.17$. For $\boldsymbol{M}^{(1)}$ we get an eigenvector $(\ell_1^{(1)}, \ell_2^{(1)}) = (\lambda - 3, 1) = (2\sqrt{2}, 1)$ and therefore we have $\mathcal{L}^{(1)} = \{r\lambda + s \mid r, s \in \mathbb{Z}\} = \mathbb{Z}[\lambda]$. We observe that $\mathcal{L}^{(1)}$ is a lattice in $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{2})$ (see Definition 3.88) with basis $\{1, \lambda\}$ (respectively with basis $\{1, 2\sqrt{2}\}$). By Lemma 3.97, we calculate the volume of the fundamental domain of its geometric image in $\mathbb{R}^2$ as $\mu_{\mathbb{R}^2}\left(\mathrm{FD}(\tilde{\tilde{\mathcal{L}}}^{(1)})\right) = \mu_{\mathbb{R}^2}\left(\mathrm{FD}(\widetilde{\widetilde{\mathbb{Z}[\lambda]}})\right) = \sqrt{32} = 4\sqrt{2}$

We will now show that $\mathcal{L}^{(2)} \neq \mathbb{Z}[\lambda]$: For $\boldsymbol{M}^{(2)}$ the (left) eigenvector of the eigenvalue λ is given by the multiples of $(\frac{1}{2}\lambda - \frac{3}{2}, 1) = (\sqrt{2}, 1)$. So, we may multiply this vector with any number $a + b\sqrt{2} \in \mathbb{Q}(\lambda)$ (where $a, b \in \mathbb{Q}$) to obtain $(a\sqrt{2} + 2b, a + b\sqrt{2})$. We want to have $\ell_1^{(2)}, \ell_2^{(2)} \in \mathbb{Z}[\lambda]$, wherefore $a, b \in 2\mathbb{Z}$. Again, we calculate the volume of the fundamental domain as $\mu_{\mathbb{R}^2}\left(\mathrm{FD}(\tilde{\tilde{\mathcal{L}}}^{(2)})\right) = 2\sqrt{2}\,|a^2 - 2b^2|$. For any choice $a, b \in 2\mathbb{Z}$, one has $\mu_{\mathbb{R}^2}\left(\mathrm{FD}(\tilde{\tilde{\mathcal{L}}}^{(2)})\right) = 2\sqrt{2}\,|a^2 - 2b^2| > \mu_{\mathbb{R}^2}\left(\mathrm{FD}(\tilde{\tilde{\mathcal{L}}}^{(1)})\right)$.

We note that the maximal order here is $\mathfrak{o}_{\mathbb{Q}(\lambda)} = \mathbb{Z}[\sqrt{2}]$, which is spanned by the eigenvector $(\sqrt{2}, 1)$ of $\boldsymbol{M}^{(2)}$; but there is no eigenvector of $\boldsymbol{M}^{(1)}$ which spans $\mathbb{Z}[\sqrt{2}]$. Moreover, we note that this example also shows that $\mathbb{Z}[\lambda]$ is in general not the maximal order for a PV-number λ.

So, the invariants det and tr are not sufficient to characterise the lattice $\mathcal{L}$. For $\boldsymbol{M} \in SL(2, \mathbb{Z})$, Rademacher [302, Satz 7] defines another (besides det and tr) invariant Ψ of the equivalence class of similar matrices ($\boldsymbol{M}'$, $\boldsymbol{M}$ are similar if there exists a $\boldsymbol{U} \in SL(2, \mathbb{Z})$ such that $\boldsymbol{M}' = \boldsymbol{U}^{-1}\boldsymbol{M}\boldsymbol{U}$) by

$$\Psi(\boldsymbol{M}) = \Psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \frac{b}{d}, & \text{if } c = 0, \\ \frac{a+d}{c} - 12 \cdot \operatorname{sgn}(c) \cdot s(a, |c|) - 3 \cdot \operatorname{sgn}(c \cdot (a + d)), & \text{if } c \neq 0, \end{cases}$$

where the function s is defined by

$$s(\tilde{h}, \tilde{k}) = s(h, k) = \frac{h\,(k-1)\,(2k-1)}{6k} - \frac{k-1}{4} - \frac{1}{k}\sum_{i=0}^{k-1} i \cdot \left[\frac{i \cdot h}{k}\right],$$

[7] A similar example can be found in [208, Example 3].

where $h = \tilde{h}/\gcd(\tilde{h}, \tilde{k})$ and $k = \tilde{k}/\gcd(\tilde{h}, \tilde{k})$. We get $\Psi(\boldsymbol{M}^{(1)}) = -3$ and $\Psi(\boldsymbol{M}^{(2)}) = 0$ (note that $\Psi(\boldsymbol{M}) \in \mathbb{Z}$ and $\Psi(\boldsymbol{M}) = \Psi(-\boldsymbol{M}) = -\Psi(\boldsymbol{M}^{-1}) = \Psi(\boldsymbol{M}^t)$ for $\boldsymbol{M} \in SL(2, \mathbb{Z})$, see [302, Equations (12), (14) and (14a)] for these claims). However, note that for $(\boldsymbol{M}^{(i)})^t$ we get the same lattice[8] $\mathcal{L}^{(i)}$ as for $\boldsymbol{M}^{(i)}$, but $\Psi((\boldsymbol{M}^{(i)})^t) = 3 = -\Psi(\boldsymbol{M}^{(1)})$ (and $\Psi((\boldsymbol{M}^{(2)})^t) = 0 = \Psi(\boldsymbol{M}^{(2)})$).

Thus, the following question arises: Do det, tr and Ψ (or even $|\Psi|$) determine $\mathcal{L}$? We first observe that these three invariants of the equivalence class of similar matrices do not determine the equivalence class, *e.g.*, for the following matrices[9] we have $\det \boldsymbol{M}^{(i)} = 1$, $\operatorname{tr} \boldsymbol{M}^{(i)} = 18$ and $\Psi(\boldsymbol{M}^{(i)}) = 0$ for $i = 3, 4$ (and eigenvalues $9 \pm 4 \cdot \sqrt{5}$):

$$\boldsymbol{M}^{(3)} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{M}^{(4)} = \begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix}.$$

They are not similar: Otherwise, there would be a matrix $\boldsymbol{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in SL(2, \mathbb{Z})$ that solves the following linear system of equations (see [143, Section VI.8.1]) $\boldsymbol{U}\boldsymbol{M}^{(3)} - \boldsymbol{M}^{(4)}\boldsymbol{U} = \boldsymbol{0}$. This system is singular with solutions

$$u_{11} = \frac{1}{2}\, u_{21} + \frac{5}{2}\, u_{22} \qquad \text{and} \qquad u_{12} = \frac{1}{2}\, u_{21} + \frac{1}{2}\, u_{22}.$$

We recall that the numbers $u_{11}, \ldots, u_{22}$ are integers and that $\det \boldsymbol{U} = \frac{1}{2}\,(-u_{21}^2 + 5\, u_{22}^2)$. Thus, the condition $\det \boldsymbol{U} = 1$ yields $-u_{21}^2 + 5\, u_{22}^2 = 2$, which has no solution modulo 5.

Again, we show that we do not get lattices with the same volume: For $\boldsymbol{M}^{(3)}$ we obtain for the left eigenvector $(\frac{1+\sqrt{5}}{2}\,, 1)$, wherefore a possible lattice is given by the maximal order $\mathfrak{o}_{\mathbb{Q}(\sqrt{5})}$ with associated volume $\sqrt{5}$ of its fundamental domain. For $\boldsymbol{M}^{(4)}$, a left eigenvector is given (by the multiples of) $(\sqrt{5}\,, 1)$. As before, we may multiply this vector by $\frac{a}{2} + \frac{b}{2}\sqrt{5}$ (with $a, b \in \mathbb{Z}$), and obtain as possible volumes for the fundamental domain $\sqrt{5}\,\frac{1}{2}\,|a^2 - 5b^2|$, which – as our previous calculation shows – never equals $\sqrt{5}$.

Similar matrices, however, yield lattices with the same volumes of their fundamental domains and their left eigenvector are connected *via* the transformation matrix $\boldsymbol{U} \in SL(2\,, \mathbb{Z})$ which establishes the similarity, wherefore one can choose the same lattice for similar lattices. But since the equivalence class is not uniquely determined by the invariants det, tr and Ψ, this invariants do not determine the lattice.

6.3. Pisot Substitutions Generate Meyer Sets

In this section, we show that every Pisot substitution with $\operatorname{card} \mathcal{A} = n$ in its representation by natural intervals is a subset of a model set with internal space $\mathbb{R}^{n-1}$. Consequently, it is a multi-component Meyer set (see Corollary 5.14) and FLC.

By Lemma 6.23, $\mathcal{L}$ is complete module in $\mathbb{Q}(\lambda)$ for every Pisot substitution. Thus, recalling the definition of a geometric image (see Definition 3.94), we obtain the following statement from Lemma 3.97.

[8] We have: If (ℓ_1, ℓ_2) is an eigenvector for $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, then (ℓ_2, ℓ_1) is an eigenvector for $\boldsymbol{M}^t$.

[9] We have obtained these two matrices by the unique representation of a representant in the class described in [302, Section 8], where we here have looked for representants with $\Psi(\boldsymbol{M}) = 0$ and the same trace but different length ν of the representants. See also [302, Satz 8].

Corollary 6.37. *Let $\mathbb{Q}(\lambda)$ be an algebraic number field of degree n and signature $[r\,,s]$. Let $\mathcal{L}$ be a complete module in $\mathbb{Q}(\lambda)$. Then, its geometric image $\tilde{\tilde{\mathcal{L}}} \subset \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n$ is a lattice. Moreover, if the discriminant of this complete module is d, then the Lebesgue measure of any of the fundamental domains of $\tilde{\tilde{\mathcal{L}}}$ is given by $2^{-s} \cdot \sqrt{|d|}$.* □

Remark 6.38. One can also do explicit calculations. We first note how the volume of the fundamental domain in Lemma 3.97 (also see the above corollary) is obtained.

We set

$$\boldsymbol{V} = \begin{pmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \lambda_r^2 & \cdots & \lambda_r^{n-1} \\ 1 & \mathrm{Re}(\lambda_{r+1}) & \mathrm{Re}(\lambda_{r+1}^2) & \cdots & \mathrm{Re}(\lambda_{r+1}^{n-1}) \\ 0 & \mathrm{Im}(\lambda_{r+1}) & \mathrm{Im}(\lambda_{r+1}^2) & \cdots & \mathrm{Im}(\lambda_{r+1}^{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{Re}(\lambda_{r+s}) & \mathrm{Re}(\lambda_{r+s}^2) & \cdots & \mathrm{Re}(\lambda_{r+s}^{n-1}) \\ 0 & \mathrm{Im}(\lambda_{r+s}) & \mathrm{Im}(\lambda_{r+s}^2) & \cdots & \mathrm{Im}(\lambda_{r+s}^{n-1}) \end{pmatrix}.$$

Then, the volume of the geometric image of $\mathbb{Z}[\lambda]$ in $\mathbb{R}^r \times \mathbb{C}^s$ is given by $|\det \boldsymbol{V}|$. Using that $\det \boldsymbol{V}$ is basically a *Vandermonde determinant* (see [226, p. 516]), one derives

$$\det \boldsymbol{V} = \left(\frac{i}{2}\right)^s \prod_{i<j} (\lambda_j - \lambda_i).$$

We now establish the connection between this product and the discriminant $d(1, \lambda, \ldots, \lambda^{n-1})$ of the order $\mathbb{Z}[\lambda]$ since

$$d(1, \lambda, \ldots, \lambda^{n-1}) = \prod_{i<j} (\lambda_j - \lambda_i)^2.$$

Note that $d(1, \lambda, \ldots, \lambda^{n-1})$ is related to the discriminant $d_{\mathbb{Q}(\lambda)}$ of the maximal order $\mathfrak{o}_{\mathbb{Q}(\lambda)}$ by (see Lemma 3.45)

$$d(1, \lambda, \ldots, \lambda^{n-1}) = (\mathfrak{o}_{\mathbb{Q}(\lambda)} : \mathbb{Z}[\lambda])^2 \cdot d_{\mathbb{Q}(\lambda)}.$$

(Thus, if $d(1, \lambda, \ldots, \lambda^{n-1})$ is square-free, $\mathbb{Z}[\lambda]$ is the maximal order. Also note that for discriminants d of orders we get $d \equiv 0, 1 \bmod 4$, see [211, Proposition 1.4].)

If we choose $\ell_i \in \mathbb{Z}[\lambda]$, then there exists a matrix $\boldsymbol{A} \in GL_n(\mathbb{Z})$ such that

$$(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})\, \boldsymbol{A} = (\ell_1, \ell_2, \ldots, \ell_n).$$

Then, $d(\ell_1, \ldots, \ell_n) = (\det \boldsymbol{A})^2 \cdot d(1, \lambda, \ldots, \lambda^{n-1})$ (see Lemma 3.45), respectively $\det(\boldsymbol{V}\boldsymbol{A}) = (\det \boldsymbol{V}) \cdot (\det \boldsymbol{A})$ establishes Lemma 3.97 (respectively the above corollary).

Now, λ is a PV-number, wherefore multiplication by λ of $x \in \mathcal{L}$ corresponds to applying the matrix $\boldsymbol{M}(\lambda)$ to the geometric image $\tilde{\tilde{x}} \in \tilde{\tilde{\mathcal{L}}}$, where $\boldsymbol{M}(\lambda)$ is given as in Equation (3.1) on

p. 38, *i.e.*,

$$\boldsymbol{M}(\lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda_r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{Re}(\lambda_{r+1}) & -\mathrm{Im}(\lambda_{r+1}) & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{Im}(\lambda_{r+1}) & \mathrm{Re}(\lambda_{r+1}) & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & \mathrm{Re}(\lambda_{r+s}) & -\mathrm{Im}(\lambda_{r+s}) \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & \mathrm{Im}(\lambda_{r+s}) & \mathrm{Re}(\lambda_{r+s}) \end{pmatrix},$$

where we use the notation $\lambda_i = \sigma_i(\lambda)$ (where σ_i is a Galois automorphism, with the usual ordering as in Definition 3.94). We can also interpret this as (diagonal) linear mapping in $\boldsymbol{M}(\lambda) : \mathbb{R}^r \times \mathbb{C}^s \to \mathbb{R}^r \times \mathbb{C}^s$,

$$(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) \mapsto (\lambda\, x_1, \lambda_2\, x_2, \ldots, \lambda_r\, x_r; \lambda_{r+1}\, x_{r+1}, \ldots, \lambda_{r+s}\, x_{r+s}).$$

By the definition of a PV-number, only the multiplication in the first coordinate is an expansion, in all other multiplications are contractions. We therefore write

$$\lambda^{\tilde{\star}} = (\lambda_2, \ldots, \lambda_{r+s}), \text{ wherefore } \boldsymbol{M}(\lambda) = \begin{pmatrix} \lambda & \boldsymbol{0} \\ \boldsymbol{0} & \lambda^{\tilde{\star}} \end{pmatrix}$$

if we interpret $\lambda^{\tilde{\star}}$ as diagonal matrix. In particular, $\lambda^{\tilde{\star}}$ is a contraction on $\mathbb{R}^{r-1} \times \mathbb{C}^s \cong \mathbb{R}^{n-1}$. From this it follows directly that we have a CPS $(\mathbb{R}\,, \mathbb{R}^{r-1} \times \mathbb{C}\,, \tilde{\tilde{\mathcal{L}}})$. In fact, we even have a *symmetric* CPS, *i.e.*,

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times (\mathbb{R}^{r-1} \times \mathbb{C}^s) & \xrightarrow{\pi_2} & \mathbb{R}^{r-1} \times \mathbb{C}^s \cong \mathbb{R}^{n-1} \\ \text{dense} \cup & & \cup & & \cup \text{ dense} \\ \mathcal{L} & \xleftrightarrow{1-1} & \tilde{\tilde{\mathcal{L}}} & \xleftrightarrow{1-1} & \pi_2(\tilde{\tilde{\mathcal{L}}}) = \mathcal{L}^{\tilde{\star}} \end{array}$$

where the bijectivity of π_1 and π_2 on the lattice $\tilde{\mathcal{L}}$ follows by construction, the denseness of $\mathcal{L} \subset \mathbb{R}$ is clear by the rational independence of the natural lengths ℓ_i (we assume $\operatorname{card} \mathcal{A} \geq 2$), and the denseness of $\pi_2(\tilde{\tilde{\mathcal{L}}}) \subset \mathbb{R}^{n-1}$ follows by $\lambda\,\mathcal{L} \subset \mathcal{L}$ (see proof of Lemma 6.23), which yields $\lambda^{\tilde{\star}}\,\mathcal{L}^{\tilde{\star}} \subset \mathcal{L}^{\tilde{\star}}$ and therefore the denseness since $\lambda^{\tilde{\star}}$ is a contraction. Moreover, the map $^{\tilde{\star}}$ is the star-map in this CPS and is explicitly given by

$$x^{\tilde{\star}} = \big(\sigma_2(x), \ldots, \sigma_r(x), \mathrm{Re}(\sigma_{r+1}(x)), \mathrm{Im}(\sigma_{r+1}(x)), \ldots, \mathrm{Re}(\sigma_{r+s}(x)), \mathrm{Im}(\sigma_{r+s}(x))\big)\,. \qquad (6.5)$$

We also have the following geometric interpretation of $\lambda\,\mathcal{L} \subset \mathcal{L}$.

Lemma 6.39. *Let σ be a Pisot substitution with PV-number λ. The geometric image of $\lambda\,\mathcal{L}$ is a sublattice of $\tilde{\tilde{\mathcal{L}}}$ of index $|\det \boldsymbol{S}\sigma|$.*

Proof. The set $\lambda\,\mathcal{L}$ is a complete module generated by $\lambda\,\ell_j = \sum_{i=1}^{n} (\boldsymbol{S}\sigma)_{ij}\,\ell_i$. Thus, its geometric image is a lattice, and the index follows similar to the considerations above. □

Applying the star-map in Equation (6.5) to the EMFS $\boldsymbol{\Theta}$, yields the following statement.

Proposition 6.40. *Let σ be a Pisot substitution with fixed point u. Denote the representation with natural lengths by $\underline{\Lambda}$ and the corresponding EMFS by $\boldsymbol{\Theta}$. We define an IFS $\boldsymbol{\Theta}^{\tilde{\star}}$ on $\mathbb{R}^{r-1} \times \mathbb{C}^s$ by (compare Equation (6.2))*

$$\Theta_{ij}^{\tilde{\star}} = \bigcup_{\{w \mid wi \triangleleft \sigma(j)\}} \{t_{(l'(w)^{\tilde{\star}})} \circ f_0^{\tilde{\star}}\} = \bigcup_{\{t_{(a)} \circ f_0 \in \Theta_{ij}\}} \{t_{(a^{\tilde{\star}})} \circ f_0^{\tilde{\star}}\},$$

where $f_0^{\tilde{\star}}(x) = \lambda^{\tilde{\star}}\, x^{\tilde{\star}}$. We denote the attractor of this IFS (the unique family of compact sets satisfying $\boldsymbol{\Theta}^{\tilde{\star}}$) by $(\mathcal{Q}_i)_{i=1}^{n}$. Then, $\underline{\Lambda}$ is a subset of $\Lambda(\underline{\mathcal{Q}})$, i.e., $\Lambda_i \subset \Lambda(\mathcal{Q}_i)$ for $1 \le i \le n$. Moreover, these statements are independent of the the chosen fixed point u.

Proof. It is clear that $\boldsymbol{\Theta}^{\tilde{\star}}$ is an IFS on $\mathbb{R}^{r-1} \times \mathbb{C}^s$. Moreover, $\boldsymbol{S\Theta}^{\tilde{\star}} = \boldsymbol{S\Theta}$ is primitive, thus the existence and uniqueness of the set $\underline{\mathcal{Q}}$ follows by Proposition 4.89. In particular, this also implies that $\underline{\mathcal{Q}}$ is independent of the chosen fixed point u, it only depends on the EMFS $\boldsymbol{\Theta}$ respectively the IFS $\boldsymbol{\Theta}^{\tilde{\star}}$.

The statement that $\underline{\Lambda}$ is a subset of $\Lambda(\underline{\mathcal{Q}})$ is an argument like on p. 173 (see Equation (5.12)). Also compare Lemma 5.139. □

This last proposition has the following immediate consequences.

Corollary 6.41. *Every Pisot substitution generates an multi-component Meyer set. In particular, $\underline{\Lambda}$ is FLC. Consequently, the height group is trivial, i.e., $\mathcal{L} = \mathcal{L}'$, and $\underline{\Lambda}$ is a repetitive UCF set. Moreover, the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), \mathbb{R})$ is strictly ergodic.*

Proof. By construction, any Λ_i is relatively dense and a subset of a model set $\Lambda(\mathcal{Q}_i)$. Consequently, every component Λ_i and $\operatorname{supp}\underline{\Lambda} \subset \Lambda(\operatorname{supp}\underline{\mathcal{Q}})$ are Meyer sets by Corollary 5.14. Thus, they are FLC and by Lemma 6.34 the height group is trivial. Repetitivity, UCF and strict ergodicity follows by Corollary 6.10 (also see Corollary 5.89). □

As last point in this section, we would like to argue that there is no Salem substitution that generates a model set.

We note that the term *geometric realisation* in [181, 182] is a generalisation of what we call "representation with natural intervals" to the eigenspace of all eigenvalues of the substitution matrix. Consequently, [181, Theorem 2.3] reads in our setting.

Proposition 6.42. *Let σ be a primitive substitution on the alphabet $\mathcal{A}$ with fixed point u. Let $\lambda_i \in \mathbb{C}$ be an eigenvalue of the matrix $\boldsymbol{S}\sigma$ and let ℓ_{λ_i} be a corresponding left eigenvector. Using the homomorphism of Abelianisation l (see Definition 6.6), we define*

$$\begin{aligned}\Lambda_{\lambda_i} &= \langle \ell_{\lambda_i}, l(u) \rangle \\ &= \{0\} \cup \left\{ \sum_j (\ell_{\lambda_i})_j \cdot (l(w))_j \mid w = u_0 \dots u_N \text{ for all } N \in \mathbb{Z}_{\ge 0} \right\} \\ &\quad \cup \left\{ -\sum_j (\ell_{\lambda_i})_j \cdot (l(w))_j \mid w = u_{-N} \dots u_{-1} \text{ for all } N \in \mathbb{N} \right\}.\end{aligned}$$

Then, we have

(i) *If* $|\lambda_i| > 1$*, the set* $\varLambda_{\lambda_i}$ *is unbounded.*

(ii) *If* $\lambda_i = 0$*, the set* $\varLambda_{\lambda_i}$ *is a finite set.*

(iii) *If* λ_i *is a root of unity, the set* $\varLambda_{\lambda_i}$ *is either unbounded or is a finite set.*

(iv) *If* $|\lambda_i| = 1$ *and* λ_i *is not a root of unity, the set* $\varLambda_{\lambda_i}$ *is unbounded.*

(v) *If* $0 < |\lambda_i| < 1$*, the set* $\varLambda_{\lambda_i}$ *is bounded. Moreover, its closure in* $\mathbb{C}$ *is a perfect subset of* $\mathbb{C}$*.* □

We note the case (i) is clear and (v) is simply the statement that the set $\mathrm{cl}_{\mathbb{C}}\, \varLambda_{\lambda_i}$ is an attractor of a self-similar IFS, compare Proposition 4.99. In (ii), the set $\varLambda_{\lambda_i}$ is given by the set of translational parts of the maps in the corresponding MFS (with "singular contraction" $f_0(x) = 0 \cdot x$). Interesting are the cases (iii) & (iv), we indicate (the hard case) (iv): There is an $0 \neq x \in \varLambda_{\lambda_i}$ and thus (if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$) a dense set on the circle of radius $|x|$ is contained in $\varLambda_{\lambda_i}$ (since $\{\lambda_i^k\, x\}_{k\in\mathbb{N}} \in \varLambda_{\lambda_i}$). Some component of ℓ_{λ_i} is nonzero, wherefore for every $x \in \varLambda_{\lambda_i}$ there is also a $y \in \varLambda_{\lambda_i}$ such that $|x| < |y|$. This then establishes the claim.

We compare this last proposition to Propositions 5.13 & 6.16:

- By Proposition 5.13(ii), a Meyer set is a subset of a model set. In fact, for Pisot substitutions, we have constructed such a model set $\Lambda(\underline{Q})$ above.
- Proposition 6.16 leaves open the possibility that a Salem substitution also generates a Meyer set. We might proceed analogously as for a Pisot substitution (compare the following example), but (iv) implies that one does not get an appropriate compact set which may serve as window.

Consequently, we only consider Pisot substitutions here.

Example 6.43. Consider the following substitution $a \mapsto ab$, $b \mapsto ac$, $c \mapsto d$, $d \mapsto b$, see [181, Example 2.9]. This substitution is a Salem substitution with inflation factor $\eta \approx 1.72208$, the minimal Salem number with $\mathrm{Irr}(\eta, \mathbb{Q}, x) = x^4 - x^3 - x^2 - x + 1$. One obtains the left PF eigenvector $(\eta^3 - 1, \eta^2, \eta, 1)$, thus $\mathcal{L}$ is a complete module in $\mathbb{Q}(\eta)$. Consequently, the geometric image $\tilde{\tilde{\mathcal{L}}}$ is a lattice in $\mathbb{R} \times (\mathbb{R} \times \mathbb{C})$ by Corollary 6.37 (we note that the algebraic conjugates of η are approximately given by $\eta_2 \approx 0.58069$ and $\eta_{3,4} \approx -0.65139 \pm i\, 0.75874$ with $|\eta_{3,4}| = 1$; obviously, $\eta_{3,4}$ are not roots of unity). Now, Proposition 6.42(iv) shows that we do not get a compact window within this CPS $(\mathbb{R}\,, \mathbb{R} \times \mathbb{C}\,, \tilde{\tilde{\mathcal{L}}})$. We note that by (v), the set $\mathrm{cl}_{\mathbb{R}}\, \varLambda_{\eta_2}$ is a perfect compact subset of $\mathbb{R}$. However, for this "window" we do not have an appropriate lattice (and thus no CPS). The above proposition indicates that Salem substitutions behave like non-Pisot substitutions; however, for non-Pisot substitutions the set $\varDelta'$ is (often) "eventually dense" (see p. 362), while we do not know if this also holds for Salem substitutions (since one has [68, Theorem 5.5.1]).

6.4. A CPS for Pisot Substitutions

So far we have the following ingredients for Pisot substitutions:

- If card $\mathcal{A} \geq 2$, then any fixed point u of σ is aperiodic (Lemma 6.26). Consequently, also the representation in natural intervals $\mathcal{T} = \underline{\Lambda} + \underline{A}$, respectively $\underline{\Lambda}$ is aperiodic.
- The multi-component set $\underline{\Lambda}$ is, by construction, a primitive representable substitution multi-component Delone set with finite seed consisting of two elementary multi-component point sets. Moreover, one can choose this finite seed to be legal, wherefore every cluster is legal (see Remark 5.91).
- Since $\underline{\Lambda}$ is a FLC multi-component set, it has trivial height group and it is repetitive by Corollary 6.41. In fact, $\underline{\Lambda}$ is even a multi-component Meyer set.
- Each Λ_i is a subset of $\mathbb{Q}(\Lambda)$. By multiplication, we can achieve that $\Lambda_i \subset \mathbb{Z}[\lambda]$ and therefore also $\Delta_i, \Delta', \Delta \subset \mathbb{Z}[\lambda]$ and $\mathcal{L} = \mathcal{L}'$ is a subgroup of $\mathbb{Z}[\lambda]$.
- We have $f_0(x) = \lambda \cdot x$. Assuming that $\underline{\Lambda}$ admits an algebraic coincidence, we have that $\{f_0^m(\Delta') \mid m \in \mathbb{Z}_{\geq 0}\}$ is a countable base at 0 for $\mathcal{L}$ equipped with the AC topology (see Corollary 5.135).

It is hard to prove directly that $\underline{\Lambda}$ admits an algebraic coincidence. Therefore, we use a different approach in this section:

(i) We first "refine" and extend the CPS used in the last section for a Pisot substitution.

(ii) With this CPS, we obtain conditions under which a Pisot substitution can be described as model set, see Theorem 5.147.

This is the program of this and the next two sections.

We begin with a "heuristic" construction of a CPS for an FLC multi-component Delone set $\underline{\Lambda}$ associated to a (Pisot) substitution σ.

Completions of a Subgroup of an Algebraic Number Field. If $\underline{\Lambda}$ is a model set, then it is also an FLC set and satisfies assumptions **(As)**; consequently, the internal space H and the lattice $\tilde{\mathcal{L}}$ can be recovered by the construction in Section 5.3. More precisely, the internal space H' is obtained as Hausdorff completion of $\mathcal{L}' = \langle \Delta' \rangle_{\mathbb{Z}}$ with respect to the AC topology (defined through the neighbourhood base given by the sets P'_ε respectively by the maximum variogram $\varrho_{\underline{\Lambda}}$) by Proposition 5.48. Of course, the difficulty is to obtain the maximum variogram $\varrho_{\underline{\Lambda}}$.

We therefore look for a "canonical choice" for such a completion if $\mathcal{L}' \subset \mathcal{L} \subset K = \mathbb{Q}(\lambda)$ and without explicit knowledge of the AC topology respectively the maximum variogram. Here, the construction using the geometric image (also known as Minkowski embedding) in the last section leads the way and we observe the following:

- As topological group, the AC topology of $\mathcal{L}'$ (respectively, $\mathcal{L}$) is generated by a family of invariant (pseudo)metrics, see Remark 2.37 (in fact, the maximum variogram $\varrho_{\underline{\Lambda}}$ is such a *single* pseudometric, but we are now looking for a different family). Thus, as stated in Remark 2.42, we can "complete" the group $\mathcal{L}'$ with respect to each of these pseudometrics and form their product. The closure of the image of the group in this product is then its completion.

- By Lemma 2.69 we have: The completion of a subspace Y of a uniform space X can be identified with $\mathrm{cl}_{\hat{X}}\, Y$.

- We denote the degree of the algebraic number field $K = \mathbb{Q}(\lambda)$ by $n = [K : \mathbb{Q}]$ and its signature by $[r\,, s]$ (wherefore $r + 2s = n$). Moreover, we denote by $\mathbb{P}_K$ the set of all prime ideals of $\mathfrak{o}_{\mathbb{Q}(\lambda)}$. Then, there are $r + s$ non-equivalent embeddings of K into $\mathbb{C}$, exactly r of them are real, *i.e.*, there exist r non-equivalent completions of K which are isomorphic to $\mathbb{R}$ and s non-equivalent completions which are isomorphic to $\mathbb{C}$ (compare Lemmas 3.57 & 3.57' and Definition 3b.1). If we denote the n different Galois automorphisms by σ_i as in Definition 3.21 and order them such that $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_r$ are the real embeddings and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ are the (pair-wise non-equivalent) complex embeddings (and $\sigma_{r+s+i} = \overline{\sigma}_{r+i}$ for $1 \leq i < s$), then the canonical homomorphisms of $K = \mathbb{Q}(\lambda)$ (which is also commutative topological field) into its completion is given by σ_i. Furthermore, to every prime ideal $\mathfrak{p} \in \mathbb{P}_K$, there exists a $\mathfrak{p}$-adic number field $\mathbb{Q}_{\mathfrak{p}}$ which is the completion of K with respect to the (non-Archimedean) absolute value $\|\cdot\|_{\mathfrak{p}}$. Moreover, $\mathbb{Q}_{\mathfrak{p}}$ and $\mathbb{Q}_{\mathfrak{p}'}$ are not topologically isomorphic if $\mathfrak{p} \neq \mathfrak{p}'$ (and, of course, neither topologically isomorphic to $\mathbb{R}$ nor $\mathbb{C}$). We recall that these are all non-trivial (and non-equivalent) completions of the algebraic number field K.

- Consequently, the set of places yields a (or even *the*) family of invariant (non-trivial) metrics on K (which are even compatible with the field structure).

While the completion of $\mathcal{L}'$ (respectively $\mathcal{L}$) with respect to an Archimedean place is therefore either $\mathbb{R}$ (if the associated Galois automorphism yields a real embedding) or $\mathbb{C}$, we introduce the following notation for a non-Archimedean place $\mathfrak{p} \in \mathbb{P}_K$: Denote by $v_{\mathfrak{p}}$ the valuation on $\mathbb{Q}_{\mathfrak{p}}$ (see Definition 3.54) and set (recall that $f_{\mathfrak{p}|(p)}$ is the residue degree of $\mathfrak{p}$ in $\mathbb{Q}(\lambda)/\mathbb{Q}$)

$$\delta_{\mathfrak{p}}^{\mathcal{L}} = \min\{v_{\mathfrak{p}}(x) \mid x \in \mathcal{L}\} \quad \text{respectively} \quad p^{-f_{\mathfrak{p}|(p)} \cdot \delta_{\mathfrak{p}}^{\mathcal{L}}} = \max\{\|x\|_{\mathfrak{p}} \mid x \in \mathcal{L}\}, \text{ where } \mathfrak{p} \mid (p)$$

(note that we use normalised absolute values, see p. 30 and Definition 3b.10). Then, one also has[‡] $\mathcal{L} \subset \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$. Moreover, by the strong triangle inequality (see Lemma 2.26(v) respectively the definition of an ultrametric absolute value in Definition 3.60), we also have $\delta_{\mathfrak{p}}^{\mathcal{L}} = \min\{v_{\mathfrak{p}}(x) | x \in \varDelta\}$ and it is even enough to take the minimum over the valuation of the generators of $\mathcal{L}$. Obviously, we have similar statements for $\mathcal{L}'$, and the completion of $\mathcal{L}$ (of $\mathcal{L}'$) with respect to a non-Archimedean place $\mathfrak{p}$ is isomorphic to the compact group $\mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$ (respectively $\mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}'}}$).

In view of Remark 2.42 (see the first point above), we have "guessed" the possible factors of the product that yields the completion. One now uses the structure of the substitution to decide which of them one should choose.

[‡] Actually, it would be better to define the compactly generated subspace $H_{\mathfrak{p}}$ of $\mathbb{Q}_{\mathfrak{p}}$ by $\mathcal{L}$ for each $\mathfrak{p}$. Obviously, one then has $\mathcal{L} \subset H_{\mathfrak{p}} \subset \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$, but in general not $H_{\mathfrak{p}} = \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$. For example, the unramified extension $\mathbb{Q}_2(\sqrt{5})$ of $\mathbb{Q}_2$ is one (the contractive one) of the non-Archimedean completions we can associate with the substitution $a \mapsto aaaaab$ and $b \mapsto ab$ (with $\ell_a = 2 - \sqrt{5}$ and $\ell_b = 1$) with PV-number $\lambda = 3 + \sqrt{5}$. For $\mathbb{Q}_2(\sqrt{5})$, the uniformiser is 2, the residue field has the representatives $\{0, 1, \tau, \tau + 1\}$ (with the golden mean $\tau = \frac{1+\sqrt{5}}{2}$) wherefore for $\mathfrak{p} = (2)$ one has $\widehat{\mathfrak{o}_{\mathfrak{p}}} = \mathfrak{p} \cup \mathfrak{p}+1 \cup \mathfrak{p}+\tau \cup \mathfrak{p}+\tau+1$. Now, here one obtains $H_{\mathfrak{p}} = H_{(2)} = \mathfrak{p} \cup \mathfrak{p}+1$, but $\delta_{\mathfrak{p}}^{\mathcal{L}} = 0$, *i.e.*, $\mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}} = \widehat{\mathfrak{o}_{\mathfrak{p}}}$. It would be better to work with $H_{\mathfrak{p}}$ instead of $\mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$ in all the following statements!

Contractive Completions. Ultimately, we would like to have property **(PLT+)**, wherefore we assume that $\underline{\Lambda}$ is a representable aperiodic primitive substitution multi-component Delone set. In particular, the associated EMFS $\boldsymbol{\Theta}$ satisfies property **(LT)**, and we have that $f_0(\operatorname{supp} \underline{\Lambda}) \subset \operatorname{supp} \underline{\Lambda}$. While f_0 is an expansion on the direct space, its action $\hat{f}_0$ on H (obtained through Proposition 2.47 and Lemma 2.68) should be a contraction, compare Proposition 5.131. Obviously, we are therefore interested in the answer of the following question:

In which local fields associated to $K = \mathbb{Q}(\lambda)$ is f_0 a contraction (or an expansion)?

Since $f_0(x) = \lambda x$ is simply a multiplication by the PV-number λ (an algebraic integer), one has that f_0 is a contraction on the complete local field K_ν (where we denote the place by ν) iff $|\eta|_\nu < 1$. In particular, by the definition of a PV-number, this includes all infinite places but the one associated to $\sigma_1 = \mathrm{id}$ (all of its algebraic conjugates are less than 1 in modulus).

For the finite places, we apply Theorem 3.72 to obtain the following statement.

Lemma 6.44. *Let λ be a PV-number. Then, $\|\lambda\|_{\mathfrak{p}} \leq 1$ for all prime ideals $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$. Furthermore, $\|\lambda\|_{\mathfrak{p}} < 1$ only if the prime number $p \in \mathfrak{p}$ divides $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda)$.*

Proof. Since λ is an algebraic integer, one has $\lambda \in \widehat{\mathfrak{o}_{\mathfrak{p}}}$ for all prime ideals $\mathfrak{p}$ (also compare Lemma 3b.8) and the first claim follows (*i.e.*, λ is also a $\mathfrak{p}$-adic integer). By Theorem 3.72(iii) we have

$$N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) = \prod_{\mathfrak{p}|(p)} N_{\mathbb{Q}_{\mathfrak{p}}/\mathbb{Q}_p}(\lambda).$$

We have $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) \in \mathbb{Q} \subset \mathbb{Q}_p$ and $N_{\mathbb{Q}_{\mathfrak{p}}/\mathbb{Q}_p}(\lambda) \in \mathbb{Q}_p$ (see Lemma 3.39). In fact, since λ is an algebraic integer, one even has $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) \in \mathbb{Z} \subset \mathbb{Z}_p$ and $N_{\mathbb{Q}_{\mathfrak{p}}/\mathbb{Q}_p}(\lambda) \in \mathbb{Z}_p$ (compare the definition of the norm *via* the characteristic polynomial in Definition 3.36). We can therefore take the p-adic absolute value of this last equation and obtain by Corollary 3.76 (we use normalised absolute values here)

$$\|N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda)\|_p = \prod_{\mathfrak{p}|(p)} \|N_{\mathbb{Q}_{\mathfrak{p}}/\mathbb{Q}_p}(\lambda)\|_p = \prod_{\mathfrak{p}|(p)} \|\lambda\|_{\mathfrak{p}}.$$

The factors on the right hand side are less than or equal to 1, and the number on the left hand side is (also) less than or equal to 1 and less than 1 only if p divides $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda)$. This establishes the lemma. □

For a Pisot substitution σ, one has $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) = \pm \det \boldsymbol{S}\sigma$, wherefore we obtain the following statement.

Corollary 6.45. [345, Corollaire 6.3.1] and [346, Corollaire 4.3] *Let σ be a Pisot substitution with PV-number λ. Then, $\|\lambda\|_{\mathfrak{p}} \leq 1$ for all prime ideals $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$, and $\|\lambda\|_{\mathfrak{p}} < 1$ only if the prime number $p \in \mathfrak{p}$ divides* $\det \boldsymbol{S}\sigma$. □

Note that there are only finitely many prime ideals $\mathfrak{p}$ such that $\|\lambda\|_{\mathfrak{p}} < 1$. We also note that the subgroup $\boldsymbol{\mathcal{L}}$ is not only a complete module in $\mathbb{Q}(\lambda)$, but might even be a fractional ideal.

We have therefore answered the questions about "contractive" and "expansive" local fields (the only "expansive" local field is $\mathbb{R}$, the one where our representation with natural intervals lives and which is associated with the Galois automorphism id).

A Possible CPS. By Corollary 6.41, we have for a Pisot substitution that $\mathcal{L} = \mathcal{L}'$, wherefore we also have $\mathcal{L}_{\text{ext}} = \mathcal{L}'_{\text{ext}}$. We recall the definition of $\mathcal{L}_{\text{ext}} = \langle \bigcup_{m \geq 0} f_0^{-m}(\varDelta) \rangle_{\mathbb{Z}}$. Since $|\eta|_\nu < 1$ iff $|1/\eta|_\nu > 1$, the completion of $\mathcal{L}_{\text{ext}}$ with respect to a non-Archimedean place is $\mathbb{Q}_\mathfrak{p}$ iff $\|\eta\|_\mathfrak{p} < 1$ (and still $\mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}$ iff $\|\eta\|_\mathfrak{p} = 1$).

Thus, we have "guessed" the following internal spaces:

$$H \cong \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ \|\lambda\|_\mathfrak{p} < 1}} \mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}, \qquad H_{\text{ext}} \cong \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ \|\lambda\|_\mathfrak{p} < 1}} \mathbb{Q}_\mathfrak{p}. \tag{6.6}$$

We will have a closer look at H_{ext} in Section 6.5.

Now, condition **(LT)**(iii), *i.e.*, that $\mu(f_0(W)) = \alpha \cdot \mu(W)$ if f_0 is an expansion and $\mu(f_0(W)) = \frac{1}{\alpha} \cdot \mu(W)$ if f_0 is a contraction, with the *same* constant α for $\boldsymbol{\varTheta}$ and $\boldsymbol{\varTheta}^\star$ (where we replace f_0 by $\hat{f}_0$) is again Artin's product formula, see Proposition 4.38 and Lemma 3b.11. Furthermore, we stress that if $\underline{\varLambda}$ is a model set then

- the internal (and product) space H cannot contain[10] a factor $\mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}$ with $\|\lambda\|_\mathfrak{p} = 1$: Consider any ball $B_{\leq r}(0) \subset H$ and its projection on the $\mathfrak{p}$th coordinate (which, by abuse of notation, we may again denote by $B_{\leq r}(0)$). Since $\|\lambda\|_\mathfrak{p} = 1$, the projection of $\lambda^m \, B_{\leq r}(0)$ on the $\mathfrak{p}$th coordinate coincide for all $m \in \mathbb{N}$. But then, the countable neighbourhood base $\{\text{cl}_H \, f_0^m(\varDelta) \mid m \in \mathbb{Z}_{\geq 0}\}$ of 0 for H (see Proposition 5.48 and Corollary 5.135) does not define a Hausdorff topology (see Lemma 2.17), which contradicts the construction of H in Section 5.3.

- the internal space H must contain all factors $\mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}$ with $\|\lambda\|_\mathfrak{p} < 1$: This follows by the uniqueness of the Hausdorff completion (also see Remark 2.42). More precisely, let H be the Hausdorff completion of $\mathcal{L}$ and assume there is a factor $\mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}$ not contained in H. Then, f_0 is a continuous homomorphism from $\mathcal{L}$ to the product space of H and this factor (and this product space is Hausdorff and complete). Then, we apply Proposition 2.47 to this situation and recall that different local fields are not topologically isomorphic. This yields a contradiction, wherefore all such factors are contained in H.

So far, we have justified the choice of the internal space H.

We now show that the diagonal embedding of $\mathcal{L}$ is lattice $\tilde{\mathcal{L}} \subset \mathbb{R} \times H$. But this follows easily from Definition 3.94 and Lemma 3.97.

Lemma 6.46. *Let* $\mathbb{Q}(\lambda)$ *be an algebraic number field of degree* n *with signature* $[r,s]$*. Let* $\mathcal{L} = \{\sum_{i=1}^n c_i \cdot \ell_i \mid c_i \in \mathbb{Z}\}$ *be a lattice (i.e., a complete module) in* $\mathbb{Q}(\lambda)$ *where* $\ell_1, \ldots, \ell_n$ *is a* $\mathbb{Q}$*-basis of* $\mathbb{Q}(\lambda)$*. Then the diagonal embedding of* $\mathcal{L}$ *in* H *is a lattice, where* H *is as in Equation* (6.6)*. Moreover, if* $d = d_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\ell_1, \ldots, \ell_n)$*, then the volume of any of its fundamental domains* $\text{FD}(\tilde{\mathcal{L}})$ *(i.e., the Haar measure of* $\text{FD}(\tilde{\mathcal{L}})$ *in* H*) is given by* $2^{-s} \cdot \sqrt{|d|} \cdot \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_\mathfrak{p} < 1} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_\mathfrak{p}^\mathcal{L}}$.

Proof. Lemma 3.97 (also see Corollary 6.37) establishes that the geometric image of $\mathcal{L}$ in $\mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n$ is a lattice. Thus, we only have to argue that the additional $\mathfrak{p}$-adic factors (a

[10] Here, we assume that H is given by a product of finitely many local fields, or as restricted product of infinitely many local fields (then we recall Lemma 3b.15). In fact, in the next section, we will consider the internal space H essentially given by the adele group of $\mathbb{Q}(\lambda)$ (more precisely, $\mathbb{R} \times H \cong \mathbb{A}_{\mathbb{Q}(\lambda)}$).

compact space!) do not destroy this property. Obviously, since $\mathcal{L}$ is a subgroup of $\mathbb{Q}(\lambda)$, so is $\tilde{\mathcal{L}}$ in H. Moreover, relative denseness of $\tilde{\mathcal{L}} \subset H$ follows from the relative denseness of the geometric image in $\mathbb{R}^n$, since the additional $\mathfrak{p}$-adic factors is a product of compact spaces.

Similarly, uniform discreteness follows: Let $U \subset \mathbb{R}^r \times \mathbb{C}^s$ be the open neighbourhood of 0 for the geometric image of $\mathcal{L}$. We claim that for every $\tilde{x} \in \tilde{\mathcal{L}}$ one has

$$\tilde{\mathcal{L}} \cap \tilde{x} + \left(U \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ \|\lambda\|_{\mathfrak{p}} < 1}} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}} \right) = \{\tilde{x}\}.$$

But this follows in one direction by the choice of the numbers $\delta_{\mathfrak{p}}^{\mathcal{L}}$ and the diagonal embedding and in the other direction by the choice of U.

The Haar measure of the fundamental domain follows by the normalisation of the absolute values, see Example 4.37. □

Remark 6.47. Up to now, one may also easily treat more general situations like product spaces K^d (with $d \geq 1$, see [292, Section 3]), respectively $\prod_i K_i$, *i.e.*, the situation that the substitution multi-component set $\underline{\Lambda}$ lives on a product of some algebraic number fields $K_i = \mathbb{Q}(a_i)$, *i.e.*, $\operatorname{supp} \underline{\Lambda} \subset \prod_i K_i$. Furthermore, we assume that each K_i is minimal in the sense that there is no subfield K_i' with $\mathbb{Q} \subset K_i' \subset K_i$ such that $\langle \Delta \rangle_{\mathbb{Z}} = \mathcal{L} \subset \prod_i K_i'$.

We again observe that the multiplication f_0 on a number field K extends to a multiplication on any local field of K (note that the case where $f_0(x) = \boldsymbol{M}x$ with $x \in K^d$ can be treated similarly, see Remark 3b.18). If $\underline{\Lambda}$ is a multi-component model set, then – by Proposition 5.131 – we expect that the expansion f_0 on the direct space extends to a contraction on the internal space. Consequently, our previous considerations suggest the following structure of the direct and the internal space for a possible CPS for $\underline{\Lambda}$:

> The internal space is formed by all local fields associated to K on which f_0 acts as contraction; moreover, the direct space – where $\underline{\Lambda}$ lives – must consist of all local fields on which f_0 acts as expansion.

For the second statement, we recall the definition of a *Pisot family* in [364, p. 712]: A set of algebraic integers $S = (\lambda_1, \ldots, \lambda_r)$ is a Pisot family if, for all $1 \leq i \leq r$, every Galois conjugate γ of λ_i with $|\gamma| > 1$ is in S. Of course, this definition lines up with our previous considerations: The direct space corresponding to such a Pisot family is given by $\mathbb{R}^{d_1} \times \mathbb{C}^{d_2} \cong \mathbb{R}^r$ (since the complex conjugate of a proper complex number in S also belongs to S). In fact, the intention of the definition of a “Pisot family” is the following: Let $f_0 = T$ be a linear map on $\mathbb{M} = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \ldots \times \mathbb{Q}_{\mathfrak{p}_k}$ as in Equation (4.5) on p. 101 with algebraic numbers $a_1, \ldots, a_{r+s+k}$. Then, $S = (a_1, \ldots, a_{r+s+k})$ is a *generalised Pisot family* if, for all $1 \leq i \leq r+s+k$, all local fields associated to $\mathbb{Q}(a_i)$, on which multiplication by a_i acts as expansion, are contained (at least as subfields) in the product $\mathbb{M}$. Then, the product of all corresponding local fields, where a_i acts as contraction, form the internal space. Again, for more general situations where f_0 is not a “diagonal” map, see Remark 3b.18.

Remark 6.48. That this heuristic construction indeed yields the correct CPS, will later follow from the considerations on the possible eigenvalues of the corresponding dynamical systems, see p. 378.

$\mathcal{M}^\star$ is a Lattice. Inside $\tilde{\mathcal{L}}$, we have $\tilde{\mathcal{M}} = \{(x, x^\star) | x \in \mathcal{M}\}$. Since $\mathcal{M}$ has $n-1$ generators, it is not a complete module and thus *not* a lattice in $\mathbb{R} \times H$. Of interest is the object $\mathcal{M}^\star \subset \mathcal{L}^\star \subset H$.

We note that each set $\varLambda_i^\star \subset H$ is relatively compact, since

$$\varLambda_i^\star \subset \mathcal{Q}_i \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ \|\lambda\|_\mathfrak{p} < 1}} \mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}},$$

where $\underline{\mathcal{Q}}$ is the attractor of $\boldsymbol{\Theta}^{\tilde\star}$ in Proposition 6.40 (*i.e.*, each $\varLambda_i^\star$ is contained in a compact set). Since $\mathcal{L}^\star$ is dense in H, then – by $\mathcal{L}^\star = (\operatorname{supp} \underline{\varLambda})^\star \dot{+} \mathcal{M}^\star$, see Proposition 6.33 – the set $\mathcal{M}^\star$ is relatively dense in H. Moreover, since $\operatorname{rank}(\mathcal{M}) = n-1 = \dim(\mathbb{R}^{n-1})$ and $\mathbb{R}^{n-1} \cong \mathbb{R}^{r-1} \times \mathbb{C}^s$, the set $\mathcal{M}^{\tilde\star}$ must be uniformly discrete in $\mathbb{R}^{r-1} \times \mathbb{C}^s$. But then – similarly to the argument in the proof of Lemma 6.46 – the set $\mathcal{M}^\star$ is also uniformly discrete in H, and therefore a Delone subgroup of H, *i.e.*, a lattice. We have therefore derived the following statement.

Lemma 6.49. *The set $\mathcal{M}^\star$ is a lattice in H.* □

Our next goal is to calculate $\mu_H(\mathrm{FD}(\mathcal{M}^\star))$. Again, we look at the Archimedean and non-Archimedean coordinates in H separately, since by the strong triangle inequality in $\mathfrak{p}$-adic fields one immediately establishes $\|\ell_i - \ell_1\|_\mathfrak{p} = \max\{\|\ell_i\|_\mathfrak{p}, \|\ell_1\|_\mathfrak{p}\}$. Therefore, the contribution to $\mu_H(\mathrm{FD}(\mathcal{M}^\star))$ that arises from the $\mathfrak{p}$-adic components is again given by the product $\prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_\mathfrak{p} < 1} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_\mathfrak{p}^\mathcal{L}}$. Thus, we only have to calculate the contribution from the Archimedean local fields.

By Corollary 6.11, the frequency ϱ_i of a letter i are given by the (positive) i-th coordinate of the right eigenvector ϱ of $\boldsymbol{S}\sigma$ to the eigenvalue λ with appropriate normalisation, *i.e.*, $(\boldsymbol{S}\sigma)\varrho = \lambda \cdot \varrho$. This eigenvector ϱ is perpendicular to all left eigenvectors except $\ell = (\ell_1, \dots, \ell_n)$, the left eigenvector to λ, because for a left eigenvector v to the eigenvalue $\lambda_i \neq \lambda$ we get (no eigenvalue vanishes)

$$\lambda \cdot \langle v^t, \varrho \rangle = \langle v^t, (\boldsymbol{S}\sigma)\varrho \rangle = \langle (\boldsymbol{S}\sigma)^t v^t, \varrho \rangle = \lambda_i \cdot \langle v^t, \varrho \rangle.$$

We get the average length $\bar\ell$ as $\bar\ell = \sum_{i=1}^n \varrho_i \cdot \ell_i = \sum_{i=1}^n \varrho_i \cdot g_i(\lambda)$ (recall that $\ell_i = g_i(\lambda)$ where $g_i \in \mathbb{Z}[x]$ with $\deg g_i \le n-1$). Our previous observation shows that $\sum_{i=1}^n \varrho_i \, g_i(\lambda_j) = 0$ for all eigenvalues $\lambda_j \neq \lambda$ of $\boldsymbol{S}\sigma$. Using the matrices $\boldsymbol{V}$, $\boldsymbol{A}$ from Remark 6.38, we can rewrite this as

$$\begin{pmatrix} \bar\ell \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \ell_1 = g_1(\lambda) & \cdots & \ell_n = g_n(\lambda) \\ g_1(\lambda_2) & \cdots & g_n(\lambda_2) \\ \vdots & \ddots & \vdots \\ g_1(\lambda_r) & \cdots & g_n(\lambda_r) \\ \mathrm{Re}(g_1(\lambda_{r+1})) & \cdots & \mathrm{Re}(g_n(\lambda_{r+1})) \\ \mathrm{Im}(g_1(\lambda_{r+1})) & \cdots & \mathrm{Im}(g_n(\lambda_{r+1})) \\ \vdots & \ddots & \vdots \\ \mathrm{Re}(g_1(\lambda_{r+s})) & \cdots & \mathrm{Re}(g_n(\lambda_{r+s})) \\ \mathrm{Im}(g_1(\lambda_{r+s})) & \cdots & \mathrm{Im}(g_n(\lambda_{r+s})) \end{pmatrix} \varrho = \begin{pmatrix} \ell_1 & \cdots & \ell_n \\ \ell_1^{\tilde\star} & \cdots & \ell_n^{\tilde\star} \end{pmatrix} \varrho = \boldsymbol{V}\boldsymbol{A}\varrho,$$

where we use the notation $x^{\tilde\star}$ for $x \in \mathbb{Q}(\lambda)$ as in Equation (6.5) on p. 227 (thus, this is the "Archimedean part" of the star-map). Next, we observe that

$$\begin{pmatrix} \ell_i - \ell_1 \\ \ell_i^{\tilde\star} - \ell_1^{\tilde\star} \end{pmatrix} = \boldsymbol{V}\boldsymbol{A}(e_i - e_1),$$

where e_i denotes the i-th unit vector (*i.e.*, $e_1 = (1, 0, \ldots, 0)^t$ *etc.*).

With all this we are set for the following calculation:

$$\bar{\ell} \cdot \mu_{\mathbb{R}^{n-1}}(\mathrm{FD}(\tilde{\mathcal{M}^\star})) = \bar{\ell} \cdot \left|\det \begin{pmatrix} \tilde{\ell_2^\star} - \tilde{\ell_1^\star} & \cdots & \tilde{\ell_n^\star} - \tilde{\ell_1^\star} \end{pmatrix}\right| = \left|\det \begin{pmatrix} \bar{\ell} & \ell_2 - \ell_1 & \cdots & \ell_n - \ell_1 \\ 0 & \tilde{\ell_2^\star} - \tilde{\ell_1^\star} & \cdots & \tilde{\ell_n^\star} - \tilde{\ell_1^\star} \end{pmatrix}\right|$$

$$= \left|\det \boldsymbol{V} \ \det \boldsymbol{A} \ \det \begin{pmatrix} \varrho_1 & -1 & -1 & \cdots & -1 & -1 \\ \varrho_2 & 1 & 0 & \cdots & 0 & 0 \\ \varrho_3 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \varrho_{n-1} & 0 & 0 & \ddots & 1 & 0 \\ \varrho_n & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right|$$

$$= |\det \boldsymbol{V} \ \det \boldsymbol{A} \ \cdot (\varrho_1 + \ldots + \varrho_n)| = |\det \boldsymbol{V} \ \det \boldsymbol{A}|$$

Consequently, we have established the following.

Lemma 6.50. *Let σ be a Pisot substitution. Then, for the representation with natural intervals for any of its fixed points u, we obtain*

$$\bar{\ell} \cdot \mu_H(\mathrm{FD}(\mathcal{M}^\star)) = \mu_{\mathbb{R}\times H}(\mathrm{FD}(\tilde{\mathcal{L}}))$$

respectively

$$\frac{\mu_H(\mathrm{FD}(\mathcal{M}^\star))}{\mu_{\mathbb{R}\times H}(\mathrm{FD}(\tilde{\mathcal{L}}))} = \frac{1}{\bar{\ell}} = \mathrm{dens}(\mathrm{supp}\, \underline{\Lambda}).$$ □

6.5. The Adelic Viewpoint and an Extended CPS

We recall and reinterpret our findings in Remarks 4.108 & 4.109:

- On the n-dimensional adele $\mathbb{A}_{\mathbb{Q}}^n$ of $\mathbb{Q}$, the substitution matrix $\boldsymbol{S}\sigma$ is a principal lattice transformation, see Definition 3b.16. Therefore, with respect to the matrix norm $\|\!| \cdot |\!\|_{\mathcal{G}}$ defined in Remark 3b.18, we have

$$1 = \|\!| \boldsymbol{S}\sigma |\!\|_{\mathcal{G}} = |\det \boldsymbol{S}\sigma| \cdot \prod_{p\in\mathbb{P}} |\det \boldsymbol{S}\sigma|_p,$$

  which generalises Artin's product formula (see Lemma 3b.11). Moreover, we note that $|\det \boldsymbol{S}\sigma| \geq 1$ while $|\det \boldsymbol{S}\sigma|_p \leq 1$.

- Let $J = \mathbb{P} \cup \{\infty\}$ the set of places in $\mathbb{Q}$ and $J' = \mathbb{P}_{\mathbb{Q}(\lambda)} \cup \{\nu_1, \ldots, \nu_{r+s}\}$ the set of places of $\mathbb{Q}(\lambda)$, where ν_i denotes an infinite place and the signature of $\mathbb{Q}(\lambda)$ is $[r, s]$. Then, we can identify the principal lattice transform $(\boldsymbol{S}\sigma)_{\nu\in J}$ on $\mathbb{A}_{\mathbb{Q}}^n$ with the multiplication map $T(x) = (\lambda \cdot x_{\nu'})_{\nu'\in J'}$ on $\mathbb{A}_{\mathbb{Q}(\lambda)}$ (where the diagonal embedding $(\lambda, \lambda, \ldots)$ has to be read as $(\lambda, \lambda_2, \ldots, \lambda_{r+s}, \lambda, \lambda, \ldots)$ as usual). Moreover, we have $|\det \boldsymbol{S}\sigma|_p = \|\lambda\|_{\mathfrak{p}_1} \cdots \|\lambda\|_{\mathfrak{p}_\ell}$, where the product runs over all prime ideals $\mathfrak{p}_i$ of (p). This also confirms Corollary 6.45. Similarly, we have $|\det \boldsymbol{S}\sigma| = |\lambda| \cdot |\lambda_2| \cdots |\lambda_{r+s}|$ (noting that in the s complex cases we use the square of the usual absolute value).

- By the definition of a Pisot substitution, only the absolute value $|\lambda|$ is greater than 1. The corresponding local field $\mathbb{R}$ is the direct space of the CPS. Similarly, the product of all local fields for which $|\lambda|_\nu < 1$ (the "contracting" part of $\mathbb{A}_{\mathbb{Q}(\lambda)}$) yields the internal space H_{ext} of the CPS associated to a Pisot substitution. We have $H_{\text{ext}} = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\, \|\lambda\|_\mathfrak{p}<1} \mathbb{Q}_\mathfrak{p}$, compare Equation (6.6) on p. 233.

We now show how the "indifferent" part of the adele $\mathbb{A}_{\mathbb{Q}(\lambda)}$, *i.e.*, $\prod'_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\, \|\lambda\|_\mathfrak{p}=1} \mathbb{Q}_\mathfrak{p}$, is associated to the lattice $\tilde{\mathcal{L}}$ respectively ($\tilde{\mathcal{L}}_{\text{ext}}$) in the CPS.

Proposition 6.51. *Let σ be a Pisot substitution with PV-number λ (where the signature of $\mathbb{Q}(\lambda)$ is $[r\,, s]$). Let $\underline{\Lambda}$ be the representation with natural intervals $[0\,, \ell_i]$ (where $\ell_i \in \mathbb{Q}(\lambda)$) of any of its fixed points. Define $\mathcal{L}$, $\mathcal{L}_{\text{ext}} = \bigcup_{k\geq 0} \mathcal{L}/\lambda^k$ and $\delta_\mathfrak{p}^\mathcal{L}$ (where $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$) as before. Then, in the adele*

$$\mathbb{A}_{\mathbb{Q}(\lambda)} = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)}}{}' \mathbb{Q}_\mathfrak{p}$$

the diagonal embedding of $\mathbb{Q}(\lambda)$ is a lattice. Thus, if we set

$$G_{\mathbb{A}_{\mathbb{Q}(\lambda)}} = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_\mathfrak{p}<1}} \mathbb{Q}_\mathfrak{p} = \mathbb{R} \times H_{\text{ext}} \qquad \textit{and} \qquad H_{\mathbb{A}_{\mathbb{Q}(\lambda)}} = \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_\mathfrak{p}=1}}{}' \mathbb{Q}_\mathfrak{p},$$

then $\mathbb{A}_{\mathbb{Q}(\lambda)} = G_{\mathbb{A}_{\mathbb{Q}(\lambda)}} \times H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$ and $\left(G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}\,, H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}\,, \mathbb{Q}(\lambda)\right)$ is a CPS. Set

$$\Omega_\mathcal{L} = \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\, \|\lambda\|_\mathfrak{p}=1} \mathfrak{p}^{\delta_\mathfrak{p}^\mathcal{L}}$$

and denote by $\tilde{\mathcal{L}}_{\text{ext}}$ the diagonal embedding of $\mathcal{L}_{\text{ext}}$ in $G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$. Then, $\tilde{\mathcal{L}}_{\text{ext}}$ is a sublattice of the lattice $\Lambda(\Omega_\mathcal{L})$. Moreover, if $\mathcal{L}$ is a fractional ideal, then we have $\tilde{\mathcal{L}}_{\text{ext}} = \Lambda(\Omega_\mathcal{L})$.

Proof. That $\left(G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}\,, H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}\,, \mathbb{Q}(\lambda)\right)$ is a CPS, is established as in Chapter 5a (see p. 201) by the strong approximation theorem (see Proposition 3b.12). So, we only have to prove the statements about $\tilde{\mathcal{L}}_{\text{ext}}$ and $\Lambda(\Omega_\mathcal{L})$.

To this end, we observe that $\Omega_\mathcal{L}$ is a compact set in $H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$ (the topology of $H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$ is the topology of a restricted product and, as subspace of $\mathbb{A}_{\mathbb{Q}(\lambda)}$, coincides with the induced topology of $\mathbb{A}_{\mathbb{Q}(\lambda)}$). This is easily seen by the metric given in Lemma 3b.15: Since $\delta_\mathfrak{p}^\mathcal{L} \geq 0$ for all but[11] finitely many $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$, the set $\Omega_\mathcal{L}$ is closed and bounded (and therefore compact) with respect to this metric. Moreover, we note that actually for all but finitely many $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$ we have $\delta_\mathfrak{p}^\mathcal{L} = 0$: one has $p^{-f_{\mathfrak{p}|(p)}\cdot\delta_\mathfrak{p}^\mathcal{L}} = \max\{\|\ell_i\|_\mathfrak{p} \mid 1 \leq i \leq n\}$ and Artin's product formula (Lemma 3b.11) yields $\prod_{\nu\in J'} |\ell_i|_\nu = 1$ for all i (where J' denotes the set of places of $\mathbb{Q}(\lambda)$). Consequently, this establishes that we have $\{x \in \mathbb{Q}_\mathfrak{p} | v_\mathfrak{p}(x) \leq \delta_\mathfrak{p}^\mathcal{L}\} = \hat{\mathfrak{o}}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, and – by the definition of the topology of a restricted product – the set $\Omega_\mathcal{L}$ has nonempty interior.

[11] If $\mathcal{L} \subset \mathbb{Z}[\lambda]$, then one has $\delta_\mathfrak{p}^\mathcal{L} \geq 0$ for all $\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$. Otherwise, there is a number $c \in \mathbb{Q}(\lambda)$ such that $c \cdot \mathcal{L} \subset \mathbb{Z}[\lambda]$ (note that $\mathcal{L}$ has n generators).

Thus, $\Lambda(\Omega_{\mathcal{L}})$ is a Delone set by Lemma 5.8. It is also a group: Take $x, y \in \Lambda(\Omega_{\mathcal{L}})$, then we have $\|x\|_{\mathfrak{p}}, \|y\|_{\mathfrak{p}} \leq \delta_{\mathfrak{p}}$ and therefore (by the strong triangle inequality) also $\|x-y\|_{\mathfrak{p}} \leq \delta_{\mathfrak{p}}$ for all $\mathfrak{p}$, which establishes $x - y \in \Lambda(\Omega_{\mathcal{L}})$. Consequently, $\Lambda(\Omega_{\mathcal{L}})$ is a lattice in $G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$.

By construction, we have $\tilde{\mathcal{L}}_{\text{ext}} \subset \Lambda(\Omega_{\mathcal{L}})$. Thus, $\tilde{\mathcal{L}}_{\text{ext}}$ is uniformly discrete. Since $\lambda \mathcal{L} \subset \mathcal{L}$, $\mathcal{L}_{\text{ext}}$ and therefore also $\tilde{\mathcal{L}}_{\text{ext}}$ is also a group: Take $x, y \in \mathcal{L}_{\text{ext}}$. Then, we may write $x = (a_1 \ell_1 + \ldots + a_n \ell_n)/\lambda^k$ and $y = (b_1 \ell_1 + \ldots + b_n \ell_n)/\lambda^{k'}$ with $a_i, b_i \in \mathbb{Z}$ (for $1 \leq i \leq n$) and $k, k' \geq 0$. Without loss of generality, we assume that $k' < k$, wherefore one also has $y = (a'_1 \ell_1 + \ldots + a'_n \ell_n)/\lambda^k$ for some $a'_i \in \mathbb{Z}$. Consequently, one has $x - y \in \mathcal{L}_{\text{ext}}$, wherefore it is a subgroup of $\Lambda(\Omega_{\mathcal{L}})$.

The group $\tilde{\mathcal{L}}_{\text{ext}}$ is also relatively dense in $G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$: By Lemma 6.46, $\tilde{\mathcal{L}}$ is a lattice in $\mathbb{R} \times H$; moreover, the proof of that lemma explicitly gives $W = W' \times \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_{\mathfrak{p}} < 1} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$ as (possible) compact set in the definition of relative denseness, where $W' \subset \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n$ denotes the corresponding compact set for the geometric image of $\mathcal{L}$. We now argue that one can also use this set W for $\tilde{\mathcal{L}}_{\text{ext}}$. For this, we note that, for every $x \in \tilde{\mathcal{L}}_{\text{ext}}$, one has $x + \tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}_{\text{ext}}$. Thus, we have

$$\tilde{\mathcal{L}}_{\text{ext}} + W \;\supset\; x + \tilde{\mathcal{L}} + W \;=\; x + (\mathbb{R} \times H),$$

and the claim follows since the projection of $\tilde{\mathcal{L}}_{\text{ext}}$ on the "p-adic part" $\prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_{\mathfrak{p}} < 1} \mathbb{Q}_{\mathfrak{p}}$ of the internal space H_{ext} (respectively, $G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$) is dense. Consequently, $\tilde{\mathcal{L}}_{\text{ext}}$ is a lattice in $G_{\mathbb{A}_{\mathbb{Q}(\lambda)}} \cong \mathbb{R} \times H_{\text{ext}}$ (and thus a sublattice of $\Lambda(\Omega_{\mathcal{L}})$).

We now proof the last claim. An application of the density formula (Corollary 5.27) together with Lemmas 3b.9 & 3.97 yields

$$\mu_{G_{\mathbb{A}_{\mathbb{Q}(\lambda)}}}(\mathrm{FD}(\Lambda(\Omega_{\mathcal{L}}))) = \frac{\mu_{\mathbb{A}_{\mathbb{Q}(\lambda)}}(\mathrm{FD}(\mathbb{Q}(\lambda)))}{\mu_{H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}}(\Omega_{\mathcal{L}})} = \frac{\mu_{\mathbb{R}^r \times \mathbb{C}^s}(\mathrm{FD}(\mathbb{Q}(\lambda)_\infty))}{\prod_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \\ \|\lambda\|_{\mathfrak{p}} = 1}} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_{\mathfrak{p}}^{\mathcal{L}}}} = \frac{2^{-s} \cdot \sqrt{|d_{\mathbb{Q}(\lambda)}|}}{\prod_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \\ \|\lambda\|_{\mathfrak{p}} = 1}} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_{\mathfrak{p}}^{\mathcal{L}}}}.$$

Now, if $\mathcal{L}$ is a fractional ideal in $\mathbb{Q}(\lambda)$, it possesses a unique prime ideal factorisation $\mathcal{L} = \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}} \mathfrak{p}^{n_{\mathfrak{p}}}$ (see Lemma 3.34). Moreover, by the multiplicativity of the ideal norm (see Lemma 3.45) and the definition of the normalised absolute value respectively the numbers $\delta_{\mathfrak{p}}^{\mathcal{L}}$, one has

$$N\mathcal{L} = \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}} (N\mathfrak{p})^{n_{\mathfrak{p}}} = \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}} p^{f_{\mathfrak{p}|(p)} \cdot n_{\mathfrak{p}}} = \frac{1}{\prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_{\mathfrak{p}}^{\mathcal{L}}}},$$

wherefore by Lemma 3.45 we have

$$d_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\ell_1 \ldots, \ell_n) = \left(\frac{1}{\prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}} p^{-f_{\mathfrak{p}|(p)} \cdot \delta_{\mathfrak{p}}^{\mathcal{L}}}} \right)^2 \cdot d_{\mathbb{Q}(\lambda)}.$$

Thus, Lemma 6.46 establishes that the measure of the fundamental domains of $\tilde{\mathcal{L}}$ and $\Lambda(\Omega_{\mathcal{L}})$ are equal, wherefore these two lattices are equal. □

We note that one may also use the notation $\mathcal{L}_{\text{ext}} = \mathcal{L} \cdot \mathbb{Z}[\frac{1}{\lambda}]$.

Remark 6.52. The previous considerations also yield the following interpretation, why the matrices $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ in Remark 6.36 cannot define the same lattice (note that here $\mathcal{L} = \mathcal{L}_{\mathrm{ext}}$): Both matrices have the same PF-eigenvalue $\lambda = 3 + 2\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. The prime ideals of $\mathbb{Q}(\sqrt{2})$ can be calculated by Lemma 3.82: The prime 2 is ramified (namely, $(2) = (\sqrt{2})^2$), the primes $p \equiv 1 \bmod 4$ are splitting and the primes $p \equiv 3 \bmod 4$ are inert.

An left PF-eigenvector of $\boldsymbol{M}^{(1)}$ is given by $(\ell_1^{(1)}, \ell_2^{(1)}) = (\lambda - 3, 1) = (2\sqrt{2}, 1)$. One calculates the discriminant $d_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\ell_1^{(1)}, \ell_2^{(1)}) = 32$. Similarly, an left PF-eigenvector of $\boldsymbol{M}^{(2)}$ is given by $(\ell_1^{(2)}, \ell_2^{(2)}) = (\frac{1}{2}(\lambda - 3), 1) = (\sqrt{2}, 1)$, wherefore we obtain $d_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\ell_1^{(2)}, \ell_2^{(2)}) = d_{\mathbb{Q}(\sqrt{2})} = 8$.

Obviously, we have $\mathcal{L}^{(2)} = \mathbb{Z}[\sqrt{2}] = \mathfrak{o}_{\mathbb{Q}(\sqrt{2})}$, wherefore we have $\delta_{\mathfrak{p}}^{(2)} = 0$ for all prime ideals $\mathfrak{p}$. Moreover, $\tilde{\mathcal{L}}^{(1)}$ is a sublattice of $\tilde{\mathcal{L}}^{(2)}$ of index 2. Assume that there is a factor $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ such that $((a+b\sqrt{2}) \cdot \ell_1^{(2)}, (a+b\sqrt{2}) \cdot \ell_2^{(2)})$ spans the same lattice as $\mathcal{L}^{(1)} = \{c_1 + 2c_2\sqrt{2} \,|\, c_i \in \mathbb{Z}\}$. Then, we have $(a + b\sqrt{2})\mathcal{L}^{(2)} = \mathcal{L}^{(1)}$, and therefore $|N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(a + b\sqrt{2})| = |a^2 - 2b^2| = 2$ (this implies that $\|a + b\sqrt{2}\|_{(\sqrt{2})} = \frac{1}{2}$ and $\|a + b\sqrt{2}\|_{\mathfrak{p}} = 1$ otherwise). But $1, \sqrt{2} \in \mathcal{L}^{(2)}$, which yields $a, b \in 2\mathbb{Z}$. But this contradicts $|a^2 - 2b^2| = 2$, wherefore no such factor exists and our findings in Remark 6.36 are confirmed.

Alternatively, we can simply confirm that $\mathcal{L}^{(2)} = \mathfrak{o}_{\mathbb{Q}(\sqrt{2})}$ is (trivially) an ideal, while $\mathcal{L}^{(1)}$ is not[12] a (fractional) ideal. But the assumption that there is a factor $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ as above implies that $\mathcal{L}^{(1)}$ is also a fractional ideal.

Note that the lattice $\tilde{\mathcal{L}}^{(2)}$ can be written as model set with window $\varOmega_{\mathcal{L}^{(2)}} = \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\sqrt{2})}} \widehat{\mathfrak{o}_{\mathfrak{p}}}$ relative to the CPS $(\mathbb{R}^2, \prod_{\mathfrak{p} \in \mathbb{P}_{QQ(\sqrt{2})}} \mathbb{Q}_{\mathfrak{p}}, \mathbb{Q}(\sqrt{2}))$ (see Proposition 6.51). We observe that for the complete module $\mathcal{L}^{(1)}$, one obtains $\delta_{\mathfrak{p}}^{(1)} = 0 = \delta_{\mathfrak{p}}^{(2)}$; thus, $\tilde{\mathcal{L}}^{(1)}$ – being a sublattice of index 2 of $\tilde{\mathcal{L}}^{(2)}$ – is not a model set relative to the CPS $(\mathbb{R}^2, \prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\sqrt{2})}} \mathbb{Q}_{\mathfrak{p}}, \mathbb{Q}(\sqrt{2}))$, since it would otherwise have the same window $\varOmega_{\mathcal{L}^{(1)}} = \varOmega_{\mathcal{L}^{(2)}}$.

We now show that we have a CPS $(\mathbb{R}, H_{\mathrm{ext}}, \tilde{\mathcal{L}}_{\mathrm{ext}})$.

Lemma 6.53. $\tilde{\mathcal{L}}_{\mathrm{ext}} = (\widetilde{\lambda \mathcal{L}_{\mathrm{ext}}})$.

Proof. We have $\lambda \mathcal{L}_{\mathrm{ext}} \subset \mathcal{L}_{\mathrm{ext}}$, and thus $\lambda^\star \mathcal{L}_{\mathrm{ext}}^\star = (\lambda \mathcal{L}_{\mathrm{ext}})^\star \subset \mathcal{L}_{\mathrm{ext}}^\star$. Therefore, we have $(\widetilde{\lambda \mathcal{L}_{\mathrm{ext}}}) \subset \tilde{\mathcal{L}}_{\mathrm{ext}}$. Obviously, $(\widetilde{\lambda \mathcal{L}_{\mathrm{ext}}})$ is a sublattice of $\tilde{\mathcal{L}}_{\mathrm{ext}}$ (compare Lemma 6.39). We compare $\mu_{\mathbb{R} \times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}}))$ and $\mu_{\mathbb{R} \times H_{\mathrm{ext}}}(\mathrm{FD}((\widetilde{\lambda \mathcal{L}_{\mathrm{ext}}})))$: Artin's product formula (see Lemma 3b.11), yields, in particular, $(\prod_{i=1}^n |\sigma_i(\lambda)|) \cdot \left(\prod_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_{\mathfrak{p}} < 1} \|\lambda\|_{\mathfrak{p}}\right) = 1$. With Proposition 4.38, we obtain

$$\begin{aligned}
\mu_{\mathbb{R} \times H_{\mathrm{ext}}}(\mathrm{FD}((\widetilde{\lambda \mathcal{L}_{\mathrm{ext}}}))) &= \left(\prod_{i=1}^n |\sigma_i(\lambda)|\right) \cdot \left(\prod_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \\ \|\lambda\|_{\mathfrak{p}} < 1}} \|\lambda\|_{\mathfrak{p}}\right) \cdot \mu_{\mathbb{R} \times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}})) \\
&= \mu_{\mathbb{R} \times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}})).
\end{aligned}$$

This establishes the claim □

[12] Assume that $\mathcal{L}^{(1)}$ is a (fractional) ideal. From $[\mathfrak{o}_{\mathbb{Q}(\sqrt{2})} : \mathcal{L}^{(1)}] = 2$ (thus $N\mathcal{L}^{(1)} = 2$) it would follow that $\mathcal{L}^{(1)} = (\sqrt{2})$, which is false (*e.g.*, one has $1 \in \mathcal{L}^{(1)}$ but $1 \notin (\sqrt{2})$).

This statement has the following consequence, which generalises Corollary 6.24.

Corollary 6.54. *One has* $\lambda\,\mathcal{L}_{\text{ext}} = \mathcal{L}_{\text{ext}}$. □

Lemma 6.55. $\mathcal{L}^\star_{\text{ext}}$ *is dense in* H_{ext}.

Proof. We have $\lambda\mathcal{L}_{\text{ext}} = \mathcal{L}_{\text{ext}}$, and thus $\lambda^\star\mathcal{L}^\star_{\text{ext}} = (\lambda\mathcal{L}_{\text{ext}})^\star = \mathcal{L}^\star_{\text{ext}}$. We note that by the definition of H_{ext}, $f_0(x) = \lambda^\star \cdot x$ is a contraction on H_{ext}.

Since $\tilde{\mathcal{L}}_{\text{ext}}$ is a lattice in $\mathbb{R}\times H_{\text{ext}}$, hence relatively dense, $\mathcal{L}^\star_{\text{ext}}$ must also be relatively dense in H_{ext}. Therefore, there is a radius $R > 0$ such that $B_{\leq R} + \mathcal{L}^\star_{\text{ext}} = H_{\text{ext}}$. Multiplying this equation by $\lambda^\star$, *i.e.*, applying the contraction f_0, consequently establishes the denseness. □

Altogether, we have now the following symmetric CPS $(\mathbb{R}\,, H_{\text{ext}}\,, \tilde{\mathcal{L}}_{\text{ext}})$, with the additional property that $\hat{f}_0 : H_{\text{ext}} \to H_{\text{ext}}$, $\hat{f}_0(z) = \lambda^\star\, z$ is a homeomorphism:

$$
\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\;\pi_1\;} & \mathbb{R}\times H_{\text{ext}} & \xrightarrow{\;\pi_2\;} & H_{\text{ext}} \cong \mathbb{R}^{r-1}\times\mathbb{C}^s\times \prod\limits_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)}\\ \|\lambda\|_{\mathfrak{p}}<1}} \mathbb{Q}_{\mathfrak{p}} \\
\text{dense}\,\cup & & \cup & & \cup\,\text{dense} \\
\mathcal{L}_{\text{ext}} & \xleftrightarrow{\;1-1\;} & \tilde{\mathcal{L}}_{\text{ext}} & \xleftrightarrow{\;1-1\;} & \pi_2(\tilde{\mathcal{L}}_{\text{ext}}) = \mathcal{L}^\star_{\text{ext}}
\end{array}
\tag{6.7}
$$

We may therefore apply the theory of iterated function systems established in Section 4.8 to the situation here. This will be done in the next section. By $\mathcal{L}\subset\mathcal{L}_{\text{ext}}$ and $H\subset H_{\text{ext}}$, we may also often switch between the two cut and project schemes $(\mathbb{R}\,, H\,, \tilde{\mathcal{L}})$ and $(\mathbb{R}\,, H_{\text{ext}}\,, \tilde{\mathcal{L}}_{\text{ext}})$. This is justified by the following statement.

Corollary 6.56. *We assume the setting of Proposition* 6.51. *Then,*

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\text{ext}} \cap (\mathbb{R}\times H)\,.$$

Proof. The implication $\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}_{\text{ext}} \cap (\mathbb{R}\times H)$ is obvious. For the converse, we first show that $\tilde{\mathcal{L}} = (\widetilde{\tfrac{1}{\lambda}\,\mathcal{L}}) \cap (\mathbb{R}\times H)$: From Lemma 6.39, we conclude that $\frac{1}{\lambda}\,\mathcal{L}$ can be written as disjoint union of $|\det \boldsymbol{S}\sigma|$ copies of $\mathcal{L}$, *i.e.*, there are $t_i \in \frac{1}{\lambda}\mathcal{L}$ with $1 \leq i \leq |\det \boldsymbol{S}\sigma| - 1 = m$ such that $\frac{1}{\lambda}\,\mathcal{L} = \mathcal{L} \;\cup\; \mathcal{L}+t_1 \;\cup\; \ldots \;\cup\; \mathcal{L}+t_m$ and this union is disjoint. But we also have $[\frac{1}{\lambda}\,H : H] = |\det \boldsymbol{S}\sigma|$, since $N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda) = |\det \boldsymbol{S}\sigma| = \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_{\mathfrak{p}}<1} \|\frac{1}{\lambda}\|_{\mathfrak{p}}$ (see proof of Lemma 6.44 and Corollary 6.45). Thus, one has $\tilde{\mathcal{L}} + \tilde{t}_i \subset (\mathbb{R}\times H) + \tilde{t}_i$, and the set $\{0, \tilde{t}_1, \ldots, \tilde{t}_m\}$ is a set of representatives of the factor group $(\mathbb{R}\times\frac{1}{\lambda}\,H)/(\mathbb{R}\times H) \cong (\frac{1}{\lambda}\,H)/H$ (we recall that H – and therefore also $\frac{1}{\lambda}\,H$ – is a compactly generated group). But this establishes $\tilde{\mathcal{L}} = (\widetilde{\tfrac{1}{\lambda}\,\mathcal{L}}) \cap (\mathbb{R}\times H)$.

Similarly, it follows that $\tilde{\mathcal{L}} = (\widetilde{\tfrac{1}{\lambda^m}\,\mathcal{L}}) \cap (\mathbb{R}\times H)$ for any $m\in\mathbb{N}$. But this also proves the claim. □

We also note that any (compact) set $W \subset H$ is embedded into H_{ext} in the obvious way, and we normalise the Haar measure such that $\mu_H(W) = \mu_{H_{\text{ext}}}(W)$ for any measurable set W. Obviously, one has $\mu_{\mathbb{R}\times H}(\text{FD}(\tilde{\mathcal{L}})) = \mu_{\mathbb{R}\times H_{\text{ext}}}(\text{FD}(\tilde{\mathcal{L}}_{\text{ext}}))$, since one may choose the same fundamental domain for the lattice $\tilde{\mathcal{L}}\subset\mathbb{R}\times H$ and its "extended" version $\tilde{\mathcal{L}}_{\text{ext}}\subset\mathbb{R}\times H_{\text{ext}}$

We now construct the dual lattice of $\tilde{\mathcal{L}}_{\text{ext}}$, which gives us information about the diffraction spectrum (see Folklore Theorem 5a.10) provided we have a model set. The observation that we can sometimes write the lattice $\tilde{\mathcal{L}}_{\text{ext}}$ as model set in the adele ring $\mathbb{A}_{\mathbb{Q}(\lambda)}$, gives us also an easy method to calculate its annihilator $\left(\tilde{\mathcal{L}}_{\text{ext}}\right)^{\perp}$ by Folklore Theorem 5a.10.

Lemma 6.57. *Assume the setting of Proposition* 6.51*. If* $\mathcal{L}$ *is a fractional ideal and* $\mathfrak{D}_{\mathbb{Q}(\lambda)/\mathbb{Q}}$ *is the different with prime ideal factorisation* $\mathfrak{D}_{\mathbb{Q}(\lambda)/\mathbb{Q}} = \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)}} \mathfrak{p}^{n_\mathfrak{p}}$*, then the annihilator* $\left(\tilde{\mathcal{L}}_{\text{ext}}\right)^{\perp}$ *is given as model set* $\Lambda(\Omega_{\mathcal{L}}^{\perp})$ *where*

$$\Omega_{\mathcal{L}}^{\perp} = \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_\mathfrak{p}=1}} \mathfrak{p}^{n_\mathfrak{p}-\delta_\mathfrak{p}^{\mathcal{L}}}.$$

Proof. By Lemma 3b.13, $\mathbb{A}_{\mathbb{Q}(\lambda)}$ is self-dual and the annihilator of $\mathbb{Q}(\lambda)$ is $\mathbb{Q}(\lambda)$ again. So, we have to calculate the Fourier transform of the characteristic function $\chi_{\Omega_{\mathcal{L}}}$ of the window $\Omega_{\mathcal{L}}$ in $H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$. Observing that the Fourier transform is given as product of the Fourier transform in the components of $H_{\mathbb{A}_{\mathbb{Q}(\lambda)}}$, an application of Lemma 5a.13 yields

$$\chi_{\Omega_{\mathcal{L}}}^{*} = \left(\prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_\mathfrak{p}=1}} p^{f_{\mathfrak{p}|(p)}\cdot\delta_\mathfrak{p}^{\mathcal{L}}-\frac{n_\mathfrak{p}}{2}} \right) \cdot \chi_{\Omega_{\mathcal{L}}^{\perp}}.$$

Consequently, the diffraction spectrum of the lattice $\tilde{\mathcal{L}} = \Lambda(\Omega_{\mathcal{L}})$ is concentrated on the lattice $\Lambda(\Omega_{\mathcal{L}}^{\perp})$. From Proposition 5a.11 we conclude that $\tilde{\mathcal{L}}^{\perp} = \Lambda(\Omega_{\mathcal{L}}^{\perp})$. □

In general, the annihilator $\left(\tilde{\mathcal{L}}_{\text{ext}}\right)^{\perp}$ is calculated *via* the codifferent (see Definition 3.50) as the following statement shows.

Proposition 6.58. *Let* σ *be a Pisot substitution with PV-number* λ *(where the signature of* $\mathbb{Q}(\lambda)$ *is* $[r\,,s]$*). Let* $\underline{\Lambda}$ *the representation with natural intervals* $[0\,,\ell_i]$ *(where* $\ell_i\in\mathbb{Q}(\lambda)$*) of any of its fixed points. Define* $\mathcal{L}=\langle\ell_1,\dots,\ell_n\rangle_{\mathbb{Z}}$*,* $\mathcal{L}_{\text{ext}}=\bigcup_{k\geq 0}\mathcal{L}/\lambda^k$ *and* $\tilde{\mathcal{L}}_{\text{ext}}$ *as before. Let* $\mathcal{L}^{\wedge}$ *be the codifferent of* $\mathcal{L}$*. Then, the annihilator of* $\tilde{\mathcal{L}}_{\text{ext}}$ *is given by*

$$\left(\tilde{\mathcal{L}}_{\text{ext}}\right)^{\perp} = \left(\widetilde{\bigcup_{k\geq 0}\frac{1}{\lambda^k}\,\mathcal{L}^{\wedge}}\right),$$

i.e., the diagonal embedding of $\bigcup_{k\geq 0}\mathcal{L}^{\wedge}/\lambda^k$ *in* $G^{*}\times H_{\text{ext}}^{*}\cong\mathbb{R}^r\times\mathbb{C}^s\times\prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\,\|\lambda\|_\mathfrak{p}<1}\mathbb{Q}_\mathfrak{p} = G\times H_{\text{ext}}$*. The annihilator is again a lattice in* $G\times H_{\text{ext}}$*. If* λ *is a unit, one has* $\tilde{\mathcal{L}}^{\perp}=(\mathcal{L}_{\text{ext}})^{\perp}=\widetilde{\mathcal{L}^{\wedge}}$*.*

Proof. The statement that $(\tilde{\mathcal{L}}_{\text{ext}})^{\perp}$ is a lattice in $G\times H_{\text{ext}}$, follows from Corollary 3.124 and the observation that $G\times H_{\text{ext}}$ is self-dual (see Remark 3.121).

If λ is a unit in $\mathbb{Q}(\lambda)$, we have $\mathcal{L}=\mathcal{L}_{\text{ext}}$ (see Corollary 6.24) and there are no $\mathfrak{p}$-adic components in $H_{\text{ext}}=\mathbb{R}^{r-1}\times\mathbb{C}^s=H$. By Definition 3.93, $\mathcal{L}^{\wedge}$ is the dual lattice of $\mathcal{L}$ and this is consistent with the use of word “dual lattice” in Remark 3.125 for $\widetilde{\mathcal{L}^\wedge}$. Comparing definitions, one sees that $\widetilde{\mathcal{L}^\wedge} = \tilde{\mathcal{L}}^\perp$ (see Definition 3.122). We observe that $\mathcal{L}^\wedge$ is again a complete module in $\mathbb{Q}(\lambda)$, wherefore – by Corollary 6.24 and since $\widetilde{a \cdot \chi}(\tilde{x}) = \tilde{\chi}(\widetilde{a \cdot x})$ by self-duality ($a, \chi, x \in \mathbb{Q}(\lambda)$) – one has $\frac{1}{\lambda}\mathcal{L}^\wedge = \mathcal{L}^\wedge$. This proves the claim if λ is a unit.

Now we consider the general case. We note that a $\mathbb{Z}$-basis of $\mathcal{L}$ is given by $\{\ell_1, \dots, \ell_n\}$. We denote the (dual) $\mathbb{Z}$-basis of $\mathcal{L}$ (*i.e.*, the $\mathbb{Z}$-basis of $\mathcal{L}^\wedge$, see Lemma 3.51) by $\{\ell_i^\perp, \dots, \ell_n^\perp\}$ (where $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\ell_i^\perp \cdot \ell_j) = 1$ if $i = j$ and 0 otherwise).

We first show

$$\left(\tilde{\mathcal{L}}_{\text{ext}}\right)^\perp \supset \left(\widetilde{\bigcup_{k \geq 0} \frac{1}{\lambda^k}\mathcal{L}^\wedge}\right).$$

To this end, choose $x \in \mathcal{L}_{\text{ext}}$ and $\chi \in \bigcup_{k\geq 0} \mathcal{L}^\wedge/\lambda^k$ arbitrary. We have to show that $\tilde{\chi}(\tilde{x}) = \langle \tilde{\chi}, \tilde{x} \rangle = 1$. We can write $x = (a_1 \ell_1 + \dots + a_b \ell_n)/\lambda^k$ and $\chi = (b_1 \ell_1^\perp + \dots + b_n \ell_n^\perp)/\lambda^{k'}$ with $a_i, b_i \in \mathbb{Z}$ (for all $1 \leq i \leq n$) and $k, k' \geq 0$. Noting that $\chi \in \mathbb{Q}(\lambda)$, one readily establishes

$$\chi \cdot x = \left(\sum_{i=1}^n a_i\, b_i\right) / \lambda^{k+k'}.$$

Using Lemma 3.116 (and Proposition 3.120), one now has to calculate $\langle \tilde{\chi}, \tilde{x} \rangle$. But this reduces to showing that

$$\mathbb{Z} \ni \left(\sum_{i=1}^n a_i\, b_i\right) \left(\frac{1}{\lambda^{k+k'}} + \frac{1}{\lambda_2^{k+k'}} + \frac{1}{\lambda_r^{k+k'}} + \frac{1}{\lambda_{r+1}^{k+k'}} + \frac{1}{\overline{\lambda}_{r+1}^{k+k'}} + \dots + \frac{1}{\lambda_{r+s}^{k+k'}} + \frac{1}{\overline{\lambda}_{r+s}^{k+k'}} \right.$$
$$\left. - \sum_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)} \\ \|\lambda\|_\mathfrak{p} < 1}} \vartheta_p \left(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}\left(\frac{1}{\lambda^{k+k'}}\right)\right)\right).$$

The first factor is an integer, while the first part of the second factor is simply the trace of $1/\lambda^{k+k'}$. Thus, we are done if we can show that

$$T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\left(\frac{1}{\lambda^{k+k'}}\right) - \sum_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)} \\ \|\lambda\|_\mathfrak{p} < 1}} \vartheta_p \left(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}\left(\frac{1}{\lambda^{k+k'}}\right)\right) \in \mathbb{Z}.$$

But this follows from Theorem 3.72(iii) (note that $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(1/\lambda^{k+k'}) \in \mathbb{Z}/(\det \boldsymbol{S}\sigma)^{k+k'}$) and the definition of ϑ_p as “p-adic fractional part”, wherefore $\vartheta_p(T_{\mathbb{Q}_\mathfrak{p}/\mathbb{Q}_p}(1/\lambda^{k+k'})) = 0$ for all prime ideals $\mathfrak{p}$ with $\|\lambda\|_\mathfrak{p} = 1$.

We only sketch the reverse implication. Suppose $\tilde{\chi} \in \left(\tilde{\mathcal{L}}_{\text{ext}}\right)^\perp$. Obviously, we have

$$\tilde{\chi} \in \widetilde{\widetilde{\mathbb{Q}(\lambda)}} \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)} \\ \|\lambda\|_\mathfrak{p} < 1}} \mathbb{Q}_\mathfrak{p} = \langle \widetilde{\widetilde{\ell_1^\perp}}, \dots, \widetilde{\widetilde{\ell_n^\perp}} \rangle_\mathbb{Q} \times \prod_{\substack{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)} \\ \|\lambda\|_\mathfrak{p} < 1}} \mathbb{Q}_\mathfrak{p}$$

by Proposition 3.120, where $\tilde{\tilde{x}}$ denotes the geometric image of $x \in \mathbb{Q}(\lambda)$, see Definition 3.94. Writing $\tilde{\chi} = \left((q_1\,\ell_1^\perp + \ldots + q_n\,\ell_n^\perp)^{\widetilde{\ }}, q_{\mathfrak{p}_1}, \ldots, q_{\mathfrak{p}_k}\right)$ (with $q_1, \ldots, q_n \in \mathbb{Q}$ and $q_{\mathfrak{p}_i} \in \mathbb{Q}_{\mathfrak{p}_i}$, where we have enumerated the $\mathfrak{p}$-adic fields appearing in H_{ext} from $\mathbb{Q}_{\mathfrak{p}_1}$ to $\mathbb{Q}_{\mathfrak{p}_k}$), a calculation as above shows that

$$\tilde{\chi}(\widetilde{\ell_j}) = q_j - \sum_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)} \\ \|\lambda\|_{\mathfrak{p}}<1}} \vartheta_p\left(T_{\mathbb{Q}_{\mathfrak{p}}/\mathbb{Q}_p}\left(q_{\mathfrak{p}} \cdot \ell_j\right)\right)$$

for every $1 \leq j \leq n$. Since $\tilde{\chi} \in \left(\tilde{\mathcal{L}}_{\text{ext}}\right)^\perp$ and by the definition of ϑ_p, one immediately follows that the denominator of q_j divides some power of $\det \boldsymbol{S}\sigma$. We now look at the numbers $q_{\mathfrak{p}}$: Obviously, the choice $q_{\mathfrak{p}} = q_1\,\ell_1^\perp + \ldots + q_n\,\ell_n^\perp$ for every $\mathfrak{p}$ would yield $\tilde{\chi}(\widetilde{\ell_j}) \in \mathbb{Z}$ (by $q_{\mathfrak{p}}\,\ell_j = q_j$ and an argument like above). By the definition of ϑ_p as p-adic "fractional part", one may equally well chose a $q_{\mathfrak{p}} \in \mathbb{Q}_{\mathfrak{p}}$ such that $q_{\mathfrak{p}}\,\ell_j \in q_j + \widehat{\mathfrak{o}_{\mathfrak{p}}}$ for every $\mathfrak{p}$. Thus, the next goal is to fix the digits of the $\mathfrak{p}$-adic series expansion of $q_{\mathfrak{p}}$ (see Lemma 3.68). This is achieved by considering $\tilde{\chi}(\widetilde{\ell_j/\lambda^m})$ for all $m \in \mathbb{Z}_{\geq 0}$, which establishes $q_{\mathfrak{p}} = q_1\,\ell_1^\perp + \ldots + q_n\,\ell_n^\perp$ for every $\mathfrak{p}$ in question. Moreover, this also establishes the reverse implication and therefore the claim. □

Remark 6.59. The previous proposition is the statement of [142, Eq. (4.5) on p. 161].

From Lemma 3.51 we deduce[13] that the codfferent of $\mathbb{Z}[\lambda]$ is given by $\mathbb{Z}[\lambda]/p'(\lambda)$, where p' is the formal derivative of $p(x) = \text{Irr}(\lambda, \mathbb{Q}, x)$. We have chosen (respectively, we may choose) the natural lengths such that $\mathcal{L} \subset \mathbb{Z}[\lambda]$. To calculate the codifferent of $\mathcal{L}$, we observe Remark 6.38 where we noted that there is a matrix $\boldsymbol{A} \in GL_n(\mathbb{Z})$ (or, more general by $\boldsymbol{A} \in GL_n(\mathbb{Q})$) such that $(1, \ldots, \lambda^{n-1})\,\boldsymbol{A} = (\ell_1, \ldots \ell_n)$. Also, one may calculate the dual basis to $\{1, \lambda, \ldots, \lambda^{n-1}\}$ (which is given by some basis transformation of $\{1/p'(\lambda), \ldots, \lambda^{n-1}/p'(\lambda)\}$); we denote the dual basis by $\{e_i^\perp\}_{i=1}^n$, where $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(e_i^\perp \cdot \lambda^j) = 1$ if $i-1=j$ and 0 otherwise (where $0 \leq j \leq n-1$ and $1 \leq i \leq n$). Then, the dual basis $\{\ell_i^\perp\}_{i=1}^n$ of $\mathcal{L}^\perp$ to the basis $\{\ell_i\}_{i=1}^n$ of $\mathcal{L}$ is given by

$$(e_1^\perp, \ldots, e_n^\perp)\,(\boldsymbol{A}^{-1})^t = (\ell_1^\perp, \ldots, \ell_n^\perp).$$

Remark 6.60. For more on the relationship between the above CPS and the dual lattice (respectively the CPS appearing in the calculation of the diffraction pattern, see Folklore Theorem 5a.10), see Definition 7.17.

Noting that $f_0^{-1}(x) = \frac{1}{\lambda}x$, we have

$$\mathcal{L}_{\text{ext}} = \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m}\mathcal{L} = \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m}\langle \ell_1, \ldots, \ell_n\rangle_{\mathbb{Z}} \subset \bigcup_{m=0}^{\infty} \frac{1}{|\det \boldsymbol{S}\sigma|^m}\mathcal{L} = \bigcup_{m=0}^{\infty} \frac{1}{|\det \boldsymbol{S}\sigma|^m}\langle \ell_1, \ldots, \ell_n\rangle_{\mathbb{Z}},$$

where we use $\text{Irr}(\lambda, \mathbb{Q}, x) = x^n + \ldots + (-1)^n \cdot \det \boldsymbol{S}\sigma = \det(x \cdot \boldsymbol{E} - \boldsymbol{S}\sigma)$. Note that this also establishes $\mathcal{L}_{\text{ext}} = \mathcal{L}$ if $|\det \boldsymbol{S}\sigma| = 1$, *i.e.*, if λ is a unit (compare Corollary 6.24).

Remark 6.61. One may also define the maps ϑ and χ (see Definition 6.30; note that χ does not denote a character here) over the rational span $\langle \ell_1, \ldots, \ell_n\rangle_{\mathbb{Q}}$ (then ϑ takes values in $\mathbb{Q}^n$, while χ takes values in $\mathbb{Q}$), where we note that $\langle \ell_1, \ldots, \ell_n\rangle_{\mathbb{Q}} = \mathbb{Q}(\lambda)$.

Moreover, one might hope that defining $\mathcal{M}_{\text{ext}} = \{x \in \mathcal{L}_{\text{ext}} \,|\, \chi(x) = 0\}$, one obtains a periodic tiling $\mathcal{M}_{\text{ext}}^\star + \text{supp}\,\underline{\Omega}$ of H_{ext}. But the following example shows that this is in general not the

[13] One also has $\mathbb{Z}[\lambda] \subset \mathfrak{o}_{\mathbb{Q}(\lambda)}$ and – by the definition of the different (see Definition 3.50) – that $\mathfrak{o}_{\mathbb{Q}(\lambda)}^\wedge = \mathfrak{D}_{\mathbb{Q}(\lambda)/\mathbb{Q}}^{-1}$ (also observe the appearance of the conductor $\mathfrak{F}_{\mathbb{Z}[\lambda]}$ in Lemma 3.51).

case: Consider the Pisot substitution $a \mapsto aab$ and $b \mapsto abab$ with PV-number $\lambda = 2 + \sqrt{2}$ with natural lengths $\ell = (1\,, \sqrt{2})$ (this example will be considered in Example 6.112 as σ_{BMS}). Then, $\mathcal{L} = \mathbb{Z}[\sqrt{2}] = \mathfrak{o}_{\mathbb{Q}(\lambda)}$ and $\frac{1}{\lambda} = \frac{1}{2}\,(2 - \sqrt{2})$. We note that $H_{\mathrm{ext}} = \mathbb{R} \times \mathbb{Q}_2(\sqrt{2})$. By

$$\frac{1}{\lambda}\,(m + \sqrt{2}\,n) = (m - n) + \sqrt{2}\,\left(\frac{2\,n - m}{2}\right),$$

one obtains $\mathcal{L}_{\mathrm{ext}} \cap (\mathbb{R} \times \frac{1}{\sqrt{2}}\mathbb{Z}_2[\sqrt{2}]) = \mathbb{Z} \oplus \sqrt{2}\,\frac{\mathbb{Z}}{2}$. But this shows that

$$\mathcal{M}_{\mathrm{ext}} \cap (\mathbb{R}\times\frac{1}{\sqrt{2}}\mathbb{Z}_2[\sqrt{2}]) = \{x \in \mathcal{L}_{\mathrm{ext}} \cap (\mathbb{R}\times\frac{1}{\sqrt{2}}\mathbb{Z}_2[\sqrt{2}])\,|\,\chi(x) = 0\} = \{x \in \mathcal{L}\,|\,\chi(x) = 0\} = \mathcal{M},$$

wherefore $\mathcal{M}^{\star}_{\mathrm{ext}} + \operatorname{supp} \underline{\Omega}$ has empty intersection with $(\mathbb{R} \times \frac{1}{\sqrt{2}}\mathbb{Z}_2[\sqrt{2}]) + (\frac{1}{2}\,\sqrt{2})^{\star}$.

Because of these finding, one might either modify the definitions of the maps ϑ and χ to obtain a "good" $\mathcal{M}_{\mathrm{ext}}$ (we have not come up with one so far), or – as we will do in the following – one might not consider their extensions to $\mathcal{L}_{\mathrm{ext}}$ respectively $\mathbb{Q}(\lambda)$. Moreover, if we speak of the "periodic tiling", we will always mean the periodic tiling $\mathcal{M}^{\star} + \operatorname{supp} \underline{\Omega}$ of H. However, we note that this periodic tiling might be easily extended to a periodic tiling of H_{ext}. Actually, there are infinitely many possibilities (in the non-unimodular case) to obtain a periodic tiling, *e.g.*, in the previous example one has

$$H_{\mathrm{ext}} = H \,\dot\cup\, H + \frac{1}{2}\sqrt{2} \,\dot\cup\, H + \frac{1}{2} \,\dot\cup\, H + \frac{1}{2}(1 + \sqrt{2}) \,\dot\cup\, \ldots,$$

wherefore, for any choice $t_1, t_2, \ldots \in H$, we define $\mathcal{M}' = \mathcal{M} \cup \mathcal{M} + t_1 + \frac{1}{2}\sqrt{2} \cup \mathcal{M} + t_2 + \frac{1}{2} \cup \ldots$ and obtain the tiling $(\mathcal{M}')^{\star} + \operatorname{supp} \underline{\Omega}$ of H_{ext}. However, there is no choice $\mathcal{M}' \subset \mathcal{L}_{\mathrm{ext}}$ which is also a group.

Example 6.62. The inclusion sign in

$$\bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}} \subset \bigcup_{m=0}^{\infty} \frac{1}{|\det \boldsymbol{S}\sigma|^m} \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}},$$

is in general proper, as the following example shows: Let σ be a Pisot substitution with

$$\boldsymbol{S}\sigma = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

(one has $\lambda \approx 5.1508$ and $\lambda_{2,3} \approx -0.0754 \pm i \cdot 0.8780$). Let ℓ be a left eigenvector (*e.g.*, $\ell = (\lambda^2 - 3\lambda - 2, -\lambda^2 + 7\lambda - 2, 4)$). Using the Parikh map one obtains $\vartheta(\lambda \cdot x) = (\boldsymbol{S}\sigma)\vartheta(x)$. Moreover, extending the Parikh map to the rational span of the lengths ℓ_i, *i.e.*, $\vartheta : \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Q}} \to \mathbb{Q}^n$ (which is possible since the ℓ_i's are rationally independent), we also have $\vartheta(\lambda \cdot x) = (\boldsymbol{S}\sigma)\,\vartheta(x)$ and $\vartheta(\frac{1}{\lambda} \cdot x) = (\boldsymbol{S}\sigma)^{-1}\,\vartheta(x)$. Moreover, we observe that for every $z \in \bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}}$ there is an $m \in \mathbb{Z}_{\geq 0}$ such that $\vartheta(\lambda^m \cdot z) \in \mathbb{Z}^n$. But one easily confirms that multiplication with powers of λ of $\frac{1}{2}\ell_1 + \frac{1}{2}\ell_2 + \frac{1}{2}\ell_3$ never yields integer coefficients (this is always the case if the denominators of all three coefficients or exactly one coefficient are 2, where in the latter case the remaining two coefficients are integers). Consequently, $\frac{1}{2}\ell_1 + \frac{1}{2}\ell_2 + \frac{1}{2}\ell_3$ is not in $\bigcup_{m=0}^{\infty} \frac{1}{\lambda^m} \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}}$, but obviously in $\bigcup_{m=0}^{\infty} \frac{1}{|\det \boldsymbol{S}\sigma|^m} \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}}$.

We now do some number theoretic calculations to gain some deeper insight into this example and into local fields and their extensions in general. The maximal order of $\mathbb{Q}(\lambda)$ is given by $\mathfrak{o}_{\mathbb{Q}(\lambda)} = \langle 1, \lambda, \frac{1}{4}(\lambda^2 - 3\lambda - 2)\rangle_{\mathbb{Z}}$. The ideal (2) has the prime ideal factorisation (obtained by KANT [293])

$$(2) = \left(\frac{1}{2}\lambda^2 - \frac{5}{2}\lambda\right) \cdot \left(\frac{1}{4}(\lambda^2 - 3\lambda - 2)\right)^2 = \mathfrak{p}_1 \cdot \mathfrak{p}_2^2.$$

One has $\|\lambda\|_{\mathfrak{p}_1} = 1$ and $\|\lambda\|_{\mathfrak{p}_2} = \frac{1}{4}$.

In view of Theorem 3.72, we note that $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$ factors as

$$\begin{aligned}\mathrm{Irr}(\lambda, \mathbb{Q}, x) = x^3 - 5\,x^2 - 4 = (x - .100100110011\ldots)\cdot \\ (x^2 + .001000110011\ldots x + .001001111010\ldots) = p_1(x)\cdot p_2(x)\end{aligned}$$

in $\mathbb{Q}_2$ (the existence of such a factorisation is confirmed by Proposition 3.98 with $\overline{g}(x) = (x+1)$ and $\overline{h}(x) = x^2$, respectively Proposition 3.99 with $g_0(x) = (x+1)$, $h_0(x) = x^2$ and therefore $r = 0$). By Theorem 3.72, we therefore have at least two different prolongations of the 2-adic valuation to $\mathbb{Q}(\lambda)$. Since $p_1(x) = (x - .100100110011\ldots)$ is irreducible, we have $\mathbb{Q}_2(x)/p_1(x) \cong \mathbb{Q}_2$. For $p_2(x)$, one observes that it must have a root in either $\mathbb{Q}_2$ (then it is a reducible polynomial in $\mathbb{Q}_2$) or in one of the seven non-isomorphic quadratic extension of $\mathbb{Q}_2$, see Proposition 3.109. Observing that $\mathbb{Q}_2(\sqrt{10}) \cong \mathbb{Q}_2(\sqrt{-6})$, one then confirms that

$$.011110011001\ldots \pm \sqrt{10}\cdot .001010011101\ldots = .011110011001\ldots \pm \sqrt{-6}\cdot .001000010101\ldots$$

are the roots[14] of $p_2(x)$ (which we therefore identify[15] with λ). Consequently, $p_2(x)$ is irreducible over $\mathbb{Q}_2$ and $\mathbb{Q}_2(x)/p_2(x) \cong \mathbb{Q}_2(\sqrt{10}) \cong \mathbb{Q}_2(\sqrt{-6})$. This also establishes $\|\lambda\|_{\mathbb{Q}_2} = 1$ and $\|\lambda\|_{\mathbb{Q}_2(\sqrt{10})} = \frac{1}{4}$ (see Corollary 3.76) without explicit knowledge of the prime ideals of $\mathbb{Q}(\lambda)$. Moreover, from Proposition 3.109, we also know that $\mathbb{Q}_2(\sqrt{10})$ is fully ramified, wherefore $f_{\mathbb{Q}_2(\sqrt{10})/\mathbb{Q}_2} = 1$ and $e_{\mathbb{Q}_2(\sqrt{10})/\mathbb{Q}_2} = 2$. From this, we can also establishes that the prime ideal factorisation of (2) has the form $(2) = \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ in $\mathbb{Q}(\lambda)$.

Now, the deeper reason why we only have $\bigcup_{m=0}^{\infty} \frac{1}{\lambda^m}\mathcal{L} \subset \bigcup_{m=0}^{\infty} \frac{1}{|\det \boldsymbol{S}\sigma|^m}\mathcal{L}$ and not equality is the following: The factor 2 (respectively $4 = |\det \boldsymbol{S}\sigma|$) is a contraction on both $\mathbb{Q}_{\mathfrak{p}_1} \cong \mathbb{Q}_2$ and $\mathbb{Q}_{\mathfrak{p}_2} \cong \mathbb{Q}_2(\sqrt{10})$, while the factor λ is only a contraction on $\mathbb{Q}_2(\sqrt{10})$.

6.6. An IFS in the Extended Internal Space

Since we have established the CPS $(\mathbb{R}, H_{\mathrm{ext}}, \tilde{\mathcal{L}}_{\mathrm{ext}})$ for a Pisot substitution, we now look at the (nonsingular primitive) IFS $\boldsymbol{\Theta}^\star$ in H_{ext}. By construction, the map $x \mapsto \lambda^\star \cdot x$ is a contraction on H_{ext}. We therefore obtain the following consequence of Proposition 4.97.

[14] Note that Cardano's formula for solving the cubic $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$ includes expressions with $\sqrt{-3}$ (see [396, Section 2.1.6.2]), wherefore it is not surprising that we find solutions in $\mathbb{Q}_2(\sqrt{-3\cdot 2})$.

[15] In connection with Remark 4.108, we note that $.01111\ldots \pm \sqrt{10}\cdot .00101\ldots$ (respectively $.01111\ldots \pm \sqrt{-6}\cdot .00100\ldots$) is a pair of algebraic conjugate numbers in $\mathbb{Q}_2^a$. Thus, with respect to algebraic conjugation over $\mathbb{Q}_2$, the roots of $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$ form 2 classes, namely the class formed by the root $.10010\ldots$ of $p_1(x)$ and the algebraic conjugate roots of $p_2(x)$. Consequently, in the parlance of Remark 4.108, we identify the application of $\boldsymbol{S}\sigma$ in $\mathbb{Q}_2^3$ with the diagonal map $x \mapsto ((.10010\ldots)\cdot x, (.01111\ldots + \sqrt{10}\cdot .00101\ldots)\cdot x)$ in $\mathbb{Q}_2(.10010\ldots) \times \mathbb{Q}_2(.01111\ldots + \sqrt{10}\cdot .00101\ldots) \cong \mathbb{Q}_2 \times \mathbb{Q}_2(\sqrt{10}) \cong \mathbb{Q}_{\mathfrak{p}_1} \times \mathbb{Q}_{\mathfrak{p}_2}$, *i.e.*, we identify λ with $.10010\ldots$ in $\mathbb{Q}_{\mathfrak{p}_1}$ and with (one member of the conjugate pair) $.01111\ldots + \sqrt{10}\cdot .00101\ldots$ in $\mathbb{Q}_{\mathfrak{p}_2}$.

Corollary 6.63. *Let u be any fixed point of a Pisot substitution σ with* $\operatorname{card}\mathcal{A} \geq 2$. *Let $\underline{\Lambda}$ be the representation with natural lengths of u. Then, there is a family $\underline{\Omega} \subset (\mathcal{K}H_{\text{ext}})^n$ of nonempty compact subsets of H_{ext} such that $\underline{\Lambda} \subset \Lambda(\underline{\Omega})$, namely the attractor of $\boldsymbol{\Theta}^\star$. Moreover, if there is an i such that Ω_i has interior points, then all Ω_i have interior points.*

Proof. The statement that $\underline{\Lambda} \subset \Lambda(\underline{\Omega})$ follows as in the proof of Lemma 5.139 (since we have a finite seed; note that we do not assume FLC here). The remaining part is Proposition 4.97. □

We remark that by the uniqueness of the attractor of the IFS, the sets Ω_i are independent of the used fixed point u (obviously, they only depend on the IFS $\boldsymbol{\Theta}^\star$ and thus on the EMFS $\boldsymbol{\Theta}$).

We note that the family of sets $\underline{\mathcal{Q}}$ in Proposition 6.40 is simply the projection of $\underline{\Omega}$ to the Euclidean subspace $\mathbb{R}^{r-1} \times \mathbb{C}^s$ of H_{ext}. Consequently, if σ is a unimodular Pisot substitution, we have $\underline{\mathcal{Q}} = \underline{\Omega}$. Thus, we may interpret $\underline{\Omega}$ as "refinement" of $\underline{\mathcal{Q}}$ by including (appropriate) $\mathfrak{p}$-adic fields. In particular, Proposition 6.40 also holds for $\boldsymbol{\Theta}^\star$, wherefore one has $\Lambda_i \subset \Lambda(\Omega_i)$ for $1 \leq i \leq n$ (note that in general one has $\Lambda(\Omega_i) \subset \Lambda(\mathcal{Q}_i)$ with equality for the unimodular case).

By Lemma 6.49, $\mathcal{M}^\star \subset H$ is a lattice (in H). Moreover, we derive the following statement.

Lemma 6.64. *We have* $\mathcal{L}^\star = (\operatorname{supp}\underline{\Lambda})^\star \dot{+} \mathcal{M}^\star$ *and* $H = \Omega + \mathcal{M}^\star$, *where* $\Omega = \operatorname{supp}\underline{\Omega}$.

Proof. The first statement follows from Proposition 6.33, *i.e.*, from $\mathcal{L} = (\operatorname{supp}\underline{\Lambda}) \dot{+} \mathcal{M}$, and the bijectivity of the star-map. The second statement follows by the denseness of $\mathcal{L}^\star \subset H$ (see Lemma 6.55) and $\operatorname{cl}(\operatorname{supp}\underline{\Lambda})^\star \subset \Omega$. □

The last lemma also implies that there is a $t \in \mathcal{M}^\star$ such that Ω and $\Omega + t$ do overlap (they cannot have a positive distance by the relative denseness of $\mathcal{L}^\star$). Also, since Ω is compact there are only finitely many $t \in \mathcal{M}^\star$ such that Ω overlaps with $\Omega + t$. We would like to have that they only have boundary points in common, *i.e.*, that they are measure-disjoint. Note that if we have a point $x \in \operatorname{int}\Omega \cap \operatorname{int}(\Omega + t)$ then there exists a neighbourhood U of x such that $U \subset \operatorname{int}\Omega \cap \operatorname{int}(\Omega + t)$.

The proof of the following statement with Baire's category theorem (see Proposition 2.14) also appears in [88, Corollary 3.9].

Lemma 6.65. *There is at least one set Ω_i with nonempty interior.*

Proof. First suppose Ω is nowhere dense (*i.e.*, $\operatorname{cl}\operatorname{int}\Omega = \varnothing$. Note that $\operatorname{cl}\Omega = \Omega$). Then the set $\Omega + \mathcal{M}^\star$ is meager, which contradicts Lemma 6.64, because H is of second category. Therefore Ω has nonempty interior. But then at least one Ω_i has nonempty interior (and therefore all Ω_i have nonempty interior). □

We recall that we will often switch between the CPS $(\mathbb{R}, H, \tilde{\mathcal{L}})$ and the CPS $(\mathbb{R}, H_{\text{ext}}, \tilde{\mathcal{L}}_{\text{ext}})$ in the following, see Corollary 6.56.

We now establish that we are in the situation of Proposition 4.99: The PF-eigenvalue of $\boldsymbol{\Theta}^\star$ is $\lambda = |\lambda|$ and Artin's product formula (see Lemma 3b.11), yields, in particular, $(\prod_{i=1}^n |\sigma_i(\lambda)|) \cdot \left(\prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\, \|\lambda\|_\mathfrak{p}<1} \|\lambda\|_\mathfrak{p}\right) = 1$. With Proposition 4.38, we obtain $\mu_{H_{\text{ext}}}(W) = \lambda \cdot \mu_{H_{\text{ext}}}(f_0(W))$ for all compact sets $W \subset H_{\text{ext}}$. Moreover, we note that f_0 is a homeomorphism in H_{ext}.

Therefore, we can apply Proposition 4.99 to the sets Ω_i and obtain the following statement (note that parts of this corollary can be found, *e.g.*, in [381, Prop. 10.7], [359, Theorem 4.1], [221, Proof of Theorem 5.5(vi)⇒(vii)]).

Corollary 6.66. *Let u be any fixed point of a Pisot substitution σ with* $\operatorname{card}\mathcal{A} \geq 2$. *Let $\underline{\Lambda}$ be the representation with natural lengths of u and $\underline{\Omega}$ the attractor of $\boldsymbol{\Theta}^\star$. Then the following hold:*

(i) *All Ω_i have non-zero Haar measure.*

(ii) *The unions in the IFS $\boldsymbol{\Theta}^\star$ are measure disjoint.*

(iii) *The boundaries $\partial\Omega_i$ are of Haar measure zero .*

(iv) *The sets Ω_i are perfect sets and regularly closed.* □

Remark 6.67. We also get that $(\mu_{H_{\text{ext}}}(\Omega_1, \dots, \mu_{H_{\text{ext}}}(\Omega_n))^t$ is a right eigenvector of $\boldsymbol{S}\sigma$ for the eigenvalue λ, see Equation (4.2) on p. 98. The same holds for ϱ, the vector whose coordinates are the frequencies of the symbols. So, we have $(\mu_{H_{\text{ext}}}(\Omega_1), \dots, \mu_{H_{\text{ext}}}(\Omega_n))^t = C \cdot \varrho$ for some constant $C > 0$.

A consequence of the density formula (see Corollary 5.27) and Lemma 6.50 is the following statement.

Proposition 6.68. *Let σ be a Pisot substitution (with* $\operatorname{card}\mathcal{A} \geq 2$*). Let $\underline{\Lambda}$ be the representation with natural intervals for any of its fixed points u and $\underline{\Omega}$ the attractor of $\boldsymbol{\Theta}^\star$. Then $\underline{\Lambda}$ is a regular multi-component (inter) model set iff $\bigcup_{i=1}^n \Omega_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^\star$.*

Proof. Similarly to the discussion on pp. 172–174, we have: If $\underline{\Lambda}$ is a subsitution multi-component Delone set that is also a a multi-component (inter) model set, then its window is given by $\underline{\Omega}$. Since the sets Λ_i are disjoint, the sets Ω_i are measure-disjoint and the density formula establishes that $\bigcup_{i=1}^n \Omega_i$ is a fundamental domain of $\mathcal{M}^\star$ (see Lemma 6.50, since $\bar{\ell}$ and $\mu_{\mathbb{R}\times H}(\mathrm{FD}(\tilde{\mathcal{L}}))$ are given).

For the converse, we observe that by $(\operatorname{supp}\underline{\Lambda})^\star \subset \bigcup_{i=1}^n \Omega_i$ we have: If $\bigcup_{i=1}^n \Omega_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^\star$, then by Lemma 6.50 and the previous remark, we have that Λ_i and $\Lambda(\Omega_i)$ have the same density □

Note that for the σLCAG H_{ext} and the self-affine IFS $\boldsymbol{\Theta}^\star$ our findings in the Sections 4.9 & 4.10 apply. In particular, we may calculate the (lower) affinity dimension of $\boldsymbol{\Theta}^\star$: The singular values of $f_0^\star$ (where $f_0^\star(x) = \lambda^\star \cdot x$) are given by $|\sigma_2(\lambda)|, \dots, |\sigma_r(\lambda)|, |\sigma_{r+1}(\lambda)|, |\sigma_{r+1}(\lambda)|, |\sigma_{r+2}(\lambda)|, \dots,$ $|\sigma_{r+s}(\lambda)|, |\sigma_{r+s}(\lambda)|, \|\lambda\|_{\mathfrak{p}_1}, \dots, \|\lambda\|_{\mathfrak{p}_k}$ (where, as usual, $\mathfrak{p}_i$ are all prime ideals of $\mathbb{Q}(\lambda)$ such that $\|\lambda\|_{\mathfrak{p}_i} < 1$). Again, an application of Artin's product formula (see Lemma 3b.11) yields in Corollary 4.118 respectively Lemma 4.117:

$$\underline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \overline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \dim_{\text{metr}} H_{\text{ext}} = r + 2s + k.$$

Since the interior of Ω_i is nonempty we consequently obtain the following statement (also, we recall the definition of the box counting definitions in Definition 4.131).

Corollary 6.69. *Let $\boldsymbol{\Theta}^\star$ be the IFS with attractor $\underline{\Omega}$ in H_{ext} associated to a Pisot substitution. Then,*

$$\underline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \dim_{\text{Hd}} \underline{\Omega} = \underline{\dim}_{\text{box}}\underline{\Omega} = \overline{\dim}_{\text{box}}\underline{\Omega} = \overline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \dim_{\text{metr}} H_{\text{ext}} = r + 2s + k. \qquad \square$$

We note that the inequality $\dim_{\text{Hd}} \underline{\Omega} \leq \overline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star$ is also established by Proposition 4.122 (also see Lemma 4.133), while the inequality $\underline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star \leq \dim_{\text{Hd}} \underline{\Omega}$ follows from Proposition 4.127 provided the intersections $\Omega_i \cap \Omega_j$ are $(r+2s+k)$-measure-disjoint for $i \neq j$. But if the sets Ω_i have no common interior points, they only intersect on the boundary. We therefore calculate the Hausdorff dimension of the boundaries $\partial\Omega_i$ next, where we show that not only the Haar measure of the boundary is zero but also that the boundary has a Hausdorff dimension which is strictly smaller than the metric dimension of H_{ext} (and thus, the Hausdorff dimension of the sets Ω_i). This strengthens Proposition 4.99(iii).

Lemma 6.70. *For a Pisot substitution we get*

$$\dim_{\text{Hd}} \partial\Omega_i \leq \underline{\dim}_{\text{box}}\partial\Omega_i \leq \overline{\dim}_{\text{box}}\partial\Omega_i < \dim_{\text{metr}} H_{\text{ext}} = r + 2s + k$$

for all $1 \leq i \leq n$. $\square$

Proof. To prove this claim, we use a variant of the "Keesling-argument", compare proof of Proposition 4.99(iii) and Remark 4.100: We have $\underline{\Omega} = \boldsymbol{\Theta}^\star(\underline{\Omega})$, $\underline{\partial\Omega} \subset \underline{\Omega}$ and $\underline{\partial\Omega} \subset \boldsymbol{\Theta}^\star(\underline{\partial\Omega})$. The same statements also hold if we replace $\boldsymbol{\Theta}^\star$ by any power $(\boldsymbol{\Theta}^\star)^m$ with $m \in \mathbb{N}$. Moreover, one has $\overline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \overline{\dim}_{\text{aff}}(\boldsymbol{\Theta}^\star)^m$ (and also $\underline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \underline{\dim}_{\text{aff}}(\boldsymbol{\Theta}^\star)^m$), compare Corollary 4.103; in fact, Corollary 6.69 yields

$$\underline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \underline{\dim}_{\text{aff}}(\boldsymbol{\Theta}^\star)^m = r + 2s + k = \overline{\dim}_{\text{aff}}\boldsymbol{\Theta}^\star = \overline{\dim}_{\text{aff}}(\boldsymbol{\Theta}^\star)^m,$$

and we obtain the following spectral radius of the powers of the substitution matrix:

$$\rho\left(\boldsymbol{S}(\boldsymbol{\Theta}^\star)^m\right) = \lambda^m.$$

We show that the boundary $\underline{\partial\Omega}$ is a subset of an attractor of an IFS whose spectral radius of its substitution matrix is smaller (but has the same map f_0). Consequently, since the singular value functions Φ^γ is strictly decreasing, the box counting and the Hausdorff dimension of the boundary is then strictly less than $r + 2s + k$ (by Corollary 4.118).

By Lemma 6.65, there is at least one $i_0 \in \mathcal{A}$ such that Ω_{i_0} has nonempty interior. Thus, there is a radius $\varepsilon > 0$ and an $x \in H_{\text{ext}}$ such that $B_{<\varepsilon}(x) \subset \operatorname{int}\Omega_{i_0}$. Now, as in the proof of Proposition 4.99, we define $D = \max_{1\leq j\leq n} \operatorname{diam}(\Omega_j)$ and we choose a k_0 such that $q(\boldsymbol{\Theta}^\star)^{k_0} \cdot D < \frac{2}{3}\,\varepsilon$ (recall Remark 4.93 for the definition of $q(\boldsymbol{\Theta}^\star)$). Then, there is a $j_0 \in \mathcal{A}$ such that $f(\Omega_{j_0}) \subset B_{<\varepsilon}(x)$ for some $f \in \Theta^{\star(k_0)}_{i_0 j_0}$, and $f(\Omega_{j_0}) \cap \partial\Omega_{i_0} = \varnothing$, *i.e.*, $f(\Omega_{j_0})$ does not contain any boundary point of Ω_{i_0}.

We now define the following IFS $\boldsymbol{\Theta}'$:

$$\Theta'_{ij} = \begin{cases} \Theta^{(k_0)}_{i_0 j_0} \setminus \{f\}, & \text{if } i = i_0 \text{ and } j = j_0, \\ \Theta^{(k_0)}_{ij}, & \text{otherwise.} \end{cases}$$

Obviously, one has $\boldsymbol{S\Theta}' < \boldsymbol{S\Theta}^{k_0}$ and thus $\rho(\boldsymbol{S\Theta}') < \lambda^{k_0}$ by Corollary 4.73. Denote the unique attractor of $\boldsymbol{\Theta}'$ by $\underline{\mathcal{R}}$. We have to argue that $\underline{\partial\Omega} \subset \underline{\mathcal{R}}$ (by construction, one has $\underline{\mathcal{R}} \subset \underline{\Omega}$).

We have constructed $\boldsymbol{\Theta}'$ in such a way that $\underline{\partial\Omega} \subset \boldsymbol{\Theta}'(\underline{\partial\Omega})$. By Proposition 4.89, one therefore has

$$\underline{\partial\Omega} \subset \mathrm{cl}_{H_{\mathrm{ext}}}\left(\bigcup_{m\geq 0} (\boldsymbol{\Theta}')^m\,(\underline{\partial\Omega})\right) = \underline{\mathcal{R}} \in (\mathcal{K}H_{\mathrm{ext}})^n.$$

Consequently, this establishes the claim. □

Remark 6.71. This last result can also be found in [129, Theorem 3.2] for unimodular Pisot substitutions.

6.7. Dual Tilings

We start this section with a multi-component Delone set that now lives in the internal space H_{ext}. We observe that the CPS in Equation (6.7) on p. 240 is symmetric, wherefore we may also consider the CPS $(H_{\mathrm{ext}}\,, \mathbb{R}\,, \tilde{\mathcal{L}}_{\mathrm{ext}})$ (where $\mathbb{R}$ is the internal space). We denote the star-map of the CPS $(H_{\mathrm{ext}}\,, \mathbb{R}\,, \tilde{\mathcal{L}}_{\mathrm{ext}})$ by $\cdot^{\star} : \mathcal{L}^{\star}_{\mathrm{ext}} \to \mathbb{R}$.

From Lemma 5.142 and Remark 5.143 we obtain for Pisot substitutions the following statement.

Proposition 6.72. *Let u be any fixed point of a Pisot substitution σ with* $\operatorname{card}\mathcal{A} \geq 2$. *Let $\underline{\Lambda}$ be the representation with natural intervals $[0\,,\ell_i]$ of u and $\underline{\Omega}$ the attractor of $\boldsymbol{\Theta}^{\star}$. Define the following regular multi-component inter model set $\underline{\Upsilon}$ in H_{ext} relative to the CPS $(H_{\mathrm{ext}}\,, \mathbb{R}\,, \tilde{\mathcal{L}}_{\mathrm{ext}})$:*

$$\Upsilon_i = \Lambda([0\,,\ell_i[).$$

Then, we have the following statements:

(i) *$\underline{\Upsilon}$ is a primitive substitution multi-component Delone set with EMFS $\boldsymbol{\Theta}^{\star\#}$ (which satisfies* **(LT)***).*

(ii) *The tiles of the associated tiling are given by $\underline{\Omega}$ and the tile substitution is given by*

$$\Omega_i + x \qquad \mapsto \qquad \left(\Omega_j + f(x) \mid 1 \leq j \leq n, f \in \Theta^{\star\#}_{ji}\right).$$

(iii) *$\underline{\Upsilon}$ is repetitive.*

(iv) *Suppose $\underline{\Upsilon}$ is representable, i.e., $\underline{\Upsilon} + \underline{\Omega}$ is a tiling, then every cluster is legal.*

(v) *$\underline{\Upsilon} + \underline{\Omega}$ is a tiling (in H_{ext}) iff $\mathcal{M}^{\star} + \underline{\Omega}$ is a tiling (in H).*

Proof. (i): Primitivity is clear since the Pisot substitution is primitive. Checking that it is a substitution multi-component Delone set, is a calculation like in Equation (5.19) on p. 189, where we here observe that

$$[0\,,\ell_i[= \bigcup_{j=1}^{n} \bigcup_{f\in\Theta^{\#}_{ij}} f([0\,,\ell_i[)$$

and these unions are disjoint (wherefore there is equality throughout the corresponding equation). Consequently, this induces $\underline{\Upsilon} = \boldsymbol{\Theta}^{\star\#}(\underline{\Upsilon})$ and that the unions in this EMFS are disjoint.

(ii) is clear by construction (since $\boldsymbol{\Theta}^{\star}$ is the adjoint IFS).

(iii): Let W be a compact set in H_{ext}. We have to show that the (translates of the) cluster $\underline{\mathcal{P}} = W \cap \underline{\Upsilon}$ occurs relatively dense in H_{ext}. Let $\mathcal{P}_i^{\star} = (W \cap \Upsilon_i)^{\star} \subset [0\,,\ell_i[$ be the components of the star image of this cluster. Then, there is an $\varepsilon_1 = \varepsilon_1(W) > 0$ such that for the interval $[0\,,\varepsilon_1[$ the following holds: For every $1 \le i \le n$, one has $\mathcal{P}_i^{\star} + [0\,,\varepsilon_1[\subset [0\,,\ell_i[$. Consequently, one has that $\underline{\mathcal{P}} + t \subset \underline{\Upsilon}$ for all $t \in \Lambda([0\,,\varepsilon_1[)$. As model set, $\Lambda([0\,,\varepsilon_1[)$ is relatively dense. Thus, we have shown that the set

$$\{t \in H_{\mathrm{ext}} \mid (\underline{\Upsilon} - t) \cap W \subset \underline{\Upsilon} \cap W = \underline{\mathcal{P}}\}$$

is relatively dense. This is nearly repetitivity, compare Definition 5.34. We note that $\Lambda([0\,,\varepsilon_1[) \subset \operatorname{supp}\underline{\Upsilon}$, wherefore – by FLC – there are only finitely many different patterns $(\underline{\Upsilon} - t) \cap W$ with $t \in \operatorname{supp}\underline{\Upsilon}$, *i.e.*, there is a finite set Y such that for every given $t \in \operatorname{supp}\underline{\Upsilon}$ there is a $t' \in Y$ with $(\underline{\Upsilon} - t) \cap W = (\underline{\Upsilon} - t') \cap W$. We denote the component-wise union of these patterns as $\underline{\mathcal{P}}_{\mathrm{all}}$, *i.e.*,

$$\underline{\mathcal{P}}_{\mathrm{all}} = \bigcup_{t' \in Y} (\underline{\Upsilon} - t') \cap W = \left(\bigcup_{t' \in Y} \underline{\Upsilon} - t' \right) \cap W = \left(\bigcup_{t' \in Y} \Upsilon_1 - t', \dots, \bigcup_{t' \in Y} \Upsilon_1 - t' \right) \cap W.$$

Obviously, one has $\underline{\mathcal{P}} = W \cap \underline{\Upsilon} \subset \underline{\mathcal{P}}_{\mathrm{all}}$. We now compare the star images of $\underline{\mathcal{P}}^{\star}$ and $\underline{\mathcal{P}}_{\mathrm{all}}^{\star}$: If $x^{\star} \in \left(\mathcal{P}_{\mathrm{all}}^{\star}\right)_i \setminus \mathcal{P}_i^{\star}$, then one has $x^{\star} \notin [0\,,\ell_i[$. Since there are only finitely many points in $\underline{\mathcal{P}}_{\mathrm{all}}$, there is an $0 < \varepsilon_2 = \varepsilon_2(W) < \varepsilon_1$ (therefore, one has $x^{\star} + [0\,,\varepsilon_2[\in [0\,,\ell_i[$ for all $x^{\star} \in \mathcal{P}_i^{\star}$) such that $\left(x^{\star} + [0\,,\varepsilon_2[\right) \cap [0\,,\ell_i[= \varnothing$ for all $x^{\star} \in \left(\mathcal{P}_{\mathrm{all}}^{\star}\right)_i \setminus \mathcal{P}_i^{\star}$ (there is nothing to show for $x^{\star} \ge 0$, otherwise, $x^{\star}$ has positive distance from the origin). As before, $\Lambda([0\,,\varepsilon_2[)$ is relatively dense; moreover, since we have an symmetric CPS and therefore a bijective star-map ${}^{\star} : \mathcal{L}_{\mathrm{ext}} \to \mathcal{L}_{\mathrm{ext}}^{\star}$ (respectively, "inverse" star-map ${}^{\star} : \mathcal{L}_{\mathrm{ext}}^{\star} \to \mathcal{L}_{\mathrm{ext}}$), one now has $\underline{\mathcal{P}} = (\underline{\Upsilon} - t) \cap W$ for all $t \in \Lambda([0\,,\varepsilon_2[)$. Since this holds for all compact sets W, repetitivity is established.

(iv) is simply Corollary 5.89.

(v): This is either Proposition 5.144 together with Proposition 6.68 or it can be established directly by a density argument as follows:
By the density formula (see Corollary 5.27) we have

$$\operatorname{dens}\Upsilon_i = \frac{\ell_i}{\mu_{H_{\mathrm{ext}} \times \mathbb{R}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}}))}.$$

By Remark 6.67, we also have

$$(\mu_{H_{\mathrm{ext}}}(\Omega_1), \dots, \mu_{H_{\mathrm{ext}}}(\Omega_n)) = C \cdot \mu_H(\mathrm{FD}(\mathcal{M}^{\star})) \cdot (\varrho_1, \dots, \varrho_n) \tag{6.8}$$

for some constant $C > 0$.

If $\mathcal{M}^{\star} + \underline{\Omega}$ is a (periodic) tiling, then we have $C = 1$ in Equation (6.8). Therefore the covering degree of $\underline{\Upsilon} + \underline{\Omega}$ is $\mu_{H_{\mathrm{ext}}}$-a.e. given by

$$\sum_{i=1}^{n} \mu_{H_{\mathrm{ext}}}(\Omega_i) \cdot \operatorname{dens}\Upsilon_i = \sum_{i=1}^{n} \mu_H(\mathrm{FD}(\mathcal{M}^{\star})) \cdot \varrho_i \cdot \frac{\ell_i}{\mu_{H_{\mathrm{ext}} \times \mathbb{R}}(\mathrm{FD}(\tilde{\mathcal{L}}))} = \frac{\mu_H(\mathrm{FD}(\mathcal{M}^{\star})) \cdot \bar{\ell}}{\mu_{H_{\mathrm{ext}} \times \mathbb{R}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}}))} = 1,$$

where the last equality follows by Lemma 6.50.

For the other direction, we first observe the following: If $\underline{\varUpsilon} + \underline{\varOmega}$ is a tiling, then $\bigcup_{i=1}^{n} \varOmega_i$ is measure-disjoint (since $0 \in \varUpsilon_i$ for all $1 \le i \le n$). Then, we have

$$1 = \sum_{i=1}^{n} \mu_{H_{\mathrm{ext}}}(\varOmega_i) \cdot \operatorname{dens} \varUpsilon_i = \sum_{i=1}^{n} C \cdot \mu_H(\mathrm{FD}(\mathcal{M}^\star)) \cdot \varrho_1 \frac{\ell_i}{\mu_{H\times\mathbb{R}}(\mathrm{FD}(\tilde{\mathcal{L}}))} = C,$$

i.e., we have $C = 1$ and $\bigcup_{i=1}^{n} \varOmega_i$ is a fundamental domain of $\mathcal{M}^\star$ (compare Lemma 6.64 and Proposition 6.68). □

Remark 6.73. The density argument in (v) can also be found in [192, Prop. 3.5].

Remark 6.74. Let $\ell_{\min} = \min\{\ell_i \mid 1 \le i \le n\}$. Then, one has $\Lambda([0\,,\ell_{\min}[) \subset \varUpsilon_i$ for all $1 \le i \le n$. In fact, one may order the lengths with increasing value, wherefore one gets a total ordering of the sets $\varUpsilon_i$ by set inclusion; in particular, one has $\operatorname{supp} \underline{\varUpsilon} = \Lambda([0\,,\ell_{\max}[)$ with $\ell_{\max} = \max\{\ell_i \mid 1 \le i \le n\}$.

Remark 6.75. In the terminology of [221], $\underline{\varUpsilon}$ is a (repetitive) substitution Delone set family (see [221, Definition 2.8]). However, $\underline{\varUpsilon}$ might be decomposable (see Definition 5.73 and [221, Definition 2.8]), as the following example shows: Consider the unimodular Pisot substitution $a \mapsto abc$, $b \mapsto ba$, and $c \mapsto b$ and let β be a complex root of $x^3 - 2x^2 - 1 = 0$. Then, $\underline{\varUpsilon} = \boldsymbol{\varTheta}^{\star\#}(\underline{\varUpsilon})$ reads

$$\begin{array}{rcl} \varUpsilon_a &=& \beta^{-1}\varUpsilon_a \;\cup\; \beta^{-1}\varUpsilon_b + \beta - 1 \;\cup\; \beta^{-1}\varUpsilon_c + \beta \\ \varUpsilon_b &=& \beta^{-1}\varUpsilon_b \;\cup\; \beta^{-1}\varUpsilon_a + 1 \\ \varUpsilon_c &=& \beta^{-1}\varUpsilon_b \end{array}$$

Starting the iteration with $\underline{\varUpsilon}_1^{(0)} = \omega_a(0)$ and with $\underline{\varUpsilon}_2^{(0)} = \omega_b(0)$, we obtain two disjoint point sets $\underline{\varUpsilon}_1 = \lim_{m\to\infty} \left(\boldsymbol{\varTheta}^{\star\#}\right)^m (\underline{\varUpsilon}_1^{(0)})$ and $\underline{\varUpsilon}_2$ (actually, one has $\underline{\varUpsilon} = \underline{\varUpsilon}_1 \cup \underline{\varUpsilon}_2$). We note that in this example, some (and by primitivity all) of the components of both $\underline{\varUpsilon}_1$ and $\underline{\varUpsilon}_2$ cannot be Delone sets (observe [221, Theorem 7.1]), since they are not relatively dense. Therefore, the above substitution is – in the terminology of [221] (see there for definitions) – a "reducible weak Delone multiset family" ("reducible" and "decomposable" are synonymous here by [221, Theorem 3.3]) satisfaying an "inflation functional equation" $\underline{\varUpsilon} = \boldsymbol{\varTheta}^{\star\#}(\underline{\varUpsilon})$, where the subdivision matrix $\boldsymbol{S\varTheta}^{\star\#}$ is primitive; moreover, it is also a "self-replicating multi-tiling family" (compare with Lemma 5.76). Because "reducibility" respectively "decomposability" occurs for Pisot substitutions, one can not – in general – apply the results of [221] to Pisot substitutions.

Remark 6.76. Obviously, one has $x^{\star\star} = x$ for all $x \in \mathcal{L}_{\mathrm{ext}}$ (and therefore also $\boldsymbol{\varTheta}^{\star\star} = \boldsymbol{\varTheta}$). Moreover, if one applies the heuristic construction of a CPS in Section 6.4 to the (aperiodic[16]) substitution multi-component set $\underline{\varUpsilon}$, one obtains $\mathcal{L}^\star = \langle \bigcup_i \varUpsilon_i - \varUpsilon_i \rangle_{\mathbb{Z}}$ and consequently recovers again the CPS $(H_{\mathrm{ext}}\,, \mathbb{R}\,, \tilde{\mathcal{L}}_{\mathrm{ext}})$. Thus, it is justified to call the tiling $\underline{\varUpsilon} + \underline{\varOmega}$ (of H_{ext}) the *dual tiling* to $\underline{\varLambda} + \underline{A}$ (of $\mathbb{R}$), where $A_i = [0\,,\ell_i]$ and provided they are tilings (*i.e.*, if Proposition 5.144 holds). In connection with Remark 6.47, we note that $(1/\lambda)^\star$ yields a generalised Pisot family.

Together with Theorem 5.154, we now have the following equivalent statements, where we again make use of the fact that we have a symmetric CPS and, in particular, of the duality inasmuch as we can obviously formulate the condition in Corollary 5.145 either for $\underline{\varLambda}$ or for

[16] That $\underline{\varUpsilon}$ is aperiodic follows from the bijectivity of the $\star$-map and the definition as multi-component model set: If $t^\star$ is a period of $\underline{\varUpsilon}$, then we also have $t\mathbb{Z} \subset [0\,,\ell_i[$ for all i, and the only possibility is $t^\star = 0$.

$\underline{\varUpsilon}$. Thus, we use the notations $\varXi_{\underline{\varOmega}}(i,k,x) = \varOmega_i \cap \varOmega_k + x$ (which, as set, is a subset of H_{ext}) and $\varXi_{\underline{A}}(i,k,x) = A_i \cap A_k + x = [0\,,\ell_i] \cap [0\,,\ell_k] + x$ (which is a subset of $\mathbb{R}$). Note that clearly, $\mathcal{L} = \mathcal{L}^{\Gamma}$ is finitely generated by $\varDelta'$.

Theorem 6.77. *Let σ be a Pisot substitution with fixed point u. Denote the representation with natural intervals $A_i = [0\,,\ell_i]$ of u by $\underline{\varLambda}$. Then, $\underline{\varLambda}$ is a representable repetitive aperiodic multi-component Delone set (which, in particular, satisfies* **(PLT+)***). Denote the attractor of the IFS $\boldsymbol{\varTheta}^\star$ by $\underline{\varOmega}$ and set $\varUpsilon_i = \Lambda([0\,,\ell_i[)$. Then, the substitution multi-component Delone set $\underline{\varUpsilon}$ is repetitive, and the following statements are equivalent:*

(i) *$\underline{\varLambda}$ is a multi-component inter model set with $\Lambda(\operatorname{int}\underline{\varOmega}) \subset \underline{\varLambda} \subset \Lambda(\underline{\varOmega})$.*

(ii) *$\bigcup_{i=1}^n \varOmega_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^\star$.*

(iii) *$\underline{\varUpsilon} + \underline{\varOmega}$ is a tiling of H_{ext}.*

(iv) *$\bigcup_{i=1}^n [-\ell_i\,,0] \times \varOmega_i$ is a measure-disjoint union and a fundamental domain of $\widetilde{\mathcal{L}}_{\text{ext}}$.*

(v) *The only strong overlaps $\varXi_{\underline{A}}(i,k,x)$ with $x \in \Lambda(\operatorname{int}\varOmega_k - \operatorname{int}\varOmega_i)$ are coincidences.*

(vi) $\lim_{m\to\infty} \operatorname{dens}^{\mathit{ovlap}}_{\underline{\varLambda}+\underline{A}}(f_0^m(x)) = 1$ *for every $x \in \varDelta'$.*

(vii) $1 - \operatorname{dens}^{\mathit{ovlap}}_{\underline{\varLambda}+\underline{A}}(f_0^m(x)) \le C \cdot r^m$ *for an $m \in \mathbb{N}$, every $x \in \varDelta'$ and some constants (independent of x) $C > 0$ and $r \in]0\,,1[$.*

(viii) *$\underline{\varLambda} + \underline{A}$ admits an overlap coincidence.*

(ix) *$\underline{\varLambda}$ admits an algebraic coincidence.*

(x) *The only strong overlaps $\varXi_{\underline{\varOmega}}(i,k,x)$ with $x \in \Lambda(]-\ell_i\,,\ell_k[)$ are coincidences.* □

We note that the possible values for x with $x \in \Lambda(]-\ell_i\,,\ell_k[)$ are much easier to calculate than the ones with $x \in \Lambda(\operatorname{int}\varOmega_k - \operatorname{int}\varOmega_i)$ since the sets $\underline{\varOmega}$ are only implicitly given as attractor of an IFS. We will make use of this in Section 6.9.

Remark 6.78. The klotz tiling by $\bigcup_{i=1}^n [-\ell_i\,,0] \times \varOmega_i$ already appears in [294, 295], since it is connected to so-called "Markov partitions", see Definitions 7.63 & 7.64. The equivalence of (iii) and (iv) can also be found on [192, Prop. 3.7] for unimodular Pisot substitutions.

Remark 6.79. We observe that we do not make full use of the duality between the tilings in the above theorem, since we "broke the symmetry" provided by the CPS by defining $\underline{\varUpsilon}$ already as multi-component sets. However, if there is an $i_0 \in \{1,\dots,n\}$ such that 0 is[17] an inner point of $\varOmega_{i_0}$ – or, more precisely, if 0 is an inner point of the support of some legal patch $\mathcal{P}$ of (non-overlapping) tiles of $\underline{\varUpsilon} + \underline{\varOmega}$ – we may define

$$\underline{\varUpsilon}' = \lim_{m\to\infty} \left(\boldsymbol{\varTheta}^{\star\#}\right)^m \left(\omega_{i_0}(0)\right)$$

(as usual, the limit is taken with respect to the local topology, and – as in Remark 6.4 – one may have, for some $N \in \mathbb{N}$, to consider the Nth power of the IFS $\boldsymbol{\varTheta}^{\star\#}$ for this limit to exist). By construction, one has $\underline{\varUpsilon}' \subset \underline{\varUpsilon}$ and $\underline{\varUpsilon}' + \underline{\varOmega}$ is a tiling of H_{ext}, and we may add the following equivalent conditions to Theorem 6.77:

[17] The Pisot substitution σ_{Dirk2} in Example 6c.16 shows that there are Pisot substitutions where 0 is not even an inner point of $\operatorname{supp}\underline{\varOmega}$, see the figure on the right side on p. 333.

(xi) $\underline{\Upsilon} = \underline{\Upsilon}'$.

(xii) $\lim_{m'\to\infty} \mathrm{dens}^{\mathit{ovlap}}_{\underline{\Upsilon}'+\underline{\Omega}}(\hat{f}_0^{-m'}(x)) = 1$ for every $x \in \bigcup_{i=1}^n (\Upsilon_i' - \Upsilon_i')$.

(xiii) $1 - \mathrm{dens}^{\mathit{ovlap}}_{\underline{\Upsilon}'+\underline{\Omega}}(\hat{f}_0^{-m'}(x)) \le C' \cdot (r')^{m'}$ for an $m' \in \mathbb{N}$, every $x \in \bigcup_{i=1}^n (\Upsilon_i' - \Upsilon_i')$ and some constants (independent of x) $C' > 0$ and $r' \in]0\,,1[$.

(xiv) $\underline{\Upsilon}' + \underline{\Omega}$ admits an overlap coincidence.

(xv) $\underline{\Upsilon}'$ admits an algebraic coincidence.

We think that one always has (at least) one such $\underline{\Upsilon}'$, but will not pursue this question further here (the problem, of course, is to prove that such a *legal* patch $\mathcal{P}$ contained in the *covering* $\underline{\Upsilon} + \underline{\Omega}$ exists).

6.8. Tilings by (Hyper-)Polygons

We have obtained that a Pisot substitution can be described as multi-component inter model set iff $\bigcup_{i=1}^n [-\ell_i\,, 0] \times \Omega_i$ is a measure-disjoint union and a fundamental domain of $\tilde{\mathcal{L}}_{\mathrm{ext}}$. We now replace this fundamental domain by a finite union of hyperpolygons and then obtain the "stepped surface" in H_{ext}, also compare [57, p. 61ff], [18, Section 2] and [19, Sections 3–5].

Again, we treat the Archimedean and non-Archimedean coordinates of H_{ext} separately and therefore again use the notation $x^{\tilde{\star}}$ as in Equation (6.5) on p. 227. Since $\mathcal{L} = \langle \ell_1, \dots, \ell_n \rangle_{\mathbb{Z}}$ and the ℓ_i are rationally independent, we obtain the following fundamental domain $\tilde{P}$ of $\tilde{\mathcal{L}}$ (respectively $\tilde{\mathcal{L}}_{\mathrm{ext}}$):

$$\tilde{P} = \left\{ \sum_{i=1}^n a_i \cdot (\ell_i\,, \ell_i^{\tilde{\star}})^t \mid 0 \le a_i \le 1 \right\} \times \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_{\mathfrak{p}}<1}} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}.$$

The boundary of the "hyperpolygon" $\tilde{P}$ is given by

$$\begin{aligned}\partial\tilde{P} &= \partial \left\{ \sum_{i=1}^n a_i \cdot (\ell_i, \ell_i^{\tilde{\star}})^t \mid 0 \le a_i \le 1 \right\} \times \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_{\mathfrak{p}}<1}} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}} \\ &= \left(\bigcup_{j=1}^n \left\{ \sum_{i=1}^n a_i \cdot (\ell_i, \ell_i^{\tilde{\star}})^t \mid 0 \le a_i \le 1 \text{ for } i \ne j \text{ and } a_j \in \{0\,,1\} \right\} \right) \times \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_{\mathfrak{p}}<1}} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}},\end{aligned}$$

i.e., its $2n$ "hyperfaces" are given by

$$\tilde{P}_j = \left\{ \sum_{i=1}^n a_i \cdot (\ell_i, \ell_i^{\tilde{\star}})^t \mid 0 \le a_i \le 1 \text{ for } i \ne j \text{ and } a_j = 0 \right\} \times \prod_{\substack{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\\ \|\lambda\|_{\mathfrak{p}}<1}} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$$

for $1 \le j \le n$ and the translates $\tilde{P}_j + (\ell_j, \ell_j^{\star})^t$. Note that $\bigcap_{j=1}^n \tilde{P}_j = \{0\}$.

Now, we look at the points in the multi-component Delone set given by $\Upsilon_i = \Lambda([0\,,\ell_i[)$: If $x^\star \in \Upsilon_i \subset H_{\mathrm{ext}}$, then $(x\,,x^\star) \in \tilde{\mathcal{L}}_{\mathrm{ext}}$, and the hyperface $\tilde{P}_j + (x\,,x^\star)^t$ belongs to the hyperpolygon $\tilde{P}+(x-\ell_j\,,x^\star-\ell_j^\star)^t$. Moreover, the projection $\pi_1 : H_{\mathrm{ext}} \to \mathbb{R}$ (projection on the first coordinate) of that hyperpolygon is such that 0 is an interior point of $\pi_1\left(\tilde{P} + (x-\ell_j\,,x^\star-\ell_j^\star)^t\right)$ and the image of the hyperface lies in the nonnegative half-space, *i.e.*, $\pi_1\left(\tilde{P}_j + (x\,,x^\star)^t\right) \subset \mathbb{R}_{\geq 0}$. Alternatively, we may say that $\{0\} \times H_{\mathrm{ext}}$ intersects $\tilde{P} + (x-\ell_j\,,x^\star-\ell_j^\star)^t$ in its interior such that the hyperface $\tilde{P}_j + (x\,,x^\star)^t \subset \mathbb{R}_{\geq 0} \times H_{\mathrm{ext}}$.

Definition 6.80. The collection of all hyperfaces $\tilde{P}_j + (x\,,x^\star)^t$ with $(x\,,x^\star) \in [0\,,\ell_j[\times H_{\mathrm{ext}}$ is called a *stepped surface* of $\mathbb{R} \times H_{\mathrm{ext}}$. We also call the collection of their projections $P_j + x^\star = \pi_2(\tilde{P}_j + (x\,,x^\star)^t) = \pi_2(\tilde{P}_j) + x^\star$ into H_{ext} the *stepped surface* of H_{ext}.

See Figure 6.1 for an example of this two notions of a stepped surface.

Remark 6.81. Note that P_i is denoted $[i^*]$ respectively $(\mathbf{0}, i^*)$ in [192] and [298, Chapter 8].

The following is a well-known result, at least in the unimodular case with 3 symbols (where $H_{\mathrm{ext}} \cong \mathbb{R}^2$), *e.g.*, see [20], [18, Corollary 1], [19, Theorem 3.4] and [67, Theorem 9] (also compare the domino and lozenge tilings in [373, Section 6]).

Lemma 6.82. *The stepped surface $\underline{\Upsilon} + \underline{P}$ of H_{ext} is a tiling*[§] *of H_{ext}.*

Sketch of Proof. The main idea is the following: One "fills up" the half-space $\mathbb{R}_{\leq 0} \times H_{\mathrm{ext}}$ with hyperpolygonal fundamental domain $\tilde{P}$ of $\mathcal{L}_{\mathrm{ext}}$. The stepped surface of $\mathbb{R} \times H_{\mathrm{ext}}$ is then the surface of this "filled up" half-space where all points $\mathbb{R}_{\leq 0} \times H_{\mathrm{ext}}$ are covered by some $\tilde{P} + z$ with $z \in \mathcal{L}_{\mathrm{ext}}$, but removing one such $\tilde{P} + z$ destroys this covering of the half-space.

Then, one checks that the projection onto H_{ext} (*i.e.*, the "top view") yields $\underline{\Upsilon} + \underline{P}$, and this is a tiling of H_{ext}. □

We now calculate the $\mu_{H_{\mathrm{ext}}}(P_i)$, where we observe that the contribution from the $\mathfrak{p}$-adic components is always given by $I_{H_{\mathrm{ext}}} = \prod_{\mathfrak{p}\in\mathbb{P}_{\mathbb{Q}(\lambda)},\, \|\lambda\|_{\mathfrak{p}}<1} p^{-f_{\mathfrak{p}|(p)}\cdot\delta_{\mathfrak{p}}^{\mathcal{L}}}$. We use the matrices $\boldsymbol{V}$ and $\boldsymbol{A}$ as in Remark 6.38 and observe that $\mu_{\mathbb{R}\times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}})) = |\det(\boldsymbol{V}\,\boldsymbol{A})| \cdot I_{H_{\mathrm{ext}}}$.

The Archimedean contribution of $\mu_{H_{\mathrm{ext}}}(P_i)$ is given by

$$|\det(\ell_1^{\tilde{\star}}\ \cdots\ \ell_{i-1}^{\tilde{\star}}\ \ell_{i+1}^{\tilde{\star}}\ \cdots \ell_n^{\tilde{\star}})|.$$

We make the following calculation:

$$\begin{aligned}\bar{\ell} \cdot \mu_{H_{\mathrm{ext}}}(P_i) &= \bar{\ell} \cdot |\det(\ell_1^{\tilde{\star}}\ \cdots\ \ell_{i-1}^{\tilde{\star}}\ \ell_{i+1}^{\tilde{\star}}\ \cdots \ell_n^{\tilde{\star}})| \cdot I_{H_{\mathrm{ext}}} \\ &= \left|\det\begin{pmatrix}\ell_1 & \cdots & \ell_{i-1} & \bar{\ell} & \ell_{i+1} & \cdots \ell_n \\ \ell_1^{\tilde{\star}} & \cdots & \ell_{i-1}^{\tilde{\star}} & 0 & \ell_{i+1}^{\tilde{\star}} & \cdots \ell_n^{\tilde{\star}}\end{pmatrix}\right| \cdot I_{H_{\mathrm{ext}}}\end{aligned}$$

§We also note that $\mathcal{M}^\star + \underline{P}$ is a tiling of H.

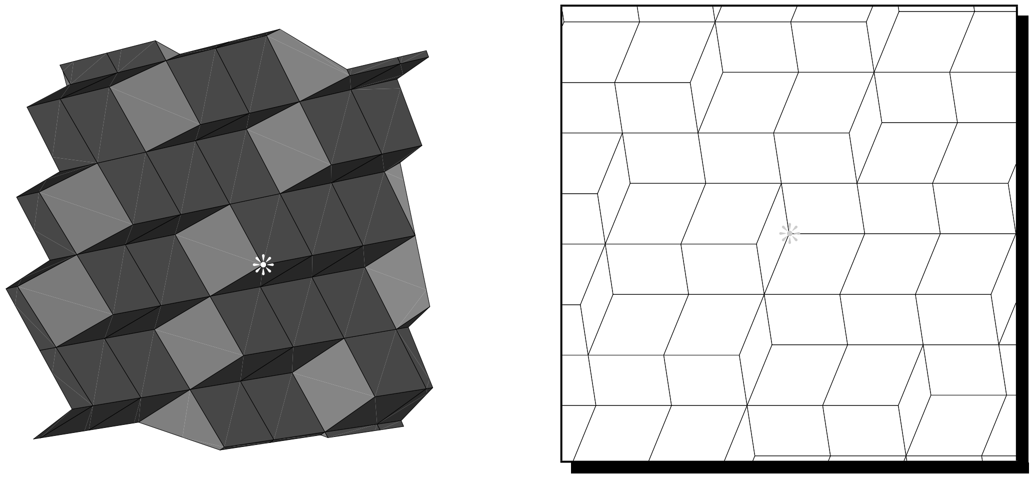


Figure 6.1.: A part of a stepped surface in $\mathbb{R}^3$ (to get the perspective right: the light source is in the lower right, the dark sides face up, *i.e.*, "$\circledast$" is the point of least first coordinate) and its projection onto $\mathbb{R}^2$.

$$= \left| \det \boldsymbol{A} \, \det \boldsymbol{V} \, \det \begin{pmatrix} 1 & 0 & \cdots & 0 & \varrho_1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 & \varrho_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & \varrho_{i-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \varrho_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \varrho_{i+1} & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \varrho_n & 0 & \cdots & 1 \end{pmatrix} \right| \cdot I_{H_{\mathrm{ext}}}$$

$$= |\det(\boldsymbol{A}\,\boldsymbol{V})| \cdot \varrho_i \cdot I_{H_{\mathrm{ext}}} = \mu_{\mathbb{R}\times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}})) \cdot \varrho_i$$

and therefore by Lemma 6.50

$$\mu_{H_{\mathrm{ext}}}(P_i) = \frac{\mu_{\mathbb{R}\times H_{\mathrm{ext}}}(\mathrm{FD}(\tilde{\mathcal{L}}_{\mathrm{ext}}))}{\bar{\ell}} \cdot \varrho_i = \mu_H(\mathrm{FD}(\mathcal{M}^\star)) \cdot \varrho_i .$$

Thus, we deduce the following statement, also compare [345, Chap. 5 & Section 7.2.1].

Lemma 6.83. *One has* $\mu_{H_{\mathrm{ext}}}(P_i) = \varrho_i \cdot \mu_H(\mathrm{FD}(\mathcal{M}^\star))$*, wherefore* $(\mu_{H_{\mathrm{ext}}}(P_1), \ldots, \mu_{H_{\mathrm{ext}}}(P_n))^t$ *is a left PF-eigenvector of* $\boldsymbol{S}\sigma$. □

We observe that the measure of the hyperpolygons P_i in H_{ext} is exactly the measure we want to have for the sets Ω_i. Moreover, we have the following substitution on the stepped surface (it is called "$*$-substitution" in [192], also see [17, Section 3] where this substitution is used)

$$P_i + x \qquad \mapsto \qquad \Big(P_j + f(x) \,|\, 1 \le j \le n,\, f \in \Theta_{ji}^{\star\#}\Big) . \tag{6.9}$$

Since $\underline{\varUpsilon}$ is a substitution multi-component Delone set, the tiles of the right hand side are measure-disjoint. Of course, the support of the collection of hyperpolygons on the right hand side does *not* equal $(f_0^\star)^{-1}(P_i + x)$. In fact, applying $\hat{f}_0$ to the right hand side is simply the IFS $\boldsymbol{\Theta}^\star$ applied to $\underline{P}$. We note that

- with respect to the Hausdorff metric one has $\lim_{m\to\infty}(\boldsymbol{\Theta}^\star)^m(\underline{P}) = \underline{\Omega}$ (by Proposition 4.89).
- if we set $(\boldsymbol{\Theta}^\star)^m(\underline{P}) = \underline{P}^{(m)}$, then – by the measure disjointness of the tiles of the stepped surfaces followed by the "rescaling" $\hat{f}_0$ – one has $\mu_{H_{\text{ext}}}(P_i) = \mu_{H_{\text{ext}}}(P_i^{(m)})$ for all $m \in \mathbb{N}$ (since their measure is given as component of the left PF-eigenvector of $\boldsymbol{S}\sigma$). We also call the sets $\underline{P}^{(m)}$ (the m-th iterate of the) *hyperpolygons*, although they are not necessarily connected.
- not only $\underline{\Upsilon} + \underline{P}$ is a tiling of H_{ext} but also $\underline{\Upsilon} + \underline{P}^{(m)}$ for every $m \in \mathbb{N}$. The last observation follows again from measure-disjointness of the tile substitution in Equation (6.9) and since $\underline{\Upsilon} + \underline{P}$ is a tiling.

Unfortunately, this last finding does not hold in the limit $m \to \infty$, but we note the following statement.

Corollary 6.84. *With the previous notation, we always have* $\mu_{H_{\text{ext}}}(\Omega_i) \geq \mu_{H_{\text{ext}}}(P_i^{(m)}) = \mu_{H_{\text{ext}}}(P_i)$.

Proof. Define $Q_i = \bigcup_{m=1}^{\infty}(\boldsymbol{\Theta}^\star)^m(\underline{P})$. Then, one has $\boldsymbol{\Theta}^\star(\underline{Q}) \subset \underline{Q}$, wherefore the attractor of the IFS is given by $\underline{\Omega} = \bigcap_{m\in\mathbb{N}}(\boldsymbol{\Theta}^\star)^m(\underline{Q})$, see Proposition 4.89. But we have (we understand unions and intersections component-wise)

$$\begin{aligned}\underline{\Omega} &= \bigcap_{m\in\mathbb{N}}(\boldsymbol{\Theta}^\star)^m(\underline{Q}) = \bigcap_{m=1}^{\infty}(\boldsymbol{\Theta}^\star)^m\left(\bigcup_{i=1}^{\infty}(\boldsymbol{\Theta}^\star)^i(\underline{P})\right)\\ &= \bigcap_{m=1}^{\infty}\bigcup_{i=m+1}^{\infty}(\boldsymbol{\Theta}^\star)^i(\underline{P}) = \bigcap_{m=1}^{\infty}\bigcup_{i=m+1}^{\infty}\underline{P}^{(i)} = \limsup_m \underline{P}^{(m)}.\end{aligned}$$

Then, the claim follows by Proposition 4.5 since $\bigcup_{m\in\mathbb{N}}\underline{P}^{(m)}$ is a family of compact sets (since $\underline{P}^{(m)}$ is convergent in the Hausdorff metric, there is, for every $\delta > 0$, an $N \in \mathbb{N}$ such that $P_i^{(m)}$ is contained in the closed δ-fringe (see Definition 4.61) of Ω_i (which is compact) for all $m \geq N$. Consequently, $\bigcup_{m\in\mathbb{N}} P_i^{(m)}$ is contained in a compact set (which is obtained as finite union of compact sets) and has therefore finite Haar measure). □

Remark 6.85. We note that we cannot replace lim sup by lim inf to estimate the measure in the previous proof, since in that case the attractor $\underline{\Omega}$ is obtained only after taking the closure, see Proposition 4.89.

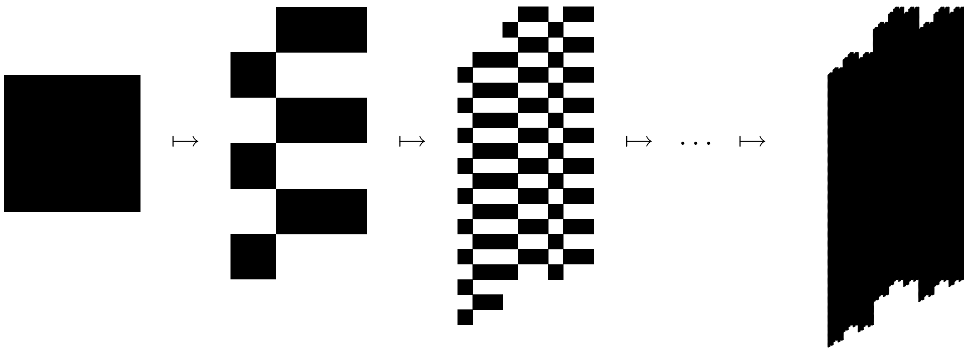
Remark 6.86. For an IFS, the situation $\mu_{H_{\text{ext}}}(\Omega_i) > \mu_{H_{\text{ext}}}(P_i^{(m)}) = \mu_{H_{\text{ext}}}(P_i)$ may occur as the following following example shows (it is taken from [381, Example 4.2]): In $\mathbb{R}^2$, set $\boldsymbol{Q} = \operatorname{diag}(\frac{1}{3}, \frac{1}{3})$ and

$$D = \{(-1\,,-3)^t, (-1\,,-1)^t, (-1\,,1)^t, (0\,,-2)^t, (0\,,0)^t, (0\,,2)^t, (1\,,-2)^t, (1\,,0)^t, (1\,,2)^t\}.$$

Define the IFS

$$\Theta(x) = \bigcup_{d\in D}\{\boldsymbol{Q}(x+d)\},$$

and start with the unit square $P^{(0)} = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. We obtain the following sequence of sets:



Then, we have $1 = \mu_{\mathbb{R}^2}(P^{(1)}) = \mu_{\mathbb{R}^2}(\Theta^m)(P^{(1)}))$ for all $m \in \mathbb{N}$, but the attractor has Lebesgue measure 2. A similar example can be found in [380, Fig. 3].

The example in this last remark indicates what the problem might be: In the limit $m \to \infty$, the boundary $\partial P_i^{(m)}$ acquires positive measure. More precisely, we have the following statement.

Proposition 6.87. [192, Theorem 3.3 (i), (iii)–(v)] *The following are equivalent:*

(i) $\underline{\Upsilon} + \underline{\Omega}$ *is a tiling.*

(ii) *There is an i (by primitivity, it then holds for all i) such that* $\lim_{m\to\infty} \partial P_i^{(m)} = \partial \Omega_i$.

(iii) *There is an i (by primitivity, it then holds for all i) such that* $\lim_{m\to\infty} \partial P_i^{(m)}$ *is not* space-filling. *More precisely, we have* $\lim_{m\to\infty} \partial P_i^{(m)} \neq \Omega_i$, *and we say that a sequence of curves* $\partial P_i^{(m)}$ *is space-filling, if* $\mu_{H_{\mathrm{ext}}}(\lim_{m\to\infty} \partial P_i^{(m)}) > 0$.

(iv) *Let* $r_n^{(i)}$ *be the radius of the largest ball* $B_{<r_n^{(i)}}$ *such that*

$$B_{<r_n^{(i)}} \subset \operatorname{supp}\left(\underline{P} + (\boldsymbol{\Theta}^{\star\#})^n(\omega_i(0))\right).$$

Then, there is an i (by primitivity, it then holds for all i) such that $r_n^{(i)} \to \infty$ *for* $n \to \infty$.

Proof. We will use the following easy results about convergence in the Hausdorff metric (also see [380, Proof of Theorem 3]): The proof of Corollary 6.84 shows that for any sequence of iterates $\underline{S}^{(m)} = \boldsymbol{\Theta}^m(\underline{S})$ of an IFS $\boldsymbol{\Theta}$ (with some starting multi-component set $\underline{S}$), the attractor $\underline{S}'$ is obtained as $\limsup_{m\to\infty} \underline{S}^{(m)}$. If $x \notin S_i'$, then there exists a radius $r > 0$ and an $N \in \mathbb{N}$ such that $S_i^{(m)} \cap B_{\leq r}(x) = \varnothing$ for all $m \geq N$. If $x \in S_i'$, then, for any $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $S_i^{(m)} \cap B_{\leq \varepsilon}(x) \neq \varnothing$ for all $m \geq N$.
We will use the notation $\check{P}_i = \lim_{m\to\infty} \partial P_i^{(m)}$.

(i)⇒(ii): We first show that $\partial\Omega_i \subset \check{P}_i$. Assume that $x \in \partial\Omega_i \subset \Omega_i$. Then, for any $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that the ball $B_{\leq\varepsilon}(x)$ contains a point in $P_i^{(m)}$ for all $m \geq N$. Since $x \in \partial\Omega_i$, the interior $\operatorname{int} B_{\leq\varepsilon}(x)$ contains a point not in Ω_i. Therefore, there exists an $N' = N'(\varepsilon) \in \mathbb{N}$ such that the ball $B_{\leq\varepsilon}(x)$ also contains a point in the complement of $P_i^{(m)}$ for $m \geq N'$. Thus, for any $\varepsilon > 0$ and $m \geq \max\{N(\varepsilon), N'(\varepsilon)\}$, $B_{\leq\varepsilon}(x)$ contains points in both

$P_i^{(m)}$ and its complement; this implies that $x \in \check{P}_i$.
We prove $\check{P}_i \subset \partial\Omega_i$ by contraposition. To this end, we assume that there exists a point $x \in \check{P}_i \setminus \partial\Omega_i$. Obviously, we have $x \in \Omega_i$ since $\limsup_m P_i^{(m)} = \Omega_i$. Thus, we also have $x \in \operatorname{int}\Omega_i$. Now consider any ball $B_{\leq\varepsilon}(x)$ contained in Ω_i. By assumption, $B_{\leq\varepsilon}(x) \cap \partial P_i^{(m)} \neq \varnothing$ for m sufficiently large, say $m \geq N$. Since $\underline{\varUpsilon} + \underline{P}^{(m)}$ are tilings of H_{ext} for all m, there exists, for all $m \geq N$, a $j_m \in \mathcal{A}$ and a $y_{j_m} \in \varUpsilon_{j_m}$ (where $y_{j_m} \neq 0$ if $i = j_m$) such that $P_{j_m}^{(m)} + y_{j_m} \subset \underline{\varUpsilon} + \underline{P}^{(m)}$ and $B_{\leq\varepsilon}(x) \cap P_{j_m}^{(m)} + y_{j_m} \neq \varnothing$. By the compactness of the sets $\bigcup_{m\in\mathbb{N}} P_i^{(m)}$ and since the sets $\varUpsilon_i$ are Meyer sets, the set $\{y_{j_m} \mid m \geq N\}$ is finite. Thus, by taking an appropriate subsequence $\{m_k\}_{k\in\mathbb{N}}$, there exists a single $j \in \mathcal{A}$ and a single point $y \in \varUpsilon_j$ such $B_{\leq\varepsilon} \cap P_j^{(m_k)} + y \neq \varnothing$. Since $\limsup_m P_j^{(m)} + y = \Omega_j + y$, it follows that $x \in \Omega_j + y$. Therefore, we have $x \in \operatorname{int}\Omega_i \cap \Omega_j + y$ (with $y \neq 0$ if $i = j$). We claim that the regular closedness of Ω_j implies that $\operatorname{int}\Omega_i \cap \operatorname{int}\Omega_j + y \neq \varnothing$, wherefore, consequently, $\underline{\varUpsilon} + \underline{\Omega}$ is not a tiling (the covering degree in some open set is greater than 1): Let U and V be two sets where V is regularly closed. Suppose there exists an $x \in \operatorname{int} U \cap V$. Then there is a radius $r > 0$ such that $B_{<r}(x) \subset \operatorname{int} U$. Moreover, we have $(B_{<r/2}(x) \cap V) \subset (B_{<r}(x) \cap V) \neq \varnothing$. Assume that $B_{<r}(x)$ contains no interior points of V, *i.e.*, $B_{<r}(x) \cap \operatorname{int} V = \varnothing$. But then the compact set $B_{\leq r/2}(x) \subset B_{<r}(x)$ has positive distance from the closed set $\operatorname{cl}\operatorname{int} V = V$, wherefore $B_{\leq r/2}(x) \cap V = \varnothing$, a contradiction.

(ii)⇒(iii): Clear, since $\partial\Omega_i$ is not space-filling.

(iii)⇒(i): Suppose $\underline{\varUpsilon} + \underline{\Omega}$ is not a tiling, then it is a covering of H_{ext} with almost everywhere constant covering degree greater than 1. So there are tiles $\Omega_i + x$ and $\Omega_j + y$ (with $x \in \varUpsilon_i$ and $y \in \varUpsilon_j$) which are not measure-disjoint, *i.e.*, $\mu_{H_{\text{ext}}}((\Omega_i + x) \cap (\Omega_j + y)) > 0$. We will show that $(\Omega_i + x) \cap (\Omega_j + y) \subset (\check{P}_i + x) \cap (\check{P}_j + y)$, which proves the claim (since then $\check{P}_i$ has positive measure).
We proceed by contradiction and assume that $(\Omega_i + x) \cap (\Omega_j + y) \not\subset (\check{P}_i + x) \cap (\check{P}_j + y)$. Then – by $\lim_{m\to\infty}(\partial P_i^{(m)} + x) \cap (\partial P_j^{(m)} + y) = (\check{P}_i + x) \cap (\check{P}_j + y)$ – there is a ball $B_{\leq r}(z)$ centred at some point $z \in \Big((\Omega_i + x) \cap (\Omega_j + y)\Big) \setminus \Big((\check{P}_i + x) \cap (\check{P}_j + y)\Big)$ such that $B_{\leq r}(z) \cap \Big((\partial P_i^{(m)} + x) \cap (\partial P_j^{(m)} + y)\Big) = \varnothing$ for sufficiently large m. But – by $\limsup_m P_i^{(m)} = \Omega_i$ – the ball $B_{\leq r}(z)$ must contain points of both $P_i^{(m)} + x$ and $P_j^{(m)} + y$ for sufficiently large m. Now, $\underline{\varUpsilon} + \underline{P}^{(m)}$ is a tiling, therefore $B_{\leq r}$ must contain a point of $(\partial P_i^{(m)} + x) \cap (\partial P_j^{(m)} + y)$, a contradiction.

(i)⇒(iv): We have

$$\hat{f}_0^{-1}(\Omega_i) = \operatorname{supp}\Big(\underline{\Omega} + (\boldsymbol{\Theta}^{\star\#})^n(\omega_i(0))\Big)$$

and $\underline{\Omega} + (\boldsymbol{\Theta}^{\star\#})^n(\omega_i(0))$ is a patch of $\underline{\varUpsilon} + \underline{\Omega}$. Denote by $R_n^{(i)}$ the radius of the largest ball contained in $B_{<R_n^{(i)}} \subset \hat{f}_0^{-1}(\Omega_i)$. Since Ω_i has nonempty interior, one has $R_n^{(i)} \to \infty$ for $n \to \infty$. We recall that $\underline{\varUpsilon} + \underline{P}$ is a tiling of H_{ext}. If $\underline{\varUpsilon} + \underline{\Omega}$ is also a tiling, then $\mu_{H_{\text{ext}}}(\Omega_i) = \mu_{H_{\text{ext}}}(P_i)$ and, by the compactness of the sets Ω_i and P_i, we have that there is a constant $C > 0$ such that $r_n^{(i)} \geq R_n^{(i)} - C$ (the constant C takes "boundary" effects of the support of the considered patches into account).

(iv)⇒(i): By the construction of the stepped surface, any patch of $\underline{\Upsilon}+\underline{P}$ has covering degree at most 1 $\mu_{H_{\text{ext}}}$-a.e. Moreover, by the measure-disjointness of the tile substitution, the covering degree in the patch (of $\underline{\Upsilon}+\underline{\Omega}$)

$$\underline{\Omega} + (\boldsymbol{\Theta}^{\star\#})^n(\omega_i(0)) \tag{6.10}$$

is at most 1 $\mu_{H_{\text{ext}}}$-a.e. Now, the existence of the balls in (iv) also ensures that there is an open set in H_{ext} in which the covering degree of $\underline{\Upsilon}+\underline{\Omega}$ is 1 $\mu_{H_{\text{ext}}}$-a.e. (since, similar to the converse direction, there is a constant $C' > 0$ such that $R_n^{(i)} \geq r_n^{(i)} - C'$, and inside this maximal $B_{R_n^{(i)}}$ the tiling $\underline{\Upsilon}+\underline{\Omega}$ and the patch of Equation (6.10) coincide). Note that in this direction it is enough if $r_n^{(i)}$ is greater than a certain constant $C' > 0$. □

Remark 6.88. We remark that this is a generalisation of statements about "digit tiles", see [380, Theorem 3], [110, Theorem 1] and [381, Theorem 4.3]. Also see Remark 6.118.

Remark 6.89. Using the hyperpolygons $P_i^{(m)}$ to approximate Ω_i appeared in [243, pp. 1985–1988 & Figures 9–13]. This construction and the stepped surface is extensively studied by S. Ito *et al.* (*e.g.*, see [20, 191, 192, 322], [298, Chapter 8] and references therein). Also compare [18, 19] and see [205] and [364, Section 7.2].

Moreover, we note that (iv) can be used to formulate a (sufficient) condition under which a Pisot substitution is a multi-component model set: Informally, one has to exclude the existence of "holes" as in the example of Remark 6.86 if one applies the "hyperpolygon substitution" of Equation (6.9) on p. 255. This is called[18] "ring condition" by [65, 344].

6.9. Coincidence Conditions and an IFS for the Boundary

The so-called "strong", respectively "geometric" or "super" coincidence conditions are closely related to condition in Theorem 6.77(x) respectively to Lemma 5.152. The goal is to show that $\underline{\Upsilon}+\underline{\Omega}$ is a tiling. We take a step back and first look at the union $\bigcup_{i=1}^n \Omega_i$: Is this union measure-disjoint? A sufficient condition for this measure-disjointness is the "strong coincidence condition", see [20, Section 6] (also compare [298, Section 7.5] and [88, Section 4]).

Definition 6.90. A substitution σ over $\mathcal{A}$ satisfies the *strong coincidence condition*, or *SCC* for short, if, for every pair of letters $i, j \in \mathcal{A}$, there is an $N \in \mathbb{N}$ and a $k \in \mathcal{A}$ such that

$$\sigma^N(i) = w_1\, k\, v_1 \text{ and } \sigma^N(j) = w_2\, k\, v_2, \text{ with } l(w_1) = l(w_2), \tag{6.11}$$

where $w_{1,2}, v_{1,2} \in \mathcal{A}^{\text{fin}}$ (maybe the empty word) and l is the homomorphism of Abelianisation (see Definition 6.6).

Furthermore, we say that a substitution σ satisfies a *coincidence condition for the pair* $i, j \in \mathcal{A}$ if Equation (6.11) holds (only) for that pair.

Remark 6.91. More precisely, one may call the previous condition the *strong coincidence condition on prefixes*, while we get the *strong coincidence condition on suffixes* if we replace $l(w_1) = l(w_2)$ in Equation 6.11 by $l(v_1) = l(v_2)$, see [20, Remark 7] and [298, Definition 7.5.7]. One may work with either one, *mutatis mutandis* all statements remain the same, also see Remark 6.9.

[18]We suggest the even more figurative term "no-fiords condition" for this.

The SCC is connected to the sets Ω_i as follows.

Lemma 6.92. *Let σ be a Pisot substitution. If the pair $i, j \in \mathcal{A}$ ($i \neq j$) satisfies a coincidence condition, then Ω_i and Ω_j are measure-disjoint. Consequently, if σ satisfies the SCC, the union $\bigcup_{k=1}^{n} \Omega_k$ is measure-disjoint.*

Proof. Using the representation with natural intervals, the coincidence condition for the pair $i, j \in \mathcal{A}$ reads: There is an $N \in \mathbb{N}$, a $k \in \mathcal{A}$ and a $z \in \mathbb{R}$ (with $z = \sum_{m=1}^{n} \ell_m \cdot (l(w_1))_m$ in the notation of Definition 6.90) such that

$$\omega_k(z) \in \boldsymbol{\Theta}^N(\omega_i(0)) \sqcap \boldsymbol{\Theta}^N(\omega_j(0)).$$

Applying Equation (5.21) on p. 194 and the star-map yields

$$\Omega_i + \left(\left(\frac{z}{\lambda}\right)^N\right)^\star, \Omega_j + \left(\left(\frac{z}{\lambda}\right)^N\right)^\star \in \left(\boldsymbol{\Theta}^{\#\star}\right)^N (\omega_k(0)) + \underline{\Omega}, \tag{6.12}$$

and thus the measure-disjointness of Ω_i and Ω_j (one may also say that, by the measure-disjointness of the unions in the IFS $\boldsymbol{\Theta}^\star$, there is a z' such that the sets $(\lambda^N)^\star\, \Omega_i + z'$ and $(\lambda^N)^\star\, \Omega_j + z'$ appear in the kth component of $(\boldsymbol{\Theta}^\star)^N\, (\underline{\Omega})$). The other statements are clear. □

It is unknown, whether every (unimodular) Pisot substitution satisfies the SCC. But the following is known.

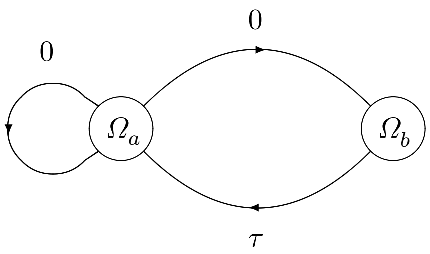
Proposition 6.93. [47, Theorem 1] *Let σ be a Pisot substitution. Then there exist distinct $i, j \in \mathcal{A}$ that satisfies a coincidence condition.* □

We only add some remarks on the proof of this statement. There are two observations that play a crucial role: Firstly, for every $x \in \Lambda_i$, the internal coordinate $x^\star$ is bounded (we have an IFS $\boldsymbol{\Theta}^\star$!); moreover, $(x\,, x^\star)$ is also a lattice point of $\tilde{\mathcal{L}}_{\text{ext}}$. Consequently, there are only finitely many points in any compact interval of $\mathbb{R}$ that might belong to $\underline{\Lambda}$. Secondly, given $x \in \Lambda_i$, one also knows that there is a $k \in \mathcal{A}$ such that $x + \ell_i \in \Lambda_k$ (one knows the position, but maybe not the type, of the "successor" of any point of $\underline{\Lambda}$, where we use the natural order on the points of $\underline{\Lambda}$ given by $\mathbb{R}$). A clever use of these two observations proves the above proposition. We also remark that the proof in [47] actually makes use of so-called "strands" and therefore might seem to apply only to the unimodular case, but one can easily modify the proof for the general case.

An immediate consequence of the last proposition is the following statement.

Corollary 6.94. *The SCC holds for Pisot substitutions over 2 symbols, i.e., if σ is a Pisot substitution over $\mathcal{A} = \{a, b\}$, then Ω_a and Ω_b are measure-disjoint.* □

Remark 6.95. We may also formulate the SCC using the graph $G(\boldsymbol{\Theta}^\star)$ defined in Remark 4.104. Recall that the vertices of $G(\boldsymbol{\Theta}^\star)$ are (labelled by) the components Ω_i of the attractor of the IFS $\boldsymbol{\Theta}^\star$, and there is a directed edge from Ω_i to Ω_j labelled f if $f \in \Theta_{ij}^\star$, wherefore $f(\Omega_j) \subset \Omega_i$. Since all maps in the IFS can be written as $f = t_{(x^\star)} \circ \hat{f}_0$ for some $x^\star \in H_{\text{ext}}$ (with $x \in \mathcal{L}_{\text{ext}}$), we may simply label the edges by the (inverse of star-map of the) translational part x (we recall that here the star-map is bijective). As an example, we give the corresponding graph for the Fibonacci substitution $a \mapsto ab$ and $b \mapsto a$, which we will discus in detail in Section 6.10.1 (for details, *e.g.*, Equation (6.13) on p. 268, see there; observe that $\tau^\star = -\frac{1}{\tau}$, where the golden mean τ is the corresponding PV-number of this Pisot substitution).



$$\begin{aligned} \Omega_a &= -\frac{1}{\tau}\,\Omega_a \cup -\frac{1}{\tau}\,\Omega_b \\ \Omega_b &= -\frac{1}{\tau}\,\Omega_a - \frac{1}{\tau} \end{aligned}$$

Now, the SCC reads in terms of this graph as follows: A Pisot substitution σ satisfies the SCC iff, for every pair of vertices i, j (more precisely, we would have to write vertices Ω_i, Ω_j) in the directed graph $G(\boldsymbol{\Theta}^\star)$, there exists a vertex k and walks w and w' along directed edges such that

- w and w' have the same length.
- w starts in k and ends in i, while w' starts in k and ends in j.
- $F(w) = F(w')$, where F is the following function from the sequence of edge labels of the walk to $\mathbb{R}$: As before (see our example), we label an edge $(w_m w_{m+1})$ of w by the inverse of the star-map of the translational part. It is therefore a real number which we denote by $e(w_m w_{m+1})$. Then, F is given by

$$\begin{aligned} F(w) = F(w_0 \dots w_N) &= e(w_0 w_1)\,\lambda^0 + e(w_1 w_2)\,\lambda^1 + \dots + e(w_{N-1} w_N)\,\lambda^{N-1} \\ &= \sum_{m=0}^{N-1} e(w_m w_{m+1})\,\lambda^m. \end{aligned}$$

(For the Fibonacci substitution we only have to check the pair Ω_a and Ω_b; obviously, already the two possible walks of length 1 starting at Ω_a – both labelled by 0 – satisfy the SCC). Note that $F(w)$ is exactly the number z^N in Equation (6.12) (this observation also proves the this is the correct formulation in terms of the graph here). Obviously, one might also state the SCC using the graph where all edges have the opposite direction than in $G(\boldsymbol{\Theta}^\star)$; then the walks start at the vertices i, j, and one looks for an appropriate common terminating vertex k.

Remark 6.96. One might wonder if it is possible that the sets Ω_i are not only pairwise measure-disjoint but even pairwise disjoint. We show that often the origin belongs to more than one set Ω_i (it always is an adherence point of at least one Ω_i). For this, we use the graph $G(\boldsymbol{\Theta}^*)$ and the map F defined in Remark 6.95. In fact, we observe the following: If w is a one-sided infinite walk along directed in edges starting in k, then $F(w)^\star \in \Omega_k$. This follows by the definition of the graph $G(\boldsymbol{\Theta}^\star)$, compare Remark 4.104.

Thus, if there are distinct $i, j \in \mathcal{A}$ and an infinite walk w (respectively w') starting at i (respectively j) such that $F(w)^\star = F(w')^\star$, then the point $z = F(w)^\star$ belongs to both Ω_i and Ω_j.

We only look at the first symbol of each substitute, because if $\sigma^{(+)}(i) = j$, then one has $\hat{f}_0(\Omega_i) \subset \Omega_j$ or, alternatively, there is a directed edge from j to i labelled 0. Consequently, for every letter i, there is an incoming edge labelled 0. Moreover, our considerations in Remark 6.4 show that these edges with label 0 yield a cycle in $G(\boldsymbol{\Theta}^\star)$. We now consider the case that the edges with label 0 form more than one cycle or the length of the cycle is greater than one. In this case, chose two arbitrary vertices $i \neq j$ of these cycles respectively this cycle. Then, there is an infinite walk w starting at i (and a walk w' starting at j) running only through edges labelled 0. Consequently, one has $0 \in \Omega_i \cap \Omega_j$.

Using (essentially) the "torus parametrisation" (see Section 7.3), it is claimed in [181, Proposition 3.17] that there are always distinct letters $i, j \in \mathcal{A}$ such that $\Omega_i \cap \Omega_j \neq \varnothing$.

We now turn our attention on $\underline{\Upsilon} + \underline{\Omega}$. The condition that ensures the tiling property, is called "super-coincidence condition" in [192, Definition 4.2] respectively "geometric coincidence condition" in [50, Definition 7.1]. Close observation shows that this condition is simply the overlap coincidence condition in the formulation of Corollary 5.153 (also see [50, Prop. 17.1]). Using the equivalences in Theorem 6.77, we therefore obtain the following formulation of this condition in our notation (also compare [347, Section 5] and [192, Section 4.2] where the equivalent formulation – the second formulation in the following definition – is used; observe the "sign change" in the allowed values for "x" respectively "z" because of Lemma 5.152).

Definition 6.97. Let σ be a Pisot substitution with representation by natural intervals $\underline{\Lambda}$. Let $\underline{\Upsilon} = \Lambda(\underline{A})$ as before. We say that σ satisfies the *geometric coincidence condition*, or *GCC* for short (also called the *super-coincidence condition*), if it satisfies one of the following equivalent conditions:

- For all $i, j \in \mathcal{A}$ and all $x \in]-\ell_j\,, \ell_i[\cap \mathcal{L}_{\text{ext}}$, the strong overlaps $\Xi_{\underline{A}}(i, j, x)$ lead to an overlap coincidence.
- For all $i, k \in \mathcal{A}$ and all $z \in \Lambda(]-\ell_i\,, \ell_k[)$, the only strong overlaps $\Xi_{\underline{\Omega}}(i, k, z)$ are coincidences, *i.e.*, the sets Ω_i and $\Omega_k + z$ are measure-disjoint (where $i \neq k$ if $z = 0$).

Remark 6.98. As in Remark 6.95, we may use the labelled graph $G(\boldsymbol{\Theta}^\star)$ to state the GCC: A Pisot substitution σ satisfies the GCC, iff, for every pair of vertices $i, j \in \mathcal{A}$ in the directed graph $G(\boldsymbol{\Theta}^\star)$ and all $x \in [0\,, \ell_i[\cap \mathcal{L}_{\text{ext}}$ and $y \in [0\,, \ell_j[\cap \mathcal{L}_{\text{ext}}$, there exists a vertex k and walks w and w' along directed edges such that

- w and w' have the same length $N \in \mathbb{N}$ (*i.e.*, $w = w_0 \dots w_N$).
- w starts in k and ends in i, while w' starts in k and ends in j.
- $\lambda^{-N} \cdot (F(w) - F(w')) = x - y$ (or $F(w) - \lambda^N x = F(w') - \lambda^N y$).

In this case, one has

$$\Omega_i + x^\star + \left(\frac{F(w)}{\lambda^N}\right)^\star, \Omega_i + y^\star + \left(\frac{F(w')}{\lambda^N}\right)^\star \in \left(\boldsymbol{\Theta}^{\#\star}\right)^N (\omega_k(0)) + \underline{\Omega}.$$

We also note that both formulations of the GCC are "present" here!

Remark 6.99. Obviously, the SCC is a special case of the GCC. While the SCC at first sight only seems to be a sufficient condition for the measure-disjointness of the union $\bigcup_i \Omega_i$, the GCC – which is a necessary and sufficient condition for the tiling property of $\underline{\Upsilon} + \underline{\Omega}$ – shows that the SCC is in fact also a necessary condition for this!

Again, Pisot substitutions over two symbols play a special role, since one can show – using Proposition 6.93 respectively Corollary 6.94 – that they satisfy the GCC.

Proposition 6.100. [50, Prop. 19.3] and [192, Theorem 1.4] *The GCC holds for Pisot substitutions over* 2 *symbols.*

Sketch of Proof. The proofs in [50, 192] are formulated in the unimodular setting, but they carry over to the general setting since here it is enough to consider only the Euclidean part of the internal space; consequently, we basically assume that we have a CPS as in Section 6.3.

The proof is based on the following observations:

- In the local topology, iteration of $\boldsymbol{\Theta}(\omega_i(-x))$ with $x \in \operatorname{int} A_i =]0\,,\ell_i[$ converges to an element of $\mathbb{X}(\underline{\Lambda})$.
- The family $\boldsymbol{\Theta}^N(\omega_i(-x))$ can be interpreted as connected broken line in $\mathbb{R}^n \cong \mathbb{R}^r \times \mathbb{C}^s$, where we replace $\omega_j(y)$ be the line segment joining $(y\,,\tilde{y^\star})$ with $(y+\ell_j\,,\tilde{(y+\ell_j)^\star})$. Such a connected broken line is called a "strand" in [50]. Note that, by construction, such a strand is function $\gamma : \mathbb{R} \to \mathbb{R}^n$.

Now, the GCC is proved as follows: Assume that the GCC does not hold. Then, there are at least two sequences $\underline{\Lambda}', \underline{\Lambda}'' \in \mathbb{X}(\underline{\Lambda})$ (which, by the unique composition property, see Proposition 5.120, we may assume that they are obtained as some limit in the local topology as above), such that $\operatorname{supp} \underline{\Lambda}' \cap \operatorname{supp} \underline{\Lambda}'' = \varnothing$ (otherwise, one obtains a coincidence from each point they have in common by Corollary 6.94). Consequently, the corresponding strands $\gamma_{\underline{\Lambda}'}, \gamma_{\underline{\Lambda}''}$ are separated by some positive $\mathbb{R}^{n-1}$-distance, where we measure the distance between two strands as follows: The $\mathbb{R}^{n-1}$-distance is the minimum of the Euclidean distance of $\pi_2(\gamma_{\underline{\Lambda}'}(z))$ and $\pi_2(\gamma_{\underline{\Lambda}''}(z))$ over all $z \in \mathbb{R}$, where $\pi_2 : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the canonical projection.

Now, a specific two-dimensional argument/observation establishes the claim in the case $n = \operatorname{card} \mathcal{A} = 2$:

- The strands are separated, wherefore w.l.o.g. $\gamma_{\underline{\Lambda}'}$ "lies above" $\gamma_{\underline{\Lambda}''}$, in the sense that for every $z \in \mathbb{R}$ one has $\pi_2(\gamma_{\underline{\Lambda}'}(z)) > \pi_2(\gamma_{\underline{\Lambda}''}(z))$.
- Since $\mathbb{X}(\underline{\Lambda})$ is minimal, there is a sequence of numbers $(t_n)_{n\in\mathbb{N}}$ such that $\underline{\Lambda}' + t_n \to \underline{\Lambda}''$ in the local topology. But then (using the same sequence (t_n)), the strand γ' corresponding to the limit of $\underline{\Lambda}'' + t_n$ has twice the $\mathbb{R}$-distance from $\gamma_{\underline{\Lambda}'}$ than the strand $\gamma_{\underline{\Lambda}'}$ has from $\gamma_{\underline{\Lambda}''}$. Moreover, $\gamma_{\underline{\Lambda}'}$ also lies above γ'.
- Iterating this procedure yields elements of $\mathbb{X}(\underline{\Lambda})$ whose strands are separated by a bigger and bigger $\mathbb{R}$-distance. But this contradicts the convergence of the IFS $\tilde{\boldsymbol{\Theta}^\star}$ to a compact set $\underline{\Omega}$. This proves the proposition. □

On the basis of the second formulation of the GCC, *i.e.*, the criterion that all strong coincidences $\Xi_{\underline{\Omega}}(i,k,z)$ with $z \in \Lambda(]\ell_i\,,\ell_k[)$ are coincidences, we will now construct graphs similar to $G^{ovlap}_{str}(\underline{\Upsilon}+\underline{\Omega})$ respectively $G^{ovlap}_{weak}(\underline{\Upsilon}+\underline{\Omega})$ in Section 5.7.1 (see Definition 5.126) that also gives us an IFS for the boundary of the tiles Ω_i. Variants of the graphs we obtain by this procedure already appeared in [345, Chapitre 4] and [347] in terms of automata.

Definition 6.101. Let σ be a Pisot substitution. Then, we define two graphs. The *overlap graph* $G_\sigma^{\mathit{ovlap}}(\underline{\Upsilon}+\underline{\Omega})$ associated to σ is the following subgraph of the weak overlap graph $G_{\mathit{weak}}^{\mathit{ovlap}}(\underline{\Upsilon}+\underline{\Omega})$ (see Definition 5.126): The vertices are the overlaps $\Xi_{\underline{\Omega}}(i,k,x^\star)$ with $x^\star \in \Lambda(]\ell_i\,,\ell_k[)$. Since the star-map is bijective, we usually write $\Xi_{\underline{\Omega}}(i,k,x)$ instead of $\Xi_{\underline{\Omega}}(i,k,x^\star)$. We put a directed edge from $\Xi_{\underline{\Omega}}(i,k,x)$ to $\Xi_{\underline{\Omega}}(j,m,y)$ if $\Xi_{\underline{\Omega}}(j,m,y)$ appears[19] on the right hand side of the corresponding substitution of $\Xi_{\underline{\Omega}}(i,k,x)$ as in Equation (5.13) on p. 175. The edge is labelled by the (inverse of the star-map of the) corresponding translational part of the map $f \in \boldsymbol{\Theta}^{\#\star}$.

The *boundary graph* $G_\sigma^{\mathit{bd}}(\underline{\Upsilon}+\underline{\Omega})$ associated to σ is the subgraph of the overlap graph, where only vertices and edges are considered which can be reached by a directed walk from a coincidence $\Xi_{\underline{\Omega}}(i,i,0)$

Remark 6.102. We may view the set of vertices of the overlap graph as the collection of all possibilities how two tiles of $\underline{\Upsilon}+\underline{\Omega}$ are neighbouring, *e.g.*, the tiles $\Omega_i + x^\star$ and $\Omega_k + y^\star$ with $x^\star \in \Upsilon_i = \Lambda([0\,,\ell_i[)$ and $y^\star \in \Upsilon_k = \Lambda([0\,,\ell_k[)$ are represented by the vertex $\Xi_{\underline{\Omega}}(i,k,y-x)$ provided $\Omega_i + x^\star \cap \Omega_k + y^\star \neq \varnothing$.

However, the question is if $\underline{\Upsilon}+\underline{\Omega}$ *is* a tiling, or only a covering. Thus, the boundary graph is the collection of all *legal* neighbouring patches consisting of two tiles. Consequently, the vertices of the overlap graph consists – besides the coincidences – only of weak overlaps.

Note that by the compactness of the sets Ω_i, the number x in $\Xi_{\underline{\Omega}}(i,k,x)$ not only has to fulfil the condition $x \in]-\ell_i\,,\ell_k[\cap\mathcal{L}_{\mathrm{ext}}$, but also the distance of $x^\star$ to the origin in H_{ext} (*e.g.*, measured with the distance $d_{\mathbb{M}}$ as on p. 85) has, at least, to be less than or equal to twice the maximal diameter of the sets Ω_i – otherwise $\Xi_{\underline{\Omega}}(i,k,x)$ is empty (consequently, both graphs are finite). Moreover, a bound on the diameter is easily established from the knowledge of the IFS $\boldsymbol{\Theta}^\star$, see[20] the examples in the next section (the determination of $\check{R}$), and one may apply this condition to each “coordinate” of the space $H_{\mathrm{ext}} \cong \mathbb{R}^{r-1} \times \mathbb{C} \times \mathbb{Q}_{\mathfrak{p}_1} \times \ldots \times \mathbb{Q}_{\mathfrak{p}_k}$ separately.

Remark 6.103. Usually, only the essential part (see p. 179) of the graphs $G_\sigma^{\mathit{ovlap}}(\underline{\Upsilon}+\underline{\Omega})$ and $G_\sigma^{\mathit{bd}}(\underline{\Upsilon}+\underline{\Omega})$ is considered. Recall that the essential part of a graph is unique, and obtained by removing (iteratively) all stranded vertices, *i.e.*, vertices with no outgoing or incoming edges. In the following, if we speak of these graphs we always mean their essential part.

The following statement is follows easily from the construction of the graphs. This is also the reason why there will only be one graph for each example in the next section.

Corollary 6.104. *A Pisot substitution σ satisfies the GCC (i.e., $\underline{\Lambda}$ is an IMS) iff the overlap graph and the boundary graph are equal, $G_\sigma^{\mathit{ovlap}}(\underline{\Upsilon}+\underline{\Omega}) = G_\sigma^{\mathit{bd}}(\underline{\Upsilon}+\underline{\Omega})$.* □

Remark 6.105. That there are two graphs associated to a self-replicating covering/tiling, is also observed in [324, Section 2] for the case of so-called “digit tiles”, also compare Remark 6.118.

Trivially, the overlap and the boundary graph certainly contain (an isomorphic copy) of $G(\boldsymbol{\Theta}^\star)$ as subgraph (in fact, by the construction of $G(\boldsymbol{\Theta}^\star)$ respectively $\boldsymbol{\Theta}^{\star\#}$ from $\boldsymbol{\Theta}^\star$, the direction of the edges is actually consistent, but the edge labels differ by a factor of λ). We now remove this subgraph from $G_\sigma^{\mathit{ovlap}}(\underline{\Upsilon}+\underline{\Omega})$ and $G_\sigma^{\mathit{bd}}(\underline{\Upsilon}+\underline{\Omega})$ and again consider only the essential part obtained

[19] Since $\underline{\Upsilon}$ is substitution multi-component Delone set, one also has $y \in]\ell_j\,,\ell_m[$ for $\Xi_{\underline{\Omega}}(j,m,y)$.

[20] Explicitly, this bound on the diameter is for example given in [204, Proof of Theorem 10], [181, Proof of Theorem 2.3(5)] or [129, Lemma 1.3]. However, it will be clear from our examples how to calculate this bound.

after this operation. We denote the graphs obtained by this procedure by $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ and $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$. With the remarks in Section 5.7.1, we note the following:

- $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ is basically[21] the graph of an IFS for the boundaries $\partial\Omega_i$, see Remark 4.104 (one easily obtains an IFS from the graph). Consequently, we can now give explicit bounds on the Hausdorff dimension of the boundary using the methods established in Section 4.10. In fact, from Corollary 4.118, it follows that we only have to know the spectral radius of the adjacency matrix of $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$.
- If the overlap graph differs from the boundary graph, then some vertices of $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})$ are strong overlaps (but not coincidences). Consequently, as sets, they have positive Haar measure and their Hausdorff dimension equals the metric dimension of the underlying space. Thus, the IFS defined by the graph $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ has an affinity dimension that equals (at least) the metric dimension.

These considerations lead to the following statements.

Proposition 6.106. *Let σ be a Pisot substitution with PV-number λ, overlap graph $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})$ and boundary graph $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})$. Denote the essential part of the induced graph obtained by removing the coincidences by $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ respectively $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$. Denote by $\boldsymbol{M}^{ovlap}_{\sigma}$ respectively $\boldsymbol{M}^{bd}_{\sigma}$ the adjacency matrix of $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ respectively $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$. Then, the following holds.*

(i) *The spectral radius of $\boldsymbol{M}^{bd}_{\sigma}$ satisfies $\rho(\boldsymbol{M}^{bd}_{\sigma}) < \lambda$. In particular, the affinity dimension of the IFS defined by $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ is given by the unique $\gamma > 0$ such that $\Phi^{\gamma}(T_{f_0}) \cdot \rho(\boldsymbol{M}^{bd}_{\sigma}) = 1$ and is less than the metric dimension of the space H_{ext}. Consequently, the Hausdorff dimension of the boundaries $\partial\Omega_i$ is less than the metric dimension of H_{ext}.*

(ii) *The Pisot substitution satisfies the GCC iff $\rho(\boldsymbol{M}^{ovlap}_{\sigma}) < \lambda$.*

Proof. (i): We already know from Lemma 6.70 that the Hausdorff dimension of the boundary is less than the metric dimension of H_{ext}. We may apply the findings in the proof of that lemma to the situation here (for notations, see that proof): By the "Keesling argument", we have constructed an attractor $\underline{\mathcal{R}}$ of an IFS $\boldsymbol{\Theta}'$ such that $\partial\underline{\Omega} \subset \underline{\mathcal{R}} \subset \underline{\Omega}$, and one has for the spectral radius $\rho(\boldsymbol{S\Theta}') < \lambda^{k_0}$. Consequently, the corresponding upper net measure (see Lemma 4.116) of the IFS $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ that yields (part of the) boundary $\partial\underline{\Omega}$ is less than or equal to the upper net measure of this latter IFS $\boldsymbol{\Theta}'$. Thus, by the definition of the affinity dimension (see Lemma 4.117, compare Corollary 4.118), the spectral radius $\boldsymbol{M}^{bd}_{\sigma}$ satisfies $\rho(\boldsymbol{M}^{bd}_{\sigma}) < \lambda$. The other statement follow directly from Corollary 4.118 and Proposition 4.122.

(ii): If σ satisfies the GCC, the graphs $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ and $G^{bd}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$, and thus the adjacency matrices $\boldsymbol{M}^{ovlap}_{\sigma}$ and $\boldsymbol{M}^{bd}_{\sigma}$ coincide and the statement follows from (i).

Conversely, if σ does not satisfy the GCC, there are sets in the IFS $G^{ovlap}_{\sigma}(\underline{\Upsilon} + \underline{\Omega})^*$ whose Hausdorff dimension equals the metric dimension of the underlying space. But then Artin's product formula establishes that the spectral radius $\rho(\boldsymbol{M}^{ovlap}_{\sigma})$ must be at least λ (since Artin's product formula states that the product over all local fields $\prod_{\nu} |\lambda|_{\nu} = 1$, and by construction all $|\lambda|_{\nu} < 1$ appear as singular values). □

[21] Since we restrict ourselves to the essential part of the graphs, parts of boundaries might be missing, but are easily recovered using the "non-essential versions" of the graphs in question, see the Example $\sigma_{\clubsuit}$ in the next section (Figure 6.2).

Remark 6.107. The statement in (ii) also sheds light on Corollary 6.84 (and Proposition 6.87(ii)), since $G^{\mathit{ovlap}}_\sigma(\underline{\varUpsilon}+\underline{\varOmega})^*$ is the IFS whose attractor is (basically) $\left(\lim_{m\to\infty}\partial P_i^{(m)}\right)_{i=1}^n$.

Moreover, we also note that, for "digit tiles", (ii) is [155, Theorem 4.8], also see Remark 6.118.

Remark 6.108. The result (ii) of the last proposition can – for the unimodular case – be found in [129, Theorem 1]. The argument of [129, Theorem 3.2] is essentially the one we used here, also using the graph $G^{\mathit{bd}}_\sigma(\underline{\varUpsilon}+\underline{\varOmega})^*$. We note that for so-called "digit tiles", the adjacency matrix $\boldsymbol{M}^{\mathit{bd}}_\sigma$ is called "contact matrix", see [155], [110] and [381, Section 6].

We also note that often (*e.g.*, see [182, Theorem 4.4]) an upper bound for the Hausdorff dimension is obtained by replacing $\Phi^\gamma(T_{f_0})$ in $\Phi^\gamma(T_{f_0})\cdot\rho(\boldsymbol{M}^{\mathit{bd}}_\sigma)=1$ simply by α_1^γ, where α_1 is the greatest singular value. However, if some of the singular values (respectively, algebraic conjugates) are close to 1 while others are close to 0 in modulus, this bound can easily yield values much bigger than the metric dimension of the underlying space and is then of no practical use.

We also note that there is a result about the boundary of so-called "Markov partitions" (see Remark 7.65), from which one can deduce that the Hausdorff dimension of the boundary $\partial\underline{\varOmega}$ is (usually) not an integer. In the card $\mathcal{A}=3$ case, this can be found in [79, Theorem] (also see [364, p. 702]). Generalisations to higher dimensions can be found in [92]. Basically, one has to establish a lower bound on the Hausdorff dimension such that $\dim_{\text{metr}} H_{\text{ext}}-1<\dim_{\text{Hd}}\partial\underline{\varOmega}$. Unfortunately, there seems to be no straightforward way to use the lower affinity dimension to obtain such a general bound: On the one hand, Lemma 4.126 respectively Propositions 4.127 & 4.129 relay on some disjointness condition of the IFS in question, on the other hand, explicit knowledge of the spectral radius (or, at least, a good lower bound on this spectral radius) of the adjacency matrix of this IFS is needed. While for a given example one may calculate the lower affinity dimension, an argument for the general case is out of sight.

Remark 6.109. In two dimensions the boundary can alternatively be construct explicitly by the boundary generating method of Dekking, see [381, Examples 10.4] and [54, 102, 103]. In [190] this method is used to construct the boundary of the "Rauzy fractal" (the attractor $\underline{\varOmega}$ of the unimodular Pisot substitution $a\mapsto ab$, $b\mapsto ac$ and $c\mapsto a$). Also compare [205] and [364, Section 7.2].

Example 6.110. We give some examples of the calculation of the Hausdorff dimension of the boundaries:

- For the "Rauzy substitution" (see [306], it is also called "tribonacci substitution"), given by $a\mapsto ab$, $b\mapsto ac$ and $c\mapsto a$, the PV-number is given by $\operatorname{Irr}(\lambda,\mathbb{Q},x)=x^3-x^2-x-1$ (which yields approximately $\lambda\approx 1.839$) and the internal space is given by $\mathbb{C}\cong\mathbb{R}^2$. The corresponding adjacency matrix $\boldsymbol{M}^{\mathit{bd}}_{\sigma_{\text{Rauzy}}}$ is given by

$$\boldsymbol{M}^{\mathit{bd}}_{\sigma_{\text{Rauzy}}}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&1\\1&1&0&0\end{pmatrix},$$

see [190, Theorem 3.1] and [356, Theorem 4.1] (also [129, Example 4.2]). The Perron-Frobenius eigenvalue α of this matrix is approximately $\alpha\approx 1.395$ ($\operatorname{Irr}(\alpha,\mathbb{Q},x)=x^4-2x-1$), wherefore – by the open set condition (OSC), see Remarks 4.135 & 4.136 – the boundary $\partial \underline{\Omega}$ has Hausdorff and affinity dimension $\log(\alpha)/\log(\sqrt{\lambda}) \approx 1.093$. We also note that the structure of this boundary is further explored in [355–357].

- The following family of Pisot substitutions are considered in [190, Example 6.1]: σ : $a \mapsto a^m b$, $b \mapsto c$ and $c \mapsto a$ with $m \in \mathbb{N}$. The corresponding PV-number λ is solution of the equation $x^3 - mx^2 - 1 = 0$, again, the internal space is $\mathbb{C}$. Here, the corresponding adjacency matrix is given by

$$\boldsymbol{M}_\sigma^{bd} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & m & 0 \end{pmatrix},$$

  and, denoting the PF-eigenvalue of this matrix by α, the Hausdorff dimension of the boundary is given (again, by an application of the OSC) by $2 \cdot \log(\alpha)/\log(\lambda)$, which yields approximately 1.471 for $m = 1$ and tends to 1 for $m \to \infty$. For a candidate of a family of Pisot substitutions (also with internal space $\mathbb{C}$), where the Hausdorff dimension of the boundary tends to 2 for $m \to \infty$ (for $m = 1$, one has dimension approximately 1.217), see Example 6.114.

- For further examples, with internal space $\mathbb{C}$ and thus self-similar IFS, and their corresponding boundary graphs see [129, Section 4]. *E.g.*, for the "flipped tribonacci substitution", given by $a \mapsto ab$, $b \mapsto ca$ and $c \mapsto a$ (thus, it has the same substitution matrix as the above Rauzy/tribonacci substitution), the Hausdorff (and affinity) dimension of the boundaries is approximately given by 1.792, see [129, Example 4.3].

- Similar treatments of the boundaries of "digit tiles" can also be found in [10, 165, 227, 269, 323, 324] .

Remark 6.111. We also note that this algorithm of establishing an IFS for the boundary of the tiles Ω_i (and thus implicit knowledge of the boundaries) can easily be extended such that one obtains an IFS for the points where more than two tiles meet: The same procedure as before for the boundary graph applied to

$$\Xi(i, j, k; x, y) = \Omega_i \cap \left(\Omega_j + x^\star \cap \Omega_k + y^\star\right),$$

with the convention $j < k$ lexicographically and $x \leq y$ if $j = k$ to force a unique representation, yields an IFS for the points which are common to three tiles (note that boundary graph is a subgraph of the graph constructed with these sets, namely the subgraph induced by the vertices $\Xi(i, j, j; x, x)$). The generalisation to points common to more than three tiles is straightforward.

Again, this is already known for "digit tiles", see [324, Section 6] and [10, Section 6]. For the Rauzy substitution, the points that belong to more than two tiles are calculated in [319].

6.10. A Fully Worked Example

Before we look at a (non-unimodular) Pisot substitution, we first explain our methods on the maybe best-known Pisot substitution: the Fibonacci sequence. We note that both examples are Pisot substitutions over 2 symbols and are therefore model sets by Proposition 6.100. However, we use these examples to show how the explicit constructions in the last section work and since, unfortunately, these constructions become more extensive the bigger the alphabet of the corresponding Pisot substitution is.

6.10.1. Apéritif: Fibonacci Substitution

The Fibonacci substitution σ_{Fib} on $\mathcal{A} = \{a, b\}$ is given[22] by $a \mapsto ab$ and $b \mapsto a$, wherefore we obtain for the substitution matrix:

$$\boldsymbol{S}\sigma_{\mathrm{Fib}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

eigenvalues	left PF-eigenvector	right PF-eigenvector
$\tau = \frac{1+\sqrt{5}}{2} \approx 1.618$ $\frac{1-\sqrt{5}}{2} = -\frac{1}{\tau} \approx -0.618$	$\ell = (\tau, 1)$	$\varrho = \begin{pmatrix} \frac{1}{\tau} \\ \frac{1}{\tau^2} \end{pmatrix} \approx \begin{pmatrix} 0.618 \\ 0.382 \end{pmatrix}$

Consequently, using the natural intervals $A_a = [0\,, \tau]$ and $A_b = [0\,, 1]$, we obtain the following EMFS and its adjoint IFS for the Fibonacci substitution:

$$\boldsymbol{\Theta} = \begin{pmatrix} \{f_0\} & \{f_0\} \\ \{f_\tau\} & \varnothing \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Theta}^{\#} = \begin{pmatrix} \{g_0\} & \{g_1\} \\ \{g_0\} & \varnothing \end{pmatrix},$$

where $f_0(x) = \tau \cdot x$, $f_\tau(x) = \tau \cdot x + \tau$, $g_0(x) = \frac{1}{\tau} \cdot x$ and $g_1(x) = \frac{1}{\tau} \cdot x + 1$, and one easily checks that $\underline{A} = (A_a, A_b) = ([0\,, \tau], [0\,, 1])$ is the attractor of $\boldsymbol{\Theta}^{\#}$. Moreover, one has $\varLambda_a = \{\ldots, 0, \tau + 1, 2\tau + 1, 3\tau + 2, \ldots\}$ and $\varLambda_b = \{\ldots, \tau, 3\tau + 1, 4\tau + 2, \ldots\}$, wherefore one establishes $\mathcal{L} = \mathcal{L}' = \mathbb{Z}[\tau]$ (*i.e.*, the set of algebraic integers of $\mathbb{Q}(\sqrt{5})$). Since $\det \boldsymbol{S}\sigma_{\mathrm{Fib}} = -1$, this is a unimodular Pisot substitution, and we obtain $H = \mathbb{R}$, the star-map is given by the Galois-automorphism which maps τ to $-\frac{1}{\tau}$ and

$$\tilde{\mathcal{L}} = \left\{ m_a \cdot \begin{pmatrix} \tau \\ -\frac{1}{\tau} \end{pmatrix} + m_b \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; m_a, m_b \in \mathbb{Z} \right\}.$$

Consequently, the corresponding IFS and EMFS in internal space read as follows:

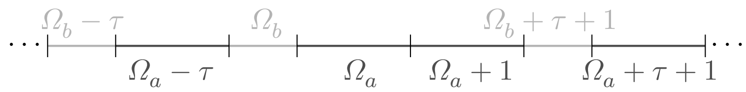
$$\boldsymbol{\Theta}^{\star} = \begin{pmatrix} \{f_0^\star\} & \{f_0^\star\} \\ \{f_\tau^\star\} & \varnothing \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Theta}^{\#\star} = \begin{pmatrix} \{g_0^\star\} & \{g_1^\star\} \\ \{g_0^\star\} & \varnothing \end{pmatrix},$$

where $f_0^\star(y) = -\frac{1}{\tau} \cdot y$, $f_\tau^\star(y) = -\frac{1}{\tau} \cdot y - \frac{1}{\tau}$, $g_0^\star(y) = -\tau \cdot y$ and $g_1^\star(y) = -\tau \cdot y + 1$. Since the tiles $\varOmega_a, \varOmega_b$ of the tiling of $H = \mathbb{R}$ are given as attractor of the IFS $\boldsymbol{\Theta}^\star$, we are looking for a solution of

$$\begin{aligned} \varOmega_a &= -\tfrac{1}{\tau}\varOmega_a \,\cup\, -\tfrac{1}{\tau}\varOmega_b \\ \varOmega_b &= -\tfrac{1}{\tau}\varOmega_a - \tfrac{1}{\tau}. \end{aligned} \tag{6.13}$$

Here, one can check that $\varOmega_a = [-\frac{1}{\tau^2}\,, \frac{1}{\tau}]$ and $\varOmega_b = [-1\,, -\frac{1}{\tau^2}]$ is a solution of this IFS and therefore the unique attractor. Since $\mathcal{M} = (\ell_a - \ell_b)\mathbb{Z} = \frac{1}{\tau}\mathbb{Z}$, we have $\mathcal{M}^\star = -\tau\mathbb{Z} = \tau\mathbb{Z}$, and one immediately establishes that $\mathcal{M}^\star + \underline{\varOmega}$ is a tiling of $\mathbb{R}$ (since $\varOmega_a \cup \varOmega_b = [-1\,, \frac{1}{\tau}]$, this union being measure-disjoint, and the length of this interval is exactly τ). Alternatively, one might calculate $\varUpsilon_a = \Lambda([0\,, \ell_a[) = \Lambda([0\,, \tau[) = \{\ldots, -\tau, 0, 1, \tau + 1, \ldots\}$ and $\varUpsilon_b = \Lambda([0\,, \ell_b[) = \Lambda([0\,, 1[) = \{\ldots, -\tau, 0, \tau + 1, \ldots\}$ (which one also obtains as fixed point of the EMFS $\boldsymbol{\Theta}^{\#\star}$ starting with the seed $\omega_a(0)$, wherefore $\underline{\varUpsilon}$ repetitive and every cluster is legal) within the CPS $(H = \mathbb{R}\,, \mathbb{R}\,, \tilde{\mathcal{L}})$. Then, one can check that $\underline{\varUpsilon} + \underline{\varOmega}$ yields a tiling of $H = \mathbb{R}$ (or, at least, that its covering degree is 1 inside some open set), as the following picture indicates:

[22] This yields the sequence $b\dot{a} \mapsto a\dot{a}b \mapsto ab\dot{a}ba \mapsto aba\dot{a}baab \mapsto abaab\dot{a}baababa \mapsto \ldots \mapsto \ldots aba{\bullet\bullet}\dot{a}baab\ldots$, where $\bullet\bullet$ is either ab or ba (thus, we have a – respectively two, but they only differ on two positions – fixed point under σ_{Fib}^2 and not under σ_{Fib}).



So, we have established that the Fibonacci sequence is a multi-component model set.

Here, however, the situation is particularly easy since one explicitly knows the attractor $\underline{\varOmega}$ (and this attractor is simply given by intervals). Suppose, we do not know this explicit form of this attractor. Then, we construct the graphs $G^{ovlap}_{\sigma_{\mathrm{Fib}}}(\underline{\varUpsilon} + \underline{\varOmega})$ respectively $G^{bd}_{\sigma_{\mathrm{Fib}}}(\underline{\varUpsilon} + \underline{\varOmega})$ (both, of course, are the same) to decide the model set property. The IFS $\boldsymbol{\varTheta}^\star$ yields the following tile substitutions for the sets $\varOmega_i$:

$$\begin{array}{lcllcl} \varOmega_a + x^\star & \mapsto & \varOmega_a - \tau \cdot x^\star \,\cup\, \varOmega_b - \tau \cdot x^\star & = & \varOmega_a + (\frac{x}{\tau})^\star \,\cup\, \varOmega_b + (\frac{x}{\tau})^\star \\ \varOmega_b + x^\star & \mapsto & \varOmega_a + 1 - \tau \cdot x & = & \varOmega_a + (\frac{x}{\tau} + 1)^\star \end{array}$$

Consequently, we obtain for $\varXi(i,j,x) = \varOmega_i \cap \varOmega_j + x^\star$ the following substitutions (of course, one has $(1)^\star = 1$, but we use this notation since t and not $t^\star$ will be the corresponding edge label in the graph):

$$\begin{array}{lcl} \varXi(a,a,x) & \mapsto & \varXi(a,a,\frac{x}{\tau}) \,\cup\, \varXi(a,b,\frac{x}{\tau}) \,\cup\, \varXi(b,a,\frac{x}{\tau}) \,\cup\, \varXi(b,b,\frac{x}{\tau}) \\ \varXi(a,b,x) & \mapsto & \varXi(a,a,\frac{x}{\tau}+1) \,\cup\, \varXi(b,a,\frac{x}{\tau}+1) \\ \varXi(b,a,x) & \mapsto & \varXi(a,a,\frac{x}{\tau}-1) + (1)^\star \,\cup\, \varXi(a,b,\frac{x}{\tau}-1) + (1)^\star \\ \varXi(b,b,x) & \mapsto & \varXi(a,a,\frac{x}{\tau}) + (1)^\star \end{array} \tag{6.14}$$

We have to determine the admissible x in $\varXi(i,j,x)$. For this, we first estimate the diameter of the sets $\varOmega_i$. The contraction factor of $\boldsymbol{\varTheta}^\star$ is $\frac{1}{\tau}$, and the maximal translational part in modulus of the maps in $\boldsymbol{\varTheta}^\star$ is the translational part of $f^\star_\tau$ (see Equation (6.13)), which is (in modulus) also $\frac{1}{\tau}$. Consequently, the sets $\varOmega_i$ are in the compact ball $B_{\leq \breve{R}}(0)$ with

$$\breve{R} = \sum_{k=0}^{\infty} \left(\frac{1}{\tau}\right)^k \cdot \frac{1}{\tau} = \frac{1}{\tau - 1} = \tau.$$

Thus, their diameter is bounded by $2\,\breve{R}$, and the admissible values for x in $\varXi(i,j,x)$ is given by the (finite) set[23]

$$\varGamma_{ij} = \{x \in \mathcal{L} \mid x \in]-\ell_j\,,\ell_i[\text{ and } x^\star \in [-4\,\breve{R}\,, 4\,\breve{R}]\},$$

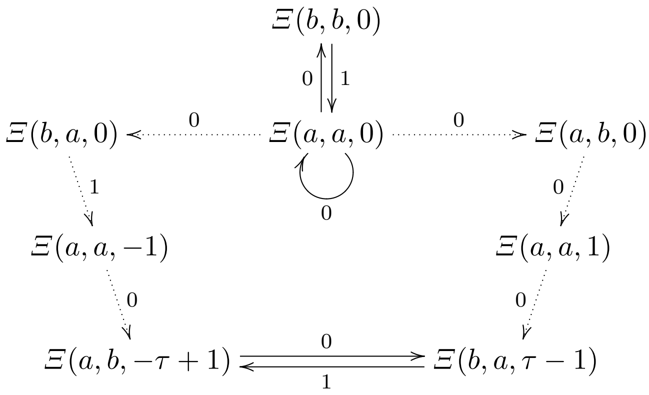
where we here obtain

$$\begin{array}{lcl} \varGamma_{bb} & = & \{-3\tau+4, -2\tau+3, -2\tau+4, -\tau+1, -\tau+2, 0, \tau-1, \tau-2, 2\tau-3, 2\tau-4, 3\tau-4\} \\ \varGamma_{ab} & = & \varGamma_{bb} \,\cup\, \{-\tau+3, 1, 2\tau-2\} \\ \varGamma_{ba} & = & \varGamma_{bb} \,\cup\, \{-2\tau+2, -1, \tau-3\} \\ \varGamma_{aa} & = & \varGamma_{bb} \,\cup\, \{-2\tau+2, -\tau+3, -1, 0, 1, \tau-3, 2\tau-2\}. \end{array}$$

This yields the vertices of our graphs under consideration, namely, the set of vertices is given by $\{\varXi(i,j,x) \,|\, x \in \varGamma_{ij},\ i,j \in \{a,b\}\}$. We now connect these vertices according to the substitution rules of Equation (6.14) (note that many vertices have no children, *e.g.*, $\varXi(b,b,-2\tau+3)$ would

[23] Actually, it would be enough to check all $x^\star \in [-3\,\breve{R}\,, 3\,\breve{R}]$.

yield $\Xi(a,a,3\tau-5)$ which is not an admissible vertex). The (unique) essential part of the graph we obtain by this procedure, looks as follows :



All vertices can be reached by a directed path from $\Xi(a,a,0)=\Omega_a$ respectively $\Xi(b,b,0)=\Omega_b$, wherefore the two graphs, namely the overlap and the boundary graph, coincide (and we have confirmed that we have a model set). Moreover, all vertices different from $\Xi(a,a,0)$ and $\Xi(b,b,0)$ represent subsets of the common boundary $\partial\Omega_a\cup\partial\Omega_b$ (which, given the explicit form, one easily calculates as $\{-1,-\frac{1}{\tau^2},\frac{1}{\tau}\}$). This union of boundaries can be calculated using the IFS formed by the vertices $\Xi(b,a,\tau-1)$ and $\Xi(a,b,-\tau+1)$ (we have indicated this IFS and the IFS for the sets Ω_a by using bold arrows "→", while otherwise we use dotted arrows "⇢"). As sets, *i.e.*, disregarding the order of the intersection, we have

$$\Xi(b,a,\tau-1)=\Omega_b\cap\Omega_a+(\tau-1)^\star=(\Omega_a\cap\Omega_b+(-\tau+1)^\star)+(\tau-1)^\star=\Xi(a,b,-\tau+1)+(\tau-1)^\star,$$

wherefore the IFS

$$\Xi(b,a,\tau-1)\;=\;-\frac{1}{\tau}\left(\Xi(a,b,-\tau+1)+(1)^\star\right)\qquad \Xi(a,b,-\tau+1)\;=\;-\frac{1}{\tau}\Xi(b,a,\tau-1)$$

reduces[24] to

$$\Xi(a,b,-\tau+1)\;=\;-\frac{1}{\tau}\Xi(a,b,-\tau+1)+1\quad\text{(respectively, }\Xi(b,a,\tau-1)\;=\;-\frac{1}{\tau}\Xi(b,a,\tau-1)-\tau).$$

This yields the solution $\Xi(a,b,-\tau+1)=\{\sum_{k=0}^{\infty}\left(\frac{-1}{\tau}\right)^k\cdot 1\}=\{\frac{1}{\tau}\}$ (respectively, similarly, $\Xi(b,a,\tau-1)=\{-1\}$). Thus, one can determine all sets in the above graph, and therefore all nonempty sets $\Xi(i,j,x)$ with $x\in\Gamma_{ij}$, namely (observing that $\Xi(a,a,1)=-\frac{1}{\tau}\left(\Xi(b,a,\tau-1)+0^\star\right)$ *etc.*)

$$\begin{array}{lllll}\{-1\} &=& \Xi(b,a,\tau-1) && \\ \{\frac{1}{\tau}\} &=& \Xi(a,b,-\tau+1) &=& \Xi(a,a,1)\\ \{-\frac{1}{\tau^2}\} &=& \Xi(a,b,0) &=& \Xi(b,a,0) \;=\; \Xi(a,a,-1).\end{array}$$

The boundary of Ω_i is given by the union of all weak overlaps $\Xi(i,k,x)$, which here yields – as expected –

$$\begin{array}{lllll}\partial\Omega_a &=& \Xi(a,b,-\tau+1)\cup\Xi(a,a,1)\cup\Xi(a,b,0)\cup\Xi(a,a,-1) &=& \{-\frac{1}{\tau^2},\frac{1}{\tau}\}\\ \partial\Omega_b &=& \Xi(b,a,\tau-1)\cup\Xi(b,a,0) &=& \{-1,-\frac{1}{\tau^2}\}\end{array}$$

[24] One can also solve $\Xi(b,a,\tau-1)\;=\;-\frac{1}{\tau}\left(\Xi(a,b,-\tau+1)+(1)^\star\right)\;=\;\frac{1}{\tau^2}\Xi(b,a,\tau-1)-\frac{1}{\tau}$.

6.10.2. The Clubsuit Substitution

We consider the following Pisot substitution $\sigma_\clubsuit$ on $\mathcal{A} = \{a, b\}$ given by $a \mapsto aaba$ and $b \mapsto aa$ which yields the fixed point $\ldots abaaabaa\dot{a}abaaab \ldots$ and which – for no particular reason – we call the clubsuit substitution or $\clubsuit$-substitution. Here, we have

$$\boldsymbol{S}\sigma_\clubsuit = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix},$$

eigenvalues	left PF-eigenvector	right PF-eigenvector
$\lambda = \frac{3+\sqrt{17}}{2} \approx 3.562$ $\lambda_2 = \frac{3-\sqrt{17}}{2} \approx -0.562$	$\ell = (\frac{\lambda}{2}, 1)$	$\varrho = \begin{pmatrix} \frac{\lambda}{2} - 1 \\ 2 - \frac{\lambda}{2} \end{pmatrix} \approx \begin{pmatrix} 0.781 \\ 0.219 \end{pmatrix}$

Thus, we obtain the following EMFS and its adjoint IFS for the $\clubsuit$-substitution:

$$\boldsymbol{\Theta} = \begin{pmatrix} \{f_0, f_{\frac{\lambda}{2}}, f_{\lambda+1}\} & \{f_0, f_{\frac{\lambda}{2}}\} \\ \{f_\lambda\} & \varnothing \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Theta}^{\#} = \begin{pmatrix} \{g_0, g_{\frac{1}{2}}, g_{\frac{\lambda-1}{2}}\} & \{g_1\} \\ \{g_0, g_{\frac{1}{2}}\} & \varnothing \end{pmatrix},$$

where we use the obvious notations $f_0(x) = \lambda \cdot x$, $f_{\frac{\lambda}{2}}(x) = \lambda \cdot x + \frac{\lambda}{2}$, $f_\lambda(x) = \lambda \cdot x$, $f_{\lambda+1}(x) = \lambda \cdot x + \lambda + 1$, $g_0(x) = \frac{1}{\lambda} \cdot x$, $g_{\frac{1}{2}}(x) = \frac{1}{\lambda} \cdot x + \frac{1}{2}$, $g_1(x) = \frac{1}{\lambda} \cdot x + 1$ and $g_{\frac{\lambda-1}{2}}(x) = \frac{1}{\lambda} \cdot x + \frac{\lambda-1}{2}$. Already from $\dot{a} \mapsto \dot{a}aba$ (since this means that $\varLambda_a = \{\ldots, 0, \frac{\lambda}{2}, \lambda - 1, \ldots\}$), we conclude that $\mathcal{L} = \mathcal{L}' = \langle \frac{\lambda}{2}, 1 \rangle_{\mathbb{Z}}$. The reason why we have chosen $(\frac{\lambda}{2}, 1)$ (and not $(\lambda, 2)$) as left eigenvector of $\boldsymbol{S}\sigma_\clubsuit$ is the following: We have $\det \boldsymbol{S}\sigma_\clubsuit = -2$, wherefore the internal space also contains a 2-adic component (more precisely, a prolongation of $\mathbb{Q}_2$). Indeed, Lemma 3.82 establishes that the prime 2 splits in $\mathbb{Q}(\sqrt{17})$, say as $(2) = \mathfrak{P}_1 \cdot \mathfrak{P}_2$. Therefore, both $\mathbb{Q}_{\mathfrak{P}_1}$ and $\mathbb{Q}_{\mathfrak{P}_2}$ are "extensions" of degree 1 of $\mathbb{Q}_2$ and we may identify them simply with $\mathbb{Q}_2$. In fact, we obtain the following factorisation in $\mathbb{Q}_2$ of the minimal polynomial of λ (also see Propositions 3.98 & 3.99):

$$\begin{aligned} \operatorname{Irr}(\lambda, \mathbb{Q}, x) &= \det(x\boldsymbol{E} - \boldsymbol{S}\sigma_\clubsuit) = x^2 - 3x - 2 \\ &= (x - .10110001001101100110\ldots) \cdot (x - .01101110110010011001\ldots) \end{aligned}$$

But this shows[25] that $\|\lambda\|_{\mathfrak{P}_1} = 1$, while $\|\lambda\|_{\mathfrak{P}_2} = \frac{1}{2}$ (so, although we identify $\mathbb{Q}_{\mathfrak{P}_1}$ and $\mathbb{Q}_{\mathfrak{P}_2}$ with $\mathbb{Q}_2$, we have to be careful what we mean if we simply write $\lambda \in \mathbb{Q}_2$). Moreover, we note that $\|\frac{\lambda}{2}\|_{\mathfrak{P}_2} = 1$, wherefore $\mathcal{L} \subset \widehat{\mathfrak{o}_{\mathfrak{P}_2}}$ and we may identify $\widehat{\mathfrak{o}_{\mathfrak{P}_2}}$ with $\mathbb{Z}_2$. Consequently, the internal space(s) are given by $H = \mathbb{R} \times \widehat{\mathfrak{o}_{\mathfrak{P}_2}} \cong \mathbb{R} \times \mathbb{Z}_2$ and $H_{\text{ext}} = \mathbb{R} \times \mathbb{Q}_{\mathfrak{P}_2} \cong \mathbb{R} \times \mathbb{Q}_2$. In this example and from now on, we will always mean $\mathbb{Q}_{\mathfrak{P}_2}$ (respectively $\widehat{\mathfrak{o}_{\mathfrak{P}_2}}$) if we write $\mathbb{Q}_2$ (respectively $\mathbb{Z}_2$), wherefore we always have $\lambda = .01101\ldots$ as 2-adic expansion. Note that we have chosen $(\frac{\lambda}{2}, 1)$ as left eigenvector of $\boldsymbol{S}\sigma_\clubsuit$ since we then have $\delta_{\mathfrak{P}_2} = 1$, *i.e.*, $\mathcal{L} \subset \widehat{\mathfrak{o}_{\mathfrak{P}_2}}$ (although we have $\mathfrak{o}_{\mathbb{Q}(\sqrt{17})} = \mathbb{Z}[\frac{\sqrt{17}}{2}] \subsetneq \mathcal{L}$ here, as also $\|\frac{\lambda}{2}\|_{\mathfrak{P}_1} = 2$ confirms).

Here, the star-map $^\star : \mathbb{Q}(\lambda) \to \mathbb{R} \times \mathbb{Q}_2$ is explicitly given[26] by $q_1 + q_2 \cdot \sqrt{17} \mapsto (q_1 - q_2 \cdot \sqrt{17}, q_1 + q_2 \cdot .10010\ldots)$ (where $q_1, q_2 \in \mathbb{Q}$), which, of course, is the action of the corresponding Galois

[25]Here, we have taken $\mathfrak{P}_1 = (2, \frac{1+\sqrt{17}}{2})$ and $\mathfrak{P}_2 = (2, \frac{-1+\sqrt{17}}{2})$ as Lemma 3.82 suggests. Indeed, one can check that $\lambda \notin \mathfrak{P}_1$, but $\lambda = 2 + \frac{-1+\sqrt{17}}{2} \in \mathfrak{P}_2$. Using KANT [293], one can also establish that $\mathfrak{P}_2 = (\lambda)$ and $\mathfrak{P}_1 = (3 - \lambda)$; note that $\mathfrak{o}_{\mathbb{Q}(\sqrt{17})}$ is an Euclidean ring and therefore, in particular, a principal ideal domain, *i.e.*, every ideal is principal. Equivalently, the class number of $\mathbb{Q}(\sqrt{17})$ is 1.

[26]From the above 2-adic expansion of $\lambda = .01101\ldots$ one also derives the 2-adic expansion of $\sqrt{17}$ as $.100101110110\ldots$.

automorphism $\lambda \mapsto \lambda_2$ in the first coordinate and the diagonal embedding into $\mathbb{Q}_{\mathfrak{P}_2} \cong \mathbb{Q}_2$ in the second coordinate. The lattice $\tilde{\mathcal{L}}$ in $\mathbb{R}^2 \times \mathbb{Z}_2$ is therefore given by (note that $\frac{\lambda}{2} = .11011\ldots$)

$$\tilde{\mathcal{L}} = \left\{ m_1 \cdot \begin{pmatrix} \frac{3+\sqrt{17}}{4} \\ \frac{3-\sqrt{17}}{4} \\ .11011\ldots \end{pmatrix} + m_2 \cdot \begin{pmatrix} 1 \\ 1 \\ .1 \end{pmatrix} \;\middle|\; m_1, m_2 \in \mathbb{Z} \right\},$$

and similar for the extended version $\tilde{\mathcal{L}}_{\text{ext}}$ in $\mathbb{R}^2 \times \mathbb{Q}_2$ (one may explicitly check that $(\frac{1}{\lambda}\mathcal{L} \setminus \mathcal{L}) \cap \mathbb{Z}_2 = \varnothing$, since $\frac{1}{\lambda}\mathcal{L} \subset \mathbb{Z}_2$ iff $m_1 \equiv 0 \equiv m_2 \bmod 2$; but $\frac{2}{\lambda} = \lambda - 3$, wherefore one gets an element of $\mathcal{L}$ if $m_1 \equiv 0 \equiv m_2 \bmod 2$).

We construct the graphs $G^{ovlap}_{\sigma_\clubsuit}(\varUpsilon + \varOmega)$ respectively $G^{bd}_{\sigma_\clubsuit}(\varUpsilon + \varOmega)$ (they are, of course, the same) to establish that the $\clubsuit$-sequence is a multi-component model set. The IFS $\boldsymbol{\Theta}^\star$ yields the following tile substitution for the sets $\varOmega_i$:

$$\begin{aligned} \varOmega_a + x^\star \;\; &\mapsto \;\; \varOmega_a + \left(\tfrac{x}{\lambda}\right)^\star \cup \varOmega_b + \left(\tfrac{x}{\lambda}\right)^\star \cup \\ &\qquad \varOmega_a + \left(\tfrac{x}{\lambda} + \tfrac{1}{2}\right)^\star \cup \varOmega_b + \left(\tfrac{x}{\lambda} + \tfrac{1}{2}\right)^\star \cup \varOmega_a + \left(\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \\ \varOmega_b + x^\star \;\; &\mapsto \;\; \varOmega_a + \left(\tfrac{x}{\lambda} + 1\right)^\star \end{aligned}$$

From these, we obtain the following substitutions for the sets $\varXi(i,j,x)$:

$$\begin{aligned}
\varXi(a,a,x) \;\; \mapsto \;\; & \varXi(a,a,\tfrac{x}{\lambda}) \cup \varXi(a,a,\tfrac{x}{\lambda}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda}) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \cup \varXi(a,a,\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - 1) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} - \tfrac{\lambda}{2} + 1) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \cup \varXi(a,b,\tfrac{x}{\lambda}) \cup \\
& \varXi(a,b,\tfrac{x}{\lambda}) + \left(\tfrac{1}{2}\right)^\star \cup \varXi(a,b,\tfrac{x}{\lambda} - \tfrac{\lambda}{2} + 1) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \cup \\
& \varXi(b,a,\tfrac{x}{\lambda}) \cup \varXi(b,a,\tfrac{x}{\lambda}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(b,a,\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - 1) + \left(\tfrac{1}{2}\right)^\star \cup \varXi(b,b,\tfrac{x}{\lambda}) \cup \\
& \varXi(b,b,\tfrac{x}{\lambda}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} + \tfrac{1}{2}) \cup \varXi(a,a,\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - \tfrac{1}{2}) \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} - \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \cup \varXi(a,a,\tfrac{x}{\lambda} - \tfrac{\lambda}{2} + \tfrac{1}{2}) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,b,\tfrac{x}{\lambda} + \tfrac{1}{2}) \cup \varXi(a,b,\tfrac{x}{\lambda} - \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,b,\tfrac{x}{\lambda} - \tfrac{\lambda}{2} + \tfrac{1}{2}) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \cup \varXi(b,a,\tfrac{x}{\lambda} - \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(b,a,\tfrac{x}{\lambda} + \tfrac{1}{2}) \cup \varXi(b,a,\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - \tfrac{1}{2}) \cup \\
& \varXi(b,b,\tfrac{x}{\lambda} + \tfrac{1}{2}) \cup \varXi(b,b,\tfrac{x}{\lambda} - \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \\
\varXi(a,b,x) \;\; \mapsto \;\; & \varXi(a,a,\tfrac{x}{\lambda} + 1) \cup \varXi(b,a,\tfrac{x}{\lambda} + 1) \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} + \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \cup \varXi(b,a,\tfrac{x}{\lambda} + \tfrac{1}{2}) + \left(\tfrac{1}{2}\right)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} - \tfrac{\lambda}{2} + \tfrac{3}{2}) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star \\
\varXi(b,a,x) \;\; \mapsto \;\; & \varXi(a,a,\tfrac{x}{\lambda} - 1) + (1)^\star \cup \varXi(a,b,\tfrac{x}{\lambda} - 1) + (1)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} - \tfrac{1}{2}) + (1)^\star \cup \varXi(a,b,\tfrac{x}{\lambda} - \tfrac{1}{2}) + (1)^\star \cup \\
& \varXi(a,a,\tfrac{x}{\lambda} + \tfrac{\lambda}{2} - \tfrac{3}{2}) + (1)^\star \\
\varXi(b,b,x) \;\; \mapsto \;\; & \varXi(a,a,\tfrac{x}{\lambda}) + (1)^\star
\end{aligned} \tag{6.15}$$

Again, we have to determine the admissible x for these sets $\varXi(i,j,x)$. The easy part is the bound in the 2-adic component: Since $\varLambda_i \subset \mathcal{L} \subset \mathbb{Z}_2$ and $\mathbb{Z}_2 + \mathbb{Z}_2 = \mathbb{Z}_2$, we must certainly have

$|x|_2 \leq 1$. The bound in the Euclidean coordinate on Ω_i follows as in the Fibonacci case as (noting the contraction factor is $|3-\lambda| = \lambda - 3$ and the, in modulus, maximal translational part that occurs in the IFS $\boldsymbol{\Theta}^\star$ is also $\max\{0, |\frac{\lambda^\star}{2}|, |\lambda^\star + 1|, |\lambda^\star|\} = \lambda - 3$)

$$\breve{R} = \sum_{k=0}^{\infty} (\lambda - 3)^{k+1} = \frac{1}{2}(\lambda - 1) \approx 1.281.$$

Thus, we are looking for the sets (note that since $x \in \mathbb{Z}_2$ it is enough to consider the lattice points in $\mathbb{R}^2 \times \mathbb{Z}_2$ and therefore we simply have to consider the lattice $\tilde{\mathcal{L}}$)

$$\Gamma_{ij} = \{x \in \mathcal{L} \mid x \in]-\ell_j, \ell_i[\text{ and } x^\star \in [-4\,\breve{R}, 4\,\breve{R}] \times \mathbb{Z}_2\},$$

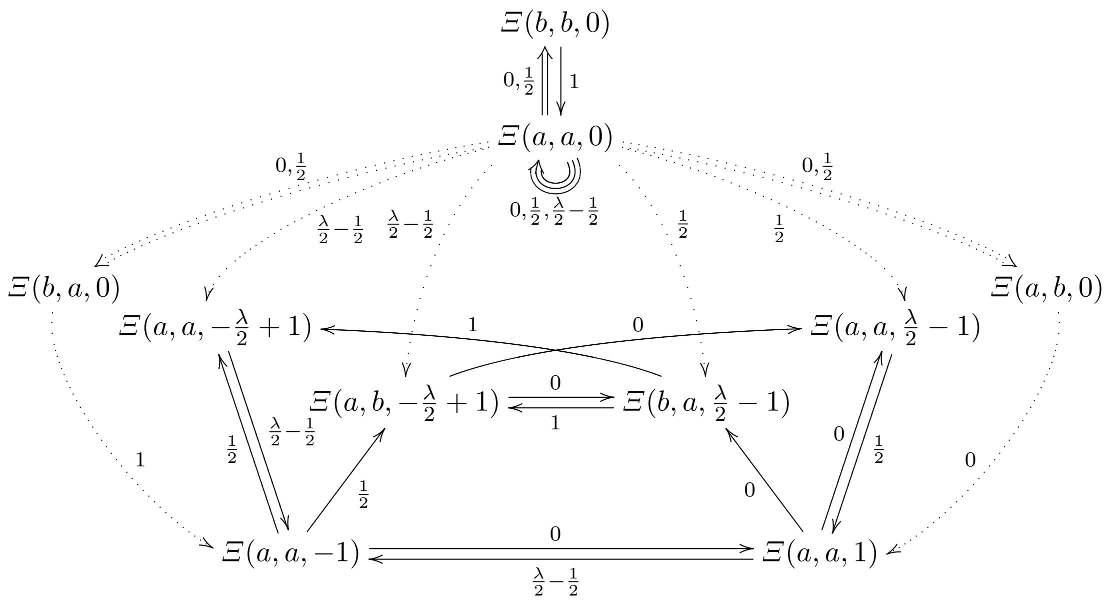
which are explicitly given by

$$\begin{aligned}
\Gamma_{bb} &= \{-\lambda + 4, -\lambda + 3, -\tfrac{\lambda}{2} + 2, -\tfrac{\lambda}{2} + 1, 0, \tfrac{\lambda}{2} - 1, \tfrac{\lambda}{2} - 2, \lambda - 3, \lambda - 4\} \\
\Gamma_{ab} &= \Gamma_{bb} \cup \{-\tfrac{\lambda}{2} + 3, 1, \lambda - 2, \tfrac{3\lambda}{2} - 4\} \\
\Gamma_{ba} &= \Gamma_{bb} \cup \{-\tfrac{3\lambda}{2} + 4, -\lambda + 2, -1, \tfrac{\lambda}{2} - 3\} \\
\Gamma_{aa} &= \Gamma_{ab} \cup \Gamma_{ba}.
\end{aligned}$$

This yields the vertices of our graphs under consideration. The reason why we have grouped the substitutes in Equation (6.15) into two sets (for $\Xi(a,a,x)$ the first 7 lines *vs.* lines 8–13, and for $\Xi(a,b,x)$ respectively for $\Xi(b,a,x)$ the first *vs.* the second and third line) is the following: if $|x|_2 < 1$, then only sets of the respective first group can appear as children, since then $|\frac{x}{\lambda}|_2 \leq 1$ but in the second group one always adds a number y with $|y|_2 > 1$ to $\frac{x}{\lambda}$ (thus, one would then have $|\frac{x}{\lambda} + y|_2 > 1$). Similarly, if $|x|_2 = 1$, only substitutes of the second group can appear. For example, one has

$$\begin{aligned}
\Xi(a,b,0) &\mapsto \Xi(a,a,1) \\
\Xi(a,b,1) &\mapsto \Xi(a,a,\tfrac{\lambda}{2} - 1) + \left(\tfrac{1}{2}\right)^\star \cup \Xi(b,a,\tfrac{\lambda}{2} - 1) + \left(\tfrac{1}{2}\right)^\star \cup \Xi(a,a,0) + \left(\tfrac{\lambda}{2} - \tfrac{1}{2}\right)^\star.
\end{aligned}$$

We again only look at the essential part of the graph obtained by this procedure, which here looks as follows:



All vertices can be reached by a directed path from $\mathit{\Xi}(a,a,0)=\mathit{\Omega}_a$ respectively $\mathit{\Xi}(b,b,0)=\mathit{\Omega}_b$, wherefore the boundary and the overlap graph coincide; this proves that we have a model set. Consequently, the sets $\mathit{\Omega}_a$ and $\mathit{\Omega}_b$, which are visualised in Figure 6.2, are the windows for this multi-component model set.

Let us now determine the boundaries $\partial\mathit{\Omega}_a$ and $\partial\mathit{\Omega}_b$. The corresponding IFS is given by the subgraph consisting of the 6 vertices $\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)$, $\mathit{\Xi}(a,a,\frac{\lambda}{2}-1)$, $\mathit{\Xi}(a,a,1)$, $\mathit{\Xi}(a,a,-1)$, $\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)$ and $\mathit{\Xi}(a,b,-\frac{\lambda}{2}+1)$. The adjacency matrix of the subgraph induced by these vertices (the lower part, connected by bold directed edges "$\twoheadrightarrow$") is an irreducible but not primitive matrix (its index of imprimitivity is 2, see Lemma 5.85) of spectral radius 2. This is already enough to calculate the affinity dimension by Corollary 4.118, but we can further simplify this IFS (and then also calculate the lower affinity dimension) by observing that we have the following relationships between the $\mathit{\Xi}(i,j,x)$ interpreted as sets:

$$\mathit{\Xi}(a,a,1)=\mathit{\Xi}(a,a,-1)+(1)^\star\,,\qquad \mathit{\Xi}(a,a,\frac{\lambda}{2}-1)=\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)+\left(\frac{\lambda}{2}-1\right)^\star$$
$$\text{and}\quad \mathit{\Xi}(a,b,-\frac{\lambda}{2}+1)=\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)+\left(-\frac{\lambda}{2}+1\right)^\star$$

Consequently, the IFS for the boundaries reduces to

$$\begin{aligned}
\mathit{\Xi}(a,a,-1)&=\lambda^\star\,\left(\mathit{\Xi}(a,a,-1)+(1)^\star\right)\,\cup\,\lambda^\star\left(\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)+(-\frac{\lambda}{2}+\frac{3}{2})^\star\right)\,\cup\\
&\qquad\lambda^\star\left(\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)+(\frac{1}{2})^\star\right)\\
\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)&=\lambda^\star\left(\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)+(-\frac{\lambda}{2}+2)^\star\right)\,\cup\,\lambda^\star\left(\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)+(1)^\star\right)\\
\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)&=\lambda^\star\left(\mathit{\Xi}(a,a,-1)+(\frac{\lambda}{2}-\frac{1}{2})^\star\right)
\end{aligned}\tag{6.16}$$

where we denote by $\lambda^\star$ the map $\mathbb{R}\times\mathbb{Q}_2\to\mathbb{R}\times\mathbb{Q}_2$, $(y,z)\mapsto(\lambda_2\cdot y,\lambda\cdot z)=(\frac{3-\sqrt{17}}{2}\cdot y,.01101\ldots\cdot z)$ (the "diagonal embedding" of the multiplication by λ). The adjacency matrix of this IFS is primitive, again with spectral radius 2. In Figure 6.2, we indicate how the sets on the right hand side of this IFS are mapped into (subsets of) the corresponding sets on the left hand side, the components of the attractor of this IFS. This is indicated by arrows. Moreover, this figure also indicates that the sets $\mathit{\Xi}(a,a,-1)$, $\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)$ and $\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)$ are pairwise disjoint and all unions in the IFS of Equation (6.16) are disjoint, *e.g.*, the set $\mathit{\Xi}(a,a,-\frac{\lambda}{2}+1)$ is enclosed in the compact set $[\frac{1}{2}\,,1]\times(2\mathbb{Z}_2+1)$, which is mapped to $[2\,\lambda_2\,,\frac{3}{2}\,\lambda_2]\times(4\mathbb{Z}_2)\approx[-1.123\,,-0.842]\times(4\mathbb{Z}_2)$ under the IFS. But this set encloses the "lower" part of $\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)$ (the remaining part of $\mathit{\Xi}(b,a,\frac{\lambda}{2}-1)$ is, *e.g.*, enclosed by $[-0.9\,,-0.6]\times(4\mathbb{Z}_2+2)$ and therefore disjoint from the former one). One observes, that the total disconnectedness of the 2-adic spaces makes it particularly easy to check that the components of the attractor are pairwise disjoint and all unions in this IFS are disjoint. Consequently, the lower affinity dimension of this IFS gives a lower bound on the Hausdorff dimension of these sets by Proposition 4.129. The upper bound, of course, is given by the affinity dimension, see Proposition 4.122.

Since $0.562\approx|\lambda_2|=(\lambda-3)>|\lambda|_2=\frac{1}{2}$, the (second) singular value function for $\gamma\in]1\,,2]$

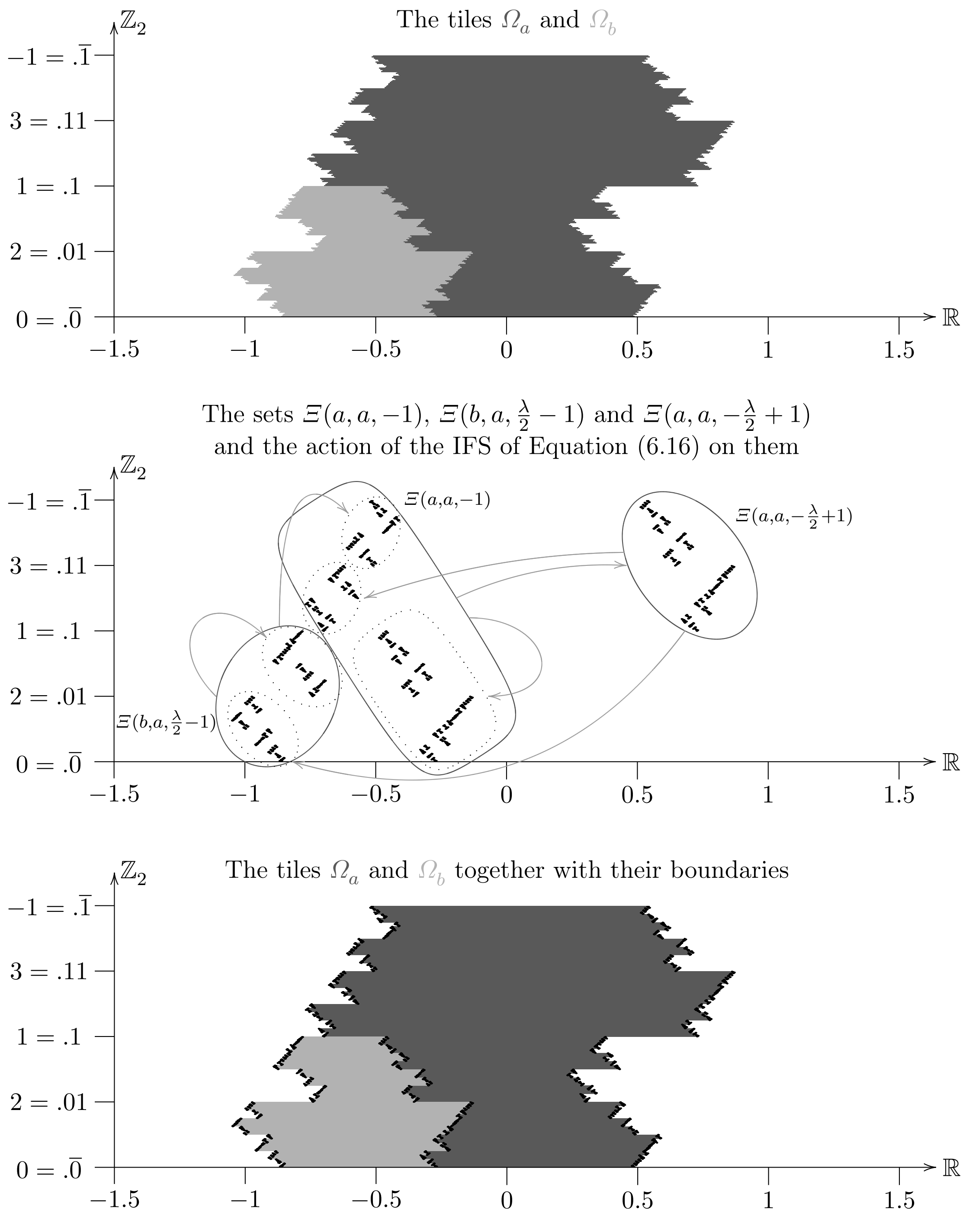


Figure 6.2.: The tiles Ω_a and Ω_b, *i.e.*, the windows for the ♣-sequence represented by natural intervals, are shown in the top picture. In the middle picture, the action of the IFS given in Equation (6.16) on the components of its attractor is shown. These sets make up part of the boundaries, which are fully shown – together with the tiles Ω_a and Ω_b – in the bottom picture.

is given by

$$\Phi^{\gamma}(\lambda^{\star}) = (\lambda - 3) \cdot \left(\frac{1}{2}\right)^{\gamma-1} \qquad \left(\text{respectively, } \Psi^{\gamma}(\lambda^{\star}) = \left(\frac{1}{2}\right) \cdot (\lambda - 3)^{\gamma-1}\right),$$

see Definition 4.111. Now, by Corollary 4.118 we can calculate the affinity dimension as $2 + \frac{\log(\lambda-3)}{\log(2)} \approx 1.167$ and, by Lemma 4.120, the lower affinity dimension as 1. Consequently, since these sets make up (part of) the boundaries of Ω_a and Ω_b, we have obtained (also see Lemma 4.133)

$$1 \;\le\; \dim_{\mathrm{Hd}} \partial \underline{\Omega} \;\le\; \underline{\dim}_{\mathrm{box}} \partial \underline{\Omega} \;\le\; \overline{\dim}_{\mathrm{box}} \partial \underline{\Omega} \;\le\; 2 + \frac{\log(\lambda - 3)}{\log(2)} \;\approx\; 1.167.$$

We now turn our attention on the periodic tiling $\operatorname{supp} \underline{\Omega} + \mathcal{M}^{\star}$ of $H = \mathbb{R} \times \mathbb{Z}_2$ and the aperiodic tiling $\underline{\Upsilon} + \underline{\Omega}$ of $H_{\mathrm{ext}} = \mathbb{R} \times \mathbb{Q}_2$, see Figure 6.3. Since $\ell_b - \ell_a = .11011\ldots - .1 = .01011\ldots$, one has $\ell_b - \ell_a \in 2\,\mathbb{Z}_2$ and thus $\mathcal{M}^{\star} = ((\ell_b - \ell_a)\,\mathbb{Z})^{\star} \subset \mathbb{R} \times 2\,\mathbb{Z}_2$. Since one also has $\Omega_b \subset \mathbb{R} \times 2\,\mathbb{Z}_2$, one obtains $\Omega_b + \mathcal{M}^{\star} \subset \mathbb{R} \times 2\,\mathbb{Z}_2$. Consequently, the translates of Ω_b are only found in the "lower half" of the periodic tiling (the periodic tiling is also a good example that subsequent translations act as "rotation" on a $\mathfrak{p}$-adic space, wherefore – in the classical sense – this tiling does "not look" periodic).

For the aperiodic tiling, we first consider the stepped surface, also shown in Figure 6.3. The model sets $\Upsilon_i = \Lambda([0\,,\ell_i[)$ yield the stepped surface $\underline{\Upsilon} + \underline{P}$: Here, $P_a = [0\,,1] \times \mathbb{Z}_2$ and $P_b = [\lambda_2\,,0] \times \mathbb{Z}_2$, wherefore – in the visualisation – to every point of $\Upsilon_a = \Lambda([0\,,\ell_a[)$ one attaches a "big rectangle P_a to the right", while to every point of $\Upsilon_b = \Lambda([0\,,\ell_b[)$ one attaches a "small rectangle P_b to the left". Replacing the hyperpolygons $\underline{P}$ by the sets $\underline{\Omega}$ yields the aperiodic tiling $\underline{\Upsilon} + \underline{\Omega}$ of H_{ext}. For an example with purely Euclidean internal space, see Figure 6.9 of Example 6.114.

Since we have constructed the stepped surface now, we also show the approximation of the sets Ω_i by the iterates $\underline{P}^{(m)} = (\boldsymbol{\Theta}^{\star})^m\,(\underline{P})$ of the $\underline{P}$ with the corresponding IFS in Figure 6.4. This iteration converges exponentially fast (with respect to the Hausdorff metric) to the attractor $\underline{\Omega}$ (already $\underline{P}^{(1)}$ captures the most prominent feature that $\Omega_b \subset \mathbb{R} \times 2\,\mathbb{Z}_2$).

As last point, we now calculate the dual lattice of $\tilde{\mathcal{L}}_{\mathrm{ext}}$: Note that for $a, b \in \mathbb{Q}$ one has $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(a + b\,\lambda) = 2\,a + 3\,b$. Thus, to derive the dual basis $\{e_1^{\perp}, e_2^{\perp}\}$ of $\{\frac{\lambda}{2}, 1\}$, we have to solve the following linear systems (we use the notation $e_i^{\perp} = a_i + b_i\,\lambda$):

$$1 = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\left(\frac{\lambda}{2} \cdot e_1^{\perp}\right) = \frac{3}{2}\,a_1 + \frac{13}{2}\,b_1 \qquad 0 = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\left(\frac{\lambda}{2} \cdot e_2^{\perp}\right) = \frac{3}{2}\,a_2 + \frac{13}{2}\,b_2$$

$$0 = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\left(1 \cdot e_1^{\perp}\right) = 2\,a_1 + 3\,b_1 \qquad 1 = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\left(1 \cdot e_2^{\perp}\right) = 2\,a_2 + 3\,b_2$$

This yields $e_1^{\perp} = -\frac{6}{17} + \frac{4}{17}\,\lambda = \frac{2}{17}\sqrt{17}$ and $e_2^{\perp} = \frac{13}{17} - \frac{3}{17}\,\lambda = \frac{4}{17} - \frac{3}{34}\sqrt{17}$. Alternatively, we know that the codifferent of $\mathbb{Z}[\lambda]$ is $\mathbb{Z}[\lambda]^{\wedge} = \mathbb{Z}[\lambda]/\sqrt{17}$. One derives for the dual basis (of $\mathbb{Z}[\lambda]^{\wedge}$) to $\{1, \lambda\}$ the set $\{\frac{13}{17} - \frac{3}{17}\,\lambda, -\frac{3}{17} + \frac{2}{17}\,\lambda\}$. From $(1, \lambda)\operatorname{diag}(1, \frac{1}{2}) = (1, \frac{\lambda}{2})$ one then also calculates $\{e_1^{\perp}, e_2^{\perp}\}$ as on p. 243. Note that $\langle e_1^{\perp}\,, e_2^{\perp}\rangle_{\mathbb{Z}} = \langle 1 + 5\,\lambda\,, 7 + \lambda\rangle_{\mathbb{Z}}/17$, wherefore

$$\mathcal{L}_{\mathrm{ext}}^{\wedge} = \bigcup_{k\ge 0} \frac{1}{\lambda^k}\,\mathcal{L}^{\perp} = \bigcup_{k\ge 0} \left(\frac{\lambda-3}{2}\right)^k \langle e_1^{\perp}\,, e_2^{\perp}\rangle_{\mathbb{Z}} = \bigcup_{k\ge 0} \frac{1}{17}\left(\frac{\lambda-3}{2}\right)^k \langle 1 + 5\,\lambda\,, 7 + \lambda\rangle_{\mathbb{Z}}. \quad (6.17)$$

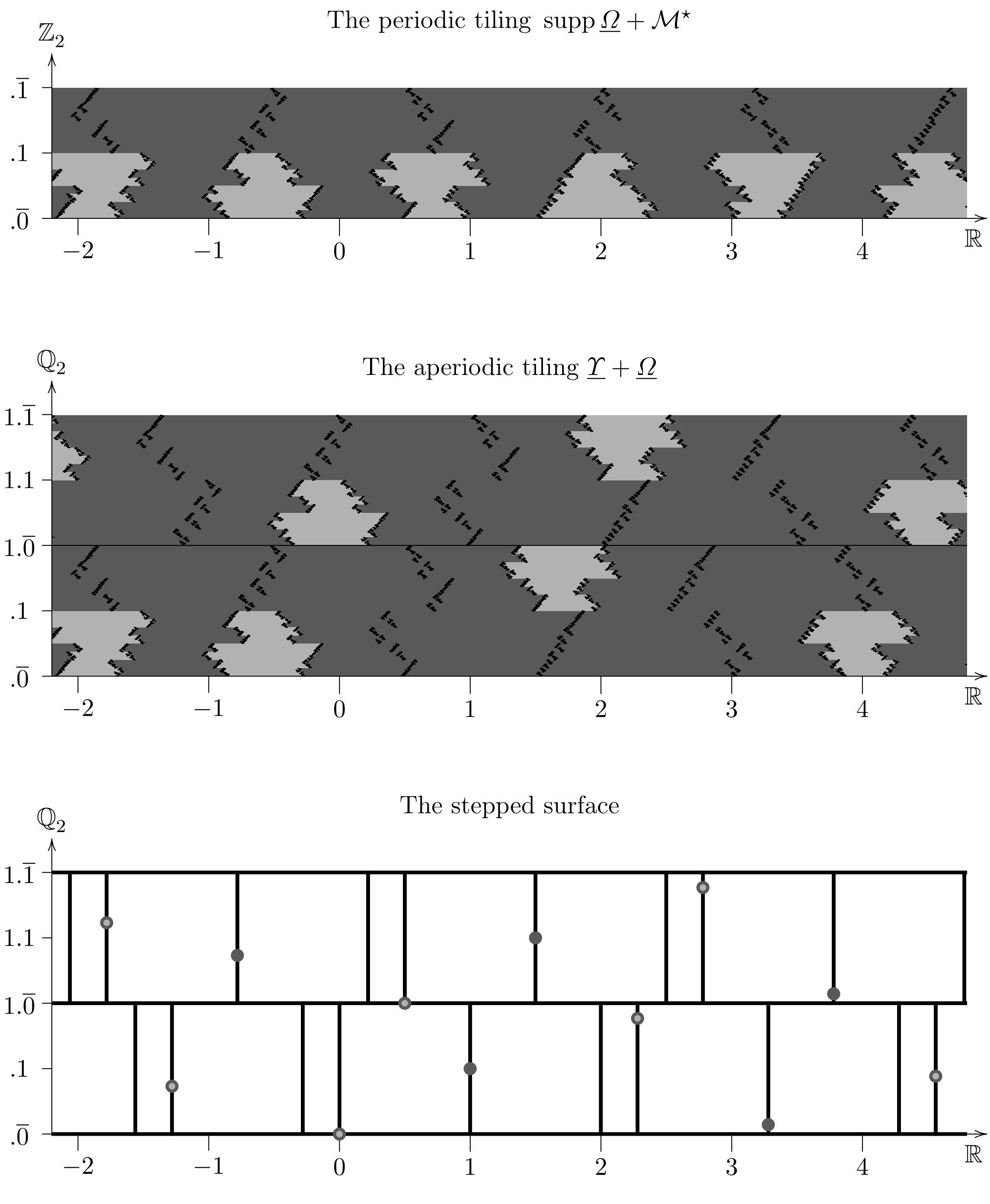


Figure 6.3.: The periodic (top) and the aperiodic (middle) tiling as well as the stepped surface (bottom) for the ♣-substitution. The tiles of the tilings are coloured as follows: Ω_a in dark gray and Ω_b in light gray. In the picture of the stepped surface, the sets $\underline{\Upsilon}$ are also shown: $\Upsilon_a = \Lambda([0, \frac{\lambda}{2}[)$ using big points and $\Upsilon_b = \Lambda([0, 1[)$ using small points.

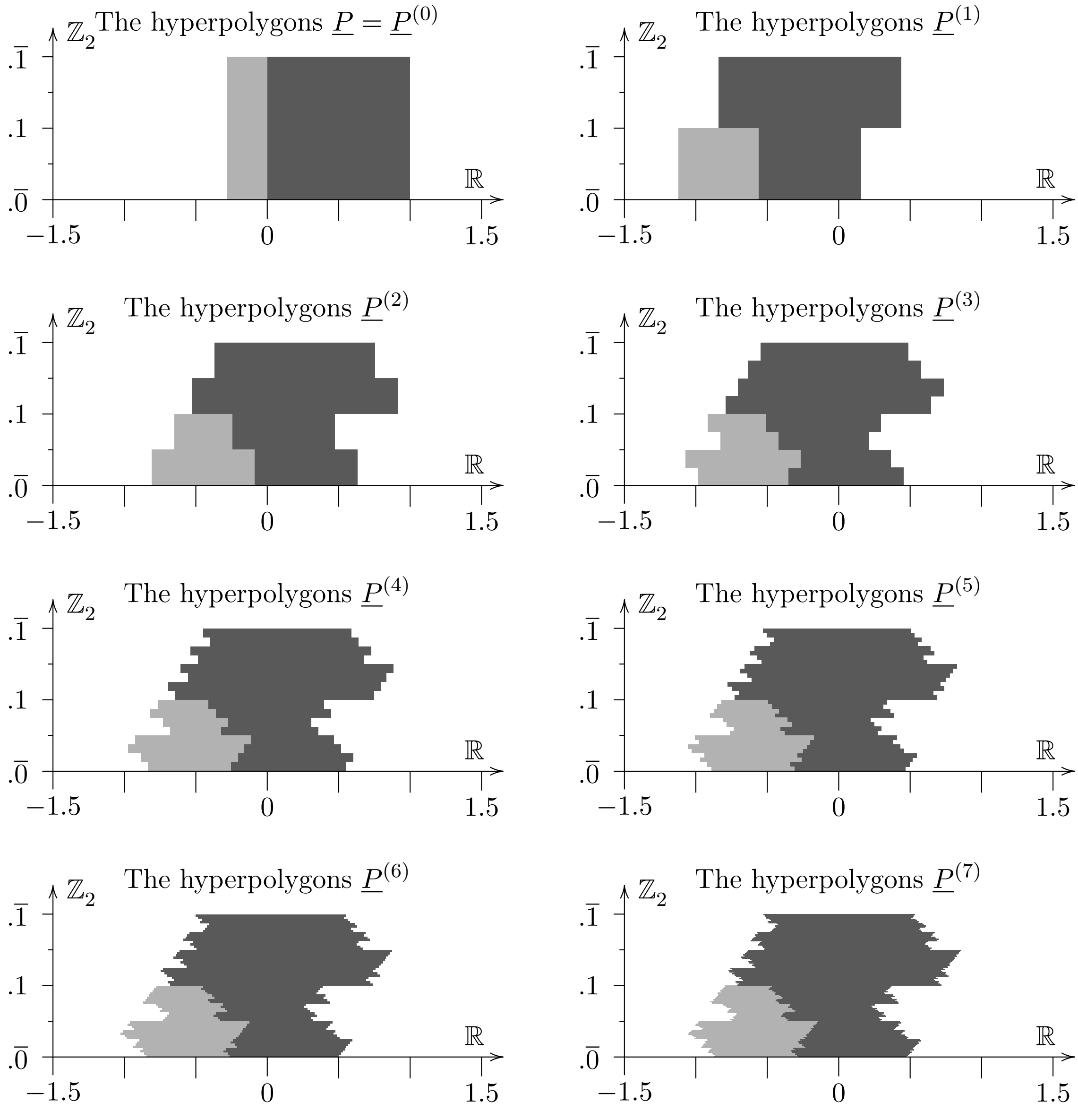


Figure 6.4.: The iterates $\underline{P}^{(m)} = (\boldsymbol{\Theta}^{\star})^m\,(\underline{P})$ are shown for $m = 0, \ldots, 7$. The colours $P_a^{(m)}$ (dark gray) and $P_b^{(m)}$ (light gray) are used.

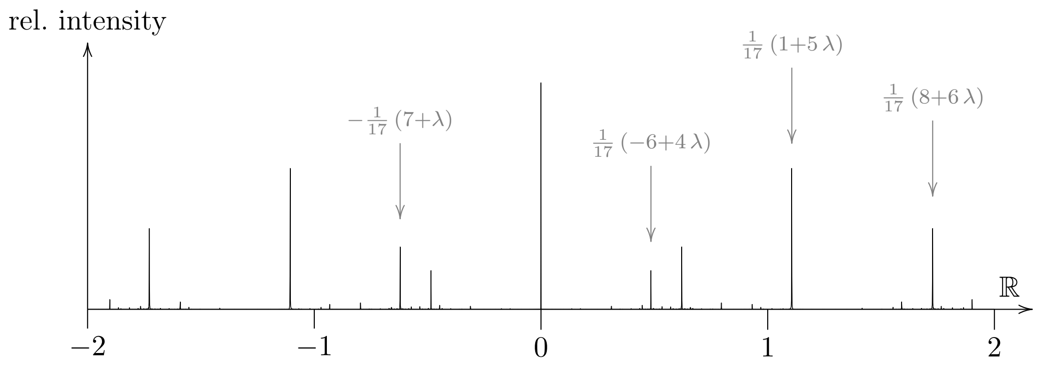


Figure 6.5.: The relative intensity in the diffraction pattern of the clubsuit sequence (in its natural interval representation, all points of the multi-component Delone set have the same scattering strength). The diffraction pattern is numerically calculated from the first 634 points in $\operatorname{supp}\underline{\varLambda}$ (note that $\#\left(\sigma_{\clubsuit}^{5}(a)\right) = 634$). For some of the most prominent so-called "Bragg peaks" (see Section 7.1), we have indicated at which position they (supposedly) reside. From Folklore Theorem 5a.10 one knows that these positions are elements of $\mathcal{L}_{\mathrm{ext}}^{\wedge}$, compare Equation (6.17).

(We also note that $\frac{1}{\lambda^2}\,\langle 1+5\,\lambda\,, 7+\lambda\rangle_{\mathbb{Z}} = \frac{1}{\lambda}\,\langle 1+5\,\lambda\,, \frac{7}{2}+\frac{\lambda}{2}\rangle_{\mathbb{Z}} = \langle \frac{5}{4}-\frac{9}{4}\,\lambda\,, \frac{7}{2}+\frac{\lambda}{2}\rangle_{\mathbb{Z}}$.)

By Folklore Theorem 5a.10, the intensity in the diffraction pattern (which lives on $\mathbb{R}^* = \mathbb{R}$) of the clubsuit sequence is concentrate on $\mathcal{L}_{\mathrm{ext}}^{\wedge}$. We show a numerically calculated intensity diagram in Figure 6.5 (also compare Remark 7.9). Note that diffraction patterns are inversion symmetric in the origin (this is known as "Friedel's rule" in crystallography).

6.11. Further Examples

Example 6.112. We begin with two "classical" examples: The substitution σ_{BMS} given by $a \mapsto aab$ and $b \mapsto abab$ which appears in [39], and the substitution σ_{Siegel} given by $a \mapsto aaab$ and $b \mapsto ab$ which appears[27] in [345, Section 7.1.2] and in connection with [346, Fig. 5.1] (also see [66, Section 1.3] and [348, Fig. 6]). For both, the determinant of their substitution matrix is given by 2, and the eigenvalues of these matrices are $\lambda = 2+\sqrt{2}$ and $2-\sqrt{2}$. We make the following choice on the left PF-eigenvalue:

- For σ_{BMS}, we choose $\ell = (1, \lambda - 2) = (1, \sqrt{2})$.
- For σ_{Siegel}, we choose $\ell = (\lambda - 1, 1) = (\sqrt{2}+1, 1)$.

This has the consequence that we obtain $\mathcal{L} = \mathbb{Z}[\sqrt{2}]$ in both cases (we also have $\mathcal{L}^{\wedge} = \mathbb{Z}[\sqrt{2}]^{\wedge} = \frac{1}{2}\,\mathbb{Z}[\sqrt{2}]$ in both cases). Consequently, the internal space is given by $H = \mathbb{R} \times \mathbb{Z}_2[\sqrt{2}]$ (we observe[28] that $\mathbb{Q}_2(\sqrt{2})$ is a quadratic extension of $\mathbb{Q}_2$ by Proposition 3.109; a uniformiser is obviously given by $\sqrt{2}$).

[27] The description of the "torus parametrisation" (see Section 7.3) for this substitution is already indicated in [307, Section 4].

[28] By Lemma 3.82, the ideal (2) is ramified in $\mathbb{Q}(\sqrt{2})$, namely, $(2) = (\sqrt{2})^2$.

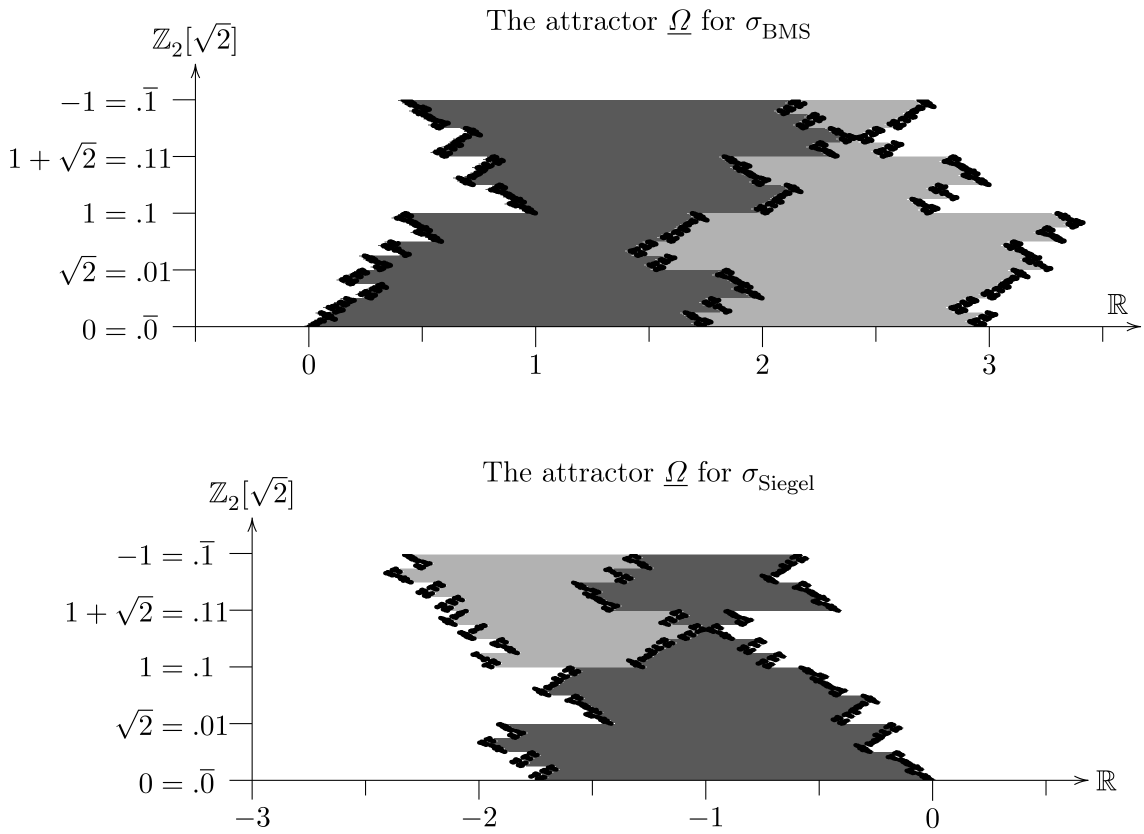


Figure 6.6.: The tiles Ω_a and Ω_b together with their respective boundaries are shown for the substitutions σ_{BMS} (top) and σ_{Siegel} (bottom).

Instead of given lengthy calculations, we simply visualise the sets Ω_a and Ω_b in Figure 6.6. We remark that every number $x \in \mathbb{Z}[\sqrt{2}]$ can be written as $x = \sum_{k=0}^{\infty} s_k \sqrt{2}^k$ with $s_k \in \{0, 1\}$, from which the visualisation of such a space follows. We also note that (parts of) the boundaries are given by the following iterated function systems:

- in case of σ_{BMS} by

$$\Xi(b, a, 0) \;=\; \lambda^{\star}\, \Xi(b, a, 0) + (\ell_a)^{\star} \;\cup\; \lambda^{\star}\, \Xi(b, a, 0) + (2\ell_a + \ell_b)^{\star}$$

- in case of σ_{Siegel} by

$$\Xi(a, a, -1) \;=\; \lambda^{\star}\, \Xi(a, a, -1) + (\ell_a)^{\star} \;\cup\; \lambda^{\star}\, \Xi(a, a, -1) + (2\ell_a)^{\star}$$

Consequently, using Corollary 4.118 and Lemma 4.120, as well as Propositions 4.122 & 4.129, we obtain in both cases the following estimate for the Hausdorff dimension of the boundaries (note that $\|\lambda\|_{\mathbb{Q}_2(\sqrt{2})} = \frac{1}{2} \leq 2 - \sqrt{2}$):

$$1 \leq \dim_{\mathrm{Hd}} \partial\Omega_a = \dim_{\mathrm{Hd}} \partial\Omega_b \leq 2 - \frac{\log(2-\sqrt{2})}{\log(2)} \approx 1.228$$

Of course, the sequences obtained by these substitutions are multi-component model sets[29] and the sets Ω_i in Figure 6.6 are the corresponding windows.

Example 6.113. We compile some further examples of non-unimodular Pisot substitutions over 2 symbols. Pictures of the corresponding attractors $\underline{\Omega}$ may be found in Figures 6.7.

- Consider the substitution σ given by $a \mapsto aaaab$ and $b \mapsto ab$. The determinant of the substitution matrix $\boldsymbol{S}\sigma$ is 3, its eigenvalues are $\lambda = \frac{1}{2}(5+\sqrt{13})$ and $\frac{1}{2}(5-\sqrt{13})$. We observe that the minimal polynomial splits in $\mathbb{Q}_3$ as follows:

  $$\mathrm{Irr}(\lambda, \mathbb{Q}, x) = x^2 - 5x + 3 = (x - .220112102101\ldots)\cdot(x - .022110120121\ldots)$$

  Consequently, we have the internal space $H_{\mathrm{ext}} \cong \mathbb{R}\times\mathbb{Q}_3$ (similarly as for the clubsuit-substitution, the ideal (3) splits in $\mathbb{Q}(\sqrt{13})$, both corresponding $\mathfrak{p}$-adic fields can be identified with $\mathbb{Q}_3$, and on one of them λ acts as contraction). We choose the left PF-eigenvector $\ell = (\lambda - 1, 1)$, wherefore one also has $H \cong \mathbb{R}\times\mathbb{Z}_3$. Thus, we also have $\mathcal{L} = \mathbb{Z}[\lambda]$ and $\mathcal{L}^{\wedge} = \mathbb{Z}[\lambda]/\sqrt{13}$. This is an example with a 3-adic internal space.

- Given the substitution[30] $a \mapsto aab$ and $b \mapsto aa$, the corresponding substitution matrix has determinant -2 and eigenvalues $\lambda = 1+\sqrt{3}$ and $1-\sqrt{3}$. A left PF-eigenvector is given by $\ell = (1+\lambda, \lambda)$, which establishes $\mathcal{L} = \mathbb{Z}[\sqrt{3}]$ (so, we also have $\mathcal{L}^{\wedge} = \mathbb{Z}[\sqrt{3}]/(2\sqrt{3})$). We note that $\mathbb{Q}_2(\sqrt{3})$ is a quadratic extension of $\mathbb{Q}_2$ (see Proposition 3.109), wherefore the internal space is given by $H = \mathbb{R}\times\mathbb{Z}_2[\sqrt{3}]$. We note that $1+\sqrt{3}$ is a uniformiser for $\mathbb{Z}_2[\sqrt{3}]$ respectively $\mathbb{Q}_2(\sqrt{3})$. This is an example where the uniformiser is not simply a root of a prime number.

- The substitution matrix of the substitution $a \mapsto aaab$ and $b \mapsto aaa$ has determinant -3, its eigenvalues are $\lambda = \frac{1}{2}(3+\sqrt{21})$ and $\frac{1}{2}(3-\sqrt{21})$ and a left PF-eigenvector is $\ell = (\lambda, \lambda+3)$. Thus, one obtains $\mathcal{L} = \langle\lambda, 3\rangle_{\mathbb{Z}}$ and $\mathcal{L}^{\wedge} = \langle \frac{2}{21} + \frac{1}{21}\lambda, \frac{1}{21} - \frac{3}{21}\lambda\rangle_{\mathbb{Z}}$. The minimal polynomial of λ is $\mathrm{Irr}(\lambda,\mathbb{Q},x) = x^2 - 3x - 3$ and therefore an Eisenstein polynomial (for $3\mathbb{Z}$). Consequently, $\mathbb{Q}_3(\lambda) = \mathbb{Q}_3(\sqrt{21})$ is a quadratic extension of $\mathbb{Q}_3$ (one also checks that the prime 3 is ramified in $\mathbb{Q}(\sqrt{21})$, see Lemma 3.82), and one confirms that it is isomorphic with $\mathbb{Q}_3(\sqrt{3})$ (compare Proposition 3.109). Actually, the 3-adic expansion of λ in $\mathbb{Q}_3(\sqrt{3})$ reads

  $$\begin{aligned}\lambda = \frac{3}{2} \pm \sqrt{3}\,.202121120221\ldots &= .02\bar{1} \pm \sqrt{3}\,.202121120221\ldots\\ &= .02\bar{1} \mp \sqrt{3}\,.120101102001\ldots\end{aligned}$$

  (where we observe that $.20212\ldots + .12010\ldots = 0$, *i.e.*, one of these two numbers is the "negative" of the other). Obviously, a uniformiser for $\mathbb{Q}_3(\sqrt{3})$ is $\sqrt{3}$. Here, λ in terms of the uniformiser is is a non-trivial expression in the $\mathfrak{p}$-adic field under consideration.

- Let σ be the substitution given by $a \mapsto aaaaaaab$ and $b \mapsto ab$. The determinant of the substitution matrix is 6, its eigenvalues are $\lambda = 4+\sqrt{10}$ and $4-\sqrt{10}$. We observe that the minimal polynomial splits in $\mathbb{Q}_3$ as follows:

  $$\mathrm{Irr}(\lambda, \mathbb{Q}, x) = x^2 - 8x + 6 = (x - .212121001201\ldots)\cdot(x - .011101221021\ldots)$$

[29] For σ_{Siegel} this is proven in the cited literature. Here, it is an application of Proposition 6.100.

[30] This example can also be found in [300, Section 7.3.1], where – using the "balanced pair algorithm" (see Section 6c.5) – it is shown that it is a multi-component model set.

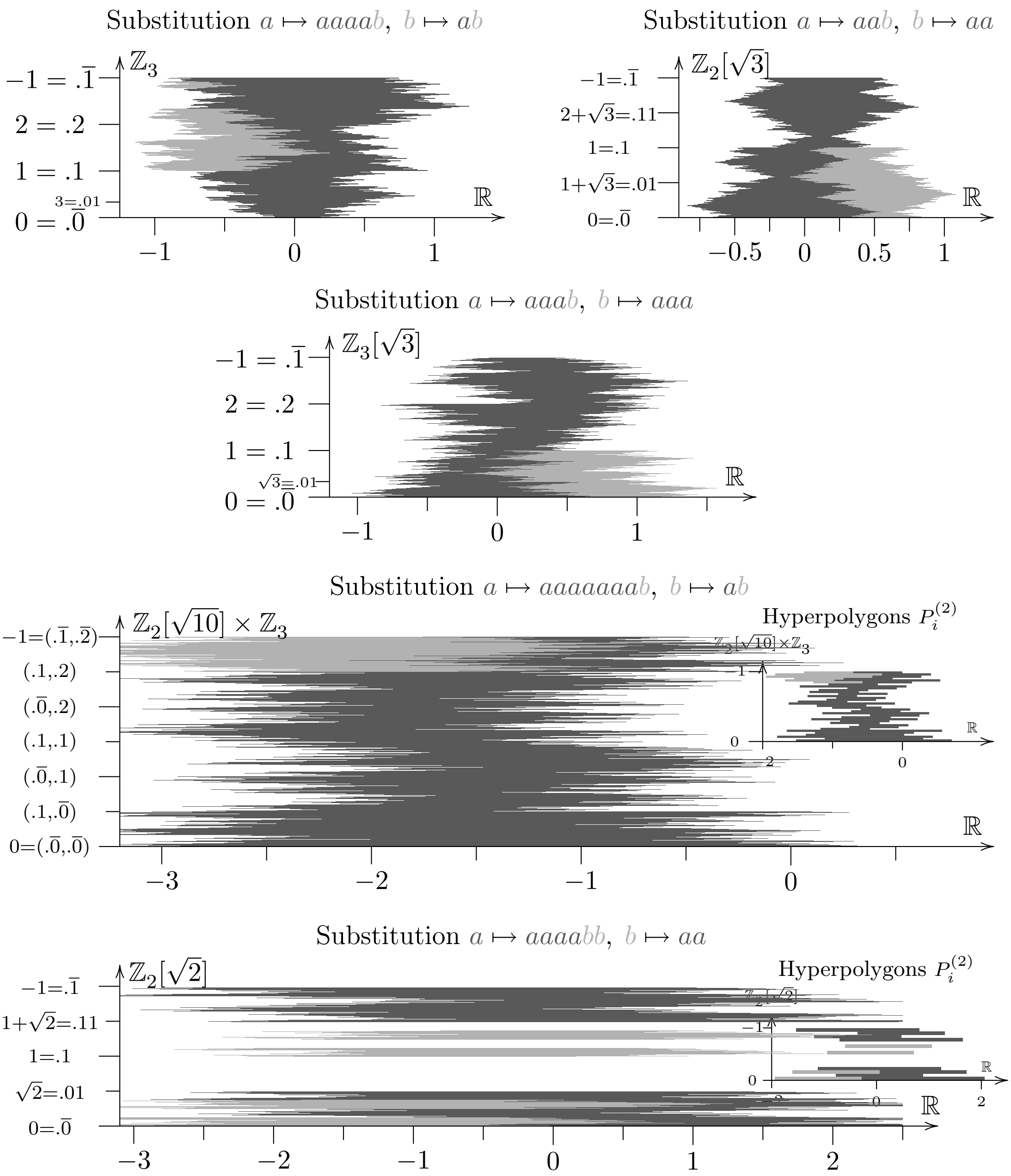


Figure 6.7.: The tiles Ω_a (in dark gray) and Ω_b (in light gray) for various non-unimodular substitutions are shown. The same colouring is used for the approximating hyperpolygons, namely, $P_a^{(2)}$ and $P_b^{(2)}$.

while it does not split[31] in $\mathbb{Q}_2$ (again, it is an Eisenstein polynomial); in fact, one immediately has that $\mathbb{Q}_2(\lambda) \cong \mathbb{Q}_2(\sqrt{10})$ and $\sqrt{10}$ is a uniformiser of $\mathbb{Q}_2(\sqrt{10})$ (by Proposition 3.109, this is a quadratic extension of $\mathbb{Q}_2$). Choosing the left PF-eigenvalue $\ell = (\lambda - 1, 1)$ yields the internal space $H \cong \mathbb{R} \times \mathbb{Z}_2(\sqrt{10}) \times \mathbb{Z}_3$ (one also has $\mathcal{L} = \mathbb{Z}[\lambda]$ and $\mathcal{L}^{\wedge} = \mathbb{Z}[\lambda]/(2\sqrt{10})$). For the visualisation of $\mathbb{Z}_2(\sqrt{10}) \times \mathbb{Z}_3$, we recall that one can embed this product in $\mathbb{R}$, see p. 68: A set of representatives for the 3-adic expansion is given by $\{0, 1, 2\}$, while the corresponding set for $\mathbb{Z}_2[\sqrt{10}]$ is given by $\{0, 1\}$. Thus, every number $x \in \mathbb{Z}[\sqrt{10}]$ can be uniquely written as $x = \sum_{k=0}^{\infty} a_k \sqrt{10}^k$ and as $x = \sum_{k=0}^{\infty} b_k 3^k$ with $a_k \in \{0, 1\}$ and $b_k \in \{0, 1, , 2\}$. Comparing with Section 3c.3 and using the notation of that section, we define the following "embedding" of any number $x \in \mathbb{Z}_2[\sqrt{10}] \times \mathbb{Z}_3$ (not just $x \in \mathbb{Z}[\sqrt{10}]$) into $[0\,, 1]$:

$$\varphi(x) = \frac{1}{6} \sum_{k=0}^{\infty} (a_k + 2 \cdot b_k) \cdot 6^{-k}.$$

We have included this example to show how one can treat products of $\mathfrak{p}$-adic spaces. However, since the sets Ω_a and Ω_b are quite "ragged" in this example, we also show a small picture of the approximating hyperpolygons $P_a^{(2)}$ and $P_b^{(2)}$ in Figure 6.7.

- As last example, we consider the substitution $a \mapsto aaaabb$ and $b \mapsto aa$, whose substitution matrix has determinant -4 and eigenvalues $\lambda = 2\,(1 + \sqrt{2})$ and $2\,(1 - \sqrt{2})$. A left PF-eigenvector is $(\frac{\lambda}{2}, 1)$, and we have $\mathcal{L} = \mathbb{Z}[\sqrt{2}]$ and $H = \mathbb{R} \times \mathbb{Z}_2[\sqrt{2}]$ as for σ_{BMS} and σ_{Siegel}. But here, we observe that $\|\lambda\|_{(\sqrt{2})} = \frac{1}{4}$, which is one reason why we choose this example. The other can be seen from the visualisation in Figure 6.7: There are $(\sqrt{2})$-adic "gaps" in the tiles Ω_a and Ω_b. For better representation, we again included the approximation by (the second iteration of) the hyperpolygons $P_i^{(2)}$.

Of course, the picture of the (possible) windows do not prove that these examples are really multi-component model sets; this is already done by Proposition 6.100. The purpose of these examples is to show that mixed Euclidean and $\mathfrak{p}$-adic spaces are nothing to be "afraid of" and – after some basic algebraic number theory – one might apply our algorithm to construct the overlap respectively boundary graph in a straightforward manner.

Example 6.114. Historically (see [352]), our favourite examples are so-called Kolakoski sequences. In fact, using the periodic tiling of H with $\operatorname{supp} \underline{\Omega}$, it has been established in [41] (also compare [352, Section 2.3]) that the Pisot substitution $\sigma_{\mathrm{Kol}(3,1)}$ associated with the so-called Kolakoski(3, 1) sequence is a (multi-component) model set. This substitution $\sigma_{\mathrm{Kol}(3,1)}$ is given by $a \mapsto abc$, $b \mapsto ab$ and $c \mapsto b$, whose substitution matrix – as the (maybe second best studied Pisot substitution after Fibonacci) tribonacci substitution $a \mapsto ab$, $b \mapsto ac$ and $c \mapsto a$ originally analysed in [306] – is unimodular and has one real and a complex conjugate pair of eigenvalues. More precisely, its eigenvalues have the minimal polynomial $x^3 - 2\,x^2 - 1$ over $\mathbb{Q}$ and one obtains approximately $\lambda \approx 2.206$ and $-0.103 \pm i \cdot 0.665$. A left PF-eigenvector of the substitution matrix is given by $(\lambda^2 - \lambda, \lambda, 1)$. Consequently, the internal space is given by $H_{\mathrm{ext}} = H = \mathbb{C}$ and one has $\mathcal{L} = \mathbb{Z}[\lambda]$ and $\mathcal{M} = \langle \lambda^2 - 2\,\lambda, \lambda - 1 \rangle_{\mathbb{Z}}$ (therefore, one also has $\mathcal{L}^{\wedge} = \mathbb{Z}[\lambda]/(3\,\lambda^2 - 4\,\lambda) = \frac{1}{59}(-6 + 25\,\lambda - 8\,\lambda^2)\,\mathbb{Z}[\lambda]$). We show the periodic tiling $\operatorname{supp} \underline{\Omega} + \mathcal{M}^{\star}$

[31] Using Lemma 3.82, one confirms that the ideal (3) splits in $\mathbb{Q}(\sqrt{10})$, while the ideal $(2) = (\sqrt{10})^2$ is ramified.

Kolakoski(3, 1): The periodic tiling supp $\underline{\Omega} + \mathcal{M}^\star$

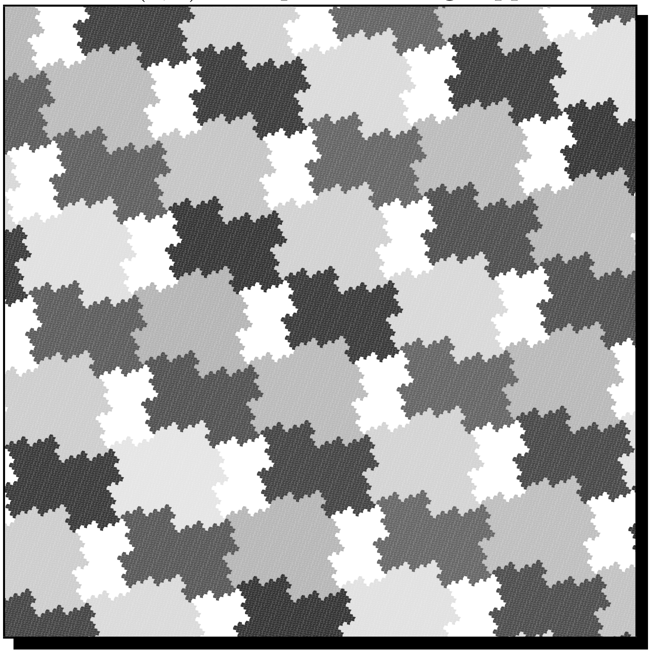


Figure 6.8.: The periodic tiling of $\mathbb{C}$ for Kolakoski(3, 1) in the range $-3 \leq \mathrm{Re}(x) \leq 3$, $-3 \leq \mathrm{Im}(x) \leq 3$. The colouring is as follows: Ω_a, Ω_b and Ω_c in white.

in Figure 6.8, the sets $\underline{\Upsilon} = \Lambda([0\,, \ell_i[)$ and the stepped surface, and the aperiodic tiling $\underline{\Upsilon} + \underline{\Omega}$ in Figure 6.9. One should compare these pictures with the corresponding pictures for the ♣-substitution in Figure 6.3 (basically, one should notice that, after customising to the $\mathfrak{p}$-adic topology, (how) both cases are the "same").

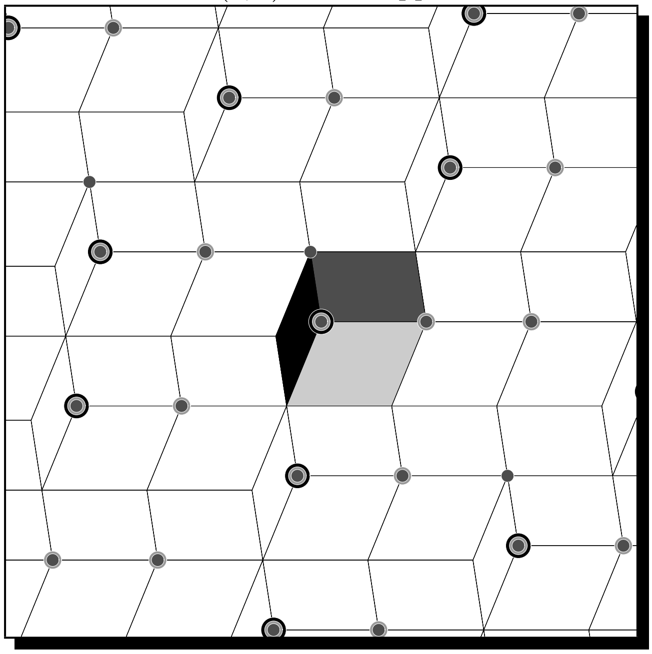
We remark that here the IFS for the boundaries of the sets Ω_a, Ω_b and Ω_c is self-similar, wherefore the OSC (open set condition, see Remarks 4.135 & 4.136) can be used to determine their Hausdorff dimension exactly, which here turns out to be $-\log(\frac{1+\sqrt{5}}{2})/\log(|\lambda^\star|) \approx 1.217$, see [41]. Moreover, there is a whole family of unimodular Pisot substitutions associated with Kolakoski sequences; more precisely, Kolakoski$(2m+1, 2m-1)$ sequences with $m \in \mathbb{N}$ provide examples of unimodular Pisot substitutions given by the substitution matrix

$$\begin{pmatrix} m & m & m-1 \\ 1 & 1 & 1 \\ m & m-1 & m-1 \end{pmatrix}$$

and the substitutions $a \mapsto a \ldots abc \ldots c$, $b \mapsto a \ldots abc \ldots c$ and $c \mapsto a \ldots abc \ldots c$ (with the appropriate number of a's and c's in each case). One may conjecture, that all are multi-component model sets and the Hausdorff dimension of the respective boundaries increases as function of m, in fact, approaching 2 in the limit $m \to \infty$, compare [352, Section 2.3.7].

There are also (actually, also infinitely many) non-unimodular Pisot substitutions associated to Kolakoski sequences. The one with the least number of symbols (*i.e.*, with $\sum_{i \in \mathcal{A}} \#\sigma(i)$ minimal) is the Pisot substitution associated to Kolakoski(7, 3) (compare [352, Section 2.3.8]), which is given by $a \mapsto aaabccc$, $b \mapsto aaabc$ and $c \mapsto abc$. Accidently, the substitution matrix happens to be the one already considered in Example 6.62. Consequently, the internal space

Kolakoski(3, 1): The stepped surface



Kolakoski(3, 1): The aperiodic tiling $\underline{\Upsilon} + \underline{\Omega}$

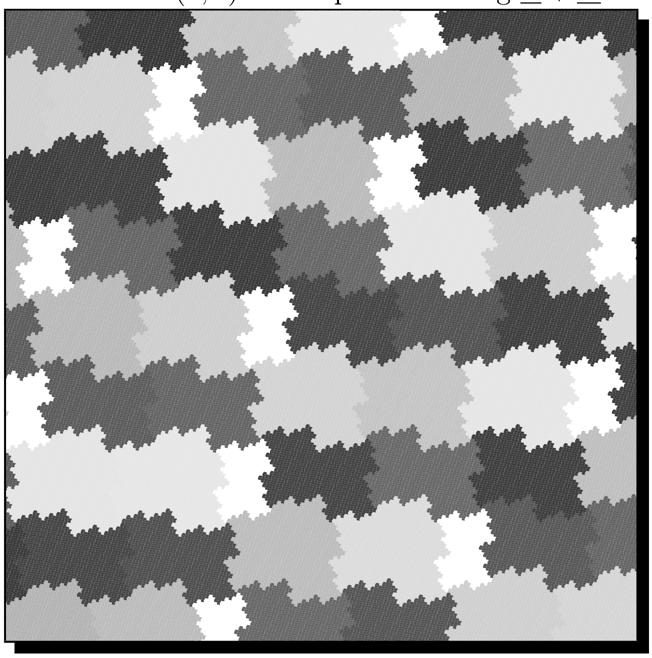
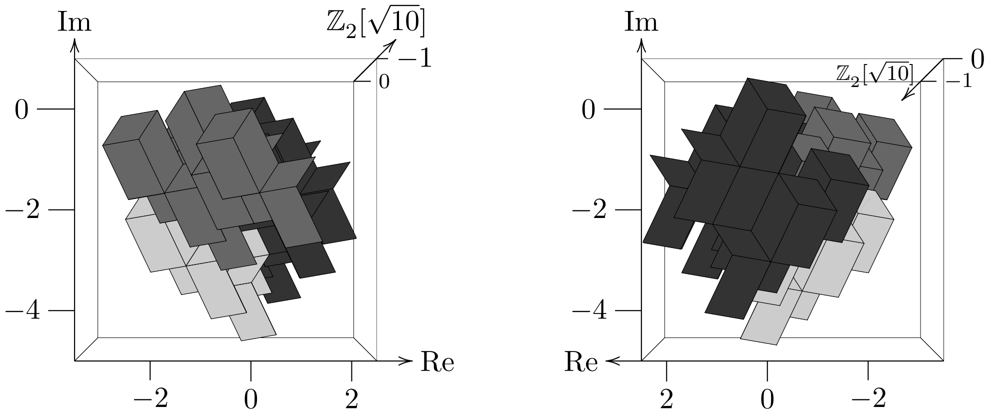


Figure 6.9.: The stepped surface and the aperiodic tiling in $\mathbb{C}$ for Kolakoski(3, 1) are shown, both in the range $-3 \leq \mathrm{Re}(x) \leq 3$, $-3 \leq \mathrm{Im}(x) \leq 3$. In the top picture, the stepped surface together with the sets $\Upsilon_a = \Lambda([0\,,\ell_a[)$ (represented by a small point), $\Upsilon_b = \Lambda([0\,,\ell_b[)$ (represented by a medium point) and $\Upsilon_c = \Lambda([0\,,\ell_c[)$ (represented by a big point) is shown. Also, we highlight one set of polygons P_a, P_b and P_c, which form the stepped surface by $\underline{\Upsilon} + \underline{P}$. In the bottom picture, the aperiodic tiling of $\mathbb{C}$ with the following colouring is shown: Ω_a, Ω_b and Ω_c in white.

is here given by $H_{\text{ext}} \cong \mathbb{C} \times \mathbb{Q}_2(\sqrt{10})$. Using the left PF-eigenvector normalised such that $\ell_c = 1$, one obtains the following "top" and "bottom" views of the approximation of the sets Ω_i by the hyperpolygons $P_a^{(2)}$, $P_b^{(2)}$ and $P_c^{(2)}$ (here, one can also see how they are obtained from the hyperpolygons $P_i = P_i^{(0)}$).



This, of course, may only serve as starting point of a more thorough analysis of this[32] example.

Example 6.115. As last example of this section, we treat the following Pisot substitution, which appears in [346, Fig. 4.2 & Example[33] in Section 4.4]:

$$a \mapsto aaaabbbbcc, \qquad b \mapsto a \quad \text{and} \quad c \mapsto ab.$$

The characteristic polynomial of its substitution matrix is given by $p(x) = x^3 - 4\,x^2 - 6\,x - 2$ and has one real root, the PV-number λ given by $\lambda = \frac{4}{3} + \frac{1}{3}\,(199 - 3\,\sqrt{33})^{\frac{1}{3}} + \frac{1}{3}\,(199 + 3\,\sqrt{33})^{\frac{1}{3}} \approx 5.222$, and a complex conjugate pair $\lambda_{2,3} \approx -0.611 \pm i\,0.097$. Moreover, it is an Eisenstein polynomial for $2\,\mathbb{Z}$ and $2\,\mathbb{Z}_2$, wherefore $\mathbb{Q}_2(\lambda)$ is a totally ramified extension of degree 3 of $\mathbb{Q}_2$ by Proposition 3.106 (it is also tamely ramified). We have checked with KANT [293] that we have the following ideal factorisation: $(2) = (2, \lambda)^3 = (6 + 9\,\lambda - 2\,\lambda^2)^3$ (we have also calculated that $\mathfrak{o}_{\mathbb{Q}(\lambda)} = \mathbb{Z}[\lambda]$). Moreover, since λ is a root $p(x)$, Corollary 3.76 yields $\|\lambda\|_{\mathbb{Q}_2(\lambda)} = |2|_2 = \frac{1}{2}$ and λ is a uniformiser for $\mathbb{Q}_2(\lambda)$ (similarly, $6 + 9\,\lambda - 2\,\lambda^2$ with minimal polynomial $x^3 + 2\,x^2 + 2\,x + 2$ is also a uniformiser).

To be specific, the λ-adic expansion of any number $x = m_1 + m_2\,\lambda + m_3\,\lambda^2$ can be obtained as follows: We observe that

$$\frac{m_1 + m_2\,\lambda + m_3\,\lambda^2}{\lambda} = (-3\,m_1 + m_2) + (-2\,m_1 + m_3)\,\lambda + \frac{1}{2}\,m_1\,\lambda^2,$$

[32] A similar example can be found in [358, Example 6.1]: The substitution matrix of the non-unimodular Pisot substitution $a \mapsto ab$, $b \mapsto bbc$ and $c \mapsto aa$ has determinant 2, its eigenvalues are given by $\lambda \approx 2.521$ and the complex conjugate pair $\approx 0.239 \pm i \cdot 0.858$. Its characteristic polynomial splits in $\mathbb{Q}_2$ as

$$x^3 - 3\,x^2 + 2\,x - 2 = (x - .111010\ldots) \cdot (x^2 + .001010\ldots\, x + .011100\ldots),$$

where the second polynomial has the following ("contractive") root in $\mathbb{Q}_2(\sqrt{6})$: $.011011\ldots \pm \sqrt{6}\,.101001\ldots = .011011\ldots \mp \sqrt{6}\,.110110\ldots$. Consequently, the internal space is given by $\mathbb{C} \times \mathbb{Q}_2(\sqrt{6})$ and $\|\lambda\|_{\mathbb{Q}_2(\sqrt{6})} = \frac{1}{2}$.

[33] The example with four letters in that section is not a Pisot substitution; two eigenvalues are of modulus greater than 1.

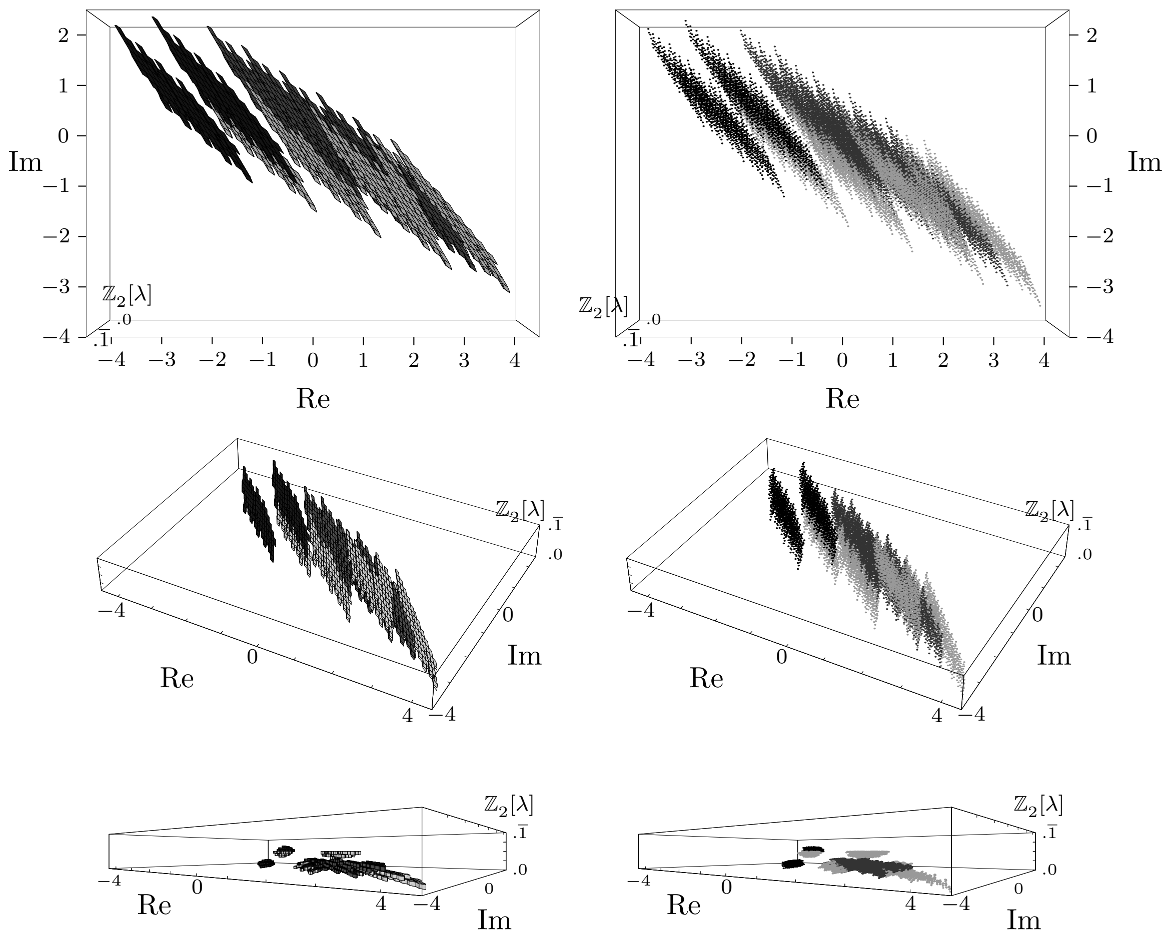


Figure 6.10.: Approximation on the sets $\underline{\Omega}$ associated to the Pisot substitution $a \mapsto aaaabbbbcc$, $b \mapsto a$ and $c \mapsto ab$ of Example 6.115: Left the hyperpolygons $\underline{P}^{(4)}$, right the star-map of the first 10 000 points in $\underline{\Lambda}$, both shown from (the same) three perspectives. We use the colouring Ω_a (in dark gray), Ω_b (in light gray) and Ω_c (in black).

wherefore $x = m_1 + m_2\,\lambda + m_3\,\lambda^2 \in \mathbb{Z}[\lambda]$ is divisible by λ iff $m_1 \equiv 0 \bmod 2$ (otherwise, $x - 1$ is divisible by π). Iteration of this "transformation formula" yields the π-adic expansion, *e.g.*,

$$1 = .1, \quad 2 = .000111110001\ldots, \quad 10 = .000111110111\ldots \quad \lambda = .01, \quad \lambda^2 = .001.$$

As natural lengths, we may choose $\ell_a = -2 - 4\,\lambda + \lambda^2 \approx 4.383$, $\ell_b = 2 + 5\,\lambda - \lambda^2 \approx 0.839$ and $\ell_c = 1$. In Figure 6.10, we compare the two "approximations" of the attractor $\underline{\Omega}$: the hyperpolygons $\underline{P}^{(4)}$ and the star-map applied to the first 10 000 (starting from 0) points of $\underline{\Lambda}$. Obviously, both methods yield similar pictures.

6.12. Pisot Substitutions: The Theorem

We recall Theorem 6.77, Propositions 6.87 & 6.100 & 6.106(ii) and Corollary 6.104 and obtain the following list of equivalent conditions.

Theorem 6.116. *Let σ be a Pisot substitution with fixed point u. Denote the representation with natural intervals $A_i = [0\,,\ell_i]$ of u by $\underline{\varLambda}$. Then, $\underline{\varLambda}$ is a representable repetitive aperiodic multi-component Delone set (which, in particular, satisfies* **(PLT+)***). Denote the attractor of the IFS $\boldsymbol{\Theta}^\star$ by $\underline{\varOmega}$ and set $\varUpsilon_i = \Lambda([0\,,\ell_i[)$. Then, the substitution multi-component Delone set $\underline{\varUpsilon}$ is repetitive, and the following statements are equivalent:*

(i) *$\underline{\varLambda}$ is a multi-component inter model set with $\Lambda(\operatorname{int}\underline{\varOmega}) \subset \underline{\varLambda} \subset \Lambda(\underline{\varOmega})$.*

(ii) *$\bigcup_{i=1}^n \varOmega_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^\star \subset H$.*

(iii) *$\underline{\varUpsilon} + \underline{\varOmega}$ is a tiling of H_{ext}.*

(iv) *$\bigcup_{i=1}^n [-\ell_i\,,0] \times \varOmega_i$ is a measure-disjoint union and a fundamental domain of $\widetilde{\mathcal{L}}_{\text{ext}}$.*

(v) *The only strong overlaps $\varXi_{\underline{\varLambda}}(i,k,x)$ with $x \in \Lambda(\operatorname{int}\varOmega_k - \operatorname{int}\varOmega_i)$ are coincidences.*

(vi) *$\lim_{m\to\infty} \operatorname{dens}^{\text{ovlap}}_{\underline{\varLambda}+\underline{\varLambda}}(f_0^m(x)) = 1$ for every $x \in \varDelta'$.*

(vii) *$1 - \operatorname{dens}^{\text{ovlap}}_{\underline{\varLambda}+\underline{\varLambda}}(f_0^m(x)) \le C \cdot r^m$ for an $m \in \mathbb{N}$, every $x \in \varDelta'$ and some constants (independent of x) $C > 0$ and $r \in]0\,,1[$.*

(viii) *$\underline{\varLambda} + \underline{\varLambda}$ admits an overlap coincidence.*

(ix) *$\underline{\varLambda}$ admits an algebraic coincidence.*

(x) *The only strong overlaps $\varXi_{\underline{\varOmega}}(i,k,x)$ with $x \in \Lambda(]-\ell_i\,,\ell_k[)$ are coincidences.*

(xi) *There is an i such that $\lim_{m\to\infty} \partial P_i^{(m)} = \partial\varOmega_i$.*

(xii) *There is an i such that $\lim_{m\to\infty} \partial P_i^{(m)}$ is not space-filling.*

(xiii) *Let $r_n^{(i)}$ be the radius of the largest ball $B_{<r_n^{(i)}}$ such that*

$$B_{<r_n^{(i)}} \subset \operatorname{supp}\left(\underline{P} + (\boldsymbol{\Theta}^{\star\#})^n(\omega_i(0))\right).$$

Then, there is an i such that $r_n^{(i)} \to \infty$ for $n \to \infty$.

(xiv) *The GCC is satisfied.*

(xv) *The overlap graph $G_\sigma^{\text{ovlap}}(\underline{\varUpsilon} + \underline{\varOmega})$ and boundary graph $G_\sigma^{\text{bd}}(\underline{\varUpsilon} + \underline{\varOmega})$ coincide.*

(xvi) *The spectral radius of the adjacency matrix $\boldsymbol{M}_\sigma^{\text{ovlap}}$ of the graph $G_\sigma^{\text{ovlap}}(\underline{\varUpsilon} + \underline{\varOmega})^*$ (i.e., the essential graph induced from $G_\sigma^{\text{ovlap}}(\underline{\varUpsilon} + \underline{\varOmega})$ by removing the vertices which are coincidences) satisfies $\rho(\boldsymbol{M}_\sigma^{\text{ovlap}}) < \lambda$.* □

Moreover, we obtain further equivalent conditions from the statements in Chapter 7, see Theorem 7.42 (respectively, Proposition 7.11), Theorem 7.29, Proposition 7.45(iii) and Lemma 7.56 (respectively, Proposition 7.55).

Continuation of Theorem 6.116. *Let $(a_i)_{i=1}^n$ be a family of complex numbers and define a weighted Dirac comb $\nu = \sum_i a_i \cdot \delta_{\Lambda_i}$ associated to $\underline{\Lambda}$. Then, we also have the following additional equivalent statements.*

(xvii) *The ε-almost period $P_\varepsilon = P_\varepsilon(\underline{\Lambda})$ is relatively dense for every $\varepsilon > 0$, i.e., $\underline{\Lambda}$ is a SPPD set.*

(xviii) *The autocorrelation measure γ_ν is norm almost periodic (for any choice $(a_i)_{i=1}^n$).*

(xix) *The autocorrelation measure γ_ν is strongly almost periodic (for any choice $(a_i)_{i=1}^n$).*

(xx) *The diffraction measure $\hat{\gamma}_\nu$ is pure point (for any choice $(a_i)_{i=1}^n$).*

(xxi) *The diffraction measure $\hat{\gamma}_{\delta_{\Lambda_i}}$ is a pure point measure for every $1 \le i \le n$.*

(xxii) *The measure theoretic dynamical system $(\mathbb{X}(\underline{\Lambda}), \mathfrak{B}, \mu_{\mathbb{R}}, \mathbb{R})$ has pure point spectrum.*

(xxiii) *The topological dynamical system $(\mathbb{X}(\underline{\Lambda}), \mathbb{R})$ has pure point dynamical spectrum with continuous eigenfunctions, and these eigenfunctions separate almost all points of $\mathbb{X}(\underline{\Lambda})$.*

(xxiv) *The autocorrelation hull $\mathbb{A}(\underline{\Lambda})$ is compact.*

(xxv) *There is a torus parametrisation $\beta_{\underline{\Lambda}} : \mathbb{X}(\underline{\Lambda}) \to \mathbb{A}(\underline{\Lambda}) \cong (\mathbb{R} \times H_{\text{ext}})/\tilde{\mathcal{L}}_{\text{ext}}$ which is one-to-one a.e. with respect to the Haar measure on $\mathbb{A}(\underline{\Lambda})$.* □

Furthermore, we recall Remark 6.79 and can therefore also add the following statements.

Continuation of Theorem 6.116. *Assume that there is a legal cluster $\underline{\mathcal{P}}$ contained in $\underline{\Upsilon}$ such that $0 \in \operatorname{int}\operatorname{supp}(\underline{\mathcal{P}} + \underline{\Omega})$. Define (with respect to the local topology)*

$$\underline{\Upsilon}' = \lim_{m\to\infty} \left(\boldsymbol{\Theta}^{\star\#}\right)^m (\underline{\mathcal{P}}) .$$

Then, by construction, $\underline{\Upsilon}'$ is a representable substitution multi-component Delone set (with $\underline{\Upsilon}' \subset \underline{\Upsilon}$), every patch is legal (thus, $\underline{\Upsilon}' + \underline{\Omega}$ is repetitive by Corollary 5.89), and we obtain the following additional equivalent statements.

(xxvi) *$\underline{\Upsilon} = \underline{\Upsilon}'$.*

(xxvii) *$\lim_{m'\to\infty} \operatorname{dens}^{\text{ovlap}}_{\underline{\Upsilon}'+\underline{\Omega}}(\hat{f}_0^{-m'}(x)) = 1$ for every $x \in \bigcup_{i=1}^n (\Upsilon_i' - \Upsilon_i')$.*

(xxviii) *$1 - \operatorname{dens}^{\text{ovlap}}_{\underline{\Upsilon}'+\underline{\Omega}}(\hat{f}_0^{-m'}(x)) \le C' \cdot (r')^{m'}$ for an $m' \in \mathbb{N}$, every $x \in \bigcup_{i=1}^n (\Upsilon_i' - \Upsilon_i')$ and some constants (independent of x) $C' > 0$ and $r' \in]0\,,1[$.*

(xxix) *$\underline{\Upsilon}' + \underline{\Omega}$ admits an overlap coincidence.*

(xxx) *$\underline{\Upsilon}'$ admits an algebraic coincidence.*

Moreover, let $(b_i)_{i=1}^n$ *be a family of complex numbers and define a weighted Dirac comb* $\nu' = \sum_i b_i \cdot \delta_{\varUpsilon'_i}$ *associated to* $\underline{\varUpsilon}'$. *Then, the following conditions extend our list of equivalent statements.*

(xxxi) *The* ε*-almost period* $P_\varepsilon(\underline{\varUpsilon}')$ *is relatively dense for every* $\varepsilon > 0$*, i.e.,* $\underline{\varUpsilon}'$ *is a SPPD set.*

(xxxii) *The autocorrelation measure* $\gamma_{\nu'}$ *is norm almost periodic (for any choice* $(b_i)_{i=1}^n$*).*

(xxxiii) *The autocorrelation measure* $\gamma_{\nu'}$ *is strongly almost periodic (for any choice* $(b_i)_{i=1}^n$*).*

(xxxiv) *The diffraction measure* $\hat{\gamma}_{\nu'}$ *is pure point (for any choice* $(b_i)_{i=1}^n$*).*

(xxxv) *The diffraction measure* $\hat{\gamma}_{\delta_{\varUpsilon'_i}}$ *is a pure point measure for every* $1 \le i \le n$.

(xxxvi) *The measure theoretic dynamical system* $(\mathbb{X}(\underline{\varUpsilon}')\,, \mathfrak{B}\,, \mu_{H_{\text{ext}}}\,, H_{\text{ext}})$ *has pure point spectrum.*

(xxxvii) *The topological dynamical system* $(\mathbb{X}(\underline{\varUpsilon}')\,, H_{\text{ext}})$ *has pure point dynamical spectrum with continuous eigenfunctions, and these eigenfunctions separate almost all points of* $\mathbb{X}(\underline{\varUpsilon}')$.

(xxxviii) *The autocorrelation hull* $\mathbb{A}(\underline{\varUpsilon}')$ *is compact.*

(xxxix) *There is a torus parametrisation* $\beta_{\underline{\varUpsilon}'} : \mathbb{X}(\underline{\varUpsilon}') \to \mathbb{A}(\underline{\varUpsilon}') \cong (\mathbb{R} \times H_{\text{ext}})\,/\tilde{\mathcal{L}}_{\text{ext}}$ *which is one-to-one a.e. with respect to the Haar measure on* $\mathbb{A}(\underline{\varUpsilon}')$. □

Remark 6.117. This impressive list of equivalent conditions and the lack of counterexamples supports the following “Pisot Conjecture”:

Conjecture. *Any Pisot substitution generates a multi-component inter model set.*

This conjecture can be traced back to at least the early eighties. To our knowledge, some of the first – at least implicit – statements in the literature can be attribute it to E. Bombieri and J.E. Taylor, see [71, 72], respectively A.N. Livshits, see [366, Section 4]. However, as the work of G. Rauzy indicates, see [306], it might be known to the experts much longer.

Remark 6.118. As indicated in Sections 6.8 & 6.9, some of the above statements are generalisations of statements about *digit tiles*, see [155, Section 4], [368, Sections 2 & 3], [110, Section 3] and [381, Section 5].

Definition. Let $L \subset \mathbb{R}^d$ be a lattice, $\boldsymbol{A}$ an expansive linear map such that $\boldsymbol{A}\,L \subset L$. A set D of coset representatives of the quotient group $L/\boldsymbol{A}\,L$ is called a *digit set*. Obviously, one has $\operatorname{card} D = |\det \boldsymbol{A}|$. The attractor $\varOmega$ of the IFS

$$\varOmega = \bigcup_{d \in D} \boldsymbol{A}^{-1}\,(\varOmega + d) = \varTheta(\varOmega) \tag{6.18}$$

is called a *digit tile*. Moreover, one usually assumes that $\boldsymbol{A}$ is an integer matrix, $L = \mathbb{Z}^d$, D spans L and $0 \in \varOmega$. If this assumptions are satisfied, $\varOmega$ is called a *basic digit tile*.

There is only little loss of generality by restricting to basic digit tiles, see [381, Theorem 2.7] (also see [220, 379]). Now, the theorem about (basic) digit tiles reads as follows.

Theorem. [381, Theorem 4.3] *Let Ω be a basic digit tile with integer matrix $\boldsymbol{A}$ and lattice $L = \mathbb{Z}^d$. Let P_0 be the unit d-cube centred at the origin with edges parallel to the axes, i.e., $P = [-\frac{1}{2}, \frac{1}{2}] \times \ldots \times [-\frac{1}{2}, \frac{1}{2}]$, and define $P_m = \Theta^m(P_0)$ with IFS Θ as in Equation* (6.18). *Then the following statements are equivalent, where limits are with respect to the Hausdorff metric.*

(i) $\Omega + L$ *is a tiling of* $\mathbb{R}^d$.

(ii) $\Omega + L$ *is a self-replicating tiling of* $\mathbb{R}^d$.

(iii) $\mu_{\mathbb{R}^d}(\Omega) = 1$.

(iv) *The characteristic function*

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega, \end{cases}$$

is a scaling function of a multi-resolution analysis.

(v) $\lim_{m\to\infty} \partial P_m = \partial \Omega$.

(vi) $\lim_{m\to\infty} \partial P_m$ *is not space-filling.*

(vii) $L = \bigcup_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \boldsymbol{A}^i D \right)$.

(viii) *Every lattice point has a unique finite address.*

(ix) *By the compactness of* Ω*, every lattice point in a certain ball has a finite address.*

(x) *The spectral radius of the contact matrix is less than* $|\det \boldsymbol{A}|$. □

The generalisations for Pisot substitutions are as follows: The first two statements generalise to the statement that $\operatorname{supp} \underline{\Omega} + \mathcal{M}^\star$ respectively $\underline{\Upsilon} + \underline{\Omega}$ are tilings. The third statements is simply the statement that the sets Ω_i have the right measure. The fourth statement, a statement about wavelet analysis (see [156]), does not seem[34] to have any importance in the multi-component case. The fifth and sixth statement are clear. The seventh to ninth statement state in our "language" that the dual EMFS $\Theta^\#$ with finite seed $\{0\}$ (note that here 0 is an inner point of Ω) yields a substitution (uniqueness!) Delone set $\Upsilon' = \lim_{m\to\infty} \Theta^m(\{0\})$ with $\Theta = L$, wherefore every lattice point can be reached by a walk of finite length (the finite address!) from 0 on the graph $G(\Upsilon)$ introduced in Definition 5.68. The corresponding statement for Pisot substitutions is therefore the condition $\underline{\Upsilon}' = \underline{\Upsilon}$. The last statement, of course, is the statement about $\rho(\boldsymbol{M}_\sigma^{\text{ovlap}})$.

More on digit tiles can also be found in [10, 46, 204, 269, 323, 324, 380]

[34] However, there are some considerations to apply wavelets to the analysis of substitution tilings which are model sets, see [214] and references therein.

Remark 6.119. We only note that, after one has established that $\underline{\Lambda}$ is a multi-component inter model set, one may also check if it is a generic model set. A sufficient condition for this is given in [42, Section 4]: If $c\dot{a}$ respectively $\omega_c(-\ell_c) \cup \omega_a(0)$ is the corresponding finite seed of the sequence u respectively the multi-component Delone set $\underline{\Lambda}$ as in Remark 6.4, then one basically only has to check if $0 \in \operatorname{int} \Omega_a$ and $-\ell_c^\star \in \operatorname{int} \Omega_c$ (also compare to Lemma 5.151). For details we refer to [42, Section 4], whose statements can easily be generalised.

Many of the above conditions in Theorem 6.116 do not rely on the special structure provided by a Pisot substitution, wherefore one may also apply these conditions in more general situations. This will be indicated in the following sections for two-dimensional tilings, lattice substitution systems, β-substitutions and reducible Pisot substitutions.

In view of this, we observe that the characteristic condition for a Pisot substitution seems to be that $\operatorname{supp} \underline{\Omega} + \mathcal{M}^\star$ is a tiling of H. But is this enough to prove the Pisot Conjecture?

6a. Internal Space for Two-Dimensional Substitution Tilings

Die Sonne schien aufs Meer herab,
Sie schien mit aller Macht;
Gab sich die allergrößte Müh,
Daß sie das Meer zum Glitzern bracht' –
Und das war seltsam, denn es war
Schon kurz nach Mitternacht.

Alice hinter den Spiegeln – *Lewis Carroll*

The full machinery we have developed so far, works not only for one-dimensional point sets. In the higher dimensional case (besides the realm of lattice substitution systems discussed in the next chapter), however, only model sets in the purely Euclidean setting are known. This chapter serves as starting point for the analysis of some substitution tilings which might have a description as model sets with mixed Euclidean and $\mathfrak{p}$-adic internal space.

6a.1. Cyclotomic Prerequisites

Two-dimensional tilings are often defined on cyclotomic fields, by which we mean that, for example, the position of vertices of the (polygonal) tiles are elements of the corresponding cyclotomic integers. The reason, why one uses cyclotomic fields, is that one automatically has an inherent rotational symmetry which becomes obvious in the diffraction pattern (it was the unusual 5-fold respectively 12-fold symmetry that led to the discovery of quasicrystals, see [189, 342]).

Definition 6a.1. Let $n \in \mathbb{N}$. By a *primitive n-th root of unity* we mean a generator of the cyclic group of all complex n-th roots of unity, *e.g.*, the complex number $\xi_n = \exp(2\pi i/n)$ is one primitive n-th root of unity (the others are given by ξ_n^h where $\gcd(h\,,n) = 1$, wherefore there are $\phi(n)$ primitive n-th roots of unity, where ϕ denotes[1] the *Euler ϕ-function*). The algebraic number field $\mathbb{Q}(\xi_n)$ generated by any primitive n-root of unity is called the *n-th cyclotomic field*. It is the splitting field of the (monic) *n-th cyclotomic polynomial* $\mathrm{Irr}(\xi_n, \mathbb{Q}, x) = \prod_{\gcd(h,n)=1}(x - \xi_n^h) \in \mathbb{Z}[x]$, and therefore a Galois extension of degree $\phi(n)$.

More on cyclotomic fields may be found in the books [225, 385], especially, and in any book about algebraic number theory. Here, we are interested in the factorisation of some ideals, wherefore we note the following important statement.

Proposition 6a.2. [385, Theorem 2.6] and [211, Theorem 1.61] *The order $\mathbb{Z}[\xi_n]$ is the ring of algebraic integers of $\mathbb{Q}(\xi_n)$, i.e., $\mathfrak{o}_{\mathbb{Q}(\xi_n)} = \mathbb{Z}[\xi_n]$.* □

[1]Properties of the Euler ϕ-function may be found in [14].

Note that this establishes that we can use Proposition 3.79 to obtain the prime ideal decomposition of the ideals (p) (where $p \in \mathbb{P}$) in $\mathbb{Q}(\xi_n)$ (note that the splitting of a prime $p \not| n$ is answered by [385, Theorem 2.13], otherwise observe [385, Prop. 2.3]). Especially, we have the following:

- For[2] $\mathbb{Q}(\xi_5) = \mathbb{Q}(\xi_{10})$, the 10-th cyclotomic polynomial is given by $\mathrm{Irr}(\xi_{10}, \mathbb{Q}, x) = x^4 - x^3 + x^2 - x + 1$. We note that $\mathrm{Irr}(\xi_{10}, \mathbb{Q}, x) \equiv (x+1)^4 \bmod 5$, wherefore by Proposition 3.79 we have $(5) = \mathfrak{P}_5^4$ and $\mathbb{Q}_{\mathfrak{P}_5}$ is a totally and tamely ramified extension of $\mathbb{Q}_5$.
- For $\mathbb{Q}(\xi_8)$, the 8-th cyclotomic polynomial is given by $\mathrm{Irr}(\xi_8, \mathbb{Q}, x) = x^4 + 1$. Similarly as before, we note that $\mathrm{Irr}(\xi_8, \mathbb{Q}, x) \equiv (x+1)^4 \bmod 2$, wherefore we have $(2) = \mathfrak{P}_8^4$ and $\mathbb{Q}_{\mathfrak{P}_8}$ is a totally and wildly ramified extension of $\mathbb{Q}_2$.
- For $\mathbb{Q}(\xi_{12})$, the 12-th cyclotomic polynomial is given by $\mathrm{Irr}(\xi_{12}, \mathbb{Q}, x) = x^4 - x^2 + 1$. Here, we have $\mathrm{Irr}(\xi_{12}, \mathbb{Q}, x) \equiv (x^2 + x + 1)^2 \bmod 2$, wherefore one has $(2) = \mathfrak{P}_{12}^2$ and $\mathbb{Q}_{\mathfrak{P}_{12}}$ is an extension of degree 4 of $\mathbb{Q}_2$ with ramification index 2 and residue degree 2 (and therefore wildly ramified).

In connection with Proposition 3.106, we note that the polynomial $\mathrm{Irr}(\xi_8, \mathbb{Q}, x)$ is an Eisenstein polynomial (for $2\mathbb{Z}$).

We are now looking for uniformisers of $\mathbb{Q}_{\mathfrak{P}_5}$, $\mathbb{Q}_{\mathfrak{P}_8}$ and $\mathbb{Q}_{\mathfrak{P}_{12}}$. The following results applies to p-th cyclotomic fields, where p is a prime number.

Corollary 6a.3. [385, Lemma 1.4] *Let $p \in \mathbb{P}$ be an odd prime. Then, the ideal $(1 - \xi_p)$ is a prime ideal of $\mathbb{Z}[\xi_p]$ and $(1 - \xi_p)^{p-1} = (p)$.* □

Moreover, we have: Let $f(x) = \mathrm{Irr}(\xi_p, \mathbb{Q}, x) = x^{p-1} + x^{p-2} + \ldots + x + 1$ be the p-th cyclotomic polynomial, then $f(x+1)$ is an Eisenstein polynomial for $p\mathbb{Z}$ (see [153, p. 174]), wherefore $\|\xi_p - 1\|_{(\xi_p - 1)} = \frac{1}{p}$ and $\xi_p - 1$ is a uniformiser for $\mathbb{Q}_{(\xi_p - 1)}$. Consequently, $\xi_5 - 1$ is a uniformiser for $\mathbb{Q}_{\mathfrak{P}_5}$. However, we also observe the following: Let $f_{10}(x) = \mathrm{Irr}(\xi_{10}, \mathbb{Q}, x)$. Then, $f_{10}(x-1) = x^4 - 5x^4 + 10x^2 - 10x + 5$ is an Eisenstein polynomial for $5\mathbb{Z}$; consequently, we also have $\|1 + \xi_{10}\|_{\mathfrak{P}_5} = \frac{1}{5}$ and $\pi_5 = 1 + \xi_{10}$ is also a uniformiser for $\mathbb{Q}_{\mathfrak{P}_5}$.

Similarly, let $f_8(x) = \mathrm{Irr}(\xi_8, \mathbb{Q}, x)$. Then, $f_8(x-1) = x^4 - 4x^3 + 6x^2 - 4x + 2$ is an Eisenstein polynomial for $2\mathbb{Z}$, wherefore $\|\xi_8 + 1\|_{\mathfrak{P}_8} = \frac{1}{2}$ and $\pi_8 = 1 + \xi_8$ is a uniformiser for $\mathbb{Q}_{\mathfrak{P}_8}$.

Now, for $\mathbb{Q}(\xi_{12})$, the situation is different, since we are not in the fully ramified case. In fact, the above trick does not yield an Eisenstein polynomial for $\xi_{12} + 1$. For simplicity, we check the numbers $\xi_{12}^j + 1$ $(1 \leq j \leq 11)$: Since $\xi_{12}^3 = i$, we define $f_{12}(x) = \mathrm{Irr}(i, \mathbb{Q}, x) = x^2 + 1$. Then, $f_{12}(x-1) = x^2 - 2x + 2$ is an Eisenstein polynomial for $2\mathbb{Z}$, therefore irreducible and one establishes $N_{\mathbb{Q}_{\mathfrak{P}_{12}}/\mathbb{Q}_2}(1 + i) = 2^2 = 4$. Consequently, by Corollary 3.76, one obtains $\|1 + i\|_{\mathfrak{P}_{12}} = \frac{1}{4} = 1/2^{f_{\mathbb{Q}_{\mathfrak{P}_{12}}/\mathbb{Q}_2}}$. But this establishes that $\pi_{12} = 1 + i$ is a uniformiser for $\mathbb{Q}_{\mathfrak{P}_{12}}$ (since it has maximal (normalised) absolute value less than 1).

To summarise, we have the following situation:

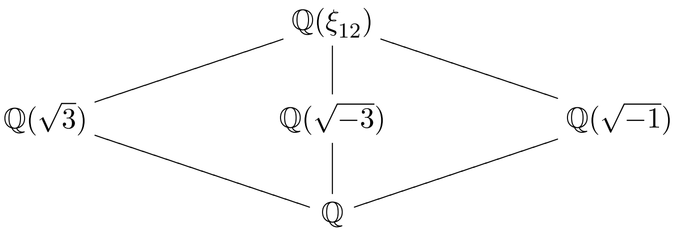
- In $\mathbb{Q}(\xi_5)$, we have the ideal factorisation $(5) = \mathfrak{P}_5^4$, and a uniformiser for $\mathbb{Q}_{\mathfrak{P}_5}$ is $\pi_5 = 1 + \xi_{10}$.

[2] Note that for $2\,n \equiv 2 \bmod 4$ one always has $\mathbb{Q}(\xi_n) = \mathbb{Q}(\xi_{2n})$ since one always has $-1 \in \mathbb{Q}(\xi_n)$, see [385, p. 10].

- In $\mathbb{Q}(\xi_8)$, we have the ideal factorisation $(2) = \mathfrak{P}_8^4$, and a uniformiser for $\mathbb{Q}_{\mathfrak{P}_8}$ is $\pi_8 = 1 + \xi_8$.

- In $\mathbb{Q}(\xi_{12})$, we have the ideal factorisation $(2) = \mathfrak{P}_{12}^2$, and a uniformiser for $\mathbb{Q}_{\mathfrak{P}_{12}}$ is $\pi_{12} = 1 + i$.

Remark 6a.4. For $\mathbb{Q}_{\mathfrak{P}_5}$ (respectively $\mathbb{Q}_{\mathfrak{P}_8}$) a system of representatives of $\widehat{\mathfrak{o}_{\mathfrak{P}_5}}/\mathfrak{P}_5$ (respectively $\widehat{\mathfrak{o}_{\mathfrak{P}_8}}/\mathfrak{P}_8$) is given by $\{0,1,2,3,4\}$ (respectively $\{0,1\}$). For $\mathbb{Q}_{\mathfrak{P}_{12}}$ we observe the following: The Galois group of $\mathbb{Q}(\xi_{12})$ over $\mathbb{Q}$ is given by $\mathrm{Gal}(\mathbb{Q}(\xi_{12}/\mathbb{Q}) \cong \mathsf{C}_2 \times \mathsf{C}_2$ (see [385, Theorem 2.5]) and we have three quadratic subfields as follows:



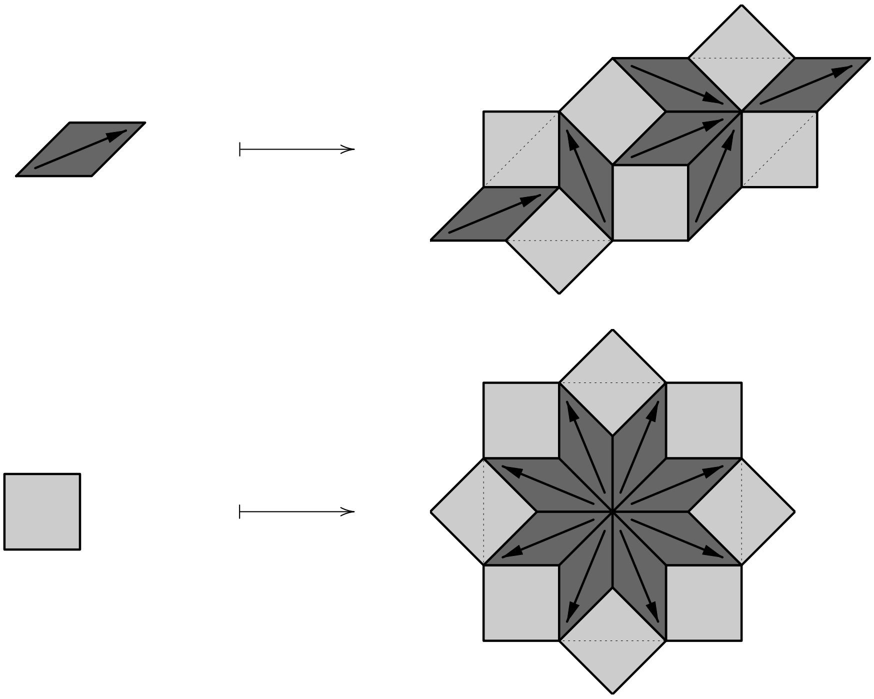
In fact, $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\xi_{12} + \bar{\xi}_{12})$ is the maximal real subfield of $\mathbb{Q}(\xi_{12})$, see [385, p. 15].

Now, in view of Proposition 3.106, we observe that $\mathbb{Q}_2(\sqrt{3})$, $\mathbb{Q}_2(\sqrt{-3})$ and $\mathbb{Q}\mathbb{Q}_2(\sqrt{-1})$ are non-isomorphic quadratic extensions of $\mathbb{Q}_2$, where $\mathbb{Q}_2(\sqrt{-3})$ is a unramified extension, see Proposition 3.109. Consequently, $\mathbb{Q}_2(\sqrt{-3})$ is the (unique) intermediate field of $\mathbb{Q}_{\mathfrak{P}_{12}}/\mathbb{Q}_2$ such that $\mathbb{Q}_2(\sqrt{-3})/\mathbb{Q}_2$ is unramified and $\mathbb{Q}_{\mathfrak{P}_{12}}/\mathbb{Q}_2(\sqrt{-3})$ is totally ramified. Consequently, the system of representatives of $\widehat{\mathfrak{o}_{\mathfrak{P}_{12}}}/\mathfrak{P}_{12}$ (and $\widehat{\mathfrak{o}_{\mathbb{Q}_2(\sqrt{-3})}}/(2)$, but then with uniformiser 2) is given by the 3rd roots of unity and 0, *i.e.*, $\{0,1,\xi_3,\xi_3^2\} = \{0,1,\xi_{12}^4,\xi_{12}^8\}$.

6a.2. The Watanabe-Ito-Soma Tiling(s)

Maybe the first example of a two-dimensional substitution tiling with an inflation factor that is a PV-number but not an algebraic unit, is given in [386]. The prototiles are given by a square and a 45° rhombus (and their rotated versions). We observe that the square has (formally) to be divided into two half-squares and that the substitution rule of the rhombus is asymmetric, wherefore we use markings to break the symmetry of the rhombi, compare [142, p. 170 & Fig. 7]. A possible choice is the following substitution rule (as already observed in [386], compare [386, Fig. 1], the substitute of the rhombus includes a regular octagon (in its centre) formed by two squares and four rhombi; thus, one can already derive eight different substitution rules by rotating this octagon, choosing different directions of the arrow markings of the rhombi increases the number of different substitution rules and patterns further), which we chose in accordance with [386, Fig. 1]. The inflation factor is $\lambda = 2 + \sqrt{2} \approx 3.4142$, which is a PV-number (since $2 - \sqrt{2} \approx 0.5858$) but not a unit (we note that $N_{\mathbb{Q}(\xi_8)/\mathbb{Q}}(2 + \sqrt{2}) =$

$\left(N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(2+\sqrt{2})\right)^2 = 4$; $\mathbb{Q}(\sqrt{2})$ is the maximal real subfield of $\mathbb{Q}(\xi_8)$).



Removing the arrow markings of the rhombi, this substitution rule produces the tiling shown in Figure 6a.1. If one chooses edge length 1 for the tiles (the square and the rhombus), then one can achieve that all vertices of this tiling lie in $\mathbb{Z}[\xi_8]$.

Consequently, one is interested if the point set formed by the vertices is a model set (as it is the case for the famous Penrose and Ammann-Beenker tilings, [12, 58, 82, 83, 284] and [157, p.550 ff]). We note that one has $\varDelta \subset \mathbb{Z}[\xi_8]$ and therefore $\mathcal{L} = \mathbb{Z}[\xi_8]$. To obtain a CPS, we have to decide in which local fields associated to $\mathbb{Q}(\xi_8)$ the inflation factor $\lambda = 2+\sqrt{2}$ is an expansion/contraction. Since $N_{\mathbb{Q}(\xi_8)/\mathbb{Q}}(2+\sqrt{2}) = 4$ and $(2) = \mathfrak{P}_8^4$ we have:

- The only local field on which λ acts as an expansion is $\mathbb{C} \cong \mathbb{R}^2$. (This local field is associated with the pair of Galois automorphisms id and complex conjugation).
- The factor $\lambda = 2-\sqrt{2}$ acts as a contraction on $\mathbb{C}$, the local field associated with the pair of Galois automorphisms given by $\xi_8 \mapsto \xi_8^3$ and its complex conjugate partner given by $\xi_8 \mapsto \xi_8^5$. Moreover, it is also a contraction on $\mathbb{Q}_{\mathfrak{P}_8}$, where $\lambda = .00111100111000001111\ldots$ is the $\mathfrak{P}_8$-adic expansion with respect to the uniformiser $1+\xi_8$ and therefore $\|\lambda\|_{\mathfrak{P}_8} = \frac{1}{4}$.

Thus, we expect the following CPS for a Watanabe-Ito-Soma tiling:

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{\pi_1} & \mathbb{C}\times\mathbb{C}\times\widehat{\mathfrak{o}_{\mathfrak{P}_8}} & \xrightarrow{\pi_2} & \mathbb{C}\times\widehat{\mathfrak{o}_{\mathfrak{P}_8}} \\ \text{dense}\ \cup & & \cup & & \cup\ \text{dense} \\ \mathbb{Z}[\xi_8] & \xleftrightarrow{\text{bijective}} & \tilde{\mathcal{L}} & \xleftrightarrow{\text{bijective}} & \mathcal{L}^\star \end{array}$$

where the star-map is given by

$$x = m_1 + m_2\cdot\xi_8 + m_3\cdot\xi_8^2 + m_3\cdot\xi_8^3 \;\mapsto\; x^\star = (m_1 + m_2\cdot\xi_8^3 + m_3\cdot\xi_8^6 + m_3\cdot\xi_8, x),$$

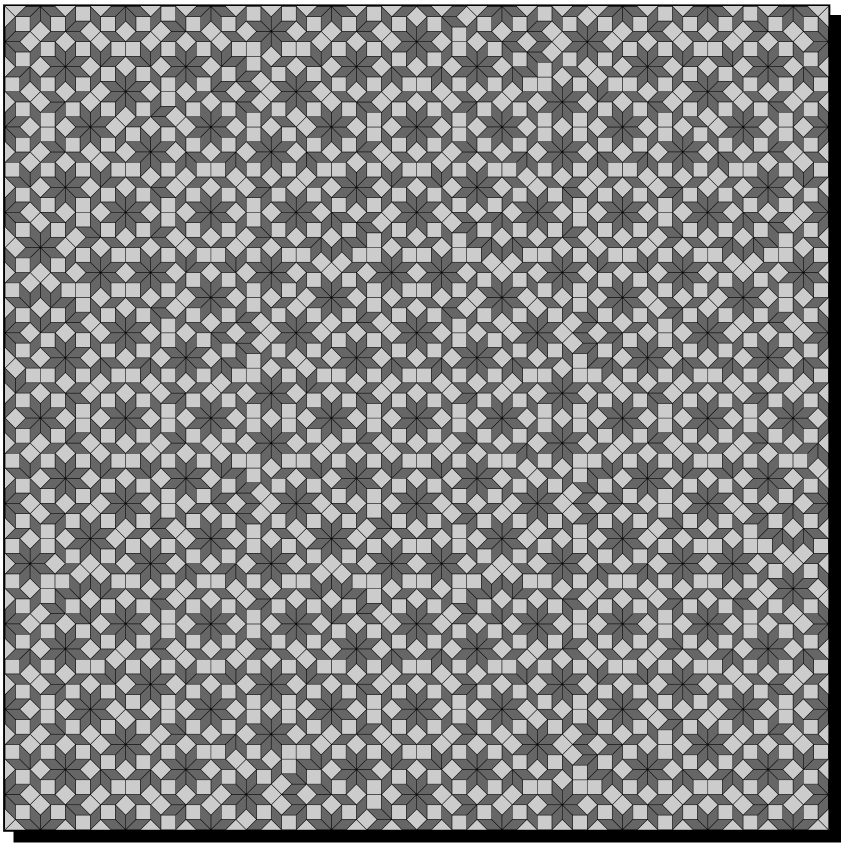


Figure 6a.1.: A patch of a Watanabe-Ito-Soma tiling.

i.e., by the Galois automorphism $\xi_8 \mapsto \xi_8^3$ (one can also use $\xi_8 \mapsto \xi_8^5$ by complex conjugation) and the embedding $\mathbb{Z}[\xi_8] \hookrightarrow \widehat{\mathfrak{o}_{\mathfrak{P}_8}}$.

Remark 6a.5. More precisely, one would have to derive a substitution for the vertices from the known substitutions of the tilings such that one obtains the vertices as substitution (multi-component) Delone set. This is in principle possible by the methods used in [163] (or by trial-and-error), but often very laborious. Moreover, we note that one really obtains a multi-component Delone set that way, since one has to differentiate the vertices according to their surrounding: One is basically looking for a (multi-component) Delone set that is LI to the tiling and has a substitution rule that corresponds to the tile substitution.

Note that it might happen that $\mathcal{L}'$ is a proper subgroup of $\mathcal{L}$ for the substitution multi-component Delone set derived that way. In fact, this happens for the well-known Penrose tiling, where the CPS is given by $(\mathbb{C}, \mathbb{C} \times \mathsf{C}_5, \widetilde{\mathcal{L}})$ with $\mathcal{L} = \mathbb{Z}[\xi_5]$ and the star-map

$$x = m_1 + m_2\,\xi_5 + m_3\,\xi_5^2 + m_4\,\xi_5^3 \mapsto x^\star = (m_1 + m_2\,\xi_5^2 + m_3\,\xi_5^4 + m_4\,\xi_5, (m_1 + m_2 + m_3 + m_4) \bmod 5)$$

(in the first component, this is the Galois automorphism $\xi_5 \mapsto \xi_5^2$, and the other three Galois automorphisms are given by the identity, complex conjugation and the composition of the complex conjugation with the first one), see [31]. Here, one has $\mathcal{L}/\mathcal{L}' \cong \mathsf{C}_5$, and the vertices of the Penrose tiling only belong to four of the five possible cosets (and in the literature are often differentiated on this basis, compare the use of "star/triangle/square/circle vertices" in [188, 394]).

Having established a CPS for the vertices, one is then interested if they can be described as multi-component model set. More precisely, the program is as follows:

- From the tile substitutions, derive an EMFS $\boldsymbol{\Theta}$ such that the vertices (respectively some different points) are (primitive) substitution multi-component Delone sets thereof.
- Calculate $\mathcal{L}'$ and confirm/establish the CPS (or, alternatively, the extended CPS).
- Using $\boldsymbol{\Theta}^{\#}$, $\boldsymbol{\Theta}^{\star}$ and $\boldsymbol{\Theta}^{\star\#}$ calculate the associated tiling (note that the tiles may in general differ[3] from the tiles used originally), the attractor of the IFS $\boldsymbol{\Theta}^{\star}$ and its corresponding tiling in internal space.
- Use any criterion of Theorem 5.154 (also compare Theorem 6.116) to decide if one has a model set.

We leave this program for future work and here only speculate about the possible shape of the window. For this we simple apply the star-map to the vertices (of a big patch, in fact we use a circular patch with 329 281 vertices, corresponding to fifth iterate of the substitution of the square), and look at the structure we obtain in the internal space by this procedure. We also note that the $\mathfrak{P}_8$-adic expansion of an $x \in \mathbb{Z}[\xi_8]$ can be iteratively obtained using

$$\begin{aligned}\frac{m_1 + m_2 \cdot \xi_8 + m_3 \cdot \xi_8^2 + m_4 \cdot \xi_8^3}{1 + \xi_8} =& \frac{m_1 + m_2 - m_3 + m_4}{2} + \frac{-m_1 + m_2 + m_3 - m_4}{2} \cdot \xi_8 + \\ & \frac{m_1 - m_2 + m_3 + m_4}{2} \cdot \xi_8^2 + \frac{-m_1 + m_2 - m_3 + m_4}{2} \cdot \xi_8^3,\end{aligned}$$

since $x = m_1 + m_2 \cdot \xi_8 + m_3 \cdot \xi_8^2 + m_4 \cdot \xi_8^3$ is divisible by $1 + \xi_8$ iff $m_1 + m_2 + m_3 + m_4 \equiv 0 \bmod 2$ (otherwise, $x - 1$ is divisible by $1 + \xi_8$). This then yields the $\mathfrak{P}_8$-adic expansion using the uniformiser $1 + \xi_8$.

The result of this application of the star-map is shown in Figure 6a.2, where we look at projections to the complex subspace $\mathbb{C}$ of the internal space of all points that lie in (one of the eight disjoint) balls of diameter (respectively radius) $\frac{1}{8}$ in $\widehat{\mathfrak{o}_{\mathfrak{P}_8}}$. This should give an impression of the window that one expects for a Watanabe-Ito-Soma tiling. In particular, one immediately notices some symmetries present in these plots: If the $\mathfrak{P}_8$-adic expansion starts with .1, then the corresponding four parts of the possible window seem to be mirror-symmetric with a two-fold axis, moreover, these four parts seem to be rotated versions of each other. If the expansion starts with .01, then we also seem to have a four-fold axis, and if it starts with .00 then we get the (maximal) 8-fold symmetry. Note that this symmetry comes from the symmetry of the tiling (respectively its translates) with respect to the origin. It would be good to see these symmetries already in the corresponding IFS $\boldsymbol{\Theta}^{\star}$ (*i.e.*, we hope that one can use/find an appropriate EMFS $\boldsymbol{\Theta}$).

6a.3. Further Examples

Similar examples (respectively, a simple variant of the above Watanabe-Ito-Soma tiling) can be found in [141] (also see [274–276]). Again, one considers 2-dimensional substitution tilings

[3] In fact, they usually have a self-similar structure and therefore a fractal-shaped boundary (*i.e.*, the Hausdorff dimension of the boundary is not an integer); thus, they are called “fractalized tilings” in the literature, see [45], [158, Section 3.3] and [146]. Note that in [146] also, in principle, the attractor of $\boldsymbol{\Theta}^{\star}$ for the Penrose tiling is considered (also compare [134, 136] for some recent work on different examples).

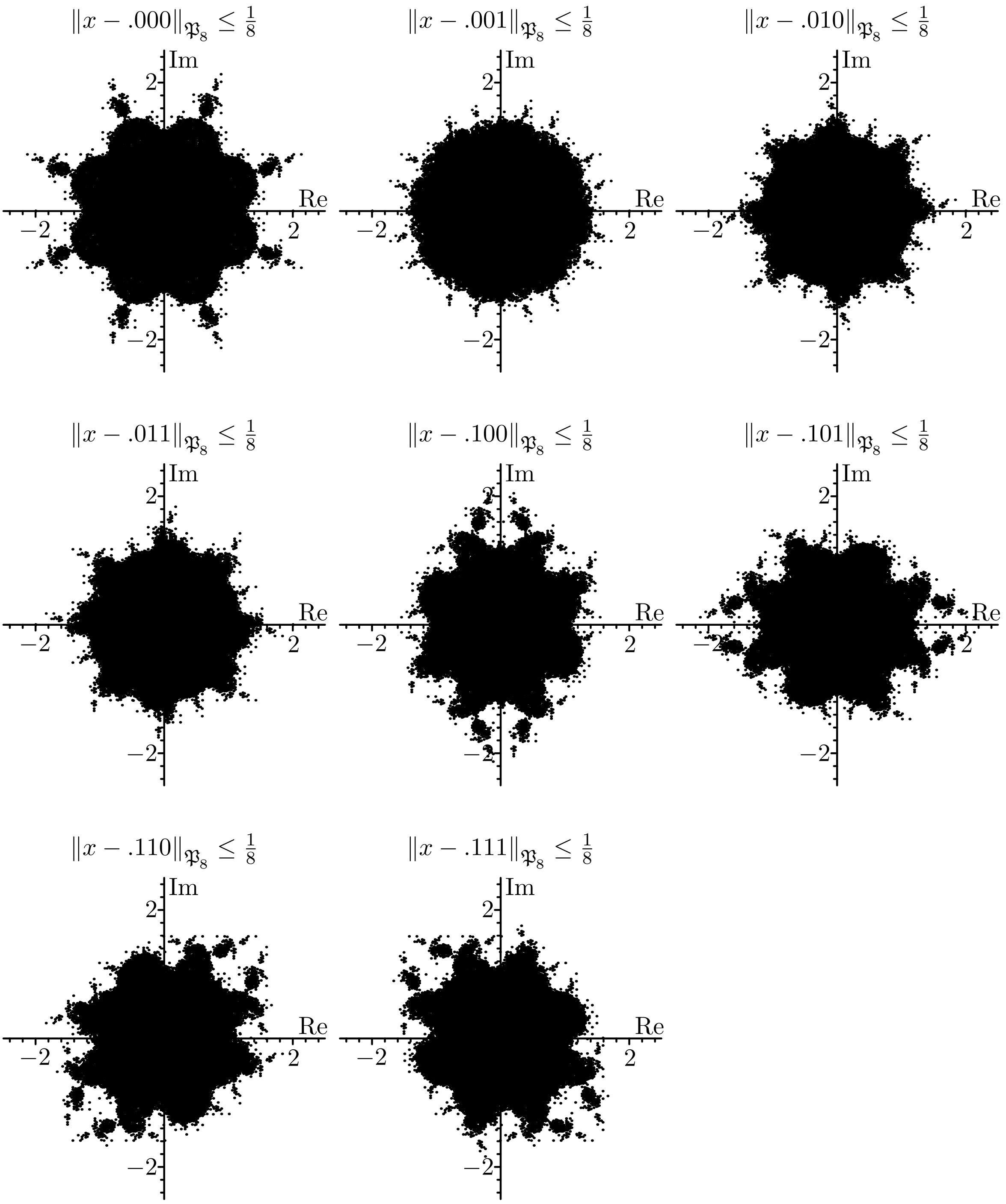


Figure 6a.2.: Plots of the image of the vertices of a circular patch of a Watanabe-Ito-Soma tiling with 329 281 vertices under the star-map. Shown is the projection to $\mathbb{C} \cong \mathbb{R}^2$ of all points that are contained in one of the eight disjoint balls of radius $\frac{1}{8}$ in $\widehat{\mathfrak{o}_{\mathfrak{P}_8}}$.

on a cyclotomic field $\mathbb{Q}(\xi_n)$ respectively its maximal order $\mathbb{Z}[\xi_n]$ with some inflation factor λ, which is a PV-number (more precisely, an inflation factor $\lambda' \cdot \xi_n$ is used such that its n-th power is a Pisot number). Let us look at the cyclotomic fields and the inflation factors and derive the possible internal space.

- On $\mathbb{Z}[\xi_8]$, the inflation factor $\lambda' = \sqrt{\lambda} \cdot \exp(2\pi i/16)$ with $\lambda = 2 + \sqrt{2}$ is used. Consequently, its square (for which one derives a fixed point) is essentially the inflation factor from the Watanabe-Ito-Soma tiling. In fact, the example discussed there is a simple variant of it, using the octagon in the substitute of the rhombus as another prototile (this removes the symmetry breaking of the rhombus!). Obviously, one obtains the same CPS as for a Watanabe-Ito-Soma tiling.

- On $\mathbb{Z}[\xi_{10}]$, a tiling with inflation factor $\lambda' = \sqrt{\lambda} \cdot \exp(2\pi i/20)$ is used, where $\lambda = \frac{1}{2}(25 + 11\sqrt{5})$ is the PV-number corresponding to the polynomial $x^2 - 25x + 5$. Consequently, one has to check the factorisation of the ideal (5) in $\mathbb{Q}(\xi_{10})$, and one therefore obtains a CPS with internal space $\mathbb{C} \times \widehat{\mathfrak{o}_{\mathfrak{P}_5}}$. The lattice is given by the diagonal embedding of $\mathbb{Z}[\xi_{10}]$ (where the action of the star-map on the complex coordinate is given by the Galois-automorphism $\xi_{10} \mapsto \xi_{10}^3$). Moreover, one obtains the $\mathfrak{P}_5$-adic expansion of a number in $\mathbb{Z}[\xi_{10}]$ using

$$\begin{aligned}\frac{m_1 + m_2\,\xi_{10} + m_3\,\xi_{10}^2 + m_4\,\xi_{10}^3}{1+\xi_{10}} =& \frac{4\,m_1 + m_2 - m_3 + m_4}{5} + \\ & \frac{-3\,m_1 + 3\,m_2 + 2\,m_3 - 2\,m_4}{5} \cdot \xi_{10} + \\ & \frac{2\,m_1 - 2\,m_2 + 2\,m_3 + 3\,m_4}{5} \cdot \xi_{10}^2 + \\ & \frac{-m_1 + m_2 - m_3 + m_4}{5} \cdot \xi_{10}^3,\end{aligned}$$

  wherefore (one can convince oneself that) $x = m_1 + m_2\,\xi_{10} + m_3\,\xi_{10}^2 + m_4\,\xi_{10}^3$ is divisible by $1+\xi_{10}$ iff $m_1 - m_2 + m_3 - m_4 \equiv 0 \bmod 5$ (otherwise $x - ((m_1 - m_2 + m_3 - m_4) \bmod 5)$ is divisible by $1 + \xi_{10}$).

- On $\mathbb{Z}[\xi_{12}]$, a tiling with inflation factor $\lambda' = \sqrt{\lambda}\cdot\exp(2\pi i/24)$ is used, where $\lambda = 14+8\sqrt{3})$ is the PV-number corresponding to the polynomial $x^2 - 28x + 4$. Consequently, one has to check the factorisation of the ideal (2) in $\mathbb{Q}(\xi_{12})$, and one therefore obtains a CPS with internal space $\mathbb{C} \times \widehat{\mathfrak{o}_{\mathfrak{P}_{12}}}$. The lattice is given by the diagonal embedding of $\mathbb{Z}[\xi_{12}]$ (where the action of the star-map on the complex coordinate is given by the Galois-automorphism $\xi_{12} \mapsto \xi_{12}^5$). Moreover, one obtains the $\mathfrak{P}_{12}$-adic expansion of a number in $\mathbb{Z}[\xi_{12}]$ using

$$\begin{aligned}\frac{m_1 + m_2\,\xi_{12} + m_3\,\xi_{12}^2 + m_4\,\xi_{12}^3}{1+i} =& \frac{m_1 + m_2 + m_4}{2} + \frac{m_2 + m_3}{2} \cdot \xi_{12} + \\ & \frac{-m_2 + m_3}{2} \cdot \xi_{12}^2 + \frac{-m_1 - m_3 + m_4}{2} \cdot \xi_{12}^3\end{aligned}$$

  by observing the following:

  - If $m_1 + m_2 + m_3 \equiv 0 \bmod 2$ and $m_2 + m_3 \equiv 0 \bmod 2$, then $x = m_1 + m_2\,\xi_{12} + m_3\,\xi_{12}^2 + m_4\,\xi_{12}^3$ is divisible by $1 + i$.

- If $m_1 + m_2 + m_3 \equiv 1 \bmod 2$ and $m_2 + m_3 \equiv 0 \bmod 2$, then $x - 1$ is divisible by $1 + i$.
- If $m_1 + m_2 + m_3 \equiv 0 \bmod 2$ and $m_2 + m_3 \equiv 1 \bmod 2$, then $x - \xi_3^2 = x + \xi_{12}^2$ is divisible by $1 + i$.
- If $m_1 + m_2 + m_3 \equiv 1 \bmod 2$ and $m_2 + m_3 \equiv 1 \bmod 2$, then $x - \xi_3 = x - \xi_{12}^2 + 1$ is divisible by $1 + i$.

This (iteratively) establishes the $\mathfrak{P}_{12}$-adic expansion of a number in $\mathbb{Z}[\xi_{12}]$.

We note that these conditions for the respective $\mathfrak{P}$-adic expansions are also obtained by inspection of the transformation of the Archimedean part of the lattice $\tilde{\mathcal{L}}$ under multiplication with λ' on [141, p. 589].

Again, it would be useful to conduct the program mentioned in the last section also with these tilings. Moreover, we also mention that the star-map of the vertices (together with the centres[4] of the decagons respectively dodecagons) of the tilings in question again show a similar symmetry behaviour as our considered Watanabe-Ito-Soma tiling in Figure 6a.2, see [141, Fig. 2].

Remark 6a.6. As in Section 6.5, one can easily derive the dual lattice of $\tilde{\mathcal{L}}_{\mathrm{ext}}$. Consequently, one can explicitly calculate the diffraction pattern from Folklore Theorem 5a.10 of these tilings if they are model sets.

We end this section with patches of these three tilings, see Figure 6a.3 (compare [141, Fig. 1]).

[4]The deeper reason, why one to include these centres is (or better: might be) the following: We want to have property **(LT)**(iii) for the corresponding EMFS $\boldsymbol{\Theta}$, *i.e.*, a substitution which yields a Delone set. If one tries to find a substitution on the vertices only, one does not obtain an EMFS for a Delone set, also compare Lemma 5.93.

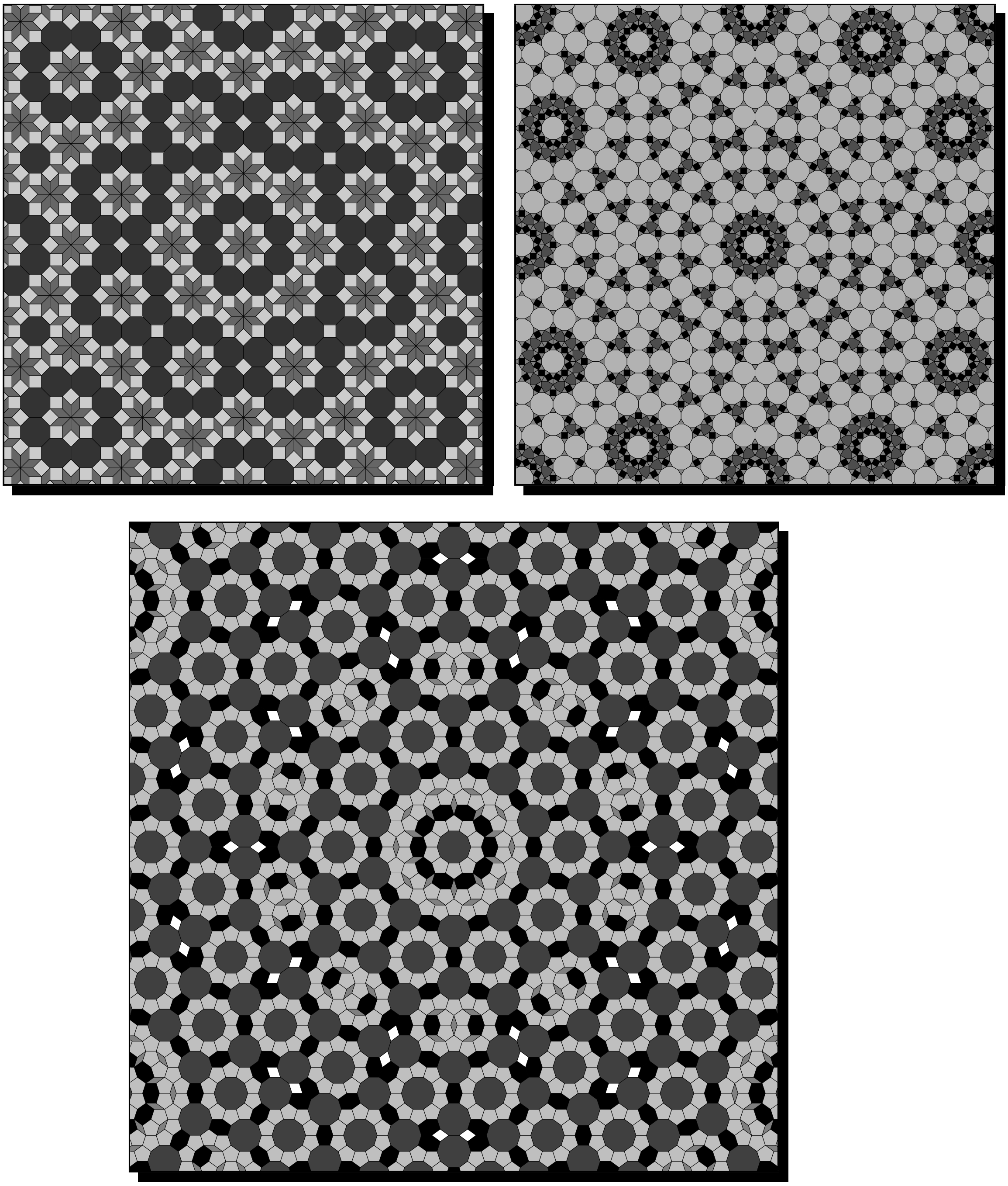


Figure 6a.3.: The tilings of Fujita and Niizeki that can be obtained by a substitution with an inflation factor that is not an algebraic unit: Vertices of the tiles lie in $\mathbb{Z}[\xi_8]$ for the upper left tiling, vertices and centres of the dodecagons lie in $\mathbb{Z}[\xi_{12}]$ for the upper right tiling, and vertices and the centres of the decagons lie in $\mathbb{Z}[\xi_{10}]$ for the lower tiling (where one chooses edge length 1 for all appearing tiles).

6b. Lattice Substitution Systems

> Die Geschichten, die die anderen über einen erzählen, und die Geschichten, die man über sich selbst erzählt: welche kommen der Wahrheit näher?
>
> Nachtzug nach Lissabon – *Pascal Mercier*

In this chapter, it is shown that lattice substitution systems find their place within the framework we have constructed. Moreover, it should also serve as an advertisement for our method of visualising (see Chapter 3c) the windows of the model sets that arise here.

6b.1. The Cut and Project Scheme

We first define what we understand under a lattice substitution system.

Definition 6b.1. A *lattice substitution system* (or *LSS* for short) $(\underline{\Lambda}, \boldsymbol{\Theta})$ is a substitution multi-component Delone set $\underline{\Lambda}$ in a Euclidean space $\mathbb{R}^d$ (for some $d \in \mathbb{N}$) such that $\operatorname{supp} \underline{\Lambda}$ is a lattice in $\mathbb{R}^d$ and the sets Λ_i are mutually disjoint, together with its associated EMFS $\boldsymbol{\Theta}$.

We can therefore identify $\operatorname{supp} \underline{\Lambda}$ with $\mathbb{Z}^d$ (more precisely, $\operatorname{supp} \underline{\Lambda}$ is an affine image of $\mathbb{Z}^d$). Moreover, this definition has the implication that $\underline{\Lambda}$ is an FLC multi-component set. We note that $\underline{\Lambda}$ gives a partition of the lattice.

Example 6b.2. We consider the following EMFS

$$\boldsymbol{\Theta} = \begin{pmatrix} \{f_0\} & \{f_1\} \\ \{f_1\} & \{f_0\} \end{pmatrix}$$

in $\mathbb{R}$, where $f_0(x) = 2x$ and $f_1(x) = t_{(1)} \circ f_0(x) = 2x + 1$. Then, $\underline{\Lambda} = (\mathbb{Z}\,, \mathbb{Z})$ is a fixed point of $\boldsymbol{\Theta}$ such that the unions in the components of $\boldsymbol{\Theta}(\underline{\Lambda})$ are disjoint. Thus, this is a substitution multi-component Delone set (with $\operatorname{supp} \underline{\Lambda}$ a lattice), but does not yield an LSS.

We note that the Thue-Morse sequence is another fixed point of this EMFS, but there one obtains a partition of the lattice $\mathbb{Z}$, see Example 5.64. We observe that the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$ is given by $([0,1]\,, [0,1])$. So, while the Thue-Morse sequence is representable, the fixed point $\underline{\Lambda} = (\mathbb{Z}\,, \mathbb{Z})$ is not.

We recall that for a primitive substitution FLC multi-component Delone sets $\underline{\Lambda}$ with **(LT)** and which is representable, one can easily decide if it is repetitive: It is repetitive iff every cluster of it is legal, see Corollary 5.90. We therefore assume the following:

- Each Θ_{ij} is a finite (possibly empty) set of mappings $x \mapsto \boldsymbol{Q}x + t_{ijk}$, where $\boldsymbol{Q}$ (a $d \times d$-matrix over $\mathbb{R}$) is an expansive linear map and $t_{ijk} \in \mathbb{R}^d$ (with $1 \le i, j \le n$, $1 \le k \le \operatorname{card} \Theta_{ij}$). Since $\boldsymbol{Q}$ is an expansion, one has $|\det \boldsymbol{Q}| > 1$. In our previous notation, we have $f_0(x) = \boldsymbol{Q}x$.

- $\boldsymbol{S\Theta}$ is primitive. Then – since we have a multi-component Delone set – $|\det \boldsymbol{Q}|$ equals the PF-eigenvalue of $\boldsymbol{S\Theta}$ by Lemma 5.93.

These are the standing assumptions in this chapter.

We now introduce a subclass of lattice substitution systems which are representable by definition.

Definition 6b.3. Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a primitive LSS. Denote by $L = \operatorname{supp} \underline{\Lambda}$ the lattice. A primitive LSS $(\underline{\Lambda}, \boldsymbol{\Theta})$ is called *admissible*, if the attractor of the adjoint IFS $\boldsymbol{\Theta}^{\#}$ is given by $\underline{A} = (\operatorname{cl} P)_{i=1}^{n}$ where P (with $\operatorname{int} \operatorname{cl} P \subset P \subset \operatorname{cl} P$ and $\operatorname{cl} P$ is regularly closed with boundary of Lebesgue measure 0) is a fundamental domain of L.

We note that by $L - L = L$, one also obtains $\boldsymbol{Q}L \subset L$. Consequently, $\boldsymbol{Q}P \cap L$ is a system of $(|\det \boldsymbol{Q}|)$ representatives of the factor group $L/\boldsymbol{Q}L$. Therefore, we obtain the following alternative definition of admissibility.

Definition 6b.3'. Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a primitive LSS. Denote by $L = \operatorname{supp} \underline{\Lambda}$ the lattice and let S be a fixed system of representatives of $L/\boldsymbol{Q}L$. A primitive LSS $(\underline{\Lambda}, \boldsymbol{\Theta})$ is called *admissible*, if all substitutions $f \in \boldsymbol{\Theta}$ are of the form $f(x) = \boldsymbol{Q}x + t$ with $t \in S$.

Note that by the disjointness in the definition of an LSS, each map $f(x) = \boldsymbol{Q}x + t$ appears exactly once in each column of $\boldsymbol{\Theta}$ (wherefore one indeed gets the same set P in each component of the attractor of the adjoint IFS).

Remark 6b.4. A (primitive) admissible LSS in one dimension (*i.e.*, on $L = \mathbb{Z}$) is known as a primitive *substitution of constant length* q, see [101, 298, 299]. We note that $(1, \ldots, 1)$ is a left eigenvector of $\boldsymbol{S\Theta}$ (which has column sum q for all columns, wherefore its PF-eigenvalue is also q), which lines up with the attractor $\underline{A} = ([0\,, 1])_{i=1}^{n}$. Thus, the mappings of the EMFS $\boldsymbol{\Theta}$ in this case are of the form $x \mapsto q \cdot x + t$ where $t \in \{0, \ldots, q-1\}$. Consequently, using an alphabet $\mathcal{A}$ with $\operatorname{card} \mathcal{A} = n$, a substitution of constant length is symbolically given by $i \mapsto \sigma(i)$ where $\sigma(i)$ is a word of length q.

Given an arbitrary LSS $(\underline{\Lambda}', \boldsymbol{\Theta}')$, one is interested under which conditions one finds an admissible LSS $(\underline{\Lambda}, \boldsymbol{\Theta})$ such that $\underline{\Lambda}'$ and $\underline{\Lambda}$ are MLD (see 5.114). Such a condition is given by the following definition.

Definition 6b.5. Let $(\underline{\Lambda}', \boldsymbol{\Theta}')$ be a primitive LSS and P' a fundamental domain of the lattice $L = \operatorname{supp}(\underline{\Lambda})$. We say that $(\underline{\Lambda}', \boldsymbol{\Theta}')$ is *nicely growing*, if there is an $R > 0$ such that, for all R–clusters $\underline{\Lambda} \cap B_{\leq R}(x)$ one has

$$L \cap (\boldsymbol{Q}P' + B_{\leq R}(\boldsymbol{Q}x) \subset \operatorname{supp}\left(\boldsymbol{\Theta}(\underline{\Lambda} \cap B_{\leq R}(x))\right). \tag{6b.1}$$

In plain words: We require that the action of $\boldsymbol{\Theta}$ maps the set of points of $\underline{\Lambda}' \cap B_{\leq R}(x)$ to the points inside a region which is considerably larger, namely, $\underline{\Lambda}' \cap (\boldsymbol{Q}P' + B_{\leq R}(0))$ (and maybe to some more points). And in general, the points of $\underline{\Lambda}'$ inside any ball of radius R, centred at x, are mapped to the points in the set $\underline{\Lambda} \cap (\boldsymbol{Q}P' + B_{\leq R}(\boldsymbol{Q}x))$, "centred" at $\boldsymbol{Q}x$ (and maybe to some more points).

Corollary 6b.6. [138, Remark after Def. 4.10] *All admissible LSS are nicely growing.*

Proof. If $(\underline{\Lambda}, \boldsymbol{\Theta})$ is admissible, then $\boldsymbol{\Theta}$ maps $\underline{\Lambda} \cap B_{\leq R}(0)$ onto $\underline{\Lambda} \cap \boldsymbol{Q}B_{\leq R}(0)$ (and possibly more), at least for R large enough. Then the number of points in the support of the left hand side of (6b.1) grows approximately like $R^d + cR^{d-1}$, where the number of points in the support of the right hand side grows like $R^d|\det \boldsymbol{Q}|$. Since $|\det \boldsymbol{Q}| > 1$, Equation (6b.1) is fulfilled for R large enough. □

Proposition 6b.7. [138, Theorem 4.11] *Let $(\underline{\Lambda}', \boldsymbol{\Theta}')$ be a nicely growing (primitive) LSS. Then there exists an LSS $(\underline{\Lambda}, \boldsymbol{\Theta})$, such that $\underline{\Lambda}'$ and $\underline{\Lambda}$ are MLD and $(\underline{\Lambda}, \boldsymbol{\Theta})$ is admissible.* □

One can even show that every representable primitive LSS $(\underline{\Lambda}', \boldsymbol{\Theta}')$ is nicely growing, at least if every cluster is legal [137]. Therefore, all lattice substitution systems we are interested in are nicely growing.

We now construct a cut and project scheme for an LSS. We have already defined $L = \operatorname{supp} \underline{\Lambda} = L - L$. Since Λ_i is a partition of L, one has $\Delta = L$, where $\Delta = (\operatorname{supp} \underline{\Lambda}) - (\operatorname{supp} \underline{\Lambda})$. As usual, we set $L' = \langle \Delta' \rangle_{\mathbb{Z}} = L_1 + \ldots + L_n$, where $L_i = \langle \Delta_i \rangle_{\mathbb{Z}}$, $\Delta' = \bigcup_{i=1}^n \Delta_i$ and $\Delta_i = \Lambda_i - \Lambda_i$. Consequently, L' – as subgroup of L – is a sublattice of L and we have $L' - L' = L'$. The height group L/L' is a finite group (where $\operatorname{card} L/L' = [L : L'] = |\det \boldsymbol{A}|$ with $\boldsymbol{A} \in GL(n, \mathbb{Z})$ is such that $\boldsymbol{A}L = L'$). Moreover, one has $\boldsymbol{Q}L' \subset L'$, and Δ' is, as subset of a lattice, uniformly discrete. If $\underline{\Lambda}$ is repetitive, then Δ' is also relatively dense and actually a Meyer set. Before we apply Corollary 5.135, we note that there is an alternative form of algebraic coincidence in case of an LSS.

Lemma 6b.8. *Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a primitive LSS such that $\underline{\Lambda}$ is aperiodic, representable and every cluster of it is legal (equivalently, it is repetitive). Then $\underline{\Lambda}$ admits an algebraic coincidence (i.e., there exists an $1 \le i \le n$, an $m \in \mathbb{Z}_{\ge 0}$ and a $t \in \Lambda_i$ such that $t + \boldsymbol{Q}^m \Delta' \subset \Lambda_i$) iff there exists a $1 \le j \le n$, an $m' \in \mathbb{Z}_{\ge 0}$ and a $t' \in \Lambda_j$ such that $t' + \boldsymbol{Q}^{m'} L' \subset \Lambda_j$.*

Proof. Since $\Delta' \subset L'$, one direction is trivial. For the other direction, we recall Equation (5.17) on p. 184: If $\underline{\Lambda}$ admits an algebraic coincidence, then there is an $M \in \mathbb{Z}_{\ge 0}$ such that $\boldsymbol{Q}^M(\Delta' + \Delta') \subset \Delta'$.

The lattice $L' = \langle \Delta' \rangle_{\mathbb{Z}} \subset \mathbb{R}^d$ is finitely generated (by d generators); moreover, there is a finite set $F \subset \Delta'$ (whose cardinality might be bigger than d, *e.g.*, consider the case $\Delta' = 2\mathbb{Z} \cup 3\mathbb{Z}$) such that $L' = \langle F \rangle_{\mathbb{Z}}$: Take d linear independent elements of Δ' (which exists by the relative denseness), denote this set by F'. These elements span a sublattice of L' (of finite index). If $L' \ne \langle F' \rangle_{\mathbb{Z}}$, then add an element $v \in (\Delta' \setminus \langle F' \rangle_{\mathbb{Z}})$ to F'. Then, $F' \cup \{v\}$ spans a sublattice of L' of smaller index than F'. After finitely many such steps, one obtains such a set F one is looking for.

The difference set Δ' is relatively dense, wherefore there is a compact set W such that $W + \Delta' = \mathbb{R}^d$. But every element of $W \cap L'$ (a finite set!) can (trivially) be written as linear combination of elements of F with finite coefficients. Consequently, one has $W \cap L' \subset \Delta' + \ldots + \Delta'$ with (at most) $C = C(F, W) > 0$ many sets Δ' on the right hand side. Thus, one also has $L' \subset \Delta' + \ldots + \Delta' + \Delta'$ (with $C+1$ many Δ'). Then, $\boldsymbol{Q}^M(\Delta' + \Delta') \subset \Delta'$ shows that there is an M' such that $\boldsymbol{Q}^{M'} L' \subset \Delta'$. This proves the claim. □

Proposition 6b.9. *Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a primitive LSS such that $\underline{\Lambda}$ is aperiodic, representable and every cluster of it is legal. Suppose there is a $1 \le j \le n$, an $m' \in \mathbb{Z}_{\ge 0}$ and a $t' \in \Lambda_j$ such that $t' + \boldsymbol{Q}^{m'} L' \subset \Lambda_j$ (or, alternatively, an algebraic coincidence). Then $\{\boldsymbol{Q}^m L' \mid m \in \mathbb{Z}_{\ge 0}\}$ is a countable base at 0 for the AC topology on L.* □

We can now easily establish a CPS, see [235, p. 542]: Let $q = |\det \boldsymbol{Q}| = [L'{:}\boldsymbol{Q}L'] = [L{:}\boldsymbol{Q}L] > 1$. We define the "$\boldsymbol{Q}$-adic" completions

$$H = \varprojlim_k L/\boldsymbol{Q}^k L' \qquad \text{and} \qquad H' = \varprojlim_k L'/\boldsymbol{Q}^k L'.$$

We note that both are profinite groups (see Definition 3a.6), wherefore they are compact Hausdorff and totally disconnected by Lemma 3a.8. By construction, they are complete (in the sense of Definition 3a.10, also see Remark 3a.11). Moreover, as "pro-Abelian" groups, *i.e.*, as projective limits of finite Abelian groups, they are necessarily Abelian, since they are closed subgroups of a direct product of Abelian groups (by Lemma 3a.5), see [311, Exercise 2.1.7]. Obviously, one can identify H and $H' \times L/L'$, and $\{\mathrm{cl}(\boldsymbol{Q}^m L') = \boldsymbol{Q}^m H' \mid m \in \mathbb{Z}_{\geq 0}\}$ forms a base of clopen sets at 0 in the profinite topology. We note that one might define a metric on H respectively on H' such that $\boldsymbol{Q}^m H'$ is simply the ball of radius q^{-m} around 0. This is then a ultrametric, H and H' are ultrametric compact Abelian groups, for which consequently Lemma 2.26 applies.

We normalise the Haar measure μ on H such that $\mu(H) = 1$ (recall that H is compact). Then, for any $x \in H$ and $k \in \mathbb{Z}_{\geq 0}$, we have

$$\mu(x + \boldsymbol{Q}^k H') = \frac{1}{q^k \cdot \mathrm{card}\, L/L'}. \tag{6b.2}$$

This, of course, calls the p-adic integers $\mathbb{Z}_p$ to mind. Since L respectively L' are finitely generated (they are free groups with d generators), the finite groups $L'/\boldsymbol{Q}^k L'$ have (at most) d generators, wherefore H' has (at most) d generators by Lemma 3a.17. Moreover, H' is torsion-free. Therefore, we may apply Proposition 3a.20, where we note that the Haar measure indicates that only p-adic integers $\mathbb{Z}_p$ arise in the direct sum with $p \,|\, q$; consequently, the direct sum in Proposition 3a.20 is a finite product and we have

$$H' \cong \prod_{p|q} \prod_{m(p)} \mathbb{Z}_p,$$

where each $m(p)$ is a natural number with $m(p) \leq d$. It also follows that L/L' is a (finite) direct product of finite cyclic p-groups (where, of course, $p \mid \mathrm{card}\, L/L'$), where there are at most d such cyclic p-groups for each p. Note that the above Haar measure on H is simply the product measure of the Haar measures on the groups $\mathbb{Z}_p$ by Lemma 4.39. In particular, we can use the methods from Chapter 3c to visualise the profinite groups H and H' (see the remark on p. 68).

Thus, the CPS corresponding to an LSS is given by

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \cong H' \times L/L' \cong \prod_{p|q} \prod_{m(p)} \mathbb{Z}_p \times L/L' \\ \cup & & \cup & & \cup \text{ dense} \\ L & \longleftrightarrow & \tilde{L} & \longleftrightarrow & L \end{array} \tag{6b.3}$$

where $\tilde{L}$ is the diagonal embedding of L. Moreover, it is easy to extend this CPS. Set

$$L_{\mathrm{ext}} = \bigcup_{m \geq 0} \boldsymbol{Q}^{-m} L \qquad \text{and} \qquad L'_{\mathrm{ext}} = \bigcup_{m \geq 0} \boldsymbol{Q}^{-m} L'.$$

Note that it might happen that some of the cyclic p-groups in L/L' might be embedded *via* $\mathbb{Z}_p \cong \mathsf{C}_{p^b} \times p^b\mathbb{Z}_p \cong \mathsf{C}_{p^b} \times \mathbb{Z}_p$ (compare Lemma 3a.19) into H' (respectively H'_{ext}) if $p \,|\, q$ (in one dimension $d = 1$, where $L/L' \cong \mathsf{C}_{p_1^{b_1}} \times \ldots \times \mathsf{C}_{p_\ell^{b_\ell}}$ with $p_i \neq p_j$ for $i \neq j$, this is always the case, wherefore $L_{\mathrm{ext}}/L'_{\mathrm{ext}} = \prod_{p_i \nmid q} \mathsf{C}_{p_i^{b_i}}$). Thus, $L_{\mathrm{ext}}/L'_{\mathrm{ext}}$ is a subgroup of L/L', and the extended internal space is given[1] by $H'_{\mathrm{ext}} \cong \prod_{p|q} \prod_{m(p)} \mathbb{Q}_p$, and the extended CPS (where $L_{\mathrm{ext}} \subset \mathbb{R}^d$ is also dense) is given by $(\mathbb{R}^d \,, \prod_{p|q} \prod_{m(p)} \mathbb{Q}_p \times L_{\mathrm{ext}}/L'_{\mathrm{ext}} \,, \tilde{L}_{\mathrm{ext}})$.

6b.2. Model Sets & Modular Coincidence

One might now establish that a given LSS is a model set. We observe the following easy facts:

- Since $\operatorname{supp} \underline{\Lambda} = L$, one has $\operatorname{cl}(\operatorname{supp} \underline{\Lambda})^\star = H$ (where the star-map is just the embedding $L \hookrightarrow H$).
- The set $\operatorname{cl}(x + \boldsymbol{Q}^k L')^\star$ (with $x \in L$) is an clopen ball in H, and any other ball is either disjoint from this ball or one is contained in the other.
- Consequently, if there is an algebraic coincidence $x + \boldsymbol{Q}^k L' \subset \Lambda_i$, then the clopen ball $\operatorname{cl}(x + \boldsymbol{Q}^k L')^\star$ is contained in $\operatorname{cl} \Lambda_i^\star$, which therefore contains interior points. Moreover, one has $\operatorname{cl}(x + \boldsymbol{Q}^k L')^\star \cap \operatorname{cl} \Lambda_j^\star = \varnothing$ for $j \neq i$, *i.e.*, these interior points are *exclusive*. This, after the following observation, is basically [235, Theorem 5.8] (respectively, [232, Lemma 4]).
- The IFS $\boldsymbol{\Theta}^\star$ satisfies the conditions of Proposition 4.99 (since we have the Haar measure μ given as in Equation (6b.2) on p. 306). Consequently, we have for its attractor $\underline{\Omega}$: All Ω_i have nonzero Haar measure (since $\bigcup_{i=1}^n \Omega_i = H$) and are perfect and regularly closed; moreover, their boundaries $\partial\Omega_i$ have zero Haar measure (obviously, $x \mapsto \boldsymbol{Q}x$ is not a homeomorphism on H, but the statements of Proposition 4.99 also hold in this setting; alternatively, one may embed H into H_{ext} where it is a homeomorphism). Moreover, the unions in the IFS $\boldsymbol{\Theta}^\star$ are disjoint. Note that this is [235, Theorem 5.7] (respectively, [232, Theorem 1]).

Since balls are clopen in ultrametric spaces, the boundary of regularly closed sets in such spaces consists usually of only a finite number of points, see the examples in the next section. Consequently, the calculation of the Hausdorff dimension in such spaces is rather boring: While each Ω_i has full dimension (*i.e.*, $\sum_{p|q} m(p)$), their boundaries $\partial\Omega_i$ have dimension 0.

We make the following observation, which (almost) establishes that an algebraic coincidence ensures the model set property of an LSS:

> If there is an algebraic coincidence and since $\operatorname{cl} \Lambda_i^\star = \Omega_i$, one has for the preimage[2] of an exclusive inner point $x \in \operatorname{int} \Omega_i \cap L$ that $x \in \Lambda_i \subset \Lambda(\Omega_i)$ and $x \notin \Lambda_j \subset \Lambda(\Omega_j)$ (for all $j \neq i$). Under admissibility, the tiling $\underline{\Lambda} + (P)_{i=1}^n$ has covering degree 1 inside $x + P$.

[1] One may simple view H'_{ext} as set of all two-sided sequences $a = (a_n)_{n=-\infty}^{+\infty}$ over the alphabet $\mathcal{A} = \{x \mid x$ is a representative of $L'/\boldsymbol{Q}L'\}$ such that there is a number $N = N(a) \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$. Thus, we may identify H'_{ext} (after properly defining an addition) with the the Abelian group of q-adic numbers (and H' with the q-adic integers), see [169, Definition 10.2].

[2] By bijectivity in the CPS, we identify this point x with its preimage.

In view of Proposition 5.144 we note the following

- Using the lattice $L = \mathbb{Z}^d$ in $\mathbb{R}^d$, we may assume that $P = [0,1[\times \ldots \times [0,1[= \{x \in \sum_{i=1}^d \gamma_i \cdot e_i \mid 0 \leq \gamma_i < 1\}$, where e_i are the canonical basis vectors. Consequently, we have $L \cap P = \{0\}$.

- Relative to the CPS $(H_{\text{ext}}, \mathbb{R}^d, \tilde{L}_{\text{ext}})$, the model set[3] $\Lambda(\underline{A})$ with $\underline{A} = (P)_{i=1}^n$ is repetitive: Let $F^\star$ be the (pre)image of a (finite) cluster $F \subset \Lambda(\underline{A})$. Then, there is always an $\varepsilon > 0$ such that $F^\star + t \subset P$ for every $t \in ([0,\varepsilon[\times \ldots [0,\varepsilon[) = \{x \in \sum_{i=1}^d \gamma_i \cdot e_i \mid 0 \leq \gamma_i < \varepsilon\} = C_{<\varepsilon}$ (the (half-open) cube of sidelength ε). Consequently, we have $x + F \subset \Lambda(\underline{A})$ for every $x \in \Lambda(C_{<\varepsilon})$. But $\Lambda(C_{<\varepsilon})$ is a relatively dense set by Lemma 5.8, hence – since the cluster was chosen to be arbitrary – $\Lambda(\underline{A})$ is repetitive.

- Moreover, with respect to $\boldsymbol{\Theta}^{\star\#}$, $\Lambda(\underline{A})$ is a substitution multi-component Delone set (although a rather boring one, since all components are equal), also compare Proposition 6.72.

- Now, inside (the ultrametric space) H_{ext}, the compact space H is simply the compact ball of radius 1 around 0 (we put the issue of the height groups L/L' and $L_{\text{ext}}/L'_{\text{ext}}$ aside; they only add some technicalities). Moreover, in each ball $x + H$ (where $x \in H_{\text{ext}}$), there is exactly one point of $\Lambda(P)$: By $L \cap P = \{0\}$, this is clear for H itself. Otherwise, it follows by the construction of H_{ext} as "Abelian group of $\boldsymbol{Q}$-adic numbers" (where H is the group of "$\boldsymbol{Q}$-adic integers") and the properties of the "$\boldsymbol{Q}$-adic" ultrametric.

- Thus, $\Lambda(P)$ is almost a lattice in H_{ext}: The set H (*i.e.*, the clopen unit ball) is both, a packing and a covering set of H_{ext} for $\Lambda(P)$, *i.e.*, $H \dot{+} \Lambda(P) = H_{\text{ext}}$ (so, H is something like a fundamental domain). But $\Lambda(P)$ is no subgroup of H_{ext} (take $x, y \in \Lambda(P)$, then $x^\star \pm y^\star$ is in general not an element of P).

Now, $\Lambda(\underline{A})$ is a repetitive substitution multi-component Delone set, wherefore $\Lambda(\underline{A}) + \underline{\Omega}$ is a self-replicating tiling of almost everywhere constant covering degree (by Lemma 5.79). If there is an exclusive inner points, then there is an open set where the covering degree is 1. But this is the case iff we have an algebraic coincidence, and Proposition 5.144 then establishes the following statement.

Proposition 6b.10. *Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be an admissible primitive LSS such that $\underline{\Lambda}$ is aperiodic, representable and every cluster of it is legal. Then, equivalent are:*

(i) *$\underline{\Lambda}$ is a regular multi-component model set.*

(ii) *There is an $1 \leq i \leq n$, an $x \in \mathbb{R}^d$ and a $k \in \mathbb{N}$ such that $x + \boldsymbol{Q}^k L' \subset \Lambda_i$.* □

In fact, one may drop the prerequisite "admissibility" in this statement, since it is enough that each component in $\underline{A}$, the attractor of $\boldsymbol{\Theta}^{\#}$, is some fundamental domain of L.

Usually, one does not use the algebraic coincidence, but the so-called modular coincidence ([232, 235]) respectively the coincidence introduced by Dekking ([101]) in the one-dimensional

[3] To our knowledge, the first appearance of a model set $\Lambda([0,1[)$ relative to the CPS $(\mathbb{Q}_p, \mathbb{R}, \tilde{\mathcal{L}})$, where $\tilde{\mathcal{L}}$ denotes the diagonal embedding of $\mathcal{L} = \mathbb{Z}[\frac{1}{p}] = \bigcup_{m=0}^{\infty} \frac{1}{p^m}\mathbb{Z}$, can be found in [332–334, Prop. 1] (*sic*, always Prop. 1). Higher dimensional generalisations may be found in [335–337]. Also see [256, Prop. II.9].

case to decide if an LSS is a model set or not. Basically, they are a reformulation of the algebraic coincidence making explicit use of the EMFS $\boldsymbol{\Theta}$. However, the formulation of modular coincidence is rather technical.

Definition 6b.11. Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a (primitive) LSS and define L and L' as before. For $x \in L$, let

$$\begin{aligned}\Theta_{ij}[x] &= \{f \in \Theta_{ij} \mid f(y) \equiv x \bmod \boldsymbol{Q}L', \text{ where } \Lambda_j \subset y + L'\}\\ &= \{f \in \Theta_{ij} \mid f(\Lambda_j) \subset x + \boldsymbol{Q}L'\}.\end{aligned}$$

Furthermore, let

$$\Theta[x] = \bigcup_{1 \le i,j \le m} \Theta_{ij}[x].$$

We say that $(\underline{\Lambda}, \boldsymbol{\Theta})$ admits a *modular coincidence relative to* $\boldsymbol{Q}L'$, if $\Theta[x]$ is contained entirely in one row of $\boldsymbol{\Theta}$ for some $x \in L$, *i.e.*, if $(x + \boldsymbol{Q}L') \subset \Lambda_i$ for some $1 \le i \le m$.

Examples how to compute modular coincidences may be found in [232, Sections 5 & 7], [235, Example 5.14] and [138, Example 6.4]. The formulation gets considerably easier, if one only considers admissible LSS. In one dimension, this is criterion stated in [101] – we call it "Dekking coincidence"– which can be generalised to higher dimensions, see [138].

Definition 6b.12. Let σ be a substitution of constant length q on $L = \mathbb{Z}$. Suppose, $L' = r \cdot \mathbb{Z}$. We call $h = \max\{n \in \mathbb{N} \mid \gcd(n, q) = 1 \text{ and } n|r\}$ the *height* of the substitution. If $h = 1$, then we say that σ (respectively, its fixed point u) admits a *Dekking coincidence* if there exist a $k \in \mathbb{N}$ and a $0 \le j < q^k$ such that the j-th symbol of $\sigma^k(i)$ is the same for all $i \in \mathcal{A}$. If $h > 1$, define a new substitution $\tilde{\sigma}$ as follows: Take as new alphabet the (finite) set of all words $\{u_{m \cdot h} \cdots u_{m \cdot h + h - 1} \mid m \in \mathbb{Z}\}$ (these are the words of length h, which start at position $m \cdot h$ in u). Let $\tilde{\sigma}$ be the induced substitution by σ on this new alphabet. Then, $\tilde{\sigma}$ has constant length q and height $h = 1$. Consequently, for $h > 1$, we say that σ has a *Dekking coincidence* if $\tilde{\sigma}$ has a Dekking coincidence.

Similarly, for an (admissible) LSS $(\underline{\Lambda}, \boldsymbol{\Theta})$ we define the following subsets of $\mathcal{A}$:

$$\Psi_k[x] = \{i \in \mathcal{A} \mid (x + \boldsymbol{Q}^k L') \cap \Lambda_i \neq \varnothing\}$$

We say that the LSS has a *generalised Dekking coincidence* if[4] there is a $k \in \mathbb{N}$ and an $x \in L$ such that $\operatorname{card} \Psi_k[x] = 1$.

The convenient property of these coincidences is that they can easily be read off from the substitution σ respectively the EMFS $\boldsymbol{\Theta}$ (in the admissible case), as the following examples show.

Example 6b.13. Let σ be the following constant length substitution: $a \mapsto aba$, $b \mapsto bcc$ and $c \mapsto abc$. Then, since there are two subsequent c's in $\sigma(b)$, we have height $h = 1$ and we have

$$\begin{array}{ccccc} a & \overset{\sigma}{\mapsto} & aba & \overset{\sigma}{\mapsto} & aba\ bc\boldsymbol{c}\ \boldsymbol{ab}a \\ b & \overset{\sigma}{\mapsto} & bcc & \overset{\sigma}{\mapsto} & bcc\ ab\boldsymbol{c}\ \boldsymbol{ab}c \\ c & \overset{\sigma}{\mapsto} & abc & \overset{\sigma}{\mapsto} & abc\ bc\boldsymbol{c}\ \boldsymbol{ab}c \end{array}$$

[4] Note that the fundamental domain P' of L' is given by the (disjoint) union of $\operatorname{card} L/L'$ many fundamental domains of L. Moreover, $\boldsymbol{Q}^k P' + \boldsymbol{Q}^k L'$ is a tiling of $\mathbb{R}^d$ and $\boldsymbol{Q}^k P' \cap L$ a system of representatives of $L/\boldsymbol{Q}^k L'$. In the admissible case, such a system of representatives coincides with the set of all translational parts of the maps in $\boldsymbol{\Theta}^k$. Thus, the maps in $\boldsymbol{\Theta}^k$ are in close relation with the representatives x used in $\Psi_k[x]$, also see the following example.

wherefore the 6th (respectively the 7th, respectively the 8th) symbol of $\sigma^2(i)$ is the same for all $i \in \mathcal{A}$.

Similarly, for an LSS on $\mathbb{Z}^2$ with $\boldsymbol{Q} = \mathrm{diag}(r,s)$, we may use the notation

$$i \quad\mapsto\quad \begin{array}{ccccc} j_{0,s} & j_{1,s} & j_{2,s} & \cdots & j_{r-1,s} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ j_{0,1} & j_{1,1} & j_{2,1} & \cdots & j_{r-1,1} \\ j_{0,0} & j_{1,0} & j_{2,0} & \cdots & j_{r-1,0} \end{array}$$

if $f(x) = \boldsymbol{Q}x + (k,\ell)^t \in \Theta_{j_{k,\ell}i}$.

Now, consider the LSS given implicitly by

$$a \mapsto \begin{array}{cc} d & a \\ a & b \end{array}, \qquad b \mapsto \begin{array}{cc} b & c \\ a & b \end{array}, \qquad c \mapsto \begin{array}{cc} d & c \\ c & b \end{array}, \qquad d \mapsto \begin{array}{cc} d & c \\ a & d \end{array}.$$

Although one has $L' = \{(x,y) \in L \mid x \equiv y \bmod 2\}$ (the "checkerboard" sublattice (of index 2) of $\mathbb{Z}^2$), we even have $x + \boldsymbol{Q}^2 L \subset \Lambda_i$ for some x and i, as the following establishes (note that here one has $L_{\mathrm{ext}} = L'_{\mathrm{ext}}$):

$$a \;\mapsto\; \begin{array}{cc} d & a \\ a & b \end{array} \;\mapsto\; \begin{array}{cccc} d & \boldsymbol{c} & \boldsymbol{d} & a \\ \boldsymbol{a} & d & a & \boldsymbol{b} \\ \boldsymbol{d} & a & b & \boldsymbol{c} \\ a & \boldsymbol{b} & \boldsymbol{a} & b \end{array} \qquad b \;\mapsto\; \begin{array}{cc} b & c \\ a & b \end{array} \;\mapsto\; \begin{array}{cccc} b & \boldsymbol{c} & \boldsymbol{d} & c \\ \boldsymbol{a} & b & c & \boldsymbol{b} \\ \boldsymbol{d} & a & b & \boldsymbol{c} \\ a & \boldsymbol{b} & \boldsymbol{a} & b \end{array}$$

$$c \;\mapsto\; \begin{array}{cc} d & c \\ c & b \end{array} \;\mapsto\; \begin{array}{cccc} d & \boldsymbol{c} & \boldsymbol{d} & c \\ \boldsymbol{a} & d & c & \boldsymbol{b} \\ \boldsymbol{d} & c & b & \boldsymbol{c} \\ c & \boldsymbol{b} & \boldsymbol{a} & b \end{array} \qquad d \;\mapsto\; \begin{array}{cc} d & c \\ a & d \end{array} \;\mapsto\; \begin{array}{cccc} d & \boldsymbol{c} & \boldsymbol{d} & c \\ \boldsymbol{a} & d & c & \boldsymbol{b} \\ \boldsymbol{d} & a & d & \boldsymbol{c} \\ a & \boldsymbol{b} & \boldsymbol{a} & d \end{array}$$

Consequently, this (implicit) EMFS yields an LSS which is a (multi-component) model set.

Remark 6b.14. This last example indicates that under certain circumstances one does not have to care much about L' (respectively L/L') if one reads off the coincidences from the EMFS. In one-dimension, this is the case iff the height $h = 1$ (although one might have $L' = r \cdot \mathbb{Z}$ with $r > 1$). In that case, the substitution rule already lines up with the coset structure of L/L' in such a way, that the above procedure ensures the existence of an algebraic coincidence, see the remarks in [138, Appendix A]. A higher dimensional equivalent of this applies to the above two-dimensional example.

Let us have a closer look at the one-dimensional case: We set $L = \mathbb{Z}$ and $L' = r \cdot \mathbb{Z}$ with $r \geq 1$. Let q be the length of the substitution (*i.e.*, the inflation factor). Then, we may define "extended" versions of L and L' as before by $L_{\mathrm{ext}} = \bigcup_{k \geq 0} L/q^k = \bigcup_{k \geq 0} \mathbb{Z}/q^k$ and $L'_{\mathrm{ext}} = \bigcup_{k \geq 0} L'/q^k = \bigcup_{k \geq 0} r\,\mathbb{Z}/q^k$. But then, it is easy to see that the factor groups L/L' and $L_{\mathrm{ext}}/L'_{\mathrm{ext}}$ are isomorphic iff r and q are relatively prime. But then, one has $h = r$ and $L/L' \cong \mathsf{C}_r$. Moreover, with this interpretation one now establishes the following interpretation of the height:

> The height h of a substitution of constant length is simply the cardinality of the "extended" height group $L_{\mathrm{ext}}/L'_{\mathrm{ext}}$ (and one has $L_{\mathrm{ext}}/L'_{\mathrm{ext}} \cong \mathsf{C}_h$).

This is easily proven by looking at the factorisations of q and r. We look at the following examples (which indicate how to prove this claim):

- Chose $r = 2$ and $q = 6$ (and therefore $h = 1$). Then, one has $L_{\text{ext}} = \bigcup_{k \geq 0} \mathbb{Z}/6^k$ and $L'_{\text{ext}} = \bigcup_{k \geq 0} 2\,\mathbb{Z}/6^k = L_{\text{ext}}$ (the last equality follows since $\bigcup_{k=0}^{N} \mathbb{Z}/6^k \subset \bigcup_{k=0}^{N+1} 2\,\mathbb{Z}/6^k \subset \bigcup_{k=0}^{N+1} \mathbb{Z}/6^k$ for all $N \in \mathbb{N}$).
- Chose $r = 2$ and $q = 3$ (thus, $h = 2$). Then, one has $L_{\text{ext}} = \bigcup_{k \geq 0} \mathbb{Z}/3^k$ and $L'_{\text{ext}} = \bigcup_{k \geq 0} 2\,\mathbb{Z}/3^k = 2 \bigcup_{k \geq 0} \mathbb{Z}/3^k = 2\, L_{\text{ext}}$.
- Chose $r = 6$ and $q = 10$ (wherefore, $h = 3$). Then, one has $L_{\text{ext}} = \bigcup_{k \geq 0} \mathbb{Z}/10^k$ and $L'_{\text{ext}} = \bigcup_{k \geq 0} 6\,\mathbb{Z}/10^k = 3 \bigcup_{k \geq 0} 2\,\mathbb{Z}/10^k$, where we observe that $\bigcup_{k=0}^{N} \mathbb{Z}/10^k \subset 2 \bigcup_{k=0}^{N+1} \mathbb{Z}/10^k \subset \bigcup_{k=0}^{N+1} \mathbb{Z}/10^k$ for all $N \in \mathbb{N}$. Therefore, one has $L'_{\text{ext}} = 3\,\mathcal{L}_{\text{ext}}$.

More examples using the Dekking coincidence can be found in [101], [299], [298] and references therein for substitutions of constant length. Examples in the higher dimensional case, using the generalised criterion, can be found in [138] (also compare [131, 132]). There, also a graph-theoretic formulation of the generalised Dekking coincidence is established, and the connections to the overlap coincidence are established. Moreover, one can even establish that one only has to check σ^k respectively $\boldsymbol{\Theta}^k$ up to a certain k to obtain a generalised Dekking coincidence.

Proposition 6b.15. [138, Theorem 4.5] *Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be an admissible primitive LSS with $\operatorname{card} \mathcal{A} = m > 2$. If $\boldsymbol{\Theta}^k$ admits no modular coincidence (respectively no generalised Dekking coincidence) for any $k \leq 2^m - m - 2$, then $\boldsymbol{\Theta}^k$ admits no modular coincidence (and therefore also no algebraic coincidence) for any $k \in \mathbb{N}$.* □

If an LSS has no algebraic coincidence, then the associated tiling of H_{ext} has covering degree strictly greater than 1 (almost everywhere), wherefore one does not have a multi-component model set. Consequently, one has the following statement, see [235, Theorem 5.12] (compare [232, Theorem 3]).

Theorem 6b.16. *Let $(\underline{\Lambda}, \boldsymbol{\Theta})$ be a (primitive) LSS such that $\underline{\Lambda}$ is aperiodic, representable and every cluster of it is legal. Set $L = \operatorname{supp} \underline{\Lambda}$ and $L' = \langle \bigcup_{i=1}^{n} \Lambda_i - \Lambda_i \rangle_{\mathbb{Z}}$. The following are equivalent:*

(i) *$\underline{\Lambda}$ is a regular multi-component model set within the CPS of Equation* (6b.3) *on p. 306.*

(ii) *$\underline{\Lambda}$ admits an algebraic coincidence, respectively a modular coincidence, respectively, in its admissible version, a (generalised) Dekking coincidence.* □

We note that this theorem is the generalisation of a corresponding theorem about constant length substitutions. Reformulating [101, Theorem II.13 & Theorem III.7] (compare [299, Theorem VI.13 & Theorem VI.24] and [298, Theorem 7.3.1 & Theorem 7.3.6]), the corresponding statement for constant length substitutions reads as follows.

Lemma 6b.17. *Let σ be a substitution of constant length q and height h. Let u be the sequence generated by σ. Then u is a regular model set iff σ admits a Dekking coincidence. The internal space H is given by $H = \mathbb{Z}_q \times \mathsf{C}_h$, where $\mathbb{Z}_q$ denotes the product over the distinct primes p dividing q of the p-adic integers $\mathbb{Z}_p$ and C_h is the cyclic group of order h.* □

6b.3. Examples and Further Considerations

Example 6b.18. We have already introduced the Thue-Morse sequence(s) in Example 5.64. A Thue-Morse sequence is generated by the constant length substitution $a \mapsto ab$ and $b \mapsto ba$. It follows immediately that there is no Dekking coincidence. Alternatively, we observe the following: We have already remarked that the symbol on a position $x \in \mathbb{Z}$ is simply determined by the binary expansion $x = \sum_{k=0}^{N} a_k \cdot 2^k$ (respectively $x = -\sum_{k=0}^{N} a_k \cdot 2^k$ if $x < 0$). Moreover, it follows that if there is an a (respectively a b) on x, then there is a b (respectively an a) on $x+2^{N+1}$ (respectively $x-2^{N+1}$ if $x < 0$). But this establishes that, for every $z \in H = \mathbb{Z}_2$, there are points of $\varLambda_a^\star$ and $\varLambda_b^\star$ in every neighbourhood of z. Consequently, we have $\mathrm{cl}\,\varLambda_a^\star = \mathbb{Z}_2 = \mathrm{cl}\,\varLambda_b^\star$, *i.e.*, the covering degree of the tiling in internal space H respectively H_{ext} is 2 everywhere.

A similar example is the *Rudin-Shapiro sequence* (see [105, Section 1.6] and [11, Example I.4.3]) generated by the constant length $a \mapsto ac$, $b \mapsto dc$, $c \mapsto ab$ and $d \mapsto db$. One easily establishes that it has no Dekking coincidence (note that $L' = 2L$ but the height $h = 1$). The internal space is given by $H = 2\mathbb{Z}_2 \times \mathsf{C}_2 \cong \mathbb{Z}_2$, and one easily establishes that $\mathrm{cl}\,\varLambda_a^\star = 2\mathbb{Z}_2 = \mathrm{cl}\,\varLambda_d^\star$ and $\mathrm{cl}\,\varLambda_b^\star = 2\mathbb{Z}_2 + 1 = \mathrm{cl}\,\varLambda_c^\star$, wherefore the covering degree of the associated tiling in internal space is again 2 everywhere.

Example 6b.19. Some examples of substitutions of constant length that admit a Dekking coincidence are:

- The period-doubling sequence $a \mapsto ab$ and $b \mapsto aa$, see Example 5.64. One has $H = \mathbb{Z}_2$.
- The paperfolding sequence $a \mapsto ab$, $b \mapsto cb$, $c \mapsto ad$ and $d \mapsto cd$, see Example 5.158. One has $H = 2\mathbb{Z}_2 \times \mathsf{C}_2 \cong \mathbb{Z}_2$.
- The substitution $a \mapsto aba$, $b \mapsto bcc$ and $c \mapsto abc$ given in Example 6b.13. One has $H = \mathbb{Z}_3$. This substitution is related to the so-called Kolakoski$(4, 2)$ sequence, see [352, 353]. We note that there is a whole family of Kolakoski sequences which provide examples for substitutions of constant length, see [353] and [352, Section 2.2].
- The substitution $a \mapsto aba$, $b \mapsto cab$ and $c \mapsto bac$ given in [101, Example after Definition II.20] (also see [298, Exercises 7.3.3 & 7.3.7]). Note that this substitution has height $h = 2$, wherefore $H = \mathbb{Z}_3 \times \mathsf{C}_2$. Its associated height $h = 1$ substitution is given by $a \mapsto aab$ and $b \mapsto aba$.
- The substitution $a \mapsto abc$, $b \mapsto dcb$, $c \mapsto cda$ and $d \mapsto dab$ given in [235, Example 5.14]. This is again an example of height $h = 2$, wherefore $H = \mathbb{Z}_3 \times \mathsf{C}_2$.

With the methods from Chapter 3c, we visualise the corresponding attractors $\underline{\varOmega}$ of $\boldsymbol{\varTheta}^\star$ in the respective internal space in Figure 6b.1. Note that with some practise one might read off the first terms of equations like $\varLambda_b = \bigcup_{m \geq 1} (2^{m+2}\mathbb{Z} + 2^m - 1)$ in case of the paperfolding sequence (see Equation (5.22) on p. 197) from these pictures. Moreover, one can also (often) read off the boundary points of the sets $\varOmega_i$ (where, consequently, the covering degree equals not 1) from these pictures, *e.g.*, $\partial\varOmega_a = \{-1\} = \partial\varOmega_b$ in case of the period-doubling sequence.

Example 6b.20. The *chair tiling* and the *table tiling* (also called the *domino tiling*) and their spectral properties are discussed in detail in [364], [315] and [39]. They are generated by the following tile substitutions (on the left hand side the chair tile substitution, on the right hand side the table tile substitution):

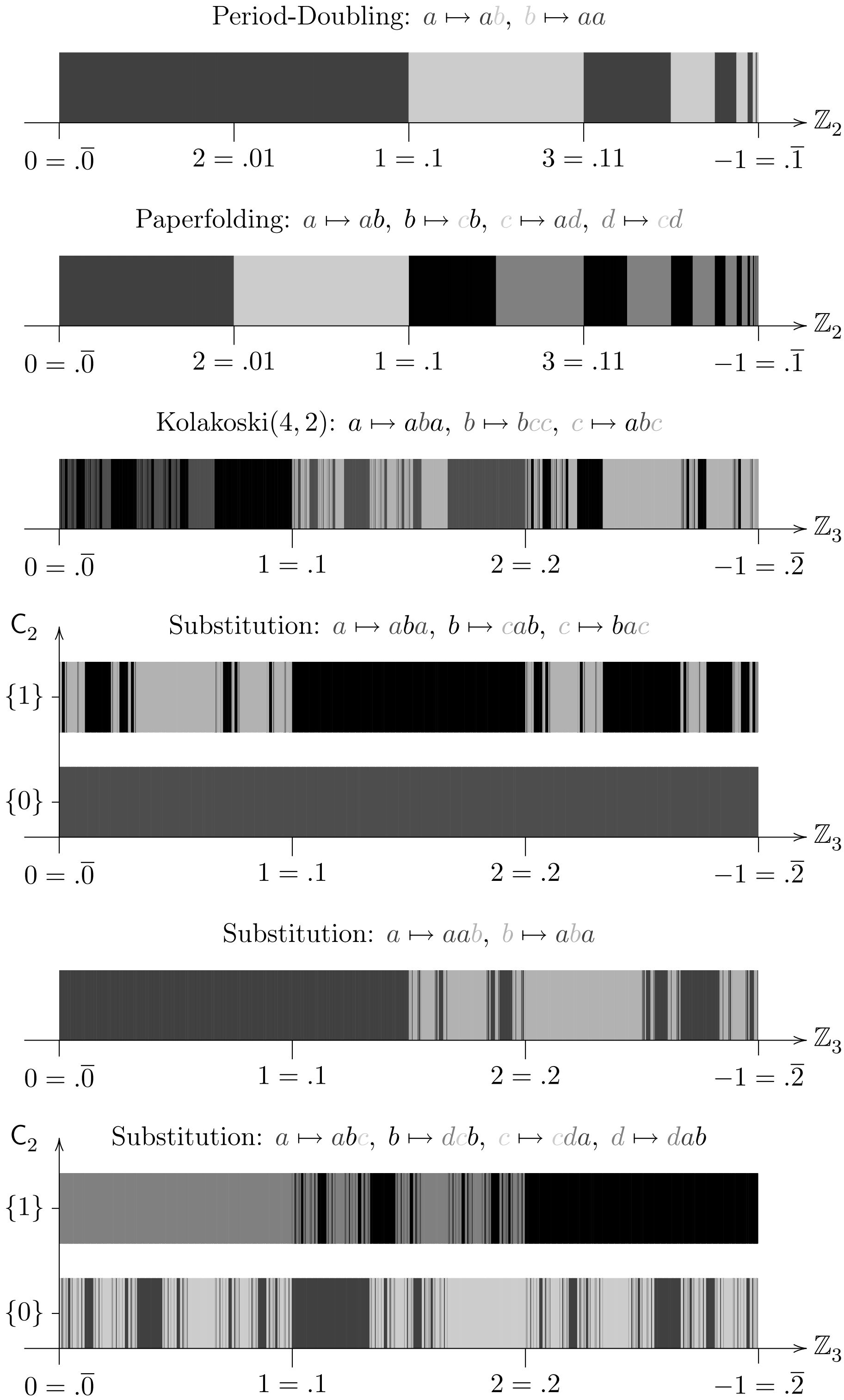
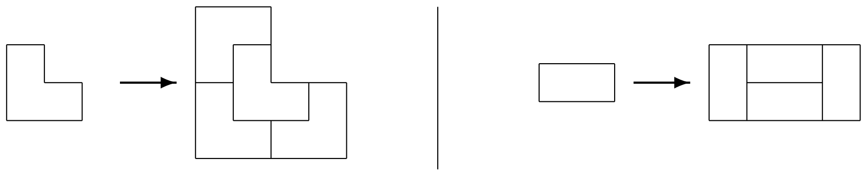
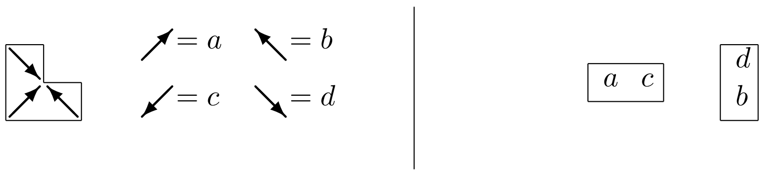


Figure 6b.1.: Visualisation of the windows of the constant length substitutions given in Example 6b.19. We use as colouring for a window $\Omega_i \subset H$ the same gray level as indicated in the corresponding symbolical substitution.



The expansion $\boldsymbol{Q}$ is given by $\boldsymbol{Q} = \mathrm{diag}(2,2)$, and the rotated versions of the shown tiles are substituted analogously. Both tile substitutions can be described by substitutions over the alphabet $\{a,b,c,d\}$ on $L = \mathbb{Z}^2$, where one uses the following labellings.



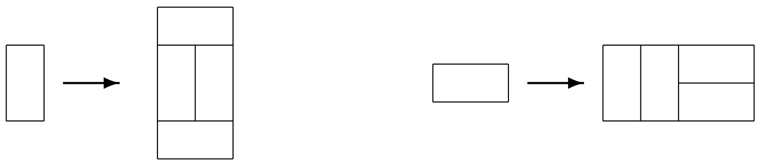
This yields the following EMFS for an admissible LSS:

$$a \mapsto \begin{matrix} d & a \\ a & b \end{matrix} \qquad b \mapsto \begin{matrix} b & c \\ a & b \end{matrix} \qquad \Big| \qquad a \mapsto \begin{matrix} d & a \\ b & a \end{matrix} \qquad b \mapsto \begin{matrix} b & b \\ a & c \end{matrix}$$

$$c \mapsto \begin{matrix} d & c \\ c & b \end{matrix} \qquad d \mapsto \begin{matrix} d & c \\ a & d \end{matrix} \qquad \Big| \qquad c \mapsto \begin{matrix} c & d \\ c & b \end{matrix} \qquad d \mapsto \begin{matrix} a & c \\ d & d \end{matrix}$$

The substitutions for the chair tiling are the ones of the two-dimensional example in Example 6b.13, wherefore it can be described as multi-component model set. For the table tiling one checks (see [138, Section 6.7]) that there is no modular/algebraic coincidence, wherefore it cannot[5] be described as multi-component model set. These findings for the chair and table tiling have originally been proven in [315, Theorems 8.1 & 8.3] respectively [364, Examples 7.1 & 7.3], also see [316, Section 8.4]). We also show the diffraction pattern of the chair tiling in Figure 6b.4.

However, there is a modified version of the table substitution which admits a coincidence[6], see [315, Section 13].



Thus, one uses different substitutions for the two (rotated) versions of the table. With the

[5]In fact, since $\mathrm{card}\, \Psi_k[x] = 4 = \mathrm{card}\{a,b,c,d\}$ for any $k \in \mathbb{N}$ and any $x \in L$, one immediately has that the covering degree of the associated tiling in internal space is 4 and all components of the attractor of the IFS $\boldsymbol{\Theta}^\star$ are given by $\Omega_i = H = \mathbb{Z}_2^2$.

[6]Another example, with the same tiles as the table substitution but with $\boldsymbol{Q} = \mathrm{diag}(2,3)$ and which also admits a coincidence, is shown in [316, Fig. 5] and given by the symbolic substitutions

$$a,b \mapsto \begin{matrix} d & a & c \\ b & a & c \end{matrix} \quad \text{and} \quad c,d \mapsto \begin{matrix} a & c & d \\ a & c & b \end{matrix}.$$

Consequently, the internal space is here given by $H = \mathbb{Z}_2 \times \mathbb{Z}_3$. Obviously, one can easily find more such "table tilings", even three-dimensional examples (see [316, Fig. 5]).

same labelling as above, the admissible substitution now reads as follows.

$$a \mapsto \begin{matrix} d & d \\ b & b \end{matrix} \qquad b \mapsto \begin{matrix} b & b \\ a & c \end{matrix} \qquad c \mapsto \begin{matrix} a & c \\ a & c \end{matrix} \qquad d \mapsto \begin{matrix} a & c \\ d & d \end{matrix}$$

The chair, table and the modified table tiling are shown in Figure 6b.2. The internal space for both the chair tiling and the modified table tiling is given by $H = \mathbb{Z}_2^2$. Consequently, we can also visualise the windows for these multi-component model sets. This is done in Figure 6b.3. Note that the structure of the windows for the modified table tiling is more complicated than that for the chair tiling (we note that for the chair tiling there is a (more precisely, eight different) modular coincidence relative to $\boldsymbol{Q}^2L$, while[7] the first (more precisely, there are eight different) modular coincidence for the modified table tiling occurs relative to $\boldsymbol{Q}^3L$).

Remark 6b.21. For a given LSS (or a lattice tiling), it can be quite laborious to check for coincidences respectively non-coincidences; see [232, Section 5] for the sphinx tiling, which is a multi-component model set, and [138, Section 6.9] for the semi-detached house tiling, which is not a model set.

We also note that (the direction of) the eigenvectors respectively (the direction of) the principle semiaxes (see Remark 4.110) of $\boldsymbol{Q}$ do in general not yield a basis of the underlying lattice L as the example

$$\boldsymbol{Q} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$$

on $L = \mathbb{Z}^2$ shows [133]. The eigenvectors (and the direction of the principle semiaxes) are given by $(\tau\,, 1)^t$ and $(-\frac{1}{\tau}\,, 1)^t$ where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Obviously, this (or any multiple thereof) is not a basis of $\mathbb{Z}^2$. Still, any point $x \in \mathbb{Z}^2$ admits a $\boldsymbol{Q}$-adic expansion $x = (a_1, a_2, a_3, \ldots)$ with $a_i \in \{(0\,,0)^t, (1\,,0)^t, (0\,,1)^t, (1\,,1)^t\}$ and $x \in a_1 + \boldsymbol{Q}L$, $x - a_1 \in \boldsymbol{Q}(a_2 + \boldsymbol{Q}L)$, $x - a_1 - \boldsymbol{Q}a_2 \in \boldsymbol{Q}^2(a_3 + \boldsymbol{Q}L)$, *etc.*

The definition of an LSS (see Definition 6b.1) is very restrictive in the sense that $\operatorname{supp} \underline{\varLambda}$ has to be a lattice. With our methods, one can relax this condition a little, as the following example show.

Example 6b.22. Let σ be the (non-constant length) substitution $a \mapsto aabb$ and $b \mapsto ab$. The lengths of the natural intervals for σ are given by $\ell_a = 2$ and $\ell_b = 1$, wherefore the fixed point $u = \ldots aabbab\dot{a}abbaabb \ldots$ in its representation with natural intervals yields sets $\varLambda_a$ and $\varLambda_b$ with $\varLambda_a \cup \varLambda_b \subset \mathbb{Z}$ (and their union is a proper subset of $\mathbb{Z}$). The PF-eigenvalue of $\boldsymbol{S}\sigma$ is 3, wherefore one expects for the internal spaces that $H = H' = \mathbb{Z}_3$ respectively $H_{\text{ext}} = H'_{\text{ext}} = \mathbb{Q}_3$ (where we have used that $L = \mathbb{Z} = L'$). We still might look at the corresponding tiling $\Lambda(\underline{A}) + \underline{\varOmega}$ in H_{ext}. However, contrary to our previous considerations, we now have $A_i = [0\,, \ell_i[$, *i.e.*, $A_a = [0\,,2[$ but $A_b = [0\,,1[$ (again, one uses half-open intervals to ensure that $\Lambda(\underline{A})$ is a repetitive substitution multi-component Delone set). Still, it is enough to consider (one of) the balls $x + H$ in H_{ext}: there are always one point arising from $\Lambda(A_b)$ but two points arising from $\Lambda(A_a)$ in each such ball. In particular, one has $\Lambda(A_b) \cap H = \{0\}$

[7] This is also the reason, why the visualisation of the windows for the sphinx tiling [149] is not helpful: For the sphinx tiling, there are modular coincidences relative to $\boldsymbol{Q}^8L'$ (see [232, p. 190]), which correspond to balls of radius $2^{-8} = 1/256$ in the internal space H. Thus, one would need a good magnification and a very good resolution (*i.e.*, many points) to judge from such a visualisation that one has a model set. An opposite (two dimensional, also on the hexagonal lattice like the sphinx tiling) example, which already shows a coincidence relative to $\boldsymbol{Q}L'$, is the half-hex tiling, see [135, Section 4.1].

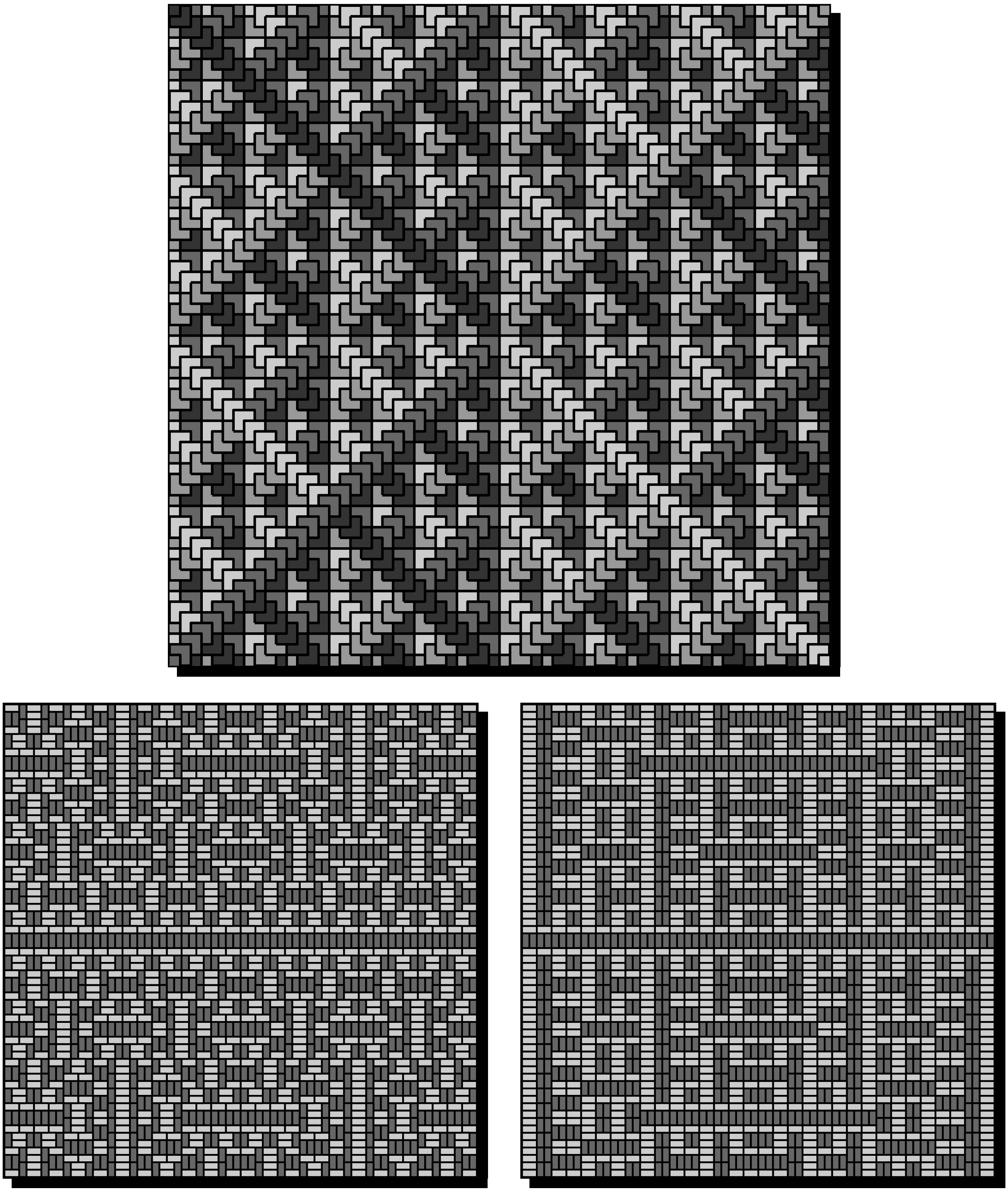


Figure 6b.2.: The chair tiling, the table tiling (lower left) and the modified table tiling (lower right). The tiles are coloured according to their orientation.

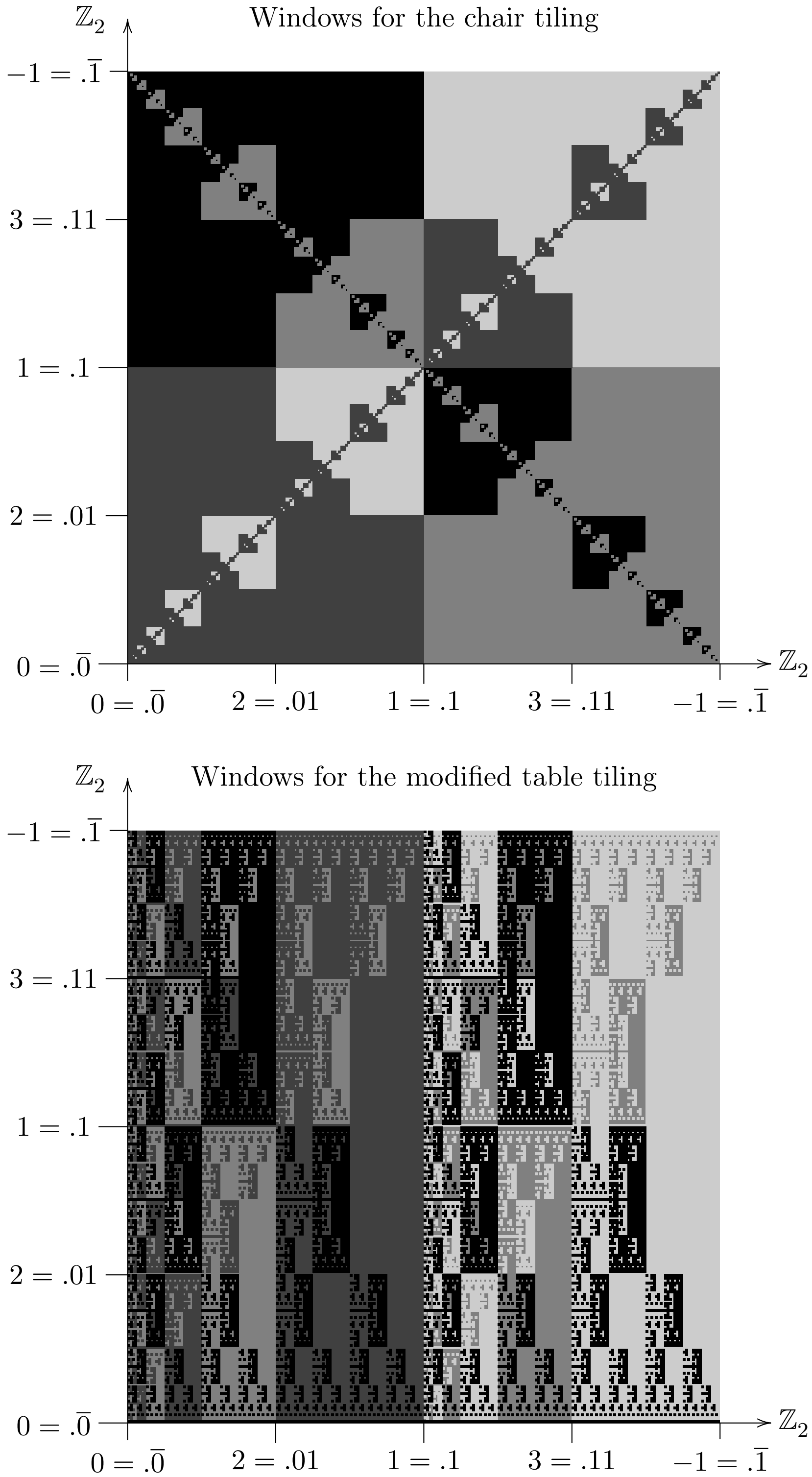
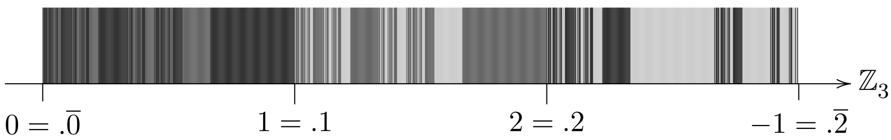


Figure 6b.3.: Visualisation of the windows for the chair tiling and the modified table tiling. The internal space is $H = \mathbb{Z}_2^2$. With the labelling $\{a, b, c, d\}$ as on p. 314, the colouring is chosen as follows: Ω_a in dark gray, Ω_b in medium gray, Ω_c in light gray and Ω_d in black.

and $\Lambda(A_a) \cap H = \{0, 1 = .1\}$. Again, we visualise this situation and obtain the following:



where the following gray levels are used: Ω_a, $\Omega_a + 1 = \Omega_a + .1$ and Ω_b. Consequently, the covering degree of this tiling is again 1 almost everywhere (and therefore we have a multi-component model set). Moreover, one easily checks that there are algebraic coincidences (*e.g.*, $3^2\mathbb{Z} + 6 \subset \varLambda_a$ and $3^2\mathbb{Z} + 5 \subset \varLambda_b$).

We also note that, in this example, one might also replace the letter a by two letters $a_1 a_2$ and derive a substitution of constant length 3 on the alphabet $\{a_1, a_2, b\}$ (and therefore a proper LSS). In fact, with the identifications $a_1 \to a$, $a_2 \to b$ and $b \to c$, this simply yields the Kolakoski$(4, 2)$ substitution, see Example 6b.19. Also observe that the visualisation for the tiling/windows in H are the same under this identification (compare Figure 6b.1).

The previous is an example of the following statement, also compare [258, Section III.].

Lemma 6b.23. [101, Theorem V.1] *Let σ be a primitive substitution of non-constant length $\ell = (\#\sigma(i))_{i=1}^n$ (i.e., there is at least one pair i, j such that $\ell_i \neq \ell_j$), where $\mathcal{A} = \{1, \ldots, n\}$. If ℓ is a left eigenvector of $\boldsymbol{S}\sigma$, then, given any fixed point u of σ, the dynamical system $(\mathbb{X}(u)\,, S)$ (where S is the shift map) is topologically conjugate to a topological dynamical system generated by a substitution of constant length.* □

Obviously, one can think of higher dimensional generalisations of this statement. But they get more technical, wherefore we will not do so here. But we remark that the previous lemma actually also involves a statement about (the dynamical system of) so-called "deformed model sets", since – classically – the sequence u is always thought of as being realised on $\mathbb{Z}$ (*i.e.*, with intervals $[0\,, 1[$, and not necessarily the natural intervals), also see Definition 7.36.

Remark 6b.24. Since we have $H \cong \prod_{p|q} \prod_{m(p)} \mathbb{Z}_p \times L/L'$ (see Equation (6b.3) on p. 306), one can easily calculate its character group using Lemma 3.119 respectively Proposition 3a.21 as

$$H^* \cong \prod_{p|q} \prod_{m(p)} \mathsf{C}_{p^\infty} \times L/L'$$

(recall that any finite Abelian group F – as product of cyclic groups – is self-dual $F^* \cong F$, see [169, Example 23.27(d)]). However, we note that H_{ext} is self-dual, but these observations are consistent: One may interpret H^* as fractional part and H as integer part of H_{ext}. Thus, in view of Proposition 5a.11 and the Folklore Theorem 5a.10, the situation can be interpreted as follows: Using the (compact) internal space H to calculate the diffraction pattern of a LSS which admits an algebraic coincidence, and the above H^*, one basically calculates the diffraction pattern on the d-dimensional torus $\mathbb{R}^d/L^\perp \cong \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$, *i.e.*, only on a fundamental domain of the dual lattice $L^\perp$ (by Proposition 5a.11, the diffraction pattern is periodic with $L^\perp$ as lattice of periods). On the other hand, if one uses H_{ext} to calculate the diffraction pattern, then the periodicity of the diffraction pattern with $L^\perp$ might at first sight not be obvious, but is easily established using the Fourier transform of balls inside (the compact unit ball) H (more precisely, inside H_{ext}), see Lemma 5a.12 and compare with the calculations for the visible lattice points following that lemma. Also compare with the diffraction pattern for the chair tiling in Figure 6b.4.

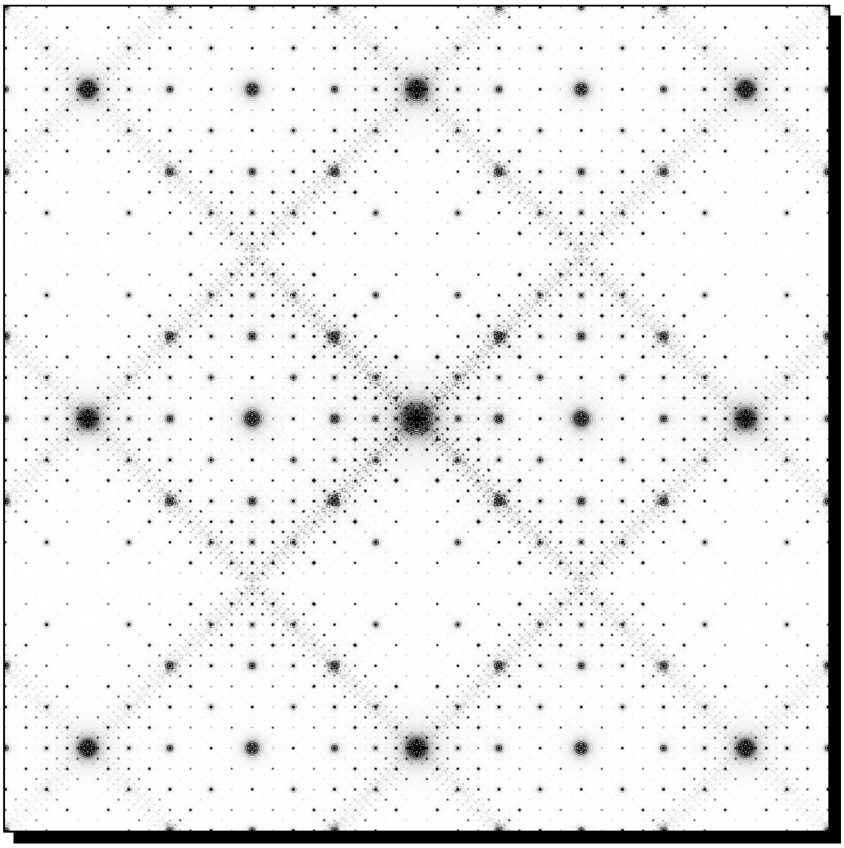


Figure 6b.4.: The part $[-1.25\,,1.25]\times[-1.25\,,1.25]$ of the diffraction pattern of the chair tiling. More precisely, the diffraction pattern of the a and b points in $\mathbb{Z}^2$ is shown. Numerically calculated from a patch of radius 100 on a 901×901-grid using DISCUS [296]. Observe that the intensity – the Bragg peaks – is (up to numerics) concentrated on $\mathbb{Z}[\frac{1}{2}]\times\mathbb{Z}[\frac{1}{2}]$ (which – by the $(\mathbb{Z}^2)^{\perp}=\mathbb{Z}^2$-periodicity of the diffraction pattern according to Proposition 5a.11 – we may also interpret as being concentrated on $\mathsf{C}_{2^\infty}\times\mathsf{C}_{2^\infty}\cong(\mathbb{Z}[\frac{1}{2}]\times\mathbb{Z}[\frac{1}{2}])/(\mathbb{Z}\times\mathbb{Z})$). Also note that the "checkerboard" $L'=\{(x\,,y)\in\mathbb{Z}^2=L\mid x\equiv y \bmod 2\}$ manifests itself in the diffraction pattern as "diagonal structure".

6c. Additional Topics

> Die Klassifikatoren von Dingen, also jene Wissenschaftler, deren Wissenschaft nur im Klassifizieren besteht, wissen im allgemeinen nicht, daß das Klassifizierbare unendlich ist und also nicht klassifiziert werden kann.
>
> Das Buch der Unruhe – *Fernando Pessoa*

This chapter discusses some topics that are closely connected to Pisot substitution, *e.g.*, reducible Pisot substitution or substitutions with cobounds. Comparing this with our findings for Pisot substitutions, opens the eyes for those properties characteristic to Pisot substitutions.

6c.1. Connectivity

Already in the article of Rauzy [306, Lemme 5.2] the question of connectivity of the Rauzy fractal, *i.e.*, of the attractor $\underline{\Omega}$ of the IFS $\boldsymbol{\Theta}^\star$ in the internal space $H = \mathbb{C}$ associated to the tribonacci substitution $a \mapsto ab$, $b \mapsto ac$ and $c \mapsto a$, is treated.

For unimodular Pisot substitution satisfying the SCC (so, one only needs the measure-disjointness of the sets Ω_i), Rauzy's method was generalised by Canterini [87]. The following is a reformulation of [87, Theorem 3.1].

Lemma 6c.1. *Let σ be a unimodular Pisot substitution satisfying the SCC. Denote the associated EMFS of its representation with natural intervals by $\boldsymbol{\Theta}$, and the attractor of the IFS $\boldsymbol{\Theta}^\star$ in internal space H by $\underline{\Omega}$. Define the collections $\mathcal{B}_k = \left\{ f(\Omega_j) \mid f \in \left[(\boldsymbol{\Theta}^\star)^k\right]_{ij},\ 1 \le i,j \le n \right\}$. Then, the set* $\operatorname{supp}\underline{\Omega}$ *is connected iff*

$$\begin{gathered}\forall k \geq 0,\ \forall B_1, B_2 \in \mathcal{B}_k :\ \exists \check{B}_1, \ldots, \check{B}_m \in \mathcal{B}_k \text{ such that} \\ B_1 \cap \check{B}_1 \neq \varnothing,\ \check{B}_1 \cap \check{B}_2 \neq \varnothing,\ \ldots,\ \check{B}_{m-1} \cap \check{B}_m \neq \varnothing, \check{B}_m \cap B_2 \neq \varnothing,\end{gathered}$$

i.e., for every k and given any two sets $B_1, B_2 \in \mathcal{B}_k$ there is a "finite chain" of sets "leading" from one B_i to the other.

Proof. If $\operatorname{supp}\Omega$ is connected, obviously the condition is fulfilled (note that each $\mathcal{B}_k$ is a finite set).

Conversely, suppose $\operatorname{supp}\underline{\Omega}$ is not connected. It is compact, so the union of at least two compact connected components. For simplicity, we assume that there are exactly two connected components, say K_1 and K_2. By compactness, there is a $\delta > 0$ such that $d(K_1, K_2) = \delta$. Choose $k \geq 0$ such that $\operatorname{diam}\check{B}_i < \delta$ for all $\check{B}_i \in \mathcal{B}_k$ (which is possible, since $\lambda^\star$ is a contraction). Then, one can partition $\mathcal{B}_k$ in two disjoint subsets $\mathcal{B}_k^{(1)}$, $\mathcal{B}_k^{(2)}$ such that $K_j = \bigcup_{\check{B}_i \in \mathcal{B}_k^{(j)}} \check{B}_i$. Consequently, one cannot choose sets $\check{B}_i \in \mathcal{B}_k$ of the above type for any pair $B_1 \in \mathcal{B}_k^{(1)}$ and $B_2 \in \mathcal{B}_k^{(2)}$. This proves the lemma. □

From this, one obtains the following sufficient condition for connectedness, see [87, Theorem 3.2].

Lemma 6c.2. *Assume the setting of Lemma 6c.1. Then the set* $\operatorname{supp} \underline{\Omega}$ *is connected if the following two conditions hold:*

- $\forall i, j \in \{1, \dots, n\}$: $\exists m$ *and* $\exists i_k \in \{1, \dots, n\}$ *such that* $\Omega_i \cap \Omega_{i_1} \neq \varnothing, \dots, \Omega_{i_m} \cap \Omega_j \neq \varnothing$.
- $\forall i \in \{1, \dots, n\}$ *and* $\forall B_1, B_2 \in \mathcal{B}^{(i)}$: $\exists \check{B}_1, \dots, \check{B}_m \in \mathcal{B}^{(i)}$ *such that* $B_1 \cap \check{B}_1 \neq \varnothing, \dots,$ $\check{B}_m \cap B_2 \neq \varnothing$, *where* $\mathcal{B}^{(i)} = \{f(\Omega_j) \mid f \in [\boldsymbol{\Theta}^\star]_{ij}\,,\ 1 \le j \le n\}$.

Moreover, in this case, each Ω_i *is connected.*

Proof. Clear by the iteration of the IFS $\boldsymbol{\Theta}^\star$ and the previous lemma. □

Remark 6c.3. One can also easily modify the last lemma to obtain a bound on the number of connected components of each Ω_i, see [87, Theorem 3.3]. The idea in that statement is the following: One replaces the Pisot substitution σ by a substitution over a bigger alphabet with the same natural length representation $\operatorname{supp} \underline{\Lambda}$ (wherefore the characteristic polynomial of the substitution matrix is not irreducible anymore) such that each connected component corresponds to (at most) one letter of the alphabet.

Remark 6c.4. Obviously, $\operatorname{supp} \Omega$ can be disconnected, see the examples in [243, Section 4] for the substitution matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

(the square and the cube of the substitution matrix of the Fibonacci substitution).

The Pisot substitution $a \mapsto abcab$, $b \mapsto acb$ and $c \mapsto b$ is an example where Ω_c consists of two connected components, while Ω_a, Ω_b and $\operatorname{supp} \underline{\Omega}$ are connected, see [87, Example 3 & Figure 4].

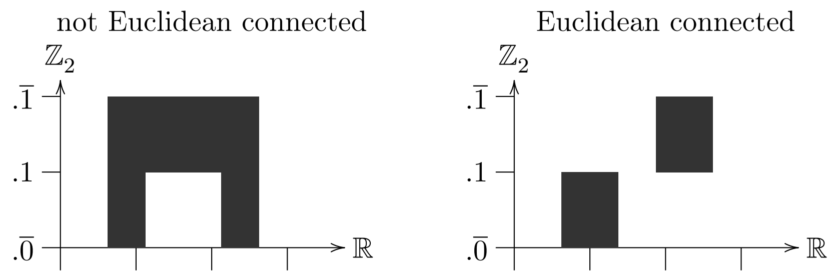
We also note that Lemma 6c.2 is used to prove the connectedness of a certain class of Pisot substitutions over 3 letters in [86, Theorem 5.1]: the (primitive) Arnoux-Rauzy substitutions [21] (the SCC for these Arnoux-Rauzy substitutions is shown in [20, Example 5]).

Remark 6c.5. For unimodular Pisot substitutions over two symbols, say $\mathcal{A} = \{a, b\}$, there is an algebraic criterion for connectedness (*e.g.*, see [223, Theorem 1.5] and [64, Theorem 3]): The sets Ω_a, Ω_b and $\Omega_a \cup \Omega_b$ are connected, *i.e.*, are intervals, iff the substitution σ is *invertible*, *i.e.*, iff σ is an automorphism of the free group generated by the alphabet $\mathcal{A} = \{a, b\}$ (also see [116] on a characterisation of invertible substitutions over two letters).

For non-unimodular Pisot substitution, the internal space H (respectively H_{ext}) has $\mathfrak{p}$-adic components. Since $\mathfrak{p}$-adic fields $\mathbb{Q}_\mathfrak{p}$ are totally disconnected, the attractor $\underline{\Omega}$ corresponding to a non-unimodular Pisot substitution can, of course, not be connected. But with the following weak form of connectedness, the previous lemmas also apply to non-unimodular Pisot substitutions.

Definition 6c.6. Let H be the product of a connected space H_1, *e.g.*, $\mathbb{R}^{r-1} \times \mathbb{C}$, and a totally disconnected space H_2, *e.g.*, $\mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}$. Denote by $\pi : H \to H_1$ the canonical projection of H to the connected subspace. We note that the connected component of a point $x = (x_1, x_2) \in H = H_1 \times H_2$ is given by $x + H_1 \times \{0\} = H_1 \times \{x_2\}$. We say, that a compact set $W \subset H$ is *Euclidean connected* if the projection $\pi(W \cap (H_1 \times \{y\})$ is connected for every $y \in H_2$.

With our visualisation of $\mathfrak{p}$-adic spaces, we have the following situation for compact subsets of $\mathbb{R} \times \mathbb{Z}_2$:



6c.2. Cobounds

A one-dimensional sequence u obtained by a substitution σ (over some alphabet $\mathcal{A}$) is usually studied from a dynamical systems point of view. Here, one is interested in the spectrum of the dynamical system $(\mathbb{X}(u), S)$ (where S denotes the action given by the shift) and, in particular, its eigenvalues. We refer to Section 7.2 for definitions and statements about the spectral theory of dynamical systems.

Instead of the general case (see [130, Prop. 3]) of a one-dimensional substitution σ, we will restrict ourselves to the case that every eigenvalue of the substitution matrix $\boldsymbol{S}\sigma$ of modulus greater or equal to one is a simple eigenvalue. We need the following definitions.

Definition 6c.7. Let u be an aperiodic[1] fixed point of a primitive substitution σ on the alphabet $\mathcal{A}$ with $n = \operatorname{card} \mathcal{A}$. Then, there exist an integer $1 < r \leq n$ and an integer N such that, for every $m \geq N$, the set $\sigma^m(\mathcal{A})$ has exactly r elements. Thus, we identify letters which have essentially the same substitution in the following (note that such letters are, in particular, represented by the same natural interval).

Any word $w = w_1 \dots w_s$ appearing in u and satisfying, for all $m \geq N$,

$$\sigma^m(w_s) = \sigma^m(w_1), \quad \text{and } \sigma^m(w_j) \neq \sigma^m(w_1) \text{ for all } 1 < j < s,$$

is called a *return word*. For a given return word w, we define the associated *return time sequence* by $r_m(w) = \#\sigma^m(w)$ for all $m \geq 1$.

By repetitivity, there exist only finitely many return words. Moreover, the return time sequence is easily calculated: With the homomorphism of Abelianisation l (see Definition 6.6), one has $r_m(w) = e\,(\boldsymbol{S}\sigma)^m\, l(w)$, where $e = (1, \dots, 1)$. Thus, if we have q return words $\{w^{(j)} \mid 1 \leq j \leq q\}$ and define the matrix $m \times q$-matrix

$$\boldsymbol{L} = \begin{pmatrix} l(w^{(1)}) & \cdots & l(w^{(q)}) \end{pmatrix},$$

then the vector $r_m = \left(r_m(w^{(1)}), \dots, r_m(w^{(q)})\right)$ is given by $r_m = e\,(\boldsymbol{S}\sigma)^m\, \boldsymbol{L}$. With these notations, the eigenvalues of $(\mathbb{X}(u), S)$ are calculated as follows.

Proposition 6c.8. [130, Prop. 4] *Let σ be a primitive substitution with aperiodic fixed point u such that every eigenvalue of modulus greater or equal to one of its substitution matrix $\boldsymbol{S}\sigma$ is*

[1] By Proposition 5.120, the fixed point u is UCP (respectively "recognisable").

a simple eigenvalue. Denote by $\lambda_1, \ldots, \lambda_k$ *the eigenvalues of* $\boldsymbol{S}\sigma$ *and by* $D = \{w^{(j)} \,|\, 1 \le j \le q\}$ *the (finite) set of return words.*

For $w^{(j)} \in D$*, let* $h(i, w^{(j)})$ *be the* j*-th coordinate of the (row) vector*

$$e \left(\prod_{t \neq i} (\boldsymbol{S}\sigma - \lambda_t \boldsymbol{E}) \right) \boldsymbol{L},$$

and let

$$A(w^{(j)}) = \{ i \in \{1, \ldots, k\} \,|\, |\lambda_i| \ge 1 \text{ and } h(i, w^{(j)}) \neq 0 \}.$$

Then, α *is an eigenvalue of the dynamical system* $(\mathbb{X}(u)\,, S)$ *iff, for every* $w \in D$ *and every* $i \in A(w)$*, there exists a polynomial* $Q_w^{(i)} \in \mathbb{Z}[x]$ *such that*

$$\alpha = \frac{1}{h(i,w)}\, \lambda_i^{n-1} \cdot Q_w^{(i)} \left(\frac{1}{\lambda_i} \right). \qquad \square$$

Example 6c.9. We look at the following substitution (see [298, Example 7.3.14.3] and [348, Section 3.3]): $a \mapsto abca$, $b \mapsto bcb$ and $c \mapsto cabc$. For this substitution, one establishes that there are six return words, namely

$$D = \{abc, bca, cab, abcbc, bc, cb\}.$$

Thus, we have the following matrices:

$$\boldsymbol{S}\sigma = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{L} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{S}\sigma$ are $\lambda_1 = \frac{1}{2}(5 + \sqrt{5})$, $\lambda_2 = 1$ and $\lambda_3 = \frac{1}{2}(5 - \sqrt{5})$. Therefore, one calculates

$$\begin{aligned}
\big(h(1,w^{(j)})\big)_{j=1}^{6} &= \big(9{+}4\sqrt{5} \quad 9{+}4\sqrt{5} \quad 9{+}4\sqrt{5} \quad \tfrac{1}{2}(29{+}13\sqrt{5}) \quad \tfrac{1}{2}(11{+}5\sqrt{5}) \quad \tfrac{1}{2}(11{+}5\sqrt{5})\big)\,, \\
\big(h(2,w^{(j)})\big)_{j=1}^{6} &= \big(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\big)\,, \\
\big(h(3,w^{(j)})\big)_{j=1}^{6} &= \big(9{-}4\sqrt{5} \quad 9{-}4\sqrt{5} \quad 9{-}4\sqrt{5} \quad \tfrac{1}{2}(29{-}13\sqrt{5}) \quad \tfrac{1}{2}(11{-}5\sqrt{5}) \quad \tfrac{1}{2}(11{-}5\sqrt{5})\big)\,,
\end{aligned}$$

and thus $A(w^{(j)}) = \{1\}$ for $1 \le j \le 6$.

So, we can conclude that α is an eigenvalue of the dynamical system iff there exist polynomials $p_\alpha^{(1)}, p_\alpha^{(2)}, p_\alpha^{(3)} \in \mathbb{Z}[x]$ such that

$$\alpha = -\frac{5}{2}\,(-7 + 3\sqrt{5}) \cdot p_\alpha^{(1)} \left(\frac{1}{\lambda_1} \right) = \frac{5}{2}\,(-11 + 5\sqrt{5}) \cdot p_\alpha^{(2)} \left(\frac{1}{\lambda_1} \right) = 5\,(-2 + \sqrt{5}) \cdot p_\alpha^{(3)} \left(\frac{1}{\lambda_1} \right),$$

where we observe that $1/\lambda_1 = \frac{\sqrt{5}-5}{10}$ and $\lambda_1^2/(9 + 4\sqrt{5}) = -\frac{5}{2}\,(-7 + 3\sqrt{5})$ *etc.* One can show that

$$-\frac{5}{2}\,(-7 + 3\sqrt{5}) \cdot \mathbb{Z} \left[\frac{\sqrt{5}-5}{10} \right] = \frac{5}{2}\,(-11 + 5\sqrt{5}) \cdot \mathbb{Z} \left[\frac{\sqrt{5}-5}{10} \right] = 5\,(-2 + \sqrt{5}) \cdot \mathbb{Z} \left[\frac{\sqrt{5}-5}{10} \right].$$

Therefore, the group of eigenvalues of the dynamical system $(\mathbb{X}(u)\,, S)$ is here given by $(-10 + 5\sqrt{5}) \cdot \mathbb{Z} \left[\frac{1}{\lambda_1} \right]$.

We will argue in Remark 7.40 that the group of eigenvalues of the discrete dynamical system $(\mathbb{X}(u)\,,S)$ equals the group of eigenvalues of the continuous dynamical system $(\mathbb{X}(\underline{\varLambda})\,,\mathbb{R})$, where the average length of the intervals in $\underline{\varLambda}$ is $\bar{\ell}=1$. But the latter group is generated by the set of Bragg peaks (compare Corollary 7.30) and thus simply equals the projection of the annihilator $\tilde{\mathcal{L}}_{\text{ext}}^{\perp}$ of the lattice $\tilde{\mathcal{L}}_{\text{ext}}$ on $\mathbb{R}$. But this projected set is given by $\bigcup_{k\geq 0}(\mathcal{L}')^{\wedge}/\lambda^k$, compare Proposition 6.58.

We continue with the above example.

Continuation of Example 6c.9. The frequency of the letters is given by $\varrho_a = 1/\lambda_1^3$ and $\varrho_b = 1/\lambda_2^2 = \varrho_c$, and we obtain the following lengths of the natural intervals, normalised such that $\bar{\ell} = \sum_{i\in\{a,b,c\}} \varrho_i\,\ell_i = 1$:

$$\ell_a = \ell_c = \frac{1}{2}\,(1+3\,\frac{1}{\sqrt{5}}) = \frac{1}{10}\,(5+3\sqrt{5}) \quad\text{and}\quad \ell_b = \frac{1}{2}+\frac{1}{\sqrt{5}} = \frac{1}{10}\,(5+\sqrt{5}).$$

One checks that

$$\mathcal{L} = \mathcal{L}' = \langle \ell_a\,,\ell_b\rangle_{\mathbb{Z}} = \frac{1}{2}\langle 1\,,\frac{1}{\sqrt{5}}\rangle_{\mathbb{Z}}.$$

The codifferent of $\mathcal{L}$ (over $\mathbb{Q}$) is given by

$$\mathcal{L}^{\wedge} = \langle 1\,,\sqrt{5}\rangle_{\mathbb{Z}},$$

since one easily checks that the dual basis $e_1^{\perp}, e_2^{\perp}$ of the lattice spanned by

$$e_1 = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \quad\text{and}\quad e_2 = \frac{1}{2}\begin{pmatrix}\frac{1}{\sqrt{5}}\\ -\frac{1}{\sqrt{5}}\end{pmatrix}$$

is given by

$$e_1^{\perp} = \begin{pmatrix}1\\1\end{pmatrix} \quad\text{and}\quad e_2^{\perp} = \begin{pmatrix}\sqrt{5}\\ -\sqrt{5}\end{pmatrix}.$$

We also note that $\det \boldsymbol{S}\sigma = 5$, wherefore one establishes a CPS with $H_{\text{ext}} = \mathbb{R}\times\mathbb{Q}_5(\sqrt{5})$ (and $H = \mathbb{R}\times\frac{1}{\sqrt{5}}\,\mathbb{Z}_5[\sqrt{5}]$) for this substitution. Consequently, one has

$$\mathcal{L}_{\text{ext}} = \bigcup_{k\geq 0}\frac{1}{\lambda_1^k}\,\frac{1}{2}\langle 1\,,\frac{1}{\sqrt{5}}\rangle_{\mathbb{Z}} \quad\text{and}\quad \mathcal{L}_{\text{ext}}^{\wedge} = \bigcup_{k\geq 0}\frac{1}{\lambda_1^k}\,\langle 1\,,\sqrt{5}\rangle_{\mathbb{Z}} = \bigcup_{k\geq 0}\left(\frac{\sqrt{5}-5}{10}\right)^k\langle 1\,,\sqrt{5}\rangle_{\mathbb{Z}}.$$

We may also write this as $\mathcal{L}_{\text{ext}}^{\wedge} = \mathbb{Z}[\frac{1}{\lambda_1}]\cdot\mathbb{Z}[\sqrt{5}]$.

We claim that $\mathcal{L}_{\text{ext}}^{\wedge}$ is exactly the group of eigenvalues of the dynamical system $(\mathbb{X}(u)\,,S)$. *i.e.*, we claim that

$$(5\sqrt{5}-10)\,\mathbb{Z}[\frac{1}{\lambda_1}] = \mathbb{Z}[\sqrt{5}]\cdot\mathbb{Z}[\frac{1}{\lambda_1}].$$

Obviously, one has the inclusion "$\subset$", but one also checks that

$$(5\sqrt{5}-10)\left(-1+2\left(\frac{1}{\lambda_1}\right)^2\right) = 1 \quad\text{and}\quad (5\sqrt{5}-10)\left(-3-4\,\frac{1}{\lambda_1}\right) = \sqrt{5},$$

which consequently establishes the equality. Thus, $\mathcal{L}_{\text{ext}}^{\wedge}$ is indeed the group of eigenvalues for the dynamical system. Note that we have *not* proven pure pointedness for this sequence (and we will not prove it here).

Comparing the two methods of calculating the eigenvalues for the dynamical system, it should be clear that the method *via* the CPS is computationally easier. We now would like to establish a connection between the matrix $\boldsymbol{L}$ and the height group $\mathcal{L}/\mathcal{L}'$. To clarify $\boldsymbol{L}$, we introduce the following definition, see [183, Section 1.3], [299, Def. VI.2] and [298, Def. 7.3.13].

Definition 6c.10. A function $h : \mathcal{A} \to \mathbb{T}$ is called a *cobound* (or[2] *cobord* or *coboundary*) of σ if, for every word $w = w_1 \dots w_{s-1} w_s$ with $w_1 = w_s$ of u, one has

$$h(w_1) \cdot h(w_2) \cdot \ldots \cdot h(w_{s-1}) = 1.$$

Equivalently, h is a cobound of σ iff there exists a function $g : \mathcal{A} \to \mathbb{T}$ such that $g(b) = g(a) \cdot h(a)$ for every two-letter word ab of u. We say that the cobound h is *trivial* if the only possible choice is $h \equiv 1$.

Obviously, it is enough to look at the return words in the definition of the cobound. Moreover, it is clear that if $\mathcal{L}/\mathcal{L}'$ is non-trivial, so is the cobound: Let $M = \operatorname{card} \mathcal{L}/\mathcal{L}'$. Then, we can enumerate the cosets $x + \mathcal{L}' \subset \mathcal{L}$ bijectively by $\{0, \dots, M-1\}$. Thus, we have a function $\check{M} : \mathcal{L} \to \{0, \dots, M-1\}$ which is constant on each coset $x + \mathcal{L}'$ and, in particular, on each set $\varLambda_i$ (since each set $\varLambda_i$ of the representation with natural intervals of u belongs to exactly one coset). We now set

$$h(a) = \exp\left(2\pi i \, \frac{\check{M}(\varLambda_a + \ell_a) - \check{M}(\varLambda_a)}{M}\right)$$

for every $a \in \mathcal{A}$. This is well defined, a cobound and non-trivial if $\mathcal{L}/\mathcal{L}'$ is non-trivial.

Continuation of Example 6c.9. If $\mathcal{L}/\mathcal{L}'$ is trivial, one may still obtain a non-trivial cobound (compare [348, Section 3.3]): For the above substitution one can choose $h(a) = 1$ and $h(b) = \exp(2\pi i \, \kappa) = 1/h(c)$ for every $\kappa \in [0\,,1[$.

The cobound can be used to characterise the spectrum of the dynamical system $(\mathbb{X}(u)\,, S)$ by the following statement.

Lemma 6c.11. [183, Théoreme 1.4(ii)] and [130, Prop. 1] *Let σ be a primitive substitution over the alphabet $\mathcal{A}$ with aperiodic fixed point u. A number $\alpha \in \mathbb{R}$ is an eigenvalue of* $(\mathbb{X}(u)\,, S)$ *iff*

(i) *there exists an integer $q > 0$ such that for every $a \in \mathcal{A}$ the limit*

$$h(a) = \lim_{m\to\infty} \exp(2\pi i \, \alpha \cdot \# \sigma^{q^m}(a))$$

exists and h is a cobound.

(ii) $\lim_{m\to\infty} \exp(2\pi i \, \alpha \cdot r_m(w)) = 1$ *for every return word w.* □

This result should be compared with its generalisation in Proposition 7.43. One may say that the use of "cobounds" and "return words" is a way to accommodate to the situation that one does not work with the natural lengths here, but forces all tiles to have length 1. Also see the concept of a "deformed model set" (Definition 7.36) and Remark 7.40.

[2] Since the terms "cobord(ism)" and "coboundary" are already used in algebraic topology, we prefer the name cobound here.

Remark 6c.12. The cobound (and thus also the return words) have been introduced to generalise the notion of the "height" of a constant length substitution (see Definition 6b.12) to the non-constant case. Consequently, the "unifying" notion is actually th height group $\mathcal{L}/\mathcal{L}'$ respectively $\mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}}$.

In [298, p. 228] and [348, Section 3.3], it is noted that the spectrum of a substitutive system can be divided into two parts: On the hand, the "arithmetic spectrum" which only depends on the substitution matrix and, on the other hand, the "combinatorial spectrum" which depends on the return words (and therefore the order of the letters in the substitutes $\sigma(i)$). Of course, this is simply the statement that $\mathcal{L}$ is determined by the left PF-eigenvector of the substitution matrix while for $\mathcal{L}'$ one has to know the sequence (respectively $\varDelta'$).

So far, we are not aware of a one-dimensional substitution σ which yields a regular multi-component model set with non-trivial height group but has an inflation factor which is not an integer (consequently, such an example has a non-trivial cobound and pure point dynamical spectrum). We now give such an example.

Example 6c.13. We consider the following primitive substitution σ_{cob} over $\mathcal{A} = \{a, A, b, B\}$:

$$a \mapsto aBbAa, \quad A \mapsto BbAaA, \quad b \mapsto a, \quad B \mapsto A.$$

We have

$$\boldsymbol{S}\sigma_{\text{cob}} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

eigenvalues	left PF-eigenvector	right PF-eigenvector
$\lambda = \frac{3+\sqrt{17}}{2}$ $\lambda_2 = \frac{3-\sqrt{17}}{2}$ 1 and 0	$\ell = (\lambda, \lambda, 1, 1)$	$\varrho = \frac{1}{\lambda+2} \begin{pmatrix} \frac{\lambda}{2} \\ \frac{\lambda}{2} \\ 1 \\ 1 \end{pmatrix}$

which should be compared with $\sigma_{\clubsuit}$. We write down part of fixed point u of this substitution:

$$\ldots aBbAaAaBbAaAaBbAaBbAaA\dot{a}BbAaAaBbAaAaBbAaBbAaAaBbAa \ldots$$

Obviously, Aa and Bb always come in pairs[3] here, wherefore one has $\varLambda_a = \varLambda_A + \ell_A = \varLambda_a + \lambda$ and $\varLambda_b = \varLambda_B + \ell_B = \varLambda_B + 1$. Moreover, one easily establishes

$$\mathcal{L} = \langle \lambda\,, 1 \rangle_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{L}' = \{m + n \cdot \lambda \in \mathbb{Z}[\lambda] \mid m \equiv 0 \equiv n \bmod 2\} = \langle 2\,\lambda\,, 2 \rangle_{\mathbb{Z}}$$

and therefore $\operatorname{card} \mathcal{L}/\mathcal{L}' = 4$. In fact, one has $\varLambda_a \subset \mathcal{L}'$, $\varLambda_b \subset \lambda + 1 + \mathcal{L}'$ and $\varLambda_A, \varLambda_B \subset \lambda + \mathcal{L}'$ (note that there are no points in $1 + \mathcal{L}'$) and one has $\mathcal{L}/\mathcal{L}' \cong \mathsf{C}_2 \times \mathsf{C}_2$.

Thus, a possible choice for a non-trivial cobound h is

$$h(a) = \exp(2\pi i \, \frac{1}{4}), \quad h(A) = \exp(2\pi i \, \frac{3}{4}), \quad h(b) = \exp(2\pi i \, \frac{1}{2}) = h(B),$$

or, more generally, $h(a) = \exp(2\pi i \, \kappa_1) = 1/h(A)$ and $h(b) = \exp(2\pi i \, \kappa_2) = 1/h(B)$ for any $\kappa_1, \kappa_2 \in [0\,, 1[$.

[3] Therefore, one may also define a Pisot substitution over two letters $\mathcal{A} = \{Aa, Bb\}$ whose substitution matrix is the transpose of the substitution matrix of $\sigma_{\clubsuit}$. However, we will analyse σ_{cob} here.

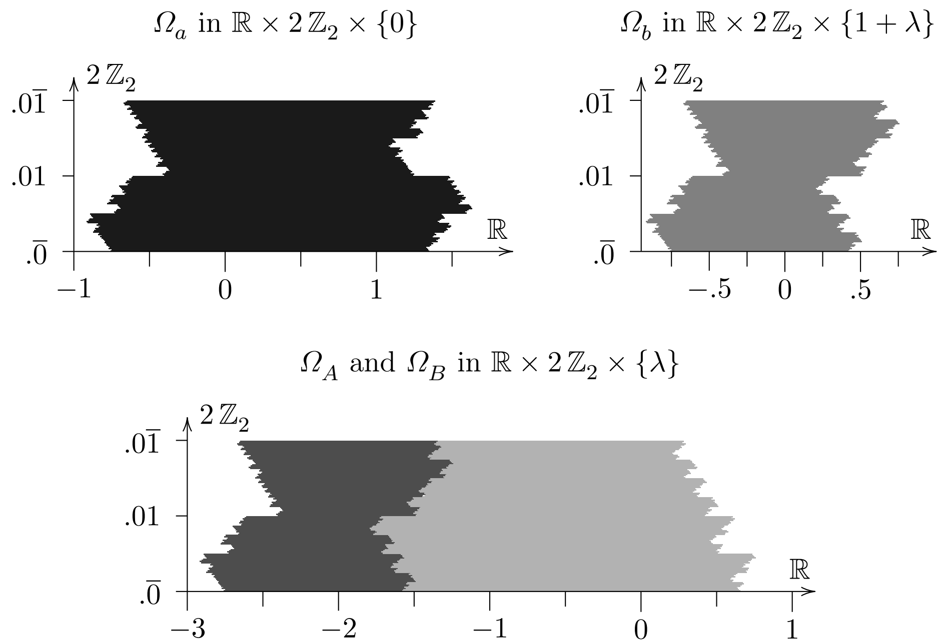


Figure 6c.1.: The tiles for the σ_{cob}-substitution in $H = \mathbb{R} \times 2\,\mathbb{Z}_2 \times \mathsf{C}_2 \times \mathsf{C}_2$: $\varOmega_a$ in very dark gray, $\varOmega_A$ in light gray, $\varOmega_b$ in medium gray and $\varOmega_B$ in dark gray. For the height group $\mathcal{L}/\mathcal{L}' \cong \mathsf{C}_2 \times \mathsf{C}_2$, the representatives $\{0, 1, \lambda, \lambda+1\}$ are chosen. Consequently, one has to translate $\varOmega_b$ by $(1+\lambda)^\star$ to obtain the tile marked as $\varOmega_b + 0$ in Figure 6c.2.

We now want to establish internal spaces for this substitution and (a)periodic tilings thereof. Similarly as for the clubsuit substitution, we identify λ with $.0110111\ldots \in \mathbb{Z}_2$ and therefore obtain $H' \cong \mathbb{R} \times 2\,\mathbb{Z}_2$ (the completion of $\mathcal{L}'$ as in Section 6.4) and $H = H' \times (\mathcal{L}/\mathcal{L}') \cong \mathbb{R} \times 2\,\mathbb{Z}_2 \times \mathsf{C}_2 \times \mathsf{C}_2$. For the height group $\mathcal{L}/\mathcal{L}' \cong \mathsf{C}_2 \times \mathsf{C}_2$, we use the set $\{0, 1, \lambda, \lambda+1\}$ as set of representatives (thus, one may say that the height group is the Abelian group generated by $\{1, \lambda\}$ with the relations $1+1 = 0$ and $\lambda + \lambda = 0$). The pictures of the sets $\varOmega_i$ with $i \in \mathcal{A}$ are shown in Figure 6c.1. We note that

$$\varOmega_a = \varOmega_A + (\lambda_2, \lambda, \lambda) = \varOmega_A + \lambda^\star \quad \text{and} \quad \varOmega_b = \varOmega_B + (1,1,1) = \varOmega_B + 1^\star.$$

We note that – using the argumentation in Section 6.4 – the completion of $\mathcal{L}$ relative to the (countable) base $\breve{\mathcal{B}} = \{f_0^m(\varDelta) \,|\, m \in \mathbb{Z}_{\geq 0}\}$ at 0 yields $\mathbb{R} \times \mathbb{Z}_2$, while, as we have seen above, the completion relative to the base $\mathcal{B} = \{f_0^m(\varDelta') \mid m \in \mathbb{Z}_{\geq 0}\}$ at 0 yields $H = \mathbb{R} \times 2\,\mathbb{Z}_2 \times \mathsf{C}_2 \times \mathsf{C}_2$ (here, $f_0(x) = \lambda\, x$). Thus, we may say that one factor C_2 arises from $\mathbb{Z}_2/2\,\mathbb{Z}_2$ (thus, we may also use $H = \mathbb{R} \times \mathbb{Z}_2 \times \mathsf{C}_2$). The other factor of C_2 comes from $\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}$, as we will demonstrate next.

The next step is the calculation of $\mathcal{L}'_{\mathrm{ext}}$ and $\mathcal{L}_{\mathrm{ext}}$: Obviously, $\mathcal{L}_{\mathrm{ext}} = \mathbb{Q}(\lambda)$. Using $\frac{1}{\lambda} = \frac{1}{2}(\lambda - 3)$ (wherefore $\frac{1}{\lambda}\,(m + n\,\lambda) = \frac{1}{2}\,(2\,n - 3\,m) + \frac{1}{2}\,m\,\lambda$), one obtains (for all $k \in \mathbb{N}$)

$$\mathbb{Z}[\lambda] \cap \frac{1}{\lambda}\mathcal{L}' = \mathbb{Z}[\lambda] \cap \frac{1}{\lambda^k}\mathcal{L}' = \{m + n \cdot \lambda \in \mathbb{Z}[\lambda] \mid m \equiv n \bmod 2\}.$$

Thus, we may also write

$$\mathcal{L}'_{\mathrm{ext}} = \varinjlim_{k\in\mathbb{Z}_{\geq 0}} \frac{1}{\lambda^k}\,\{m + n\cdot\lambda \in \mathbb{Z}[\lambda] \mid m \equiv n \bmod 2\},$$

and one obtains $\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}} \cong \mathsf{C}_2$; in fact, one has $\mathcal{L}_{\mathrm{ext}} = \mathcal{L}'_{\mathrm{ext}} \,\dot\cup\, \mathcal{L}'_{\mathrm{ext}} + 1$. As in Section 6.4, the completion of $\mathcal{L}'_{\mathrm{ext}}$ is $\mathbb{R}\times\mathbb{Q}_2$. Therefore, we obtain the internal space

$$H_{\mathrm{ext}} \cong H'_{\mathrm{ext}} \times \bigl(\mathcal{L}_{\mathrm{ext}}/\mathcal{L}'_{\mathrm{ext}}\bigr) \cong \mathbb{R}\times\mathbb{Q}_2\times\mathsf{C}_2.$$

We now look at the tiling $\underline{\varOmega} + \underline{\varUpsilon}$ of H_{ext} in Figure 6c.2: We have $\varUpsilon_a = \Lambda([0\,,\lambda[) = \varUpsilon_A$, $\varUpsilon_b = \Lambda([0\,,1[) = \varUpsilon_B$ and $\varOmega_a, \varOmega_b \subset \mathbb{R}\times\mathbb{Q}_2\times\{0\}$, while $\varOmega_A, \varOmega_B \subset \mathbb{R}\times\mathbb{Q}_2\times\{1\}$. We also note that Figure 6c.2 confirms that $\underline{\varOmega} + \underline{\varUpsilon}$ is a tiling (and thus we have a regular multi-component inter model set $\Lambda(\operatorname{int}\underline{\varOmega}) \subset \underline{\varLambda} \subset \Lambda(\underline{\varOmega})$).

We also point out that there is also a periodic tiling of $H = \mathbb{R}\times\mathbb{Z}_2\times\mathsf{C}_2$ by the tiles $\varOmega_i$, see Figure 6c.2. Here, the observation from Footnote 3 on p. 327 comes into play, namely, the tiling obtained by σ_{cob} is LI (by a rather simple rule) to a Pisot substitution (over the alphabet $\{Aa, Bb\}$). Since this latter Pisot substitution admits a periodic tiling H, this induces also a periodic tiling of H with lattice of periods $(\ell_b - \ell_a)^\star\,\mathbb{Z} = (1-\lambda)^\star\,\mathbb{Z}$. Note that $(1-\lambda)^\star\,\mathbb{Z} \subset \mathbb{R}\times\mathbb{Z}_2\times\{0\}$, wherefore the subspace $\mathbb{R}\times\mathbb{Z}_2\times\{0\}$ only contains translates of the tiles $\varOmega_a$ and $\varOmega_b$, while the subspace $\mathbb{R}\times\mathbb{Z}_2\times\{1\}$ only contains translates of the tiles $\varOmega_A$ and $\varOmega_B$.

As last remark to this example, we note that the group of eigenvalues of the associated dynamical system is now easily established: One has $\mathcal{L} = \mathbb{Z}[\lambda]$ and $\mathcal{L}' = 2\,\mathbb{Z}[\lambda]$, wherefore one has $\mathcal{L}^\wedge = \mathbb{Z}[\lambda]/(2\lambda - 3)$ (by Lemma 3.51) and $(\mathcal{L}')^\wedge = \frac{1}{2}\mathcal{L}^\wedge$ (observe that this factor $\frac{1}{2}$ arises from the group C_2). Now, the group of eigenvalues is given by $\bigcup_{k\geq 0} (\mathcal{L}')^\wedge/\lambda^k = \frac{1}{2}\,\mathbb{Z}[\frac{1}{\lambda}]\,\mathbb{Z}[\lambda]/(2\lambda-3)$ (note that $\mathbb{Z}[\frac{1}{\lambda}]\,\mathbb{Z}[\lambda] \neq \mathbb{Q}(\lambda)$).

6c.3. Reducible Substitutions

Our definition of a Pisot substitution is very restrictive, since it implies that the characteristic polynomial $p(x) = \det(x\boldsymbol{E} - \boldsymbol{S}\sigma)$ is irreducible over $\mathbb{Q}$. We now relax this condition.

Definition 6c.14. We say that the (one-dimensional) substitution σ is a *reducible Pisot substitution* if the substitution matrix $\boldsymbol{S}\sigma$ has a dominant simple eigenvalue $\lambda > 1$ which is a PV-number.

As the Thue-Morse sequence shows, there are reducible Pisot substitutions which are not multi-component model sets, see Example 6b.18. An example with a PV-number that is not an integer is the following, which we have taken from [43, Example 5.3]

Example 6c.15. We consider two substitutions σ. The first one is given by

$$a \mapsto acbaaa, \qquad b \mapsto acbbaa \quad\text{and}\quad c \mapsto cbbaa.$$

Its substitution matrix has eigenvalues $\{3 \pm 2\sqrt{2}, 1\}$, and a left eigenvector to the PV-number $3 + 2\sqrt{2}$ is given by $(\frac{1}{2} + \frac{1}{2}\sqrt{2}, \frac{1}{2} + \frac{1}{2}\sqrt{2}, 1)$ (so, we have $\ell_a = \ell_b$). Thus, it is a reducible Pisot

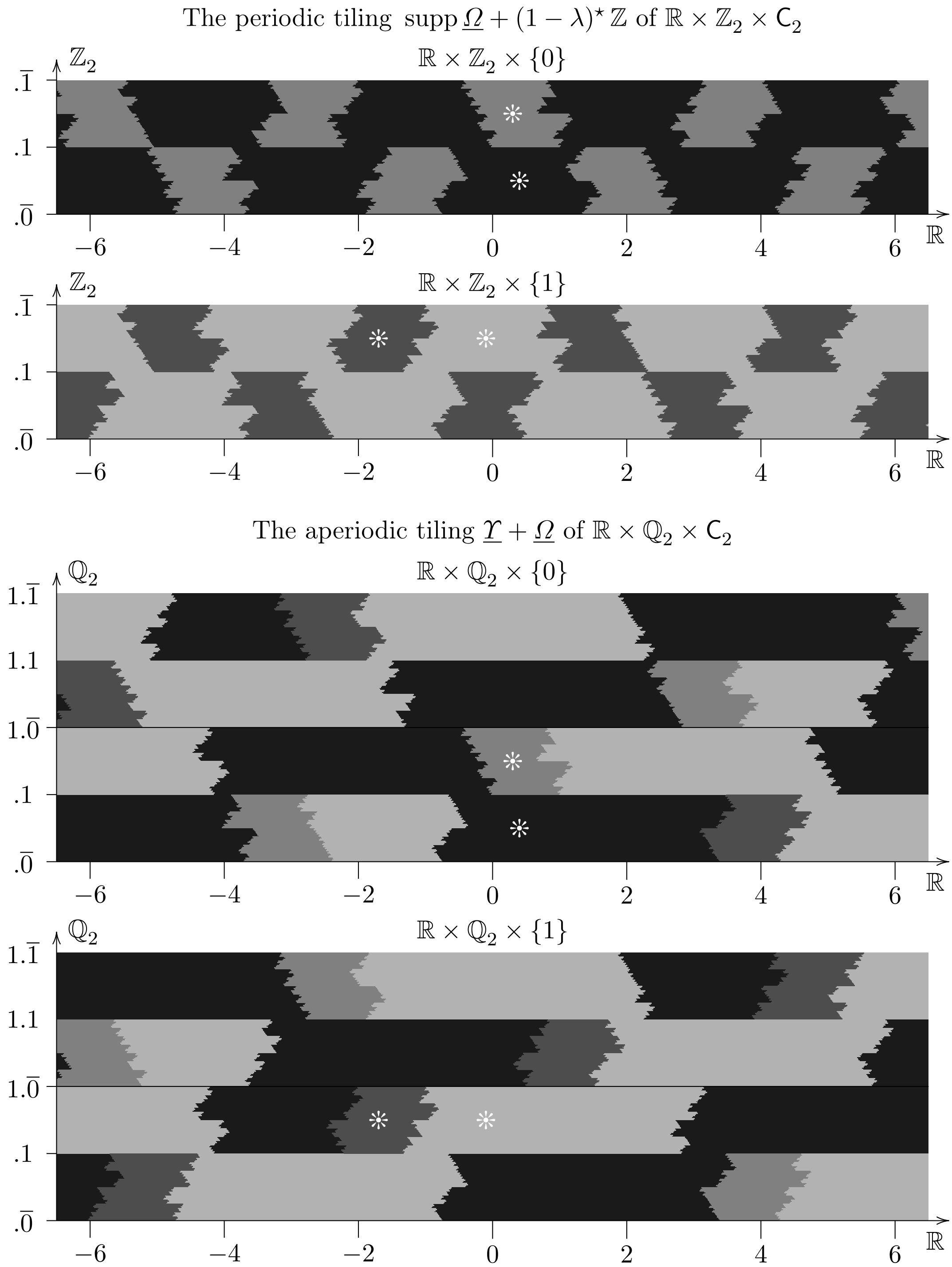


Figure 6c.2.: A periodic (top) and the aperiodic (bottom) tiling for the σ_{cob}-substitution. The tiles of the tilings are coloured as follows: Ω_a in very dark gray, Ω_A in light gray, Ω_b in medium gray and Ω_B in dark gray. The tiles $\Omega_i + 0$ are marked by "✻". We have chosen the representatives $\{0, 1\}$ for the height group $\mathcal{L}_{\text{ext}}/\mathcal{L}'_{\text{ext}} \cong \mathsf{C}_2$.

substitution with inflation factor $3+2\sqrt{2}$. One also establishes that the solution of the IFS (in the one-dimensional) internal space is given by

$$\varOmega_a = [0\,,\tfrac{1}{2}\sqrt{2}],\quad \varOmega_b = [\tfrac{1}{2}\sqrt{2}\,,1]\quad\text{and}\quad \varOmega_c = [\tfrac{1}{2}\sqrt{2}-\tfrac{1}{2}\,,0] \approx [-0.207\,,0].$$

Moreover, we obtains the following sets $\varUpsilon_i = ([0\,,\ell_i[)$:

$$\varUpsilon_a = \varUpsilon_b = \left\{\ldots,\,-\sqrt{2}-1,\,-\tfrac{1}{2}\sqrt{2}-\tfrac{3}{2},\,0,\,1,\,\tfrac{1}{2}\sqrt{2}+\tfrac{3}{2},\ldots\right\}\quad\text{and}$$
$$\varUpsilon_c = \left\{\ldots,\,-\sqrt{2}-1,\,-\tfrac{1}{2}\sqrt{2}-\tfrac{3}{2},\,0,\,\tfrac{1}{2}\sqrt{2}+\tfrac{3}{2},\ldots\right\},$$

wherefore $\underline{\varUpsilon}+\underline{\varOmega}$ is a tiling of covering degree 1 a.e. Consequently, this substitution yields a multi-component model set, even though it is a reducible Pisot substitution.

We now extend the alphabet of this substitution in such a way that the covering degree of the internal space is 2 a.e. This can be achieved by

$$a \mapsto a\tilde{c}baaa,\quad b \mapsto \tilde{a}c\tilde{b}baa\quad c \mapsto c\tilde{b}baa\quad \tilde{a} \mapsto \tilde{a}c\tilde{b}\tilde{a}\tilde{a}\tilde{a},\quad \tilde{b} \mapsto a\tilde{c}b\tilde{b}\tilde{a}\tilde{a}\quad\text{and}\quad \tilde{c} \mapsto \tilde{c}b\tilde{b}\tilde{a}\tilde{a}.$$

The eigenvalues of the corresponding substitution matrix are here given by $\{3\pm 2\sqrt{2}, 2\pm\sqrt{3}, 1\}$ (the eigenvalue 1 occurs twice), where $3+2\sqrt{2}$ is the dominant eigenvalue. Moreover, a left eigenvector to this dominant eigenvalue is here given by $\ell = (\ell_a, \ell_a, 1, \ell_a, \ell_a, 1)$ with $\ell_a = \frac{1}{2}+\frac{1}{2}\sqrt{2}$ as before. Furthermore, this substitution is constructed in such a way that $\varOmega_i = \varOmega_{\tilde{i}}$ and $\varUpsilon_i = \varUpsilon_{\tilde{i}}$, where $\varOmega_a$, $\varOmega_b$, $\varOmega_c$ (respectively, $\varUpsilon_a$, $\varUpsilon_b$, $\varUpsilon_c$) are given as in the previous substitution. Therefore, the covering degree of $\underline{\varUpsilon}+\underline{\varOmega}$ is now 2 a.e. Consequently, it is not a multi-component model set.

As we will see, β-substitutions (see Section 6c.4) sometimes provide examples of reducible Pisot substitutions. To our knowledge, these have been the first examples of reducible Pisot substitutions which have been considered. It has also been observed in [117, 118] that – in our language – $\underline{\varUpsilon}+\underline{\varOmega}$ has constant covering degree a.e., where $\varUpsilon_i = \Lambda([0\,,\ell_i[)$ (thus, that Theorem 5.147 is not restricted only to the "irreducible" case), but one may also find appropriate tilings by hyperpolygons (compare Section 6.8). However, there is not such a canonical choice for the hyperpolygons as in Section 6.8, and they are essentially found[4] by trial and error in [117]. The biggest difference of the reducible case is that one does (in general[5]) not have a periodic tiling of the internal space by $\operatorname{supp}\underline{\varOmega}$.

If one has a reducible Pisot substitution σ over n symbols, one may also derive a corresponding (irreducible) Pisot substitution as follows, [43, Section 2] and [118, Section 2.1]:

- The characteristic polynomial $p(x)$ of $\boldsymbol{S}\sigma$ (with inflation factor λ) decomposes into irreducible (over $\mathbb{Q}$) can be factored as follows:

$$p(x) = \det(x\,\boldsymbol{E} - \boldsymbol{S}\sigma) = \operatorname{Irr}(\lambda, \mathbb{Q}, x)\cdot q(x),$$

[4] Actually, the existence of such a tiling by hyperpolygons might not be a big surprise: We know the sets $\varUpsilon_i = \Lambda([0\,,\ell_i[)$ and the measures of the polygons $\mu(P_i)$ up to a multiplicative constant (since they should be components of a left PF-eigenvector of the substitution matrix, see Lemma 6.83) and therefore, since they should tile the internal space, also their exact expected values. So the task is to find such polygons of given measures such that $\underline{\varUpsilon}+\underline{P}$ is a tiling. Note that we have FLC for $\underline{\varUpsilon}+\underline{P}$ and that *any* choice can be used since it automatically fulfils Equation (6.9) on p. 255 and – by uniqueness – one always has $\underline{P}^{(m)} \to \underline{\varOmega}$.

[5] In [117, p. 114 & Fig. 16] an example is treated where $\operatorname{supp}\underline{\varOmega}$ does not admit a periodic tiling but the union of $\operatorname{supp}\underline{\varOmega}$ with a translate of its reflected version $-\operatorname{supp}\underline{\varOmega}$ admits a periodic tiling.

where $q(x) \in \mathbb{Q}[x]$ is a monic but maybe reducible polynomial, which is relatively prime to $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$ (since, by assumption, λ is a simple eigenvalue of $\boldsymbol{S}\sigma$). Using the extended Euclidean algorithm, one may find polynomials $r(x), s(x) \in \mathbb{Q}[x]$ with $\deg r(x), \deg s(x) \leq \min\{\deg \mathrm{Irr}(\lambda, \mathbb{Q}, x), \deg q(x)\}$ such that $r(x) \cdot \mathrm{Irr}(\lambda, \mathbb{Q}, x) + s(x) \cdot q(x) = 1$.

- The matrices $\boldsymbol{P}_1 = r(\boldsymbol{S}\sigma) \cdot \mathrm{Irr}(\lambda, \mathbb{Q}, \boldsymbol{S}\sigma)$ and $\boldsymbol{P}_2 = s(\boldsymbol{S}\sigma) \cdot q(\boldsymbol{S}\sigma)$ are the complementary projections on the eigenspace V of λ and its algebraic conjugates and on its orthogonal complement W, *i.e.*, we have $\mathbb{R}^n = V \oplus W$ with $V = \ker(\boldsymbol{P}_1)$ and $W = \ker(\boldsymbol{P}_2)$. Moreover, the dimension of V equals the degree of $\mathrm{Irr}(\lambda, \mathbb{Q}, x)$.
- Using the restriction of $\boldsymbol{S}\sigma$ to V is then the substitution matrix of a Pisot substitution with the PV-number λ. More precisely, V is spanned by $\deg \mathrm{Irr}(\lambda, \mathbb{Q}, x)$ vectors. Applying $\boldsymbol{S}\sigma$ to such a vector yields a linear combination of these vectors (since $V \oplus W$ is an $\boldsymbol{S}\sigma$-invariant splitting), which gives the substitution rule.

It is the hope that the (irreducible) Pisot substitution obtained by this method still has a connection to the original reducible Pisot substitution. For the following example, which appears in [136, Sections 2.2 & 2.3], this is indeed the case. For an example on three letters that reduces to two letters, see [43, Example 3.5].

Example 6c.16. We consider the following reducible Pisot substitution σ_{Dirk1}:

$$a \mapsto d, \qquad b \mapsto cd, \qquad c \mapsto bcd \quad \text{and} \quad d \mapsto abcd.$$

The minimal polynomial $p(x)$ of $\boldsymbol{S}\sigma_{\mathrm{Dirk1}}$ factors as $p(x) = (x^3 - 3\,x^2 + 1) \cdot (x + 1)$, wherefore the eigenvalues of $\boldsymbol{S}\sigma_{\mathrm{Dirk1}}$ are given by $\{-1, \lambda, \lambda_2, \lambda_3\}$ where λ, λ_2 and λ_3 are the roots of the irreducible polynomial $x^3 - 3\,x^2 + 1 = \mathrm{Irr}(\lambda, \mathbb{Q}, x)$ and are approximately given by $\lambda \approx 2.879$, $\lambda_2 \approx 0.653$ and $\lambda_3 \approx -0.532$. A left PF-eigenvector of $\boldsymbol{S}\sigma_{\mathrm{Dirk1}}$ is given by

$$\ell = \left(\frac{1}{3}(2\,\lambda^2 - 5\,\lambda - 1), \frac{1}{3}(-\lambda^2 + 4 - 1), 1, \frac{1}{3}(\lambda^2 - \lambda - 2)\right) \approx (0.395, 0.742, 1, 1.137).$$

We observe that $\ell_a + \ell_b = \ell_d$. Furthermore, we observe that we have $\mathcal{L} = \mathcal{L}' = \langle \ell_a, \ell_b, \ell_c, \ell_d \rangle_{\mathbb{Z}} = \langle \ell_b, \ell_c, \ell_d \rangle_{\mathbb{Z}}$, and that we may construct a CPS with $H = \mathbb{R}^2$, where one $\mathbb{R}$ arises from the Galois conjugate λ_2 and the other from the Galois conjugate λ_3 of λ (consequently, the star-map $\mathbb{Q}(\lambda) \to \mathbb{R}^2$ is given by $x \mapsto (\sigma_2(x), \sigma_3(x))$, where σ_i is the Galois automorphism given by the action $\lambda \mapsto \lambda_i$).

We now apply the above method to derive a corresponding (irreducible) Pisot substitution: The extended Euclidean algorithm yields

$$p(x) = -\frac{1}{3} \cdot (x^3 - 3\,x^2 + 1) + \frac{1}{3}\,x^2 - 4\,x + 4) \cdot (x + 1).$$

Consequently, the projection on V is given by

$$\boldsymbol{P}_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix},$$

and the kernel of $\boldsymbol{P}_1$ (applied to left, *i.e.*, to row vectors) is given by $v_1 = \frac{1}{3}(-1,2,0,1)$, $v_2 = (0,0,1,0)$ and $v_3 = \frac{1}{3}(1,1,0,2)$. One has

$$v_1(\boldsymbol{S}\sigma_{\mathrm{Dirk1}}) = v_2 + v_3, \qquad v_2(\boldsymbol{S}\sigma_{\mathrm{Dirk1}}) = v_1 + v_2 + v_3 \quad \text{and} \quad v_3(\boldsymbol{S}\sigma_{\mathrm{Dirk1}}) = v_2 + 2\,v_3,$$

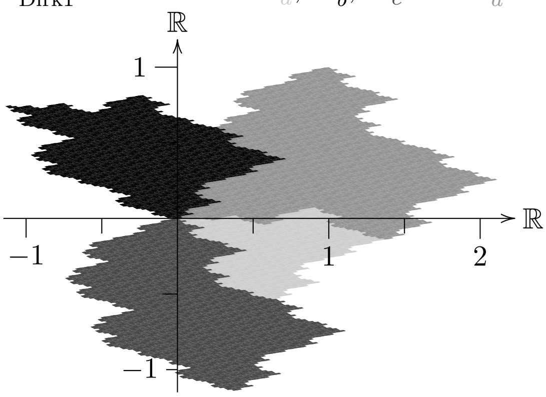
wherefore – with this choice of the basis vectors v_i – the restriction of $\boldsymbol{S}\sigma_{\mathrm{Dirk1}}$ to V is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

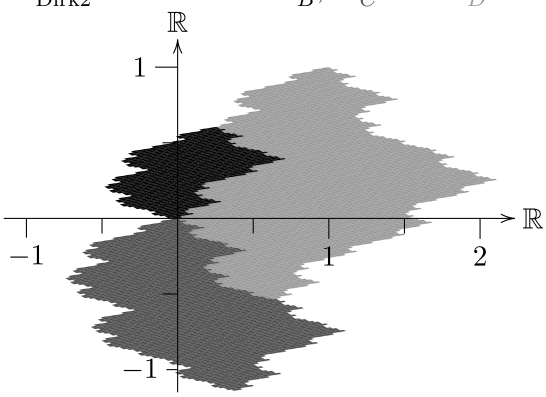
Indeed, the characteristic polynomial of this matrix is given by $x^3 - 3\,x^2 + 1$ and a left PF-eigenvector by $(\ell_b\,,\ell_c\,,\ell_d)$. Consequently, a possible Pisot substitution having this last matrix as substitution matrix is given by $\sigma_{\mathrm{Dirk2}} : B \mapsto CD$, $C \mapsto BCD$ and $D \mapsto DCD$ (and one then has $\ell_B = \ell_b$, $\ell_C = \ell_c$ and $\ell_D = \ell_d$). Moreover, one obtains the same CPS for both substitutions.

As observed in [136, Section 2.2], the connection between this two substitutions is even closer: in a fixed point of σ_{Dirk1} (more precisely, of $\sigma^2_{\mathrm{Dirk1}}$), the symbol a is always followed by b, $ab \mapsto dcd$ and $\ell_a + \ell_b = \ell_d$. Consequently, identifying C with c, "projecting" d and all pairs ab to D and the remaining b's to B, one obtains the substitution σ_{Dirk2} from σ_{Dirk1}. This procedure, however, has the effect that $\Lambda_c = \Lambda_C$, $\Lambda_a \cup \Lambda_d = \Lambda_D$ and $\Lambda_b \supset \Lambda_B$. But the corresponding inclusions also hold for the attractors of the corresponding IFS in internal space, where we therefore have $\Omega_c = \Omega_C$, $\Omega_a \cup \Omega_d = \Omega_D$ and $\Omega_b \supset \Omega_B$ as the following pictures confirm (the horizontal axis corresponds to the real space arising from λ_2):

σ_{Dirk1}-substitution: Ω_a, Ω_b, Ω_c and Ω_d



σ_{Dirk2}-substitution: Ω_B, Ω_C and Ω_D



Even more surprising is the effect on the tilings $\underline{\Upsilon} + \underline{\Omega}$: One has $\Upsilon_b = \Lambda([0\,,\ell_b[) = \Upsilon_B$, $\Upsilon_c = \Lambda([0\,,\ell_c[) = \Upsilon_C$ and $\Upsilon_d = \Lambda([0\,,\ell_d[) = \Upsilon_D$, still one can confirm that both collections $\underline{\Upsilon}+\underline{\Omega}$ *are* tilings of $\mathbb{R}^2$, see Figure 6c.3 (*e.g.*, we observe that the collection $\Upsilon_c + \Omega_c = \Upsilon_C + \Omega_C$ can be found in in both tilings; amazingly, everything matches!). Consequently, both substitutions yield multi-component model sets.

6c.4. β-Substitutions

A certain class of (in general reducible) Pisot substitutions is connected to the study of β-expansions. Rauzy himself used β-expansion in his famous article [306, Section 2] (see Example

The aperiodic tiling associated to σ_{Dirk1}

The aperiodic tiling associated to σ_{Dirk2}

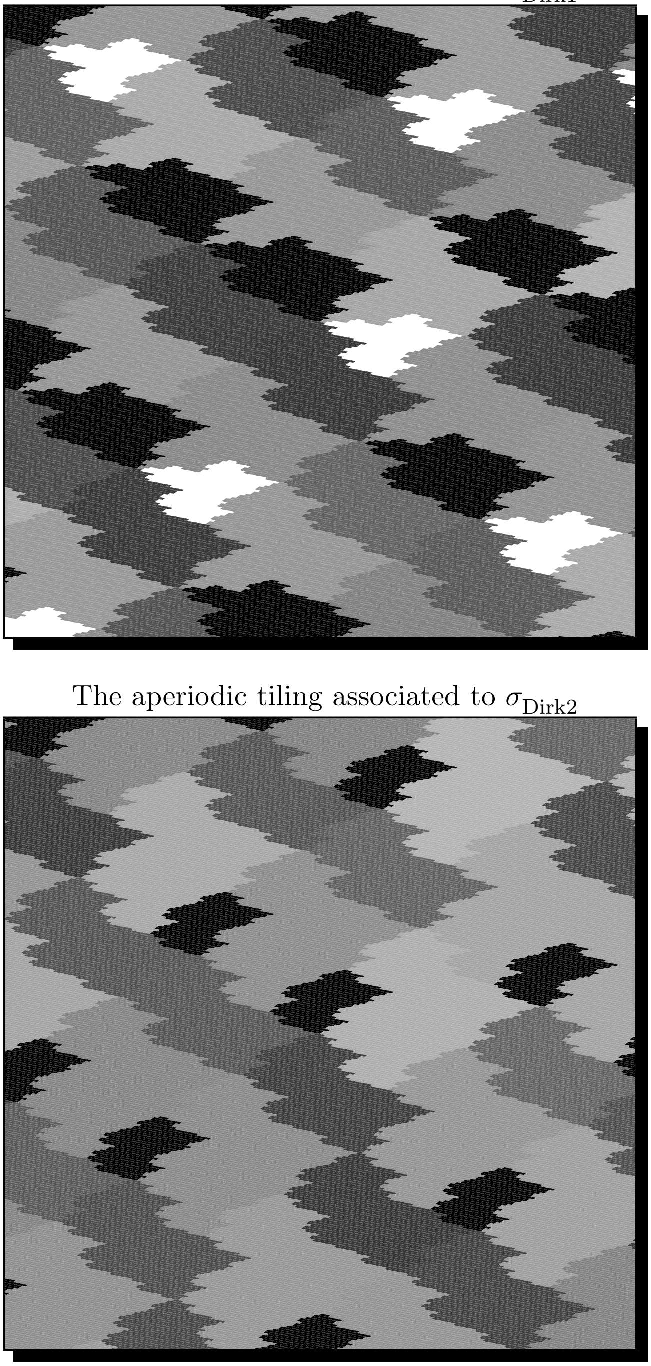


Figure 6c.3.: The aperiodic tilings in $\mathbb{R}^2$ for σ_{Dirk1} and σ_{Dirk2} are shown, both in the range $[-3\,,3]\times[-3\,,3]$ (the horizontal axis corresponds to the real space arising from $\lambda_2 \approx 0.653$, the vertical axis to the real space arising from $\lambda_3 \approx -0.532$). We use the following colouring for the (proto)tiles: Ω_a in white, Ω_b and Ω_B in black, Ω_c and Ω_C in dark gray and Ω_d and Ω_D in light gray. Note that we have $\Upsilon_b = \Upsilon_B$, $\Upsilon_c = \Upsilon_C$ and $\Upsilon_d = \Upsilon_D$, as well as $\Omega_c = \Omega_C$, $\Omega_d \cup \Omega_a = \Omega_D$ and $\Omega_b \supset \Omega_B$.

6c.19 below) to study the tribonacci substitution $a \mapsto ab$, $b \mapsto ac$ and $c \mapsto a$. Consequently, the attractor $\underline{\Omega}$, or any attractor of an associated IFS $\boldsymbol{\Theta}^\star$ of a Pisot substitution, is often called a *Rauzy fractal* (although only its boundary is a fractal!).
We first define the β-transformation, which can also be found in books about symbolic dynamics, *e.g.*, [81, Section 12.1] and [108, Beispiel 3]. Also compare [8, 70, 280, 310]

Definition 6c.17. Let $\beta > 1$ be a real number and set $b = \lfloor \beta \rfloor$ (if $\beta \in \mathbb{N}$, then $b = \beta = 1$). We obtain the *β-expansion* of any nonnegative number x as follows: The *β-transformation* is the piecewise linear transformation on $[0\,,1[$ defined by $T_\beta :\ x \mapsto \beta\, x - \lfloor \beta\, x \rfloor = \beta\, x \bmod 1$. By iterating this map and considering its trajectory[6]

$$x \xrightarrow{a_1} T_\beta(x) \xrightarrow{a_2} T_\beta^2(x) \xrightarrow{a_3} \dots$$

with $a_i = \lfloor \beta\, T_\beta^{i-1}(x) \rfloor \in \{0, \dots, b\}$, we obtain the expansion of $x = a_1\, \beta^{-1} + a_2\, \beta^{-2} + \dots = .a_1 a_2 \dots$ for $x \in [0\,,1[$ (we use "." to mark the zeroth position of the expansion, like in the decimal case). We also use the notation $d_\beta(x) = .a_1 a_2 \dots$ to denote the β-expansion of x.

Alternatively (*e.g.*, [81, Theorem 12.1.3]), the β-expansion of $x \in [0\,,1[$ coincides with the *greedy expansion of x in base β*, *i.e.*, one has

$$|x - \sum_{i=L}^{N} a_i \beta^{-i}| < \beta^{-N}$$

for any $N \geq 0$. By using the *greedy algorithm*, such an expansion always exists for any $x \in [0\,,1[$.

Formally, we may consider the trajectory of 1:

$$1 \xrightarrow{a_1} T_\beta(1) \xrightarrow{a_2} T_\beta^2(1) \xrightarrow{a_3} \dots$$

We call $a_1 a_2 a_3 \dots$ the expansion of one and denote it by $d_\beta(1)$. Define[7]

$$d_\beta^*(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is not finite} \\ .\overline{a_1 \dots a_{d-1}(a_d - 1)} & \text{if } d_\beta(1) = a_1 \dots a_d \text{ with } a_d \neq 0, \end{cases}$$

where $.\overline{a_1 \dots a_k} = .a_1 \dots a_k a_1 \dots a_k a_1 \dots$ is the periodic expansion. One can also say that $d_\beta^*(1) = \lim_{x \nearrow 1} d_\beta(x)$ in the product topology of one-sided sequences over $\mathcal{A} = \{0, \dots, b\}$.

A sequence (finite or infinite) $a_1 a_2 a_3 \dots$ over the alphabet $\mathcal{A} = \{0, 1, \dots, b\}$ is said to be *admissible* if all its *right truncations*, *i.e.*, all (sub-)sequences $a_n a_{n+1} a_{n+2} \dots$ for $n \geq 0$ (respectively $n \in \mathbb{Z}$), are lexicographically less than $d_\beta^*(1)$.

We now define the β-expansion for any real number $x \geq 0$. Let $x \geq 0$, then there is an $L \geq 0$ such that $\beta^{-L+1} x \in [0, 1[$, and we can express x in the form

$$x = a_{-L}\, \beta^L + a_{-L+1} \beta^{L-1} + \dots + a_M \beta^{-M} = (a_{-L} a_{-L+1} \dots a_M)_\beta \tag{6c.1}$$

where $a_i \in \{0, \dots, b\}$, $a_{-L} \neq 0$, $L \in \mathbb{Z}$ and $M \in \mathbb{Z} \cup \{\infty\}$ (sometimes, one calls $\deg_\beta(x) = L$ the *degree* and $\operatorname{ord}_\beta(x) = -M$ the *order* of the β-expansion of x). One can show that a sequence is the β-expansion of some real number iff it is admissible, see [280, Theorem 3] (compare [193]).

[6]Thus, β-expansions arise from orbits of the piecewise-monotone β-transformation of the unit interval.
[7]We note that $d_\beta^*(1)$ is denoted carry(β) in [373, Section 9] and [295, p. 3337].

Remark 6c.18. With regard to symbolic dynamics, we note that the β-transformation T_β is a subshift of finite type iff $d_\beta(1)$ is finite [70, Prop. 4.1] and a sofic shift iff $d_\beta(1)$ is ultimately periodic [70, Prop. 4.2]. We note (see the following paragraph) if β is a PV-number, then T_β is sofic [70, Prop. 4.3]. Conversely, if T_β is sofic, then β is a *Perron number* [70, Prop. 4.4], *i.e.*, β is an algebraic integer greater than 1 and all its conjugates have modulus less than β. Also see Section 7.4 on "shift of finite type" and "sofic shift".

We now make the connection to PV-numbers. We use the notation $M_{\geq 0} = \{x \in M \mid x \geq 0\}$ for a set $M \subset \mathbb{R}$. Then, one has:

- Let $\mathbb{Q}(\beta)$ be the smallest subfield of the reals containing β and $\mathrm{Per}(\beta)$ the set of (eventually) periodic points of the β-transformation, *i.e.*, the set of points whose orbits under T_β are finite. If β is a PV-number, then[8] $\mathrm{Per}(\beta) = \mathbb{Q}(\beta)_{\geq 0}$ by [330, Theorem 3.1] and [69, Prop. 2]. Consequently, PV-numbers behave exactly like (ordinary) integers with respect to β-expansion: if $x, y \in \mathrm{Per}(\beta)$, then one also has $x + y \in \mathrm{Per}(\beta)$.

- Let $\mathrm{Fin}(\beta)$ be the set of points with finite β-expansion, *i.e.*, the set of points whose orbits under T_β end up at zero. If $\mathbb{Z}_{\geq 0} \subset \mathrm{Fin}(\beta)$, then β is a PV-number, see [5, Proposition 1] (compare [140, Lemma 1(a)]). If $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}$, then β is a PV-number (see [140, Lemma 1(b)]), but not all PV-numbers have this property; *e.g.*, let β be the PV-number associated to the polynomial $x^3 - 3x^2 + 2x - 2$, then $d_\beta(1) = .2102$, but $d_\beta(6) = 20.21\overline{00112}$, see [140, Example 1]. In fact, if the irreducible polynomial $\mathrm{Irr}(\beta, \mathbb{Q}, x)$ of a PV-number β is 1 at 0, then $d_\beta(1)$ is not finite, see [7] and [3, Prop. 1].

  In view of the above "additive" property for numbers in $\mathrm{Per}(\beta)$, we note that one has a similar property for numbers in $\mathrm{Fin}(\beta)$: Let β be a PV-number and $x, y \in \mathrm{Fin}(\beta)$ with $x > y$ and both with β-expansion of order $\mathrm{ord}_\beta(x), \mathrm{ord}_\beta(y) \geq N$ (*i.e.*, for the β-expansion of $x = .a_1a_2a_3\ldots$ one has $a_k = 0$ for all $k > N$, and similar for y). Then there exists a constant $C = C(\beta) \in \mathbb{N}$, such that if $x+y \in \mathrm{Fin}(\beta)$ (respectively $x-y \in \mathrm{Fin}(\beta)$), then $\mathrm{ord}_\beta(x + y) \geq N - C$ (respectively $\mathrm{ord}_\beta(x - y) \geq N - C$), see [140, Prop. 2].

- The condition $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}$ is equivalent to $\mathrm{Fin}(\beta) \supset \mathbb{Z}[\beta^{-1}]_{\geq 0}$ (see [140]) and, for a PV-number β, to $\mathrm{Fin}(\beta) \supset \mathbb{Z}[\beta]_{\geq 0}$ (see [4, Theorem 1]). Note that for an algebraic integer one has $\mathbb{Z}[\beta] \subset \mathbb{Z}[\beta, \beta^{-1}] = \mathbb{Z}[\beta^{-1}]$.

For later use, we say that a PV-number β has property **(F)** (the "finiteness property") if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0} \tag{F}$$

holds.

Suppose β is a PV-number. We now show how β-expansions are connected to (reducible) Pisot substitutions, and thus deduce statements about β-expansions.

We define

$$S_{(a_La_{L+1}\cdots a_M)_\beta} = \{x \in \mathrm{Fin}(\beta) \mid x = (a_Ka_{K+1}\ldots a_{L-1}a_La_{L+1}\ldots a_M)_\beta\},$$

[8] Furthermore, if $\mathbb{Q} \cap [0,1[\subset \mathrm{Per}(\beta)$, then β is either a PV- or a Salem number [330, Theorem 2.4], while $\mathrm{Per}(\beta) \cap \mathbb{Q}$ is nowhere dense in $[0,1[$ if $\beta > 1$ is an algebraic integer but neither a PV- or a Salem-number.

i.e., all elements in $\mathrm{Fin}(\beta)$ whose greedy expansion has the same *tail part* $(a_L a_{L+1} \ldots a_M)_\beta$. A special role is played by the set

$$S_\varnothing = \{x \in \mathrm{Fin}(\beta) \mid x = a_K a_{K+1} \ldots a_{-1} a_0 .\},$$

i.e., all elements in $\mathrm{Fin}(\beta)$ whose greedy expansion involves no terms of the form β^{-m} for $m > 0$. Before we proceed, we look at an example.

Example 6c.19. [361, pp. II/17–20] Let β be the PV-number if characteristic polynomial $\mathrm{Irr}(\lambda, \mathbb{Q}, x) = x^3 - x^2 - x - 1$. One has $\beta \approx 1.839$ (and the algebraic conjugates are a complex conjugate pair $-0.420 \pm i\,0.606$) and, since $1 = \beta^{-1} + \beta^{-2} + \beta^{-3}$, one obtains $d_\beta(1) = .111$ and $d^*_\beta = .\overline{110}$. Consequently, all sequences over $\mathcal{A} = \{0, 1\}$ with no three consecutive 1's are admissible.

It is therefore reasonable, to consider the following three sets $S_{0.}$, $S_{01.}$ and $S_{011.}$, *i.e.*, the subsets of $\mathrm{Fin}(\beta)$ which end in a 0, in 01 respectively in 011. Moreover, one has $S_\varnothing = S_{0.} \cup S_{01.} \cup S_{011.}$. Since $\beta\, S_{\ldots a_0.} = S_{\ldots a_0 0.}$, one can derive an EMFS for these sets:

$$\begin{aligned} S_{0.} &= \beta\, S_\varnothing = \beta\, S_{0.} \;\cup\; \beta\, S_{01.} \;\cup\; \beta\, S_{011.} \\ S_{01.} &= \beta\, S_{0.} + 1 \\ S_{011.} &= \beta\, S_{01.} + 1 \end{aligned}$$

But this is simply the EMFS associated to the tribonacci substitution $a \mapsto ab$, $b \mapsto ac$ and $c \mapsto a$, with the normalisation of the natural lengths $\ell_a = 1$ (then $\ell_b = \beta - 1$ and $\ell_c = \beta^2 - \beta - 1$). Consequently, we can identify $\Lambda_a = S_{0.}$, $\Lambda_b = S_{01.}$ and $\Lambda_c = S_{011.}$.

We also note that we can now define a CPS and a star-map for such β-expansions and therefore obtain $\Omega_a = \mathrm{cl}_\mathbb{C}\, S^\star_{0.}$, $\Omega_b = \mathrm{cl}_\mathbb{C}\, S^\star_{01.}$, $\Omega_c = \mathrm{cl}_\mathbb{C}\, S^\star_{011.}$ and $\mathrm{supp}\, \underline{\Omega} = \mathrm{cl}_\mathbb{C}\, S^\star_\varnothing$.

Actually, the method in the last example generalises if β is a PV-number, wherefore we can associate to a β-expansion a (reducible) Pisot substitution, see [373, Section 9], [294, p. 94 ff.] and [295, p. 3336]: Suppose $d^*_\beta(1) = a_1 a_2 \ldots a_q \overline{a_{q+1} \ldots a_{q+p}}$ and set $\ell_1 = 1$. Then, $\beta \cdot [0\,, \ell_1[= \beta\,[0\,,1[$ can be decomposed into $a_1 = b$ intervals of length 1 and one of length $\beta - b = \beta - a_1$, *i.e.*,

$$\beta\,[0\,,1[= \;[0\,,\beta[= \;[0\,,1[\cup [1\,,2[\cup \;\ldots\; \cup\; [a_1 - 1\,, a_1[\cup [a_1\,, \beta[.$$

Note that the left endpoints of all these intervals have β-expansion $x_0.$ where $0 \le x_0 \le b$; therefore they are admissible and in $S_\varnothing$. We set $\ell_2 = \beta - a_1$, then $\beta\,\ell_2$ can be decomposed into a_2 intervals of length 1 and one of length $\beta^2 - a_1\,\beta - a_2$. We repeat this procedure and note that ℓ_{p+q} can be subdivided (by the periodicity) in a_{q+p} intervals of length 1 and one interval ℓ_{q+1}. Therefore $S_\varnothing$ is the set of left endpoints of the substitution over $\mathcal{A} = \{1, \ldots, (q+p)\}$

$$\begin{array}{lcl} 1 & \mapsto & 1^{a_1}\, 2 \\ 2 & \mapsto & 1^{a_2}\, 3 \\ & \vdots & \\ (q+p-1) & \mapsto & 1^{a_{q+p-1}}\, (q+p) \\ (q+p) & \mapsto & 1^{a_{q+p}}\, (q+1) \end{array}$$

using the normalisation of the natural lengths such that $\ell_1 = 1$. We also use the short-hand notation 1^a for $1 \ldots 1$ (repeated a times).

For the attractor $\underline{\Omega}$ in internal space H or, more precisely, in H_{ext}, we note that we can make the following identifications: $\Omega_2 = \text{cl}_H\, S^{\star}_{0a_1.}$, $\Omega_3 = \text{cl}_H\, S^{\star}_{0a_1a_2.}$, $\dots$, $\Omega_{q+p} = \text{cl}_H\, S^{\star}_{0a_1\dots a_{q+p-1}}$, with the exception of $\Omega_{q+1} = \text{cl}_H\, S^{\star}_{0a_1\dots a_q.} \cup \text{cl}_H\, S^{\star}_{0a_1\dots a_{q+p}.}$ and (the "remaining" parts of $S_{\varnothing}$)

$$\Omega_1 = \text{cl}_H\, S^{\star}_{0.} \cup \text{cl}_H\, S^{\star}_{01.} \cup \dots \text{cl}_H\, S^{\star}_{0(a_1-1).} \cup \dots \cup \text{cl}_H\, S^{\star}_{0a_1\dots(a_{q+p}-1).}.$$

We note that the EMFS $\boldsymbol{\Theta}$ (and therefore also the IFS $\boldsymbol{\Theta}^{\star}$) has a particular nice form since all translational parts $t_{(a)}$ of the maps $f \in \boldsymbol{\Theta}$ with $f = t_{(a)} \circ f_0$ (where $f_0(x) = \beta\, x$) are translations by an integer in $\{0, \dots, b\}$. More precisely, the EMFS $\boldsymbol{\Theta}$ associated to a PV-number β with $d^{*}_{\beta}(1) = .a_1 \dots a_q\overline{a_{q+1} \cdots a_{q+p}}$ is given as follows, where we use the notation $f_a = t_{(a)} \circ f_0$ and a set with element f_{-1} is considered to be the empty set:

$$\begin{pmatrix}
\{f_0,\dots,f_{a_1-1}\} & \{f_0,\dots,f_{a_2-1}\} & \{f_0,\dots,f_{a_3-1}\} & \cdots & \{f_0,\dots,f_{a_q-1}\} & \{f_0,\dots,f_{a_{q+1}-1}\} & \cdots & \{f_0,\dots,f_{a_{q+p-1}-1}\} & \{f_0,\dots,f_{a_{q+p}-1}\} \\
\{f_{a_1}\} & \varnothing & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varnothing \\
\varnothing & \{f_{a_2}\} & \varnothing & \cdots & \cdots & \cdots & \cdots & \cdots & \varnothing \\
\vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\varnothing & \cdots & \varnothing & \{f_{a_q}\} & & \varnothing & \cdots & \varnothing & \{f_{a_{q+p}}\} \\
\vdots & \cdots & \cdots & \ddots & & \ddots & & \cdots & \vdots \\
\varnothing & \cdots & \cdots & \cdots & \cdots & \varnothing & & \{f_{a_{q+p-1}}\} & \varnothing
\end{pmatrix}$$

Thus, nonempty sets appear on the first subdiagonal, at the entry in the $(q+1)$-st row and $(q+p)$-th column and (possibly) in the first row. The property that only translational parts by integers appear allows us to derive further results for such (reducible) Pisot substitutions that are associated to β-expansions, and which we now call *β-substitutions*.

We now want to apply our criteria to deduce that $S_{\varnothing} = \varLambda_1 \cup \dots \cup \varLambda_{p+q}$ is a model set in $\mathbb{R}$. This, of course, is equivalent to the statement that we have a tiling of H_{ext} by the sets $\varOmega_1, \dots, \varOmega_{p+q}$. We note that by construction we have $\mathcal{L} = \mathbb{Z}[\beta]$, wherefore the lattice is given by the diagonal embedding[9] of $\mathbb{Z}[\beta]$ into $\mathbb{R} \times H$ in the unimodular case. Otherwise, one has $\mathcal{L}_{\text{ext}} = \mathbb{Z}[\beta, \beta^{-1}] = \mathbb{Z}[\beta^{-1}]$ and its corresponding diagonal embedding into $\mathbb{R} \times H_{\text{ext}}$. Consequently, the sets $\varUpsilon_i$ are given by (noting that in the unimodular case one has $\mathbb{Z}[\beta] = \mathbb{Z}[\beta^{-1}]$)

$$\varUpsilon_i = \Lambda([0\,, \ell_i[) = \big(\mathbb{Z}[\beta^{-1}] \cap [0\,, \ell_i[\big)^{\star}\,,$$

and $S_{\varnothing}$ is a model set iff $\underline{\varUpsilon} + \underline{\varOmega}$ is a tiling of H_{ext}. We recall that by construction the natural lengths ℓ_i are pairwise different and $0 < \ell_i \leq 1$ (with $\ell_1 = 1$) for all $i \in \mathcal{A}$. We note the following observations:

- The sets $\varOmega_1, \dots, \varOmega_{p+q}$ are pairwise measure-disjoint. This is easily seen using the strong coincidence condition, see Definition 6.90: Since any power k of $\sigma^k(1)$ starts with 1, we establish that for every $i \in \mathcal{A}$ there is a $k_i \in \mathbb{N}$ such that $\sigma^{n_i}(i)$ starts with 1 for all $k_i \geq n_i$. But this is certainly the case if $q = 0$ in $d^{*}_{\beta}(1) = .a_1 \dots a_q\overline{a_{q+1} \cdots a_{q+p}}$ (since,

[9] We implicitly assume that the height group is trivial; however, here we deal with reducible Pisot substitutions, wherefore Lemma 6.34 does *not* hold. But the proof of that lemma indicates that it is also hard for a reducible Pisot substitution to have a non-trivial height group (since the symbol frequencies and the natural lengths have to "match"). Moreover, if there is a 2 in the β-expansion $d^{*}_{\beta}(1)$ of 1, then $1 \in \mathcal{L}'$ and thus also $1, \beta, \beta^2, \dots \in \mathcal{L}' = \mathcal{L}$.

at least, $\sigma(p) = 1^{a_{q+p+1}}$, and then $\sigma^2(p-1)$, $\sigma^3(p-2)$ *etc.* start with 1). Otherwise, at least one of the numbers $a_{q+1}, \ldots, a_{q+p}$ is greater than 0, and, again iteratively, the claim follows. Consequently, there is an power $K > 0$ such that $\sigma^K(i)$ starts with 1 for all $i \in \mathcal{A}$, wherefore the sets Ω_i are pairwise measure-disjoint.

- Let β be a PV-number with property **(F)**. Then, $x \in \mathbb{Z}[\beta^{-1}]_{\geq 0}$ has a finite β-expansion $x = a_L a_{L+1} \ldots a_{-1} a_0 . a_1 \ldots a_M$ and we set $x_1 = a_L a_{L+1} \ldots a_{-1} a_0.$ and $x_2 = .a_1 \ldots a_M$ (the "fractional part" of x). Obviously, one has $x_1 \in S_\varnothing$ and $x_2 \in \mathbb{Z}[\beta^{-1}] \cap [0\,, 1[$. Thus, we can associate x to some (and at least one) tiles in $\underline{\varUpsilon} + \underline{\varOmega}$: On the one hand, $x_1^\star \in \operatorname{supp} \underline{\varOmega}$ and therefore $x_1^\star \in \varOmega_j$ for at least one j, on the other hand, $x_2 \in \mathbb{Z}[\beta^{-1}] \cap [0\,, \ell_k[$ for some k (at least for 1). Moreover, there is at least one letter of $m \in \mathcal{A}$ such that both $x_1^\star \in \varOmega_m$ and $x_2 \in \mathbb{Z}[\beta^{-1}] \cap [0\,, \ell_m[$, otherwise the β-expansion of x would not be admissible.

- This last observation also shows the following: We may write

$$\mathbb{Z}[\beta^{-1}]_{\geq 0} = \operatorname{Fin}(\beta) = \bigcup_{\substack{.a_1 a_2 \ldots a_M \\ \text{admissible with } M > 0}} S_{.a_1 a_2 \ldots a_M}.$$

  Applying the star-map and taking the closure, one obtains a covering of H_{ext} by the tiles $\operatorname{cl}_{H_{\text{ext}}} S_{.a_1 a_2 \ldots a_M}$, which are given as translates the prototiles $\bigcup_{1 \leq \pi(i) \leq k} \varOmega_i$ with $1 \leq k \leq p+q$ and π denotes the permutation such that $\ell_1 = \ell_{\pi(1)} > \ell_{\pi^{-1}(2)} > \ldots > \ell_{\pi^{-1}(p+q)} > 0$ (in plain words, π orders the natural lengths ℓ_i in descending order). Note that these prototiles are actually the one used in [4, 6, 7, 9]. Note that $\operatorname{cl}_{H_{\text{ext}}} S_\varnothing^\star = \operatorname{cl}_{H_{\text{ext}}} S_{.0}^\star = \operatorname{supp} \underline{\varOmega}$ is also called the *central tile*.

- If β is a PV-number and an algebraic unit, then β has property **(F)** iff 0 is an exclusive inner point of the central tile, where we mean by "exclusive" that 0 does not belong to any other tile $\operatorname{cl}_{H_{\text{ext}}} S_{.a_1 a_2 \ldots a_M}$, see [4, Theorem 2], [5, Prop. 2], [6, Theorem 1] and their proofs. Moreover, this result also holds in the non-unimodular case.

- We now consider the tile substitution of the tiles $\varOmega_i$ in internal space H_{ext}, which can be deduced from $\boldsymbol{\varTheta}^\star$ respectively $\boldsymbol{\varTheta}^{\star\#}$. Since the sets $\varOmega_i$ are pairwise measure-disjoint, we may actually apply the tile substitution repeatedly to the collection $(\varOmega_i \mid i \in \mathcal{A})$ to obtain a bigger and bigger patch of measure-disjoint tiles. This is equivalent to saying that one considers $\underline{\check{\varUpsilon}}^k = \left(\boldsymbol{\varTheta}^{\star\#}\right)^k (\underline{\{0\}})$ (where $\underline{\{0\}} = (\{0\})_{i \in \mathcal{A}}$); the patch after k tile substitutions is given by $\underline{\varOmega} + \underline{\check{\varUpsilon}}^k$.

  We now consider the limit $\underline{\check{\varUpsilon}} = \lim_{k \to \infty} \underline{\check{\varUpsilon}}^k$ (in the local topology, see Definition 5.102). By the construction of the β-substitution, one has $\check{\varUpsilon}_i = (\operatorname{Fin}(\beta) \cap [0\,, \ell_i[)^\star$. Consequently, one has the following:

  - If β satisfies **(F)**, then 0 is an inner point of the central tile, wherefore $\underline{\varOmega} + \underline{\check{\varUpsilon}}$ is a tiling of H_{ext}. Moreover, all $\mathcal{L} = \mathbb{Z}[\beta^{-1}]$ have finite β-expansion, wherefore one has $\check{\varUpsilon}_i = \left(\mathbb{Z}[\beta^{-1}] \cap [0\,, \ell_i[\right)^\star = \varUpsilon_i$. Thus, as remarked in [294, p. 95] and [295, p. 3338], PV-numbers satisfying **(F)** are multi-component model sets.
  - Otherwise, we may proceed by using our algorithm to construct the overlap respectively boundary graph with the sets $\varXi(i, j, x)$ (if 0 is not an inner point of the central tile, $\underline{\{0\}}$ is not a finite seed for the sets $\varUpsilon_i = \left(\mathbb{Z}[\beta^{-1}] \cap [0\,, \ell_i[\right)^\star$, *i.e.*, $\underline{\varOmega} + \underline{\check{\varUpsilon}}$

is *not* a tiling but, for example, only a tiling of a half-space of H_{ext}). But since $\Omega_i + z^\star \cap \Omega_j + y^\star = \Xi(i,j,y-z) + z^\star$, we obtain the tiling property iff every $x \in \mathbb{Z}[\beta^{-1}]$ (more precisely, every $\mathbb{Z}[\beta^{-1}] \cap]-1\,,1[$, and, by symmetry, it suffices to consider only the nonnegative elements $\mathbb{Z}[\beta^{-1}] \cap [0\,,1[$) is the difference of two numbers in $\mathrm{Fin}(\beta) \cap [0\,,1[$. Thus, the following condition **(W)** ensures that the β-substitution yields a multi-component model set.

It was already observed in [177, Theorem 5.2.1] that the following property **(W)** (the "weak finiteness property") ensures that the dynamical system defined by the β-substitutions is pure point (on pure pointedness of a dynamical system, see Section 7.2). Also see [6, 8] on equivalent formulations of this condition.

We say that a PV-number β has property **(W)** (the "weak finiteness property") if

$$((\mathrm{Fin}(\beta) \cap [0\,,1[) - (\mathrm{Fin}(\beta) \cap [0\,,1[)) \cap [0\,,1[= \mathbb{Z}[\beta^{-1}] \cap [0\,,1[\qquad \textbf{(W)}$$

or, alternatively, if

$$\text{for any } x \in \mathbb{Z}[\beta^{-1}] \cap [0\,,1[, \text{ there exists } y, z \in \mathrm{Fin}(\beta) \cap [0\,,1[\text{ such that } x = y - z \qquad \textbf{(W)}$$

holds.

We list some PV-numbers where either **(F)** or **(W)** hold (and whose β-substitutions are therefore multi-component model sets):

- **(F)** is satisfied if β is the dominant root of $p(x) = x^m - a_1\,x^{m-1} - a_2\,x^{m-2} - \ldots - a_m$ with $a_1 \geq a_2 \geq \ldots a_m > 0$ (then $d_\beta(1) = .a_1 \ldots a_m$), see [140, Theorem 2] (also see [363, Main Theorem] and [364, Example 7.9 & Prop. 7.10] for the statement that the dynamical spectrum is pure point).

- **(F)** is satisfied if β is a root (respectively the dominant root) of $p(x) = x^m - a_1\,x^{m-1} - a_2\,x^{m-2} - \ldots - a_m$ with $a_1 > \sum_{i=2}^m a_i$ and $a_i \geq 0$ for all $1 \leq i \leq m$, see [177, Theorem 3.4.2].

- **(W)** is satisfied if β is a cubic PV-number which is a unit, see [8, Theorem 3].

- **(W)** is satisfied if β is the dominant root of $p(x) = x^m - a_1\,x^{m-1} - a_2\,x^{m-2} - \ldots - a_m$ with $a_i \geq 0$ $(2 \leq i \leq m-1)$, $a_1 > 0$ and $a_m > 0$ and the algebraic degree d of β satisfies $d > m/p$ where $p > 1$ is the smallest prime divisor of m, see [43, Theorem 7.1].

- **(W)** is satisfied if β is the dominant root of $p(x) = x^m - a_1\,x^{m-1} - \ldots - a_{m-1}\,x - 1$ with $a_i \geq 0$ $(2 \leq i \leq m-1)$, $a_1 > 0$ and $a_r > 0$ where m is prime and $r = \max\{k \mid a_{m=k+1} = \ldots = a_{m-1} = 0\}$, see [49, Theorem 3.2].

There are examples which are of neither of these forms, *e.g.*, the PV-number β with characteristic polynomial $\mathrm{Irr}(\beta, \mathbb{Q}, x) = x^5 - 2\,x^4 - x^3 - 2\,x^2 - 1$ where $d^*_\beta(1) = .\overline{21200}$ (this example is taken from [49, Section 3]).
So, β-substitutions corresponding to PV-numbers β in the previous list yield multi-component model sets. We end this section with some concrete examples of β-substitutions and some remarks.

Example 6c.20. [66, Section 1.3] The Fibonacci substitution (where $d_\tau^*(1) = .\overline{10}$, and therefore $a \mapsto ab$, $b \mapsto a$) as well as the tribonacci substitution (where $d_\beta^*(1) = .\overline{110}$ and therefore $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$) are examples of β-substitutions. Observe that both are also (irreducible) Pisot substitutions.

Let $\beta = 2 + \sqrt{2}$ be the Pisot root of $x^2 - 4x + 2$. Then we have $d_\beta^*(1) = .3\overline{1}$ and therefore the substitution $a \mapsto aaab$, $b \mapsto ab$. This is a non-unimodular example, see σ_{Siegel} in Example 6.112.

Let β be the Pisot root of $x^3 - x - 1$ (the smallest PV-number). Then one has $d_\beta^*(1) = .\overline{10000}$ (where $x^5 - x^4 - 1 = (x^2 - x + 1) \cdot (x^3 - x - 1)$, *e.g.*, see [17, Section 6 & Figs. 7 & 9 & 10]), and the associated substitutions is $a \mapsto ab$, $b \mapsto c$, $c \mapsto d$, $d \mapsto e$ and $e \mapsto a$. Note that this is a reducible Pisot substitution (a Pisot substitution with this PV-number is given by $a \mapsto b$, $b \mapsto ac$ (or $b \mapsto ca$) and $c \mapsto a$. We also quote [17, p. 73]: there seems to be no connection between this latter reduced substitutions (quote: "a quite wild tiling, with disconnected tiles") with the original β-substitutions (contrary to our findings with the particularly nice examples $\sigma_{\text{Dirk1,2}}$).

Remark 6c.21. If β is a PV-number of degree n and $d_\beta^*(1) = a_1 \dots a_q \overline{a_{q+1} \dots a_{q+p}}$ is the expansion of 1, then $p+q$ can be arbitrarily big for a given n. This is shown in [7, Example 2], where a certain class of PV-numbers of degree 4 (which are also algebraic units) is considered. Consequently, the cardinality of the alphabet for the associated β-substitution is not bounded with the degree of the PV-number in question.

Remark 6c.22. There are also results regarding the (arcwise) connectedness[10] of the tiles $\text{cl}_H\, S^\star_{.a_{-1} \cdots a_M}$. In fact, each such tile corresponding to a PV-number which is a unit is connected if $d_\beta(1)$ terminates with 1 (see [4, Theorem 3] and [7, Theorem 1.2]). Moreover, all unimodular β-substitutions where β is a PV-number of degree[11] 2 or 3 are connected (for degree 3, see [7, Theorems 1.3 & 4.5]). For a PV-number β of degree 4, let $\text{Irr}(\beta, \mathbb{Q}, x) = x^4 - a\,x^3 - b\,x^2 - c\,x \pm 1$. If the constant is $+1$, then connectedness follows. Otherwise, each tile is connected iff $a + c - 2\lfloor \beta \rfloor \neq 1$ (note that always $a + c - 2\lfloor \beta \rfloor \leq 1$), see [7, Theorems 1.3 & 4.6 & 4.8 & 4.9]. We also note that [7] also characterises all PV-numbers which are units according to their β-expansion of 1 in the cases where β has degree 3 and 4, see [7, Theorems 4.3 & 4.4 & 4.7 & 4.10 & Prop. 4.1] (also see [52] for all PV-numbers of degree 3).

According to these findings, the β-substitution corresponding to the PV-number β with $\text{Irr}(\beta, \mathbb{Q}, x) = x^4 - 3\,x^3 + x^2 - 2\,x - 1$ is an example for disconnectedness; here, $d_\beta^* = .22\overline{20101}$.

6c.5. The Balanced Pair Algorithm

The balanced pair algorithm is basically the symbolic version of the overlap algorithm in the one-dimensional case. To see this, we begin with the definition of a *geometric balanced pair*, compare [358, Section 5].

Definition 6c.23. Let σ be a primitive substitution on a alphabet $\mathcal{A}$ with fixed point u and $\underline{\Lambda}$ its realisation with natural intervals $A_i = [0\,, \ell_i]$ (whose lengths are given by the left

[10] By a result in [164], arcwise connectedness and connectedness are equivalent for the tiles we are considering here.

[11] All PV-numbers of degree 2 are classified in [140, Lemma 3]. The ones which are units have irreducible polynomial $x^2 - a\,x - 1$ (with $a \geq 1$) or $x^2 - a\,x + 1$ (with $a \geq 3$), wherefore we either have $.a1$ or $.(a-1)\overline{(a-2)}$ as corresponding β-expansion of 1, and one immediately checks that the corresponding β-substitution is not only a irreducible Pisot substitution but also invertible, see Remark 6c.5.

PF-eigenvector ℓ of the substitution matrix). A *geometric balanced pair* is a (unordered) pair of patches $[\mathcal{P}_1\,;\mathcal{P}_2]$ where $\mathcal{P}_i$ is a collection of consecutive tiles of $\mathcal{T} = \underline{\Lambda} + \underline{A}$ such that $\operatorname{supp}\mathcal{P}_1 = \operatorname{supp}\mathcal{P}_2 + y$ for some $y \in \Delta'$ (note that $\operatorname{supp}\mathcal{P}_i$ are simply compact intervals).

We say that a geometric balanced pair is *reducible* if each patch can be divided into two (nonempty) subpatches $\mathcal{P}_1 = \mathcal{P}_{11} \cup \mathcal{P}_{12}$, $\mathcal{P}_2 = \mathcal{P}_{21} \cup \mathcal{P}_{22}$ such that the subpatches form geometric balanced pairs $[\mathcal{P}_{11}\,;\mathcal{P}_{21}]$ and $[\mathcal{P}_{12}\,;\mathcal{P}_{22}]$ (where $\operatorname{supp}\mathcal{P}_{11} = \operatorname{supp}\mathcal{P}_{21} + y$ and $\operatorname{supp}\mathcal{P}_{12} = \operatorname{supp}\mathcal{P}_{22} + y$). In this case we say that the geometric balanced pair $[\mathcal{P}_1\,;\mathcal{P}_2]$ *splits*, the process of splitting a geometric balanced pair into irreducible geometric balanced subpairs is called *reduction*. A geometric balanced pair $[A_i\,;A_i]$ is called a *coincidence*. We say that the balanced pair $[\mathcal{P}_1\,;\mathcal{P}_2]$ *leads to a coincidence* if there exists an m such that the reduction of $[\mathcal{P}_1^{(m)}\,;\mathcal{P}_2^{(m)}]$ contains a coincidence, where we denote by $\mathcal{P}_i^{(m)}$ the patch one obtains if one obtains the tile substitution to the patch $\mathcal{P}_i$ m times.

Obviously, the following two statements are reformulations of one another: “every geometric balanced pair leads to a coincidence” and “for every strong overlap there is a directed path to a coincidence in $G_{str}^{ovlap}(\mathcal{T})$” (recall Definition 5.126).

We now describe this algorithm in the symbolic case, also see [177, Section 4.2], [178, Section 3] and [358, Section 3].

Definition 6c.24. Let σ be a primitive substitution on a alphabet $\mathcal{A}$. A pair of words $v, \tilde{v} \in \mathcal{A}^{\mathrm{fin}}$ is a *balanced pair* if they have the same length and the same occurrence of symbols, *i.e.*, if $\vartheta(v) = \vartheta(\tilde{v})$ (recall Definition 6.30). We denote a balanced pair by $[v\,;\tilde{v}]$. It *splits* if $[v_1 \dots v_m\,;\tilde{v}_1 \dots \tilde{v}_m]$ with $1 \le m < \#v = \#\tilde{v}$ is a balanced pair. A balanced pair is called *irreducible* if it does not split nontrivially. The process of splitting a balanced pair into irreducible balanced subpairs is called *reduction*. Clearly, if $[v\ ,\tilde{v}]$ is a balanced pair, $[\sigma(v)\ ,\sigma(\tilde{v}]$ is also a balanced pair.

A balanced pair $[i\ ,i]$ $(i \in \mathcal{A})$ is called a *coincidence*. A balanced pair $[v\ ,\tilde{v}]$ *leads to a coincidence* if there exists an m such that the reduction of $[\sigma^m(v)\ ,\sigma^m(\tilde{v})]$ contains a coincidence.

For Pisot substitutions, where the lengths are rationally independent and $\mathcal{L} = \mathcal{L}'$, a geometric balanced pair leads to a coincidence iff the corresponding balanced pair leads to a coincidence (note that length of the tiles of the patches bijectively code the letter in the words).

With the above definitions, we now describe the balanced pair algorithm. Let u be a fixed point of the primitive substitution σ. Let w be a nonempty “prefix” of u, *i.e.*, if $u = \dots u_{-2}u_{-1}\dot{u}_0u_1u_2\dots$ then $w = u_0u_1\dots u_m$ for some $m \ge 0$. Since σ is primitive, u is repetitive, wherefore w occurs with finite gaps in u, *i.e.*, $u = \dots w\tilde{w}_i w\tilde{w}_{i+1}w\dots$ where $\tilde{w}_i$ are words of bounded length. Consequently, there are only finitely many different such words $\tilde{w}_i$, wherefore there are only finitely many balanced pairs $[w\tilde{w}_i\,;\tilde{w}_i w]$. The set of all irreducible balanced pairs arising in the reduction of these latter balanced pairs is denoted by $I_1(w)$ and called *the set of initial balanced pairs* (one can also say that these are the irreducible balanced pairs arising from the reduction of $[u\,;S^{\#w}u]$ where S denotes the shift map). We next define inductively

$$I_n(w) = \{[v'\,;\tilde{v}'] \mid [v'\,;\tilde{v}'] \text{ is an irreducible subpair of } [\sigma(v)\,;\sigma(\tilde{v})],\ \text{for some } [v\,;\tilde{v}] \in I_{n-1}(w)\}.$$

Equivalently, $I_n(w)$ is the set of all irreducible balanced pairs that arise from the reduction of $[u, S^{\#\sigma^{n-1}(w)}u]$. We define the set of balanced pairs as $I = I(w) = \bigcup_{n=1}^{\infty} I_n(w)$.

One calls this process the *balanced pair algorithm* associated to the prefix w, or *bpa-w* for short. We say that the bpa-w *terminates* if $I(w)$ is finite.

We have not defined the spectrum of a dynamical system so far. This will be done in Section 7.2. For now we only remark that here u (respectively $\underline{\Lambda}$) is a model set iff the dynamical system $(\mathbb{X}(u), S)$ has pure point spectrum.

The following statement should therefore be no surprise. It can originally be found in [241, Theorem 2], [242, Theorem 1] and [377, Theorem 3] (it goes back to [258]). However, we will use the formulation of [358, Theorem 3.1] (also compare [177, Theorem 4.2.1] and [178, Theorem 3.1]).

Proposition 6c.25. *Let σ be a primitive substitution with fixed point u (such that $\sigma(u_0) = u_0 \ldots$).*

(i) *If for some prefix w the bpa-w terminates and every balanced pair in $I(w)$ leads to a coincidence, then $(\mathbb{X}(u), S)$ has pure point spectrum.*

(ii) *If the bpa-w terminates for some prefix $w = u_0 \ldots u_m$ such that $u_{m+1} = u_0$, and $(\mathbb{X}(u), S)$ has pure point spectrum, then every balanced pair in $I(w)$ leads to a coincidence.* □

We add some remarks to this proposition:

- The second case of the proposition basically is a criterion for the presence of a continuous spectral component, *i.e.*, for the case where u is *not* a model set.
- We note that the balanced pair algorithm is not restricted to Pisot substitutions.
- In view of our remarks in Section 6c.2, the condition "$w = u_0 \ldots u_m$ such that $u_{m+1} = u_0$" is basically due to a possible nontrivial height group $\mathcal{L}/\mathcal{L}'$.
- Usually, the balanced pair algorithm does not terminate (and then the above proposition cannot be applied) if the natural lengths are rationally dependent, since in that case a balanced pair $[v\,; v']$ can be irreducible although the corresponding geometric balanced pair $[\mathcal{P}\,; \mathcal{P}']$ is reducible, see the following example. However, it is straightforward to modify the balanced pair algorithm in this case (*e.g.*, by introducing an equivalence relation on words), this can be found in [249] (also compare [48, Section 4]). Basically, one then uses a corresponding "geometric balanced pair algorithm" (which, as we have indicated, is equivalent to checking for overlap coincidences).
- Examples of the balanced pair algorithm can be found in [258, Section III.], [177, p. 57], [178, Section 3], [358, Sections 3 & 6] and [249]. In [358], the URL of a Mathematica program is given in which the balanced pair algorithm is implemented.
- For Pisot substitutions, [50, Prop. 17.4] indicates that it is actually enough to choose $I_1 = \{[ij\,; ji]\}$ for any two letters $i, j \in \mathcal{A}$ with $i \neq j$.

Example 6c.26. Consider the period-doubling substitution $a \mapsto ab$ and $b \mapsto aa$. We simply take $w = a$. One then easily sees that $I_1(a) = \{[a\,; a], [ab\,; ba]\}$, and the iteration yields $I_2(a) = \{[a\,; a], [ab\,; ba], [b\,; b], [baa\,; aab]\}$, $[a\,\sigma(aa)\,; b\,\sigma(ab)] \in I_3(a)$, $[b\,\sigma^2(aa)\,; a\,\sigma^2(ab)] \in I_4(a)$ *etc.* Thus, for this prefix, the balanced pair algorithm does not terminate. But since it is a substitution of constant length, the only irreducible geometric balanced pairs are given by (symbolically, so we set i for $[0\,,\ell_i]$ where here all $\ell_i = 1$) $[a\,;a]$, $[b\,;b]$ and $[a\,;b] = [b\,;a]$, and the only non-coincidence $[a\,;b]$ leads immediately to a coincidence.

Also, this examples shows that the (geometric) balanced pair algorithm for a constant length substitution simply reduces to the algorithm to check for Dekking coincidences (recall Definition 6b.12): Check if *each pair* $i, j \in \mathcal{A}$ has a Dekking coincidence (and the obvious modifications if the height is greater than 1).

We now return to Pisot substitutions: It is easy to see that if the balanced pair algorithm terminates and σ satisfies the SCC (see Definition 6.90), then every balanced pair leads to a coincidence, see [178, Remark 1 after Theorem 3.1]. Now, on the one hand, one can show that the set of irreducible balanced pairs for a Pisot substitution σ over two symbols ($\operatorname{card} \mathcal{A} = 2$) is finite (see [177, Prop. 4.4.1] and [178, Prop. 4.4]), and on the other hand, one can also show that every Pisot substitution over two symbols satisfies the SCC (by [47, Theorem 1], see Proposition 6.93). Thus, we once again arrive at the following statement.

Corollary 6c.27. [178, Theorem 2.2] *Every Pisot substitution over two symbols is a regular multi-component model set.* □

Actually, the proof of Proposition 6.100 may be viewed as (geometric) re-interpretation of this result (note that in the last corollary we also do not assume unimodularity).

7. Diffraction and Dynamical Systems

Man läßt nicht auf allen Weiden, die der Geist geschaffen hat, Kühe grasen.
Am Ende der Geduld beginnt der Himmel.
Die Nacht dauert lange, aber schließlich kommt der Tag.

Die Nächte des grossen Jägers – *Ahmadou Kourouma*

So far, we have mainly been concerned with the "geometrical" part of the theory. Thus, this chapter – which is less formal in style than the previous ones – establishes the link to the theory of diffraction and to dynamical systems. We mainly review the results from the literature, their straightforward application to Pisot substitutions is already included in Theorem 6.116.

7.1. Diffraction of Meyer Sets

The basic object of interest in this section is a set of "scatterers" in a σLCAG and their diffraction pattern. For this, we first recall the definition of the Fourier transform of a (complex) measure and then apply the theory to the point sets of our interest.

Throughout this chapter, μ_G denotes the (unique) Haar measure on G.

7.1.1. Measures and Their Fourier Transform

We first define a certain function space and its dual.

Definition 7.1. Let G be a σLCAG. We define the following function space and its dual:

- Let $\mathscr{K}(G)$ denote the set of all complex-valued continuous functions f on G such that there exists a compact subset $W \subset G$ (depending on f) such that $f(x) = 0$ for all $x \in W^c$. We also define the sup-norm $\|f\| = \sup\{|f(x)| \mid x \in G\}$.
- The dual of $\mathscr{K}(G)$ is denoted by $\mathfrak{M}(G)$. It consists of all complex Borel measures on G with its vague topology.

We now clarify the notions in the previous definition in the following, see [53, 59, 60, 147, 159, 169, 272, 309] for background material and compare [26, 30, 36].

By the σ-compactness of G, there exists a sequence of $\{U_n\}_{n\in\mathbb{N}}$ of relatively compact subsets of G which cover G and with the property $\operatorname{cl} U_n \subset U_{n+1}$. Let $\mathscr{K}_n(G)$ denote the subspace of $\mathscr{K}(G)$ of functions which vanish outside $\operatorname{cl} U_n$. Then, each $\mathscr{K}_n(G)$ is a Banach space in its sup-norm topology and each $\mathscr{K}_n(G)$ is a topological subspace of $\mathscr{K}_{n+1}(G)$. Moreover, each $\mathscr{K}_n(G)$ is a *locally convex vector space*, *i.e.*, there is a neighbourhood base at 0 consisting of convex sets. For every n, we denote by j_n the embedding $j_n : \mathscr{K}_n(G) \hookrightarrow \mathscr{K}(G)$. The finest topology on $\mathscr{K}(G)$ such that all j_n are continuous is called the *strict inductive limit topology* (it does not depend on the choice of the sets U_n).

The *(topological) dual* X' of a topological vector space X (here over $\mathbb{C}$) is the space of all continuous linear functions into its scalar field; an elements of X' is called a *(continuous) functional*. The dual of $\mathscr{K}(G)$ is denoted by $\mathfrak{M}(G)$ and can be identified *via* the Riesz-Markov representation theorem with the set of all *complex* regular Borel measures on G, *i.e.*, every functional $I \in \mathfrak{M}(G)$ can (uniquely) be identified with a regular Borel measure ν_I by $I(f) = \int_G f(x)\, \mathrm{d}\nu_I(x)$ for all $f \in \mathscr{K}(G)$. Here, a complex measure on a σ-algebra $\mathfrak{C}$ is a complex valued function $\mu : \mathfrak{C} \to \mathbb{C}$ satisfying $\mu(\varnothing) = 0$ and $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for any pairwise disjoint sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ from $\mathfrak{C}$ (compare with Definition 4.4). In the following, a "measure" is always a complex regular Borel measure (respectively, a functional from $\mathfrak{M}(G)$).

Consequently, under the identification of the Riesz-Markov representation theorem, we simply write $\nu(f)$ for a regular Borel measure ν and $f \in \mathscr{K}(G)$ where we actually mean the associated functional evaluated at f. The space $\mathfrak{M}(G)$ is equipped with the *vague topology*, *i.e.*, a sequences of measures $\{\nu_n\}$ converges vaguely to ν if $\lim_{n\to\infty} \nu_n(f) = \nu(f)$ in $\mathbb{C}$ for all $f \in \mathscr{K}(G)$. This is just the *weak-* topology* on $\mathfrak{M}(G)$. Recall that for each $f \in X$, the function $p_f(x) = |f(x)|$ $(x \in X)$ is a seminorm[1] on X, and the topology defined by the family $\{p_f \mid f \in X'\}$ is called the *weak topology* of X. Similarly, for each $x \in X$, the function $p_x(f) = |f(x)|$ for every $f \in X'$ is a seminorm on X', and the topology defined by $\{p_x \,|\, x \in X\}$ is called the *weak-* topology* of X'.

For every complex measure ν and every compact set $W \subset G$, there is a constant a_W such that $|\nu(f)| \leq a_W \cdot \sup\{|f(x)| \mid x \in W\}$ for all $f \in \mathscr{K}(G)$. A measure ν is called *positive* if $\nu(f) \geq 0$ for all $f \geq 0$. For every measure ν, there is a smallest positive measure, denoted by $|\nu|$ and called the *total variation* of ν, such that $|\nu(f)| \leq |\nu|(f)$ for all $f \geq 0$. Equivalently, one has $|\nu|(A) = \sup\{\sum_k |\mu(A_k)| \mid \{A_k\}$ is a finite partition of $A\}$ where A and the A_ks are Borel sets. A measure ν is *bounded* if $|\nu|(G) < \infty$. A measure ν is called *translation bounded* if for every compact set $W \subset G$ there is a constant b_W such that

$$\|\nu\|_W = \sup\{|\nu|(W + x) \mid x \in G\} \leq b_W .$$

We denote the set of all translation bounded measures on G by $\mathfrak{M}^\infty(G)$. We also introduce a certain topology on $\mathfrak{M}^\infty(G)$.

Definition 7.2. Let W be a compact neighbourhood of $0 \in G$. We call the topology on $\mathfrak{M}^\infty(G)$ defined by the norm $\|\cdot\|_W$ (where $\|\nu\|_W = \sup\{|\nu|(W + x) \mid x \in G\}$) the *(local) norm topology* on $\mathfrak{M}^\infty(G)$. Note that any other compact neighbourhood W' would lead to an equivalent norm and hence to the same topology. Moreover, this norm makes $\mathfrak{M}^\infty(G)$ into a Banach space, see [147, Example 2.13] and [36, Remark on p. 86].

We also introduce the following notation: If ν is a measure, the conjugate of ν is defined by the mapping $f \mapsto \overline{\nu\left(\overline{f}\right)}$. It is again a measure and denoted by $\overline{\nu}$. A measure ν is called *real* (or *signed*) if $\overline{\nu} = \nu$ or, equivalently, $\nu(f)$ is real for all real valued $f \in \mathscr{K}(G)$. Note that every complex measure ν can uniquely be written as $\nu = \nu_1 + i\,\nu_2$, where ν_1, ν_2 are real measures. For $f \in \mathscr{K}(G)$, we denote by $\check{f}$ the function $\check{f}(x) = f(-x)$ and by $\tilde{f}$ the function

[1] A *seminorm* is a map of a vector space into the nonnegative reals which is absolutely homogeneous and subadditive.

$\tilde{f}(x) = \overline{f(-x)}$. Consequently, we denote by $\check{\nu}$ the measure defined by $f \mapsto \nu(\check{f})$ and by $\tilde{\nu}$ the measure defined by $f \mapsto \overline{\nu(\tilde{f})}$.

We now define the convolution of two measures. Recall that for functions $f, g \in \mathscr{K}(G)$ the *convolution* is given by

$$(f * g)(x) = \int_G f(x-y)\, g(y)\; \mathrm{d}\mu_G(x).$$

Note that the convolution of functions is commutative and can be extended to the case that one function is bounded while the other is integrable.

Definition 7.3. We say that the *convolution* $\mu * \nu$ of two positive measures μ and ν on G exists (or that μ and ν are *convolable*) if

$$(\mu * \nu)(f) = \int_G \int_G f(x+y)\; \mathrm{d}\mu(x)\; \mathrm{d}\nu(y) < \infty$$

for all $f \geq 0$. We say that convolution of two (complex) measures μ and ν exists, if the convolution of $|\mu|$ and $|\nu|$ exists; in that case, the convolution $\mu * \nu$ is defined as in the previous equation.

Not all measures are convolable, but there are sufficient conditions for convolability.

Lemma 7.4. [60, Prop. 1.13] and [147, p. 23] *The convolution of a translation bounded measure μ and a finite measure ν exists. Similarly, the convolution of a measure μ with a measure ν of compact support exists.* □

We also have (respectively set)

$$(\mu * \nu)(f) = \mu(\check{\nu} * f) = \nu(\check{\mu} * f) \tag{7.1}$$

for all $f \in \mathscr{K}(G)$ if $\mu * \nu$ exists. Consequently, $\nu * f$ is a bounded continuous complex function for all $f \in \mathscr{K}(G)$ and $\nu \in \mathfrak{M}(G)$.

Our next goal is to introduce the Fourier transform of a certain subclass of $\mathfrak{M}^\infty(G)$. We first recall Definition 5a.8 and Remark 5a.9: Let G be an LCAG and denote by G^* its character group. Let $f \in L^1(G)$. Then we define $(\mathscr{F}_G f) = (\mathscr{F} f) = f^* : G^* \to \mathbb{C}$, the Fourier transform of f, by the formula

$$(\mathscr{F} f)(x^*) = f^*(x^*) = \int_G f(y)\overline{\langle x^*, y\rangle}\, \mathrm{d}\mu(y)$$

for $x^* \in G^*$. We also define the *co-Fourier transformation* $(\overline{\mathscr{F}} f)$ by

$$(\overline{\mathscr{F}} f)(x^*) = \int_G f(y)\langle x^*, y\rangle\, \mathrm{d}\mu(y).$$

We note that one can extend the Fourier transform to an isometry of $L^2(G)$ onto $L^2(G^*)$ (one also has $\mathscr{F}_G^{-1} = \overline{\mathscr{F}}_{G^*}$). Obviously, all $f \in \mathscr{K}(G)$ are also integrable.
We now define the subclass of positive definite measures. We recall that a continuous function $g : G \to \mathbb{C}$ is *positive definite* iff

$$\int_G g(x)\, (f * \tilde{f})(x)\; \mathrm{d}\mu_G(x) \geq 0$$

for all $f \in \mathscr{K}(G)$, see [60, Prop. 4.1]. Note that this implies $g = \check{g}$ and $|g(x)| \leq g(0)$.

Definition 7.5. A measure ν on G is called *positive definite* if $\nu(f * \tilde{f}) \geq 0$ for all $f \in \mathscr{K}(G)$. We denote the set of positive measures on G by $\mathfrak{M}_p(G)$ and the set of positive and positive definite measure on G by $\mathfrak{M}_p^+(G)$.

We note that $\nu \in \mathfrak{M}_p(G)$ implies $\check{\nu} \in \mathfrak{M}_p(G)$ and $\overline{\nu} \in \mathfrak{M}_p(G)$, *i.e.*, $\mathfrak{M}_p(G)$ is stable under reflection and conjugation. Moreover, it is a vaguely closed convex cone in $\mathfrak{M}(G)$. Furthermore, if ν is a measure on G such that the convolution of ν and $\tilde{\nu}$ exists, then the measure $\nu * \tilde{\nu}$ is positive definite. One also uses the notation $\nu * \tilde{\nu} = \nu \tilde{*} \nu$, see [115].

The following statements will be used to define the Fourier transform of a measure.

Proposition 7.6. [60, Theorems 3.12 & 4.7 & 4.16 & Prop. 4.4 & Def. 4.15]

(i) *A positive and positive definite measure μ on G is translation bounded, i.e., $\mathfrak{M}_p^+(G) \subset \mathfrak{M}^\infty(G)$.*

(ii) *A measure μ on G is positive definite iff $\mu * f * \tilde{f}$ is a continuous positive definite function for all $f \in \mathscr{K}(G)$.*

(iii) *A continuous function g on G is a positive definite function iff there exists a positive bounded measure υ on G^* such that*

$$g(y) = \int_{G^*} \langle x^* \,, y \rangle \, \mathrm{d}\upsilon(x^*)$$

for all $y \in G$. The measure υ is uniquely determined by this equation.

(iv) *A measure μ on G is positive definite iff there exists a positive measure υ on G^* such that*

$$\mu(f * \tilde{f}) = \int_{G^*} \left| \overline{\mathscr{F}} f)(x^*) \right|^2 \, \mathrm{d}\upsilon(x^*)$$

for all $f \in \mathscr{K}(G)$. The positive measure υ is uniquely determined and called the Fourier transform of μ *and denoted by $\mathscr{F}\mu$ or $\mathscr{F}_G\mu$ or $\hat{\mu}$.*

(v) *The Fourier transformation $\mathscr{F}_G$ is a bijective mapping of $\mathfrak{M}_p^+(G)$ onto $\mathfrak{M}_p^+(G^*)$, the inverse mapping is $\mathscr{F}_{G^*}$. Furthermore, $\mathscr{F}_G$ is a homeomorphism if $\mathfrak{M}_p^+(G)$ and $\mathfrak{M}_p^+(G^*)$ are carrying the vague topologies.* □

We note that (ii) is also called the *Bochner's theorem*.

So, the Fourier transform of a positive definite measure ν is again a measure and we may decompose it according to Lebesgue's Theorem. We first note the following definitions.

Definition 7.7. Let μ and ν be complex measures on a measurable space on $(G\,, \mathfrak{B})$, where G is a LCAG and $\mathfrak{B}$ denotes the σ-algebra of Borel sets. If $\mu(A) = 0$ implies $\nu(A) = 0$ for $A \in \mathfrak{B}$, then the measure ν is said to be *absolutely continuous with respect to the measure* μ. If ν is absolutely continuous with respect to the Haar measure μ_G, then we simply call ν *absolutely continuous.*

Let μ and ν be complex measures on G. If there exists an $A \in \mathfrak{B}$ such that $|\mu|(A) = 0$ and $|\nu|(A^c) = 0$, then the measures μ and ν are said to be *mutually singular.* If ν and the Haar measure μ_G are mutually singular, we simply call ν *singular.*

Denote by $\nu|_B$ the *restriction* of ν to a Borel set B, *i.e.*, $\nu|_B(A) = \nu(A \cap B)$ for all $A \in \mathfrak{B}$. If $\nu = \nu|_B$, then one says that ν is *concentrated* in B. Also compare to Footnote 22 on p. 165. Note that the *support* $\operatorname{supp} \nu$ of a measure ν is the smallest closed subset of G in which the measure ν is concentrated.

A complex measure ν is said to be *pure point* (or *discrete* or *purely discontinuous*) if there is a (at most) countable subset $E \subset G$ such that $|\nu|(E^c) = 0$ (then, ν is concentrated[2] on the countable set E). In this case, ν can be written as a (at most) countable sum of Dirac measures, *i.e.*, $\nu = \sum_{i \in I} a_i \, \delta_{x_i}$ where I is a (at most) countable set, $a_i \in \mathbb{C}$ and $x_i \in G$ for all $i \in I$. A measure ν is *continuous* if $\nu(\{g\}) = 0$ for all $g \in G$.

We now state *Lebesgue's Theorem* together with the *Radon-Nikodym theorem*, see for example [159, p. 197] and [169, Theorem 14.22]. We note that one may replace the Haar measure μ_G by any positive σ-finite measure.

Proposition 7.8. *Let ν be a complex measure on $(G, \mathfrak{B}, \mu_G)$. Then there are (unique) complex measures ν_{pp}, ν_{ac} and ν_{sc} such that*

(i) $\nu = \nu_{\mathrm{pp}} + \nu_{\mathrm{ac}} + \nu_{\mathrm{sc}}$.

(ii) ν_{pp} *is a pure point measure.*

(iii) ν_{ac} *is absolutely continuous (with respect to μ_G), wherefore, by the Radon-Nikodym theorem, there exists (a uniquely determined) measurable function f – called the* Radon-Nikodym derivative *and denoted by* $\mathrm{d}\nu_{\mathrm{ac}} / \mathrm{d}\mu_G$ *– such that*

$$\nu_{\mathrm{ac}}(A) = \int_A f(x) \; \mathrm{d}\mu_G(x)$$

for every $A \in \mathfrak{B}$.

(iv) ν_{sc} *is singular and continuous.*

(v) $\nu_{\mathrm{pp}} + \nu_{\mathrm{sc}}$ *is singular.*

(vi) $\nu_{\mathrm{ac}} + \nu_{\mathrm{sc}}$ *is continuous.*

If $|\nu|(G)| < \infty$, then ν_{pp}, ν_{ac} and ν_{sc} are also bounded measures and the Radon-Nikodym derivative of ν_{ac} with respect to μ_G belongs to $L^1(G)$; moreover, in this case, one has $|\nu|(G)| = |\nu_{\mathrm{pp}} + \nu_{\mathrm{sc}}|(G) + |\nu_{\mathrm{ac}}|(G)$. □

We also note that ν is pure point (respectively continuous) iff $|\nu|$ is pure point (respectively continuous) iff $f\nu$ is pure point (continuous) for every $f \in \mathscr{K}(G)$, see [147, Prop. 10.3]. Here, $f\nu$ denotes the absolutely continuous measure with respect to ν with Radon-Nikodym derivative f.

[2] Note that the set $\{x \in G \mid \nu(\{x\}) \neq 0\}$ is at most countable for any complex measure, see [309, p. 22 in Section I.4].

7.1.2. Diffraction of a Set of Scatterers

We can now describe the distribution of "matter" in a mathematically adequate way and calculate its diffraction pattern.

Since G is σ-compact, a uniformly discrete set $\varLambda$ (respectively a multi-component uniformly discrete set $\underline{\varLambda}$) is countable (see Remark 5.3). We therefore specialise on the situation of a countable set S of scatterers in G with (bounded complex) scattering strengths $v(x)$ for $x \in S$. Moreover, we assume that S is a uniformly locally finite set. It can be represented as a complex measure in the form of a *weighted Dirac comb*

$$\nu = \sum_{x \in S} v(x)\, \delta_x,$$

where δ_x is the unit point (or Dirac) measure located at x, *i.e.*, $\delta_x(f) = f(x)$ for continuous functions f. For later reference, we write $\omega = v_a \cdot \delta_S$ if $v(x) = v_a$ for all $x \in S$. Consequently, ν is a pure point measure, we have

$$|\nu| = \sum_{x \in S} |v(x)|\, \delta_x,$$

wherefore ν is positive if $v(x) \geq 0$ for all $x \in S$, and ν is translation bounded since S is uniformly locally finite. Thus, on the physical side, atoms of the structure S are modeled by their position x and scattering strengths $v(x)$. Convolutions with more realistic profiles are not considered here, but can easily be treated by the convolution theorem, see [60, Prop. 4.10].

To calculate the diffraction spectrum, we need the *autocorrelation measure* γ_ν attached to ν. The *diffraction spectrum* is then the Fourier transform $\hat{\gamma}_\nu$ of the autocorrelation measure. Here, $\hat{\gamma}_\nu(E)$ is the total intensity scattered into the volume element E, and thus describes the outcome of a diffraction experiment, compare [97]. It can uniquely be decomposed by the Lebesgue decomposition theorem. Here, $(\hat{\gamma}_\nu)_{\text{pp}}$, the pure point part, is usually called the *Bragg* part of the diffraction spectrum, while the absolutely continuous part $(\hat{\gamma}_\nu)_{\text{ac}}$ is identified with diffuse scattering. The singular continuous part $(\hat{\gamma}_\nu)_{\text{sc}}$ is usually not considered in classical crystallography (and is somewhat intermediate between Bragg and diffuse scattering, see the scaling arguments in [22, 23, 174, 176, 240]). Often, the pure pointedness of the diffraction spectrum $\hat{\gamma}_\nu$ of ν is in question, *i.e.*, whether the Lebesgue decomposition reduces to $\hat{\gamma}_\nu = (\hat{\gamma}_\nu)_{\text{pp}}$ (then, the diffraction spectrum consists of Bragg peaks only).

Remark 7.9. The diffraction spectrum is experimentally analysed by looking at the *diffraction pattern* of the structure. Here, the (positive) diffraction measure is represented as an "intensity diagram": A Bragg peak (at x) corresponds to a sharp "peak" respectively a disc whose area is proportional to $\hat{\gamma}_\nu(\{x\})$, while the "diffuse scattering" (the absolutely continuous part) is represented in gray scale proportional to its Radon-Nikodym derivative. As noted above, the singular continuous part is somewhat intermediate between the pure point and the absolutely continuous part, but usually the diffraction pattern is (numerically) calculated (or experimentally obtained) from a finite patch of the structure and the diffraction spectrum of a finite structure is always absolutely continuous. Thus, one may view the diffraction pattern as approximation of the diffraction measure by an absolutely continuous measure. With this in mind, one should look at Figures 5a.3 & 6b.4 which show the diffraction pattern of pure point diffractive structures and at Figure 7.1, where the diffraction measure consists of a pure point measure $A \cdot \delta_{\mathbb{Z}^2}$ (with some constant $A > 0$) plus an absolutely continuous measure.

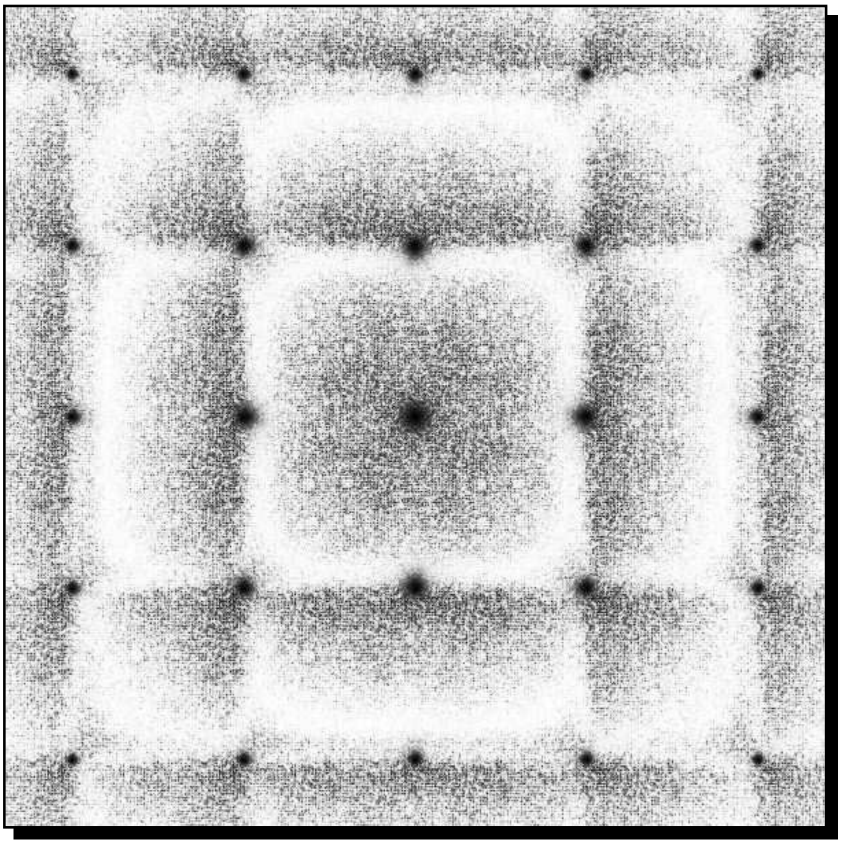


Figure 7.1.: The part $[-2.4\,,2.4] \times [-2.4\,,2.4]$ of the diffraction pattern of a structure with "cubic symmetry" and some disorder. More precisely, two types of atoms are distributed on $\mathbb{Z}^2$ (*i.e.*, one has two scattering strengths and the set of scatterers $S = \mathbb{Z}^2$), then a physical motivated relaxation between the two types of atoms takes place such that the atoms are not necessarily at a lattice point $(x\,,y) \in \mathbb{Z}^2$ any more but "distorted" from this ideal position by a small amount depending on its neighbouring atoms. This yields diffuse scattering (absolutely continuous spectrum) outside the Bragg peaks at $(\mathbb{Z}^2)^\perp = \mathbb{Z}^2$ (where one also has a pure point measure). The above diffraction pattern is numerically calculated from a big patch on a 2400×2400-grid using DISCUS [296]. Also see [85, 166, 354, 389, 390] on a discussion of this disorder phenomenon called the "size-effect". This picture already appeared in [354, Fig. 3].

After this "physical motivation", we now calculate the autocorrelation measure γ_ν. To this end, we proceed as in Sections 5.1 & 5.2: Let $\{A_n\}_{n\in\mathbb{N}}$ be a van Hove sequence (with property **(vH)**). We set $\nu^{(n)} = \nu|_{A_n}$, *i.e.*, $\nu^{(n)}$ is the restriction of ν to A_n. Then, the volume averaged convolution of the two measure with compact support (respectively, since the scattering strengths $v(x)$ are assumed to be bounded, of the two bounded measures)

$$\gamma_\nu^{(n)} = \frac{1}{\mu_G(A_n)}\,\nu^{(n)} * \widetilde{\nu^{(n)}} = \frac{1}{\mu_G(A_n)}\,\nu^{(n)} \tilde{*} \nu^{(n)}$$

is well defined. We note that $\tilde{\nu} = \sum_{x\in S} \overline{v(x)}\,\delta_{-x}$ and that $\delta_x * \delta_y = \delta_{x+y}$. Consequently, one obtains

$$\gamma_\nu^{(n)} = \sum_{z\in\Delta} \eta_\nu^{(n)}(z)\cdot\delta_z,$$

where $\Delta = S - S$ is the difference set and

$$\eta_\nu^{(n)}(z) = \frac{1}{\mu_G(A_n)} \sum_{\substack{x,y\in S\cap A_n\\ x-y=z}} v(x)\,\overline{v(y)}.$$

If the limit $\eta_\nu(z) = \lim_{n\to\infty}\eta_\nu^{(n)}(z)$ exists for all $z\in G$, we call $\eta_\nu(z)$ the *autocorrelation coefficient* of S (respectively of ν) at z. In this case, the autocorrelation measure γ_ν of ν exists, since $(\gamma_\omega^{(n)})_{n\in\mathbb{N}}$ converges in the vague topology, compare [172]. Moreover, by construction, it is then a positive definite measure ($\eta_\nu^{(n)}$ is a positive definite function on G, wherefore $\gamma_\nu^{(n)}$ is a positive definite measure by Proposition 7.6(ii)).

Thus, Definition 5.19 is the special case where $v\equiv 1$, *i.e.*, where we calculate the autocorrelation coefficients of δ_S (respectively δ_Λ). We also recall Remark 5.20 respectively [36, Remark to Axiom 2 (p. 65)]: The sequence of measures $\gamma_\nu^{(n)}$ always has points of accumulation in the vague topology. For any such limit point, one can select a suitable subsequence which converges to this limit point. In that case, one can achieve that the autocorrelation coefficients along this subsequence converge pointwise for all $z\in\Delta$. Therefore, we may assume (by choosing an appropriate van Hove sequence) that we have a autocorrelation measure γ_ν. We note that if only one point of accumulation exists (independent of the chosen van Hove sequence), the autocorrelation is unique and called the *natural autocorrelation*. However, even if S is a FLC set, the autocorrelation may not be unique, see [219].

As in Definition 5.38, one can define a translation-invariant pseudometric on G respectively $\mathcal{L} = \langle\Delta^{\mathrm{ess}}\rangle_\mathbb{Z}$ if S is an FLC set. Since $\eta_\nu(z)$ can now be a complex number, the appropriate choice is

$$\varrho_S(x\,,y) = \left|1 - \frac{\eta_\nu(x-y)}{\eta_\nu(0)}\right|^{\frac{1}{2}},$$

see [36, p. 66 and Fact 1]. We note that $\varrho_S(0\,,0) = 0$, that $\varrho(x\,,y)$ is bounded by $\sqrt{2}$ (if $v(x)\in\mathbb{R}$ for all $x\in S$, then it is bounded by 1) and $\eta_\nu(0)$ is the "volume average" of the square modulus of the scattering strengths.

With this pseudometric, we can define the set of ε-almost periods of S as before, *i.e.*, $P_\varepsilon = \{t\in G \mid \varrho_S(t\,,0) < \varepsilon\}$. We note that the case studied in Section 5.2, where $v\equiv 1$ and the sets P_ε are defined through the variogram, is simply a special case of our situation here. In fact, with $v\equiv 1$, the pseudometric defined here and the variogram only differ by the power $\frac{1}{2}$.

We can now formulate conditions under which $\hat{\gamma}_\nu$ is a pure point measure. The first statement indicates the use of "strictly pure point diffractive set" in Definition 5.40.

Theorem 7.10. [36, Theorem 1] *Let G be a $\sigma LCAG$. Assume that the weighted Dirac comb $\nu = \sum_{x \in S} v(x)\,\delta_x$ is a translation bounded measure (i.e., that S is uniformly locally finite), that S is an FLC set and that the ε-almost periods P_ε are Delone sets for every $\varepsilon > 0$. Then the autocorrelation measure γ_ν (relative to a fixed van Hove sequence) exists and is a positive definite translation bounded measure on G. Moreover, its Fourier transform $\hat{\gamma}_\nu$, the* diffraction measure, *is a translation bounded positive and pure point measure on G^*.* □

Proposition 7.11. [36, Theorem 5] *Let ν be a translation bounded measure on a $\sigma LCAG$ G whose autocorrelation γ_ν exists (relative to a van Hove sequence) and is a pure point measure with a support Δ such that $\Delta - \Delta$ is also uniformly discrete. Then, the corresponding diffraction measure $\hat{\gamma}_\nu$ exists and the following statements are equivalent.*

(i) *The ε-almost period P_ε is relatively dense for every $\varepsilon > 0$.*

(ii) *The autocorrelation measure γ_ν is* norm almost periodic, *i.e., for every $\varepsilon > 0$, the set $\{t \mid \|\delta_t * \gamma_\nu - \gamma_\nu\|_K < \varepsilon\}$ is relatively dense in G.*

(iii) *The autocorrelation measure γ_ν is strongly almost periodic, see Remark* 7.12.

(iv) *The diffraction measure $\hat{\gamma}_\nu$ is pure point.* □

Remark 7.12. Recall from Equation (7.1) on p. 347 that $\mu * f$ is a bounded continuous complex function if $\mu \in \mathfrak{M}(G)$ and $f \in \mathscr{K}(G)$. Moreover, for $\mu \in \mathfrak{M}^\infty(G)$, $\mu * f$ is a bounded, uniformly continuous complex function. If we denote the space of bounded, uniformly continuous functions on G by $C_U(G)$, then a measure $\mu \in \mathfrak{M}^\infty(G)$ can be identified with an element of the Cartesian product space $[C_U(G)]^{\mathscr{K}(G)}$ *via* (we also note that the mapping $f \mapsto \mu * f$ is continuous, see [60, Prop. 1.12])

$$\mu \to \{\mu * f\}_{f \in \mathscr{K}(G)},$$

see [147, Example 2.15], [36, p. 87] and [370, Def. 2.5 & Rem. 2.6]. Giving $[C_U(G)]^{\mathscr{K}(G)}$ the usual product topology, the induced topology on $\mathfrak{M}^\infty(G)$ is called the *strong topology.* A fundamental system of seminorms on $\mathfrak{M}^\infty(G)$ is provided by $\{\|\cdot\|_f \mid f \in \mathscr{K}(G)\}$ with $\|\mu\|_f = \|\mu * f\|_\infty$. However, $\mathfrak{M}^\infty(G)$ is not a complete subspace of $[C_U(G)]^{\mathscr{K}(G)}$, but every closed and bounded subset of $\mathfrak{M}^\infty(G)$ is complete in $[C_U(G)]^{\mathscr{K}(G)}$ (see [147, Theorem 2.4 & Corollary 2.1]).

A measure $\mu \in \mathfrak{M}^\infty(G)$ is *strongly almost periodic* if $\{\delta_t * \mu \mid t \in G\}$ is relatively compact in the strong topology. Similarly, a measure $\mu \in \mathfrak{M}^\infty(G)$ is called *weakly almost periodic* if $\{\delta_t * \mu | t \in G\}$ is relatively compact in the vague topology. We denote by $\mathcal{SAP}(G)$ respectively $\mathcal{WAP}(G)$ the strongly respectively weakly almost periodic measures of $\mathfrak{M}^\infty(G)$. Clearly, one has $\mathcal{SAP}(G) \subset \mathcal{WAP}(G)$; moreover, every $\mu \in \mathcal{WAP}(G)$ can uniquely be written in the form $\mu = \mu_S + \mu_0$ with $\mu_S \in \mathcal{SAP}(G)$ and $\mu_0 \in \mathcal{WAP}_0(G)$, where $\mathcal{WAP}_0(G)$ denotes the space of *null weakly almost periodic measures* of $\mathfrak{M}^\infty(G)$. Basically, $\mathcal{WAP}_0(G)$ is a subspace of $\mathcal{WAP}(G)$ where a certain mean is 0, see [147] and [370, Section 2 & 5] for exact definitions (compare [159, Section 2.§6]).

The autocorrelation measure γ_ν is positive definite and thus has a Fourier transform $\hat{\gamma}_\nu$. According to [147, Theorem 11.2], $\hat{\gamma}_\nu \in \mathcal{WAP}(G^*)$ and, if γ_ν is a pure point measure, one even has $\hat{\gamma}_\nu \in \mathcal{SAP}(G^*)$. An application of the inverse Fourier transform yields.

Proposition 7.13. [36, Theorem 4] and [147, Corollary 11.1] *Let $\nu \in \mathfrak{M}^\infty(G)$ and assume that (with respect to some van Hove sequence) its autocorrelation measure γ_ν exists. Then, $\gamma_\nu \in \mathfrak{M}^\infty(G) \cap \mathfrak{M}_p(G)$ and $\hat{\gamma}_\nu$ is a positive measure in $\mathfrak{M}^\infty(G)$. Moreover, $\hat{\gamma}_\nu$ is a pure point measure (and we say that ν is* pure point diffractive*) iff $\gamma_\nu \in \mathcal{SAP}(G)$. Similarly, $\hat{\gamma}_\nu$ is a continuous measure iff $\gamma_\nu \in \mathcal{WAP}_0(G)$.* □

For Meyer sets (and therefore also for multi-component Meyer sets), one now obtains the following statement.

Proposition 7.14. [370, Prop. 3.12] *Let Λ be a Meyer set in a σLCAG G and suppose that its autocorrelation measure γ_ν exists (with respect to some van Hove sequence). Then, the set of* Bragg peaks, *i.e., the set $\mathscr{P}'(\Lambda) = \{x \in G^* \mid \hat{\gamma}_\nu(\{x\}) > 0\}$, is relatively dense. Moreover, if Λ is not pure point diffractive, it has a relatively dense support for the continuous spectrum as well.* □

Further statements about the diffraction measure of Meyer sets and subsets of FLC Delone sets can be found in [369, 370].

For model sets, we recall Folklore Theorem 5a.10.

Folklore Theorem 5a.10. *Let $\Lambda = \Lambda(\Omega)$ be a regular model set in the CPS $(G\,, H\,, \tilde{L})$. Denote by $\tilde{L}^\perp$ the annihilator of $\tilde{L}$, which is a lattice in $G^* \times H^*$. Denote by $\pi_1^* : G^* \times H^* \to G^*$ and $\pi_2^* : G^* \times H^* \to H^*$ the canonical projections. Then, the diffraction measure exists and is a pure point measure, which is concentrated on $\pi_1^*(\tilde{L}^\perp)$. Assuming that the projection π_1^* is bijective on the lattice $\tilde{L}^\perp$, the intensity at a point $k \in \pi_1^*(\tilde{L}^\perp)$ is given by the square of the* Fourier-Bohr coefficient *$a(k)$ which is given by*

$$a(k) = \frac{\operatorname{dens} \Lambda}{\mu_H(\Omega)} \int_\Omega \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, y\rangle \, \mathrm{d}\mu_H(y)$$

We note that $\operatorname{dens} \Lambda / \mu_H(\Omega) = 1/\mu_{G\times H}(\mathrm{FD}(\tilde{L}))$ *by the density formula (Corollary 5.27).* □

Remark 7.15. For the convolution of two functions $f, g \in L^1(G)$, one has $\mathscr{F}(f * g) = (\mathscr{F} f) \cdot (\mathscr{F} g)$, wherefore $\mathscr{F}(f * \tilde{f}) = |\mathscr{F} f|^2$. The above statement about the Fourier-Bohr coefficients, can be interpreted as pointwise version of this for a translation bounded measure with unique autocorrelation, since it reads

$$\hat{\gamma}_\nu(\{k\}) = |a(k)|^2,$$

i.e., the diffraction measure of a regular model set is given by

$$\hat{\gamma}_\nu = \sum_{k \in \pi_1^*(\tilde{L}^\perp)} |a(k)|^2 \, \delta_k.$$

It is shown in [172, Theorem 5.4] that one can calculate the diffraction measure of a regular model set using Fourier-Bohr coefficients. Actually, the pure point part of the diffraction measure is given by such a sum over the Fourier-Bohr coefficients, provided the Fourier-Bohr coefficients ($\{A_n\}$ denotes a van-Hove sequence)

$$a(k) = \lim_{n\to\infty} \frac{1}{\mu_G(A_n)} \int_{t+A_n} \overline{\langle k\,, x\rangle} \; \nu(x)$$

exists uniformly in $t \in G$ for all $k \in G^*$, see [172, Theorem 3.4] for the case $G = \mathbb{R}^d = G^*$.

We also make the following formal calculation which shows how the Fourier-Bohr coefficients at $k \in \pi_1^*(\tilde{L}^\perp)$ in Folklore Theorem 5a.10 are obtained from their actually definition in the last equation (for regular model sets uniform existence is clear):

$$\begin{aligned}
a(k) &= \lim_{n\to\infty} \frac{1}{\mu_G(A_n)} \int_{A_n} \overline{\langle k\,, x\rangle}\; \nu(x) \\
&\overset{\nu \text{ is pure point}}{=} \lim_{n\to\infty} \frac{1}{\mu_G(A_n)} \sum_{x\in A_n \cap \Lambda(\Omega)} \overline{\langle k\,, x\rangle} \\
&= \lim_{n\to\infty} \frac{1}{\mu_G(A_n)} \sum_{x\in A_n \cap \Lambda(\Omega)} \overline{\langle k\,, x\rangle}\; \overline{\langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, x^\star\rangle}\, \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, x^\star\rangle \\
&\overset{(x,x^\star)\in\tilde{\mathcal{L}},\,(k,\pi_1^*)^{-1}(k))\in\tilde{\mathcal{L}}^\perp}{=} \lim_{n\to\infty} \frac{1}{\mu_G(A_n)} \sum_{x\in A_n \cap \Lambda(\Omega)} \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, x^\star\rangle \\
&\overset{\text{Weyl}}{=} \frac{\operatorname{dens}\Lambda}{\mu_H(\Omega)} \int_\Omega \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, y\rangle \,\mathrm{d}\mu_H(y),
\end{aligned}$$

where the last step follows by *Weyl's Theorem* for general model sets, see [35, Theorem 6.2].

Remark 7.16. For a regular multi-component model set $\Lambda(\underline{\Omega})$, Folklore Theorem 5a.10 remains virtually the same with the following modification of the Fourier-Bohr coefficients: Let $(a_i)_{i=1}^n$ be any choice of complex numbers and let $\nu = \sum_{i=1}^n a_i\, \delta_{\Lambda_i}$ be the translation bounded measure associated to $\Lambda(\underline{\Omega})$ (where $\Lambda_i = \Lambda(\Omega_i)$), *i.e.*, all points in Λ_i have scattering strength a_i. Then, the Fourier-Bohr coefficients for the diffraction measure $\hat{\gamma}_\nu$ are given by

$$a(k) = \frac{\operatorname{dens}\Lambda}{\mu_H(\operatorname{supp}\underline{\Omega})} \sum_{i=1}^n a_i \cdot \int_{\Omega_i} \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, y\rangle \,\mathrm{d}\mu_H(y).$$

This follows from [257, Theorem 7] and [260, Theorem 14], where the scattering strength $v(x)$ of a point x in a model set is given by a continuous function $H \to \mathbb{C}$, $x^\star \mapsto v(x)$. Also see Theorem 7.29 and compare this formula for the Fourier-Bohr coefficients with the one for the Fourier-Bohr coefficients of a so-called "deformed" model set, see Remark 7.38.

In view of the CPS $(G^*\,, H^*\,, \tilde{L}^\perp)$ appearing in Folklore Theorem 5a.10, we make the following definition.

Definition 7.17. We call the CPS $(G^*\,, H^*\,, \tilde{L}^\perp)$ to *(Pontryagin) dual CPS* to the CPS $(G\,, H\,, \tilde{L})$. We use the following notations for the canonical projections: $\pi_1 : G \times H \to G$, $\pi_2 : G \times H \to H$, $\pi_1^* : G^* \times H^* \to G^*$ and $\pi_2^* : G^* \times H^* \to G^*$.

The following statement can also be found in [257, Section 4] and [259, p. 418].

Lemma 7.18. [256, Lemma II.9] *Let $(G\,, H\,, \tilde{L})$ be a CPS where π_1 is bijective on $\tilde{L}$ and $\pi_2(\tilde{L})$ is dense in H. Then, for the Pontryagin dual CPS $(G^*\,, H^*\,, \tilde{L}^\perp)$, the projection π_2^* is bijective on $\tilde{L}^\perp$ and $\pi_1^*(\tilde{L}^\perp)$ is dense in G^*.* □

Thus, in particular, the set of Bragg peaks is usually dense in G^*, also see [314, Corollary 2.9].

Remark 7.19. The previous lemma is also interesting in connection with the definition of ε-*dual sets*, see [256, Chapter II], [257, Section 4] and [259, Section 6]: Let $\varLambda \subset G$ be any set and let $\varepsilon > 0$. The ε-dual set $\varLambda^{\varepsilon} \subset G^*$ of $\varLambda$ is defined by

$$\varLambda^{\varepsilon} = \{y \in G^* \mid |\langle y\,, x\rangle - 1| \le \varepsilon \text{ for all } x \in \varLambda\}.$$

Evidently, one has $\varLambda^{\varepsilon_1} \subset \varLambda^{\varepsilon_2}$ for $\varepsilon_1 \le \varepsilon_2$ (and, in fact, $\varLambda^{\varepsilon} = G^*$ for $\varepsilon \ge 2$). Moreover, by the Pontryagin duality, one obtains $\varLambda \subset (\varLambda^{\varepsilon})^{\varepsilon} = \varLambda^{\varepsilon\varepsilon}$ and $\varLambda^{\varepsilon} = \varLambda^{\varepsilon\varepsilon\varepsilon}$. In fact, Meyer defined in [256] the sets now bearing his name through these ε-duals. One can show that $\varLambda$ is a Meyer set iff $\varLambda^{\varepsilon}$ is relatively dense for all $\varepsilon > 0$ (actually, for some $\varepsilon \in [0\,, \frac{1}{2}]$ suffices), see [259, Theorem 9.1]. For the diffraction of Meyer sets the ε-dual sets play an important role, see [369], but the interesting property here is that $\varLambda^{\varepsilon}$ is a model set for $\varepsilon \in]0\,, 2[$, see [256, Theorem II.I], [257, p. 12] and [259, Prop. 10.2]. Thus, the following question arises:

Given a model set $\varLambda$, is there an $\varepsilon > 0$ such that $\varLambda = \varLambda^{\varepsilon\varepsilon}$?

For a class of model sets where this holds with $\varepsilon = 1$, see [257, Theorem 3]. We also note that for a lattice $L \subset G$, one has $L^{\perp} = L^{\varepsilon}$ if $\varepsilon < \sqrt{2}$.

7.2. Spectral Theory of Dynamical Systems

We now continue with the study of dynamical systems, see Section 5.5. In particular, we are here interested with spectral properties of dynamical systems.

7.2.1. General Dynamical Systems

Let $(X\,, \mathfrak{B}\,, \nu\,, G)$ be a measure theoretical dynamical system. We assume that X is a compact topological Hausdorff space, $\mathfrak{B}$ is the σ-algebra of Borel sets of X, ν is a Borel probability measure and the G-action is the canonical action of G on X, *i.e.*, $T_g(x) = x - g$ for all $g \in G$. Moreover, we assume that the measure ν is G-invariant.

In this section, we are in particular interested in $L^2(X\,, \mathfrak{B}\,, \nu)$, the *separable Hilbert space* of measurable complex square-integrable functions on X. As usual, $L^2(X\,, \mathfrak{B}\,, \nu)$ is equipped with the scalar product

$$\langle f_1\,, f_2\rangle_2 = \int_X \overline{f_1(x)}\, f_2(x)\ \mathrm{d}\nu(x)$$

yielding the norm $\|f\|_2 = \sqrt{\langle f\,, f\rangle_2}$ (completeness and separability follow as usual).

We now study the following family $\{U_g\}$ of unitary operators on $L^2(X\,, \mathfrak{B}\,, \nu)$ defined by

$$(U_g f)(x) = f(T_g x)$$

for $x \in X$ and $f \in L^2(X\,, \mathfrak{B}\,, \nu)$. The G-invariance of ν is equivalent to the isometricity of the members U_g (*i.e.*, for every $g \in G$, one has $\|U_g f\|_2 = \|f\|_2$ for all $f \in L^2(X\,, \mathfrak{B}\,, \nu)$), and one checks that $U_g^* = U_{-g} = U_g^{-1}$ (where U^* denotes the *adjoint* of U, *i.e.*, $\langle f_1\,, U f_2\rangle_2 = \langle U^* f_1\,, f_2\rangle_2$).

Properties of the measure theoretic dynamical system $(X\,, \mathfrak{B}\,, \nu\,, G)$ which are expressed through properties of $\{U_g\}_{g\in G}$ are called *spectral properties*. From functional analysis (*e.g.*, see [395]) one knows that the eigenvalues of unitary operators are complex numbers of absolute value one. We define eigenfunctions and eigenvalues using Proposition 7.6, compare [96, Section 1.§7 & Appendix 2].

For any nonzero element $f \in L^2(X, \mathfrak{B}, \nu)$, consider the scalar product

$$b_f(g) = \langle f, U_g f\rangle_2, \quad g \in G.$$

One checks that this is a continuous positive definite function. Consequently, using Proposition 7.6, $b_f(g)$ can be represented as

$$\langle f, U_g f\rangle_2 = b_f(g) = \int_{G^*} \langle x^*, g\rangle \, \mathrm{d}\upsilon_f(x^*)$$

for all $g \in G$, where υ_f is a positive bounded measure on G^*. Actually, one has $\upsilon_f(G^*) = \|f\|_2^2$. The measure υ_f is called the *spectral measure* of the element f with respect to the corresponding dynamical system.

Definition 7.20. A nonzero function $f \in L^2(X, \mathfrak{B}, \nu)$ is an *eigenfunction* for the G-action of the measure theoretic dynamical system $(X, \mathfrak{B}, \nu, G)$ corresponding to the *eigenvalue* $\lambda \in G^*$ if the spectral measure υ_f of f is concentrated at λ, *i.e.*,

$$\upsilon_f = \|f\|_2^2 \, \delta_\lambda.$$

Equivalently, f is an eigenfunction with eigenvalue λ iff

$$(U_g f)(x) = \langle \lambda, -g\rangle f(x)$$

for all $x \in X$ and all $g \in G$. Note that $(U_g f)(x) = f(x - g)$.

Eigenfunctions to different eigenvalues are orthogonal. Moreover, eigenfunctions to the same eigenvalue span a subspace of $L^2(X, \mathfrak{B}, \nu)$ (*e.g.*, if f is an eigenfunction with eigenvalue λ, so is $c \cdot f$ for any $c \in \mathbb{C}$), and we may obtain orthonormal system spanning this subspace (densely) by Gram-Schmidt. We note that any Hilbert space has at least one complete orthonormal system [395, Theorem III.4.1], wherefore it is natural to consider the following property.

Definition 7.21. We say that the measure theoretic dynamical system $(X, \mathfrak{B}, \nu, G)$ has *pure point (dynamical) spectrum* if there is a complete orthonormal system V consisting of orthogonal eigenfunctions, *i.e.*, if the linear combinations of the elements $f \in V$ are dense in $L^2(X, \mathfrak{B}, \nu)$.

We denote by $\mathcal{H}_{\mathrm{pp}}(G) = \mathcal{H}_{\mathrm{pp}}(\{U_g\})$ the smallest closed subspace of $L^2(X, \mathfrak{B}, \nu)$ generated by all eigenfunctions. Consequently, $(X, \mathfrak{B}, \nu, G)$ has pure point dynamical spectrum iff $\mathcal{H}_{\mathrm{pp}}(G) = L^2(X, \mathfrak{B}, \nu)$. In general, however, it is only a closed linear subspace, wherefore we can consider its *orthogonal complement* $\mathcal{H}_{\mathrm{c}}(G) = \mathcal{H}_{\mathrm{pp}}(G)^\perp$, *i.e.*, the totality of functions in $L^2(X, \mathfrak{B}, \nu)$ orthogonal to every function of $\mathcal{H}_{\mathrm{pp}}(G)$. Thus, we have

$$L^2(X, \mathfrak{B}, \nu) = \mathcal{H}_{\mathrm{pp}}(G) \oplus \mathcal{H}_{\mathrm{c}}(G).$$

Now the following holds: If $f \in \mathcal{H}_{\mathrm{pp}}(G)$, then its spectral measure υ_f is a pure point measure, while in the case $f \in \mathcal{H}_{\mathrm{c}}(G)$ the spectral measure υ_f is continuous. Moreover, we may call υ_f *absolutely continuous* if it is absolutely continuous with respect to the Haar measure μ_{G^*}. The terms *singular* respectively *singular continuous* are defined similarly.

We note that $\mathcal{H}_{\mathrm{pp}}(G)$ is nonempty since it always contains the subspace of constants (with eigenvalue 0). Therefore, the following definition is used.

Definition 7.22. Let $\mathcal{H}_c(G) \neq \varnothing$. If $\mathcal{H}_{pp}(G)$ is spanned by the constants, then we say that the measure theoretic dynamical system $(X, \mathfrak{B}, \nu, G)$ is *weakly mixing* (or – neglecting the constants – *continuous*). Otherwise, we say that $(X, \mathfrak{B}, \nu, G)$ has *mixed (dynamical) spectrum.*

The next statement implies that every weakly mixing dynamical system is ergodic.

Lemma 7.23. [96, Theorem 12.§1.2], [287, Theorem 2.4.2] and [384, Theorem 3.1] *The measure theoretic dynamical system $(X, \mathfrak{B}, \nu, G)$ is ergodic iff* 0 *is a simple eigenvalue. Moreover, if $(X, \mathfrak{B}, \nu, G)$ is ergodic, every eigenvalue is simple and $|f|$ is constant a.e. for every eigenfunction f. Moreover, the set of eigenvalues is a subgroup of G^*.* □

Remark 7.24. Let $G = \mathbb{R}^d$ and let $\{A_n\}$ be a van Hove sequence, *e.g.*, the balls of radius n. Then, ergodicity and weak mixing are often defined as follows (compare [96, Sections 1.§2 & 1.§6], [108, Section 5.3], [287, Chapter 2] and [316, Section 7.1]):

- A measure theoretic dynamical system is *strongly mixing* if for any two functions $f_1, f_2 \in L^2(X, \mathfrak{B}, \nu)$ one has
$$\lim_{|g|\to\infty} \langle U_g f_1, f_2\rangle_2 = \langle f_1, 1\rangle_2 \langle 1, f_2\rangle_2.$$
- A measure theoretic dynamical system is *weakly mixing* if for any two functions $f_1, f_2 \in L^2(X, \mathfrak{B}, \nu)$ one has
$$\lim_{n\to\infty} \frac{1}{\mu_{\mathbb{R}^d}(A_n)} \int_{A_n} \left|\langle U_g f_1, f_2\rangle_2 - \langle f_1, 1\rangle_2 \langle 1, f_2\rangle_2\right|^2 \, \mathrm{d}\mu_{\mathbb{R}^d}(g) = 0.$$
- A measure theoretic dynamical system is *ergodic* if for any two functions $f_1, f_2 \in L^2(X, \mathfrak{B}, \nu)$ one has
$$\lim_{n\to\infty} \frac{1}{\mu_{\mathbb{R}^d}(A_n)} \int_{A_n} \langle U_g f_1, f_2\rangle_2 \, \mathrm{d}\mu_{\mathbb{R}^d}(g) = \langle f_1, 1\rangle_2 \langle 1, f_2\rangle_2.$$

Obviously, strong mixing implies weak mixing implies ergodicity. We note that strong mixing does not occur for the dynamical systems we are interested in (namely, tiling dynamical systems where the tiling is obtained by a self-affine tile substitution), see [364, Theorem 4.1] (compare [104] and [299, Theorem VI.22]).

Using the main theorem of the canonical form of a unitary operator, one may also define the *maximal spectral type* and the *multiplicity function* of $(X, \mathfrak{B}, \nu, G)$, see [96, Appendix 2], [298, Sections 1.5.2 & 5.2.2], [281, Appendix], [108, Satz 100], [299, Chapters II & XI] and [351, pp. 23–25].

Definition 7.25. Given the family of unitary operators $\{U_g\}$ of the the Hilbert space $\mathcal{H} = L^2(X, \mathfrak{B}, \nu)$ and a function $f \in \mathcal{H}$, the smallest closed subspace $\mathscr{C}(f) \subset \mathcal{H}$ containing f and invariant with respect to the G-action $\{U_g\}$ is called a *cyclic subspace* of $\mathcal{H}$. Alternatively, $\mathscr{C}(f)$ is the closure of the set generated by $\{U_g f \mid g \in G\}$.

We note the following properties and definitions:

- Let $f_1, f_2 \in \mathcal{H}$ with $\mathscr{C}(f_1) = \mathscr{C}(f_2)$. Then, υ_{f_1} and υ_{f_2} are *equivalent measures*, *i.e.*, υ_{f_1} is absolutely continuous with respect to υ_{f_2} and *vice versa*. Consequently, one may call the class of equivalent bounded measures the *spectral type*, wherefore every cyclic subspace $\mathscr{C}(f)$ is characterised by its spectral type, see [96, p. 454].

- If $\mathscr{C}(f_1)$ and $\mathscr{C}(f_2)$ are cyclic spaces whose spectral types υ_{f_1} and υ_{f_2} are mutually singular, then they are orthogonal and $\mathscr{C}(f_1) \oplus \mathscr{C}(f_2)$ is a cyclic space (of spectral type $\frac{1}{2}(\upsilon_{f_1} + \upsilon_{f_2})$). Moreover, this statement is still valid for a countable sum of cyclic subspaces. Consequently, if $\mathcal{H} = \mathcal{H}_{\mathrm{pp}}$ has pure point spectrum and all its eigenvalues λ_i ($i \in I$) are simple, then $\mathcal{H}$ itself is a cyclic space whose spectral type is equivalent to the weighted sum $\sum_{i \in I} a_i \, \delta_{\lambda_i}$, where $a_i > 0$ for all $i \in I$ and $\sum_{i \in I} a_i = 1$, see [298, Lemma 5.2.11 & Corollary 5.2.12].

- If $\mathcal{H}$ is a cyclic subspace for some $f \in \mathcal{H}$, then we say that $\mathcal{H}$ possesses a *simple spectrum*.

- By the main theorem of the canonical form of a unitary operator (see its "first formulation" in [96, Appendix 2] and [299, Theorem II.4]), there exists a (up to equivalence) unique spectral measure υ_{f_1} (with some $f_1 \in \mathcal{H}$) such that all spectral measures υ_f are absolutely continuous with respect to υ_{f_1} (*i.e.*, υ_{f_1} *dominates* the family $\{\upsilon_f \mid f \in \mathcal{H}\}$). The spectral type of υ_{f_1} is called the *maximal spectral type*.

- Obviously, $(X, \mathfrak{B}, \nu, G)$ has pure point spectrum (respectively is weakly mixing) iff the maximal spectral type is a pure point measure (respectively a continuous measure plus a Dirac measure at 0).

- The *multiplicity* of the spectrum of $(X, \mathfrak{B}, \nu, G)$ is defined *via* the second formulation of the main theorem of the canonical form of a unitary operator (see [96, Appendix 2] and [299, Theorem II.11]). We note that the multiplicity is 1 if the spectrum is simple and that the spectrum is of *multiplicity at most k* if $\mathcal{H}$ is the direct sum of k cyclic subspaces, see [298, Definition 5.2.9].

Remark 7.26. Let $(X, \mathfrak{B}, \nu, \mathbb{R})$ and $(X', \mathfrak{B}', \nu', \mathbb{R})$ be two ergodic measure theoretic dynamical systems with pure point dynamical spectrum. By the *Halmos-von Neumann Theorem* the eigenvalues determine completely if they are measure theoretically conjugate: They are measure theoretically conjugate iff they have the same eigenvalues. In particular, every such pure point measure theoretic dynamical system is measure theoretical conjugate to a group translation along the one-parameter subgroup of the character group (a "rotation on a torus"). An (ergodic) measure theoretic dynamical system with pure point spectrum is also called a *Kronecker system*. See [96, Theorem 12.§3.2], [161, pp. 46–51], [351, Lecture 5] and [384, Theorems 3.4 & 3.6], also compare [316, Section 8.1].

Similarly, one can define the *topological spectrum* of a topological dynamical system (X, G), see [384, Section §5.5] and [299, Section IV.1]: Let X be a compact metric space with canonical G-action. Suppose that (X, G) is *topologically transitive*, *i.e.*, there exists an $x \in X$ such that $\mathbb{X}(x) = X$ (in particular, every minimal topological dynamical system is topologically transitive). Then, a nonzero complex valued continuous function $f \in C(X)$ is an *eigenfunction* for the G-action of (X, G) corresponding to the *eigenvalue* λ if

$$(U_g f)(x) = \langle \lambda, -g \rangle f(x)$$

for all $x \in X$ and all $g \in G$. Moreover, in this case the eigenvalues are simple and the eigenspaces one-dimensional, see [384, Theorem 5.17].

We say that (X, G) has *topological pure point dynamical spectrum* if the smallest closed subspace of $C(X)$ containing the eigenfunctions equals $C(X)$, see [384, Definition 5.9]. Similarly as before, there is also a variant of the Halmos-von Neumann Theorem, see [384, Theorem 5.18]. We call (X, G) *topologically weakly mixing* if it has no non-constant continuous eigenfunctions. Note that Kolmogorov constructed an example of measure theoretical dynamical systems with pure point spectrum but no non-constant continuous eigenfunctions, see [281, Section 5.4].

For $G = \mathbb{R}^d$, we also define *topologically strongly mixing*: The topological dynamical system $(X, \mathbb{R}^d)$ is said to be *topologically (strongly) mixing* if for any two nonempty open sets $V_1, V_2 \subset X$, there exists an $R > 0$ such that

$$U \cap T_g(V) \neq \varnothing, \text{ for all } g \in \mathbb{R}^d, |g| \geq R.$$

As before, there are equivalent characterisations of topological transitivity and topologically strongly and weakly mixing, see [287, Section 4.2.B] and [109, Section 6]. Moreover, topologically strongly mixing implies topologically weakly mixing implies topologically transitive.

Remark 7.27. Let (X, G) is a topological dynamical system and ν a G-invariant Borel probability measure on X with $\operatorname{supp} \nu = X$. If $(X, \mathfrak{B}, \nu, G)$ is ergodic, then (X, G) is topologically transitive. If $(X, \mathfrak{B}, \nu, G)$ is strongly (respectively weakly) mixing, then (X, G) is topologically strongly (respectively weakly) mixing. The converse of these implications are not true, even when (X, G) is uniquely ergodic, see [287, Prop. 4.2.5 & Remark 4.2.6] and [109, Prop. 6.7]. Moreover, neither does weakly mixing imply topologically strongly mixing nor *vice versa*, see [287, Section 4.5].

7.2.2. Pure Pointedness of Spectra

The basic observation which relates the diffraction spectrum of a multi-component Delone set and the spectrum of the associated point set dynamical system is known as *Dworkin's argument*, see [111] (also compare [299, Sections IV.3.3 & IV.3.4], [173, Section 4.2], [329, p. 154], [234, Lemma 3.4] and [32, Theorem 6]). We use the following reformulation of [234, Lemma 3.4].

Proposition 7.28. *Let $\underline{\varLambda}$ be an FLC multi-component Delone set in a σLCAG G with UCF. Then, the point set dynamical system $(\mathbb{X}(\underline{\varLambda}), G)$ is uniquely ergodic by Lemma 5.110. Denote by $(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu, G)$ the corresponding measure theoretic dynamical system supported by $(\mathbb{X}(\underline{\varLambda}), G)$. Let $\nu = \sum_{i=1}^{n} a_i \, \delta_{\varLambda_i}$ be a weighted Dirac comb associated with $\underline{\varLambda}$ where $a_i \in \mathbb{C}$ for all i (i.e., points belonging to the same $\varLambda_i$ have the same scattering strength).*
Then, the[3] diffraction measure $\hat{\gamma}_\nu$ is a spectral measure of $(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu, G)$ and, in particular, absolutely continuous with respect to the maximal spectral type of $(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu, G)$. □

Thus, in particular, pure pointedness of $(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu, G)$ implies pure pointedness of the diffraction measure $\hat{\gamma}_\nu$. Interestingly, the converse is also true, also see [299, Prop. IV.21], [32, Theorem 7], [152, Theorem 1.2].

Theorem 7.29. [234, Theorem 3.2] *Let $\underline{\varLambda}$ be an FLC multi-component Delone set in a σLCAG G with UCF. Then, the following statements are equivalent.*

[3] Under UCF, the autocorrelation measure is unique.

(i) *The measure theoretic dynamical system* $(\mathbb{X}(\underline{\varLambda})\,,\mathfrak{B}\,,\mu\,,G)$ *has pure point spectrum, where* $(\mathbb{X}(\underline{\varLambda})\,,\mathfrak{B}\,,\mu\,,G)$ *is as in Proposition* 7.28.

(ii) *The diffraction measure* $\hat{\gamma}_\nu$ *of* $\nu = \sum_{i=1}^{n} a_i\,\delta_{\varLambda_i}$ *is a pure point measure for any choice of complex numbers* $(a_i)_{i=1}^{n}$.

(iii) *The diffraction measure* $\hat{\gamma}_{\delta_{\varLambda_i}}$ *is a pure point measure for every* $1 \le i \le n$. □

The following statement shows that the group of eigenvalues of the dynamical system is generated by the "position" of the Bragg peaks of the diffraction spectrum, see [32, Theorem 9] (compare [34, Lemma 5]). We formulate this result in the multi-component case.

Corollary 7.30. *Assume the setting of Theorem 7.29. Let* $(\mathbb{X}(\underline{\varLambda})\,,\mathfrak{B}\,,\mu\,,G)$ *be a pure point measure theoretic dynamical system with group of eigenvalues* $\mathscr{P}(\underline{\varLambda})$. *Set*

$$\mathscr{P}'(\underline{\varLambda}) = \{x \in G^* \mid \hat{\gamma}_\nu(\{x\}) > 0, \text{ with } \nu = \sum_{i=1}^{n} a_i\,\delta_{\varLambda_i} \text{ for some choice } a_i \in \mathbb{C}\ (1 \le i \le n)\}.$$

Then, one has $\mathscr{P}(\underline{\varLambda}) = \langle \mathscr{P}'(\underline{\varLambda})\rangle_{\mathbb{Z}}$. □

One might ask if one can replace the measure theoretic dynamical system $(\mathbb{X}(\underline{\varLambda})\,,\mathfrak{B}\,,\mu\,,G)$ by the point dynamical system $(\mathbb{X}(\underline{\varLambda})\,,G)$ in the last theorem, *i.e.*, if the eigenfunctions can be chosen to be continuous. And indeed, in the cases we are interested in, one can as the following statements show.

Lemma 7.31. [33, Theorem 5] (respectively, [329, Theorem 4.5] together with [33, Lemma 1]) *Regular (multi-component) model sets are pure point diffractive. In fact,* $(\mathbb{X}(\underline{\varLambda})\,,G)$ *is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.* □

Moreover, there is also the following generalisation of [183, Prop. 1]. Note that the statement can be generalised to all σLCAG G.

Lemma 7.32. [364, Theorem 5.1(i)] and [360, Theorem 2.13]. *Let* $\mathcal{T}$ *be a primitive substitution tiling with FLC in* $G = \mathbb{R}^d$. *Then every measurable eigenfunction for* $(\mathbb{X}(\mathcal{T})\,,\mathfrak{B}\,,\mu\,,\mathbb{R}^d)$ *coincides with a continuous function* μ*-a.e.* □

In particular, the last result can be rephrased as: For primitive substitution tilings (respectively, representable primitive substitution Delone sets), topologically weakly mixing is equivalent to weakly mixing.

In connection with Propositions 6.16 & 6.42 we note the following statement.

Lemma 7.33. [360, Theorem 2.15] *Let* $\eta > 1$ *and let* $f_0(x) = \eta\, x$ *on* $\mathbb{R}^d$ *be the expansion map. Let* $\mathcal{T}$ *be the fixed point of a primitive tile substitution with FLC and expansion* f_0. *Then, the associated measure theoretical dynamical system is not weakly mixing iff* η *is a PV-number.* □

Thus, in particular, we have for one-dimensional sequences generated by substitutions: the dynamical system associated to the representation with natural intervals is not weakly mixing (and therefore has either pure point spectrum or mixed spectrum) iff the spectral radius (and therefore the inflation factor) of the substitution matrix is a PV-number. In fact, if the spectral radius is not[4] a PV-number, one often has a topologically strongly mixing dynamical system, see [207]: $(\mathbb{X}(\mathcal{T}), \mathbb{R})$ is topologically strongly mixing iff $\varDelta'$ is *eventually dense* in $\mathbb{R}$, *i.e.*, for any $\varepsilon > 0$, there exists an $R > 0$ such that the ε-neighbourhood of $\varDelta'$ covers $\mathbb{R} \setminus [-R, R]$ (*i.e.*, $(\mathbb{R} \setminus [-R, R]) \subset (B_{<\varepsilon}(0) + \varDelta')$).

Although Proposition 7.28 is often only applied to the pure point part of the spectrum, one can also compare the whole dynamical and diffraction spectrum. If both are pure point, then the dynamical spectrum is usually easier to calculate since the eigenvalues form a group while the set of Bragg peaks $\{x \in G^* \,|\, \hat{\gamma}_\nu(\{x\}) > 0\}$ is usually not a group (however, the set of Bragg peaks often generates the group of eigenvalues, see [32, Theorem 9]). For the continuous part of the spectrum, their relationship is less understood, wherefore we continue Example 6b.18 here.

Example 7.34. We first consider the Thue-Morse substitution $a \mapsto ab$ and $b \mapsto ba$: Using the scattering strengths $v_a = +1$ and $v_b = -1$, it is already proven in [245] that $\nu = \delta_{\varLambda_a} - \delta_{\varLambda_b}$ (where $\ell_a = 1 = \ell_b$) has a purely singular continuous diffraction measure $\hat{\gamma}_\nu$ (also compare [196], [11, Section IV.5] and [142, pp. 164–165]). We may define $\hat{\gamma}_\nu$ as weak-* limit of a sequence of absolute continuous measures by (observing that $\mathbb{R}^* = \mathbb{R}$, wherefore $\mu_{\mathbb{R}}$ is the one-dimensional Lebesgue measure)

$$\hat{\gamma}_\nu(f) = \lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, 2^n \prod_{j=0}^{n-1} \sin^2(\pi\, 2^j\, x)\ \mathrm{d}\mu_{\mathbb{R}}(x)$$

for all $f \in \mathscr{K}(G)$, see [299, Section VIII.2.1] (also compare [93, Section 2]). If one considers the diffraction measure of the weighted Dirac comb $\nu' = v_a\, \delta_{\varLambda_a} + v_b\, \delta_{\varLambda_b}$ with arbitrary $v_a, v_b \in \mathbb{C}$, one obtains

$$\hat{\gamma}_{\nu'} = \left| \frac{v_a + v_b}{2} \right|^2 \delta_{\mathbb{Z}} + \left| \frac{v_a - v_b}{2} \right|^2 \hat{\gamma}_\nu .$$

The dynamical spectrum, on the other hand, is simple, *i.e.*, $L^2(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu_{\mathbb{R}})$ is a cyclic space. Moreover, the spectrum is mixed. Every $\lambda \in \mathbb{Z}[\frac{1}{2}]$ is a (simple) eigenvalue, see [298, Prop. 5.2.21]. Note that this is the set of possible Bragg peaks provided it would be[5] a model set (see Folklore Theorem 5a.10 and Remark 6b.24). Moreover, $f \in \mathcal{H}_c(\mathbb{R})$ iff v_f is absolutely continuous with respect to $\hat{\gamma}_\nu$, see [299, Corollary VIII.3]. Thus, the maximal spectral type is given as

$$\frac{1}{2}(\hat{\gamma}_\nu + \sum_{\lambda \in \mathbb{Z}[\frac{1}{2}]} a_\lambda\, \delta_\lambda),$$

[4] An explicit class of one-dimensional primitive substitutions σ over two symbols $\{a, b\}$ which is weakly mixing is given in [300, Result 7.2.1.2]: If both eigenvalues of the substitution matrix $\boldsymbol{S}\sigma$ are in modulus greater than or equal to one (non-Pisot condition), $\det \boldsymbol{S}\sigma$ and $\operatorname{tr} \boldsymbol{S}\sigma$ are relatively prime, and $\#\sigma(a)$ and $\#\sigma(b)$ are relatively prime, then the corresponding dynamical system is weakly mixing (has no eigenvalues besides 0).

[5] A nice interpretation of the difference between the pure point part of the diffraction and the dynamical spectrum appears in [142, p. 165]: The extinctions in the pure point part of the diffraction spectrum are a consequence of the identical decoration of all points of a given type irrespective of their neighbourhood. If one allows the scattering strength of each point to depend on its neighbourhood, some of the possible Bragg peaks reappear. If each scattering strength depends on the neighbourhood up to infinity, then the pure point part of the diffraction spectrum spans all of the pure point dynamical spectrum. Thus, we may say that the pure point part of the dynamical spectrum is sensitive to the neighbourhood of each point up to infinity.

where $a_\lambda > 0$ and $\sum_\lambda a_\lambda = 1$.

As second example, we consider the Rudin-Shapiro sequence $a \mapsto ac$, $b \mapsto dc$, $c \mapsto ab$ and $d \mapsto db$ followed by the the projections $\{a, c\} \mapsto A$ and $\{b, d\} \mapsto B$: Choosing $v_A = +1$ and $v_B = -1$, the diffraction measure is simply the Lebesgue measure, *i.e.*, $\hat{\gamma}_\nu = \mu_{\mathbb{R}}$ where $\nu = (\delta_{\varLambda_a} + \delta_{\varLambda_c}) - (\delta_{\varLambda_b} + \delta_{\varLambda_d})$, see [299, Corollary VIII.5]. Thus, for arbitrary $v_A, v_B \in \mathbb{C}$, one obtains (with the obvious definition of ν')

$$\hat{\gamma}_{\nu'} = \left|\frac{v_A + v_B}{2}\right|^2 \delta_{\mathbb{Z}} + \left|\frac{v_A - v_B}{2}\right|^2 \mu_{\mathbb{R}}.$$

The dynamical spectrum is again mixed and has the same pure point part as the Thue-Morse sequence (and, in fact, as every substitution[6] of constant length 2 and height 1). Similarly as above, $f \in \mathcal{H}_c(\mathbb{R})$ iff v_f is absolutely continuous (with respect to $\hat{\gamma}_\nu = \mu_{\mathbb{R}}$), see [299, Corollary VIII.4]. Therefore the maximal spectral type is now

$$\frac{1}{2}(\mu_{\mathbb{R}} + \sum_{\lambda \in \mathbb{Z}[\frac{1}{2}]} a_\lambda \, \delta_\lambda).$$

But the Hilbert space $L^2(\mathbb{X}(\underline{\varLambda}), \mathfrak{B}, \mu_{\mathbb{R}})$ (with $\underline{\varLambda} = (\varLambda_A, \varLambda_B)$) is not cyclic but has multiplicity 2; in fact, this Hilbert space is the sum of two cyclic subspaces, one having the maximal spectral type (by definition), the other having the Lebesgue measure $\mu_{\mathbb{R}}$ as spectral type, see [299, Theorem XI.8] and [298, Section 5.3.2].

Remark 7.35. For more examples, also see [300, Sections 6.3 & 7.2]. We remark that substitutions with the same substitution matrix might have different spectrum, *e.g.*, consider the following substitutions:

$$\begin{array}{lll} a \mapsto ab & \quad a \mapsto ab & \quad a \mapsto ab \\ b \mapsto baab & \quad b \mapsto baba & \quad b \mapsto bbaa \end{array}$$

Then, the first two substitution have pure point (dynamical and diffraction) spectrum, while the third one has mixed dynamical spectrum, see [258, Section IV.] (compare [300, Section 7.2.2.1]). Obviously, a similar example is the Thue-Morse substitution and the substitution $a \mapsto ab$ and $b \mapsto ab$ which yields a periodic (and thus pure point diffractive) sequence.

As last point of this section, we discus (multi-component) Delone sets which are pure point diffractive but not necessarily FLC. For simplicity, we only consider the single component case. However, the generalisation to the multi-component case is straightforward.

Definition 7.36. Let $\varLambda = \Lambda(\varOmega)$ be a regular model set with respect to the CPS $(G, H, \tilde{L})$. Let $\vartheta : H \to G$ be a continuous function with compact support (which includes $\varOmega$). Define

$$\varLambda_\vartheta = \{x + \vartheta(x^\star) \mid x \in \varLambda\} = \{x + \vartheta(x^\star) \mid x \in L \text{ and } x^\star \in \varOmega\}.$$

If $\varLambda_\vartheta$ is still a Delone set, we call $\varLambda_\vartheta$ a *deformed model set* with *deformation* ϑ.

[6] The topological dynamical system consisting of the compact group $\mathbb{Z}_2$ together with the action $f : x \mapsto x+1$ is called the 2-*odometer*. The group of eigenvalues for this dynamical system is given by $\mathbb{Z}[\frac{1}{2}]$, thus, by the Halmos-von Neumann Theorem (see Remark 7.26), every substitution of constant length 2 and height 1 is conjugate to the 2-odometer. Similarly, one may define more general odometers for substitutions of constant length, see [300, Section 5.4] and [298, Sections 1.6.2 & 5.2.3]. Actually, such an odometer is the dynamical system $\mathbb{T}(\underline{\varLambda})$ appearing in Proposition 7.48.

The following is a generalisation of [63, Theorem 4.4].

Proposition 7.37. [33, Theorem 6] *Let $\varLambda_\vartheta$ be a deformed model set of a regular model set $\varLambda$ with continuous deformation ϑ. Then $\varLambda_\vartheta$ is pure point diffractive. In fact, the dynamical system $(\mathbb{X}(\varLambda_\vartheta)\,, G)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.* □

Remark 7.38. In particular, the condition that $\varLambda_\vartheta$ is still a Delone set implies that its density is well defined and one has $\operatorname{dens} \varLambda_\vartheta = \operatorname{dens} \varLambda$. Moreover, unless ϑ is affine over $\varOmega$, the difference set $\varLambda_\vartheta - \varLambda_\vartheta$ is not a Delone set anymore. In fact, one can choose deformations ϑ such that $\varLambda_\vartheta - \varLambda_\vartheta$ is dense in some ball in G. Consequently, the assumption that a point set is FLC is not necessary to obtain a pure point diffraction measure.

We also note that the diffraction measure is still concentrated on $\pi_1^*(\tilde{L}^\perp)$, see Folklore Theorem 5a.10. Furthermore, with the notation of that theorem, the Fourier-Bohr coefficient of $\varLambda_\vartheta$ at $k \in \pi_1^*(\tilde{L}^\perp)$ is given by

$$a(k) = \frac{\operatorname{dens} \varLambda}{\mu_H(\varOmega)} \int_\varOmega \langle \pi_2^* \circ (\pi_1^*)^{-1}(k)\,, y\rangle\, \overline{\langle k\,, \vartheta(y)\rangle}\; \mathrm{d}\mu_H(y).$$

We also note that the above proposition actually is proven by showing that $(\mathbb{X}(\varLambda_\vartheta)\,, G)$ is a factor of $(\mathbb{X}(\varLambda)\,, G)$ (recall Definition 5.101). Consequently, this yields the pure pointedness of $(\mathbb{X}(\varLambda_\vartheta)\,, G)$. Also compare [173, Section 4.3.1] and [152, Theorems 1.3 & 1.4 & Section 5]. Furthermore, we observe that the autocorrelation measure of a deformed model set is in general not a pure point measure anymore, but still a strongly almost periodic measure (see Proposition 7.13). In particular, Theorem 7.10 and Proposition 7.11 are in general not applicable to deformed model sets.

We also note that, depending on the direct and internal space of the model set, there might (besides the constants) be no deformed model sets. *E.g.*, consider the constant length substitutions over two symbols $\{a, b\}$ (which have p-adic internal space, see Chapter 6b), then one does not find a non-trivial continuous deformation. This might also be deduced from [95, Theorem 2.5], where the support of the pure point part of the diffraction spectrum of such constant length substitutions are considered depending on the ratio of the interval lengths ℓ_a/ℓ_b (note that for the natural interval lengths one always obtains $\ell_a = \ell_b$): in the case $\ell_a = \ell_b$ one obtains the result of Chapter 6b, but the support in general differs from the one for the case $\ell_a/\ell_b \in \mathbb{Q} \setminus \{1\}$. Even more surprisingly, the pure point part is always trivial if $\ell_a/\ell_b \notin \mathbb{Q}$.

Remark 7.39. Examples of deformed model sets can be found in the following articles: In [63, Section 5], deformations of the Fibonacci sequences are considered. In [42], the Kolakoski-$(3, 1)$ sequence (see Example 6.114) and a certain deformation thereof is considered. And a deformation the silver mean chain, *i.e.*, the sequence defined through the Pisot substitution $a \mapsto aba$ and $b \mapsto a$, is studied in [33, Section 7].

A different approach is taken in [354]: There, structures obtain from a regular model set (namely, the two-dimensional Penrose tiling) by a nearest neighbour interaction using a Monte-Carlo simulation are considered. It is argued that such "distorted" structures are close to deformed model sets. We note that this actually has applications in crystallography, more precisely, to the understanding of the so-called "size-effect" in quasicrystals (also compare Figure 7.1).

Remark 7.40. There are similarities between symbolic dynamical systems given through a one-dimensional symbolic sequence $u \in \mathcal{A}^{\mathbb{Z}}$, *i.e.*, the dynamical system $(\mathbb{X}(u)\,, S)$ where S denotes the shift (a $\mathbb{Z}$-action), and the dynamical system defined by the corresponding representation $\underline{\Lambda}$ with natural intervals, *i.e.*, the dynamical system $(\mathbb{X}(\underline{\Lambda})\,, \mathbb{R})$, see for example [95, 320, 358].

In fact, if $\underline{\Lambda}$ is a model set, one may use a linear deformation ϑ such that all intervals/tiles in $\underline{\Lambda}_\vartheta$ have the same length – namely, in case of a Pisot substitution, the average length $\bar{\ell} = \sum_{i=1}^{n} \varrho_i \cdot \ell_i$, see the examples in [42, 63] (in fact, it is easy to see from these examples that such a deformation for a Pisot substitution is always possible). Moreover, we may right from the start work with the normalisation of the ℓ_is such that the average length is $\bar{\ell} = 1$. Consequently, the support of the corresponding deformed model $\underline{\Lambda}_\vartheta$ is then $\mathbb{Z}$. In topological respectively dynamical systems terms, one has the following picture: The hull $\mathbb{X}(\underline{\Lambda})$ can be deformed into $\mathbb{X}(\underline{\Lambda}_\vartheta)$, and the latter hull is topologically conjugate to the "1-fold suspension" of the $\mathbb{Z}$-shift on $\mathbb{X}(u)$, *i.e.*, to a dynamical system $(\mathbb{X}(u)\,, S)$, we might associate the flow given by the $\mathbb{R}$ action $S^{\mathbb{Z}} \times \mathbb{R}/\sim$ with equivalence relation

$$(u = \ldots u_{-2}u_{-1}\dot{u}_0u_1u_2\ldots, \ell_{u_0}) \sim (S(u) = \ldots u_{-1}u_0\dot{u}_1u_2u_3\ldots, 0).$$

Consequently, one may deduce the pure pointedness of $(\mathbb{X}(u)\,, S)$ from the pure pointedness of $(\mathbb{X}(\underline{\Lambda})\,, \mathbb{R})$, although their relation to one another is in general subtle, *e.g.*, compare the statement of [95, Theorem 2.5] (see Remark 7.38).

An example where there is (or seems to be?) an additional feature in the $\mathbb{Z}$-action that is not present in the $\mathbb{R}$-action, actually arises for Pisot substitutions.

Remark 7.41. Let σ be a Pisot substitution with fixed point u. We consider the discrete dynamical system $(\mathbb{X}(u)\,, S)$, where S denotes the shift, and the attractor $\underline{\Omega}$ of $\boldsymbol{\Theta}^\star$. Recall that we use the left endpoints of the intervals $[0\,, \ell_i]$ as "control points", *i.e.*, we the following representation $\underline{\Lambda} + \underline{A}$ by natural intervals $A_i = [0\,, \ell_i]$ for u. However, as noted in Remark 6.9, we might also use the prototiles $A_i' = [-\ell_i\,, 0]$ (thus, the right endpoints of the intervals are used as "control points") to represent u as $\underline{\Lambda}' + \underline{A}' = \underline{\Lambda} + \underline{A}$. We note that one has $\Lambda_i = \Lambda_i' - \ell_i$, wherefore, if $\underline{\Lambda}$ is an IMS with windows $\underline{\Omega}$, the family $\underline{\Lambda}'$ is also an IMS where the window for Λ_i' is given by $\Omega_i - \ell_i^\star$ (note that by $\operatorname{supp} \underline{\Lambda} = \underline{\Lambda}'$ one also has $\operatorname{supp} \underline{\Omega} = \bigcup_{i=1}^{n} \Omega_i = \bigcup_{i=1}^{n} \Omega_i - \ell_i^\star$).

Now, if one only looks at the ordering of the points in $\underline{\Lambda}$ and $\underline{\Lambda}'$, then $\underline{\Lambda}$ corresponds to $\ldots u_{-2}u_{-1}\dot{u}_0u_1u_2\ldots$, while $\underline{\Lambda}'$ corresponds to $\ldots u_{-3}u_{-2}\dot{u}_{-1}u_0u_1\ldots$. Consequently, one may take the point of view that the shift S on $\mathbb{X}(u)$ corresponds[7] to a *domain exchange transformation* on $\underline{\Omega}$ defined by $\Omega_i \to \Omega_i - \ell_i^\star$, see [191, Section 3], [20], [88, Section 4], [345, Section 3.3], [298, Sections 7.4.4 & 7.5.4 & 8.3.1].

However, this structure is destroyed if one considers the $\mathbb{R}$-action $(\mathbb{X}(\underline{\Lambda})\,, \mathbb{R})$, where $\underline{\Lambda}$ is the representation by natural intervals.

7.2.3. Substitutions

For one-dimensional sequences u over an alphabet $\mathcal{A}$, the equivalence between the relative denseness of the ε-almost periods and pure point diffractivity (respectively pure point dynamical spectrum due to Theorem 7.29) in Proposition 7.11 can already been found in [299]: There, the statement that the ε-almost periods are relatively dense for every $\varepsilon > 0$ is stated as "u is *mean-almost periodic*" (see [299, Definition VI.4]).

[7] By the similarities to the well-known *interval exchange transformation* in one-dimensional ergodic theory, see [200, 376], [244, Section II.4D], [96, Chapter 5], [351, pp. 24–27] and [198, Section 14.5].

Now, for a sequence u obtained by a substitution σ, the statement of [299, Lemma VI.27] (which is attributed to B. Host) connects this property of mean-almost periodicity with the geometric convergence against zero of the non-overlap densities (in fact, they are equivalent), also see [363, Theorem 2.1] and [362, Lemma 3.2]. Generalising to our ("continuous") situation, the statement reads as follows, see [235, Theorem 4.7] (compare [364, Prop. 6.7]).

Theorem 7.42. *Let $\underline{\Lambda}$ be a repetitive substitution multi-component Delone set that satisfies* **(PLT)**. *Suppose that Δ' is a Meyer set and let $x \in \Delta'$. Then the following are equivalent:*

(i) *The ε-almost period P_ε is relatively dense for every $\varepsilon > 0$.*

(ii) $1 - \mathrm{dens}^{\mathit{ovlap}}_{\mathcal{T}}(f_0^m(x)) \leq C \cdot r^m$ *for $m \in \mathbb{N}$ and some constants (independent of x) $C > 0$ and $r \in]0\,,1[$.*

Consequently, in this case any of the equivalent properties in Lemma 5.129 *is equivalent to any of the equivalent properties in Proposition* 7.11 *(and thus also to the properties in Theorem* 7.29*).*

Proof. The case $G = \mathbb{R}^d$ is proven in [235, Theorem 4.7] (see proofs of [235, Prop. 4.5 & Lemma A.9] together with [364, Theorem 6.1]). The general case follows analogously. □

The proof of the previous statement also makes use of the following characterisation of the eigenvalues of the dynamical system $(\mathbb{X}(\underline{\Lambda})\,, \mathfrak{B}\,, \mu\,, G)$, which generalises Lemma 6c.11.

Proposition 7.43. *Let $\mathcal{T}$ be a primitive substitution tiling with FLC with expansion map f_0. Then, the following statements are equivalent for $\lambda \in G^*$.*

(i) *λ is an eigenvalue for the (uniquely ergodic) measure theoretic dynamical system $(\mathbb{X}(\mathcal{T})\,, \mathfrak{B}\,, \mu\,, G)$.*

(ii) *λ is an eigenvalue for the topological dynamical system $(\mathbb{X}(\mathcal{T})\,, G)$.*

(iii) *λ satisfies the following two conditions:*

$$\lim_{m\to\infty} \langle \lambda\,, f_0^m(x)\rangle = 1 \quad \text{for all } x \in \Delta',$$

and

$$\langle \lambda\,, z\rangle = 1 \quad \text{for all } z \in \mathcal{P},$$

where $\mathcal{P}$ denotes the group of periods of $\mathcal{T}$, i.e., $\mathcal{P} = \{t \in G \mid \mathcal{T} - t = \mathcal{T}\}$.

In particular, a corresponding statement holds for a repetitive substitution multi-component Delone set $\underline{\Lambda}$ that satisfies **(PLT)** *and where Δ' is a Meyer set.*

Proof. In the case $G = \mathbb{R}^d$, this is [360, Theorem 2.14] (also see [364, Theorem 5.1(i)]). The proof in our situation is analogous.

For the equivalence of the first two properties, also see Lemma 7.32. □

7.3. Model Sets and Torus Parametrisation

In Section 5.6, we have shown that the hull $\mathbb{X}(\underline{\Lambda})$ is compact in the local topology iff $\underline{\Lambda}$ is an FLC multi-component Delone set, see Lemma 5.104. Consequently, one obtains the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), G)$, see Corollary 5.105. We now define another topology on $\mathcal{D}_m$, the set of all multi-component uniformly discrete subsets of an LCAG G with m components, see [34, 230, 231, 233, 262, 263]. Note that often results are only formulated in the single-component case, but they usually also hold in the multi-component case (only the proofs get more technical but are – *mutatis mutandis* – the same). Therefore, we formulate statements in this section in the multi-component setting.

Definition 7.44. Let $\mathcal{D}_m$ be the set of all multi-component uniformly discrete subsets of an LCAG G with m components and denote by $\mathcal{D}_m^U$ the subset thereof consisting of those multi-component sets which are uniformly discrete for the set U (*i.e.*, where in the definition of uniformly discrete one can take the set U, see Definition 5.1).

We equip $\mathcal{D}_m$ respectively $\mathcal{D}_m^U$ with the *autocorrelation topology* (or *AC topology* for short) defined *via* the pseudometric (see Footnote 6 on p. 142)

$$d_{\mathrm{AC}}(\underline{\Lambda}', \underline{\Lambda}'') = \limsup_{m\to\infty} \frac{\sum_{i=1}^{n} \operatorname{card}\left((\Lambda_i' \,\Delta\, \Lambda_i'') \cap A_m\right)}{\mu_G(A_m)},$$

where $\underline{\Lambda}', \underline{\Lambda}'' \in \mathcal{D}_m$ (respectively $\underline{\Lambda}', \underline{\Lambda}'' \in \mathcal{D}_m^U$) and $\{A_m\}_{m\in\mathbb{N}}$ is a van Hove sequence in G.

One obtains a metric space by defining the equivalence relation "$\equiv$" on $\mathcal{D}_m$ (respectively $\mathcal{D}_m^U$) by

$$\underline{\Lambda}' \equiv \underline{\Lambda}'' \quad \text{if} \quad \tilde{d}_{\mathrm{AC}}(\underline{\Lambda}', \underline{\Lambda}'') = 0.$$

Consequently, one obtains a metric space $\hat{\mathcal{D}}_m = (\mathcal{D}_m/{\equiv}) \cong (\mathcal{D}_m/\operatorname{cl}_{\mathrm{AC}}\{0\})$ (respectively $\hat{\mathcal{D}}_m^U = (\mathcal{D}_m^U/{\equiv}) \cong (\mathcal{D}_m^U/\operatorname{cl}_{\mathrm{AC}}\{0\})$), where $\operatorname{cl}_{\mathrm{AC}}$ denotes closure with respect to the autocorrelation topology, with metric d_{AC} (by abuse of notation, we use the same symbol for the induced metric).

One now equips $\hat{\mathcal{D}}_m^U$ with a uniform structure defined through the entourages

$$U_{\mathrm{MT}}(V, \varepsilon) = \{(\underline{\Lambda}', \underline{\Lambda}'') \in \hat{\mathcal{D}}_m^U \times \hat{\mathcal{D}}_m^U \mid d_{\mathrm{AC}}(v + \underline{\Lambda}', \underline{\Lambda}'') < \varepsilon \text{ for some } v \in V\} \tag{7.2}$$

for every neighbourhood V of $0 \in G$ and every $\varepsilon > 0$. The topology on $\hat{\mathcal{D}}_m^U$ defined by this uniform structure is called[8] the *mixed topology.*

In plain words: Two multi-component point sets are close in the AC topology if they agree up to points of small density. And they are close in the mixed topology if they agree after a "small" (global) translation up to points of small density. We also note that in general the local topology (Definition 5.102) and the autocorrelation topology are not connected (see [263, Example 2.6] and [262, Example 2.1]), however, for multi-component model sets they will be. We note the following properties, see [263, Prop. 2.1 & Corollary 3.10], [34, Prop. 1 & Lemma 5], [262, Props. 2.2 & 3.1(ii)], [233, Prop 3.1]

[8] Alternatively, one may obtain the mixed topology as induced topology on $\hat{\mathcal{D}}_m^U$ of the uniform structure on $\mathcal{D}_m$ given by the entourages

$$U'_{\mathrm{MT}}(V, \varepsilon) = \{(\underline{\Lambda}', \underline{\Lambda}'') \in \mathcal{D}_m \times \mathcal{D}_m \mid d_{\mathrm{AC}}(v + \underline{\Lambda}', \underline{\Lambda}'') < \varepsilon \text{ for some } v \in V\}.$$

Proposition 7.45. (i) $\hat{\mathcal{D}}_m$ *(respectively* $\hat{\mathcal{D}}_m^U$*) is the Hausdorff completion of* $\mathcal{D}_m$ *(respectively* $\mathcal{D}_m^U$*) in the AC topology.*

(ii) $\hat{\mathcal{D}}_m^U$ *is complete in the mixed topology.*

(iii) *Let* $\mathbb{A}(\underline{\Lambda}) = \mathrm{cl}_{\mathrm{MT}}\, \underline{\Lambda} + G = \mathrm{cl}_{\mathrm{MT}}\, \mathcal{O}(\underline{\Lambda})$*. Then,* $\mathbb{A}(\underline{\Lambda})$ *is compact (in the mixed topology) iff, for every* $\varepsilon > 0$*, the* ε*-almost period* P'_ε *is relatively dense in* G*.* □

One uses the symbol "$\mathbb{A}(\underline{\Lambda})$" to indicate that one speaks about the *autocorrelation hull* (and not the hull $\mathbb{X}(\underline{\Lambda})$ defined *via* the local topology which is compact iff $\underline{\Lambda}$ is FLC by Lemma 5.104). Consequently, under the assumptions of Theorem 7.10 (note that in this case $\underline{\Lambda}$ is pure point diffractive), one has two topological dynamical system on compact spaces, namely

$$(\mathbb{X}(\underline{\Lambda})\,, G) \qquad \text{and} \qquad (\mathbb{A}(\underline{\Lambda})\,, G).$$

Note that the mixed topology takes small shifts into account in order to make the G-action continuous (thus, it "mixes" the autocorrelation topology and the usual topology of G). The interplay between these two dynamical systems is now of interest. Since there is a natural map $\beta : \mathcal{D}_m^U \to \hat{\mathcal{D}}_m^U$, $\underline{\Lambda} \to [\underline{\Lambda}]$ (where $[\underline{\Lambda}]$ denotes the equivalence class of $\underline{\Lambda}$ in $\hat{\mathcal{D}}_m^U$), one might hope that $(\mathbb{A}(\underline{\Lambda})\,, G)$ is a factor of $(\mathbb{X}(\underline{\Lambda}\,, G)$. However, it is not clear that β is continuous. But if β is continuous and $\mathbb{X}(\underline{\Lambda})$ is compact, then one actually[9] has $\beta(\mathbb{X}(\underline{\Lambda})) = \mathbb{A}(\underline{\Lambda})$, wherefore $\mathbb{A}(\underline{\Lambda})$ is also compact.

Moreover, since $\mathbb{A}(\underline{\Lambda})$ is compact (essentially) iff $\underline{\Lambda}$ is pure point diffractive (and therefore has pure point dynamical spectrum), one might also expect by the Halmos-von Neumann Theorem (see Remark 7.26) that $(\mathbb{A}(\underline{\Lambda})\,, G)$ is topologically conjugate to a "rotation on a torus". In fact, we recall the use of the name "mixed topology" in Definition 5.50: Define a pseudo-metric $d_{\underline{\Lambda}}(x\,, y)$ on G (relative to $\underline{\Lambda}$) by

$$d_{\underline{\Lambda}}(x\,, y) = d_{\mathrm{AC}}(x + \underline{\Lambda}\,, y + \underline{\Lambda}).$$

In fact, up to equivalence, this is simply the maximum variogram $\varrho_{\underline{\Lambda}}(x\,, y)$ of Definition 5.38 (see Footnote 6 on p. 142). Thus, pulling back the mixed topology from $\mathcal{D}_m^U$ to G (by $x \mapsto x + \underline{\Lambda}$), one may equip G with a corresponding uniformity defined by the entourages (compare Equation (7.2) on p. 367)

$$\begin{aligned} \check{U}_{\mathrm{MT}}(V, \varepsilon) &= \{(x\,, y) \in G \times G \mid \varrho_{\underline{\Lambda}}(x + v\,, y) < \varepsilon \text{ for some } v \in V\} \\ &= \{(x\,, y) \in G \times G \mid x - y \in (V + P'_\varepsilon)\}. \end{aligned}$$

Obviously, this is the mixed topology of G as defined in Definition 5.50. Thus, (for a fixed $\underline{\Lambda}$) one may identify $\mathbb{A}(\underline{\Lambda})$ and the completion of G with respect to the mixed topology. Consequently, $\mathbb{A}(\underline{\Lambda})$ carries a natural Abelian group structure and $(\mathbb{A}(\underline{\Lambda})\,, G)$ is minimal. Moreover, if one constructs a CPS as in Section 5.3, Lemma 5.51 applies and one has $\mathbb{A}(\underline{\Lambda}) \cong (G \times H)/\tilde{\mathcal{L}}$, *i.e.*, $\mathbb{A}(\underline{\Lambda})$ can be identified with a compact Abelian group (which actually is a torus if $G \times H \cong \mathbb{R}^d$). We use the shorthand notation $\mathbb{T}(\underline{\Lambda}) = (G \times H)/\tilde{\mathcal{L}}$.
In view of these considerations, we make the following definition.

Definition 7.46. Let X be a compact space and $(X\,, G)$ a topological dynamical system under the G-action. Let Y be a compact Abelian group and $(Y\,, G)$ be a minimal topological dynamical system. If $(Y\,, G)$ is a factor of $(X\,, G)$, we call the factor map $\beta_Y : X \to Y$ a *torus parametrisation.*

[9] By construction, $(\mathbb{A}(\underline{\Lambda})\,, G)$ and $(\mathbb{X}(\underline{\Lambda})\,, G)$ are topologically transitive.

The name comes from [29], also see [34, 329].

We recall that space X with a G-action (thus, if one has a topological dynamical system $(X\,,G)$) is also called a *G-space*. Moreover, a map $\phi : X \to X'$ between two G-spaces is called a *G-map* if for any $g \in G$ and any $x \in X$ one has $\phi \circ T_g(x) = T'_g \circ \phi(x)$ (where T_g respectively T'_g denote the G-action associated with $g \in G$ on X respectively X'). Thus, a factor map is a surjective continuous G-map, and a torus parametrisation is a factor map where the factor is a minimal topological dynamical system.

We are now first concerned with the question under which condition a torus parametrisation exists. Afterwards, we justify the word "torus" parametrisation.

Lemma 7.47. [34, Theorem 7] *Let $\underline{\Lambda}$ be a multi-component Meyer set with UCF; thus, $(\mathbb{X}(\underline{\Lambda})\,,G)$ is uniquely ergodic (see Lemma* 5.110*) and we denote the G-invariant Borel measure by ν. Then, the following assertions are equivalent.*

(i) *There exists a torus parametrisation $\beta_{\underline{\Lambda}} : \mathbb{X}(\underline{\Lambda}) \to \mathbb{A}(\underline{\Lambda})$.*

(ii) *$(\mathbb{X}(\underline{\Lambda})\,,\mathfrak{B}\,,\nu\,,G)$ has pure point dynamical spectrum with continuous eigenfunctions, i.e., $(\mathbb{X}(\underline{\Lambda})\,,G)$ has pure point topological dynamical spectrum.*

In this case, two sets $\underline{\Lambda}', \underline{\Lambda}'' \in \mathbb{X}(\underline{\Lambda})$ satisfy $\beta_{\underline{\Lambda}}(\underline{\Lambda}') = \beta_{\underline{\Lambda}}(\underline{\Lambda}'')$ iff $f(\underline{\Lambda}') = f(\underline{\Lambda}'')$ for every eigenfunction f. □

Proposition 7.48. [34, Theorem 2] and [233, Corollary 3.3] *Let $\underline{\Lambda}$ be a multi-component Meyer set with UCF. Suppose that there exists a continuous G-map $\beta_{\underline{\Lambda}} : \mathbb{X}(\underline{\Lambda}) \to \mathbb{A}(\underline{\Lambda})$. Then, there is a CPS*[10] *$(G\,,H\,,\mathcal{L})$, and there is a topological isomorphism $\mathbb{A}(\underline{\Lambda}) \cong \mathbb{T}(\underline{\Lambda}) = (G\times H)/\tilde{\mathcal{L}}$ which is a G-map sending $\underline{\Lambda} \in \mathbb{A}(\underline{\Lambda})$ to $0 \in \mathbb{T}(\underline{\Lambda})$. In particular, $\beta_{\underline{\Lambda}}$ is a torus parametrisation and there is also a torus parametrisation $\beta'_{\underline{\Lambda}} : \mathbb{X}(\underline{\Lambda}) \to \mathbb{T}(\underline{\Lambda})$.* □

Remark 7.49. Our findings about $\mathbb{A}(\underline{\Lambda})$ (its connection with the sets P'_ε and with $\mathbb{T}(\underline{\Lambda})$) should be compared with the following statement about a discrete topological dynamical system $(X\,,f)$.

Lemma. [287, Theorem 4.2.11] *The following statements about a minimal topological dynamical system $(X\,,f)$ on a metric space $(X\,,d)$ are equivalent:*

(i) *$(X\,,f)$ is* equicontinuous, *i.e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x\,,y) < \delta$ then $d(f^n(x)\,,f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$ (thus, $\{f^n \mid n \in \mathbb{Z}\}$ forms an equicontinuous family of maps $X \to X$).*

(ii) *$(X\,,f)$ is* uniformly almost periodic, *i.e., for any $\varepsilon > 0$ there is a relatively dense set $S \subset \mathbb{Z}$ such that $d(x\,,f^n(x)) < \varepsilon$ for all $n \in S$.*

(iii) *X can be given a group structure which makes it a compact Abelian topological group, and there is an element $x_0 \in X$ such that $\{n\cdot x_0 \mid n \in \mathbb{Z}\}$ is dense in X and $f(x) = x_0 + x$ for all $x \in X$. Thus, $(X\,,f)$ has topological pure point spectrum.* □

[10] Meyer and unique ergodicity imply that the Dirac comb associated to $\underline{\Lambda}$ is a translation bounded measure, that the autocorrelation measure exists and is pure point for every member of $\mathbb{X}(\underline{\Lambda})$. Continuity of the G-map $\beta_{\underline{\Lambda}}$ yields the compactness of $\mathbb{A}(\underline{\Lambda})$ and thus the relative denseness of P'_ε for all $\varepsilon > 0$, see [34, Section 5.2]. Thus, assumption **(As)** in Section 5.3 is satisfied and one can construct a CPS.

We also note that every topological dynamical system (X, f) admits a "largest" equicontinuous factor, say (Y, f'), in the sense that any topological equicontinuous factor of (X, f) is a topological factor of (Y, f'). Consequently, (Y, f') is called the *maximal equicontinuous factor* of (X, f). In fact, the maximal equicontinuous factor of a uniquely ergodic dynamical system (X, f), where all eigenfunctions can be chosen to be continuous, is nothing else but a rotation on a compact Abelian group determined by the pure point part of its spectrum (thus, again an application of the Halmos-von Neumann Theorem), see [298, Section 1.6.4]. On[11] equicontinuity and its connection to so-called distal topological dynamical systems, see [287, Section 4.2], [80, Section 2.7] and [108, Section 3.1].

Definition 7.50. Let $\beta : X \to Y$ be a torus parametrisation. For $y \in Y$, we call the inverse image $\beta^{-1}(\{y\})$ the *fibre* over y. Then, $x \in X$ is called *singular* if the fibre over $\beta(x)$ consists of more than one element. Otherwise, x is called *non-singular*. In this case, $\{x\} = \beta^{-1}(\beta(x))$ is called a *singleton fibre*.

We note the following abstract result about the set of singleton fibres (which should be compared to Remark 5.10) and the following characterisation in the case of (inter) model sets.

Lemma 7.51. [34, Corollary 1] *Let $\beta : X \to Y$ be a torus parametrisation. Assume that the topology on X stems*[12] *from a metric. Then the set of singleton fibres in X is a $\mathfrak{G}_\delta$-set.* □

Lemma 7.52. [34, Prop. 8] and compare [233, Props. 4.3 & 5.5] *Let $(G, H, \tilde{\mathcal{L}})$ be a CPS, and $\underline{\Omega}$ be a family of nonempty, regularly closed compact subsets of H. Let $\underline{\Lambda}$ be a muli-component IMS, i.e., $\Lambda(\operatorname{int}\underline{\Omega}) \subset \underline{\Lambda} \subset \Lambda(\underline{\Omega})$. Assume that there exists a torus parametrisation $\beta'_{\underline{\Lambda}} : \mathbb{X}(\underline{\Lambda}) \to \mathbb{T}(\underline{\Lambda})$ with $\beta'_{\underline{\Lambda}}(\underline{\Lambda}) = (0, 0) + \tilde{\mathcal{L}}$. Then, for any $c \in H$, the following properties are equivalent.*

(i) $\Lambda(-c + \operatorname{int}\underline{\Omega}) = \Lambda(-c + \underline{\Omega})$.

(ii) $\partial(-c + \Omega_i) \cap \mathcal{L}^\star = \varnothing$ *for every* $1 \le i \le n$.

(iii) *The fibre over* $(0, c) + \tilde{\mathcal{L}}$ *is non-singular.*

In this case, $\underline{\Lambda}' = \Lambda(-c + \operatorname{int}\underline{\Omega}) = \Lambda(-c + \underline{\Omega})$ is the fibre over $(0, c) + \tilde{\mathcal{L}}$, a generic multi-component (inter) model set and one has $\mathbb{X}(\underline{\Lambda}') \subset \mathbb{X}(\underline{\Lambda})$. □

Thus, the torus parametrisation tells us if a given IMS is generic. It also tells us if the IMS is regular. In view of Remark 5.10, this statement fully justifies the use of the word "generic" model set.

Lemma 7.53. [34, Theorem 5] and [233, Props. 4.7 & 5.7] *Assume the setting of Lemma 7.52. Then, the boundaries of $\underline{\Omega}$ have Haar measure* 0, *i.e., $\mu_H(\partial\Omega_i) = 0$ for all $1 \le i \le n$, iff the torus parametrisation $\beta'_{\underline{\Lambda}}$ is one-to-one μ_H-a.e.* □

Obviously, the property that the torus parametrisation is one-to-one plays a special role. We also recall that an FLC multi-component Delone set is repetitive iff its associated point set dynamical system is minimal, see Lemma 5.108. Moreover, every generic model set is repetitive, see Lemma 5.118. Thus, if the torus parametrisation is one-to-one a.e., one has "lots of" repetitive model sets (also compare [34, Prop. 2 & Theorem 3]). Therefore, for a given regularly closed set Ω, repetitivity (and thus minimality of the dynamical system) is the "generic case" for inter model sets defined by the window $\Omega + t$ for some translation $t \in H$.

[11] The attentive reader will have noticed that a statement paralleling "equicontinuity" is missing for $(\mathbb{A}(\underline{\Lambda}), G)$.
[12] For $\mathbb{X}(\underline{\Lambda})$ this is the case essentially if G is metrisable, see Definition 5.102.

Proposition 7.54. [34, Theorem 6] and [233, Theorem 4.8] *Let G be a $\sigma LCAG$ and $\underline{\mathit{\Lambda}}$ be a multi-component Meyer set of G such that the canonical*[13] *map $\beta : \mathbb{X}(\underline{\mathit{\Lambda}}) \to \mathbb{A}(\underline{\mathit{\Lambda}})$ is continuous and one-to-one a.e. with respect to the Haar measure on $\mathbb{A}(\underline{\mathit{\Lambda}}) = \beta(\mathbb{X}(\underline{\mathit{\Lambda}}))$. Then, $\mathbb{X}(\underline{\mathit{\Lambda}})$ is uniquely ergodic and $\underline{\mathit{\Lambda}}$ agrees with a regular multi-component model set up to a (multi-component) set of density zero. Furthermore, if $\underline{\mathit{\Lambda}}$ is repetitive, then there is a regular generic multi-component model set $\underline{\mathit{\Lambda}}'$ such that $\mathbb{X}(\underline{\mathit{\Lambda}}) = \mathbb{X}(\underline{\mathit{\Lambda}}')$. In this case, for each $\underline{\mathit{\Lambda}}'' \in \mathbb{X}(\underline{\mathit{\Lambda}})$, there exists $(x\,, -y) \in G \times H$ such that*

$$x + \Lambda(y + \operatorname{int} \underline{\mathit{\Omega}}) \subset \underline{\mathit{\Lambda}}'' \subset x + \Lambda(y + \underline{\mathit{\Omega}}),$$

where $\underline{\mathit{\Omega}}$ is a family of nonempty, regularly closed and compact subset of H whose boundaries are of Haar measure zero. □

The main characterisation of a multi-component model set in terms of dynamical systems is now the following statement (we recall that [34] is concerned with the single-component case).

Proposition 7.55. [34, Theorem 1] *Let G be a $\sigma LCAG$ and $(X\,,G)$ a point set dynamical system on G. Then, for $(X\,,G)$ to be the dynamical system associated to a repetitive regular multi-component model set $\underline{\mathit{\Lambda}}$, i.e., one has $X = \mathbb{X}(\underline{\mathit{\Lambda}})$, it is necessary and sufficient that the following four conditions are satisfied.*

(i) *All elements of X are multi-component Meyer sets (with the same number of nonempty components).*

(ii) *$(X\,,G)$ is strictly ergodic.*

(iii) *$(X\,,G)$ has pure point dynamical spectrum with continuous eigenfunctions.*

(iv) *The eigenfunctions of $(X\,,G)$ separate almost all points of X, i.e., the set $\{\underline{\mathit{\Lambda}}' \in X \mid$ there exists $\underline{\mathit{\Lambda}}'' \neq \underline{\mathit{\Lambda}}'$ with $f(\underline{\mathit{\Lambda}}'') = f(\underline{\mathit{\Lambda}}')$ for all eigenfunctions $f\}$ has measure zero (recall, that by unique ergodicity there is a unique Borel probability measure on X).* □

We note that if we replace the last condition by the statement that the eigenfunctions separate all points of X, then this characterisation is equivalent to the associated torus parametrisation being bijective (and $\underline{\mathit{\Lambda}}$ being Meyer) and equivalent to $\underline{\mathit{\Lambda}}$ being *crystallographic* (or *fully periodic*), *i.e.*, the set of periods $\{t \in G \mid \underline{\mathit{\Lambda}} + t = \underline{\mathit{\Lambda}}\}$ forms a lattice, see [34, Theorem 10].

The next statement can now be interpreted as corollary to the above characterisation (also compare Proposition 7.54).

Lemma 7.56. [233, Theorem 1.1] and [237, Theorem 7] *Let $\underline{\mathit{\Lambda}}$ be a repetitive multi-component Meyer set (with UCF) in G. Then, there is a torus parametrisation $\beta_{\underline{\mathit{\Lambda}}} : \mathbb{X}(\underline{\mathit{\Lambda}}) \to \mathbb{A}(\underline{\mathit{\Lambda}})$ which is one-to-one a.e. with respect to the Haar measure of $\mathbb{A}(\underline{\mathit{\Lambda}})$ iff $\underline{\mathit{\Lambda}}$ (or, equivalently, each element of $\mathbb{X}(\underline{\mathit{\Lambda}})$) is a regular multi-component model set. Moreover, this is also equivalent to conditions* (iii) *&* (iv) *in Proposition* 7.55. □

[13] *I.e.*, β is the map $\beta : \mathcal{D}_m^U \to \hat{\mathcal{D}}_m^U$, $\beta(\underline{\mathit{\Lambda}}) = [\underline{\mathit{\Lambda}}]$ defined on p. 368.

7.4. Symbolic Dynamics

Let $\mathcal{A}$ be an alphabet with $m = \operatorname{card}\mathcal{A}$ letters. The set of all sequences in $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{Z}}$. As in Section 6.1, $\mathcal{A}^{\mathbb{Z}}$ is a compact space and the shift $S((u_n)_{n\in\mathbb{Z}}) = S(\ldots u_{-2}u_{-1}\dot{u}_0u_1u_2\ldots) = \ldots u_{-2}u_{-1}u_0\dot{u}_1u_2\ldots = (u_{n+1})_{n\in\mathbb{Z}}$ a natural $\mathbb{Z}$-action, *i.e.*, $(\mathcal{A}^{\mathbb{Z}}, S)$ is a topological dynamical system. Recall that $\mathcal{A}^{\mathbb{Z}}$ is topologised by a ultrametric and therefore the topology has a base of clopen sets called cylinders, see Remark 6.2.

Definition 7.57. A *subshift* is a closed subset $X \subset \mathcal{A}^{\mathbb{Z}}$ invariant under the shift S and its inverse. We also call $\mathcal{A}^{\mathbb{Z}}$ the *full m-shift*.

Note that for any subshift X there is always a countable set $C \subset \mathcal{A}^{\mathrm{fin}}$ of *forbidden words* $w \in C$ such that no sequence in X contains a forbidden word (*i.e.*, a member of C) and each sequence $\mathcal{A}^{\mathbb{Z}} \setminus X$ contains at least one forbidden word.

Definition 7.58. A subshift X is called a *subshift of finite type* or *SFT* for short if the set of forbidden words C is finite (possibly empty, in which case one has the full shift). A factor of a subshift of finite type is called a *sofic shift*.

We note the following properties:

- A subshift that is topologically conjugate to an SFT is itself an SFT, see [239, Theorem 2.1.10].
- Sofic shifts are subshifts, see [239, Theorem 3.1.4].
- Every SFT is sofic, see [239, Theorem 3.1.5].
- A factor of a sofic shift is sofic, see [239, Corollary 3.2.2].
- A subshift that is topologically conjugate to a sofic shift is sofic, see [239, Corollary 3.2.3].

Suppose we label the edges in a directed (multi-)graph with symbols from an alphabet $\mathcal{A}$, where two or more edges could have the same label. Then, the set of all bi-infinite walks on the graph (coded by the labels) is a sofic shift.

Lemma 7.59. [239, Definition 3.1.3 & Theorem 3.2.1] and [80, Prop. 3.7.1] *Let $G(V, \vec{E})$ be a directed multigraph (where V denotes the finite vertex set and $\vec{E}$ the set of directed edges). Assign to each edge $e \in \vec{E}$ a label from a finite alphabet $\mathcal{A}$. Then, the set of labels of all bi-infinite (directed) walks on $G(V, \vec{E})$ is a sofic shift. Conversely, every sofic shift can be obtained in such a way from a directed multigraph $G(V, \vec{E})$. In this case, the graph $G(V, \vec{E})$ is called the* presentation *of the sofic shift.* □

Remark 7.60. We have an obvious connection between a sofic shift and an IFS $\boldsymbol{\Theta}$ (respectively an MFS), see Remark 4.104: If we label the edges of the graph $G(\boldsymbol{\Theta})$ by the maps $f \in \boldsymbol{\Theta}$, then the net measures[14] in Lemma 4.116 are defined *via* (the functions Φ^γ and Ψ^γ on) the cylinder sets of the cylinder sets on the corresponding sofic shift. This should only serve as a first indication of the connection; a more thorough analysis will follow in Section 7.5.2.

[14] One may therefore regard the net measures as a generalisation of the so-called *Parry measure*. See [109, Definition 17.15], [281, Section 3.4], [244, Section IV.6] and [384, p. 194] on the Parry measure.

Symbolic dynamical systems (*i.e.*, shift spaces) are used to study more complex discrete dynamical system: Let $(X\,,f)$ be a be a discrete topological dynamical system and consider a partition $X_1 \cup \ldots \cup X_m = X$ of X. For each $x \in X$, let $\psi_i(x)$ be the index of the element of the partition $\{X_1, \ldots, X_m\}$ containing $f^i(x)$. The sequence $(\psi_i(x))_{i\in\mathbb{Z}}$ is called the *itinerary* of x. This defines a map $\psi : X \to \{1, \ldots, m\}^{\mathbb{Z}}$ which satisfies $\psi \circ f = S \circ \psi$, wherefore ψ is a factor map (provided $\psi(X)$ is actually a shift space!). Now, the question is if for a given topological dynamical system such a factor is a sofic shift or even an SFT. We first given some further examples of dynamical systems.

Definition 7.61. Let $\boldsymbol{M}$ be an $n \times n$-matrix with integer coefficients such that $|\det \boldsymbol{M}| = 1$. Then $\boldsymbol{M}$ preserves the Haar measure and also the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$; furthermore, it induces a group automorphism (again called $\boldsymbol{M}$) on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. If $\boldsymbol{M}$ has no eigenvalues of unit modulus, then $\boldsymbol{M} : \mathbb{T}^n \to \mathbb{T}^n$ (respectively $\boldsymbol{M} : \mathbb{R}^n \to \mathbb{R}^n$) has expanding and contracting subspaces (explicitly given by the eigenspaces of the eigenvalues of modulus greater and less than one) of complementary dimensions and is called a *hyperbolic toral automorphism*. Moreover, the expanding as well as the contracting subspace of a hyperbolic toral automorphism is dense in $\mathbb{T}^n$ (see [80, Theorem 5.10.3]). We note that for a hyperbolic toral automorphism, the (discrete) measure theoretic dynamical system $(\mathbb{T}^n\,, \mathfrak{B}\,, \mu_{\mathbb{T}^n}\,, \boldsymbol{M})$ is ergodic and even strongly mixing (see [109, Prop. 24.1], [161, pp. 53–55] and [287, Section 2.5]) and that the periodic points – which are exactly the points with rational coordinates – are dense (see [109, Prop. 24.7]).

A hyperbolic toral automorphism on $\mathbb{T}^n$ is a special case of an *Anosov diffeomorphism* (see [109, p. 239] and [108, Beispiel 67]): In short, a diffeomorphism $\Phi : M \to M$, where M is a compact connected differentiable manifold, is said to be an *Anosov diffeomorphism* if there exists a splitting of the tangent bundle $\mathrm{T}\,M$ into a continuous sum of subbundles $\mathrm{T}M = E^+ \oplus E^-$ such that the corresponding tangent map preserves the splitting and Φ contracts along the direction of E^+ and expands along the direction of E^- (one also says that the manifold M is *hyperbolic*), see [109, Definition 23.9], [80, Sections 5.2 & 5.10], [244, Section III.2] and [108, Definition 51] for further details. We also remark that Anosov diffeomorphism are a special case of so-called *axiom A diffeomorphisms*, also called *Smale diffeomorphisms* (here, the denseness of the periodic points enters in the definition), see [78, Chapter 3], [109, Definition 23.11], [80, Section 5.11], [343, Definition 8.9][15] and [108, Definition 51] for exact definitions.

Our considerations in Remarks 4.108 & 4.109 (recall Chapter 3b) together with the considerations in Section 6.5 show that one can extend the definition of a hyperbolic toral automorphism.

Continuation of Definition 7.61. Let $\boldsymbol{M}$ be a nonsingular $n \times n$-matrix with rational coefficients. Then, $\boldsymbol{M}$ is a principle lattice transformation on the n-dimensional adele $\mathbb{A}^n_{\mathbb{Q}}$, see Definition 3b.16. We assume that the characteristic polynomial of $\boldsymbol{M}$ is irreducible over $\mathbb{Q}$, wherefore the principle lattice transformation $\boldsymbol{M}$ on $\mathbb{A}^n_{\mathbb{Q}}$ can be identified with the multiplication by λ on $\mathbb{A}_{\mathbb{Q}(\lambda)}$, where λ is a root of the characteristic polynomial, see Remark 4.109. Consequently, there is a splitting of $\mathbb{A}_{\mathbb{Q}(\lambda)}$ in its expanding, contracting and indifferent part,

$$\mathbb{A}_{\mathbb{Q}(\lambda)} = \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\mathrm{exp}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\mathrm{con}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\mathrm{ind}}$$

[15] For the curious reader: there are also "axiom B diffeomorphisms", see [343, Exercise 9.3].

with

$$\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{exp}} = \prod_{\substack{\nu \in J \\ |\lambda|_\nu > 1}} \hat{\mathbb{Q}}_\nu, \qquad \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{con}} = \prod_{\substack{\nu \in J \\ |\lambda|_\nu < 1}} \hat{\mathbb{Q}}_\nu, \quad \text{and} \quad \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{ind}} = \prod_{\substack{\nu \in J \\ |\lambda|_\nu = 1}}{}' \hat{\mathbb{Q}}_\nu$$

where J denotes the set of places. Denote the restriction of $\boldsymbol{M}$ respectively of the multiplication by λ to $\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{exp}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{con}}$ by T. Let L be a lattice in $\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{exp}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{con}}$ on which T acts[16] as group automorphism. Then, T also induces a group automorphism on the "torus"

$$\left(\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{exp}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{con}}\right) / L,$$

and consequently call T a *hyperbolic toral automorphism*.

Remark 7.62. It would be interesting if one derive statements similar to the ones known for hyperbolic toral automorphisms on $\mathbb{T}^n$ also for this generalised hyperbolic toral automorphisms, *e.g.*, is the topological entropy again given by $\sum_{\nu \in J, |\lambda|_\nu > 1} \log(|\lambda|_\nu)$? Compare [109, Theorem 24.5] and [287, Section 6.1 A].

Obviously, we have defined "hyperbolic toral automorphism" in such a way that the substitution matrix of a Pisot substitution respectively the multiplication by the corresponding PV-number λ induces such hyperbolic toral automorphism on $(\mathbb{R} \times H_{\text{ext}}) / \tilde{\mathcal{L}}_{\text{ext}}$.

As indicated above, we may now use shift space to study the dynamical system given by a hyperbolic toral automorphism. This leads to so-called "Markov partitions", for which we give two definitions: an explicit one (here, restricted to a hyperbolic toral automorphism) according to [108, Definition 60] (also compare [78, Section 3C], [109, Definition 25.7], [80, Section 5.12], [244, p. 182], [1, Appendix C, Section IIb] and [343, Definition 10.15]) and a weak one as in [239, Definition 6.5.6] (also see [298, Definition 7.1.11]).

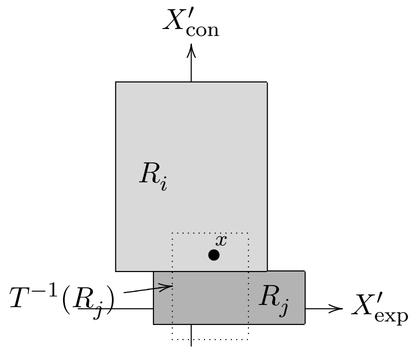
Definition 7.63. Let (X, T) be a hyperbolic toral automorphism, where the "torus" X is given by $\left(X'_{\text{exp}} \times X'_{\text{con}}\right) / L$ as in Definition 7.61 (thus, the action of T can also be interpreted as $T : X'_{\text{exp}} \times X'_{\text{con}} \to X'_{\text{exp}} \times X'_{\text{con}}$, $x \mapsto Tx \bmod L$). We denote the canonical projections to the expansive (respectively contracting) subspace X'_{exp} (respectively X'_{con}) by π_1 (respectively π_2). A subset $R \subset X$ is called a *rectangle* if $(\pi_1(x), \pi_2(y)) \in R$ whenever $x, y \in R$, and if $R \subset \operatorname{cl} \operatorname{int} R$. A partition of X by a finite number of rectangles $\{R_1, \dots R_N\}$, wherefore one has $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ if $i \neq j$ and $\bigcup_i R_i = X$, is called a *Markov partition* if, for all $1 \leq i, j \leq N$ and all $x \in \operatorname{int} R_i \cap T^{-1}(\operatorname{int} R_j)$, the conditions

$$\begin{aligned} &\{(\pi_1(y), \pi_2(Tx)) \mid y \in R_j\} \subset T(\{(\pi_1(y), \pi_2(x)) \mid y \in R_i\}) \quad \text{and} \\ &\qquad T(\{(\pi_1(x), \pi_2(z)) \mid z \in R_i\}) \subset \{(\pi_1(Tx), \pi_2(z)) \mid z \in R_j\} \end{aligned} \tag{7.3}$$

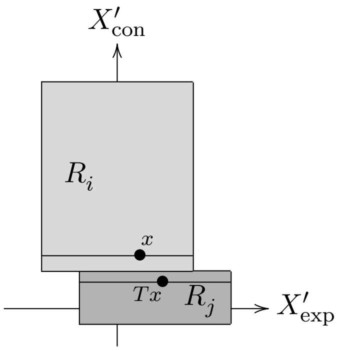
hold.

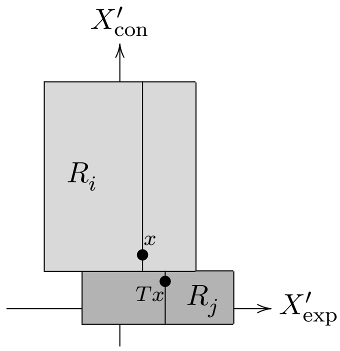
[16] As our considerations in Section 6.5 show, such a lattice exists if there are no infinite places in $\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{ind}}$: Given the CPS $\left(\left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{exp}} \oplus \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{con}}, \left(\mathbb{A}_{\mathbb{Q}(\lambda)}\right)_{\text{ind}}, \mathbb{Q}\right)$, any model set to a window $\prod'_{\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}, \|\lambda\|_{\mathfrak{p}} = 1} \mathfrak{p}^{n_{\mathfrak{p}}}$, where all be finitely many $n_{\mathfrak{p}} \in \mathbb{Z}$ are zero, is such a lattice.

Regarding the two conditions, suppose we have the following situation (we will see in a moment that this situation is actually not possible):



Then, the first condition (respectively, the second condition) states that the image under T of the line through x covers (respectively, is contained in) the line through Tx in the image on the left (respectively, right):





In plain language, the conditions state: If $T(R_i)$ (respectively, $T^{-1}(R_i)$) intersects R_j, then it has to stretch all the way through R_j along the direction of X'_{exp} (respectively, of X'_{con}).

We now state a weaker form for the definition of a Markov partition.

Definition 7.64. Let $(X\,, T)$ be an invertible dynamical system. Let $\{P_1, \dots, P_N\}$ be a finite partition of X (where $P_i \subset \operatorname{cl}\operatorname{int} P_i$), and define for each sequence $u = \dots u_{-1}\dot{u}_0 u_1 \dots \in \{1, \dots, N\}^{\mathbb{Z}}$ the map $D : \{1, \dots, N\}^{\mathbb{Z}} \to X$,

$$D(u) = \bigcap_{n=0}^{\infty} \operatorname{cl}\left(\bigcap_{k=-n}^{n} T^{-k}(\operatorname{int} P_{u_k})\right).$$

Then, $Y = \{u \in \{1, \dots, N\}^{\mathbb{Z}} \mid D(u) \neq \varnothing\}$ is a subshift. We say that the partition gives a *symbolic representation* of $(X\,, T)$ if $\operatorname{card} D(u) = 1$ for all $u \in Y$. In this case, the points in Y correspond to points in X. We call $\{P_1, \dots, P_N\}$ a *Markov partition* for $(X\,, T)$ if it gives a symbolic representation such that Y is a SFT.

We note that the two conditions of Equation 7.3 secure that $D(u)$ consists of exactly one point if it is nonempty. Moreover, in the case of a symbolic representation, the definitions are set up so that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ X & \xrightarrow{T} & X \end{array}$$

In fact D is even continuous and onto, and therefore a factor map, see [239, Prop. 6.5.8] and [109, Remark after Theorem 25.8].

Remark 7.65. Markov partitions – or more generally, a symbolic representation where, say, the subshift Y is sofic – allow a combinatorial understanding of a dynamical system. The existence of Markov partitions for hyperbolic toral automorphisms, respectively more generally for Anosov (and axiom A) diffeomorphisms is proven in [77, 349, 350], also compare [108, Satz 80], [343, Theorem 10.28] and [244, Theorem 9.5] for details. Unfortunately, this existence proof is not constructive. For a linear automorphisms on $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, a Markov partition is constructed in [2], but there were only few explicit higher dimensional examples, *e.g.*, see [54], especially since the boundary of the tiles in for Markov partitions on $\mathbb{T}^3$ is known to have non-integral Hausdorff dimension by [79] (also see [92] for higher dimensions). Using (unimodular) β-substitutions (see Section 6c.4), however, B. Praggastis (see [294, 295]) constructed tiles which yield to Markov partitions also in higher dimensional spaces. Consequently, this indicates that and how Markov partitions for Pisot substitutions might be obtained, also see [208, 228], [345, Chapitre 5] and our considerations in Section 7.5.2. Also see [298, Section 7.1.2] for a historical overview and further references.

7.5. Applications to Pisot Substitutions

Many statements of the previous sections can be directly applied to Pisot substitutions, and are therefore already stated in Theorem 6.116. Consequently, we only concentrate on two topics here: the eigenvalues associated to the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), \mathbb{R})$ and the torus parametrisation for this dynamical system.

7.5.1. Eigenvalues

Let $\underline{\Lambda}$ the point set in $\mathbb{R}$ derived from the representation with natural intervals of a fixed point u of a Pisot substitution σ. Then, $\underline{\Lambda}$ is a an aperiodic repetitive substitution multi-component Delone set that satisfies **(PLT)** and where Δ' is a Meyer set. Thus, Proposition 7.43 reads for a Pisot substitution.

Proposition 7.66. *Let σ be a Pisot substitution with PV-number λ and* $\operatorname{card}\mathcal{A} \geq 2$. *Let $\underline{\Lambda}$ the Delone set derived from the representation with natural intervals of an fixed point u of σ. Then, $\alpha \in \mathbb{R}$ is an eigenvalue of the point set dynamical system*[17] $(\mathbb{X}(\underline{\Lambda}), \mathbb{R})$ *iff* $\lim_{m\to\infty}\langle \alpha , \lambda^m x\rangle = 1$ *for all $x \in \Delta' = \Delta$.* □

Since $\langle \alpha , \lambda^m x\rangle = \exp(2\pi i\, \lambda^m \alpha\, x)$, we are thus interested in characterising the set

$$\mathscr{X}_\lambda = \{z \in \mathbb{R} \mid \lim_{m\to\infty} \lambda^m z \equiv 0 \bmod 1\}.$$

[17] The point set dynamical system is strictly ergodic, and every measurable eigenfunctions of the corresponding measure theoretic dynamical system coincides with a continuous function $\mu_{\mathbb{R}}$-a.e. In particular, every eigenvalue is simple.

In fact, here PV-numbers arise naturally, since one has the following statement.

Lemma 7.67. [290, 378], [321, Theorem B], [90, Theorem VIII.I] and [68, Theorems 5.4.1 & 5.6.2] *An algebraic number $\lambda > 1$ is a PV-number iff there exists a nonzero real x such that* $\lim_{m\to\infty} \lambda^m x \equiv 0 \bmod 1$. *Moreover, an algebraic number $\lambda > 1$ is a PV-number iff there exists a nonzero real x such that $\lim_{m\to\infty} \lambda^m x$ has finitely many limit points modulo* 1. □

We have included the second characterisation in view of the statement for the cobound in Lemma 6c.11. We now characterise the set $\mathscr{X}_\lambda$, also compare [90, Theorem VIII.I].

Proposition 7.68. [216, Theorem 2] *Suppose λ is a PV-number. Let $p(x) = \mathrm{Irr}(\lambda, \mathbb{Q}, x) \in \mathbb{Z}[x]$ and let $p'(x)$ be the formal derivative of $p(x)$. The codifferent of $\mathbb{Z}[\lambda]$ is $\mathbb{Z}[\lambda]^\wedge = \frac{1}{p'(x)}\,\mathbb{Z}[\lambda]$ (see Lemma* 3.51*). Then, $x \in \mathscr{X}_\lambda$ iff $\lambda^m x \in \mathbb{Z}[\lambda]^\wedge$ for some $m \geq 0$, i.e.,*

$$\mathscr{X}_\lambda = \bigcup_{k\geq 0} \frac{1}{\lambda^k}\,\mathbb{Z}[\lambda]^\wedge = \mathbb{Z}\left[\frac{1}{\lambda}\right] \cdot \mathbb{Z}[\lambda]^\wedge. \qquad \square$$

We observe that $\mathbb{Z}[\lambda] \subset \frac{1}{\lambda}\mathbb{Z}[\lambda]$, wherefore one also has $\mathbb{Z}[\lambda]^\wedge \subset \frac{1}{\lambda}\mathbb{Z}[\lambda]^\wedge$. We also note that $\mathscr{X}_\lambda \subset \mathbb{Q}(\lambda)$, and for every $x \in \mathbb{Q}(\lambda)$ one has (we denote the Galois automorphisms in $\mathbb{Q}(\lambda)$ – as usual and with the usual ordering – by σ_i)

$$T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^m x) = \lambda^m x + \left(\sum_{i=2}^{r} \lambda_i^m\, \sigma_i(x)\right) + \left(\sum_{i=r+1}^{r+s} \left(\lambda_i^m\, \sigma_i(x) + \overline{\lambda}_i^m\, \overline{\sigma_i(x)}\right)\right),$$

wherefore $\lambda^m x$ converges[18] geometrically to $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^m x) \in \mathbb{Q}$ (since one has $|\lambda_i| < 1$ for $2 \leq i \leq r+s$). This might indicate that there is a connection to the statements of Lemma 5.129(i) & (ii).

We now identify the set $\bigcup_{k\geq 0} \mathcal{L}^\wedge/\lambda^k$ as group of eigenvalues for a Pisot substitution, also compare [50, Theorem 9.3] and [43, Theorem 5.1].

Proposition 7.69. *Let σ be a Pisot substitution with PV-number λ and $\operatorname{card}\mathcal{A} \geq 2$. Let $\underline{\Lambda}$ the Delone set derived from the representation with natural intervals of an fixed point u of σ. Then, the group of eigenvalues of the point set dynamical system $(\mathbb{X}(\underline{\Lambda}), \mathbb{R})$ is given by $\bigcup_{k\geq 0} \mathcal{L}^\wedge/\lambda^k$.*

Proof. We first show that every $\alpha \in \bigcup_{k\geq 0} \mathcal{L}^\wedge/\lambda^k = \mathbb{Z}\left[\frac{1}{\lambda}\right] \cdot \mathcal{L}^\wedge$ is an eigenvalue. To this end, we observe that one derives $\mathcal{L}\,\mathbb{Z}[\lambda] = \mathcal{L}$ since $\lambda\,\mathcal{L} \subset \mathcal{L}$, wherefore, for $y \in \mathcal{L}^\wedge$, one has

$$\mathbb{Z} \supset T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\,(y\,\mathcal{L}) = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\,(y\,(\mathcal{L}\,\mathbb{Z}[\lambda])) = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}\,((y\,\mathcal{L})\,\mathbb{Z}[\lambda])\,.$$

By definition, we have $\mathcal{L}^\wedge\,\mathcal{L} \subset \mathbb{Z}[\lambda]^\wedge$ (compare with Lemma 3.51) and consequently

$$\mathbb{Z}\left[\frac{1}{\lambda}\right]\,\mathcal{L}^\wedge\,\mathcal{L} \subset \mathbb{Z}\left[\frac{1}{\lambda}\right]\,\mathbb{Z}[\lambda]^\wedge = \mathscr{X}_\lambda.$$

Observing that $\mathcal{L} = \langle \varDelta \rangle_{\mathbb{Z}} = \langle \varDelta' \rangle_{\mathbb{Z}}$ establishes the claim.

[18] In fact, a real number greater than zero is a PV-number iff there is a nonzero real x such that the fractional parts of the numbers in the set $\{\lambda^m x\}$ converge to zero with $o(m^{-1/2})$, see [68, Theorems 5.4.2 – 5.4.4]. In particular, $\{\lambda^m x\}$ is not uniformly distributed modulo 1.

For the other direction, we use the notation of Proposition 7.66 and observe that from $x \in \varDelta \subset \mathbb{Q}(\lambda)$ and $\mathscr{X}_\lambda \subset \mathbb{Q}(\lambda)$ we also have $\alpha \in \mathbb{Q}(\lambda)$ for every eigenvalue α. By the explicit form of the set $\mathscr{X}_\lambda$, we know that, for every eigenvalue α and every $x \in \varDelta \subset \mathcal{L}$, there is a $k \geq 0$ (depending on x and α) such that $\lambda^k \alpha x \in \mathbb{Z}[\lambda]^\wedge$. Thus,

$$\mathbb{Z} \supset T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^k \alpha x \mathbb{Z}[\lambda]) \subset T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^k \alpha \varDelta \mathbb{Z}[\lambda]) \subset T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^k \alpha \mathcal{L} \mathbb{Z}[\lambda]) = T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^k \alpha \mathcal{L}).$$

By the $\mathbb{Q}$-linearity of the trace (compare Lemma 3.39), it follows that, if $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(S) \subset \mathbb{Z}$ for some set S, then one also has $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\langle S \rangle_{\mathbb{Z}}) \subset \mathbb{Z}$. Since $\mathcal{L}$ is generated by n generators (belonging to $\varDelta$), there is a number $k' \geq 0$ such that $T_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\lambda^{k'} \alpha \mathcal{L}) \subset \mathbb{Z}$, and by the definition of the codifferent one has $\lambda^{k'} \alpha \in \mathcal{L}^\wedge$ and therefore $\alpha \in \mathcal{L}^\wedge / \lambda^{k'}$. This proves the proposition. □

Note that this is exactly the group of eigenvalues one expects from Folklore Theorem 5a.10 and Proposition 6.58 *via* Corollary 7.30. This identifies the CPS in Equation (6.7) on p. 240 as *the* CPS for a Pisot substitution.

Remark 7.70. This also indicates how model sets (respectively a CPS) naturally arise for pure point dynamical systems: Given a pure point dynamical system $(\mathbb{X}(\underline{\varLambda}), G)$ with group of eigenvalues $\mathscr{P} \subset G^*$, one looks for a (dual) internal space H^* such that the diagonal embedding of $\mathscr{P}$ into $G^* \times H^*$ is a lattice (of course, one has to ask if or under which conditions this is possible). One then has to identify the usual CPS (for $\underline{\varLambda}$) with the (Pontryagin) dual CPS of $(G^*, H^*, \tilde{\mathscr{P}})$. Obviously, regular model sets can be thought of as a special case of the Halmos-von Neumann Theorem.

Moreover, this point of view might be useful for a deformed model sets $\varLambda_\vartheta$, since – for a "generic" deformation ϑ – the group $\mathscr{P}$ generated by the set of Bragg peaks $\mathscr{P}'(\varLambda_\vartheta)$ of $\varLambda_\vartheta$ equals the group of eigenvalues for the corresponding (undeformed) regular model set $\varLambda$, compare Remark 7.38. Thus, one might derive the CPS for $\varLambda$ from the diffraction spectrum of $\varLambda_\vartheta$.

7.5.2. Torus Parametrisation

We now explicitly construct the torus parametrisation for Pisot substitutions (provided it is a model set). As usual, we denote by λ the PV-number and by $\boldsymbol{\Theta}$ the EMFS associated with a given Pisot substitution. Then we note that we have the following Markov partition-like property (see Definitions 7.63 & 7.64) for the partition $\{[-\ell_1, 0] \times \varOmega_1, \dots, [-\ell_n, 0] \times \varOmega_n\}$ of the fundamental domain of $\mathcal{L}_{\text{ext}}$:

- Let $f = t_{(a)} \circ f_0 \in \varTheta_{kj}$ and denote the corresponding map in $\varTheta^\star_{kj}$ by $\hat{f}$. Then, $f([-\ell_j, 0]) = [-\lambda \ell_j, 0] + a$ and $\hat{f}(\varOmega_j) = \lambda^\star \varOmega_j + a^\star$. We recall Equations (6.1) & (6.2) on p. 215, wherefore $[0, \lambda \ell_j] = \bigcup_{\{w | wi \triangleleft \sigma(j)\}} [0, \ell_i] + l'(w)$ and thus

$$[-\lambda, \ell_j, 0] = \bigcup_{\{w | wi \triangleleft \sigma(j)\}} [-\ell_i, 0] - l'(w).$$

Observing that $\widetilde{l'(w)} \in \tilde{\mathcal{L}}$ (and therefore also $(a, a^\star) \in \tilde{\mathcal{L}}$) for all words w, we can make the following calculation:

$$\begin{aligned} f([-\ell_j\,,0]) \times \hat{f}(\Omega_j) &= \big([-\lambda\,\ell_j\,,0] \times \lambda^\star\,\Omega_j\big) + (a, a^\star) \;\equiv\; [-\lambda\,\ell_j\,,0] \times \lambda^\star\,\Omega_j \\ &= \bigcup_{\{w \mid wi \triangleleft \sigma(j)\}} \big([-\ell_i\,,0] - l'(w)\big) \times \lambda^\star\,\Omega_j \\ &\equiv \bigcup_{\{w \mid wi \triangleleft \sigma(j)\}} [-\ell_i\,,0] \times \big(\lambda^\star\,\Omega_j + \big(l'(w)\big)^\star\big) \qquad \bmod \tilde{\mathcal{L}}. \end{aligned}$$

By the definition of the IFS $\boldsymbol{\Theta}^\star$, one has (using sloppy notation)

$$\Omega_i = \bigcup_{\{w \mid wi \triangleleft \sigma(j)\}} \lambda^\star\,\Omega_j + \big(l'(w)\big)^\star\,,$$

wherefore $f([-\ell_j\,,0]) \times \hat{f}(\Omega_j)$ stretches all the way, in the direction of the direct space $\mathbb{R}$, through every tile $[-\ell_m\,,0] \times \Omega_m + \tilde{u}$ (with $\tilde{u} \in \tilde{L}$ and therefore also $\tilde{u} \in \tilde{L}_{\text{ext}}$) it intersects. We note that, by calculating modulo $\tilde{\mathcal{L}}_{\text{ext}}$, the result is independent of the translational part $t_{(a)}$ of the map f, *i.e.*, one gets the same result if one simply uses the map f_0 (respectively, any map $t_{(u)} \circ f_0$ with a $u \in \mathcal{L}_{\text{ext}}$).

- One may make a similar calculation for the maps in $\boldsymbol{\Theta}^{\#}$ respectively $\boldsymbol{\Theta}^{\#\star}$ (respectively, simply for g_0 and $\hat{g}_0$). Then, for $g \in \boldsymbol{\Theta}^{\#}$, the "deformed tile" $g([-\ell_j\,,0]) \times \hat{g}(\Omega_j)$ stretches all the way, in the direction of the internal space H_{ext}, through every tile $[-\ell_k\,,0] \times \Omega_k + \tilde{t}$ it intersects. Here, one has to note that one has for a translational part b of a map $g = t_{(b)} \circ f_0^{-1} \in \boldsymbol{\Theta}^{\#}$ always $b \in \mathcal{L}_{\text{ext}}$ (although it might happen, in the non-unimodular case, that $b \notin \mathcal{L}$).

- Moreover, observing that if $f = t_{(a)} \circ f_0 \in \boldsymbol{\Theta}$ then $f^{-1} = f_0^{-1} \circ t_{(-a)}$ (and similar for $g \in \boldsymbol{\Theta}^{\#}$), a similar calculation – and therefore analogous findings – also applies to $f^{-1}([-\ell_i\,,0]) \times \hat{f}^{-1}(\Omega_i)$ (respectively $g^{-1}([-\ell_i\,,0]) \times \hat{g}^{-1}(\Omega_i)$). In fact, to every map $f = t_{(a)} \circ f_0 \in \Theta_{ij}$, there exists a $g = t_{(a/\lambda)} \circ f_0^{-1} \in \Theta^{\#}_{ji}$ and *vice versa*, and one observes that $f^{-1} = t_{(-a/\lambda)} \circ f_0^{-1}$ (respectively $g^{-1} = t_{(-a)} \circ f_0$) in this case.

For an explicit calculation, we look at the following example.

Example 7.71. We look at the Fibonacci substitution σ_{Fib} and assume the notations used in Section 6.10.1: The (partition) of the fundamental domain of the lattice

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\text{ext}} = \left\{ k \cdot \begin{pmatrix} \tau \\ -\frac{1}{\tau} \end{pmatrix} + m \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid k, m \in \mathbb{Z} \right\}$$

is given by

$$[-\ell_a\,,0] \times \Omega_a \;\cup\; [-\ell_b\,,0] \times \Omega_b \;=\; [-\tau\,,0] \times \left[-\frac{1}{\tau^2}\,,\frac{1}{\tau}\right] \;\cup\; [-1\,,0] \times \left[-1\,,-\frac{1}{\tau^2}\right].$$

Modulo lattice vectors, one calculates

$$
\begin{aligned}
f_0([-\ell_a\,,0])\times f_0^\star(\varOmega_a) &= [-\tau^2\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\\
&\equiv [-\tau\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right] \quad \bmod \tilde{\mathcal{L}}\\
f_\tau([-\ell_a\,,0])\times f_\tau^\star(\varOmega_a) &= \left([-\tau^2\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\right)+\left(\tau,-\frac{1}{\tau}\right)\\
&\equiv [-\tau\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right] \quad \bmod \tilde{\mathcal{L}}\\
f_0([-\ell_b\,,0])\times f_0^\star(\varOmega_b) &= [-\tau\,,0]\times\left[\frac{1}{\tau^3}\,,\frac{1}{\tau}\right]\\
g_0([-\ell_a\,,0])\times g_0^\star(\varOmega_a) &= [-1\,,0]\times\left[-1\,,\frac{1}{\tau}\right]\\
&= [-1\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right]\\
g_1([-\ell_b\,,0])\times g_1^\star(\varOmega_b) &= \left(\left[-\frac{1}{\tau}\,,0\right]\times\left[\frac{1}{\tau}\,,\tau\right]\right)+(1,1)\\
&\equiv [-\tau\,,-1]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau}\right] \quad \bmod \tilde{\mathcal{L}}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
f_0^{-1}([-\ell_a\,,0])\times\left(f_0^{-1}\right)^\star(\varOmega_a) &= [-1\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right]\\
f_\tau^{-1}([-\ell_b\,,0])\times\left(f_\tau^{-1}\right)^\star(\varOmega_b) &= [-\tau\,,-1]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau}\right]\\
g_0^{-1}([-\ell_a\,,0])\times\left(g_0^{-1}\right)^\star(\varOmega_a) &\equiv [-\tau\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right] \quad \bmod \tilde{\mathcal{L}}\\
g_0^{-1}([-\ell_b\,,0])\times\left(g_0^{-1}\right)^\star(\varOmega_b) &= [-\tau\,,0]\times\left[\frac{1}{\tau^3}\,,\frac{1}{\tau}\right]\\
g_1^{-1}(([-\ell_a\,,0])\times\left(g_1^{-1}\right)^\star(\varOmega_a) &\equiv [-\tau\,,0]\times\left[-\frac{1}{\tau^2}\,,\frac{1}{\tau^3}\right]\,\cup\,[-1\,,0]\times\left[-1\,,-\frac{1}{\tau^2}\right] \quad \bmod \tilde{\mathcal{L}}.
\end{aligned}
$$

We observe that the result is independent of the translational part, and in fact the first set of equalities is depicted in Figure 7.2 (the second set may easily derived from them) and confirms our previous findings.

We now recall Remark 6.95 (also see Remark 4.104). Obviously, the graph $G(\boldsymbol{\varTheta}^\star)$ yields a sofic shift[19] on the alphabet given by the edge labels, see Lemma 7.59. Consequently, the set of all two-sided infinite walks indexed by the edge labels yields a (sofic) topological dynamical system, which we denote by $(X_{G(\boldsymbol{\varTheta}^\star)}\,,S)$. We denote the alphabet by $\mathcal{A}'$, and recall

[19] It is even *left resolving* (see [239, p. 76]), *i.e.*, all incoming edges to a vertex carry different labels.

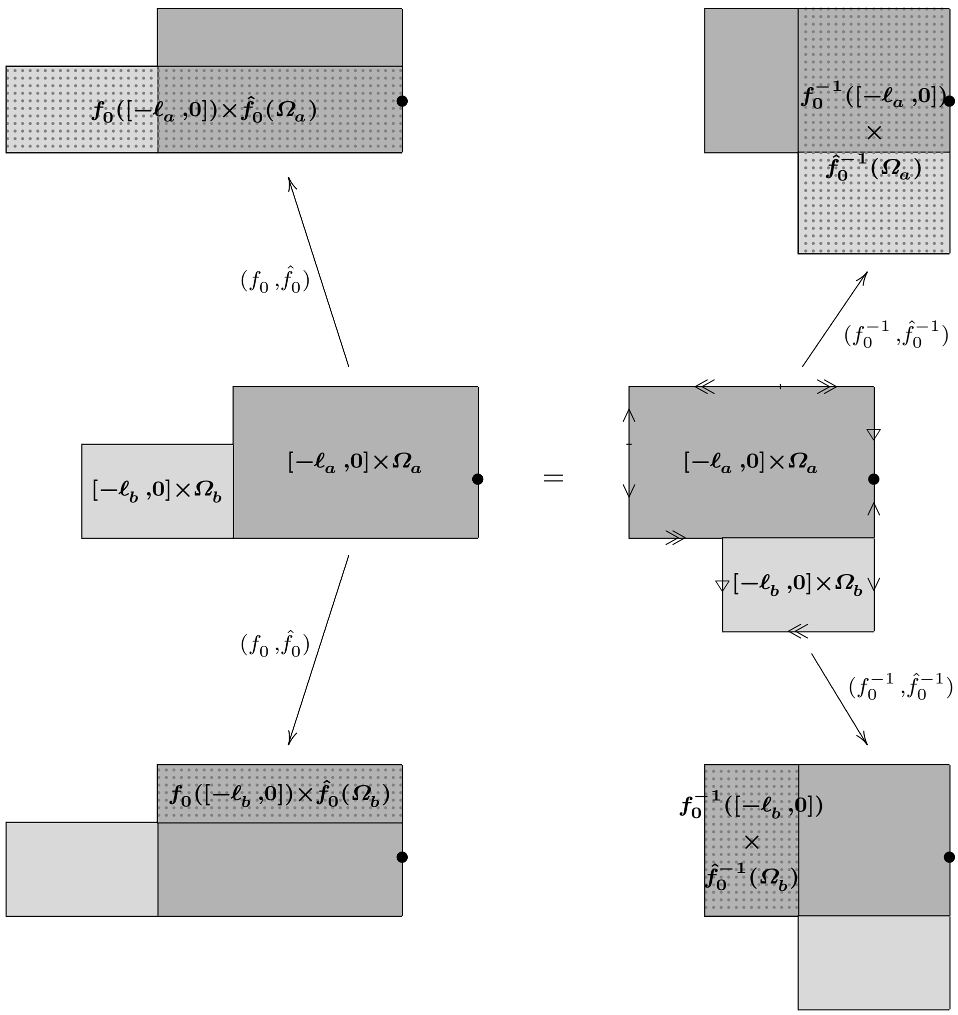


Figure 7.2.: The action of the maps in $\boldsymbol{\Theta}$ respectively in $\boldsymbol{\Theta}^{\#}$ on the tiles $[-\ell_i, 0] \times \Omega_i$ that partition the fundamental domain of $\tilde{\mathcal{L}}_{\mathrm{ext}}$ is shown for the Fibonacci substitution, see Example 7.71. Since the result is independent of the translational part, we denote the maps simply by f_0 respectively f_0^{-1}. The location of the lattice point is given by "•", and we show two possible representations of the fundamental domain (the tiles beside respectively above each other), the identifications of the (parts of the) edges on the torus $\mathbb{R}^2/\tilde{\mathcal{L}}$ are indicated by "≪", "≫", "∧", "∨" and "▽" in the middle right figure.

the definition of the map F in Remark 6.95 respectively the following star version of F for one-sided infinite walks in $G(\boldsymbol{\Theta}^\star)$:

$$F^\star : \mathcal{A}^{\mathbb{N}} \to H_{\mathrm{ext}},\ w = e_0e_1e_2\ldots \mapsto \lim_{N\to\infty} (F(e_0\ldots e_N))^\star = \lim_{N\to\infty} \left(\sum_{m=0}^{N} e_m\,\lambda^m\right)^\star .$$

We recall that, if a one-sided infinite walk w starts at vertex i in the graph, then one has $F^\star(w) \in \Omega_i$. A similar consideration applies to the IFS $\boldsymbol{\Theta}^{\#}$, where we observe that the graph $G(\boldsymbol{\Theta}^{\#})$ is obtained from $G(\boldsymbol{\Theta}^{\star})$ by reversing all arrows and dividing all edge labels by λ. Consequently, we define the following map J on the "negative" part of the sequences in $X_{G(\boldsymbol{\Theta}^{\star})}$:

$$J : \mathcal{A}^{\mathbb{N}} \to \mathbb{R},\ w = \ldots e_{-3}e_{-2}e_{-1} \mapsto \sum_{m=1}^{\infty} e_{-m}\, \lambda^{-m}.$$

Now, if a walk w ends in the vertex i, then $J(w) \in [0\,, \ell_i] = A_i$. Consequently, we have the following "torus parametrisation" β'' for $X_{G(\boldsymbol{\Theta}^{\star})}$:

$$\beta'' : X_{G(\boldsymbol{\Theta}^{\star})} \to \mathbb{R} \times H_{\text{ext}},\ w = \ldots e_{-2}e_{-1}\dot{e}_0 e_1 \ldots \mapsto \bigl(-J(\ldots e_{-2}e_{-1}), F^{\star}(e_0 e_1 \ldots)\bigr)\,, \tag{7.4}$$

where $\beta''(w) \in [-\ell_i\,, 0] \times \Omega_i$ if w passes through i between e_{-1} and e_0 (in fact, if W_i denotes the subset of walks in $X_{G(\boldsymbol{\Theta}^{\star})}$ that pass through he vertex i between e_{-1} and e_0, then one has $\text{cl}_{G\times H_{\text{ext}}}\, \beta''(W_i) = [-\ell_i\,, 0] \times \Omega_i$).

Since both, J and $F^{\star}$ are continuous, the map β'' is continuous. For $x \in H_{\text{ext}}$, let $\lambda\, x$ respectively $\frac{1}{\lambda}\, x$ denote the multiplication by λ respectively $\frac{1}{\lambda}$ in every component of the product space H_{ext}. Then, one has

$$\beta''(Sw) = \frac{1}{\lambda}\, \bigl(\beta''(w) - \tilde{e}_0\bigr) \quad \text{and} \quad \beta''(S^{-1}w) = \lambda\, \beta''(w) + \tilde{e}_{-1}$$

for all $w \in X_{G(\boldsymbol{\Theta}^{\star})}$, wherefore, by observing that $\tilde{e}_{-1}, -\frac{1}{\lambda}\, \tilde{e}_0 \in \tilde{\mathcal{L}}_{\text{ext}}$ for all edge labels, one in fact has

$$\beta''(S^k w) \equiv \lambda^{-k}\, \beta''(w) \bmod \tilde{\mathcal{L}}_{\text{ext}}$$

for $k \in \mathbb{Z}$.

We now compare the above definition of β'' with the following form which corresponds to the form of the map D in Definition 7.64. To this end, we introduce the following notations: We again denote a walk w on $G(\boldsymbol{\Theta}^{\star})$ by the sequence of edges $w = \ldots e_{-2}e_{-1}\dot{e}_0 e_1 e_2 \ldots$ it runs through, and use the notation $v_i = v(e_{i-1}e_i)$ for the vertex this walk w passes through between the edges e_{i-1} and e_i. Let f_e denotes the map that corresponds to the edge with label e in the graph $G(\boldsymbol{\Theta}^{\star})$. Then, we may also define the map β'' *via*

$$\begin{aligned} \beta''(w) =& [-\ell_{v_0}\,, 0] \times \Omega_{v_0} \\ & \cap \bigcap_{n=1}^{\infty} \text{cl}\left[\bigcap_{k=1}^{n} \Bigl(f_{e_0} \circ \ldots \circ f_{e_{k-1}}(]-\ell_{v_k}\,, 0[)\Bigr) \times \Bigl(\hat{f}_{e_0} \circ \ldots \circ \hat{f}_{e_{k-1}}(\text{int}\, \Omega_{v_k})\Bigr)\right] \\ & \cap \bigcap_{n=1}^{\infty} \text{cl}\left[\bigcap_{k=1}^{n} \Bigl(f_{e_{-1}}^{-1} \circ \ldots \circ f_{e_{-k}}^{-1}(]-\ell_{v_{-k}}\,, 0[)\Bigr) \times \Bigl(\hat{f}_{e_{-1}}^{-1} \circ \ldots \circ \hat{f}_{e_{-k}}^{-1}(\text{int}\, \Omega_{v_{-k}})\Bigr)\right] \end{aligned} \tag{7.5}$$

In fact, one has

$$\begin{aligned} F^{\star}(e_0 e_1 \ldots) &= \lim_{N\to\infty} \hat{f}_{e_0} \circ \hat{f}_{e_1} \circ \ldots \circ \hat{f}_{e_{N-1}}(W) \\ -J(e_{-1}e_{-2}\ldots) &= \lim_{N\to\infty} f_{e_{-1}}^{-1} \circ f_{e_{-2}}^{-1} \circ \ldots \circ f_{e_{-N}}^{-1}(W') \end{aligned}$$

for any compact sets $W \subset H_{\text{ext}}$ and $W' \subset \mathbb{R}$ (note that the minus sign in the second equality arises because of $f_e^{-1} = t_{(-e/\lambda)} \circ f_0^{-1}$), and this establishes the equality of the two statements.

Consequently, we have established the following statement.

Proposition 7.72. *Let σ be a Pisot substitution (to the PV-number λ) such that $\underline{\Lambda}$ is a multi-component inter model set. In particular, $\bigcup_{i=1}^{n}[-\ell_i\,,0]\times\Omega_i$ is a measure-disjoint union and a fundamental domain of $\tilde{\mathcal{L}}_{\text{ext}}$. Then, the topological dynamical system $(\mathbb{R}\times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}\,,1/\lambda)$ given by the hyperbolic toral automorphism (the multiplication by) $1/\lambda$ is a factor of the sofic shift $(X_{G(\boldsymbol{\Theta}^\star)}\,,S)$, where the factor map is given by β'' in Equation* (7.4). □

Considering the space $\tilde{X}_{G(\boldsymbol{\Theta}^\star)}$ of all two-sided infinite walks opposite to the direction of the edges of $G(\boldsymbol{\Theta}^\star)$, yields a dynamical $(\tilde{X}_{G(\boldsymbol{\Theta}^\star)}\,,S)$ with factor $(\mathbb{R}\times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}\,,\lambda)$. Moreover, reversing the arrows in $G(\boldsymbol{\Theta}^\star)$ has the advantage that the corresponding graph is *right resolving* (see [239, Definition 3.3.1]), *i.e.*, the edge labels of the edges starting at a vertex are pairwise different. Most results in [239] are formulated for right resolving presentations of sofic shifts. Furthermore, in this case, the corresponding definition of the map β'' in Equation (7.5) would be analogous to definition of the map D in Definition 7.64, where the maps f_e^{-1} act on the sets that belong to the "positive part" of the corresponding walk.

Remark 7.73. Similar formulations of the last statement may also be found in [208, Theorem 3], also see [88, Section 4.3] and the remarks in [298, Section 7.6.2]. Also compare [55, 191, 228, 247, 294, 295] and [345, Chapitre 5]. The map F^* also appears, in principle, in [181, Section 3].

Remark 7.74. We may also state the above proposition by saying that the partition by the sets $\{[-\ell_i\,,0]\times\Omega_i\}_{i=1}^{n}$ yields *almost* a Markov partition (it is a Markov partition, if we use a more general definition of Markov partition allowing sofic shifts). In fact, the most often used explicit Markov partition is a variant of our findings for Fibonacci in Example 7.71 (compare Figure 7.2), see [239, Figs. 6.5.1 & 6.5.2], [198, Sections 1.8 & 2.5d] and [80, Section 5.12].

Remark 7.75. We also note that, by construction, the topological entropy of the sofic shift $(X_{G(\boldsymbol{\Theta}^\star)}\,,S)$ and of the hyperbolic toral automorphism $(\mathbb{R}\times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}\,,\lambda)$ coincide (at least in the unimodular case), both are given by $\log(\lambda)$. For the entropy of a sofic shift, see [239, Theorem 4.3.3] (compare [109, Prop. 25.17]). For the entropy of a hyperbolic toral automorphism, see [109, Theorem 24.5], [384, Theorem 8.15], [287, p. 249 (Section 6.1A)] and [244, Corollary IV.10.3]

Of course, we are more interested in a torus parametrisation for $\mathbb{X}(\underline{\Lambda})$ than for $(X_{G(\boldsymbol{\Theta}^\star)}\,,S)$. To this end, we make of the aperiodicity of $\underline{\Lambda}$, and therefore the unique composition property (UCP) according to Proposition 5.120 (recall Definition 5.119), and define a continuous surjective map $\mathbb{X}(\underline{\Lambda})\to X_{G(\boldsymbol{\Theta}^\star)}$ as follows: To remove ambiguity, we consider, for any $\underline{\Lambda}'\in\mathbb{X}(\underline{\Lambda})$, the tiling $\underline{\Lambda}'+\underline{\hat{A}}$ where $\hat{A}_i=[0\,,\ell_i[$ denotes the half-open interval. Thus, the covering degree of $\underline{\Lambda}'+\underline{\hat{A}}$ is *everywhere* one. We now consider the origin 0: for every $\underline{\Lambda}'\in\mathbb{X}(\underline{\Lambda})$, there is exactly one $\omega_i(x)\in\underline{\Lambda}'$ (*i.e.*, one $x\in\mathbb{R}$ and one $1\le i\le n$) such that $0\in[0\,,\ell_i[+x=\hat{A}_i+x$. We apply the tile substitution to $\underline{\Lambda}'$ and obtain, say, $\underline{\Lambda}''\in\mathbb{X}(\underline{\Lambda})$, wherefore there is an $\omega_j(y)\in\underline{\Lambda}''$ such that $0\in[0\,,\ell_j[+y=\hat{A}_j+y$. Moreover, there is exactly one map $f\in\Theta_{ji}$ such that $f(x)=y$. By the UCP, wherefore the tile substitution on $\mathbb{X}(\underline{\Lambda})$ is bijective, we therefore obtain for any $\underline{\Lambda}'\in\mathbb{X}(\underline{\Lambda})$ a two-sided infinite walk on $G(\boldsymbol{\Theta}^\star)$ (with our convention of the direction of the edges, the previous map $f\in\Theta_{ji}$ such that $f(x)=y$ yields the edge e_{-1}). This yields the sought after map, and therefore, composing it with β'', also the torus parametrisation $\beta':\mathbb{X}(\underline{\Lambda})\to\mathbb{R}\times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}=\mathbb{T}(\underline{\Lambda})\cong\mathbb{A}(\underline{\Lambda})$.

Note that with this identification, the fixed points (periodic points) of the tile substitution on $\mathbb{X}(\underline{\Lambda})$ correspond to fixed points (periodic points) of the multiplication by λ (or, we may say, the application of the substitution matrix $\boldsymbol{S}\sigma$) on torus $\mathbb{R} \times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}$, also compare the example at the end of Remark 6.13.

Remark 7.76. If we assume that the torus parametrisation is one-to-one almost everywhere on $\mathbb{T}(\underline{\varLambda}) = \mathbb{R} \times H_{\text{ext}}/\tilde{\mathcal{L}}_{\text{ext}}$, then this equivalent to $\underline{\varLambda}$ being a multi-component IMS by Lemma 7.56 (here we may establish the equivalence *via* the statement that $\bigcup_{i=1}^{n}[-\ell_i\,, 0] \times \varOmega_i$ is a fundamental domain of $\tilde{\mathcal{L}}_{\text{ext}}$). Moreover, in this case, not only the set of singleton fibres on $\underline{\varLambda}$ is a $\mathfrak{G}_\delta$-set by Lemma 7.51, but also its image (having full measure) on $\mathbb{T}(\underline{\varLambda})$ by Lemma 2.13 and Remark 5.10 (taking the intersection of open sets that contain the image of the singleton fibres), also recall Lemma 7.53.

While Proposition 7.48 derives the torus parametrisation $\beta' : \mathbb{X}(\underline{\varLambda}) \to \mathbb{T}(\underline{\varLambda})$ from the torus parametrisation $\beta : \mathbb{X}(\underline{\varLambda}) \to \mathbb{A}(\underline{\varLambda})$, we now consider properties of the map $\beta' : \mathbb{X}(\underline{\varLambda}) \to \mathbb{T}(\underline{\varLambda})$. For a Pisot substitution σ, we define

$$m_\sigma = \min\{\text{card}(\beta')^{-1}(\{y\}) \mid y \in \mathbb{T}(\underline{\varLambda})\}.$$

We may think of m_σ as minimal covering degree of $(\bigcup_{i=1}^{n}[-\ell_i\,, 0] \times \varOmega_i) + \tilde{\mathcal{L}}_{\text{ext}}$ (note that β' is finite-to-one, also compare [50, Theorem 6.1]). Moreover, one can show that the set

$$\{y \in \mathbb{T}(\underline{\varLambda}) \mid \text{card}(\beta')^{-1}(\{y\}) = m_\sigma\}$$

is a full measure $\mathfrak{G}_\delta$-set of $\mathbb{T}(\underline{\varLambda})$, see [50, Theorem 7.3] and [43, Theorem 4.2]. Consequently, the case $m_\sigma = 1$ is equivalent to the statement that $\underline{\varLambda}$ is a multi-component IMS. Together with the explicit characterisation of the eigenvalues in Proposition 7.69 (which implies that the toral flow $(\mathbb{T}(\underline{\varLambda})\,, \mathbb{R})$ is the maximal equicontinuous factor, see Remark 7.49 and compare [50, Theorem 9.3], [43, Theorem 5.1]), one obtains the following statement.

Proposition 7.77. [50, Corollary 9.4] and [43, Corollary 5.2] *Let σ be a Pisot substitution. Then, $\underline{\varLambda}$ is a multi-component IMS iff $m_\sigma = 1$. Moreover, one has $m_\sigma = 1$ iff $(\mathbb{X}(\underline{\varLambda})\,, \mathfrak{B}\,, \mu_{\mathbb{X}(\underline{\varLambda})}\,, \mathbb{R})$ has pure point spectrum.*

Sketch of Proof. We refer the reader to the cited literature for details and only remark:

- If $m_\sigma = 1$, the statement follows from the above considerations (since an IMS has pure point spectrum).
- For the other implication, assume that $m_\sigma > 1$ and $(\mathbb{X}(\underline{\varLambda})\,, \mathfrak{B}\,, \mu_{\mathbb{X}(\underline{\varLambda})}\,, \mathbb{R})$ has pure point spectrum. Then, one explicitly constructs a non-zero function $f \in L^2(\mathbb{X}(\underline{\varLambda})\,, \mathfrak{B}\,, \mu_{\mathbb{X}(\underline{\varLambda})})$, which is orthogonal to all eigenfunctions (we also recall that by ergodicity, all eigenvalues are simple). The contradiction arises because the eigenfunctions are constant on the fibres of β' and thus cannot form a dense subset of $L^2(\mathbb{X}(\underline{\varLambda})\,, \mathfrak{B}\,, \mu_{\mathbb{X}(\underline{\varLambda})})$ (consequently, the eigenfunctions do not "separated" enough points). □

Note that is result is complementary to our previous findings, see Theorem 7.42.

Remark 7.78. We conjecture that a corresponding description of the eigenvalues (and thus also eigenfunctions) as for $\underline{\varLambda}$ in Proposition 7.69 also holds for the multi-component Delone set $\underline{\varUpsilon}'$ described in Theorem 6.116. In fact, these eigenvalues should essentially be given *via* $(\mathscr{X}_\lambda)^\star$, compare Proposition 7.66.

8. Conclusion

> There's an old joke – um... two elderly women are at a Catskill mountain resort, and one of 'em says, "Boy, the food at this place is really terrible." The other one says,"Yeah, I know; and such small portions." Well, that's essentially how I feel about life – full of loneliness, and misery, and suffering, and unhappiness, and it's all over much too quickly.
>
> ANNIE HALL – *Woody Allen & Marshall Brickman*

On balance the result of the previous investigations might be formulated as follows: Given a substitution Delone set, one can (under primitivity, repetitivity *etc.*) expect the equivalence of the following statements.

- The Delone set is a regular model set.
- The Delone set admits some sort of coincidence.
- The Delone set is pure point diffractive.
- The Delone set has pure point dynamical spectrum.
- The internal space in the CPS can be tiled aperiodically.
- The product of the direct and the internal space in the CPS admits a klotz tiling.

We have deliberately left the formulation here vague, since we hope that this might serve as a guideline also for structures beyond the scope of this thesis.

However, we have made the above list precise in the case of Pisot substitutions; this can be found as Theorem 6.116. We have also, exemplary, indicated in Chapters 6a & 6b & 6c that (and how) this list can be used for structures that are *not* obtained by Pisot substitutions, *e.g.*, tilings, lattice substitution systems and reducible Pisot substitutions.

In the course of our considerations, we also came across a variety of themes not directly connected to Pisot substitutions. Here, we want to explicitly mention the following three topics:

- The way "to think geometrically" about $\mathfrak{p}$-adic fields in Chapter 3c.
- Iterated function systems on product spaces of local fields and obtaining bounds on the corresponding Hausdorff dimension in Chapter 4.
- The description of visible lattice points as (weak) model sets in Chapter 5a.

It is therefore our hope that this thesis is not only stimulating for a reader interested in Pisot substitutions or, more generally, aperiodic structures, but also for a reader how is more oriented towards fractal geometry and/or algebraic number theory.

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Extras: Summarium & Disputation (Slides)

Bernd Sing

Pisot Substitutions and Beyond

Summarium der Ergebnisse

Die vorliegende Arbeit besteht (hauptsächlich) aus drei Teilen: Zum einen werden Hausdorff-Maße auf Produkträumen über lokalen Körpern definiert. Hier werden dann iterierte Funktionensysteme betrachtet, für deren Attraktoren die Hausdorff-Dimension abgeschätzt werden kann. Danach werden Streifenprojektionsmengen (`cut and project sets`) betrachtet; für Punktmengen, die sich mittels einer Substitutionsregel erzeugen lassen, werden Bedingungen angegeben, die garantieren, daß sie sich auch als Streifenprojektionsmenge schreiben lassen. Angewandt werden die so entwickelten Methoden und Ergebnisse auf Pisot Substitutionen. Es wird vermutet, daß sich alle durch Pisot Substitutionen erzeugten (1-dimensionale) Punktmengen als Streifenprojektionsmengen schreiben lassen. Dazu wird eine Liste äquivalenter Bedingungen formuliert.

In Kapitel 4 wird zunächst gezeigt, daß das Haar-Maß auf einem Produktraum über lokalen Körpern ein Hausdorff-Maß ist (Theorem 4.56). Danach werden iterierte Funktionensysteme und ihr jeweiliger Attraktor betrachtet. Indem Ergebnisse von K.J. Falconer (über $\mathbb{R}^n$) verallgemeinert werden, erhält man obere (Prop. 4.122) bzw. in manchen Fällen auch untere (Lemma 4.126 & Props. 4.127 & 4.129) Schranken für die Hausdorff-Dimension (wie auch die Box-Counting Dimension (Lemma 4.133)) dieser Attraktoren.

Kapitel 5 steht im Zeichen von Streifenprojektionsmengen und Delone-Punktmengen, die durch Substitutionen erzeugt werden, sowie ihrer Beziehungen zueinander. In Abschnitt 5.3 wird gezeigt, wie man – nach Baake-Moody – insbesondere im Fall einer Delone-Menge mit mehreren Komponenten ein dazugehörendes Streifenprojektionsschema konstruiert. Diese Konstruktion läßt sich im Fall einer Substitutionsmenge übertragen und erweitern (Abschnitt 5.7.3), falls eine sogenannte algebraische oder Überlapp-Koinzidenz vorliegt; tatsächlich erhält man hier einen "vergrößerten" internen Raum (Prop. 5.137). Zentrale Aussage ist nun Theorem 5.154, das äquivalente Bedingungen dafür angibt, daß eine Substitutionsmenge eine Streifenprojektionsmege ist: Entweder besitzt die Substitution eine algebraische oder Überlapp-Koinzidenz, oder es existiert eine bestimmte aperiodische Parkettierungen des (erweiterten) internen Raumes.

Angewandt wird dies in Kapitel 6 auf Sequenzen, die von Pisot Substitutionen herrühren (Theorem 6.77). Allerdings kommen hier weitere äquivalente Bedingungen dazu, so z.B. daß eine bestimmte periodische Parkettierung des internen Raumes existiert (siehe Prop. 6.72), oder daß die sogenannte "geometrische Koinzidenz-Bedingung" (`GCC`) erfüllt ist (dazu wird auch eine graphentheoretische Formulierung angegeben, siehe Abschnitt 6.9). Die komplette Liste aller somit gefundenen äquivalenten Bedingungen – und somit die zentrale Aussage dieser Arbeit – ist dann Theorem 6.116. Dabei sind auch schon diejenigen äquivalente Aussagen mitaufgenommen, die sich erst im Kapitel 7 aus allgemeinen Betrachtungen über Diffraktionsmaße von Delone-Mengen, dem Spektrum von durch Delone-Mengen erzeugten dynamischen Systemen oder der sogenannten Torus-Parametrisierung für Streifenprojektionsmengen ergeben (Kapitel 7 ist hauptsächlich ein Überblick über schon bekannte Ergebnisse in der Literatur).

Um die erhaltenen Aussagen in den größeren Zusammenhang einzuordnen, werden außerdem die "sichtbaren Gitterpunkte" (Kapitel 5a), Parkettierungen in der Ebene (Kapitel 6a), Gittersubstitutionssysteme (Kapitel 6b), sowie reduzible Pisot Substitutionen und β-Subsitutionen (Kapitel 6c) besprochen und mit den erarbeiteten Methoden behandelt.

Pisot Substitutions and Beyond

Bernd Sing

Disputation – Bielefeld, 4.4.2007 1

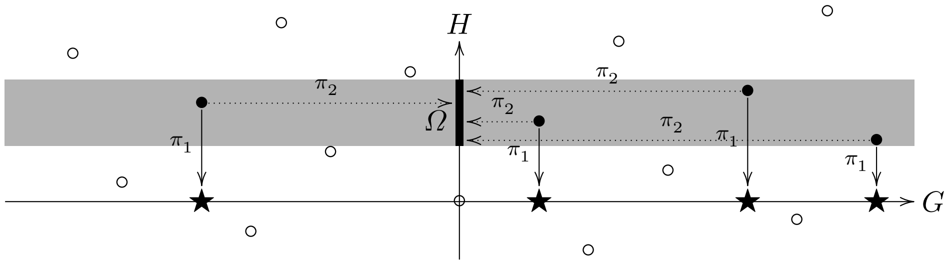
Cut and Project Scheme & Model Sets

- Cut and project scheme:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \text{ dense} \\ L = \pi_1(\tilde{L}) & \overset{\text{bijective}}{\longleftrightarrow} & \tilde{L} & \longrightarrow & L^\star = \pi_2(\tilde{L}) \end{array}$$

- Model set: $\varLambda = \Lambda(\varOmega) = \{x \in L \mid x^\star \in \varOmega\}$
  ($\varOmega$ compact und regularly closed $\varOmega = \text{cl int}\, \varOmega$)

- Regular model set if $\mu_H(\partial \varOmega) = 0$.



Disputation – Bielefeld, 4.4.2007 2

Why Model Sets?

Regular model set $\varLambda = \Lambda(\varOmega)$

Hof '95/'98, Schlottmann '00 (↓) | Baake-Lenz-Moody '07 (↑)

Dynamical system $(\mathbb{X}(\varLambda), G)$ has pure point spectrum.

Dworkin '93 (↓) | Lee-Moody-Solomyak '02, Gouéré '05, Baake-Lenz '04 (↑)

Moody '02 (from regular model set ↓ to diffraction)

Diffraction measure of $\varLambda$ is pure point / $\varLambda$ is pure point diffractive.

Task

- Given a point set $\varLambda$:

  Can it be described as model set?

- At least since the beginning of the '80s:

  Every (1-dim.) sequence generated by a Pisot substitution seems to be pure point.

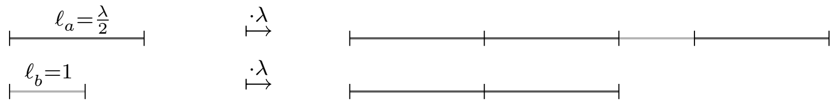
- There are explicit "geometrical" criteria for such Pisot substitution sequences to decide if one has a model set.

Pisot Substitutions

- Finite alphabet & substitution rule σ
  e.g. $\{a, b\}$ $\quad a \overset{\sigma}{\mapsto} aaba,\ b \overset{\sigma}{\mapsto} aa \qquad a.a \overset{\sigma}{\mapsto} aaba.aaba \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} \ldots baaaaaba.aabaaaba \ldots$

- Inflation factor λ is Perron-Frobenius eigenvalue of substitution matrix
  $$\mathbf{S}\sigma = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \qquad \lambda = \tfrac{3+\sqrt{17}}{2} \approx 3.562$$

- $\mathbf{S}\sigma$ primitive: components of (left) PF-Eigenvector are positive
  $\Rightarrow$ lengths of intervals for geometric representation



- σ is a Pisot substitution if $\mathbf{S}\sigma$ has a dominant eigenvalue $\lambda > 1$ and all other eigenvalues λ' satisfy $0 < |\lambda'| < 1$.
  An algebraic integer $\lambda > 1$ is a Pisot-Vijayaraghavan number if all its algebraic conjugates are contained inside the unit circle.

CPS for Pisot Substitution Sequences

Inflation factor λ is algebraic integer

Sequence $\underline{\Lambda} = (\Lambda_i)_{i=1}^n$ (left endpoints of intervals of length ℓ_i)
is contained in $\mathbb{Q}(\lambda)$, more precisely in

$$\mathcal{L} = \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}} = \langle (\bigcup_i \Lambda_i) - (\bigcup_i \Lambda_i) \rangle_{\mathbb{Z}} \overset{!}{=} \langle \bigcup_i (\Lambda_i - \Lambda_i) \rangle_{\mathbb{Z}}$$

Local fields:

- s (non-equiv.) complex embeddings
- r real embeddings
  $\|\lambda\|_\infty > 1$ in exactly one $\mathbb{R}$, otherwise $\|\lambda'\|_\infty < 1$

$\Big\}\ 2s + r = n$

- $\mathfrak{p}$-adic local fields $\mathbb{Q}_\mathfrak{p}$, always $\|\lambda\|_\mathfrak{p} \le 1$
  $\|\lambda\|_\mathfrak{p} < 1$ only if $\mathfrak{p}$ lies above a prime dividing $|\det \mathbf{S}\boldsymbol{\sigma}|$

CPS for Pisot Substitution Sequences

Inflation factor λ is algebraic integer

Sequence $\underline{\varLambda} = (\varLambda_i)_{i=1}^n$ (left endpoints of intervals of length ℓ_i) is contained in $\mathbb{Q}(\lambda)$, more precisely in

$$\mathcal{L} = \langle \ell_1, \ldots, \ell_n \rangle_{\mathbb{Z}} = \langle (\textstyle\bigcup_i \varLambda_i) - (\bigcup_i \varLambda_i) \rangle_{\mathbb{Z}} \overset{!}{=} \langle \bigcup_i (\varLambda_i - \varLambda_i) \rangle_{\mathbb{Z}}$$

- Direct space: $G = \mathbb{R}$
- Internal space: $H_{\text{ext}} = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod\limits_{\mathfrak{p}:\, \|\lambda\|_{\mathfrak{p}} < 1} \mathbb{Q}_{\mathfrak{p}}$
- Lattice: diagonal embedding of $\mathcal{L}_{\text{ext}} = \bigcup\limits_{m=0}^{\infty} \frac{1}{\lambda^m}\, \mathcal{L}$

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\;\pi_1\;} & \mathbb{R} \times H_{\text{ext}} & \xrightarrow{\;\pi_2\;} & H_{\text{ext}} \\
\text{dense} \cup & & \cup & & \cup \text{ dense} \\
\mathcal{L}_{\text{ext}} & \overset{\text{bijective}}{\longleftrightarrow} & \widetilde{\mathcal{L}_{\text{ext}}} & \overset{\text{bijective}}{\longleftrightarrow} & \mathcal{L}_{\text{ext}}^{\star}
\end{array}$$

IFS & EMFS

One has $\mathbb{R} = \bigcup\limits_{i=1}^{n} [0, \ell_i] + \varLambda_i = \underline{A} + \underline{\varLambda}$

where $\varLambda_i = \bigcup_j \lambda\, \varLambda_j + T_{ij}$ resp. $\underline{\varLambda} = \boldsymbol{\Theta}(\underline{\varLambda})$

and $[0, \ell_i] = \bigcup_j \frac{1}{\lambda}\, [0, \ell_j] + \frac{1}{\lambda} T_{ji}$ resp. $\underline{A} = \boldsymbol{\Theta}^{\#}(\underline{A})$.

On $\mathbb{R}$: $\boxed{\underline{\varLambda} = \boldsymbol{\Theta}(\underline{\varLambda})}$ $\boxed{\underline{A} = \boldsymbol{\Theta}^{\#}(\underline{A})}$

On H_{ext}: $\boxed{\underline{\varOmega} = \boldsymbol{\Theta}^{\star}(\underline{\varOmega})}$

By constuction: $\varOmega_i = \operatorname{cl} \varLambda_i^{\star}$

Theorem. Suppose one set $\varOmega_i$ has interior points. Then:

- All $\varOmega_j$ have Haar measure bigger than 0.
- The boundaries $\partial \varOmega_j$ have Haar measure 0.
- All $\varOmega_j$ are perfect sets and regularly closed. □

Dual Tiling

On $\mathbb{R}$: $\boxed{\underline{\Lambda} = \boldsymbol{\Theta}(\underline{\Lambda})}$ $\quad$ $\boxed{\underline{A} = \boldsymbol{\Theta}^{\#}(\underline{A})}$

On H_{ext}: $\boxed{\underline{\Omega} = \boldsymbol{\Theta}^{\star}(\underline{\Omega})}$ $\quad$ $\boxed{\underline{\Upsilon} = \boldsymbol{\Theta}^{\#\star}(\underline{\Upsilon})}$

$\Upsilon_i = \Lambda([0, \ell_i[)$ is a possible solution of $\underline{\Upsilon} = \boldsymbol{\Theta}^{\#\star}(\underline{\Upsilon})$ which is also Delone and repetitive.

Theorem. $\underline{\Omega} + \underline{\Upsilon}$ is self-replicating (w.r.t. inflation by $\left(\frac{1}{\lambda}\right)^{\star}$) and has constant covering degree a.e. □

Theorem. $\underline{\Lambda}$ is a model set iff $\underline{\Omega} + \underline{\Upsilon}$ is a tiling. □

Periodic Tiling

Let H be the compactly generated subspace of H_{ext} by $\underline{\Omega}$

$$H = \mathbb{R}^{r-1} \times \mathbb{C}^{s} \times \prod_{\mathfrak{p}:\, \|\lambda\|_{\mathfrak{p}} < 1} \mathfrak{p}^{\delta_{\mathfrak{p}}^{\mathcal{L}}}$$

e.g. for $a \mapsto aaba, b \mapsto aa$ with $\ell_a = \frac{\lambda}{2}, \ell_b = 1$: $\quad H = \mathbb{R} \times \mathbb{Z}_2 \quad (\|\ell_i\|_2 = 1)$

$\mathcal{M} = \langle \ell_2 - \ell_1, \ldots, \ell_n - \ell_1 \rangle_{\mathbb{Z}}$ is a $\mathbb{Z}$-modul of rank $n-1$.

One has: $\quad \mathcal{L} = (\bigcup_i \Lambda_i) \dot{+} \mathcal{M} \quad \overset{\star\&\,\mathrm{cl}}{\Longrightarrow} \quad H = (\bigcup_i \Omega_i) + \mathcal{M}^{\star}$

- ⇝ $\mathcal{M}^{\star}$ is a lattice in H.
- ⇝ Every Ω_i has interior points.

Check: $\quad \mathrm{dens}(\underline{\Lambda}) \overset{!}{=} \frac{\mu_H(\mathrm{FD}(\mathcal{M}^{\star}))}{\mu_{\mathbb{R}\times H}(\mathrm{FD}(\widetilde{\mathcal{L}}))} \leq \frac{\sum_i \mu_H(\Omega_i)}{\mu_{\mathbb{R}\times H}(\mathrm{FD}(\widetilde{\mathcal{L}}))} = \mathrm{dens}(\Lambda(\underline{\Omega}))$

Theorem. $\underline{\Lambda}$ is a model set iff $\bigcup_i \Omega_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^{\star}$. □

Pisot Substitution Sequences

Main Theorem. Let σ be a Pisot substitution. Let $\underline{\mathit{\Lambda}}$ be the representation of its fix point by intervals $[0, \ell_i]$ and $\mathit{\Upsilon}_i = \Lambda([0, \ell_i[)$. Then, the following statements are equivalent:

- $\underline{\mathit{\Lambda}}$ is an (inter-) model set, i.e., $\Lambda(\operatorname{int} \underline{\mathit{\Omega}}) \subset \underline{\mathit{\Lambda}} \subset \Lambda(\underline{\mathit{\Omega}})$.
- $\bigcup_i \mathit{\Omega}_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{M}^\star \subset H$.
- $\underline{\mathit{\Omega}} + \underline{\mathit{\Upsilon}}$ is a tiling of H_{ext}.
- $\bigcup_i [-\ell_i, 0] \times \mathit{\Omega}_i$ is a measure-disjoint union and a fundamental domain of $\mathcal{L}_{\text{ext}} \subset \mathbb{R} \times H_{\text{ext}}$.
- No overlaps, satisfies coincidence condition(s) (“GCC”)
- Iteration of certain polygons coverges “well-behaved” to $\underline{\mathit{\Omega}}$ (“stepped surface”)
- Pure point diffractive, pure point dynamical system… □

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