

Universität Bielefeld
Fakultät für Mathematik

Decomposition and stability of multifronts
and multipulses

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Introduction

In this thesis we consider time dependent reaction diffusion systems that have multiple pulse or front solutions. We develop a new numerical method that decomposes the solutions into their single pulses or fronts and in addition one computes the speeds and the positions of the single pulses and fronts. We show that the method is numerically feasible and prove stability results for the multiple pulse and front solutions if the distance of the pulses or fronts is sufficiently large and they interact only through their small tails.

The underlying nonlinear time dependent reaction diffusion system in one space dimension for functions $u(x, t) \in \mathbb{R}^m$ is of the form

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R}, t \geq 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad (1)$$

where the diffusion matrix $A \in \mathbb{R}^{m,m}$ is assumed to be positive definite and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is assumed to be smooth.

Reaction diffusion equations describe dynamical processes in chemistry, physics and biology. A prominent example is the class of equations that describe propagation in nerve axons, see [15] or [18].

Traveling waves (w, c) are solutions of the reaction diffusion equation (1) of the form

$$u(x, t) = w(x - ct), \quad (2)$$

i.e. the solution has the profile w and moves with the velocity c in space, see the left picture of Figure 1 for an illustration.

Traveling waves describe natural, ubiquitous phenomena in excitable media. They arise in a lot of natural applied phenomena within the nonlinear sciences, for instance, population dynamics in mathematical biology, see [25], in chemical reactions, cf. [38] or in the context of combustion, see [34].

On the mathematical side there is a well developed stability theory for traveling waves, see [30], [38].

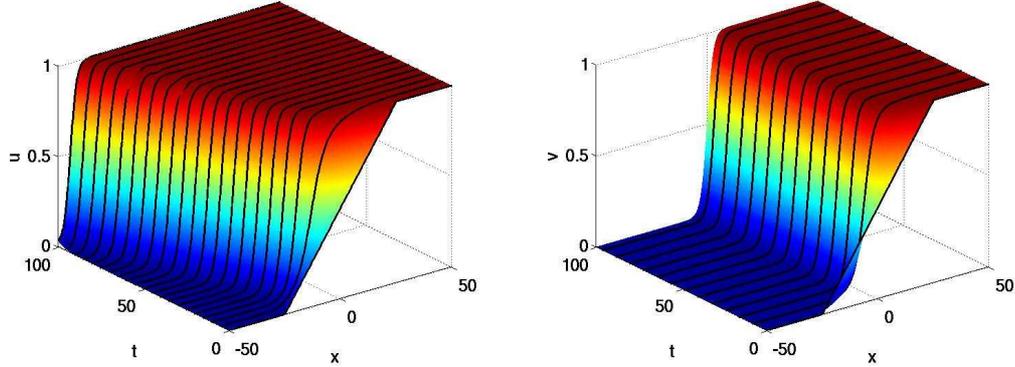


Figure 1: Front moving in the Nagumo-equation, $u_t = u_{xx} + u(1 - u)(u - a)$ with $a = 0.25$ and result of single freezing method applied to the front.

Consider a finite computational domain, traveling wave solutions will always leave such a finite computational domain. If this domain is too small, the solution of the parabolic equation (1) may leave the domain before the steady profile appears. This problem was the main motivation behind the freezing method developed in [5], [6]. Let us first briefly describe this method: The main idea is to separate the shape dynamics from the dynamics of the position of the traveling wave. Let us write the solution of (1) in the following form $u(x, t) = v(x - g(t), t)$, where $g(t)$ denotes the position of the profile v at the time t . Inserting the ansatz into (1) yields the following partial differential algebraic equation system (PDAE)

$$v_t = Av_{\xi\xi} + \mu v_{\xi} + f(v), \quad v(\cdot, 0) = u_0, \quad (3)$$

$$0 = \psi(v, g), \quad (4)$$

where $\mu(t) = g_t(t)$, $g(0) = 0$ and (4) denotes a phase condition defined by a functional $\psi(v, g)$. The extra phase condition is added to compensate the extra degree of freedom introduced by the new variable g . In practice the choice of the phase condition can be derived from minimization or orthogonality principles, see [5]. A numerical solution $v(x, t)$ of the Nagumo-equation (see Chapter 2) computed using this method is displayed in the right picture of Figure 1 for the Nagumo-equation. A traveling wave (w, c) is a stationary solution of the system (3) - (4) if the traveling wave satisfies the phase condition (4). The freezing method computes a comoving coordinate frame in which traveling waves become stationary. In fact, it is shown in [36] and [37] that the traveling wave (w, c) becomes an asymptotically stable steady state for the PDAE (3) - (4) if the linearization of (1) at the wave satisfies certain conditions.

This thesis deals with nonlinear time dependent reaction diffusion systems in one space dimension that have multipulse or multifront solutions, i.e. solutions that look like a superposition of several single waves traveling at different speeds, see Figure 2. In order to have a general term we use the expression "multistructures" to describe multifronts or multipulses.

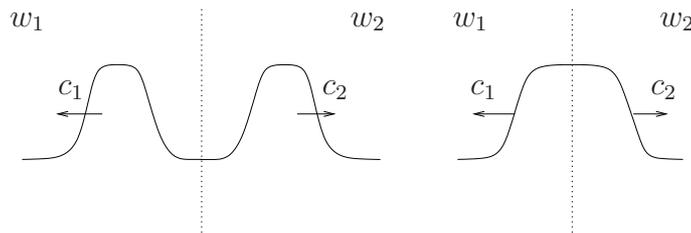


Figure 2: Multipulse and multifront

Recently the study of these interactions of pulses and fronts created a lot of attention and there exist quite a few analytical, numerical and experimental studies, see e.g. [40], [31], [10], [11], [13], [41], [19], [26] and [32]. In these investigations one finds different types of interaction called weak and strong interaction. In the theoretical part of this thesis we consider the case of weak interaction. We investigate the interaction of localized pulses or fronts when their distance is sufficiently large and they interact only through their small tails, i.e. the pulses or fronts are well separated in space during a certain interval of time. If the pulses or fronts are close to each other we call the phenomenon strong interaction, see e.g. [26], they interact strongly and may annihilate or reflect after collision. In the applications of this thesis we consider both kinds of interactions.

In the following we give a more technical outline of the topics of this thesis. Let us assume that the system (1) has several traveling waves $w_j, j = 1, \dots, N$ with different speeds $c_j, j = 1, \dots, N$ and assume that the left limits $\lim_{x \rightarrow -\infty} w_j(x) := w_j^-$ and the right limits $\lim_{x \rightarrow \infty} w_j(x) := w_j^+$ of the traveling waves w_j satisfy $w_j^+ = w_{j+1}^-$ for $j = 1, \dots, N - 1$. Consequently, if we suitably shift the traveling waves in space they fit together after summation and we obtain a multistructure.

To handle such multistructure solutions, we develop a numerical method which extends the freezing method to a 'decompose and freeze method' and furthermore we provide an analysis for the case of weak interaction. If the multistructure solutions travel at different speeds, the freezing method can freeze only one pulse or front of the superposition. A typical example is shown in Figure 3. An initial perturbation creates two pulses traveling in opposite directions. The right figure

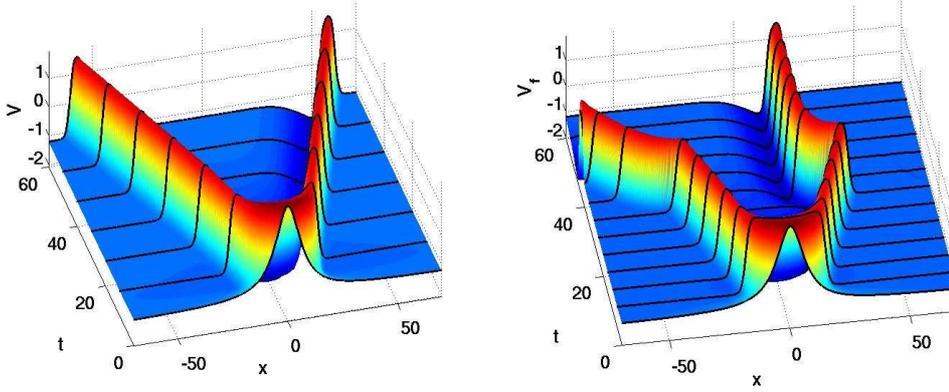


Figure 3: Double pulse in the V -component of the FitzHugh-Nagumo-equation, $V_t = V_{xx} + V - \frac{1}{3}V^3 - R$, $R_t = \varepsilon(V + a - bR)$, $a = 0.7$, $b = 0.8$, $\varepsilon = 0.08$ and result of single freezing applied to the double pulse.

shows the result of the single freezing method applied to this double pulse. Again the initial hump splits into two traveling components. It is shown in Figure 3 that in this application the single freezing method can only freeze the right pulse, i.e. the right pulse stabilizes and the left pulse leaves the computational domain.

The idea of the 'decompose and freeze method' is to decompose the solution of the Cauchy problem (1) into a finite superposition of single profiles v_j that asymptotically assume the shape of shifted single traveling waves. We assume the decomposition of the solution of (1) to be of the form

$$u(x, t) = \sum_{j=1}^N v_j(x - g_j(t), t), \quad (5)$$

where the new variables g_j denote the time-dependent position of the patterns v_j . This idea goes back to [4].

We insert (5) into (1), substitute $\mu_j = g_{j,t}$. Let u be given by (5), then u solves (1) provided $(v_1, \dots, v_N, g_1, \dots, g_N, \mu_1, \dots, \mu_N)$ solves the coupled PDAE system for $j = 1, \dots, N$ of the form

$$v_{j,t} = Av_{j,\xi\xi} + \mu_j v_{j,\xi} + F_j(v_1, \dots, v_N, g_1, \dots, g_N, \mu_j), \quad v_j(0) = v_j^0, \quad (6)$$

$$g_{j,t} = \mu_j, \quad g_j(0) = g_j^0, \quad (7)$$

$$0 = \psi_j(v_j, g_j). \quad (8)$$

For details on the computation of the nonlinear and nonlocal coupling term F_j , we refer to Chapter 1, in particular equation (1.15). It is important to note that the

coupling term F_j depends on all patterns v_k , all positions g_k and on the velocity μ_j . Note that there are differences in the nonlinear term F_j used in (6) when compared with [4].

The decomposed system (6) - (8) uses a partition of unity and is not unique. Again we add extra phase conditions (8) to make the solution of (6), (7) unique. We expect that the profiles v_j converge towards shifted traveling waves and that the superposition of the shifted traveling waves satisfies (1) in an asymptotic sense.

Assuming that the velocities are ordered according to $c_1 < \dots < c_N$, we introduce the notion of joint asymptotic stability for the system $(w_j, c_j), j = 1, \dots, N$ of traveling waves in some exponentially weighted Sobolev space. The idea of using exponentially weighted spaces is a common tool for handling stability problems on the infinite axis, see e.g. [31], [40], [42], [21].

The main result of this thesis is a stability theorem for the PDAE system (6) - (8) for the case of weak interaction of multistructures. If the traveling waves interact only weakly, i.e. if $g_1^0 < \dots < g_N^0$ and the minimal distance $\min_{j=1, \dots, N-1} |g_{j+1} - g_j|$ is sufficiently large, and if the initial functions v_j^0 are close enough to w_j (up to a shift), then there exists a solution (v_j, g_j, μ_j) of the system of PDAEs (6) - (8) for all times. Moreover, the profiles v_j converge exponentially fast towards a suitable shift of w_j in the exponentially weighted space, see Theorem 1.13.

Figure 4 shows the result of a corresponding numerical computation obtained from (6) - (8), when applied to the FitzHugh-Nagumo-system from Figure 3. The frozen profiles V_j in the comoving frames are displayed as functions of time. In the moment of separation of the original hump, see Figure 3, small additional pulses appear that vanish and the profiles V_j stabilize very rapidly.

As a consequence of our theorem the superposition (5) of the shifted profiles is a solution of (1) that converges as $t \rightarrow \infty$ to a superposition of the shifted wave profiles in some exponentially weighted space (Corollary 1.15). Results of this type has been proven for the case of multipulses in [40]. Note, however, that the analytical approach taken in [40] uses explicit knowledge of the single waves and does not directly lead to an implementable form. Our main concern here is with stability and numerical solution of the extended PDAEs (6) - (8) rather than the given system (1).

The main difficulty in the proof consists of the proper analysis of the nonlinear and nonlocal coupling terms F_j and the side constraints (8). Furthermore, we use well known stability techniques like semigroup theory and the variation of constants formula, see e.g. [17], [23]. Essentially we use spectral properties of and resolvent

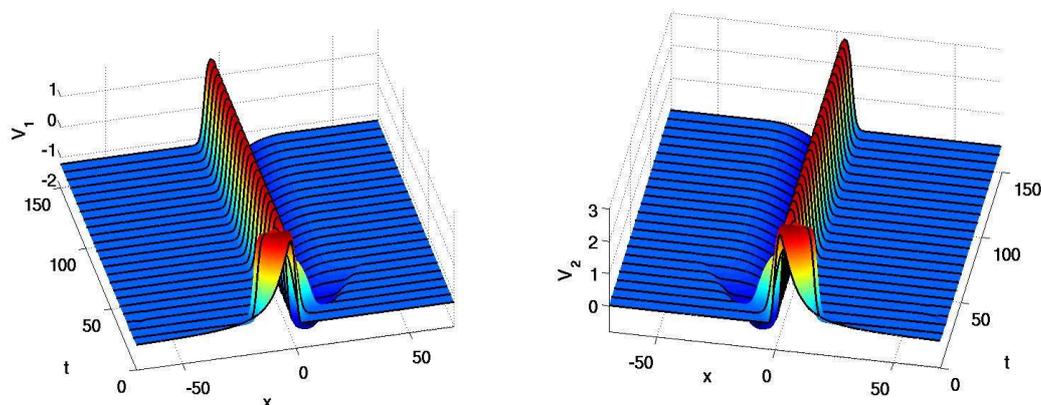


Figure 4: Splitting of the V component into two pulses in the FitzHugh-Nagumo-equations $V_t = V_{xx} + V - \frac{1}{3}V^3 - R$, $R_t = \varepsilon(V + a - bR)$, $a = 0.7$, $b = 0.8$, $\varepsilon = 0.08$, evolution of left and right traveling pulses V_1 and V_2 .

estimates for the operators

$$\Lambda_j u = Au_{\xi\xi} + c_j u_{\xi} + Df(w_j)u$$

which we obtain by linearizing (3) at the traveling wave w_j . Another important property is the fact that the eigenvalue zero of Λ_j , which is always present, is removed by the phase condition.

In Chapter 1 we present in detail the 'decompose and freeze method' and we introduce the notion of joint asymptotic stability. Based on these notions we formulate the stability theorem which is the main theorem of this thesis (Theorem 1.13).

In Chapter 2 we demonstrate the 'decompose and freeze method' on three examples of weak interactions: a multifront solution for the Nagumo-equation (2.8), a multipulse solution for the FitzHugh-Nagumo-equations (2.10) - (2.11), and the three component system (2.12) - (2.14) introduced in [32].

In Chapter 3 we prove the stability theorem. A particular difficulty arises from the fact that the nonlinear and nonlocal PDAE system has to be linearized and delicate estimates are needed that use the shape and location of the bump function that occurs in the decomposition.

Chapter 4 contains more numerical computations. The reaction diffusion systems considered there are the Nagumo-equation (2.8) and the FitzHugh-Nagumo-equations (2.10) - (2.11). Some of the examples show that the 'decompose and freeze method' gives also interesting results for the strong interaction case.

Appendix A contains more technical estimates for the nonlinear coupling terms.

Moreover, Appendix A summarizes some functional analytic notions and results, facts about exponential dichotomies and exponentially weighted spaces. Important notation used in this thesis is listed in the Appendix B.

In summary, this thesis brings together the idea of separating the shape dynamics from the underlying group dynamics with the stability analysis of multi-pulse and multifront solutions of reaction diffusion systems. Our main result shows feasibility of the decomposition method in the case of weak interaction and contains the study of existence and stability of multiple pulse and front solutions if the single traveling pulses or fronts are well separated in space during a certain interval of time.

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Chapter 1

Stability of multifronts and multipulses

We consider nonlinear time dependent reaction diffusion systems in one space dimension that have multipulse or multifront solutions, i.e. solutions that look like a superposition of several waves.

To handle multipulse or multifront solutions, we develop a numerical method which extends the freezing method for single traveling waves, see [5], to a 'decompose and freeze method'. The method separates the group dynamics from the shape dynamics of the single pulses and fronts. The idea is to decompose the solution of the Cauchy problem into a finite superposition of single profiles that asymptotically assume the shape of suitably shifted single traveling waves. Each of the single waves has its own moving coordinate frame. We derive a system of partial differential algebraic equations (PDAEs) coupled by nonlinear and nonlocal terms.

We introduce the notion of joint asymptotic stability and present a stability theorem for multipulse and multifront solutions which shows that the shifted traveling waves are asymptotically stable solutions of the PDAE system. Furthermore, the superposition of the profiles, when suitably shifted, converges towards the solution of the parabolic system.

1.1 Decomposition of multifronts

Consider a parabolic system for a function $u(x, t) \in \mathbb{R}^m$ on the real line

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R}, t \geq 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad (1.1)$$

where $A \in \mathbb{R}^{m,m}$ is assumed to be positive definite and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is assumed to be sufficiently smooth.

Assume that the system (1.1) has several traveling wave solutions (w_j, c_j) , $j = 1, \dots, N$ of the form

$$u_j(x, t) = w_j(x - c_j t), \quad j = 1, \dots, N \quad (1.2)$$

traveling at different speeds c_j with $c_1 < \dots < c_N$ and with limiting behavior

$$w_j^- = \lim_{\xi \rightarrow -\infty} w_j(\xi), \quad w_j^+ = \lim_{\xi \rightarrow \infty} w_j(\xi). \quad (1.3)$$

We assume that the left and right limits of the single waves match in the sense that

$$w_j^+ = w_{j+1}^-, \quad j = 1, \dots, N - 1. \quad (1.4)$$

Using this assumptions we want to patch these solutions together to multipulses and multifronts. Recall that solutions of (1.1) that look like a superposition of several possibly shifted waves traveling at different speeds are usually called multifronts or multipulses depending on whether the limits at $\pm\infty$ agree or disagree. To have a general term we use the expression multistructures to describe multipulses or multifronts. Figure 1.1 is an illustration of a multipulse and a multifront in the case $N = 2$, $c_1 < 0 < c_2$ satisfying the condition $w_1^+ = w_2^-$. Compare the Figure 2 in the Introduction, it has the same form, but in Figure 1.1 more details are given.

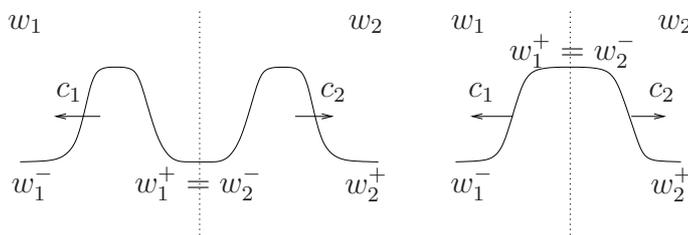


Figure 1.1: Multipulse and multifront

We consider the superposition

$$W^k(x, t) = \sum_{j=1}^N \hat{w}_j(x - c_j t - k_j) \quad (1.5)$$

for some $k = (k_1, \dots, k_N) \in \mathbb{R}^N$, where we have subtracted left limits so that the shifted profiles \hat{w}_j , defined by

$$\hat{w}_j(\xi) = w_j(\xi) - \tilde{w}_j^-, \quad \tilde{w}_j^- = \begin{cases} 0, & j = 1 \\ w_j^-, & j \geq 2, \end{cases} \quad (1.6)$$

fit together upon summation, this is illustrated in Figure 1.2. Note that for some $\xi \in \mathbb{R}$ and $s \in \{1, \dots, N\}$ the superposition $\sum_{j=1}^N \hat{w}_j(\xi)$ can be equivalently written as

$$\sum_{j=1}^N \hat{w}_j(\xi) = \sum_{j=1}^{s-1} (w_j(\xi) - w_j^+) + w_s(\xi) + \sum_{j=s+1}^N (w_j(\xi) - w_j^-). \quad (1.7)$$

Therefore, the superposition W^k has the properties

$$\lim_{x \rightarrow -\infty} W^k(x, t) = \sum_{j=1}^N w_j^- - \sum_{j=2}^N w_j^- = w_1^-, \quad \lim_{x \rightarrow \infty} W^k(x, t) = \sum_{j=1}^N w_j^+ - \sum_{j=2}^N w_j^- = w_N^+. \quad (1.8)$$

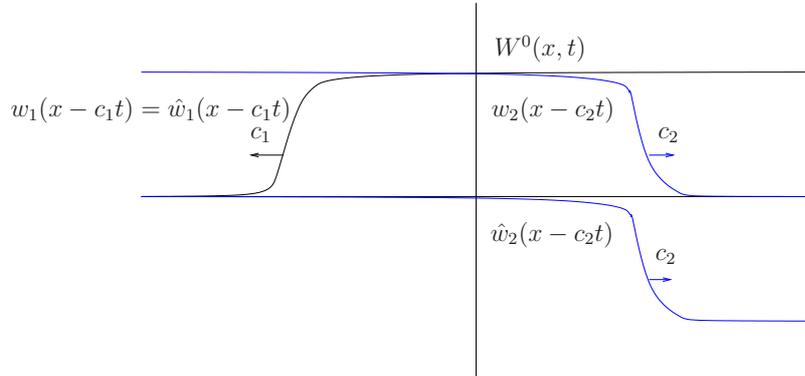


Figure 1.2: The modified profiles $\hat{w}_j(x - c_j t)$, $c_1 < 0 < c_2$

Remark 1.1. Notice that instead of the superposition W^k we could also consider the superposition

$$\check{W}^k(x, t) = \sum_{j=1}^N \check{w}_j(x - c_j t - k_j)$$

with

$$\check{w}_j(\xi) = w_j(\xi) - \bar{w}_j^+, \quad \bar{w}_j^+ = \begin{cases} w_j^+, & j \leq N-1, \\ 0 & j = N. \end{cases} \quad (1.9)$$

Again the \check{w}_j fit together upon summation. This superposition \check{W}^k satisfies also the properties $\lim_{x \rightarrow -\infty} \check{W}^k(x, t) = w_1^-$, $\lim_{x \rightarrow \infty} \check{W}^k(x, t) = w_N^+$. Consequently, the decomposition of the solution is not unique.

We are interested in solutions $u(x, t)$ of (1.1) that asymptotically assume the shape of W^k for some $k \in \mathbb{R}^N$. We follow an idea of [4] and we write the solution $u(x, t)$ of (1.1) in the following form

$$u(x, t) = \sum_{j=1}^N v_j(x - g_j(t), t). \quad (1.10)$$

Here the function $g_j : \mathbb{R} \rightarrow \mathbb{R}$ denotes the time-dependent position of the pattern $v_j : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}^m$, $(\xi, t) \mapsto v_j(\xi, t)$. We develop a numerical method that decomposes solutions of (1.1) into a finite superposition of functions $v_j(\cdot, t)$, where the functions $v_j(\cdot, t)$ should approximate the shape of the shifted waves \hat{w}_j for large times, we expect $v_j(\cdot, t)$ to be constant outside a small region.

For the decomposition we use the idea of partition of unity. Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a positive bump function with its main mass located near zero. We are interested in solutions of the form (1.10) and insert this into (1.1). We suppress the arguments $(x - g_j(t), t)$ of v_j and obtain

$$\begin{aligned} u_t &= \sum_{j=1}^N [v_{j,t} - v_{j,\xi} g_{j,t}] = \sum_{j=1}^N A v_{j,\xi\xi} + f\left(\sum_{k=1}^N v_k\right) \\ &= \sum_{j=1}^N \left[A v_{j,\xi\xi} + \frac{\varphi(\cdot - g_j(t))}{\sum_{k=1}^N \varphi(\cdot - g_k(t))} f\left(\sum_{k=1}^N v_k\right) \right] \\ &\quad + \sum_{j=1}^N \frac{\varphi(\cdot - g_j(t))}{\sum_{k=1}^N \varphi(\cdot - g_k(t))} \left(\sum_{k=1}^N [f(v_k + \tilde{w}_k^-) - f(v_k + \tilde{w}_k^-)] \right) \\ &= \sum_{j=1}^N \left[A v_{j,\xi\xi} + \frac{\varphi(\cdot - g_j(t))}{\sum_{k=1}^N \varphi(\cdot - g_k(t))} \left(f\left(\sum_{k=1}^N v_k\right) - \sum_{k=1}^N f(v_k + \tilde{w}_k^-) \right) \right] \\ &\quad + \sum_{j=1}^N f(v_j + \tilde{w}_j^-) \\ &= \sum_{j=1}^N \left[A v_{j,\xi\xi} + f(v_j + \tilde{w}_j^-) + \frac{\varphi(\cdot - g_j(t))}{\sum_{k=1}^N \varphi(\cdot - g_k(t))} \left(f\left(\sum_{k=1}^N v_k\right) - \sum_{k=1}^N f(v_k + \tilde{w}_k^-) \right) \right] \end{aligned} \quad (1.11)$$

Note that the quotients

$$\frac{\varphi(x - g_j(t))}{\sum_{k=1}^N \varphi(x - g_k(t))},$$

have non-vanishing denominators and form a time-dependent partition of unity.

We substitute $\xi = x - g_j(t)$ and $\mu_j = g_{j,t}$ and obtain the following coupled system for $j = 1, \dots, N, \xi \in \mathbb{R}, t \geq 0$

$$\begin{aligned} v_{j,t}(\xi, t) = & Av_{j,\xi\xi}(\xi, t) + v_{j,\xi}(\xi, t)\mu_j(t) + f(v_j(\xi, t) + \tilde{w}_j^-) + \\ & + \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi - g_k(t) + g_j(t))} \left[f \left(\sum_{k=1}^N v_k(\xi - g_k(t) + g_j(t), t) \right) \right. \\ & \left. - \sum_{k=1}^N f(v_k(\xi - g_k(t) + g_j(t), t) + \tilde{w}_k^-) \right] \end{aligned} \quad (1.12)$$

and the simple set of ODEs

$$g_{j,t} = \mu_j(t), \quad j = 1, \dots, N. \quad (1.13)$$

Let u be given by (1.10), then u is a solution of the parabolic system $u_t = Au_{xx} + f(u)$ provided the set $v_j, \mu_j, g_j, j = 1, \dots, N$ solves the system (1.12), (1.13).

For simplicity of notation, we write $v = (v_1, \dots, v_N)$, $g = (g_1, \dots, g_N)$, $\mu = (\mu_1, \dots, \mu_N)$ and we abbreviate the nonlinear terms in (1.12) as follows

$$F_j(v, g)(\xi, t) = f(v_j(\xi, t) + \tilde{w}_j^-) \quad (1.14)$$

$$+ Q_j^{g(t)}(\xi) \left[f \left(\sum_{k=1}^N v_k(\xi_{kj}^{g(t)}, t) \right) - \sum_{k=1}^N f(v_k(\xi_{kj}^{g(t)}, t) + \tilde{w}_k^-) \right], \quad (1.15)$$

$$Q_j^g(\xi) = \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi_{kj}^g)}, \quad \xi_{kj}^g = \xi - g_k + g_j. \quad (1.16)$$

Note $0 \leq Q_j^g(\xi) \leq 1$ for all $\xi \in \mathbb{R}, j = 1, \dots, N$. The important point to note here is that the nonlinear terms $F_j(v, g)$ couple the single functions $v_k, k = 1, \dots, N$ in a nonlocal fashion.

Remark 1.2. Note the difference in the nonlinear term F_j used in (1.15) when compared with [4]. The calculation in (1.11) is a modification to the calculation in [4], Section 2. The numerical computations in Chapter 2 show that the modified method works just as well as the method proposed in [4].

The system will be completed by initial conditions for the functions v_j, g_j

$$v_j(0) = v_j^0, \quad g_j(0) = g_j^0, \quad j = 1, \dots, N \quad (1.17)$$

that satisfy $u^0(x) = \sum_{j=1}^N v_j^0(x - g_j^0), x \in \mathbb{R}$, see [4]. Further we have to add phase conditions that compensate the extra degrees of freedom introduced by the

new variables μ_j . There are different possibilities for deriving appropriate phase conditions, see [5]. We use the fixed phase condition, i.e. the v_j should stay as close as possible to given reference functions $\hat{v}_j, j = 1, \dots, N$. Consequently we require the distance function $d_j(g) = \|v_j(\cdot, t) - \hat{v}_j(\cdot - g)\|_{\mathcal{L}^2}$ to achieve its minimum at $g = 0$ for all times. If we differentiate the distance with respect to g we obtain the N different phase conditions

$$\langle v_j - \hat{v}_j, \hat{v}_{j,\xi} \rangle = 0, \quad j = 1, \dots, N, \quad (1.18)$$

where $\langle \cdot, \cdot \rangle$ denotes the \mathcal{L}_2 inner-product, i.e. $\langle u, v \rangle := \int_{\mathbb{R}} u(\xi)^T v(\xi) d\xi$.

In summary, the coupled PDAE system (1.12), (1.13) together with (1.18) as phase conditions and the initial conditions (1.17) has to be solved.

1.2 Notations and definitions - part 1

In this section we introduce some notations and definitions.

Given a norm $\|\cdot\|_*$, we define for $s = (s_1, \dots, s_N)$

$$\|s\|_* := \max_{1 \leq j \leq N} \|s_j\|_*.$$

Let $0 \leq \tau < \infty$, X Banach space, $\|\cdot\|_*$ norm, we define for a function $u : [0, \tau] \rightarrow X$

$$\|u\|_*^\tau = \sup_{0 \leq t \leq \tau} \|u(t)\|_*. \quad (1.19)$$

We consider functions in the Banach spaces $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^m)$, $\mathcal{H}^1(\mathbb{R}, \mathbb{R}^m)$ and $\mathcal{H}^2(\mathbb{R}, \mathbb{R}^m)$. In the following we omit the spaces \mathbb{R}, \mathbb{R}^m and simply write $\mathcal{L}_2, \mathcal{H}^1, \mathcal{H}^2$.

We use exponentially weighted spaces and semigroup theory to handle stability problems of the system (1.12), (1.13), (1.17), (1.18) on the infinite axis. Define for $b \geq 0$ the weight function θ_b by

$$\theta_b(\xi) = \cosh(b\xi) = \frac{e^{b\xi} + e^{-b\xi}}{2}. \quad (1.20)$$

Additionally we define weighted spaces together with weighted norms

$$\mathcal{L}_{2,b} := \{u | \theta_b u \in \mathcal{L}_2\}, \quad \|u\|_{\mathcal{L}_{2,b}} := \|\theta_b u\|_{\mathcal{L}_2},$$

$$\mathcal{H}^{1,b} := \{u | \theta_b u \in \mathcal{H}^1\}, \quad \|u\|_{\mathcal{H}^{1,b}} := \|\theta_b u\|_{\mathcal{H}^1}$$

and analogously

$$\mathcal{H}^{2,b} := \{u | \theta_b u \in \mathcal{H}^2\}, \quad \|u\|_{\mathcal{H}^{2,b}} := \|\theta_b u\|_{\mathcal{H}^2}.$$

The norms satisfy the following estimates

$$\|u\|_{\mathcal{L}_2} \leq \|u\|_{\mathcal{L}_{2,b}}, \quad \|u\|_{\mathcal{H}^1} \leq \|u\|_{\mathcal{H}^{1,b}} \quad \text{and} \quad \|u\|_{\mathcal{H}^2} \leq \|u\|_{\mathcal{H}^{2,b}}. \quad (1.21)$$

For abbreviation, we write $w = (w_1, \dots, w_N)$, $\hat{w} = (\hat{w}_1, \dots, \hat{w}_N)$, $c = (c_1, \dots, c_N)$. Let $\varrho > 0$ and $u = (u_1, \dots, u_n) \in (\mathcal{H}^{1,b})^N$, $r = (r_1, \dots, r_N)$, $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ with $u_j \in \mathcal{H}^{1,b}$, $j = 1, \dots, N$, we define the ball around zero with radius ϱ by

$$B_{\varrho,b}(0) = \{(u, r, \lambda) : \|u\|_{\mathcal{H}^{1,b}} + \|r\| + \|\lambda\| \leq \varrho, u_j \in \mathcal{H}^{1,b}, r_j, \lambda_j \in \mathbb{R}, j = 1, \dots, N\}. \quad (1.22)$$

Let $a \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and define the punctured sector $\bar{S}_{a,\theta} \subset \mathbb{C}$ by

$$\bar{S}_{a,\theta} = \{s \in \mathbb{C} : |\arg(s + a)| \leq \theta, s \neq -a\}$$

and the open sector $S_{a,\theta} \subset \mathbb{C}$ by

$$S_{a,\theta} = \{s \in \mathbb{C} : |\arg(s + a)| < \theta, s \neq -a\}.$$

We recall the definition of a sectorial operator in a Banach space X , see Figure 1.3.

Definition 1.3 (sectorial operator in X). *Let X be a Banach space and let $\Lambda : \mathcal{D}(\Lambda) \rightarrow X$ be a linear operator on X . Λ is called sectorial if*

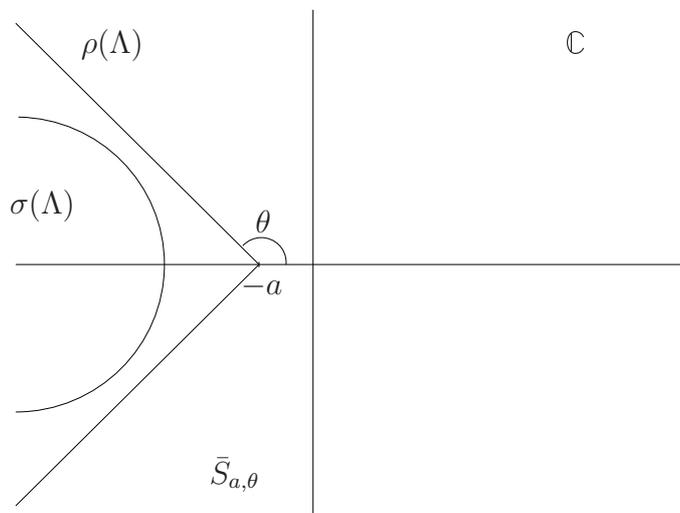
1. Λ is closed and densely defined,
2. there exists $\theta \in (\frac{\pi}{2}, \pi)$, $M > 1$, $a \in \mathbb{R}$ such that the sector $\bar{S}_{a,\theta}$ is contained in the resolvent set $\rho(\Lambda)$ and the following estimate holds

$$\|(sI - \Lambda)^{-1}\| \leq \frac{M}{|s + a|}, \quad \forall s \in \bar{S}_{a,\theta}.$$

1.3 The main stability theorem and joint asymptotic stability

Before we formulate the main result of this chapter, we formulate assumptions on the function f and on the traveling waves (w, c) .

Hypothesis 1.4. *Assume $f \in \mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m)$ and $A > 0$, i.e. $\langle Av, v \rangle > 0$ for all $v \in \mathbb{R}^m$, $v \neq 0$.*

Figure 1.3: Sectorial operator Λ with sector $\bar{S}_{a,\theta} \subset \rho(\Lambda)$

Hypothesis 1.5. Let (w, c) be a set of traveling waves with $w_j \in \mathcal{C}_b^2$, $j = 1, \dots, N$ for the system (1.1) satisfying the conditions (1.2) - (1.4). Let \hat{v}_j , $j = 1, \dots, N$ be given reference functions with $\hat{v}_j - \hat{w}_j \in \mathcal{H}^2$, $j = 1, \dots, N$ such that

$$0 = \langle \hat{w}_j - \hat{v}_j, \hat{v}_{j,\xi} \rangle. \quad (1.23)$$

and

$$\langle w_{j,\xi}, \hat{v}_{j,\xi} \rangle \neq 0 \quad \forall j = 1, \dots, N.$$

The assumption $\hat{v}_j - \hat{w}_j \in \mathcal{H}^2$ means that $w_j - (\hat{v}_j + \tilde{w}_j) \in \mathcal{H}^2$ and that the functions \hat{v}_j and \hat{w}_j have the same limiting behavior. Furthermore, for $\hat{w}_{j,\xi} \in \mathcal{L}_2$ we conclude $\hat{v}_{j,\xi} \in \mathcal{L}_2$. In summary, it follows that the integral (1.23) exists.

Hypothesis 1.6. Let (w, c) be a set of traveling waves with $w_j \in \mathcal{C}_b^2$, $j = 1, \dots, N$ that satisfy for some constants $C_\eta, \eta > 0$ and $j = 1, \dots, N$ the following estimates

$$c_1 < c_2 < \dots < c_N, \quad (1.24)$$

$$\|w_j(\xi) - w_j^+\| \leq C_\eta e^{-\eta\xi}, \quad \xi \in \mathbb{R}_+ \quad (1.25)$$

$$\|w_j(\xi) - w_j^-\| \leq C_\eta e^{\eta\xi}, \quad \xi \in \mathbb{R}_- \quad (1.26)$$

$$\|w_{j,\xi}(\xi)\| + \|w_{j,\xi\xi}(\xi)\| + \|w_{j,\xi\xi\xi}(\xi)\| \leq C_\eta e^{-\eta|\xi|}, \quad \xi \in \mathbb{R} \quad (1.27)$$

$$w_k^+ = w_{k+1}^-, \quad k = 1, \dots, N-1. \quad (1.28)$$

Remark 1.7. Instead of exponential decay it is sufficient to have $w_j \in \mathcal{C}_b^2$ and $w_j(\xi) \rightarrow w_j^\pm$ as $\xi \rightarrow \pm\infty$ for $j = 1, \dots, N$. Together with Hypothesis 1.9 below

this will imply exponential decay (1.25) - (1.27), see Remark 3.19 following Lemma 3.18.

Since the functions w_j for $j = 1, \dots, N$ are traveling waves, they solve the following stationary equation

$$0 = Aw_{j,\xi\xi} + c_j w_{j,\xi} + f(w_j). \quad (1.29)$$

The linearization of the right hand side of (1.29) at the traveling wave profile (w_j, c_j) is given by

$$\Lambda_j v = Av_{\xi\xi} + B_j v_{\xi} + C_j v \quad (1.30)$$

with

$$B_j = c_j I, \quad C_j(\xi) = Df(w_j(\xi)).$$

Note that C_j converges as $\xi \rightarrow \pm\infty$ to

$$\lim_{\xi \rightarrow \pm\infty} C_j(\xi) = Df(w_j^{\pm}) =: C_{j,\pm}.$$

Using Hypotheses 1.4 and 1.6 we obtain that there exist constants $\bar{B}, \bar{C} > 0$ such that for all $\xi \in \mathbb{R}, j = 1, \dots, N$ holds

$$\|B_j\| \leq \bar{B}, \quad \|C_j(\xi)\| \leq \bar{C}. \quad (1.31)$$

Let $j \in \{1, \dots, N\}$, the function $w_j(\cdot + q), q \in \mathbb{R}$ is also a solution of (1.29): We insert $w_j(\cdot + q)$ into (1.29) and differentiate w.r.t. q at $q = 0$, then we conclude that the function $w_{j,\xi}$ is in the null space of Λ_j . The following eigenvalue and spectral conditions are the main assumptions on the operator $\Lambda_j, j = 1, \dots, N$ to obtain a stability result, compare e.g. [6], [40], [13]:

Hypothesis 1.8 (Eigenvalue condition). *For $j = 1, \dots, N$ the function $w_{j,\xi}$ spans the null space of Λ_j in \mathcal{L}_2 and the eigenvalue 0 of Λ_j is algebraically simple. There exists $\bar{\kappa} > 0$ such that for all $j = 1 \dots, N$ there is no other isolated eigenvalue s of the operators Λ_j of finite multiplicity with $\Re s \geq -\bar{\kappa}$.*

Hypothesis 1.9 (Spectral condition). *There exists $\sigma, \bar{\kappa} > 0$ such that for s with $\Re s \geq -\bar{\kappa}$ the solutions λ of the quadratic eigenvalue problems*

$$\det(\lambda^2 A + \lambda B_j + C_{j,\pm} - sI) = 0 \quad (1.32)$$

for some $j = 1, \dots, N$ satisfy: $|\Re \lambda| \geq \sigma$.

The spectral condition 1.9 ensures that the essential spectrum $\sigma_{ess}(\Lambda_j)$ is contained in the left half plane, compare [36], Theorem 1.3 or [17], Chapter 5, Theorem A.2. From Hypothesis 1.8 we obtain that the point spectrum of $\Lambda_j, j = 1, \dots, N$, i.e. all isolated eigenvalues of finite multiplicity, except for the eigenvalue 0 have real part less than $-\bar{\kappa} < 0$.

We impose some conditions on the bump function φ :

Hypothesis 1.10. *There exist constants $0 \leq C_\varphi, C_0, C_1, C_\beta$ and $\beta > 0$ such that the function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies*

$$0 < \varphi(\xi) \leq C_\varphi \quad \forall \xi \in \mathbb{R}, \quad (1.33)$$

$$C_0 e^{-\beta|\xi|} \leq \varphi(\xi) \leq C_1 e^{-\beta|\xi|}, \quad \xi \in \mathbb{R}. \quad (1.34)$$

and the derivative of the bump function satisfies

$$|\varphi'(\xi)| \leq C_\beta e^{-\beta|\xi|}, \quad \xi \in \mathbb{R}. \quad (1.35)$$

A typical function φ that satisfies (1.33) - (1.35) is $\varphi(\xi) = \operatorname{sech}(\beta\xi) = \frac{2}{e^{-\beta\xi} + e^{\beta\xi}}$ for some small $\beta > 0$. The numerical experiments (see Chapter 2) will show that non-smooth bump functions such as $\varphi(\xi) = e^{-\beta|\xi|}$, $\beta > 0$ work equally well.

We assume that the initial conditions satisfy $g_1^0 < \dots < g_N^0$ and we denote the minimal distance by $G^0 = \min_{k \in \{1, \dots, N-1\}} |g_{k+1}^0 - g_k^0|$.

Recall the elements \tilde{w}_j^- in (1.9) and the coupled PDAE system (1.12), (1.13), (1.17), (1.18) for $j = 1, \dots, N$ and $t \geq 0, \xi \in \mathbb{R}$

$$\begin{aligned} v_{j,t}(\xi, t) = & A v_{j,\xi\xi}(\xi, t) + v_{j,\xi}(\xi, t) \mu_j(t) + f(v_j(\xi, t) + \tilde{w}_j^-) + Q_j^{g(t)}(\xi) \\ & \cdot \left[f \left(\sum_{k=1}^N v_k(\xi_{k,j}^{g(t)}, t) \right) - \sum_{k=1}^N f \left(v_k(\xi_{k,j}^{g(t)}, t) + \tilde{w}_k^- \right) \right], \quad v_j(\xi, 0) = v_j^0, \end{aligned} \quad (1.36)$$

$$g_{j,t}(t) = \mu_j(t), \quad g_j(0) = g_j^0, \quad (1.37)$$

$$0 = \langle v_j(t) - \hat{v}_j, \hat{v}_{j,\xi} \rangle. \quad (1.38)$$

Before we present the main Stability Theorem 1.13, we introduce the following definition of a solution of the coupled PDAE system (1.36) - (1.38). Note that this is a modified version of the solution concept used in [23].

Definition 1.11. *Let $b \geq 0, \tau \in (0, \infty]$ be given. For $j = 1, \dots, N$ let Λ_j be sectorial operators in $\mathcal{L}_{2,b}$ with $\mathcal{D}(\Lambda_j) = \mathcal{H}^{2,b}$, $\hat{\psi}_j \in \mathcal{L}_{2,\eta_j}$, $\eta_j \in \mathbb{R}$ and $k_j : [0, \tau) \times (\mathcal{H}^{1,b})^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathcal{L}_{2,b}$. Then $(v, g, \mu) : [0, \tau) \rightarrow (\mathcal{H}^{1,b})^N \times \mathbb{R}^N \times \mathbb{R}^N$ is called a solution of the system*

$$v_{j,t}(t) = \Lambda_j v_j(t) + k_j(t, v(t), g(t), \mu_j(t)), \quad v_j(0) = v_j^0 \in \mathcal{H}^{1,b}, \quad j = 1, \dots, N,$$

$$g_{j,t}(t) = \mu_j(t), \quad g_j(0) = g_j^0,$$

$$\eta_j = \langle \hat{\psi}_j, v_j(t) \rangle$$

in $[0, \tau)$ if the following conditions are satisfied for each $j = 1, \dots, N$:

1. $k_j(\cdot, v(\cdot), g(\cdot), \mu_j(\cdot)) : [0, \tau) \rightarrow \mathcal{L}_{2,b}$ is continuous,
2. $v_j : [0, \tau) \rightarrow \mathcal{H}^{1,b}$ is continuous, $v_j(t) \in \mathcal{H}^{2,b}$ for $t \in (0, \tau)$ and $v_j(0) = v_j^0$,
3. g_j is continuously differentiable in $(0, \tau)$, $g_{j,t}(t) = \mu_j(t)$ for $t \in (0, \tau)$ and $g_j(0) = g_j^0$,
4. μ_j is continuous in $[0, \tau)$,
5. $v_{j,t}(t) \in \mathcal{L}_{2,b}$ exists and $v_{j,t}(t) = \Lambda_j v_j(t) + k_j(t, v(t), g(t), \mu_j(t))$ for $t \in (0, \tau)$,
6. $\langle \hat{\psi}_j, v_j(t) \rangle = \eta_j$ for all $t \in [0, \tau)$.

To characterize the long time behavior of the solution of the coupled PDAE system (1.36) - (1.38) we give the following definition:

Definition 1.12 (Joint asymptotic stability). *The waves (w, c) are called jointly asymptotically stable with respect to the norm $\|\cdot\|_{\mathcal{H}^{1,b}}$ in the Banach space $\mathcal{H}^{1,b}$, if for each $\varepsilon > 0$ there exists $G^0, \delta > 0$ such that for each solution (v, g, μ) of (1.36) - (1.38) with $v_j(\cdot, 0) \in \mathcal{H}^{1,b}$, $j = 1, \dots, N$, $g_1^0 < \dots < g_N^0$ and*

$$\begin{aligned} \|v(\cdot, 0) - \hat{w}\|_{\mathcal{H}^{1,b}} + \|\mu(0) - c\| &\leq \delta, \\ |g_j^0 - g_i^0| &\geq G^0, \quad i, j = 1, \dots, N, j \neq i \end{aligned}$$

there exist phase shifts $\tau_j \in \mathbb{R}$, $j = 1, \dots, N$ such that for all $j = 1, \dots, N$

$$\|v_j(t) - \hat{w}_j\|_{\mathcal{H}^{1,b}} + |\mu_j(t) - c_j| + |g_j(t) - c_j t - g_j^0 - \tau_j| \leq \varepsilon \quad \forall t \geq 0$$

and

$$\|v_j(t) - \hat{w}_j\|_{\mathcal{H}^{1,b}} + |\mu_j(t) - c_j| + |g_j(t) - c_j t - g_j^0 - \tau_j| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

We can now formulate the following main stability result:

Theorem 1.13 (Stability Theorem). *Assume that Hypotheses 1.4 and 1.10 hold. Let (w, c) be a set of traveling waves that satisfies Hypotheses 1.5, 1.6, 1.8 and 1.9. Then the waves (w, c) are jointly asymptotically stable with respect to $\|\cdot\|_{\mathcal{H}^{1,b}}$. More precisely, there exist $b > 0$ and $G^0, \delta > 0$ such that for (v^0, g^0) with*

$$\|v^0 - \hat{w}\|_{\mathcal{H}^{1,b}} \leq \delta, \quad \langle \hat{v}_{j,\xi}, v_j^0 - \hat{w}_j \rangle = 0, \quad j = 1, \dots, N$$

and

$$g_1^0 < g_2^0 < \dots < g_N^0, \quad G^0 \leq |g_j^0 - g_i^0|, \quad j \neq i,$$

there exists a unique solution $(v(t), g(t), \mu(t))$ of (1.36) - (1.38) on $[0, \infty)$ and the following exponential estimate is satisfied for some $C, \nu, \gamma > 0$, $\tau_j \in \mathbb{R}$ and $j = 1, \dots, N$, $t \geq 0$

$$\|v_j(t) - \hat{w}_j\|_{\mathcal{H}^{1,b}} + |g_j(t) - c_j t - g_j^0 - \tau_j| + |\mu_j(t) - c_j| \leq C e^{-\nu t} (\|v^0 - \hat{w}\|_{\mathcal{H}^{1,b}} + e^{-\gamma G^0}). \quad (1.39)$$

Remark 1.14.

1. An important point to note here is that the estimates in the $\mathcal{H}^{1,b}$ norm are stronger than in the Sobolev space \mathcal{H}^1 norm. To obtain the estimate (1.39) we have to assume that the differences of the initial functions and the shifted traveling waves lie in the weighted space $\mathcal{H}^{1,b}$.
2. The proof will show how the constants b, ν, γ, C depend on the parameters β, η and on the operator $\Lambda_j, j = 1, \dots, N$.

The goal of this thesis is to show that the 'decompose and freeze method' can be implemented numerically and to show that the single profiles assume asymptotically the shape of the suitably shifted traveling waves \hat{w}_j . We emphasize that the stability theorem yields estimates for each component of the superposition W defined by (1.5). As a by-product we obtain the result of the following corollary:

Corollary 1.15. *Let the assumptions of Theorem 1.13 hold.*

There exist $b > 0$ and $G^0, \delta > 0$ such that that $u(t)$, given by (1.10), is a solution of the PDE $u_t = Au_{xx} + f(u)$ with initial data $u(x, 0) = u^0(x) := \sum_{j=1}^N v_j^0(x - g_j^0)$ on $[0, \infty)$ if (v^0, g^0) satisfies

$$\|v^0 - \hat{w}\|_{\mathcal{H}^{1,b}} \leq \delta, \quad \langle \hat{v}_{j,\xi}, v_j^0 - \hat{w}_j \rangle = 0, \quad j = 1, \dots, N$$

and

$$g_1^0 < g_2^0 < \dots < g_N^0, \quad G^0 \leq |g_j^0 - g_i^0|, \quad j \neq i.$$

Furthermore, there exists $\tau = (\tau_1, \dots, \tau_N) \in \mathbb{R}^N$ such that the following estimate is satisfied for some $C, \nu > 0$ and $t \geq 0$

$$\|u(t) - W^{g^0 + \tau}(t)\|_{\mathcal{H}^{1,b}} \leq Ce^{-\nu t} (\|v^0 - \hat{w}\|_{\mathcal{H}^{1,b}} + e^{-\gamma G^0}). \quad (1.40)$$

Remark 1.16. *A result of this type (1.40) for a multipulse consisting of two pulses has been proven in [40], Theorem 4. The proof in [40] uses a decomposition that requires explicit knowledge of the single waves w_j . Therefore, it cannot be employed directly for numerical computations. We emphasize that the PDAE approach proposed here aims at a system of equations which can be solved numerically and which provides access to all single waves that form the multistructure by superposition.*

Chapter 2

Numerical applications - Weak interaction

We test the 'decompose and freeze method' on several well known examples which possess multipulse and multifront solutions for the weak interaction case, where the single pulses or fronts are well separated in space and interact only through their small tails. We illustrate our results on the Nagumo-equation [20], the FitzHugh-Nagumo-equations [24], and the three component system introduced by [16].

We consider the case of a multipulse or multifront consisting of two profiles, i.e. $N = 2$. We recall the coupled PDAE system (1.36) - (1.38) for the case $N = 2$ and $t \geq 0, \xi \in \mathbb{R}$, we set $dg := g_2 - g_1$ and obtain

$$v_{1,t} = Av_{1,\xi\xi} + v_{1,\xi}\mu_1 + f(v_1) + \frac{\varphi}{\varphi + \varphi(\cdot - dg)} \\ * [f(v_1 + v_2(\cdot - dg)) - f(v_1) - f(v_2(\cdot - dg) + \tilde{w}_2^-)], \quad v_1(0) = v_1^0, \quad (2.1)$$

$$v_{2,t} = Av_{2,\xi\xi} + v_{2,\xi}\mu_2 + f(v_2 + \tilde{w}_2^-) + \frac{\varphi}{\varphi + \varphi(\cdot + dg)} \\ * [f(v_1(\cdot + dg) + v_2) - f(v_1(\cdot + dg)) - f(v_2 + \tilde{w}_2^-)], \quad v_2(0) = v_2^0, \quad (2.2)$$

$$g_{j,t} = \mu_j, \quad g_j(0) = g_j^0, \quad j = 1, 2 \quad (2.3)$$

$$0 = \langle v_j - \hat{v}_j, \hat{v}_{j,\xi} \rangle, \quad j = 1, 2. \quad (2.4)$$

We solve the system (2.1) - (2.4) on a finite spatial computational domain $[-L, L]$ and use Neumann boundary conditions, i.e. $v_{j,\xi}(\pm L) = 0$. Since we consider the system (2.1) - (2.4) on a finite interval, we cannot expect the (v_j, μ_j) to converge towards the (\hat{w}_j, c_j) how it was shown in the Stability Theorem 1.13. Instead we assume that the (v_j, μ_j) converge to an approximation $(\hat{w}_{j,L}, c_{j,L})$ which solve the following system of stationary boundary value problems on the

interval $[-L, L]$

$$0 = A(\hat{w}_{j,L})_{\xi\xi} + c_{j,L}(\hat{w}_{j,L})_{\xi} + f(\hat{w}_{j,L} + \tilde{w}_j^-) \quad (2.5)$$

$$0 = R(\hat{w}_{j,L}(-L), \hat{w}_{j,L}(L)) \quad (2.6)$$

for $j = 1, \dots, N$, where (2.6) denotes the boundary condition for the stationary boundary value problem.

We proceed similarly to the numerical applications of [4] to solve (2.1) - (2.4). We use the finite element package Comsol MultiphysicsTM [1] with second order elements in space. In time we apply a BDF method with the absolute tolerance 10^{-4} and relative tolerance 10^{-2} .

In the examples below we will specify the initial values v_j^0, g_j^0 . The initial values will be add up such that the multipulse or multifront starts with the initial function

$$u^0(x) = \sum_{j=1}^N v_j^0(x - g_j^0), \quad x \in \mathbb{R}. \quad (2.7)$$

As reference functions \hat{v}_j we use the initial functions v_j^0 . Recall that we have nonlocal terms in the nonlinearity f . Therefore we will interpolate them inside the computational domain $[-L, L]$ and extrapolate them outside this interval with the constant boundary values $v_j(\pm L)$. We use the bump function $\varphi(x) = \frac{2}{e^{\beta x} + e^{-\beta x}}, x \in \mathbb{R}$ with $\beta = 0.5$. We will demonstrate that certain other bump functions may be used as well and that the computation gives quite similar results.

2.1 The Nagumo-equation

One standard example of a traveling wave is the Nagumo-equation (cf. [20])

$$u_t = u_{xx} + u(1 - u)(u - a), \quad u(x, t) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0, \quad a \in \left(0, \frac{1}{2}\right). \quad (2.8)$$

An explicit traveling wave solution $u_1(x, t) = w_1(x - c_1 t)$ connecting $w_1^- = 0$ and $w_1^+ = 1$ is given by

$$w_1(x) = \left(1 + e^{\frac{-x}{\sqrt{2}}}\right)^{-1}, \quad c_1 = -\sqrt{2} \left(\frac{1}{2} - a\right)$$

and a traveling wave solution $u_2(x, t) = w_2(x - c_2 t)$ connecting $w_2^- = 1$ and $w_2^+ = 0$

is given by

$$w_2(x) = 1 - \left(1 + e^{\frac{-x}{\sqrt{2}}}\right)^{-1}, \quad c_2 = \sqrt{2} \left(\frac{1}{2} - a\right).$$

In the following example we choose $a = 0.25$ and the computational domain $[-L, L]$ with $L = 50$. We set initial conditions $g_1^0 = -50, g_2^0 = 50$ and spatial step size $\Delta\xi = 0.1$.

Figure 2.1 shows the superposition

$$u_L(x, t) = v_1(x - g_1(t), t) + v_2(x - g_2(t), t) \quad (2.9)$$

together with the velocities $\mu_j, j = 1, 2$ as a function of time. The darker shaded domains show the intervals $g_j(t) + [-L, L]$, where v_j contributes to the superposition. By a slight abuse of notation we will call $g_j + [-L, L]$ the support of the function v_j . The lighter shaded domains of the superposition u_L indicate that extrapolation with the boundary values of v_j has been used. The single frozen profiles $v_j, j = 1, 2$ are displayed in Figure 2.2.

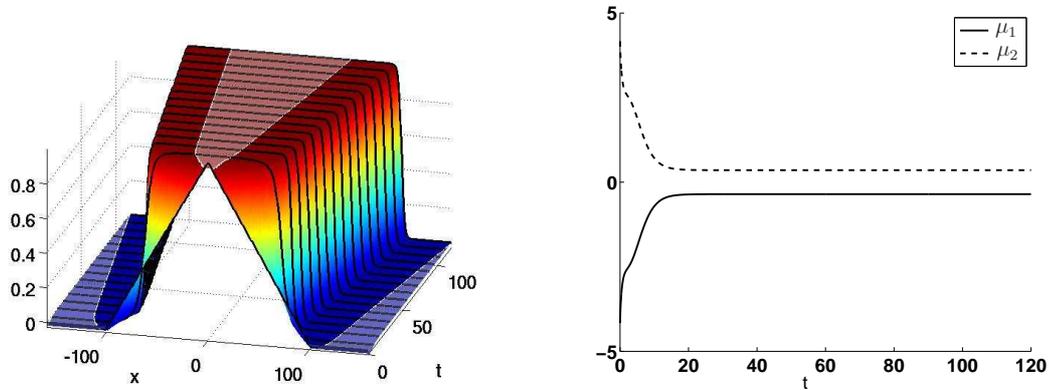


Figure 2.1: Fronts moving in opposite directions in the Nagumo-equation, evolution of superposition u_L and velocities μ_1, μ_2 .

As a result we see that after a short time the frozen profiles v_j stabilize and the superposition u_L gets a broadening plateau moving with opposite velocities μ_j to the left and to the right. The velocities μ_j converge after a short transient period. In this computation we have used Neumann boundary conditions. A simulation with Dirichlet boundary conditions, i.e. $v_1(-L) = 0, v_1(L) = 1, v_2(-L) = 0, v_2(L) = -1$, yields almost identical results.

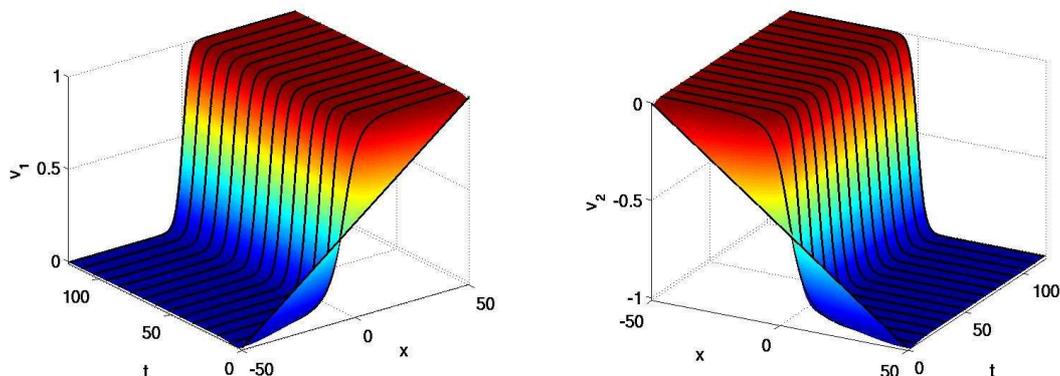


Figure 2.2: Fronts moving in opposite directions in the Nagumo-equation, evolution of frozen v_1, v_2 .

Let u_l be the numerical solution of the Nagumo-equation (2.8) on a sufficiently large interval. We compare the superposition u_L with the solution u_l . The comparison in absolute values is displayed in the left picture of Figure 2.3 and in the \mathcal{L}_2 -norm, i.e. $\text{dist} = \|u_L(\cdot, t) - u_l(\cdot, t)\|_{\mathcal{L}_2}$, as function of time is pictured in the right picture. We see in the left picture that the two solutions almost agree except for a small domain near the single fronts. The right picture shows that the \mathcal{L}_2 -distance becomes constant which is caused by a single phase shift. As explained in [4] the condition $\langle \hat{w}_j - \hat{v}_j, \hat{v}_{j,\xi} \rangle_{\mathcal{L}_2} = 0$ is not satisfied, but there exists $\delta_j \in \mathbb{R}$ with $\langle \hat{w}_j(\cdot - \delta_j) - \hat{v}_j, \hat{v}_{j,\xi} \rangle_{\mathcal{L}_2} = 0$. The traveling waves $(w_j(\cdot - \delta_j), c_j)$ satisfy the assumption of the Stability Theorem 1.13, therefore we obtain the convergence

$$\|\hat{w}_j(\cdot - \delta_j) - v_j(\cdot, t)\|_{\mathcal{L}_2} \rightarrow 0, \quad |c_j - \mu_j| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

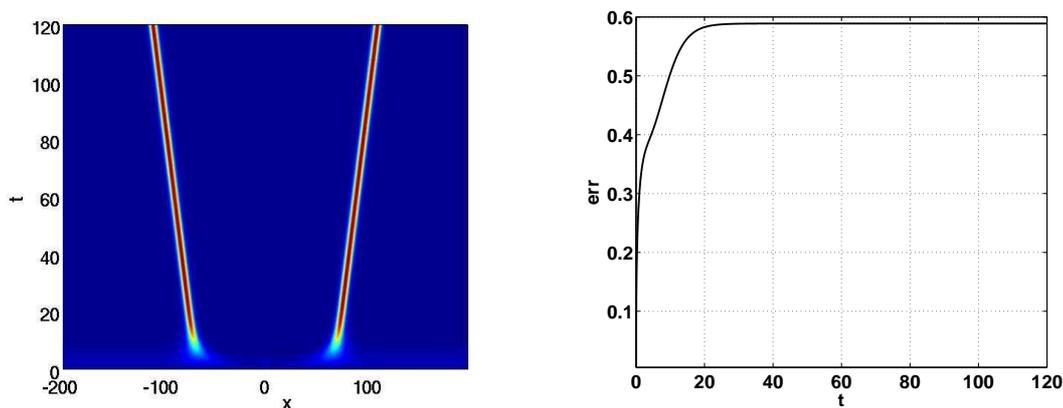


Figure 2.3: Evolution of the absolute-error and the \mathcal{L}_2 -error for a double front moving in opposite directions in the Nagumo-equation.

In order to investigate the influence of the chosen bump function φ , we perform the above numerical computations with two alternative bump functions $\tilde{\varphi}(\xi) = \exp(-0.5|\xi|)$ and $\bar{\varphi}(\xi) = \exp(-0.05\xi^2)$, see Figure 2.4.

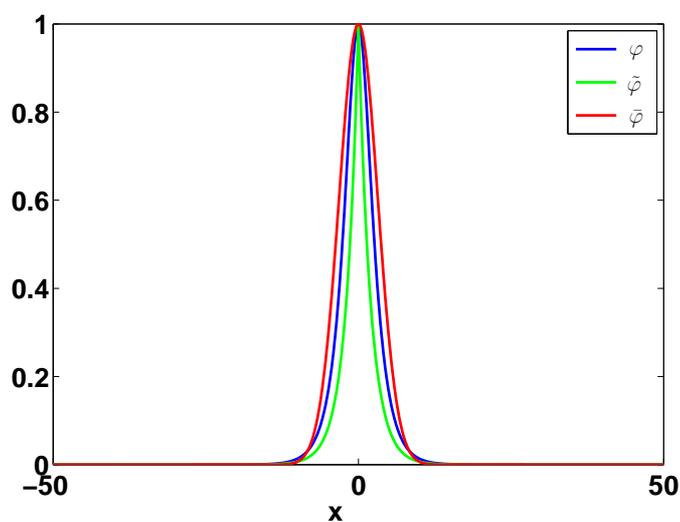


Figure 2.4: Different bump functions $\varphi(\xi) = \text{sech}(0.5\xi)$, $\tilde{\varphi}(\xi) = \exp(-0.5|\xi|)$ and $\bar{\varphi}(\xi) = \exp(-0.05\xi^2)$.

Note that $\tilde{\varphi}$ and $\bar{\varphi}$ do not satisfy all conditions of Hypothesis 1.10. The results nearly agree with the ones pictured in Figures 2.1 and 2.2. Figure 2.5 compares the evolution of the time derivatives of $\|u_t\|_{\mathcal{L}_2}$ and $\|\mu_t\|$ in a logarithmic scale for the different bump functions. (In the following we omit the \mathcal{L}_2 -symbol and write $\|u_t\|$ instead of $\|u_t\|_{\mathcal{L}_2}$). All time derivatives decay exponentially fast and the rate of decay is almost identical.

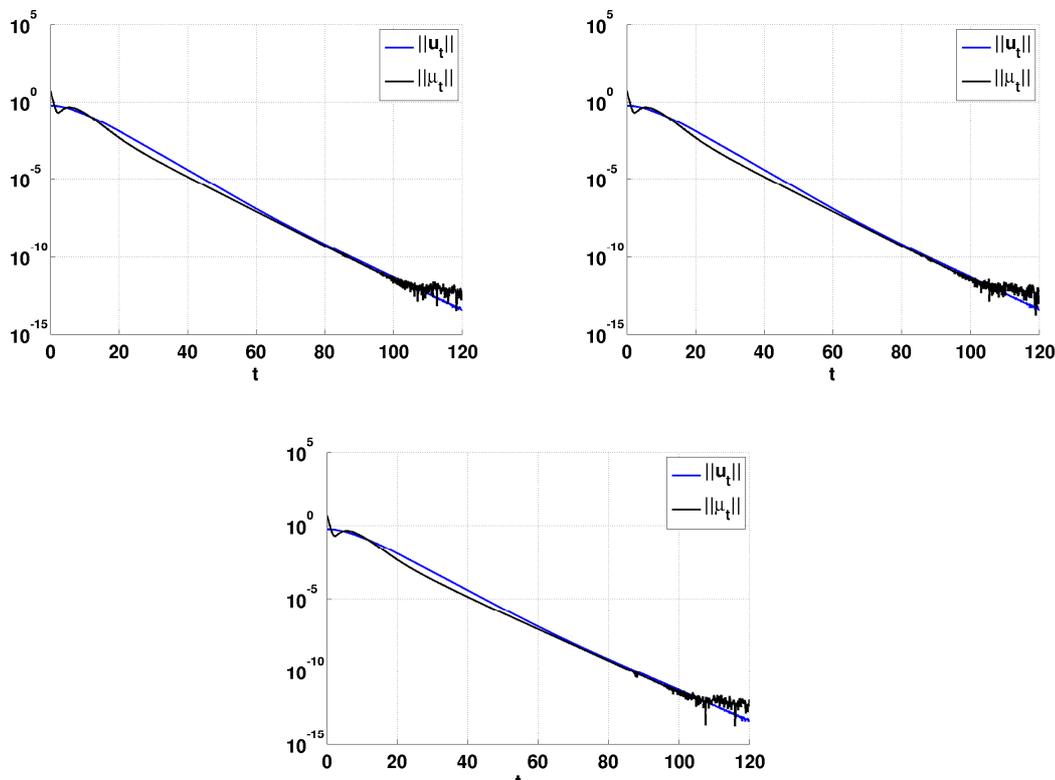


Figure 2.5: Fronts moving in opposite direction in the Nagumo-equation: $\|u_t\|$ and $\|\mu_t\|$ (logarithmic scale) for different bump functions $\varphi(\xi) = \text{sech}(0.5\xi)$ (left), $\tilde{\varphi}(\xi) = \exp(-0.5|\xi|)$ (right) and $\tilde{\varphi}(\xi) = \exp(-0.05\xi^2)$ (bottom).

2.2 FitzHugh-Nagumo-equations

As our second example choose the the well-known FitzHugh-Nagumo-equations, see [15],

$$V_t = V_{xx} + V - \frac{1}{3}V^3 - R, \quad (2.10)$$

$$R_t = \gamma(V + a - bR). \quad (2.11)$$

The component V is called the activator and the component R is called the inhibitor. The FitzHugh-Nagumo-equations model nerve conduction. We use the parameters $a = 0.7, b = 0.8, \gamma = 0.08$ for which traveling multipulse solution exist, see [24]. Note that in the inhibitor component the diffusion term is missing, i.e. the FitzHugh-Nagumo-equations is a mixed parabolic-hyperbolic system, so the theory does not apply.

For the numerical computation we use the interval $[-L, L]$ with $L = 70$, relative tolerance 10^{-6} and absolute tolerance $2 * 10^{-7}$. We set the initial data $g_1^0 = g_2^0 = 0$, the spatial step size $\Delta\xi = 0.2$ and employ Neumann boundary conditions.

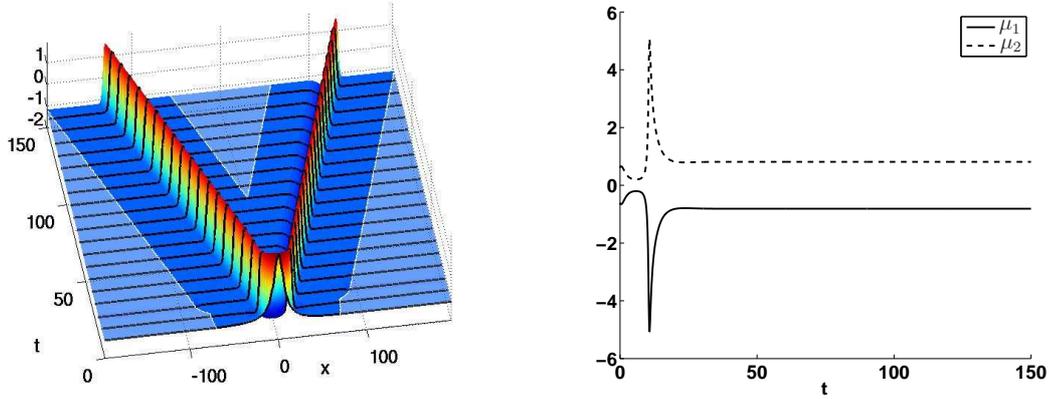


Figure 2.6: Splitting of a single pulse into a two-pulses in the FitzHugh-Nagumo-equations, evolution of V_L and of the velocities μ_1 and μ_2 .

Figure 2.6 shows the time evolution of the first component of the sum $u_L = (V_L, R_L)^T$ defined by (2.9), and the evolution of the velocities $\mu_j, j = 1, 2$. The initial profile of the component V_L is shown in the figure, the one of the R_L component is given by the stationary value $\bar{R} = -0.62426$. The initial pulse splits into two pulses moving with opposite velocities μ_1 and μ_2 to the left and to the right. As in Figure 2.1 the darker shaded domains show the supports $g_j + [-L, L]$ of the profiles V_1 and V_2 and the lighter shaded domains show the extrapolated boundary values of V_1 and V_2 . The velocities converge very fast resulting in opposite values $\mu_1 = -\mu_2$ and μ_1 converges to -0.8118 .

In Figure 2.7, the frozen profiles V_j in the comoving frame are displayed as functions of time. We see that at the moment of separation small additional pulses appear which decay and vanish in time. When these small pulses have decayed the profiles $V_j, j = 1, 2$ rapidly become stationary.

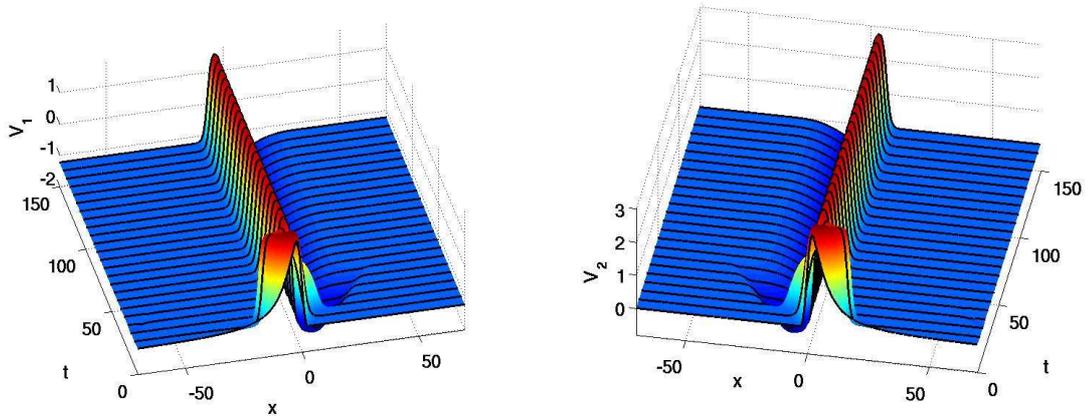


Figure 2.7: Splitting of the V_L component into a two-pulses in the FitzHugh-Nagumo-equations, evolution of the frozen pulses V_1 and V_2 .

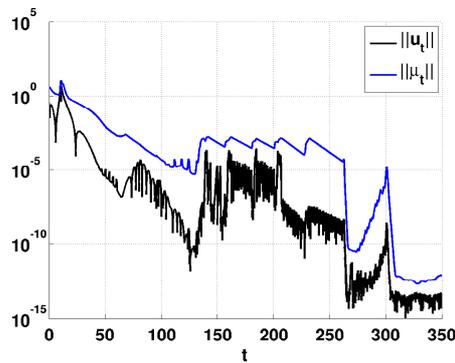


Figure 2.8: Splitting of a single pulse into a two-pulses in the FitzHugh-Nagumo-equations: rates of decay $\|(V_t, R_t)^T\|$ and $\|\mu_t\|$ (logarithmic scale).

Figure 2.8 shows the rate of decay of the solution u_L and of the velocities μ_j in a logarithmic scale. Although the time derivatives $\|u_t\| := \|(V_t, R_t)^T\|$ and $\|\mu_t\|$ do not decay exponentially fast we conclude that the profiles $v_j = (V_j, R_j)$ and the velocities μ_j become stationary as we have already seen in Figure 2.7 and 2.6.

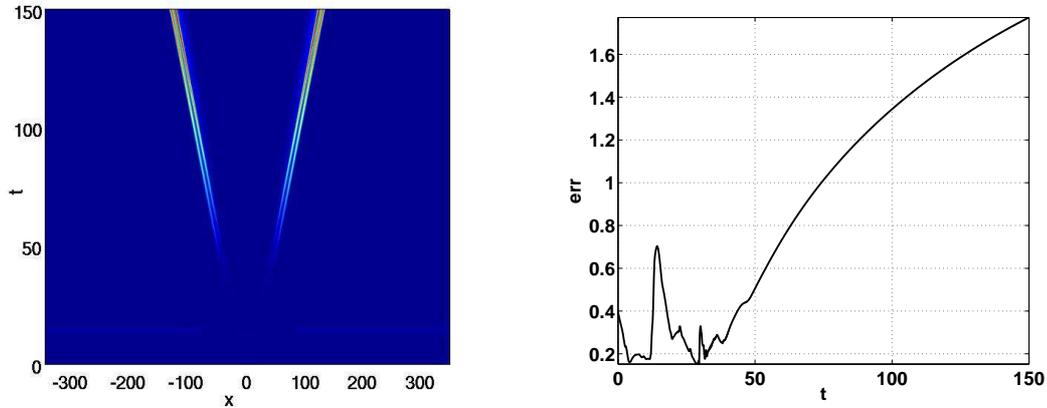


Figure 2.9: Splitting of a single pulse into a two-pulses in the FitzHugh-Nagumo-equations, difference of the the superposition $u_L = (V_L, R_L)^T$ and the two-pulse computed on a large domain.

In contrast to the Nagumo example above, the absolute value-distance and \mathcal{L}_2 -distance between the superposition $u_L = (V_L, R_L)^T$ and the solution of the FitzHugh-Nagumo-equations (2.10) - (2.11) solved on a large domain grows and a slight drift remains. The discussion in [4] suggests that this behavior is caused by the mixed-parabolic-hyperbolic character of system (2.10) - (2.11). If we add a small diffusion term in (2.11), e.g. $0.01R_{xx}$, the system becomes parabolic.

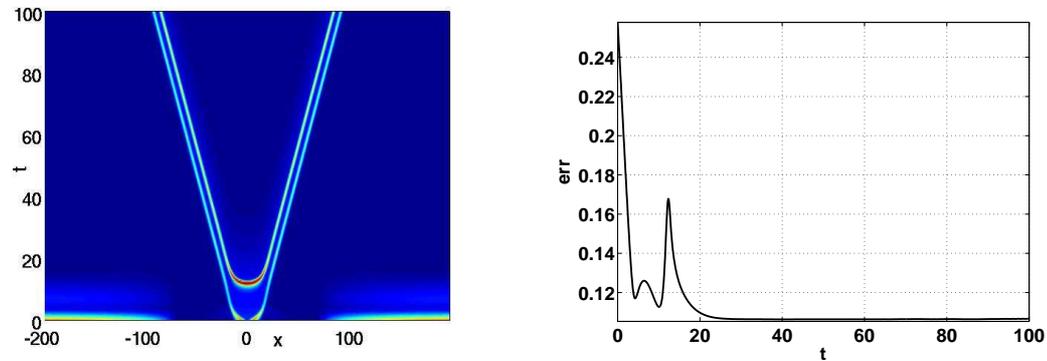


Figure 2.10: Splitting of a single pulse into a two-pulses in the FitzHugh-Nagumo-equations with extra diffusion term, difference of the the superposition $u_L = (V_L, R_L)^T$ and the two-pulse computed on a large domain.

In Figure 2.10, the absolute value-distance and the \mathcal{L}_2 -distance between the superposition u_L and the solution of the modified FitzHugh-Nagumo system (2.10) - (2.11) is displayed. The solutions agree except for a single domain close to the pulses and the \mathcal{L}_2 -distance becomes almost constant except for very small variation caused by the numerical discretization.

2.3 Three-component-system

As a third numerical example we consider the three-component system introduced in [32] and [16]. This system is a paradigm model, because it supports a rich variety of front, pulse and spot dynamics. There are extensive numerical simulations of this system, see [26], [7]. In addition, there is a theory on its qualitative behavior based on a singular perturbation analysis, see [11], [12]. We consider the three-component system

$$U_t = D_U U_{xx} + F(U) - \kappa_3 V - \kappa_4 Z + \kappa_1, \quad (2.12)$$

$$\tau V_t = D_V V_{xx} + U - V, \quad (2.13)$$

$$\theta Z_t = D_Z Z_{xx} + U - Z \quad (2.14)$$

in one space dimension. The system consists of the activator component $U(x, t)$ and the two inhibitor components $V(x, t), Z(x, t)$ with $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. The nonlinearity is defined as $F(U) = \lambda U - U^3$ with $\lambda > 0$. The diffusion coefficients D_U, D_V, D_Z are positive, the positive constants τ, θ denote the ratio of the characteristic times of both inhibitors. The parameter κ_1 has arbitrary sign and denotes the constant source term, whereas κ_3, κ_4 are positive and denote the reaction rates.

2.3.1 Scaling of the three-component-system

We consider the case $\tau = \theta$. The three-component-system (2.12) - (2.14) may be interpreted as a modulated FitzHugh-Nagumo system coupled with a second inhibitor component. In [12] the system is scaled to obtain the singular perturbation form, here we perform a different scaling that reveals similarities of this system to the FitzHugh-Nagumo system (2.10) - (2.11):

Let $F(U) = \lambda U - U^3$ with $\lambda = 3^{\frac{1}{3}}$ and define

$$\alpha = \frac{\kappa_3}{3^{\frac{1}{3}} \tau D_U}, \quad \beta = \frac{\kappa_4}{3^{\frac{1}{3}} \tau D_U}, \quad a = \frac{-\kappa_1 3^{\frac{1}{3}}}{\kappa_3 + \kappa_4}, \quad b = \frac{1}{\tau D_U}, \quad \gamma = \frac{D_U}{3^{\frac{1}{3}}}, \quad \delta_1 = \frac{D_V}{\tau D_U}, \quad \delta_2 = \frac{D_Z}{\tau D_U}.$$

We introduce the following scaling

$$\tilde{x} = \frac{1}{\sqrt{\gamma}} x, \quad \tilde{t} = 3^{\frac{1}{3}} t, \quad (\tilde{U}, \tilde{V}, \tilde{Z}) = \left(3^{\frac{1}{3}} U, \frac{a + 3^{\frac{1}{3}} V}{b}, \frac{a + 3^{\frac{1}{3}} Z}{b} \right)$$

and obtain the system

$$\tilde{U}_{\tilde{t}} = \tilde{U}_{\tilde{x}\tilde{x}} + \tilde{U} - \frac{1}{3} \tilde{U}^3 - \alpha \tilde{V} - \beta \tilde{Z}, \quad (2.15)$$

$$\tilde{V}_{\tilde{t}} = \delta_1 \tilde{V}_{\tilde{x}\tilde{x}} + \gamma \tilde{U} + \gamma a - \gamma b \tilde{V}, \quad (2.16)$$

$$\tilde{Z}_{\tilde{t}} = \delta_2 \tilde{V}_{\tilde{x}\tilde{x}} + \gamma \tilde{U} + \gamma a - \gamma b \tilde{Z}. \quad (2.17)$$

For the sake of readability we suppress the tilde-symbols.

For the numerical computation we use the parameters $\alpha = \beta = 0.5$, $\delta_1 = 0.1, \delta_2 = 0.6$ and $a = 0.7, b = 0.8, \gamma = 0.08$. We consider the finite interval $[-L, L]$ with $L = 300$ and spatial step size $\Delta\xi = 0.5$, fix the absolute tolerance $2 \cdot 10^{-7}$ and the relative tolerance 10^{-6} , use the initial data $g_1^0 = g_2^0 = 0$ and impose Neumann boundary conditions.

Figure 2.11 shows the time evolution of the components U_L and V_L of the sum $u_L = (U_L, V_L, Z_L)^T$ defined by (2.9). Analogously, Figure 2.12 shows the time evolution of the component Z_L of the sum $u_L = (U_L, V_L, Z_L)^T$ and the evolution of the velocities $\mu_j, j = 1, 2$ as functions of time. The initial profile of the component U is a little hump while the profiles of the V and the Z components are initially set to their stationary value $\bar{V} = \bar{Z} = -0.62426$. The initial pulse splits into a two-pulse moving with opposite velocities μ_1 and μ_2 to the left and to the right. As in Figure 2.1 the darker shaded domains show the moving profiles and the lighter shaded domains show the extrapolated boundary values.

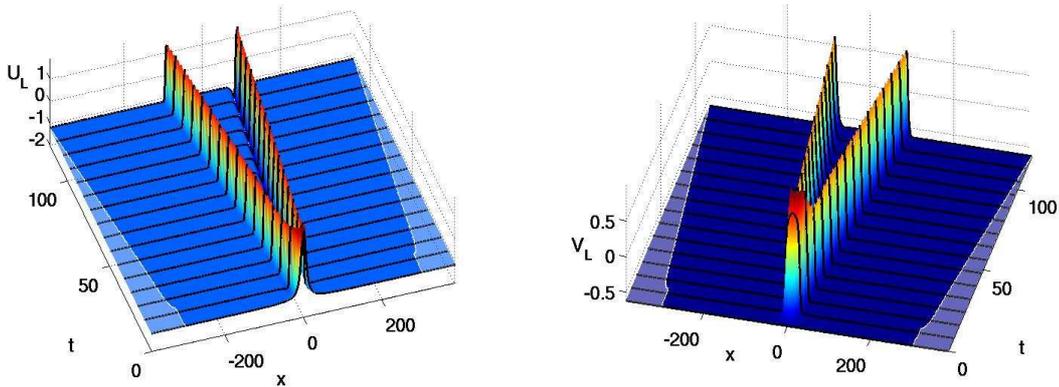


Figure 2.11: Two pulses moving in opposite directions in the three-component-system, evolution of U_L and V_L .

The following Figures 2.13, 2.14 and 2.15 display the frozen profiles U_j, V_j and Z_j in their comoving frame as function of time. In Figure 2.12 we see at the moment of separation that for all profiles small additional pulses appear which travel toward the boundary and decay. At the time around 450 this very small decaying pulses have reached the boundary and disappear. Therefore, all the profiles become stationary, also the the velocities converge very fast resulting in opposite values $\mu_1 = -\mu_2$.

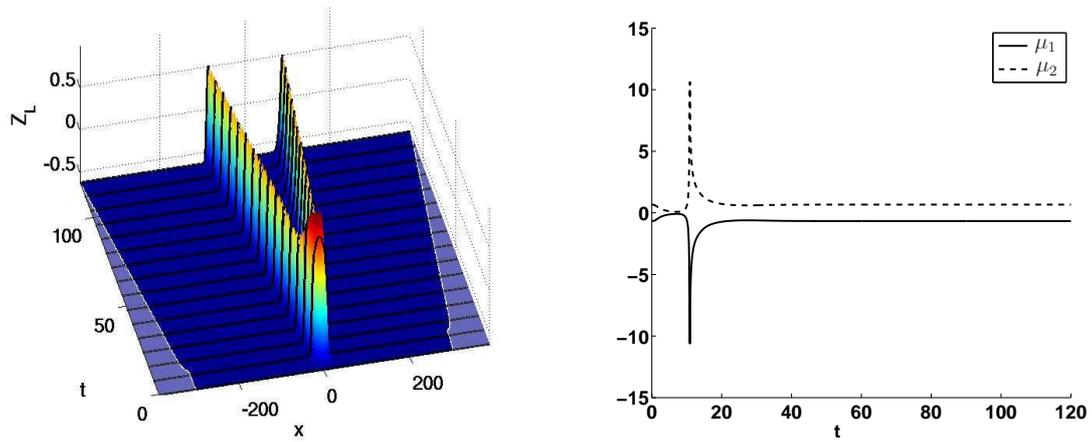


Figure 2.12: Two pulses moving in opposite directions in the three-component-system, evolution of Z_L and of the velocities μ_1 and μ_2 .

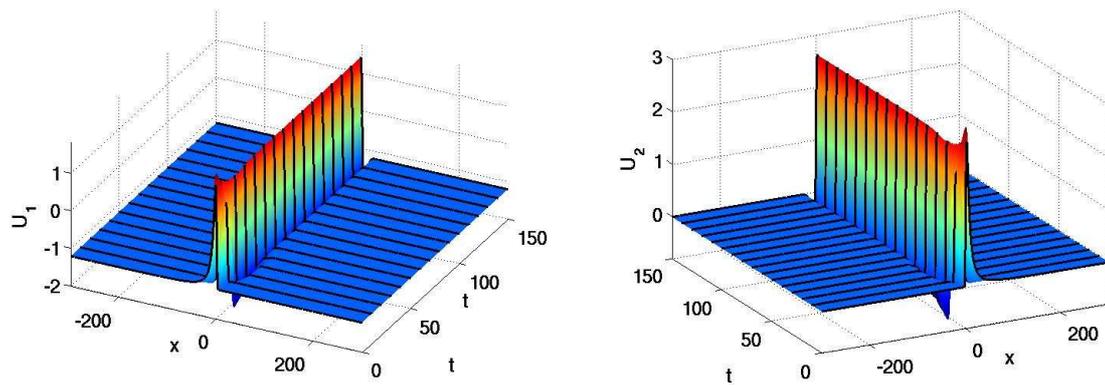


Figure 2.13: Two pulses moving in opposite directions in the three-component-system, evolution of the frozen pulses U_1 and U_2 .

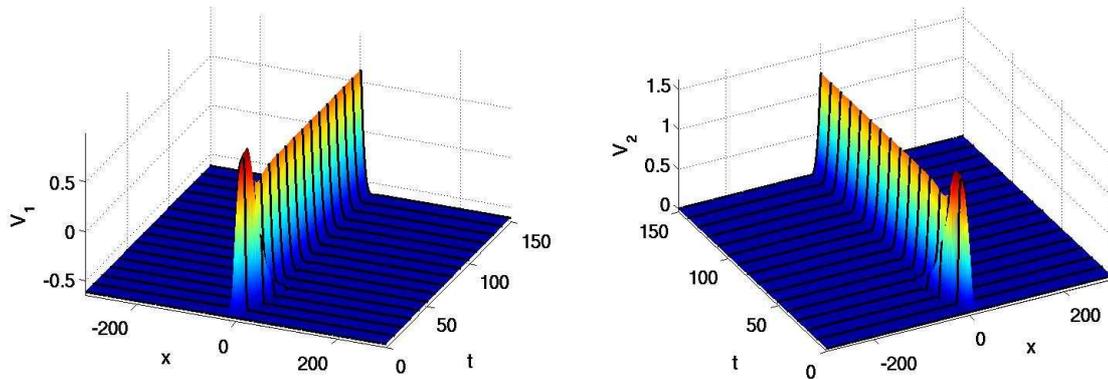


Figure 2.14: Two pulses moving in opposite directions in the three-component-system, evolution of the frozen pulses V_1 and V_2 .

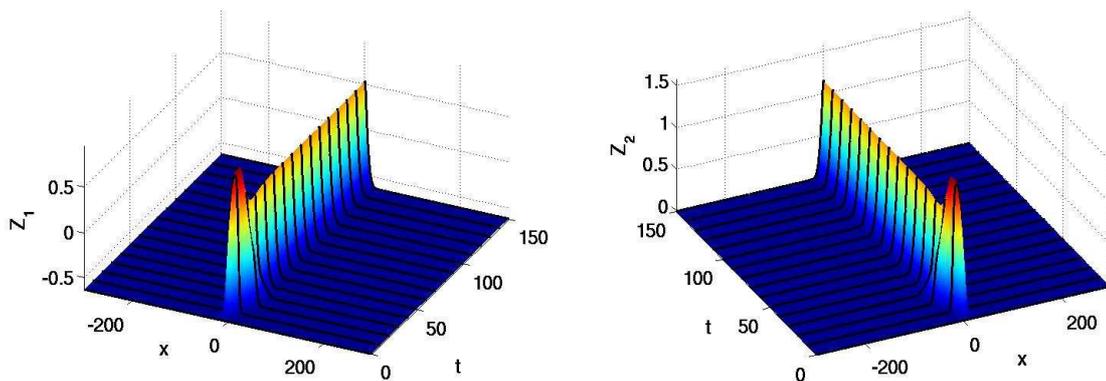


Figure 2.15: Two pulses moving in opposite directions in the three-component-system, evolution of the frozen pulses Z_1 and Z_2 .

As displayed in the left picture of Figure 2.16 we observe that on the logarithmic scale the time derivatives $\|u_t\| := \|(U_t, V_t, Z_t)^T\|$ and $\|\mu_t\|$ of the solution decay to some very small value in the initial phase. Remember that small additional pulses appear and travel toward the boundary for a certain time, in this time interval the time derivatives stay constant and after that period the time derivatives decay again. Therefore, we see that the single profiles of the system (2.15) - (2.17) obtained from the decompose and freeze method stabilize. In contrast, we consider the solution $u_l := (U_l, V_l, Z_l)^T$ of the system (2.15) - (2.17) solved on a large domain. We see that the time derivative $\|u_{l,t}\| := \|(U_{l,t}, V_{l,t}, Z_{l,t})^T\|$ on the logarithmic scale converges to a fixed positive value, compare the right picture of Figure 2.16.

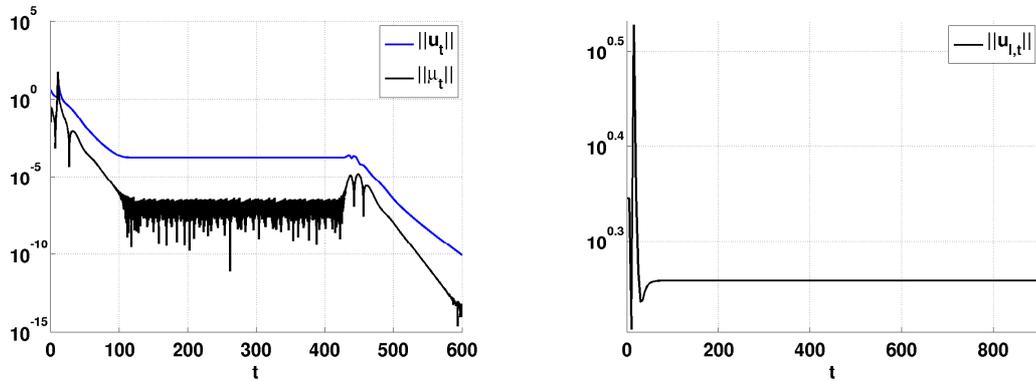


Figure 2.16: Two pulses moving in opposite directions in the three-component-system, rates of decay $\|(U_t, V_t, Z_t)^T\|$ and $\|\mu_t\|$ (on logarithmic scale) (left), rates of decay $\|(U_{l,t}, V_{l,t}, Z_{l,t})^T\|$ (on logarithmic scale) (right).

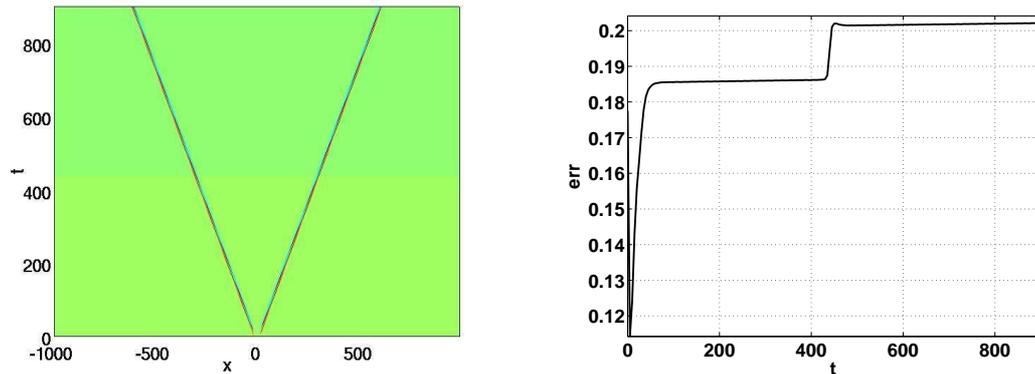


Figure 2.17: Two pulses moving in opposite directions in the three-component-system, difference of the the superposition $u_L = (U_L, V_L, Z_L)^T$ and the two-pulse computed on a large domain.

The left picture of Figure 2.17 shows the absolute value-distance of the superposition $u_L = (U_L, V_L, Z_L)^T$ and the solution $u_l := (U_l, V_l, Z_l)^T$ of the three-

component-system (2.12) - (2.14) solved on a large domain. The solutions agree except for small regions. Again we see the influence of the small additional pulses that travel toward the boundary. In the right picture of Figure 2.17 we consider the \mathcal{L}_2 -difference of $u_L = (U_L, V_L, Z_L)^T$ and $u_l := (U_l, V_l, Z_l)^T$. In the initial phase the \mathcal{L}_2 -difference seems to become constant. There is a little jump in the \mathcal{L}_2 -difference in the moment when the small additional pulses have reached the boundary ($t \approx 450$). Finally the L_2 -difference becomes almost constant, only some small variations remain. We believe that these variations are due to interpolation and boundary effects.

Chapter 3

Proof of the main stability theorem

The proof of the Stability Theorem 1.13 falls naturally into four parts. First we use the transformation $u_j = v_j - \hat{w}_j, r_j(t) = g_j(t) - c_j t - g_j^0, \lambda_j = \mu_j - c_j$ and get an equivalent formulation of the coupled system (1.36) - (1.38) which has zero as a stable solution. It is then sufficient to consider this system and we restate the Stability Theorem. In the next step we estimate the nonlinear coupling terms with respect to exponentially weighted norms. Third we consider the corresponding linear decoupled system. Using semigroup theory, the variation of constants formula we show resolvent estimates in the exponentially weighted spaces. In the last step we apply the estimates of the nonlinear terms to the coupled system and show existence, uniqueness and stability of the solution.

3.1 Transformation of nonlinear systems

We want to control small perturbations of the shifted traveling waves \hat{w}_j , the velocities c_j and the time-dependent position $c_j t + g_j^0$ for $j = 1, \dots, N$. For this reason we introduce new variables

$$u_j = v_j - \hat{w}_j, r_j(t) = g_j(t) - c_j t - g_j^0, \lambda_j = \mu_j - c_j, \quad j = 1, \dots, N, \quad t \geq 0. \quad (3.1)$$

Using this transformation we get an equivalent formulation of (1.36) - (1.38), namely

$$u_{j,t}(t) = \Lambda_j u_j(t) + \lambda_j(t) w_{j,\xi} + h_j(t, u(t), r(t), \lambda_j(t)), \quad u_j(0) = v_j^0 - \hat{w}_j =: u_j^0, \quad (3.2)$$

$$r_{j,t}(t) = \lambda_j(t), \quad r_j(0) = 0, \quad (3.3)$$

$$0 = \langle \hat{v}_{j,\xi}, u_j(t) \rangle \quad (3.4)$$

for $j = 1, \dots, N$ and $t \geq 0$, the argument ξ is suppressed. Here Λ_j is the linearization defined by (1.30) and

$$\begin{aligned} h_j(t, u(t), r(t), \lambda_j(t)) &= \lambda_j(t)u_{j,\xi}(t) - f(w_j) + f(u_j(t) + w_j) - Df(w_j)u_j(t) \\ &\quad + Q_j^{r(t)+ct+g^0} f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{r(t)+ct+g^0}, t) \right) \\ &\quad - Q_j^{r(t)+ct+g^0} \sum_{k=1}^N f \left((u_k + w_k)(\cdot_{kj}^{r(t)+ct+g^0}, t) \right) \end{aligned}$$

again the argument ξ is suppressed. Using (1.29) we find that the equations (3.2) and (1.36) are equivalent, since

$$\begin{aligned} &\Lambda_j u_j(t) + \lambda_j(t)w_{j,\xi} + h_j(t, u(t), r(t), \lambda_j(t)) \\ &= (A + c_j I + Df(w_j))(v_j(t) - \hat{w}_j) + (\mu_j(t) - c_j)v_{j,\xi}(t) - f(w_j) + f(v_j(t) + \tilde{w}_j^-) \\ &\quad - Df(w_j)(v_j(t) - \hat{w}_j) + Q_j^{g(t)} \left[f \left(\sum_{k=1}^N v_k(\cdot_{kj}^{g(t)}, t) \right) - \sum_{k=1}^N f \left((v_k + \tilde{w}_k^-)(\cdot_{kj}^{g(t)}, t) \right) \right]. \end{aligned}$$

We will show that zero is a jointly asymptotically stable solution of the PDAE system (3.2) - (3.4). Using such a result, the transformation (3.1) and the assumptions on the set (w, c) of traveling waves in Hypotheses 1.5 and 1.6, we conclude that (w, c) is a jointly asymptotically stable solution of the PDAE system (1.36) - (1.38). Therefore, Stability Theorem 1.13 follows from:

Theorem 3.1. *Let the assumptions of Theorem 1.13 hold.*

Then zero is a jointly stable stationary solution of the PDAE system (3.2) - (3.4). More precisely, there exists $b > 0$ and $G^0, \tilde{\delta} > 0$ such that for g^0 with

$$g_1^0 < g_2^0 < \dots < g_N^0, \quad G^0 \leq |g_j^0 - g_i^0|, \quad j \neq i$$

and for u^0 with

$$\|u^0\|_{\mathcal{H}^{1,b}} < \tilde{\delta}, \quad \langle \hat{v}_{j,\xi}, u_j^0 \rangle = 0, \quad j = 1, \dots, N$$

there exists a unique solution $(u(t), r(t), \lambda(t))$ of (3.2) - (3.4) on $[0, \infty)$ which satisfies the following exponential estimates for some $C, \nu, \gamma > 0, \tau_j \in \mathbb{R}$ and $j = 1, \dots, N, t \geq 0$

$$\|u_j(t)\|_{\mathcal{H}^{1,b}} + |r_j(t) - \tau_j| + |\lambda_j(t)| \leq C e^{-\nu t} (\|u^0\|_{\mathcal{H}^{1,b}} + e^{-\gamma G^0}). \quad (3.5)$$

To examine the nonlinear terms of the coupled nonlinear system (3.2) - (3.4) we rewrite the system as

$$u_{j,t}(t) = \Lambda_j u_j(t) + \lambda_j(t) w_{j,\xi} + E_j(t, u(t)) + T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)), \quad (3.6)$$

$$r_{j,t}(t) = \lambda_j(t), \quad (3.7)$$

$$0 = \langle \hat{v}_{j,\xi}, u_j(t) \rangle \quad (3.8)$$

for $j = 1, \dots, N$ with initial condition

$$u_j(0) = u_j^0, \quad r_j(0) = 0, \quad j = 1, \dots, N, \quad (3.9)$$

where we will explain the terms E_j, T_j, N_j term by term. Therefore we define for $j = 1, \dots, N$ the operator $\tilde{G}_j : \mathbb{R}^{Nm} \rightarrow \mathbb{R}^m$ as

$$\tilde{G}_j(v) = f \left(\sum_{k=1}^N v_k \right) - \sum_{l=1}^N f((v_l + \tilde{w}_l^-)). \quad (3.10)$$

Furthermore, for $t \in \mathbb{R}_+$, $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{Nm}$, $r : \mathbb{R}_+ \rightarrow \mathbb{R}^N$ and $\xi \in \mathbb{R}$ we define

$$G_j(t, u, r)(\xi) = Q_j^{r(t)}(\xi) \tilde{G}_j \left(\left[u_k(\xi_{kj}^{r(t)}, t) \right]_{k=1}^N \right). \quad (3.11)$$

Let the time-dependent function $r^0 : \mathbb{R}_+ \rightarrow \mathbb{R}^N$ be given by $r^0(t) = ct + g^0$, then we define the operators E_j, T_j, N_j for $t \in \mathbb{R}_+$, $u : \mathbb{R} \rightarrow \mathbb{R}^{Nm}$, $r : \mathbb{R} \rightarrow \mathbb{R}^n$ and $\lambda_j \in \mathbb{R}$ by

$$E_j(t, u) = Q_j^{r^0(t)} D_v \tilde{G}_j \left(\left[\hat{w}_k(\cdot_{kj}^{r^0(t)}) \right]_{k=1}^N \right) \left[u_k(\cdot_{kj}^{r^0(t)}) \right]_{k=1}^N, \quad (3.12)$$

$$T_j(t) = G_j(t, \hat{w}, r^0) \quad (3.13)$$

and

$$\begin{aligned} N_j(t, u, r, \lambda_j) = & f(u_j + w_j) - f(w_j) - Df(w_j)u_j + \lambda_j u_{j,\xi} \\ & + G_j(t, u + \hat{w}, r + r^0) - G_j(t, \hat{w}, r^0) - E_j(t, u). \end{aligned} \quad (3.14)$$

Note that for $\xi \in \mathbb{R}$ the term $E_j(t, u)(\xi)$ is linear in u and equal to

$$E_j(t, u)(\xi) = \left[Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi_{kj}^{ct+g^0}) \right) - Df \left(w_s(\xi_{sj}^{ct+g^0}) \right) \right] u_s(\xi_{sj}^{ct+g^0}) \right]_{s=1}^N.$$

Remark 3.2. *The idea of rewriting the system to control the nonlinear terms is similar to [40]. In the next section we show "quadratic estimates" for these nonlinear terms with respect to exponentially weighted norms, i.e. we show that these terms can be estimates by the variables u, r, λ and decay exponentially in time and in the minimal distance G^0 of the initial data g^0 .*

3.2 Properties of the nonlinear operators T_j, N_j, E_j

Some of the more technical estimates for the operators N_j, E_j, T_j for $j = 1, \dots, N$ will be deferred to the Appendix A, Section A.4 and Section A.5. In particular, we show "quadratic estimates" for the nonlinear terms in the coupled system (3.6) - (3.8). For related estimates in a simpler context see [17], Theorem 5.1.1. and [30]. Additionally we will show that the nonlinear terms are locally Lipschitz in all components and locally Lipschitz in time.

In the following we denote generic constants by $C, \tilde{C} > 0$.

3.2.1 Estimates of spatial terms

Lemma 3.3. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min \left\{ \frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\})$.

Then there exists constant $C_T > 0$ such that for all b with $0 \leq b < \min(\frac{\eta q}{1+q}, \frac{1}{2}\eta, \frac{1}{2}\beta)$ there exists $\gamma_T > 0$ such that the following estimate is satisfied for all $j = 1, \dots, N$, $t \geq 0$ and g^0

$$\|T_j(t)\|_{\mathcal{L}_{2,b}} \leq C_T e^{-\gamma_T t} e^{-\gamma_T G^0}. \quad (3.15)$$

Proof. Let $t \geq 0$ and $j = 1, \dots, N$. Note $f(w_j^\pm) = 0$ for $j = 1, \dots, N$ and recall $\xi_{jj}^g = \xi$ from (1.15). We decompose T_j into two parts

$$\begin{aligned} \|T_j(t)\|_{\mathcal{L}_{2,b}} &= \|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - \sum_{l=1}^N f \left(w_l(\cdot, {}_{lj}^{ct+g^0}) \right) \right]\|_{\mathcal{L}_{2,b}} \\ &\leq C \left(\|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - f(w_j) \right]\|_{\mathcal{L}_{2,b}} \right. \\ &\quad \left. + \sum_{l \neq j} \|Q_j^{ct+g^0} f \left(w_l(\cdot, {}_{lj}^{ct+g^0}) \right)\|_{\mathcal{L}_{2,b}} \right) =: I_1 + I_2. \end{aligned}$$

From Hypothesis 1.6, Lemma A.12 (with $u = 0$) and Lemma A.10 (with $r = g = 0$) we conclude that there exists $\gamma > 0$ such that the following estimates are satisfied

$$I_1^2 = \|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - f(w_j) \right]\|_{\mathcal{L}_{2,b}}^2 \leq C e^{-\gamma t} e^{-\gamma G^0}$$

and

$$\begin{aligned}
I_2^2 &= \sum_{l \neq j} \|Q_j^{ct+g^0} f(w_l(\cdot^{ct+g^0}))\|_{\mathcal{L}_{2,b}}^2 \\
&= \sum_{l \neq j} \|Q_j^{ct+g^0} (f(w_l(\cdot^{ct+g^0})) - f(w_l^\pm))\|_{\mathcal{L}_{2,b}}^2 \\
&\leq C \sum_{l \neq j} \int_{\mathbb{R}} \left(Q_j^{ct+g^0}(\xi) e^{-\eta|\xi|^{ct+g^0}} |e^{b|\xi}| \right)^2 d\xi \\
&\leq C e^{-\gamma t} e^{-\gamma G^0}.
\end{aligned}$$

Note, we use w_l^- for $\xi_{lj}^{ct+g^0} < 0$ and w_l^+ for $\xi_{lj}^{ct+g^0} \geq 0$. ■

Lemma 3.4. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\})$ and let $0 \leq \varrho < \infty$.

Then there exist constants $C_N, \tilde{C}_N > 0$ such that for all b with $0 \leq b < \min(\frac{\eta q}{1+q}, \frac{\beta}{2}, \frac{\eta}{2})$ there exists $\gamma_N > 0$ such that the following estimate is satisfied for all g^0 with $G^0 \geq 12\varrho$ and for all $j = 1, \dots, N, t \geq 0$ and $(u, r, \lambda) \in B_{\varrho,b}(0)$ with $\|u\|_{\mathcal{H}^{1,b}} \leq 1$

$$\begin{aligned}
\|N_j(t, u, r, \lambda_j)\|_{\mathcal{L}_{2,b}} &\leq C_N (\|u\|_{\mathcal{H}^{1,b}}^2 + \|u\|_{\mathcal{H}^{1,b}} |\lambda_j|) + C_N e^{\tilde{C}_N \|r\|} \|r\| \|u\|_{\mathcal{H}^{1,b}} \\
&\quad + C_N e^{\tilde{C}_N \|r\|} \|r\| e^{-\gamma_N t} e^{-\gamma_N G^0} (1 + \|u\|_{\mathcal{H}^{1,b}}). \tag{3.16}
\end{aligned}$$

Proof. Let $j \in \{1, \dots, N\}, t \geq 0$. Define $r^0(t) = ct + g^0$. We consider the operator N_j

$$\begin{aligned}
&\|N_j(t, u, r, \lambda_j)\|_{\mathcal{L}_{2,b}} \\
&\leq \|f(u_j + w_j) - f(w_j) - Df(w_j)u_j\|_{\mathcal{L}_{2,b}} + \|\lambda_j u_{j,\xi}\|_{\mathcal{L}_{2,b}} \\
&\quad + \|G_j(t, u + \hat{w}, r^0) - G_j(t, \hat{w}, r^0) - E_j(t, u)\|_{\mathcal{L}_{2,b}} \\
&\quad + \|G_j(t, u + \hat{w}, r + r^0) - G_j(t, u + \hat{w}, r^0)\|_{\mathcal{L}_{2,b}}.
\end{aligned}$$

We estimate each term separately:

$$\begin{aligned}
\|f(u_j + w_j) + f(w_j) - Df(w_j)u_j\|_{\mathcal{L}_{2,b}} &= \left\| \int_0^1 (Df(w_j + \tau u_j) - Df(w_j)) u_j d\tau \right\|_{\mathcal{L}_{2,b}} \\
&\leq C \|u\|_{\mathcal{H}^{1,b}}^2,
\end{aligned}$$

$$\|\lambda_j u_{j,\xi}\|_{\mathcal{L}_{2,b}} \leq C |\lambda_j| \|u\|_{\mathcal{H}^{1,b}}.$$

Note that $G_j(t, u + \hat{w}, r^0)(\xi) - G_j(t, \hat{w}, r^0)(\xi)$ for $\xi \in \mathbb{R}$ is equal to

$$\int_0^1 \left[Q_j^{ct+g^0}(\xi) \left[Df \left(\sum_{k=1}^N (\hat{w}_k + \tau u_k)(\xi_{kj}^{ct+g^0}) \right) - Df \left((w_s + \tau u_s)(\xi_{sj}^{ct+g^0}) \right) \right] u_s(\xi_{sj}^{ct+g^0}) \right]_{s=1}^N d\tau.$$

Therefore we obtain

$$\begin{aligned} & \|G_j(t, u + \hat{w}, r^0) - G_j(t, \hat{w}, r^0) - E_j(t, u)\|_{\mathcal{L}_{2,b}}^2 \\ & \leq C \max_{1 \leq s \leq N} \sum_{k=1}^N \int_{\mathbb{R}} \left(Q_j^{ct+g^0}(\xi) \|u_s(\xi_{sj}^{ct+g^0}) u_k(\xi_{kj}^{ct+g^0})\| \theta_b(\xi) \right)^2 d\xi. \end{aligned}$$

Using Lemma A.8 (with $r = 0$), the Sobolev Imbedding estimate (A.5) and (1.21) we conclude for $s \in \{1, \dots, N\}$

$$\begin{aligned} & \int_{\mathbb{R}} \left(Q_j^{ct+g^0}(\xi) \|u_s(\xi_{sj}^{ct+g^0}) u_k(\xi_{kj}^{ct+g^0})\| \theta_b(\xi) \right)^2 d\xi \\ & \leq C \|u_s(\cdot_{sj}^{ct+g^0})\|_{\infty}^2 \int_{\mathbb{R}} \left(Q_j^{ct+g^0}(\xi) \|u_k(\xi_{kj}^{ct+g^0})\| \frac{\theta_b(\xi_{kj}^{ct+g^0})}{\theta_b(\xi_{sj}^{ct+g^0})} e^{b|\xi|} \right)^2 d\xi \\ & \leq C \|u_s\|_{\mathcal{H}^{1,b}}^2 \int_{\mathbb{R}} \|u_k(\xi_{kj}^{ct+g^0})\|^2 \theta_b(\xi_{kj}^{ct+g^0})^2 d\xi \sup_{\xi \in \mathbb{R}} |Q_j^{ct+g^0}(\xi)|^2 e^{-b|\xi_{kj}^{ct+g^0}| + b|\xi|} \\ & \leq C \|u\|_{\mathcal{H}^{1,b}}^4. \end{aligned}$$

From Hypothesis 1.10 and Lemma A.6 (with $g = 0$) we conclude that for $\xi \in \mathbb{R}$ holds

$$\begin{aligned} |Q_j^{r+ct+g^0}(\xi) - Q_j^{ct+g^0}(\xi)| &= \frac{\varphi(\xi) \left(\sum_{k=1}^N \varphi(\xi_{kj}^{ct+g^0}) - \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0}) \right)}{\sum_{k=1}^N \varphi(\xi_{kj}^{ct+g^0}) \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0})} \\ &\leq C \frac{\varphi(\xi) \sum_{k \neq j} \int_0^1 |\varphi'(\xi_{kj}^{r-hr+ct+g^0})| dh}{\sum_{k=1}^N \varphi(\xi_{kj}^{ct+g^0}) \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0})} \|r\| \\ &= C \frac{\sum_{k \neq j} \int_0^1 |\varphi'(\xi_{kj}^{r-hr+ct+g^0})| dh}{\sum_{k=1}^N \varphi(\xi_{kj}^{ct+r+g^0})} Q_j^{ct+g^0}(\xi) \|r\| \\ &\leq C e^{\tilde{C}\|r\|} Q_j^{ct+g^0}(\xi) \|r\|. \end{aligned}$$

Further follows from Lemma A.8 and Lemma A.10 (with $g = 0$) and from Lemma

A.12 that there exists $\gamma > 0$ such that the following estimate is satisfied

$$\begin{aligned}
& \|G_j(t, u + \hat{w}, r + r^0) - G_j(t, u + \hat{w}, r^0)\|_{\mathcal{L}_{2,b}} \\
& \leq C \left(\left\| \left(Q_j^{r+ct+g^0} - Q_j^{ct+g^0} \right) \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(u_j + w_j) \right] \right\|_{\mathcal{L}_{2,b}} \right. \\
& \quad \left. + \left\| \left(-Q_j^{r+ct+g^0} + Q_j^{ct+g^0} \right) \sum_{l \neq j} f \left((u_l + w_l)(\cdot_{lj}^{ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}} \right. \\
& \quad \left. + \left\| Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{r+ct+g^0}) \right) - f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) \right. \right. \right. \\
& \quad \left. \left. - \sum_{l=1}^N \left(f \left((u_l + w_l)(\cdot_{lj}^{r+ct+g^0}) \right) - f \left((u_l + w_l)(\cdot_{lj}^{ct+g^0}) \right) \right) \right] \right\|_{\mathcal{L}_{2,b}} \right) \\
& \leq C e^{\tilde{C}\|r\|} \|r\| \left(\left\| Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(u_j + w_j) \right] \right\|_{\mathcal{L}_{2,b}} \right. \\
& \quad \left. + \left\| Q_j^{ct+g^0} \sum_{l \neq j} f \left((u_l + w_l)(\cdot_{lj}^{ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}} \right) \\
& \quad + C \left(\sum_{k=1}^N \left\| Q_j^{r+ct+g^0} \left(u_k(\cdot_{kj}^{r+ct+g^0}) - u_k(\cdot_{kj}^{ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}} \right. \\
& \quad \left. + \sum_{k=1}^N \left\| Q_j^{r+ct+g^0} \left(w_k(\cdot_{kj}^{r+ct+g^0}) - w_k(\cdot_{kj}^{ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}} \right) \\
& \leq C e^{\tilde{C}\|r\|} \|r\| \left(\|u\|_{\mathcal{H}^{1,b}} + e^{-\gamma t} e^{-\gamma G^0} + \|u\|_{\mathcal{H}^{1,b}} e^{-\gamma t} e^{-\gamma G^0} \right),
\end{aligned}$$

since we obtain as in the proof of Lemma 3.3 and from Lemma A.8 and Lemma A.10 (with $r = g = 0$)

$$\begin{aligned}
& \left\| \sum_{l \neq j} Q_j^{ct+g^0} f \left((u_l + w_l)(\cdot_{lj}^{ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}}^2 \\
& \leq \sum_{l \neq j} \left\| Q_j^{ct+g^0} \left(f \left((u_l + w_l)(\cdot_{lj}^{ct+g^0}) \right) - f(w_l^\pm) \right) \right\|_{\mathcal{L}_{2,b}}^2 \\
& \leq C \sum_{l \neq j} \left[\int_{\mathbb{R}} \left(Q_j^{ct+g^0}(\xi) e^{-\eta|\xi_{lj}^{ct+g^0}|} e^{b|\xi|} \right)^2 d\xi \right. \\
& \quad \left. + \int_{\mathbb{R}} \left\| Q_j^{ct+g^0}(\xi) u(\xi_{lj}^{ct+g^0}) \theta_b(\xi_{lj}^{ct+g^0}) e^{-b|\xi_{lj}^{ct+g^0}|} e^{b|\xi|} \right\|^2 d\xi \right] \\
& \leq C e^{-\gamma t} e^{-\gamma G^0} + C \|u\|_{\mathcal{L}_{2,b}}^2
\end{aligned}$$

and from Lemma A.8 and Lemma A.10 (with $g = 0$)

$$\begin{aligned}
& \|Q_j^{r+ct+g^0} \left(u_k(\cdot_{kj}^{r+ct+g^0}) - u_k(\cdot_{kj}^{ct+g^0}) \right)\|_{\mathcal{L}_{2,b}}^2 \\
& \leq \int_{\mathbb{R}} \|Q_j^{r+ct+g^0} \left(u_k(\xi_{kj}^{r+ct+g^0}) - u_k(\xi_{kj}^{ct+g^0}) \right) \theta_b(\xi)\|^2 d\xi \\
& \leq \int_{\mathbb{R}} \int_0^1 \|Q_j^{r+ct+g^0} u_{k,\xi}(\xi_{kj}^{hr+ct+g^0}) e^{b|\xi|}\|^2 dh d\xi \|r\|^2 \\
& \leq C \int_{\mathbb{R}} \int_0^1 \|u_{k,\xi}(\xi_{kj}^{hr+ct+g^0}) \theta_b(\xi_{kj}^{hr+ct+g^0})\|^2 \sup_{\xi \in \mathbb{R}} |Q_j^{r+ct+g^0} e^{-b|\xi_{kj}^{hr+ct+g^0} + b|\xi|}|^2 dh d\xi \|r\|^2 \\
& \leq C e^{2\tilde{C}\|r\|} \|u_k\|_{\mathcal{H}^{1,b}}^2 \|r\|^2
\end{aligned} \tag{3.17}$$

and analogously

$$\begin{aligned}
& \|Q_j^{r+ct+g^0} \left(w_k(\cdot_{kj}^{r+ct+g^0}) - w_k(\cdot_{kj}^{ct+g^0}) \right)\|_{\mathcal{L}_{2,b}}^2 \\
& \leq \int_{\mathbb{R}} \int_0^1 \|Q_j^{r+ct+g^0} w_{k,\xi}(\xi_{kj}^{hr+ct+g^0}) e^{b|\xi|}\|^2 dh d\xi \|r\|^2 \\
& \leq C \int_{\mathbb{R}} \int_0^1 |e^{-\eta|\xi_{kj}^{hr+ct+g^0} + b|\xi|}|^2 dh d\xi \|r\|^2 \\
& \leq C e^{2\tilde{C}\|r\|} e^{-\gamma t} e^{-\gamma G^0} \|r\|^2.
\end{aligned} \tag{3.18}$$

■

It is worth pointing out that in the estimates (3.15), (3.16) of the lemmas above γ_T, γ_N depend on b , in particular, $\gamma_T, \gamma_N \rightarrow 0$ as $b \rightarrow \min(\frac{\eta q}{1+q}, \frac{1}{2}\eta, \frac{1}{2}\beta)$. If b is sufficiently small γ_T, γ_N can be estimated by a constant, compare Remark A.14. Therefore, the estimates (3.15), (3.16) are satisfied for all sufficiently small b . In particular, the estimates are true for $b = 0$, i.e. dealing with these nonlinear part can be handled as usually. To handle the nonlinear term $E_j, j = 1, \dots, N$ and show quadratic estimates it is important that $b > 0$. The estimates work for exponentially weighted functions.

Lemma 3.5. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min\left\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\right\})$.

Then there exists a constant $C_E > 0$ such that for all b with $0 < b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$ there exists $\gamma_E > 0$ such that the following estimate is satisfied for all $g^0, j = 1, \dots, N, t \geq 0$ and $u = (u_1, \dots, u_N)$ with $u_k \in \mathcal{L}_{2,b}, k = 1, \dots, N$

$$\|E_j(t, u)\|_{\mathcal{L}_{2,b}} \leq C_E e^{-\gamma_E t} e^{-\gamma_E G^0} \|u\|_{\mathcal{L}_{2,b}}. \tag{3.19}$$

Furthermore, there exists a constant $C_e > 0$ such that for all $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$ and $g^0, u = (u_1, \dots, u_N)$ with $u_k \in \mathcal{L}_{2,b}, k = 1, \dots, N$ and for all $j = 1, \dots, N, t \geq 0$ the following estimate is satisfied

$$\|E_j(t, u)\|_{\mathcal{L}_{2,b}} \leq C_e \|u\|_{\mathcal{L}_{2,b}}. \quad (3.20)$$

Proof. Let $t \geq 0$. Let $j \in \{1, \dots, N\}$ and $b \geq 0$. We estimate

$$\begin{aligned} & \|E_j(t, u)\|_{\mathcal{L}_{2,b}} \\ & \leq C \max_{1 \leq s \leq N} \|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot_{kj}^{ct+g^0}) \right) - Df \left(w_s(\cdot_{sj}^{ct+g^0}) \right) \right] u_s(\cdot_{sj}^{ct+g^0})\|_{\mathcal{L}_{2,b}} \\ & \leq C \max_{1 \leq s \leq N} \|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot_{kj}^{ct+g^0}) \right) - Df \left(w_s(\cdot_{sj}^{ct+g^0}) \right) \right] \\ & \quad \cdot e^{b|\cdot|} \frac{\theta_b(\cdot_{sj}^{ct+g^0})}{\theta_b(\cdot_{sj}^{ct+g^0})} u_s(\cdot_{sj}^{ct+g^0})\|_{\mathcal{L}_2} \\ & \leq C \max_{1 \leq s \leq N} \sup_{\xi \in \mathbb{R}} \|Q_j^{ct+g^0}(\xi) \left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi_{kj}^{ct+g^0}) \right) - Df \left(w_s(\xi_{sj}^{ct+g^0}) \right) \right] e^{b|\xi| - b|\xi_{sj}^{ct+g^0}|}\| \\ & \quad \cdot \|u_s(\cdot_{sj}^{ct+g^0})\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

Let $d_k(t) = (c_k - c_j)t + g_k^0 - g_j^0$. We estimate the term

$$A_s := \|Q_j^{ct+g^0}(\xi) \left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi - d_k(t)) \right) - Df \left(w_s(\xi - d_s(t)) \right) \right] e^{b|\xi| - b|\xi - d_s(t)|}\|$$

for all $\xi \in \mathbb{R}$.

Note that the quotient

$$Q_j^{ct+g^0}(\xi) = \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi - d_k(t))}$$

is always positive and satisfies $Q_j^{ct+g^0}(\xi) \leq 1$. Furthermore, using Hypothesis 1.10 the quotient is estimated by

$$Q_j^{ct+g^0}(\xi) \leq \frac{\varphi(\xi)}{\varphi(\xi - d_k(t))} \leq \frac{C_1}{C_0} e^{-\beta|\xi| + \beta|\xi - d_k(t)|}$$

for some $k \in \{1, \dots, N\}$.

Using Hypothesis 1.4 on the nonlinearity f and Hypothesis 1.6 on the bounded traveling waves w_j we conclude that $\left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi - d_k(t)) \right) - Df \left(w_s(\xi - d_s(t)) \right) \right]$

is bounded.

Using this argument and that $Q_j^{ct+g^0}$ is bounded we conclude that A_s is bounded for the case $b = 0$ and (3.20) follows for $b = 0$. Showing that the estimate (3.19) is valid for all $0 < b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$ we conclude that (3.20) holds for all $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$.

We proceed for the case $b > 0$ by considering different cases for $1 \leq s \leq N$.

Define $q := \min(\min \left\{ \frac{c_s - c_{s-1}}{2|c_s + c_{s-1} - 2c_j|} : 1 \leq j \leq N, 2 \leq s \leq N, s-1 \neq j \right\}, \frac{1}{4})$.

Case 1: $s-1 > j, j = 1, \dots, N-2$:

From the definition of q we obtain

$$c_s - c_{s-1} - q|2c_j - c_{s-1} - c_s| > 0$$

and this clearly forces that there exists $T_s(g^0) \in \mathbb{R}$ with

$$g_{s-1}^0 - g_s^0 + q|2g_j^0 - g_{s-1}^0 - g_s^0| = (c_s - c_{s-1} - q|2c_j - c_{s-1} - c_s|) T_s(g^0)$$

such that for all $t > T_s(g^0)$

$$g_{s-1}^0 - g_s^0 + q|2g_j^0 - g_{s-1}^0 - g_s^0| < (c_s - c_{s-1} - q|2c_j - c_{s-1} - c_s|) t.$$

This implies $(1+q)d_{s-1}(t) < (1-q)d_s(t)$ for all $t > T_s(g^0)$.

Case 1a: Let $t > T_s(g^0)$. We estimate A_s on the subintervals

$$I_1 = (-\infty, 0], I_2 = [0, \frac{1}{2}d_{s-1}(t)], I_3 = [\frac{1}{2}d_{s-1}(t), d_{s-1}(t)],$$

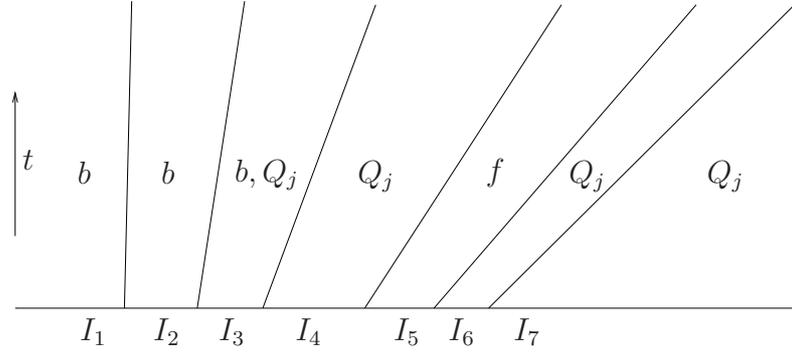
$$I_4 = [d_{s-1}(t), (1+q)d_{s-1}(t)], I_5 = [(1+q)d_{s-1}(t), (1-q)d_s(t)],$$

$$I_6 = [(1-q)d_s(t), d_s(t)], I_7 = [d_s(t), \infty)$$

which form a partition of \mathbb{R} . We use the assumptions on the nonlinearity, on the weight or on the time dependent partition of unity to estimate A_s , the term used are indicated in Figure 3.1.

From Hypotheses 1.4, 1.6, 1.10 and (A.13) we obtain the following estimates: Consider the case $\xi \in I_1$. We estimate A_s using as noted above that $Q_j^{ct+g^0}(\xi)$ and $\left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi - d_k(t)) \right) - Df(w_s(\xi - d_s(t))) \right]$ are bounded:

$$A_s \leq C e^{-b\xi + b\xi - bd_s(t)} = C e^{-bd_s(t)} = C e^{-b(c_s - c_j)t} e^{-b(g_s^0 - g_j^0)} \leq C e^{-\gamma t} e^{-\gamma G^0} \quad (3.21)$$

Figure 3.1: Decomposition of \mathbb{R} for $t \geq 0$.

for some $\gamma > 0$ which depends on b . In the following the arguments will not be repeated and we obtain similarly for $\xi \in I_2$:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t)} \leq C e^{-bd_s(t) + bd_{s-1}(t)}.$$

For $\xi \in I_3$:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t) - \beta\xi - \beta\xi + \beta d_{s-1}(t)} \leq C e^{-bd_s(t) + bd_{s-1}(t)}.$$

For $\xi \in I_4$:

$$A_s \leq C e^{b\xi - \beta\xi + \beta\xi - \beta d_{s-1}(t)} \leq C e^{(-\beta + b + qb)d_{s-1}(t)} \leq C e^{(-\frac{1}{2}\beta + \frac{1}{4}b)d_{s-1}(t)}.$$

Recall equation (1.7), for $\xi \in I_5$ we obtain the estimate if $s \neq N$:

$$\begin{aligned} A_s &\leq \|Df\left(\sum_{k < s} (w_k(\xi_{kj}^{ct+g^0}) - w_k^+) + w_s(\xi_{sj}^{ct+g^0}) + \sum_{k > s} (w_k(\xi_{kj}^{ct+g^0}) - w_k^-) - Df(w_s(\xi_{sj}^{ct+g^0}))\right)\| e^{b\xi + b\xi - bd_s(t)} \\ &\leq C e^{b\xi + b\xi - bd_s(t)} \left(\sum_{k < s} \|w_k(\xi_{kj}^{ct+g^0}) - w_k^+\| + \sum_{k > s} \|w_k(\xi_{kj}^{ct+g^0}) - w_k^-\| \right) \\ &\leq C e^{b\xi + b\xi - bd_s(t)} \left(\sum_{k < s} e^{-\eta\xi + \eta d_k(t)} + \sum_{k > s} e^{\eta\xi - \eta d_k(t)} \right) \\ &\leq C e^{b\xi + b\xi - bd_s(t)} (e^{-\eta\xi + \eta d_{s-1}(t)} + e^{\eta\xi - \eta d_{s+1}(t)}) \\ &\leq C (e^{(b+q(2b-\eta))d_{s-1}(t)} + e^{(b+q(-2b-\eta))d_s(t)}). \end{aligned}$$

For $\xi \in I_5$ and $s = N$ the last sum is empty:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t) - \eta\xi + \eta d_{s-1}(t)} \leq C e^{(b+q(2b-\eta))d_{s-1}(t)}.$$

For $\xi \in I_6$:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t) - \beta\xi - \beta\xi + \beta d_s(t)} \leq C e^{(b-\beta+q(-2b+2\beta))d_s(t)} \leq C e^{(-\frac{1}{2}\beta+q(-2b+2\beta))d_s(t)}.$$

For $\xi \in I_7$:

$$A_s \leq C e^{b\xi - b\xi + bd_s(t) - \beta\xi + \beta\xi - \beta d_s(t)} \leq C e^{(b-\beta)d_s(t)}. \quad (3.22)$$

Case 1b: $T_s(g^0) \geq 0$, i.e. $(1+q)d_{s-1}(t) \geq (1-q)d_s(t)$ for all $0 \leq t \leq T_s(g^0)$.

Case 1bi: $(1+q)d_{s-1}(t) < d_s(t)$ for all $0 \leq t \leq T_s(g^0)$, see Figure 3.2.

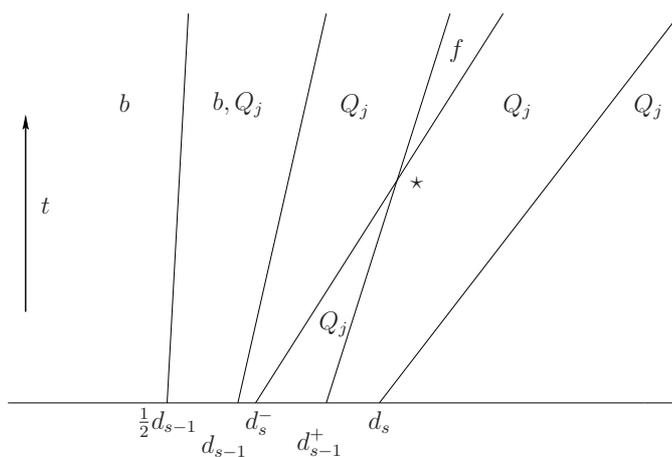


Figure 3.2: Positions of the different rays over time, where $d_k^\pm = d_k(1 \pm q)(0)$, $k = s-1, s$ and \star marks the point $d_{s-1}(1+q)(T_s(g^0)) = d_s(1-q)(T_s(g^0))$. (Note $d_s^-(t)$ could also cross $d_{s-1}(t)$ or $\frac{1}{2}d_{s-1}(t)$ for $0 \leq t < T_s(g^0)$.)

The estimate of A_s for all $\xi \in \mathbb{R}$, $0 \leq t \leq T_s(g^0)$ can be handled in much the same way as above. We estimate A_s on the subintervals I_1, \dots, I_4 , $\tilde{I}_6 := [(1+q)d_{s-1}(t), d_s(t)]$, I_7 which form a partition of \mathbb{R} and I_i , $i = 1, \dots, 4, 7$ are defined and estimated as above. For $\xi \in \tilde{I}_6$ we use $(1-q)d_s(t) \leq (1+q)d_{s-1}(t)$ and estimate:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t) - \beta\xi - \beta\xi + \beta d_s(t)} \leq C e^{(b-\beta+q(-2b+2\beta))d_s(t)}.$$

Case 1bii: Again we conclude from the definition of q that there exists $0 \leq T_{1,s}(g^0) < T_s(g^0)$ such that $(1+q)d_{s-1}(t) < d_s(t)$ holds for all $T_{1,s}(g^0) < t \leq T_s(g^0)$ and $(1+q)d_{s-1}(t) \geq d_s(t)$ for all $0 \leq t \leq T_{1,s}(g^0)$, see Figure 3.3.

For $T_{1,s}(g^0) < t \leq T_s(g^0)$ we proceed as in case 1bi.

The estimate of A_s for all $\xi \in \mathbb{R}$, $0 \leq t \leq T_{1,s}(g^0)$ can be handled in much the

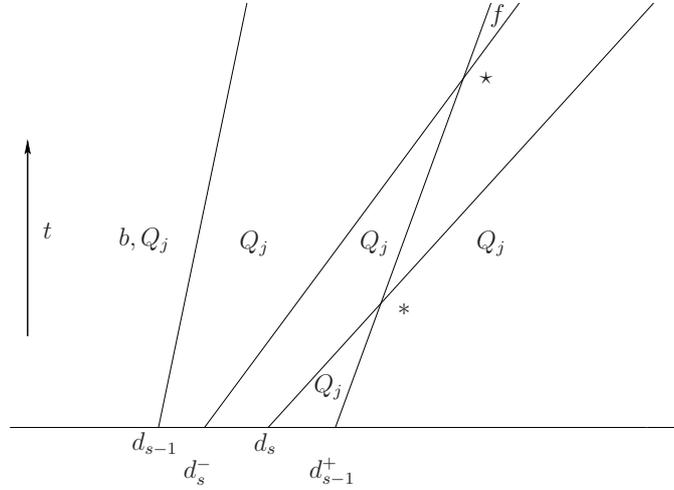


Figure 3.3: Position of the different rays over time, where $d_k^\pm := d_k(1 \pm q)(0)$, $k = s - 1, s$ and \star marks the point $d_{s-1}(1 + q)(T_s(g^0)) = d_s(1 - q)(T_s(g^0))$ and $*$ the point $d_{s-1}(1 + q)(T_{1,s}(g^0)) = d_s(T_{1,s}(g^0))$. (Note $d_{s-1}(t)$ could also cross $d_s^-(t)$ for $0 \leq t < T_s(g^0)$.)

same way as above. We estimate the term A_s on the subintervals I_1, \dots, I_4 , $\tilde{I}_7 := [(1 + q)d_{s-1}(t), \infty)$ which form a partition of \mathbb{R} and I_i , $i = 1, \dots, 4$ are defined as above. For $\xi \in \tilde{I}_7$ we use $d_s(t) \leq (1 + q)d_{s-1}(t)$ and estimate:

$$A_s \leq C e^{b\xi - b\xi + bd_s(t) - \beta\xi + \beta\xi - \beta d_s(t)} \leq C e^{(b-\beta)d_s(t)}.$$

Case 2: $s - 1 = j$, $j = 1, \dots, N - 1$:

We estimate A_s on the following subintervals

$$I_1 = (-\infty, 0], \tilde{I}_2 = [0, (\frac{1}{2} - q)d_s(t)], \tilde{I}_5 = [(\frac{1}{2} - q)d_s(t), (1 - q)d_s(t)],$$

$$I_6 = [(1 - q)d_s(t), d_s(t)], I_7 = [d_s(t), \infty)$$

which form a partition of \mathbb{R} for all $t \geq 0$. The term A_s is estimated for $\xi \in I_1, I_6, I_7$ as in the second case. For $\xi \in \tilde{I}_2$ we obtain:

$$A_s \leq C e^{2b\xi - bd_s(t)} = C e^{-2qb d_s(t)}.$$

For $\xi \in \tilde{I}_5$ we get similar to above for $s \neq N$:

$$\begin{aligned} A_s &\leq C e^{b\xi + b\xi - bd_s(t)} (e^{-\eta\xi} + e^{\eta\xi - \eta d_{s+1}(t)}) \\ &\leq C \left(e^{(-\frac{1}{2}\eta - q(2b - \eta))d_s(t)} + e^{(b + q(-2b - \eta))d_s(t)} \right) \end{aligned}$$

and for $s = N$ vanishes the last term:

$$A_s \leq C e^{b\xi + b\xi - bd_s(t)} (e^{-\eta\xi}) \leq C \left(e^{(-\frac{1}{2}\eta - q(2b-\eta))d_s(t)} \right).$$

Case 3: $s < j$, $j = 2, \dots, N$. This case may be handled in much the same way.

Case 4: $s = j$, $2 \leq j \leq N - 1$:

In this case A_s can be shortly written as

$$A_s = \|Q_j^{ct+g^0}(\xi) \left[Df \left(\sum_{k=1}^N \hat{w}_k(\xi_{kj}^{ct+g^0}) \right) - Df(w_j(\xi)) \right] \|\.$$

Let $0 < \varepsilon < \frac{1}{2}$. We estimate A_s on each of the following intervals

$$I_1 = (-\infty, d_{j-1}(t)], I_2 = [d_{j-1}(t), (\frac{1}{2} + \varepsilon)d_{j-1}(t)], I_3 = [(\frac{1}{2} + \varepsilon)d_{j-1}(t), (\frac{1}{2} + \varepsilon)d_{j+1}(t)],$$

$$I_4 = [(\frac{1}{2} + \varepsilon)d_{j+1}(t), d_{j+1}(t)], I_5 = [d_{j+1}(t), \infty)$$

which form a partition of \mathbb{R} for $t \geq 0$. Again we use either the assumptions on the nonlinearity term or on the time dependent partition of unity to estimate A_s , see Figure 3.4.

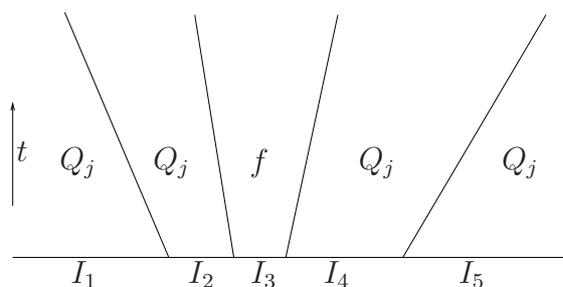


Figure 3.4: Decomposition of \mathbb{R} for $t \geq 0$.

From Hypotheses 1.4, 1.6 and 1.10 we obtain the estimates:

For $\xi \in I_1$:

$$A_s \leq C e^{\beta\xi - \beta\xi + \beta d_{j-1}(t)} = C e^{\beta d_{j-1}(t)}.$$

For $\xi \in I_2$:

$$A_s \leq C e^{2\beta\xi - \beta d_{j-1}(t)} \leq C e^{2\varepsilon\beta d_{j-1}(t)}.$$

For $\xi \in I_3$:

$$\begin{aligned}
A_s &\leq \|Df \left(\sum_{k<j} (w_k(\xi_{kj}^{ct+g^0}) - w_k^+) + w_j(\xi) + \sum_{k>j} (w_k(\xi_{kj}^{ct+g^0}) - w_k^-) \right) - Df(w_j(\xi))\| \\
&\leq C \left(\sum_{k<j} \|w_k(\xi_{kj}^{ct+g^0}) - w_k^+\| + \sum_{k>j} \|w_k(\xi_{kj}^{ct+g^0}) - w_k^-\| \right) \\
&\leq C \left(\sum_{k<j} e^{-\eta\xi + \eta d_k(t)} + \sum_{k>j} e^{\eta\xi - \eta d_k(t)} \right) \\
&\leq C \left(e^{-\eta\xi + \eta d_{j-1}(t)} + e^{\eta\xi - \eta d_{j+1}(t)} \right) \\
&\leq C \left(e^{(\frac{1}{2}-\varepsilon)\eta d_{j-1}(t)} + e^{(-\frac{1}{2}+\varepsilon)\eta d_{j+1}(t)} \right).
\end{aligned}$$

For $\xi \in I_4$:

$$A_s \leq C e^{-2\beta\xi + \beta d_{j+1}(t)} \leq C e^{-2\varepsilon\beta d_{j+1}(t)}.$$

For $\xi \in I_5$:

$$A_s \leq C e^{-\beta\xi + \beta\xi - \beta d_{j+1}(t)} = C e^{-\beta d_{j+1}(t)}.$$

For $s = j = 1$ we divide \mathbb{R} into I_3, I_4, I_5 , where I_4, I_5 are defined as before and I_3 is changed into

$$I_3 = (-\infty, (\frac{1}{2} + \varepsilon)d_2(t)].$$

We estimate for $\xi \in I_3$:

$$A_s \leq C e^{\eta\xi - \eta d_2(t)} \leq C e^{(-\frac{1}{2} + \varepsilon)\eta d_2(t)}.$$

For $s = j = N$ we divide \mathbb{R} into I_1, I_2, I_3 , where I_1, I_2 are define as before and I_3 is changed into

$$I_3 = [(\frac{1}{2} + \varepsilon)d_{N-1}(t), \infty).$$

We estimate for $\xi \in I_3$:

$$A_s \leq C e^{-\eta\xi + \eta d_{N-1}(t)} \leq C e^{(\frac{1}{2} - \varepsilon)\eta d_{N-1}(t)}.$$

■

Remark 3.6. Note that, for instance, the estimate (3.21) of the term A_s in the proof of the lemma above in the subspace $(-\infty, 0]$ needs the condition $b > 0$, otherwise the term A_s can only be estimated by a constant. In particular for $s - 1 < j$ Figure 3.1 shows the domains where the condition $b > 0$ is needed. For $s + 1 > j$ it turns out that the Figure 3.1 is nearly reflected.

We will not get an exponentially decay in time and in the minimal distance G^0 for $b = 0$, since neither the assumptions on φ nor on the traveling waves (w, c) give exponentially decay in time. Note that γ_E in (3.19) tends to zero as b tends to zero. Similarly to γ_T, γ_N in the estimates (3.15), (3.16). In addition, γ_E tends to zero as b tends to $\min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$.

Remark 3.7. Assume that the bump function φ satisfies instead of (1.34) of Hypothesis 1.10 the weaker condition

$$C_0 e^{-\beta_0 |\xi|} \leq \varphi(\xi) \leq C_1 e^{-\beta_1 |\xi|} \quad \forall \xi \in \mathbb{R}$$

with some positive constants $C_0 \leq C_1$ and $\beta_1 < \beta_0$. For the case $s - 1 > j$ we get difficulties to show the estimate (3.19): Consider the interval $I_7 = [d_s(t), \infty)$, we cannot derive an estimate like (3.22) of the term A_s .

Furthermore, we need Lipschitz estimates of the nonlinear terms. Lipschitz estimates in the parameters u, r and λ are obviously satisfied for the nonlinear terms $T_j, E_j, j = 1, \dots, N$. For the nonlinear term N_j we obtain:

Lemma 3.8. Assume that Hypotheses 1.4, 1.6, 1.10 hold.

Let $q := \min(\frac{1}{4}, \min\left\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\right\})$ and let $\varrho \geq 0$.

Then there exists constants $C_n, \tilde{C}_n > 0$ such that for all b with $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$ there exists $\gamma_n > 0$ such that the following estimate is satisfied for all G^0 with $G^0 \geq 12\varrho$ and for all $(u, r, \lambda), (v, g, \mu) \in B_{\varrho, b}(0)$ with $\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}} \leq 1$ and for all $j = 1, \dots, N, t \geq 0$

$$\begin{aligned} & \|N_j(t, u, r, \lambda_j) - N_j(t, v, g, \mu_j)\|_{\mathcal{L}_{2,b}} \\ & \leq C_n (\max(\|v\|_{\mathcal{H}^{1,b}}, \|u\|_{\mathcal{H}^{1,b}}) |\mu_j - \lambda_j| + \max(|\lambda_j|, |\mu_j|) \|v_j - u_j\|_{\mathcal{H}^{1,b}}) \\ & \quad + C_n e^{\tilde{C}_n \max(\|r\|, \|g\|)} \|v - u\|_{\mathcal{H}^{1,b}} \\ & \quad + C_n e^{\tilde{C}_n \max(\|r\|, \|g\|)} \|r - g\| \max(\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}}, e^{-\gamma_n t} e^{-\gamma_n G^0}) \max(1, \|r\|, \|g\|) \\ & \quad + C_n e^{\tilde{C}_n \max(\|r\|, \|g\|)} \|r - g\| \max(\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}}) e^{-\gamma_n t} e^{-\gamma_n G^0} \max(1, \|r\|, \|g\|). \end{aligned} \tag{3.23}$$

Proof. Let $j \in \{1, \dots, N\}, t \geq 0$. Define $r^0(t) = ct + g^0$. Since

$$\begin{aligned} & \|N_j(t, u, r, \lambda_j) - N_j(t, v, g, \mu_j)\|_{\mathcal{L}_{2,b}} \\ & \leq \|f(u_j + w_j) - f(v_j + w_j)\|_{\mathcal{L}_{2,b}} + \|Df(w_j)(u_j - v_j)\|_{\mathcal{L}_{2,b}} \\ & \quad + \|E_j(t, u - v)\|_{\mathcal{L}_{2,b}} + \|\lambda_j u_{j,\xi} - \mu_j v_{j,\xi}\|_{\mathcal{L}_{2,b}} \\ & \quad + \|G_j(t, u + \hat{w}, r + r^0) - G_j(t, v + \hat{w}, g + r^0)\|_{\mathcal{L}_{2,b}}, \end{aligned}$$

we use (A.19) and estimate each term separately:

$$\|f(u_j + w_j) - f(v_j + w_j)\|_{\mathcal{L}_{2,b}} + \|Df(w_j)(u_j - v_j)\|_{\mathcal{L}_{2,b}} \leq C\|u_j - v_j\|_{\mathcal{L}_{2,b}}$$

From Lemma 3.5 we conclude

$$\|E_j(t, u - v)\|_{\mathcal{L}_{2,b}} \leq C\|u_j - v_j\|_{\mathcal{L}_{2,b}}.$$

Furthermore we estimate

$$\begin{aligned} \|\lambda_j u_{j,\xi} - \mu_j v_{j,\xi}\|_{\mathcal{L}_{2,b}} &\leq |\lambda_j| \|u_{j,\xi} - v_{j,\xi}\|_{\mathcal{L}_{2,b}} + |\lambda_j - \mu_j| \|v_{j,\xi}\|_{\mathcal{L}_{2,b}} \\ &\leq |\lambda_j| \|u_j - v_j\|_{\mathcal{H}^{1,b}} + |\lambda_j - \mu_j| \max(\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}}). \end{aligned}$$

Using Hypothesis 1.10 and Lemma A.6 we conclude for $\xi \in \mathbb{R}$

$$\begin{aligned} |Q_j^{r+ct+g^0}(\xi) - Q_j^{g+ct+g^0}(\xi)| &= \frac{\varphi(\xi) \left(\sum_{k=1}^N \varphi(\xi_{kj}^{g+ct+g^0}) - \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0}) \right)}{\sum_{k=1}^N \varphi(\xi_{kj}^{g+ct+g^0}) \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0})} \\ &\leq C \frac{\varphi(\xi) \sum_{k \neq j} \int_0^1 |\varphi'(\xi_{kj}^{g+h(r-g)+ct+g^0})| dh}{\sum_{k=1}^N \varphi(\xi_{kj}^{g+ct+g^0}) \sum_{k=1}^N \varphi(\xi_{kj}^{r+ct+g^0})} \|r - g\| \\ &= C \frac{\sum_{k \neq j} \int_0^1 |\varphi'(\xi_{kj}^{g+h(r-g)+ct+g^0})| dh}{\sum_{k=1}^N \varphi(\xi_{kj}^{g+ct+g^0})} Q_j^{r+ct+g^0}(\xi) \|g - r\| \\ &\leq C e^{\tilde{C} \max(\|r\|, \|g\|)} Q_j^{r+ct+g^0}(\xi) \|r - g\|. \end{aligned} \quad (3.24)$$

We consider

$$\begin{aligned} &\|G_j(t, u + \hat{w}, r + r^0) - G_j(t, v + \hat{w}, g + r^0)\|_{\mathcal{L}_{2,b}} \\ &= \|G_j(t, u + \hat{w}, r + r^0) - G_j(t, v + \hat{w}, r + r^0)\|_{\mathcal{L}_{2,b}} \\ &\quad + \|G_j(t, v + \hat{w}, r + r^0) - G_j(t, v + \hat{w}, g + r^0)\|_{\mathcal{L}_{2,b}} \\ &= J_1 + J_2 \end{aligned}$$

and estimate each term separately. We use Lemma A.8 (with $g = 0, h = 0$) and

obtain

$$\begin{aligned}
J_1 &\leq C \|Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r+ct+g^0}) \right) - f \left(\sum_{k=1}^N (v_k + \hat{w}_k) (\cdot_{kj}^{r+ct+g^0}) \right) \right] \|_{\mathcal{L}_{2,b}} \\
&\quad + C \|Q_j^{r+ct+g^0} \sum_{l=1}^N \left[f \left((u_l + w_l) (\cdot_{lj}^{r+ct+g^0}) \right) - f \left((v_l + w_l) (\cdot_{lj}^{r+ct+g^0}) \right) \right] \|_{\mathcal{L}_{2,b}} \\
&\leq C \sum_{k=1}^N \left(\int_{\mathbb{R}} \|Q_j^{r+ct+g^0}(\xi) (u_k - v_k) (\xi_{kj}^{r+ct+g^0}) \theta_b(\xi_{kj}^{r+ct+g^0}) e^{-b|\xi_{kj}^{r+ct+g^0}| + b|\xi|} \|^2 d\xi \right)^{\frac{1}{2}} \\
&\leq C e^{\tilde{C}\|r\|} \|u - v\|.
\end{aligned}$$

We proceed similarly to Lemma 3.4. It follows from (3.24), Hypothesis 1.6 and Lemma A.8 - A.12 that there exists $\gamma > 0$ such that the following estimate holds

$$\begin{aligned}
J_2 &\leq C \left(\| (Q_j^{r+ct+g^0} - Q_j^{g+ct+g^0}) \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k) (\cdot_{kj}^{r+ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \right. \\
&\quad + \| (-Q_j^{r+ct+g^0} + Q_j^{g+ct+g^0}) \sum_{l \neq j} f \left((v_l + w_l) (\cdot_{lj}^{r+ct+g^0}) \right) \|_{\mathcal{L}_{2,b}} \\
&\quad + \| Q_j^{g+ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k) (\cdot_{kj}^{r+ct+g^0}) \right) - f \left(\sum_{k=1}^N (v_k + \hat{w}_k) (\cdot_{kj}^{g+ct+g^0}) \right) \right] \|_{\mathcal{L}_{2,b}} \\
&\quad \left. + \| Q_j^{g+ct+g^0} \sum_{l=1}^N \left(f \left((v_l + w_l) (\cdot_{lj}^{r+ct+g^0}) \right) - f \left((v_l + w_l) (\cdot_{lj}^{g+ct+g^0}) \right) \right) \|_{\mathcal{L}_{2,b}} \right) \\
&\leq C \left[\left(\| Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k) (\cdot_{kj}^{r+ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \right. \right. \\
&\quad \left. + \| Q_j^{r+ct+g^0} \sum_{l \neq j} f \left((v_l + w_l) (\cdot_{lj}^{r+ct+g^0}) \right) \|_{\mathcal{L}_{2,b}} \right) \|r - g\| e^{\tilde{C} \max(\|r\|, \|g\|)} \\
&\quad + \sum_{k=1}^N \| Q_j^{g+ct+g^0} \left(v_k (\cdot_{kj}^{r+ct+g^0}) - v_k (\cdot_{kj}^{g+ct+g^0}) \right) \|_{\mathcal{L}_{2,b}} \\
&\quad \left. + \sum_{k=1}^N \| Q_j^{g+ct+g^0} \left(w_k (\cdot_{kj}^{r+ct+g^0}) - w_k (\cdot_{kj}^{g+ct+g^0}) \right) \|_{\mathcal{L}_{2,b}} \right] \\
&\leq C e^{\tilde{C} \max(\|r\|, \|g\|)} \|r - g\| \max(1, \|r\|, \|g\|) \\
&\quad * (\|v\|_{\mathcal{H}^{1,b}} + e^{-\gamma t} e^{-\gamma G^0} + \|v\|_{\mathcal{H}^{1,b}} e^{-\gamma t} e^{-\gamma G^0}), \tag{3.25}
\end{aligned}$$

where the last estimate will be explained term by term. For the first term we obtain from (3.24), (3.17), (3.18) and from Lemma A.8, Lemma A.10 (with $h = 0, g = 0$)

and Lemma A.12

$$\begin{aligned}
& \|Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{r+ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \\
&= \|Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{r+ct+g^0}) \right) - f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) \right. \\
&\quad \left. + f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \\
&\leq C e^{\tilde{C}\|r\|} \|r\| \left(\|v\|_{\mathcal{H}^{1,b}} + e^{-\gamma t} e^{-\gamma G^0} \right) \\
&\quad + \|Q_j^{r+ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \\
&\leq C e^{\tilde{C}\|r\|} \|r\| \left(\|v\|_{\mathcal{H}^{1,b}} + e^{-\gamma t} e^{-\gamma G^0} \right) \\
&\quad + \| [Q_j^{r+ct+g^0} - Q_j^{ct+g^0}] \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \\
&\quad + \|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N (v_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(v_j + w_j) \right] \|_{\mathcal{L}_{2,b}} \\
&\leq C e^{\tilde{C}\|r\|} \max(\|r\|, 1) \left(\|v\|_{\mathcal{H}^{1,b}} + e^{-\gamma t} e^{-\gamma G^0} + \|v\|_{\mathcal{H}^{1,b}} e^{-\gamma t} e^{-\gamma G^0} \right). \tag{3.26}
\end{aligned}$$

Using Lemma A.8, Lemma A.10 (with $h = 0, g = 0$) we obtain as in the proof of Lemma 3.3

$$\begin{aligned}
& \left\| \sum_{l \neq j} Q_j^{r+ct+g^0} f \left((v_l + w_l)(\cdot_{lj}^{r+ct+g^0}) \right) \right\|_{\mathcal{L}_{2,b}}^2 \\
&\leq \sum_{l \neq j} \|Q_j^{r+ct+g^0} \left(f \left((v_l + w_l)(\cdot_{lj}^{r+ct+g^0}) \right) - f(w_l^\pm) \right)\|_{\mathcal{L}_{2,b}}^2 \\
&\leq C \sum_{l \neq j} \left[\int_{\mathbb{R}} \left(Q_j^{r+ct+g^0}(\xi) e^{-\eta |\xi_{lj}^{r+ct+g^0}|} e^{b|\xi|} \right)^2 d\xi \right. \\
&\quad \left. + \int_{\mathbb{R}} \|Q_j^{r+ct+g^0}(\xi) v(\xi_{lj}^{r+ct+g^0}) \theta_b(\xi_{lj}^{r+ct+g^0}) e^{-b|\xi_{lj}^{r+ct+g^0}|} e^{b|\xi|}\|^2 d\xi \right] \\
&\leq C e^{2\tilde{C}\|r\|} e^{-\gamma t} e^{-\gamma G^0} + C e^{2\tilde{C}\|r\|} \|v\|_{\mathcal{L}_{2,b}}^2. \tag{3.27}
\end{aligned}$$

The last two terms in (3.25) are estimated as in (3.17), (3.18). ■

3.2.2 Time dependent estimates

In order to apply the local existence theorem (see Lemma 3.24) we show that the operators $E_j, T_j, N_j, j = 1, \dots, N$ are locally Lipschitz in time. The proofs are similar to the proofs in Section 3.2.1, therefore we give only the main steps.

Lemma 3.9. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min \left\{ \frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\})$. Let $0 \leq \delta < \infty$.

Then there exist constants $C_{\tilde{T}}, \tilde{C}_{\tilde{T}} > 0$ such that for all $|s-t| \leq \delta, t, s \geq 0, j = 1, \dots, N$ and for all b with $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+q}, \frac{1}{2}\eta)$ and g^0 with $G^0 \geq 4\bar{B}\delta$ the following estimate is satisfied

$$\|T_j(t) - T_j(s)\|_{\mathcal{L}_{2,b}} \leq C_{\tilde{T}} e^{\tilde{C}_{\tilde{T}}|t-s|} |t-s|.$$

Proof. Let $j \in \{1, \dots, N\}, t, s \geq 0, |t-s| \leq \delta$. We estimate the difference $T_j(s) - T_j(t)$ with the help of Hypothesis 1.6, Lemma A.6, A.10 and A.12 (with $r = 0, g = cs - ct, u = 0$):

$$\begin{aligned} & \|T_j(t) - T_j(s)\|_{\mathcal{L}_{2,b}} \\ & \leq \|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N \hat{w}_k(\cdot, \frac{ct+g^0}{k_j}) \right) - f \left(\sum_{k=1}^N \hat{w}_k(\cdot, \frac{cs+g^0}{k_j}) \right) \right]\|_{\mathcal{L}_{2,b}} \\ & \quad + \|Q_j^{ct+g^0} \sum_{l=1}^N \left[f \left(w_l(\cdot, \frac{ct+g^0}{l_j}) \right) - f \left(w_l(\cdot, \frac{cs+g^0}{l_j}) \right) \right]\|_{\mathcal{L}_{2,b}} \\ & \quad + \left\| \left[Q_j^{ct+g^0} - Q_j^{cs+g^0} \right] \left(f \left(\sum_{k=1}^N \hat{w}_k(\cdot, \frac{cs+g^0}{k_j}) \right) - \sum_{l=1}^N f \left(w_l(\cdot, \frac{cs+g^0}{l_j}) \right) \right) \right\|_{\mathcal{L}_{2,b}} \\ & \leq C e^{\tilde{C}|t-s|} |t-s|. \end{aligned}$$

■

Lemma 3.10. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min \left\{ \frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\})$. Let $\delta \geq 0$.

Then there exist constants $C_{\tilde{E}}, \tilde{C}_{\tilde{E}} > 0$ such that for all $|t-h| \leq \delta, t, h \geq 0, j = 1, \dots, N$ and for all b with $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$ and g^0 with $G^0 \geq 12\bar{B}\delta$ and for all $u = (u_1, \dots, u_N)$ with $u_k(t) \in \mathcal{L}_{2,b}, k = 1, \dots, N$ the following estimate is satisfied

$$\begin{aligned} \|E_j(t, u(t)) - E_j(h, u(h))\|_{\mathcal{L}_{2,b}} & \leq C_{\tilde{E}} \|u(t) - u(h)\|_{\mathcal{L}_{2,b}} \\ & \quad + C_{\tilde{E}} e^{\tilde{C}_{\tilde{E}}|t-h|} |t-h| \max(\|u(t)\|_{\mathcal{L}_{2,b}}, \|u(h)\|_{\mathcal{L}_{2,b}}). \end{aligned}$$

Proof. Let $j \in \{1, \dots, N\}$, $t, h \geq 0, |t - h| \leq \delta$. We estimate

$$\begin{aligned}
& \|E_j(t, u(t)) - E_j(h, u(h))\|_{\mathcal{L}_{2,b}} \\
& \leq C \max_{1 \leq s \leq N} \|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - Df \left(w_s(\cdot, {}_{sj}^{ct+g^0}) \right) \right] \\
& \quad * \left(u_s(\cdot, {}_{sj}^{ct+g^0}, t) - u_s(\cdot, {}_{sj}^{ct+g^0}, h) \right) \|_{\mathcal{L}_{2,b}} \\
& \quad + C \max_{1 \leq s \leq N} \|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - Df \left(w_s(\cdot, {}_{sj}^{ct+g^0}) \right) \right] u_s(\cdot, {}_{sj}^{ct+g^0}, h) \\
& \quad - Q_j^{ch+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ch+g^0}) \right) - Df \left(w_s(\cdot, {}_{sj}^{ch+g^0}) \right) \right] u_s(\cdot, {}_{sj}^{ch+g^0}, h) \|_{\mathcal{L}_{2,b}} \\
& =: I_1 + I_2.
\end{aligned}$$

The first term I_1 is estimated analogously to Lemma 3.5 and we obtain

$$I_1 \leq C \|u(t) - u(h)\|_{\mathcal{L}_{2,b}}.$$

With the help of Lemma A.6, A.8, A.10 we obtain for the second term

$$\begin{aligned}
I_2 & \leq C \max_{1 \leq s \leq N} \left(\|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ct+g^0}) \right) - Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ch+g^0}) \right) \right. \right. \\
& \quad \left. \left. - Df \left(w_s(\cdot, {}_{sj}^{ct+g^0}) \right) + Df \left(w_s(\cdot, {}_{sj}^{ch+g^0}) \right) \right] u_s(\cdot, {}_{sj}^{ct+g^0}, h) \|_{\mathcal{L}_{2,b}} \right. \\
& \quad \left. + \|Q_j^{ct+g^0} \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ch+g^0}) \right) - Df \left(w_s(\cdot, {}_{sj}^{ch+g^0}) \right) \right] \right. \\
& \quad \left. \cdot \left(u_s(\cdot, {}_{sj}^{ct+g^0}, h) - u_s(\cdot, {}_{sj}^{ch+g^0}, h) \right) \|_{\mathcal{L}_{2,b}} \right) \\
& \quad + \| \left(Q_j^{ct+g^0} - Q_j^{ch+g^0} \right) \left[Df \left(\sum_{k=1}^N \hat{w}_k(\cdot, {}_{kj}^{ch+g^0}) \right) - Df \left(w_s(\cdot, {}_{sj}^{ch+g^0}) \right) \right] \\
& \quad \cdot u_s(\cdot, {}_{sj}^{ch+g^0}, h) \|_{\mathcal{L}_{2,b}} \\
& \leq C e^{\tilde{C}|t-h|} \|u(h)\|_{\mathcal{H}^{1,b}} |t - h|.
\end{aligned}$$

■

Lemma 3.11. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min \left\{ \frac{c_{k+1} - c_k}{2|2c_j - c_k - c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\})$. Let

$\delta, \varrho \geq 0$.

Then there exist some constants $C_{\tilde{N}}, \tilde{C}_{\tilde{N}} > 0$ such that for all $|t - s| \leq \delta, t, s \geq 0$, $j = 1, \dots, N$ and for all b with $0 \leq b < \min(\frac{\eta q}{1+2q}, \frac{1}{2}\beta)$ and g^0 with $G^0 \geq 12(\bar{B}\delta + \varrho)$ and for all $(u(t), r(t), \lambda(t)), (u(s), r(s), \lambda(s)) \in B_{\varrho, b}(0)$ with $\|u(t)\|_{\mathcal{H}^{1,b}} \leq 1, \|v(t)\|_{\mathcal{H}^{1,b}} \leq 1$ the following estimate is satisfied

$$\begin{aligned} & \|N_j(t, u(t), r(t), \lambda_j(t)) - N_j(s, u(s), r(s), \lambda_j(s))\|_{\mathcal{L}_{2,b}} \\ & \leq C_{\tilde{N}} e^{\tilde{C}_{\tilde{N}} \max(\|r(s)\|, \|r(t)\|)} \|u(s) - u(t)\|_{\mathcal{H}^{1,b}} \\ & \quad + C_{\tilde{N}} (\max(1, |\lambda_j(t)|, |\lambda_j(s)|) \|u_j(t) - u_j(s)\|_{\mathcal{H}^{1,b}} \\ & \quad + |\lambda_j(t) - \lambda_j(s)| \max(\|u(t)\|_{\mathcal{H}^{1,b}}, \|u(s)\|_{\mathcal{H}^{1,b}}) \\ & \quad + C_{\tilde{N}} e^{\tilde{C}_{\tilde{N}} \max(|t-s|, \|r(t)\|, \|r(s)\|)} \max(1, \|r(t)\|, \|r(s)\|) (\|r(t) - r(s)\| + |t - s|) \\ & \quad * (1 + \max(\|u(t)\|_{\mathcal{H}^{1,b}}, \|u(s)\|_{\mathcal{H}^{1,b}})). \end{aligned}$$

Proof. Let $j \in \{1, \dots, N\}$, $t, s \geq 0, |t - s| \leq \delta$. Define $r^0(t) = ct + g^0$. We estimate

$$\begin{aligned} & \|N_j(t, u(t), r(t), \lambda_j(t)) - N_j(s, u(s), r(s), \lambda_j(s))\|_{\mathcal{L}_{2,b}} \\ & \leq \|f(u_j(t) + w_j) - f(u_j(s) + w_j)\|_{\mathcal{L}_{2,b}} + \|Df(w_j)(u_j(t) - u_j(s))\|_{\mathcal{L}_{2,b}} \\ & \quad + \|\lambda_j(t)u_{j,\xi}(t) - \lambda_j(s)u_{j,\xi}(s)\|_{\mathcal{L}_{2,b}} + \|G_j(t, \hat{w}, r^0) - G_j(s, \hat{w}, r^0)\|_{\mathcal{L}_{2,b}} \\ & \quad + \|E_j(t, u(t)) - E_j(s, u(s))\|_{\mathcal{L}_{2,b}} \\ & \quad + \|G_j(t, u + \hat{w}, r + r^0) - G_j(s, u + \hat{w}, r + r^0)\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

Again we use (A.19) and estimate each term separately. From Lemma 3.9, 3.10 we conclude

$$\begin{aligned} & \|f(u_j(t) + w_j) - f(u_j(s) + w_j)\|_{\mathcal{L}_{2,b}} \\ & \quad + \|Df(w_j)(u_j(t) - u_j(s))\|_{\mathcal{L}_{2,b}} \leq C \|u_j(t) - u_j(s)\|_{\mathcal{L}_{2,b}}, \end{aligned}$$

$$\begin{aligned} & \|\lambda_j(t)u_{j,\xi}(t) - \lambda_j(s)u_{j,\xi}(s)\|_{\mathcal{L}_{2,b}} \\ & \leq |\lambda_j(t)| \|u_j(t) - u_j(s)\|_{\mathcal{H}^{1,b}} + |\lambda_j(t) - \lambda_j(s)| \max(\|u(t)\|_{\mathcal{H}^{1,b}}, \|u(s)\|_{\mathcal{H}^{1,b}}), \end{aligned}$$

and

$$\begin{aligned}
& \|G_j(t, \hat{w}, r^0) - G_j(s, \hat{w}, r^0)\|_{\mathcal{L}_{2,b}} + \|E(t, u(t)) - E_j(s, u(s))\|_{\mathcal{L}_{2,b}} \\
& = \|T_j(t) - T_j(s)\|_{\mathcal{L}_{2,b}} + \|E_j(t, u(t)) - E_j(s, u(s))\|_{\mathcal{L}_{2,b}} \\
& \leq C e^{\tilde{C}|t-s|} (|t-s| + |t-s| \max(\|u(t)\|_{\mathcal{L}_{2,b}}, \|u(s)\|_{\mathcal{L}_{2,b}})) + C \|u(t) - u(s)\|_{\mathcal{L}_{2,b}}.
\end{aligned}$$

Furthermore, we estimate the difference

$$\begin{aligned}
& \|G_j(t, u + \hat{w}, r + r^0) - G_j(s, u + \hat{w}, r + r^0)\|_{\mathcal{L}_{2,b}} \\
& \leq \|Q_j^{r(t)+ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(t)+ct+g^0}, t) \right) - f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right. \\
& \quad \left. - \sum_{k=1}^N f \left((u_k + w_k) (\cdot_{kj}^{r(t)+ct+g^0}, t) \right) + \sum_{k=1}^N f \left((u_k + w_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right] \|_{\mathcal{L}_{2,b}} \\
& + \|Q_j^{r(t)+ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right. \\
& \quad \left. - \sum_{k=1}^N f \left((u_k + w_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right] \\
& \quad - Q_j^{r(s)+cs+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(s)+cs+g^0}, s) \right) \right. \\
& \quad \left. + \sum_{k=1}^N f \left((u_k + w_k) (\cdot_{kj}^{r(s)+cs+g^0}, s) \right) \right] \|_{\mathcal{L}_{2,b}} \\
& =: I_1 + I_2.
\end{aligned}$$

We estimate I_1 with the help of Lemma A.8 (with $r = r(t), g = 0, h = 0$)

$$I_1 \leq C e^{\tilde{C} \max(\|r(t)\|, \|r(s)\|)} \|u(t) - u(s)\|_{\mathcal{H}^{1,b}}.$$

Furthermore, we infer from Hypothesis 1.6, Lemma A.6 - A.12

$$\begin{aligned}
I_2 &\leq \left\| \left(Q_j^{r(t)+ct+g^0} - Q_j^{r(s)+cs+g^0} \right) \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right. \right. \\
&\quad \left. \left. - \sum_{l=1}^N f \left((u_l + w_l) (\cdot_{lj}^{r(t)+ct+g^0}, s) \right) \right] \right\|_{\mathcal{L}_{2,b}} \\
&\quad + \left\| Q_j^{r(s)+cs+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(t)+ct+g^0}, s) \right) \right. \right. \\
&\quad \left. \left. - f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{kj}^{r(s)+cs+g^0}, s) \right) \right] \right\|_{\mathcal{L}_{2,b}} \\
&\quad + \left\| Q_j^{r(s)+cs+g^0} \sum_{l=1}^N \left[f \left((u_l + w_l) (\cdot_{lj}^{r(t)+ct+g^0}, s) \right) \right. \right. \\
&\quad \left. \left. - f \left((u_l + w_l) (\cdot_{lj}^{r(s)+cs+g^0}, s) \right) \right] \right\|_{\mathcal{L}_{2,b}} \\
&\leq C e^{\tilde{C} \max(|t-s|, \|r(t)\|, \|r(s)\|)} \max(1, \|r(t)\|, \|r(s)\|) \\
&\quad * (|t-s| + \|r(t) - r(s)\|) (\|u(s)\|_{\mathcal{H}^{1,b}} + 1).
\end{aligned}$$

The final estimate is a consequence of (3.26), (3.27), the estimate

$$\begin{aligned}
&|Q_j^{r(t)+ct+g^0}(\xi) - Q_j^{r(s)+cs+g^0}(\xi)| \\
&\leq C e^{\tilde{C} \max(|t-s|, \|r(t)\|, \|r(s)\|)} Q_j^{r(t)+ct+g^0}(\xi) (|t-s| + \|r(t) - r(s)\|),
\end{aligned}$$

and of the following estimate obtain with the help of Lemma A.8, A.10 (with $r = r(s)$, $g = ct - cs + r(t)$)

$$\begin{aligned}
& \|Q_j^{r(s)+cs+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{k_j}^{r(t)+ct+g^0}, s) \right) \right. \\
& \quad \left. - f \left(\sum_{k=1}^N (u_k + \hat{w}_k) (\cdot_{k_j}^{r(s)+cs+g^0}, s) \right) \right] \|_{\mathcal{L}_{2,b}} \\
& + \|Q_j^{r(s)+cs+g^0} \sum_{l=1}^N \left[f \left((u_l + w_l) (\cdot_{l_j}^{r(t)+ct+g^0}, s) \right) \right. \\
& \quad \left. - f \left((u_l + w_l) (\cdot_{l_j}^{r(s)+cs+g^0}, s) \right) \right] \|_{\mathcal{L}_{2,b}} \\
& \leq C \sum_{k=1}^N \sup_{0 \leq h \leq 1} \left(\|u_{k,\xi}(s)\|_{\mathcal{L}_{2,b}} \sup_{\xi \in \mathbb{R}} |Q_j^{r(s)+cs+g^0} e^{b|\xi|} e^{-b|\xi_{k_j}^{r(s)+cs+h(r(t)-r(s)+ct-cs)+g^0}}| \right. \\
& \quad \left. + \sup_{\xi \in \mathbb{R}} |Q_j^{r(s)+cs+g^0} e^{b|\xi|} e^{-\eta|\xi_{k_j}^{r(s)+cs+h(r(t)-r(s)+ct-cs)+g^0}}| \right) (\|r(t) - r(s)\| + |t - s|) \\
& \leq C e^{\tilde{C} \max(|t-s|, \|r(t)\|, \|r(s)\|)} (\|r(t) - r(s)\| + |t - s|) (1 + \|u(s)\|_{\mathcal{H}^{1,b}}).
\end{aligned}$$

■

Let $q := \min(\frac{1}{4}, \min \left\{ \frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\})$ and $b_1 := \min(\frac{1}{2}\beta, \frac{\eta q}{1+2q})$.

For $0 \leq b < b_1$ we define the b -dependent constant $\gamma_1 := \min(\gamma_T, \gamma_N, \gamma_n)$ and for $0 < b < b_1$ we define the b -dependent constant $\gamma := \min(\gamma_1, \gamma_E)$ obtained from Lemma 3.3 - 3.8. Recall that γ_1 and γ tend to zero as b tends to b_1 . Furthermore γ_1 tends to some positive constant as b tends to zero, but γ tends to zero if b tends to zero. Therefore it will be important to fix a small b greater than zero in the proof of the Stability Theorem 3.1.

3.3 The linear inhomogeneous decoupled system

Before we consider the nonlinear inhomogeneous coupled system (3.6) - (3.8) and apply the nonlinearity estimates in Section 3.2 above, we analyze the decoupled linear system for $j = 1, \dots, N$

$$u_{j,t} = \Lambda_j u_j + \lambda_j w_{j,\xi} + k_j, \quad (3.28)$$

$$r_{j,t} = \lambda_j, \quad (3.29)$$

$$0 = \langle \hat{v}_{j,\xi}, u_j \rangle \quad (3.30)$$

with $k_j \in \mathcal{C}([0, \tau], \mathcal{L}_{2,b})$.

We will make use of bilinear forms and projectors to derive a reduced projected decoupled system.

Define the bilinear form $a_j : \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathbb{R}, j = 1, \dots, N$ by

$$a_j(u, v) = \int_{\mathbb{R}} -u_\xi(\xi)^T A v_\xi(\xi) + u(\xi)^T (B_j v_\xi(\xi) + C_j(\xi) v(\xi)) d\xi.$$

For $u, \hat{v}_{j,\xi} \in \mathcal{H}^1, j = 1, \dots, N$ we estimate

$$|a_j(\hat{v}_{j,\xi}, u)| \leq C_v \|u\|_{\mathcal{H}^1}$$

for some $C_v > 0$. Let $u \in \mathcal{H}^2$, then we obtain for $j = 1, \dots, N$

$$a_j(\hat{v}_{j,\xi}, u) = \langle \hat{v}_{j,\xi}, \Lambda_j u \rangle.$$

Using Hypothesis 1.5 we conclude that there exists some $C_{v,w} > 0$ such that for $j = 1, \dots, N$ the following estimate is satisfied

$$|\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1}| \leq C_{v,w}. \quad (3.31)$$

The projector $P_j, j = 1, \dots, N$ onto $\hat{v}_{j,\xi}^\perp$ along $w_{j,\xi}$ is given by

$$P_j u = u - w_{j,\xi} \langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} \langle \hat{v}_{j,\xi}, u \rangle. \quad (3.32)$$

From (3.31) and from the Cauchy-Schwarz Theorem, [39], Theorem V.1.2, we conclude that the projectors $P_j, j = 1, \dots, N$ are bounded: Let $\|u\|_*$ be bounded, we obtain the estimate

$$\begin{aligned} \|P_j u\|_* &\leq \|u\|_* + C_{v,w} \|\hat{w}_{j,\xi}\|_* |\langle \hat{v}_{j,\xi}, u \rangle| \\ &\leq \|u\|_* + C_{v,w} \|\hat{w}_{j,\xi}\|_* \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|u\|_{\mathcal{L}_2} \leq C_P \|u\|_* \end{aligned} \quad (3.33)$$

with $C_P := 1 + C_{v,w} \|w_\xi\|_* \|\hat{v}_\xi\|_{\mathcal{L}_2}$ and $*$ = $\mathcal{L}_{2,b}$ if $w_j \in \mathcal{H}^{1,b}$, $*$ = $\mathcal{H}^{1,b}$ if $w_j \in \mathcal{H}^{2,b}$ and $*$ = $\mathcal{H}^{2,b}$ if $w_j \in \mathcal{H}^{3,b}$. Note that we conclude $w_{j,\xi} \in \mathcal{H}^{2,b}$ and C_P is independent of b for $0 \leq b \leq \frac{\eta}{2}$ from (1.27), the estimates (A.9), (A.10) and the calculation

$$\int_{\mathbb{R}} (e^{-\eta|\xi|} \theta_b(\xi))^2 d\xi \leq \int_{\mathbb{R}} (e^{-\eta|\xi|+b|\xi|})^2 d\xi \leq \frac{2}{\eta-b} \leq \frac{4}{\eta}.$$

The next lemma gives an equivalent formulation of the system (3.28) - (3.30). The result is proven as in [36], Lemma 1.17.

Lemma 3.12. *Let $k_j \in \mathcal{C}([0, \tau], \mathcal{L}_{2,b}), j = 1, \dots, N$ and assume that (3.31) holds. Then (u, r, λ) is a solution of (3.28) - (3.30) for $j = 1, \dots, N$ on the interval $(0, \tau)$ with consistent initial conditions*

$$u_j^0 \in \mathcal{H}^{1,b}, \langle \hat{v}_{j,\xi}, u_j^0 \rangle = 0, r_j(0) = r_j^0 \in \mathbb{R}, j = 1, \dots, N$$

if and only if $u = (u_1, \dots, u_N)$ is a solution of the PDEs

$$u_{j,t} = P_j(\Lambda_j u_j + k_j), \quad u_j(0) = u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j), \quad (3.34)$$

$\lambda = (\lambda_1, \dots, \lambda_N)$ satisfies on $[0, \tau)$

$$\lambda_j(t) = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j(t)) + \langle \hat{v}_{j,\xi}, k_j(t) \rangle) \quad (3.35)$$

and $r = (r_1, \dots, r_N)$ satisfies on $[0, \tau)$

$$r_j(t) = \int_0^t \lambda_j(s) ds + r_j^0. \quad (3.36)$$

Proof. We know that (3.29) together with the initial conditions $r_j(0) = 0, j = 1, \dots, N$ is equivalent to (3.36).

Using (3.28) and differentiating (3.30) with respect to time, we obtain (3.35). From (3.35) and (3.28) we get the differential equation (3.34).

Conversely, let u_j be a solution of (3.34) with initial condition $u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$. This implies $u_j \in \mathcal{R}(P_j)$, thus (3.30) holds. By a calculation using (3.34), (3.35) and the definition of P_j we obtain (3.28). ■

Recall the linear inhomogeneous equation (3.34). Analogous to the non-weighted case, we want to apply the variation of constants formula to obtain a formula for the solution $u_j(t), j = 1, \dots, N$:

We consider the operator $\Lambda_{P,j} := P_j \Lambda_j |_{\mathcal{R}(P_j)}$ on the exponentially weighted subspace $\mathcal{R}(P_j) \cap \mathcal{L}_{2,b}$ for $j = 1, \dots, N$. In the next sections we show that this operator is sectorial in this exponentially weighted subspace, therefore we can solve equation (3.34) with the help of the variation of constants formula via

$$u_j(t) = e^{\Lambda_{P,j} t} u_j^0 + \int_0^t e^{\Lambda_{P,j}(t-s)} P_j k_j(s) ds,$$

compare Section 3.5. The solution operator $e^{\Lambda_{P,j} t}$ on $\mathcal{R}(P_j) \cap \mathcal{L}_{2,b}$ for $j = 1, \dots, N$ is defined with the help of the resolvent $(sI - \Lambda_{P,j})^{-1}$ as the Dunford integral, see [17],

$$e^{\Lambda_{P,j} t} = \frac{1}{2\pi i} \int_{\Gamma} e^{st} (sI - \Lambda_{P,j})^{-1} ds, \quad \forall t \geq 0, \quad (3.37)$$

where Γ is a contour in $\rho(\Lambda_{P,j})$ with $\arg s \rightarrow \pm\theta$ as $|s| \rightarrow \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. We use semigroup theory to show the resolvent estimates on $\mathcal{R}(P_j) \cap \mathcal{L}_{2,b}$. We have to prove that $\Lambda_{j,P}$ is sectorial for all $j = 1, \dots, N$, we use common tools like in [17], [36]. The main difficulty we have to handle are the exponentially weighted spaces.

3.4 Sectorial operators in $\mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$

To shorten notation in most of the proofs of this section we suppress the index j . Before we specify a lemma which gives the solution of the system (3.28) - (3.30), we have to consider the resolvent operator for $P_j\Lambda_j$ on $\mathcal{R}(P_j) \cap \mathcal{L}_{2,b}$ and show resolvent estimates for each $j = 1, \dots, N$.

We begin by proving resolvent estimates for the operator $\Lambda_j, j = 1, \dots, N$. The proof of the following lemma is deferred to the Appendix A.6.

Lemma 3.13. *Assume that Hypotheses 1.4, 1.6 hold. For $j = 1, \dots, N$ there exist constants $\zeta \in (\frac{\pi}{2}, \pi), K_G, C_R > 0$ such that $v_j = R_s(\Lambda_j)\tilde{k}_j = (sI - \Lambda_j)^{-1}\tilde{k}_j$ satisfies the following estimates for all $s \in S_{0,\zeta}$ with $|s| > K_G$*

$$|s|^2 \|v_j\|_{\mathcal{L}_2}^2 + |s| \|v_j\|_{\mathcal{H}^1}^2 \leq C_R \|\tilde{k}_j\|_{\mathcal{L}_2}^2. \quad (3.38)$$

For s in a compact set $S_C \subset \rho(\Lambda_j)$ we have a uniform estimate

$$\|v_j\|_{\mathcal{H}^1} \leq C_R \|\tilde{k}_j\|_{\mathcal{L}_2}. \quad (3.39)$$

Assume further $\tilde{k}_j = (sI - \Lambda_j)v_j \in \mathcal{H}^1$ then is for each $s \in S_{0,\zeta}$ with $|s| > K_G$ the following estimate satisfied

$$|s|^2 \|v_j\|_{\mathcal{H}^1}^2 + |s| \|v_j\|_{\mathcal{H}^2}^2 \leq C_R \|\tilde{k}_j\|_{\mathcal{H}^1}^2. \quad (3.40)$$

The following lemma shows that the estimates in Lemma 3.13 are also true for slightly weighted spaces.

Lemma 3.14. *Assume that Hypotheses 1.4, 1.6 hold.*

There exists $b_2, K_G > 0, C_G, \zeta \in (\frac{\pi}{2}, \pi)$ such that $u_j = R_s(\Lambda_j)k_j = (sI - \Lambda_j)^{-1}k_j, j = 1, \dots, N$ satisfies the following estimates for all $0 \leq b < b_2, j = 1, \dots, N$ and $s \in S_{0,\zeta}$ with $|s| > K_G$

$$|s|^2 \|u_j\|_{\mathcal{L}_{2,b}}^2 + |s| \|u_j\|_{\mathcal{H}^{1,b}}^2 \leq C_G \|k_j\|_{\mathcal{L}_{2,b}}^2.$$

For s in a compact set $S_C \subset \rho(\Lambda_j)$ we have a uniform estimate

$$\|u_j\|_{\mathcal{H}^{1,b}} \leq C_G \|k_j\|_{\mathcal{L}_{2,b}}.$$

If, in addition, $k_j = (sI - \Lambda_j)u_j \in \mathcal{H}^{1,b}$, then for each $s \in S_{0,\zeta}$ with $|s| > K_G$ is the following estimate satisfied

$$|s| \|u_j\|_{\mathcal{H}^{1,b}} \leq C_G \|k_j\|_{\mathcal{H}^{1,b}}.$$

Proof. The case $b = 0$ is treated in Lemma 3.13.

Let $b_2 := \min(1, \frac{\sqrt{K_G}}{2\sqrt{C_R}(5\bar{B}+27\|A\|)}, \frac{1}{2C_R(\bar{B}+5\|A\|)})$ and $0 < b < b_2$. Define $v := \theta_b u$, $\tilde{k} := \theta_b k$. We consider the equation $u = R_s(\Lambda)k = (sI - \Lambda)^{-1}k$, which is equivalent to $0 = (sI - \Lambda)u - k$. We consider the right hand side in the weighted space $\mathcal{L}_{2,b}$

$$\begin{aligned} \|(sI - \Lambda)u - k\|_{\mathcal{L}_{2,b}} &= \|\theta_b((sI - \Lambda)u - \tilde{k})\|_{\mathcal{L}_2} \\ &= \|sv - \theta_b\Lambda(\theta_b)^{-1}v - \tilde{k}\|_{\mathcal{L}_2} \\ &= \|(s - \Lambda)v - R_bv - \tilde{k}\|_{\mathcal{L}_2}, \end{aligned}$$

where

$$R_bv = B\theta_b(\theta_b^{-1})_\xi v + A\theta_b(\theta_b^{-1})_{\xi\xi}v + 2A\theta_b(\theta_b^{-1})_\xi v_\xi.$$

The derivative of R_bv with respect to ξ for $v \in \mathcal{H}^2$ has the form

$$\begin{aligned} (R_bv)_\xi &= B(\theta_b)_\xi(\theta_b^{-1})_\xi v + B\theta_b(\theta_b^{-1})_{\xi\xi}v + B\theta_b(\theta_b^{-1})_\xi v_\xi \\ &\quad + A(\theta_b)_\xi(\theta_b^{-1})_{\xi\xi}v + A\theta_b(\theta_b^{-1})_{\xi\xi\xi}v + 3A\theta_b(\theta_b^{-1})_{\xi\xi}v_\xi \\ &\quad + 2A(\theta_b)_\xi(\theta_b^{-1})_\xi v_\xi + 2A\theta_b(\theta_b^{-1})_\xi v_{\xi\xi}. \end{aligned}$$

Using (A.6) - (A.12) and (1.31) we estimate for $v \in \mathcal{H}^1$

$$\begin{aligned} \|R_bv\|_{\mathcal{L}_2} &\leq \bar{B}b\|v\|_{\mathcal{L}_2} + \|A\|3b^2\|v\|_{\mathcal{L}_2} + \|A\|2b\|v_\xi\|_{\mathcal{L}_2} \\ &\leq b(\bar{B} + (3b + 2)\|A\|)\|v\|_{\mathcal{H}^1} \leq b(\bar{B} + 5\|A\|)\|v\|_{\mathcal{H}^1}, \end{aligned}$$

and further for $v \in \mathcal{H}^2$,

$$\begin{aligned} \|(R_bv)_\xi\|_{\mathcal{L}^2} &\leq b\bar{B}(1 + 4b)\|v\|_{\mathcal{H}^1} + \|A\|b(14b^2 + 11b + 2)\|v\|_{\mathcal{H}^2} \\ &\leq b(5\bar{B} + 27\|A\|)\|v\|_{\mathcal{H}^2} \end{aligned}$$

and thus

$$\|R_bv\|_{\mathcal{H}^1} \leq b\sqrt{2}(5\bar{B} + 27\|A\|)\|v\|_{\mathcal{H}^2}.$$

We apply Lemma 3.13 to v and $\tilde{k} + R_bv$ and obtain that there exist constants $\zeta \in (\frac{\pi}{2}, \pi)$, $K_G, C_R > 0$ such that for each $s \in S_{0,\zeta}$ with $|s| > K_G$ the following estimate is satisfied

$$\begin{aligned} |s|^2\|v\|_{\mathcal{L}_2}^2 + |s|\|v\|_{\mathcal{H}^1}^2 &\leq C_R\|\tilde{k} + R_bv\|_{\mathcal{L}_2}^2 \\ &\leq 2C_R\|\tilde{k}\|_{\mathcal{L}_2}^2 + 2C_R(b(\bar{B} + 5\|A\|))^2\|v\|_{\mathcal{H}^1}^2. \end{aligned}$$

It follows

$$|s|^2\|v\|_{\mathcal{L}_2}^2 + (K_G - 2C_Rb^2(\bar{B} + 5\|A\|)^2)\|v\|_{\mathcal{H}^1}^2 \leq 2C_R\|\tilde{k}\|_{\mathcal{L}_2}^2$$

and

$$|s|^2 \|u\|_{\mathcal{L}_{2,b}}^2 = |s|^2 \|v\|_{\mathcal{L}_2}^2 \leq 2C_R \|\tilde{k}\|_{\mathcal{L}_2}^2 = 2C_R \|k\|_{\mathcal{L}_{2,b}}^2.$$

Furthermore,

$$\sqrt{|s|} \|v\|_{\mathcal{H}^1} \leq \sqrt{C_R} \|\tilde{k} + R_b v\|_{\mathcal{L}_2} \leq \sqrt{C_R} \left(\|\tilde{k}\|_{\mathcal{L}_2} + b(\bar{B} + 5\|A\|) \|v\|_{\mathcal{H}^1} \right)$$

and we obtain

$$\begin{aligned} \|v\|_{\mathcal{H}^1} &\leq \frac{\sqrt{C_R}}{\sqrt{|s|}} \left(\|\tilde{k}\|_{\mathcal{L}_2} + b(\bar{B} + 5\|A\|) \|v\|_{\mathcal{H}^1} \right) \\ &\leq \frac{\sqrt{C_R}}{\sqrt{|s|}} \|\tilde{k}\|_{\mathcal{L}_2} + \frac{\sqrt{C_R} b(\bar{B} + 5\|A\|)}{\sqrt{K_G}} \|v\|_{\mathcal{H}^1} \end{aligned}$$

or equivalently

$$\left(1 - \frac{\sqrt{C_R} b(\bar{B} + 5\|A\|)}{\sqrt{K_G}} \right) \|v\|_{\mathcal{H}^1} \leq \frac{\sqrt{C_R}}{\sqrt{|s|}} \|\tilde{k}\|_{\mathcal{L}_2}.$$

Thus follows

$$\begin{aligned} |s| \|u\|_{\mathcal{H}^{1,b}}^2 &= |s| \|v\|_{\mathcal{H}^1}^2 \leq \frac{K_G C_R}{(\sqrt{K_G} - \sqrt{C_R} b(\bar{B} + 5\|A\|))^2} \|\tilde{k}\|_{\mathcal{L}_2}^2 \\ &= \frac{K_G C_R}{(\sqrt{K_G} - \sqrt{C_R} b(\bar{B} + 5\|A\|))^2} \|k\|_{\mathcal{L}_{2,b}}^2. \end{aligned}$$

For s in a compact set $S_C \subset \rho(\Lambda)$ we have a uniform estimate

$$\|v\|_{\mathcal{H}^1} \leq C_R \|\tilde{k}\|_{\mathcal{L}_2} + C_R b(\bar{B} + 5\|A\|) \|v\|_{\mathcal{H}^1}.$$

It follows

$$\|v\|_{\mathcal{H}^1} \leq \frac{C_R}{1 - C_R b(\bar{B} + 5\|A\|)} \|\tilde{k}\|_{\mathcal{L}_2}$$

and therefore we obtain

$$\begin{aligned} \|u\|_{\mathcal{H}^{1,b}} &= \|v\|_{\mathcal{H}^1} \leq \frac{C_R}{1 - C_R b(\bar{B} + 5\|A\|)} \|\tilde{k}\|_{\mathcal{L}_2} \\ &= \frac{C_R}{1 - C_R b(\bar{B} + 5\|A\|)} \|k\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

Let $\tilde{k} \in \mathcal{H}^1$. Again we apply Lemma 3.13 to v and $\tilde{k} + R_b v$ and obtain that there exist constants $\zeta \in (\frac{\pi}{2}, \pi)$, $K_G, C_R > 0$ such that for each $s \in S_{0,\zeta}$ with $|s| > K_G$ holds

$$\begin{aligned} |s|^2 \|v\|_{\mathcal{H}^1}^2 + |s| \|v\|_{\mathcal{H}^2}^2 &\leq C_R \|\tilde{k} + R_b v\|_{\mathcal{H}^1}^2 \\ &\leq 2C_R \|\tilde{k}\|_{\mathcal{H}^1}^2 + 2C_R (b\sqrt{2}(5\bar{B} + 27\|A\|))^2 \|v\|_{\mathcal{H}^2}^2 \end{aligned}$$

or equivalently written

$$|s|^2 \|v\|_{\mathcal{H}^1}^2 + (K_G - 2C_R(b\sqrt{2}(5\bar{B} + 27\|A\|))^2) \|v\|_{\mathcal{H}^2}^2 \leq 2C_R \|\tilde{k}\|_{\mathcal{H}^1}^2.$$

Thus follows

$$|s|^2 \|u\|_{\mathcal{H}^{1,b}}^2 = |s|^2 \|v\|_{\mathcal{H}^1}^2 \leq 2C_R \|\tilde{k}\|_{\mathcal{H}^1}^2 = 2C_R \|k\|_{\mathcal{H}^{1,b}}^2.$$

■

The next lemma can be proven in the same way as [36], Lemma 1.20 without exponentially weighted norms.

Lemma 3.15. *Let $b \geq 0$, $k_j \in \mathcal{L}_{2,b}$, $j = 1, \dots, N$, $s \in \mathbb{C}$ and let (3.31) be satisfied. Then $(u, \lambda) \in (\mathcal{H}^{2,b})^N \times \mathbb{R}^N$ is a solution of*

$$(sI - \Lambda_j)u_j - w_{j,\xi}\lambda_j = P_j k_j, \quad (3.41)$$

$$\langle \hat{v}_{j,\xi}, u_j \rangle = 0 \quad (3.42)$$

for $j = 1, \dots, N$ if and only if $u = (u_1, \dots, u_N)$ with $u_k \in \mathcal{H}^{2,b} \cap \mathcal{R}(P_j)$, $k = 1, \dots, N$ is a solution of the resolvent equation

$$(sI - P_j \Lambda_j)u_j = P_j k_j \quad (3.43)$$

for $j = 1, \dots, N$ and $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ satisfies

$$\lambda_j = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} a_j(\hat{v}_{j,\xi}, u_j). \quad (3.44)$$

Remark 3.16. *If $s \neq 0$ and $u \in \mathcal{H}^{2,b}$ is a solution of (3.43), then we conclude that $u \in \mathcal{R}(P_j)$. Let $s = 0$ and $u_j \in \mathcal{H}^{2,b}$ be a solution of (3.43), then also $\tilde{u}_j := P_j u_j$ solves (3.43) and $\tilde{u}_j \in \mathcal{H}^{2,b} \cap \mathcal{R}(P_j)$.*

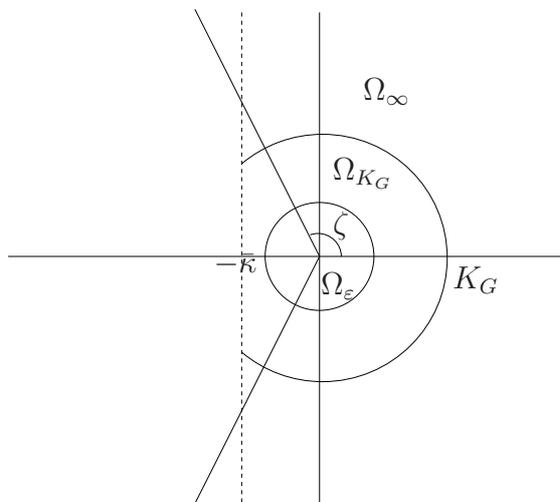
To obtain resolvent estimates for the projected PDAE system (3.34), (3.35) and in particular for the operator $\Lambda_{P,j}$, we have to consider the system (3.43), (3.44). In the next part of this section we show resolvent estimates in different domains of \mathbb{C} for the system (3.41), (3.42) which is equivalent to (3.43), (3.44) by the lemma above. For these estimates we make use of these resolvent estimates in Lemma 3.14 for the operators Λ_j , $j = 1, \dots, N$ in exponentially weighted norms.

Let $\bar{\kappa}$ be given by the eigenvalue Condition 1.8 and the spectral Condition 1.9, let

$K_G > 0$, $\zeta \in (\frac{\pi}{2}, \pi)$ be defined as in Lemma 3.14 and let $0 < \varepsilon < K_G$. We define the subsets

$$\Omega_\varepsilon = \{s \in \mathbb{C} \mid |s| < \varepsilon, \Re s \geq -\bar{\kappa}\},$$

$$\Omega_{K_G} = \{s \in \mathbb{C} \mid \varepsilon \leq |s| \leq K_G, \Re s \geq -\bar{\kappa}\},$$

Figure 3.5: Sections $\Omega_{K_G}, \Omega_\infty, \Omega_\varepsilon \subset \mathbb{C}$

$$\Omega_\infty = \{s \in \mathbb{C} \mid |s| > K_G, |\arg(s)| < \zeta\},$$

see Figure 3.5.

To obtain some resolvent estimates in $\Omega_\varepsilon, \Omega_{K_G}, \Omega_\infty$ we use Lemma 3.15 and Lemma 3.14 and consider the system (3.41), (3.42) instead of (3.43), (3.44). Note that the idea of proving resolvent estimates in domains like $\Omega_\varepsilon, \Omega_{K_G}, \Omega_\infty$ is well known, cf. [3], [36]. The difference here is to prove the resolvent estimates for slightly weighted norms.

Lemma 3.17. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold.*

There exists $b_2 > 0$ and a constant $C_A > 0$ such that for all $j = 1, \dots, N$, $0 \leq b < \min(b_2, \frac{\eta}{2})$ and for each $s \in \Omega_{K_G} \cup \Omega_\infty$ there exists a solution (u, λ) of (3.41), (3.42) for which the following estimates are satisfied

$$\|u_j\|_{\mathcal{H}^{1,b}} + |\lambda_j| \leq C_A \|k_j\|_{\mathcal{L}_{2,b}} \quad \text{for } s \in \Omega_{K_G}$$

and

$$|s|^2 \|u_j\|_{\mathcal{L}_{2,b}}^2 + |s| \|u_j\|_{\mathcal{H}^{1,b}}^2 + |\lambda_j|^2 \leq C_A \|k_j\|_{\mathcal{L}_{2,b}}^2 \quad \text{for } s \in \Omega_\infty.$$

If $k_j \in \mathcal{H}^{2,b}$, the following estimate holds

$$|s| \|u_j\|_{\mathcal{H}^{1,b}} + |\lambda_j| \leq C_A \|k_j\|_{\mathcal{H}^{1,b}} \quad \text{as } s \in \Omega_\infty.$$

Proof. The proof is similar to [36], Lemma 1.21. If we define $u = (sI - \Lambda)^{-1}k$ then we conclude from Lemma 3.14 and Hypotheses 1.8, 1.9 that there exist $b_2 > 0$ and

constants $K_G, C_G > 0, \zeta \in (\frac{\pi}{2}, \pi)$ such that we obtain for each $0 \leq b < \min(\frac{\eta}{2}, b_2)$ and $s \in C_\infty$ the following estimate

$$|s|^2 \|u\|_{\mathcal{L}_{2,b}}^2 + |s| \|u\|_{\mathcal{H}^{1,b}}^2 \leq C_G \|k\|_{\mathcal{L}_{2,b}}^2. \quad (3.45)$$

and for s in the compact set $C_{K_G} \subset \mathbb{C}$ we have a uniform estimate

$$\|u\|_{\mathcal{H}^{1,b}} \leq C_G \|k\|_{\mathcal{L}_{2,b}}. \quad (3.46)$$

Let $s \in \rho(\Lambda)$, we obtain a solution of (3.41) by taking the part $w_\xi \lambda$ to the right hand side and conclude

$$u = R_s(\Lambda)(Pk + w_\xi \lambda).$$

We use equation (3.42) and get

$$\lambda = -\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1} \langle \hat{v}_\xi, R_s(\Lambda)Pk \rangle$$

and

$$u = HR_s(\Lambda)Pk,$$

where H denotes the projector defined by

$$Hw = w - R_s(\Lambda)w_\xi \langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1} \langle \hat{v}_\xi, w \rangle.$$

Further we conclude from Hypothesis 1.8

$$\langle \hat{v}_\xi, w_\xi \rangle = \langle \hat{v}_\xi, R_s(\Lambda)(sw_\xi - \Lambda w_\xi) \rangle = s \langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle$$

and consequently we obtain

$$|\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle|^{-1} = |s| |\langle \hat{v}_\xi, w_\xi \rangle|^{-1} \leq |s| C_{v,w}.$$

As we have seen above, for each $s \in \Omega_\infty$ the following estimates are satisfied

$$\|R_s(\Lambda)k\|_{\mathcal{L}_{2,b}} \leq \frac{\sqrt{C_G}}{|s|} \|k\|_{\mathcal{L}_{2,b}}, \quad \|R_s(\Lambda)k\|_{\mathcal{H}^{1,b}} \leq \frac{\sqrt{C_G}}{\sqrt{|s|}} \|k\|_{\mathcal{L}_{2,b}}.$$

Using (1.21) we estimate for $z \in \mathcal{L}_{2,b}$

$$\begin{aligned} \|Hz\|_{\mathcal{L}_{2,b}} &\leq \|z\|_{\mathcal{L}_{2,b}} + \|R_s(\Lambda)w_\xi\|_{\mathcal{L}_{2,b}} |\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1}| |\langle \hat{v}_\xi, z \rangle| \\ &\leq \|z\|_{\mathcal{L}_{2,b}} + \|R_s(\Lambda)w_\xi\|_{\mathcal{L}_{2,b}} |s| C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|z\|_{\mathcal{L}_2} \\ &\leq \|z\|_{\mathcal{L}_{2,b}} + \sqrt{C_G} \|w_\xi\|_{\mathcal{L}_{2,b}} C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|z\|_{\mathcal{L}_2} \\ &\leq C_H \|z\|_{\mathcal{L}_{2,b}} \end{aligned}$$

for some $C_H > 0$ and thus

$$\|u\|_{\mathcal{L}_{2,b}} = \|HR_s(\Lambda)Pk\|_{\mathcal{L}_{2,b}} \leq C_H \|R_s(\Lambda)Pk\|_{\mathcal{L}_{2,b}} \leq \frac{C_H \sqrt{C_G} C_P}{|s|} \|k\|_{\mathcal{L}_{2,b}}.$$

Using $R_s(\Lambda)w_\xi = s^{-1}w_\xi$ we obtain $\|R_s(\Lambda)w_\xi\|_{\mathcal{H}^{1,b}} = |s|^{-1} \|\theta_b w_\xi\|_{\mathcal{H}^1}$. From $b < \min(1, \frac{\eta}{2})$, Hypothesis 1.6 and (A.9) follows $|(\theta_b)_\xi(\xi)(\theta_b)^{-1}(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$ and $\|R_s(\Lambda)w_\xi\|_{\mathcal{H}^{1,b}} \leq C_{\tilde{w}} |s|^{-1}$ for some $C_{\tilde{w}} > 0$. Consequently we obtain for $z \in \mathcal{H}^{1,b}$ the estimate

$$\begin{aligned} \|Hz\|_{\mathcal{H}^{1,b}} &\leq \|z\|_{\mathcal{H}^{1,b}} + \|R_s(\Lambda)w_\xi\|_{\mathcal{H}^{1,b}} |\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1}| |\langle \hat{v}_\xi, z \rangle| \\ &\leq \|z\|_{\mathcal{H}^{1,b}} + \|R_s(\Lambda)w_\xi\|_{\mathcal{H}^{1,b}} |s| C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|z\|_{\mathcal{L}_2} \\ &\leq \|z\|_{\mathcal{H}^{1,b}} + C_{\tilde{w}} C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|z\|_{\mathcal{L}_2} \\ &\leq C_H \|z\|_{\mathcal{H}^{1,b}} \end{aligned}$$

for some $C_H > 0$ and thus

$$\|u\|_{\mathcal{H}^{1,b}} = \|HR_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq C_H \|R_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq \frac{C_H \sqrt{C_G} C_P}{\sqrt{|s|}} \|k\|_{\mathcal{L}_{2,b}}.$$

It remains to estimate λ :

$$\begin{aligned} |\lambda| &= |\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1}| |\langle \hat{v}_\xi, R_s(\Lambda)Pk \rangle| \\ &\leq |s| C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|R_s(\Lambda)Pk\|_{\mathcal{L}_{2,b}} \\ &\leq C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \sqrt{C_G} C_P \|k\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

Assume further $k \in \mathcal{H}^{1,b}$. Again we use the results of Lemma 3.14 and obtain

$$\|u\|_{\mathcal{H}^{1,b}} = \|HR_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq C_H \|R_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq \frac{C_H C_G C_P}{|s|} \|k\|_{\mathcal{H}^{1,b}}.$$

For s in a compact set $S_C \subset \rho(\Lambda)$, i.e. $|s| \leq C$, we estimate the operator H by

$$\begin{aligned} \|Hw\|_{\mathcal{H}^{1,b}} &\leq \|w\|_{\mathcal{H}^{1,b}} + \|R_s(\Lambda)w_\xi\|_{\mathcal{H}^{1,b}} |\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1}| |\langle \hat{v}_\xi, w \rangle| \\ &\leq \|w\|_{\mathcal{H}^{1,b}} + C_G \|w_\xi\|_{\mathcal{L}_{2,b}} |s| C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|w\|_{\mathcal{L}_2} \\ &\leq C_H \|w\|_{\mathcal{H}^{1,b}} \end{aligned}$$

for some $C_H > 0$ and therefore we conclude

$$\|u\|_{\mathcal{H}^{1,b}} = \|HR_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq C_H \|R_s(\Lambda)Pk\|_{\mathcal{H}^{1,b}} \leq C_H C_G C_P \|k\|_{\mathcal{L}_{2,b}}$$

and

$$\begin{aligned} |\lambda| &= |\langle \hat{v}_\xi, R_s(\Lambda)w_\xi \rangle^{-1}| |\langle \hat{v}_\xi, R_s(\Lambda)Pk \rangle| \\ &\leq |s| C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} \|R_s(\Lambda)Pk\|_{\mathcal{L}_{2,b}} \\ &\leq C C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_2} C_G C_P \|k\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

■

It remains to prove the resolvent estimate for the system (3.41), (3.42) in the domain Ω_ε . The first part of the proof is similar to [36], Lemma 1.22.

Lemma 3.18. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold.*

Then there exist $b_3, \varepsilon > 0$ and some constant $C_\varepsilon > 0$ such that for all $0 \leq b < b_3$ the system (3.41), (3.42) possesses a unique solution (u, λ) for $s \in B_\varepsilon(0)$ which satisfies the following estimate for $j = 1, \dots, N$

$$\|u_j\|_{\mathcal{H}^{1,b}} + |\lambda_j| \leq C_\varepsilon \|k_j\|_{\mathcal{L}_{2,b}}. \quad (3.47)$$

Proof. Let $j \in \{1, \dots, N\}$ and $z_j = (u_j, u_{j,\xi})$. We transform (3.41), (3.42) to

$$L_j(s)z_j = R_j - \Phi_j \lambda_j \quad (3.48)$$

$$0 = \langle \Psi_j, z_j \rangle, \quad (3.49)$$

where

$$L_j(s)z_j = z_{j,\xi} - M_j(\cdot, s)z_j, \quad M_j(\xi, s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C_j(\xi)) & -A^{-1}B_j \end{pmatrix},$$

$$R_j = \begin{pmatrix} 0 \\ -A^{-1}P_j k_j \end{pmatrix}, \quad \Phi_j = \begin{pmatrix} 0 \\ A^{-1}w_{j,\xi} \end{pmatrix}, \quad \Psi = \begin{pmatrix} \hat{v}_{j,\xi} \\ 0 \end{pmatrix}.$$

Using Hypothesis 1.9 and [33], Lemma 2.27 we conclude that $\lim_{\xi \rightarrow \pm\infty} M_j(\xi, s)$ is hyperbolic for all $s \in \mathbb{C}$ with $\Re s > -\bar{\kappa}$. An application of Corollary A.2 shows that the operator $L_j(s)$ has an exponential dichotomy on \mathbb{R}_\pm with data $(K_j^\pm, \alpha_j^\pm, \pi_j^\pm)$. It remains so show the solvability of (3.48), (3.49) for $s = 0$. A regular perturbation argument, [9], Theorem 9.3, yields the solvability for $s \in B_\varepsilon(0)$ for small $\varepsilon > 0$.

Let $0 \leq b < \frac{1}{4} \min(\alpha_j^\pm, j = 1, \dots, N) =: b_3$.

From now on we suppress j in the proof.

Let $s = 0$. From (A.3) we obtain that the solutions on \mathbb{R}^\pm of (3.48) are given by

$$z^\pm = S(\cdot, 0)z^{0,\pm} + s^\pm(R - \Phi\lambda),$$

where S is the solution operator of (A.1),

$$s^-(g)(\xi) := \int_{-\infty}^0 G(\xi, x)g(x) dx, \quad s^+(g)(\xi) := \int_0^\infty G(\xi, x)g(x) dx$$

and the Green's function G is defined by (A.2).

The function $z(\xi) = \begin{cases} z^+(\xi), & \xi \geq 0 \\ z^-(\xi), & \xi < 0 \end{cases}$ is a solution of (3.48), (3.49) if z solves the phase condition (3.49) and $z^-(0) = z^+(0) \in \mathcal{N}(\pi^-(0)) \cap \mathcal{R}(\pi^+(0))$ holds. In operator form this is equivalent to

$$T(z^{0,-}, z^{0,+}, \lambda) = \begin{pmatrix} \rho \\ \delta \end{pmatrix}, \quad (3.50)$$

where $T : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^{2m} \times \mathbb{R}$ is given by

$$T = \begin{pmatrix} I & -I & \Sigma \\ \Theta & \Lambda & \Xi \end{pmatrix}$$

with

$$\begin{aligned} \Sigma &= s^+(\Phi)(0) - s^-(\Phi)(0), \\ \Theta &= \int_{-\infty}^0 \Psi(\xi)^T S(\xi, 0) d\xi, \quad \Lambda = \int_0^{\infty} \Psi(\xi)^T S(\xi, 0) d\xi, \\ \Xi &= - \int_{-\infty}^0 \Psi(\xi)^T s^-(\Phi)(\xi) d\xi - \int_0^{\infty} \Psi(\xi)^T s^+(\Phi)(\xi) d\xi \end{aligned}$$

and

$$\begin{aligned} \rho &= s^+(R)(0) - s^-(R)(0), \\ \delta &= - \int_{-\infty}^0 \Psi(\xi)^T s^-(R)(\xi) d\xi - \int_0^{\infty} \Psi(\xi)^T s^+(R)(\xi) d\xi. \end{aligned}$$

It remains to show that T is injective, i.e. the equation $T(z^{0,-}, z^{0,+}, \lambda) = 0$ implies $(z^{0,-}, z^{0,+}, \lambda) = 0$.

Let $T(z^{0,-}, z^{0,+}, \lambda) = 0$, we construct $z = (u, u_\xi)$ with $z(\xi) = S(\cdot, 0)z^{0,\pm} + s^\pm(-\Phi\lambda)$ for $\pm\xi \geq 0$. Then z is a bounded solution of

$$z_\xi - M(0)z = -\Phi\lambda, \quad \langle \Psi, z \rangle = 0.$$

The equation $z_\xi - M(0)z = -\Phi\lambda$ is equivalent to

$$Au_{\xi\xi} + Bu_\xi + Cu + \lambda w_\xi = 0.$$

Using the Hypothesis 1.8 we conclude $\lambda = 0$ and $u = ew_\xi$ for some $e \in \mathbb{R}$.

The conditions $\langle \hat{v}_{k,\xi}, w_{k,\xi} \rangle \neq 0, k = 1 \dots, N$ and $\langle \Psi, z \rangle = 0$ give $e = 0$. So we have $\lambda = 0, u = 0$ and therefore $z = 0$ and $z^{0,\pm} = 0$. It follows that T is injective

and invertible. Therefore, there exists a solution of (3.50) which is estimated for $R \in \mathcal{L}_{2,b}$ by

$$\|z^{0,-}\| + \|z^{0,+}\| + |\lambda| = \|T^{-1} \begin{pmatrix} \rho \\ \delta \end{pmatrix}\| \leq C_T(\|\rho\| + \|\delta\|)$$

for some $C_T > 0$. We have to estimate $\|\rho\|, \|\delta\|$ and use (A.4):

$$\|\rho\| = \|s^+(R)(0) - s^-(R)(0)\| \leq C_\rho \|R\|_{\mathcal{L}_2}$$

for some $C_\rho > 0$ and for some $C_\delta > 0$ we obtain

$$\begin{aligned} \|\delta\| &\leq \left\| \int_{-\infty}^0 \Psi(\xi)^T s^-(R)(\xi) d\xi \right\| + \left\| \int_0^\infty \Psi(\xi)^T s^+(R)(\xi) d\xi \right\| \\ &\leq \|\Psi\|_{\mathcal{L}_2(-\infty,0]} \|s^-(R_j)\|_{\mathcal{L}_2(-\infty,0]} + \|\Psi\|_{\mathcal{L}_2[0,\infty)} \|s^+(R)\|_{\mathcal{L}_2[0,\infty)} \\ &\leq C_\delta \|\Psi\|_{\mathcal{L}_2} \|R\|_{\mathcal{L}_2}. \end{aligned}$$

In summary, we get for some $C_0 > 0$

$$\|z^{0,-}\| + \|z^{0,+}\| + |\lambda| \leq C_0 \|R\|_{\mathcal{L}_2}.$$

Finally we have to estimate z^\pm . We estimate z^- for $\xi < 0$, z^+ is handled similarly. Since $z^{0,-} \in \mathcal{N}(\pi^-(0))$ we obtain the estimates

$$\begin{aligned} \|S(\cdot, 0)z^{0,-}\|_{\mathcal{L}_{2,b}(-\infty,0]}^2 &= \int_{-\infty}^0 e^{-2b\xi} \|S(\xi, 0)z^{0,-}\|^2 d\xi \\ &\leq \int_{-\infty}^0 e^{-2b\xi} \|S(\xi, 0)(I - \pi^-(0))z^{0,-}\|^2 d\xi \\ &\leq (K^-)^2 \int_{-\infty}^0 e^{2(\alpha^- - b)\xi} \|z^{0,-}\|^2 dx \\ &\leq \frac{(K^-)^2 \|z^{0,-}\|^2}{2(\alpha^- - b)} \leq \frac{4(K^-)^2 \|z^{0,-}\|^2}{3\alpha^-} \end{aligned}$$

and

$$\begin{aligned}
& \|s^-(R - \Phi\lambda)\|_{\mathcal{L}_{2,b}(-\infty,0]}^2 \\
& \leq (K^-)^2 \int_{-\infty}^0 e^{-2b\xi} \left(\int_{-\infty}^0 e^{-\alpha^-|\xi-x|} \|(R(x) - \Phi(x)\lambda_j)\| dx \right)^2 d\xi \\
& \leq (K^-)^2 \int_{-\infty}^0 e^{-2b\xi} \left(\int_{-\infty}^0 e^{-\frac{\alpha^-}{2}|\xi-x|} (e^{-\frac{\alpha^-}{2}|\xi-x|} \|R(x) - \Phi(x)\lambda\|) dx \right)^2 d\xi \\
& \leq (K^-)^2 \int_{-\infty}^0 e^{-2b\xi} \int_{-\infty}^0 e^{-\alpha^-|\xi-x|} dx \int_{-\infty}^0 e^{-\alpha^-|\xi-x|} \|R(x) - \Phi(x)\lambda\|^2 dx d\xi \\
& \leq \frac{2(K^-)^2}{\alpha^-} \int_{-\infty}^0 \int_{-\infty}^0 e^{2b|\xi|} e^{-\alpha^-|\xi-x|} \|R(x) - \Phi(x)\lambda\|^2 dx d\xi \\
& \leq \frac{2(K^-)^2}{\alpha^-} \int_{-\infty}^0 \int_{-\infty}^0 e^{(2b-\alpha^-)|\xi-x|} e^{2b|x|} \|R(x) - \Phi(x)\lambda\|^2 dx d\xi \\
& \leq \frac{4(K^-)^2}{(\alpha^-)(\alpha^- - 2b)} \int_{-\infty}^0 e^{2b|x|} \|R(x) - \Phi(x)\lambda\|^2 dx d\xi \\
& \leq \frac{8(K^-)^2}{\alpha^-} (\|R\|_{\mathcal{L}_{2,b}} + \|\Phi\|_{\mathcal{L}_{2,b}}|\lambda|).
\end{aligned}$$

So we obtain for some $C_1 > 0$ the following estimate

$$\|z\|_{\mathcal{L}_{2,b}} + |\lambda| \leq C_1 \|R\|_{\mathcal{L}_{2,b}}.$$

Since $z = (u, u_\xi)$ and $|(\theta_b)_\xi(\theta_b)^{-1}|$ is bounded by (A.9), we conclude that some $C_\varepsilon > 0$ exists such that (3.47) is satisfied.

Note that C_ε can be chosen independently of j by taking the maximum over a finite number of constants. \blacksquare

Remark 3.19. Using the fact that $w_j(\xi) \rightarrow w_j^\pm$ as $\xi \rightarrow \pm\infty$ and $w_j \in \mathcal{C}_b^2(\mathbb{R})$ we conclude as in the proof of Lemma 3.18 above that the operator $L_j(0)$ has an exponential dichotomy on \mathbb{R}_\pm with some data $(K_j^\pm, \alpha_j^\pm, \pi_j^\pm)$.

Let $j \in \{1, \dots, N\}$, note that the functions $w_{j,\xi}, w_{j,\xi\xi}$ are bounded. The function $(w_{j,\xi}, w_{j,\xi\xi})$ is a bounded solution of $L_j(0)z_j = 0$ on \mathbb{R}_\pm . Therefore we obtain that constants $\eta, C_\eta > 0$ exist such that the following estimate is satisfied:

$$\|w_{j,\xi}(\xi)\| + \|w_{j,\xi\xi}(\xi)\| \leq C_\eta e^{-\eta|\xi|} \quad \forall \xi \in \mathbb{R}$$

with $\eta = \min\{\alpha_j^\pm | j = 1, \dots, N\}$. Furthermore, the estimates (1.25) - (1.26) result from the estimates

$$\|w_j(\xi) - w_j^+\| \leq \int_\xi^\infty \|w_{j,\xi}(\tau)\| d\tau \quad \text{and} \quad \|w_j(\xi) - w_j^-\| \leq \int_{-\infty}^\xi \|w_{j,\xi}(\tau)\| d\tau.$$

It remains to show the sectorial estimates:

Lemma 3.20. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold.*

Let $j \in \{1, \dots, N\}$ and $k_j \in \mathcal{L}_{2,b}$. There exists $b_4 > 0$ and furthermore there exist some $C_s > 0$ and a sector $\bar{S}_{a,\theta} \subset \rho(\Lambda_{P,l})$ for all $l = 1, \dots, N$ and $0 < a < \min(K_G, \bar{\kappa}), \theta \in (\frac{\pi}{2}, \pi)$ with the following properties: The solution

$$u_j = (sI - P_j \Lambda_j)^{-1} P_j k_j, \quad u_j \in \mathcal{R}(P_j), \quad \lambda_j = -\langle \hat{v}_{j,\xi} w_{j,\xi} \rangle^{-1} a_j(\hat{v}_{j,\xi}, u_j)$$

satisfies the following estimates

$$\|u_j\|_{\mathcal{L}_{2,b}} \leq \frac{C_s}{|s + \alpha|} \|k_j\|_{\mathcal{L}_{2,b}}, \quad \|u_j\|_{\mathcal{H}^{1,b}} \leq \frac{C_s}{\sqrt{|s + \alpha|}} \|k_j\|_{\mathcal{L}_{2,b}} \quad (3.51)$$

for all $s \in \bar{S}_{a,\theta}$, $0 < \alpha \leq a$, $0 \leq b < b_4$.

If, in addition $k_j \in \mathcal{H}^{1,b}$, then the following estimate holds

$$\|u_j\|_{\mathcal{H}^{1,b}} \leq \frac{C_s}{|s + \alpha|} \|k_j\|_{\mathcal{H}^{1,b}}. \quad (3.52)$$

Proof. Using Lemma 3.15 we find that the system (3.43), (3.44) is equivalent to (3.41), (3.42) and we apply Lemma 3.17, 3.18. Therefore there exist $b_4 := \min(b_2, b_3, \frac{\eta}{2})$ and $\check{C} := \max(C_\varepsilon, C_A)$. Let $0 \leq b < b_4$, $0 < a < \min(\bar{\kappa}, K_G)$, $0 < \alpha \leq a$. Let $j \in \{1, \dots, N\}$, again we suppress j in the proof.

Let $s \in \Omega_\varepsilon \cup \Omega_{K_G}$, i.e. $|s| \leq K_G$, and $s \neq -\alpha$. We get

$$\|u\|_{\mathcal{L}_{2,b}} \leq \check{C} \|k\|_{\mathcal{L}_{2,b}} \leq \check{C} \frac{2K_G}{|s + \alpha|} \|k\|_{\mathcal{L}_{2,b}},$$

in an analogous fashion

$$\|u\|_{\mathcal{H}^{1,b}} \leq \check{C} \|k\|_{\mathcal{L}_{2,b}} \leq \check{C} \frac{\sqrt{2K_G}}{\sqrt{|s + \alpha|}} \|k\|_{\mathcal{L}_{2,b}}.$$

For $s \in \Omega_\infty$, i.e. $|s| > K_G$ we obtain

$$\|u\|_{\mathcal{L}_{2,b}} \leq \frac{\sqrt{\check{C}}}{|s|} \|k\|_{\mathcal{L}_{2,b}} \leq \frac{2\sqrt{\check{C}}}{|s + \alpha|} \|k\|_{\mathcal{L}_{2,b}}$$

and

$$\|u\|_{\mathcal{H}^{1,b}} \leq \frac{\sqrt{\check{C}}}{\sqrt{|s|}} \|k\|_{\mathcal{L}_{2,b}} \leq \frac{\sqrt{2\check{C}}}{\sqrt{|s + \alpha|}} \|k\|_{\mathcal{L}_{2,b}}.$$

For $k \in \mathcal{H}^{1,b}$ we obtain again estimates for the sectors: Let $s \in \Omega_\varepsilon \cup \Omega_{K_G}$, i.e. $|s| \leq K_G$, and $s \neq -\alpha$. We get

$$\|u\|_{\mathcal{H}^{1,b}} \leq \check{C} \frac{2K_G}{|s + \alpha|} \|k\|_{\mathcal{H}^{1,b}}.$$

Let $s \in \Omega_\infty$, i.e. $|s| > K_G$. We obtain

$$\|u\|_{\mathcal{H}^{1,b}} \leq \frac{2\sqrt{\check{C}}}{|s + \alpha|} \|k\|_{\mathcal{H}^{1,b}}.$$

As shown in Figure 3.6 we can choose $0 < a < \min(K_G, \bar{\kappa})$, $\theta \in (\frac{\pi}{2}, \pi)$, $\theta \leq \zeta$ such that a sector $\bar{S}_{a,\theta} \subset \rho(\Lambda_k)$ for all $k = 1, \dots, N$ and some $C_s > 0$ exist such that for all $0 < \alpha \leq a$ the estimates (3.51), (3.52) are satisfied. ■

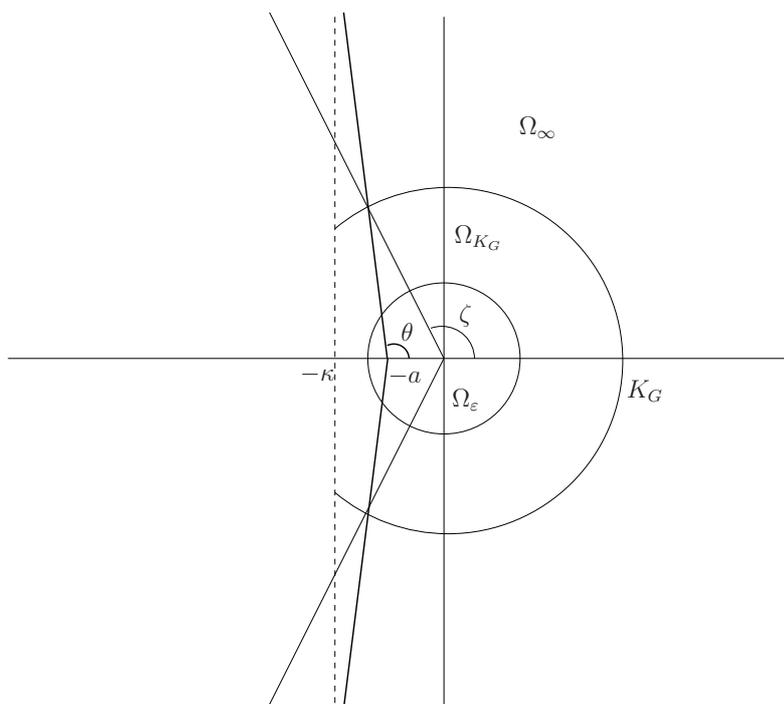


Figure 3.6: Sector $\bar{S}_{a,\theta} \subset \mathbb{C}$ together with the sections $\Omega_{K_G}, \Omega_\infty, \Omega_\varepsilon \subset \mathbb{C}$

Let $j \in \{1, \dots, N\}$, $t \geq 0$. Recall the operator $e^{\Lambda_{P,j}t}$ is defined using the Dunford integral, see [17],

$$e^{\Lambda_{P,j}t} = \frac{1}{2\pi i} \int_{\Gamma} e^{st} (sI - \Lambda_{P,j})^{-1} ds, \quad \forall t \geq 0, \quad (3.53)$$

where Γ is a contour in $\rho(\Lambda_{P,j})$ with $\arg s \rightarrow \pm\theta$ as $|s| \rightarrow \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. Let us now mention an important consequence of the last lemma: The essential estimates of the operator $e^{\Lambda_{P,j}t}$, $j = 1, \dots, N$:

Theorem 3.21. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold.*

Then there exist $b_4 > 0, K \geq 1, \alpha > 0$ such that for all $0 \leq b < b_4, j = 1, \dots, N, k_j \in \mathcal{L}_{2,b}$ the following exponential estimates are satisfied

$$\|e^{\Lambda_{P,j}t} P_j k_j\|_{\mathcal{L}_{2,b}} \leq K e^{-\alpha t} \|P_j k_j\|_{\mathcal{L}_{2,b}}, \quad \|e^{\Lambda_{P,j}t} P_j k_j\|_{\mathcal{H}^{1,b}} \leq K e^{-\alpha t} t^{-\frac{1}{2}} \|P_j k_j\|_{\mathcal{L}_{2,b}}.$$

If in addition $k_j \in \mathcal{H}^{1,b}$, then the following estimate holds

$$\|e^{\Lambda_{P,j}t} P_j k_j\|_{\mathcal{H}^{1,b}} \leq K e^{-\alpha t} \|P_j k_j\|_{\mathcal{H}^{1,b}}.$$

Further holds

$$\|e^{\Lambda_{P,j}t} P_j k_j - P_j k_j\|_{\mathcal{H}^{1,b}} \rightarrow 0 \text{ as } t \rightarrow 0^+. \quad (3.54)$$

Proof. Let C be some positive constant. Let $j \in \{1, \dots, N\}$, again we suppress j in the proof.

Let $0 \leq b < b_4$, where $b_4 > 0$ is chosen as in Lemma 3.20. Further let a, θ be chosen as in Lemma 3.20. We know that the estimates (3.51), (3.52) hold for $s \in \bar{S}_{a,\theta}$ with $C_s > 0$ and $0 < \alpha \leq a$.

Let $0 < \alpha < a$. Let Γ be a path around the eigenvalues of Λ_P with $\Re s < 0$ for all $s \in \Gamma$. We choose Γ to be the sum of two rays q_1 and q_2 with $q_1(\lambda) = -p - \lambda |\cos \theta| + i\lambda \sin \theta$, $q_2(\lambda) = -p - \lambda |\cos \theta| - i\lambda \sin \theta$ for $\lambda \in [0, \infty)$, $0 < p < a - \alpha$, see Figure 3.7. We can move Γ to $\Gamma - \alpha$ such that it is also a path around the eigenvalues of $P\Lambda$ and the integral defined in (3.53) does not change.

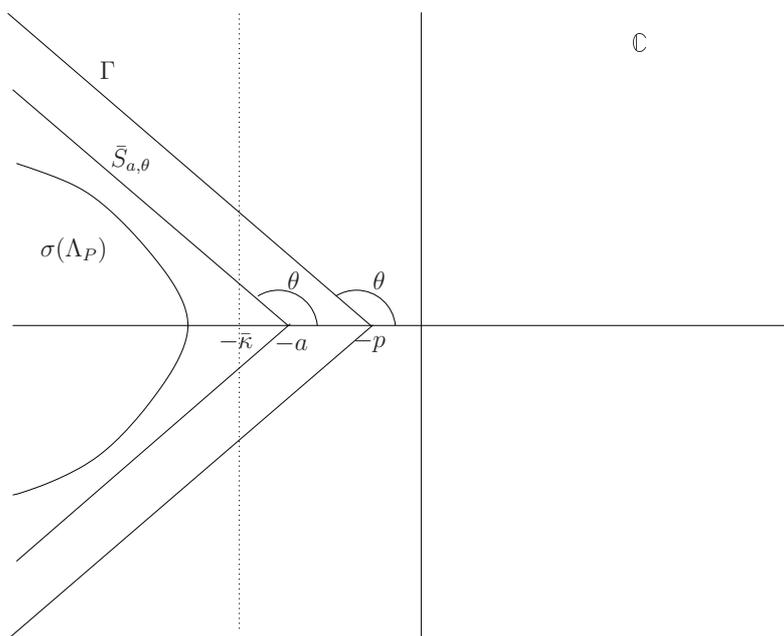
Let $* \in \{\mathcal{L}_{2,b}, \mathcal{H}^{1,b}\}$. We get

$$\begin{aligned} \|e^{P\Lambda t} Pk\|_* &= \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{st} (sI - P\Lambda)^{-1} Pk ds \right\|_* \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma - \alpha} e^{st} (sI - P\Lambda)^{-1} Pk ds \right\|_* \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{(s-\alpha)t} ((s-\alpha)I - P\Lambda)^{-1} Pk ds \right\|_* \\ &\leq \frac{e^{-\alpha t}}{2\pi} \int_{\Gamma} |e^{st}| \|((s-\alpha)I - P\Lambda)^{-1} Pk\|_* |ds|. \end{aligned}$$

Note $s - \alpha \neq 0$ and $u = (s - \alpha)^{-1} P(k + \Lambda u)$, hence $u = ((s - \alpha)I - P\Lambda)^{-1} Pk \in \mathcal{R}(P)$. Using Lemma 3.20 we obtain

$$\|e^{P\Lambda t} Pk\|_* \leq \frac{C e^{-\alpha t}}{2\pi} \|Pk\|_* \int_{\Gamma} \frac{|e^{st}|}{|s|} |ds| = \frac{C e^{-\alpha t}}{2\pi} \|Pk\|_* \int_{\Gamma} \frac{|e^q|}{|q|} |dq|.$$

The last integral is bounded since $\Re q < 0$ and $|q|^{-1}$ is bounded. Further we conclude

Figure 3.7: Γ - path of integration

$$\|e^{P\Lambda t}Pk\|_{\mathcal{H}^{1,b}} \leq \frac{Ce^{-\alpha t}}{2\pi} \|Pk\|_{\mathcal{L}_{2,b}} \int_{\Gamma} \frac{|e^{st}|}{\sqrt{|s|}} |ds| = \frac{Ce^{-\alpha t}}{2\pi\sqrt{t}} \|Pk\|_{\mathcal{L}_{2,b}} \int_{\Gamma} \frac{|e^q|}{\sqrt{|q|}} |dq|,$$

again the last integral is bounded.

The proof of last claim is missing. From Lemma A.5 we obtain that $\mathcal{H}^{2,b}$ is dense in $\mathcal{L}_{2,b}$. For this reason there exists for $k \in \mathcal{H}^{1,b}$ some $h \in \mathcal{H}^{2,b}$ sufficiently close to k , that is why we only have to show the estimate (3.54) for $h \in \mathcal{H}^{2,b}$, since

$$\begin{aligned} \|e^{\Lambda_P t}Pk - Pk\|_{\mathcal{H}^{1,b}} &\leq \|e^{\Lambda_P t}(Pk - Ph)\|_{\mathcal{H}^{1,b}} + \|e^{\Lambda_P t}Ph - Ph\|_{\mathcal{H}^{1,b}} \\ &\quad + \|Ph - Pk\|_{\mathcal{H}^{1,b}}. \end{aligned}$$

We only estimate the second term for $h \in \mathcal{H}^{2,b}$, which implies $P\Lambda Ph \in \mathcal{L}_{2,b}$. We follow an idea of [17], Theorem 1.3.4. and use the Gauss integral theorem, [27],

Theorem 5.4:

$$\begin{aligned}
\|e^{\Lambda_P t} P h - P h\|_{\mathcal{H}^{1,b}} &= \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{st} [(sI - P\Lambda)^{-1} - s^{-1}] P h ds \right\|_{\mathcal{H}^{1,b}} \\
&= \left\| \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{st}}{s} (sI - P\Lambda)^{-1} P\Lambda P h ds \right\|_{\mathcal{H}^{1,b}} \\
&= \left\| \frac{1}{2\pi i} \int_{\Gamma_{-\alpha}} \frac{e^{st}}{s} (sI - P\Lambda)^{-1} P\Lambda P h ds \right\|_{\mathcal{H}^{1,b}} \\
&\leq \frac{C}{2\pi} \int_{\Gamma} \frac{|e^{(s-\alpha)t}|}{|s-\alpha|\sqrt{|s|}} |ds| \|P\Lambda P h\|_{\mathcal{L}_{2,b}} \\
&\leq \frac{C t^{\frac{3}{2}}}{2\pi} \int_{\Gamma} \frac{|e^{st}|}{|st|\sqrt{|st|}} |ds| \|P\Lambda P h\|_{\mathcal{L}_{2,b}} \\
&\leq \frac{C t^{\frac{1}{2}}}{2\pi} \int_{\Gamma} \frac{|e^q|}{|q|\sqrt{|q|}} |ds| \|P\Lambda P h\|_{\mathcal{L}_{2,b}}.
\end{aligned}$$

Similarly to above the last integral is bounded. ■

Let $j \in \{1, \dots, N\}$. We proceed with the main result of this section which states that the operators $\Lambda_{P,j} = P_j \Lambda_j : \mathcal{H}^{2,b} \cap \mathcal{R}(P_j) \rightarrow \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ $j = 1, \dots, N$, are sectorial:

Theorem 3.22. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold.*

There exist $b_4 > 0$ such that for all $0 \leq b < b_4$ and $j \in \{1, \dots, N\}$ the operator $\Lambda_{P,j}$ is sectorial in $\mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$.

Proof. Let $0 \leq b < b_4$, where b_4 is chosen as in Theorem 3.21. Let $C > 0$ be some generic constant and $j \in \{1, \dots, N\}$.

First step: $\mathcal{H}^{2,b} \cap \mathcal{R}(P_j)$ is dense in $\mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$.

Using Lemma A.5 we know that $\mathcal{H}^{2,b}$ is dense in $\mathcal{L}_{2,b}$. Let $u \in \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$. From Lemma A.5 we conclude that there exists $u_n \in \mathcal{H}^{2,b}$ with $\|u_n - u\|_{\mathcal{L}_{2,b}} \rightarrow 0$ as $n \rightarrow \infty$. From Hypothesis 1.6 and the estimates in the Appendix A.3 we infer $w_{j,\xi} \in \mathcal{H}^{2,b}$. Therefore $P_j u_n \in \mathcal{H}^{2,b} \cap \mathcal{R}(P_j)$ and $\lim_{n \rightarrow \infty} P_j u_n = u$, since

$$\|P_j u_n - u\|_{\mathcal{L}_{2,b}} = \|P_j(u_n - u)\|_{\mathcal{L}_{2,b}} \leq C_P \|u_n - u\|_{\mathcal{L}_{2,b}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Second step: $P_j \Lambda_j$ is closed in $\mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$:

Let $u_n \in \mathcal{H}^{2,b} \cap \mathcal{R}(P_j)$, $n \in \mathbb{N}$ converge to $u \in \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ and $P_j \Lambda_j u_n$ to $v \in \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$. We show $u \in \mathcal{H}^{2,b}$ and $P_j \Lambda_j u = v$:

For $s \in \bar{S}_{a,\theta}$ holds by the resolvent estimates for the domains $\Omega_\varepsilon, \Omega_{K_G}$ and Ω_∞

$$|s|^2 \|u_n - u_m\|_{\mathcal{L}_{2,b}}^2 + |s| \|u_n - u_m\|_{\mathcal{H}^{1,b}}^2 \leq C \|(sI - P_j \Lambda_j)(u_n - u_m)\|_{\mathcal{L}_{2,b}}^2$$

Since the term on the right side converges to zero follows u_n is a Cauchy sequence in $\mathcal{H}^{1,b}$ and $u_n \rightarrow \bar{u}$ in $\mathcal{H}^{1,b}$ as $n \rightarrow \infty$ as well as $\bar{u} = u$.

We obtain $\hat{v}_{j,\xi\xi} \in \mathcal{L}_2$ from Hypotheses 1.5, 1.6. We use the assumption $P_j \Lambda_j u_n$ converges to $v \in \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$, further we conclude $P_j B_j u_{n,\xi} \rightarrow P_j B_j u_\xi$ and $P_j C_j u_n \rightarrow P_j C_j u$ as $n \rightarrow \infty$ from above. We derive

$$\begin{aligned} Au_{n,\xi\xi} &= P_j \Lambda_j u_n - P_j B_j u_{n,\xi} - P_j C_j u_n + w_{j,\xi} \langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} \langle \hat{v}_{j,\xi}, Au_{n,\xi\xi} \rangle \\ &\rightarrow v - P_j B_j u_\xi - P_j C_j u + w_{j,\xi} \langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} \langle \hat{v}_{j,\xi\xi}, Au_\xi \rangle \end{aligned}$$

as $n \rightarrow \infty$, hence $u_{n,\xi\xi} \rightarrow \tilde{u}$ in $\mathcal{L}_{2,b}$ as $n \rightarrow \infty$ and $\tilde{u} = u_{\xi\xi}$, $u \in \mathcal{H}^{2,b}$ as well as $P_j \Lambda_j u = v$.

Third step: $\bar{S}_{a,\theta} \subset \rho(\Lambda_{P,j})$ and the resolvent estimate holds:

The resolvent estimate (3.51) is given by Lemma 3.20.

The operator $R_s(\Lambda_{P,j})$ exists in $\bar{S}_{a,\theta}$, since $sI - \Lambda_{P,j}$ is in this sector by the resolvent estimate in Theorem 3.21 and Lemma 3.20 injective, thus invertible.

$R_s(\Lambda_{P,j})$ is bounded and continuous, since for every $k \in \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ there exists $M > 0$ such that for all $s \in \bar{S}_{a,\theta}$ the estimates (3.45), (3.46) and (3.47) give

$$\|R_s(\Lambda_{P,j})k\|_{\mathcal{L}_{2,b}} \leq M \|k\|_{\mathcal{L}_{2,b}}.$$

■

3.5 The solution operator

Recall the decoupled projected system (3.34) - (3.36). Using the resolvent estimates of the section above we are able to solve this system with the help of the variation of constants formula.

Let L be sectorial in a Banach space X and $\Re \lambda < -\kappa$ for all $\lambda \in \sigma(L)$. Let U be a nonempty open subset of X . Let $T \in (0, \infty]$. Let $F : [0, T) \times U \rightarrow X$ be locally Lipschitz. i.e. for every $t \in [0, T)$ and $u \in U$ there exists $C, \delta > 0$ such that

$$\|F(t_1, u_1) - F(t_2, u_2)\|_X \leq C(|t_1 - t_2| + \|u_1 - u_2\|_U)$$

whenever $t_1, t_2 \in [0, T)$, $u_1, u_2 \in U$ and $|t_i - t| + \|u_i - u\|_U < \delta$ for $i = 1, 2$.

We use [23], Chapter 6.4 to obtain an equivalent formulation of the solution of a single PDE of the form $u'(t) = Lu(t) + F(t, u(t))$:

Definition 3.23. For $\tau \in (0, T]$ let $\mathcal{S}(\tau)$ denote the collection of $u \in C([0, \tau), U)$ such that $u'(t)$ exists, $u(t) \in \mathcal{D}(L)$ and $u'(t) = Lu(t) + F(t, u(t)) \forall t \in (0, \tau)$.

Note that this definition is in agreement with [23], Definition 6.4. An important consequence is the following variation of constants formula, see [23], Lemma 6.4.3.

Lemma 3.24. *Suppose $\tau \in (0, T]$. Then $u \in \mathcal{S}(\tau)$ if and only if $u \in C([0, \tau], U)$ and*

$$u(t) = e^{Lt}u(0) + \int_0^t e^{L(t-s)}F(s, u(s))ds \quad \forall t \in [0, \tau). \quad (3.55)$$

Let $0 \leq b < b_4$, where $b_4 > 0$ is chosen as in Theorem 3.21. We apply the lemma above to the decoupled system (3.34) - (3.36) and set $X = (X_1, \dots, X_N), U = (U_1, \dots, U_N)$, where $X_j = \mathcal{L}_{2,b} \cap \mathcal{R}(P_j), U_j = \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$ for $j = 1, \dots, N$. To utilize the lemma we have to show that for $g_j : [0, \tau) \rightarrow \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ locally Lipschitz the term $\int_0^t e^{\Lambda_{P,j}(t-s)}g_j(s)ds$ is continuous and in $\mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$.

Lemma 3.25. *Let $T \in (0, \infty]$, $0 \leq b < b_4$ and $j \in \{1, \dots, N\}$. Assume $g_j : [0, T) \rightarrow \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ is locally Lipschitz. For $0 \leq t < T$ define*

$$G_j(t) = \int_0^t e^{\Lambda_{P,j}(t-s)}g_j(s)ds$$

Then $G_j \in C([0, \tau), \mathcal{H}^{1,b} \cap \mathcal{R}(P_j))$ for $\tau \in (0, T]$.

Proof. The proof is similar to the proof of [17], Lemma 3.2.1, but uses weighted norms. Let $j \in \{1, \dots, N\}$. To shorten notation in this proof we suppress j . Let $g(s) = 0$ for $s < 0$. Define for small $\rho > 0$

$$G_\rho(t) = \int_0^{t-\rho} e^{\Lambda_P(t-s)}g(s)ds$$

with $G_\rho(t) = 0$ for $0 \leq t \leq \rho$.

Let $t \in [0, \tau)$. Using assumption $g(s) \in \mathcal{R}(P)$ for $s \in [0, t]$ we infer that the operator $e^{\Lambda_P(t-s)}$ maps into $\mathcal{R}(P)$, $G_\rho(t) \in \mathcal{R}(P)$ and also $G(t) \in \mathcal{R}(P)$.

We apply Theorem 3.21 and obtain $G(t) \in \mathcal{H}^{1,b}$ for $0 \leq t < \tau$:

$$\begin{aligned} \|G(t)\|_{\mathcal{H}^{1,b}} &\leq \int_0^t \|e^{\Lambda_P(t-s)}g(s)\|_{\mathcal{H}^{1,b}}ds \\ &\leq \int_0^t \frac{e^{-\alpha(t-s)}}{\sqrt{t-s}} \|g(s)\|_{\mathcal{L}_{2,b}}ds \\ &\leq \sup_{s \in [0,t]} \|g(s)\|_{\mathcal{L}_{2,b}} \int_0^t \frac{e^{-\alpha s}}{\sqrt{s}}ds \\ &\leq \frac{\sqrt{\pi}}{\sqrt{\alpha}} \sup_{s \in [0,t]} \|g(s)\|_{\mathcal{L}_{2,b}}. \end{aligned}$$

Similarly we obtain $G_\rho(t) \in \mathcal{H}^{1,b}$ for $0 \leq t < \tau$. We show that G is continuous by estimating $\|G(t) - G_\rho(t)\|_{\mathcal{H}^{1,b}}$ and $\|G_\rho(t+h) - G_\rho(t)\|_{\mathcal{H}^{1,b}}$.

First step: Let $0 \leq t \leq \tau_1, \tau_1 < \tau$. We apply Theorem 3.21 and obtain

$$\begin{aligned} \|G(t) - G_\rho(t)\|_{\mathcal{H}^{1,b}} &\leq \int_{t-\rho}^t \|e^{\Lambda_P(t-s)} g(s)\|_{\mathcal{H}^{1,b}} ds \\ &\leq \sup_{s \in [t-\rho, t]} \|g(s)\|_{\mathcal{L}_{2,b}} \int_0^\rho \frac{e^{-\alpha s}}{\sqrt{s}} ds. \end{aligned}$$

This integral tends to 0 for $\rho \rightarrow 0^+$ uniformly in $0 \leq t \leq \tau_1$:

$$\begin{aligned} \int_0^\rho \frac{e^{-\alpha s}}{\sqrt{s}} ds &= \frac{1}{\sqrt{\alpha}} \int_0^{\alpha\rho} \frac{e^{-s}}{\sqrt{s}} ds = \frac{2}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha\rho}} e^{-u^2} du \\ &\leq \frac{2}{\sqrt{\alpha}} \sqrt{\int_{B_{\sqrt{2\alpha\rho}}(0)} e^{-(u_1^2+u_2^2)} du} \\ &= \frac{2}{\sqrt{\alpha}} \sqrt{\int_0^{2\pi} \int_0^{\sqrt{2\alpha\rho}} e^{-r^2} r dr d\theta} = \frac{2}{\sqrt{\alpha}} \sqrt{\pi(1 - e^{-2\alpha\rho})}. \end{aligned}$$

Second step: Let $h > 0$ be small and $0 \leq t \leq t+h \leq \tau_1$, we obtain:

$$\begin{aligned} &\|G_\rho(t+h) - G_\rho(t)\|_{\mathcal{H}^{1,b}} \\ &\leq \|(e^{\Lambda_P h} - I) \int_0^{t-\rho} e^{\Lambda_P(t-s)} g(s) ds\|_{\mathcal{H}^{1,b}} + \left\| \int_{t-\rho}^{h+t-\rho} e^{\Lambda_P(h+t-s)} g(s) ds \right\|_{\mathcal{H}^{1,b}} \\ &\leq \|(e^{\Lambda_P h} - I) \int_0^{t-\rho} e^{\Lambda_P(t-s)} g(s) ds\|_{\mathcal{H}^{1,b}} + \sup_{s \in [t-\rho, t+h-\rho]} \|g(s)\|_{\mathcal{L}_{2,b}} \int_\rho^{h+\rho} \frac{e^{-\alpha s}}{\sqrt{s}} ds. \end{aligned}$$

We conclude from Theorem 3.21 that the first part becomes small for $h \rightarrow 0$, the second part becomes small using the argument above. \blacksquare

To obtain stability estimates like (1.39), we estimate the solution $u_j(t)$, $j = 1, \dots, N$ of the system (3.28) - (3.30) with respect to an exponentially weighted norm in time. Therefore we define for $u : [0, \tau) \rightarrow X$ and $\nu > 0, t \in [0, \tau)$

$$\|u\|_{t,\nu,X} := e^{\nu t} \|u(t)\|_X, \quad (3.56)$$

where $X \in \{\mathcal{L}_{2,b}, \mathcal{H}^{1,b}\}$. Furthermore, compare Definition 1.19, we define the supremum over the time interval $[0, t]$ by

$$\|u\|_{\nu,X}^t := \sup_{0 \leq s \leq t} \|u\|_{s,\nu,X}. \quad (3.57)$$

Consider the system (3.28) - (3.30) with appropriate initial conditions. Using the equivalence from Lemma 3.12 and the solution operator $e^{\Lambda_{p,j}}$ for $j = 1, \dots, N$, the

above lemma will show that we can determine the unique solution of the system (3.28) - (3.30).

Lemma 3.26. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9 hold. Let $0 < \tau < \infty$ and $0 \leq b < b_4$. For $j = 1, \dots, N$ let $k_j : [0, \tau) \rightarrow \mathcal{L}_{2,b}$ be locally Lipschitz, assume $u_j(0) = u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$, $r_j^0 \in \mathbb{R}$.*

Then the following PDAE

$$u_{j,t} = \Lambda_j u_j + \lambda_j w_{j,\xi} + k_j, \quad u_j(0) = u_j^0, \quad (3.58)$$

$$r_{j,t} = \lambda_j, \quad r_j(0) = r_j^0, \quad (3.59)$$

$$0 = \langle \hat{v}_{j,\xi}, u_j \rangle \quad (3.60)$$

for $j = 1, \dots, N$ has a unique solution (u^e, r^e, λ^e) on $[0, \tau)$, namely

$$u_j^e(t) = e^{\Lambda_{P,j} t} u_j^0 + \int_0^t e^{\Lambda_{P,j}(t-s)} P_j k_j(s) ds, \quad (3.61)$$

$$r_j^e(t) = \int_0^t \lambda_j^e(s) ds + r_j^0, \quad (3.62)$$

$$\lambda_j^e(t) = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j \langle \hat{v}_{j,\xi}, u_j^e(t) \rangle + \langle \hat{v}_{j,\xi}, k_j(t) \rangle), \quad t \in [0, \tau). \quad (3.63)$$

The following estimate is satisfied for all $0 \leq t < \tau$, $j = 1, \dots, N$, $0 < \nu < \alpha$:

$$e^{\nu t} \|u_j^e(t)\|_{\mathcal{H}^{1,b}} \leq K \|u_j^0\|_{\mathcal{H}^{1,b}} + \frac{2C_P K \sqrt{\pi}}{\sqrt{\alpha - \nu}} \|k_j\|_{\nu, \mathcal{L}_{2,b}}^t.$$

Proof. From Lemma 3.12 we conclude that solving the system (3.58) - (3.60) is equivalent to solving the system (3.34) - (3.36). Hence we apply Lemma 3.24 to the equation (3.34) with $L = (L_1, \dots, L_N)$, $F = (F_1, \dots, F_N)$, where $L_j := P_j \Lambda_j : \mathcal{H}^{2,b} \cap \mathcal{R}(P_j) \rightarrow \mathcal{L}_{2,b} \cap \mathcal{R}(P_j)$ and $F_j(t, u(t)) := P_j k_j(t)$ for $j = 1, \dots, N$. Using Theorem 3.21 and Lemma 3.25 we obtain that $u_j, j = 1, \dots, N$ defined by (3.55) is in $\mathcal{C}([0, \tau), \mathcal{H}^{1,b} \cap \mathcal{R}(P_j))$. It follows from Lemma 3.24 applied to (3.58) that the conditions 2., 5. and 6. of Definition 1.11 are satisfied. Since a_j and k_j are continuous for $j = 1, \dots, N$, conditions 1., 3. and 4. are also satisfied.

Let $j \in \{1, \dots, N\}, t \in [0, \tau), 0 < \nu < \alpha$. From Theorem 3.21 we obtain the

following estimate

$$\begin{aligned}
e^{\nu t} \|u_j^e(t)\|_{\mathcal{H}^{1,b}} &\leq e^{\nu t} \|e^{\Lambda_{P,j} t} u_j^0\|_{\mathcal{H}^{1,b}} + \int_0^t e^{\nu t} \|e^{\Lambda_{P,j}(t-s)} P_j k_j(s)\|_{\mathcal{H}^{1,b}} ds \\
&\leq K e^{(\nu-\alpha)t} \|u_j^0\|_{\mathcal{H}^{1,b}} + C_P K \int_0^t \frac{e^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} e^{\nu s} \|k_j(s)\|_{\mathcal{L}_{2,b}} ds \\
&\leq K e^{(\nu-\alpha)t} \|u_j^0\|_{\mathcal{H}^{1,b}} + C_P K \int_0^t \frac{e^{(\nu-\alpha)s}}{\sqrt{s}} ds \|k_j\|_{\nu, \mathcal{L}_{2,b}}^t \\
&\leq K \|u_j^0\|_{\mathcal{H}^{1,b}} + \frac{C_P K \sqrt{\pi}}{\sqrt{\alpha-\nu}} \|k_j\|_{\nu, \mathcal{L}_{2,b}}^t.
\end{aligned}$$

■

For the nonlinear terms $E_j, j = 1, \dots, N$ in the PDAE system (3.6) - (3.9) we need $b > 0$, otherwise we cannot use the estimates of the operator $E_j, j = 1, \dots, N$ in Section 3.2. Recall b_1 is the upper bound for b calculated in Section 3.2. Furthermore, recall Lemma 3.10, the operator $E_j(s, u(s))$ is continuous in time for given $\delta > 0, 0 < b < b_1$ and $G^0 \geq 12\delta\bar{B}$. Therefore, we assume $\delta \leq 1$, in the following define $G_{1,1}^0 = 12\bar{B}$.

Lemma 3.27. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9, 1.10 hold. Let $0 < \tau < \infty, 0 < b < \min(b_1, b_4), \varrho > 0$ and $G^0 \geq G_{1,1}^0$. For $j = 1, \dots, N$ let $\tilde{k}_j : [0, \tau] \rightarrow \mathcal{L}_{2,b}$ be locally Lipschitz, assume $u_j(0) = u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$. Let (u, r, λ) be a solution of the system*

$$u_j(t) = e^{\Lambda_{P,j} t} u_j^0 + \int_0^t e^{\Lambda_{P,j}(t-s)} P_j (\tilde{k}_j(s) + E_j(s, u(s))) ds, \quad (3.64)$$

$$r_j(t) = \int_0^t \lambda_j(s) ds, \quad (3.65)$$

$$\lambda_j(t) = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j(t)) + \langle \hat{v}_{j,\xi}, \tilde{k}_j(t) + E_j(t, u(t)) \rangle), \quad t \in [0, \tau]. \quad (3.66)$$

Let G^0 be sufficiently large, i.e. $\sqrt{\alpha-\nu} - C_P K C_E \sqrt{\pi} e^{-\gamma G^0} > 0$ is satisfied. Then u satisfies for $t \in [0, \tau], 0 < \nu < \alpha$ the following estimate

$$\|u\|_{\nu, \mathcal{H}^{1,b}}^t \leq \left(1 + \frac{C_P K C_E \sqrt{\pi}}{\sqrt{\alpha-\nu} - C_P K C_E \sqrt{\pi} e^{-\gamma G^0}} \right) (K \|u^0\|_{\mathcal{H}^{1,b}} + \frac{C_P K \sqrt{\pi}}{\sqrt{\alpha-\nu}} \|\tilde{k}\|_{\nu, \mathcal{L}_{2,b}}^t). \quad (3.67)$$

Proof. The system (3.64) - (3.66) can equivalently be written as

$$\begin{aligned}
u_j(t) &= u_j^e(t) + u_j^p(t), \\
u_j^e(t) &= e^{\Lambda_{P,j}t} u_j^0 + \int_0^t e^{\Lambda_{P,j}(t-s)} P_j \tilde{k}_j(s) ds, \\
u_j^p(t) &= \int_0^t e^{\Lambda_{P,j}(t-s)} P_j (E_j(s, u^e(s) + u^p(s))) ds, \\
r_j(t) &= \int_0^t \lambda_j(s) ds, \\
\lambda_j(t) &= \lambda_j^e(t) + \lambda_j^p(t), \\
\lambda_j^e(t) &= -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j^e(t)) + \langle \hat{v}_{j,\xi}, \tilde{k}_j(t) \rangle), \\
\lambda_j^p(t) &= -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j^p(t)) + \langle \hat{v}_{j,\xi}, E_j(t, u^p(t) + u^e(t)) \rangle), \quad t \in [0, \tau].
\end{aligned}$$

Let $0 < \nu < \alpha$, $j \in \{1, \dots, N\}$, $t \in [0, \tau]$. We obtain from Theorem 3.21, Lemma 3.26 and Lemma 3.5 the estimates

$$\begin{aligned}
e^{\nu t} \|u_j^e(t)\|_{\mathcal{H}^{1,b}} &\leq K \|u_j^0\|_{\mathcal{H}^{1,b}} + \frac{C_P K \sqrt{\pi}}{\sqrt{\alpha - \nu}} \|\tilde{k}_j\|_{\nu, \mathcal{L}_{2,b}}^t, \\
e^{\nu t} \|u_j^p(t)\|_{\mathcal{H}^{1,b}} &\leq \int_0^t e^{\nu s} \|e^{\Lambda_{P,j}(t-s)} P_j (E_j(s, u^e(s) + u^p(s)))\|_{\mathcal{H}^{1,b}} ds \\
&\leq C_P K \int_0^t \frac{e^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} e^{\nu s} \|E_j(s, u^e(s) + u^p(s))\|_{\mathcal{L}_{2,b}} ds \\
&\leq C_P K C_E \int_0^t \frac{e^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} e^{\nu s} e^{-\gamma G^0} (\|u^e(s)\|_{\mathcal{H}^{1,b}} + \|u^p(s)\|_{\mathcal{H}^{1,b}}) ds \\
&\leq \frac{C_P K C_E \sqrt{\pi} e^{-\gamma G^0}}{\sqrt{\alpha - \nu}} \|u^e\|_{\nu, \mathcal{H}^{1,b}}^t + \frac{C_P K C_E \sqrt{\pi}}{\sqrt{\alpha - \nu}} e^{-\gamma G^0} \|u^p\|_{\nu, \mathcal{H}^{1,b}}^t.
\end{aligned}$$

Let G^0 be so large that $\sqrt{\alpha - \nu} - C_P K C_E \sqrt{\pi} e^{-\gamma G^0} > 0$ is satisfied, then

$$\|u^p\|_{\mathcal{H}^{1,b}}^t \leq \frac{C_P K C_E \sqrt{\pi} e^{-\gamma G^0}}{\sqrt{\alpha - \nu} - C_P K C_E \sqrt{\pi} e^{-\gamma G^0}} \|u^e\|_{\nu, \mathcal{H}^{1,b}}^t$$

and

$$\|u\|_{\nu, \mathcal{H}^{1,b}}^t \leq \left(1 + \frac{C_P K C_E \sqrt{\pi}}{\sqrt{\alpha - \nu} - C_P K C_E \sqrt{\pi} e^{-\gamma G^0}} \right) \left(K \|u^0\|_{\mathcal{H}^{1,b}} + \frac{C_P K \sqrt{\pi}}{\sqrt{\alpha - \nu}} \|\tilde{k}\|_{\nu, \mathcal{L}_{2,b}}^t \right).$$

■

Now we can give an equivalent formulation of the coupled PDAE system (3.6) - (3.8) using the variation of constants formula. We apply the lemmas above and

set $k_j(t) := T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)) + E_j(t, u(t))$. Recall that we have shown in Section 3.2 that the operator T_j, N_j, E_j are continuous and Lipschitz for given $\varrho, \delta > 0$ and $0 < b < b_4$, $G^0 \geq 12(\bar{B}\delta + \varrho)$. W.l.o.g. we assume $\delta, \varrho \leq 1$ and define $G_u^0 = 12(\bar{B} + 1)$. Furthermore, recall the Definition (1.22) of the ball $B_{b,\varrho}$ around zero with radius ϱ .

Lemma 3.28. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9, 1.10 hold. Let $0 \leq b < \min(b_1, b_4)$, $0 < \varrho, \delta \leq 1$, $G^0 \geq G_u^0$ and $0 < \nu < \alpha$.*

For $j = 1, \dots, N$ assume $u_j(0) = u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$, $r_j^0 \in \mathbb{R}$.

Any solution (u, r, λ) of the PDAE system

$$u_{j,t}(t) = \Lambda_j u_j(t) + \lambda_j(t) w_{j,\xi} + T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)) + E_j(t, u(t)), u_j(0) = u_j^0, \quad (3.68)$$

$$r_{j,t}(t) = \lambda_j(t), \quad r_j(0) = r_j^0, \quad (3.69)$$

$$0 = \langle \hat{v}_{j,\xi}, u_j(t) \rangle \quad (3.70)$$

on $[0, \tau]$ for $j = 1, \dots, N$ with $(u(t), r(t), \lambda(t)) \in B_{\varrho,b}(0)$ for all $0 \leq t < \tau$ satisfies

$$u_j(t) = e^{\Lambda_j t} u_j^0 + \int_0^t e^{\Lambda_j(t-s)} P_j(T_j(s) + N_j(s, u(s), r(s), \lambda_j(s)) + E_j(s, u(s))) ds, \quad (3.71)$$

$$r_j(t) = \int_0^t \lambda_j(s) ds, \quad (3.72)$$

$$\begin{aligned} \lambda_j(t) = & - \langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} a_j(\hat{v}_{j,\xi}, u_j(t)) \\ & - \langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} \langle \hat{v}_{j,\xi}, T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)) + E_j(t, u(t)) \rangle. \end{aligned} \quad (3.73)$$

Conversely, if $u_j : [0, \tau] \rightarrow \mathcal{H}^{1,b}$, $\lambda_j, r_j : [0, \tau] \rightarrow \mathbb{R}$ are continuous for $j = 1, \dots, N$, $(u(t), r(t), \lambda(t)) \in B_{\varrho,b}(0)$ for all $0 \leq t < \tau$ and if (3.71) - (3.73) holds on $[0, \tau]$, then (u, r, λ) is a solution of (3.68) - (3.70) on $[0, \tau]$.

Fix $0 < b < \min(b_1, b_4)$. Choose G^0 such that in addition $\frac{\sqrt{\alpha-\nu}}{2} > C_P K C_E \sqrt{\pi} e^{-\gamma G^0}$ holds, then $u(t)$ with $t \in [0, \tau]$ satisfies the following estimate for some $K_\nu, \tilde{K}_\nu > 0$

$$\|u\|_{\nu, \mathcal{H}^{1,b}}^t \leq K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + \tilde{K}_\nu \|T + N(\cdot, u, r, \lambda)\|_{\nu, \mathcal{L}_{2,b}}^t. \quad (3.74)$$

Proof. To prove the first part of this lemma we apply Definition 1.11 and Lemma 3.26 with $k_j(t) = T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)) + E_j(t, u(t))$ for $j = 1, \dots, N$. Let $(u, r, \lambda) \in B_{\varrho,b}(0)$ and $0 < \delta \leq 1$. We conclude from Lemma 3.9, 3.10 and 3.11 that the functions T_j, N_j, E_j are continuous in time for $j = 1, \dots, N$, hence there exists a constant $\tilde{C}_\varrho > 0$, where \tilde{C}_ϱ depends on the size of ϱ , and it holds for all $s, t \geq 0$ with $|t - s| \leq \delta$ the following estimate

$$\begin{aligned} \|k_j(t) - k_j(s)\|_{\mathcal{L}_{2,b}} & \leq \tilde{C}_\varrho e^{\tilde{C}_\varrho |t-s|} (|t - s| + \|r(t) - r(s)\| \\ & \quad + \|\lambda(t) - \lambda(s)\| + \|u(t) - u(s)\|_{\mathcal{H}^{1,b}}). \end{aligned}$$

We obtain that k_j is locally Lipschitz in time for all $j = 1, \dots, N$ and the claim follows from Lemma 3.26.

Conversely, we know from Lemma 3.12 that solving the given system (3.68) - (3.70) is equivalent to solving the system (3.34) - (3.36) for $k_j(t) := T_j(t) + N_j(t, u(t), r(t), \lambda_j(t))$. Thus we apply Lemma 3.24 with $L = (L_1, \dots, L_N)$, $F = (F_1, \dots, F_N)$, where $L_j := \Lambda_{P,j}$ and $F_j(t, u(t)) := P_j(T_j(t) + N_j(t, u(t), r(t), \lambda_j(t)) + E_j(t, u(t)))$ for $j = 1, \dots, N$. Hence we know if (u, r, λ) is a solution of (3.71) - (3.73) then u_j is a solution of (3.68) and Lemma 3.12 yields that (u, r, λ) is a solution of (3.68) - (3.70).

Fix $0 < b < \min(b_1, b_4)$. Let G^0 satisfy $\frac{\sqrt{\alpha-\nu}}{2} > C_P K C_E \sqrt{\pi} e^{-\gamma G^0}$. Applying Lemma 3.27 we obtain the following estimate

$$\|u\|_{\nu, \mathcal{H}^{1,b}}^t \leq \left(1 + \frac{2C_P K C_E \sqrt{\pi}}{\sqrt{\alpha-\nu}}\right) \left(K \|u^0\|_{\mathcal{H}^{1,b}} + \frac{C_P K \sqrt{\pi}}{\sqrt{\alpha-\nu}} \|T + N(\cdot, u, r, \lambda)\|_{\nu, \mathcal{L}^{2,b}}^t\right). \quad (3.75)$$

■

Remark 3.29. Note that the equivalence statement is satisfied for all $0 \leq b < \min(b_1, b_4)$, in particular for $b = 0$. Only the estimate (3.74) needs the assumption $0 < b < \min(b_1, b_4)$.

3.6 Notations and definitions - part 2

In this section we give useful definitions and notations that will be needed to handle the difficulties in the proof of the Stability Theorem 3.1 introduced by the exponentially weighted norms.

Let $b \geq 0, \tau > 0$ be given parameters. Furthermore, let the functions $(u(t), r(t), \lambda(t)) = (u_1(t), \dots, u_N(t), r_1(t), \dots, r_N(t), \lambda_1(t), \dots, \lambda_N(t))$ be given with $u_j : [0, \tau) \rightarrow \mathcal{H}^{1,b}$, $r_j : [0, \tau) \rightarrow \mathbb{R}$, $\lambda_j : [0, \tau) \rightarrow \mathbb{R}$, $j = 1, \dots, N$. Note that $(u(t), r(t), \lambda(t))$ behave differently, therefore we attach in the following proofs different weights to $(u(t), r(t), \lambda(t))$. Let $\omega_1, \omega_2 > 0$ be weights, where ω_1 has to be large to handle the influence of $\lambda(t)$ and ω_2 has to be small to show that $r(t)$ is bounded. Furthermore, we multiply the $u(t)$ and $\lambda(t)$ with $e^{\nu t}$ for $\nu > 0$ to prove the exponentially decay in time as stated in (3.5).

Let $0 \leq t < \tau$, then we define the weighted norm expression for $(u(t), r(t), \lambda(t)) = (u_1(t), \dots, u_N(t), r_1(t), \dots, r_N(t), \lambda_1(t), \dots, \lambda_N(t))$ by

$$\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} = e^{\nu t} \omega_1 \|u(t)\|_{\mathcal{H}^{1,b}} + \omega_2 \|r(t)\| + e^{\nu t} \|\lambda(t)\|$$

and for $(u(t), r(t), \lambda_j(t)) = (u_1(t), \dots, u_N(t), r_1(t), \dots, r_N(t), \lambda_j(t))$ by

$$\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} = e^{\nu t} \omega_1 \|u(t)\|_{\mathcal{H}^{1,b}} + \omega_2 \|r(t)\| + e^{\nu t} |\lambda_j(t)|.$$

We define by

$$B_{\delta, \omega_1, \omega_2, \nu, b}^\tau(u, r, \lambda) = \{(v, g, \mu) : \sup_{0 \leq t < \tau} \|(v - u, g - r, \mu - \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \delta, \\ v_j(t) \in \mathcal{H}^{1, b}, g_j(t), \mu_j(t) \in \mathbb{R}, l = 1, \dots, N\}$$

the ball of radius δ in the weighted norm around $(u(t), r(t), \lambda(t))$.

It is convenient to introduce the following abbreviations, we define for $u : [0, \tau) \rightarrow \mathcal{H}^{1, b}, \lambda : [0, \tau) \rightarrow \mathbb{R}, r \in \mathbb{R}, t \geq 0$, compare Definition (3.56),

$$\|u\|_{t, \omega_1, \nu, \mathcal{H}^{1, b}} := \omega_1 e^{\nu t} \|u(t)\|_{\mathcal{H}^{1, b}}, \quad |\lambda|_{t, \nu} := e^{\nu t} |\lambda(t)| \quad \text{and} \quad |r|_{\omega_2} := \omega_2 |r|. \quad (3.76)$$

Furthermore, for $\varrho > 0$ we define the Ball B_ϱ around zero with radius ϱ by

$$B_\varrho = \{\mu \in \mathbb{R} : |\mu| \leq \varrho\}.$$

3.7 Local existence and uniqueness

Before we proceed with the proof of the Stability Theorem 3.1 we make use of Lemma 3.28 to show a local existence and uniqueness result. The proof of Theorem 3.30 has some similarities to [17], Theorem 3.3.3. and [36], Lemma 1.27.

Recall the parameter $b_1 > 0$ as defined in Section 3.2 and $b_4 > 0$ given by Theorem 3.21.

Theorem 3.30. *Assume that Hypotheses 1.4, 1.5, 1.6, 1.8, 1.9, 1.10 hold.*

Let $0 < b < \min(b_1, b_4)$. Then there exist weights $\nu > 0, \omega_1 > 1, 0 < \omega_2 < 1$ such that for any $0 < \varrho < \omega_2$ there exist constants $G^0, \varrho_1, \tau_1 > 0$ with the following property:

For any consistent initial values $u^0 = (u_1^0, \dots, u_N^0), u_j^0 \in \mathcal{H}^{1, b} \cap \mathcal{R}(P_j)$, $j = 1, \dots, N$ with $\|u^0\|_{\mathcal{H}^{1, b}} \leq \varrho_1$, $r^0 = (r_1^0, \dots, r_N^0)$ with $\|r^0\| \leq \frac{\varrho_1}{\omega_2}$ and g^0 with $|g_j^0 - g_i^0| > G^0, j \neq i$ we have the following existence results:

(i) *There exists a unique solution $\lambda_j^0 \in \mathcal{C}_b([0, \infty), B_{\frac{\varrho}{16}})$ of the consistency condition*

$$\lambda_j^0(\cdot) = -\langle \hat{v}_{j, \xi}, w_{j, \xi} \rangle^{-1} (a_j(\hat{v}_{j, \xi}, u_j^0) + \langle \hat{v}_{j, \xi}, N_j(\cdot, u^0, r^0, \lambda_j^0(\cdot)) + T_j(\cdot) + E_j(\cdot, u^0) \rangle) \quad (3.77)$$

for $j = 1, \dots, N$.

(ii) *The system (3.68) - (3.70) has a unique solution (u, r, λ) on $[0, \tau_1)$ with*

$$\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \varrho \quad \forall t \in [0, \tau_1). \quad (3.78)$$

Proof. Fix $0 < b < \min(b_1, b_4)$ and let $0 < \nu < \min(\alpha, \gamma)$.

First take $0 < \omega_2 < 1$ with $\frac{\omega_2}{\nu} < \frac{1}{16}$ and then choose $\omega_1 > 1$ such that

$$\omega_1 > 8C_{v,w}(C_v + \|\hat{v}_\xi\|_{\mathcal{L}_2}(5C_n e^{\tilde{C}_n} + C_e)), \quad (3.79)$$

$$\omega_1 > 2C_{v,w} \left(C_v + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \left(8C_T + 3C_N e^{\frac{\tilde{C}_N}{8}} + C_e \right) \right), \quad (3.80)$$

$$\omega_1 > \frac{1}{\omega_2}. \quad (3.81)$$

Consider an arbitrary ϱ with $0 < \varrho < \omega_2$.

Then let $G_u^0 = 12(\bar{B} + 1)$. Choose $G^0 \geq G_u^0$ such that

$$\frac{\omega_1 e^{-\gamma_1 G^0}}{\omega_2} \leq \varrho. \quad (3.82)$$

Choose $0 < \varrho_1 < \frac{\varrho}{4\omega_1(K+2)}$ and find $\varrho_1 < \frac{\varrho}{8\omega_1}$, since $K > 1$.

Define $\bar{B}_{\frac{\varrho}{16}} = \mathcal{C}_b([0, \infty), [-\frac{\varrho}{16}, \frac{\varrho}{16}])$. The task is now to show the solvability of the consistency equation (3.77) for u^0 with $u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j)$, $j = 1, \dots, N$ and $\|u^0\|_{\mathcal{H}^{1,b}} \leq \varrho_1$ and $r^0 \in \mathbb{R}^N$ with $\|r^0\| \leq \frac{\varrho_1}{\omega_2}$. We use the Banach fixed point Theorem, [14], Theorem 9.2.1 and solve (3.77).

For $j = 1, \dots, N$ we define the operator $l_j : \bar{B}_{\frac{\varrho}{16}} \rightarrow \mathcal{C}_b([0, \infty), \mathbb{R})$ by

$$l_j(\lambda_j)(t) = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle (a_j(\hat{v}_{j,\xi}, u_j^0) + \langle \hat{v}_{j,\xi}, T_j(t) + N_j(t, u^0, r^0, \lambda_j(t)) + E_j(t, u^0) \rangle)$$

Let $\lambda_j(\cdot) \in \bar{B}_{\frac{\varrho}{16}}$. From (1.21) and Lemma 3.3 - 3.5 we obtain for $t \geq 0$ the estimate

$$\begin{aligned} |l_j(\lambda_j)(t)| &\leq |\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1}| |a_j(\hat{v}_{j,\xi}, u_j^0) + \langle \hat{v}_{j,\xi}, T_j(t) + N_j(t, u^0, r^0, \lambda_j(t)) + E_j(t, u^0) \rangle| \\ &\leq C_{v,w} (C_v \|u^0\|_{\mathcal{H}^1} + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|T_j(t) + N_j(t, u^0, r^0, \lambda_j(t)) + E_j(t, u^0)\|_{\mathcal{L}_2}) \\ &\leq C_{v,w} (C_v \|u^0\|_{\mathcal{H}^{1,b}} \\ &\quad + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} (C_T e^{-\gamma_1 G^0} + C_N \|u^0\|_{\mathcal{H}^{1,b}}^2 + C_N \|u^0\|_{\mathcal{H}^{1,b}} |\lambda_j(t)|) \\ &\quad + C_N e^{\tilde{C}_N \|r^0\|} \|r^0\| \|u^0\|_{\mathcal{H}^{1,b}} + C_N e^{\tilde{C}_N \|r^0\|} \|r^0\| e^{-\gamma_1 G^0} (1 + \|u^0\|_{\mathcal{H}^{1,b}}) \\ &\quad + C_e \|u^0\|_{\mathcal{H}^{1,b}}) \\ &\leq C_{v,w} \left(C_v \frac{\varrho}{8\omega_1} + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \left(C_T \frac{\varrho^2}{\omega_1} \right. \right. \\ &\quad \left. \left. + C_N e^{\tilde{C}_N \frac{\varrho}{8\omega_1 \omega_2}} \left(\frac{\varrho^2}{64\omega_1^2} + \frac{\varrho^2}{128\omega_1} + \frac{\varrho^2}{64\omega_1^2 \omega_2} + \frac{\varrho^2}{4\omega_1^2} \right) + C_e \frac{\varrho}{8\omega_1} \right) \right) \\ &\leq \frac{C_{v,w} \varrho}{8\omega_1} \left(C_v + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \left(8C_T + 3C_N e^{\frac{\tilde{C}_N}{8}} + C_e \right) \right) \\ &< \frac{\varrho}{16}. \end{aligned}$$

Furthermore, using Lemma 3.9 - 3.11 we conclude that $l_j(\lambda_j)(t)$ is continuous in t . Therefore, $l_j(\bar{B}_{\frac{\varrho}{16}}) \subset B_{\frac{\varrho}{16}}$. For $t \geq 0$ we estimate the difference

$$\begin{aligned} & |l_j(\mu_j)(t) - l_j(\lambda_j)(t)| \\ & \leq |\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1}| \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|N_j(t, u^0, r^0, \lambda_j(t)) - N_j(t, u^0, r^0, \mu_j(t))\|_{\mathcal{L}_2} \\ & \leq C_{v,w} \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} C_n \frac{\varrho}{8\omega_1} |\lambda_j(t) - \mu_j(t)| < \frac{\varrho}{16} |\lambda_j(t) - \mu_j(t)|. \end{aligned}$$

Hence the function $l_j(\cdot)$ is contracting and has a fixed point $\lambda_j^0 \in \bar{B}_{\frac{\varrho}{16}}$.

Choose $\tau_1 > 0$ such small that the following conditions are satisfied

$$\begin{aligned} & e^{\nu\tau_1} < 2, \\ & \tau_1 < \frac{1}{16\omega_2}, \\ & C_P K \left(5C_N e^{\tilde{C}_N} + C_T + C_e \right) \int_0^{\tau_1} e^{(\nu-\alpha)s} \frac{1}{\sqrt{s}} ds < \frac{1}{16}, \\ & C_P K \left(5C_n e^{\tilde{C}_n} + C_e \right) \int_0^{\tau_1} e^{(\nu-\alpha)s} \frac{1}{\sqrt{s}} ds < \frac{1}{4}. \end{aligned}$$

From Theorem 3.21 we obtain for all $j = 1, \dots, N, 0 \leq t \leq \tau_1$

$$e^{\nu t} \|(e^{\Lambda_{P,j}t} - I)u_j^0\|_{\mathcal{H}^{1,b}} \leq \|e^{\Lambda_{P,j}t}u_j^0\|_{\mathcal{H}^{1,b}} + e^{\nu\tau_1} \|u_j^0\|_{\mathcal{H}^{1,b}} \leq (K+2)\varrho_1 < \frac{\varrho}{4\omega_1}. \quad (3.83)$$

We define the weighted ball around $(u^0, r^0, \lambda^0(\cdot))$ by

$$\begin{aligned} B = \{ & (u, r, \lambda) \in \mathcal{C}([0, \tau_1], (\mathcal{H}^{1,b})^N \times \mathbb{R}^N \times \mathbb{R}^N) : u_j \in \mathcal{R}(P_j), j = 1, \dots, N, \\ & \sup_{0 \leq t < \tau_1} \|(u - u^0, r - r^0, \lambda - \lambda^0)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \frac{\varrho}{2} \} \end{aligned}$$

and the weighted ball around $(u^0, r^0, \lambda_j^0(\cdot))$ for $j = 1, \dots, N$ by

$$\begin{aligned} B_j = \{ & (u, r, \lambda_j) \in \mathcal{C}([0, \tau_1], (\mathcal{H}^{1,b})^N \times \mathbb{R}^N \times \mathbb{R}) : u_j \in \mathcal{R}(P_j), j = 1, \dots, N, \\ & \sup_{0 \leq t < \tau_1} \|(u - u^0, r - r^0, \lambda_j - \lambda_j^0)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \frac{\varrho}{2} \}. \end{aligned}$$

Using the above assumption on the parameter τ_1 and $\|u^0\|_{\mathcal{H}^{1,b}} \leq \varrho_1$, $\lambda_j^0(t) \leq \frac{\varrho}{16}$ we conclude that $\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} < \varrho$ holds for $(u, r, \lambda) \in B$ and $0 \leq t < \tau_1$, since

$$\begin{aligned} \|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} &\leq \|(u - u^0, r - r^0, \lambda_j - \lambda_j^0)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\ &\quad + \omega_1 \|u^0\| e^{\nu t} + \|r^0\| \omega_2 + \|\lambda^0(t)\| e^{\nu t} \\ &\leq \frac{\varrho}{2} + \frac{\omega_1 \varrho}{8\omega_1} e^{\nu \tau_1} + \frac{\varrho \omega_2}{8\omega_1 \omega_2} + \frac{\varrho}{16} e^{\nu \tau_1} \\ &< \varrho. \end{aligned}$$

We use again the Banach fixed point Theorem to show that the system (3.68) - (3.70) has a unique solution (u, r, λ) on $[0, \tau_1)$. We define the function H for $(u, r, \lambda) \in B$, $0 \leq t < \tau_1$ by

$$H(u, r, \lambda)(t) = \begin{pmatrix} H_1(u, r, \lambda_1)(t) \\ \vdots \\ H_N(u, r, \lambda_N)(t) \end{pmatrix}$$

and for $(u, r, \lambda_j) \in B_j$ the functions $H_j(u, r, \lambda_j)(t)$ by

$$\begin{pmatrix} e^{\Lambda_{P,j} t} u_j^0 + \int_0^t e^{\Lambda_{P,j}(t-s)} P_j (E_j(s, u_j(s)) + T_j(s) + N(s, u(s), r(s), \lambda_j(s))) ds \\ \int_0^t \lambda_j(s) ds + r_j^0 \\ -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j(t)) + \langle \hat{v}_{j,\xi}, E_j(t, u(t)) + T_j(t) + N(t, u(t), r(t), \lambda_j(t)) \rangle) \end{pmatrix}.$$

We need estimates of the operators $N_j, T_j, j = 1, \dots, N$ for (u, r, λ_j) in B_j . From Lemma 3.3 we obtain for $t \in [0, \tau)$, $j = 1, \dots, N$ the following estimate

$$e^{\nu t} \|T_j(t)\|_{\mathcal{L}_{2,b}} \leq C_T e^{(\nu - \gamma_1)t} e^{-\gamma_1 G^0} \leq C_T e^{-\gamma_1 G^0}. \quad (3.84)$$

Using Lemma 3.4 we estimate the expression $N_j(t, u(t), r(t), \lambda_j(t))$ for $t \in [0, \tau_1)$, $j = 1, \dots, N$

$$\begin{aligned} &e^{\nu t} \|N_j(t, u(t), r(t), \lambda_j(t))\|_{\mathcal{L}_{2,b}} \\ &\leq C_N e^{\tilde{C}_N \|r(t)\|} \frac{\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}}{\omega_1} \\ &\quad \cdot \left(\|u(t)\|_{\mathcal{H}^{1,b}} + |\lambda_j(t)| + \frac{\|r(t)\|_{\omega_2}}{\omega_2} + \frac{\omega_1 e^{-\gamma_1 G^0}}{\omega_2} + \frac{\|r(t)\|_{\omega_2} e^{-\gamma_1 G^0}}{\omega_2} \right) \\ &\leq C_N e^{\tilde{C}_N} \frac{\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}}{\omega_1} \left(\frac{\varrho}{\omega_1} + \varrho + \frac{\varrho}{\omega_2} + \varrho + \frac{\varrho e^{-\gamma_1 G^0}}{\omega_2} \right) \\ &\leq \left(3 + \frac{2}{\omega_2} \right) \frac{\varrho}{\omega_1} C_N e^{\tilde{C}_N} \|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}. \end{aligned} \quad (3.85)$$

Combined with (3.84) this gives the following estimate

$$\|T_j + N_j(\cdot, u, r, \lambda_j)\|_{t, \omega_1, \nu, \mathcal{L}_{2,b}} \leq (C_T + 5C_N e^{\tilde{C}_N}) \varrho. \quad (3.86)$$

We use the estimates in Lemma 3.8 and we find that the following difference $N_j(t, u(t), r(t), \lambda_j(t)) - N_j(t, v(t), g(t), \mu_j(t))$ for $t \geq 0$, $j = 1, \dots, N$ and $\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho$, $\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho$ can be estimated by

$$\begin{aligned} & e^{\nu t} \|N_j(t, u(t), r(t), \lambda_j(t)) - N_j(t, v(t), g(t), \mu_j(t))\|_{\mathcal{L}_{2,b}} \\ & \leq \frac{C_n e^{\tilde{C}_n \max(\|r(t)\|, \|g(t)\|)}}{\omega_1} \|(u - v, r - g, \lambda_j - \mu_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\ & \quad \left(\max(\|u(t)\|_{\omega_1, \mathcal{H}^{1,b}}, \|v(t)\|_{\omega_1, \mathcal{H}^{1,b}}) + \max(|\lambda_j(t)|, |\mu_j(t)|) + 1 \right. \\ & \quad + \max(\|u\|_{t, \nu, \mathcal{H}^{1,b}}, \|v\|_{t, \nu, \mathcal{H}^{1,b}}, e^{-\gamma_1 G^0}) \frac{\omega_1}{\omega_2} \max(1, \|r(t)\|, \|g(t)\|) \\ & \quad \left. + \max(\|u(t)\|_{\omega_1, \mathcal{H}^{1,b}}, \|v(t)\|_{\omega_1, \mathcal{H}^{1,b}}) \frac{e^{-\gamma_1 G^0}}{\omega_2} \max(1, \|r(t)\|, \|g(t)\|) \right) \\ & \leq \frac{C_n e^{\tilde{C}_n}}{\omega_1} \|(u - v, r - g, \lambda_j - \mu_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\ & \quad * \left(2\varrho + 1 + \max\left(\frac{\varrho}{\omega_2}, \frac{\omega_1 e^{-\gamma_1 G^0}}{\omega_2}\right) + \frac{\varrho e^{-\gamma_1 G^0}}{\omega_2} \right) \\ & \leq \frac{5}{\omega_1} C_n e^{\tilde{C}_n} \|(u - v, r - g, \lambda_j - \mu_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}. \end{aligned} \quad (3.87)$$

Let $t \in [0, \tau_1)$ and $j \in \{1, \dots, N\}$. We use these estimates, Lemma 3.5, Theorem 3.21, (3.83) and the fact that $\lambda_j^0(t), j = 1, \dots, N$ is a solution of the consistency

equation (3.77) such that we obtain for $(u, r, \lambda_j) \in B_j$

$$\begin{aligned}
& \|H_j(u, r, \lambda_j) - (u_j^0, r_j^0, \lambda_j^0)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \\
& \leq e^{\nu t} \omega_1 \|(e^{\Lambda_{P, j} t} - I)u_j^0\|_{\mathcal{H}^{1, b}} + \omega_1 C_P K \int_0^\tau \frac{e^{(\nu - \alpha)(t-s)}}{\sqrt{t-s}} e^{\nu s} (\|T_j(s) \\
& \quad + N_j(s, u(s), r(s), \lambda_j(s))\|_{\mathcal{L}_{2, b}} + \|E_j(s, u(s))\|_{\mathcal{L}_{2, b}}) ds \\
& \quad + \omega_2 \left| \int_0^t \lambda_j(s) ds \right| + e^{\nu t} | - \langle \hat{v}_{j, \xi}, w_{j, \xi} \rangle^{-1} (a_j(\hat{v}_{j, \xi}, u_j(t)) \\
& \quad + \langle \hat{v}_{j, \xi}, N_j(t, u(t), r(t), \lambda_j(t)) + T_j(t) + E_j(t, u(t)) \rangle) - \lambda_j^0(t) | \\
& \leq e^{\nu t} \omega_1 \|(e^{\Lambda_{P, j} t} - I)u_j^0\|_{\mathcal{H}^{1, b}} \\
& \quad + C_P K \int_0^{\tau_1} \frac{e^{(\nu - \alpha)(t-s)}}{\sqrt{t-s}} ds (C_T + 5C_N e^{\tilde{C}_N} + C_e) \varrho \\
& \quad + \omega_2 \varrho \int_0^t e^{-\nu s} ds + e^{\nu t} | \langle \hat{v}_{j, \xi}, w_{j, \xi} \rangle^{-1} \|a_j(\hat{v}_{j, \xi}, u_j(t) - u_j^0)\| \\
& \quad + e^{\nu t} | \langle \hat{v}_{j, \xi}, w_{j, \xi} \rangle^{-1} | \langle \hat{v}_{j, \xi}, N_j(t, u(t), r(t), \lambda_j(t)) - N_j(t, u^0, 0, \lambda_j^0(t)) \rangle | \\
& \quad + e^{\nu t} | \langle \hat{v}_{j, \xi}, w_{j, \xi} \rangle^{-1} | \langle \hat{v}_{j, \xi}, E_j(t, u(t)) - E_j(t, u^0) \rangle | \\
& \leq \frac{\varrho}{4} + C_P K \int_0^{\tau_1} \frac{e^{(\nu - \alpha)s}}{\sqrt{s}} ds (C_T \varrho + 5C_N \rho + C_e \varrho) + \frac{\omega_2 \varrho}{\nu} \\
& \quad + e^{\nu t} C_{v, w} C_v \|u_j(t) - u_j^0\|_{\mathcal{H}^{1, b}} \\
& \quad + C_{v, w} \|\hat{v}_{j, \xi}\|_{\mathcal{L}_2} \frac{5}{\omega_1} C_n e^{\tilde{C}_n} \|(u - u^0, r, \lambda_j - \lambda_j^0)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1, b}}^t \\
& \quad + e^{\nu t} C_{v, w} C_e \|\hat{v}_{j, \xi}\|_{\mathcal{L}_2} \|u(t) - u^0\|_{\mathcal{H}^{1, b}} \\
& \leq \frac{3\varrho}{8} + C_{v, w} (C_v + (C_e + 5C_n e^{\tilde{C}_n}) \|\hat{v}_{j, \xi}\|_{\mathcal{L}_2}) \frac{\varrho}{2\omega_1} < \frac{\varrho}{2}.
\end{aligned}$$

Taking the supremum over all $t \in [0, \tau_1)$ and $j = 1, \dots, N$ gives

$$\sup_{0 \leq t < \tau_1} \|H(u, r, \lambda) - (u^0, r^0, \lambda^0)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \frac{\varrho}{2}.$$

From Lemma 3.9 - 3.11 and Lemma 3.25 we obtain that $H_j(u, r, \lambda_j)(t)$ is continuous in t .

Let $t \in [0, \tau_1)$, we use the estimate (3.87), from the following computation we

see that $H_j, j = 1, \dots, N$ are contractive

$$\begin{aligned}
& \|H_j(u, r, \lambda_j) - H_j(v, g, \mu_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& \leq e^{\nu t} \omega_1 \int_0^t \|e^{\Lambda_{P,j}(t-s)} P_j(E_j(s, u(s) - v(s)) \\
& \quad + N_j(s, u(s), r(s), \lambda_j(s)) - N_j(s, v, g(s), \mu_j(s))\|_{\mathcal{H}^{1,b}} ds \\
& \quad + \omega_2 \left| \int_0^t \lambda_j(s) - \mu_j(s) ds \right| + e^{\nu t} |\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1}| (|a_j(\hat{v}_{j,\xi}, u_j(t) - v_j(t))| \\
& \quad + \langle \hat{v}_{j,\xi}, E_j(t, u(t) - v(t)) + N_j(t, u(t), r(t), \lambda_j(t)) - N_j(t, v(t), g(t), \mu_j(t)) \rangle) \\
& \leq \omega_1 C_P K \int_0^t \frac{e^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} e^{\nu s} (C_e \|u(s) - v(s)\|_{\mathcal{L}_{2,b}} \\
& \quad + \|N_j(s, u(s), r(s), \lambda_j(s)) - N_j(s, v(s), g(s), \mu_j(s))\|_{\mathcal{L}_{2,b}} ds \\
& \quad + \omega_2 \tau_1 \max_{s \in [0, \tau]} |\lambda_j(s) - \mu_j(s)| + e^{\nu t} C_{v,w} (C_v \|u_j(t) - v_j(t)\|_{\mathcal{H}^{1,b}} \\
& \quad + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} (C_e \|u(t) - v(t)\|_{\mathcal{H}^{1,b}} \\
& \quad + \|N_j(t, u(t), r(t), \lambda_j(t)) - N_j(t, v(t), g(t), \mu_j(t))\|_{\mathcal{L}_{2,b}}) \\
& \leq C_P K \int_0^{\tau_1} \frac{e^{(\nu-\alpha)s}}{\sqrt{s}} ds (5C_n e^{\tilde{C}_n} + C_e) \sup_{0 \leq t < \tau_1} \|(u, r, \lambda) - (v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& \quad + \omega_2 \tau_1 \sup_{0 \leq t < \tau_1} \|(u, r, \lambda_j) - (v, g, \mu_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& \quad + \frac{1}{\omega_1} C_{v,w} (C_v + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} (5C_n e^{\tilde{C}_n} + C_e)) \|(u, r, \lambda) - (v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& \leq \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{8}\right) \|(u, r, \lambda) - (v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& < \frac{1}{2} \sup_{0 \leq t < \tau_1} \|(u, r, \lambda) - (v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}.
\end{aligned}$$

By taking the supremum over all $t \in [0, \tau_1)$ and $j = 1, \dots, N$ we obtain

$$\begin{aligned}
& \sup_{0 \leq t < \tau_1} \|H(u, r, \lambda) - H(v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\
& \leq \frac{1}{2} \sup_{0 \leq t < \tau_1} \|(u, r, \lambda) - (v, g, \mu)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}.
\end{aligned}$$

By the Banach fixed point theorem exists a fixed point $(u, r, \lambda) \in B$, especially $(u, r, \lambda) \in B_{\varrho, \omega_1, \omega_2, \nu, b}^{\tau_1}(0)$ and therefore $(u(t), r(t), \lambda(t)) \in B_{1,b}(0)$ for all $t \in [0, \tau_1)$. Using Lemma 3.28 we conclude that (u, r, λ) solves (3.68) - (3.70) on $[0, \tau_1)$. ■

Remark 3.31. Note that the existence and uniqueness result (3.30) holds also for $b = 0$ if we choose $\nu < \min(\alpha, \gamma_1)$ in the proof. Therefore, the existence and uniqueness result is satisfied in a small time-dependent subspace interval for all small b -dependent exponentially weighted spaces, including the case $b = 0$. For the long time analysis we require $0 < b < \min(b_1, b_4)$ and $0 < \nu < \min(\alpha, \gamma)$.

3.8 Proof of the Stability Theorem 3.1

With all the technical preparation at hand we proceed with the proof of Theorem 3.1. This final step has similarities to the analysis of single traveling waves in [17], Theorem 5.1.1, but is considerably more involved.

Proof.

Step 1: Fix $0 < b < \min(b_1, b_4)$, where b_1 is defined in Section 3.2 and b_4 is given by Theorem 3.21. Choose ν, ω_1, ω_2 as in Theorem 3.30 and select ϱ such that $0 < \varrho < \omega_2$ and the following condition is satisfied

$$\tilde{K}_\nu \varrho C_N e^{\tilde{C}N} \left(3 + \frac{2}{\omega_2}\right) < \frac{1}{4}, \quad (3.88)$$

Then there exist constants $\tau_1, \varrho_1, G^0 > 0$ such that for any consistent initial values $u^0 = (u_1^0, \dots, u_N^0), u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j), j = 1, \dots, N$ with $\|u^0\|_{\mathcal{H}^{1,b}} \leq \varrho_1$, $r^0 = (r_1^0, \dots, r_N^0)$ with $\|r^0\| \leq \frac{\varrho_1}{\omega_2}$ and g^0 with $|g_j^0 - g_i^0| > G^0, j \neq i$ the following existence results hold:

- (i) There exists a unique solution $\lambda_j^0 \in \mathcal{C}_b([0, \infty), B_{\frac{\varrho}{16}})$ of the consistency condition

$$\lambda_j^0(\cdot) = -\langle \hat{v}_{j,\xi}, w_{j,\xi} \rangle^{-1} (a_j(\hat{v}_{j,\xi}, u_j^0) + \langle \hat{v}_{j,\xi}, N_j(\cdot, u^0, r^0, \lambda_j^0(\cdot)) + T_j(\cdot) + E_j(\cdot, u^0) \rangle)$$

for $j = 1, \dots, N$.

- (ii) The system (3.68) - (3.70) has a unique solution (u, r, λ) on $[0, \tau_1]$ with

$$\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho \quad \forall t \in [0, \tau_1].$$

Let

$$\varrho_2 \leq \max\left(\varrho_1, \frac{\varrho_1}{4K_\nu \omega_1}\right). \quad (3.89)$$

Using the existence results we conclude that for any consistent initial condition $u^0 = (u_1^0, \dots, u_N^0), u_j^0 \in \mathcal{H}^{1,b} \cap \mathcal{R}(P_j), j = 1, \dots, N$ with $\|u^0\|_{\mathcal{H}^{1,b}} \leq \varrho_2$, $r^0 = (0, \dots, 0)$, and g^0 with $|g_j^0 - g_i^0| > G^0, j \neq i$ the following holds:

From (ii) we obtain that the system (3.68) - (3.70) has a unique solution (u, r, λ) on $[0, \tau_1)$ with

$$\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \varrho \quad \forall t \in [0, \tau_1). \quad (3.90)$$

Increase G^0 such that in addition the following holds:

$$\frac{\omega_1 e^{-\gamma_1 G^0}}{\omega_2} \leq \varrho_1, \quad (3.91)$$

$$C_P K C_E \sqrt{\pi} e^{-\gamma G^0} < \frac{\sqrt{\alpha - \nu}}{2}, \quad (3.92)$$

$$C_{v, w} \|\hat{v}_\xi\|_{\mathcal{L}_2} C_T e^{-\gamma_1 G^0} \leq \frac{\varrho_1}{8}, \quad (3.93)$$

$$\tilde{K}_\nu C_T \omega_1 e^{-\gamma_1 G^0} \leq \frac{\varrho_1}{8}. \quad (3.94)$$

We define

$$\tau_\infty = \sup\{\tau > 0 : \text{There exists a unique solution } (u, r, \lambda) \text{ of (3.68) - (3.70) on } [0, \tau) \text{ with } \sup_{0 \leq t < \tau} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \varrho\}.$$

We prove $\tau_\infty = \infty$. Assume $\tau_\infty < \infty$, i.e. there exists a unique solution (u, r, λ) of (3.68) - (3.70) on $[0, \bar{\tau})$ with

$$\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1, b}}^{\bar{\tau}} \leq \varrho \quad (3.95)$$

for all $0 < \bar{\tau} < \tau_\infty$. Note from Theorem 3.30 we conclude $\tau_\infty \geq \tau_1$.

Step 2: Let $\bar{\tau} \in [0, \tau_\infty)$. Assume (u, r, λ) is a solution of (3.68) - (3.70) on $[0, \bar{\tau})$. Assume for $0 < \tau < \bar{\tau}$,

$$\sup_{0 \leq t \leq \tau} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \varrho_1 \quad (3.96)$$

with ϱ_1 from Step 1.

Then we claim that the following estimate holds

$$\sup_{0 \leq t \leq \tau} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1, b}} \leq \frac{15\varrho_1}{16}. \quad (3.97)$$

The system (3.68) - (3.70) has an equivalent formulation (3.71) - (3.73), compare Lemma 3.28. We need an estimate of the operator $N_j, j = 1, \dots, N$. Using Lemma 3.4 and (3.91) we estimate the terms $N_j(t, u(t), r(t), \lambda_j(t))$ for $t \in [0, \tau]$,

$j = 1, \dots, N$

$$\begin{aligned}
& e^{\nu t} \|N_j(t, u(t), r(t), \lambda_j(t))\|_{\mathcal{L}_{2,b}} \\
& \leq C_N e^{\tilde{C}_N \|r(t)\|} \frac{\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}}{\omega_1} \\
& \quad \cdot \left(\|u(t)\|_{\mathcal{H}^{1,b}} + |\lambda_j(t)| + \frac{\|r(t)\|_{\omega_2}}{\omega_2} + \frac{\omega_1 e^{-\gamma_1 G^0}}{\omega_2} + \frac{\|r(t)\|_{\omega_2} e^{-\gamma_1 G^0}}{\omega_2} \right) \\
& \leq C_N e^{\tilde{C}_N} \frac{\|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}}{\omega_1} \left(\frac{\varrho_1}{\omega_1} + \varrho_1 + \frac{\varrho_1}{\omega_2} + \varrho_1 + \frac{\varrho_1 e^{-\gamma_1 G^0}}{\omega_2} \right) \\
& \leq \left(3 + \frac{2}{\omega_2} \right) \frac{\varrho_1}{\omega_1} C_N e^{\tilde{C}_N} \|(u, r, \lambda_j)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}.
\end{aligned}$$

From the conditions on the weights given in Theorem 3.30, (3.88), (3.89), (3.92) - (3.94), (3.74), Lemma 3.3 and Lemma 3.5 we obtain for $t \in [0, \tau]$, $j = 1, \dots, N$

$$\begin{aligned}
e^{\nu t} \omega_1 \|u_j(t)\|_{\mathcal{H}^{1,b}} & \leq \omega_1 K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + \omega_1 \tilde{K}_\nu \|T_j + N_j(\cdot, u, r, \lambda_j)\|_{t, \mathcal{L}_{2,b}}^t \\
& \leq \omega_1 K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + \tilde{K}_\nu C_T \omega_1 e^{-\gamma_1 G^0} \\
& \quad + \tilde{K}_\nu \left(3 + \frac{2}{\omega_2} \right) \varrho_1 C_N e^{\tilde{C}_N} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \\
& < \frac{3\varrho_1}{8} + \frac{\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t}{4},
\end{aligned}$$

$$\begin{aligned}
\|r_j(t)\|_{\omega_2} & \leq \omega_2 \left| \int_0^t \lambda_j(s) ds \right| \leq \omega_2 \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \int_0^\infty e^{-\nu s} ds \\
& \leq \frac{\omega_2}{\nu} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \leq \frac{\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t}{16},
\end{aligned}$$

$$\begin{aligned}
e^{\nu t} \|\lambda_j(t)\| & \leq C_{v,w} (C_v \|u\|_{t, \nu, \mathcal{H}^{1,b}} + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|T_j + N_j(\cdot, u, r, \lambda_j)\|_{t, \nu, \mathcal{L}_{2,b}} \\
& \quad + C_e \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|u\|_{t, \nu, \mathcal{L}_{2,b}}) \\
& \leq C_{v,w} (C_v \|u\|_{t, \nu, \mathcal{H}^{1,b}} + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} C_T e^{-\gamma_1 G^0} \\
& \quad + \left(3 + \frac{2}{\omega_2} \right) \frac{\varrho_1}{\omega_1} C_N e^{\tilde{C}_N} \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \\
& \quad + C_e \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \|u\|_{t, \nu, \mathcal{H}^{1,b}}) \\
& \leq \frac{C_{v,w}}{\omega_1} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t (C_v + (5C_N e^{\tilde{C}_N} + C_e) \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2}) \\
& \quad + C_{v,w} \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} C_T e^{-\gamma_1 G^0} \\
& \leq \frac{1}{8} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t + \frac{\varrho_1}{8}.
\end{aligned}$$

Summarizing the terms and using (3.96) we obtain the estimate (3.97).

Step 3: Define

$$s_\infty = \sup\{s \in [0, \tau_\infty) : \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho_1 \quad \forall 0 \leq t \leq s\}.$$

Then $\tau_\infty = s_\infty$ is satisfied.

Assume $s_\infty < \tau_\infty$. From the continuity of the solution (u, r, λ) in $[0, s_\infty]$ we obtain the estimate $\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho_1$ for all $0 \leq t \leq s_\infty$. We infer from Step 2 that even $\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \frac{15}{16}\varrho_1$ is satisfied for $0 \leq t \leq s_\infty$. Furthermore, we conclude from the continuity of the solution (u, r, λ) in $[0, \tau_\infty) \supset [0, s_\infty]$ that there exists $\varepsilon > 0$ such that even $\|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho_1$ holds for $0 \leq t \leq s_\infty + \varepsilon$, which is a contradiction to the definition of s_∞ . Therefore, $\tau_\infty = s_\infty$.

Step 4: Since $\tau_\infty \geq \tau_1$ we conclude $\tau_\infty - \frac{1}{4}\tau_1 > 0$. By definition of τ_∞ and Step 3 there exists a unique solution (u, r, λ) of (3.68) - (3.70) on $[0, \tau_\infty - \frac{1}{4}\tau_1]$ with $\sup_{0 \leq t < \tau_\infty - \frac{1}{4}\tau_1} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho_1$.

Consider $\tilde{\tau} := \tau_\infty - \frac{1}{2}\tau_1$, we conclude $\|u(\tilde{\tau})\|_{\mathcal{H}^{1,b}} \leq \varrho_1$ and $\|r(\tilde{\tau})\| \leq \frac{\varrho_1}{\omega_2}$. We apply Theorem 3.30 with initial data $u(\tilde{\tau})$ and $r(\tilde{\tau})$ and obtain a solution on $[\tilde{\tau}, \tilde{\tau} + \tau_1)$. Gluing the solutions together we obtain a unique solution (u, r, λ) of (3.68) - (3.70) on $[0, \tau_\infty + \frac{1}{2}\tau_1)$ with $\sup_{0 \leq t < \tau_\infty + \frac{1}{2}\tau_1} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho$.

This is a contradiction to the definition of τ_∞ . Therefore, we conclude that there exists a solution (u, r, λ) of (3.68) - (3.70) on $[0, \infty)$ with $\sup_{0 \leq t < \infty} \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \leq \varrho$.

Step 5: Similar to the above arguments we estimate for $t \geq 0$

$$\begin{aligned} & \|(u, r, \lambda)\|_{t, \omega_1, \omega_2, \nu, \mathcal{H}^{1,b}} \\ & \leq \omega_1 K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + \tilde{K}_\nu C_T \omega_1 e^{-\gamma_1 G^0} + \tilde{K}_\nu \left(3 + \frac{2}{\omega_2}\right) \varrho C_N e^{\tilde{C}_N} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \\ & \quad + \frac{\omega_2}{\nu} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t + C_{v,w} \left(C_v \frac{\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t}{\omega_1} + \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} C_T e^{-\gamma_1 G^0} \right. \\ & \quad \left. + (5C_N e^{\tilde{C}_N} + C_e) \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2} \frac{\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t}{\omega_1} \right) \\ & \leq \omega_1 K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + (\tilde{K}_\nu \omega_1 + C_{v,w} \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2}) C_T e^{-\gamma_1 G^0} + \frac{7}{16} \|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \end{aligned}$$

Taking the supremum over $[0, t]$ we obtain for all $t \geq 0$

$$\|(u, r, \lambda)\|_{\omega_1, \omega_2, \nu, \mathcal{H}^{1,b}}^t \leq \frac{16}{9} (\omega_1 K_\nu \|u^0\|_{\mathcal{H}^{1,b}} + (\tilde{K}_\nu \omega_1 + C_{v,w} \|\hat{v}_{j,\xi}\|_{\mathcal{L}_2}) C_T e^{-\gamma_1 G^0}).$$

Choosing $\tilde{C} := \frac{16}{9} \max(\omega_1 K_\nu, \tilde{K}_\nu C_T \omega_1 + C_{v,w} \|\hat{v}_\xi\|_{\mathcal{L}_{2,b}} C_T)$ gives the stability estimate

$$\omega_1 \|u(t)\|_{\mathcal{H}^{1,b}} + e^{-\nu t} \omega_2 \|r(t)\| + \|\lambda(t)\| \leq \tilde{C} e^{-\nu t} (\|u^0\|_{\mathcal{H}^{1,b}} + e^{-\gamma_1 G^0}).$$

Therefore $\tau_j := \int_0^\infty \lambda_j(s) ds$ is finite, we estimate $r_j(t) - \tau_j$ for $j = 1, \dots, N$:

$$\begin{aligned} |r_j(t) - \tau_j| &\leq \int_t^\infty \|\lambda(s)\| ds \leq \int_t^\infty \tilde{C} e^{-\nu s} (\|u^0\|_{\mathcal{H}^{1,b}} + e^{-\gamma_1 G^0}) ds \\ &\leq \frac{\tilde{C}}{\nu} e^{-\nu t} (\|u^0\|_{\mathcal{H}^{1,b}} + e^{-\gamma_1 G^0}). \end{aligned}$$

In summary, for $j = 1, \dots, N$ exist $C, \nu > 0, \tau_j \in \mathbb{R}$ such that the following estimate is satisfied

$$\|u_j(t)\|_{\mathcal{H}^{1,b}} + |r_j(t) - \tau_j| + |\lambda_j(t)| \leq C e^{-\nu t} (\|u^0\|_{\mathcal{H}^{1,b}} + e^{-\gamma_1 G^0}).$$

■

Chapter 4

Numerical applications - Strong interaction

We proceed with more numerical applications of the 'decompose and freeze method'. We consider multipulse or multifront solutions, where the single traveling pulses or fronts interact strongly.

4.1 Multipulse of multifront consisting of two components

We consider the case $N = 2$ and the system (2.1) - (2.4) given in Chapter 2 and proceed analogously to Chapter 2. We test our method again on the standard example, the Nagumo-equation (2.8) with the parameter $a = 0.25$. The examples presented here have also been considered in [4].

Again we choose as a bump function $\varphi = \operatorname{sech}(\beta\xi)$ with $\beta = 0.5$, Neumann boundary conditions and the finite computational domain $[-L, L]$ with $L = 50$.

4.1.1 Fronts moving in the same direction in the Nagumo-equation

The first numerical example shows two fronts moving in the same same direction. As displayed in Figure 4.1, after collision of the fronts the multifront becomes a single front moving in the same direction. Though N is larger than necessary, i.e. the number of components in the 'decompose and freeze ansatz' is larger than the components that constitute the solution, the method creates no problems and gives reasonable results. Both velocities μ_j converge to the same value \bar{c}_1 , the single frozen profiles $v_j, j = 1, 2$, displayed in Figure 4.2, become stationary

and the superposition u_L becomes a single traveling front moving with velocity $\mu_1 = \mu_2 = \bar{c}_1$ to the left. The positions $g_j(t), j = 1, 2$ tend to $\bar{c}_1 t + K_j$ with some constants $K_j, j = 1, 2$.

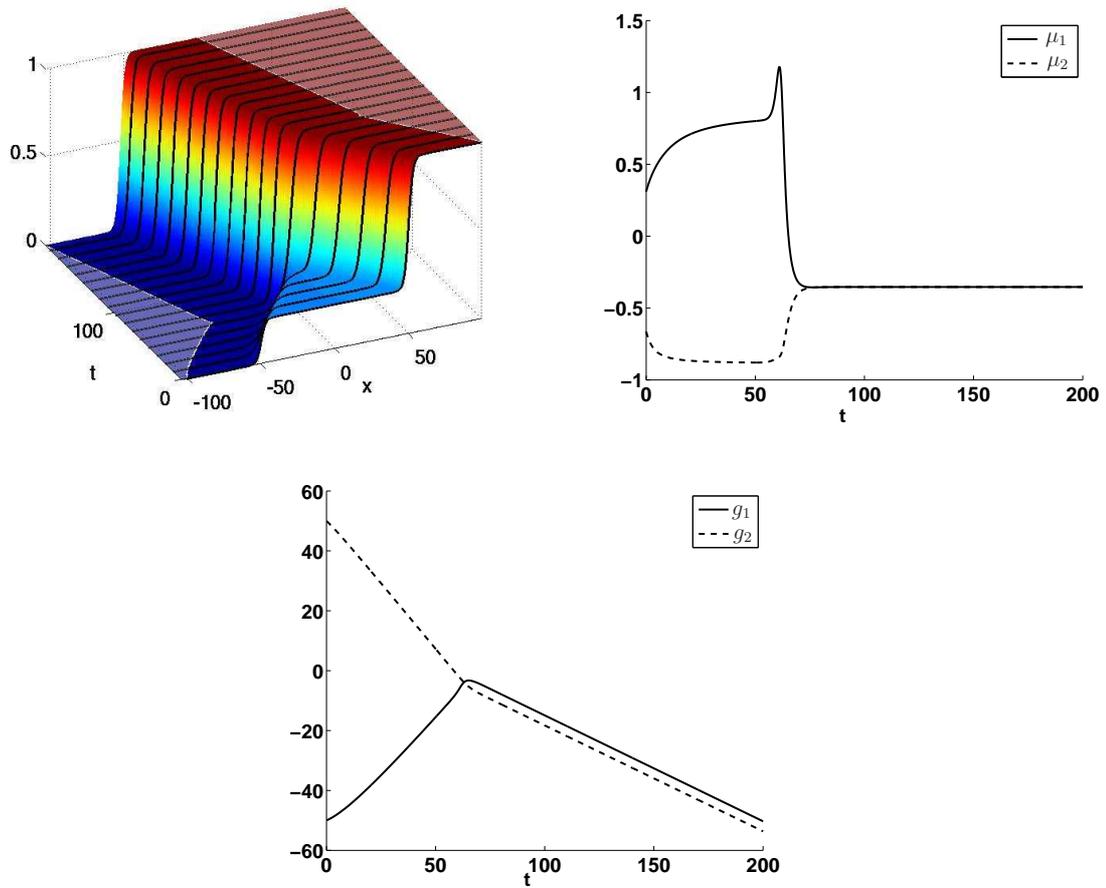


Figure 4.1: Fronts moving in the same directions in the Nagumo-equation, evolution of superposition u_L , the velocities μ_1, μ_2 and the positions g_1, g_2 .

In numerical tests we observe that asymptotically the traveling wave solution $u(x, t) = \bar{w}(x - \bar{c}_1 t)$ of the Nagumo-equation (2.8) is given by $u(x, t) = z_1(x - \bar{c}_1 t - K_1) + z_2(x - \bar{c}_1 t - K_2)$ with constants K_1, K_2 and

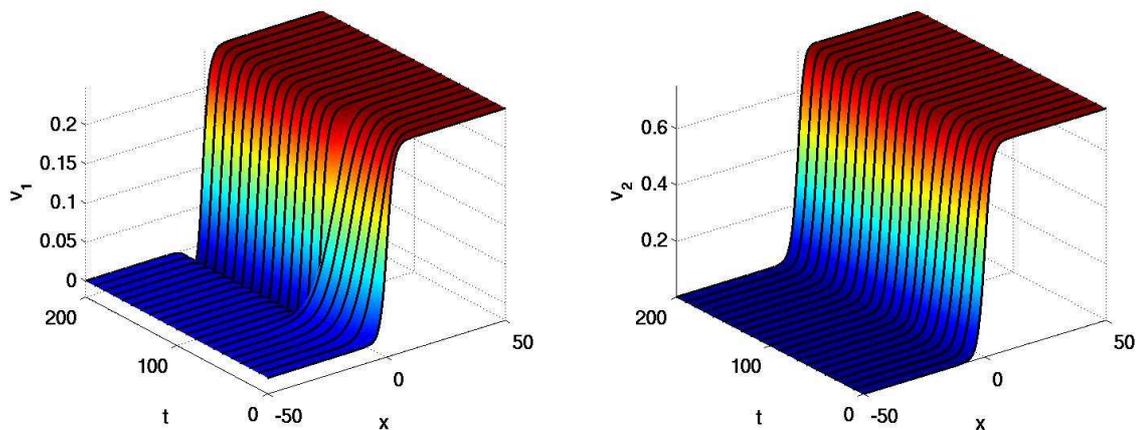


Figure 4.2: Fronts moving in the same directions in the Nagumo-equation, evolution of frozen v_1, v_2 .

$v_j(\cdot, t) \rightarrow z_j(\cdot), j = 1, 2$. The $z_j, j = 1, 2$ satisfy the system

$$\begin{aligned} 0 &= Az_{1,\xi\xi} + \bar{c}_1 z_{1,\xi} + f(z_1) + \frac{\varphi}{\varphi + \varphi(\cdot - K_2 + K_1)} \\ &\quad * (f(\bar{w}_1(\cdot + K_1)) - f(z_1) - f(\bar{w}_1(\cdot + K_1) - z_1) + a) \\ 0 &= Az_{2,\xi\xi} + \bar{c}_1 z_{2,\xi} + f(z_2 + a) + \frac{\varphi}{\varphi + \varphi(\cdot - K_1 + K_2)} \\ &\quad * (f(\bar{w}_1(\cdot + K_2)) - f(\bar{w}_1(\cdot + K_2) - z_2) - f(z_2 + a)). \end{aligned}$$

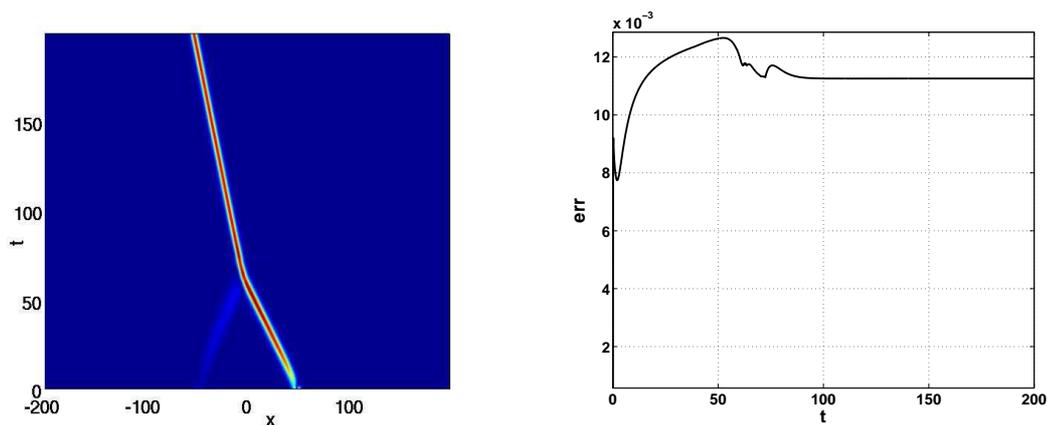


Figure 4.3: Evolution of the absolute-error and the \mathcal{L}_2 -error for fronts moving in the same directions in the Nagumo-equation.

In Figure 4.3 we show the comparison of u_L , defined by (2.9), with the solution u_l of the Nagumo-equation (2.8) for an sufficiently large interval in absolute values

and in the \mathcal{L}_2 -norm as functions of time. We see that the error gets asymptotically constant and the two solutions agree except for a small domain.

4.1.2 Collision of two traveling waves in the Nagumo-equation

Another very interesting example of strongly interacting pulses or fronts is the annihilation of two traveling fronts.

Again Figure 4.4 and Figure 4.5 shows the superposition (2.9) consisting of the moving profiles v_j together with the velocities $\mu_j, j = 1, 2$ and the positions $g_j, j = 1, 2$ as functions of time. We start with a downward hat function and obtain as a result that the two profiles v_j annihilate each other. Although the solution of (2.8) is constant after collision, the single profiles remain fronts. The two single frozen profiles $v_j, j = 1, 2$ displayed in Figure 4.6 become stationary. The velocities μ_j and the position g_j converge after a short transient period towards zero.

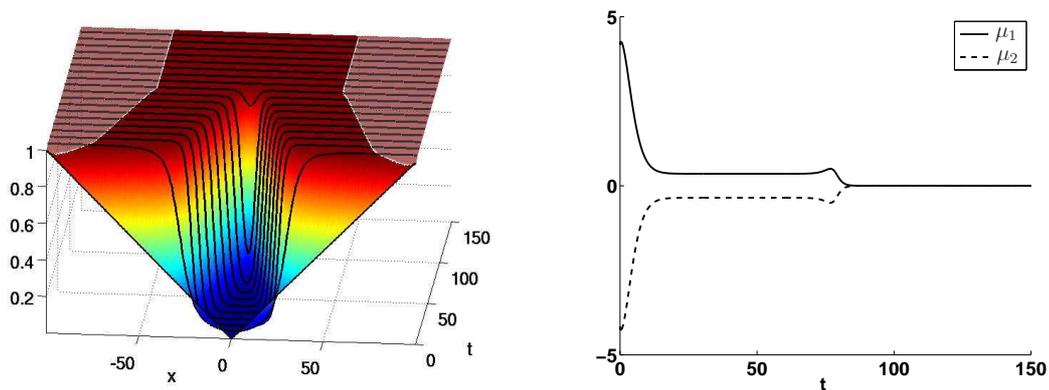


Figure 4.4: Annihilating fronts in the Nagumo-equation, evolution of superposition u_L , the velocities μ_1, μ_2 .

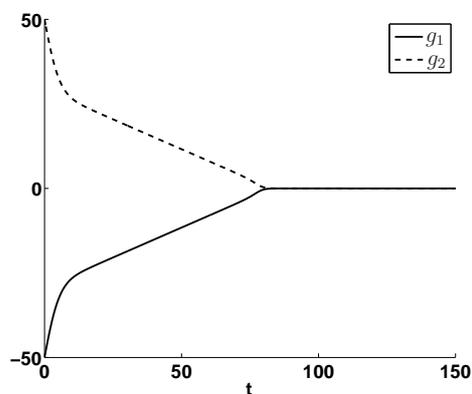


Figure 4.5: Annihilating fronts in the Nagumo-equation, evolution of the positions g_1, g_2 .

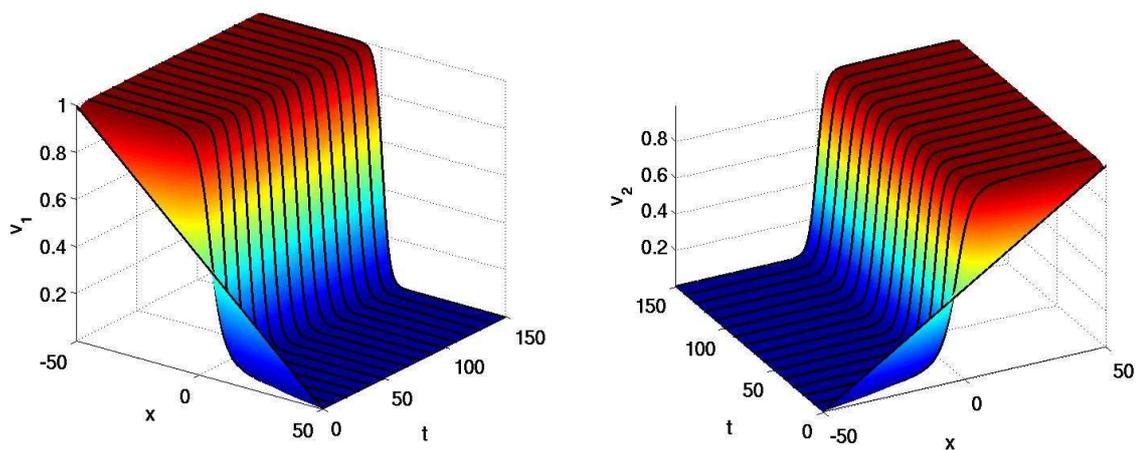


Figure 4.6: Annihilating fronts moving in the Nagumo-equation, evolution of frozen v_1, v_2 .

Numerically we observe that the solution $u(x, t)$ of the Nagumo-equation (2.8) is asymptotically given by $u(x, t) = 1 = z_1(x) + z_2(x)$, because $v_j(\cdot, t) \rightarrow z_j(\cdot), j = 1, 2$. The $z_j, j = 1, 2$ satisfy the system

$$0 = Az_{j,\xi\xi} + f(z_j) - \frac{1}{2}(f(z_j) + f(1 - z_j)), j = 1, 2.$$

Figure 4.7 compares the superposition u_L with the solution u_l of the Nagumo-

equation (2.8) for an sufficiently large interval in absolute values and in the \mathcal{L}_2 -norm as functions of time. In the moment of strong interaction the error, which was before constant because of a phase shift, tends to zero.

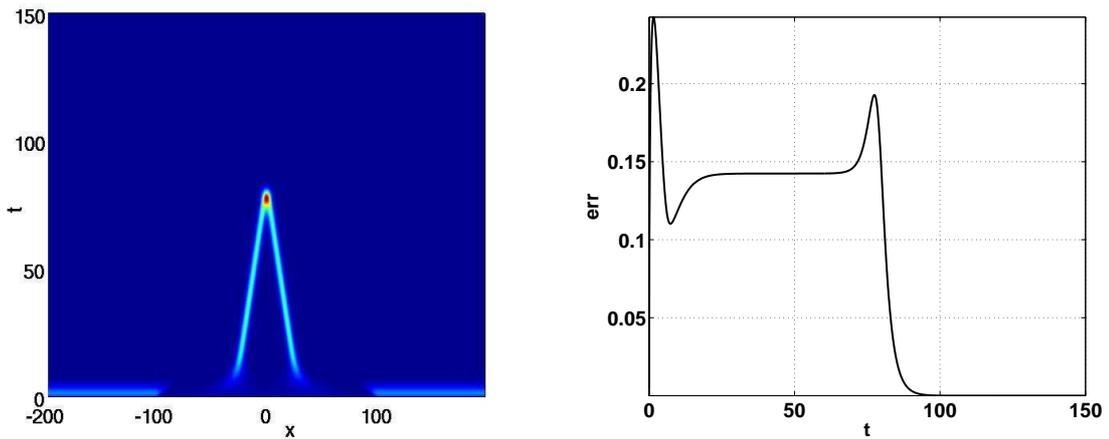


Figure 4.7: Evolution of the absolute-error and the \mathcal{L}_2 -error for the annihilation of two fronts in the Nagumo-equation.

4.2 Multipulse or multifront consisting of three components

We consider the case of a multipulse or multifront consisting of three profiles, i.e. N is equal to 3.

We recall the coupled PDAE system (1.36) - (1.38) for the case $N = 3$ and

$t \geq 0, \xi \in \mathbb{R}$

$$v_{1,t} = Av_{1,\xi\xi} + v_{1,\xi}\mu_1 + f(v_1) + \frac{\varphi}{\varphi + \varphi(\cdot - g_2 + g_1) + \varphi(\cdot - g_3 + g_1)} \\ * [f(v_1 + v_2(\cdot - g_2 + g_1) + v_3(\cdot - g_3 + g_1)) - f(v_1) - f(v_2(\cdot - g_2 + g_1) + \tilde{w}_2^-) \\ - f(v_3(v_3(\cdot - g_3 + g_1) + \tilde{w}_3^-))], \quad v_1(0) = v_1^0, \quad (4.1)$$

$$v_{2,t} = Av_{2,\xi\xi} + v_{2,\xi}\mu_2 + f(v_2 + \tilde{w}_2^-) + \frac{\varphi}{\varphi + \varphi(\cdot - g_1 + g_2) + \varphi(\cdot - g_3 + g_2)} \\ * [f(v_1(\cdot - g_1 + g_2) + v_2 + v_3(\cdot - g_3 + g_2)) - f(v_1(\cdot - g_1 + g_2)) - f(v_2 + \tilde{w}_2^-) \\ - f(v_3(\cdot - g_3 + g_2) + \tilde{w}_3^-)], \quad v_2(0) = v_2^0, \quad (4.2)$$

$$v_{3,t} = Av_{3,\xi\xi} + v_{3,\xi}\mu_3 + f(v_3 + \tilde{w}_3^-) + \frac{\varphi}{\varphi + \varphi(\cdot - g_1 + g_3) + \varphi(\cdot - g_2 + g_3)} \\ * [f(v_1(\cdot - g_1 + g_3) + v_2(\cdot - g_2 + g_3) + v_3) - f(v_1(\cdot - g_1 + g_3)) \\ - f(v_2(\cdot - g_2 + g_3) + \tilde{w}_2^-) - f(v_3 + \tilde{w}_3^-)], \quad v_3(0) = v_3^0, \quad (4.3)$$

$$g_{j,t} = \mu_j, \quad g_j(0) = g_j^0, \quad j = 1, 2, 3, \quad (4.4)$$

$$0 = \langle v_j - \hat{v}_j, \hat{v}_{j,\xi} \rangle, \quad j = 1, 2, 3. \quad (4.5)$$

Note that the argument ξ is suppressed.

4.2.1 Three interacting traveling waves in the Nagumo-equation

Again we consider the standard numerical example, the Nagumo-equation (2.8) with parameter $a = 0.25$ and Neumann boundary conditions on the computational domain $[-50, 50]$.

We are interested in the sum

$$u_L(x, t) = v_1(x - g_1(t), t) + v_2(x - g_2(t), t) + v_3(x - g_3(t), t) \quad (4.6)$$

for $t \geq 0$. Figure 4.8 shows this sum u_L as functions of time together with the velocities $\mu_j, j = 1, \dots, 3$ and the time-dependent positions $g_j, j = 1, \dots, 3$, similarly to Figure 2.1.

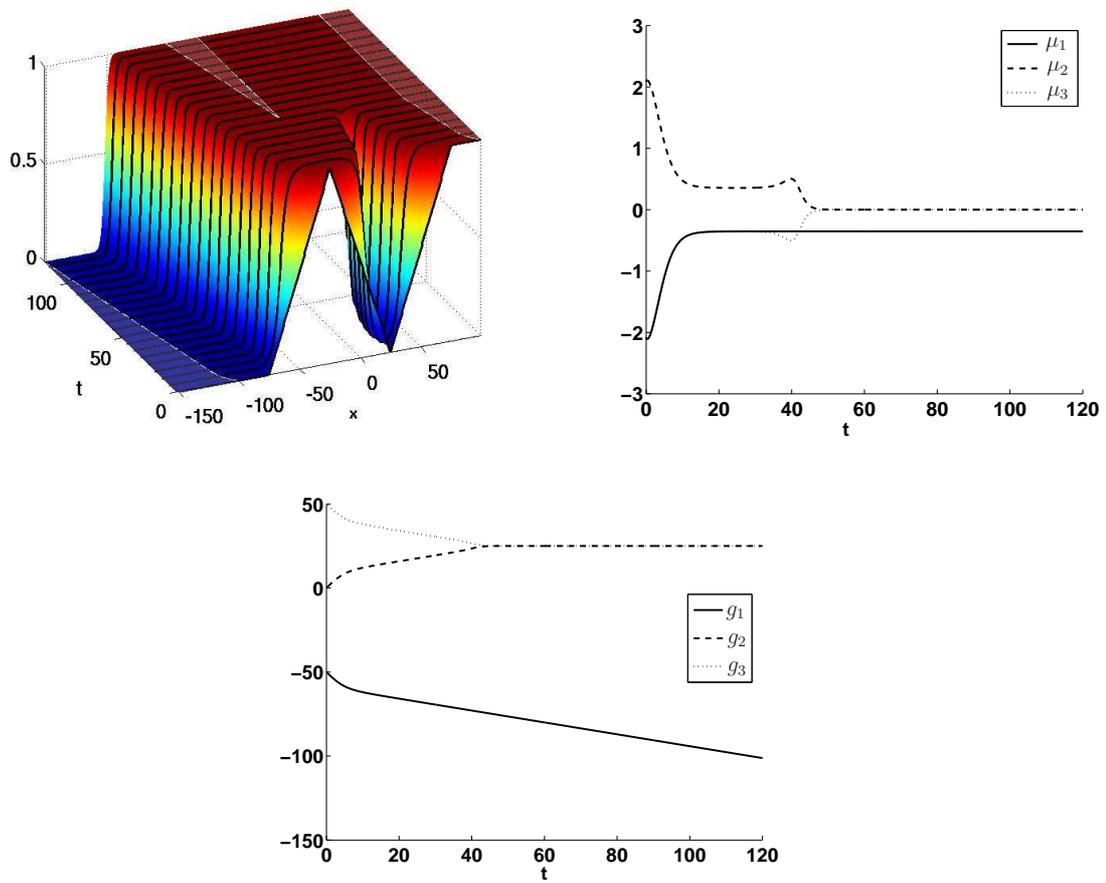


Figure 4.8: Multifront in the Nagumo-equation, evolution of superposition u_L , the velocities μ_1, μ_2, μ_3 and the positions g_1, g_2, g_3 .

The initial profile turns into one traveling front, where the profiles v_2 and v_3 annihilate each other. After a transient time, the single frozen profiles become stationary, see Figure 4.9. As a consequence of the annihilation of profiles v_2 and v_3 , the velocities μ_2 and μ_3 tend to zero and the velocity μ_1 tends to some \bar{c}_1 . The positions g_2 and g_3 tend to the same constant K_2 , whereas the position g_1 tends to $\bar{c}_1 t + K_1$ for some constant K_1 .

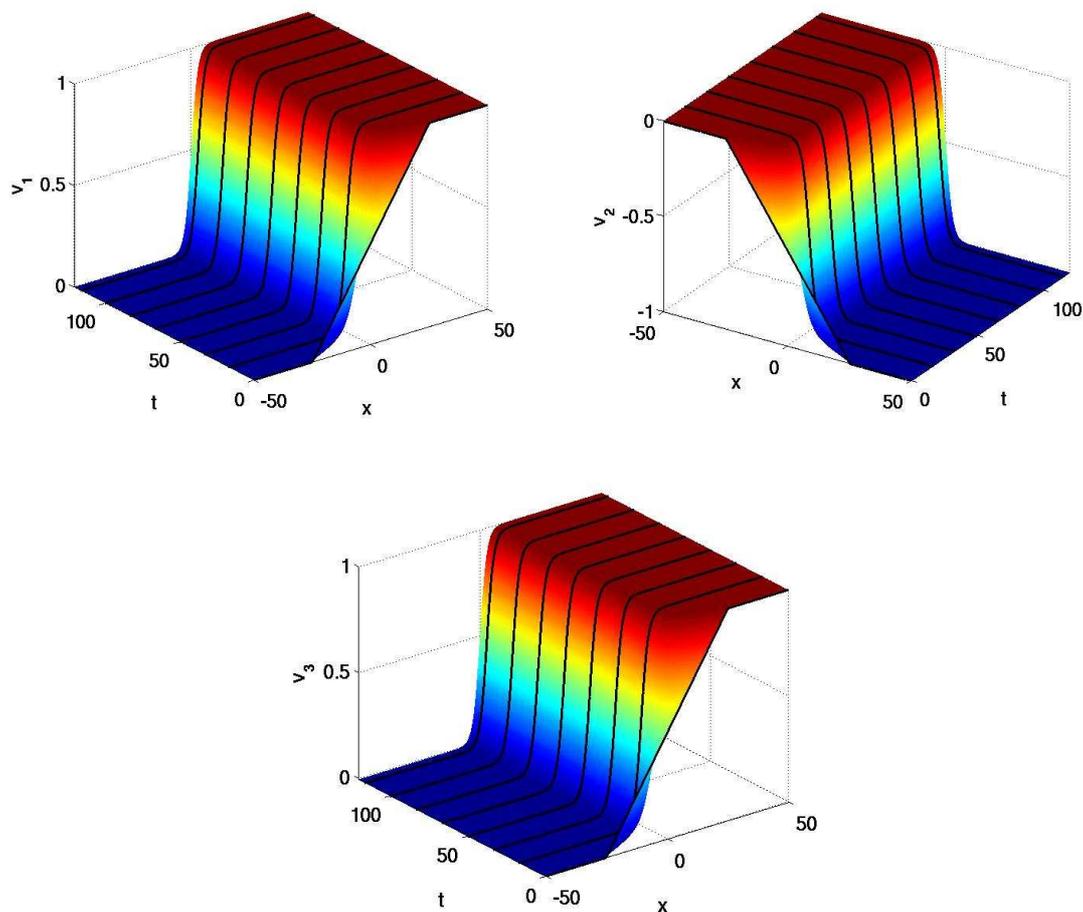


Figure 4.9: Multifront in the Nagumo-equation, evolution of frozen profiles v_1 , v_2 , v_3 .

We compare the superposition u_L and the solution u_l of the Nagumo-equation (2.8) on a large interval. Figure 4.10 shows that the two resulting fronts agree except for a small domain. Further the \mathcal{L}_2 -distance gets asymptotically constant caused by the single phase shift.

We obtain analogous results, see Figure 4.11, if we start with similar initial functions u^0 given by the sum of some initial profiles v_j^0 and the initial positions g_j^0 , compare (2.7). These initial profiles could be defined on larger intervals and have different slopes.

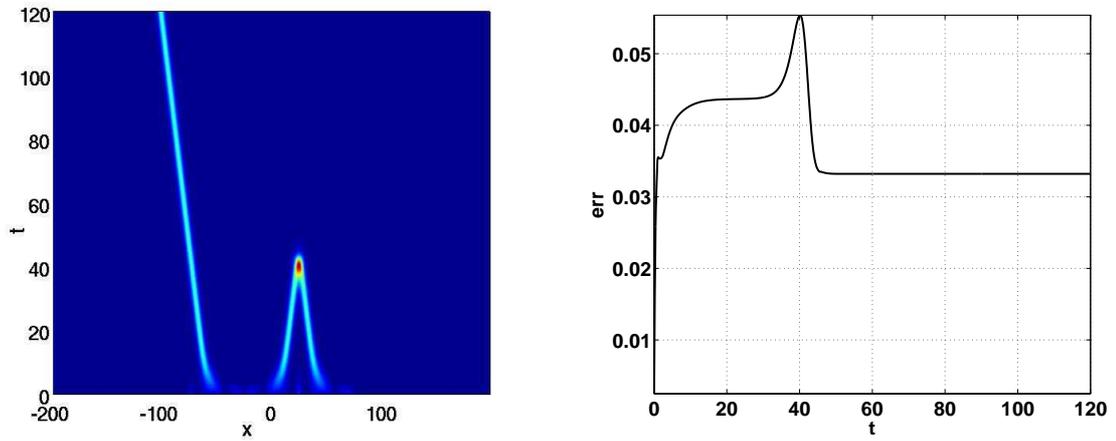


Figure 4.10: Evolution of the absolute-error and the \mathcal{L}_2 -error for the annihilation of three fronts in the Nagumo-equation.

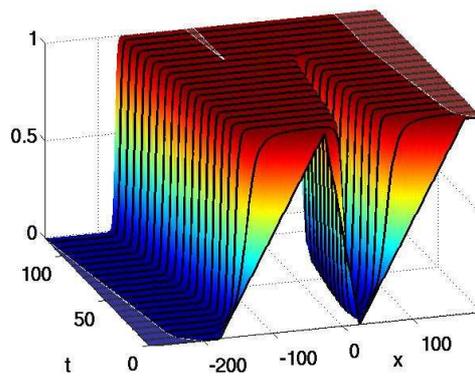


Figure 4.11: Multifront in the Nagumo-equation.

4.3 Open problems in case of collision

We test our method on a collision case in the FitzHugh-Nagumo-equations (2.10) - (2.11) for the case $N = 2$, i.e. for our numerical computations we consider the system (2.1) - (2.4) on the interval $[-L, L]$ with $L = 70$. We choose the relative tolerance 10^{-5} , the absolute tolerance $2 * 10^{-6}$, the spatial step size $\Delta\xi = 0.2$ and we impose Neumann boundary conditions.

We start with the traveling pulses computed in the FitzHugh-Nagumo example in Chapter 2, Section 2.2 and interchange the g_j to use them as new initial positions g_j^0 . The pulses move towards each other and the frozen profiles seems to stabilize in the initial phase. In contrast to the Nagumo collision example the velocities tend to infinity in the moment of collision and the pulses of the profiles V_j become rapidly constant, see Figure 4.12 and Figure 4.13.

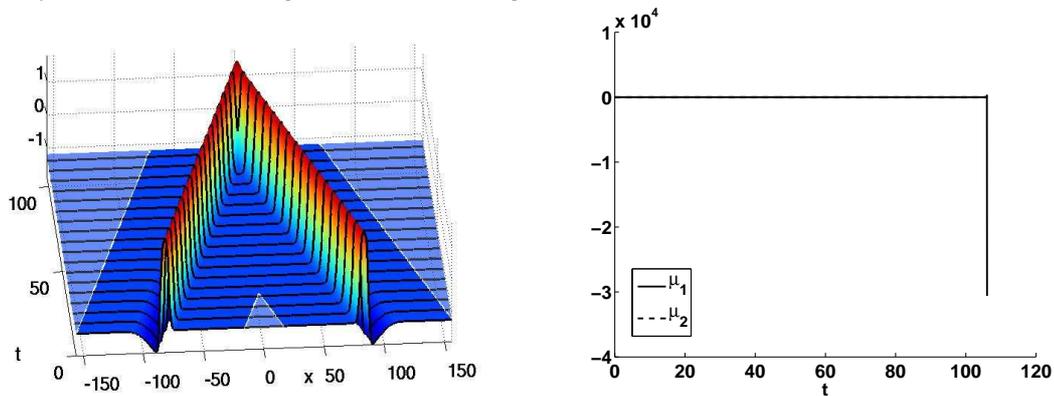


Figure 4.12: Collision of two pulses in the FitzHugh-Nagumo-equations, evolution of V_L and of the velocities μ_1 and μ_2 .

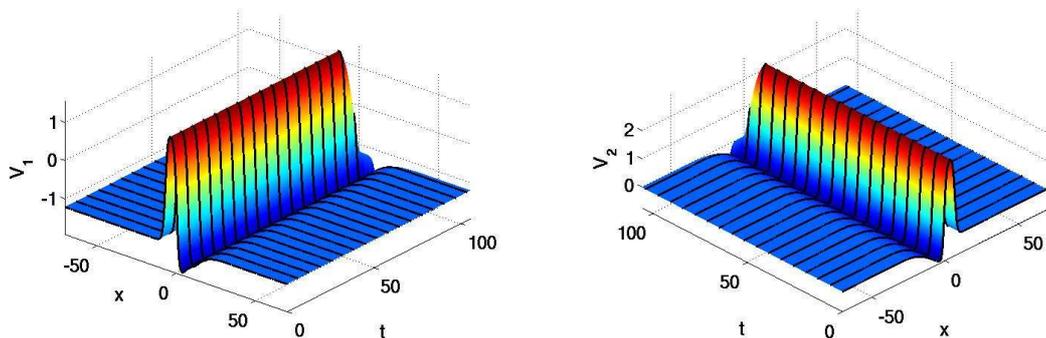


Figure 4.13: Collision of two pulses of the V_L component in the FitzHugh-Nagumo-equations, evolution of the frozen pulses V_1 and V_2 .

Note that in the moment of collision the profiles V_1, V_2, R_1, R_2 become constant

in space and hence $V_{1,\xi}, V_{2,\xi}, R_{1,\xi}, R_{2,\xi}$ approach zero. For the calculation of μ_j the phase condition gives the following terms $\mu_j \langle \hat{v}_{j,\xi}, v_{j,\xi} \rangle$. Therefore, the problem becomes ill-posed and the μ_j explode. A deeper analysis of this kind of problem is still ahead.

Note the difference to the Nagumo collision case, where we have considered two colliding fronts. There $v_{1,\xi}, v_{2,\xi}$ do not approach zero and the terms $\mu_j \langle \hat{v}_{j,\xi}, v_{j,\xi} \rangle$ do not create problems for the calculation of μ_1, μ_2 . The fact that we consider here a multipulse creates the above discussed ill-posed problem.

This example shows that there are still open problems for this method. Currently, the Stability Theorem 1.13 is proven only for weakly interacting pulses or fronts. But we hope that it will be possible to extend it to strongly interacting pulses or fronts in a reasonable way in order to be able to apply our method to more complicated phenomena.

Appendix A

Auxiliary results

A.1 Exponential dichotomies

This section contains a brief summary of exponential dichotomies. For a deeper discussion of this topic we refer the reader to [8], [28] and [30].

Definition A.1. *The linear differential operator*

$$Lz = z_\xi - Mz, \quad \xi \in J \subset \mathbb{R}, M : J \rightarrow \mathbb{R}^{l,l} \quad (\text{A.1})$$

with solution operator $S(\xi, x)$ has an **exponential dichotomy** on the interval J with data (K, α, π) if there exists a bound $K > 0$, a rate $\alpha > 0$ and a projector valued function $\pi : J \ni \xi \mapsto \pi(\xi)$ such that

$$S(\xi, x)\pi(x) = \pi(\xi)S(\xi, x)$$

holds and that the Green's function

$$G(\xi, x) = \begin{cases} S(\xi, x)\pi(x), & \xi \geq x, \\ -S(\xi, x)(I - \pi(x)), & \xi < x, \end{cases} \quad (\text{A.2})$$

satisfies the exponential estimate

$$\|G(\xi, x)\| \leq Ke^{-\alpha|\xi-x|}, \quad \xi, x \in J.$$

Using the definition of G the solution of

$$Lz = r, \quad \xi \in J, \quad z(\xi_0) = z_{\xi_0}$$

is given by

$$z(\xi) = S(\xi, \xi_0)z_{\xi_0} + s_J(r)(\xi), \quad (\text{A.3})$$

where $s_J(r)(\xi) = \int_J G(\xi, x)r(x)dx$. From [35], Lemma 3.10 we conclude that there exists some $C_s > 0$ such that the following estimate is satisfied

$$\|s_J(g)(\xi)\| + \|s_J(g)\|_{\mathcal{L}_2} \leq C_s \|g\|_{\mathcal{L}_2}, \quad \forall \xi \in J. \quad (\text{A.4})$$

The operator L given by (A.1) has exponential dichotomies on \mathbb{R}_- and on \mathbb{R}_+ if the boundary matrices $M_{\pm} = \lim_{\xi \rightarrow \pm\infty} M(\xi)$ are hyperbolic. For the proof we refer to [35], Corollary 3.8 and [2], Lemma 2.1.

Corollary A.2. *Let L be given by (A.1). Let $M \in C(\mathbb{R}, \mathbb{R}^{l,l})$. The following limits exist*

$$M_{\pm} = \lim_{\xi \rightarrow \pm\infty} M(\xi)$$

and the matrices M_{\pm} are hyperbolic. Let X_{\pm}^s be the stable subspace of M_{\pm} and X_{\pm}^u be the unstable subspace of M_{\pm} .

Then L has an exponential dichotomy on \mathbb{R}_- and an exponential dichotomy on \mathbb{R}_+ and the projectors satisfy

$$\lim_{\xi \rightarrow -\infty} (I - \pi_-(\xi)) = E_-^u, \quad \lim_{\xi \rightarrow \infty} (\pi_+(\xi)) = E_-^s,$$

where E_-^u denotes the projector onto X_-^u and E_-^s the projector onto X_-^s . If the number of stable and unstable eigenvalues of M_{\pm} is equal to m , we obtain

$$\dim \mathcal{N}(\pi_-(0)) = \dim \mathcal{R}(E_-^u) \quad \text{and} \quad \dim \mathcal{R}(\pi_+(0)) = \dim \mathcal{R}(E_+^s).$$

A.2 Functional analytic notions and results

In this section we recall the notion of the resolvent set and the spectrum of an operator $T : \mathcal{D}(T) \subset X \rightarrow X$. Further we present some functional analytic results.

Definition A.3 (Resolvent and spectrum). *Let X be a Banach space and $T : X \supseteq \mathcal{D}(T) \rightarrow X$ be a linear operator. Let $T_{\lambda} = \lambda I - T, \lambda \in \mathbb{C}$.*

1. *The operator $R_{\lambda} = (\lambda I - T)^{-1}$ with domain $\mathcal{D}(R_{\lambda})$ is called the **resolvent** of T at the point λ . The mapping $R(\lambda) = R_{\lambda}$ is called the **resolvent function** of T .*
2. *The **resolvent set** $\rho(T)$ contains all points $\lambda \in \mathbb{C}$ for which the following holds:*
 - $R(\lambda)$ exists,
 - $R(\lambda)$ is continuous,

- $\mathcal{D}(R(\lambda)) = R(T_\lambda)$ is dense in X .

3. The complement of the resolvent set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the **spectrum**. It is divided in two subsets $\sigma(T) = \sigma_{ess}(T) \cup \sigma_{pt}(T)$, where the **point spectrum** $\sigma_{pt}(T)$ contains all isolated eigenvalues of finite multiplicity and $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_{pt}(T)$ is called the **essential spectrum**.

The following functional analytic result is called Sobolev Imbedding theorem, see, for instance, [29], Theorem 6.91.

Theorem A.4 (The Sobolev Imbedding Theorem). *Let $s > \frac{m}{2}$. Then*

$$H^s(\mathbb{R}^m) \subset \mathcal{C}_b(\mathbb{R}^m).$$

Moreover, this imbedding is continuous, i.e., there is a constant C such that

$$\|u\|_\infty \leq C\|u\|_{\mathcal{H}^s} \quad (\text{A.5})$$

for every $u \in \mathcal{H}^s(\mathbb{R}^m)$.

A.3 The weighted spaces $\mathcal{L}_{2,b}$, $\mathcal{H}^{1,b}$ and $\mathcal{H}^{2,b}$

The definition of the weighted spaces $\mathcal{L}_{2,b}$, $\mathcal{H}^{1,b}$ and $\mathcal{H}^{2,b}$ for $b > 0$ is given in Chapter 1, Section 1.2 with weight function

$$\theta_b(\xi) = \frac{1}{2}(e^{b\xi} + e^{-b\xi}) = \cosh(b\xi), \quad \forall \xi \in \mathbb{R}.$$

It follows

$$\theta_b^{-1}(\xi) = \frac{2}{e^{b\xi} + e^{-b\xi}} = \operatorname{sech}(b\xi), \quad \forall \xi \in \mathbb{R}.$$

For the resolvent estimate in Chapter 3, Section 3.4 we need estimates of products of the functions θ_b , $(\theta_b)^{-1}$ and its derivatives. We calculate

$$\begin{aligned} (\theta_b^{-1})_\xi(\xi) &= -b \operatorname{sech}(b\xi) \tanh(b\xi), & (\theta_b^{-1})_{\xi\xi}(\xi) &= -b^2 \operatorname{sech}(b\xi)(1 - 2 \tanh^2(b\xi)), \\ (\theta_b^{-1})_{\xi\xi\xi}(\xi) &= b^3 \operatorname{sech}(b\xi) \tanh(b\xi)(5 - 6 \tanh^2(b\xi)), \\ (\theta_b)_\xi(\xi) &= b \sinh(b\xi), & (\theta_b)_{\xi\xi}(\xi) &= b^2 \cosh(b\xi) \end{aligned}$$

and estimate

$$|\theta_b(\xi)(\theta_b^{-1})_\xi(\xi)| = |b \tanh(b\xi)| \leq b, \quad (\text{A.6})$$

$$|\theta_b(\xi)(\theta_b^{-1})_{\xi\xi}(\xi)| = |-b^2 + 2b^2 \tanh(b\xi)| \leq 3b^2, \quad (\text{A.7})$$

$$|\theta_b(\xi)(\theta_b^{-1})_{\xi\xi\xi}(\xi)| = |b^3 \tanh(b\xi)(5 - 6 \tanh^2(b\xi))| \leq 11b^3, \quad (\text{A.8})$$

$$|(\theta_b)_\xi(\xi)\theta_b^{-1}(\xi)| = |b \tanh(b\xi)| \leq b, \quad (\text{A.9})$$

$$|(\theta_b)_{\xi\xi}(\xi)\theta_b^{-1}(\xi)| = b^2, \quad (\text{A.10})$$

$$|(\theta_b)_\xi(\xi)(\theta_b^{-1})_\xi(\xi)| = |b^2 \tanh^2(b\xi)| \leq b^2, \quad (\text{A.11})$$

$$|(\theta_b)_\xi(\xi)(\theta_b^{-1})_{\xi\xi}(\xi)| = |-b^3 \tanh(b\xi) + 2b^3 \tanh^3(b\xi)| \leq 3b^3. \quad (\text{A.12})$$

The density of the space $H^{k,b}$ is classical for the case $b = 0$ (see [29], Corollary 6.72). Since we could not find a proper reference for the case $b > 0$ we prove it here for completeness.

Lemma A.5. *Let $b \geq 0$. $\mathcal{H}^{2,b}$ is dense in $\mathcal{L}_{2,b}$.*

Proof. For $b = 0$ apply [29], Corollary 6.72: \mathcal{C}_0^∞ is dense in \mathcal{H}^2 and in \mathcal{L}_2 . From $\mathcal{H}^2 \subset \mathcal{L}_2$ follows the claim.

Let $b > 0$ and $u \in \mathcal{L}^{2,b}$. From above and $\theta_b u \in \mathcal{L}_2$ we conclude that there exist $\tilde{u}_n \in \mathcal{H}^2$ with $\|\tilde{u}_n - \theta_b u\|_{\mathcal{L}_2} \rightarrow 0$ as $n \rightarrow \infty$. Define $u_n := (\theta_b)^{-1} \tilde{u}_n \in \mathcal{H}^{2,b}$. We obtain for $n \rightarrow \infty$

$$\|u_n - u\|_{\mathcal{L}_{2,b}} = \|\theta_b u_n - \theta_b u\|_{\mathcal{L}_2} = \|\theta_b (\theta_b)^{-1} \tilde{u}_n - \theta_b u\|_{\mathcal{L}_2} = \|\tilde{u}_n - \theta_b u\|_{\mathcal{L}_2} \rightarrow 0.$$

■

A.4 Estimates of the bump function

The next two sections present some preliminary estimates of the operators T_j, N_j, E_j for $j = 1, \dots, N$ in Chapter 3, Section 3.2.

Lemma A.6. *Assume that Hypothesis 1.10 holds.*

Then there exist constants $C, \tilde{C} > 0$ such that for all $j, k \in \{1, \dots, N\}$, $t \geq 0, \xi \in \mathbb{R}$, $h \in [0, 1]$ and $r, g \in \mathbb{R}^N$ the following estimate is satisfied

$$\frac{\int_0^1 |\varphi'(\xi_{kj}^{r+h(g-r)+ct+g^0})| dh}{\varphi(\xi_{kj}^{r+ct+g^0})} \leq C e^{\tilde{C} \max(\|g\|, \|r\|)}.$$

Proof. In the following we use C to denote a generic constant. Let $t \geq 0, h \in [0, 1]$. Let $d_k(t) = (c_k - c_j)t + g_k^0 - g_j^0 + r_k - r_j$ and $d_k^h(t) = d_k(t) + h(g_k - g_j - r_k + r_j)$. We estimate the term

$$I := \frac{\int_0^1 |\varphi'(\xi - d_k^h(t))| dh}{\varphi(\xi - d_k(t))}$$

on three different subintervals that form a partition of \mathbb{R} .

Let $\xi \geq d_k(t) + 4 \max(\|r\|, \|g\|)$. We use Hypothesis 1.10 and estimate

$$I \leq C \int_0^1 e^{-\beta(\xi - d_k^h(t))} dh e^{\beta(\xi - d_k(t))} \leq C \sup_{h \in [0, 1]} e^{\beta h |g_k - g_j + r_j - r_k|} \leq C e^{4\beta \max(\|r\|, \|g\|)}.$$

Let $d_k(t) - 4 \max(\|r\|, \|g\|) \leq \xi \leq d_k(t) + 4 \max(\|r\|, \|g\|)$.

$$I \leq C \int_0^1 e^{-\beta|\xi - d_k^h(t)|} dh e^{\beta|\xi - d_k(t)|} \leq C e^{4\beta \max(\|r\|, \|g\|)}.$$

Let $\xi \leq d_k(t) - 4 \max(\|r\|, \|g\|)$. As for the first term we obtain

$$I \leq C \sup_{h \in [0, 1]} e^{\beta h |g_k - g_j + r_j - r_k|} \leq C e^{\beta 4 \max(\|r\|, \|g\|)}.$$

■

Remark A.7. In the following proofs we estimate the terms $Q_j^{ct+r+g^0}(\xi)$ given by (1.16) for $j = 1, \dots, N$ and $r = (r_1, \dots, r_N), r_j \in \mathbb{R}, j = 1, \dots, N$ bounded. Let $k \in \{1, \dots, N\}$. We conclude from Hypothesis 1.10

$$Q_j^{ct+r+g^0}(\xi) = \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi_{kj}^{ct+r+g^0})} \leq \frac{\varphi(\xi)}{\varphi(\xi_{kj}^{ct+r+g^0})} \leq \frac{C_1}{C_0} e^{-\beta|\xi| + \beta|\xi_{kj}^{ct+r+g^0}|}. \quad (\text{A.13})$$

Lemma A.8. Assume that Hypotheses 1.6 and 1.10 hold.

Given $\beta > 0, \varrho > 0$ there exist constants $C, \tilde{C} > 0$ such that the following estimate holds for all $0 \leq b \leq \beta, r, g \in \mathbb{R}^N$ with $\|r\|, \|g\| \leq \varrho, h \in [0, 1], G^0 \geq 12\varrho, j, k \in \{1, \dots, N\}$ and $t \geq 0$

$$\sup_{\xi \in \mathbb{R}} |Q_j^{ct+r+g^0}(\xi) e^{b|\xi| - b|\xi_{kj}^{ct+r+g^0} + h(g-r)|}| \leq C e^{\tilde{C} \max(\|g\|, \|r\|)}. \quad (\text{A.14})$$

Proof. In the following we use C to denote a generic constant. Let $t \geq 0, h \in [0, 1]$. Define $d_k(t) = (c_k - c_j)t + r_k - r_j + g_k^0 - g_j^0, \bar{d}_k(t) = d_k(t) + 4 \max(\|r\|, \|g\|), \tilde{d}_k(t) = d_k(t) - 4 \max(\|r\|, \|g\|), d_k^h(t) = d_k(t) + h(g_k - g_j - r_k + r_j)$. Note $\tilde{d}_k(t) \geq 0$ for all $t \geq 0$, since $\tilde{d}_k(t) \geq (c_k - c_j)t - 2\varrho + G^0 - 4\varrho \geq G^0 - 6\varrho$.

The case $b = 0$ is clear since $Q_j^{ct+r+g^0} < 1$. Let $0 < b \leq \beta$. The case $k = j$ is obvious. Assume $k > j$.

We estimate the term

$$I := Q_j^{ct+r+g^0}(\xi) e^{b|\xi| - b|\xi - d_k^h(t)|}$$

on six different subintervals which form a partition of \mathbb{R} . We use Hypotheses 1.6 and 1.10, $0 \leq Q_j^{ct+r+g^0} \leq 1$, (A.13) and $b \leq \beta$ and obtain the estimates:

For $\xi \geq \bar{d}_k(t)$:

$$I \leq C e^{-\beta\xi + \beta\xi - \beta d_k(t) + b\xi - b\xi + b d_k^h(t)} \leq C e^{4b \max(\|r\|, \|g\|)} e^{(b-\beta)d_k(t)}.$$

For $d_k(t) \leq \xi \leq \bar{d}_k(t)$:

$$I \leq C e^{-\beta\xi + \beta\xi - \beta d_k(t) + b\xi - b|\xi - d_k^h(t)|} \leq C e^{4b \max(\|r\|, \|g\|)} e^{(b-\beta)d_k(t)}.$$

For $\tilde{d}_k(t) \leq \xi \leq d_k(t)$:

$$I \leq C e^{-\beta\xi - \beta\xi + \beta d_k(t) + b\xi - b|\xi - d_k^h(t)|} \leq C e^{4(2\beta+b) \max(\|r\|, \|g\|)} e^{(b-\beta)d_k(t)}.$$

From the choice of G^0 we conclude that the interval $\frac{1}{2}d_k^h(t) \leq \xi \leq \tilde{d}_k(t)$ exists, since

$$\begin{aligned} \tilde{d}_k(t) - \frac{1}{2}d_k^h(t) &\geq \frac{1}{2}(g_k^0 - g_j^0 + r_k - r_j - h(g_k - g_j - r_k + r_j)) - 4 \max(\|r\|, \|g\|) \\ &\geq \frac{1}{2}G^0 - 6 \max(\|r\|, \|g\|). \end{aligned}$$

We estimate:

$$I \leq C e^{-\beta\xi - \beta\xi + \beta d_k(t) + b\xi + b\xi - b d_k^h(t)} \leq C e^{4\beta \max(\|r\|, \|g\|)} e^0. \quad (\text{A.15})$$

For $0 \leq \xi \leq \frac{1}{2}d_k^h(t)$:

$$I \leq C e^{b\xi + b\xi - b d_k^h(t)} \leq C e^0.$$

For $\xi \leq 0$:

$$I \leq C e^{-b\xi + b\xi - b d_k^h(t)} \leq C e^{4b \max(\|r\|, \|g\|)} e^{-b d_k(t)}.$$

Assume $k < j$.

Again we estimate the term I on six different subintervals which form a partition of \mathbb{R} , use Hypotheses 1.6 and 1.10, $0 \leq Q_j^{ct+r+g^0} \leq 1$, (A.13) and $b \leq \beta$ and obtain

the estimates:

For $\xi \geq 0$:

$$I \leq C e^{b\xi - b\xi + bd_k^h(t)} \leq C e^{4b \max(\|r\|, \|g\|)} e^{bd_k(t)}.$$

For $\frac{1}{2}d_k^h(t) \leq \xi \leq 0$:

$$I \leq C e^{-b\xi - b\xi + bd_k^h(t)} \leq C.$$

From the choice of G^0 we obtain that the interval $\bar{d}_k(t) \leq \xi \leq \frac{1}{2}d_k^h(t)$ exists and estimate:

$$I \leq C e^{\beta\xi + \beta\xi - \beta d_k(t) - b\xi - b\xi + bd_k^h(t)} \leq C e^{4\beta \max(\|r\|, \|g\|)}. \quad (\text{A.16})$$

For $d_k(t) \leq \xi \leq \bar{d}_k(t)$:

$$I \leq C e^{\beta\xi + \beta\xi - \beta d_k(t) - b\xi - b|\xi - d_k^h(t)|} \leq C e^{4(2\beta+b) \max(\|r\|, \|g\|)} e^{(-b+\beta)d_k(t)}.$$

For $\tilde{d}_k(t) \leq \xi \leq \bar{d}_k(t)$:

$$I \leq C e^{\beta\xi - \beta\xi + \beta d_k(t) - b\xi - b|\xi - bd_k^h(t)|} \leq C e^{4b \max(\|r\|, \|g\|)} e^{(-b+\beta)d_k(t)}.$$

For $\xi \leq \tilde{d}_k(t)$:

$$I \leq C e^{\beta\xi - \beta\xi + \beta d_k(t) - b\xi + b\xi - bd_k^h(t)} \leq C e^{4b \max(\|r\|, \|g\|)} e^{(-b+\beta)d_k(t)}.$$

■

Remark A.9. If we replace the condition (1.34) in Hypothesis 1.10 by the weaker condition

$$\varphi(\xi) \leq C_r e^{-\beta_r \xi} \quad \forall \xi \geq 0 \quad \text{and} \quad \varphi(\xi) \leq C_l e^{\beta_l \xi} \quad \forall \xi \leq 0$$

for some $C_l, C_r, \beta_l, \beta_r > 0$, we run into problems with the estimate (A.14) for $\beta_l \neq \beta_r$. More precisely, the estimate (A.15) (for $k > j$) will only hold for $\beta_l \leq \beta_r$, but the estimate (A.16) (for $k < j$) will instead only hold for $\beta_r \geq \beta_l$.

Lemma A.10. Assume that Hypotheses 1.6 and 1.10 hold.

Given $\eta > 0, q = \frac{1}{4}, \varrho > 0$ there exist constants $C, \tilde{C} > 0$ such that for all $0 \leq b < \min(\frac{q\eta}{1+q}, \frac{1}{2}\beta, \frac{1}{2}\eta)$ there exist $\gamma > 0$ such that the following estimate holds for all $r, g \in \mathbb{R}^N$ with $\|r\|, \|g\| \leq \varrho$, $G^0 \geq 4\varrho$, $0 \leq h \leq 1$, $j, k \in \{1, \dots, N\}$ with $k \neq j$ and $t \geq 0$

$$\sup_{\xi \in \mathbb{R}} |Q_j^{ct+r+g^0}(\xi) e^{b|\xi| - \eta |\xi_{kj}^{ct+r+g^0+h(g-r)}|}| \leq C e^{\tilde{C} \max(\|g\|, \|r\|)} e^{-\gamma t} e^{-\gamma G^0}, \quad (\text{A.17})$$

$$\|Q_j^{ct+r+g^0}(\xi) e^{b|\xi| - \eta |\xi_{kj}^{ct+r+g^0+h(g-r)}|}\|_{\mathcal{L}_2} \leq C e^{\tilde{C} \max(\|g\|, \|r\|)} e^{-\gamma t} e^{-\gamma G^0}. \quad (\text{A.18})$$

Proof. In the following we use C to denote a generic constant. For $t \geq 0, h \in [0, 1]$ let $d_k(t) = (c_k - c_j)t + r_k - r_j + g_k^0 - g_j^0$, $d_k^h(t) = d_k(t) + h(g_k - g_j - r_k + r_j)$. Note that $d_k^h(t) \geq G^0 - 4 \max(\|r\|, \|g\|) \geq 0$.

Assume $k > j$. The case $k < j$ is treated analogously (see proof of Lemma A.8). We estimate the term

$$J := Q_j^{ct+r+g^0}(\xi) e^{b|\xi| - \eta|\xi - d_k^h(t)|}$$

on four different subintervals which form a partition of \mathbb{R} . We use Hypotheses 1.6, 1.10, (A.13), $0 \leq Q_j^{ct+r+g^0} \leq 1$ and $0 \leq b < \min(\frac{q\eta}{1+q}, \frac{1}{2}\beta, \frac{1}{2}\eta)$. The proof is very similar to the proof above but now η instead of b . We obtain the following estimates:

For $\xi \leq 0$:

$$\begin{aligned} J &\leq C e^{-b\xi + \eta\xi - \eta d_k^h(t)} \leq C e^{-\eta d_k^h(t)} \\ &\leq C e^{-\eta(c_k - c_j)t} e^{-\eta(g_k^0 - g_j^0)} e^{-\eta(r_k - r_j + h(g_k - g_j - r_k + r_j))} \\ &\leq C e^{-\eta(c_k - c_j)t} e^{-\eta G^0} e^{2\eta \max(\|r\|, \|g\|)}. \end{aligned}$$

The term J is estimated on the other intervals very similarly.

For $0 \leq \xi \leq (1 - q)d_k^h(t)$:

$$J \leq C e^{b\xi + \eta\xi - \eta d_k^h(t)} \leq C e^{((1-q)b - q\eta)d_k^h(t)}.$$

For the interval $(1 - q)d_k^h(t) \leq \xi \leq (1 + q)d_k^h(t)$:

$$\begin{aligned} J &\leq C e^{b\xi - \beta\xi + \beta|\xi - d_k(t)|} \\ &\leq C e^{4\beta \max(\|r\|, \|g\|)} e^{(b(1-q) - \beta(1-q) + q\beta)d_k^h(t)} \\ &\leq C e^{4\beta \max(\|r\|, \|g\|)} e^{(b - bq - \beta + 2q\beta)d_k^h(t)}. \end{aligned}$$

For $\xi \geq (1 + q)d_k^h(t)$:

$$J \leq C e^{b\xi - \eta\xi + \eta d_k^h(t)} \leq C e^{(b(1+q) - q\eta)d_k^h(t)}.$$

Further we have to estimate the following integral for all $t \geq 0$

$$\begin{aligned} I &:= \|J^2\|_{\mathcal{L}_2}^2 \\ &\leq C \left(\int_{-\infty}^0 J^2 d\xi + \int_0^{(1-q)d_k^h(t)} J^2 d\xi + \int_{(1-q)d_k^h(t)}^{(1+q)d_k^h(t)} J^2 d\xi + \int_{(1+q)d_k^h(t)}^{\infty} J^2 d\xi \right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For $i = 1, \dots, 4$ we obtain estimates for the integrals I_i :

$$I_1 \leq C \int_{-\infty}^0 e^{-2b+2\eta-2\eta d_k^h(t)} d\xi \leq \frac{C}{2(\eta-b)} e^{-2\eta d_k^h(t)} \leq C \frac{1}{\eta} e^{-2\eta d_k^h(t)}.$$

$$I_2 \leq C \int_0^{(1-q)d_k^h(t)} e^{2b\xi+2\eta\xi-2\eta d_k^h(t)} d\xi \leq \frac{C}{2(\eta+b)} e^{2((1-q)b-q\eta)d_k^h(t)}.$$

$$\begin{aligned} I_3 &\leq C \int_{(1-q)d_k^h(t)}^{(1+q)d_k^h(t)} e^{2b\xi-2\beta\xi+2\beta|\xi-d_k(t)|} d\xi \\ &\leq \frac{C}{2(\beta-b)} e^{8\beta \max(\|r\|, \|g\|)} e^{2(b(1-q)-\beta(1-q)+q\beta)d_k^h(t)} \\ &\leq \frac{C}{\beta} e^{8\beta \max(\|r\|, \|g\|)} e^{2(b(1-q)-\beta(1-q)+q\beta)d_k^h(t)}. \end{aligned}$$

$$I_4 \leq C \int_{(1+q)d_k^h(t)}^{\infty} e^{2b\xi-2\eta\xi+2\eta d_k^h(t)} d\xi \leq \frac{C}{2(\eta-b)} e^{2(b(1+q)-q\eta)d_k^h(t)} \leq \frac{C}{\eta} e^{2(b(1+q)-q\eta)d_k^h(t)}.$$

■

A.5 Estimates of nonlinearities

In this section we prove some estimates of the bump function and the nonlinearities together in weighted norms.

Lemma A.11. *Assume $g \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$. Given $\varrho > 0$.*

Then there exist constants $C_\varrho, \tilde{C}_\varrho > 0$ such that for all $b \geq 0, x, \xi \in \mathbb{R}$ and $u, v \in \mathcal{H}^{1,b}$ with $\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}} \leq \varrho$ the following estimates are satisfied

$$\|g(u(\xi)) - g(v(x))\| \leq C_\varrho \|u(\xi) - v(x)\| \quad (\text{A.19})$$

$$\|g(u(\xi)) - g(v(x))\| \leq \tilde{C}_\varrho (\|u\|_{\mathcal{H}^{1,b}} + \|v\|_{\mathcal{H}^{1,b}}). \quad (\text{A.20})$$

Proof. Consider $\|g(u(\xi)) - g(v(x))\|$ with $\xi, x \in \mathbb{R}$ and $u, v \in \mathcal{H}^{1,b}$, where $\|u\|_{\mathcal{H}^{1,b}}, \|v\|_{\mathcal{H}^{1,b}} \leq \varrho$. This term is estimated by

$$\|g(u(\xi)) - g(v(x))\| \leq \int_0^1 \|Dg(u(\xi) + \tau(v(x) - u(\xi)))\| d\tau \|u(\xi) - v(x)\|.$$

Using (1.21) we conclude $\|u\|_{\mathcal{H}^1} \leq \|u\|_{\mathcal{H}^{1,b}} \leq \varrho$, $\|v\|_{\mathcal{H}^1} \leq \|v\|_{\mathcal{H}^{1,b}} \leq \varrho$, thus we estimate the term $Dg(u(\xi) + \tau(v(x) - u(\xi)))$ for $\tau \in [0, 1]$ by a constant C_ϱ which depends on the size of ϱ .

Furthermore we use the Sobolev Imbedding Theorem A.4, in particular (A.5), and (1.21) to estimate $\|u(\xi) - v(x)\|$

$$\|u(\xi) - v(x)\| \leq \|u\|_\infty + \|v\|_\infty \leq \tilde{C}(\|u\|_{\mathcal{H}^1} + \|v\|_{\mathcal{H}^1}) \leq \tilde{C}(\|u\|_{\mathcal{H}^{1,b}} + \|v\|_{\mathcal{H}^{1,b}}).$$

for some $\tilde{C} > 0$. ■

Lemma A.12. *Assume that Hypotheses 1.4, 1.6, 1.10 hold.*

Let $q := \min(\frac{1}{4}, \min\left\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\right\})$. Given $0 \leq \varrho \leq 1$.

Then there exist constant $C > 0$ such that for all $0 \leq b < \min(\frac{1}{2}\beta, \frac{1}{2}\eta, \frac{\eta q}{1+q})$ there exists $\gamma > 0$ such that the following estimate is satisfied for all $u = (u_1, \dots, u_N)$ with $\|u_l\|_{\mathcal{H}^{1,b}} \leq \varrho, l = 1, \dots, N$ and for all $j = 1, \dots, N, t \geq 0$

$$\begin{aligned} & \|Q_j^{ct+g^0} \left[f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(u_j + w_j) \right] \|_{\mathcal{H}^{1,b}} \\ & \leq C \left(e^{-\gamma t} e^{-\gamma G^0} + \|u\|_{\mathcal{L}_{2,b}} + e^{-\gamma t} e^{-\gamma G^0} \|u\|_{\mathcal{H}^{1,b}} \right). \end{aligned} \quad (\text{A.21})$$

Remark A.13. *In the applications in Section 3.2 of the lemma above we choose a bounded parameter $0 < \varrho \leq 1$ and obtain a constant $C > 0$ in the estimate (A.21) that is independent of ϱ , compare the proof of Lemma A.11.*

Remark A.14. *Note that in this lemma and in the lemma A.10 above γ depends on the size of b and tends to zero as b tends to $\min(\frac{1}{2}\beta, \frac{1}{2}\eta, \frac{\eta q}{1+q})$ for $q := \min(\frac{1}{4}, \min\left\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\right\})$. If we set $q := \min(\frac{1}{5}, \min\left\{\frac{c_{k+1}-c_k}{2|2c_j-c_k-c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j\right\})$ and let $0 \leq b < \min(\frac{1}{2}\beta, \frac{\eta q}{2(1+q)})$, then the constant $\gamma > 0$ in the estimates (A.17), (A.18) and (A.21) can be chosen independently of b .*

Proof. The proof is based on the concept of [4], Theorem 4.2.

In the following we use C to denote a generic constant.

Let $t \geq 0, j = 1, \dots, N$. In the following j is fixed. We write I for the value of

$$I := \|Q_j^{ct+g^0} f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\cdot_{kj}^{ct+g^0}) \right) - f(u_j + w_j)\|_{\mathcal{L}_{2,b}}^2.$$

We partition \mathbb{R} into subintervals, on each of these subintervals we use either the smallness of f or of $Q_j^{ct+g^0}$ to estimate I .

For $0 < q \leq q_1 := \frac{1}{4}$ sufficiently small we define

$$d_k(t) = (c_k - c_j)t + g_k^0 - g_j^0, \quad d_k^\pm(t) = (1 \pm q)d_k(t).$$

Note $d_k(t) \leq d_{k+1}(t)$ and $d_k^\pm(t) \leq d_{k+1}^\pm(t)$ for all $t \geq 0$ and $k = 1, \dots, N$. Further holds $d_j(t) = d_j^\pm(t) = 0$ for all $t \geq 0$. Let $\zeta > 0$ then there exist $\tilde{\zeta} > 0$ such that holds

$$e^{-\zeta d_k(t)} = e^{-\zeta(c_k - c_j)t} e^{-\zeta(g_k^0 - g_j^0)} \leq e^{-\tilde{\zeta}t} e^{-\tilde{\zeta}G^0} \quad \text{for } k > j, \quad (\text{A.22})$$

$$e^{\zeta d_k(t)} = e^{\zeta(c_k - c_j)t} e^{\zeta(g_k^0 - g_j^0)} \leq e^{-\tilde{\zeta}t} e^{-\tilde{\zeta}G^0} \quad \text{for } k < j. \quad (\text{A.23})$$

Case 1: We consider t -values for which the numbers $d_k(t), d_k^\pm(t)$ are ordered as follows

$$\begin{aligned} -\infty < d_1^+ < d_1 < d_1^- < d_2^+ < \dots < d_{j-1}^+ < d_{j-1} < d_{j-1}^- < 0 \\ < d_{j+1}^- < d_{j+1} < d_{j+1}^+ < \dots < d_N^- < d_N < d_N^+ < \infty. \end{aligned} \quad (\text{A.24})$$

For $N \geq 3$ consider the relations $d_k^-(t) < d_{k+1}^+(t)$ for $k \leq j-1$ and $d_k^+(t) < d_{k+1}^-(t)$ for $k \geq j+1$ which are equivalent to

$$g_k^0 - g_{k+1}^0 + q|2g_j^0 - g_k^0 - g_{k+1}^0| < (c_{k+1} - c_k - q|2c_j - c_k - c_{k+1}|)t.$$

For every k there exist $q_{2,k}$ such that the term on the right side is greater than zero, $q_{2,k}$ has to satisfy

$$q_{2,k} < \frac{c_{k+1} - c_k}{|2c_j - c_k - c_{k+1}|}.$$

We introduce $T_k(g^0)$ by

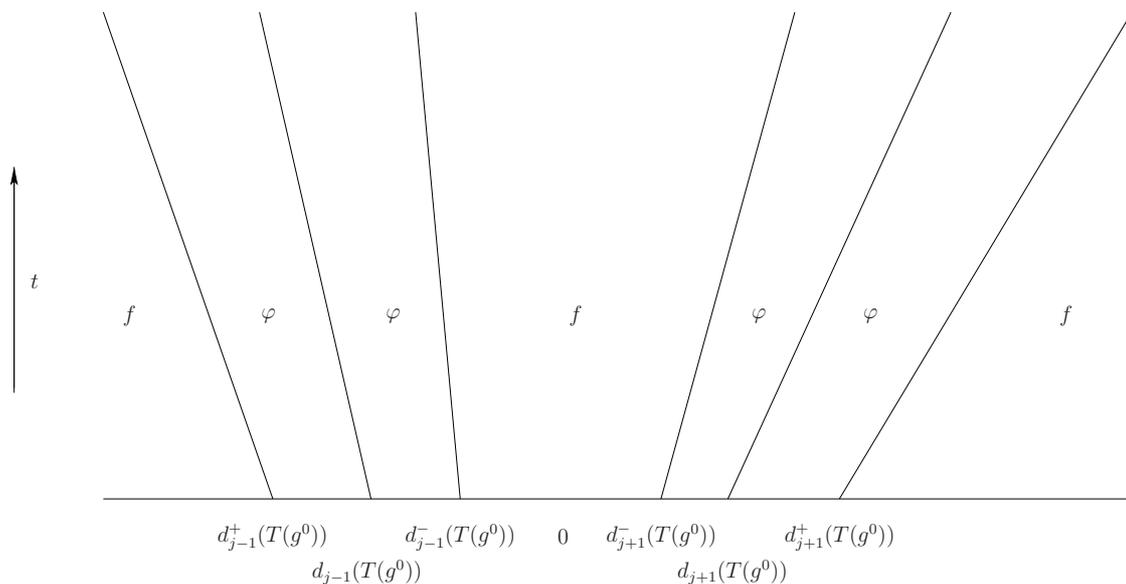
$$g_k^0 - g_{k+1}^0 + q|2g_j^0 - g_k^0 - g_{k+1}^0| = (c_{k+1} - c_k - q|2c_j - c_k - c_{k+1}|)T_k(g^0).$$

and define $q := \min(q_1, \frac{1}{2}q_2)$ with

$$q_2 := \min \left\{ \frac{c_{k+1} - c_k}{|2c_j - c_k - c_{k+1}|} : 1 \leq j \leq N, 1 \leq k \leq N-1, k \neq j \right\},$$

then the relation (A.24) holds for all $t > T(g^0) = \max_{k \neq j} T_k(g^0)$.

For all $t > T(g^0)$ we partition \mathbb{R} as in (A.24) and use on each of these subintervals either the conditions on f or on the bump function φ to estimate I as indicated in Figure A.1.

Figure A.1: Decomposition of \mathbb{R} for $t > T(g^0)$.

Throughout the proof we estimate the term

$$M_j(\xi) = Q_j^{ct+g^0}(\xi)^2 \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) - f((u_j + w_j)(\xi)) \right\|^2 \theta_b^2(\xi).$$

Note that the quotient

$$Q_j^{ct+g^0}(\xi) = \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi - d_k(t))}$$

is always positive and less than one. Furthermore, using (A.13) (with $r = 0$) the quotient can be estimated by

$$Q_j^{ct+g^0}(\xi) \leq \frac{\varphi(\xi)}{\varphi(\xi - d_k(t))} \leq \frac{C_1}{C_0} e^{-\beta|\xi| + \beta|\xi - d_k(t)|} \quad (\text{A.25})$$

for all $k \in \{1, \dots, N\}$.

Using $f(w_l^\pm) = 0, l = 1, \dots, N$ and Hypothesis 1.4 we conclude that there exists a

Lipschitz constant $L > 0$ such that

$$\begin{aligned}
& \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) \right\| \\
&= \left\| f \left(\sum_{k=1}^N u_k(\xi - d_k(t)) + \sum_{k=1}^N w_k(\xi - d_k(t)) + \sum_{k=2}^N w_k^- \right) - f(w_l^+) \right\| \\
&\leq L \left\| \sum_{k=1}^N u_k(\xi - d_k(t)) \right\| + L \left\| \sum_{k=1}^N w_k(\xi - d_k(t)) - \sum_{k=2}^N w_k^- - w_l^+ \right\| \\
&\leq L \left\| \sum_{k=1}^N u_k(\xi - d_k(t)) \right\| \\
&\quad + L \left\| \sum_{k=1}^l (w_k(\xi - d_k(t)) - w_k^+) + \sum_{k=l+1}^N (w_k(\xi - d_k(t)) - w_k^-) \right\|. \tag{A.26}
\end{aligned}$$

From the Sobolev imbedding estimate (A.5) we conclude that the first term is always bounded

$$\left\| \sum_{k=1}^N u_k(\xi - d_k(t)) \right\| \leq \sum_{k=1}^N \|u_k\|_\infty \leq C \|u\|_{\mathcal{H}^1} \leq C \|u\|_{\mathcal{H}^{1,b}}.$$

The second term is also bounded by some constants, since the traveling waves w_j are bounded functions. To obtain better estimates we can choose l appropriately on each subinterval of the partition (A.24).

We obtain

$$\begin{aligned}
I &\leq C \left(\int_{-\infty}^{d_1^+(t)} M_j(\xi) d\xi + \sum_{l=1}^{j-1} \int_{d_l^+(t)}^{d_l(t)} M_j(\xi) d\xi + \sum_{l=1}^{j-1} \int_{d_l(t)}^{d_l^-(t)} M_j(\xi) d\xi \right. \\
&\quad + \sum_{l=1}^{j-2} \int_{d_l^-(t)}^{d_{l+1}^+(t)} M_j(\xi) d\xi + \int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} M_j(\xi) d\xi + \sum_{l=j+1}^N \int_{d_l^-(t)}^{d_l(t)} M_j(\xi) d\xi \\
&\quad \left. + \sum_{l=j+1}^N \int_{d_l(t)}^{d_l^+(t)} M_j(\xi) d\xi + \sum_{l=j+1}^{N-1} \int_{d_l^+(t)}^{d_{l+1}^-(t)} M_j(\xi) d\xi + \int_{d_N^+(t)}^\infty M_j(\xi) d\xi \right) \\
&=: I^b + \sum_{l=1}^{j-1} I_l^{1-} + \sum_{l=1}^{j-1} I_l^{2-} + \sum_{l=1}^{j-2} I_l^{3-} + I^c \\
&\quad + \sum_{l=j+1}^N I_l^{1+} + \sum_{l=j+1}^N I_l^{2+} + \sum_{l=j+1}^{N-1} I_l^{3+} + I^e.
\end{aligned}$$

To estimate the u_j components in the weighted space $\mathcal{H}^{1,b}$ we use Lemma A.8. Let $1 < j < N$. From Hypotheses 1.4, 1.6, 1.10, Theorem A.4 and (A.20) we obtain the estimates:

$$I^b \leq \int_{-\infty}^{d_1^+(t)} Q_j^{ct+g^0}(\xi)^2 \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) - f(w_1^-) \right. \\ \left. + f(w_j^-) - f(u_j(\xi) + w_j(\xi)) \right\|^2 \theta_b^2(\xi) d\xi.$$

As noted above we estimate

$$Q_j^{ct+g^0}(\xi)^2 \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) - f(w_1^-) \right. \\ \left. + f(w_j^-) - f(u_j(\xi) + w_j(\xi)) \right\|^2 \theta_b^2(\xi) \\ \leq C Q_j^{ct+g^0}(\xi)^2 \left(\left\| \sum_{k=1}^N u_k(\xi - d_k(t)) \right\|^2 e^{2b|\xi|} + \left\| \sum_{k=1}^N w_k(\xi - d_k(t)) - \sum_{k=1}^N w_k^- \right\|^2 e^{2b|\xi|} \right).$$

For the first part we use Lemma A.8 (with $r = g = 0$) and obtain

$$Q_j^{ct+g^0}(\xi)^2 \left\| \sum_{k=1}^N u_k(\xi - d_k(t)) \right\|^2 e^{2b|\xi|} \\ \leq C Q_j^{ct+g^0}(\xi)^2 \sum_{k=1}^N \|u_k(\xi - d_k(t))\|^2 e^{2b|\xi|} \theta_b(\xi - d_k(t))^2 (\theta_b(\xi - d_k(t)))^{-2} \\ \leq C \max_{k=1, \dots, N} \|u_k(\xi - d_k(t))\|^2 \theta_b(\xi - d_k(t))^2 |Q_j^{ct+g^0}(\xi) e^{b|\xi| - b|\xi - d_k(t)|}|^2 \\ \leq C \max_{k=1, \dots, N} \|u_k(\xi - d_k(t))\|^2 \theta_b(\xi - d_k(t))^2.$$

For the second part we use the estimates in Hypothesis 1.6 and $Q_j^{ct+g^0}(\xi)^2 \leq 1$:

$$\begin{aligned}
I^b &\leq C \left(\int_{-\infty}^{d_1^+(t)} \sum_{k=1}^N e^{2\eta(\xi-d_k(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{-\infty}^{d_1^+(t)} e^{2\eta(\xi-d_1(t))} e^{-2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2\eta-2b} e^{2((\eta-b)q-b)d_1(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{\eta} e^{2((\eta-b)q-b)d_1(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{\eta} e^{2((\eta-b)q-b)(c_1-c_j)t} e^{2((\eta-b)q-b)(g_1^0-g_j^0)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{\eta} e^{-\gamma t} e^{-\gamma G^0} + \|u\|_{\mathcal{L}_{2,b}}^2 \right),
\end{aligned}$$

for some $\gamma > 0$ (Note, using the assumption $b < \frac{\eta q}{1+q}$ we conclude $(\eta-b)q-b > 0$).

For $l \in \{1, \dots, j-1\}$ we estimate:

We use (A.22) and (A.23). As noted above the f -terms in $M_j(\xi)$ are bounded by $C \max(1, \|u\|_{\mathcal{H}^{1,b}})$ for some positive constant. Furthermore we use the assumptions on the bump function φ in Hypothesis 1.10 and obtain

$$\begin{aligned}
I_l^{1-} &\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l^+(t)}^{d_l(t)} \frac{\varphi(\xi)^2 e^{2b|\xi|}}{\varphi(\xi-d_l(t))^2} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l^+(t)}^{d_l(t)} e^{2(-b\xi-\beta\xi+\beta d_l(t))} e^{2\beta\xi} d\xi.
\end{aligned}$$

Here we estimate $e^{2\beta\xi}$ by the maximal value in the given interval, then the integral is found to be

$$e^{2\beta d_l(t)} \frac{1}{2(\beta+b)} e^{2((-\beta-b)q-b)d_l(t)} \leq \frac{1}{2\beta} e^{2((-\beta-b)q-b+\beta)d_l(t)}.$$

Note $((-\beta-b)q-b+\beta) > 0$, since $q \leq \frac{1}{4}$ and $b < \frac{1}{2}\beta$. In the following the arguments used will not be repeated.

For $l \in \{1, \dots, j-1\}$:

$$\begin{aligned}
I_l^{2-} &\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l(t)}^{d_l^-(t)} \frac{\varphi(\xi)^2 e^{2b|\xi|}}{\varphi(\xi - d_l(t))^2} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l(t)}^{d_l^-(t)} e^{2(-b\xi + 2\beta\xi - \beta d_l(t))} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{2(2\beta - b)} e^{2((-2\beta + b)q + \beta - b)d_l(t)} \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{3\beta} e^{2((-2\beta + b)q + \beta - b)d_l(t)},
\end{aligned}$$

for $l \in \{1, \dots, j-2\}$ we use (A.26) and obtain:

$$\begin{aligned}
I_l^{3-} &\leq \int_{d_l^-(t)}^{d_{l+1}^+(t)} \|f\left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t))\right) - f(w_l^+) \\
&\quad + f(w_j^-) - f(u_j(\xi) + w_j(\xi))\|^2 \theta_b^2(\xi) d\xi \\
&\leq C \left[\int_{d_l^-(t)}^{d_{l+1}^+(t)} \sum_{k=1}^l \|w_k(\xi - d_k(t)) - w_k^+\|^2 e^{2b|\xi|} d\xi \right. \\
&\quad \left. + \int_{d_l^-(t)}^{d_{l+1}^+(t)} \sum_{k=l+1}^N \|w_k(\xi - d_k(t)) - w_k^-\|^2 e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right] \\
&\leq C \left(\int_{d_l^-(t)}^{d_{l+1}^+(t)} \sum_{k=1}^l e^{-2\eta(\xi - d_k(t))} e^{-2b\xi} d\xi \right. \\
&\quad \left. + \int_{d_l^-(t)}^{d_{l+1}^+(t)} \sum_{k=l+1}^N e^{2\eta(\xi - d_k(t))} e^{-2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_l^-(t)}^{d_{l+1}^+(t)} e^{-2b\xi} e^{-2\eta(\xi - d_l(t))} + e^{-2b\xi} e^{2\eta(\xi - d_{l+1}(t))} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta + b)} e^{2(q(b+\eta) - b)d_l(t)} + \frac{1}{2(\eta - b)} e^{2(-b - bq + \eta q)d_{l+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2\eta} e^{2(qb + q\eta - b)d_l(t)} + \frac{1}{\eta} e^{2(-b - bq + \eta q)d_{l+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right),
\end{aligned}$$

$$\begin{aligned}
I^c &\leq C \left(\int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} \left(\sum_{k=1}^{j-1} \|w_k(\xi - d_k(t)) - w_k^+\|^2 e^{2b|\xi|} \right. \right. \\
&\quad \left. \left. + \sum_{k=j+1}^N \|w_k(\xi - d_k(t)) - w_k^-\|^2 e^{2b|\xi|} \right) d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} \sum_{k=1}^{j-1} e^{-2\eta(\xi - d_k(t))} e^{2b|\xi|} d\xi \right. \\
&\quad \left. + \int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} \sum_{k=j+1}^N e^{2\eta(\xi - d_k(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} e^{-2\eta(\xi - d_{j-1}(t))} e^{2b|\xi|} d\xi + \int_{d_{j-1}^-(t)}^{d_{j+1}^-(t)} e^{2\eta(\xi - d_{j+1}(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta + b)} e^{2(-b+bq+\eta q)d_{j-1}(t)} + \frac{1}{2(\eta - b)} e^{2\eta d_{j-1}(t)} \right. \\
&\quad \left. + \frac{1}{2(\eta + b)} e^{2(b-bq-\eta q)d_{j+1}(t)} + \frac{1}{2(\eta - b)} e^{-2\eta d_{j+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2\eta} e^{2(-b+bq+\eta q)d_{j-1}(t)} + \frac{1}{\eta} e^{2\eta d_{j-1}(t)} \right. \\
&\quad \left. + \frac{1}{2\eta} e^{2(b-bq-\eta q)d_{j+1}(t)} + \frac{1}{\eta} e^{-2\eta d_{j+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right),
\end{aligned}$$

for $l \in \{j+1, \dots, N\}$:

$$\begin{aligned}
I_l^{1+} &\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l^-(t)}^{d_l(t)} \frac{\varphi(\xi)^2 e^{2b|\xi|}}{\varphi(\xi - d_l(t))^2} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l^-(t)}^{d_l(t)} e^{2(-2\beta\xi + \beta d_l(t))} e^{2b\xi} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{2(2\beta - b)} e^{2((2\beta - b)q - \beta + b)d_l(t)} \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{3\beta} e^{2((2\beta - b)q - \beta + b)d_l(t)},
\end{aligned}$$

for $l \in \{j+1, \dots, N\}$:

$$\begin{aligned}
I_l^{2+} &\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l(t)}^{d_l^+(t)} \frac{\varphi(\xi)^2 e^{2b|\xi|}}{\varphi(\xi - d_l(t))^2} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \int_{d_l(t)}^{d_l^+(t)} e^{2((\beta+b)\xi - \beta d_l(t))} e^{-2\beta\xi} d\xi \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) e^{-2\beta d_l(t)} \frac{1}{2(\beta+b)} e^{2((\beta+b)q+b)d_l(t)} \\
&\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{2\beta} e^{2((\beta+b)q+b-\beta)d_l(t)},
\end{aligned}$$

for $l \in \{j+1, \dots, N-1\}$ we use (A.26) and obtain:

$$\begin{aligned}
I_l^{3+} &\leq C \int_{d_l^+(t)}^{d_{l+1}^-(t)} \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) - f(w_l^+) \right. \\
&\quad \left. + f(w_j^+) - f(u_j(\xi) + w_j(\xi)) \right\|^2 \theta_b^2(\xi) d\xi \\
&\leq C \left(\int_{d_l^+(t)}^{d_{l+1}^-(t)} \sum_{k=1}^l \|w_k(\xi - d_k(t)) - w_k^+\|^2 e^{2b\xi} d\xi \right. \\
&\quad \left. + \int_{d_l^+(t)}^{d_{l+1}^-(t)} \sum_{k=l+1}^N \|w_k(\xi - d_k(t)) - w_k^-\|^2 e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_l^+(t)}^{d_{l+1}^-(t)} \sum_{k=1}^l e^{-2\eta(\xi - d_k(t))} e^{2b\xi} d\xi + \int_{d_l^+(t)}^{d_{l+1}^-(t)} \sum_{k=l+1}^N e^{2\eta(\xi - d_k(t))} e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_l^+(t)}^{d_{l+1}^-(t)} e^{-2\eta(\xi - d_l(t))} e^{2b\xi} d\xi + \int_{d_l^+(t)}^{d_{l+1}^-(t)} e^{2\eta(\xi - d_{l+1}(t))} e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta - b)} e^{(2b+2bq-2\eta q)d_l(t)} + \frac{1}{2(\eta + b)} e^{(2b-2bq-2\eta q)d_{l+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{\eta} e^{(2b+2bq-2\eta q)d_l(t)} + \frac{1}{2\eta} e^{(2b-2bq-2\eta q)d_{l+1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right)
\end{aligned}$$

and finally, we get

$$\begin{aligned}
I^e &\leq C \int_{d_N^+(t)}^{\infty} \left\| f \left(\sum_{k=1}^N (u_k + \hat{w}_k)(\xi - d_k(t)) \right) - f(w_N^+) \right. \\
&\quad \left. + f(w_j^+) - f(u_j(\xi) + w_j(\xi)) \right\|^2 \theta_b^2(\xi) d\xi \\
&\leq C \left(\int_{d_N^+(t)}^{\infty} \sum_{k=1}^N \|w_k(\xi - d_k(t)) - w_k^+\|^2 e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_N^+(t)}^{\infty} \sum_{k=1}^N e^{-2\eta(\xi - d_k(t))} e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_N^+(t)}^{\infty} e^{-2\eta(\xi - d_N(t))} e^{2b\xi} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta - b)} e^{(2b+2bq-2\eta q)d_N(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{\eta} e^{2(b+bq-\eta q)d_N(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right).
\end{aligned}$$

For $j = 1$, the estimate of I has fewer terms

$$\begin{aligned}
I &\leq C \left(\int_{-\infty}^{d_2^-(t)} M_1(\xi) d\xi + \sum_{k=2}^N \int_{d_k^-(t)}^{d_k(t)} M_1(\xi) d\xi + \sum_{k=2}^N \int_{d_k(t)}^{d_k^+(t)} M_1(\xi) d\xi \right. \\
&\quad \left. + \sum_{k=2}^{N-1} \int_{d_k^+(t)}^{d_{k+1}^-(t)} M_1(\xi) d\xi + \int_{d_N^+(t)}^{\infty} M_1(\xi) d\xi \right) \\
&=: I^c + \sum_{k=2}^N I_k^{1+} + \sum_{k=2}^N I_k^{2+} + \sum_{k=2}^{N-1} I_k^{3+} + I^e.
\end{aligned}$$

$I_k^{1+}, I_k^{2+}, k \in \{2, \dots, N\}$, $I_k^{3+}, k \in \{2, \dots, N-1\}$, and I^e are estimated as before and for I^c we obtain

$$\begin{aligned}
I^c &\leq C \left(\int_{-\infty}^{d_2^-(t)} \sum_{k=2}^N e^{2\eta(\xi - d_k(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{-\infty}^{d_2^-(t)} e^{2\eta(\xi - d_2(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta + b)} e^{(2b-2bq-2\eta q)d_2(t)} + \frac{1}{2(\eta - b)} e^{-2\eta d_2(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2\eta} e^{2(b-bq-\eta q)d_2(t)} + \frac{1}{\eta} e^{-2\eta d_2(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right).
\end{aligned}$$

For $j = N$ the estimate of I has also fewer terms

$$\begin{aligned}
I &\leq C \left(\int_{-\infty}^{d_1^+(t)} M_N(\xi) d\xi + \sum_{k=1}^{N-1} \int_{d_k^+(t)}^{d_{k+1}^+(t)} M_N(\xi) d\xi + \sum_{k=1}^{N-1} \int_{d_k(t)}^{d_k^-(t)} M_N(\xi) d\xi \right. \\
&\quad \left. + \sum_{k=1}^{N-2} \int_{d_k^-(t)}^{d_{k+1}^+(t)} M_N(\xi) d\xi + \int_{d_{N-1}^-(t)}^{\infty} M_N(\xi) d\xi \right) \\
&=: I^b + \sum_{k=1}^{N-1} I_k^{1-} + \sum_{k=1}^{N-1} I_k^{2-} + \sum_{k=1}^{N-2} I_k^{3-} + I^c.
\end{aligned}$$

The terms $I^b, I_k^{1-}, I_k^{2-}, k \in \{1, \dots, N-1\}, I_k^{3-}, k \in \{1, \dots, N-2\}$ are treated as before and for I^c we have:

$$\begin{aligned}
I^c &\leq C \left(\int_{d_{N-1}^-(t)}^{\infty} \sum_{k=1}^{N-1} e^{-2\eta(\xi-d_k(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\int_{d_{N-1}^-(t)}^{\infty} e^{-2\eta(\xi-d_{N-1}(t))} e^{2b|\xi|} d\xi + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2(\eta+b)} e^{(-2b+2bq+2\eta q)d_{N-1}(t)} + \frac{1}{2(\eta-b)} e^{2\eta d_{N-1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right) \\
&\leq C \left(\frac{1}{2\eta} e^{2(-b+bq+\eta q)d_{N-1}(t)} + \frac{1}{\eta} e^{2\eta d_{N-1}(t)} + \|u\|_{\mathcal{L}_{2,b}}^2 \right).
\end{aligned}$$

Case 2: $T(g^0) \geq 0$. For at least one s we have $T_s(g^0) \geq 0$ and the ordering in (A.24) is not satisfied. Since $T_s(g^0) \leq T(g^0)$, there may be mixed cases. Note $d_k^+(t) < d_{k+1}^+(t)$ for all $k \in \{1, \dots, N-1\}, k \neq j, t \geq 0$. To estimate I we have to show estimates for the integrals

$$J_k := \int_{d_k^+(t)}^{d_{k+1}^+(t)} M_j(\xi) d\xi, \quad k \in \{1, \dots, N-1\}, k \neq j$$

and for $I^b, I_{j-1}^{1-}, I_{j-1}^{2-}, I^c, I_{j+1}^{1+}, I_{j+1}^{2+}, I^e$, the last integrals are estimated as above. Also the integrals J_k can be estimated as above if $T_k(g^0) < 0$.

Assume the case $d_s^+(t) \geq d_{s+1}^-(t)$ for $s \geq j+1$ and for $0 \leq t \leq T_s(g^0)$. The case $d_s^-(t) < d_{s+1}^+(t)$ for $s \leq j-1$ is treated analogously. Consequently it is sufficient to show estimates for the interval J_s .

Case 2a: $d_s^+(t) < d_{s+1}^-(t)$ for $0 \leq t \leq T_s(g^0)$, see Figure A.2 for an illustration.

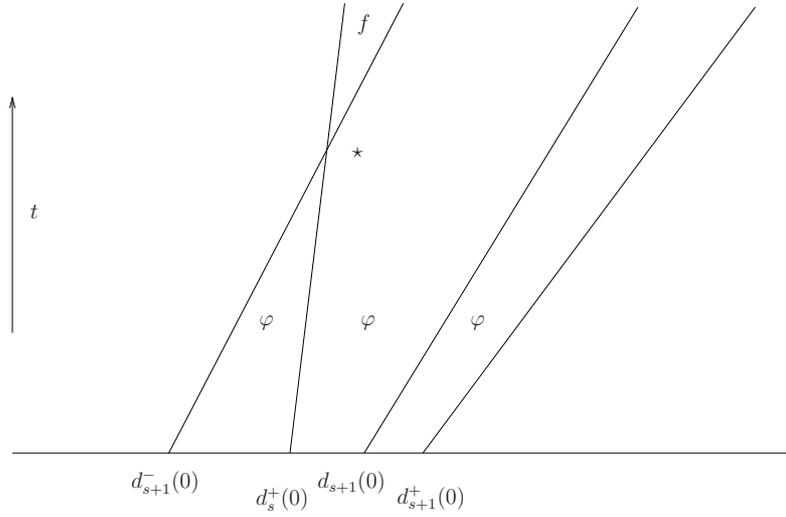


Figure A.2: Decomposition of \mathbb{R} for the set $0 \leq t \leq T_s(g^0)$, \star marks the point where $d_s^+(T_s(g^0)) = d_{s+1}^-(T_s(g^0))$.

We estimate the integral J_s for $0 \leq t \leq T_s(g^0)$ (note $d_s^+(t) \geq d_{s+1}^-(t)$ for all $0 \leq t \leq T_s(g^0)$)

$$\begin{aligned} J_s &= \int_{d_s^+(t)}^{d_{s+1}(t)} M_j(\xi) d\xi + \int_{d_{s+1}(t)}^{d_{s+1}^+(t)} M_j(\xi) d\xi \leq I_{s+1}^{1+} + I_{s+1}^{2+} \\ &\leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{3\beta} e^{2((2\beta-b)q-\beta+b)d_{s+1}(t)} \\ &\quad + C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{2\beta} e^{2((\beta+b)q+b-\beta)d_{s+1}(t)}, \end{aligned}$$

where I_{s+1}^{1+} and I_{s+1}^{2+} are defined and estimated as above.

Case 2b: Using the definition of q we conclude analogously to above that there exists $0 \leq T_s^1(g^0) < T_s(g^0)$ such that $d_s^+(t) < d_{s+1}(t)$ for $T_s^1(g^0) < t \leq T_s(g^0)$ and $d_s^+(t) \geq d_{s+1}(t)$ for $0 \leq t \leq T_s^1(g^0)$, see Figure A.3 for an illustration.

For $T_s(g^0) < t \leq T(g^0)$ we proceed analogously to Case 2a to estimate the integral J_s . Furthermore, we estimate the integral J_s for $0 \leq t \leq T_s^1(g^0)$ (note $d_s^+(t) \geq d_{s+1}(t)$ for all $0 \leq t \leq T_s^1(g^0)$)

$$J_s \leq I_{s+1}^{2+} \leq C \max(1, \|u\|_{\mathcal{H}^{1,b}}) \frac{1}{2\beta} e^{2(\beta q + b q - \beta + b)d_{s+1}(t)},$$

where I_{s+1}^{2+} is defined and estimated as above.

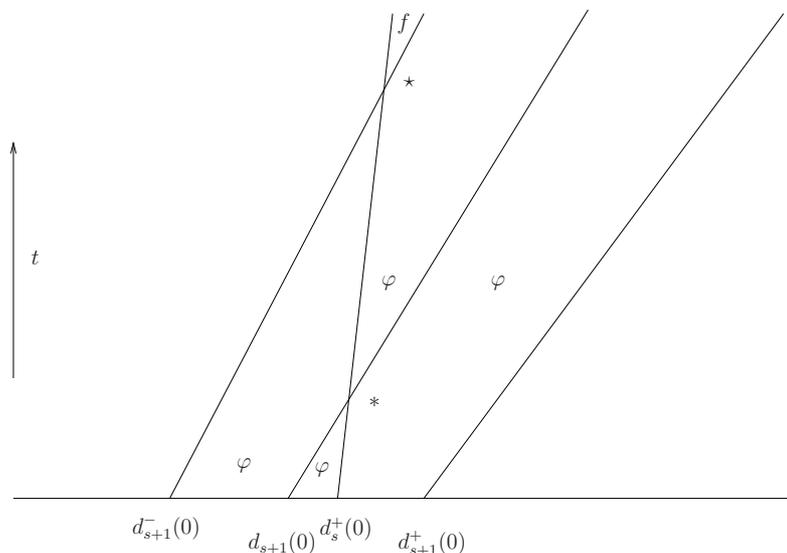


Figure A.3: Decomposition of \mathbb{R} for the set $0 \leq t \leq T_s(g^0)$, \star marks the point where $d_s^+(T_s(g^0)) = d_{s+1}^-(T_s(g^0))$ and \ast marks the point where $d_s^+(T_s^1(g^0)) = d_{s+1}^+(T_s^1(g^0))$.

■

A.6 Proof of resolvent estimates in Lemma 3.13

It remains to prove Lemma 3.13.

Proof. We begin with the first estimate. A similar proof for symmetric matrices A can be found in [35], Lemma 2.27 or [22], Lemma 2.1 and for bounded intervals in [3], Lemma 2.4.

Let $j \in \{1, \dots, N\}$. To shorten notation we suppress j in the proof.

We prove the first estimate (3.38). Consider

$$sv - \Lambda v = \tilde{k}$$

which is equivalent to

$$sv - Av_{\xi\xi} = Bv_{\xi} + Cv + \tilde{k}. \quad (\text{A.27})$$

We take the \mathcal{L}_2 inner product with v and obtain

$$s\langle v, v \rangle - \langle v, Av_{\xi\xi} \rangle = \langle v, Bv_{\xi} \rangle + \langle v, Cv \rangle + \langle v, \tilde{k} \rangle. \quad (\text{A.28})$$

Using $-\langle v, Av_{\xi\xi} \rangle = \langle v_{\xi}, Av_{\xi} \rangle$ gives

$$s\|v\|_{\mathcal{L}_2}^2 + \langle v_{\xi}, Av_{\xi} \rangle = \langle v, Bv_{\xi} \rangle + \langle v, Cv \rangle + \langle v, \tilde{k} \rangle.$$

From the assumptions $v_\xi \in \mathcal{L}_2$ and A positive definit we obtain $\Re\langle v_\xi, Av_\xi \rangle \geq \bar{\alpha}\langle v_\xi, v_\xi \rangle$ for some $\bar{\alpha} > 0$.

Consequently we estimate the real part of the equation by

$$\Re s \|v\|_{\mathcal{L}_2}^2 + \bar{\alpha} \|v_\xi\|_{\mathcal{L}_2} \leq \bar{B} \|v\|_{\mathcal{L}_2} \|v_\xi\|_{\mathcal{L}_2} + \bar{C} \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2}. \quad (\text{A.29})$$

For $\sigma > 0$ the following estimate is satisfied

$$ab \leq \frac{a^2 \sigma^2}{2} + \frac{b^2}{2\sigma^2}. \quad (\text{A.30})$$

From this and (A.29) we conclude

$$\Re s \|v\|_{\mathcal{L}_2}^2 + \bar{\alpha} \|v_\xi\|_{\mathcal{L}_2}^2 \leq \bar{B} \left(\frac{\sigma^2}{2} \|v\|_{\mathcal{L}_2}^2 + \frac{1}{2\sigma^2} \|v_\xi\|_{\mathcal{L}_2}^2 \right) + \bar{C} \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2}$$

or equivalently

$$\begin{aligned} \Re s \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2} \|v_\xi\|_{\mathcal{L}_2}^2 &\leq \bar{B} \frac{\sigma^2}{2} \|v\|_{\mathcal{L}_2}^2 + \left(\bar{B} \frac{1}{2\sigma^2} - \frac{\bar{\alpha}}{2} \right) \|v_\xi\|_{\mathcal{L}_2}^2 \\ &\quad + \bar{C} \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2}. \end{aligned}$$

With $\sigma^2 \geq \frac{1}{\bar{\alpha}} \bar{B}$ and $\tilde{K}_1 = \bar{B} \frac{\sigma^2}{2} + \bar{C}$ follows

$$\Re s \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2} \|v_\xi\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2} \|v\|_{\mathcal{L}_2}^2 \leq \frac{\bar{\alpha}}{2} \|v\|_{\mathcal{L}_2}^2 + \tilde{K}_1 \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2}$$

and therefore

$$\Re s \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2} \|v\|_{\mathcal{H}^1}^2 \leq \tilde{K}_2 \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2} \quad (\text{A.31})$$

with $\tilde{K}_2 = \tilde{K}_1 + \frac{\bar{\alpha}}{2}$.

Further we estimate the absolute value of the imaginary part of (A.28). Let $z_1 := \Re v_\xi, z_2 := \Im v_\xi$. We obtain

$$\begin{aligned} |\Im(s \|v\|_{\mathcal{L}_2}^2 + \langle v_\xi, Av_\xi \rangle)| &= |\Im s \|v\|_{\mathcal{L}_2}^2 - \langle z_2, Az_1 \rangle + \langle z_1, Az_2 \rangle| \\ &\geq |\Im s| \|v\|_{\mathcal{L}_2}^2 - |\langle z_2, Az_1 \rangle - \langle z_1, Az_2 \rangle| \\ &\geq |\Im s| \|v\|_{\mathcal{L}_2}^2 - |\langle z_2, Az_1 \rangle| - |\langle z_1, Az_2 \rangle| \\ &\geq |\Im s| \|v\|_{\mathcal{L}_2}^2 - 2\|A\| \|z_2\|_{\mathcal{L}_2} \|z_1\|_{\mathcal{L}_2} \\ &\geq |\Im s| \|v\|_{\mathcal{L}_2}^2 - 2\|A\| \|v_\xi\|_{\mathcal{L}_2}^2. \end{aligned}$$

Therefore we conclude

$$|\Im s| \|v\|_{\mathcal{L}_2}^2 \leq \bar{B} \|v\|_{\mathcal{L}_2} \|v_j\|_{\mathcal{H}^1} + \bar{C} \|v\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2} + \tilde{\alpha} \|v\|_{\mathcal{H}^1}^2, \quad (\text{A.32})$$

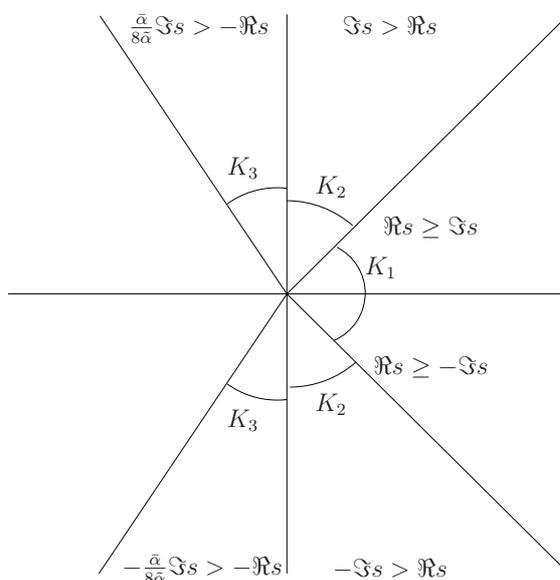


Figure A.4: Sections in the proof of lemma 3.13.

where $\tilde{\alpha} = 2\|A\|$.

We consider the areas $\Re s \geq |\Im s|$, $0 \leq \Re s < |\Im s|$ and $|\Re s| < \frac{\bar{\alpha}}{8\bar{\alpha}}|\Im s|$ with $\Re s < 0$ separately, see Figure A.4.

First case $\Re s \geq |\Im s|$ and $|s| \geq 2\sqrt{2}\tilde{K}_2 =: K_1$.

We obtain

$$0 < \Re s \leq |s| \leq \sqrt{2}\Re s, \quad \tilde{K}_2 \leq \frac{|s|}{2\sqrt{2}} \leq \frac{\Re s}{2}.$$

It follows from (A.31), rewritten as

$$(\Re s - \tilde{K}_2)\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2}\|v\|_{\mathcal{H}^1}^2 \leq \|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2},$$

and from (A.30) with $\sigma^2 = \frac{|s|}{2\sqrt{2}}$

$$\frac{|s|}{2\sqrt{2}}\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2}\|v\|_{\mathcal{H}^1}^2 \leq \frac{|s|}{4\sqrt{2}}\|v\|_{\mathcal{L}_2}^2 + \frac{\sqrt{2}}{|s|}\|\tilde{k}\|_{\mathcal{L}_2}^2.$$

Therefore we obtain

$$\frac{|s|}{4\sqrt{2}}\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{2}\|v\|_{\mathcal{H}^1}^2 \leq \frac{\sqrt{2}}{|s|}\|\tilde{k}\|_{\mathcal{L}_2}^2.$$

To obtain an estimate for s in the second and third area we multiply (A.32) with $\frac{\bar{\alpha}}{4\tilde{\alpha}}$, add equation (A.31) and estimate

$$\begin{aligned} \frac{\bar{\alpha}}{4\tilde{\alpha}}|\Im s| \|v\|_{\mathcal{L}_2}^2 + \Re s \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{4}\|v\|_{\mathcal{H}^1}^2 &\leq \frac{\bar{\alpha}}{4\tilde{\alpha}}\bar{B}\|v\|_{\mathcal{L}_2}\|v\|_{\mathcal{H}^1} + \left(\frac{\bar{\alpha}}{4\tilde{\alpha}}\bar{C} + \tilde{K}_2\right)\|v\|_{\mathcal{L}_2}^2 \\ &\quad + \left(\frac{\bar{\alpha}}{4} + 1\right)\|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2}. \end{aligned}$$

Using (A.30) we conclude $\frac{\bar{\alpha}}{4\tilde{\alpha}}\bar{B}\|v\|_{\mathcal{L}_2}\|v\|_{\mathcal{H}^1} \leq \frac{\bar{B}\bar{\alpha}}{8\tilde{\alpha}}(\sigma^2\|v\|_{\mathcal{L}_2}^2 + \frac{1}{\sigma^2}\|v\|_{\mathcal{H}^1}^2)$. Choose $\sigma > 0$ such that $\sigma^2 \geq \frac{1}{\tilde{\alpha}}\bar{B}$, then we conclude

$$\frac{\bar{\alpha}}{4\tilde{\alpha}}|\Im s| \|v\|_{\mathcal{L}_2}^2 + \Re s \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8}\|v\|_{\mathcal{H}^1}^2 \leq \tilde{K}_3\|v\|_{\mathcal{L}_2}^2 + \tilde{K}_4\|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2}, \quad (\text{A.33})$$

where $\tilde{K}_3 = \frac{\bar{\alpha}}{4\tilde{\alpha}}\bar{C} + \tilde{K}_2 + \frac{\bar{B}\bar{\alpha}}{8\tilde{\alpha}}\sigma^2$, $\tilde{K}_4 = \frac{\bar{\alpha}}{4} + 1$.

Second case $0 \leq \Re s < |\Im s|$ and $|s| \geq \frac{8\sqrt{2}\tilde{\alpha}}{\bar{\alpha}}\tilde{K}_3 =: K_2$.

From (A.33) we conclude

$$\left(\frac{\bar{\alpha}}{4\tilde{\alpha}}|\Im s| - \tilde{K}_3\right)\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8}\|v\|_{\mathcal{H}^1}^2 \leq \tilde{K}_4\|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2}. \quad (\text{A.34})$$

Further we have

$$0 < |\Im s| \leq |s| \leq \sqrt{2}|\Im s|, \quad \tilde{K}_3 \leq \frac{\bar{\alpha}|s|}{8\tilde{\alpha}\sqrt{2}} \leq \frac{\bar{\alpha}|\Im s|}{8\tilde{\alpha}}.$$

Therefore we get

$$\frac{\bar{\alpha}|s|}{8\tilde{\alpha}\sqrt{2}}\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8}\|v_j\|_{\mathcal{H}^1}^2 \leq \tilde{K}_4\|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2} \leq \tilde{K}_4\left(\frac{\sigma^2}{2}\|v_j\|_{\mathcal{L}_2}^2 + \frac{1}{2\sigma^2}\|\tilde{k}\|_{\mathcal{L}_2}^2\right).$$

With $\sigma^2 = \frac{\bar{\alpha}|s|}{8\tilde{\alpha}\sqrt{2}\tilde{K}_4}$ follows

$$\frac{\bar{\alpha}|s|}{16\tilde{\alpha}\sqrt{2}}\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8}\|v\|_{\mathcal{H}^1}^2 \leq \frac{\tilde{K}_4^2 4\tilde{\alpha}\sqrt{2}}{|s|\bar{\alpha}}\|\tilde{k}\|_{\mathcal{L}_2}.$$

Third case $\Re s < 0$, $|\Re s| < \frac{\bar{\alpha}}{8\tilde{\alpha}}|\Im s|$ and $|s| \geq \frac{\tilde{K}_3 16\sqrt{2}\tilde{\alpha}}{\bar{\alpha}} =: K_3$.

Using (A.33) we obtain

$$\left(\frac{\bar{\alpha}}{4\tilde{\alpha}}|\Im s| - |\Re s| - \tilde{K}_3\right)\|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8}\|v\|_{\mathcal{H}^1}^2 \leq \tilde{K}_4\|v\|_{\mathcal{L}_2}\|\tilde{k}\|_{\mathcal{L}_2}. \quad (\text{A.35})$$

Since $0 < |\Im s| \leq |s| \leq \sqrt{2}|\Im s|$,

$$\frac{\bar{\alpha}}{4\tilde{\alpha}}|\Im s| - |\Re s| = \frac{\bar{\alpha}}{8\tilde{\alpha}}|\Im s| + \frac{\bar{\alpha}}{8\tilde{\alpha}}|\Im s| - |\Re s| \geq \frac{\bar{\alpha}}{8\tilde{\alpha}}|\Im s| \geq \frac{\bar{\alpha}}{8\sqrt{2}\tilde{\alpha}}|s|$$

and furthermore from $\tilde{K}_3 \leq \frac{\bar{\alpha}|s|}{16\sqrt{2}\tilde{\alpha}}$ we obtain

$$\frac{\bar{\alpha}}{16\sqrt{2}\tilde{\alpha}}|s| \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8} \|v\|_{\mathcal{H}^1}^2 \leq \tilde{K}_4 \|v\|_{\mathcal{L}_2} \|\tilde{k}\|_{\mathcal{L}_2} \leq \tilde{K}_4 \left(\frac{\sigma^2}{2} \|v\|_{\mathcal{L}_2}^2 + \frac{1}{2\sigma^2} \|\tilde{k}\|_{\mathcal{L}_2}^2 \right).$$

With $\sigma^2 = \frac{|s|\bar{\alpha}}{16\sqrt{2}\tilde{K}_4\tilde{\alpha}}$ follows

$$\frac{\bar{\alpha}}{32\sqrt{2}\tilde{\alpha}}|s| \|v\|_{\mathcal{L}_2}^2 + \frac{\bar{\alpha}}{8} \|v_j\|_{\mathcal{H}^1}^2 \leq \frac{8\sqrt{2}\tilde{\alpha}\tilde{K}_4^2}{|s|\bar{\alpha}} \|\tilde{k}\|_{\mathcal{L}_2}.$$

For the proof of the second estimate (3.39) we can proceed analogous to the proof of [3], Theorem 3.1.

The last estimate (3.40) is a consequence of (3.38). Note $\tilde{k} \in \mathcal{H}^1$. We denote by $\tilde{C}_j = D^2 f(w_j) w_{j,\xi}$. Using Hypotheses 1.4 and 1.6 we obtain that there exist some constant $\tilde{C}_c > 0$ such that for all $\xi \in \mathbb{R}, j = 1, \dots, N$ holds

$$\|\tilde{C}_j(\xi)\| \leq \tilde{C}_c.$$

We consider again the equation (A.27) and differentiate it with respect to ξ

$$sz - \Lambda z = sz - Az_{\xi\xi} - Bz_{\xi} - Cz = \tilde{C}v + \tilde{k}_{\xi}$$

with $z = v_{\xi}$. We apply the estimate (3.38) and obtain

$$|s|^2 \|v_{\xi}\|_{\mathcal{L}_2}^2 + |s| \|v_{\xi}\|_{\mathcal{H}^1}^2 \leq 2C_R(\tilde{C}_c^2 \|v\|_{\mathcal{L}_2}^2 + \|\tilde{k}\|_{\mathcal{H}^1}^2).$$

If we again use (3.38) follows

$$\begin{aligned} & |s|^2 (\|v_{\xi}\|_{\mathcal{L}_2}^2 + \|v\|_{\mathcal{L}_2}^2) + |s| (\|v_{\xi}\|_{\mathcal{H}^1}^2 + \|v\|_{\mathcal{L}_2}^2) \\ & \leq 2C_R \tilde{C}_c^2 \|v\|_{\mathcal{L}_2}^2 + 2C_R \|\tilde{k}\|_{\mathcal{H}^1}^2 + C_R \|\tilde{k}\|_{\mathcal{L}_2}^2 \end{aligned}$$

and

$$\begin{aligned} |s|^2 \|v\|_{\mathcal{H}^1}^2 + |s| \|v\|_{\mathcal{H}^2}^2 & \leq 2C_R \tilde{C}_c^2 \frac{C_R}{|s|^2} \|\tilde{k}\|_{\mathcal{L}_2}^2 + 3C_R \|\tilde{k}\|_{\mathcal{H}^1}^2 \\ & \leq (2\tilde{C}_c^2 \frac{C_R^2}{K_G^2} + 3C_R) \|\tilde{k}\|_{\mathcal{H}^1}^2. \end{aligned}$$

■

Appendix B

Notation

$A > 0$	For $A \in \mathbb{R}^{m,m}$ holds $w^T A w > 0$ for all $w \in \mathbb{R}^m, w \neq 0$
$\mathcal{D}(P)$	Domain of definition of the operator P
$\mathcal{N}(P)$	Null space or kernel of the operator P
$\mathcal{R}(P)$	Image space of the operator P
$\ P\ _{X \rightarrow Y}$	Norm of the bounded operator $P : X \rightarrow Y$:

$$\|P\|_{X \rightarrow Y} = \sup_{x \in \mathcal{D}(P), x \neq 0} \frac{\|P(x)\|_Y}{\|x\|_X}$$

$\sigma(P)$	Spectrum of the operator P
$\rho(P)$	Resolvent set of the operator P
$\mathcal{C}(X, Y)$	Space of continuous operators from X to Y
$\mathcal{C}^k(X, Y)$	Space of k -times continuous differentiable operators from X to Y

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

$\mathcal{C}_b(\mathbb{R}, \mathbb{K}^m)$	Space of the continuous bounded functions from \mathbb{R} to \mathbb{K}^m
$\mathcal{C}_b^k(\mathbb{R}, \mathbb{K}^m)$	Space of functions with continuous, bounded derivatives $u^{(j)} = \frac{d^j}{d\xi^j} f$ up to order k equipped with the norm:

$$\|u\|_k := \sum_{j=0}^k \|u^{(j)}\|_\infty = \sum_{j=0}^k \sup_{\xi \in \mathbb{R}} \|u^{(j)}(\xi)\|$$

$\mathcal{C}_0(\mathbb{R}, \mathbb{K}^m)$	Space of functions from \mathbb{R} to \mathbb{K}^m with bounded support
$\mathcal{C}_0^\infty(\mathbb{R}, \mathbb{K}^m)$	Space of infinitely differentiable functions from \mathbb{R} to \mathbb{K}^m with compact support

$\mathcal{L}_p(\mathbb{R}, \mathbb{K}^m)$ Lebesgue measurable functions from \mathbb{R} to \mathbb{K}^m with $\|u\|_{\mathcal{L}_p(\mathbb{R}, \mathbb{K}^m)} < \infty$, where

$$\|u\|_{\mathcal{L}_p(\mathbb{R}, \mathbb{K}^m)} = \left(\int_{\mathbb{R}} \|u(\xi)\|^p d\xi \right)^{1/p}, \quad 1 \leq p < \infty$$

$\mathcal{L}_\infty(\mathbb{R}, \mathbb{K}^m)$ Lebesgue measurable functions from \mathbb{R} to \mathbb{K}^m with $\|u\|_{\mathcal{L}_\infty(\mathbb{R}, \mathbb{K}^m)} < \infty$, where

$$\|u\|_{\mathcal{L}_\infty(\mathbb{R}, \mathbb{K}^m)} = \operatorname{ess\,sup}_{\mathbb{R}} \|u\|$$

$\mathcal{H}^k(\mathbb{R}, \mathbb{K}^m)$ Space of Sobolev functions $u \in \mathcal{L}_2(\mathbb{R}, \mathbb{K}^m)$, which possess $\mathcal{L}_2(\mathbb{R}, \mathbb{K}^m)$ -integrable derivatives up to k equipped with the norm:

$$\|u\|_{\mathcal{H}^k(\mathbb{R}, \mathbb{K}^m)} = \left(\sum_{j=0}^k \|u^{(j)}\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{K}^m)}^2 \right)^{1/2} = \left(\int_{\mathbb{R}} \sum_{j=0}^k \|u^{(j)}(\xi)\|^2 d\xi \right)^{1/2}$$

θ_b Function $\theta_b(\xi) := \cosh(b\xi)$ for $b \geq 0$
 $\mathcal{L}_{2,b}(\mathbb{R}, \mathbb{K}^m)$ Functions u from \mathbb{R} to \mathbb{K}^m with $\theta_b u \in \mathcal{L}_2(\mathbb{R}, \mathbb{K}^m)$ equipped with the norm:

$$\|u\|_{\mathcal{L}_{2,b}(\mathbb{R}, \mathbb{K}^m)} = \left(\int_{\mathbb{R}} \|\theta_b(\xi)u(\xi)\|^2 d\xi \right)^{1/2}$$

$\mathcal{H}^{k,b}(\mathbb{R}, \mathbb{K}^m)$ Functions u from \mathbb{R} to \mathbb{K}^m with $\theta_b u \in \mathcal{H}^k(\mathbb{R}, \mathbb{K}^m)$ equipped with the norm:

$$\|u\|_{\mathcal{H}^{k,b}(\mathbb{R}, \mathbb{K}^m)} = \|\theta_b u\|_{\mathcal{H}^k(\mathbb{R}, \mathbb{K}^m)}$$

u_ξ, u_t Partial derivatives of a function $u(\xi, t)$
 $\langle u, v \rangle$ $\mathcal{L}_2(\mathbb{R}, \mathbb{K}^m)$ inner-product,

$$\langle u, v \rangle := \int_{\mathbb{R}} \bar{u}(\xi)^T v(\xi) d\xi$$

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