Fine Properties of Stochastic Evolution Equations and Their Applications

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
Wei Liu
aus Anhui, China
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Abstract

In this work, we aim to study some fine properties for a class of nonlinear SPDE within the variational framework. The results consist of three main parts. In the first part, we study the asymptotic behavior of nonlinear SPDE with small multiplicative noise. A Freidlin-Wentzell large deviation principle is established for the distributions of solutions to a large class of SPDE, which include all stochastic evolution equations with monotone coefficients. In the second part, some properties of invariant measures and transition semigroups are investigated for SPDE with additive noise. The main tool is the dimension-free Harnack inequality, which is established by using a coupling method and Girsanov transformation techniques. Subsequently, the Harnack inequality is used to derive the ergodicity, compactness and contractivity (e.g. hyperboundedness or ultraboundedness) for the associated transition semigroups. Moreover, the uniformly exponential convergence of the transition semigroup to the invariant measure and the existence of a spectral gap are also obtained. These results are first established for general stochastic evolution equations with strongly dissipative drift, e.g. stochastic reaction-diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation ($p \geq 2$) in Hilbert space. Stochastic fast diffusion equations and the singular stochastic p-Laplace equation ($1 < p < 2$) are investigated separately by using more delicate arguments due to the weak dissipativity of the drifts. In the last part, the invariance of subspaces under the solution flow of SPDE is investigated. We prove that the solution of an SPDE takes values in some suitable subspace of the state space if the initial state does so. This gives the stronger regularity estimates for the solution of an SPDE, which can be used for further study of the corresponding random dynamical system. As examples, the main results are applied to many concrete SPDEs in Hilbert space.

Keywords: Stochastic evolution equations, variational approach, large deviation principle, weak convergence approach, Harnack inequality, strong Feller property, irreducibility, ergodicity, spectral gap, coupling method, porous medium equation, fast diffusion equation, p-Laplace equation, reaction-diffusion equation.

Contents

Introduction

The theory of Itô stochastic differential equations is one of the most beautiful and fruitful areas in the theory of stochastic processes. It started to develop at the beginning of 1940s and is based on Itô's stochastic calculus (cf.[Itô46, Itô51]). However, the range of investigations in this theory before 1960s had been mainly restricted to ordinary stochastic differential equations.

The situation started to change from 1960s and 1970s. The necessity of considering equations combining the features of partial differential equations and Itô equations had appeared both in the theory of stochastic processes and related fields. In various branches of science (e.g. physics, biology and control theory), a large number of models were found that could be described by stochastic partial differential equations (SPDE) of evolutionary type. Those equations can describe the evolution (in time) of processes with values in function spaces and can be used to model all types of dynamics with random influence. For example, one can use stochastic evolution equations (SEE) to describe a free (boson) field in relativistic quantum mechanics, a hydromagnetic dynamo process in cosmology, the diffraction in random-heterogeneous media in statistical physics, and the dynamics of populations for models with a geographical structure in population genetics (cf.[Roz90, KR79]).

One powerful impetus to the development of the theory of stochastic evolution equations comes from the problem of non-linear filtering of diffusion processes. The filtering problem is one of the classical problems in the statistics of stochastic processes. The main goal is to estimate the "signal" by observing it when it is mixed with some noise. One of the key results of modern non-linear filtering theory states that the solution of the filtering problem for processes described by Itô's ordinary stochastic equations is equivalent to the solution of an equation commonly called the filtering equation, which is a typical example of a stochastic evolution equation.

The emergence of stochastic evolution equations was also simultaneously stimulated by the inner requirements of mathematics. In fact, the incentive for the first mathematical investigation of SEE was the inner needs of the theory of differential equations in infinite dimensional spaces. In the mid-sixties of the 20th century, Baklan [Bak63, Bak64] and Daletskii [Dal66] studied stochastic evolution equations with the goal of constructing a solution to the Cauchy problem for the following Kolmogorov equation

$$-\frac{\partial F(t,x)}{\partial t} = \frac{1}{2} tr[B^*(t,x)F''(t,x)B(t,x)] + A(t,x)F'(t,x);\ t \geq 0; F(0,x) = \Phi(x).$$

They used a probabilistic method for constructing the solution and the main idea was to write the solution in the form $F(t,x) = \mathbf{E}[\Phi(X(t))|X(0) = x]$, where $X(t)$ is the solution of the following stochastic evolution equation

$$\mathrm{d}X(t) = A(t, X(t))\mathrm{d}t + B(t, X(t))\mathrm{d}W(t),\ X(0) = x. \tag{0.0.1}$$

Therefore, it was necessary to study SEE (0.0.1) in order to realize this procedure in [Bak63, Bak64, Dal66]. Concerning the existence of a solution to (0.0.1), they assumed that $A(t,u) = A(t)u$ and the operator $A(t)$ generates an inhomogeneous semigroup (i.e. evolution operators) $T_{s,t}$ and B satisfies a Lipschitz condition. Then the following equation was considered

$$X(t) = T_{0,t}X(0) + \int_0^t T_{s,t}B(X(s))\mathrm{d}W(s). \tag{0.0.2}$$

The proof of existence and uniqueness of the solution for this equation is done simply by Banach's fixed-point theorem. It was then proved under additional conditions that the solution of (0.0.2) belongs to the domain of the operator A (for Lebesgue almost all time) and equation (0.0.1) is equivalent to (0.0.2). This is the main idea of the so-called semigroup (or mild solution) approach for SPDE.

In 1971, Bensoussan [Ben71] used a completely different idea to construct the solution of (0.0.1) for $B = I$ (identity operator). He formulated a coercivity condition instead of assuming that A generates a semigroup, and the method of time discretization was used to construct the solution. The coercivity condition ensures that the corresponding discrete equation is easily solvable and the solution also satisfies some a priori estimates; then a weak limiting procedure was employed to obtain the solution for the original stochastic equation. This method was used earlier by Lions for deterministic equations (cf.[Lio72]).

Stochastic Evolution Equations

In [BT72] Bensoussan and Temam studied stochastic evolution equations with a nonlinear drift A satisfying a monotonicity condition. This monotonicity method was further developed in the works of Pardoux [Par72, Par75], where he investigated a general SEE with unbounded nonlinear operators as drift and diffusion. The solution obtained in [Par75] belongs to the domains of the operator A, B in (0.0.1) (for Lebesgue almost all time) and is also measurable with respect to the σ-algebra generated by the Wiener process on a prescribed probability space, hence it is a strong solution according to the terminology of stochastic equations.

In [KR79] Krylov and Rozovskii generalized these results to general stochastic evolution equations

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{0.0.3}$$

and certain conditions (e.g. the local Lipschitz condition on B) in [Par75] were removed. The Markov property of the solution was obtained for SEE with deterministic coefficients. In particular, the results in [KR79] also generalized Itô's classical theorem on the strong solvability of finite dimensional stochastic differential equations with random coefficients satisfying Lipschitz conditions. They also proved Itô's formula for the square of the norm of a semimartingale in a rigged Hilbert space, which plays a very important role in the entire theory. This is the so-called variational approach for SPDE in the literature. This seminal work was extended later in many different aspects, we may refer to [Gyö82, GM05, RRW07, Zha08] for various generalizations.

In this work, we use the variational approach to analyze a wide class of nonlinear SPDE in a unified framework. But we should mention that there also exist many other important approaches to study SPDE in the literature: e.g. the martingale (measure) approach (cf.[Wal86]), the semigroup approach (cf.[DPZ92c, DPZ96]) and the white noise approach (cf.[DKPW02, HOUZ96]). For each approach there exist an enormous literatures which can not be listed here; hence we refer the reader to the above monographs and the references therein.

The main aim of this thesis is to investigate some fine properties for a large class of SEE within the variational framework, e.g. small noise asymptotic properties and long time behavior of the solution, ergodicity, contractivity and compactness of the associated transition semigroups, and some regularity estimates of the solution in subspace. Now we describe those results more specifically.

Large Deviation Principle

In probability theory, large deviation theory mainly concerns the asymptotic behavior of remote tails of sequences (or families) of probability distributions. The first rigorous results concerning large deviations are due to Cramér, who applied them to models in insurance business. Establishing large deviations principles is one of the most effective ways to obtain information from a probabilistic model. Some of the best known applications of large deviation theory arise in statistical mechanics, quantum mechanics, operations research, ergodic theory, information theory and risk management.

In chapter 2 we will study large deviation principle (LDP) for the solutions of general SEE driven by small noise

$$\mathrm{d}X_t^\varepsilon = A(t, X_t^\varepsilon)\mathrm{d}t + \varepsilon B(t, X_t^\varepsilon)\mathrm{d}W_t, \quad X_0^\varepsilon = x. \tag{0.0.4}$$

Roughly speaking, $\{X^\varepsilon\}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in some Polish space E (e.g. path space). Large deviation theory is mainly concerned with the tail (or deviation) events A for which probabilities $\mathbf{P}(X^\varepsilon \in A)$ converge to zero exponentially fast as $\varepsilon \to 0$. The obtained convergence results can be applied to the analysis of the destablizing effect of the noise term in (0.0.4) (cf.[DPZ92c]). The rate of such exponential decay is expressed by the rate function.

Definition 0.0.1 *(Rate function) A function $I : E \to [0, +\infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in E : I(x) \le K\}$ is compact for each $K < \infty$.*

Definition 0.0.2 *(Large deviation principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of E*

$$-\inf_{x\in A^o} I(x) \leq \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}(X^\varepsilon \in A) \leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}(X^\varepsilon \in A) \leq -\inf_{x\in \bar{A}} I(x),$$

where A^o and $\bar{A}$ are respectively the interior and the closure of A in E.

This general large deviation principle was first formulated by Varadhan [Var66] in 1966, although some basic ideas of the theory can be traced back to Laplace and Cramér. Concerning its validity for stochastic differential equations in finite dimensional case we mainly refer to the well-known Freidlin-Wentzell LDP (cf. [FW84]). The same problem was also treated by Varadhan in [Var84] and Stroock in [Str84] by a different approach, which followed the large deviation theory developed by Azencott [Aze80], Donsker and Varadhan [DV77, Var66]. In the classical paper [Fre88] Freidlin studied large deviations for the small noise limit of stochastic reaction-diffusion equations. Subsequently, many authors have endeavored to establish the large deviations results under less and less restrictive assumptions. For the extensions to infinite dimensional diffusions or SPDE under global Lipschitz condition on the nonlinear term we refer the reader to Da Prato, Zabczyk [DPZ92c] and Peszat [Pes94] (also the references therein). For the case of local Lipschitz condition we refer to the work of Cerrai and Röckner [CR04], where the case of multiplicative and degenerate noise was also studied. One should also mention the result of Cardon-Weber [CW99] on the LDP for the stochastic Burgers equations and the work of Hino and Ramirez [HR03] for Varadhan's small time estimate of large deviations for general symmetric Markov processes.

Concerning the large deviation results for SPDE within the variational framework, Chow first studied the LDP for semilinear stochastic parabolic equations on a Gelfand triple in [Cho92]. Recently, Röckner *et al* obtained the LDP in [RWW06] for the distributions of the stochastic porous media equations with additive noise. All these papers mainly followed the classical ideas of discretized approximations, which was first developed by Freidlin and Wentzell. The standard procedure to establish the small noise LDP for SPDE is as follows. One first needs to consider an approximating Gaussian model by time discretization and establish the LDP for this approximated model. Then one can derive the LDP for the original non-Gaussian model by establishing some necessary exponential continuity and tightness of the solutions in suitable spaces. But the situation becomes much involved and complicated in the infinite dimensional case since different types of SPDE need different techniques and estimates.

An alternative approach for LDP has been developed by Feng and Kurtz in [FK06], which mainly used nonlinear semigroup theory and infinite dimensional Hamilton-Jacobi equation. The techniques rely on the uniqueness theory for the infinite dimensional Hamilton-Jacobi equation and some exponential tightness estimates.

In chapter 2 we derive the Freidlin-Wentzell LDP for general SEE with monotone drifts and small multiplicative noise, which cover all types of SPDE within the variational framework (cf.[PR07, KR79]). Instead of studying different types of SPDE in infinite dimensional spaces case by case, we establish a general theorem for the large deviation principle. The main results are applied to derive the LDP for stochastic reaction-diffusion equations, stochastic porous media equations and fast diffusion equations, and the stochastic p-Laplace equation in Hilbert space *etc.* In particular, the main results generalize and improve the earlier work [Cho92] on semilinear SPDE and [RWW06] on stochastic porous media equations.

The proof of our main results on the LDP is mainly based on a weak convergence approach and some approximation techniques. In fact, it would be quite difficult to follow the classical discretization approach in the present case. Many technical difficulties would appear in the discretization arguments, e.g. it would be very difficult to obtain some regularity (Hölder) estimate for the solution w.r.t. the time variable, which is essentially required in the classical proof of the LDP by discretization techniques.

Hence in chapter 2 we adopt a stochastic control and weak convergence approach in the proof. This approach is mainly based on a variational representation formula for certain functionals of infinite dimensional Brownian motion, which was established by Budhiraja and Dupuis in [BD00]. The main advantage of the weak convergence approach is that one can avoid some exponential probability estimates, which might be very difficult to derive for infinite dimensional models. However, there are still some technical difficulties appearing in the implementation of the weak convergence approach within our variational framework. The reason is that the coefficients of SEE are nonlinear operators which are only well-defined via a Gelfand triple (so three spaces are involved). Hence we have to properly handle many estimates involving different spaces instead of just one single space. Some approximation techniques (e.g. finite dimensional approximation and truncation techniques) are also used in the proof.

Harnack Inequality and Its Applications

In chapter 3, 4 and 5 we establish the dimension-free Harnack inequality and strong Feller property for the transition semigroups associated with different types of nonlinear SPDE within the variational framework. As applications, the ergodicity, contractivity (hyperboundedness or ultraboundedness) and compactness property are derived for the associated Markov semigroups. The convergence rate of the transition semigroups to the invariant measure and the existence of a spectral gap are also investigated.

The dimension-free Harnack inequality was first introduced by F.-Y. Wang in [Wan97] for diffusions on Riemannian manifolds. Let M be a connected complete d-dimensional Riemannian manifold and $L := \Delta + Z$ for some C^1-vector field Z such that

$$Ric(X, X) - \langle \nabla_X Z, X \rangle \geq -K|X|^2, \ X \in TM$$

for some $K \in \mathbb{R}$. Then the corresponding semigroup $P_t := e^{tL}$ satisfies the following Harnack inequality: for any $p > 1$ and nonnegative $f \in C_b(M)$ we have

$$(P_t f(x))^p \leq (P_t f^p(y)) \exp\left[\frac{pK\rho(x, y)^2}{2(p-1)(1-e^{-2Kt})}\right], \ x, y \in M, \qquad (0.0.5)$$

where $\rho(x, y)$ is the Riemannian distance between x and y.

The main feature of this Harnack inequality is that the estimate (0.0.5) does not depend on the dimension of the underlying manifold M, hence it can be applied to study many infinite dimensional models. This is the key difference of this inequality from Li-Yau's parabolic Harnack inequality (cf.[LY86]). Even in finite dimensional case, there are some very useful models which satisfy the dimension-free Harnack inequality (0.0.5), but which do not satisfy Li-Yau's Harnack inequality, e.g. the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$ (cf.[LW03]).

In recent years, the dimension-free Harnack inequality turned out to be a very efficient tool for the study of finite and infinite dimensional diffusion semigroups. For example, it has been applied to study functional inequalities in [Wan99, Wan01, RW03b, RW03a]; the short time behavior of infinite dimensional diffusions in [AK01, AZ02, Kaw05]; the estimation of high order eigenvalues in [GW04, Wan00]; the transportation-cost inequality in [BGL01] and heat kernel estimates in [GW01].

Very recently, the dimension-free Harnack inequality was established in [Wan07] for a class of stochastic porous media equations and in [LW08] for stochastic fast-diffusion equations. As applications, an estimate of the transition density, ergodicity and some contractivity properties were obtained for the associated transition semigroups. The approach used in [Wan07, LW08] is mainly based on a new coupling argument developed in [ATW06], where the Harnack inequality was derived for diffusion semigroups on Riemannian manifolds with curvature unbounded below. The advantage of this approach is that one can avoid the assumption that the curvature is lower bounded, which was used in previous articles (cf.[AK01, AZ02, BGL01, RW03a, RW03b]) in an essential way and would be very hard to verify in the present framework of SPDE.

In chapter 3 we establish the Harnack inequality for a large class of SEE with additive noise. More precisely, we mainly deal with stochastic evolution equations with strongly dissipative drifts in Hilbert space, which cover many important types of SPDE such as stochastic reaction-diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation (cf.[PR07, KR79, Zha08]). The proof of the Harnack inequality and the strong Feller property is based on a coupling method and Girsanov transformation techniques. Subsequently, we investigate some properties of the invariant measures such as the existence, uniqueness and concentration property. Moreover, based on the Harnack inequality, the ergodicity, contractivity (e.g. hyperboundedness or ultraboundedness) and compactness are established for the associated transition semigroups in Hilbert space. In particular, we give a very easy proof for the (topological) irreducibility by using the established Harnack inequality. Hence the uniqueness of invariant measures for the transition semigroups is obtained without assuming strict monotonicity for the drift, which was required in many earlier works [PR07, Wan07, LW08, RRW07, DPRRW06]. We also derive the convergence rate of the transition semigroups to the invariant measure, which implies a decay estimate of the solutions for the corresponding deterministic evolution equations (e.g. p-Laplace equation, porous medium equation). This result coincides with some well-known estimates in PDE theory. Finally, some uniformly exponential ergodicity of the associated Markov semigroup and the existence of a spectral gap are also investigated.

As we mentioned before, the main results in chapter 3 are applied to many nonlinear SPDEs in Hilbert space. However, the stochastic fast diffusion equation ($0 < r < 1$)

$$\mathrm{d}X_t = \Delta(|X_t|^{r-1}X_t)\mathrm{d}t + B\mathrm{d}W_t$$

and the singular stochastic p-Laplace equation ($1 < p < 2$)

$$\mathrm{d}X_t = \mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t)\mathrm{d}t + B\mathrm{d}W_t$$

does not satisfy the strong dissipativity condition which is assumed for the drift in chapter 3, hence the general result cannot be applied to these two types of SPDE. For example, the drift of the stochastic porous media equation ($r > 1$) satisfies

$${}_{V^*}\langle \Delta(|u|^{r-1}u) - \Delta(|v|^{r-1}v), u - v\rangle_V \le -c\int_\Lambda |u-v|^{r+1}\mathrm{d}x = -c\|u-v\|_V^{r+1}.$$

But the drift of the stochastic fast diffusion equation ($0 < r < 1$) only satisfies the following weak dissipativity property (see chapter 4 for details)

$${}_{V^*}\langle \Delta(|u|^{r-1}u) - \Delta(|v|^{r-1}v), u - v\rangle_V \le -c\int_\Lambda \left(|u-v|^2\left(|u| \vee |v|\right)^{r-1}\right)\mathrm{d}x.$$

Therefore, we study the stochastic fast diffusion equations in chapter 4 and the singular stochastic p-Laplace equations ($p < 2$) in chapter 5 separately. Due to the weak dissipativity of the drift, we need to make more delicate estimates in order to establish the Harnack inequality. The strong Feller property and heat kernel estimates are also obtained for the corresponding transition semigroups. Moreover, if we have some strongly dissipative perturbations in the drift, then the ultraboundedness and compactness property can also be derived for the associated Markov semigroups. In particular, the exponential ergodicity and the existence of a spectral gap are also investigated. As applications, some explicit examples are discussed to illustrate the main results. In particular, we prove that the transition semigroup associate to a stochastic reaction-diffusion equation is ultrabounded and compact, hence its generator has only discrete spectrum.

Invariance of Subspaces under the Solution Flow

Recently, Röckner and Wang proved in [RW08] the L^2-invariance of the solution for the stochastic porous media equations ($r > 1$)

$$\mathrm{d}X_t = \Delta(|X_t|^{r-1}X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t,$$

i.e. the solution takes values in the L^2 space (note that the original state space is W^{-1}) if the initial condition does and has right continuous paths in L^2 (almost surely). Later, this property was used to investigate the existence of the random attractor (cf.[BLR08]).

Chapter 6 is devoted to establish this type of regularity properties for a large class of SPDE within the variational framework. The desired regularity property can be generally formulated as follows. Consider the Gelfand triple

$$V \subset H \equiv H^* \subset V^*$$

and the stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{0.0.6}$$

where $A : [0, T] \times V \times \Omega \to V^*$ and $B : [0, T] \times V \times \Omega \to L_2(U, H)$ are progressively measurable. Suppose $(S, \|\cdot\|_S)$ is a subspace of H and $X_0 \in S$ *a.s.*. We want to prove that the solution X_t of (0.0.6) also takes values in S for almost all path, *i.e.*

$$\mathbf{P}(\omega : X_t(\omega) \in S,\ 0 \leq t < T) = 1.$$

In fact, in chapter 6 we prove that for some $p \geq 1$

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_S^p < \infty. \tag{0.0.7}$$

The typical choice of the subspace is $S = \mathcal{D}(\sqrt{T})$, where T is a positive definite self-adjoint operator on H. This regularity estimate (0.0.7) has been used in [GM07] for deriving the convergence rate of implicit approximations for SEE and in [Cho92] for establishing the large deviation principle for semilinear type SPDE.

The main idea in the proof is to find a sequence of equivalent norms $\|\cdot\|_n$ on H satisfying

$$\forall x \in S,\ \|x\|_n \uparrow \|x\|_S (n \to \infty).$$

Then by applying Itô's formula for $\|\cdot\|_n^2$ we may prove for any time T

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_n^p \le K,\ n \ge 1$$

for some constants $p \ge 1$ and K. Then the desired result follows by taking the limit in the above estimate.

As examples, the main results are applied to stochastic reaction-diffusion equations, the stochastic p-Laplace equation, stochastic porous media and fast diffusion equations in Hilbert space.

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Chapter 1

Preliminaries on Stochastic Analysis in Infinite Dimensional Space

In this chapter, we collect some results of stochastic analysis in infinite dimensional space as preliminaries for the following chapters. We omit all proofs and refer the reader to [PR07, DPZ92c] for details. In the first part, we introduce the Wiener process and general martingales in infinite dimensional space, then we give the definition of the stochastic integral in Hilbert space and some important properties. In the second part, we recall the variational framework and some classical results for stochastic evolution equations in [KR79]. In the last part, we shortly review the different concepts of solution to stochastic equations and their relations.

1.1 Stochastic integral in Hilbert space

The theory of stochastic integration in infinite dimensional space is a very broad area in the theory of stochastic processes. The first important work in this direction was due to Daletskii [Dal66], where he constructed a Wiener process (with an identity covariance operator) in a Hilbert space and defined the stochastic integral. Kuo [Kuo75] investigated the stochastic integral with respect to an abstract Wiener process in a Banach space and Kunita [Kun70] initiated the study of the integrability w.r.t. a square-integrable martingale in a Hilbert space. Later some considerable progress was achieved by Metivier, Meyer and many others, we refer to [KR79] for more detailed exposition.

1.1.1 Infinite dimensional Wiener processes

For a fixed separable Hilbert space $(U, \langle\cdot,\cdot\rangle_U)$ we denote its Borel σ-algebra by $\mathcal{B}(U)$ and all bounded operators on U by $L(U)$.

Definition 1.1.1 *A probability measure μ on Hilbert space $(U, \mathcal{B}(U))$ is called Gaussian if for all $u \in U$ the bounded linear mapping*

$$u' : U \to \mathbb{R};\ v \mapsto \langle v, u\rangle_U$$

has a Gaussian law, i.e. for all $u \in U$ there exist $m := m(u) \in \mathbb{R}$ and $\sigma := \sigma(u) \in [0, \infty)$ such that if $\sigma(u) > 0$,

$$\mu \circ (u')^{-1}(A) = \mu(u' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} \mathrm{d}x,\ \text{for all } A \in \mathcal{B}(\mathbb{R}),$$

and if $\sigma(u) = 0$,

$$\mu \circ (u')^{-1} = \delta_{m(u)}.$$

Theorem 1.1.1 *A measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if for any $u \in U$,*

$$\hat{\mu}(u) := \int_U e^{i\langle u,v\rangle_U} \mu(\mathrm{d}v) = e^{i\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U},$$

where $m \in U$ and $Q \in L(U)$ is a non-negative, symmetric and trace class operator.

In this case μ will be denoted by $N(m, Q)$ where m and Q are called mean and covariance (operator) respectively. The measure μ is uniquely determined by m and Q.

Proposition 1.1.2 *If $Q \in L(U)$ is a non-negative, symmetric and trace class operator, then there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of U such that*

$$Qe_k = \lambda_k e_k,\ \lambda_k \geq 0, k \in \mathbb{N},$$

where $\sum\limits_{k\in\mathbb{N}} \lambda_k < \infty$ and 0 is the only accumulation point of the sequence $(\lambda_k)_{k\in\mathbb{N}}$.

Proposition 1.1.3 ***(Representation of a Gaussian random variable)*** *Suppose $m \in U$ and $Q \in L(U)$ is a non-negative, symmetric and trace class operator, $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$. Then a U-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is Gaussian with $\mathbf{P} \circ X^{-1} = N(m, Q)$ if and only if*

$$X = \sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k e_k + m,$$

where $\beta_k, k \in \mathbb{N}$, are independent real-valued Gaussian random variables with mean 0 and variance 1. And the series converges in $L^2(\Omega, \mathcal{F}, \mathbf{P}; U)$.

Now we can give the definition of the standard Q-Wiener process. To this end we fix a positive time T and a non-negative symmetric trace class operator Q on U.

Definition 1.1.2 *A U-valued stochastic process $W(t), t \in [0, T]$, on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a (standard) Q-Wiener process if:*

(i) $W(0) = 0$;

(ii) W has $\mathbf{P}$-a.s. continuous trajectories;

(iii) the increments of W are independent, i.e. the random variables

$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$$

are independent for all $0 \le t_1 < \cdots < t_n \le T,\ n \in \mathbb{N}$;

(iv) the increments have the following Gaussian laws:

$$\mathbf{P} \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q),\ 0 \le s \le t \le T.$$

Proposition 1.1.4 ***(Representation of the Q-Wiener process)*** *Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$. Then a U-valued stochastic process $W(t), t \in [0, T]$, on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a Q-Wiener process if and only if*

$$W(t) = \sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k(t) e_k,\ t \in [0, T],$$

where $\beta_k, k \in \mathbb{N}$, are independent real-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbf{P})$. The series converges in $L^2(\Omega, \mathcal{F}, \mathbf{P}; C([0,T], U))$ and thus always has a $\mathbf{P}$-a.s. continuous modification.

Definition 1.1.3 *A Q-Wiener process $W(t), t \in [0,T]$, is called a Q-Wiener process with respect to a filtration $\mathcal{F}_t, t \in [0,T]$, if:*

(i) $W(t), t \in [0,T]$, is adapted to $\mathcal{F}_t, t \in [0,T]$;

(ii) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$.

Now we consider the following cylindrical Wiener process.

Definition 1.1.4 *Suppose $Q \in L(U)$ is non-negative and symmetric, then a cylindrical Wiener process on U is defined as the following series:*

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k, \ t \in [0,T], \tag{1.1.1}$$

where $e_k, k \in \mathbb{N}$, is an orthonormal basis of $Q^{\frac{1}{2}}(U)$ and $\beta_k, k \in \mathbb{N}$, is a family of independent real-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbf{P})$.

Remark 1.1.1 *If Q is a trace class operator, then we know the series (1.1.1) converges in $L^2(\Omega, \mathcal{F}, \mathbf{P}; U)$. In the case that Q is not trace class operator then one looses this convergence. However, one can show that (1.1.1) converges in $L^2(\Omega, \mathcal{F}, \mathbf{P}; U_1)$ whenever the embedding $U_0 \subset U_1$ is Hilbert-Schmidt. And it is also easy to see that $W(t), t \in [0,T]$, is a Wiener process on U_1 with trace class covariance operator.*

1.1.2 Martingales in Banach space

We first introduce the conditional expectation of any Bochner integrable random variable with values in a separable real Banach space $(E, \|\cdot\|)$, which is similar to the real-valued case.

Proposition 1.1.5 *Let X be a Bochner integrable E-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. Then there exists a* *unique, up to a set of* $\mathbf{P}$*-probability zero, Bochner integrable* E*-valued random variable* Z*, measurable with respect to* $\mathcal{G}$ *such that*

$$\int_A X \mathrm{d}\mathbf{P} = \int_A Z \mathrm{d}\mathbf{P} \text{ for all } A \in \mathcal{G}.$$

The random variable Z *is denoted by* $\mathbf{E}(X|\mathcal{G})$ *and is called the conditional expectation of* X *w.r.t.* $\mathcal{G}$.

Definition 1.1.5 *Let* $M(t), t \geq 0$*, be a stochastic process on* $(\Omega, \mathcal{F}, \mathbf{P})$ *with values in* E *and let* $\mathcal{F}_t, t \geq 0$*, be a filtration on* $(\Omega, \mathcal{F}, \mathbf{P})$*. Then the process* M *is called a* $\mathcal{F}_t$*-martingale if:*

(i) $\mathbf{E}(\|M(t)\|) < \infty$ *for all* $t \geq 0$*;*

(ii) $M(t)$ *is* $\mathcal{F}_t$*-measurable for all* $t \geq 0$*;*

(iii) $\mathbf{E}(M(t)|\mathcal{F}_s) = M(s)$ $\mathbf{P}$*-a.s. for all* $0 \leq s \leq t < \infty$.

Now we denote the space of all E-valued continuous square integrable martingales $M(t), t \in [0,T]$ by $\mathcal{M}_T^2(E)$, which will play an important role in the definition of stochastic integral.

Proposition 1.1.6 *The space* $\mathcal{M}_T^2(E)$ *equipped with the norm*

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t \in [0,T]} \left(\mathbf{E}(\|M(t)\|^2)\right)^{1/2} = \left(\mathbf{E}(\|M(T)\|^2)\right)^{1/2}$$

is a Banach space.

Proposition 1.1.7 *Let* $W(t), t \in [0,T]$*, be a* U*-valued* Q*-Wiener process with respect to a normal filtration* $\mathcal{F}_t, t \in [0,T]$*, on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$*, then* $W(t), t \in [0,T]$ *is a continuous square integrable* $\mathcal{F}_t$*-martingale, i.e.* $W \in \mathcal{M}_T^2(U)$.

Proposition 1.1.8 ***(Burkhölder-Davis-Gundy inequality)*** *If* $M \in \mathcal{M}_T^2(E)$ *and* τ *is an a.s. finite stopping time, then*

$$\mathbf{E} \sup_{t \leq \tau} \|M(t)\| \leq 3\mathbf{E}\langle M \rangle_\tau^{1/2}.$$

1.1.3 Stochastic integral in Hilbert space

Let $(L_2(X,Y), \|\cdot\|_2)$ denote the space of all Hilbert-Schmidt operators from X to Y. Similar to the finite dimensional case, one can first consider the stochastic integral of elementary processes w.r.t. the Wiener process and establish the Itô-isometry. Then by a standard limiting procedure and localization argument one can extend the definition of stochastic integral to the following class of processes:

$$\mathcal{N}_W := \Big\{ \Phi : [0,T] \times \Omega \to L_2(Q^{\frac{1}{2}}(U), H) \Big| \ \Phi \text{ is predictable and} \\ \mathbf{P}\left(\int_0^T \|\Phi(s)\|_2^2 \mathrm{d}s < \infty \right) = 1 \Big\}.$$

Note that $U_0 := Q^{\frac{1}{2}}(U)$ is a separable Hilbert space equipped with the following inner product

$$\langle u, v \rangle_0 := \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_U, \ \ u, v \in Q^{\frac{1}{2}}(U),$$

where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Hence we know that $\|\Phi(s)\|_2 = \|\Phi(s) \circ Q^{\frac{1}{2}}\|_{L_2(U,H)}$.

Proposition 1.1.9 *Let $\Phi \in \mathcal{N}_W$ and $M(t) := \int_0^t \Phi(s)\mathrm{d}W(s), t \in [0,T]$. Define*

$$\langle M \rangle_t := \int_0^t \|\Phi(s)\|_2^2 \mathrm{d}s, \ \ t \in [0,T],$$

then $\langle M \rangle$ is the unique continuous increasing $\mathcal{F}_t$-adapted process starting at zero such that $\|M(t)\|^2 - \langle M \rangle_t, t \in [0,T]$, is a local martingale.

Remark 1.1.2 *Q is not necessarily a trace-class operator here. The case $Q = I$, i.e. W_t is a cylindrical Wiener process, is also included.*

Proposition 1.1.10 ***(Girsanov theorem)*** *Assume that $\varphi(\cdot)$ is a U_0-valued $\mathcal{F}_t$-predictable process such that*

$$\mathbf{E}\left(\exp\left(\int_0^T \langle \varphi(s), \mathrm{d}W(s) \rangle_0 - \frac{1}{2} \int_0^T \|\varphi(s)\|_0^2 \mathrm{d}s \right) \right) = 1. \tag{1.1.2}$$

Then the process

$$\tilde{W}(t) = W(t) - \int_0^t \varphi(s)\mathrm{d}s, \ \ t \in [0,T]$$

is a Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$ *on the probability space* $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ *where*

$$\mathrm{d}\tilde{\mathbf{P}}(\omega) = \exp\left(\int_0^T \langle \varphi(s), \mathrm{d}W(s)\rangle_0 - \frac{1}{2}\int_0^T \|\varphi(s)\|_0^2 \mathrm{d}s\right) \mathrm{d}\mathbf{P}(\omega).$$

Proposition 1.1.11 *Either of the following conditions is sufficient in order for (1.1.2) to hold:*

(i) $\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\varphi(s)\|_0^2 \mathrm{d}s\right)\right] < \infty$;

(ii) there exists $\delta > 0$ *such that* $\sup_{s\in[0,T]} \mathbf{E}\left(e^{\delta\|\varphi(s)\|_0^2}\right) < \infty$.

1.2 Variational approach for stochastic evolution equations

Now we describe the variational framework and the main results of [KR79] in detail. Let

$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space and identified with its dual space by the Riesz isomorphism, V is a reflexive and separable Banach space such that it is continuously and densely embedded into H. If ${}_{V^*}\langle\cdot,\cdot\rangle_V$ denotes the dualization between V and its dual space V^*, then it follows that

$${}_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H, \ \ u \in H, v \in V.$$

Let $\{W_t\}_{t\geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $(L_2(U; H), \|\cdot\|_2)$ denote the space of all Hilbert-Schmidt operators from U to H. Now we consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{1.2.1}$$

where for some fixed time T

$$A : [0, T] \times V \times \Omega \to V^*; \quad B : [0, T] \times V \times \Omega \to L_2(U; H)$$

are progressively measurable, *i.e.* for every $t \in [0, T]$, these maps restricted to $[0, t] \times V \times \Omega$ are $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable ($\mathcal{B}$ denotes the corresponding Borel σ-algebra). For the existence and uniqueness of the solution to (1.2.1) we need to assume the following conditions on A and B.

Suppose for a fixed $\alpha > 1$ there exist constants $\theta > 0$, K and a positive adapted process $f \in L^1([0,T] \times \Omega; \mathrm{d}t \times \mathbf{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t,\omega) \in [0,T] \times \Omega$.

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Monotonicity)

$$2_{V^*}\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_V + \|B(t,v_1) - B(t,v_2)\|_2^2 \le K\|v_1 - v_2\|_H^2.$$

$(H3)$ (Coercivity)

$$2_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|_2^2 + \theta\|v\|_V^\alpha \le f_t + K\|v\|_H^2.$$

$(H4)$ (Boundedness)

$$\|A(t,v)\|_{V^*} \le f_t^{(\alpha-1)/\alpha} + K\|v\|_V^{\alpha-1}.$$

Definition 1.2.1 *(Solution of SEE) A continuous H-valued $(\mathcal{F}_t)$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of* (1.2.1)*, if for its $\mathrm{d}t \otimes \mathbf{P}$-equivalent class $\bar{X}$ we have*

$$\bar{X} \in L^\alpha([0,T] \times \Omega, \mathrm{d}t \otimes \mathbf{P}; V) \cap L^2([0,T] \times \Omega, \mathrm{d}t \otimes \mathbf{P}; H)$$

and $\mathbf{P}$-a.s.

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s, \ \ t \in [0,T].$$

Theorem 1.2.1 ([KR79] Theorems II.2.1, II.2.2) *Suppose $(H1) - (H4)$ hold, then for any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbf{P})$* (1.2.1) *has a unique solution $\{X_t\}_{t\in[0,T]}$ and satisfies*

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_H^2 < \infty.$$

The proof of this theorem strongly depends on the following Itô formula for the square norm of the solution

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2_{V^*}\langle A(s,X_s), X_s\rangle_V + \|B(s,X_s)\|_2^2\right) \mathrm{d}s + 2\int_0^t \langle X_s, B(s,X_s)\mathrm{d}W_s\rangle_H. \tag{1.2.2}$$

This Itô's formula (or energy identity) was essentially used to derive some a priori estimates and to prove the uniqueness and continuity of the solution. We should remark that the proof of (1.2.2) in a rigged Hilbert space is much more difficult than the case that all components take values in a single Hilbert space.

This seminal work was extended later in various directions: e.g. (1.2.1) driven by a general martingale (not necessarily continuous) in [Gyö82]; K and θ in the assumptions $(H2)-(H4)$ are time-dependent in [GM05]; (1.2.1) with coefficients A and B related to Orlicz space framework in [RRW07]; K, θ in $(H2)-(H4)$ are random and time-dependent in [Zha08].

1.3 Different concepts of solution to stochastic equations

In this part we give a short review about the different types of solution to stochastic equations and the relations among them. Roughly speaking, there mainly exist three kinds of solution for S(P)DE in the literature: the strong, weak and martingale solution. In finite dimensional case, the corresponding definitions and their relations are well investigated. For instance, weak solution is equivalent to martingale solution due to the well-known Doob (martingale representation) theorem (cf.[Doo53, SV79]). But the analogue of this result in infinite dimensional space becomes very delicate and complicated. One purpose of this section is to clarify different concepts of solution and the relations among them in infinite dimensional space, and we also want to emphasis the differences comparing with the corresponding finite dimensional results.

1.3.1 Strong solution vs. Weak solution

For studying stochastic differential equations, one has to differentiate between strong and weak solution. A strong solution is usually defined as a measurable functional of given Wiener process (on some path space) that satisfies equation in a classical or generalized sense (cf.[IW81]). Strong solution exists for many classes of S(P)DE such as: Itô equations with Lipschitz coefficients (cf.[SV79, KS05]), stochastic evolution equations with monotone coefficients (cf.[Par75, KR79]), Kushner's and Zakai's equations of nonlinear filtering (cf.[Roz90]) and many others.

But a strong solution often fails to exist in the case of S(P)DE with non-smooth coefficients. The following simple example was given by Tanaka. Consider the equation

$$\mathrm{d}X_t = B(X_t)\mathrm{d}W_t,\ X_0 = 0, \tag{1.3.1}$$

where W_t is a 1-dimensional Brownian motion and

$$B(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ -1, & \text{if } x < 0. \end{cases}$$

One can prove that such an equation has no strong solution. On the other hand, according to the classical result in Doob's book (see [Doo53] Ch.VI, Section 3) one can show that (1.3.1) has a martingale (equivalently, weak) solution. Roughly speaking, one replaces the requirements on the integro-differential relations between the strong solution and the Brownian motion by the appropriate conditions on the probability law of the solution. The difference between these two concepts is similar to the one between a random variable and its law. In general, one could not conclude that the weak solution X' is a measurable functional of Brownian motion W' on the path space. But X' has the same probability law with the strong solution X if it exists, and in many cases the probability law is the only thing that really matters.

On the other hand, according to the famous Yamada-Watanabe theorem, there exists a unique strong solution if and only if there exists a weak solution and the pathwise uniqueness holds. This result was first proved in [YW71] for finite dimensional case, see [PR07] for a detailed proof. About some further related work we refer to [Jac80, Eng91, Che03]. In recent years, the analog result in infinite dimensional space has been established by Ondreját [Ond04] within the semigroup framework (cf.[DPZ92c]) and Röckner *et al* [RSZ08] within the variational framework (cf.[KR79]).

Remark 1.3.1 *(1) Pathwise uniqueness is obviously far from being a necessary condition for the existence of strong solution. Even in the case that the uniqueness in law does not hold, there exist some examples which show that strong solution can exist (see e.g.[Eng91] section 4). Engelbert proposed some sufficient and necessary condition for* *the existence of strong solution in [Eng91], where he used the concept of "joint solution measure" introduced by Jacod [Jac80].*

(2) In [Che03] Cherny proved that uniqueness in law together with the existence of strong solution imply the pathwise uniqueness. This is a dual result of the well-known Yamada-Watanabe theorem. The analog result for SPDE in Banach space was established by Ondreját in [Ond04].

1.3.2 Weak solution vs. Martingale solution

In finite dimensional space (e.g. $\mathbb{R}^d$), weak solution is equivalent to martingale solution due to the classical martingale representation theorem. Various classes of SDE, where strong solutions do not exist or the existence is very difficult to prove, can be handled by using the martingale problem approach. For example, S(P)DE with non-smooth coefficients arising in physics and other sciences such as stochastic hydrodynamic equations [GLP99], stochastic quantization equations in quantum field theory [JLM85].

The idea of martingale problem approach can be traced back to Doob [Doo53]. Stroock and Varadhan were the first to give the general concept of the martingale problem in finite dimensional space and developed the related techniques comprehensively in [SV79]. Skorohod also introduced another approach to the weak solution of ordinary SDE [Sko65], see also [EK85, IW81, ZK74] for more references therein.

The martingale problem approach was applied to infinite dimensional systems, in particular, to many important classes of nonlinear SPDE first by Viot [Vio76]. Further developments are due to Grigelionis, Mikulevicius, Kozlov, Kunita, Metivier, Mikulevicius, Rozovskii and many others [Kun97, MR99, GRZ08].

Concerning the equivalence between weak and martingale solution in infinite dimensional spaces, the situation becomes quite complicated because there exist various generalizations of the martingale representation theorem under different (incomparable) assumptions in infinite dimensional space. One may refer to the following references, where the infinite dimensional martingale representation theorem was established under different assumptions within different frameworks.

• Hilbert spaces: Lepingle-Ouvrard[LO73]; Ouvrard[Ouv75]; Da Prato-Zabczyk[DPZ92c];

• Complete nuclear spaces: Körezlioglu-Martias[KM88];

• Banach spaces: Dettweiler [Det90], Ondreját [Ond05];

• Topological vector spaces: Mikulevicius-Rozovskii [MR99].

Chapter 2

Freidlin-Wentzell Large Deviations for Stochastic Evolution Equations

In this chapter the Freidlin-Wentzell large deviation principle is established for the distributions of the solutions to general stochastic evolution equations with small noise. In the first section we give a short introduction to the weak convergence approach, which has been used in the proof of the LDP for general SEE. Then we formulate the main results on the LDP and the proof is divided into several steps in section 2. In the last section the main results are applied to derive the LDP for stochastic reaction-diffusion equations, stochastic porous media equations, stochastic fast diffusion equations and the stochastic p-Laplace equation in Hilbert space. The main results of this chapter have already been submitted for publication, see [Liu08c].

2.1 Introduction to weak convergence approach

Large deviations was used for the asymptotic computation of small probability events on an exponential scale. A precise calculation of the probabilities of such events turns out to be crucial in the study of many problems. For instance, it plays a key role in the study of integrals of exponential functionals of sums of random variables, which come up in probability theory, statistics, information theory, statistical mechanics and financial mathematics etc. Now let us first recall some standard definitions and results from the large deviation theory. Suppose $\{X^\varepsilon\}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in some Polish space E.

Definition 2.1.1 *(Rate function) A function $I : E \to [0, +\infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in E : I(x) \le K\}$ is compact for each $K < \infty$.*

Definition 2.1.2 *(Large deviation principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of E*

$$-\inf_{x\in A^o} I(x) \le \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}(X^\varepsilon \in A) \le \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}(X^\varepsilon \in A) \le -\inf_{x\in \bar{A}} I(x),$$

where A^o and $\bar{A}$ are respectively the interior and the closure of A in E.

The starting point of the weak convergence approach is the equivalence between the large deviation principle and the Laplace principle (LP) if E is a Polish space and the rate function is good.

Definition 2.1.3 *(Laplace principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the Laplace principle with rate function I if for each real-valued bounded continuous function h defined on E,*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{E}\left\{\exp\left[-\frac{1}{\varepsilon^2} h(X^\varepsilon)\right]\right\} = -\inf_{x\in E} \{h(x) + I(x)\}.$$

This equivalence was first formulated in [Puk93] and it is essentially a consequence of Varadhan's lemma [Var66] and Bryc's converse theorem [Bry90]. We refer to [DE97, DZ00] for an elementary proof of it.

Let $\{W_t\}_{t\ge 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ (*i.e.* the path of W take values in $C([0,T]; U_1)$, where U_1 is another Hilbert space such that the embedding $U \subset U_1$ is Hilbert-Schmidt). Suppose $g^\varepsilon : C([0,T]; U_1) \to E$ is a measurable map and $X^\varepsilon = g^\varepsilon(W_\cdot)$. Let

$$\mathcal{A} = \left\{v : v \text{ is } U\text{-valued } \mathcal{F}_t\text{-predictable process s.t. } \int_0^T \|v_s(\omega)\|_U^2 ds < \infty \ a.s.\right\},$$

$$S_N = \left\{ \phi \in L^2([0,T],U) : \int_0^T \|\phi_s\|_U^2 \mathrm{d}s \le N \right\}.$$

The set S_N endowed with the weak topology is a Polish space (we will always refer to the weak topology on S_N if we do not state it explicitly). Define

$$\mathcal{A}_N = \{ v \in \mathcal{A} : v(\omega) \in S_N, \mathbf{P} - a.s. \}.$$

Then the crucial step in the proof of the Laplace principle is based on the following variational representation formula obtained in [BD00]:

$$-\log \mathbf{E} \exp \{-f(W)\} = \inf_{v \in \mathcal{A}} \mathbf{E} \left(\frac{1}{2} \int_0^T \|v_s\|_U^2 \mathrm{d}s + f\left(W_\cdot + \int_0^\cdot v_s \mathrm{d}s \right) \right), \tag{2.1.1}$$

where f is any bounded Borel measurable function from $C([0,T];U_1)$ to $\mathbb{R}$. The connection between exponential functionals and variational representations appeared to be first exploited by Fleming in [Fle78]. The formula (2.1.1) for finite dimensional Brownian motion case was obtained in [BD98]. Now we formulate the following sufficient condition established in [BD00] for the Laplace principle (equivalently, large deviation principle) of $\{X^\varepsilon\}$ as $\varepsilon \to 0$.

(A) There exists a measurable map $g^0 : C([0,T];U_1) \to E$ such that the following two conditions hold:

(i) Let $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. If v^ε converges to v in distribution as S_N-valued random elements, then

$$g^\varepsilon \left(W_\cdot + \frac{1}{\varepsilon} \int_0^\cdot v_s^\varepsilon \mathrm{d}s \right) \to g^0 \left(\int_0^\cdot v_s \mathrm{d}s \right)$$

in distribution as $\varepsilon \to 0$.

(ii) For each $N < \infty$, the set

$$K_N = \left\{ g^0 \left(\int_0^\cdot \phi_s \mathrm{d}s \right) : \phi \in S_N \right\}$$

is a compact subset of E.

Lemma 2.1.1 ([BD00] Theorem 4.4) *If $\{g^\varepsilon\}$ satisfies $(\mathbf{A})$, then the family $\{X^\varepsilon\}$ satisfies the Laplace principle (hence large deviation principle) on E with the good rate function I given by*

$$I(f) = \inf_{\left\{\phi \in L^2([0,T];U):\ f = g^0(\int_0^\cdot \phi_s \mathrm{d}s)\right\}} \left\{ \frac{1}{2} \int_0^T \|\phi(s)\|_U^2 \mathrm{d}s \right\}, \ f \in E. \tag{2.1.2}$$

Therefore, in order to establish the LDP one only needs to verify the (weak convergence) assumption **(A)**. The main advantage of the weak convergence approach is that one can avoid some exponential probability estimates, which may be very difficult to derive for infinite dimensional models. In recent years, this approach has been used to study the large deviations for homeomorphism flows of non-Lipschitz SDE by Ren and Zhang in [RZ05a], for two-dimensional stochastic Navier-Stokes equations by Sritharan and Sundar in [SS06] and reaction-diffusion type SPDE by Budhiraja *et al* in [BDM08]. For more references on this approach we may refer to [DE97, RZ05b, DM].

2.2 Freidlin-Wentzell large deviation principle: the main results

Let

$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and V is a reflexive separable Banach space such that $V \subset H$ is continuous and dense. The dualization between V^* and V is denoted by ${}_{V^*}\langle \cdot, \cdot \rangle_V$. Let $\{W_t\}_{t\geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $(L_2(U; H), \|\cdot\|_2)$ denotes the space of all Hilbert-Schmidt operators from U to H. We use $L(X, Y)$ to denote the space of all bounded linear operators from space X to Y.

Consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{2.2.1}$$

where $A : [0, T] \times V \to V^*$ and $B : [0, T] \times V \to L_2(U; H)$ are measurable. For the large deviation principle we need to assume the following conditions on A and B.

For a fixed $\alpha > 1$, there exist constants $\delta > 0$ and K such that the following conditions hold for all $v, v_1, v_2 \in V$ and $t \in [0, T]$.

$(A1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(A2)$ (Strong monotonicity)

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \leq -\delta\|v_1 - v_2\|_V^\alpha + K\|v_1 - v_2\|_H^2.$$

$(A3)$ (Boundedness)

$$\sup_{t\in[0,T]} \|B(t,0)\|_2 < \infty;\ \|A(t,v)\|_{V^*} + \|B(t,v)\|_{L(U,V^*)} \le K(1+\|v\|_V^{\alpha-1}).$$

$(A4)$ Suppose there exists a sequence of subspaces $\{H_n\}$ of H such that

$$H_n \subseteq H_{n+1},\ H_n \hookrightarrow V \text{ is compact and } \bigcup_{n=1}^{\infty} H_n \subseteq H \text{ is dense},$$

and for any $M > 0$,

$$\sup_{(t,v)\in[0,T]\times S_M} \|P_n B(t,v) - B(t,v)\|_2 \to 0\ (n \to \infty), \tag{2.2.2}$$

where $P_n : H \to H_n$ is the projection operator and $S_M = \{v \in V : \|v\|_H \le M\}$.

Remark 2.2.1 *(1) By $(A2)$ and $(A3)$ we can obtain the coercivity and boundedness of A and B:*

$$2_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|_2^2 + \frac{\delta}{2}\|v\|_V^\alpha \le C(1+\|v\|_H^2),$$
$$\|B(t,v)\|_2^2 \le C(1+\|v\|_H^2 + \|v\|_V^\alpha).$$

Hence the boundedness of B in $(A3)$ automatically holds in the case $\alpha \ge 2$. If $1 < \alpha < 2$, the additional assumption on B in $(A3)$ is assumed for the well-posedness of the skeleton equation (2.2.5). It is easy to see from the proof that we can also replace the assumption on B in the case $1 < \alpha < 2$ by the following one

$$\|B(t,v)\|_{L(U,V^*)} \le K(1+\|v\|_H).$$

(2) Since for any $(t,v) \in [0,T] \times V$ we have

$$\|P_n B(t,v) - B(t,v)\|_2 \to 0\ (n \to \infty),$$

(2.2.2) obviously holds if $\{B(t,v) : (t,v) \in [0,T] \times S_M\}$ is a relatively compact set in $L_2(U;H)$. One simple example is

$$B(t,v) = \sum_{i=1}^{N} b_i(v) B_i(t),$$

where $b_i(\cdot) : V \to \mathbb{R}$ are Lipschitz functions and $B_i(\cdot) : [0,T] \to L_2(U;H)$ are continuous.

Another simple example for (2.2.2) holds is $B(t,v) = QB_0(t,v)$ where $Q \in L_2(H;H)$ and

$$B_0 : [0,T] \times V \to L(U;H) \text{ and } \sup_{(t,v)\in[0,T]\times S_M} \|B_0(t,v)\|_{L(U;H)} < \infty, \ \forall M > 0.$$

(3) Suppose there exists a Hilbert space H_0 such that the embedding $H_0 \subseteq H$ is compact, and there also exists $\{e_i\} \subseteq H_0 \cap V$ is an ONB in H_0 and orthogonal in H. If for all $M > 0$

$$\sup_{(t,v)\in[0,T]\times S_M} \|B(t,v)\|_{L_2(U;H_0)} < \infty,$$

then (2.2.2) holds. In fact $B(t,v) = \sum_{i,j=1}^{\infty} b_{i,j}(t,v) u_i \otimes e_j$, then by the assumption we know $\|e_j\|_H^2 \to 0$ and

$$\sup_{(t,v)\in[0,T]\times S_M} \sum_{i,j=1}^{\infty} b_{i,j}^2(t,v) < \infty.$$

Hence

$$\|P_n B(t,v) - B(t,v)\|_2^2 = \sum_{i=1}^{\infty} \sum_{j=n+1}^{\infty} b_{i,j}^2(t,v)\|e_j\|_H^2.$$

Then (2.2.2) follows from the dominated convergence theorem.

If $(A1)-(A3)$ hold, according to Theorem 1.2.1, for any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbf{P})$ (2.2.1) has a unique solution $\{X_t\}_{t\in[0,T]}$ which is an adapted continuous process on H such that $\mathbf{E}\int_0^T (\|X_t\|_V^\alpha + \|X_t\|_H^2)\,\mathrm{d}t < \infty$ and

$$\langle X_t, v\rangle_H = \langle X_0, v\rangle_H + \int_0^t {}_{V^*}\langle A(s,X_s), v\rangle_V \mathrm{d}s + \int_0^t \langle B(s,X_s)\mathrm{d}W_s, v\rangle_H, \ \mathbf{P}-a.s.$$

holds for all $v \in V$ and $t \in [0,T]$. Moreover, we have $\mathbf{E}\sup_{t\in[0,T]} \|X_t\|_H^2 < \infty$ and the crucial Itô formula

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2{}_{V^*}\langle A(s,X_s), X_s\rangle_V + \|B(s,X_s)\|_2^2\right)\mathrm{d}s + 2\int_0^t \langle X_s, B(s,X_s)\mathrm{d}W_s\rangle_H.$$

Now we consider the stochastic evolution equation with small noise:

$$\mathrm{d}X_t^\varepsilon = A(t,X_t^\varepsilon)\mathrm{d}t + \varepsilon B(t,X_t^\varepsilon)\mathrm{d}W_t, \ X_0^\varepsilon = x \in H, \ \varepsilon > 0. \tag{2.2.3}$$

Hence the unique strong solution $\{X^\varepsilon\}$ to (2.2.3) takes values in $C([0,T];H)\cap L^\alpha([0,T];V)$. It is well-known that $(C([0,T];H)\cap L^\alpha([0,T];V),\rho)$ is a Polish space with the metric

$$\rho(f,g) := \sup_{t\in[0,T]} \|f_t - g_t\|_H + \left(\int_0^T \|f_t - g_t\|_V^\alpha \mathrm{d}t\right)^{\frac{1}{\alpha}}. \tag{2.2.4}$$

It follows (from the infinite dimensional Yamada-Watanabe theorem in [RSZ08]) that there exists a Borel-measurable function

$$g^\varepsilon : C([0,T];U_1) \to C([0,T];H)\cap L^\alpha([0,T];V)$$

such that $X^\varepsilon = g^\varepsilon(W)$ *a.s.*. To state our main result, we introduce the following skeleton equation associated to (2.2.3):

$$\frac{\mathrm{d}z_t^\phi}{\mathrm{d}t} = A(t,z_t^\phi) + B(t,z_t^\phi)\phi_t, \quad z_0^\phi = x,\ \phi\in L^2([0,T];U). \tag{2.2.5}$$

An element $z^\phi\in C([0,T];H)\cap L^\alpha([0,T];V)$ is called a solution to (2.2.5) if for any $v\in V$

$$\langle z_t^\phi, v\rangle_H = \langle x,v\rangle_H + \int_0^t {}_{V^*}\langle A(s,z_s^\phi) + B(s,z_s^\phi)\phi_s, v\rangle_V \mathrm{d}s, \quad t\in[0,T]. \tag{2.2.6}$$

We will prove (see Lemma 2.3.1) that $(A1)-(A3)$ also imply the existence and uniqueness of the solution to (2.2.5) for any $\phi\in L^2([0,T];U)$.

Define $g^0 : C([0,T];U_1)\to C([0,T];H)\cap L^\alpha([0,T];V)$ by

$$g^0(h) := \begin{cases} z^\phi, & \text{if } \ h = \int_0^\cdot \phi_s \mathrm{d}s \ \text{ for some } \phi\in L^2([0,T];U); \\ 0, & \text{otherwise.}\end{cases}$$

Then it is obvious that the rate function in (2.1.2) can be written as

$$I(z) = \inf\left\{\frac{1}{2}\int_0^T \|\phi_s\|_U^2 \mathrm{d}s :\ z = z^\phi,\ \phi\in L^2([0,T],U)\right\}, \tag{2.2.7}$$

where $z\in C([0,T];H)\cap L^\alpha([0,T];V)$.

Now we formulate the main result which is the well-known Freidlin-Wentzell type estimate.

Theorem 2.2.1 *Assume $(A1)-(A4)$ hold. For each $\varepsilon>0$, let $X^\varepsilon = \{X_t^\varepsilon\}_{t\in[0,T]}$ be the solution to (2.2.3). Then as $\varepsilon\to 0$, $\{X^\varepsilon\}$ satisfies the LDP on $C([0,T];H)\cap L^\alpha([0,T];V)$ with the good rate function I which is given by (2.2.7).*

Remark 2.2.2 *(1) Note that* $(A4)$ *is assumed for establishing the convergence of* h^ε *(as elements in* $C([0,T];V)$*) in the proof. Hence we can replace* $(A4)$ *by the following simple assumption:*

$(A4')$

$$B : [0,T] \times U \to L_2(U,V); \ \|B(t,v)\|^2_{L_2(U,V)} \le K(1 + \|v\|^\alpha_V + \|v\|^2_H).$$

By using $(A4')$ *one can easily conclude* h^ε *converge to* 0 *in* $C([0,T];V)$*. Then the proof of Theorem* 2.2.1 *will be significantly simplified because we can drop section* 2.3.2 *completely and need not to use the finite dimensional approximation and truncation techniques.*

(2) According to [BDM08, Theorem 5], we can also prove the uniform Laplace principle by using the same arguments but with more involved notations.

(3) This theorem can not be applied to stochastic fast diffusion equations in [LW08, RRW07] since $(A2)$ *fails to hold. However, if we replace* $(A2)$ *by the following monotone and coercive conditions*

$(A2')$ *(Monotonicity and coercivity)*

$$2_{V^*}\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_V + \|B(t,v_1) - B(t,v_2)\|_2^2 \le K\|v_1 - v_2\|^2_H;$$
$$2_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|_2^2 + \delta\|v\|^\alpha_V \le K(1 + \|v\|^2_H).$$

Then the LDP can be established on $C([0,T];H)$ *by a similar argument.*

Theorem 2.2.2 *Assume* $(A1), (A2'), (A3)$ *and* $(A4)$ *hold. Then as* $\varepsilon \to 0$*, the solution* $\{X^\varepsilon\}$ *to* (2.2.3) *satisfies the LDP on* $C([0,T];H)$ *with the good rate function* I *which is given by* (2.2.7)*.*

Remark 2.2.3 *(1) Note that* $(A2)$ *is mainly used to prove the additional convergence in* $L^\alpha([0,T];V)$*. Hence, if we only concern the LDP on* $C([0,T];H)$*, we can prove Theorem* 2.2.2 *under the weaker assumption* $(A2')$*. Since the proof is only a small modification of the argument for Theorem* 2.2.1*, we will omit the details here.*

(2) Recently, I was informed that there are some independent work done by Ren and Zhang [RZ08] where they used some different techniques to establish the LDP for stochastic

evolution equations. Comparing with our result, they assume that B satisfies a Lipschitz condition and V is compactly embedded into H in [RZ08] instead of $(A4)$ *in our assumption. Another difference is the results in [RZ08] only work for the case $\alpha \geq 2$, while our result can also be applied to some examples with $\alpha < 2$, e.g. stochastic fast-diffusion equations and the singular p-Laplace equation (see Example 2.4.4 and Remark 2.4.4).*

The proof of the main theorem is divided into several steps. In the next section, we first prove Theorem 2.2.1 by using the weak convergence approach under additional assumption $(A5)$ on B. Afterwards, the assumption $(A5)$ can be relaxed to $(A4)$ by using some standard approximation techniques.

2.3 Proof of the large deviation principle

2.3.1 Proof of the main theorem under (A5)

In order to verify the sufficient conditions **(A)**, we need to first consider the equation (2.2.3) with finite dimensional noise, *i.e.* we approximate the diffusion coefficient B by $P_n B$. But for the simplicity of notations, we assume the following additional condition on B:

$(A5)$ $B : [0, T] \times V \to L(U; V_0)$ satisfies

$$\|B(t, v)\|^2_{L(U;V_0)} \leq C(1 + \|v\|^\alpha_V + \|v\|^2_H),$$

where $V_0 \subseteq V$ is a compact embedding and C is a constant.

For the reader's convenience, we recall two well-known inequalities which are used quite often in the proof. Throughout the paper, generic constants may change from line to line. If it is essential, we will write the dependence of the constant on parameters explicitly.

Young's inequality: If $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then for any positive number σ, a and b we have

$$ab \leq \sigma \frac{a^p}{p} + \sigma^{-\frac{q}{p}} \frac{b^q}{q}.$$

Gronwall's lemma: Let $F, \Phi, \Psi : [0,T] \to \mathbb{R}^+$ be Lebesgue measurable and Ψ be locally integrable such that $\int_0^T \Psi(s)F(s)\mathrm{d}s < \infty$. If

$$\begin{aligned} F(t) &\leq \Phi(t) + \int_0^t \Psi(s)F(s)\mathrm{d}s,\ t \in [0,T] \ \text{ or} \\ F'(t) &\leq \Phi'(t) + \Psi(t)F(t),\ t \in [0,T),\ F(0) \leq \Phi(0), \end{aligned} \tag{2.3.1}$$

then we have

$$\begin{aligned} F(t) &\leq \Phi(t) + \int_0^t \exp\left[\int_s^t \Psi(u)\mathrm{d}u\right] \Psi(s)\Phi(s)\mathrm{d}s \\ &\leq \exp\left[\int_0^t \Psi(u)\mathrm{d}u\right]\left(\Phi(0) + \int_0^t \Phi'(s)\exp\left[-\int_0^s \Psi(u)\mathrm{d}u\right]\mathrm{d}s\right),\ t \in [0,T]. \end{aligned} \tag{2.3.2}$$

Lemma 2.3.1 *Assume* $(A1) - (A3)$ *hold and*

$$\|z\| := \sup_{t\in[0,T]} \|z_t\|_H^2 + \delta \int_0^T \|z_t\|_V^\alpha \mathrm{d}t$$

for $z \in C([0,T];H) \cap L^\alpha([0,T];V)$*. Then for any* $x \in H$ *and* $\phi \in L^2([0,T];U)$ *there exists a unique solution* z^ϕ *to* (2.2.5) *and for any* $\phi, \psi \in L^2([0,T];U)$

$$\|z^\phi - z^\psi\| \leq \exp\left\{\int_0^T \left(K + \|\phi_t\|_U^2 + \|B(t,z_t^\psi)\|_2^2\right)\mathrm{d}t\right\}\int_0^T \|\phi_t - \psi_t\|_U^2 \mathrm{d}t, \tag{2.3.3}$$

where K *is a constant.*

Proof. For the existence of the solution to (2.2.5), we only need to verify the assumptions in Theorem 1.2.1. First we assume $\phi \in L^\infty([0,T];U)$ and

$$\tilde{A}(s,v) := A(s,v) + B(s,v)\phi_s.$$

Then, due to $(A1) - (A3)$, it is easy to verify that $(H1) - (H4)$ in Theorem 1.2.1 hold.

(i) Hemicontinuity of $\tilde{A}$ follow from $(A1)$ and $(A2)$.

(ii) Monotonicity and coercivity of $\tilde{A}$ follow from $(A2)$ and $(A3)$.

(iii) Boundedness of $\tilde{A}$ follows from $(A3)$.

Therefore, we know (2.2.5) has a unique solution.

For general $\phi \in L^2([0,T];U)$, we can find a sequence of $\phi^n \in L^\infty([0,T];U)$ such that

$$\phi_n \to \phi \ \text{ strongly in } \ L^2([0,T];U).$$

Let z^n be the unique solution to (2.2.5) corresponding to ϕ^n. We will show $\{z^n\}$ is a Cauchy sequence in $C([0,T];H) \cap L^\alpha([0,T];V)$. By using $(A2)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_t^n - z_t^m\|_H^2 =& 2{}_{V^*}\langle A(t,z_t^n) - A(t,z_t^m), z_t^n - z_t^m\rangle_V \\
&+ 2\langle B(t,z_t^n)\phi_t^n - B(t,z_t^m)\phi_t^m, z_t^n - z_t^m\rangle_H \\
\le& 2{}_{V^*}\langle A(t,z_t^n) - A(t,z_t^m), z_t^n - z_t^m\rangle_V + \|B(t,z_t^n) - B(t,z_t^m)\|_2^2 \\
&+ \|\phi_t^n\|_U^2\|z_t^n - z_t^m\|_H^2 + 2\langle z_t^n - z_t^m, B(t,z_t^m)\phi_t^n - B(t,z_t^m)\phi_t^m\rangle_H \\
\le& -\delta\|z_t^n - z_t^m\|_V^\alpha + (K + \|\phi_t^n\|_U^2)\|z_t^n - z_t^m\|_H^2 \\
&+ 2\|B^*(t,z_t^m)\left(z_t^n - z_t^m\right)\|_U\|\phi_t^n - \phi_t^m\|_U \\
\le& -\delta\|z_t^n - z_t^m\|_V^\alpha + \|\phi_t^n - \phi_t^m\|_U^2 \\
&+ \left(K + \|\phi_t^n\|_U^2 + \|B(t,z_t^m)\|_2^2\right)\|z_t^n - z_t^m\|_H^2,
\end{aligned} \tag{2.3.4}$$

where B^* denotes the adjoint operator of B and we also use the fact

$$\|B^*\|_{L(H;U)} = \|B\|_{L(U;H)} \le \|B\|_2.$$

Then by Gronwall's lemma we have

$$\|z^n - z^m\| \le \exp\left\{\int_0^T \left(K + \|\phi_t^n\|_U^2 + \|B(t,z_t^m)\|_2^2\right)\mathrm{d}t\right\}\int_0^T \|\phi_t^n - \phi_t^m\|_U^2\mathrm{d}t. \tag{2.3.5}$$

By a similar argument we arrive that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_t^n\|_H^2 =& 2{}_{V^*}\langle A(t,z_t^n), z_t^n\rangle_V + 2\langle B(t,z_t^n)\phi_t^n, z_t^n\rangle_H \\
\le& -\frac{\delta}{2}\|z_t^n\|_V^\alpha + C(1 + \|z_t^n\|_H^2) + \|\phi_t^n\|_U^2\|z_t^n\|_H^2.
\end{aligned} \tag{2.3.6}$$

Then by Gronwall's lemma and boundedness of ϕ^n in $L^2([0,T];U)$ we have

$$\|z^n\| \le C\exp\left\{\int_0^T \left(C + \|\phi_t^n\|_U^2\right)\mathrm{d}t\right\}\left(\|x\|_H^2 + T\right) \le \textbf{Constant} < \infty. \tag{2.3.7}$$

Therefore,

$$\int_0^T \|B(t,z_t^m)\|_2^2\mathrm{d}t \le C\int_0^T \left(1 + \|z_t^m\|_H^2 + \|z_t^m\|_V^\alpha\right)\mathrm{d}t \le \textbf{Constant} < \infty. \tag{2.3.8}$$

Combining (2.3.5),(2.3.8) and $\phi^n \to \phi$ we can conclude that $\{z^n\}$ is a Cauchy sequence in $C([0,T];H) \cap L^\alpha([0,T];V)$, and we denote the limit by z^ϕ.

Then by repeating the standard monotonicity argument (e.g.[Zei90, Theorem 30.A]) one can show that z^ϕ is the solution to (2.2.5) corresponding to ϕ. And (2.3.3) can be derived from (2.3.5).

Now the proof is complete. ∎

The following result shows that I defined by (2.2.7) is a good rate function.

Lemma 2.3.2 *Assume* $(A1)-(A3)$ *hold. For every* $N<\infty$*, the set*

$$K_N = \left\{ g^0\left(\int_0^\cdot \phi_s \mathrm{d}s\right) : \phi \in S_N \right\}$$

is a compact subset in $C([0,T];H) \cap L^\alpha([0,T];V)$*.*

Proof. Step 1: we first assume B also satisfies $(A5)$. By definition we know

$$K_N = \left\{ z^\phi : \phi \in L^2([0,T];U),\ \int_0^T \|\phi_s\|_U^2 \mathrm{d}s \le N \right\}.$$

For any sequence $\phi^n \subset S_N$, we may assume $\phi^n \to \phi$ weakly in $L^2([0,T];U)$ since S_N is weakly compact. Denote z^n and z are the solutions to (2.2.5) corresponding to ϕ^n and ϕ respectively. Now it is sufficient to show $z^n \to z$ strongly in $C([0,T];H) \cap L^\alpha([0,T];V)$.

From (2.3.4) we have

$$\begin{aligned} &\|z_t^n - z_t\|_H^2 + \delta \int_0^t \|z_s^n - z_s\|_V^\alpha \mathrm{d}s \\ &\le \int_0^t (K + \|\phi_s^n\|_U^2)\|z_s^n - z_s\|_H^2 \mathrm{d}s + 2\int_0^t \langle z_s^n - z_s, B(s,z_s)(\phi_s^n - \phi_s)\rangle_H \mathrm{d}s. \end{aligned} \tag{2.3.9}$$

Define

$$h_t^n = \int_0^t B(s,z_s)(\phi_s^n - \phi_s)\mathrm{d}s.$$

By $(A5)$ and (2.3.8) we know $h^n \in C([0,T];V_0)$ and

$$\begin{aligned} \sup_{t\in[0,T]} \|h_t^n\|_{V_0} &\le \int_0^T \|B(s,z_s)(\phi_s^n - \phi_s)\|_{V_0} \mathrm{d}s \\ &\le \left(\int_0^T \|B(s,z_s)\|_{L(U,V_0)}^2 \mathrm{d}s\right)^{1/2} \left(\int_0^T \|\phi_s^n - \phi_s\|_U^2 \mathrm{d}s\right)^{1/2} \\ &\le \mathbf{Constant} < \infty. \end{aligned} \tag{2.3.10}$$

Since the embedding $V_0 \subseteq V$ is compact and $\phi^n \to \phi$ weakly in $L^2([0,T];U)$, it is easy to show that $h^n \to 0$ in $C([0,T];V)$ by using the Arzèla-Ascoli theorem (see e.g. [BD00, Lemma 3.2]) (more precisely, this convergence may only hold for a subsequence, but it is enough for our purpose since we may denote the convergent subsequence still by h^n). In particular, $h^n \to 0$ in $C([0,T];H) \cap L^\alpha([0,T];V)$.

Moreover the derivative (w.r.t. time variable) is given by

$$(h_s^n)' = B(s,z_s)(\phi_s^n - \phi_s).$$

As in Lemma 2.3.1, we may assume $\phi^n, \phi \in L^\infty([0,T];U)$ first. Then by $(A3)$

$$\begin{aligned}\int_0^T \|(h_s^n)'\|_{V^*}^{\frac{\alpha}{\alpha-1}} \mathrm{d}s &\le \int_0^T \|B(s,z_s)(\phi_s^n - \phi_s)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \mathrm{d}s \\ &\le C \int_0^T (1 + \|z_s\|_V^\alpha)\, \mathrm{d}s \\ &\le \mathbf{Constant} < \infty.\end{aligned} \tag{2.3.11}$$

Hence $(h_\cdot^n)'$ is an element in $L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)$.

By [Zei90, Proposition 23.23] we have the following integration by parts formula

$$\langle z_t^n - z_t, h_t^n\rangle_H = \int_0^t {}_{V^*}\langle (z_s^n - z_s)', h_s^n\rangle_V \mathrm{d}s + \int_0^t {}_{V^*}\langle (h_s^n)', z_s^n - z_s\rangle_V \mathrm{d}s.$$

Hence one has

$$\begin{aligned}&\int_0^t \langle z_s^n - z_s, B(s,z_s)(\phi_s^n - \phi_s)\rangle_H \mathrm{d}s \\ =&\langle z_t^n - z_t, h_t^n\rangle_H - \int_0^t {}_{V^*}\langle (z_s^n - z_s)', h_s^n\rangle_V \mathrm{d}s \\ =&\langle z_t^n - z_t, h_t^n\rangle_H - \int_0^t {}_{V^*}\langle A(s,z_s^n) - A(s,z_s), h_s^n\rangle_V \mathrm{d}s \\ &\quad - \int_0^t \langle B(s,z_s^n)\phi_s^n - B(s,z_s)\phi_s, h_s^n\rangle_H \mathrm{d}s \\ =:&I_1 + I_2 + I_3.\end{aligned} \tag{2.3.12}$$

By using the Hölder inequality, $(A3)$ and (2.3.7) we have

$$
\begin{aligned}
I_1 &\le \|z_t^n - z_t\|_H \cdot \|h_t^n\|_H \le \frac{1}{4}\|z_t^n - z_t\|_H^2 + \|h_t^n\|_H^2,\\
I_2 &\le \int_0^t \|A(s, z_s^n) - A(s, z_s)\|_{V^*} \|h_s^n\|_V \mathrm{d}s\\
&\le \left(\int_0^t \|A(s, z_s^n) - A(s, z_s)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \left(\int_0^t \|h_s^n\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}}\\
&\le \left(\int_0^t C\left(1 + \|z_s\|_V^\alpha + \|z_s^n\|_V^\alpha\right) \mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \left(\int_0^t \|h_s^n\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}}\\
&\le C\left(\int_0^t \|h_s^n\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}},\\
I_3 &\le \int_0^t \|B(s, z_s^n)\phi_s^n - B(s, z_s)\phi_s\|_H \cdot \|h_s^n\|_H \mathrm{d}s\\
&\le \sup_{s\in[0,t]} \|h_s^n\|_H \int_0^t \|B(s, z_s^n)\phi_s^n - B(s, z_s)\phi_s\|_H \mathrm{d}s\\
&\le \sup_{s\in[0,t]} \|h_s^n\|_H \left\{N^{1/2}\left(\int_0^t \|B(s, z_s^n)\|_2^2 \mathrm{d}s\right)^{1/2} + N^{1/2}\left(\int_0^t \|B(s, z_s)\|_2^2 \mathrm{d}s\right)^{1/2}\right\}\\
&\le C \sup_{s\in[0,t]} \|h_s^n\|_H,
\end{aligned}
\tag{2.3.13}
$$

where C is a constant (changing from line to line) and we use the following estimate

$$
\int_0^t \|B(s, z_s^n)\|_2^2 \mathrm{d}s \le C\int_0^t \left(1 + \|z_s^n\|_H^2 + \|z_s^n\|_V^\alpha\right) \mathrm{d}s \le \textbf{Constant} < \infty.
$$

Combining (2.3.9) and (2.3.12)-(2.3.13) we have

$$
\begin{aligned}
&\|z_t^n - z_t\|_H^2 + \delta\int_0^t \|z_s^n - z_s\|_V^\alpha \mathrm{d}s\\
&\le C\int_0^t (1 + \|\phi_s^n\|_U^2)\|z_s^n - z_s\|_H^2 \mathrm{d}s + C\left(\sup_{s\in[0,t]} \|h_s^n\|_H + \sup_{s\in[0,t]} \|h_s^n\|_H^2 + \left(\int_0^t \|h_s^n\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}}\right).
\end{aligned}
\tag{2.3.14}
$$

Then by Gronwall's lemma and L^2-boundedness of ϕ^n, there exists a constant C such that

$$
\|z^n - z\| \le C\left(\sup_{s\in[0,T]} \|h_s^n\|_H + \sup_{s\in[0,T]} \|h_s^n\|_H^2 + \left(\int_0^T \|h_s^n\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}}\right).
$$

Since $h^n \to 0$ in $C([0,T];H) \cap L^\alpha([0,T];V)$, we know $z^n \to z$ strongly in $C([0,T];H) \cap L^\alpha([0,T];V)$ as $n \to \infty$.

Since Lemma 2.3.1 shows that the convergence of the corresponding solution z^ϕ is uniform on S_N w.r.t. the approximation on ϕ, the conclusion in the case $\phi^n, \phi \in L^2([0,T];U)$ can de derived by the proof above and a standard 3ε-argument.

Step 2: Now we prove the conclusion for general B without assuming $(A5)$. Denote $z_{t,n}^\phi$ the solution to the following equation

$$\frac{\mathrm{d}z_{t,n}^\phi}{\mathrm{d}t} = A(t, z_{t,n}^\phi) + P_n B(t, z_{t,n}^\phi)\phi_t, \quad z_{0,n}^\phi = x,$$

where P_n is the standard finite dimensional projection (see $(A4)$ and section 4 for details). By using the same argument in Lemma 2.3.1 we can prove

$$\begin{aligned} &\sup_{t\in[0,T]} \|z_n^\phi - z^\phi\|_H^2 + \delta \int_0^T \|z_{s,n}^\phi - z_s^\phi\|_V^\alpha \mathrm{d}s \\ &\leq \exp\left\{\int_0^T (K + 2\|\phi_s\|_U^2)\mathrm{d}s\right\} \int_0^T \|(I - P_n)B(s, z_s^\phi)\|_2^2 \mathrm{d}s. \end{aligned} \tag{2.3.15}$$

Since $B(\cdot,\cdot)$ are Hilbert-Schmidt (hence compact) operators, then by the dominated convergence theorem we know

$$\int_0^T \|(I - P_n)B(s, z_s^\phi)\|_2^2 \mathrm{d}s \to 0 \text{ as } n \to \infty.$$

Hence $z_n^\phi \to z^\phi$ in $C([0,T];H) \cap L^\alpha([0,T];V)$ as $n \to \infty$. Moreover, this convergence is uniform (w.r.t ϕ) on bounded set of $L^2([0,T];U)$, which follows from (2.3.15) and (2.3.8).

Note that $P_n B$ satisfies $(A5)$, by combining **Step 1** with a standard 3ε-argument we can conclude that $z^n \to z$ strongly in $C([0,T];H) \cap L^\alpha([0,T];V)$ for general B.

Now the proof is complete. ∎

Lemma 2.3.3 *Assume $(A1)-(A3)$ and $(A5)$ hold. Let $\{v^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$ for some $N < \infty$. Assume v^ε converges to v in distribution as S_N-valued random elements, then*

$$g^\varepsilon\left(W_\cdot + \frac{1}{\varepsilon}\int_0^\cdot v_s^\varepsilon \mathrm{d}s\right) \to g^0\left(\int_0^\cdot v_s \mathrm{d}s\right)$$

in distribution as $\varepsilon \to 0$.

Proof. By the Girsanov theorem and uniqueness of solution to (2.2.3), it is easy to see that $X^\varepsilon := g^\varepsilon\left(W_\cdot + \frac{1}{\varepsilon}\int_0^\cdot v_s^\varepsilon \mathrm{d}s\right)$ (the abuse of notation here is for simplicity) is the unique solution to the following equation

$$\mathrm{d}X_t^\varepsilon = \left(A(t,X_t^\varepsilon) + B(t,X_t^\varepsilon)v_t^\varepsilon\right)\mathrm{d}t + \varepsilon B(t,X_t^\varepsilon)\mathrm{d}W_t,\ X_0^\varepsilon = x. \tag{2.3.16}$$

Now we only need to show $X^\varepsilon \to z^v$ in distribution as $\varepsilon \to 0$. We may assume $\varepsilon \le \frac{1}{2}$, by using Itô's formula, Young's inequality and $(A2)$ we have

$$\begin{aligned}
\mathrm{d}\|X_t^\varepsilon - z_t^v\|_H^2 =& 2{}_{V^*}\langle A(t,X_t^\varepsilon) - A(t,z_t^v), X_t^\varepsilon - z_t^v\rangle_V \mathrm{d}t \\
&+ 2\langle X_t^\varepsilon - z_t^v, (B(t,X_t^\varepsilon) - B(t,z_t^v))v_t^\varepsilon + B(t,z_t^v)(v_t^\varepsilon - v_t)\rangle_H \mathrm{d}t \\
&+ \varepsilon^2\|B(t,X_t^\varepsilon)\|_2^2\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon - z_t^v, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H \\
\le& \left(2{}_{V^*}\langle A(t,X_t^\varepsilon) - A(t,z_t^v), X_t^\varepsilon - z_t^v\rangle_V + \|B(t,X_t^\varepsilon) - B(t,z_t^v)\|_2^2\right)\mathrm{d}t \\
&+ 2\|v_t^\varepsilon\|_U^2\|X_t^\varepsilon - z_t^v\|_H^2\mathrm{d}t + 2\langle X_t^\varepsilon - z_t^v, B(t,z_t^v)(v_t^\varepsilon - v_t)\rangle_H\mathrm{d}t \\
&+ 2\varepsilon^2\|B(t,z_t^v)\|_2^2\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon - z_t^v, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H \\
\le& \left[-\delta\|X_t^\varepsilon - z_t^v\|_V^\alpha + C(1+\|v_t^\varepsilon\|_U^2)\|X_t^\varepsilon - z_t^v\|_H^2 + 2\varepsilon^2\|B(t,z_t^v)\|_2^2\right]\mathrm{d}t \\
&+ 2\langle X_t^\varepsilon - z_t^v, B(t,z_t^v)(v_t^\varepsilon - v_t)\rangle_H\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon - z_t^v, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H.
\end{aligned} \tag{2.3.17}$$

Similarly we define

$$h_t^\varepsilon = \int_0^t B(s,z_s^v)(v_s^\varepsilon - v_s)\mathrm{d}s,$$

then we know that $h^\varepsilon \to 0$ in distribution as $C([0,T];V)$-valued random elements, consequently also in $C([0,T];H)\cap L^\alpha([0,T];V)$. Note that

$$2\langle X_t^\varepsilon - z_t^v, h_t^\varepsilon\rangle_H = \|X_t^\varepsilon - z_t^v + h_t^\varepsilon\|_H^2 - \|X_t^\varepsilon - z_t^v\|_H^2 - \|h_t^\varepsilon\|_H^2.$$

By using Itô's formula for corresponding square norm we can derive that

$$\begin{aligned}
&\int_0^t \langle X_s^\varepsilon - z_s^v, B(s,z_s^v)(v_s^\varepsilon - v_s)\rangle_H\mathrm{d}s \\
=&\langle X_t^\varepsilon - z_t^v, h_t^\varepsilon\rangle_H - \int_0^t {}_{V^*}\langle A(s,X_s^\varepsilon) - A(s,z_s^v), h_s^\varepsilon\rangle_V\mathrm{d}s \\
&- \int_0^t \langle B(s,X_s^\varepsilon)v_s^\varepsilon - B(s,z_s^v)v_s, h_s^\varepsilon\rangle_H\mathrm{d}s - \varepsilon\int_0^t\langle B(s,X_s^\varepsilon)\mathrm{d}W_s, h_s^\varepsilon\rangle_H.
\end{aligned} \tag{2.3.18}$$

By using a same argument as in (2.3.13) we obtain

$$
\begin{aligned}
&\int_0^t \langle X_s^\varepsilon - z_s^v, B(s,z_s^v)(v_s^\varepsilon - v_s)\rangle_H \mathrm{d}s \\
\leq & \frac{1}{4}\|X_t^\varepsilon - z_t^v\|_H^2 + \sup_{s\in[0,t]}\|h_s^\varepsilon\|_H^2 - \varepsilon\int_0^t \langle B(s,X_s^\varepsilon)\mathrm{d}W_s, h_s^\varepsilon\rangle_H \\
&+ C\left(\int_0^t (1+\|z_s^v\|_V^\alpha + \|X_s^\varepsilon\|_V^\alpha)\,\mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \cdot \left(\int_0^t \|h_s^\varepsilon\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} \\
&+ C\sup_{s\in[0,t]}\|h_s^\varepsilon\|_H\left\{\left(\int_0^t \|B(s,X_s^\varepsilon)\|_2^2\mathrm{d}s\right)^{1/2} + \left(\int_0^t \|B(s,z_s^v)\|_2^2\mathrm{d}s\right)^{1/2}\right\}.
\end{aligned}
\tag{2.3.19}
$$

Hence from (2.3.17)-(2.3.19) we have

$$
\begin{aligned}
&\|X_t^\varepsilon - z_t^v\|_H^2 + \delta\int_0^t \|X_t^\varepsilon - z_t^v\|_V^\alpha \mathrm{d}s \\
&\leq c_1\int_0^t (1+\|v_s^\varepsilon\|_U^2)\|X_s^\varepsilon - z_s^v\|_H^2\mathrm{d}s + c_2(\varepsilon^2 + \sup_{s\in[0,t]}\|h_s^\varepsilon\|_H^2) \\
&+ c_3\left(1+\int_0^t \|X_s^\varepsilon\|_V^\alpha \mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \cdot \left(\int_0^t \|h_s^\varepsilon\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} \\
&+ c_4\sup_{s\in[0,t]}\|h_s^\varepsilon\|_H\left\{1+\left(\int_0^t \|X_s^\varepsilon\|_H^2\mathrm{d}s\right)^{1/2}\right\} \\
&+ 4\varepsilon\int_0^t \langle X_s^\varepsilon - z_s^v - h_s^\varepsilon, B(s,X_s^\varepsilon)\mathrm{d}W_s\rangle_H,
\end{aligned}
\tag{2.3.20}
$$

where we used the estimate (see (2.3.6)-(2.3.8)) that there exists a constant C such that

$$
\int_0^T \|B(s,z_s^v)\|_2^2\mathrm{d}s + \int_0^T \|z_s^v\|_V^\alpha \mathrm{d}s \leq C, \quad a.s..
$$

By applying Gronwall's lemma we have

$$
\begin{aligned}
&\sup_{s\in[0,t]}\|X_s^\varepsilon - z_s^v\|_H^2 + \delta\int_0^t \|X_s^\varepsilon - z_s^v\|_V^\alpha \mathrm{d}s \\
&\leq C\Bigg[\varepsilon^2 + \sup_{s\in[0,t]}\|h_s^\varepsilon\|_H^2 + \left(1+\int_0^t \|X_s^\varepsilon\|_V^\alpha \mathrm{d}s\right)^{\frac{\alpha-1}{\alpha}} \left(\int_0^t \|h_s^\varepsilon\|_V^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} \\
&+ \sup_{s\in[0,t]}\|h_s^\varepsilon\|_H\left\{1+\left(\int_0^t \|X_s^\varepsilon\|_H^2\mathrm{d}s\right)^{1/2}\right\} + \sup_{u\in[0,t]}\left|\varepsilon\int_0^u \langle X_s^\varepsilon - z_s^v - h_s^\varepsilon, B(s,X_s^\varepsilon)\mathrm{d}W_s\rangle_H\right|\Bigg].
\end{aligned}
\tag{2.3.21}
$$

Define the stopping time

$$\tau^{M,\varepsilon} = \inf\left\{t \le T : \sup_{s\in[0,t]} \|X_s^\varepsilon\|_H^2 + \int_0^t \|X_s^\varepsilon\|_V^\alpha \mathrm{d}s > M\right\},$$

then by the Burkhölder-Davis-Gundy inequality one has

$$\begin{aligned}
&\varepsilon\mathbf{E} \sup_{t\in[0,\tau^{M,\varepsilon}]} \left|\int_0^t \langle X_s^\varepsilon - z_s^v - h_s^\varepsilon, B(s, X_s^\varepsilon)\mathrm{d}W_s\rangle_H\right| \\
\le & 3\varepsilon\mathbf{E}\left\{\int_0^{\tau^{M,\varepsilon}} \|X_s^\varepsilon - z_s^v - h_s^\varepsilon\|_H^2 \|B(s,X_s^\varepsilon)\|_2^2 \mathrm{d}s\right\}^{1/2} \\
\le & 3\varepsilon\mathbf{E}\left\{\sup_{s\in[0,\tau^{M,\varepsilon}]} \|X_s^\varepsilon - z_s^v - h_s^\varepsilon\|_H^2 + C\int_0^{\tau^{M,\varepsilon}} \left(1 + \|X_s^\varepsilon\|_H^2 + \|X_s^\varepsilon\|_V^\alpha\right)\mathrm{d}s\right\} \\
\le & C\varepsilon \to 0 \ (\varepsilon \to 0).
\end{aligned} \tag{2.3.22}$$

By using a similar argument with (2.3.17) we have

$$\mathrm{d}\|X_t^\varepsilon\|_H^2 \le -\frac{\delta}{2}\|X_t^\varepsilon\|_V^\alpha \mathrm{d}t + C(1 + \|X_t^\varepsilon\|_H^2 + \|v_t^\varepsilon\|_U^2\|X_t^\varepsilon\|_H^2)\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon, B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H,$$

where C is a constant. Repeat the same argument in [KR79, Theorem 3.10] we can prove

$$\sup_{\varepsilon\in[0,1)} \mathbf{E}\left\{\sup_{t\in[0,T]} \|X_t^\varepsilon\|_H^2 + \int_0^T \|X_t^\varepsilon\|_V^\alpha \mathrm{d}t\right\} < \infty.$$

Hence there exists a suitable constant C such that

$$\liminf_{\varepsilon\to 0} \mathbf{P}\{\tau^{M,\varepsilon} = T\} \ge 1 - \frac{C}{M}. \tag{2.3.23}$$

Recall that $h^\varepsilon \to 0$ in distribution in $C([0,T];H)\cap L^\alpha([0,T];V)$, combining with (2.3.21)-(2.3.23) one can conclude

$$\sup_{t\in[0,T]} \|X_t^\varepsilon - z_t^v\|_H^2 + \int_0^T \|X_t^\varepsilon - z_t^v\|_V^\alpha \mathrm{d}t \to 0 \ (\varepsilon\to 0)$$

in distribution. Hence the proof is complete. ∎

Remark 2.3.1 *According to Lemma 2.1.1, Lemma 2.3.2 and Lemma 2.3.3, we know that* $\{X^\varepsilon\}$ *satisfies the LDP provided* (A1)−(A3) *and* (A5) *hold. By using some approximation arguments in next section, we can replace* (A5) *by the weaker assumption* (A4)*.*

2.3.2 Replace (A5) by the weaker assumption (A4)

Suppose for any fixed $n \geq 1$, $H_n \subseteq V$ is compact and $P_n : H \to H_n$ is the orthogonal projection. Let $X_t^{\varepsilon,n}$ be the solution to

$$\mathrm{d}X_t^{\varepsilon,n} = A(t, X_t^{\varepsilon,n})\mathrm{d}t + \varepsilon P_n B(t, X_t^{\varepsilon,n})\mathrm{d}W_t, \quad X_0^{\varepsilon,n} = x. \tag{2.3.24}$$

Since $P_n B$ satisfies $(A5)$, according to Remark 2.3.1 we know $\{X^{\varepsilon,n}\}$ satisfies LDP provided $(A1)-(A3)$ hold. Now we prove that $\{X^{\varepsilon,n}\}$ are the exponential good approximation to $\{X^\varepsilon\}$ if the following additional assumption holds.

$(A4')$

$$a_n := \sup_{(t,v)\in[0,T]\times V} \|P_n B(t,v) - B(t,v)\|_2^2 \to 0 \ (n \to \infty).$$

Lemma 2.3.4 *If $(A1)-(A3)$ and $(A4')$ hold, then for any $\sigma > 0$*

$$\limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}\left(\rho(X^\varepsilon, X^{\varepsilon,n}) > \sigma\right) = -\infty, \tag{2.3.25}$$

where ρ is the metric on $C([0,T];H) \cap L^\alpha([0,T];V)$ defined in (2.2.4).

Proof. For $\varepsilon < \frac{1}{2}$, by using the Itô formula and $(A2)$ we have

$$\begin{aligned}
&\mathrm{d}\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 = 2{}_{V^*}\langle A(t,X_t^\varepsilon) - A(t, X_t^{\varepsilon,n}), X_t^\varepsilon - X_t^{\varepsilon,n}\rangle_V \mathrm{d}t \\
&+ \varepsilon^2 \|B(t,X_t^\varepsilon) - P_n B(t, X_t^{\varepsilon,n})\|_2^2 \mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon - X_t^{\varepsilon,n}, (B(t,X_t^\varepsilon) - P_n B(t,X_t^{\varepsilon,n}))\mathrm{d}W_t\rangle_H.
\end{aligned}$$

Define

$$\|X_t^\varepsilon - X_t^{\varepsilon,n}\| = \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 + \delta \int_0^t \|X_s^\varepsilon - X_s^{\varepsilon,n}\|_V^\alpha \mathrm{d}s.$$

Note that

$$M_t^{(n)} := \int_0^t \langle X_s^\varepsilon - X_s^{\varepsilon,n}, (B(s,X_s^\varepsilon) - P_n B(s, X_s^{\varepsilon,n}))\,\mathrm{d}W_s\rangle_H$$

is a local martingale and its quadratic variation process satisfies

$$\langle M^{(n)}\rangle_t \leq 2\int_0^t \|X_s^\varepsilon - X_s^{\varepsilon,n}\|_H^2 (\|B(s,X_s^\varepsilon) - B(s,X_s^{\varepsilon,n})\|_2^2 + a_n)\mathrm{d}s.$$

Let $\varphi_\theta(y) = (a_n + y)^\theta$ for some $\theta \le \frac{1}{4\varepsilon^2}$, then by $(A2)$

$$
\begin{aligned}
&\mathrm{d}\varphi_\theta(\|X_t^\varepsilon - X_t^{\varepsilon,n}\|) \le \theta(a_n + \|X_t^\varepsilon - X_t^{\varepsilon,n}\|)^{\theta-1}\left(\mathrm{d}\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 + \delta\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_V^\alpha \mathrm{d}t\right)\\
&\quad + 4\varepsilon^2\theta(\theta-1)(a_n + \|X_t^\varepsilon - X_t^{\varepsilon,n}\|)^{\theta-2}\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2\left(\|B(t,X_t^\varepsilon) - B(t,X_t^{\varepsilon,n})\|_2^2 + a_n\right)\mathrm{d}t\\
\le& C\theta\varphi_\theta\left(\|X_t^\varepsilon - X_t^{\varepsilon,n}\|\right)\mathrm{d}t + \mathrm{d}\beta_t,
\end{aligned}
\tag{2.3.26}
$$

where C is a constant and β_t is a local martingale. By a standard localization argument we may assume β_t is a martingale for simplicity. Let $\theta = \frac{1}{4\varepsilon^2}$ we know

$$
N_t := \exp\left[-\frac{C}{4\varepsilon^2}t\right]\varphi_{\frac{1}{4\varepsilon^2}}\left(\|X_t^\varepsilon - X_t^{\varepsilon,n}\|\right)
$$

is a supermartingale. Hence we have

$$
\begin{aligned}
&\mathbf{P}\left(\rho(X^\varepsilon, X^{\varepsilon,n}) > 2\sigma\right)\\
\le& \mathbf{P}\left(\sup_{t\in[0,T]}\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H > \sigma\right) + \mathbf{P}\left(\int_0^T \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_V^\alpha \mathrm{d}t > \sigma^\alpha\right)\\
\le& \mathbf{P}\left(\sup_{t\in[0,T]} N_t > \exp\left[-\frac{C}{4\varepsilon^2}T\right](\sigma^2 + a_n)^{\frac{1}{4\varepsilon^2}}\right) + \mathbf{P}\left(\sup_{t\in[0,T]} N_t > \exp\left[-\frac{C}{4\varepsilon^2}T\right](\delta\sigma^\alpha + a_n)^{\frac{1}{4\varepsilon^2}}\right)\\
\le& \exp\left[\frac{C}{4\varepsilon^2}T\right](\sigma^2 + a_n)^{-\frac{1}{4\varepsilon^2}}\mathbf{E}N_0 + \exp\left[\frac{C}{4\varepsilon^2}T\right](\delta\sigma^\alpha + a_n)^{-\frac{1}{4\varepsilon^2}}\mathbf{E}N_0\\
=& \exp\left[\frac{C}{4\varepsilon^2}T\right]\left\{\left(\frac{a_n}{\sigma^2 + a_n}\right)^{\frac{1}{4\varepsilon^2}} + \left(\frac{a_n}{\delta\sigma^\alpha + a_n}\right)^{\frac{1}{4\varepsilon^2}}\right\}.
\end{aligned}
$$

This implies that

$$
\limsup_{\varepsilon\to 0}\varepsilon^2 \log \mathbf{P}\left(\rho(X^\varepsilon, X^{\varepsilon,n}) > 2\sigma\right) \le \frac{CT}{4} + \max\left\{\log\frac{a_n}{\sigma^2 + a_n}, \log\frac{a_n}{\delta\sigma^\alpha + a_n}\right\}.
$$

Since $(A4')$ says $a_n \to 0$ as $n \to \infty$, (2.3.25) holds and the proof is complete. ∎

Corollary 2.3.5 *If $(A1)$–$(A3)$ and $(A4')$ hold, then $\{X^\varepsilon\}$ satisfies the LDP in $C([0,T];H) \cap L^\alpha([0,T];V)$ with the rate function (2.2.7).*

Proof. According to [Wu04, Theorem 2.1] and section 3 one can conclude $\{X^\varepsilon\}$ satisfies the LDP with the following rate function

$$
\tilde{I}(f) := \sup_{r>0}\liminf_{n\to\infty}\inf_{g\in S_r(f)} I^n(g) = \sup_{r>0}\limsup_{n\to\infty}\inf_{g\in S_r(f)} I^n(g),
$$

where $S_r(f)$ is the closed ball in $C([0,T];H) \cap L^\alpha([0,T];V)$ centered at f with radius r and I^n is given by

$$I^n(z) := \inf\left\{\frac{1}{2}\int_0^T \|\phi_s\|_U^2 \mathrm{d}s: \ z = z^{n,\phi}, \ \phi \in L^2([0,T],U)\right\}, \tag{2.3.27}$$

where $z^{n,\phi}$ is the unique solution to following equation

$$\frac{\mathrm{d}z_t^n}{\mathrm{d}t} = A(t, z_t^n) + P_n B(t, z_t^n)\phi_t, \ z_0^n = x.$$

Now we only need to prove $\tilde{I} = I$, *i.e.*

$$I(f) = \sup_{r>0} \liminf_{n\to\infty} \inf_{g\in S_r(f)} I^n(g).$$

We will first show that for any $r > 0$

$$I(f) \geq \liminf_{n\to\infty} \inf_{g\in S_r(f)} I^n(g).$$

We assume $I(f) < \infty$. By Lemma 2.3.2 there exists ϕ such that

$$f = z^\phi \ \text{ and } \ I(f) = \frac{1}{2}\int_0^T \|\phi_s\|_U^2 \mathrm{d}s.$$

Since $z^{n,\phi} \to z^\phi$, for n large enough we have

$$f_n := z^{n,\phi} \in S_r(f).$$

Noting that $I^n(f_n) \leq \frac{1}{2}\int_0^T \|\phi_s\|_U^2 \mathrm{d}s$, hence we have

$$\liminf_{n\to\infty} \inf_{g\in S_r(f)} I^n(g) \leq \liminf_{n\to\infty} I^n(f_n) \leq I(f).$$

Since r is arbitrary, we have proved the lower bound

$$I(f) \geq \sup_{r>0} \liminf_{n\to\infty} \inf_{g\in S_r(f)} I^n(g).$$

For the upper bound we can proceed as in finite dimensional case in [Str84, Lemma 4.6] to show

$$\limsup_{n\to\infty} \inf_{g\in S_r(f)} I^n(g) \geq \inf_{g\in S_r(f)} I(g)$$

Hence we have

$$\sup_{r>0} \limsup_{n\to\infty} \inf_{g\in S_r(f)} I^n(g) \geq \sup_{r>0} \inf_{g\in S_r(f)} I(g) \geq I(f).$$

Now the proof is complete. ∎

In order to replace assumption $(A4')$ by $(A4)$, we need to use some truncation techniques.

Lemma 2.3.6 *Assume* $(A1)-(A4)$ *hold, then*

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}(\sup_{t\in[0,T]} \|X_t^\varepsilon\|_H^2 + \frac{\delta}{2}\int_0^T \|X_t^\varepsilon\|_V^\alpha \mathrm{d}t > R) = -\infty. \tag{2.3.28}$$

Proof. By using the Itô formula we have

$$\mathrm{d}\|X_t^\varepsilon\|_H^2 = \left(2_{V^*}\langle A(t,X_t^\varepsilon), X_t^\varepsilon\rangle_V + \varepsilon^2 \|B(t,X_t^\varepsilon)\|_2^2 \right) \mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon, (B(t,X_t^\varepsilon)\mathrm{d}W_t\rangle_H.$$

Note that $M_t^{(n)} := \int_0^t \langle X_s^\varepsilon, B(s,X_s^\varepsilon)\mathrm{d}W_s\rangle_H$ is a local martingale and

$$\langle M^{(n)}\rangle_t \leq \int_0^t \|X_s^\varepsilon\|_H^2 \|B(s,X_s^\varepsilon)\|_2^2 \mathrm{d}s.$$

Define

$$\|X_t^\varepsilon\| := \|X_t^\varepsilon\|_H^2 + \frac{\delta}{2}\int_0^t \|X_s^\varepsilon\|_V^\alpha \mathrm{d}s, \quad \varphi_\theta(y) = (1+y)^\theta,\ \theta > 0,$$

then for $\theta \leq \frac{1}{2\varepsilon^2}$, by $(A2)$ and $(A3)$ we have

$$\begin{aligned}
\mathrm{d}\varphi_\theta(\|X_t^\varepsilon\|) \leq & \theta(1+\|X_t^\varepsilon\|)^{\theta-1} \left(\mathrm{d}\|X_t^\varepsilon\|_H^2 + \frac{\delta}{2}\|X_t^\varepsilon\|_V^\alpha \mathrm{d}t \right) \\
& + 2\varepsilon^2\theta(\theta-1)(1+\|X_t^\varepsilon\|)^{\theta-2}\|X_t^\varepsilon\|_H^2 \|B(t,X_t^\varepsilon)\|_2^2 \mathrm{d}t \\
\leq & C\theta\varphi_\theta(\|X_t^\varepsilon\|)\mathrm{d}t + \mathrm{d}\beta_t,
\end{aligned} \tag{2.3.29}$$

where β_t is a local martingale. We also omit the standard localization procedure here. Let $\theta = \frac{1}{2\varepsilon^2}$ we know

$$N_t := \exp\left[-\frac{C}{2\varepsilon^2} t \right] \varphi_{\frac{1}{2\varepsilon^2}}(\|X_t^\varepsilon\|)$$

is a supermartingale. Hence we have

$$
\begin{aligned}
&\mathbf{P}\left(\sup_{t\in[0,T]}\|X_t^\varepsilon\|_H^2+\frac{\delta}{2}\int_0^T\|X_t^\varepsilon\|_V^\alpha \mathrm{d}t>R\right)\\
\leq&\mathbf{P}\left(\sup_{t\in[0,T]}N_t>\exp\left[-\frac{C}{2\varepsilon^2}T\right](1+R)^{\frac{1}{2\varepsilon^2}}\right)\\
\leq&\exp\left[\frac{C}{2\varepsilon^2}T\right](1+R)^{-\frac{1}{2\varepsilon^2}}\mathbf{E}N_0\\
=&\exp\left[\frac{C}{2\varepsilon^2}T\right]\left(\frac{1}{1+R}\right)^{\frac{1}{2\varepsilon^2}}.
\end{aligned}
$$

This implies that

$$
\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(\sup_{t\in[0,T]}\|X_t^\varepsilon\|>R\right)\leq\frac{1}{2}\log\frac{1}{1+R}+\frac{CT}{2}.
$$

Therefore, by letting $R\to\infty$ we have (2.3.28). ∎

After all these preparations, now we can finish the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1: Define $\xi:V\to[0,1]$ be a C_0^∞-function such that

$$
\xi(v):=\begin{cases}0, & \text{if } \|v\|_H>2,\\ 1, & \text{if } \|v\|_H\leq 1.\end{cases}
$$

Let $\xi_N(v)=\xi(\frac{v}{N})$ and

$$
B_N(t,v)=\xi_N(v)B(t,v)+(1-\xi_N(v))B(t,0).
$$

Consider the mollified problem for equation (2.2.3):

$$
\mathrm{d}X_{t,N}^\varepsilon=A(t,X_{t,N}^\varepsilon)\mathrm{d}t+\varepsilon B_N(t,X_{t,N}^\varepsilon)\mathrm{d}W_t,\ X_0=x. \tag{2.3.30}
$$

It is easy to see that A,B_N satisfy $(A1)-(A3)$ and $(A4')$, since in this case $(A4)$ implies that for B_N

$$
a_n=\max\left\{\sup_{(t,v)\in[0,T]\times S_{2N}}\|(I-P_n)B(t,v)\|_2^2,\ \sup_{t\in[0,T]}\|(I-P_n)B(t,0)\|_2^2\right\}\to 0(n\to\infty).
$$

Hence by Corollary 2.3.5 we know $\{X_N^\varepsilon\}_{\varepsilon>0}$ satisfies the LDP on $C([0,T];H)\cap L^\alpha([0,T];V)$ with the following mollified rate function

$$I_N(z) := \inf\left\{\frac{1}{2}\int_0^T \|\phi_s\|_U^2 \mathrm{d}s:\ z = z_N^\phi,\ \phi\in L^2([0,T],U)\right\}, \tag{2.3.31}$$

where z_N^ϕ is the unique solution to the following equation

$$\frac{\mathrm{d}z_{t,N}}{\mathrm{d}t} = A(t,z_{t,N}) + B_N(t,z_{t,N})\phi_t,\ \ z_{0,N} = x.$$

Let $N\to\infty$, then the LDP for $\{X^\varepsilon\}$ can be derived as follows, which is similar to the finite dimensional case (cf.[Str84, Theorem 4.13]).

According to Lemma 2.3.2, I defined in (2.2.7) is a good rate function. Note $I_N(z) = I(z)$ for any $z\in C([0,T];H)\cap L^\alpha([0,T];V)$ satisfying

$$\|z\|_T := \sup_{t\in[0,T]}\|z_t\|_H \le N.$$

We first show that for any open set $G\subseteq C([0,T];H)\cap L^\alpha([0,T];V)$

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(X^\varepsilon\in G\right) \ge -\inf_{z\in G} I(z).$$

Obviously, we only need to prove that for all $\overline{z}\in G$ with $\overline{z}_0 = x$

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(X^\varepsilon\in G\right) \ge -I(\overline{z}).$$

Choose $R>0$ such that $\|\overline{z}\|_T < R$ and set

$$N_R = \{z\in C([0,T];H)\cap L^\alpha([0,T];V) : \|z\|_T < R\},$$

then we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(X^\varepsilon\in G\right) &\ge \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(X^\varepsilon\in G\cap N_R\right)\\
&= \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbf{P}\left(X_N^\varepsilon\in G\cap N_R\right)\\
&\ge -\inf_{z\in G\cap N_R} I_N(z)\\
&\ge -I(\overline{z}).
\end{aligned}$$

Finally, for any given closed set F and constant $L < \infty$, by Lemma 2.3.6 there exists R such that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}\left(X^\varepsilon \in F\right) &\leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log \left(\mathbf{P}(X^\varepsilon \in F \cap \overline{N_R}) + \mathbf{P}(X^\varepsilon \in N_R^c)\right) \\
&\leq (-\inf_{z\in F\cap \overline{N_R}} I_N(z)) \vee (-L) \\
&\leq -\left[\inf_{z\in F} I(z) \wedge L\right].
\end{aligned}$$

Taking $L \to \infty$ we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbf{P}\left(X^\varepsilon \in F\right) \leq -\inf_{z\in F} I(z).$$

Now the proof of Theorem 2.2.1 is complete.

2.4 Applications to different types of SPDE

Now we can apply the main results to different types of stochastic evolution equations as examples. In order to verify the strong monotonicity assumption $(A2)$ we need the following lemma.

Lemma 2.4.1 *Let $(E, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a Hilbert space, then for any $r \geq 0$ we have*

$$\langle \|a\|^r a - \|b\|^r b, a - b\rangle \geq 2^{-r}\|a-b\|^{r+2}, \ \ a, b \in E. \tag{2.4.1}$$

Proof. By the symmetry of (2.4.1) we may assume $\|a\| \geq \|b\|$. Then

$$\begin{aligned}
& \langle \|a\|^r a - \|b\|^r b, a - b\rangle \\
= \ & \|b\|^r\|a-b\|^2 + (\|a\|^r - \|b\|^r)\langle a, a-b\rangle \\
= \ & \|b\|^r\|a-b\|^2 + (\|a\|^r - \|b\|^r)\cdot\frac{1}{2}(\|a\|^2 + \|a-b\|^2 - \|b\|^2) \\
\geq \ & \|b\|^r\|a-b\|^2 + \frac{1}{2}(\|a\|^r - \|b\|^r)\|a-b\|^2 \\
= \ & \frac{1}{2}(\|a\|^r + \|b\|^r)\|a-b\|^2 \\
\geq \ & 2^{-r}\|a-b\|^{r+2},
\end{aligned}$$

since $\|a-b\|^r \leq 2^{r-1}(\|a\|^r + \|b\|^r)$. ∎

The first example is to obtain the LDP for a class of reaction-diffusion type SPDE within the variational framework, which improves the main result in [Cho92].

Example 2.4.2 *(Stochastic reaction-diffusion equations)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary and L be a negative definite self-adjoint operator on $H := L^2(\Lambda)$. Suppose

$$V := \mathcal{D}(\sqrt{-L}), \quad \|v\|_V := \|\sqrt{-L}v\|_H$$

is a Banach space such that $V \subseteq H$ is dense and compact, and L can be extended to a continuous operator from V to its dual space V^*. Consider the following semilinear stochastic equation

$$\mathrm{d}X_t^\varepsilon = (LX_t^\varepsilon + F(t, X_t^\varepsilon))\mathrm{d}t + \varepsilon B(t, X_t^\varepsilon)\mathrm{d}W_t, \ X_0^\varepsilon = x \in H, \tag{2.4.2}$$

where W_t is a cylindrical Wiener process on another separable Hilbert space U and

$$F : [0, T] \times V \to V^*, \quad B : [0, T] \times V \to L_2(U; V).$$

If F and B satisfy the following conditions:

$$\begin{aligned} 2{}_{V^*}\langle F(t, u) - F(t, v), u - v\rangle_V + \|B(t, u) - B(t, v)\|_2^2 \le C\|u - v\|_H^2, \\ \sup_{t\in[0,T]} \|B(t, 0)\|_2 < \infty, \ \|F(t, v)\|_{V^*} \le C(1 + \|v\|_V), \ u, v \in V, \end{aligned} \tag{2.4.3}$$

where C is a constant, then $\{X^\varepsilon\}$ satisfies the LDP on $C([0, T]; H) \cap L^2([0, T]; V)$.

Proof. From assumption (2.4.3) we can obtain that

$$\|B(t, v)\|_2 \le C(1 + \|v\|_H + \|v\|_V), \ v \in V,$$

i.e. $(A4')$ holds. And it is also easy to show $(A1) - (A3)$ hold for $\alpha = 2$. Hence the conclusion follows from Theorem 2.2.1 and Remark 2.2.2. ∎

Remark 2.4.1 *(1) We can simply take L as the Laplace operator with Dirichlet boundary condition and $F(t, X_t) = -|X_t|^{p-2}X_t (1 \le p \le 2)$ as a concrete example.*

(2) Comparing with the result in [Cho92, Theorem 4.2] (only time homogeneous case was studied), the author in [Cho92] needs to assume F is local Lipschitz and have more restricted range conditions:

$$F : [0, T] \times V \to H.$$

In our example we can allow F to be monotone and take values in V^*. Another difference is we also drop the non-degenerate condition $(A.4)$ on B in [Cho92].

(3) Note here one can also take $B : V \to L_2(U; H)$ with locally compact range (see Remark 2.2.1), which seems not allowed in [Cho92, Theorem 4.2]. In particular, $B(\cdot, u)$ may depend on the gradient of u.

The second example is stochastic porous media equations, which have been studied intensively in recent years (see e.g.[DPRRW06, RRW07, RWW06, Wan07]). The porous media equation can be used to describe the flow of an isentropic gas through a porous medium [Mus37] or to model the heat radiation in plasmas [ZR66]. Other applications have been proposed in mathematical biology, water infiltration, lubrication, boundary layer theory and other fields (cf.[Váz07, Váz06]). In the following example we use the same framework as in [RWW06, Wan07] for simplicity.

Example 2.4.3 *(Stochastic porous media equations)*
Let $(E, \mathcal{M}, \mathbf{m})$ be a separable probability space and $(L, \mathcal{D}(L))$ be a negative definite self-adjoint operator on $(L^2(\mathbf{m}), \langle\cdot,\cdot\rangle)$ with spectrum contained in $(-\infty, -\lambda_0]$ for some $\lambda_0 > 0$. Then the embedding

$$H^1 := \mathcal{D}(\sqrt{-L}) \subseteq L^2(\mathbf{m})$$

is dense and continuous, and H is defined as the dual Hilbert space of H^1 realized through this embedding.

For fixed $r > 1$, we assume L^{-1} is continuous on $L^{r+1}(\mathbf{m})$. Now we consider the following Gelfand triple

$$V := L^{r+1}(\mathbf{m}) \subseteq H \subseteq V^*$$

and the stochastic porous media equation

$$\mathrm{d}X_t^\varepsilon = (L\Psi(t, X_t^\varepsilon) + \Phi(t, X_t^\varepsilon))\mathrm{d}t + \varepsilon B(t, X_t^\varepsilon)\mathrm{d}W_t, \ X_0^\varepsilon = x \in H, \tag{2.4.4}$$

where W_t is a cylindrical Wiener process on $L^2(\mathbf{m})$, $\Psi, \Phi : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable and continuous in the second variable.

Suppose $L^2(\mathbf{m}) \subseteq H$ is compact and $B : [0,T] \times V \to L_2(L^2(\mathbf{m}))$. If there exist two *constants $\delta > 0$ and K such that*

$$
\begin{aligned}
&|\Psi(t,x)| + |\Phi(t,x)| + \|B(t,0)\|_2 \le K(1+|x|^r), \quad t\in[0,T], x\in\mathbb{R};\\
&-\langle \Psi(t,u)-\Psi(t,v), u-v\rangle - \langle \Phi(t,u)-\Phi(t,v), L^{-1}(u-v)\rangle\\
&\qquad \le -\delta\|u-v\|_V^{r+1} + K\|u-v\|_H^2;\\
&\|B(t,u)-B(t,v)\|_2^2 \le K\|u-v\|_H^2, \quad t\in[0,T], u,v\in V.
\end{aligned}
\tag{2.4.5}
$$

Then $\{X^\varepsilon\}$ satisfy the LDP on $C([0,T];H)\cap L^{r+1}([0,T];V)$.

Proof. From the assumptions and the relation

$$ {}_{V^*}\langle L\Phi(t,u)+\Phi(t,u), u\rangle_V = -\langle \Phi(t,u), u\rangle - \langle \Phi(t,u), L^{-1}u\rangle, $$

it's easy to show that $(A1)-(A4)$ hold for $\alpha = r+1$ from (2.4.5). We refer to [PR07, Example 4.1.11] for the details, see also [DPRRW06, RWW06, Wan07]. Hence the conclusion follows from Theorem 2.2.1. ∎

Remark 2.4.2 *(1) If we take L as the Laplace operator on a smooth bounded domain in a complete Riemannian manifold with Dirichlet boundary condition, then one simple example for Ψ and Φ satisfying (2.4.5) is given by*

$$ \Psi(t,x) = f(t)|x|^{r-1}x (r>1), \quad \Phi(t,x) = g(t)x $$

for some strictly positive continuous function f and bounded function g on $[0,T]$.

(2) This example generalized the main result in [RWW06, Theorem 1.1] where the LDP was obtained for stochastic porous media equations with additive noise. For the proof in [RWW06] the authors mainly used the piecewise linear approximation to the path of Wiener process and generalized contraction principle, which would be very difficult to be extended to the present multiplicative noise case.

If we assume $0 < r < 1$ in the above example (cf.[LW08, RRW07]), then the corresponding equation turns into the stochastic version of classical fast diffusion equations. The behavior of the solutions to these two types of PDE have many essentially different aspects, see e.g. [Aro86].

Example 2.4.4 *(Stochastic fast diffusion equations)*
Consider the same framework as Example 2.4.3 for $0 < r < 1$ and assume the embedding $V := L^{r+1}(\mathbf{m}) \subseteq H$ is continuous and dense. We consider the following equation

$$\mathrm{d}X_t^\varepsilon = \left\{L\Psi(t, X_t^\varepsilon) + \eta_t X_t^\varepsilon\right\}\mathrm{d}t + \varepsilon B(t, X_t^\varepsilon)\mathrm{d}W_t,\ X_0^\varepsilon = x \in H, \tag{2.4.6}$$

where $\eta : [0,T] \to \mathbb{R}$ is locally bounded and measurable, $\Psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable and continuous in the second variable, W_t is a cylindrical Wiener process on $L^2(\mathbf{m})$ and $B : [0,T] \times V \to L_2(L^2(\mathbf{m}))$ are measurable.

Suppose there exist constants $\delta > 0$ and K such that for all $x, y \in \mathbb{R}, t \in [0,T]$ and $u, v \in V$

$$\begin{aligned}
&|\Psi(t,x)| + \|B(t,0)\|_2 \le K(1 + |x|^r);\\
&(\Psi(t,x) - \Psi(t,y))(x-y) \ge \delta|x-y|^2(|x| \vee |y|)^{r-1};\\
&\|B(t,u) - B(t,v)\|_2^2 \le K\|u-v\|_H^2;\\
&\|B(t,u)\|_{L(L^2(\mathbf{m}),V^*)} \le K(1 + \|u\|_V^r).
\end{aligned} \tag{2.4.7}$$

Then $\{X^\varepsilon\}$ satisfy the LDP on $C([0,T]; H)$.

Proof. Note that

$$_{V^*}\langle L\Psi(t,u) + \eta_t u, u\rangle_V = -\langle \Psi(t,u), u\rangle_{L^2} + \langle \eta_t u, u\rangle_H,$$

then it is easy to show $(A1), (A2'), (A3)$ and $(A4)$ hold for $\alpha = r+1$ under the assumptions (2.4.7). Then the conclusion follows from Theorem 2.2.2. ∎

Remark 2.4.3 *(1) In particular, if $\eta = 0, B = 0$ and $\Psi(t,s) = |s|^{r-1}s$ for some $r \in (0,1)$, then (2.4.6) reduces back to the classical fast diffusion equations (cf.[Aro86]).*

(2) In the example we assume the embedding $L^{r+1}(\mathbf{m}) \subseteq H$ is continuous and dense only for simplicity, we refer to [LW08] and [PR07, Remark 4.1.15] for some sufficient conditions of this assumption. But in general $L^{r+1}(\mathbf{m})$ and H are incomparable, then one need to use the more general framework as in [RRW07] involving with Orlicz space.

Example 2.4.5 *(Stochastic p-Laplace equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary. We consider the triple

$$V := H_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (H_0^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t^\varepsilon = \left[\mathbf{div}(|\nabla X_t^\varepsilon|^{p-2}\nabla X_t^\varepsilon) - \eta_t |X_t^\varepsilon|^{\tilde{p}-2} X_t^\varepsilon\right] \mathrm{d}t + \varepsilon B(t, X_t^\varepsilon)\mathrm{d}W_t,\ \ X_0^\varepsilon = x \in H, \quad (2.4.8)$$

where $2 \le p < \infty, 1 \le \tilde{p} \le p$, η is a positive continuous function and W_t is a cylindrical Wiener process on H. If

$$B(t, v) = \sum_{i=1}^{N} b_i(v) B_i(t), \quad (2.4.9)$$

where $b_i(\cdot) : V \to \mathbb{R}$ are Lipschitz functions and $B_i(\cdot) : [0,T] \to L_2(H)$ are continuous, then $\{X^\varepsilon\}$ satisfy the LDP on $C([0,T]; H) \cap L^p([0,T]; V)$.

Proof. The assumptions for existence and uniqueness of the solution was verified in [PR07, Example 4.1.9] for $\alpha = p$. Hence we only need to prove $(A2)$ here. By using (2.4.1) in Lemma 2.4.1 we have

$$\begin{aligned}
& {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u) - \mathbf{div}(|\nabla v|^{p-2}\nabla v), u - v\rangle_V \\
= & -\int_\Lambda \langle |\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x) - \nabla v(x)\rangle_{\mathbb{R}^d} \mathrm{d}x \\
\le & -2^{p-2}\int_\Lambda |\nabla u(x) - \nabla v(x)|^p \mathrm{d}x \\
\le & -c\|u - v\|_V^p,
\end{aligned}$$

where c is a positive constant and we use the Poincaré inequality in last step.

By the monotonicity of function $|x|^{\tilde{p}-2}x$ we also have

$${}_{V^*}\langle |u|^{\tilde{p}-2}u - |v|^{\tilde{p}-2}v, u - v\rangle_V \ge 0.$$

Hence $(A2)$ holds and the conclusion follows from Theorem 2.2.1. ∎

Remark 2.4.4 *(1) For deriving the LDP the main assumption on B is $(A4)$, hence one can also use other types of conditions as in Remark 2.2.1 for B instead of (2.4.9).*

(2) If we take $1 < p < 2$ in (2.4.8), then the assumption $(A2)$ does not hold in this case. Hence like the case of stochastic fast diffusion equations, we should apply Theorem 2.2.2 to derive the LDP on $C([0,T]; H)$ for (2.4.8). We omit the details here.

The following SPDE has been studied in [KR79, Liu08a], in which the main part of drift in the equation is a high order generalization of the Laplace operator.

Example 2.4.6 *Let Λ be an open bounded domain in $\mathbb{R}^1$ and $m \in \mathbb{N}_+$. We consider the triple*

$$V := H_0^{m,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (H_0^{m,p}(\Lambda))^*$$

and the stochastic evolution equation

$$\begin{aligned} \mathrm{d}X_t^\varepsilon(x) = & \left[(-1)^{m+1}\frac{\partial}{\partial x^m}\left(\left|\frac{\partial^m}{\partial x^m}X_t^\varepsilon(x)\right|^{p-2}\frac{\partial^m}{\partial x^m}X_t^\varepsilon(x)\right) + F(t, X_t^\varepsilon(x))\right]\mathrm{d}t \\ & + \varepsilon B(t, X_t^\varepsilon(x))\mathrm{d}W_t, \quad X_0^\varepsilon = x \in H, \end{aligned} \tag{2.4.10}$$

where $2 \le p < \infty$, W_t is a cylindrical Wiener process on H and

$$F : [0,T] \times V \to V^*, \quad B : [0,T] \times V \to L_2(H)$$

are measurable. Suppose $B(t,v) = QB_0(t,v), Q \in L_2(H)$ and

$$\begin{aligned} 2{}_{V^*}\langle F(t,u) - F(t,v), u - v\rangle_V &\le C\|u-v\|_H^2, \\ \|B_0(t,u) - B_0(t,v)\|_{L(H)} &\le C\|u-v\|_H, \\ \|F(t,u)\|_{V^*} + \|B_0(t,0)\|_{L(H)} &\le C(1+\|u\|_V^{p-1}), \ u,v \in V, \ t \in [0,T], \end{aligned}$$

where C is a constant. Then $\{X^\varepsilon\}$ satisfy the LDP on $C([0,T];H) \cap L^p([0,T];V)$.

Proof. By Lemma 2.4.1 $(A2)$ can be verified by a similar argument as in Example 2.4.5. Note that $(A1), (A3)$ and $(A4)$ can be proved easily by using the assumptions above, hence the conclusion follows from Theorem 2.2.1.

∎

Chapter 3

Harnack Inequality and Its Applications to SEE

In this chapter we establish the dimension-free Harnack inequality and strong Feller property for the transition semigroups associated with a large class of SPDE. Then the ergodicity, contractivity (e.g. hyperboundedness and ultraboundedness) and compactness property are derived for the corresponding Markov semigroups. In particular, exponential convergence to the equilibrium (invariant measure) and the existence of a spectral gap are also investigated. The main results are applied to stochastic reaction-diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation in Hilbert space.

In the first section, we give a brief introduction to the classical Harnack inequality and the dimension-free Harnack inequality. Since the strong Feller property is proved here by using a new coupling argument, we also give a short review in section 2 on different methods of deriving the strong Feller property in the literature. In the third section, the main results on the Harnack inequality and many resulting properties for the transition semigroups and invariant measures are established. In the last section we apply these results to study many concrete SPDEs in Hilbert space as examples. Part of the results in this chapter have already been submitted for publication, see [Liu08a, Liu08b].

3.1 Introduction to Harnack inequality

These types of inequalities are named after Carl Gustav Axel von Harnack. The classical Harnack inequality was originally derived for harmonic functions in the plane and much later became a very important tool in the general theory of harmonic functions and partial differential equations, and it also plays an important role in the geometric analysis and probability theory. We refer to two survey articles [Wan06, Kas07] for more detailed exposition and references.

In [Har87] Harnack proved the following result in the case $d = 2$.

Theorem 3.1.1 [**Har87**] *Let $u : B_R(x_0) \subset \mathbb{R}^d \to \mathbb{R}$ be a harmonic function which is either non-negative or non-positive. Then the value of u at any point in $B_r(x_0)(r < R)$ is bounded from above and below by the quantities*

$$u(x_0)\frac{R-r}{R+r}\left(\frac{R}{R+r}\right)^{d-2} \quad \textit{and} \quad u(x_0)\frac{R+r}{R-r}\left(\frac{R}{R-r}\right)^{d-2}.$$

This assertion holds for any harmonic function and any ball $B_R(x_0)$. Another popular presentation in textbooks is as follows.

Corollary 3.1.2 *For any given domain $\Lambda \subset \mathbb{R}^d$ and proper subdomain $\Lambda' \subset \Lambda$ there exists a constant $C = C(d, \Lambda, \Lambda')$ such that for any non-negative harmonic function $u : \Lambda \to \mathbb{R}$*

$$\sup_{x\in\Lambda'} u(x) \leq C \inf_{x\in\Lambda'} u(x).$$

But it is very difficult to establish a analog estimate for non-negative solutions to the heat equation. Until 1954, this problem was solved independently by Pini [Pin54] and Hadamard [Had54]. The following sharp version of the result was obtained by Moser.

Theorem 3.1.3 [**Mos64**] *Let $u \in C^\infty((0,\infty) \times \mathbb{R}^d)$ be a non-negative solution of the heat equation, then*

$$u(t,x) \leq u(t+s,y)\left(\frac{t+s}{t}\right)^{d/2} \exp\left(\frac{|\, y-x\,|^2}{4s}\right), \; x,y \in \mathbb{R}^d, \; t,s > 0.$$

Note that one has to use time-shift in the comparison estimate. As in the elliptic case, a very important consequence of the Harnack inequality is that bounded weak solutions to parabolic equation are locally Hölder continuous. Another major breakthrough in Harnack inequality was obtained by Krylov and Safonov, where they established the parabolic and elliptic Harnack inequalities for partial differential operators in non-divergence form.

Now we turn to Harnack inequality in the non-Euclidean case. Bombieri and Giusti proved the Harnack inequality for elliptic differential equations on minimal surfaces using a geometric analysis technique and Yau proved the elliptic Harnack inequality for Riemannian manifolds. Here we only present the well-known Li-Yau's parabolic Harnack inequality, which was established in [LY86] for the Riemannian manifolds with Ricci curvature bounded from below. Then we will explain the reason why a new type of Harnack inequality is needed in applications, especially for infinite dimensional models.

Let M be a d-dimensional compact connected Riemannian manifold such that for some constant $K \geq 0$

$$Ric(X,X) \geq -K|X|^2, \quad X \in TM,$$

where Ric is the Ricci curvature. Let $P_t := e^{t\Delta}(t \geq 0)$ be the heat semigroup.

Theorem 3.1.4 [**LY86**] *For any $s,t > 0$, $p > 1$ and nonnegative $f \in C_b(M)$ we have*

$$P_t f(x) \leq (P_{t+s} f(y)) \left(\frac{t+s}{t}\right)^{pd/2} \exp\left[\frac{p\rho(x,y)^2}{4s} + \frac{pdKs}{4(p-1)}\right], \; x, y \in M,$$

where $\rho(x,y)$ is the Riemannian distance between x and y.

This inequality has been widely used in the geometric analysis, for instance, to estimate heat kernel, first eigenvalue and log-Sobolev constant etc. Moreover, this parabolic Harnack inequality also reflects some properties on the structure of the underlying manifold. For instance, Grigor'yan and Saloff-Coste proved in [Gri91, SC92] that Harnack inequality is equivalent to a volume doubling condition and a weak version of Poincaré's inequality.

However, Li-Yau's Harnack inequality involves the dimension of the underlying manifold explicitly in the estimate, hence it is difficult to be extended to infinite dimensional models. Moreover, the Ricci curvature condition above also excludes many important models like the Ornstein-Uhlenbeck operator $\Delta - x \cdot \nabla$ on the Euclidean space. Since the Ornstein-Uhlenbeck process plays a fundamental role in the stochastic analysis, it would be very useful to establish a new type of Harnack inequality which also works for the operators without the dimension-curvature condition (cf.[BQ99]) and for infinite dimensional models. This is the main motivation for the following dimension-free Harnack inequality, which was first introduced by Wang in [Wan97] for diffusions on Riemannian manifolds.

Consider $L := \Delta + Z$ for some C^1-vector field Z such that

$$Ric(X,X) - \langle \nabla_X Z, X\rangle \geq -K|X|^2, \ X \in TM \tag{3.1.1}$$

holds for some $K \in \mathbb{R}$, then the corresponding semigroup $P_t := e^{tL}$ satisfies the well-known gradient estimate

$$|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|, \ t > 0, \ f \in C_b^1(M).$$

Theorem 3.1.5 [**Wan97**] *The curvature condition* (3.1.1) *holds if and only if for any $p > 1$ and nonnegative $f \in C_b(M)$*

$$(P_t f(x))^p \leq (P_t f^p(y)) \exp\left[\frac{pK\rho(x,y)^2}{2(p-1)(1-e^{-2Kt})}\right], \ x, y \in M,$$

where $\rho(x,y)$ is the Riemannian distance between x and y.

Remark 3.1.1 *As we explained before, the Ornstein-Uhlenbeck semigroup does not satisfy Li-Yau's Harnack inequality (cf.[LW03]) but satisfies the present inequality for $K = -1$.*

This dimension-free Harnack inequality turned out to be a very efficient tool for the study of finite and infinite dimensional diffusion semigroups in recent years. For example, it has been applied to study functional inequalities in [Wan99, Wan01, RW03b, RW03a]; the short time behavior of infinite-dimensional diffusions in [AK01, AZ02, Kaw05]; the estimation of high order eigenvalues in [GW04, Wan00]; the transportation-cost inequality in [BGL01] and heat kernel estimates in [GW01].

Recently, the dimension-free Harnack inequality was established in [Wan07] for a class of stochastic porous media equations and in [LW08] for stochastic fast-diffusion equations. As applications, an estimate of transition density, ergodicity and some contractivity properties were obtained for the associated transition semigroups. The approach used in [Wan07, LW08] is mainly based on a new coupling argument developed in [ATW06], where Harnack inequality was derived for the diffusion semigroups on Riemannian manifolds with curvatures unbounded below. The advantage of this approach is that one can avoid the assumption that the curvature is lower bounded, which was required in many articles (cf. [AK01, AZ02, BGL01, RW03a, RW03b]) in an essential way and would be very hard to verify in the framework of nonlinear SPDE. In this chapter we will establish the Harnack inequality and many resulting properties for the transition semigroups associated with a large class of SPDE, which include stochastic reaction-diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation *etc.* In particular, it generalizes the main results obtained in [Wan07] for stochastic porous media equations.

3.2 Review on the strong Feller property and uniqueness of invariant measures

The strong Feller property (SFP) of Markov semigroup was introduced by Girsanov [Gir60] in 1960 for the connection with probabilistic potential theory. It's a very useful tool in the ergodic theory of Markov process. For example, the strong Feller property together with (topological) irreducibility imply the uniqueness of invariant measures and strong asymptotic stability, *i.e.* the probability law of the process converges to invariant measure under the total variation norm. Moreover, the strong Feller property may give a quite complete description of the long time behavior of a Markov process and can be used to establish a recurrence-transience dichotomy (cf.[MS02]). For more detailed review on the SFP and the uniqueness of invariant measures we refer to the survey articles [MS99, MS02, Hai03].

Strong Feller property The strong Feller property may hold for deterministic systems only in some very special cases, therefore this property indicates that a stochastic system is sufficiently non-degenerate. For finite dimensional non-degenerate SDE, a standard way to show the uniqueness of invariant measures is to use the correspondence between transition densities and the fundamental solution to corresponding Kolmogorov equation. The smoothing properties of Kolmogorov equation can yield the strong Feller property and the irreducibility of associated Markov process. Then the classical results in the ergodic theory of Markov processes, as developed by Doob, Maruyama, Tanaka, Khas'minskiĭ and others, can be applied to obtain the uniqueness of invariant measures as well as the strong asymptotic stability.

For infinite dimensional state spaces, there exist also several methods to establish the similar results for nonlinear stochastic systems. The SFP had been proved for semilinear systems by finite dimensional approximations in the early paper of Maslowski [Mas89]. And a controllability method to prove the irreducibility was also developed there. Later the SFP for reaction-diffusion equations with additive noise was obtained by using the smoothing properties of mild solutions to the associated backward Kolmogorov equation, which was established by Da Prato and Zabczyk (cf.[DPZ92c]). For further results we refer to the works of Goldys *et al* [GG97, CMG95], where the infinite-dimensional Kolmogorov equations and their links with invariant measure were deeply studied also. But these methods mainly works for stochastic equations with additive noise.

Another way of proving the SFP is the Bismut-Elworthy formula, which first appeared in the paper [DPEZ95] by Da Prato *et al*. They derived a formula for the directional derivatives of Markov transition semigroup involving the L^2-derivative of the solution w.r.t. initial condition. Later this approach was extended by Peszat and Zabczyk [PZ95] to stochastic parabolic equations with multiplicative noise. We refer to [MS99, MS02] for more references, where this method has been applied to investigate various important systems such as stochastic Burgers equations, stochastic Cahn-Hilliard equations, two-dimensional stochastic Navier-Stokes equations and rather general stochastic reaction-diffusion equations.

By using the Malliavin calculus and Girsanov transformation Fuhrman [Fuh96] proved the smoothing properties (in particular, SFP and irreducibility) of transition semigroup associated to stochastic equations. A probabilistic approach for SFP was developed by Maslowski and Seidler in [MS00], and the main idea is to show the SFP may be preserved under Girsanov transformations.

The strong Feller property is very efficient for studying the long time behavior of Markov processes, it usually can give a quite complete description of the qualitative behavior of the solution to the considered SPDE. But the SFP usually requires that the stochastic equations are driven by sufficiently non-degenerate noise. However, such non-degeneracy assumption is not necessary for the uniqueness and stability of invariant measures. So it is reasonable to find some other methods for studying the long time behavior of SPDE with degenerate noise.

Uniqueness of invariant measures Like the finite dimensional case, the uniqueness of invariant measures may be obtained from some pathwise stability of the process, which is often investigated by using the Lyapunov (function) techniques. This method was used by Ichikawa [Ich84] to establish the uniqueness of invariant measures for stochastic evolution equations. Later it was further developed by Maslowski, Leha and Ritter *etc* (cf.[MS99]).

The dissipativity method (remote start method) was first developed by Da Prato and Zabczyk in [DPZ92a, DPZ92b] for stochastic equations with additive noise, later it was extended to multiplicative noise case in [DPGZ92]. We refer to the monograph [DPZ96] for more systematic description.

Some analytic approaches were also used to study invariant measures for infinite dimensional stochastic systems. We refer to [Str93, Zeg95] for the log-Sobolev inequality method and [BR95, BKR96, BRZ00] for the Dirichlet form techniques.

The coupling method is also a very efficient tool for establishing the uniqueness of invariant measures for SPDE. This method can be traced back to the Doeblin's work [Doe38] on Markov chains and it is one of the main tools in particle systems (cf.[Che04]). The first use of coupling for SPDE up to our knowledge was due to Mueller [Mue93], who used this technique to prove the uniqueness of invariant measures for the stochastic heat equation. Recently, the coupling method has been used to prove the ergodicity and exponential convergence to invariant measure for the Navier-Stokes equations driven by degenerate noises [KS01, KS02, Mat99, Mat02]. This method has also been applied in [Hai02] for the stochastic reaction-diffusion equations, in [DPDT05] for the stochastic Burgers equations and in [Oda06] for the stochastic Ginzburg-Landau equations. For highly degenerate noise, by using the concept of asymptotic strong Feller property, the uniqueness of invariant measures has been established by Hairer and Mattingly [HM04, HM06] for the 2D stochastic Navier-Stokes equations. We refer to the review papers [Mat03, Hai03] on this subject for more references.

In this chapter we employ a coupling method to establish the Harnack inequality and strong Feller property for the transition semigroups of SPDE. The coupling we constructed here shows some different features with those works mentioned above. For example, the coupling time usually require to be finite almost surely in the classical coupling approach, but in our case the coupling time needs to be less than some fixed time almost surely due to the special construction.

3.3 Harnack inequality and its applications: the main results

Consider the Gelfand triple

$$V \subset H \equiv H^* \subset V^*$$

and the stochastic evolution equation with additive noise

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B_t \mathrm{d}W_t, \ X_0 = x \in H, \tag{3.3.1}$$

where W_t is a cylindrical Wiener process on U and

$$A : [0, \infty) \times V \times \Omega \to V^*; \ B : [0, \infty) \times \Omega \to L_2(U, H)$$

are progressively measurable. We intend to establish the Harnack inequality for the associated transition "semigroup"

$$P_t F(x) := \mathbf{E} F(X_t(x)), \ t \geq 0, \ x \in H,$$

where F is a bounded measurable function on H and $X_t(x)$ is the solution to (3.3.1) with starting point x. We need to assume $B_t(\omega)$ is non-degenerate for $t \geq 0$ and $\omega \in \Omega$; that is, $B_t(\omega)y = 0$ implies $y = 0$. Then for any $u \in V$

$$\|u\|_{B_t} := \begin{cases} \|y\|_U, & \text{if } y \in U, \ B_t y = u; \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 3.3.1 *Suppose A is hemicontinuous and for a fixed exponent $\alpha > 1$ we have*

$$\|A(t, v)\|_{V^*} \leq K(1 + \|v\|_V^{\alpha - 1}), \ v \in V, \tag{3.3.2}$$

where K is a constant. If there exist constant $\sigma \geq 2, \sigma > \alpha - 2$ and continuous functions $\delta, \gamma, \xi \in \mathbf{C}[0, \infty)$ such that for any $t \geq 0, \omega \in \Omega$ and $u, v \in V$ we have

$$2_{V^*}\langle A(t, u) - A(t, v), u - v \rangle_V \leq -\delta_t \|u - v\|_V^\alpha + \gamma_t \|u - v\|_H^2, \tag{3.3.3}$$

$$\|u\|_V^\alpha \geq \xi_t \|u\|_{B_t}^\sigma \|u\|_H^{\alpha-\sigma}, \tag{3.3.4}$$

where ξ, δ are strictly positive on $[0, \infty)$, then P_t is a strong Feller operator for any $t > 0$. And for any $p > 1$ and positive bounded measurable function F on H we have

$$(P_t F(y))^p \leq P_t F^p(x) \exp\left[\frac{p}{p-1} C(t,\sigma) \|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right], \ x, y \in H, \tag{3.3.5}$$

where

$$C(t,\sigma) = \frac{2(\sigma+2)^{2+\frac{2}{\sigma}} t^{\frac{\sigma-2}{\sigma}}}{(\sigma+2-\alpha)^{2+\frac{2}{\sigma}} \left[\int_0^t (\delta_s \xi_s)^{\frac{1}{\sigma}} \exp(\frac{\alpha-2-\sigma}{2\sigma} \int_0^s \gamma_u \mathrm{d}u) \mathrm{d}s\right]^2}.$$

Let us first explain the main idea of the proof. To prove the Harnack inequality (3.3.5), for any fixed time T it is sufficient to construct a coupling (X_t, Y_t), which is a continuous adapted process on $H \times H$ such that

(i) X_t solves (3.3.1) with $X_0 = x$;

(ii) Y_t solves the following equation

$$\mathrm{d}Y_t = A(t, Y_t)\mathrm{d}t + B_t \mathrm{d}\tilde{W}_t, \ Y_0 = y \in H,$$

for another cylindrical Wiener process $\tilde{W}_t$ on U under a weighted probability measure $R\mathbf{P}$, where $\tilde{W}_t$ and the density R will be constructed later by a Girsanov transformation;

(iii) $X_T = Y_T, a.s.$.

As soon as (i)-(iii) are satisfied, then we have

$$\begin{aligned} P_T F(y) &= \mathbf{E}(RF(Y_T)) = \mathbf{E}(RF(X_T)) \\ &\leq (\mathbf{E}R^{p/(p-1)})^{(p-1)/p} (\mathbf{E}F^p(X_T))^{1/p} \\ &= (\mathbf{E}R^{p/(p-1)})^{(p-1)/p} (P_T F^p(x))^{1/p}, \end{aligned} \tag{3.3.6}$$

which implies the desired Harnack inequality (3.3.5) provided $\mathbf{E}R^{p/(p-1)} < \infty$.

Now we construct the coupling process Y_t. We take $\varepsilon \in (0,1)$, $\beta \in \mathbf{C}([0,\infty); \mathbb{R}_+)$ and consider the equation

$$\mathrm{d}Y_t = \left(A(t, Y_t) + \frac{\beta_t (X_t - Y_t)}{\|X_t - Y_t\|_H^\varepsilon} \mathbf{1}_{\{t<\tau\}}\right) \mathrm{d}t + B_t \mathrm{d}W_t, \ Y_0 = y \in H, \tag{3.3.7}$$

where $X_t := X_t(x)$ and $\tau := \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time.

First we prove that (3.3.7) also has a unique strong solution $Y_t(y)$ by using a similar argument as in [Wan07, Theorem A.2].

Lemma 3.3.2 *If $\varepsilon \in (0,1)$, then there exists a unique strong solution Y_t to (3.3.7). Moreover, we have $X_t = Y_t$ for all $t \geq \tau$.*

Proof. According to Theorem 1.2.1, we only have to verify $(H1)-(H4)$ for the coefficients of (3.3.7). Let

$$\mathbb{A}(t,u) := \frac{X_t - u}{\|X_t - u\|_H^{\varepsilon}} \mathbf{1}_{\{t<\tau\}}.$$

Since $\mathbf{E} \sup\limits_{t\in[0,T]} \|X_t\|_H^2 < \infty$, $\mathbb{A}(t,u) \in H$ and

$$\|\mathbb{A}(t,u)\|_H = \|X_t - u\|_H^{1-\varepsilon} \mathbf{1}_{\{t<\tau\}}, \qquad u \in V.$$

Then it is easy to see that $(H1), (H3)$ and $(H4)$ hold.

To verify $(H2)$, it is enough to prove the following monotonicity

$$\langle \mathbb{A}(t,x) - \mathbb{A}(t,y), x-y\rangle_H \leq 0 \text{ on } \Omega, \quad x,y \in V. \tag{3.3.8}$$

By the symmetry, for a fixed $\omega \in \Omega$ it is sufficient to verify (3.3.8) for $x, y \in V$ with

$$\|X_t - x\|_H \leq \|X_t - y\|_H. \tag{3.3.9}$$

(i) If $\|X_t - x\|_H \geq \|x-y\|_H$, then by (3.3.9) and the mean-valued theorem we have

$$\begin{aligned}
&\langle \mathbb{A}(t,x) - \mathbb{A}(t,y), x-y\rangle_H \\
&= -\frac{\|x-y\|_H^2}{\|X_t - x\|_H^{\varepsilon}} + \frac{\|X_t - y\|_H^{\varepsilon} - \|X_t - x\|_H^{\varepsilon}}{\|X_t - y\|_H^{\varepsilon}\|X_t - x\|_H^{\varepsilon}} \langle X_t - y, x-y\rangle_H \\
&\leq -\frac{\|x-y\|_H^2}{\|X_t - x\|_H^{\varepsilon}} + \frac{\varepsilon\|X_t - y\|_H^{1-\varepsilon}\|x-y\|_H^2}{\|X_t - x\|_H} \\
&\leq -\frac{\|x-y\|_H^2}{\|X_t - x\|_H^{\varepsilon}} + \frac{\varepsilon 2^{-\varepsilon}(\|X_t - x\|_H^{1-\varepsilon} + \|x-y\|_H^{1-\varepsilon})\|x-y\|_H^2}{\|X_t - x\|_H} \\
&\leq -\frac{(1-\varepsilon 2^{1-\varepsilon})\|x-y\|_H^2}{\|X_t - x\|_H^{\varepsilon}} \leq 0,
\end{aligned}$$

where in the third step we use the following inequality

$$(a+b)^r \leq 2^{r-1}(a^r + b^r), \ a, b \geq 0, \ r > 0.$$

(ii) If $\|X_t - x\|_H \leq \|x-y\|_H$, (3.3.8) can be proved by a similar argument.

Therefore, (3.3.7) also has a unique strong solution Y_t. Moreover, by (3.3.3) we have

$$\|X_t - Y_t\|_H^2 \leq \|X_s - Y_s\|_H^2 + \int_s^t \left(-\delta_u \|X_u - Y_u\|_V^\alpha + \gamma_u \|X_u - Y_u\|_H^2\right) \mathrm{d}u$$

for all $0 \leq s \leq t$. Hence we have $X_t = Y_t$ for $t \geq \tau$ by using Gronwall's lemma. ∎

Now we will prove the coupling time $\tau \leq T$ *a.s.* by choosing β_t appropriately in (3.3.7).

Lemma 3.3.3 *If β satisfies $\int_0^T \beta_t e^{-\frac{\varepsilon}{2}\int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t \geq \frac{2}{\varepsilon}\|x-y\|_H^\varepsilon$, then $X_T = Y_T$, a.s..*

Proof. It is easy to show that

$$e^{-\int_0^t \gamma_s \mathrm{d}s}\|X_t - Y_t\|_H^2 \leq \|x-y\|_H^2 - \int_0^t e^{-\int_0^u \gamma_s ds}\Big(\delta_u \|X_u - Y_u\|_V^\alpha + \beta_u \|X_u - Y_u\|_H^{2-\varepsilon} \mathbf{1}_{\{u<\tau\}}\Big)\mathrm{d}u. \tag{3.3.10}$$

By (3.3.10) and the chain rule we have

$$\left\{e^{-\int_0^t \gamma_s \mathrm{d}s}\|X_t - Y_t\|_H^2\right\}^{\varepsilon/2} \leq \|x-y\|_H^\varepsilon - \frac{\varepsilon}{2}\int_0^t \beta_s e^{-\frac{\varepsilon}{2}\int_0^s \gamma_u \mathrm{d}u} \mathrm{d}s, \ t \leq \tau \wedge T.$$

If $T < \tau(\omega_0)$ for some $\omega_0 \in \Omega$, then by taking $t = T$ and using the assumption we have

$$e^{-\frac{\varepsilon}{2}\int_0^T \gamma_s \mathrm{d}s}\|X_T(\omega_0) - Y_T(\omega_0)\|_H^\varepsilon \leq \|x-y\|_H^\varepsilon - \frac{\varepsilon}{2}\int_0^T \beta_t e^{-\frac{\varepsilon}{2}\int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t \leq 0.$$

This implies $X_T(\omega_0) = Y_T(\omega_0)$, which contradicts with the assumption $T < \tau(\omega_0)$.

Hence $\tau \leq T$, *i.e.* $X_T = Y_T, a.s.$. ∎

Proof of Theorem 3.3.1 : Let $\varepsilon = 1 - \frac{\alpha}{\sigma+2} \in (0,1)$, then by (3.3.10) and (3.3.4) we have

$$\begin{aligned}
\mathrm{d}\left\{\|X_t - Y_t\|_H^2 e^{-\int_0^t \gamma_s \mathrm{d}s}\right\}^\varepsilon &\leq -\varepsilon\delta_t e^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\|X_t - Y_t\|_H^{2(\varepsilon-1)}\|X_t - Y_t\|_V^\alpha \mathrm{d}t \\
&\leq -\varepsilon\delta_t\xi_t e^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\frac{\|X_t - Y_t\|_{B_t}^\sigma}{\|X_t - Y_t\|_H^{2+\sigma-\alpha-2\varepsilon}}\mathrm{d}t \\
&= -\varepsilon\delta_t\xi_t e^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}\frac{\|X_t - Y_t\|_{B_t}^\sigma}{\|X_t - Y_t\|_H^{\sigma\varepsilon}}\mathrm{d}t \\
&= -\frac{\beta_t^\sigma \|X_t - Y_t\|_{B_t}^\sigma}{c^\sigma \|X_t - Y_t\|_H^{\sigma\varepsilon}}\mathrm{d}t,
\end{aligned} \tag{3.3.11}$$

where we take

$$\beta_t^\sigma = c^\sigma \varepsilon\delta_t\xi_t e^{-\varepsilon\int_0^t \gamma_s \mathrm{d}s}, \quad c = \frac{2\|x-y\|_H^\varepsilon}{\varepsilon\int_0^T (\varepsilon\delta_t\xi_t)^{\frac{1}{\sigma}} e^{-(\frac{1}{2}+\frac{1}{\sigma})\varepsilon\int_0^t \gamma_s \mathrm{d}s}\mathrm{d}t}.$$

Let

$$\zeta_t := \frac{\beta_t B_t^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_H^{\varepsilon}} \mathbf{1}_{\{t<\tau\}}.$$

By using Hölder's inequality and (3.3.11) we obtain

$$\begin{aligned} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t &= \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_{B_t}^2}{\|X_t - Y_t\|_H^{2\varepsilon}} \mathrm{d}t \\ &\le T^{\frac{\sigma-2}{\sigma}} \Big(\int_0^T \frac{\beta_t^{\sigma} \|X_t - Y_t\|_{B_t}^{\sigma}}{\|X_t - Y_t\|_H^{\sigma\varepsilon}} \mathrm{d}t \Big)^{\frac{2}{\sigma}} \\ &\le T^{\frac{\sigma-2}{\sigma}} \Big(c^{\sigma} \|x - y\|_H^{2\varepsilon} \Big)^{\frac{2}{\sigma}}. \end{aligned} \tag{3.3.12}$$

Hence we have

$$\mathbf{E} \exp\Big[\frac{1}{2}\int_0^T \|\zeta_t\|_U^2 \mathrm{d}t\Big] < \infty. \tag{3.3.13}$$

Therefore, we can rewrite (3.3.7) as

$$\mathrm{d}Y_t = A(t, Y_t)\mathrm{d}t + B_t \mathrm{d}\tilde{W}_t,\ Y_0 = y,$$

where

$$\tilde{W}_t := W_t + \int_0^t \zeta_s \mathrm{d}s.$$

By (3.3.13) and the Girsanov theorem (cf.[DPZ92c, Theorem 10.14, Proposition 10.17]) we know that $\{\tilde{W}_t\}$ is a cylindrical Brownian motion on U under the weighted probability measure $R\mathbf{P}$, where

$$R = \exp\Big[\int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^T \|\zeta_t\|_U^2 \mathrm{d}t\Big].$$

Therefore, the distribution of $\{Y_t(y)\}_{t\in[0,T]}$ under $R\mathbf{P}$ is same with the distribution of $\{X_t(y)\}_{t\in[0,T]}$ under $\mathbf{P}$. Let $p' = \frac{p}{p-1}$, then for any $q > 1$

$$\begin{aligned}
\mathbf{E}R^{p'} &= \exp\left[p'\int_0^T \langle\zeta_t, \mathrm{d}W_t\rangle - \frac{p'}{2}\int_0^T \|\zeta_t\|_U^2\mathrm{d}t\right] \\
&\leq \left[\mathbf{E}\exp(qp'\int_0^T \langle\zeta_t, \mathrm{d}W_t\rangle - \frac{q^2(p')^2}{2}\int_0^T \|\zeta_t\|_U^2\mathrm{d}t)\right]^{\frac{1}{q}} \\
&\quad \times \left[\mathbf{E}\exp(\frac{qp'(qp'-1)}{2(q-1)}\int_0^T \|\zeta_t\|_U^2\mathrm{d}t)\right]^{\frac{q-1}{q}} \\
&\leq \left[\mathbf{E}\exp(\frac{qp'(qp'-1)}{2(q-1)}\int_0^T \|\zeta_t\|_U^2\mathrm{d}t)\right]^{\frac{q-1}{q}} \\
&\leq \exp\left[\frac{p'(qp'-1)}{2}T^{\frac{\sigma-2}{\sigma}}\left(c^\sigma\|x-y\|_H^{2\varepsilon}\right)^{\frac{2}{\sigma}}\right].
\end{aligned} \tag{3.3.14}$$

Letting $q \downarrow 1$ we get

$$\begin{aligned}
(P_TF(y))^p &\leq P_TF^p(x)(\mathbf{E}R^{p'})^{p'-1} \\
&\leq P_TF^p(x)\exp\left[\frac{p}{p-1}C(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right],
\end{aligned} \tag{3.3.15}$$

where

$$C(t,\sigma) = \frac{2(\sigma+2)^{2+\frac{2}{\sigma}}t^{\frac{\sigma-2}{\sigma}}}{(\sigma+2-\alpha)^{2+\frac{2}{\sigma}}\left[\int_0^t(\delta_s\xi_s)^{\frac{1}{\sigma}}\exp(\frac{\alpha-2-\sigma}{2\sigma}\int_0^s\gamma_u\mathrm{d}u)\mathrm{d}s\right]^2}.$$

From (3.3.14) we know that R is uniformly integrable, then by the dominated convergence theorem we have

$$\lim_{y\to x}\mathbf{E}|R-1| = \mathbf{E}\lim_{y\to x}|R-1| = 0.$$

Hence for any bounded measurable function F on H

$$|P_TF(y) - P_TF(x)| = |\mathbf{E}RF(X_T) - \mathbf{E}F(X_T)| \leq \|F\|_\infty\mathbf{E}|R-1| \to 0\ (y\to x).$$

This implies $P_TF \in C_b(H)$. Therefore, P_T is a strong Feller operator.

Now the proof of Theorem 3.3.1 is complete.

Remark 3.3.1 *(1) Note that here we use the framework in [KR79]. One can easily formulate the similar results under more general framework in [RRW07, Zha08].*

(2) This theorem covers the main result in [Wan07] for stochastic porous media equations. Moreover, if we replace $\|\cdot\|_V^\alpha$ *in* (3.3.3) *and* (3.3.4) *by* $\mathbf{m}(\mathbf{g}(\cdot))$ *for some Young function* $\mathbf{g}$*, then this theorem can also be applied to stochastic generalized porous media equations in [RRW07] involving Orlicz spaces.*

(3) The coupling we used here only depends on the natural distance between two marginal processes. Such a stronger Harnack inequality (the estimate only depends on the usual norm) provides more information such as the hyperbounded or ultrabounded property of the associated transition semigroups (see Theorem 3.3.5).

(4) (3.3.4) *implies that* V *is contained in the range of* B_t *(as a operator from* U *to* H*) for fixed* t *and* ω*. If we assume* $V \equiv H$*, then we know* B_t *is a bijection map and its inverse operator is also continuous from* H *to* U*. Since* B_t *is a Hilbert-Schmidt operator, then* H *and* U *have to be finite-dimensional space. In this case* (3.3.4) *holds provided* B_t *are invertible.*

(5) Stochastic fast diffusion equations in [RRW07] and the singular stochastic p-Laplace equation ($1 < p < 2$) *does not satisfy the assumption* (3.3.3)*, but we will establish the Harnack inequality, strong Feller property and heat kernel estimates in the subsequent chapters by using more delicate estimates.*

Theorem 3.3.4 *Suppose the coefficients* A, B *in* (3.3.1) *are deterministic and time-independent. The embedding* $V \subseteq H$ *is compact and* A *is hemicontinuous such that* (3.3.2) *and* (3.3.3) *hold.*

(i) If $\gamma \leq 0$ *also holds in the case* $\alpha \leq 2$*, then the Markov semigroup* $\{P_t\}$ *has an invariant probability measure* μ *satisfying* $\mu\left(\|\cdot\|_V^\alpha + e^{\varepsilon_0\|\cdot\|_H^\alpha}\right) < \infty$ *for some* $\varepsilon_0 > 0$*.*

(ii) If $\alpha = 2$ *, then for any* $x, y \in H$ *we have*

$$\|X_t(x) - X_t(y)\|_H^2 \leq e^{(\gamma - c_0\delta)t}\|x - y\|_H^2, \ t \geq 0,$$

where c_0 *is the constant such that* $\|\cdot\|_V^2 \geq c_0\|\cdot\|_H^2$ *holds.*

Moreover, if $\gamma < c_0\delta$*, then there exists a unique invariant measure* μ *of* $\{P_t\}$ *and for any Lipschitz continuous function* F *on* H *we have*

$$|P_tF(x) - \mu(F)| \leq Lip(F)e^{-(c_0\delta-\gamma)t/2}(\|x\|_H + C), \ x \in H, \qquad (3.3.16)$$

where $C > 0$ is a constant and $Lip(F)$ is the Lipschitz constant of F.

(iii) If $\alpha > 2$ and $\gamma \leq 0$, then there exists a constant C such that

$$\|X_t(x) - X_t(y)\|_H^2 \leq \|x - y\|_H^2 \wedge \left\{Ct^{-\frac{2}{\alpha-2}}\right\}, \ t > 0, \ x, y \in H,$$

where $X_t(y)$ is the solution to (3.3.1) with starting point y.

Therefore, $\{P_t\}$ has a unique invariant measure μ and for any Lipschitz continuous function F on H we have

$$\sup_{x\in H} |P_t F(x) - \mu(F)| \leq C Lip(F) t^{-\frac{1}{\alpha-2}}, \ t > 0. \tag{3.3.17}$$

In particular, if $B = 0$ and Dirac measure at 0 is the unique invariant measure of $\{P_t\}$, then we can take $F(x) = \|x\|_H$ in (3.3.17) and have

$$\sup_{x\in H} \|X_t(x)\|_H \leq Ct^{-\frac{1}{\alpha-2}}, \ t > 0.$$

Proof. (i) In the present case, $\{P_t\}$ is a Markov semigroup (cf.[KR79, PR07]). The existence of an invariant measure can be proved by the standard Krylov-Bogoliubov procedure (cf. [PR07, Wan07]). Let

$$\mu_n := \frac{1}{n} \int_0^n \delta_0 P_t \mathrm{d}t, \ n \geq 1,$$

where δ_0 is the Dirac measure at 0. Recall $X_t(y)$ is the solution to (3.3.1) with starting point y, then by (3.3.3) and Gronwall's lemma we have

$$\|X_t(x) - X_t(y)\|_H^2 \leq e^{\gamma t} \|x - y\|_H^2, \quad \forall x, y \in H.$$

This implies that P_t is a Feller semigroup.

Hence for the existence of an invariant measure, it is well-known that one only needs to verify the tightness of $\{\mu_n : n \geq 1\}$.

Since $\gamma \leq 0$ in the case $\alpha \leq 2$, then by (3.3.3) and (3.3.2) we have

$$\begin{aligned} 2{}_{V^*}\langle A(x), x\rangle_V &\leq -\delta\|x\|_V^\alpha + \gamma\|x\|_H^2 + 2\ {}_{V^*}\langle A(0), x\rangle_V \\ &\leq \theta_2 - \theta_1\|x\|_V^\alpha \end{aligned} \tag{3.3.18}$$

for some constants $\theta_1, \theta_2 > 0$. By using the Itô formula we have

$$\|X_t\|_H^2 \le \|x\|_H^2 + \int_0^t (c - \theta_1 \|X_s\|_V^\alpha) \mathrm{d}s + 2\int_0^t \langle X_s, B\mathrm{d}W_s\rangle_H, \tag{3.3.19}$$

where $c > 0$ is some constant which may change from line to line.

Note that $M_t := \int_0^t \langle X_s, B\mathrm{d}W_s\rangle_H$ is a martingale, then (3.3.19) implies that

$$\mu_n(\|\cdot\|_V^\alpha) = \frac{1}{n}\int_0^n \mathbf{E}\|X_t(0)\|_V^\alpha \mathrm{d}t \le \frac{c}{\theta_1},\ n \ge 1. \tag{3.3.20}$$

Since the embedding $V \subseteq H$ is compact, (3.3.20) implies that $\{\mu_n\}$ is tight. Hence the limit of a convergent subsequence provides an invariant measure μ of $\{P_t\}$.

Now we need to prove the concentration property of μ. If ε_0 is small enough, then by (3.3.19) and Itô's formula

$$\begin{aligned}
e^{\varepsilon_0\|X_t\|_H^\alpha} \le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c - \theta_1\|X_s\|_V^\alpha + \alpha\varepsilon_0\|B\|_2^2\|X_s\|_H^\alpha\right)\frac{\alpha\varepsilon_0}{2}\|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\mathrm{d}s \\
& + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\langle X_s, B\mathrm{d}W_s\rangle_H \\
\le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c - c_1\|X_s\|_H^\alpha\right)\frac{\alpha\varepsilon_0}{2}\|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\mathrm{d}s \\
& + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^q}\langle X_s, B\mathrm{d}W_s\rangle_H \\
\le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c_2 - c_3 e^{\varepsilon_0\|X_s\|_H^\alpha}\right)\mathrm{d}s + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\langle X_s, B\mathrm{d}W_s\rangle_H
\end{aligned} \tag{3.3.21}$$

holds for some positive constants c, c_1, c_2 and c_3. Therefore

$$\mu_n(e^{\varepsilon_0\|\cdot\|_H^\alpha}) = \frac{1}{n}\int_0^n \mathbf{E}e^{\varepsilon_0\|X_t(0)\|_H^\alpha}\mathrm{d}t \le \frac{1}{c_3 n} + \frac{c_2}{c_3},\ n \ge 1.$$

Hence we have $\mu(e^{\varepsilon_0\|\cdot\|_H^\alpha}) < \infty$ for some $\varepsilon_0 > 0$. In particular, this implies $\mu(\|\cdot\|_H^2) < \infty$.

By (3.3.19) there also exists a constant C such that

$$\mathbf{E}\int_0^1 \|X_t(x)\|_V^\alpha \mathrm{d}t \le C(1 + \|x\|_H^2),\ \forall x \in H.$$

Therefore

$$\mu(\|\cdot\|_V^\alpha) = \int_H \mu(\mathrm{d}x)\int_0^1 \mathbf{E}(\|X_t(x)\|_V^\alpha)\mathrm{d}t \le C + C\int_H \|x\|_H^2\mu(\mathrm{d}x) < \infty.$$

(*ii*) If $\alpha = 2$, then for any $x, y \in H$

$$\|X_t(x) - X_t(y)\|_H^2 \le \|x - y\|_H^2 + \int_0^t \left(-\delta\|X_s(x) - X_s(y)\|_V^2 + \gamma\|X_s(x) - X_s(y)\|_H^2\right) \mathrm{d}s.$$

By the Gronwall lemma we have

$$\|X_t(x) - X_t(y)\|_H^2 \le e^{(\gamma - c_0\delta)t}\|x - y\|_H^2, \ \forall x, y \in H.$$

If $\gamma < c_0\delta$, it is easy to show (3.3.18) still holds. Hence we can show that $\{P_t\}$ has an invariant measure by repeating the argument in (*i*). And we also have

$$\lim_{t\to\infty} \|X_t(x) - X_t(y)\|_H = 0, \ \forall x, y \in H.$$

By the dominated convergence theorem we know for any invariant measure μ and any bounded continuous function F

$$|P_tF(x) - \mu(F)| \le \int_H \mathbf{E}|F(X_t(x)) - F(X_t(y))|\mu(\mathrm{d}y) \to 0 \ (t \to \infty).$$

This implies the uniqueness of invariant measures.

We denote the invariant measure by μ. By (*i*) we know $\mu(\|\cdot\|_H^2) < \infty$, hence for any bounded Lipschitz function F on H we have

$$\begin{aligned}
|P_tF(x) - \mu(F)| &\le \int_H \mathbf{E}|F(X_t(x)) - F(X_t(y))|\mu(\mathrm{d}y) \\
&\le \mathrm{Lip}(F)e^{(\gamma - c_0\delta)t/2}\int_H \|x - y\|_H\mu(\mathrm{d}y) \\
&\le \mathrm{Lip}(F)e^{(\gamma - c_0\delta)t/2}\left(\|x\|_H + C\right), \ x \in H,
\end{aligned}$$

where $C > 0$ is a constant.

(*iii*) If $\alpha > 2$ and $\gamma \le 0$, then there exists a constant $c > 0$ such that

$$\|X_t(x) - X_t(y)\|_H^2 \le \|x - y\|_H^2 - c\int_0^t \|X_s(x) - X_s(y)\|_H^\alpha \mathrm{d}s, \ t \ge 0.$$

Suppose h_t solves the equation

$$h_t' = -ch_t^{\frac{\alpha}{2}}, \ h_0 = \left(\|x - y\|_H + \varepsilon\right)^2, \tag{3.3.22}$$

where ε is a positive constant. Then by a standard comparison argument we have

$$\|X_t(x) - X_t(y)\|_H^2 \le h_t \le Ct^{-\frac{2}{\alpha-2}}, \tag{3.3.23}$$

where $C > 0$ is a constant. In fact, we can define

$$\varphi_t := h_t - \|X_t(x) - X_t(y)\|_H^2, \quad \tau := \inf\{t \ge 0 : \ \varphi_t < 0\}.$$

If $\tau < \infty$, then we know $\varphi_\tau \le 0$ by the continuity.

By the mean-value theorem we have

$$\begin{aligned}\varphi_t &\ge \varphi_0 - c\int_0^t \left(h_s^{\frac{\alpha}{2}} - \|X_s(x) - X_s(y)\|_H^\alpha\right) \mathrm{d}s\\ &\ge \varepsilon^2 - K\int_0^t \varphi_s \mathrm{d}s, \ 0 \le t \le \tau,\end{aligned}$$

where $K > 0$ is some constant. Then by the Gronwall lemma we have

$$\varphi_\tau \ge \varepsilon^2 e^{-K\tau} > 0,$$

which is contradict to $\varphi_\tau \le 0$. Hence (3.3.23) holds.

Therefore, for any $x \in H$ and bounded Lipschitz function F on H we have

$$|P_t F(x) - \mu(F)| \le \int_H \mathbf{E}|F(X_t(x) - F(X_t(y)))|\mu(\mathrm{d}y) \le C\mathrm{Lip}(F)t^{-\frac{1}{\alpha-2}}.$$

Hence (3.3.17) holds and the uniqueness of invariant measures also follows.

∎

We recall that $\{P_t\}$ is called (topologically) irreducible if $P_t 1_M(\cdot) > 0$ on H for any $t > 0$ and nonempty open set M. Let $\{P_t\}$ be a semigroup defined on $L^2(\mu)$, then $\{P_t\}$ is called hyperbounded semigroup if $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ for some $t > 0$; $\{P_t\}$ is called ultrabounded semigroup if $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty$ for any $t > 0$.

Theorem 3.3.5 *Suppose the coefficients A, B in (3.3.1) are deterministic and time-independent such that all assumptions in Theorem 3.3.1 hold.*

(i) $\{P_t\}$ is irreducible and has a unique invariant measure μ with full support on H. Moreover, μ is strong mixing and for any probability measure ν on H we have

$$\lim_{t\to\infty} \| P_t^*\nu - \mu \|_{var} = 0,$$

where $\|\cdot\|_{var}$ *is the total variation norm and* P_t^* *is the adjoint operator of* P_t*.*

(ii) For any $x \in H$*,* $t > 0$ *and* $p > 1$*, the transition density* $p_t(x, y)$ *of* P_t *w.r.t* μ *satisfies*

$$\|p_t(x,\cdot)\|_{L^p(\mu)} \leq \left\{ \int_H \exp\left[-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}} \right] \mu(\mathrm{d}y) \right\}^{-\frac{p-1}{p}}.$$

(iii) If $\alpha = 2$ *and* $\gamma \leq 0$*, then* P_t *is hyperbounded and compact on* $L^2(\mu)$ *for some* $t > 0$*.*

(iv) If $\alpha > 2$ *and* $\gamma \leq 0$*, then* P_t *is ultrabounded and compact on* $L^2(\mu)$ *for any* $t > 0$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq \exp\left[C(1 + t^{-\frac{\alpha}{\alpha-2}}) \right], \ t > 0.$$

Proof. (i) By the definition of $\|\cdot\|_B$ and (3.3.4), for any constant K there exists $\overline{K} > 0$ such that

$$\{x \in H: \ \|x\|_B \leq K\} \subseteq \{Bu: \ u \in U; \ \|u\|_U \leq \overline{K}\};$$
$$\{x \in H: \ \|x\|_V \leq K\} \subseteq \{x \in H: \|x\|_B \leq \overline{K}\}.$$

Since B is a Hilbert-Schmidt (hence compact) operator, then the following set

$$\{x \in H: \ \|x\|_V \leq K\}$$

is relatively compact in H, *i.e.* the embedding $V \subseteq H$ is compact. Hence $\{P_t\}$ has an invariant measure according to Theorem 3.3.4.

Suppose μ is an invariant measure of P_t, then by taking $p = 2$ in (3.3.5) we have

$$(P_t 1_M(x))^2 \int_H e^{-2C(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}} \mu(\mathrm{d}y) \leq \int_H P_t 1_M(y)\mu(\mathrm{d}y) = \mu(M), \tag{3.3.24}$$

where M is a Borel set in H. Hence the transition kernel $P_t(x, dy)$ is absolutely continuous w.r.t. μ, and we denote the density by $p_t(x, y)$.

If μ does not have full support on H, then there exists $x_0 \in H$ and $r > 0$ such that

$$B(x_0; r) := \{y \in H: \|y - x_0\|_H \leq r\}$$

is a null set of μ. Then (3.3.24) implies that $P_t(x_0, B(x_0;r)) = 0$, *i.e.*

$$\mathbf{P}\left(X_t(x_0) \in B(x_0;r)\right) = 0, \quad t > 0,$$

Since $X_t(x_0)$ is a continuous process on H, we have $\mathbf{P}\left(X_0 \in B(x_0;r)\right) = 0$, which is contradict with $X_0 = x_0$. Therefore, μ has full support on H.

According to the Harnack inequality (3.3.5) we have

$$(P_t 1_M)^p(x_0) \le P_t 1_M(x) \exp\left[\frac{p}{p-1} C(t,\sigma)\|x - x_0\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right], \quad x, x_0 \in H.$$

Therefore, to prove the irreducibility, one only has to show for any given nonempty open set M and $t > 0$ there exists $x_0 \in H$ such that $P_t 1_M(x_0) > 0$.

Note that the full support property of μ implies

$$\int_H P_t 1_M(x)\mu(\mathrm{d}x) = \int_H 1_M(x)\mu(\mathrm{d}x) = \mu(M) > 0,$$

so $P_t 1_M(\cdot)$ cannot be the zero function. Therefore $\{P_t\}$ is irreducible.

Since $\{P_t\}$ also have the strong Feller property, then the uniqueness of invariant measure follows from the classical Doob theorem [Doo48] (or see [Hai03, Theorem 2.1]).

Note that the solution has continuous paths on H, then the other assertions follow from the general result in the ergodic theory (cf.[Sei97, Theorem 2.2 and Proposition 2.5] or [MS02]).

(ii) For any $p > 1$ and nonnegative measurable function f with $\mu(f^{p/(p-1)}) \le 1$, by replacing p with $p/(p-1)$ in (3.3.5) we have

$$\left(P_t f(x)\right)^{p/(p-1)} \le P_t f^{p/(p-1)}(y) \exp\left[pC(t,\sigma)\|x - y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right], \quad x, y \in H.$$

Taking integration w.r.t. $\mu(\mathrm{d}y)$ on both sides we have

$$\left(P_t f(x)\right)^{p/(p-1)} \int_H \exp\left[-pC(t,\sigma)\|x - y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right] \mu(\mathrm{d}y) \le \mu(f^{p/(p-1)}) \le 1.$$

This implies

$$P_t f(x) \le \left(\int_H \exp\left[-pC(t,\sigma)\|x - y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right] \mu(\mathrm{d}y)\right)^{-(p-1)/p}.$$

Note that

$$P_t f(x) = \int_H f(y) P_t(x, \mathrm{d}y) = \int_H f(y) p_t(x, y) \mu(\mathrm{d}y),$$

hence for $q = p/(p-1)$ we have

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^p(\mu)} &= \sup_{\|f\|_{L^q(\mu)} \le 1} \left| \int_H f(y) p_t(x,y)\mu(\mathrm{d}y) \right| \\ &\le \left(\int_H \exp\left[-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}} \right] \mu(\mathrm{d}y) \right)^{-(p-1)/p}.\end{aligned}$$

(iii) If $\gamma \le 0$, then by (3.3.5) there exists a constant $c > 0$ such that

$$(P_t f)^2(x) \exp\left[-\frac{c\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}}{t^{\frac{\sigma+2}{\sigma}}} \right] \le P_t f^2(y), \quad x, y \in H,\ t > 0. \tag{3.3.25}$$

By integrating on both sides w.r.t. $\mu(dy)$ we have for $f \in L^2(\mu)$ with $\mu(f^2) = 1$

$$(P_t f)^2(x) \le \frac{1}{\mu(B(0,1))} \exp\left[\frac{c(\|x\|_H + 1)^{2+\frac{2(2-\alpha)}{\sigma}}}{t^{\frac{\sigma+2}{\sigma}}} \right], \quad x \in H, t > 0, \tag{3.3.26}$$

where $B(0;1) = \{y \in H : \|y\|_H \le 1\}$ and $\mu(B(0;1)) > 0$.

If $\alpha = 2$ then there exists $C > 0$ such that

$$\int_H (P_t f)^4(x)\mu(dx) \le \frac{C}{\mu(B(0,1))} \int_H \exp\left[\frac{C\|x\|_H^2}{t^{\frac{\sigma+2}{\sigma}}} \right] \mu(\mathrm{d}x) < \infty$$

holds for sufficiently large $t > 0$, since $\mu(e^{\varepsilon_0 \|\cdot\|_H^2})$ is finite according to Theorem 3.3.4(i).

Hence P_t is hyperbounded for sufficient large $t > 0$. Since P_t has a density w.r.t. μ, then P_t is also compact in $L^2(\mu)$ for large $t > 0$ by [Wu00, Theorem 2.3].

(iv) If $\alpha > 2$, then by (3.3.21) we have for small enough $\varepsilon_0 > 0$

$$\mathrm{d}e^{\varepsilon_0\|X_t\|_H^\alpha} \le (c - \theta\|X_t\|_H^{2\alpha-2} e^{\varepsilon_0\|X_t\|_H^\alpha})\mathrm{d}t + \alpha\varepsilon_0\|X_t\|_H^{\alpha-2} e^{\varepsilon_0\|X_t\|_H^\alpha}\langle X_t, B\mathrm{d}W_t\rangle_H, \tag{3.3.27}$$

where $c, \theta > 0$ are some constants. By Jensen's inequality we have

$$\mathbf{E}e^{\varepsilon_0\|X_t\|_H^\alpha} \le e^{\varepsilon_0\|x\|_H^\alpha} + ct - \theta\varepsilon_0^{-(2\alpha-2)/\alpha} \int_0^t \mathbf{E}e^{\varepsilon_0\|X_u\|_H^\alpha} \left(\log \mathbf{E}e^{\varepsilon_0\|X_u\|_H^\alpha} \right)^{\frac{2\alpha-2}{\alpha}} \mathrm{d}u.$$

Let $h(t)$ solves the equation

$$h'(t) = c - \theta\varepsilon_0^{-(2\alpha-2)/\alpha} h(t)\big\{\log h(t)\big\}^{(2\alpha-2)/\alpha}, \quad h(0) = \exp\left[\varepsilon_0\left(\|x\|_H^\alpha + c\right)\right]. \tag{3.3.28}$$

Then by a standard comparison argument we know

$$\mathbf{E}e^{\varepsilon_0\|X_t(x)\|_H^\alpha} \leq h(t) \leq \exp\Big[c_0\big(1 + t^{-\alpha/(\alpha-2)}\big)\Big], \quad t > 0, x \in H \tag{3.3.29}$$

hold for some constant $c_0 > 0$. By using (3.3.26) we have

$$\begin{aligned} \|P_t f\|_\infty &= \|P_{t/2}P_{t/2}f\|_\infty \\ &\leq c_1 \sup_{x\in H} \mathbf{E}\exp\left[\frac{c_1}{t^{(\sigma+2)/\sigma}}\left(1 + \|X_{\frac{t}{2}}(x)\|_H\right)^{2+\frac{2(2-\alpha)}{\sigma}}\right], \quad t > 0, \end{aligned} \tag{3.3.30}$$

where $c_1 > 0$ is a constant. By the Young inequality there exists $c_2 > 0$ such that

$$\frac{c_1}{t^{\frac{\sigma+2}{\sigma}}}(1+u)^{2+\frac{2(2-\alpha)}{\sigma}} \leq \varepsilon_0(1+u^\alpha) + c_2 t^{-\alpha/(\alpha-2)}, \quad u, t > 0.$$

Therefore, there exists a constant $C > 0$ such that

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq \exp[C(1 + t^{-\frac{\alpha}{\alpha-2}})], \ t > 0.$$

Similarly, the compactness of P_t also follows from [Wu00]. ∎

Remark 3.3.2 *(1) Based on the Harnack inequality, the irreducibility can be obtained very easily for the associated transition semigroups. Then one can conclude the uniqueness of invariant measures and some ergodic properties for the transition semigroups. Comparing with the uniqueness result for invariant measure in Theorem 3.3.4, we do not need to assume $\gamma \leq 0$ or $\gamma < c_0\delta$ in this case.*

(2) In the literature, there are different outlook for the definition of total variation norm. We recall a few equivalent representation formulas here for the reader's convenience

(cf.[CL89, Hai03, Mao06]). For any two probability measures μ and ν on $(E, \mathcal{B})$ we have

$$\begin{aligned}
\|\mu - \nu\|_{var} &= \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)| \\
&= \frac{1}{2} \sup_{\|\varphi\|_\infty \leq 1} \left| \int_E \varphi(x)\mu(\mathrm{d}x) - \int_E \varphi(x)\nu(\mathrm{d}x) \right| \\
&= \inf_{\pi \in C(\mu,\nu)} \pi(E \times E \setminus \{(x,x) : x \in E\}) (\textit{maximal coupling}) \\
&= \inf_{\pi \in C(\mu,\nu)} \left| \int_{E \times E} \rho(x,y)\pi(\mathrm{d}x, \mathrm{d}y) \right| (\textit{1-Wasserstein distance}) \\
&= \sup_{Lip(\varphi)=1} \left| \int_E \varphi(x)\mu(\mathrm{d}x) - \int_E \varphi(x)\nu(\mathrm{d}x) \right|,
\end{aligned}$$

where $C(\mu,\nu)$ is the set of all couplings between μ and ν, $\rho(\cdot,\cdot)$ is the discrete metric on E and $Lip(\varphi)$ is the Lipschitz constant of function φ (w.r.t. ρ-metric).

Let $\mathcal{L}_p$ be the generator of the semigroup $\{P_t\}$ in $L^p(\mu)$. We say that $\mathcal{L}_p$ has a spectral gap in $L^p(\mu)$ if there exists $c > 0$ such that

$$\sigma(\mathcal{L}_p) \cap \{\lambda : Re\lambda > -c\} = \{0\},$$

where $\sigma(\mathcal{L}_p)$ denotes the spectrum of $\mathcal{L}_p$. The largest constant c with this property is denoted by $gap(\mathcal{L}_p)$.

Theorem 3.3.6 *Suppose all assumptions in Theorem 3.3.5 hold and μ denotes the unique invariant measure of $\{P_t\}$.*

(i) If $\alpha = 2$ and $\gamma < c_0\delta$, then the Markov semigroup $\{P_t\}$ is V-uniformly ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \geq 0$ and $x \in H$

$$\sup_{\|F\|_V \leq 1} |P_t F(x) - \mu(F)| \leq CV(x)e^{-\eta t},$$

where we can take $V(x) = 1 + \|x\|_H^2$ and $V(x) = e^{\varepsilon_0 \|x\|_H^2}$ for some small constant $\varepsilon_0 > 0$,

$$\|F\|_V := \sup_{x \in H} \frac{|F(x)|}{V(x)}.$$

Moreover, if P_t is symmetric on $L^2(\mu)$ for all $t \geq 0$, then we have

$$\|P_t F - \mu(F)\|_{L^2(\mu)} \leq e^{-\eta t}\|F\|_{L^2(\mu)}, \ F \in L^2(\mu), \ t \geq 0.$$

(ii) If $\alpha > 2$, then the Markov semigroup $\{P_t\}$ is uniformly exponential ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \geq 0$ and $x \in H$

$$\sup_{\|F\|_\infty \leq 1} |P_t F(x) - \mu(F)| \leq Ce^{-\eta t}.$$

Moreover, for each $p \in (1, \infty]$ we have

$$\|P_t F - \mu(F)\|_{L^p(\mu)} \leq C_p e^{-(p-1)\eta t/p} \|F\|_{L^p(\mu)}, \ F \in L^p(\mu), \ t \geq 0,$$

and

$$gap(\mathcal{L}_p) \geq \frac{(p-1)\eta}{p},$$

where C_p is a constant and we set $\frac{p-1}{p} = 1$ if $p = \infty$ by convention.

Proof. The proof is based on [GM04, Theorem 2.5; 2.6; 2.7]. According to Theorem 3.3.5, we know $\{P_t\}$ is strong Feller and irreducible. Now we only need to verify the following properties:

(1) For each $r > 0$ there exist $t_0 > 0$ and a compact set $M \subset H$ such that

$$\inf_{x \in B_r} P_{t_0} \mathbf{1}_M(x) > 0,$$

where $B_r = \{y \in H : \ \|y\|_H \leq r\}$.

(2) If $\alpha > 2$, then there exist constants $K < \infty$ and $t_1 > 0$ such that

$$\mathbf{E}\|X_t(x)\|_H^2 \leq K, \ x \in H, \ t \geq t_1.$$

(3) If $\alpha = 2$, then there exist constants $K < \infty$ and $\beta > 0$ such that

$$\mathbf{E}V(X_t(x)) \leq Ke^{-\beta t} V(x) + K, \ x \in H, \ t \geq 0,$$

where $V(x) = 1 + \|x\|_H^2$ and $V(x) = e^{\varepsilon_0 \|x\|_H^2}$ for some small constant $\varepsilon_0 > 0$.

By using the Itô formula we have

$$\|X_t\|_H^2 \leq \|x\|_H^2 + \int_0^t \left(c - \frac{\delta}{2}\|X_s\|_V^\alpha + \gamma \|X_s\|_H^2\right) \mathrm{d}s + \int_0^t \langle X_s, B\mathrm{d}W_s \rangle_H.$$

If $\alpha > 2$, then there exists a constant $c_1 > 0$

$$\|X_t\|_H^2 \leq \|x\|_H^2 + \int_0^t \left(c_1 - \frac{\delta}{4} \|X_s\|_V^\alpha \right) \mathrm{d}s + \int_0^t \langle X_s, B\mathrm{d}W_s \rangle_H.$$

This implies that there exists $C > 0$ such that

$$\mathbf{E} \int_0^t \|X_s\|_V^\alpha \mathrm{d}s \leq C(t + \|x\|_H^2), \ t \geq 0. \tag{3.3.31}$$

And by using Jensen's inequality

$$\mathbf{E}\|X_t\|_H^2 \leq \|x\|_H^2 + \int_0^t \left[C_1 - C_2 \left(\mathbf{E}\|X_s\|_H^2 \right)^{\alpha/2} \right] \mathrm{d}s.$$

Then by a standard comparison argument we get

$$\mathbf{E}\|X_t(x)\|_H^2 \leq C(1 + t^{-\frac{2}{\alpha-2}}), \ x \in H, \ t > 0.$$

Hence property (2) holds.

According to (3.3.5), for the property (1) it is enough to show that there exist t_0 and a compact set M in H such that $P_{t_0}\mathbf{1}_M(x) > 0$ for some $x \in B_r$.

By (3.3.31) and a simple contradiction argument, one can show that there exists $t_0 > 0$ such that $P_{t_0}\mathbf{1}_M(x) > 0$ for the compact set $M := \left\{ y \in H : \|y\|_V \leq [C(1 + r^2)]^{1/\alpha} \right\}$ and $x \in B_r$. So property (1) also holds.

Then the assertions in (ii) hold according to [GM04, Theorem 2.5; 2.7]. The modified constant in the estimates of spectral gap and exponential convergence comes from the arguments in [GM06, Theorem 7.2](in fact, (7.10) implies that (7.4) holds with a modified constant in [GM06]).

Similarly, if $\alpha = 2$ and $\gamma < c_0\delta$, then we can prove

$$\mathbf{E}\|X_t(x)\|_H^2 \leq e^{-\beta t}\|x\|_H^2 + C, \ t \geq 0, x \in H$$

holds for some constants $\beta > 0$ and C. Moreover, by (3.3.21) there also exists a small constant $\varepsilon_0 > 0$ such that

$$\mathbf{E} \exp\left[\varepsilon_0 \|X_t(x)\|_H^2\right] \leq e^{-\beta t} e^{\varepsilon_0 \|x\|_H^2} + C, \ t \geq 0, x \in H.$$

Then the conclusions in (i) follow from [GM04, Theorem 2.5; 2.6]. ∎

Remark 3.3.3 *The V-uniformly ergodicity implies that for any probability measure ν on H we have*

$$\begin{aligned}\|P_t^*\nu - \mu\|_{var} &\leq \int_H \|P(t,x,\cdot) - \mu\|_{var}\nu(\mathrm{d}x) \\ &\leq \int_H \sup_{\|\varphi\|_V \leq 1} |P_t\varphi(x) - \mu(\varphi)|\, \nu(\mathrm{d}x) \\ &\leq \int_H CV(x)e^{-\eta t}\nu(\mathrm{d}x) = C\nu(V)e^{-\eta t},\ t \geq 0.\end{aligned}$$

And it is easy to show that the uniformly exponential ergodicity is equivalent to

$$\|P_t^*\nu - \mu\|_{var} \leq Ce^{-\eta t},\ t \geq 0.$$

3.4 Applications to SPDE with strongly dissipative drifts

To apply our main results, one has to verify condition (3.3.3) and (3.3.4). To this end, we present some simple sufficient conditions for (3.3.3) and (3.3.4). In the following examples $L(Y,Z)$ denotes the space of all bounded linear operators from Y to Z and $\mathbf{Ran}(B)$ denotes the range of operator B.

Example 3.4.1 *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary and Δ be the Laplace operator on $L^2(\Lambda)$ with Dirichlet boundary condition. Consider the following triple

$$W_0^{1,2}(\Lambda) \subseteq L^2(\Lambda) \subseteq \left(W_0^{1,2}(\Lambda)\right)^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - c|X_t|^{p-2}X_t)\mathrm{d}t + B\mathrm{d}W_t,\ X_0 = x \in L^2(\Lambda) \tag{3.4.1}$$

where $1 < p \leq 2$ and $c \geq 0$, B is a Hilbert-Schmidt operator and W_t is a cylindrical Wiener process on $L^2(\Lambda)$, then the assertions in **Theorem 3.3.4** *hold for* (3.4.1).

Moreover, if B is a one-to-one operator such that

$$W_0^{1,2}(\Lambda) \subseteq \mathbf{Ran}(B), \quad B^{-1} \in L(W_0^{1,2}(\Lambda); L^2(\Lambda)),$$

then (3.3.4) *also holds. In particular, if* $d = 1$ *and* $B := (-\Delta)^{-\theta}$ *with* $\theta \in (\frac{1}{4}, \frac{1}{2}]$, *then* B *is a Hilbert-Schmidt operator and* (3.3.4) *holds. Hence the assertions in* **Theorem 3.3.1**, **3.3.5** *and* **3.3.6** *also hold for* (3.4.1). *Particularly, the associated transition semigroup of* (3.4.1) *is hyperbounded.*

Remark 3.4.1 *Suppose that*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

are the eigenvalues of $-\Delta$ *and the corresponding eigenvectors* $\{e_i\}_{i\ge 1}$ *form an orthonormal basis on* $L^2(\Lambda)$. *If* $Be_i := b_i e_i$ *and there exists a positive constant* C *such that*

$$\sum_i b_i^2 < +\infty; \quad b_i \ge \frac{C}{\sqrt{\lambda_i}}, \quad i \ge 1, \tag{3.4.2}$$

then B *is a Hilbert-Schmidt operator on* $L^2(\Lambda)$ *and* (3.3.4) *holds.*

On the other hand, by the Sobolev inequality (see [Wan00, Corollary 1.1 and 3.1]) we know that

$$\lambda_i \ge c i^{2/d}, \quad i \ge 1,$$

hold for some constant $c > 0$. *Then* (3.4.2) *implies that the space dimension* d *is less than* 2. *However, if we consider a general negative definite self-adjoint operator* L *instead of* Δ *in* (3.4.1), *e.g.* $L := -(-\Delta)^q, q > 0$, *then, by the spectral representation theorem, our results can apply to examples on* $\mathbb{R}^d$ *with* $d \ge 2$. *For more details we refer to [LW08, Wan07].*

Example 3.4.2 *(Stochastic p-Laplace equation)*
Let Λ *be an open bounded domain in* $\mathbb{R}^d$ *with smooth boundary. Consider the triple*

$$W_0^{1,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - c|X_t|^{\tilde{p}-2}X_t\right]\mathrm{d}t + B\mathrm{d}W_t, \ X_0 = x, \tag{3.4.3}$$

where $c \ge 0$, $2 \le p < \infty, 1 \le \tilde{p} \le p$, B *is a Hilbert-Schmidt operator and* W_t *is a cylindrical Wiener process on* $L^2(\Lambda)$, *then the assertions in* **Theorem 3.3.4** *hold for* (3.4.3).

Moreover, if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then (3.3.4) also holds. Therefore the assertions in **Theorem 3.3.1, 3.3.5** *and* **3.3.6** *also hold for (3.4.3). In particular, if $p > 2$, then the associated transition semigroup of (3.4.3) is ultrabounded and compact, and its generator has a spectral gap.*

Proof. According to [PR07, Example 4.1.9], the hemicontinuity and (3.3.2) hold for the coefficient of (3.4.3). Hence we only need to verify (3.3.3) under our assumptions. By using Lemma 2.4.1 and the Poincaré inequality we have

$$\begin{aligned}
&{}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u) - \mathbf{div}(|\nabla v|^{p-2}\nabla v), u - v\rangle_V \\
&= -\int_\Lambda \langle\ |\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x) - \nabla v(x)\rangle_{\mathbb{R}^d} \mathrm{d}x \\
&\le -2^{p-2}\int_\Lambda |\nabla u(x) - \nabla v(x)|^p \mathrm{d}x \\
&\le -C\|u - v\|_{1,p}^p,\ u, v \in W_0^{1,p}(\Lambda),
\end{aligned}$$

where $C > 0$ is a constant. And it's also easy to show that

$${}_{V^*}\langle |u|^{\tilde{p}-2}u - |v|^{\tilde{p}-2}v, u - v\rangle_V \ge 0.$$

Hence (3.3.3) holds and the assertions in Theorem 3.3.4 follow.

If $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then there exists a constant $c > 0$ such that (see Remark 3.4.1)

$$\|u\|_{1,2} \ge c\|u\|_B,\ u \in W_0^{1,p}(\Lambda).$$

This implies (3.3.4) holds. ∎

Remark 3.4.2 *(1) The Harnack inequality and some consequent properties still hold if one also adds some locally bounded linear (or order less than p) perturbation in the drift. Only for certain properties (e.g. hyperboundedness or ultraboundedness) we need to require the drift is dissipative (i.e.$\gamma \le 0$).*

(2) If we take $B = 0$ in (3.4.3), then by Theorem 3.3.4(iii) we can get the following decay estimate for the solution to the classical p-Laplace equation

$$\sup_{x \in L^2(\Lambda)} \|X_t(x)\|_{L^2} \le Ct^{-\frac{1}{p-2}},\ t > 0,$$

where C is a positive constant.

Example 3.4.3 *Let Λ be an open bounded domain in $\mathbb{R}^1$ and $m \in \mathbb{N}_+$. Consider the following triple*

$$W_0^{m,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{m,p}(\Lambda))^*$$

and the following stochastic evolution equation

$$\mathrm{d}X_t(x) = \left[(-1)^{m+1}\frac{\partial^m}{\partial x^m}\left(\left|\frac{\partial^m}{\partial x^m}X_t(x)\right|^{p-2}\frac{\partial^m}{\partial x^m}X_t(x)\right) - c|X_t(x)|^{\tilde{p}-2}X_t(x)\right]\mathrm{d}t + B\mathrm{d}W_t, \tag{3.4.4}$$

where $c \geq 0$, $2 \leq p < \infty, 1 \leq \tilde{p} \leq p$, $B \in L_2(L^2(\Lambda))$ and W_t is a cylindrical Wiener process on $L^2(\Lambda)$, then the assertions in **Theorem 3.3.4** *hold for (3.4.4).*

Moreover, if B is also a one-to-one operator such that $B^{-1} \in L(W_0^{m,p}(\Lambda); L^2(\Lambda))$, then (3.3.4) is also satisfied. Hence the assertions in **Theorem 3.3.1, 3.3.5** *and* **3.3.6** *hold for (3.4.4). In particular, the associated transition semigroup is ultrabounded if $p > 2$ and hyperbounded if $p = 2$.*

Remark 3.4.3 *(i) If we assume $p > 2$ and $B = 0$ in (3.4.4), then by Theorem 3.3.4 we obtain the decay of the solution to the corresponding deterministic equation, i.e.*

$$\sup_{f\in L^2(\Lambda)} \|X_t^f\|_{L^2} \leq Ct^{-\frac{1}{p-2}}, \ t > 0,$$

where X_t^f denotes the solution to the following equation

$$\frac{\mathrm{d}X_t(x)}{\mathrm{d}t} = (-1)^{m+1}\frac{\partial^m}{\partial x^m}\left(\left|\frac{\partial^m}{\partial x^m}X_t(x)\right|^{p-2}\frac{\partial^m}{\partial x^m}X_t(x)\right) - c|X_t(x)|^{\tilde{p}-2}X_t(x), \ X_0 = f \in L^2(\Lambda).$$

(ii) Assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

are the eigenvalues of a positive definite self-adjoint operator L where $\mathcal{D}(\sqrt{L}) = W_0^{m,2}(\Lambda)$, and the corresponding eigenvectors $\{e_i\}_{i\geq 1}$ form an ONB of $L^2(\Lambda)$. Suppose $Be_i := b_i e_i$ and there exists a constant $C > 0$ such that

$$\sum_i b_i^2 < +\infty; \quad b_i \geq \frac{C}{\sqrt{\lambda_i}}, \quad i \geq 1,$$

then B is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and (3.3.4) is satisfied.

Chapter 4

Harnack Inequality for Stochastic Fast Diffusion Equations

In chapter 3 the Harnack inequality has been established for a large class of stochastic evolution equations with additive noise. However, the strong monotonicity assumption (3.3.3) excludes some important types of SPDE within the variational framework such as stochastic fast diffusion equations and the singular stochastic p-Laplace equation ($1 < p < 2$). Hence we study these two types of SPDE seperately in this and the next chapter. Due to the weak dissipativity of the drift, we need to make more delicate estimates in order to establish the Harnack inequality. The strong Feller property and heat kernel estimates are also obtained for the corresponding transition semigroups. Moreover, we also derive the ultraboundedness and compactness property for the transition semigroups if there is a nonlinear perturbation in the drift. Exponential ergodicity and the existence of a spectral gap are also investigated. As applications, the main results are used to study some explicit examples in the last section. Part of the results in this chapter have already been published in [LW08].

4.1 The main results on Harnack inequality

In the field of nonlinear PDE, fast diffusion equations have been studied intensively and we may refer to the monographs [DK07, Váz06] (see also the references therein). The fast diffusion equation can be formulated as follows

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta(|u|^{r-1}u),$$

where $r \in (0,1)$ is a constant. This equation has some different features comparing with porous media equations ($r > 1$). For example, the solution of porous media equation decays to 0 with some polynomial rate but the solution of fast diffusion equation converges to 0 in finite time at each point (cf.[Aro86]).

In this chapter we mainly study the long time behavior of the fast diffusion equations under some random perturbations. The framework can be formulated as follows. Let $(E, \mathcal{M}, \mathbf{m})$ be a separable probability space and $(L, \mathcal{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ having discrete spectrum. Let

$$(0 <)\lambda_1 \le \lambda_2 \le \cdots$$

be all eigenvalues of $-L$ with the unit eigenfunctions $\{e_i\}_{i\ge1}$.

Next, let H be the completion of $(L^2(\mathbf{m}), \|\cdot\|_2)$ under the inner product

$$\langle x, y\rangle_H := \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\langle x, e_i\rangle\langle y, e_i\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbf{m})$. Let W_t be a cylindrical Wiener process on $L^2(\mathbf{m})$ w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}_t, \mathbf{P})$.

Suppose that $\Psi : \mathbb{R} \to \mathbb{R}$ is continuous and B is a Hilbert-Schmidt operator from $L^2(\mathbf{m})$ to H. We consider the following stochastic fast diffusion equation

$$\mathrm{d}X_t = \big\{L\Psi(X_t) - \gamma\|X_t\|_H^{q-2}X_t\big\}\mathrm{d}t + B\mathrm{d}W_t,\ X_0 = x \in H, \tag{4.1.1}$$

where $q \ge 2$ and $\gamma \ge 0$ are some constants. In particular, if $\gamma = 0, B = 0$ and $\Psi(s) = |s|^{r-1}s$ for some $r \in (0,1)$, then (4.1.1) reduces back to the classical fast diffusion equation.

Now for a fixed number $r \in (0,1)$ we assume that there exist positive constants δ, η such that

$$\begin{aligned}&\Psi(0) = 0,\ |\Psi(s)| \le \eta(1 + |s|^r), \quad s \in \mathbb{R},\\ &\big(\Psi(s_1) - \Psi(s_2)\big)(s_1 - s_2) \ge \delta|s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}, \quad s_1, s_2 \in \mathbb{R}, t \ge 0.\end{aligned} \tag{4.1.2}$$

Due to the mean-valued theorem and the fact that $r < 1$, one has

$$(s_1 - s_2)(s_1^r - s_2^r) \geq r|s_1 - s_2|^2(|s_1| \vee |s_2|)^{r-1}.$$

Hence a simple example where (4.1.2) holds is $\Psi(s) = s^r$ with $\eta = 1, \delta = r$.

Consider the following Gelfand triple

$$L^{r+1}(\mathbf{m}) \cap H \subseteq H \subseteq \left(L^{r+1}(\mathbf{m}) \cap H\right)^*,$$

it is easy to show that the coefficients of (4.1.1) satisfy the well-known monotone and coercive conditions (see Theorem 1.2.1). Hence according to [Zha08, Theorem 3.6], for any $x \in H$ the equation (4.1.1) has a unique solution $X_t(x)$ with $X_0(x) = x$, which is a continuous adapted process on H and satisfies

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t(x)\|_H^2 < \infty, \quad T > 0. \tag{4.1.3}$$

Consider the corresponding transition semigroup

$$P_t F(x) := \mathbf{E} F\left(X_t(x)\right), \quad t > 0,$$

where F is a bounded measurable functions on H. In fact, one can show that $\{P_t\}$ is a Markov semigroup (cf.[KR79, RRW07]). We first investigate the existence and uniqueness of invariant measures and the convergence rate of the transition semigroup to invariant measure.

Theorem 4.1.1 *Suppose (4.1.2) holds and the embedding $L^{r+1}(\mathbf{m}) \subseteq H$ is compact.*

(i) The transition semigroup $\{P_t\}$ has an invariant probability measure. If $\gamma > 0$, then the invariant measure is unique and denoted by μ. Moreover, we have $\mu(\|\cdot\|_{r+1}^{r+1} + e^{\varepsilon_0\|\cdot\|_H^q}) < \infty$ for some $\varepsilon_0 > 0$.

(ii) If $q > 2$ and $\gamma > 0$, then for any Lipschitz continuous function F on H we have

$$\sup_{x\in H} |P_t F(x) - \mu(F)| \leq C Lip(F) t^{-\frac{1}{q-2}}, \ t > 0, \tag{4.1.4}$$

where $Lip(F)$ is the Lipschitz constant of F and C is a constant. In particular, if $B = 0$ and Dirac measure at 0 is the unique invariant measure, then we can take $F(x) = \|x\|_H$ in (4.1.4) and have

$$\sup_{x\in H} \|X_t(x)\|_H \leq C t^{-\frac{1}{q-2}}, \ t > 0.$$

(iii) If $q=2$ and $\gamma>0$, then for any $x,y\in H$

$$\|X_t(x)-X_t(y)\|_H\le e^{-\gamma t}\|x-y\|_H,\ t\ge 0.$$

And for any Lipschitz continuous function F on H we have

$$|P_tF(x)-\mu(F)|\le Lip(F)e^{-\gamma t}\left(\|x\|_H+C\right),\ x\in H, \tag{4.1.5}$$

where $C>0$ is a constant.

Proof. (i) If the embedding $L^{r+1}(\mathbf{m})\subset H$ is compact, then the existence of an invariant measure follows from the standard Krylov-Bogoliubov argument. One just needs to repeat the proof of Theorem 3.3.4(i).

If $\gamma>0$, then the uniqueness of invariant measures follows from (4.1.4) and (4.1.5). Now we prove the concentration property $\mu(e^{\varepsilon_0\|\cdot\|_H^q})<\infty$ for some $\varepsilon_0>0$.

By (4.1.2) and Itô's formula we have

$$\begin{aligned}\|X_t\|_H^2&\le\|x\|_H^2+\int_0^t(b-2\delta\|X_s\|_{r+1}^{r+1}-2\gamma\|X_s\|_H^q)\mathrm{d}s+2\int_0^t\langle X_s,B\mathrm{d}W_s\rangle_H\\&\le\|x\|_H^2+\int_0^t(b-2\gamma\|X_s\|_H^q)\mathrm{d}s+M_t,\end{aligned}\tag{4.1.6}$$

where $b=\|B\|_{HS}^2$ and $M_t=2\int_0^t\langle X_s,B\mathrm{d}W_s\rangle_H$ is a martingale. If $\gamma>0$ and ε_0 is small enough, then the Itô formula implies

$$\begin{aligned}e^{\varepsilon_0\|X_t\|_H^q}\le&e^{\varepsilon_0\|x\|_H^q}+\int_0^t(c-2\gamma\|X_s\|_H^q+\varepsilon_0qb\|X_s\|_H^q)\,\frac{\varepsilon_0q}{2}\|X_s\|_H^{q-2}e^{\varepsilon_0\|X_s\|_H^q}\mathrm{d}s\\&+\frac{\varepsilon_0q}{2}\int_0^t\|X_s\|_H^{q-2}e^{\varepsilon_0\|X_s\|_H^q}\mathrm{d}M_s\\\le&e^{\varepsilon_0\|x\|_H^q}+\int_0^t(c-\gamma\|X_s\|_H^q)\,\frac{\varepsilon_0q}{2}\|X_s\|_H^{q-2}e^{\varepsilon_0\|X_s\|_H^q}\mathrm{d}s\\&+\frac{\varepsilon_0q}{2}\int_0^t\|X_s\|_H^{q-2}e^{\varepsilon_0\|X_s\|_H^q}\mathrm{d}M_s\\\le&e^{\varepsilon_0\|x\|_H^q}+\int_0^t\left(c_1-c_2e^{\varepsilon_0\|X_s\|_H^q}\right)\mathrm{d}s+\frac{\varepsilon_0q}{2}\int_0^t\|X_s\|_H^{q-2}e^{\varepsilon_0\|X_s\|_H^q}\mathrm{d}M_s\end{aligned}\tag{4.1.7}$$

holds for some positive constants c,c_1 and c_2. Therefore

$$\mu_n(e^{\varepsilon_0\|\cdot\|_H^q})=\frac{1}{n}\int_0^n\mathbb{E}e^{\varepsilon_0\|X_t(0)\|_H^q}\mathrm{d}t\le\frac{1}{c_2n}+\frac{c_1}{c_2},$$

where $\mu_n = \frac{1}{n}\int_0^n (\delta_0 P_t)\mathrm{d}t$.

Since μ is the weak limit of a subsequence of μ_n, we have $\mu(\mathrm{e}^{\varepsilon_0\|\cdot\|_H^q}) < \infty$. In particular, this implies $\mu(\|\cdot\|_H^2) < \infty$.

By (4.1.6) there also exists a constant C such that

$$\mathbf{E}\int_0^1 \|X_t(x)\|_{r+1}^{r+1}\mathrm{d}t \leq C(1+\|x\|_H^2),\ \forall x \in H.$$

Therefore

$$\mu(\|\cdot\|_{r+1}^{r+1}) = \int_H \mu(dx)\int_0^1 \mathbf{E}(\|X_t(x)\|_{r+1}^{r+1})\mathrm{d}t \leq C + C\int_H \|x\|_H^2 \mu(dx) < \infty.$$

(ii) Recall the following inequality for $q \geq 2$ (see Lemma 2.4.1)

$$\langle \|u\|_H^{q-2}u - \|v\|_H^{q-2}v, u-v\rangle_H \geq 2^{2-q}\|u-v\|_H^q,\ \forall u, v \in H. \tag{4.1.8}$$

Hence combining with (4.1.2) and the Itô formula we have

$$\|X_t(x) - X_t(y)\|_H^2 \leq \|x-y\|_H^2 - c_0\int_0^t \|X_s(x) - X_s(y)\|_H^q \mathrm{d}s,$$

where $c_0 = 2^{3-q}\gamma$ is a constant and $X_t(y)$ denotes the solution starting from $y \in H$.

Now by the standard comparison argument (see Theorem 3.3.4(iii)) one can prove

$$\|X_t(x) - X_t(y)\|_H \leq \|x-y\|_H \wedge \left\{\frac{(q-2)c_0}{2}t\right\}^{-\frac{1}{q-2}}. \tag{4.1.9}$$

Therefore, for any Lipschitz function F on H there exists $C > 0$ such that

$$|P_t F(x) - \mu(F)| \leq \int_H \mathbf{E}|F(X_t(x)) - F(X_t(y))|\mu(\mathrm{d}y) \leq C Lip(F) t^{-\frac{1}{q-2}}$$

holds for all $x \in H$. Hence invariant measure of $\{P_t\}$ is unique.

(iii) If $q = 2$ and $\gamma > 0$, we have

$$\|X_t(x) - X_t(y)\|_H^2 \leq \|X_s(x) - X_s(y)\|_H^2 - 2\gamma\int_s^t \|X_u(x) - X_u(y)\|_H^2 \mathrm{d}u,\ t \geq s \geq 0.$$

Hence

$$\|X_t(x) - X_t(y)\|_H^2 \leq \|x-y\|_H^2 e^{-2\gamma t}.$$

Then all assertions hold. ∎

In order to establish Harnack inequality for P_t, we need to assume that B is non-degenerate ; that is, $Bx = 0$ implies $x = 0$. Then we define

$$\|x\|_B := \begin{cases} \|y\|_2, & \text{if } y \in L^2(\mathbf{m}),\ By = x, \\ \infty, & \text{otherwise.} \end{cases}$$

The proof following theorem is given in the next section.

Theorem 4.1.2 *Assume (4.1.2) holds. If there exists a constant $\sigma \geq \frac{4}{r+1}$ such that*

$$\|x\|_{r+1}^2 \cdot \|x\|_H^{\sigma-2} \geq \xi \|x\|_B^\sigma, \quad x \in L^{r+1}(\mathbf{m}) \tag{4.1.10}$$

holds with some constant $\xi > 0$, then for any $t > 0$, P_t is strong Feller and for any positive bounded measurable function F, $p > 1$ and $x, y \in H$,

$$\begin{aligned}(P_tF(y))^p \leq P_tF^p(x) \exp\Big[&\frac{p-1}{4}\Big(t\lambda_t + 1 + \|x\|_H^2 + \|y\|_H^2 + \frac{(\sigma+2)^2}{\sigma^2 t}\|x-y\|_H^2\Big) \\ &+ \lambda_t^{\frac{2-\sigma}{2}} \left(\frac{\sigma+2}{\sigma}\right)^{\sigma+1} \frac{[2p(p+1)]^{\sigma/2}}{8\delta\xi(p-1)^{\sigma-1}t^\sigma}\|x-y\|_H^\sigma\Big]\end{aligned} \tag{4.1.11}$$

holds for $\lambda_t = 2\delta e^{-(2b+1)t}$ and $b = \|B\|_{HS}^2$ (Hilbert-Schmidt norm of B).

Remark 4.1.1 *(1) In [LW08], the Harnack inequality (4.1.11) has been established for stochastic fast diffusion equations with linear perturbation in the drift. One should note that Harnack inequality and strong Feller property of the transition semigroup still hold if we take $\gamma = 0$ (i.e. without high order perturbation in the drift). But we can not prove contractivity property for the transition semigroup in [LW08]. However, we can establish the ultraboundedness and compactness for the associated transition semigroup here under the influence of the strong absorption term in the drift (see Theorem 4.1.3). We should also mention the role of this absorption term in the convergence of the transition semigroup to its equilibrium (see Theorem 4.1.1).*

(3) For simplicity, we only prove the Harnack inequality for (4.1.1) with deterministic and time-independent coefficients in this chapter. But one can easily extend these results to more general case as in [LW08, Wan07].

Now we can study the ergodicity and ultrabounded property for the associated transition semigroup. We recall that the process X is called Harris recurrent if

$$\mathbf{P}_x\left\{\int_0^\infty 1_U(X_s)\mathrm{d}s = +\infty\right\} = 1$$

holds for any starting point $x \in H$ and any Borel sets U with $\mu(U) > 0$, here 1_U denotes the indicator function of U.

Theorem 4.1.3 *Assume (4.1.2) holds and the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact.*

(i) If (4.1.10) holds, then any invariant measure of $\{P_t\}$ has full support on H and $\{P_t\}$ is irreducible. Hence $\{P_t\}$ has a unique invariant measure μ and all transition probabilities

$$P_t(x, \cdot),\ t > 0,\ x \in H$$

are equivalent to μ. Moreover, the process X is Harris recurrent and for any probability measure ν on H we have

$$\lim_{t\to\infty} \| P_t^*\nu - \mu \|_{var} = 0,$$

where $\| \cdot \|_{var}$ is the total variation norm and P_t^* is the adjoint operator of P_t.

(ii) If $q > \sigma$, $\gamma > 0$ and (4.1.10) holds, then P_t is ultrabounded (i.e.$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty$) and compact on $L^2(\mu)$ for any $t > 0$.

(iii) If $q > 2$, $\gamma > 0$ and (4.1.10) holds, then $\{P_t\}$ is uniformly exponential ergodic, i.e. there exist $C, \eta > 0$ such that for any probability measure ν on H

$$\| P_t^*\nu - \mu \|_{var} \le Ce^{-\eta t},\ t \ge 0.$$

Moreover, for each $p \in (1, \infty]$ we have

$$\|P_tF - \mu(F)\|_{L^p(\mu)} \le C_p e^{-(p-1)\eta t/p}\|F\|_{L^p(\mu)},\ F \in L^p(\mu),\ t \ge 0,$$

and

$$gap(\mathcal{L}_p) \ge \frac{(p-1)\eta}{p},$$

where C_p is a constant and $\mathcal{L}_p$ is the generator of the semigroup $\{P_t\}$ on $L^p(\mu)$.

Proof. (i) The full support of μ and the irreducibility follow from the Harnack inequality (4.1.11) by repeating the proof of Theorem 3.3.5(i).

Since $\{P_t\}$ is also strong Feller, the uniqueness of invariant measures follows from the classical theorem by Doob [Doo48] (see [Hai03, Theorem 2.1]).

Note that the solution has continuous paths on H, then the other assertions follow from the general result in ergodic theory, we refer to [Sei97, Theorem 2.2 and Proposition 2.5].

(ii) If $q > \sigma(> 2)$, then by Itô's formula and (4.1.7) we have for small enough $\varepsilon_0 > 0$

$$e^{\varepsilon_0 \|X_t\|_H^q} \le e^{\varepsilon_0 \|x\|_H^q} + \int_0^t \left(c_2 - c_1 \|X_s\|_H^{2q-2} e^{\varepsilon_0 \|X_s\|_H^q} \right) \mathrm{d}s + M_t', \tag{4.1.12}$$

where $c_1, c_2 > 0$ are constants and M' is a local martingale. By Jensen's inequality

$$\mathbf{E} e^{\varepsilon_0 \|X_t\|_H^q} \le e^{\varepsilon_0 \|x\|_H^q} + c_2 t - c_1 \varepsilon_0^{-(2q-2)/q} \int_0^t \mathbf{E} e^{\varepsilon_0 \|X_s\|_H^q} \left(\log \mathbf{E} e^{\varepsilon_0 \|X_s\|_H^q} \right)^{\frac{2q-2}{q}} \mathrm{d}s.$$

Then by a comparison argument we can get the following estimate

$$\mathbf{E} e^{\varepsilon_0 \|X_t(x)\|_H^q} \le \exp\left[c_0 \left(1 + t^{-q/(q-2)} \right) \right], \quad t > 0, x \in H, \tag{4.1.13}$$

where $c_0 > 0$ is a constant.

Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$. By (4.1.11) with $p = 2$, there exists a constant $c_t > 0$ depending on t (which may change from line to line) such that

$$(P_t f)^2(x) \exp\left[-c_t \left(1 + \|x\|_H^2 + \|y\|_H^2 + \|x-y\|_H^2 + \|x-y\|_H^\sigma \right) \right] \le P_t f^2(y), \quad x, y \in H, t > 0. \tag{4.1.14}$$

By integrating on both sides w.r.t. $\mu(\mathrm{d}y)$ we obtain

$$(P_t f)^2(x) \le \frac{1}{\mu(B(0,1))} \exp\left[c_t \left(1 + \|x\|_H^2 + \|x\|_H^\sigma \right) \right], \quad x \in H, t > 0, \tag{4.1.15}$$

where $B(0,1) := \{y \in H : \|y\|_H \le 1\}$ has positive mass with respect to μ. Hence we have

$$\begin{aligned} \|P_t f\|_\infty &= \|P_{t/2} P_{t/2} f\|_\infty \\ &\le c \sup_{x \in H} \mathbf{E} \exp\left[c_t \left(1 + \|X_{\frac{t}{2}}(x)\|_H^2 + \|X_{\frac{t}{2}}(x)\|_H^\sigma \right) \right], \quad t > 0 \end{aligned} \tag{4.1.16}$$

for some $c, c_t > 0$. Since $q > \sigma$, by the Young inequality there exists $C_t > 0$ such that

$$c_t\left(1 + u^2 + u^\sigma\right) \leq C_t + \varepsilon_0 u^q, \quad u > 0.$$

Therefore, we have

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq ce^{C_t}\exp[c_0(1 + t^{-q/(q-2)})] < \infty,\ t > 0.$$

Moreover, since P_t is uniformly integrable in $L^2(\mu)$ and has a density w.r.t. μ, the compactness of P_t follows from [GW01, Lemma 3.1].

(iii) If $q > 2$ and $\gamma > 0$, by (4.1.6) and a standard comparison argument we have

$$\mathbf{E}\|X_t(x)\|_H^2 \leq C(1 + t^{-\frac{2}{q-2}}),\ x \in H,\ t \geq 0.$$

Then the conclusion can be obtained by repeating the argument in Theorem 3.3.6(ii). ∎

Remark 4.1.2 *Note that the transition semigroup $\{P_t\}$ is non-symmetric and defined on the infinite dimensional space. But the compactness of $\{P_t\}$ in the above theorem implies the generator, i.e. the corresponding Kolmogorov operator of the stochastic fast diffusion equation has only discrete spectrum.*

4.2 Proof of the Harnack inequality

As explained in chapter 3, to prove the Harnack inequality for P_t, it suffices to construct a coupling processes (X_t, Y_t) which is a continuous adapted process on $H \times H$ such that

(i) X_t solves (4.1.1) with $X_0 = x$;

(ii) Y_t solves the equation

$$\mathrm{d}Y_t = \left\{L\Psi(Y_t) - \gamma\|Y_t\|_H^{q-2}Y_t\right\}\mathrm{d}t + B\mathrm{d}\tilde{W}_t,\ Y_0 = y$$

for another cylindrical Wiener process $\tilde{W}_t$ on $L^2(\mathbf{m})$ under a weighted probability measure $R\mathbf{P}$, where $\tilde{W}_t$ and R will be constructed later by a Girsanov transformation;

(iii) $X_T = Y_T$, a.s. for a given time T.

In order to implement the above steps, for $\varepsilon \in (0,1)$ and $\beta \in \mathbf{C}([0,\infty);\mathbb{R}^+)$, let Y_t solves the coupling equation

$$\mathrm{d}Y_t = \left\{ L\Psi(Y_t) - \gamma\|Y_t\|_H^{q-2}Y_t + \frac{\beta_t(X_t - Y_t)}{\|X_t - Y_t\|_H^{\varepsilon}}\mathbf{1}_{\{t<\tau\}} \right\} \mathrm{d}t + B\mathrm{d}W_t,\ Y_0 = y \in H, \quad (4.2.1)$$

where $X_t := X_t(x)$ is the solution to (4.1.1) and $\tau := \inf\{t \geq 0 : X_t = Y_t\}$.

According to [Zha08, Theorem 3.6], we can prove that (4.2.1) also has a unique strong solution $Y_t(y)$ by using the same argument in Lemma 3.3.2.

Let

$$\zeta_t := \frac{\beta_t B^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_H^{\varepsilon}}\mathbf{1}_{\{t<\tau\}}, \quad (4.2.2)$$

then we have

$$\mathrm{d}Y_t = (L\Psi(Y_t) - \gamma\|Y_t\|_H^{q-2}Y_t)\mathrm{d}t + B(\mathrm{d}W_t + \zeta_t\mathrm{d}t), \quad Y_0 = y.$$

According to the Girsanov theorem, $\tilde{W}_t := W_t + \int_0^t \zeta_s \mathrm{d}s$ is a cylindrical Wiener process under $R\mathbf{P}$ where

$$R := \exp\Big[- \int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t \Big]. \quad (4.2.3)$$

Therefore, to verify (ii) and (iii), we need to choose ε and β such that

(a) $X_T = Y_T$ *a.s.*;

(b) $\mathbf{E}\exp\Big[\lambda \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\Big] < \infty, \quad \lambda > 0.$

By (4.1.2) we have

$$\begin{aligned} \|X_T - Y_T\|_H^2 \leq \|x - y\|_H^2 - \int_0^T &\big[2\delta\mathbf{m}\left(|X_t - Y_t|^2(|X_t| \vee |Y_t|)^{r-1}\right) \\ &+ 2\beta_t\|X_t - Y_t\|_H^{2-\varepsilon}\mathbf{1}_{\{t<\tau\}}\big]\mathrm{d}t. \end{aligned} \quad (4.2.4)$$

This implies

$$\|X_T - Y_T\|_H^{\varepsilon} \leq \|x - y\|_H^{\varepsilon} - \varepsilon\int_0^{T\wedge\tau} \beta_t \mathrm{d}t. \quad (4.2.5)$$

Hence we have the following result.

Lemma 4.2.1 *If β satisfies $\int_0^T \beta_t \mathrm{d}t \geq \frac{1}{\varepsilon}\|x - y\|_H^{\varepsilon}$, then $X_T = Y_T$ a.s.*

We also need to have the following a priori estimates.

Lemma 4.2.2 *We have*

$$\mathbf{E}\exp\left[\lambda_T\int_0^T \|X_t\|_{r+1}^{r+1}\mathrm{d}t\right]\le\exp\left[\int_0^T be^{-2bt}\mathrm{d}t+\|x\|_H^2\right];\tag{4.2.6}$$

$$\begin{aligned}&\mathbf{E}\exp\left[\lambda_T\int_0^T \|Y_t\|_{r+1}^{r+1}\mathrm{d}t\right]\\&\le\exp\left[\int_0^T be^{-(2b+1)t}\mathrm{d}t+\|y\|_H^2+\|x-y\|_H^{2(1-\varepsilon)}\int_0^T\beta_t^2e^{-(2b+1)t}\mathrm{d}t\right],\end{aligned}\tag{4.2.7}$$

where $\lambda_T=2\delta e^{-(2b+1)T}$ *and* $b=\|B\|_{HS}^2$.

Proof. Since assumption (4.1.2) implies

$$\begin{aligned}2_{V^*}\langle L\Psi(X_t),X_t\rangle_V&=-2\langle\Psi(X_t),X_t\rangle\\&=-2\langle\Psi(X_t)-\Psi(0),X_t-0\rangle\le-2\delta\|X_t\|_{r+1}^{r+1},\end{aligned}$$

then by the Itô formula we have

$$\|X_t\|_H^2\le\|x\|_H^2+\int_0^t\left(b-2\delta\|X_s\|_{r+1}^{r+1}-2\gamma\|X_s\|_H^q\right)\mathrm{d}s+2\int_0^t\langle X_s,B\mathrm{d}W_s\rangle.\tag{4.2.8}$$

This implies

$$e^{-2bT}\|X_T\|_H^2\le\|x\|_H^2+\int_0^Te^{-2bt}\left(b-2\delta\|X_t\|_{r+1}^{r+1}-2b\|X_t\|_H^2\right)\mathrm{d}t+2\int_0^Te^{-2bt}\langle X_t,B\mathrm{d}W_t\rangle.$$

Hence

$$2\delta e^{-2bT}\int_0^T\|X_t\|_{r+1}^{r+1}\mathrm{d}t\le\int_0^Tbe^{-2bt}\mathrm{d}t+\|x\|_H^2+M_T-\int_0^T2be^{-2bt}\|X_t\|_H^2\mathrm{d}t,$$

where $M_T=2\int_0^Te^{-2bt}\langle X_t,B\mathrm{d}W_t\rangle$.

It is easy to check that M_t is a martingale from (4.2.8) and (4.1.3). By taking $\lambda_T=2\delta e^{-2bT}$ we obtain

$$\begin{aligned}&\mathbf{E}\exp\left[\lambda_T\int_0^T\|X_t\|_{r+1}^{r+1}\mathrm{d}t\right]\\&\le\exp\left[\int_0^Tbe^{-2bt}\mathrm{d}t+\|x\|_H^2\right]\mathbf{E}\exp\left[M_T-\int_0^T2be^{-2bt}\|X_t\|_H^2\mathrm{d}t\right].\end{aligned}$$

Since
$$\langle M\rangle_t \le \int_0^T 4be^{-4bt}\|X_t\|_H^2\mathrm{d}t,\quad \mathbf{E}\exp[M_t-\frac{1}{2}\langle M\rangle_t]=1,$$
we have
$$\mathbf{E}\exp\left[M_T-\int_0^T 2be^{-2bt}\|X_t\|_H^2\mathrm{d}t\right]\le 1.$$
Hence (4.2.6) holds.

Similarly, since (4.2.4) implies
$$\|X_t-Y_t\|_H^2\le\|x-y\|_H^2,\ t\ge 0,$$
then by (4.2.1) and the Itô formula we have
$$\begin{aligned}
&e^{-(2b+1)T}\|Y_T\|_H^2\\
\le\ &\|y\|_H^2+\int_0^T e^{-(2b+1)t}\Big[b-2\delta\|Y_t\|_{r+1}^{r+1}-(2b+1)\|Y_t\|_H^2+2\|Y_t\|_H\beta_t\|X_t-Y_t\|_H^{1-\varepsilon}\Big]\mathrm{d}t+M_T'\\
\le\ &\|y\|_H^2+\int_0^T e^{-(2b+1)t}\Big[b-2\delta\|Y_t\|_{r+1}^{r+1}-2b\|Y_t\|_H^2+\beta_t^2\|x-y\|_H^{2(1-\varepsilon)}\Big]\mathrm{d}t+M_T',
\end{aligned}$$
where $M_t':=\int_0^t 2e^{-(2b+1)s}\langle Y_s, B\mathrm{d}W_s\rangle$ is a martingale. This implies
$$\begin{aligned}
2\delta e^{-(2b+1)T}\int_0^T\|Y_t\|_{r+1}^{r+1}\mathrm{d}t\le\int_0^T be^{-(2b+1)t}\mathrm{d}t+\|y\|_H^2+\|x-y\|_H^{2(1-\varepsilon)}\int_0^T\beta_t^2e^{-(2b+1)t}\mathrm{d}t\\
+M_T'-\int_0^T 2be^{-(2b+1)t}\|Y_t\|_H^2\mathrm{d}t.
\end{aligned}$$
Therefore, by taking $\lambda_T=2\delta e^{-(2b+1)T}$ and noting that
$$\langle M'\rangle_T\le\int_0^T 4be^{-2(2b+1)t}\|Y_t\|_H^2\mathrm{d}t,$$
we obtain (4.2.7). ∎

Now we can give the complete proof of the main theorem.

Proof of Theorem 4.1.2 From now on, for any given time T we take $\varepsilon=\frac{\sigma}{\sigma+2}$ and
$$\beta_t=c(2\varepsilon\delta\xi)^{\frac{1}{\sigma}},\quad c=\frac{\|x-y\|_H^\varepsilon}{\varepsilon(2\varepsilon\delta\xi)^{\frac{1}{\sigma}}T}.$$
Then it is easy to show $X_T=Y_T$ $a.s.$ by Lemma 4.2.1.

Let $f_t := \left(\mathbf{m}\left[(|X_t| \vee |Y_t|)^{r+1}\right]\right)^{\frac{1-r}{1+r}}$, by the Hölder inequality we have

$$\|X_t - Y_t\|_{r+1}^{r+1} \leq \mathbf{m}\left(|X_t - Y_t|^2(|X_t| \vee |Y_t|)^{r-1}\right) \cdot \left(\mathbf{m}\left[(|X_t| \vee |Y_t|)^{r+1}\right]\right)^{\frac{1-r}{1+r}}.$$

Then by (4.2.4), (4.1.10) and Itô's formula

$$\begin{aligned}
\|X_T - Y_T\|_H^{2\varepsilon} &\leq \|x-y\|_H^{2\varepsilon} - 2\varepsilon\delta \int_0^T \|X_t - Y_t\|_H^{2(\varepsilon-1)} \mathbf{m}\left(|X_t - Y_t|^2(|X_t| \vee |Y_t|)^{r-1}\right)\mathrm{d}t \\
&\leq \|x-y\|_H^{2\varepsilon} - 2\varepsilon\delta \int_0^T \|X_t - Y_t\|_H^{2(\varepsilon-1)} \frac{\|X_t - Y_t\|_{r+1}^2}{\left(\mathbf{m}\left[(|X_t| \vee |Y_t|)^{r+1}\right]\right)^{\frac{1-r}{1+r}}}\mathrm{d}t \\
&\leq \|x-y\|_H^{2\varepsilon} - 2\varepsilon\delta\xi \int_0^T \frac{\|X_t - Y_t\|_B^\sigma}{\|X_t - Y_t\|_H^{\sigma-2\varepsilon} f_t}\mathrm{d}t \\
&= \|x-y\|_H^{2\varepsilon} - \int_0^T \frac{\beta_t^\sigma \|X_t - Y_t\|_B^\sigma}{c^\sigma \|X_t - Y_t\|_H^{\sigma\varepsilon} f_t}\mathrm{d}t.
\end{aligned}$$

Combining with (4.2.2) we arrive at

$$\begin{aligned}
\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t &= \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_B^2}{\|X_t - Y_t\|_H^{2\varepsilon}}\mathrm{d}t \\
&\leq \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \left(\int_0^T \frac{\beta_t^\sigma \|X_t - Y_t\|_B^\sigma}{\|X_t - Y_t\|_H^{\sigma\varepsilon} f_t}\mathrm{d}t\right)^{\frac{2}{\sigma}} \\
&\leq \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \left(c^\sigma \|x-y\|_H^{2\varepsilon}\right)^{\frac{2}{\sigma}} \\
&\leq \lambda \int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t + \lambda^{(2-\sigma)/2} c^\sigma \|x-y\|_H^{2\varepsilon}, \quad \lambda > 0,
\end{aligned} \tag{4.2.9}$$

where the last inequality follows from Young's inequality.

Since $\sigma \geq \frac{4}{1+r}$ implies $\frac{2}{\sigma-2} \leq \frac{1+r}{1-r}$, we have

$$f_t^{\frac{2}{\sigma-2}} \leq \mathbf{m}\big(1 + |X_t|^{r+1} \vee |Y_t|^{r+1}\big)^{\frac{2(1-r)}{(\sigma-2)(1+r)}} \leq \mathbf{m}\big(1 + |X_t|^{r+1} \vee |Y_t|^{r+1}\big).$$

Thus,

$$\begin{aligned}
&\mathbf{E}\exp\left[\lambda \int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t\right] \\
&\leq \mathbf{E}\exp\left[\lambda \int_0^T (1 + \|X_t\|_{r+1}^{r+1} + \|Y_t\|_{r+1}^{r+1})\mathrm{d}t\right], \quad \lambda > 0.
\end{aligned} \tag{4.2.10}$$

From (3.3.6) we have

$$
\begin{aligned}
&(P_TF(y))^p \le P_TF^p(x)\Big(\mathbf{E}R^{p/(p-1)}\Big)^{p-1}\\
&= P_TF^p(x)\left\{\mathbf{E}\exp\left[\frac{p}{p-1}\int_0^T\langle\zeta_t,\mathrm{d}W_t\rangle-\frac{p}{2(p-1)}\int_0^T\|\zeta_t\|_2^2\mathrm{d}t\right]\right\}^{p-1}\\
&\le P_TF^p(x)\left\{\mathbf{E}\exp\left[\frac{qp}{p-1}\int_0^T\langle\zeta_t,\mathrm{d}W_t\rangle-\frac{q^2p^2}{2(p-1)^2}\int_0^T\|\zeta_t\|_2^2\mathrm{d}t\right]\right\}^{\frac{p-1}{q}}\\
&\cdot\left\{\mathbf{E}\exp\left[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2}\int_0^T\|\zeta_t\|_2^2\mathrm{d}t\right]\right\}^{\frac{(q-1)(p-1)}{q}}\\
&= P_TF^p(x)\left\{\mathbf{E}\exp\left[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2}\int_0^T\|\zeta_t\|_2^2\mathrm{d}t\right]\right\}^{\frac{(q-1)(p-1)}{q}},\quad q>1.
\end{aligned}
\tag{4.2.11}
$$

Moreover, letting $\lambda=\frac{\lambda_T(q-1)(p-1)^2}{pq(pq-p+1)}$, by (4.2.9), (4.2.10) and Lemma 4.2.2 we obtain that

$$
\begin{aligned}
&\mathbf{E}\exp\left[\frac{qp(qp-p+1)}{2(q-1)(p-1)^2}\int_0^T\|\zeta_t\|_2^2\mathrm{d}t\right]\\
&\le \mathbf{E}\exp\left[\frac{\lambda_T}{2}\int_0^T(1+\|X_t\|_{r+1}^{r+1}+\|Y_t\|_{r+1}^{r+1})\mathrm{d}t\right.\\
&\left.+\frac{qp(qp-p+1)}{2(q-1)(p-1)^2}\Big(\frac{\lambda_T(q-1)(p-1)^2}{pq(pq-p+1)}\Big)^{\frac{2-\sigma}{2}}c^\sigma\|x-y\|_H^{2\varepsilon}\right]\\
&\le \exp\left[\frac{1}{2}\Big(2\int_0^T be^{-2bt}\mathrm{d}t+\lambda_TT+\|x\|_H^2+\|y\|_H^2+\|x-y\|_H^{2(1-\varepsilon)}\int_0^T\beta_t^2e^{-(2b+1)t}\mathrm{d}t\Big)\right.\\
&\qquad\left.+\frac{qp(qp-p+1)}{2(q-1)(p-1)^2}\Big(\frac{\lambda_T(q-1)(p-1)^2}{pq(pq-p+1)}\Big)^{\frac{2-\sigma}{2}}c^\sigma\|x-y\|_H^{2\varepsilon}\right].
\end{aligned}
\tag{4.2.12}
$$

Combining this with (4.2.11) and simply letting $q=2$ we have

$$
\begin{aligned}
&(P_TF(y))^p\le P_TF^p(x)\exp\left[\frac{p-1}{4}\Big(2\int_0^T be^{-2bt}\mathrm{d}t+\lambda_TT+\|x\|_H^2+\|y\|_H^2+\right.\\
&\left.\|x-y\|_H^{2(1-\varepsilon)}\int_0^T\beta_t^2e^{-(2b+1)t}\mathrm{d}t\Big)+\frac{p(p+1)}{2(p-1)}\Big(\frac{\lambda_T(p-1)^2}{2(p+1)}\Big)^{\frac{2-\sigma}{2}}c^\sigma\|x-y\|_H^{2\varepsilon}\right].
\end{aligned}
\tag{4.2.13}
$$

Then the desired result (4.1.11) follows from the definition of β_t and c.

Finally, since (4.2.11) implies that R is uniformly integrable for fixed x and $\{y:\|y-x\|_H\le 1\}$, then by the dominated convergence theorem we have for any bounded measurable function F on H

$$\lim_{y\to x}|P_TF(y)-P_TF(x)|\le \|F\|_\infty \lim_{y\to x}\mathbf{E}|R-1| = \|F\|_\infty \mathbf{E}\lim_{y\to x}|R-1| = 0,$$

where the last equality follows from
$\lim_{y\to x} R = 1$ due to (4.2.9). Hence P_T is strong Feller. Now the proof is complete.

4.3 Applications to explicit examples

To provide explicit sufficient conditions for (4.1.10), we need the Nash inequality:

$$\|f\|_2^{2+4/d} \le C\langle f, -Lf\rangle,\ f\in\mathcal{D}(L),\ \mathbf{m}(|f|)=1. \tag{4.3.1}$$

This inequality is equivalent to the classical Sobolev inequality with dimension d if $d>2$. Hence we can also include examples with dimension $d\le 2$ here. For example, (4.3.1) holds for the Dirichlet Laplace operator on bounded domains in a Riemannian manifold and on a whole Riemannian manifold provided the injectivity radius is infinite (cf.[Cro80]).

Lemma 4.3.1 *Let $r\in(0,1)$. Assume that $-(-L)^{1/n}$ is a Dirichlet operator for some $n\ge 1$ and (4.3.1) holds for some $d\in(0,\frac{2(r+1)}{1-r})$. Then the embedding $L^{r+1}(\mathbf{m})\subset H$ is compact. In particular,*

$$\|x\|_H = \langle x, (-L)^{-1}x\rangle^{1/2} \le c\|x\|_{r+1},\ x\in L^{r+1}(\mathbf{m})$$

holds for some $c>0$.

Proof. We take $\varepsilon\in(0,1)$ such that $d_\varepsilon := d/\varepsilon \in (d, \frac{2(r+1)}{1-r})$ and let $L_\varepsilon := -(-L)^\varepsilon$. By [BM07, Theorem 1.3] and (4.3.1) there exists a constant $C'>0$ such that

$$\|f\|_2^{2+4/d_\varepsilon} \le C'\langle f, -L_\varepsilon f\rangle,\ f\in\mathcal{D}(L_\varepsilon),\ \mathbf{m}(|f|)=1.$$

Then by [BM07, Theorem 1.3] we have

$$\|f\|_2^{2+\frac{4}{d_\varepsilon n}} \le c_0\langle f, (-L_\varepsilon)^{1/n} f\rangle,\quad f\in\mathcal{D}((-L_\varepsilon)^{1/n}),\ \mathbf{m}(|f|)=1 \tag{4.3.2}$$

for some $c_0 > 0$. Let T_t be the semigroup generated by $-(-L_\varepsilon)^{1/n}$, which is sub-Markovian since $-(-L_\varepsilon)^{1/n} = -(-L)^{\varepsilon/n}$ is a Dirichlet operator. Then it follows from (4.3.2) that (see [Dav89])

$$\|T_t\|_{1\to\infty} \le c_1 t^{-d_\varepsilon n/2}, \quad t > 0$$

holds for some constant $c_1 > 0$. Since $\lambda_1 > 0$, there exists $c_2 > 0$ such that

$$\|T_t\|_{1\to\infty} \le \|T_{t/4}\|_{1\to 2}\|T_{t/2}\|_{2\to 2}\|T_{t/4}\|_{2\to\infty} \le c_2 t^{-d_\varepsilon n/2} e^{-\lambda_1^{\frac{\varepsilon}{n}} t/2}, \quad t > 0.$$

By this and the Riesz-Thorin interpolation theorem we conclude that for any $1 < p < q$,

$$\|T_t\|_{p\to q} \le \|T_t\|_{1\to\infty}^{\frac{q-p}{pq}} \le c_3 [t^{-d_\varepsilon n/2} e^{-\lambda_1^{\frac{\varepsilon}{n}} t/2}]^{\frac{q-p}{pq}}, \quad t > 0$$

holds for some constant $c_3 > 0$. Therefore,

$$C_{p,q} := \int_0^\infty \|T_t\|_{p\to q} dt < \infty$$

provided $\frac{q-p}{pq} < \frac{2}{d_\varepsilon n}$. Thus,

$$\|(-L_\varepsilon)^{-1/n}\|_{p\to q} \le C_{p,q} < \infty, \quad \frac{q-p}{pq} < \frac{2}{d_\varepsilon n}.$$

Since $d_\varepsilon < \frac{2(r+1)}{1-r}$, by letting $p_i := \frac{r+1}{1-2(i-1)(r+1)/d_\varepsilon n}$ $(1 \le i \le n+1)$ one has

$$p_1 = r+1, \quad \frac{p_{i+1} - p_i}{p_{i+1} p_i} = \frac{2}{d_\varepsilon n} \ (1 \le i \le n) \quad \text{and} \quad p_{n+1} = \frac{r+1}{1 - 2(r+1)/d_\varepsilon} > \frac{r+1}{r}.$$

So, there exist $r+1 =: p_1' < p_2' < \cdots < p_{n+1}' := \frac{r+1}{r}$ such that $\frac{p_{i+1}' - p_i'}{p_{i+1}' p_i'} < \frac{2}{d_\varepsilon n}$, $1 \le i \le n$. Therefore,

$$c^2 := \|(-L_\varepsilon)^{-1}\|_{r+1\to(r+1)/r} \le \prod_{i=1}^n \|(-L_\varepsilon)^{-\frac{1}{n}}\|_{p_i'\to p_{i+1}'} \le \prod_{i=1}^n C_{p_i',p_{i+1}'} < \infty.$$

This implies

$$\begin{aligned}
\langle x, (-L_\varepsilon)^{-1} x\rangle &\le \|x\|_{r+1}\|(-L_\varepsilon)^{-1}x\|_{(r+1)/r} \\
&\le \|x\|_{r+1}^2 \|(-L_\varepsilon)^{-1}\|_{r+1\to(r+1)/r} = c^2\|x\|_{r+1}^2, \quad x \in L^{r+1}(\mathbf{m}).
\end{aligned}$$

Then the proof is completed since $\{x \in L^2(\mathbf{m}) : \langle x, (-L_\varepsilon)^{-1}x\rangle \le N\}$ is relatively compact in H for any $N > 0$. ∎

Corollary 4.3.2 *Let $Be_i = b_i e_i,\ i \geq 1$ with $\sum_{i=1}^\infty \frac{b_i^2}{\lambda_i} < \infty$, hence B is a Hilbert-Schmidt operator from $L^2(\mathbf{m})$ to H. If $\varepsilon \in (0,1)$ and L satisfies (4.3.1) for some $d \in (0, \frac{2\varepsilon(1+r)}{1-r})$, $-(-L)^{1/n}$ is a Dirichlet operator for some $n \geq 1$ and there exist $c > 0$, $\sigma \geq \frac{4}{1+r}$ such that*

$$b_i \geq c\lambda_i^{\frac{\sigma+2\varepsilon-2}{2\sigma}},\ i \geq 1,$$

then the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact and (4.1.10) holds for the same σ.

Proof. By Lemma 4.3.1 it suffices to verify (4.1.10). By the Hölder inequality we have

$$\begin{aligned}
\|x\|_B^\sigma &= \Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 b_i^{-2}\Big)^{\sigma/2} = \Big(\sum_{i=1}^\infty \frac{\langle x, e_i\rangle^2}{\lambda_i^{\frac{\sigma-2}{\sigma}}} \lambda_i^{\frac{\sigma-2}{\sigma}} b_i^{-2}\Big)^{\sigma/2} \\
&\leq \Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 \lambda_i^{\frac{\sigma-2}{2}} b_i^{-\sigma}\Big)\Big(\sum_{i=1}^\infty \frac{\langle x, e_i\rangle^2}{\lambda_i}\Big)^{\frac{\sigma-2}{2}} \\
&= \|x\|_H^{\sigma-2}\Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 \lambda_i^{\frac{\sigma-2}{2}} b_i^{-\sigma}\Big) \\
&\leq c^{-\sigma}\|x\|_H^{\sigma-2}\Big(\sum_{i=1}^\infty \langle x, e_i\rangle^2 \lambda_i^{-\varepsilon}\Big).
\end{aligned} \tag{4.3.3}$$

By (4.3.1) and [BM07, Theorem 1.3] there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_2^{2+4\varepsilon/d} \leq C_\varepsilon \langle f, (-L)^\varepsilon f\rangle,\ f \in \mathcal{D}((-L)^\varepsilon),\ \mathbf{m}(|f|) = 1.$$

Applying Lemma 4.3.1 to $-(-L)^\varepsilon$ in place of L, there exists a constant $c_1 > 0$ such that

$$\|x\|_{r+1}^2 \geq c_1\|(-L)^{-\varepsilon/2}x\|_2^2 = c_1 \sum_{i=1}^\infty \langle x, e_i\rangle^2 \lambda_i^{-\varepsilon}.$$

Combining this with (4.3.3) we obtain that (4.1.10) holds for some constant $\xi > 0$. ∎

Now we can give the first simple example such that all assumptions in Theorem 4.1.2 and 4.1.3 are satisfied.

Example 4.3.3 *Let $\Psi(t,x) := |x|^{r-1}x$ and $L := \Delta$ be the Laplace operator on a bounded domain in $\mathbb{R}$ with the Dirichlet boundary condition. Suppose $Be_i = b_i e_i$ such that*

$$\frac{1}{C} \cdot \lambda_i^{\frac{3-3r}{8}+\varepsilon} \leq b_i \leq C \cdot \lambda_i^{\frac{1}{4}-\varepsilon},\ i \geq 1 \tag{4.3.4}$$

hold for some positive constants C and ε, then the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact and (4.1.10) holds for $\sigma = \frac{4}{r+1}$.

In particular, if $r \in (\frac{1}{3}, 1)$ and $B := (-\Delta)^\theta$ with $\theta \in \left(\frac{3-3r}{8}, \frac{1}{4}\right)$, then the transition semigroup associated with (4.1.1) is ultrabounded and compact provided $\gamma > 0$ and $q > \frac{4}{r+1}$. Moreover, the generator of the transition semigroup has a spectral gap in this case.

Proof. It is well-known that $\lambda_i \geq ci^2$ for some constant $c > 0$ in this case, then (4.3.4) implies (4.1.10) with $\sigma = \frac{4}{r+1}$ by Corollary 4.3.2. And the other assertions follow from Theorem 4.1.3. ∎

Note that the underlying space for L is 1-dimensional in the above example. However, by using the spectral representation theorem, we can have much more choices for L such as high order differential operators on a domain or on $\mathbb{R}^d$. We only present one explicit example here, where L is a fractional power of the Laplace operator. For more general self-adjoint operators as the choices for L we refer to [LW08, Wan07].

Example 4.3.4 *Let $L := -(-\Delta)^\alpha$, where α is a constant and Δ is the Dirichlet Laplace operator on a bounded domain $\Lambda \subset \mathbb{R}^d$, and $\mathbf{m}$ be the normalized volume measure on Λ. If $Be_i = b_i e_i,\ i \geq 1$ and*

$$\frac{1}{C} \cdot \lambda_i^{\frac{(2\alpha+d)(1-r)}{8\alpha}+\varepsilon} \leq b_i \leq C \cdot \lambda_i^{\frac{2\alpha-d}{4\alpha}-\varepsilon},\ i \geq 1$$

hold for some $\varepsilon \in (0,1)$, then the embedding $L^{r+1}(\mathbf{m}) \subset H$ is compact and (4.1.10) holds. Therefore, all assertions in **Theorem 4.1.1-4.1.3** *hold for (4.1.1).*

In particular, if $\alpha > \frac{d(3-r)}{2(1+r)}$ and we take $B := (-\Delta)^\theta$ with $\frac{(1-r)(2\alpha+d)}{8} < \theta < \frac{2\alpha-d}{4}$, then the transition semigroup associated with (4.1.1) is ultrabounded and compact provided $\gamma > 0$ and $q > \frac{4}{r+1}$.

Proof. By the Sobolev inequality we have (cf. [Wan00, Corollary 1.1 and 3.1])

$$\lambda_i \geq ci^{\frac{2\alpha}{d}}, \quad i \geq 1$$

for some $c > 0$. It is well-known that Δ satisfies the Nash inequality (4.3.1). Then by [BM07, Theorem 1.3] we know L satisfies the following Nash inequality

$$\|f\|_2^{2+4\alpha/d} \leq C_\alpha \langle f, -Lf \rangle,\ f \in \mathcal{D}(L),\ \mathbf{m}(|f|) = 1,$$

where $C_\alpha > 0$ is a constant. If we take $\sigma = \frac{4}{1+r}$, then all assertions follow from Corollary 4.3.2. ∎

Chapter 5

Ergodicity for Stochastic p-Laplace Equation

In this chapter we study the singular stochastic p-Laplace equation ($1 < p \leq 2$) with some nonlinear perturbations in the drift. We first investigate the existence and uniqueness of invariant measures and the convergence of the transition semigroups to the invariant measure. Then we establish the strong Feller property and the Harnack inequality for the transition semigroups associated to the p-Laplace equation with non-degenerate noise. As consequences, the ultraboundedness, compactness and the existence of a spectral gap are also derived. In particular, the main results are also applied to stochastic reaction-diffusion equations and the ultraboundedness and compactness property are established for the associated transition semigroups, which improve the corresponding results obtained in chapter 3.

5.1 Introduction and the main results

The following p-Laplace equation

$$\partial_t u = \mathbf{div}(|\nabla u|^{p-2}\nabla u),\ 1 < p < \infty \tag{5.1.1}$$

has been studied intensively in the PDE theory. (5.1.1) describes the type of diffusion with diffusivity depending on the gradient of the main unknown, which also has a strong connection with porous media equations and fast diffusion equations (cf.[Váz07, DiB93, Váz06]). This type of equation arises from geometry, quasiregular mappings and fluid dynamics *etc* (cf.[DiB93]). In particular, Ladyzenskaja suggests (5.1.1) as a model of motion of non-newtonian fluids in [Lad67]. In stochastic case, the existence and uniqueness of solution to the stochastic p-Laplace equation follows from the general results in [KR79, RRW07, Zha08]. The large deviation principle has been established in chapter 2 for (5.1.1) with small multiplicative noise. For the degenerate case (*i.e.* $p > 2$), the Markov property of the solution and some properties of invariant measures have been studied in [PR07], the Harnack inequality and many consequent results have been established in chapter 3.

Let Λ be an open bounded domain in $\mathbb{R}^d$ with a C^1 boundary. We consider the following Gelfand triple

$$W_0^{1,p}(\Lambda) \cap L^q(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{1,p}(\Lambda) \cap L^q(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - \gamma |X_t|^{q-2}X_t\right]\mathrm{d}t + B\mathrm{d}W_t,\ X_0 = x \in L^2(\Lambda), \tag{5.1.2}$$

where $1 < p \le 2 \le q$ and $\gamma \ge 0$, B is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and W_t is a cylindrical Wiener process on $L^2(\Lambda)$ w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. One should note that we would remove $L^q(\Lambda)$ in the Gelfand triple if $\gamma = 0$ in (5.1.2).

Since Λ is a bounded domain, then by the Poincaré inequality the following norm

$$\|u\|_{1,p} := \left(\int_\Lambda |\nabla u(\xi)|^p \mathrm{d}\xi\right)^{1/p},\ u \in W_0^{1,p}(\Lambda)$$

is equivalent to the classical Sobolev norm in $W_0^{1,p}(\Lambda)$. For simplicity, we will always use this equivalent norm in this chapter. We denote the norm in $L^r(\Lambda)$ by $\|\cdot\|_r$ and the inner product in $L^2(\Lambda)$ by $\langle\cdot,\cdot\rangle$.

According to [Zha08, Theorem 3.6], for any $x \in L^2(\Lambda)$ the equation (5.1.2) has a unique solution $X_t(x)$, which is a continuous adapted process on $L^2(\Lambda)$ and satisfies

$$\mathbf{E}\left(\sup_{t\in[0,T]}\|X_t(x)\|_2^2 + \int_0^T \|X_t(x)\|_{1,p}^p \mathrm{d}t\right) < \infty, \quad T > 0. \tag{5.1.3}$$

Moreover, we have the following Itô formula

$$\|X_t\|_2^2 = \|X_0\|_2^2 + \int_0^t \left(b - 2\|X_s\|_{1,p}^p - 2\gamma\|X_s\|_q^q\right) \mathrm{d}s + 2\int_0^t \langle X_s, B\mathrm{d}W_s\rangle\,,\ t \geq 0, \quad (5.1.4)$$

where $b = \|B\|_{HS}^2$ (Hilbert-Schmidt norm).

Now we consider the associated transition semigroups

$$P_tF(x) := \mathbf{E}F\left(X_t(x)\right),\ t > 0,$$

where F is a bounded measurable function on $L^2(\Lambda)$.

Theorem 5.1.1 *Suppose the embedding $W_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ is compact.*

(i) The transition semigroup $\{P_t\}$ has an invariant probability measure.

(ii) If $\gamma > 0$, then $\{P_t\}$ has a unique invariant measure μ. Moreover, we have $\mu\left(\|\cdot\|_{1,p}^p + e^{\varepsilon_0\|\cdot\|_2^q}\right) < \infty$ for some $\varepsilon_0 > 0$.

(iii) If $\gamma > 0$ and $q = 2$, then for any Lipschitz continuous function F on $L^2(\Lambda)$ we have

$$|P_tF(x) - \mu(F)| \leq Lip(F)e^{-\gamma t}\left(\|x\|_2 + C\right),\ t \geq 0,\ x \in L^2(\Lambda), \quad (5.1.5)$$

where C is a constant and $Lip(F)$ is the Lipschitz constant of F.

(iv) If $\gamma > 0$ and $q > 2$, then for any Lipschitz continuous function F on $L^2(\Lambda)$ we have

$$\sup_{x\in L^2(\Lambda)} |P_tF(x) - \mu(F)| \leq C\,Lip(F)t^{-\frac{1}{q-2}},\ t > 0, \quad (5.1.6)$$

where C is a constant.

Proof. (i) The existence of invariant measure can be proved by the standard Krylov-Bogoliubov argument (see Theorem 3.3.4(i)).

(ii) If $\gamma > 0$, then there exist positive constants c and C such that

$$\|X_t(x)-X_t(y)\|_2^2 \leq \|x-y\|_2^2 - c\gamma\int_0^t \|X_s(x)-X_s(y)\|_q^q \mathrm{d}s \leq \|x-y\|_2^2 - C\int_0^t \|X_s(x)-X_s(y)\|_2^q \mathrm{d}s.$$

Hence we have

$$\lim_{t\to\infty} \|X_t(x) - X_t(y)\|_2 = 0,\ \forall x, y \in L^2(\Lambda).$$

This implies the uniqueness of invariant measures.

Now we need to prove the concentration property of the invariant measure. (5.1.4) implies that there exists a constant C such that

$$\mu_n(\|\cdot\|_2^2) = \frac{1}{n}\int_0^n \mathbf{E}\|X_t(0)\|_2^2 \mathrm{d}t \le C, \ n \ge 1,$$

where $\mu_n = \frac{1}{n}\int_0^n \delta_0 P_t \mathrm{d}t$. Hence $\mu(\|\cdot\|_2^2) < \infty$, since μ is the weak limit of a subsequence of μ_n.

By (5.1.4) there also exists a constant C such that

$$\mathbf{E}\int_0^1 \|X_t(x)\|_{1,p}^p \mathrm{d}t \le C(1+\|x\|_2^2), \ \forall x \in L^2(\Lambda).$$

Therefore,

$$\mu(\|\cdot\|_{1,p}^p) = \int \mu(dx)\int_0^1 \mathbf{E}(\|X_t(x)\|_{1,p}^p)\mathrm{d}t \le C + C\int \|x\|_2^2 \mu(\mathrm{d}x) < \infty.$$

If $\gamma > 0$ and ε_0 is small enough, then by Itô's formula

$$\begin{aligned} e^{\varepsilon_0\|X_t\|_2^q} \le & e^{\varepsilon_0\|x\|_2^q} + \int_0^t \left(c - 2\gamma\|X_s\|_q^q + qb\varepsilon_0\|X_s\|_2^q\right)\frac{q\varepsilon_0}{2}\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\mathrm{d}s \\ & + q\varepsilon_0\int_0^t \|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s, B\mathrm{d}W_s\rangle \\ \le & e^{\varepsilon_0\|x\|_2^q} + \int_0^t \left(c - \gamma\|X_s\|_q^q\right)\frac{q\varepsilon_0}{2}\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\mathrm{d}s \\ & + q\varepsilon_0\int_0^t \|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s, B\mathrm{d}W_s\rangle \\ \le & e^{\varepsilon_0\|x\|_2^q} + \int_0^t \left(c_1 - c_2 e^{\varepsilon_0\|X_s\|_2^q}\right)\mathrm{d}s + q\varepsilon_0\int_0^t \|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s, B\mathrm{d}W_s\rangle \end{aligned} \tag{5.1.7}$$

hold for some positive constants c, c_1 and c_2. Therefore

$$\mu_n(e^{\varepsilon_0\|\cdot\|_2^q}) = \frac{1}{n}\int_0^n \mathbf{E}e^{\varepsilon_0\|X_t(0)\|_2^q}\mathrm{d}t \le \frac{1}{c_2 n} + \frac{c_1}{c_2}.$$

Hence we have $\mu(e^{\varepsilon_0\|\cdot\|_2^q}) < \infty$ for some $\varepsilon_0 > 0$.

The proof of (iii) and (iv) are very similar to the arguments in Theorem 4.1.1, hence we omit the details here.

∎

Remark 5.1.1 *(1) If $2 \geq p > \max\{1, \frac{2d}{d+2}\}$, then the embedding $H_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ is compact according to the Rellich-Kondrachov theorem.*

(2) If $B = 0$ and Dirac measure at 0 is the unique invariant measure of $\{P_t\}$, then by taking $F(x) = \|x\|_2$ in (5.1.6) we get the following algebraically decay estimate

$$\sup_{x \in L^2(\Lambda)} \|u_t(x)\|_2 \leq Ct^{-\frac{1}{q-2}},\ t > 0,$$

where $u_t(x)$ is the solution to the following deterministic equation

$$\frac{\partial u}{\partial t} = \mathbf{div}(|\nabla u|^{p-2}\nabla u) - \gamma |u|^{q-2}u,\ u_0 = x.$$

Now we assume $ker(B) = 0$ and define the following intrinsic metric for $x \in W_0^{1,p}(\Lambda)$:

$$\|x\|_B := \begin{cases} \|y\|_2, & \text{if } y \in L^2(\Lambda),\ By = x, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 5.1.2 *If there exist constants $\sigma \geq \frac{4}{p}$ and $\xi > 0$ such that*

$$\|x\|_{1,p}^2 \cdot \|x\|_2^{\sigma-2} \geq \xi \|x\|_B^\sigma, \quad \forall x \in W_0^{1,p}(\Lambda), \tag{5.1.8}$$

then for any $t > 0$, P_t is a strong Feller operator and for any positive bounded measurable function F on $L^2(\Lambda)$, $\alpha > 1$ and $x, y \in L^2(\Lambda)$ we have

$$\begin{aligned}(P_tF(y))^\alpha \leq P_tF^\alpha(x) \exp\Big[&\frac{\alpha-1}{4}\Big(1 + 2te^{-(2b+1)t} + \|x\|_2^2 + \|y\|_2^2 + \frac{(\sigma+2)^2}{\sigma^2 t}\|x-y\|_2^2\Big) \\ &+ \Big(\frac{\sigma+2}{\sigma}\Big)^{\sigma+1} \frac{[\alpha(\alpha+1)]^{\sigma/2} e^{(2b+1)(\sigma-2)t}}{4(p-1)\xi(\alpha-1)^{\sigma-1}t^\sigma}\|x-y\|_2^\sigma\Big].\end{aligned} \tag{5.1.9}$$

Proof. For $\varepsilon \in (0,1)$ and $\beta \in \mathbf{C}([0,\infty);\mathbb{R}^+)$ we consider the following equation of Y_t

$$\mathrm{d}Y_t = \Big\{\mathbf{div}(|\nabla Y_t|^{p-2}\nabla Y_t) - \gamma|Y_t|^{q-2}Y_t + \frac{\beta_t(X_t - Y_t)}{\|X_t - Y_t\|_2^\varepsilon}\mathbf{1}_{\{t<\tau\}}\Big\}\mathrm{d}t + B\mathrm{d}W_t,\ Y_0 = y, \tag{5.1.10}$$

where $X_t := X_t(x)$ and $\tau := \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time.

According to [Zha08, Theorem 3.6], (5.1.10) has a unique strong solution Y_t (see Lemma 3.3.2). Moreover, we have

$$\|X_t - Y_t\|_2 \leq \|X_s - Y_s\|_2,\ t \geq s \geq 0.$$

By the definition of τ, we have $X_t = Y_t$ for $t \geq \tau$.

By taking

$$\zeta_t := \frac{\beta_t B^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_2^\varepsilon} \mathbf{1}_{\{t<\tau\}},$$

we can rewrite the equation (5.1.10) as

$$\mathrm{d}Y_t = (\mathbf{div}(|\nabla Y_t|^{p-2}\nabla Y_t) - \gamma |Y_t|^{q-2} Y_t)\mathrm{d}t + B(\mathrm{d}W_t + \zeta_t \mathrm{d}t), \quad Y_0 = y.$$

Now we need to choose $\varepsilon \in [2-p, 1)$ and β such that
(a) $\tau \leq T$ *a.s.*;
(b) $\mathbf{E}\exp\left[\frac{1}{2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right] < \infty$.

For (a), it is easy to prove the following result (see Lemma 4.2.1).

Lemma 5.1.3 *If β satisfies $\int_0^T \beta_t \mathrm{d}t \geq \frac{1}{\varepsilon}\|x-y\|_2^\varepsilon$, then $\tau \leq T, a.s.$*

In order to verify (b), first we need to have the following a priori estimates.

Lemma 5.1.4 *We have*

$$\mathbf{E}\exp\left[\lambda_T \int_0^T \|X_t\|_{1,p}^p \mathrm{d}t\right] \leq \exp\left[\|x\|_2^2 + \int_0^T b e^{-2bt}\mathrm{d}t\right], \tag{5.1.11}$$

$$\begin{aligned} &\mathbf{E}\exp\left[\lambda_T \int_0^T \|Y_t\|_{1,p}^p \mathrm{d}t\right] \\ &\leq \exp\left[\|y\|_2^2 + \int_0^T b e^{-(2b+1)t}\mathrm{d}t + \|x-y\|_H^{2(1-\varepsilon)} \int_0^T \beta_t^2 e^{-(2b+1)t}\mathrm{d}t\right], \end{aligned} \tag{5.1.12}$$

where $\lambda_T = 2e^{-(2b+1)T}$ and $b = \|B\|_{HS}^2$.

Proof. The proof is similar to Lemma 4.2.2. By the Itô formula (5.1.4) we have

$$e^{-2bT}\|X_T\|_2^2 \leq \|x\|_2^2 + \int_0^T e^{-2bt}\left(b - 2\|X_t\|_{1,p}^p - 2b\|X_t\|_2^2\right)\mathrm{d}t + 2\int_0^T e^{-2bt}\langle X_t, B\mathrm{d}W_t\rangle.$$

This implies

$$2e^{-2bT}\int_0^T \|X_t\|_{1,p}^p \mathrm{d}t \leq \|x\|_2^2 + \int_0^T b e^{-2bt}\mathrm{d}t + M_T - \int_0^T 2b e^{-2bt}\|X_t\|_2^2 \mathrm{d}t,$$

where $M_T = 2\int_0^T e^{-2bt}\langle X_t, B\mathrm{d}W_t\rangle$.

It is easy to check from (5.1.4) and (5.1.3) that $\{M_t\}$ is a martingale. By taking $\lambda_T = 2e^{-(2b+1)T}$ we obtain

$$
\begin{aligned}
&\mathbb{E}\exp\left[\lambda_T\int_0^T \|X_t\|_{1,p}^p \mathrm{d}t\right]\\
&\le \exp\left[\int_0^T be^{-2bt}\mathrm{d}t + \|x\|_H^2\right]\mathbb{E}\exp\left[M_T - \int_0^T 2be^{-2bt}\|X_t\|_2^2\mathrm{d}t\right].
\end{aligned}
$$

Since $\langle M\rangle_t \le \int_0^T 4be^{-4bt}\|X_t\|_2^2\mathrm{d}t$ and $\mathbb{E}\exp\left[M_t - \frac{1}{2}\langle M\rangle_t\right] = 1$, then

$$
\mathbb{E}\exp\left[M_T - \int_0^T 2be^{-2bt}\|X_t\|_2^2\mathrm{d}t\right] \le 1.
$$

Hence (5.1.11) holds.

Note that

$$
\|X_t - Y_t\|_2^2 \le \|x-y\|_2^2,\ t \ge 0,
$$

then by Itô's formula we have

$$
\begin{aligned}
&e^{-(2b+1)T}\|Y_T\|_2^2\\
\le\ & \|y\|_2^2 + \int_0^T e^{-(2b+1)t}\left[b - 2\|Y_t\|_{1,p}^p - (2b+1)\|Y_t\|_2^2 + 2\|Y_t\|_2\beta_t\|X_t - Y_t\|_2^{1-\varepsilon}\right]\mathrm{d}t + M_T'\\
\le\ & \|y\|_2^2 + \int_0^T e^{-(2b+1)t}\left[b - 2\|Y_t\|_{1,p}^p - 2b\|Y_t\|_2^2 + \beta_t^2\|x-y\|_2^{2(1-\varepsilon)}\right]\mathrm{d}t + M_T',
\end{aligned}
$$

where $M_t' := \int_0^t 2e^{-(2b+1)s}\langle Y_s, B\mathrm{d}W_s\rangle$ is a martingale. This implies

$$
\begin{aligned}
2e^{-(2b+1)T}\int_0^T \|Y_t\|_{1,p}^p\mathrm{d}t \le \|y\|_2^2 + \int_0^T be^{-(2b+1)t}\mathrm{d}t + \|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\mathrm{d}t\\
+ M_T' - \int_0^T 2be^{-(2b+1)t}\|Y_t\|_2^2\mathrm{d}t.
\end{aligned}
$$

Therefore, by a similar argument one can obtain (5.1.12). ∎

Proof of the Theorem 5.1.2: Taking $\varepsilon = \frac{\sigma}{\sigma+2}$ and

$$
\beta_t = c\left(2(p-1)\varepsilon\xi\right)^{1/\sigma},\quad c = \frac{\|x-y\|_2^{\varepsilon}}{\varepsilon\left(2(p-1)\varepsilon\xi\right)^{\frac{1}{\sigma}}T},
$$

then, according to Lemma 5.1.3, there exists a unique solution Y_t to (5.1.10) such that the coupling time $\tau \le T$, *a.s.*.

We can show that for any u, v in $W_0^{1,p}(\Lambda)$ (see Lemma 5.2.1),

$$\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u) - \mathbf{div}(|\nabla v|^{p-2}\nabla v), u-v\rangle_0 \leq -(p-1)\lambda\left(|\nabla u - \nabla v|^2 \left(|\nabla u| \vee |\nabla v|\right)^{p-2}\right),$$

where λ is the Lebesgue measure on Λ and $\langle \cdot, \cdot\rangle_0$ denotes the dualization between $W_0^{1,p}(\Lambda)$ and its dual space.

By the Itô formula there exists a constant $c > 0$ such that

$$\begin{aligned} \|X_t - Y_t\|_2^2 \leq & \|X_s - Y_s\|_2^2 - 2(p-1)\int_s^t \lambda\left(|\nabla X_u - \nabla Y_u|^2 \left(|\nabla X_u| \vee |\nabla Y_u|\right)^{p-2}\right)\mathrm{d}u \\ & - 2\int_s^t \beta_t \|X_u - Y_u\|_2^{2-\varepsilon} \mathbf{1}_{\{u<\tau\}}\mathrm{d}u - c\gamma\int_s^t \|X_u - Y_u\|_q^q \mathrm{d}u. \end{aligned} \tag{5.1.13}$$

By the Hölder inequality we have

$$\|X_t - Y_t\|_{1,p}^2 \leq \lambda\left(|\nabla X_t - \nabla Y_t|^2 \left(|\nabla X_t| \vee |\nabla Y_t|\right)^{p-2}\right) \cdot \left(\lambda\left[\left(|\nabla X_t| \vee |\nabla Y_t|\right)^p\right]\right)^{\frac{2-p}{p}}.$$

Let $f_t := \left(\lambda\left[\left(|\nabla X_t| \vee |\nabla Y_t|\right)^p\right]\right)^{\frac{2-p}{p}}$, then by Itô's formula, (5.1.13) and (5.1.8)

$$\begin{aligned} \mathrm{d}\left(\|X_t - Y_t\|_2^2\right)^\varepsilon & \leq -2(p-1)\varepsilon \|X_t - Y_t\|_2^{2(\varepsilon-1)}\lambda\left(|\nabla X_t - \nabla Y_t|^2 \left(|\nabla X_t| \vee |\nabla Y_t|\right)^{p-2}\right)\mathrm{d}t \\ & \leq -2(p-1)\varepsilon \|X_t - Y_t\|_2^{2(\varepsilon-1)} \frac{\|X_t - Y_t\|_{1,p}^2}{\left(\lambda\left[\left(|\nabla X_t| \vee |\nabla Y_t|\right)^p\right]\right)^{\frac{2-p}{p}}}\mathrm{d}t \\ & \leq -2(p-1)\varepsilon\xi \frac{\|X_t - Y_t\|_B^\sigma}{\|X_t - Y_t\|_2^{\sigma-2\varepsilon} f_t}\mathrm{d}t \\ & = -2(p-1)\varepsilon\xi \frac{\|X_t - Y_t\|_B^\sigma}{\|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\mathrm{d}t \\ & = -\frac{\beta_t^\sigma \|X_t - Y_t\|_B^\sigma}{c^\sigma \|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\mathrm{d}t. \end{aligned}$$

Combining this with the Young inequality we have

$$\begin{aligned} \int_0^T \|\zeta_t\|_2^2 \mathrm{d}t & = \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_B^2}{\|X_t - Y_t\|_2^{2\varepsilon}}\mathrm{d}t \\ & \leq \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \left(\int_0^T \frac{\beta_t^\sigma \|X_t - Y_t\|_B^\sigma}{\|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\mathrm{d}t\right)^{\frac{2}{\sigma}} \\ & \leq \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \left(c^\sigma \|x-y\|_2^{2\varepsilon}\right)^{\frac{2}{\sigma}} \\ & \leq \lambda\int_0^T f_t^{\frac{2}{\sigma-2}}\mathrm{d}t + \lambda^{(2-\sigma)/2} c^\sigma \|x-y\|_2^{2\varepsilon}, \quad \lambda > 0. \end{aligned} \tag{5.1.14}$$

Since $\sigma \geq \frac{4}{p}$ implies $\frac{2}{\sigma-2} \leq \frac{p}{2-p}$, we have

$$f_t^{\frac{2}{\sigma-2}} \leq \mathbf{m}\big(|\nabla X_t|^p \vee |\nabla Y_t|^p\big)^{\frac{2(2-p)}{(\sigma-2)p}} \leq 1 + \|X_t\|_{1,p}^p + \|Y_t\|_{1,p}^p.$$

Let $\lambda = \lambda_T$ in (5.1.14), then by Lemma 5.1.4 it is easy to show (b) holds, *i.e.*

$$\mathbf{E}\exp\left[\frac{1}{2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right] < \infty.$$

Now combining (3.3.6), (4.2.3) and Hölder's inequality we have

$$\begin{aligned}
&(P_T F(y))^\alpha \leq P_T F^\alpha(x)\Big(\mathbf{E}R^{\alpha/(\alpha-1)}\Big)^{\alpha-1} \\
&= P_T F^\alpha(x)\left\{\mathbf{E}\exp\left[\frac{\alpha}{\alpha-1}\int_0^T \langle\zeta_t, \mathrm{d}W_t\rangle - \frac{\alpha}{2(\alpha-1)}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right]\right\}^{\alpha-1} \\
&\leq P_T F^\alpha(x)\left\{\mathbf{E}\exp\left[\frac{2\alpha}{\alpha-1}\int_0^T \langle\zeta_t, \mathrm{d}W_t\rangle - \frac{2\alpha^2}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}} \\
&\quad\cdot\left\{\mathbf{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}} \\
&\leq P_T F^\alpha(x)\left\{\mathbf{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}}.
\end{aligned} \tag{5.1.15}$$

Taking $\lambda = \frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}$ in (5.1.14), by Lemma 5.1.4 we obtain that

$$\begin{aligned}
&\mathbf{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2 \mathrm{d}t\right] \\
&\leq \mathbf{E}\exp\left[\frac{\lambda_T}{2}\int_0^T (1 + \|X_t\|_{1,p}^p + \|Y_t\|_{1,p}^p)\mathrm{d}t + \frac{\alpha(\alpha+1)}{(\alpha-1)^2}\Big(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_2^{2\varepsilon}\right] \\
&\leq \exp\left[\frac{1}{2}\Big(\lambda_T T + \|x\|_2^2 + \|y\|_2^2 + 2\int_0^T b e^{-2bt}\mathrm{d}t + \|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\mathrm{d}t\Big)\right. \\
&\qquad\left. + \frac{\alpha(\alpha+1)}{(\alpha-1)^2}\Big(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_2^{2\varepsilon}\right].
\end{aligned} \tag{5.1.16}$$

Then by (5.1.15) we have

$$\begin{aligned}
(P_T F(y))^\alpha \leq P_T F^\alpha(x)\exp&\left[\frac{\alpha-1}{4}\Big(1 + \lambda_T T + \|x\|_2^2 + \|y\|_2^2 + +\|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\mathrm{d}t\Big)\right. \\
&\left. + \frac{\alpha(\alpha+1)}{2(\alpha-1)}\Big(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\Big)^{\frac{2-\sigma}{2}} c^\sigma \|x-y\|_2^{2\varepsilon}\right].
\end{aligned} \tag{5.1.17}$$

Then the desired result (5.1.9) follows.

Moreover, one can also show that P_T is a strong Feller operator (see Theorem 3.3.1). Now the proof is complete. ∎

Remark 5.1.2 *(1) Note that if $\gamma = 0$ in (5.1.2), the Harnack inequality (5.1.9) still holds in the theorem above. However, we can establish the ultraboundedness and compactness of the transition semigroup here if we have this high order absorption term ($\gamma > 0$) in the drift (see Theorem 5.1.5).*

(2) The estimate in right hand side of (5.1.9) comes from our coupling argument, which looks different with the known Gaussian type estimate in finite-dimensional case (cf. [Wan97]). However, we know that the Gaussian type estimate in Harnack inequality is equivalent to some underlying curvature lower bound condition (cf.[Wan06]). Hence it seems also reasonable to have this type of estimate (5.1.9) in the present case, which describes some worse long time behavior of the semigroup. We also refer to the estimate of a similar form obtained in [ATW06] for diffusion semigroup on manifolds with curvature unbounded below.

Theorem 5.1.5 *Assume all assumptions in Theorem 5.1.2 hold.*

(i) $\{P_t\}$ is (topologically) irreducible and has a unique invariant measure μ with full support on $L^2(\Lambda)$. Moreover, for any probability measure ν on $L^2(\Lambda)$ we have

$$\lim_{t\to\infty} \| P_t^*\nu - \mu \|_{var} = 0,$$

where $\|\cdot\|_{var}$ is the total variation norm and P_t^* is the adjoint operator of P_t.

(ii) If $p = 2$, then we have $\mu(e^{\varepsilon_0\|\cdot\|_2^2}) < \infty$ for some $\varepsilon_0 > 0$. Moreover, P_t is hyperbounded (i.e. $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$) and compact for some $t > 0$.

(iii) If $\gamma > 0$ and $q > \sigma$, then P_t is ultrabounded (i.e. $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty$) and compact on $L^2(\mu)$ for any $t > 0$.

(iv) If $\gamma > 0$ and $q > 2$, then the Markov semigroup $\{P_t\}$ is uniformly exponential ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \geq 0$ and $x \in H$

$$\sup_{\|F\|_\infty \leq 1} |P_t F(x) - \mu(F)| \leq C e^{-\eta t}.$$

And for each $p \in (1,\infty]$ we have

$$\|P_tF - \mu(F)\|_{L^p(\mu)} \le C_p e^{-(p-1)\eta t/p}\|F\|_{L^p(\mu)}, \ F \in L^p(\mu), \ t \ge 0,$$

and

$$gap(\mathcal{L}_p) \ge \frac{(p-1)\eta}{p},$$

where C_p is a constant and $\mathcal{L}_p$ is the generator of the semigroup $\{P_t\}$ on $L^p(\mu)$.

Proof. (i) The proof is standard and one just need to repeat the arguments in Theorem 3.3.5.

(ii) If $p = 2$ and ε_0 is small enough, then by Itô's formula and the Poincaré inequality

$$\begin{aligned} e^{\varepsilon_0\|X_t\|_2^2} \le & e^{\varepsilon_0\|x\|_2^2} + \int_0^t \left(c - 2\|X_s\|_{1,2}^2 + 2b\varepsilon_0\|X_s\|_2^2\right)\varepsilon_0 e^{\varepsilon_0\|X_s\|_2^2}\mathrm{d}s \\ & + 2\varepsilon_0\int_0^t e^{\varepsilon_0\|X_s\|_2^2}\langle X_s, B\mathrm{d}W_s\rangle \\ \le & e^{\varepsilon_0\|x\|_2^2} + \int_0^t \left(c_1 - c_2 e^{\varepsilon_0\|X_s\|_2^2}\right)\mathrm{d}s + 2\varepsilon_0\int_0^t e^{\varepsilon_0\|X_s\|_2^2}\langle X_s, B\mathrm{d}W_s\rangle, \end{aligned} \tag{5.1.18}$$

where c, c_1 and c_2 are positive constants.

Hence by the same argument in Theorem 5.1.1(ii) we can show $\mu(e^{\varepsilon_0\|\cdot\|_2^2}) < \infty$.

If $p = 2$, one can just repeat the proof of (5.1.9) (Lemma 5.1.4 can be omitted) and thus (5.1.14) turns to be

$$\int_0^T \|\zeta_t\|_2^2\mathrm{d}t = \int_0^T \frac{\beta_t^2\|X_t - Y_t\|_B^2}{\|X_t - Y_t\|_2^{2\varepsilon}}\mathrm{d}t \le T^{\frac{\sigma-2}{\sigma}}\left(c^\sigma\|x-y\|_2^{2\varepsilon}\right)^{\frac{2}{\sigma}}.$$

Hence we can get the following Harnack inequality

$$(P_tF)^\alpha(y) \le P_tF^\alpha(x)\exp\left[\frac{C\alpha(\alpha+1)}{(\alpha-1)t^{(\sigma+2)/\sigma}}\|x-y\|_2^2\right],$$

where C is a constant depending on σ and ξ.

Since $\mu(e^{\varepsilon_0\|\cdot\|_2^2}) < \infty$, by the same argument in Theorem 3.3.5 one can obtain the hyperboundedness and compactness property of P_t for some large $t > 0$.

(iii) If $\gamma > 0$ and $q > \sigma$, then by Itô's formula and (5.1.7) we have for small $\varepsilon_0 > 0$

$$e^{\varepsilon_0 \|X_t\|_2^q} \leq e^{\varepsilon_0 \|x\|_2^q} + \int_0^t \left(c_2 - c_1 \|X_s\|_2^{2q-2} e^{\varepsilon_0 \|X_s\|_2^q} \right) \mathrm{d}s + M'_t, \tag{5.1.19}$$

where $c_1, c_2 > 0$ and M' is a local martingale. Then by Jensen's inequality

$$\mathbf{E} e^{\varepsilon_0 \|X_t\|_2^q} \leq e^{\varepsilon_0 \|x\|_2^q} + c_2 t - c_1 \varepsilon_0^{-(2q-2)/q} \int_0^t \mathbf{E} e^{\varepsilon_0 \|X_s\|_2^q} \left(\log \mathbf{E} e^{\varepsilon_0 \|X_s\|_2^q} \right)^{\frac{2q-2}{q}} \mathrm{d}s.$$

By a standard comparison argument we have

$$\mathbf{E} e^{\varepsilon_0 \|X_t(x)\|_2^q} \leq \exp\left[c_0 \left(1 + t^{-q/(q-2)} \right) \right], \quad t > 0, \ x \in L^2(\Lambda), \tag{5.1.20}$$

where $c_0 > 0$ is a constant.

Then the ultraboundedness and compactness property can be derived for $\{P_t\}$ by using the same argument in Theorem 4.1.3.

(iv) By Theorem 5.1.2 we know $\{P_t\}$ is strong Feller and irreducible. Then, according to [GM04, Theorem 2.5; 2.7], we only need to verify the following properties:

(1) For each $r > 0$ there exists $t_0 > 0$ and compact set $M \subset H$ such that

$$\inf_{\|x\|_2 \leq r} P_{t_0} \mathbf{1}_M(x) > 0.$$

(2) There exist constants $K < \infty$ and $t_1 > 0$ such that

$$\mathbf{E} \|X_t(x)\|_2^2 \leq K, \ x \in L^2(\Lambda), \ t \geq t_1.$$

By using Itô's formula we have

$$\|X_t\|_2^2 \leq \|x\|_2^2 + \int_0^t \left(b - 2\|X_s\|_{1,p}^p - 2\gamma \|X_s\|_q^q \right) \mathrm{d}s + \int_0^t \langle X_s, B \mathrm{d}W_s \rangle.$$

This implies that there exists $C > 0$ such that

$$\mathbf{E} \int_0^t \|X_s\|_{1,p}^p \mathrm{d}s \leq C(t + \|x\|_2^2), \ t \geq 0. \tag{5.1.21}$$

And by using Jensen's inequality

$$\mathbf{E} \|X_t\|_2^2 \leq \|x\|_2^2 + \int_0^t \left[b - C \left(\mathbf{E} \|X_s\|_2^2 \right)^{q/2} \right] \mathrm{d}s.$$

Then by a standard comparison estimate we get

$$\mathbf{E}\|X_t(x)\|_2^2 \leq C(1+t^{-\frac{2}{q-2}}),\ x \in L^2(\Lambda),\ t > 0.$$

Hence the property (2) holds.

Since the embedding $W_0^{1,p}(\Lambda) \subset L^2(\Lambda)$ is compact, (5.1.21) implies that the property (1) also holds. Therefore, the conclusions follow from [GM04, Theorem 2.5; 2.7] (or [GM06, Theorem 7.2]). ∎

Remark 5.1.3 *(1) Comparing with (ii) in Theorem 5.1.1, $\gamma = 0$ is allowed in (i) here. The uniqueness of invariant measures follows from the classical Doob theorem in this case.*

(2) If $p = 2$, then (5.1.2) is stochastic reaction-diffusion equations and the hyperbounded property of the corresponding transition semigroups has been established in Theorem 3.3.5 (see Example 3.4.1). However, if $\gamma > 0$ and $q > 2$ in (5.1.2), Theorem 5.1.5 (iii) implies that the associated transition semigroups are ultrabounded and compact, which are much stronger than the hyperbounded property.

5.2 Applications to stochastic p-Laplace equation and reaction-diffusion equations

As a preparation we first prove a general inequality in Hilbert space, which implies the dissipativity of the p-Laplace operator.

Lemma 5.2.1 *Suppose $(E, \langle\cdot,\cdot\rangle_E, \|\cdot\|)$ is a Hilbert space, then for any $0 < r \leq 1$ we have*

$$\langle \|a\|^{r-1}a - \|b\|^{r-1}b, a-b\rangle_E \geq r\|a-b\|^2 \left(\|a\| \vee \|b\|\right)^{r-1},\ a, b \in E. \tag{5.2.1}$$

Proof. Without loss any generality we may assume $\|a\| \geq \|b\|$. Then (5.2.1) is equivalent to

$$(\|b\|^{r-1} - \|a\|^{r-1})\langle b, a-b\rangle_E \leq (1-r)\|a\|^{r-1}\|a-b\|^2.$$

By the Cauchy-Schwarz inequality and the Young inequality we have

$$
\begin{aligned}
& (\|b\|^{r-1}-\|a\|^{r-1})\langle b, a-b\rangle_E \\
\leq\ & (\|b\|^{r-1}-\|a\|^{r-1})\|b\|\|a-b\| \\
=\ & (\|b\|^{r}\|a\|^{1-r}-\|b\|)\|a\|^{r-1}\|a-b\| \\
\leq\ & \left(r\|b\|+(1-r)\|a\|-\|b\|\right)\|a\|^{r-1}\|a-b\| \\
\leq\ & (1-r)\|a\|^{r-1}\|a-b\|^2.
\end{aligned}
$$

Hence the proof is complete. ∎

For the application of the main results, one mainly needs to verify the assumption (5.1.8). So we first give a sufficient condition such that (5.1.8) holds.

Proposition 5.2.2 *Suppose $Be_i = b_i e_i$ for $i \geq 1$, where $\{e_i\}$ is an orthonormal basis on $L^2(\Lambda)$. If there exists a constant $\sigma \geq 2$ such that*

$$B^{-\frac{\sigma}{2}} : W_0^{1,p}(\Lambda) \to L^2(\Lambda)$$

is a bounded operator, then (5.1.8) holds for the same σ.

Proof. By the assumption there exists a constant $C > 0$ such that

$$\|B^{-\frac{\sigma}{2}}x\|_2^2 = \sum_{i=1}^{\infty}\langle x, e_i\rangle^2 b_i^{-\sigma} \leq C\|x\|_{1,p}^2, \ \ \forall x \in W_0^{1,p}(\Lambda).$$

Then by Hölder's inequality we have

$$
\begin{aligned}
\|x\|_B^\sigma &= \Big(\sum_{i=1}^{\infty}\langle x, e_i\rangle^2 b_i^{-2}\Big)^{\sigma/2} = \Big(\sum_{i=1}^{\infty}\langle x, e_i\rangle^{\frac{2\sigma-4}{\sigma}}\langle x, e_i\rangle^{\frac{4}{\sigma}} b_i^{-2}\Big)^{\sigma/2} \\
&\leq \Big(\sum_{i=1}^{\infty}\langle x, e_i\rangle^2\Big)^{\frac{\sigma-2}{2}}\Big(\sum_{i=1}^{\infty}\langle x, e_i\rangle^2 b_i^{-\sigma}\Big) \\
&= \|x\|_2^{\sigma-2}\Big(\sum_{i=1}^{\infty}\langle x, e_i\rangle^2 b_i^{-\sigma}\Big) \\
&\leq C\|x\|_2^{\sigma-2}\|x\|_{1,p}^2.
\end{aligned}
\tag{5.2.2}
$$

Hence (5.1.8) holds for the same exponent σ. ∎

Corollary 5.2.3 *If Λ is a bounded C^∞-domain in $\mathbb{R}^d$ and $B = (-\Delta)^{-\theta}$ with $\theta \in \left(\frac{d}{4}, \frac{(2+d)p-2d}{8}\right]$, then B is a Hilbert-Schmidt operator and (5.1.8) holds for $\sigma = \frac{4}{p}$.*

Proof. It is well-known that there exists an ONB $\{e_i\}$ on $L^2(\Lambda)$ such that

$$\Delta e_i = -\lambda_i e_i, \ i \geq 1,$$

where the corresponding eigenvalues satisfy

$$\lambda_i \geq c \cdot i^{2/d}, \ i \geq 1$$

for some constant $c > 0$. Hence for $\theta > \frac{d}{4}$ we have

$$\|B\|_{L_2}^2 = \sum_{i=1}^{\infty} \|Be_i\|_2^2 = \sum_{i=1}^{\infty} (\lambda_i)^{-2\theta} \leq C \sum_{i=1}^{\infty} i^{-4\theta/d} < \infty,$$

i.e. B is a Hilbert-Schmidt operator on $L^2(\Lambda)$.

By Proposition 5.2.2 it is enough to show $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $W_0^{1,p}(\Lambda)$ to $L^2(\Lambda)$.

Note that $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $H^{\sigma\theta,2}(\Lambda)$ to $L^2(\Lambda)$, where $H^{\sigma\theta,2}(\Lambda)$ is a fractional Sobolev space (cf.[RS96]).

By the general embedding theorem [RS96, Theorem 1,page 82] we have for $\theta \leq \frac{(2+d)p-2d}{2p\sigma}$ the following embedding

$$W_0^{1,p}(\Lambda) \subseteq H^{\sigma\theta,2}(\Lambda)$$

is continuous, hence $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $W_0^{1,p}(\Lambda)$ to $L^2(\Lambda)$. ∎

Remark 5.2.1 *For $d = 1$ we can take $B = (-\Delta)^{-\theta}$ with $\theta \in \left(\frac{1}{4}, \frac{3p-2}{8}\right]$, where Δ is the Dirichlet Laplace operator on a bounded interval in $\mathbb{R}$. In this case, the main results can only be applied to the case $p > \frac{4}{3}$.*

Example 5.2.4 *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary and Δ be the Laplace operator on $L^2(\Lambda)$ with the Dirichlet boundary condition. Consider the following triple

$$W_0^1(\Lambda) \cap L^q(\Lambda) \subseteq L^2(\Lambda) \subseteq \left(W_0^1(\Lambda) \cap L^q(\Lambda)\right)^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - |X_t|^{q-2}X_t)\mathrm{d}t + B\mathrm{d}W_t,\ \ X_0 = x \in L^2(\Lambda), \tag{5.2.3}$$

where $q \geq 2$, B and W_t are Hilbert-Schmidt operator and cylindrical Wiener process on $L^2(\Lambda)$ respectively, then all assertions in **Theorem 5.1.1** *hold.*

Moreover, if B is a one-to-one operator such that $B^{-1}: W_0^1(\Lambda) \to L^2(\Lambda)$ is a bounded operator, then (5.1.8) holds. Therefore, all assertions in **Theorem 5.1.2** *and* **5.1.5** *also hold for (5.2.3).*

In particular, if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then the associated transition semigroup of (5.2.3) is hyperbounded. If $q > 2$, then the corresponding transition semigroup of (5.2.3) is ultrabounded and compact.

Remark 5.2.2 *If we replace Δ in (5.2.3) by a general self-adjoint operator L and assume that*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

are the eigenvalues of $-L$ and the corresponding eigenvectors $\{e_i\}_{i\geq 1}$ form an ONB of $L^2(\Lambda)$. Suppose $Be_i := b_i e_i$ and there exists a positive constant C such that

$$\sum_i b_i^2 < +\infty \ \ \text{and} \ \ b_i \geq \frac{C}{\sqrt{\lambda_i}}, \quad i \geq 1, \tag{5.2.4}$$

then B is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and (5.1.8) holds. Similar to [LW08, Wan07] we may also discuss stochastic reaction-diffusion equations in higher dimensional case (i.e. $d > 1$). Moreover, Theorem 5.1.5 implies that the transition semigroup is ultrabounded and compact if we have a nonlinear perturbations in the drift, and we have also derived the exponential ergodicity and the existence of a spectral gap in the example.

Chapter 6

Invariance of Subspaces under The Solution Flow of SPDE

In this chapter we investigate some regularity property for solutions to SPDE. More precisely, under some additional assumptions, we prove that the solution of an SPDE takes values in some subspace of the original state space if the initial condition does. This property is useful for further study of the corresponding random dynamical systems, e.g. for studying the existence of a random attractor (cf.[BLR08]). This type of regularity has been required in [GM07] for establishing the convergence rate of implicit approximations for SEE and in [Cho92] for deriving the LDP for semilinear SPDE. As examples, the main results are applied to different types of SPDE such as stochastic reaction-diffusion equations, the stochastic p-Laplace equation, stochastic porous media and fast diffusion equations in Hilbert space.

6.1 The main results

Let $V \subset H \equiv H^* \subset V^*$ be a Gelfand triple, $\langle \cdot, \cdot \rangle_H$ and ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denote the inner product in H and the dualization between V^* and V respectively. $\{W_t\}$ is a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $L_2(U; H)$ denotes the space of all Hilbert-Schmidt operators from U to H. Consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{6.1.1}$$

where $A : [0, T] \times V \times \Omega \to V^*$ and $B : [0, T] \times V \times \Omega \to L_2(U, H)$ are progressively measurable. By assuming the coefficients A, B satisfy the standard monotone and coercive conditions (see Theorem 1.2.1) we know (6.1.1) has a unique strong solution $X_t(x)$, which is a H-valued continuous process and satisfies

$$\mathbf{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^2 + \int_0^T \|X_t\|_V^\alpha \mathrm{d}t\right) < \infty.$$

If $(S, \|\cdot\|_S)$ is a subspace of H and T_n are positive definite self-adjoint operators on H such that

$$\langle x, y\rangle_n := \langle x, T_n y\rangle_H,\ x, y \in H$$

are a sequence of new inner products on H. Suppose the induced norms $\|\cdot\|_n$ are all equivalent to $\|\cdot\|_H$ and

$$\forall x \in S,\ \|x\|_n \uparrow \|x\|_S\ (n \to \infty).$$

Let $H_n := (H, \langle\cdot,\cdot\rangle_n)$, then we get a sequence of new Gelfand triples

$$V \subseteq H_n \equiv H_n^* \subseteq V^*,$$

where we use different Riesz maps i_n to identify $H_n \equiv H_n^*$, and i is the Riesz map for identifying $H \equiv H^*$.

Lemma 6.1.1 *If $T_n : V \to V$ is continuous, then $i_n \circ i^{-1} : H^* \to H_n^*$ is continuous w.r.t. $\|\cdot\|_{V^*}$. Therefore, there exists a unique continuous extension I_n of $i_n \circ i^{-1}$ on V^* such that*

$$_{V^*}\langle I_n f, v\rangle_V = {}_{V^*}\langle f, T_n v\rangle_V, \quad f \in V^*,\ v \in V. \tag{6.1.2}$$

Proof. For any $f \in H^* \subset V^*$, we know $i_n \circ i^{-1} f \in H_n^*$ and

$$\begin{aligned}\|i_n \circ i^{-1} f\|_{V^*} &= \sup_{v\in V, \|v\|_V=1} |_{V^*}\langle i_n \circ i^{-1} f, v\rangle_V| = \sup_{v\in V, \|v\|_V=1} |\langle i^{-1} f, v\rangle_n| \\ &= \sup_{v\in V, \|v\|_V=1} |\langle i^{-1} f, T_n v\rangle_H| = \sup_{v\in V, \|v\|_V=1} |_{V^*}\langle f, T_n v\rangle_V| \\ &\le \sup_{v\in V, \|v\|_V\le c_n} |_{V^*}\langle f, v\rangle_V| \le c_n \|f\|_{V^*},\end{aligned} \tag{6.1.3}$$

where c_n is the operator norm of T_n from V to V. Obviously we also have

$$_{V^*}\langle i_n \circ i^{-1} f, v\rangle_V = {}_{V^*}\langle f, T_n v\rangle_V,\ f \in H^*,\ v \in V.$$

Then it is well-known that $i_n \circ i^{-1}$ can be uniquely extended to a continuous operator on V^* such that (6.1.2) holds. ∎

Since we want to apply the Itô formula to the solution of (6.1.1) in different Gelfand triples, we need to write down the Itô formula for the square norm of the solution in a more precise way by involving the corresponding Riesz map explicitly.

Lemma 6.1.2 [RRW07, Theorem A.2] *Let $K := L^\alpha([0,T]\times\Omega \to V; \mathrm{d}t\times\mathbf{P})(\alpha>1)$ and $X_0 \in L^2(\Omega\to H;\mathcal{F}_0;\mathbf{P})$. Suppose we have a H-valued process X_t which satisfies*

$$iX_t = iX_0 + \int_0^t Y_s \mathrm{d}s + i\left(\int_0^t Z_s \mathrm{d}W_s\right), \quad t\in[0,T],$$

where $Y \in K^* = L^{\alpha/(\alpha-1)}([0,T]\times\Omega\to V^*;\mathrm{d}t\times\mathbf{P})$ and $Z\in L^2([0,T]\times\Omega\to L_2(U;H);\mathrm{d}t\times\mathbf{P})$ are two adapted processes. If there exists an element $\bar{X}$ in K such that $X=\bar{X}$ $\mathrm{d}t\times\mathbf{P}$, a.s., then X_t is a continuous adapted process on H such that $\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_H^2<\infty$ and

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t (2{}_{V^*}\langle Y_s,\bar{X}_s\rangle_V + \|Z_s\|^2_{L_2(U;H)})\mathrm{d}s + 2\int_0^t \langle X_s, Z_s\mathrm{d}W_s\rangle_H \qquad (6.1.4)$$

holds $\mathbf{P}$-a.s. for all $t\in[0,T]$. We can replace $\bar{X}_s$ by X_s in (6.1.4) if we set ${}_{V^*}\langle Y_s,X_s\rangle_V=0$ for $X_s\notin V$.

Now we formulate the main result of this chapter.

Theorem 6.1.3 *Suppose the assumptions $(H1)-(H4)$ in Theorem 1.2.1 hold, $T_n: V\to V$ is continuous and there exist a constant C and an adapted process $f\in L^1([0,T]\times\Omega;\mathrm{d}t\times\mathbf{P})$ such that for $n\geq 1$*

$$2{}_{V^*}\langle A(t,v),T_nv\rangle_V + \|B(t,v)\|^2_{L_2(U,H_n)} \leq C\|v\|_n^2 + f_t,\ v\in V,\ 0\leq t\leq T,\ \mathbf{P}-a.s.. \quad (6.1.5)$$

(i) If $\mathbf{E}\|X_0\|_S^2<\infty$, then for any $p\in[1,2)$ we have

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^p<\infty.$$

(ii) If $\mathbf{E}\|X_0\|_S^p<\infty$ for some $p\geq 2$ and

$$\|B(t,v)\|^2_{L_2(U,H_n)} \leq C\|v\|_n^2 + f_t,\ v\in V,\ 0\leq t\leq T,\ \mathbf{P}-a.s., \qquad (6.1.6)$$

where $f \in L^{\frac{p}{2}}([0,T]\times\Omega; \mathrm{d}t\times\mathbf{P})$, *then there exists a constant* C_p *such that*

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^p \le C_p\left(\mathbf{E}\|X_0\|_S^p + \mathbf{E}\int_0^T f_t^{p/2}\mathrm{d}s\right).$$

Proof. (i) It follows from the definition that the solution X_t to (6.1.1) satisfies

$$iX_t = iX_0 + \int_0^t A(s,X_s)\mathrm{d}s + i\left(\int_0^t B(s,X_s)\mathrm{d}W_s\right),\ t\in[0,T]. \tag{6.1.7}$$

According to Lemma 6.1.1, by applying the continuous operator I_n to (6.1.7) we have

$$i_nX_t = i_nX_0 + \int_0^t I_nA(s,X_s)\mathrm{d}s + i_n\left(\int_0^t B(s,X_s)\mathrm{d}W_s\right),\ t\in[0,T].$$

By Lemma 6.1.2 we can apply the Itô formula on the new Gelfand triple $V\subseteq H_n\equiv H_n^*\subseteq V^*$ to obtain

$$\begin{aligned}\|X_t\|_n^2 =&\|X_0\|_n^2 + \int_0^t \left(2\ {}_{V^*}\langle I_nA(s,X_s),X_s\rangle_V + \|B(s,X_s)\|^2_{L_2(U;H_n)}\right)\mathrm{d}s\\ &+2\int_0^t\langle X_s, B(s,X_s)\mathrm{d}W_s\rangle_n \\ \le&\|X_0\|_n^2 + \int_0^t\left(C\|X_s\|_n^2+f_s\right)\mathrm{d}s + 2\int_0^t\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle_n.\end{aligned} \tag{6.1.8}$$

Hence

$$\mathrm{e}^{-Ct}\|X_t\|_n^2 \le \|X_0\|_n^2 + \int_0^t \mathrm{e}^{-Cs}f_s\mathrm{d}s + 2\int_0^t\mathrm{e}^{-Cs}\langle X_s,B(s,X_s)\mathrm{d}W_s\rangle_n =: N_t. \tag{6.1.9}$$

It is easy to show that N_t is a local submartingale, *i.e.* the sum of an increasing process and a local martingale. Hence by a standard localization argument we know for any $p\in[1,2)$

$$\begin{aligned}&\mathbf{P}\big(\sup_{t\in[0,T]}\|X_t\|_n^p\ge r\big) = \mathbf{P}\big(\sup_{t\in[0,T]}\|X_t\|_n^2\ge r^{2/p}\big)\\ &\le \mathbf{P}\big(\sup_{t\in[0,T]}N_t \ge \mathrm{e}^{-CT}r^{2/p}\big) \le r^{-2/p}\mathrm{e}^{CT}\mathbf{E}N_T<\infty,\end{aligned} \tag{6.1.10}$$

since $\mathbf{E}N_T\le\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \mathrm{e}^{-Cs}f_s\mathrm{d}s<\infty$. Then

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_n^p =& \int_0^\infty \mathbf{P}\big(\sup_{t\in[0,T]}\|X_t\|_n^p\ge r\big)\mathrm{d}r\\ \le& \int_0^1 \mathbf{P}\big(\sup_{t\in[0,T]}\|X_t\|_n^p\ge r\big)\mathrm{d}r + \int_1^\infty \mathbf{P}\big(\sup_{t\in[0,T]}\|X_t\|_n^p\ge r\big)\mathrm{d}r\\ \le& 1+\int_1^\infty r^{-2/p}\mathrm{e}^{CT}\mathbf{E}N_T\mathrm{d}r = 1+\frac{p}{2-p}\mathrm{e}^{CT}\mathbf{E}N_T.\end{aligned}$$

Let $n \to \infty$, by the monotone convergence theorem and Fatou's lemma we have

$$\begin{aligned}
&\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_S^p = \mathbf{E} \lim_{n\to\infty} \sup_{t\in[0,T]} \|X_t\|_n^p \\
&\le \liminf_{n\to\infty} \mathbf{E} \sup_{t\in[0,T]} \|X_t\|_n^p \le 1 + \frac{p}{2-p} \mathrm{e}^{CT} \mathbf{E} N_T < \infty.
\end{aligned}$$

(ii) By Itô's formula and Young's inequality we have

$$\begin{aligned}
\|X_t\|_n^p =& \|X_0\|_n^p + \frac{p}{2} \int_0^t \|X_s\|_n^{p-2} \cdot 2 {}_{V^*}\langle I_n A(s, X_s), X_s \rangle_V \mathrm{d}s \\
&+ p \int_0^t \|X_s\|_n^{p-2} \langle X_s, B(x, X_s) \mathrm{d}W_s \rangle_n + p(\frac{p}{2} - 1) \int_0^t \|X_s\|_n^{p-4} \|X_s \circ B(t, X_s)\|_{L_2(U,H_n)}^2 \mathrm{d}t \\
\le& \|X_0\|_n^p + \frac{p}{2} \int_0^t C(\|X_s\|_n^p + f_s \|X_s\|_n^{p-2}) \mathrm{d}s + p \int_0^t \|X_s\|_n^{p-2} \langle X_s, B(x, X_s) \mathrm{d}W_s \rangle_n \\
\le& \|X_0\|_n^p + C \int_0^t (\|X_s\|_n^p + f_s^{p/2}) \mathrm{d}s + p \int_0^t \|X_s\|_n^{p-2} \langle X_s, B(x, X_s) \mathrm{d}W_s \rangle_n,
\end{aligned} \tag{6.1.11}$$

where C is a constant which may change from line to line.

Then by the Burkhölder-Davis-Gundy inequality and (6.1.6) we have

$$\begin{aligned}
&\mathbf{E} \sup_{u\in[0,t]} \left| \int_0^u \|X_s\|_n^{p-2} \langle X_s, B(s, X_s) \mathrm{d}W_s \rangle_n \right| \\
\le& 3\mathbf{E} \left(\int_0^t \|X_s\|_n^{2p-2} \|B(s, X_s)\|_{L_2(U;H_n)}^2 \mathrm{d}s \right)^{1/2} \\
\le& 3\mathbf{E} \left(\sup_{s\in[0,t]} \|X_s\|_n^{2p-2} \int_0^t (\|X_s\|_n^2 + f_s) \mathrm{d}s \right)^{1/2} \\
\le& 3\mathbf{E} \left[\varepsilon \sup_{s\in[0,t]} \|X_s\|_n^p + C_\varepsilon \left(\int_0^t (C\|X_s\|_n^2 + f_s) \mathrm{d}s \right)^{p/2} \right] \\
\le& 3\varepsilon \mathbf{E} \sup_{s\in[0,t]} \|X_s\|_n^p + 3 \cdot 2^{p/2} C_\varepsilon \mathbf{E} \int_0^t \left(\|X_s\|_n^p + f_s^{p/2} \right) \mathrm{d}s,
\end{aligned} \tag{6.1.12}$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality.

Then combining with (6.1.11) and Gronwall's lemma we have for any stopping time $\tau \le T$

$$\mathbf{E} \sup_{t\in[0,\tau]} \|X_t\|_n^p \le C \left(\mathbf{E} \|X_0\|_n^p + \mathbf{E} \int_0^T f_s^{p/2} \mathrm{d}s \right),$$

where C is a constant independent of n.

Therefore, by using a standard localization argument we have

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^p=\sup_{n\geq 1}\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_n^p\leq C\left(\mathbf{E}\|X_0\|_S^p+\mathbf{E}\int_0^T f_s^{p/2}\mathrm{d}s\right).$$

∎

Remark 6.1.1 *The idea of using equivalent norms $\|\cdot\|_n$ to approximate $\|\cdot\|_S$ has been used in [RW08] for establishing the L^2-invariance of the solution to stochastic porous media equations. In order to apply Itô's formula to the equation on different Gelfand triples, here we introduce the continuous operator I_n to transfer the equation between different triples. In the next section this theorem will be applied to investigate the regularity for many different types of SPDE in Hilbert space as examples.*

6.2 Applications to concrete SPDEs

In this section, we only consider the additive type noise (e.g. $B\in L^2([0,T]\times\Omega,L_2(U,S))$) for simplicity. Then it is obvious that (6.1.6) holds. For the examples with multiplicative noise we refer to [RW08, Remark 2.9(iii)], where a general linear multiplicative noise satisfying (6.1.6) is discussed.

Example 6.2.1 *Let Λ be an open bounded domain in $\mathbb{R}^d$ and $L^p:=L^p(\Lambda)$. Consider the following triple*

$$L^p\subseteq L^2\subseteq(L^p)^*\equiv L^{\frac{p}{p-1}}$$

and the stochastic equation

$$\mathrm{d}X_t=\left(-|X_t|^{p-2}X_t+\eta_tX_t\right)\mathrm{d}t+B_t\mathrm{d}W_t,\ t\in[0,T], \tag{6.2.1}$$

where $p\geq 2$, η is a bounded process and W_t is a cylindrical Wiener process on L^2. If $S=W_0^{1,2}$, $X_0\in L^2(\Omega,S)$ and $B\in L^2([0,T]\times\Omega,L_2(L^2,S))$, then there exists a constant C such that

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^2\leq C\left(\mathbf{E}\|X_0\|_S^2+\mathbf{E}\int_0^T\|B_t\|_2^2\mathrm{d}t\right).$$

Proof. Note that $S = W_0^{1,2} = \mathcal{D}(\sqrt{-\Delta})$, where Δ is the Laplace operator on L^2 with the Dirichlet boundary condition. Then we define $T_n = -\Delta(1-\frac{\Delta}{n})^{-1}$ which is the Yosida approximation of Δ. It is well-known that the heat semigroup $\{P_t\}_{t\geq 0}$ (generated by Δ) is a contractive semigroup and T_n are continuous operators on L^p. Therefore, by using the Hölder inequality and the contraction property of P_t on L^p we have

$$\begin{aligned}
&{}_{V^*}\langle A(t,u), T_n u\rangle_V = {}_{V^*}\langle -|u|^{p-2}u, -\Delta(1-\frac{\Delta}{n})^{-1}u\rangle_V + \eta_t\|u\|_n^2\\
=&{}_{V^*}\langle -|u|^{p-2}u, nu - n(1-\frac{\Delta}{n})^{-1}u\rangle_V + \eta_t\|u\|_n^2\\
=&-n\int_0^\infty e^{-t}{}_{V^*}\langle |u|^{p-2}u, u - P_{\frac{t}{n}}u\rangle_V \mathrm{d}t + \eta_t\|u\|_n^2\\
\leq&-n\int_0^\infty e^{-t}\left[\int_\Lambda (|u|^p - |u|^{p-2}uP_{\frac{t}{n}}u)\mathrm{d}\xi\right]\mathrm{d}t\\
\leq& C\|u\|_n^2,\ u\in L^p,
\end{aligned} \tag{6.2.2}$$

where C is a constant.

Hence (6.1.5) holds and the conclusion follows from Theorem 6.1.3. ∎

Example 6.2.2 *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. We consider the following triple

$$W_0^{1,2}(\Lambda)\cap L^p(\Lambda)\subseteq L^2(\Lambda)\subseteq (W_0^{1,2}(\Lambda)\cap L^p(\Lambda))^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - |X_t|^{p-2}X_t + \eta_t X_t)\mathrm{d}t + B_t \mathrm{d}W_t, \tag{6.2.3}$$

where $p\geq 2$, η is a bounded process and W_t is a cylindrical Wiener process on $L^2(\Lambda)$. If $S = W_0^{1,2}(\Lambda)$, $X_0\in L^2(\Omega, S)$ and $B\in L^2([0,T]\times\Omega, L_2(L^2(\Lambda), S)$, then there exists a constant C such that

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^2 \leq C\left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \|B_t\|_2^2\mathrm{d}t\right).$$

Proof. Let Δ be the Laplace operator on $L^2(\Lambda)$ with the Dirichlet boundary condition, then we define $T_n = -\Delta(1-\frac{\Delta}{n})^{-1}$, $\{P_t\}_{t\geq 0}$ and $\mathcal{E}$ denote the corresponding semigroup and Dirichlet form of Δ. It is easy to show that T_n are continuous operators on $W_0^{1,2}(\Lambda)$ since

$$T_n = n\left(I - \left(I - \frac{\Delta}{n}\right)^{-1}\right).$$

Then we have

$$\begin{aligned} & {}_{V^*}\langle \Delta u, -\Delta(1-\frac{\Delta}{n})^{-1}u\rangle_V \\ = & {}_{V^*}\langle \Delta u, nu - n(1-\frac{\Delta}{n})^{-1}u\rangle_V \\ = & -n\int_0^\infty e^{-t}\langle \nabla u, \nabla u - \nabla P_{\frac{t}{n}}u\rangle_{L^2(\Lambda)}\mathrm{d}t \\ \le & -n\int_0^\infty e^{-t}(\mathcal{E}(u,u) - \mathcal{E}(u, P_{\frac{t}{n}}u))\mathrm{d}t \\ \le & \ 0, \end{aligned}$$

where the last step follows from the contraction property of the Dirichlet form $\mathcal{E}$.

Therefore, combining with (6.2.2) we know that (6.1.5) holds and the conclusion follows from Theorem 6.1.3.

∎

Remark 6.2.1 *(1) This regularity property is used in [GM07] (see assumption (T3)) for establishing the convergence rate of the implicit approximations for stochastic evolution equations.*

(2) In the above example one can replace Δ by a more general negative definite self-adjoint operator L and obtain a similar result for $S = \mathcal{D}(\sqrt{-L})$. This type of regularity has been used in [Cho92, Lemma 3.2] for establishing the large deviation principle for semilinear SPDEs.

Example 6.2.3 *(stochastic porous media and fast diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. For $r > 0, r \ge \frac{d-2}{d+2}$ we consider the following triple

$$V := L^{r+1}(\Lambda) \subseteq H := (W_0^1(\Lambda))^* \subseteq V^*$$

and the stochastic porous media(or fast diffusion) equation

$$\mathrm{d}X_t = \left(\Delta(|X_t|^{r-1}X_t) + \eta_t X_t\right)\mathrm{d}t + B_t \mathrm{d}W_t, \qquad (6.2.4)$$

where W_t is a cylindrical Wiener process on $L^2(\Lambda)$ and η is a bounded process. If $S = L^2(\Lambda)$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0,T] \times \Omega, L_2(L^2(\Lambda))$, then there exists a constant C such that

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_S^2 \le C \left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E} \int_0^T \|B_t\|_2^2 \mathrm{d}t \right).$$

Proof. According to [PR07, Example 4.1.11;Remark 4.1.15] we know the conditions $(H1) - (H4)$ in Theorem 1.2.1 hold for $r > 0, r \ge \frac{d-2}{d+2}$. Hence we only need to verify (6.1.5) in Theorem 6.1.3 here.

It is well-known that the heat semigroup $\{P_t\}$ is contractive on $L^p(\Lambda)$ for any $p > 1$. Now we define the Yosida approximation operator

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1} \right),$$

it's easy to show that T_n are continuous operators on $L^{r+1}(\Lambda)$ by using the formula

$$(I - \frac{\Delta}{n})^{-1} u = \int_0^\infty e^{-t} P_{\frac{t}{n}} u \mathrm{d}t.$$

Then by the Hölder inequality and the contractivity of $\{P_t\}$ on $L^{r+1}(\Lambda)$ we have

$$\begin{aligned}
& {}_{V^*}\langle \Delta(|u|^{r-1}u), -\Delta(1 - \frac{\Delta}{n})^{-1} u\rangle_V \\
= \ & \langle |u|^{r-1}u, nu - n(1 - \frac{\Delta}{n})^{-1} u\rangle_{L^2} \\
= \ & -n \int_0^\infty e^{-t} \left(\int_\Lambda |u|^{r+1} \mathrm{d}x - \int_\Lambda |u|^{r-1}u \cdot P_{\frac{t}{n}} u \mathrm{d}x \right) \mathrm{d}t \\
\le \ & 0.
\end{aligned}$$

Hence the conclusion follows from the Theorem 6.1.3. ∎

Remark 6.2.2 *Note that if $r > 1$, this result has been obtained in [RW08, Theorem 2.8] where more general stochastic porous media equations were studied. But under the present framework our proof is much simpler and the result here also holds for stochastic fast diffusion equations (i.e. $r < 1$). In the example we assume $r \ge \frac{d-2}{d+2}$ such that the embedding $L^{r+1}(\Lambda) \subseteq (W_0^1(\Lambda))^*$ is dense and continuous.*

Example 6.2.4 *(Stochastic p-Laplace equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with convex and smooth boundary. We consider the following triple

$$W^{1,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - \eta_t |X_t|^{\tilde{p}-2} X_t\right] \mathrm{d}t + B_t \mathrm{d}W_t, \tag{6.2.5}$$

where $2 \leq p < \infty, 1 \leq \tilde{p} \leq p$, W_t is a cylindrical Wiener process on $L^2(\Lambda)$ and η is a positive bounded process. If $S = W^{1,2}(\Lambda)$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0,T] \times \Omega, L_2(L^2, S))$, then there exists a constant C such that

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_S^2 \leq C \left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \|B_t\|_2^2 \mathrm{d}t \right).$$

Proof. According to the results in [PR07] (e.g. Example 4.1.9), we only need to verify the assumption (6.1.5) in Theorem 6.1.3. Since $S = W^{1,2}(\Lambda) = \mathcal{D}(\sqrt{-\Delta})$, where Δ is the Laplace operator on $L^2(\Lambda)$ with the Neumann boundary condition. It is well-known that the corresponding semigroup $\{P_t\}$ is the Neumann heat semigroup (*i.e.* the corresponding Markov process is the Brownian Motion with reflecting boundary) and $P_t : L^2(\Lambda) \to W^{1,2}(\Lambda)$. Moreover, we know that P_t maps $L^p(\Lambda)$ into $W^{1,p}(\Lambda)$ continuously (see [CR04, section 2] for more general results). Hence for all $t \geq 0$, $P_t : W^{1,p}(\Lambda) \to W^{1,p}(\Lambda)$ is continuous.

Now we define

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1}\right).$$

It is easy to show that T_n are also continuous operators on $W^{1,p}(\Lambda)$ since

$$(I - \frac{\Delta}{n})^{-1} u = \int_0^\infty e^{-t} P_{\frac{t}{n}} u \mathrm{d}t.$$

Moreover, since the boundary of the domain is convex and smooth, we have the following gradient estimate (cf.[Wan05, Theorem 2.5.1])

$$|\nabla P_t u| \leq P_t |\nabla u|, \quad u \in W^{1,p}(\Lambda).$$

Since $\{P_t\}$ is a contractive semigroup on $L^p(\Lambda)$, it is easy to see that $\{P_t\}$ is a contractive semigroup on $W^{1,p}(\Lambda)$. Therefore,

$$\begin{aligned}
& {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), -\Delta(1-\frac{\Delta}{n})^{-1}u\rangle_V \\
= & {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), nu - n(1-\frac{\Delta}{n})^{-1}u\rangle_V \\
= & n\int_0^\infty e^{-t}{}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), u - P_{\frac{t}{n}}u\rangle_V \mathrm{d}t \\
= & -n\int_0^\infty e^{-t}\left(\int_\Lambda |\nabla u|^p \mathrm{d}x - \int_\Lambda |\nabla u|^{p-2}\nabla u\cdot\nabla P_{\frac{t}{n}}u\mathrm{d}x\right)\mathrm{d}t \\
\leq & 0,
\end{aligned}$$

where in the last step we use the Hölder inequality and the contractivity of $\{P_t\}$ on $W^{1,p}(\Lambda)$ to conclude

$$\begin{aligned}
& \int_\Lambda |\nabla u|^{p-2}\nabla u\cdot\nabla P_s u\mathrm{d}x \\
\leq & \left(\int_\Lambda |\nabla u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\cdot\left(\int_\Lambda |\nabla P_s u|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
\leq & \left(\int_\Lambda |\nabla u|^p\right)^{\frac{p-1}{p}}\cdot\left(\int_\Lambda |P_s|\nabla u||^p\mathrm{d}x\right)^{\frac{1}{p}} \\
\leq & \int_\Lambda |\nabla u|^p\mathrm{d}x.
\end{aligned}$$

Then the conclusion follows from Theorem 6.1.3. ∎

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