Full, Exact Subcategories of Hereditary Categories which Are Tubes

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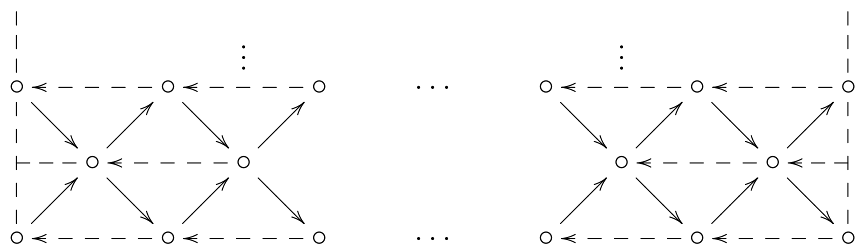
Contents

1 Introduction

Let k be an algebraically closed field. Auslander and Reiten considered the module category of a finite-dimensional k-algebra A and introduced an associated translation quiver Γ, today called the Auslander-Reiten quiver. A component of such an Auslander-Reiten quiver containing neither projectives nor injectives which is of the following shape is called a stable tube:



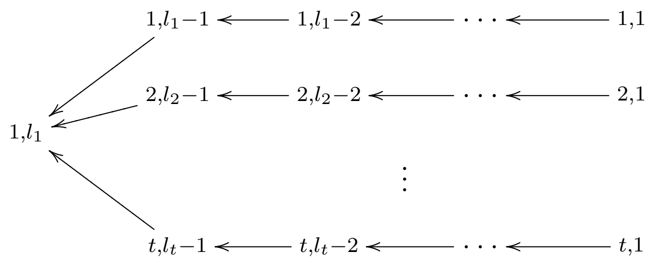
where the left and right dashed lines are identified and the picture continues infinitely in the upper direction. The number of points in each row of this picture is then called the rank of the tube.

If A is connected, hereditary and representation infinite, Γ consists of the preprojective, the preinjective and infinitely many regular components. The latter are either all of type $\mathbb{Z}A_\infty$ or all of them are stable tubes. In the first case A is called wild hereditary, in the latter case it is called tame hereditary. One reason why these names are used, is that only in the tame cases we can recover the module category from Γ in the following sense. To any translation quiver Γ we can associate a category, called its mesh category, only depending on k and Γ. The mesh category of the regular components of the Auslander-Reiten quiver of a tame hereditary algebra A is equivalent to the category of regular modules of A. This does not hold in any of the wild cases.

However, in the wild cases the module category of A can admit extension-closed, full, exact subcategories closed under direct summands which have stable tubes as Auslander-Reiten quivers. We will call such subcategories stable pseudo-tubes. These stable pseudo-tubes are called standard, if they are equivalent to the mesh categories of their Auslander-Reiten quivers. The examples which we exhibit will be standard.

In section 2 we will give first examples of them arising from embeddings of tame categories into wild ones or arising from tilting tame categories to wild ones, where in both cases the images of the tubes will be pseudo-tubes. After that we will give a criterion (theorem 2.13) to determine whether a subcategory is a pseudo-tube of rank n which only depends on finding n modules in the subcategory with certain properties. With the help of this theorem we can prove that each standard stable pseudo-tube of rank n contains exactly $2^n - 1$ standard stable pseudo-tubes (2.16) and we can prove that a representation X with $\mathrm{End}(X) = k$ and $\mathrm{Ext}^1(X, X) = k$ always lies in a pseudo-tube (2.19).

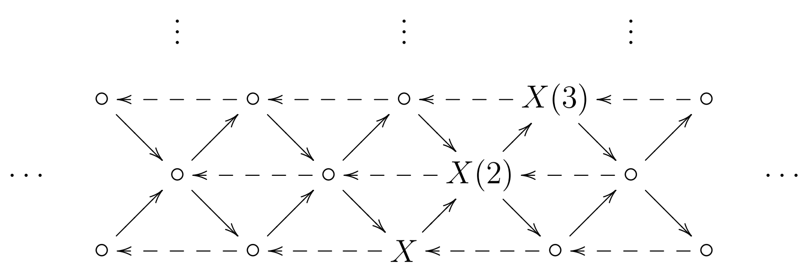
In the last section we will focus on representations of wild star quivers with subspace orientation. These are quivers of the shape:



for some numbers $l_1, \ldots, l_t > 1$ which we call the lengths of arms.

We will deal with concrete examples of star quivers with arms of length $(2, 3, n)$ for $n > 5$ and subspace orientation. For these we will examine the representations whose dimension at the unique sink of the quiver is less or equal than 3 and determine which of them are preprojective, regular and preinjective. For this classification we will define embeddings of the representations of star quivers with arms of lengths $(2, 3, n)$ into the category of representations of star quivers with arm lengths $(2, 3, n + 1)$. We will then prove that these embeddings map regular representations to regular representations (3.9). These embeddings are of interest, because the classification of representations of smaller star quivers provides us information on the representations of larger star quivers. After establishing the classification we obtain new examples of pseudo-tubes. The star quiver with arms of lengths $(2, 3, 6)$ is tame and possesses tubes, which are mapped to pseudo-tubes under the embeddings into larger star quivers. However we will find more than these examples of pseudo-tubes. The additional pseudo-tubes are of interest, because with their help we will answer two questions posed in [Ker92]. For wild acyclic quivers Kerner introduced the notion of exceptional components which are regular components of the Auslander-Reiten quiver and he defined the following two invariants for each exceptional component.

Let C be an exceptional component. Then C has the following shape:



This picture continues infinitely in the left, right and upper direction. The module X does not admit self-extensions. Let $l > 1$ be the smallest number such that $\mathrm{Ext}^1(X(l), X(l)) \neq 0$. Define

$$s := \min\{m \geq l \mid \mathrm{Hom}(X, \tau^m X) \neq 0, \mathrm{Hom}(X, \tau^{m+1} X) = 0\}.$$

Then Kerner asked whether for each component $s = l$ always holds. We will give an example where this is not the case. In this example there is a representation whose support is a representation finite subquiver, which answers the second question of [Ker92].

We will follow the conventions of [ASS06].

I thank all members of the representation theory group in Bielefeld. Especially, I thank my supervisor, Claus Michael Ringel, for giving me the advice and freedom I needed to write this thesis. I would also like to thank my family for their support during my education.

1.1 Overview of Statements

Let A be a finite-dimensional, connected algebra over k. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be orthogonal bricks with finite-dimensional Ext-spaces. For each $d \geq 0$ let $\mathcal{F}_d(\mathcal{S})$ be the full subcategory of all left A-modules M admitting a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$ with all factors $M_i/M_{i-1} \in \operatorname{add}(\mathcal{S})$. Let $\mathcal{F}(\mathcal{S}) = \bigcup_{d \geq 0} \mathcal{F}_d(\mathcal{S})$.

Lemma 2.8. *The category $\mathcal{F}(\mathcal{S})$ is an abelian category closed under extensions and the set $\mathcal{S}$ is the set of all simple objects of $\mathcal{F}(\mathcal{S})$.*

Lemma 2.9. *There is a bound for the length of local objects in $\mathcal{F}_d(\mathcal{S})$.*

For $d \in \mathbb{N}$ and $i = 1, \ldots, n$ define $P_{d,i}$ to be the longest local object in $\mathcal{F}_d(\mathcal{S})$ with top S_i.

Lemma 2.12. *The objects $P_{d,i}$ are projective in $\mathcal{F}_d(\mathcal{S})$ and $\mathcal{F}_d(\mathcal{S})$ is equivalent to* $\operatorname{mod} \operatorname{End}(\bigoplus_{i=1}^n P_{d,i})$.

Theorem 2.13. *Let $\mathcal{C}$ be a full, exact subcategory of $A \operatorname{mod}$ closed under extensions and direct summands. Then the following are equivalent:*

1. *There are $X_1, \ldots, X_n \in \mathcal{C}$ orthogonal bricks with*

$$\operatorname{Ext}^1(X_i, X_j) \cong \begin{cases} k & \text{if } i = j+1 \\ k & \text{if } i = 1 \text{ and } j = n \\ 0 & \text{else} \end{cases}$$

   and $\mathcal{C} = \mathcal{F}(X_1, \ldots, X_n)$.

2. *The Auslander-Reiten quiver of the category $\mathcal{C}$ is a standard stable tube of rank n.*

Theorem 2.15. *Let $\mathcal{C}$ be a full, exact subcategory of $A \operatorname{mod}$ closed under extensions and direct summands. Assume the Auslander-Reiten quiver Γ of $\mathcal{C}$ is a stable tube of rank n. Then the following are equivalent:*

1. *The categories $\mathcal{C}$ and the mesh category of Γ are equivalent.*

2. *The stable pseudo-tube $\mathcal{C}$ is standard.*

Proposition 2.16. *Let $\mathcal{C}$ be a standard stable pseudo-tube of rank $n \geq 1$ and $1 \leq r \leq n$. Then there are precisely $\begin{pmatrix} n \\ r \end{pmatrix}$ standard stable pseudo-tubes of rank r contined in $\mathcal{C}$.*

Let A be a connected, wild, hereditary algebra with quadratic form q.

Lemma 2.17. *Three equivalent conditions for a standard wing.*

Lemma 2.18. *Let X be a quasi-simple regular brick such that $[m]X$ is a brick for some $m > 0$, too. Then we have* $\dim \operatorname{Ext}^1(\tau^{m-1}X, X) = 1 - q(\mathbf{dim}[m]X)$.

Proposition 2.19. *Let X be a regular brick with $q(\mathbf{dim}\, X) = 0$. Then X lies in a pseudo-tube which contains the wing of X.*

For $t > 2$ let $l_i > 1$ for $i = 1, \ldots, t$ and let $\mathbb{T}^{l_1,\ldots,l_t}$ be the star star quiver with t arms which are of length $l_1, \ldots, l_t$ with subspace orientation. For this quiver call the Coxeter transformation C, the Auslander-Reiten shift τ and the quadratic form q. Let g be the g-duality (see definition 3.1).

Lemma 3.2. *Let $d \in \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}}$. Then $q(d) = q(g(d))$.*

Lemma 3.4. $\mathbf{dim}\, \tau M = C(\mathbf{dim}\, M)$

Lemma 3.5. $gCg = C^{-1}$

Proposition 3.6. *If $\mathbb{T}^{l_1,\ldots,l_t}$ is representation infinite, g induces a bijection between the dimension vectors of indecomposable preprojectives and the dimension vectors of indecomposable preinjectives which do not have coordinate 0 at the unique sink of $\mathbb{T}^{l_1,\ldots,l_t}$.*

Lemma 3.8. *Regular components admit representations M with $q(\mathbf{dim}\, M) \leq 0$.*

Lemma 3.9. *Let Q, Q' be wild acyclic quivers with quadratic forms $q_Q, q_{Q'}$. Let $F : \operatorname{rep}_k Q \longrightarrow \operatorname{rep}_k Q'$ be a faithful and exact functor mapping indecomposables to indecomposables, such that $q_Q(\mathbf{dim}\, M) = q_{Q'}(\mathbf{dim}\, F(M))$ for all $M \in \operatorname{rep}_k Q$. Then for any regular indecomposable representation M of Q the representation $F(M)$ is regular.*

Proposition 3.11. *For $n > 5$ let M be a preprojective representation of $\mathbb{T}^{n,3,2}$. Then $\dim M_* \leq \dim(\tau^- M)_*$, where $*$ is the unique sink of $\mathbb{T}^{n,3,2}$.*

For $d \in \mathbb{N}_0$ let $\mathcal{R}^d(\mathbb{T}^{l_1,\ldots,l_t})$ be the indecomposable regular objects with a d-dimensional vector space at the unique sink. For the definition of the functors $F_i : \operatorname{rep}_k \mathbb{T}^{n,3,2} \longrightarrow \operatorname{rep}_k \mathbb{T}^{n+1,3,2}$ see definition 3.10. For the definition of an exceptional component and its invariants s and l see definition 3.14.

Lemma 3.13. *For $d = 1, 2, 3$ and $n > d + 4$ we have*

$$\bigcup_{i=0}^{n} F_i(\mathcal{R}^d(\mathbb{T}^{n,3,2})) = \mathcal{R}^d(\mathbb{T}^{n+1,3,2}).$$

Lemma 3.15. *The Auslander-Reiten quiver of the path algebra of $\mathbb{T}^{7,3,2}$ has an exceptional component with $s = 7$ and $l = 4$.*

Lemma 3.16. *The Auslander-Reiten quiver of the path algebra of $\mathbb{T}^{7,3,2}$ has an exceptional component which contains a quasi-simple module whose support is representation finite.*

2 Pseudo-Tubes

2.1 Definitions

We assume all algebras to be connected. The Auslander-Reiten quiver of a module category of finite-dimensional algebras is a famous tool in representation theory which is explained in detail in [ARS97]. The existence of Auslander-Reiten sequences in subcategories of a module category has been treated in [AS81], the definition of the Auslander-Reiten quiver of a Krull-Remak-Schmidt category with exact sequences can be found in [Rin84]. We will follow the conventions of the latter.

The following definitions are taken from sections 2.2 and 2.3 in [Rin84].

Definition 2.1. Let $\mathcal{C}$ be a Krull-Remak-Schmidt category with an object X. A *source map* for X is a map $f : X \longrightarrow Y$ satisfying:

- f is not split mono.
- for any $f' : X \longrightarrow Y'$, not split mono, there is a morphism $g : Y \longrightarrow Y'$ with $f' = gf$ and
- for $g \in \operatorname{End}(Y)$ with $gf = f$, we have that g is an automorphism.

A morphism with the dual properties is called a *sink map.*

For indecomposable objects X, Y in a Krull-Remak-Schmidt category $\mathcal{C}$ we denote by $\operatorname{rad}(X,Y)$ the set of non-invertible morphisms. For objects $X = \bigoplus_{i=1}^{t} X_i$ and $Y = \bigoplus_{i=1}^{s} Y_i$ with X_i and Y_i indecomposable any map $f : \bigoplus_{i=1}^{t} X_i \longrightarrow \bigoplus_{i=1}^{s} Y_i$ can be witten as (f_{ij}) with $f_{ij} \in \operatorname{Hom}(X_i, Y_j)$. Then f is defined to lie in $\operatorname{rad}(X,Y)$, if each f_{ij} lies in $\operatorname{rad}(X_i, Y_j)$.

Now we can define the radical square of $\mathcal{C}$. A morphism $f : X \longrightarrow Y$ is in $\operatorname{rad}^2(X,Y)$, if there is an object M and morphisms $g \in \operatorname{rad}(X,M)$ and $h \in \operatorname{rad}(M,Y)$ with $f = hg$. Then let

$$\operatorname{Irr}(X,Y) = \operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$$

as $\operatorname{End}(Y)$-$\operatorname{End}(X)$-bimodule.

Let $\mathcal{C}$ be an additive, full, exact subcategory of a module category. We want to define the Auslander-Reiten quiver of $\mathcal{C}$. The underlying quiver will be $\Delta(\mathcal{C})$ the quiver which has as vertices the isomorphism classes $[X]$ of indecomposable objects X and $\dim_k \operatorname{Irr}(X,Y)$ arrows from $[X]$ to $[Y]$. For this quiver we will additionally define a translation. For this we need:

Definition 2.2. Let $\mathcal{C}$ be an additive, full, exact subcategory of a module category. Then an *Auslander-Reiten sequence in* $\mathcal{C}$ is a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

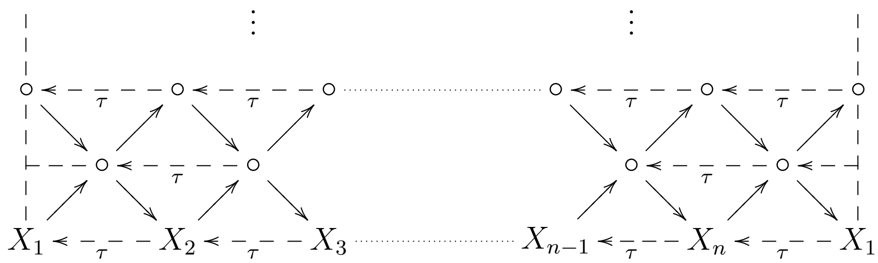
such that the map $A \longrightarrow B$ is a source map and $B \longrightarrow C$ is a sink map.

By a remark on page 61 in [Rin84] we know that if we have an Auslander-Reiten sequence as above, A and C are indecomposable and $[A]$ is uniquely determined by $[C]$ and vice versa. Define $\Delta'(\mathcal{C}) \subseteq \Delta(\mathcal{C})$ to be the full subquiver of isomorphism classes of objects C for which we have an Auslander-Reiten sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then define $\tau : \Delta_0'(\mathcal{C}) \longrightarrow \Delta_0(\mathcal{C})$ by $\tau([C]) = [A]$. By abuse of notation we will also write $\tau(C) = A$.

Definition 2.3. A *stable tube of rank* n is a component of an Auslander-Reiten quiver of the shape $\mathbb{Z}A_\infty/\tau^n$:

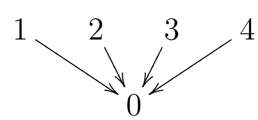


where the left and right dashed lines are identified. The set $\{X_1, \dots, X_n\}$ is the *mouth* of the tube.

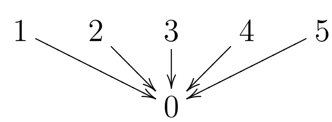
Let $\mathcal{C}$ be an abelian category. An full, exact subcategory $\mathcal{P}$ which is closed under extensions and direct summands is called a *stable pseudo-tube of rank* $n \geq 1$ provided $\mathcal{P}$ possesses an Auslander-Reiten quiver which is a tube of rank n. In addition $\mathcal{P}$ is called *standard*, if the mouth of the tube consists of pairwise orthogonal bricks (i.e. for $i \neq j$ we have $\operatorname{Hom}(X_i, X_j) = 0$ and $\operatorname{End}(X_i) \cong k$).

We will later prove a criterion, when a subcategory is a standard stable tube (see theorem 2.13), but first let us look at some examples. There are examples of pseudo-tubes which are the images of tubes under embedding or tilting functors.

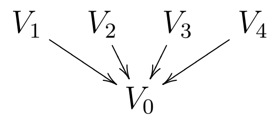
Example 2.4. Let Q be the four subspace quiver:



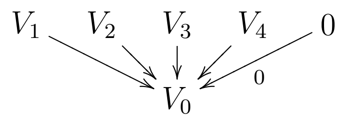
and let Q' be the five subspace quiver:



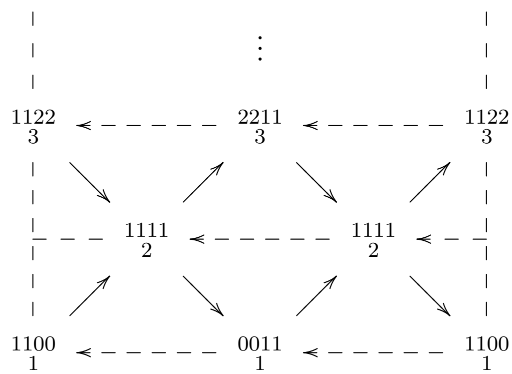
Then we can define a functor F from $\operatorname{rep}_k Q$ to $\operatorname{rep}_k Q'$ mapping a representation



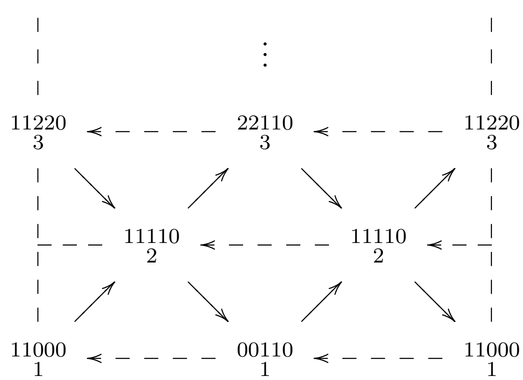
to



and mapping morphisms analogously. The quiver Q is of extended Dynkin type $\widetilde{D_4}$ and hence the Auslander-Reiten quiver of $\operatorname{rep}_k Q$ admits stable tubes. For example, there is a stable tube of rank 2. We denote its objects by dimension vectors.



The objects isomorphic to the vertices of this tube form a full, exact, extension-closed subcategory of $\operatorname{rep}_k Q$. This tube is mapped to a full, exact, extension-closed subcategory T of $\operatorname{rep}_k Q'$. The Auslander-Reiten quiver of T then is:



Hence T is a pseudo-tube.

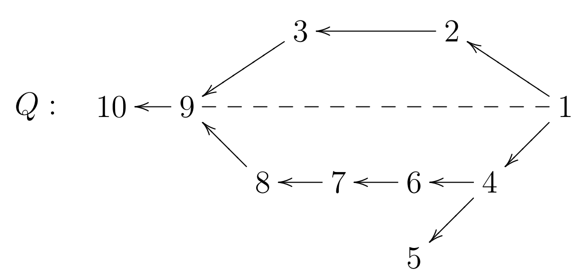
With the example in mind we can now prove that one regular module can lie in many pseudo-tubes.

Lemma 2.5. *For $n \in \mathbb{N}$ there is a wild algebra with an indecomposable module lying in n different pseudo-tubes.*

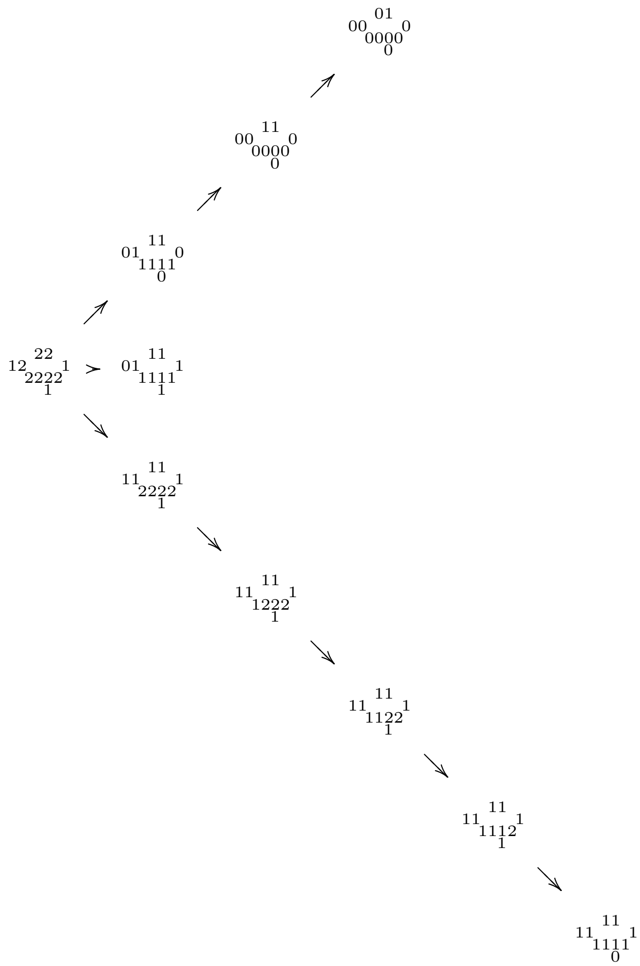
Proof. Without loss of generality we can assume $n > 1$. Define Q to be the $n+3$-subspace quiver, i.e. there are vertices $0, 1, \ldots, n+3$ and arrows from each vertex $1, \ldots, n+3$ to the vertex 0. Up to isomorphism there is exactly one representation M of dimension vector v with $v_0 = v_1 = v_2 = 1$ and all other entries 0. This representation M lies in n pseudo-tubes of rank 2. To see this fix a number $i = 3, \ldots, n+2$. There is exactly one representation M' of dimension vector v' with $v'_0 = v'_i = v'_{i+1} = 1$ and all other entries 0. The representations M and M' both lie in the mouth of a pseudo-tube as we have seen in the example above. Since for each i we get a different M' the module M lies in n pseudo-tubes. Then the path algebra of Q fulfills the assertion of the lemma, because its module category is equivalent to the $\operatorname{rep}_k Q$. □

Of course this example can be generalized to other embeddings of tame quivers in wild quivers, easily. We want to look at another example where a tame category is tilted to a wild category.

Example 2.6. In [Rin80], starting on page 220 there is a detailed construction of the Auslander-Reiten quiver Γ of the following quiver Q with relations.



The dashed line indicates a commutativity relation for the two paths from 1 to 9. Let A be the path algebra of Q modulo the ideal generated by this relation. There is a component C containing the following slice:



Let T be the sum of the modules of this slice. We claim that T is a tilting module, that is projdim $T \leq 1$, $\mathrm{Ext}^1(T,T) = 0$ and T has as many nonisomorphic direct summands as the quiver Q has vertices. The last statement is obviously true.

The extension property follows with the help of the Auslander-Reiten formula (see theorem IV.2.13 of [ASS06]):

$$\mathrm{Ext}^1(T,T) \cong \mathrm{Hom}(T, \tau T).$$

The latter is 0 because C is a standard component and there are no maps from a module to a proper predecessor.

The projective dimension of a module M is less or equal to 1, if and only if there are no non-zero maps from the indecomposable injectives to τM by statement (1) on page 74 in [Rin84]. There are exactly ten indecomposable injectives. For the eight injectives in the preinjective component it is clear that they do not map to τT, because there are no maps from the preinjective component to C. The two indecomposable injectives in the component C are proper successors of the summands of τT and since the component C is standard, there are no maps from these injectives to τT.

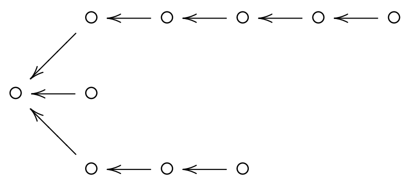
Hence T is a tilting module and there is a full subcategory

$$\mathcal{G} := \{M \mid \operatorname{Ext}^1(T, M) = 0\} \subseteq A \operatorname{mod}.$$

By the Brenner-Butler theorem (see theorem VI.3.8 of [ASS06]) there is a full, faithful and exact functor with an extension-closed image:

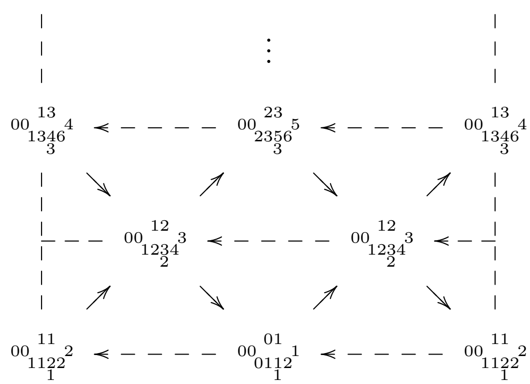
$$\operatorname{Hom}_{A \operatorname{mod}}(T, -) : \mathcal{G} \longrightarrow \operatorname{mod} \operatorname{End}(T)^{\mathrm{op}}.$$

The category $\operatorname{mod} \operatorname{End}(T)^{\mathrm{op}}$ is equivalent to the category of representations of the wild quiver Q':

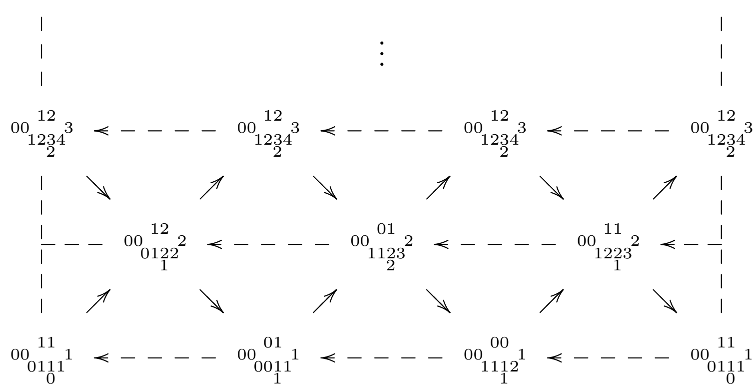


This follows from theorem II.3.7 in [ASS06] and its proof. We will now look for pseudo-tubes of $\operatorname{rep}_k Q'$ coming from $\operatorname{rep}_k Q$. There are three tubes of rank $n > 1$ which do not admit maps to C in Γ, namely:

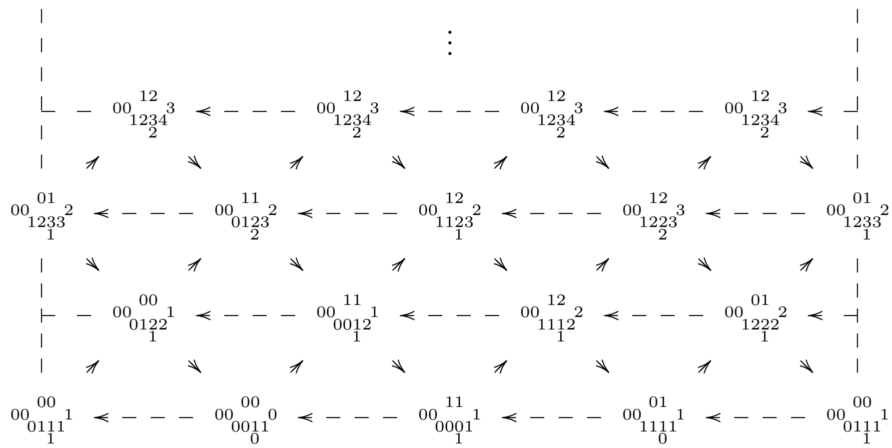
- G_1:



- G_2:

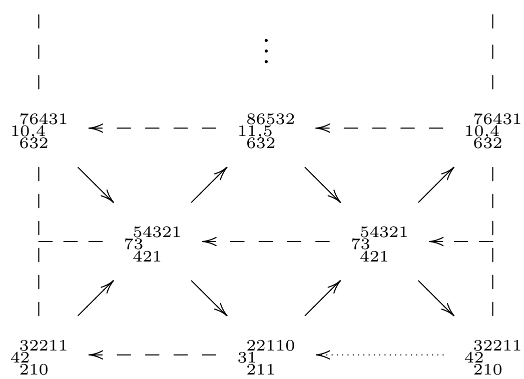


• G_3:



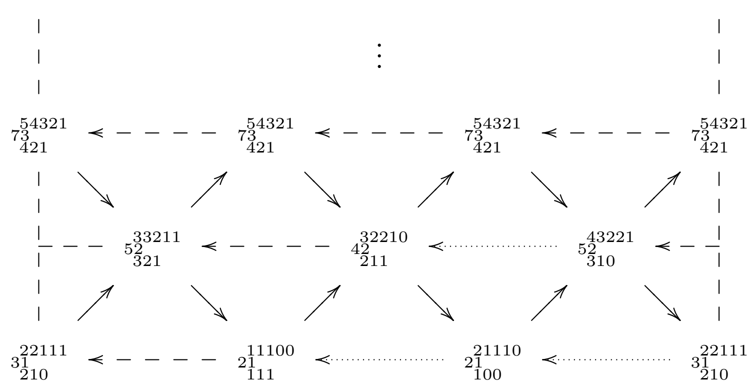
By the Auslander-Reiten formula G_1, G_2 and G_3 are contained in $\mathcal{G}$, because there are no maps from these tubes to the component C. Hence by the Brenner-Butler theorem the images of G_1, G_2 and G_3 are pseudo-tubes with the following Auslander-Reiten quivers. For this calculation we only need to compute $\operatorname{Hom}(T, X)$ for X on the mouth of a tube. The other dimension vectors can then be calculated using the Auslander-Reiten sequences in the pseudo-tube.

• G_1':



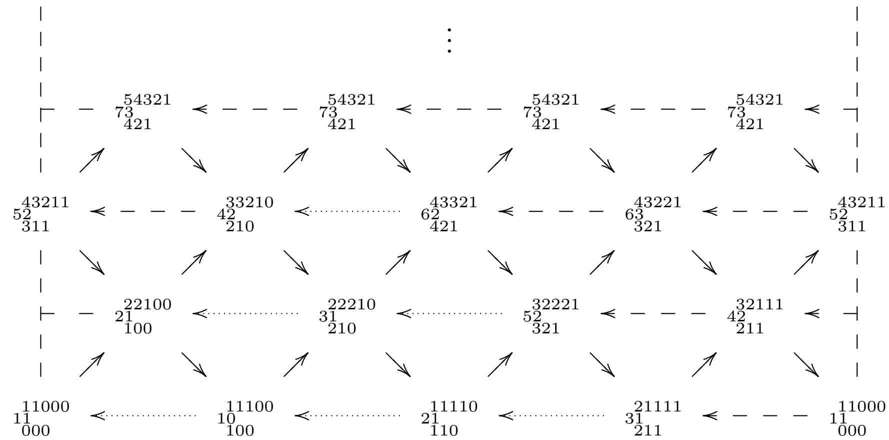
Here the dashed arrows are the Auslander-Reiten shift of the pseudo-tube. The dotted arrow is the Auslander-Reiten shift of the pseudo-tube and of the whole module category $\operatorname{rep}_k Q'$

• G_2':



Again the dotted arrows are the Auslander-Reiten shift in the pseudo-tube and in $\mathrm{rep}_k\, Q'$.

• G_3':



Again the dotted arrows are the Auslander-Reiten shift in the pseudo-tube and in $\mathrm{rep}_k\, Q'$.

Here apparently the mouth of each tube is mapped to the same component of the Auslander-Reiten quiver of Q'. This is not the case when we consider the Ext-functor belonging to T.

There is a full subcategory

$$\mathcal{F} := \{M \,|\, \mathrm{Hom}(T, M) = 0\} \subseteq A\,\mathrm{mod}\,.$$

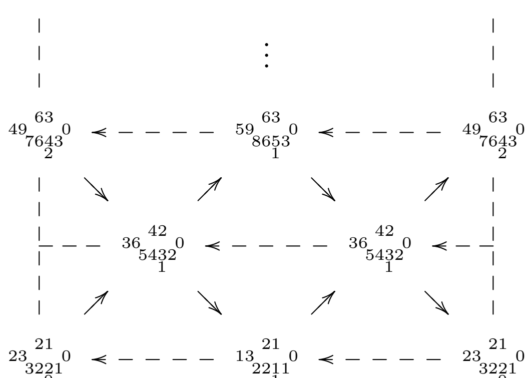
By the Brenner-Butler theorem there is a full, faithful and exact functor with an extension-closed image:

$$\mathrm{Ext}^1_{A\,\mathrm{mod}}(T, -) : \mathcal{F} \longrightarrow \mathrm{mod}\,\mathrm{End}(T)^{\mathrm{op}}.$$

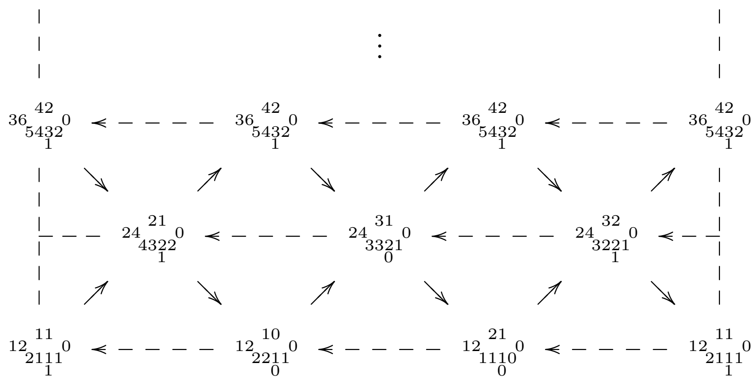
Again we will not work in $\mathrm{mod}\,\mathrm{End}(T)^{\mathrm{op}}$, but in the equivalent category $\mathrm{rep}_k\, Q'$.

In Γ there are three tubes of rank $n > 1$ which do not admit maps from C, namely:

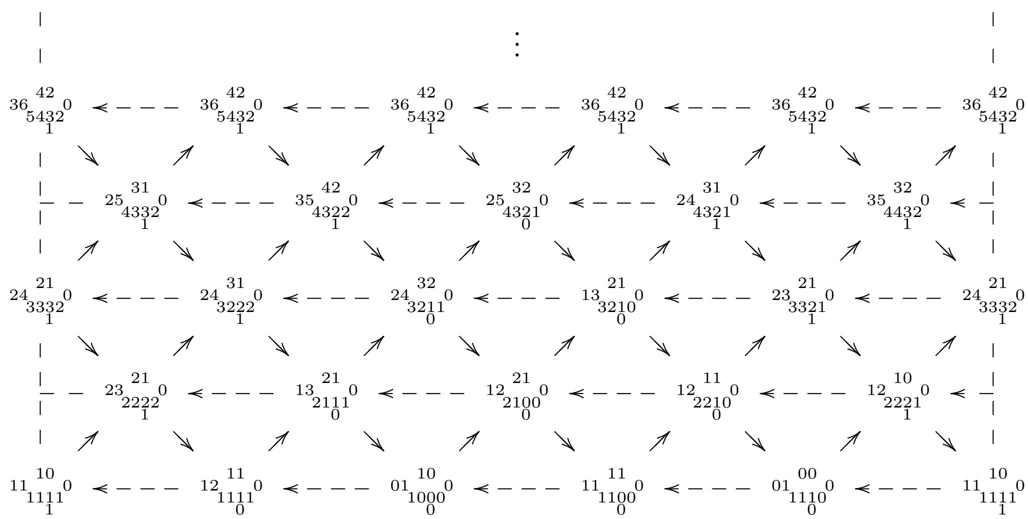
• F_1:



- F_2:

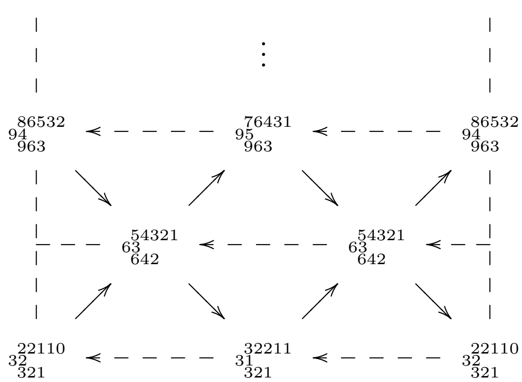


- F_3:

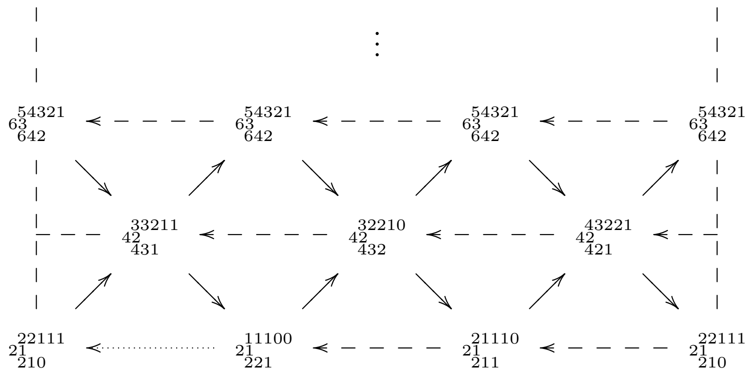


These tubes are mapped to pseudo-tubes by $\mathrm{Ext}^1(T,-)$, which have the following Auslander-Reiten quivers. Again we denote the Auslander-Reiten shift by a dotted arrow, if it is also the Auslander-Reiten shift of $\mathrm{rep}_k\, Q'$.

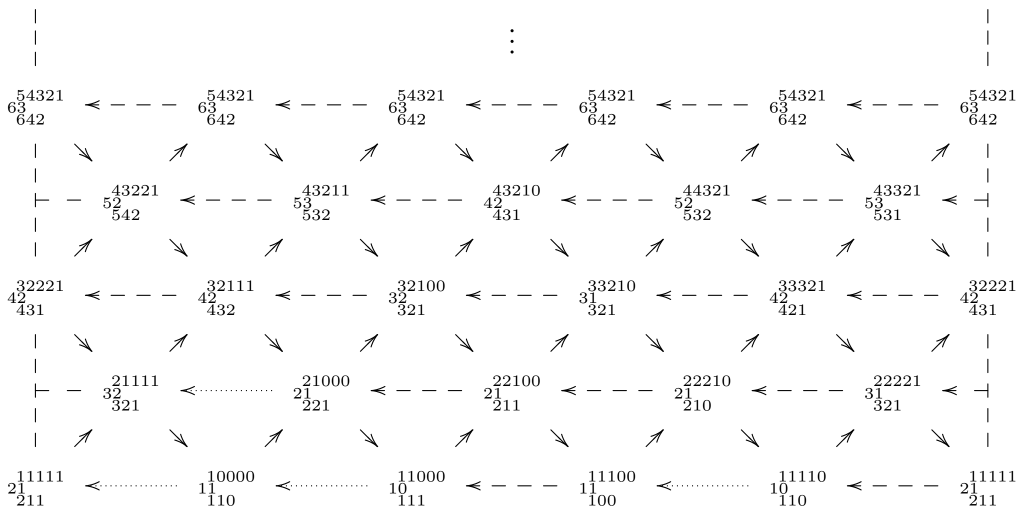
- F_1':



- F_2':



- F_3':



Hence the mouth of each of these pseudo-tubes has representations from precisely two different components of the Auslander-Reiten quiver of $\mathrm{rep}_k\, Q'$.

2.2 A Criterion for Standard Stable Pseudo-Tubes

Let A be a finite-dimensional algebra over k. We will prove a criterion when a subcategory of $A\,\mathrm{mod}$ is a pseudo-tube. For this we will need some results from [Rin76]. Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be orthogonal bricks with finite-dimensional Ext-spaces, that is there exists a number E, such that for all i and j we have $\dim \mathrm{Ext}^1(S_i, S_j) \leq E$. Let $\mathrm{add}(\mathcal{S})$ be the full subcategory of objects which are finite direct sums of objects in $\mathcal{S}$.

Definition 2.7. For each $d \geq 0$ let $\mathcal{F}_d(\mathcal{S})$ be the full subcategory of all left A-modules M admitting a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$$

with all factors $M_i/M_{i-1} \in \mathrm{add}(\mathcal{S})$. Let

$$\mathcal{F}(\mathcal{S}) = \bigcup_{d \geq 0} \mathcal{F}_d(\mathcal{S}).$$

The category $\mathcal{F}(\mathcal{S})$ consists of all A-module admitting a filtration whose quotients are in the set $\mathcal{S}$. We will now investigate this category further. It was shown in [Rin76], section 1.2 that this category is abelian:

Lemma 2.8. *Let $\mathcal{S}$ be as above. Then $\mathcal{F}(\mathcal{S})$ is an abelian category closed under extensions and the set $\mathcal{S}$ is the set of all simple objects of $\mathcal{F}(\mathcal{S})$.*

We will need a more precise description of this category. For this we need the following lemmas and definitions.

Lemma 2.9. *Let $\mathcal{S}$ be as above and M be a local object in $\mathcal{F}_d(\mathcal{S})$. Then the length l of M in the category $\mathcal{F}(\mathcal{S})$ is bounded, namely*

$$l \leq (nE+1)^{d-1}.$$

Proof. First we will show that $\mathcal{F}(\mathcal{S})$ is a length category. For this we need the theorem of Jordan-Hölder in its formulation in [Ben98], theorem 1.1.4 for the category $\mathcal{F}(\mathcal{S})$: Let N be an object in $\mathcal{F}(\mathcal{S})$ with two series of subjects in the category $\mathcal{F}(\mathcal{S})$:

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_s = N$$
$$0 = N'_0 \subseteq N'_1 \subseteq \cdots \subseteq N'_{s'} = N$$

Then these series may be refined to two series of equal length:

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = N$$

$$0 = L'_0 \subseteq L'_1 \subseteq \cdots \subseteq L'_t = N$$

such that the factors L_i/L_{i-1} are a permutation of the factors L'_j/L'_{j-1}. The statement in [Ben98] is formulated for the whole module category, but the same proof works for our statement for the subcategory as well, when one uses the fact that finite sums and intersections of subobjects in $\mathcal{F}(\mathcal{S})$ give rise to objects in $\mathcal{F}(\mathcal{S})$. Now using the fact that for an object N in $\mathcal{F}(\mathcal{S})$ we can refine its filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_d = N$$

with $N_i/N_{i-1} \in \operatorname{add}(\mathcal{S})$ to a finite composition series of N we obtain that each object has a well defined length. Denote the length of N in the subcategory by $|N|$. It is then easy to show by induction that for a short exact sequence $0 \to N' \to N \to N'' \to 0$ we have $|N| = |N'| + |N''|$

Next we prove that for a module N in $\mathcal{F}(\mathcal{S})$ and a module $S_i \in \mathcal{S}$ we have

$$\dim \operatorname{Ext}^1(N, S_i) \leq E|N|.$$

We proceed by induction on $|N|$. For $|N| = 1$ the module N is in $\mathcal{S}$ and the claim follows from the definition of E. So assume $|N| > 1$. Then there is a submodule $S_j \subseteq N$ in $\mathcal{S}$ and $|N/S_j| = |N| - 1$ and by induction we can assume that

$$\dim \operatorname{Ext}^1(N/S_j, S_i) \leq E(|N| - 1)$$

Applying $\operatorname{Hom}(-, S_i)$ to the sequence

$$0 \longrightarrow S_j \longrightarrow N \longrightarrow N/S_j \longrightarrow 0$$

we get a long exact sequence containing

$$\operatorname{Ext}^1(N/S_j, S_i) \longrightarrow \operatorname{Ext}^1(N, S_i) \longrightarrow \operatorname{Ext}^1(S_j, S_i).$$

The claim then follows from:

$$\begin{aligned}\dim \operatorname{Ext}^1(N, S_i) &\leq \dim \operatorname{Ext}^1(N/S_j, S_i) + \dim \operatorname{Ext}^1(S_j, S_i)\\ &\leq E(|N| - 1) + E \leq E|N|\end{aligned}$$

Now we can prove the lemma. Let M be a local object in $\mathcal{F}_d(\mathcal{S})$ which is not in $\mathcal{F}_{d-1}(\mathcal{S})$. We will proceed by induction on d. If $d = 1$ the object M is in $\operatorname{add}(\mathcal{S})$. Since M is local, it is indecomposable and hence already simple. This proves the assertion. Now assume $d > 1$. Then there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$$

with $M_i/M_{i-1} \in \operatorname{add}(\mathcal{S})$. Note that M_{d-1} is the unique maximal subobject of M and M/M_1 is a local object in $\mathcal{F}_{d-1}(\mathcal{S})$. Hence we can assume by induction that $|M/M_1| \leq (nE + 1)^{d-2}$. We have a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0.$$

Applying $\operatorname{Hom}(-, S_i)$ to this yields a long exact sequence:

$$0 \longrightarrow \operatorname{Hom}(M/M_1, S_i) \longrightarrow \operatorname{Hom}(M, S_i) \longrightarrow \operatorname{Hom}(M_1, S_i) \longrightarrow \operatorname{Ext}^1(M/M_1, S_i)$$

for any i. The first map is not only injective but also surjective. To see this assume f to be a nonzero homomorphism from M to S_i. Since S_i is simple in the category $\mathcal{F}(\mathcal{S})$, the map f is surjective. Hence $M/\ker f$ is isomorphic to S_i. Then $\ker f$ is a maximal subobject of M and hence the unique one, namely M_{d-1}. Since $M_1 \subseteq M_{d-1} = \ker f$ we get that f maps M_1 to 0 and thus f factors through M/M_1. This proves the surjectivity.

Then the next map in the long exact Ext-sequence is 0 and we get that

$$\dim \operatorname{Hom}(M_1, S_i) \leq \dim \operatorname{Ext}^1(M/M_1, S_i).$$

As M_1 is semisimple $\dim \operatorname{Hom}(M_1, S_i)$ is the number of summands in M_1 isomorphic to S_i and hence

$$|M_1| = \sum_{i=1}^{n} \dim \operatorname{Hom}(M_1, S_i) \leq \sum_{i=1}^{n} \dim \operatorname{Ext}^1(M/M_1, S_i) \leq nE|M/M_1|.$$

By induction hypothesis we have $|M/M_1| \leq (nE+1)^{d-2}$ and thus

$$|M_1| \leq nE(nE+1)^{d-2}.$$

Because of the above short exact sequence we now get

$$|M| = |M_1| + |M/M_1| \leq nE(nE+1)^{d-2} + (nE+1)^{d-2} = (nE+1)^{d-1}.$$

This proves our lemma. □

Definition 2.10. Define $\overline{\mathrm{rad}}\, M$ to be the intersection of all maximal subobjects of M in the category $\mathcal{F}_d(\mathcal{S})$. Then we define $\overline{\mathrm{top}}\, M$ to be $M/\overline{\mathrm{rad}}\, M$.

Note that $\overline{\mathrm{rad}}\, M$ is the same as the radical of M in the category $\mathcal{F}(\mathcal{S})$, but may differ from the usual radical which is the intersection of all submodules of M in the whole module category. With this definition we are ready to determine the indecomposable projective objects in $\mathcal{F}_d(\mathcal{S})$.

Definition 2.11. Let $\mathcal{S}$ be as above. Then for $d \in \mathbb{N}$ and $i = 1, \ldots, n$ define $P_{d,i}$ to be the longest local object in $\mathcal{F}_d(\mathcal{S})$ with $\overline{\mathrm{top}}\, P_{d,i} = S_i$. This has to exist by the last lemma. Then define $P_d = \bigoplus_{i=1}^n P_{d,i}$.

Lemma 2.12. *Let $\mathcal{S}$ be as above. Then for arbitrary i and d the object $P_{d,i}$ is projective in $\mathcal{F}_d(\mathcal{S})$ and $\mathcal{F}_d(\mathcal{S})$ is equivalent to the category of right* $\mathrm{End}(P_d)$*-modules.*

Proof. For a simple object S_i we will prove the existence of a projective cover in $\mathcal{F}_d(\mathcal{S})$ by induction on d. For $d = 1$ all objects in the category are already projective and there is nothing to show. Assume that we have $d > 1$. According to lemma 2.9 among the local objects which have S_1 as top in $\mathcal{F}_d(\mathcal{S})$ we have chosen one of maximal length, namely $P_{d,1}$. We will prove that for any j a short exact sequence of the form

$$0 \longrightarrow S_j \longrightarrow M \longrightarrow P_{d,1} \longrightarrow 0$$

in $\mathcal{F}_d(\mathcal{S})$ splits. Without loss of generality we can assume that S_j is a subobject of M and call π the canonical projection onto M/S_j.

First we will show that S_j does not lie in $\overline{\mathrm{rad}}\, M$. Assume the contrary: $S_j \subseteq \overline{\mathrm{rad}}\, M$. Then $\pi(\overline{\mathrm{rad}}\, M) = (\overline{\mathrm{rad}}\, M)/S_j$ and we have

$$\frac{M/S_j}{(\overline{\mathrm{rad}}\, M)/S_j} \cong M/(\overline{\mathrm{rad}}\, M)$$

is semisimple. Since M/S_j is local, this quotient is already simple, but this implies that M is local. This way we have constructed a local module in $\mathcal{F}_d(\mathcal{S})$ of greater length than $P_{d,1} \cong M/S_j$, a contradiction. Hence we know that $S_j \not\subseteq \overline{\mathrm{rad}}\, M$.

By the definition of the radical we now know that there is a maximal subobject M' of M with $S_j \nsubseteq M'$. Since S_j is simple, we get that $M = S_j \oplus M'$, that is our sequence splits.

We have just shown that there are no non-split short exact sequences starting in a simple object and ending in $P_{d,1}$. By induction on the length of an object this statement does not only hold for simples but for all objects in $\mathcal{F}_d(\mathcal{S})$. That is why the objects $P_{d,1}$ are projective. Analogously, all $P_{d,i}$ are projective and $P_d = \bigoplus_{i=1}^{n} P_{d,i}$ is projective in $\mathcal{F}_d(\mathcal{S})$.

Next we will prove that the functor

$$\mathrm{Hom}_{\mathcal{F}_d(\mathcal{S})}(P_d, -) : \mathcal{F}_d(\mathcal{S}) \longrightarrow \mathrm{mod}\,\mathrm{End}_{\mathcal{F}_d(\mathcal{S})}(P_d)$$

is an equivalence. It is clear, that this functor is k-linear. It suffices to show that it is dense and fully faithful.

Claim 1. The functor $\mathrm{Hom}_{\mathcal{F}_d(\mathcal{S})}(P_d, -)$ is faithful.

First we need to prove that for any object M of $\mathcal{F}_d(\mathcal{S})$ there is a natural number m such that there is an epimorphism $P_d^m \longrightarrow M$. This follows by induction on $|M|$. If M is simple, by definition of P_d there is a summand of P_d with top ismorphic to M. If $|M| > 1$ we can write M as extension of two shorter objects and get the assertion using the horseshoe lemma and induction.

Now we can prove faithfulness. Let $f : M \longrightarrow N$ be a morphism in $\mathcal{F}_d(\mathcal{S})$. We know that there is an epimorphism $\varepsilon : P_d^m \longrightarrow M$. Assume that $\mathrm{Hom}(P_d, f) = 0$, then also

$$0 = \mathrm{Hom}(P_d^m, f) : \mathrm{Hom}(P_d^m, M) \longrightarrow \mathrm{Hom}(P_d^m, N).$$

Hence

$$0 = \mathrm{Hom}(P_d^m, f)(\varepsilon) = f \circ \varepsilon$$

and since ε is an epimorphism we get that $f = 0$. This proves claim 1.

To prove that $\mathrm{Hom}(P_d, -)$ is full and dense we will first investigate its behavior with respect to projectives. First note that according to [ASS06], lemma I.5.3(b) in $\mathrm{mod}\,\mathrm{End}(P_d)$ every projective module is a direct sum of summands of $\mathrm{End}(P_d)$. So we need to determine the indecomposable summands of $\mathrm{End}(P_d)$.

Claim 2. There is a decomposition of right $\mathrm{End}(P_d)$-modules

$$\mathrm{End}(P_d) = \bigoplus_{i=1}^{n} \mathrm{Hom}(P_d, P_{d,i})$$

where the summands on the right hand side are indecomposable.

It is clear that this composition exists. We need to prove that the summands are indecomposable. For this we will to investigate $\mathrm{End}(P_{d,i})$ further for all i. The following facts in this paragraph are analogues to [ASS06], lemma I.4.8(b). The ring $\mathrm{End}(P_{d,i})$ is local. If it was not, there would be a pair of idempotents e

and $(\mathrm{id}-e)$ (see [ASS06], lemma I.4.6(d)) and then $P_{d,i} = \operatorname{im} e \oplus \operatorname{im}(\mathrm{id}-e)$ would be a nontrivial decomposition of a local module, a contradiction. Then we get according to [ASS06], lemma I.4.6(c), if the sum of two elements of $\operatorname{End}(P_{d,i})$ is the identity, one of these elements has to be invertible.

For a given i let $\iota : P_{d,i} \longrightarrow P_d$ and $\pi : P_d \longrightarrow P_{d,i}$ be the canonical inclusion and projection, respectively. Assume that we have a decomposition of right $\operatorname{End}(P_d)$-modules $\operatorname{Hom}(P_d, P_{d,i}) = X \oplus Y$. Then there are $x \in X$ and $y \in Y$, such that $\pi = x + y$. In $\operatorname{End}(P_{d,i})$ we now have

$$\mathrm{id} = \pi\iota = (x+y)\iota = x\iota + y\iota$$

Now we can assume without loss of generality that $x\iota$ is invertible. We then get

$$X \ni x(\iota(x\iota)^{-1}\pi) = (x\iota)(x\iota)^{-1}\pi = \pi,$$

because X is a right $\operatorname{End}(P_d)$-module. Now if we have $f \in \operatorname{Hom}(P_d, P_{d,i})$ arbitrary, then

$$X \ni \pi(\iota f) = (\pi\iota)f = f.$$

This implies that $Y = 0$ and hence $\operatorname{Hom}(P_d, P_{d,i})$ is indecomposable. This proves claim 2.

Now, trivially the indecomposable summands of $\operatorname{End}(P_d)$ lie in the image of the functor $\operatorname{Hom}(P_d, -)$. Since the functor is compatible with finite direct sums, for all projectives there is an isomorphic object in the image. Next we will prove fullness of the functor restricted to the projectives.

Claim 3. For $i = 1, \ldots, n$ and $M \in \mathcal{F}_d(\mathcal{S})$ the map

$$\operatorname{Hom}(P_{d,i}, M) \longrightarrow \operatorname{Hom}_{\operatorname{End}(P_d)}(\operatorname{Hom}(P_d, P_{d,i}), \operatorname{Hom}(P_d, M))$$

is surjective.

Again let $\iota : P_{d,i} \longrightarrow P_d$ and $\pi : P_d \longrightarrow P_{d,i}$ be the canonical inclusion and the canonical projection, respectively. We will show that each $\phi \in \operatorname{Hom}_{\operatorname{End}(P_d)}(\operatorname{Hom}(P_d, P_{d,i}), \operatorname{Hom}(P_d, M))$ has as preimage the morphism $\phi(\pi)\iota \in \operatorname{Hom}(P_{d,i}, M)$. To prove this we need to check that

$$\phi = \operatorname{Hom}(P_d, \phi(\pi)\iota)$$

holds. For any $g \in \operatorname{Hom}(P_d, P_{d,i})$ we have:

$$\operatorname{Hom}(P_d, \phi(\pi)\iota)(g) = (\phi(\pi)\iota)g = \phi(\pi)(\iota g) = \phi(\pi(\iota g)) = \phi((\pi\iota)g) = \phi(g).$$

This proves claim 3.

Claim 4. The functor $\operatorname{Hom}_{\mathcal{F}_d(\mathcal{S})}(P_d, -)$ is dense.

Let C be a right $\mathrm{End}(P_d)$-module. Then there is a projective resolution of C. We already know that the projectives and the homomorphisms between them can be chosen to lie in the image of our functor. So there is a morphism $f : P \longrightarrow P'$ between projectives in $\mathcal{F}_d(\mathcal{S})$, such that we have a projective resolution of C:

$$\mathrm{Hom}(P_d, P) \longrightarrow \mathrm{Hom}(P_d, P') \longrightarrow C \longrightarrow 0$$

Now define M to be the cokernel of f. Then $C \cong \mathrm{Hom}(P_d, M)$ follows, because $\mathrm{Hom}(P_d, -)$ is exact. This proves denseness.

Claim 5. The functor $\mathrm{Hom}_{\mathcal{F}_d(\mathcal{S})}(P_d, -)$ is full.

We have already seen it is dense so let M and $\widetilde{M}$ be objects in $\mathcal{F}_d(\mathcal{S})$ and $f : \mathrm{Hom}(P_d, M) \longrightarrow \mathrm{Hom}(P_d, \widetilde{M})$ be a morphism. As we have already shown there are natural numbers $m, m', \widetilde{m}, \widetilde{m}'$, such that there are projective resolutions of M and $\widetilde{M}$:

$$P_d^{m'} \longrightarrow P_d^{m} \longrightarrow M \longrightarrow 0$$

$$P_d^{\widetilde{m}'} \longrightarrow P_d^{\widetilde{m}} \longrightarrow \widetilde{M} \longrightarrow 0$$

When we apply $\mathrm{Hom}(P_d, -)$ to these sequences we obtain again projective resolutions:

$$\begin{array}{ccccccc} \mathrm{Hom}(P_d, P_d^{m'}) & \longrightarrow & \mathrm{Hom}(P_d, P_d^{m}) & \longrightarrow & \mathrm{Hom}(P_d, M) & \longrightarrow & 0 \\ & & & & \downarrow f & & \\ \mathrm{Hom}(P_d, P_d^{\widetilde{m}'}) & \longrightarrow & \mathrm{Hom}(P_d, P_d^{\widetilde{m}}) & \longrightarrow & \mathrm{Hom}(P_d, \widetilde{M}) & \longrightarrow & 0 \end{array}$$

For these projective resolutions there are maps which make the following diagram commute:

$$\begin{array}{ccccccc} \mathrm{Hom}(P_d, P_d^{m'}) & \longrightarrow & \mathrm{Hom}(P_d, P_d^{m}) & \longrightarrow & \mathrm{Hom}(P_d, M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow f & & \\ \mathrm{Hom}(P_d, P_d^{\widetilde{m}'}) & \longrightarrow & \mathrm{Hom}(P_d, P_d^{\widetilde{m}}) & \longrightarrow & \mathrm{Hom}(P_d, \widetilde{M}) & \longrightarrow & 0 \end{array}$$

As these new maps have projective domains they are in the image of the functor $\mathrm{Hom}(P_d, -)$ as we have seen above. Hence there is a diagram

$$\begin{array}{ccccccc} P_d^{m'} & \longrightarrow & P_d^{m} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow g' & & \downarrow g & & & & \\ P_d^{\widetilde{m}'} & \longrightarrow & P_d^{\widetilde{m}} & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \end{array}$$

in $\mathcal{F}_d(\mathcal{S})$, where g and g' are mapped to the above maps. This diagram commutes, because $\mathrm{Hom}(P_d, -)$ is faithful. Then there is a morphism h in $\mathcal{F}_d(\mathcal{S})$ making the diagram

$$\begin{array}{ccccccc} P_d^{m'} & \longrightarrow & P_d^{m} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ P_d^{\widetilde{m}'} & \longrightarrow & P_d^{\widetilde{m}} & \longrightarrow & \widetilde{M} & \longrightarrow & 0 \end{array}$$

commutative. Then h is mapped to f, because the functor $\mathrm{Hom}(P_d, -)$ is exact and the map f is uniquely determined by the above diagram. This proves claim 5.

Now we have shown the functor is dense, full and faithful, so it is an equivalence of categories. □

With this lemma we can now prove the following theorem which is the main tool to check that a subcategory is a pseudo-tube.

Theorem 2.13. *Let $\mathcal{C}$ be a full, exact subcategory of $A\,\mathrm{mod}$ closed under extensions and direct summands. Then the following are equivalent:*

1. *There are $X_1, \ldots, X_n \in \mathcal{C}$ orthogonal bricks with*

$$\mathrm{Ext}^1(X_i, X_j) \cong \begin{cases} k & \text{if } i = j+1 \\ k & \text{if } i = 1 \text{ and } j = n \\ 0 & \text{else} \end{cases}$$

   and $\mathcal{C} = \mathcal{F}(X_1, \ldots, X_n)$.

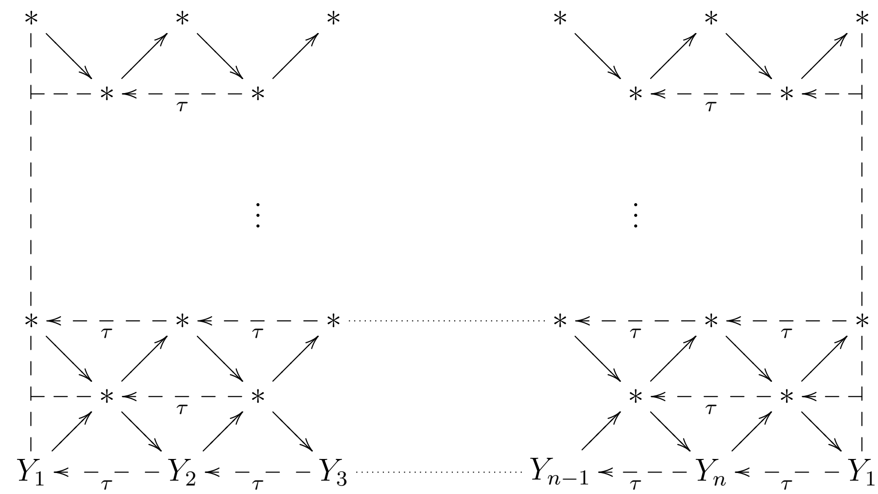
2. *The Auslander-Reiten quiver of the category $\mathcal{C}$ is a standard stable tube of rank n.*

Proof. We will first show that 1 implies 2. Assume that 1 holds. Then we know that

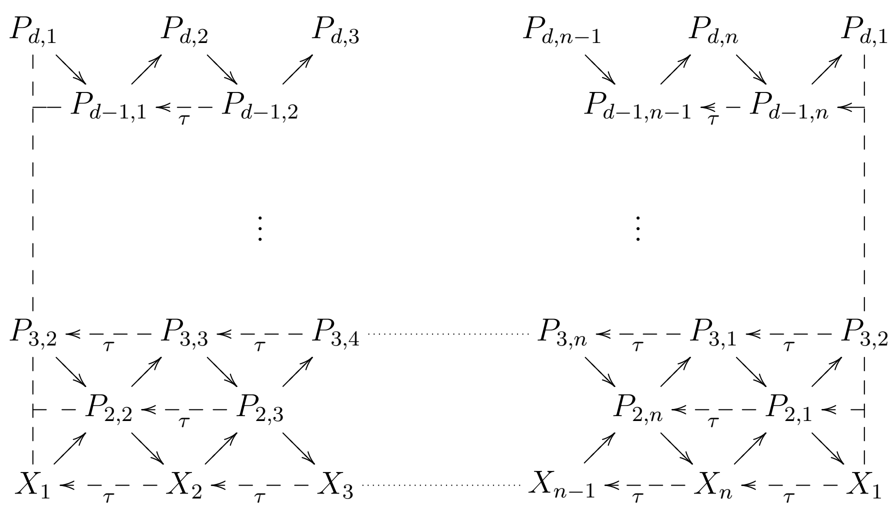
$$\mathcal{F}(X_1, \ldots, X_n) = \bigcup_d \mathcal{F}_d(X_1, \ldots, X_n)$$

and each $\mathcal{F}_d(X_1, \ldots, X_n)$ is equivalent to $\mathrm{mod}\,\mathrm{End}(P_d)$ by lemma 2.12. We will now investigate this category.

For $d \geq 2$ we have that $\mathrm{End}(P_d)$ is isomorphic to the path algebra of a cycle of length n modulo the ideal generated by all paths of length d. This follows from theorem II.3.7 in [ASS06] and its proof using the set of idempotents corresponding to the decomposition of P_d in its definition. This path algebra is a so called Nakayama algebra by proposition V.3.8 in [ASS06] and its Auslander-Reiten quiver is known (see theorem V.4.1 of [ASS06]):



where the left and right dashed lines have to be identified. Note that all modules in $\operatorname{mod}\operatorname{End}(P_d)$ are uniserial and in the Auslander-Reiten quiver the bottom $Y_1, \ldots, Y_n$ contains the simple objects. The row above contains the indecomposables of length 2 and so on. The top row contains the indecomposable projectives which are the indecomposables of length d. We will now use the isomorphism of algebras inducing an isomorphism of the Auslander-Reiten quivers of the algebras which leads to the Auslander-Reiten quiver of $\mathcal{F}_d(X_1, \ldots, X_n)$:



where the left hand side and the right hand side have to be identified. The bottom row has to contain the simples $X_1, \ldots, X_n$ and the fact that they appear in this order is implied by the dimensions of their Ext-spaces. The row above has to contain all indecomposables of length 2. Since all indecomposables in $\mathcal{F}_d(X_1, \ldots, X_n)$ are uniserial these are precisely the indecomposable projectives in $\mathcal{F}_2(X_1, \ldots, X_n)$, namely $P_{2,1}, \ldots, P_{2,n}$. Their positions with respect to the bottom line follow from the fact that $\operatorname{Hom}(P_{2,i}, X_j) = 0$ for $i \neq j$. The arrows are residue classes of inclusions and projections, because the Hom spaces are at most one dimensional and therefore generated by any nonzero map. Similarly, one obtains the indecomposables in the higher rows of the Auslander-Reiten quiver and the arrows are residue classes of inclusions and projections, too.

Claim. The Auslander-Reiten quiver of $\mathcal{F}(X_1, \dots, X_n)$ is the union of Auslander-Reiten quivers of $\mathcal{F}_d(X_1, \dots, X_n)$ for all d.

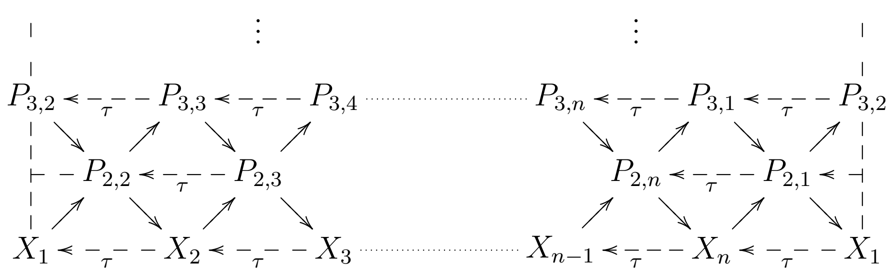
Since all sink and source maps are inclusions and projections any source or sink map in $\mathcal{F}_{d-1}(X_1, \dots, X_n)$ is again a source or sink map in $\mathcal{F}_d(X_1, \dots, X_n)$, respectively and any Auslander-Reiten sequence in $\mathcal{F}_{d-1}(X_1, \dots, X_n)$ is again an Auslander-Reiten sequence in $\mathcal{F}_d(X_1, \dots, X_n)$. To obtain the Auslander-Reiten quiver of $\mathcal{F}(X_1, \dots, X_n)$ we need to determine its Auslander-Reiten sequences. For an Auslander-Reiten sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

in $\mathcal{F}(X_1, \dots, X_n)$ there is a d such that the sequence is in $\mathcal{F}_d(X_1, \dots, X_n)$. We need to prove that the map $f : A \longrightarrow B$ is a source map in $\mathcal{F}_d(X_1, \dots, X_n)$. The three properties of the definition of a source map all follow from the fact that $\mathcal{F}_d(X_1, \dots, X_n)$ is a full subcategory of $\mathcal{F}(X_1, \dots, X_n)$. The proof that the map $B \longrightarrow C$ is a sink map is dual. Conversely, we need to show that an Auslander-Reiten sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

in $\mathcal{F}_d(X_1, \dots, X_n)$ is an Auslander-Reiten sequence in $\mathcal{F}(X_1, \dots, X_n)$. Since exactness is clear, we are left to show that $f : A \longrightarrow B$ is a source map in $\mathcal{F}(X_1, \dots, X_n)$. Again the proof that the map $B \longrightarrow C$ is a sink map is dual. The first and last property of the definition of a source map are again implied by the fullness of $\mathcal{F}_d(X_1, \dots, X_n)$ in $\mathcal{F}(X_1, \dots, X_n)$. For the second property fix an $f' : A \longrightarrow B'$ not a split mono. Then there is a $d' \geq d$ such that f' is in $\mathcal{F}_{d'}(X_1, \dots, X_n)$. By the above remark f is a source map in $\mathcal{F}_{d'}(X_1, \dots, X_n)$ and we get a map g in $\mathcal{F}_{d'}(X_1, \dots, X_n)$ with $f' = gf$. This g of course is in $\mathcal{F}(X_1, \dots, X_n)$ as well and we have shown our assertion. This proves that the Auslander-Reiten quiver of $\mathcal{F}(X_1, \dots, X_n)$ is just the union of the Auslander-Reiten quivers of $\mathcal{F}_d(X_1, \dots, X_n)$ for all d which is the following tube:



This proves our claim and hence it proves assertion 2 of the theorem.

Now we will prove the converse implication. Assume that 2 holds. Then we will name the orthogonal bricks on the mouth of the tubes $X_1, \dots, X_n$. Let us first show that $\mathcal{C} = \mathcal{F}(X_1, \dots, X_n)$.

Fix an object M in the category $\mathcal{C}$. As $\mathcal{C}$ is closed under extensions, we can assume that M is indecomposable. Then the isomorphism class of M appears in the Auslander-Reiten quiver of $\mathcal{C}$ in the layer d. If $d = 1$, the object M is simple and obviously in $\mathcal{F}_d(X_1, \dots, X_n)$. If $d > 1$, the object M appears as summand of the middle term of an Auslander-Reiten sequence with end terms in the $d-1$-st layer of the Auslander-Reiten quiver. Then by induction we can assume that these end terms are in $\mathcal{F}_d(X_1, \dots, X_n)$ and as this category is closed under extensions and direct summands M has to lie in it as well.

Now take an arbitrary object $M \in \mathcal{F}_d(X_1, \dots, X_n)$. We know by definition that there is an $\operatorname{add}(X_1, \dots, X_n)$-filtration of X which can be refined to a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$$

with X_i/X_{i-1} being isomorphic to a module in $\{X_1, \dots, X_n\}$. The assertion that $M \in \mathcal{C}$ then follows by induction on t.

It is clear that $\dim \operatorname{Ext}^1(X_i, \tau X_i) \geq 1$, because we have the Auslander-Reiten sequences which are non split. To see that there are not more extensions between the X_i assume we have a non split short exact sequence

$$0 \longrightarrow X_i \longrightarrow X \longrightarrow X_j \longrightarrow 0.$$

Then we will show that it is isomorphic to the Auslander-Reiten sequence

$$0 \longrightarrow X_i \longrightarrow M \longrightarrow \tau^- X_i \longrightarrow 0$$

starting in X_i. As the first map of an Auslander-Reiten sequence is a source map there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_i & \longrightarrow & M & \longrightarrow & \tau^- X_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \\ 0 & \longrightarrow & X_i & \longrightarrow & X & \longrightarrow & X_j & \longrightarrow & 0 \end{array}$$

Then there is an induced map f making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_i & \longrightarrow & M & \longrightarrow & \tau^- X_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & X_i & \longrightarrow & X & \longrightarrow & X_j & \longrightarrow & 0 \end{array}$$

This map f is a map between simple objects in $\mathcal{F}(X_1, \dots, X_n)$. Hence it is just the multiplication by a scalar, which implies that all non split extensions between the X_i are just multiples of Auslander-Reiten sequences. This finishes the proof. $\square$

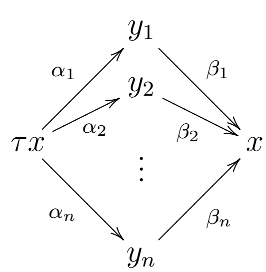
2.3 Pseudo-Tubes in Pseudo-Tubes

We will exhibit all standard stable pseudo-tubes of smaller rank in a standard stable pseudo-tube. To do this we will prove that a standard stable pseudo-tube is equivalent to the mesh category of its Auslander-Reiten quiver. This will help us, because then a standard stable pseudo-tube is equivalent to the mesh category of the Auslander-Reiten quiver of any standard stable pseudo-tube of the same rank. Then we can restrict to one special pseudo-tube, which will already be a tube, and prove the result for this tube. First we will need the mesh category of a component of an Auslander-Reiten quiver. The definition is a simplified dual version of the definition in section 2.1 of [Rin84]. We will only need the simplified version, because we do not take into account the possibility of multiple arrows. We dualize, because the composition of morphisms in [Rin84] is dual to our composition.

Definition 2.14. Let C be a component of an Auslander-Reiten quiver. For simplicity we will assume that C has no multiple arrows.

The copath category kC has by definition as objects formal direct sums of the vertices of C and as morphisms from $x \in C_0$ to $y \in C_0$ all k-linear combinations of copaths from x to y in C. Here a copath is a path in the opposite quiver. For simplicity we will denote the arrows in the quiver and its opposite by the same symbol. The composition of morphisms is induced by the concatenation of copaths.

For each non-projective point x we have a mesh



ending in x. For this x we can define the mesh element:

$$m_x = \sum_{i=1}^{n} \beta_i \alpha_i$$

which is a morphism of the copath category. Denote by M_C the ideal generated by all mesh elements m_x.

The mesh category is the quotient category kC/M_C.

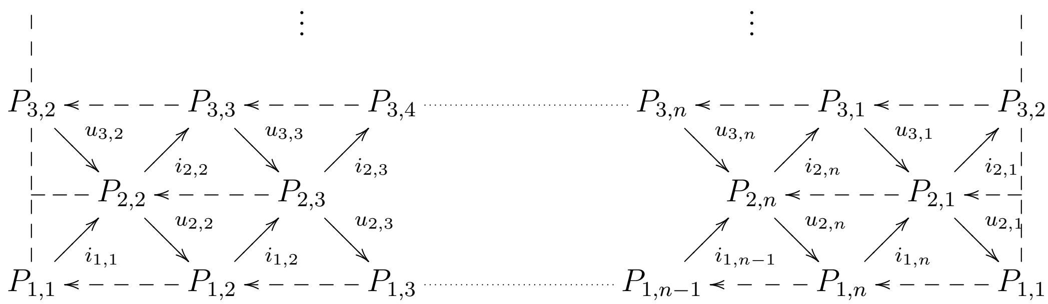
Let A be a finite-dimensional algebra.

Theorem 2.15. *Let $\mathcal{C}$ be a full, exact subcategory of A mod closed under extensions and direct summands. Assume the Auslander-Reiten quiver Γ of $\mathcal{C}$ is a stable tube of rank n. Then the following are equivalent:*

1. *The categories $\mathcal{C}$ and the mesh category of Γ are equivalent.*

2. *The stable pseudo-tube $\mathcal{C}$ is standard.*

Proof. The proof is inspired by the proof of lemma 3.1.2 in [Rin84]. We will show that $\mathcal{C}$ is equivalent to the mesh category of its Auslander-Reiten quiver, if $\mathcal{C}$ is standard. First observe that by theorem 2.13 and its proof we know, that Γ is of the shape:



where we can choose all objects and maps in such a way, that the maps $i_{*,*}$ are inclusions and the maps $u_{*,*}$ are projections. Define an additive functor F from the mesh category to $\mathcal{C}$ as follows: A point of Γ is mapped to the corresponding indecomposable representation we have chosen. An irreducible map $u_{s,t}$ is mapped to $u_{s,t}$ and an irreducible map $i_{s,t}$ is mapped to $(-1)^s i_{s,t}$. The sign is needed, because without it F would not be well-defined. That is because the mesh elements which were factored out in the mesh category are the sum of two copaths which belong to a square which we assumed to be commutative by the choice of $i_{*,*}$ and $u_{*,*}$. It is clear that F is dense, because all indecomposables lie in the image of F. Analogously, F is full, because all irreducible maps lie in the image.

What remains to be shown is that F is faithful. If we have a copath in Γ it is equivalent to 0 or to a multiple of a copath of the form

$$i_{s-l-1+m,t+m} \dots i_{s-l,t+1} i_{s-l-1,t} u_{s-l,t} \dots u_{s-1,t} u_{s,t}$$

for some l, m, s, t with respect to the equivalence relation generated by the mesh elements. We call a copath standard, if it is of the above form. The standard copaths generate the Hom-spaces between indecomposables in the mesh category. To prove that F is faithful, let $d, d' = 1, \dots, n$ and t, t' be natural numbers. Then $\mathrm{Hom}(P_{d,t}, P_{d',t'})$ is generated by standard copaths f_l where l denotes the number of coarrows $u_{*,*}$ contained in f_l. Then $\mathrm{im}(F(f_l)) = P_{d-l,t}$. If we have a non-zero morphism in the mesh category $f = \sum_l \lambda_l f_l$, where l runs over a finite set of natural numbers and $\lambda_l \in k$, there is a minimal m such that $\lambda_m \neq 0$. Then

$$\mathrm{im}\left(F\left(\sum_{l\neq m} \lambda_l f_l\right)\right) \subsetneq P_{d-m,t} = \mathrm{im}(F(f_m)).$$

Hence the image of $F(f)$ cannot be 0. This proves faithfulness.

We have shown that F is a full, faithfull and dense functor. Hence it is an equivalence of categories.

The other implication of the theorem is clear, because in the mesh category of $\mathbb{Z}A_\infty/\tau^n$ the mouth of the tube consists of orthogonal bricks. $\square$

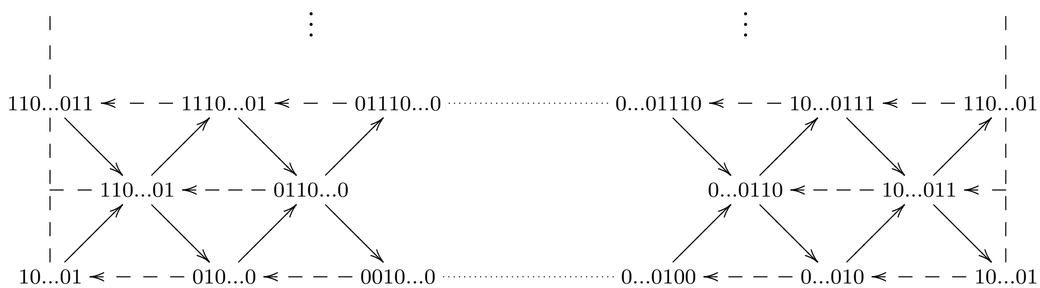
A pseudo-tube can contain smaller ones as we will see now.

Proposition 2.16. *Let $\mathcal{C}$ be a standard stable pseudo-tube of rank $n \geq 1$ and $1 \leq r \leq n$. Then there are precisely $\binom{n}{r}$ standard stable pseudo-tubes of rank r contined in $\mathcal{C}$. Hence there are precisely $2^n - 1$ standard stable pseudo-tubes of arbitrary rank contained in $\mathcal{C}$.*

Proof. According to theorem 2.15 we know that a standard stable pseudo-tube of rank n is equivalent to the mesh category of its Auslander-Reiten quiver. Theorem XIII.2.5 of [SS07] tells us that the Auslander-Reiten quiver of the category of representations of the quiver $\widetilde{A}_{1n}$:

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow n-1 \longleftarrow n \quad (\text{with an arrow } n \to 0)$$

contains a standard stable tube of rank n. In the following picture we will denote the isomorphism classes of indecomposables by their dimension vectors.



To prove the assertion we will exhibit all standard stable pseudo-tubes of rank r contained in the category $\mathcal{T}$ corresponding to this tube. This suffices because both categories are equivalent to the mesh category of the same Auslander-Reiten quiver. Note that the n lowest rows of this Auslander-Reiten quiver contain bricks and that all bricks are contained in these rows as well. There are precisely n bricks in this tube admitting self-extensions. These are the modules in the n-th row of the tube having dimension vector d which is the vector with all entries being 1. Since this argument holds for any pseudo-tube, we must have that any pseudo-tube $\mathcal{P}$ of rank r contained in $\mathcal{T}$ admits exactly r bricks with self-extensions and these are in the r-th row of the Auslander-Reiten quiver of $\mathcal{P}$.

We claim that there is a bijection between the set of pseudo-tubes of rank r and subsets with r elements of $N := \{1, 2, \ldots, n\}$. For $0 < i < n$ let us denote the simple representation corresponding to i by X_i and let X_n be the unique indecomposable representation of dimensionvector $10\dots01$. Then for $0 < i \leq n$ and $m > 0$ there is a unique object $X_i(m)$ in $\mathcal{T}$ with socle X_i and length m. This object is the codomain of the composition of precisely $m-1$ inclusions in the above tube starting with the simple object X_i.

Now we will define a map f mapping a subset $S := \{n_1 < \cdots < n_r\}$ of N to a pseudo-tube in the following way: The objects

$$X_{n_1}(n_2 - n_1), \dots, X_{n_{r-1}}(n_r - n_{r-1}), X_{n_r}(n + n_1 - n_r)$$

are bricks, because they lie in the lowest n rows of the tube. We want to apply theorem 2.13 to show that

$$\mathcal{F}(X_{n_1}(n_2 - n_1), \dots, X_{n_{r-1}}(n_r - n_{r-1}), X_{n_r}(n + n_1 - n_r))$$

is a pseudo-tube. The bricks are orthogonal, because the supports of their dimensionvectors are pairwise disjoint. To see that the dimensions of Ext-spaces are as required by theorem 2.13 we can use the dimension vectors, too or use the Auslander-Reiten formula. Now define

$$f(S) = \mathcal{F}(X_{n_1}(n_2 - n_1), \dots, X_{n_{r-1}}(n_r - n_{r-1}), X_{n_r}(n + n_1 - n_r)).$$

To see that f is a bijection we will construct its inverse g. Let $\mathcal{P}$ be a pseudo-tube with modules $Y_1, \dots, Y_r$ on the mouth. Then denote the socle of Y_i by $\overline{\text{soc}}(Y_i)$ when we consider Y as object of $\mathcal{T}$. This socle will be some X_{n_i}. Then map $\mathcal{P}$ to $\{n_1, \dots, n_r\}$. This set consists of r different numbers, because otherwise some Y_i would be a submodule of some Y_j for $i \neq j$.

By definition it is clear that gf is the identity. Now let us prove that fg is the identity, too. Let $\mathcal{P}$ be a pseudo-tube with modules $Y_1, \dots, Y_r$ on the mouth. Then the sum of dimension vectors of all Y_i is d, because in the r-th row of the Auslander-Reiten quiver of $\mathcal{P}$ we have bricks with self-extensions. These need to have dimension vector d as we have seen before. The pseudo-tube is then mapped to $\{n_1 < \cdots < n_r\}$ via g. We need to show that

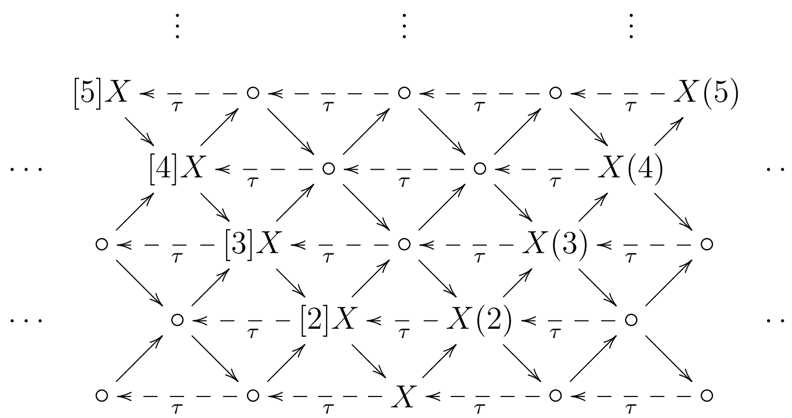
$$X_{n_1}(n_2 - n_1) = Y_1, \dots, X_{n_{r-1}}(n_r - n_{r-1}) = Y_{n-1}, X_{n_r}(n + n_1 - n_r) = Y_r.$$

It suffices to show the first equality, the others follow analogously. By definition, $\overline{\text{soc}}(Y_1) = X_{n_1} = \overline{\text{soc}}(X_{n_1}(n_2 - n_1))$ holds and thus these two modules lie on the same ray in the tube. If Y_1 was of longer length then X_{n_1}, then the modules Y_1, Y_2 would not be orthogonal, a contradiction. Analogously, if Y_1 was of shorter length, then Y_1, Y_2 would have no extensions. Hence the maps f and g are inverses. □

2.4 Standard Wings in Regular Components

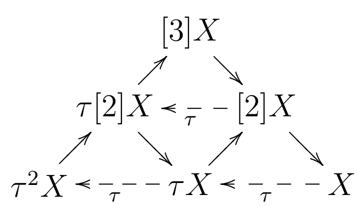
We will introduce the notion of standard wings which are subquivers of Auslander-Reiten quivers. For a wild hereditary algebra A, for example the path algebra of a wild acyclic quiver, some examples of pseudo-tubes contain standard wings which are also standard wings in regular components of the Auslander-Reiten quiver of A. To find these examples we will look for standard wings in regular components. We will follow the notation of [Ker92], section 1.

Let C be a regular component of the Auslander-Reiten quiver of A. Then C has the following shape:



This picture continues infinitely in the left, right and upper direction. The modules in the bottom τ-orbit are called quasi-simple. Once we fix a quasi-simple module X for each $i > 1$ we will denote by $X(i)$ and $[i]X$ the modules indicated in the above picture and we define $X = X(1) = [1]X$. Note that for any i we have $[i]X = \tau^{i-1}(X(i))$.

Then for a regular module Y there is a number i and a quasi-simple regular module X, such that $Y = [i]X$. We define the wing $W(Y)$ to be the mesh complete full subquiver of C defined to contain the vertices $\tau^r([s]X)$ for $1 \leq s \leq i$ and $0 \leq r \leq i - s$. For example the wing of $[3]X$ is:



The following lemma is a part of proposition 1.1 in [Ker92].

Lemma 2.17. *For a quasi-simple regular module X and $m > 1$ the following conditions are equivalent:*

- $X(m)$ *is a brick.*
- $X(m-1)$ *is a brick without self-extensions.*
- $X(1), X(2)/X(1), \ldots, X(m)/X(m-1)$ *are pairwise orthogonal bricks without self-extensions.*

If X satisfies the above conditions we will call $W(X(m))$ a standard wing.

Let Q be a wild acyclic quiver. Let q denote the corresponding quadratic form. Let τ denote the AR-translate of the category $\operatorname{rep}_k Q$

Lemma 2.18. *Let X be a quasi-simple regular brick such that $[m]X$ is a brick for some $m > 0$, too. Then we have* $\dim \operatorname{Ext}^1(\tau^{m-1}X, X) = 1 - q(\mathbf{dim}[m]X)$.

Proof. The case $m = 1$ is clear, assume $m > 1$. By the above lemma we have that the wing $W([m]X)$ is standard. Hence $X, \tau X, \ldots, \tau^{m-1}X$ are pairwise orthogonal bricks without self-extensions. Observe that by lemma 2.2 in [Rin76] we have:

$$\begin{aligned} q(\mathbf{dim}[m]X) &= q(\sum_{i=0}^{m-1} \mathbf{dim}\, \tau^i X) \\ &= \sum_{i,j=0}^{m-1} (\dim \operatorname{Hom}(\tau^i X, \tau^j X) - \dim \operatorname{Ext}^1(\tau^i X, \tau^j X)) \end{aligned}$$

By the Auslander-Reiten-formula the dimensions of the Ext-spaces can be replaced by the dimensions of Hom-spaces as follows:

$$\begin{aligned} q(\mathbf{dim}[m]X) &= \sum_{i,j=0}^{m-1} (\dim \operatorname{Hom}(\tau^i X, \tau^j X) - \dim \operatorname{Hom}(\tau^j X, \tau^{i+1} X)) \\ &= \sum_{i,j=0}^{m-1} \dim \operatorname{Hom}(\tau^i X, \tau^j X) - \sum_{i,j=0}^{m-1} \dim \operatorname{Hom}(\tau^i X, \tau^{j+1} X) \\ &= \sum_{i=0}^{m-1} \dim \operatorname{Hom}(\tau^i X, X) - \sum_{i=0}^{m-1} \dim \operatorname{Hom}(\tau^i X, \tau^m X) \end{aligned}$$

By lemma 2.17 we have that the $\tau^i X$ are orthogonal. Hence,

$$q(\mathbf{dim}[m]X) = \dim \operatorname{Hom}(X, X) - \dim \operatorname{Hom}(X, \tau^m X).$$

Applying the Auslander-Reiten-formula again now yields:

$$q(\mathbf{dim}[m]X) = 1 - \dim \operatorname{Ext}^1(\tau^{m-1}X, X)$$

This proves the assertion. □

Now we can prove a proposition which says that for certain bricks we have pseudo-tubes.

Proposition 2.19. *Let X be a regular brick with $q(\mathbf{dim}\, X) = 0$. Then X lies in a pseudo-tube which contains $W(X)$.*

Proof. The brick X lies in a regular component. Denote by $Y, \tau Y, \dots, \tau^{m-1}Y$ the quasi-simple modules in the wing $W(X)$. Then we can apply lemma 2.18 and obtain that $\dim \operatorname{Ext}^1(\tau^{m-1}Y, Y) = 1$. By the Auslander-Reiten-formula and orthogonality we conclude that

$$\operatorname{Ext}^1(\tau^i Y, \tau^j Y) \cong \begin{cases} k & \text{if } j = i+1 \\ k & \text{if } i = m-1 \text{ and } j = 0 \\ 0 & \text{else} \end{cases}$$

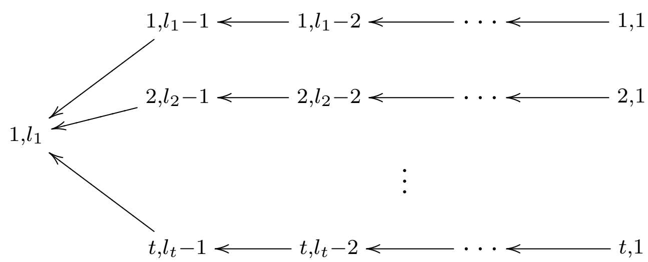
Hence we can apply theorem 2.13 and obtain that $\mathcal{F}(Y, \tau Y, \dots, \tau^{m-1}Y)$ is a pseudo-tube containing $W(X)$. □

3 Certain Star Quivers

3.1 g-Duality for Star Quivers

Let Q be an acyclic quiver. Let $K_0(\operatorname{rep}_k Q)$ be the Grothendieck group of $\operatorname{rep}_k Q$. Then there is an isomorphism $\mathbf{dim} : K_0(\operatorname{rep}_k Q) \longrightarrow \mathbb{Z}^{Q_0}$ which maps the simples on the canonical basis of $\mathbb{Z}^{Q_0}$ (see [ASS06], theorem III.3.5).

For $t > 2$ let $l_i > 1$ for $i = 1, \ldots, t$ and let $\mathbb{T}^{l_1,\ldots,l_t}$ be the star star quiver with t arms which are of length $l_1, \ldots, l_t$ with subspace orientation:



To the quiver we associate the quadratic form $q : \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}} \times \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}} \longrightarrow \mathbb{Z}$:

$$q\begin{pmatrix} & d_{1,l_1-1} & d_{1,l_1-2} & \ldots & d_{11} \\ n & d_{2,l_2-1} & d_{2,l_2-2} & \ldots & d_{21} \\ & & \vdots & & \\ & d_{l,l_t-1} & d_{l,l_t-2} & \ldots & d_{t1} \end{pmatrix}$$

$$= n^2 + \sum_{i=1}^{t}\sum_{j=1}^{l_i-1} d_{ij}^2 - \sum_{i=1}^{t}\left(nd_{i,l_i-1} + \sum_{j=1}^{l_i-2} d_{ij}d_{i,j+1}\right)$$

Definition 3.1. Let $g : \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}} \longrightarrow \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}}$ be the linear map mapping vectors as follows:

$$g\begin{pmatrix} & d_{1,l_1-1} & d_{1,l_1-2} & \ldots & d_{11} \\ n & d_{2,l_2-1} & d_{2,l_2-2} & \ldots & d_{21} \\ & & \vdots & & \\ & d_{l,l_t-1} & d_{l,l_t-2} & \ldots & d_{t1} \end{pmatrix} = \begin{pmatrix} & d'_{1,l_1-1} & d'_{1,l_1-2} & \ldots & d'_{11} \\ n & d'_{2,l_2-1} & d'_{2,l_2-2} & \ldots & d'_{21} \\ & & \vdots & & \\ & d'_{l,l_t-1} & d'_{l,l_t-2} & \ldots & d'_{t1} \end{pmatrix}$$

with $d'_{i,j} = n - d_{i,l_i-j}$. We will call g the *g-duality*, because it will help us to classify the preinjective representations when we have classified the preprojectives.

Lemma 3.2. *Let $d \in \mathbb{Z}^{\mathbb{T}_0^{l_1,\ldots,l_t}}$. Then $q(d) = q(g(d))$.*

Proof. Let

$$d = \begin{pmatrix} & d_{1,l_1-1} & d_{1,l_1-2} & \ldots & d_{11} \\ n & d_{2,l_2-1} & d_{2,l_2-2} & \ldots & d_{21} \\ & & \vdots & & \\ & d_{t,l_t-1} & d_{l,l_t-2} & \ldots & d_{t1} \end{pmatrix}$$

Then we can calculate the quadratic form:

$$q(g(d)) = n^2 + \sum_{i=1}^{t}\sum_{j=1}^{l_i-1}(n-d_{ij})^2 - \sum_{i=1}^{t}\left(n(n-d_{i1}) + \sum_{j=1}^{l_i-2}(n-d_{ij})(n-d_{i,j+1})\right)$$

$$= n^2 + \sum_{i=1}^{t}\sum_{j=1}^{l_i-1}(n^2 - 2nd_{ij} + d_{ij}^2)$$

$$- \sum_{i=1}^{t}\left(n^2 - nd_{i1} + \sum_{j=1}^{l_i-2}(n^2 - nd_{ij} - nd_{i,j+1} + d_{ij}d_{i,j+1})\right)$$

$$= n^2 + \sum_{i=1}^{t}\sum_{j=1}^{l_i-1}(-2nd_{ij} + d_{ij}^2) - \sum_{i=1}^{t}\left(-nd_{i1} + \sum_{j=1}^{l_i-2}(-nd_{ij} - nd_{i,j+1} + d_{ij}d_{i,j+1})\right)$$

$$= n^2 + \sum_{i=1}^{t}\sum_{j=1}^{l_i-1}d_{ij}^2 - \sum_{i=1}^{t}\left(nd_{i,l_i-1} + \sum_{j=1}^{l_i-2}d_{ij}d_{i,j+1}\right)$$

$$= q(d)$$

This proves the lemma. □

Definition 3.3. For the quiver $\mathbb{T}^{l_1,\dots,l_t}$ there is the *Coxeter transformation* $C : \mathbb{Z}^{\mathbb{T}_0^{l_1,\dots,l_t}} \longrightarrow \mathbb{Z}^{\mathbb{T}_0^{l_1,\dots,l_t}}$ as defined in [ASS06], p.271 which is in this special case the composition of maps

$$(s_{t,1} \circ \cdots \circ s_{t,l_t-1}) \circ \cdots \circ (s_{1,1} \circ \cdots \circ s_{1,l_1-1}) \circ s_{1,l_1}$$

where for any vertex a the map $s_a : \mathbb{Z}^{\mathbb{T}_0^{l_1,\dots,l_t}} \longrightarrow \mathbb{Z}^{\mathbb{T}_0^{l_1,\dots,l_t}}$ only changes the coordinate d_a of a vector at the vertex a by replacing it by

$$-d_a + \sum d_b$$

where the sum is taken over all vertices b adjacent to a.

Note that all these maps s_a are involutions and hence

$$C^{-1} = s_{1,l_1} \circ (s_{1,l_1-1} \circ \cdots \circ s_{1,1}) \circ \cdots \circ (s_{t,l_t-1} \circ \cdots \circ s_{t,1})$$

The Coxeter transformation will be used to calculate the Auslander-Reiten shift τ in $\operatorname{rep}_k \mathbb{T}^{l_1,\dots,l_t}$ with the help of lemmas VII.5.8(a) and VII.5.9(b) of [ASS06]:

Lemma 3.4. *Let M be an indecomposable representation of $\mathbb{T}^{l_1,\dots,l_t}$ such that $C(\mathbf{dim}\, M)$ has only non-negative coordinates. Then $\mathbf{dim}\, \tau M = C(\mathbf{dim}\, M)$.*

Lemma 3.5. *Let C be the Coxeter transformation of $\mathbb{T}^{l_1,\dots,l_t}$. Then*

$$gCg = C^{-1}.$$

Proof. It is easily checked that the inverse of g is g. So instead of proving the equation in the lemma we will prove that

$$Cg = gC^{-1}$$

holds. The dimension vectors of the indecomposable projectives of $\mathbb{T}^{l_1,\dots,l_t}$ form a basis of $\mathbb{Z}^{\mathbb{T}_0^{l_1,\dots,l_t}}$. Hence it suffices to show that this basis is mapped to the same vectors by Cg and gC^{-1}, respectively. We start with the dimension vector corresponding to the simple projective and compute

$$C^{-1}\begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & 1 & 0 & \dots & 0 & 0 \\ (t-1) & 1 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$gC^{-1}\begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & t-1 & t-1 & \dots & t-1 & t-2 \\ t-1 & t-1 & t-1 & \dots & t-1 & t-2 \\ & & \vdots & & & \\ & t-1 & t-1 & \dots & t-1 & t-2 \end{pmatrix}.$$

On the other hand:

$$g\begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ & & \vdots & & & \\ & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

and

$$Cg\begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & t-1 & t-1 & \dots & t-1 & t-2 \\ t-1 & t-1 & t-1 & \dots & t-1 & t-2 \\ & & \vdots & & & \\ & t-1 & t-1 & \dots & t-1 & t-2 \end{pmatrix}.$$

This proves that the first basis vector is mapped on the same vector by gC^{-1} and Cg.

Next we consider the dimension vector with i ones in the upper arm for $i = 1, \dots, l_1 - 2$.

$$C^{-1}\begin{pmatrix} & \overbrace{1\dots1}^{i} & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} & \overbrace{1\dots1}^{i+1} & 0 & \dots & 0 \\ t-1 & 1 & 0 & \dots & 0 \\ & & \vdots & & \\ & 1 & 0 & \dots & 0 \end{pmatrix}$$

and

$$gC^{-1}\begin{pmatrix} & \overbrace{1\dots1}^{i} & 0\dots0 \\ 1 & 0 & 0\dots0 \\ & \vdots & \\ & 0 & 0\dots0 \end{pmatrix} = \begin{pmatrix} & \overbrace{t-1\dots t-1}^{l_1-i-2} & t-2 & \dots & t-2 \\ t-1 & t-1 & \dots & t-1 & t-2 \\ & & & \vdots & \\ & t-1 & \dots & t-1 & t-2 \end{pmatrix}.$$

On the other hand:

$$g\begin{pmatrix} & \overbrace{1\dots1}^{i} & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} & \overbrace{1\dots1}^{l_1-i-1} & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ & & \vdots & & \\ & 1 & 1 & \dots & 1 \end{pmatrix}$$

and

$$Cg\begin{pmatrix} & \overbrace{1\dots1}^{i} & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} & \overbrace{t-1\dots t-1}^{l_1-i-2} & t-2 & \dots & t-2 \\ t-1 & t-1 & \dots & t-1 & t-2 \\ & & & \vdots & \\ & t-1 & \dots & t-1 & t-2 \end{pmatrix}.$$

This proves that these basis vectors are mapped on the same vector by gC^{-1} and Cg.

Now we calculate the image of the dimension vector of the projective corresponding to the last vertex in the upper arm.

$$C^{-1}\begin{pmatrix} & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ t-2 & 1 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$gC^{-1}\begin{pmatrix} & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & t-2 & t-2 & \dots & t-2 & t-2 \\ t-2 & t-2 & t-2 & \dots & t-2 & t-3 \\ & & \vdots & & & \\ & t-2 & t-2 & \dots & t-2 & t-3 \end{pmatrix}.$$

On the other hand:

$$g\begin{pmatrix} & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ & & \vdots & & & \\ & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

and

$$Cg\begin{pmatrix} & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} & t-2 & t-2 & \dots & t-2 & t-2 \\ t-2 & t-2 & t-2 & \dots & t-2 & t-3 \\ & & & \vdots & & \\ & t-2 & t-2 & \dots & t-2 & t-3 \end{pmatrix}.$$

This proves that all basis vectors of projectives corresponding to vertices in the upper arm are mapped on the same vector by gC^{-1} and Cg. For the projectives corresponding to vertices in the other arms of $\mathbb{T}^{l_1,\dots,l_t}$ the calculation is similar. Hence all projectives are mapped to the same objects by Cg and gC^{-1}. This proves our lemma. □

Proposition 3.6. *Assume that $\mathbb{T}^{l_1,\dots,l_t}$ is representation infinite. Then g induces a bijection between the dimension vectors of indecomposable preprojectives and the dimension vectors of indecomposable preinjectives which do not have coordinate 0 at the unique sink of $\mathbb{T}^{l_1,\dots,l_t}$.*

Proof. Let d be the dimension vector of an indecomposable preprojective representation. There is a natural number n such that we can apply the Coxeter transformation n times and obtain a dimension vector p of an indecomposable projective representation, that is $p = C^n d$. Without loss of generality we can assume that p is the dimension vector of an indecomposable projective corresponding to a vertex in the upper arm of the quiver. Hence there is an $i = 0, \dots, l_1 - 1$ with

$$p = \begin{pmatrix} & \overbrace{1\dots1}^{i} & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ & 0 & 0 & \dots & 0 \end{pmatrix}$$

and we thus have

$$g(p) = \begin{pmatrix} & \overbrace{1\dots1}^{l_1-i-1} & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ & & \vdots & & \\ & 1 & 1 & \dots & 1 \end{pmatrix}.$$

We claim that this is preinjective. It is easily checked that indeed $C^{-i}g(p)$ is the dimension vector of an indecomposable injective. With lemma 3.5 we can now conclude:

$$C^{-i}g(p) = C^{-i}gC^n(d) = C^{-i}C^{-n}g(d) = C^{-i-n}g(d).$$

Hence $g(d)$ is the dimension vector of an indecomposable preinjective.

Since the map g is its own inverse, it is clear that the restriction of g on dimension vectors of indecomposable preprojectives is an injective map. To see that it is surjective one needs to show that the dimension vectors d of indecomposable preinjectives which do not have coordinate 0 at the unique sink are mapped to indecomposable preprojectives. This calculation is similar to the one above. We can apply C^{-1} several times and get the image of a projective indecomposable. This will be on the same C-orbit as the image of the dimension vector d. This proves the proposition. □

Example 3.7. Let Q be the quiver $\mathbb{T}^{5,2,5}$. We can knit the preprojective component of its Auslander-Reiten quiver which can be found in the appendix. There the representations are denoted by their dimension vectors and this component continues infinitely to the right. Dually we can knit the preinjective component which is also in the appendix. Again we denote the representations by their dimension vectors and this component continues infinitely to the left. The dashed line separates the dimension vectors which lie in the image of the preprojective ones under the map g from those which do not lie in the image. On the right hand side of the dashed line there are those dimension vectors with 0 at the sink.

We can now directly read off that the projective dimension vector

$$p = \begin{pmatrix} & 1 & 1 & 0 & 0 \\ 1 & 0 & & & \\ & 0 & 0 & 0 & 0 \end{pmatrix}$$

is mapped by g to

$$g(p) = \begin{pmatrix} & 1 & 1 & 0 & 0 \\ 1 & 1 & & & \\ & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$Cg(p) = \begin{pmatrix} & 2 & 1 & 1 & 1 \\ 2 & 1 & & & \\ & 2 & 2 & 2 & 1 \end{pmatrix}.$$

We can also read off

$$C^{-1}(p) = \begin{pmatrix} & 1 & 1 & 1 & 0 \\ 2 & 1 & & & \\ & 1 & 0 & 0 & 0 \end{pmatrix}$$

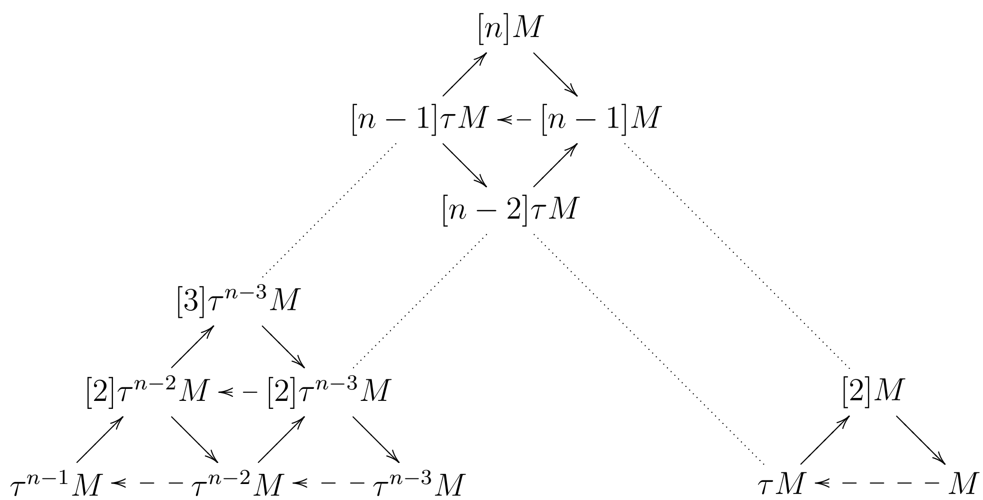
and then see that $gC^{-1}(p) = Cg(p)$.

3.2 A Criterion for Regularity

We will determine for certain dimension vectors of indecomposable representations of star quivers whether they are preprojective, regular or preinjective. For this will establish a criterion in this section. Let Q be a wild acyclic quiver.

Lemma 3.8. *Let C be a regular component of the Auslander-Reiten quiver of* $\mathrm{rep}_k\, Q$. *Then C contains a representation M with $q(\mathbf{dim}\, M) \leq 0$.*

Proof. Let M be quasi-simple in C. Then for any n the component C contains the following wing:



Observe that

$$\begin{aligned} q(\mathbf{dim}[n]M) &= q(\sum_{i=0}^{n-1} \mathbf{dim}\, \tau^i M) \\ &= \sum_{i,j=0}^{n-1} (\dim \mathrm{Hom}(\tau^i M, \tau^j M) - \dim \mathrm{Ext}^1(\tau^i M, \tau^j M)) \end{aligned}$$

holds, because of lemma 2.2 in [Rin76]. By the Auslander-Reiten formula (theorem IV.2.13 in [ASS06]) the dimensions of the Ext-spaces can be replaced by the dimensions of Hom-spaces as follows:

$$\begin{aligned} q(\mathbf{dim}[n]M) &= \sum_{i,j=0}^{n-1} (\dim \mathrm{Hom}(\tau^i M, \tau^j M) - \dim \mathrm{Hom}(\tau^j M, \tau^{i+1} M)) \\ &= \sum_{i,j=0}^{n-1} \dim \mathrm{Hom}(\tau^i M, \tau^j M) - \sum_{i,j=0}^{n-1} \dim \mathrm{Hom}(\tau^i M, \tau^{j+1} M) \\ &= \sum_{i=0}^{n-1} \dim \mathrm{Hom}(\tau^i M, M) - \sum_{i=0}^{n-1} \dim \mathrm{Hom}(\tau^i M, \tau^n M) \end{aligned}$$

By lemma 1.4 in [Ker92] which is due to Baer and Kerner the first sum tends to some finite constant for n approaching ∞, whereas the second sum tends to ∞. Hence $\lim_{n\to\infty} q(\mathbf{dim}[n]M) = -\infty$. This proves the assertion. □

Let Q, Q' be wild acyclic quivers and let $q_Q, q_{Q'}$ denote the corresponding quadratic forms.

Lemma 3.9. *Let $F : \operatorname{rep}_k Q \longrightarrow \operatorname{rep}_k Q'$ be a faithful and exact functor mapping indecomposables to indecomposables, such that*

$$q_Q(\mathbf{dim}\, M) = q_{Q'}(\mathbf{dim}\, F(M))$$

for all $M \in \operatorname{rep}_k Q$. Then for any regular indecomposable representation M of Q the representation $F(M)$ is regular.

Proof. Let M be a regular indecomposable representation of Q. By Kac's Theorem $q_Q(\mathbf{dim}\, M) \leq 1$ (see theorem in §1.10 of [Kac83] and lemma 2.1(b) of [Kac80]). If $q_Q(\mathbf{dim}\, M) \leq 0$, then $q_{Q'}(\mathbf{dim}\, F(M)) \leq 0$ and hence $F(M)$ is regular, because according to lemma VIII.2.7 in [ASS06] we have $q_{Q'}(\mathbf{dim}\, N) = 1$ for an indecomposable preprojective or preinjective N.

Now assume that $q_Q(\mathbf{dim}\, M) = 1$. First we treat the case that M is quasi-simple. By lemma 3.8 there is an integer $i > 1$ such that $q_Q(\mathbf{dim}[i]M) \leq 0$. Then we have a short exact sequence of regular representations:

$$0 \longrightarrow [i-1]\tau M \longrightarrow [i]M \longrightarrow M \longrightarrow 0$$

Applying F to this sequence we obtain a short exact sequence in $\operatorname{rep}_k Q'$:

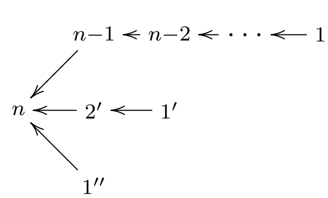
$$0 \longrightarrow F([i-1]\tau M) \longrightarrow F([i]M) \longrightarrow F(M) \longrightarrow 0$$

whose middle term is regular, because $q_{Q'}(\mathbf{dim}\, F([i]M)) \leq 0$. Since there is an epimorphism from a regular object to $F(M)$ and there are no morphisms from regular representations to preprojective representations by corollary VIII.2.6 of [ASS06], $F(M)$ cannot be preprojective. Dually it cannot be preinjective and has to be regular.

Now assume that $q_Q(\mathbf{dim}\, M) = 1$ and M is not quasi-simple. In this case there is a non-zero map from a regular quasi-simple to M. By the previous arguments this regular quasi-simple representation is mapped to a regular representation via F. Since F is faithful, there is is a non-zero map from a regular representation to $F(M)$. Dually there is a non-zero map from $F(M)$ to a regular representation. Hence $F(M)$ is regular. □

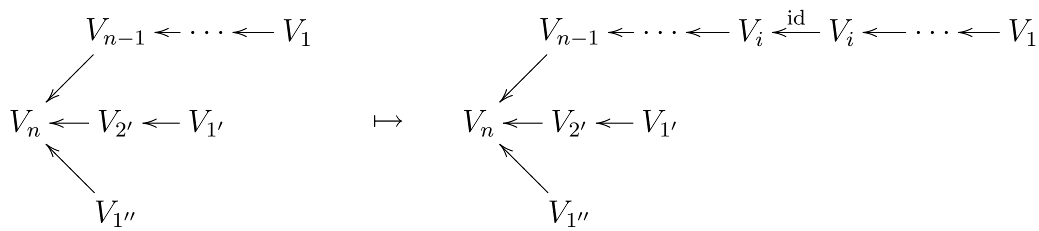
3.3 Star Quivers with Arms of Length n, 3 and 2

Let $n > 2$ and $\mathbb{T}^{n,3,2}$ be the star star quiver with three arms which are of length $n, 3, 2$ with subspace orientation. We will denote it by:



We are interested in this particular family, because it contains an extended Dynkin quiver, namely $\mathbb{T}^{6,3,2}$ which is of type $\widetilde{\mathbb{E}}_8$. We know that the category $\operatorname{rep}_k \mathbb{T}^{6,3,2}$ admits tubes and we will see that their images under certain embeddings into $\operatorname{rep}_k \mathbb{T}^{n,3,2}$ for $n > 6$ will result in pseudo-tubes.

Definition 3.10. For $i = 1, \dots n$ let $F_i : \operatorname{rep}_k \mathbb{T}^{n,3,2} \longrightarrow \operatorname{rep}_k \mathbb{T}^{n+1,3,2}$ be the functor defined on objects by



A morphism $(f_a)_{a \in \mathbb{T}_0^{n,3,2}}$ is mapped in the same way, namely the map f_i appears twice, whereas the other maps stay the same.

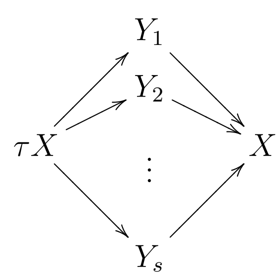
Let $F_0 : \operatorname{rep}_k \mathbb{T}_{n,3,2} \longrightarrow \operatorname{rep}_k \mathbb{T}_{n+1,3,2}$ be the functor adding the zero vector space at the end of the longest arm and keeping the other vector spaces fixed. On morphisms the functor F_0 adds the zero map at the new vertex.

It is straight forward to check that the functors F_i fulfill the conditions of lemma 3.9 and thus map regular representations to regular representations.

To check that a representation is preprojective the next proposition will be an important tool.

Proposition 3.11. *For $n > 5$ let M be a preprojective representation of $\mathbb{T}^{n,3,2}$. Then $\dim M_n \leq \dim(\tau^- M)_n$.*

Proof. The proof is inspired by [BB83]. Let f be the function mapping a representation to the dimension of the vector space at the unique sink, that is $f(M) = \dim M_n$. This function is obviously additive on exact sequences. Hence for a mesh



we have that

$$f(X) + f(\tau X) = \sum_{i=1}^{s} f(Y_i).$$

Fix $t \geq 0$ an integer. We can now define a dimension vector $d[t]$, which has at a vertex a the coordinate $f(\tau^{-t}P(a))$, where $P(a)$ is the projective cover of the simple representation corresponding to a.

Direct computation as in section 4 of [BB83] shows that $d[t+1] = C(d[t])$, where C is the Coxeter transformation.

Knitting the preprojective component of the Auslander-Reiten quiver of the category $\operatorname{rep}_k \mathbb{T}^{n,3,2}$ directly yields that

$$d[0] = \begin{pmatrix} \begin{smallmatrix} & 1...1 \\ 111 & \\ 1 & \end{smallmatrix} \end{pmatrix}.$$

Let

$$r = \begin{pmatrix} \begin{smallmatrix} & 1...10 \\ 110 & \\ 0 & \end{smallmatrix} \end{pmatrix}.$$

Then we also have $d[1] = d[0] + r$.

By induction and linearity of C it follows that

$$d[t+1] = d[t] + C^t(r)$$

and we are left to show that $C^t(r)$ never has negative entries. This follows from the fact that there is a regular representation of $\mathbb{T}^{6,3,2}$ with dimension vector

$$\begin{pmatrix} \begin{smallmatrix} & 11110 \\ 110 & \\ 0 & \end{smallmatrix} \end{pmatrix}.$$

This is mapped to a regular representation with dimension vector r by the functor $(F_2)^{n-6}$ by lemma 3.9. Hence the Coxeter-orbit of r contains only non-negative vectors. This proves the proposition. □

Definition 3.12. For Q a star let $\mathcal{P}(Q), \mathcal{R}(Q), \mathcal{I}(Q)$ denote the preprojective, regular and preinjective indecomposable objects in $\operatorname{rep}_k Q$, respectively. For $d \in \mathbb{N}_0$ let $\mathcal{P}^d(Q), \mathcal{R}^d(Q), \mathcal{I}^d(Q)$ be the corresponding indecomposable objects with a d-dimensional vector space at the unique sink.

Lemma 3.13. *With the notation above, we obtain:*

- *For $n > 5$ we have*

$$\bigcup_{i=0}^{n} F_i(\mathcal{R}^1(\mathbb{T}^{n,3,2})) = \mathcal{R}^1(\mathbb{T}^{n+1,3,2}).$$

- *For $n > 6$ we have*

$$\bigcup_{i=0}^{n} F_i(\mathcal{R}^2(\mathbb{T}^{n,3,2})) = \mathcal{R}^2(\mathbb{T}^{n+1,3,2}).$$

- *For $n > 7$ we have*

$$\bigcup_{i=0}^{n} F_i(\mathcal{R}^3(\mathbb{T}^{n,3,2})) = \mathcal{R}^3(\mathbb{T}^{n+1,3,2}).$$

Proof. By lemma 3.9 we obtain that for $d = 1, 2, 3$ we have $F_i(\mathcal{R}^d(\mathbb{T}^{n,3,2})) \subseteq \mathcal{R}^d(\mathbb{T}^{n+1,3,2})$. This implies that $\bigcup_{i=0}^{n} F_i(\mathcal{R}^d(\mathbb{T}^{n,3,2})) \subseteq \mathcal{R}^d(\mathbb{T}^{n+1,3,2})$. Note that to prove this inclusion we did not need any assumption on n. To prove the inclusion of the opposite direction we need to classify the preinjective and preprojective indecomposables in the first step.

For $n > 5$ the dimension vectors of representations in $\mathcal{P}^1$ can be found in the appendix. They are always at the beginning of a τ-orbit, because the function mapping a representation to the dimension of the vector space at the unique sink is increasing in the direction of τ^- by proposition 3.11. In the appendix one sees that there are exactly $n + 10$ different dimension vectors of representations of $\mathcal{P}^1$. By Kac's Theorem (theorem in §1.10 of [Kac83]) there are precisely $n + 10$ isomorphism classes in $\mathcal{P}^1$. Using the g-dual we get that there are precisely $n+10$ isomorphism classes in $\mathcal{I}^1$

It holds that

$$\begin{aligned}
\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) &= \mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \operatorname{rep}_k \mathbb{T}^{n+1,3,2} \\
&= \mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \bigcup_{i=0}^{n} F_i(\operatorname{rep}_k \mathbb{T}^{n,3,2}) \\
&= \mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^1(\mathbb{T}^{n,3,2})) \cup \bigcup_{i=0}^{n} F_i(\mathcal{R}^1(\mathbb{T}^{n,3,2})) \cup \bigcup_{i=0}^{n} F_i(\mathcal{I}^1(\mathbb{T}^{n,3,2})) \right) \\
&= \left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^1(\mathbb{T}^{n,3,2})) \right) \right) \\
&\quad \cup \left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{R}^1(\mathbb{T}^{n,3,2})) \right) \right) \\
&\quad \cup \left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{I}^1(\mathbb{T}^{n,3,2})) \right) \right).
\end{aligned}$$

Hence it suffices to show that

$$\left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^1(\mathbb{T}^{n,3,2})) \right) \right),$$

$$\left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{R}^1(\mathbb{T}^{n,3,2})) \right) \right)$$

and

$$\left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{I}^1(\mathbb{T}^{n,3,2}))\right)\right)$$

are contained in $\bigcup_{i=0}^{n} F_i(\mathcal{R}^1(\mathbb{T}^{n,3,2}))$. For the second class this is clear. For the first class this can be checked using the list in the appendix. Each representation is mapped to exactly two different representations by the functors F_i for $i = 0, \ldots, n$. For example an indecomposable representation M with dimension vector

$$\left(\begin{array}{l} \;11000 \\ 100 \\ \;0 \end{array}\right)$$

is mapped to objects with dimension vector

$$\mathbf{dim}(F_0(M)) = \mathbf{dim}(F_1(M)) = \mathbf{dim}(F_2(M)) = \mathbf{dim}(F_3(M)) = \left(\begin{array}{l} \;110000 \\ 100 \\ \;0 \end{array}\right)$$

and

$$\mathbf{dim}(F_4(M)) = \mathbf{dim}(F_5(M)) = \mathbf{dim}(F_6(M)) = \left(\begin{array}{l} \;111000 \\ 100 \\ \;0 \end{array}\right).$$

In each of these cases the images are preprojective. Calculating all images of objects in $\mathcal{P}^1$ under F_i yields that there are exactly four isomorphism classes in

$$\left(\mathcal{R}^1(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^1(\mathbb{T}^{n,3,2}))\right)\right).$$

They have dimension vectors:

$$\left(\begin{array}{l} \;10...0 \\ 110 \\ \;1 \end{array}\right), \left(\begin{array}{l} \;110...0 \\ 111 \\ \;0 \end{array}\right), \left(\begin{array}{l} \;1110...0 \\ 100 \\ \;1 \end{array}\right), \left(\begin{array}{l} \;11110...0 \\ 110 \\ \;0 \end{array}\right).$$

All of these four isomorphism classes are also in the class $F_0(\mathcal{R}^1(\mathbb{T}^{n,3,2}))$. This is clear, because according to lemma XIII.2.22 in [SS07] there are quasi-simple regular representation of $\mathbb{T}^{6,3,2}$ with dimension vectors:

$$\left(\begin{array}{l} \;10000 \\ 110 \\ \;1 \end{array}\right), \left(\begin{array}{l} \;11000 \\ 111 \\ \;0 \end{array}\right), \left(\begin{array}{l} \;11100 \\ 100 \\ \;1 \end{array}\right), \left(\begin{array}{l} \;11110 \\ 110 \\ \;0 \end{array}\right).$$

Using the g-dual we get the same result for the preinjectives, so the first assertion of the lemma is proven.

For $n > 6$ the dimension vectors of representations in $\mathcal{P}^2$ can be found in the appendix. They are always proper successors of objects in $\mathcal{P}^1$, because the function mapping a representation to the dimension of the vector space at the unique sink is increasing in the direction of τ^- by proposition 3.11. In the appendix one sees that there are exactly $n+10$ different dimension vectors of representations of $\mathcal{P}^2$. By Kac's Theorem (theorem in §1.10 of [Kac83]) there are precisely $n+10$ isomorphism classes in $\mathcal{P}^2$. Using the g-dual we get that there are precisely $n+10$ isomorphism classes in $\mathcal{I}^2$

Analogously to the first case, it suffices to show that

$$\left(\mathcal{R}^2(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^2(\mathbb{T}^{n,3,2}))\right)\right),$$

$$\left(\mathcal{R}^2(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{R}^2(\mathbb{T}^{n,3,2}))\right)\right)$$

and

$$\left(\mathcal{R}^2(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{I}^2(\mathbb{T}^{n,3,2}))\right)\right)$$

are contained in $\bigcup_{i=0}^{n} F_i(\mathcal{R}^2(\mathbb{T}^{n,3,2}))$. For the second class this is clear. For the first class this can be checked using the list in the appendix. Each representation is mapped to exactly three different representations by the functors F_i for $i = 0, \dots, n$. Calculating all images of objects in $\mathcal{P}^2$ under F_i yields that there are exactly fourteen isomorphism classes in

$$\left(\mathcal{R}^2(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^2(\mathbb{T}^{n,3,2}))\right)\right).$$

They have dimension vectors:

$$\begin{pmatrix} 21110\dots0 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 221110\dots0 \\ 210 \\ 1 \end{pmatrix}, \begin{pmatrix} 21\dots10 \\ 210 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 21\dots1 \\ 210 \\ 1 \end{pmatrix}, \begin{pmatrix} 110\dots0 \\ 221 \\ 1 \end{pmatrix}, \begin{pmatrix} 2110\dots0 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 22110\dots0 \\ 210 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 210\dots0 \\ 221 \\ 1 \end{pmatrix}, \begin{pmatrix} 2210\dots0 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 22210\dots0 \\ 210 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 111110\dots0 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 2111110\dots0 \\ 210 \\ 1 \end{pmatrix}, \begin{pmatrix} 211110\dots0 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 2211110\dots0 \\ 210 \\ 1 \end{pmatrix}.$$

All of these fourteen isomorphism classes are also in the class $\bigcup_{i=0}^{n} F_i(\mathcal{R}^2(\mathbb{T}^{n,3,2}))$, because for $\mathbb{T}^{7,3,2}$ there are regular objects with dimension vectors:

$$\begin{pmatrix} 211000 \\ 211 \\ 1 \end{pmatrix}, \begin{pmatrix} 221100 \\ 210 \\ 1 \end{pmatrix}, \begin{pmatrix} 211111 \\ 210 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;211111 \\ 210 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;110000 \\ 221 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;211000 \\ 211 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;221100 \\ 210 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;210000 \\ 221 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;221000 \\ 211 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;222100 \\ 210 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;111110 \\ 211 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;211111 \\ 210 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;211110 \\ 211 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;221111 \\ 210 \\ \;1 \end{pmatrix}.$$

To see that these are regular, it suffices to check that the function mapping each representation to the dimension at the unique sink is neither increasing nor decreasing along the τ-orbits of these representations. Hence, the first class is contained in $\bigcup_{i=0}^{n} F_i(\mathcal{R}^2(\mathbb{T}^{n,3,2}))$. Using the g-dual we get the same result for the preinjectives, so the second assertion of the lemma is proven.

For $n > 7$ the dimension vectors of representations in $\mathcal{P}^3$ can be found in the appendix. They are always proper successors of objects in $\mathcal{P}^2$, because the function mapping a representation to the dimension of the vector space at the unique sink is increasing in the direction of τ^- by proposition 3.11. In the appendix one sees that there are exactly $n+5$ different dimension vectors of representations of $\mathcal{P}^3$. By Kac's Theorem (theorem in §1.10 of [Kac83]) there are precisely $n+5$ isomorphism classes in $\mathcal{P}^3$. Using the g-dual we get that there are precisely $n+5$ isomorphism classes in $\mathcal{I}^3$

Analogously to the first case, it suffices to show that

$$\left(\mathcal{R}^3(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^3(\mathbb{T}^{n,3,2}))\right)\right),$$

$$\left(\mathcal{R}^3(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{R}^3(\mathbb{T}^{n,3,2}))\right)\right)$$

and

$$\left(\mathcal{R}^3(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{I}^3(\mathbb{T}^{n,3,2}))\right)\right)$$

are contained in $\bigcup_{i=0}^{n} F_i(\mathcal{R}^3(\mathbb{T}^{n,3,2}))$. For the second class this is clear. For the first class this can be checked using the list in the appendix. Each representation is mapped to exactly four different representations by the functors F_i for $i = 0, \ldots, n$. Calculating all images of objects in $\mathcal{P}^2$ under F_i yields that there are exactly $2n+5$ isomorphism classes in

$$\left(\mathcal{R}^3(\mathbb{T}^{n+1,3,2}) \cap \left(\bigcup_{i=0}^{n} F_i(\mathcal{P}^3(\mathbb{T}^{n,3,2}))\right)\right).$$

They have dimension vectors:

$$\begin{pmatrix} \;\;2110...0 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;32110...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;222110...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;22111110...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2221110...0 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;22111110...0 \\ 211 \\ \;1 \end{pmatrix}, \ldots, \begin{pmatrix} \;\;221...10 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;221...1 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;2210...0 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;32210...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;222210...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;22211110...0 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;322110...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3221110...0 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;321110...0 \\ 321 \\ \;1 \end{pmatrix}, \ldots, \begin{pmatrix} \;\;321...100 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;321...10 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;321...1 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3210...0 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;33210...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;322210...0 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;32211110...0 \\ 321 \\ \;1 \end{pmatrix}.$$

All of these $2n+5$ isomorphism classes are also in the class $\bigcup_{i=0}^{n} F_i(\mathcal{R}^3(\mathbb{T}^{n,3,2}))$, because for $\mathbb{T}^{8,3,2}$ there are regular objects with dimension vectors:

$$\begin{pmatrix} \;\;2110000 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;3211000 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2221100 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2211111 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2221110 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;2211111 \\ 211 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;2210000 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;3221000 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2222100 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;2221111 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;3221100 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3221110 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;3211100 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3211111 \\ 321 \\ \;1 \end{pmatrix},$$

$$\begin{pmatrix} \;\;3211111 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3210000 \\ 321 \\ \;2 \end{pmatrix}, \begin{pmatrix} \;\;3321000 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3221000 \\ 321 \\ \;1 \end{pmatrix}, \begin{pmatrix} \;\;3221111 \\ 321 \\ \;1 \end{pmatrix}.$$

To see that these are regular, it suffices to check that the function mapping each representation to the dimension at the unique sink is neither increasing nor decreasing along the τ-orbits of these representations. Hence, the first class is contained in $\bigcup_{i=0}^{n} F_i(\mathcal{R}^3(\mathbb{T}^{n,3,2}))$. Using the g-dual we get the same result for the preinjectives, so the third assertion of the lemma is proven. $\square$

With the lemma we can find new examples of pseudo-tubes, because if we have found a pseudo-tube in $\operatorname{rep}_k \mathbb{T}^{n,3,2}$ for $n > 5$ their images under any of the F_i are again pseudo-tubes and for the tame category $\operatorname{rep}_k \mathbb{T}^{6,3,2}$ we know all standard stable pseudo-tubes by using the classification of lemma XII.2.22 of [SS07] and proposition 2.16.

The bounds in the lemma cannot be improved. This is clear for the first assertion, because for $n = 5$ the left hand side of the equality is empty, whereas the right hand side is not.

To see that the second assertion does not hold for $n = 6$ observe that there is an indecomposable regular representation M of $\mathbb{T}^{7,3,2}$ with dimension vector:

$$\begin{pmatrix} \;\;221100 \\ 210 \\ \;1 \end{pmatrix}$$

and there are indecomposable preprojective representations M_1, M_2 and M_3 of $\mathbb{T}^{6,3,2}$ with:

$$\mathbf{dim}\, M_1 = \begin{pmatrix} \;\;21100 \\ 210 \\ \;1 \end{pmatrix}$$

$$\mathbf{dim}\, M_2 = \begin{pmatrix} \;\;22100 \\ 210 \\ \;1 \end{pmatrix}$$

$$\mathbf{dim}\, M_3 = \begin{pmatrix} \;\;22110 \\ 210 \\ \;1 \end{pmatrix}.$$

Obviously, we have that

$$F_0(M_3) = F_1(M_3) = F_3(M_2) = F_5(M_1) = F_6(M_1) = M$$

and M does not have any other preimages in $\operatorname{rep}_k \mathbb{T}^{6,3,2}$ under the functors F_i.

To see that the third assertion does not hold for $n = 7$ observe that there is an indecomposable regular representation M of $\mathbb{T}^{8,3,2}$ with dimension vector:

$$\begin{pmatrix} \;\;3211000 \\ 321 \\ \;1 \end{pmatrix}$$

and there are indecomposable preprojective representations M_1, M_2 and M_3 of $\mathbb{T}^{7,3,2}$ with:

$$\mathbf{dim}\, M_1 = \begin{pmatrix} \;\;321100 \\ 321 \\ \;1 \end{pmatrix}$$

$$\mathbf{dim}\, M_2 = \begin{pmatrix} \;\;321000 \\ 321 \\ \;1 \end{pmatrix}$$

$$\mathbf{dim}\, M_3 = \left(\begin{array}{l} \;\;211000 \\ 321 \\ \;1 \end{array} \right).$$

Obviously, we have that

$$F_0(M_1) = F_1(M_1) = F_2(M_1) = F_4(M_2) = F_7(M_3) = M$$

and M does not have any other preimages in $\mathrm{rep}_k\, \mathbb{T}^{6,3,2}$ under the functors F_i.

The fact that for $n = 6$ the second assertion does not hold is of interest, because in the category $\mathrm{rep}_k\, \mathbb{T}^{7,3,2}$ there is a standard stable pseudo-tube which helps us to answer two questions posed by Kerner in [Ker92] on exceptional components. For this we first will repeat some definitions from [Ker92].

Let A be a finite-dimensional wild hereditary algebra. Let C be a regular component of the Auslander-Reiten quiver Γ of C containing a brick without self-extensions. According to lemma 2.17 there is a quasi-simple module X in C which is a brick without self-extensions.

Definition 3.14. Let $l > 1$ be the smallest number such that $\mathrm{Ext}^1(X(l), X(l)) \neq 0$. The component C is called *exceptional* if there is a number $m > l$ such that $\mathrm{Hom}(X, \tau^m X) = 0$. In this case define

$$s := \min\{m \geq l \,|\, \mathrm{Hom}(X, \tau^m X) \neq 0, \mathrm{Hom}(X, \tau^{m+1} X) = 0\}.$$

Kerner asks whether $s = l$ always holds. This is not the case as we will see now.

Lemma 3.15. *The Auslander-Reiten quiver of the path algebra of $\mathbb{T}^{7,3,2}$ has an exceptional component with $s = 7$ and $l = 4$.*

Proof. Let X be the indecomposable representation of $\mathbb{T}^{7,3,2}$ with dimension vector

$$\mathbf{dim}\, X = \left(\begin{array}{l} \;\;221000 \\ 321 \\ \;2 \end{array} \right).$$

This exists and is uniquely determined by Kac's Theorem (see [Kac83]). Then X lies on a τ-orbit with representations with dimension vectors:

$$\mathbf{dim}\, \tau X = \left(\begin{array}{l} \;\;321111 \\ 321 \\ \;1 \end{array} \right), \quad \mathbf{dim}\, \tau^2 X = \left(\begin{array}{l} \;\;211110 \\ 321 \\ \;2 \end{array} \right),$$

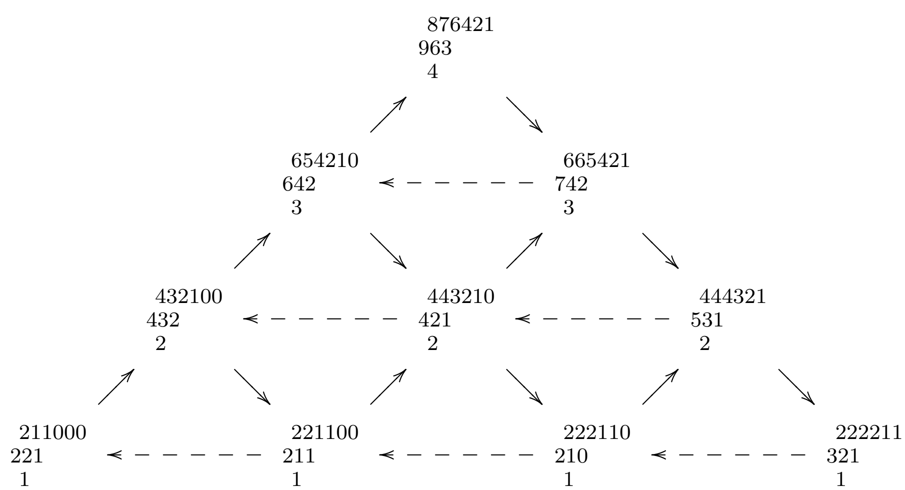
$$\mathbf{dim}\, \tau^3 X = \left(\begin{array}{l} \;\;222211 \\ 321 \\ \;1 \end{array} \right), \quad \mathbf{dim}\, \tau^4 X = \left(\begin{array}{l} \;\;222110 \\ 210 \\ \;1 \end{array} \right),$$

$$\mathbf{dim}\, \tau^5 X = \left(\begin{array}{l} \;\;221100 \\ 211 \\ \;1 \end{array} \right), \quad \mathbf{dim}\, \tau^6 X = \left(\begin{array}{l} \;\;211000 \\ 221 \\ \;1 \end{array} \right),$$

$$\mathbf{dim}\,\tau^7 X = \begin{pmatrix} \;221111 \\ 321 \\ 2 \end{pmatrix}.$$

Since the function mapping a representation to the dimension of the vector space at the unique sink is neither increasing nor decreasing along this τ-orbit these modules have to be regular by proposition 3.11 and its dual. They are even quasi-simple, because it is impossible to write $\tau^6 X$ as middle term of an Auslander-Reiten sequence. This can be checked by proving that its dimension vector is not the sum of two dimension vectors where one is the Coxeter transform of the other.

Now we will show that this component is exceptional. The component containing X contains the wing $W([4]\tau^3 X)$:



Direct calculation shows:

$$q(\mathbf{dim}[3]\tau^4 X) = q\begin{pmatrix} \;654210 \\ 642 \\ 3 \end{pmatrix} = 1.$$

Hence $[3]\tau^4 X$ is uniquely determined by its dimension vector by Kac's Theorem. There is an indecomposable representation M of $\mathbb{T}^{5,3,2}$ with dimension vector

$$\mathbf{dim}\,M = \begin{pmatrix} \;5421 \\ 642 \\ 3 \end{pmatrix}$$

which is a brick without self-extensions, because $\mathbb{T}^{5,3,2}$ is representation finite. Thus $[3]\tau^4 X = F_0(F_5(M))$ is a brick without self-extensions, too. Then by lemma 2.17 the above wing is standard and $[4]\tau^3 X$ is a brick. Hence by lemma 2.2 in [Rin76] we have:

$$0 = q(\mathbf{dim}[4]\tau^3 X) = \dim \mathrm{Hom}([4]\tau^3 X, [4]\tau^3 X) - \dim \mathrm{Ext}^1([4]\tau^3 X, [4]\tau^3 X)$$

$$= 1 - \dim \operatorname{Ext}^1([4]\tau^3 X, [4]\tau^3 X).$$

Thus $[4]\tau^3 X$ admits self-extensions and the invariant $l = 4$ for the component.

To check that the component is exceptional and to calculate s we need to calculate $\operatorname{Hom}(X, \tau^i X)$ for $i = 1, \dots, 8$. By lemma 2.17 we have that

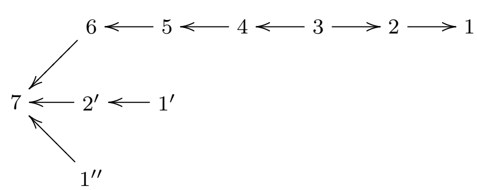
$$\operatorname{Hom}(X, \tau^i X) = 0$$

for $i = 1, \dots, 3$. To see that $\operatorname{Hom}(X, \tau^i X) \neq 0$ for $i = 4, \dots, 7$ we can use the fact that the support of X is contained in $\mathbb{T}^{5,3,2}$ which is representation finite. For $i = 4, \dots, 7$ we have a nonzero morphism from the representation X restricted to $\mathbb{T}^{5,3,2}$ to the representation $\tau^i X$ restricted to $\mathbb{T}^{5,3,2}$. This morphism induces a nonzero morphism in $\operatorname{rep}_k \mathbb{T}^{7,3,2}$. Now we are left to show that $\operatorname{Hom}(X, \tau^8 X) = 0$ which provides that the component is exceptional and $s = 7$. We use a similar trick as in the four cases before, but apply it to $\tau^- X$ instead of X. We use that the support of $\tau^- X$ is $\mathbb{T}^{5,3,2}$ to calculate $\operatorname{Hom}(\tau^- X, \tau^7 X) = 0$. Then $\operatorname{Hom}(X, \tau^8 X) = 0$ follows from corollary IV.2.15 in [ASS06]. This finishes the proof. □

To pose and answer the second question in [Ker92] we need to adjust the example in the proof to fit with Kerner's notation for the following reason: With the notation of the proof Kerner proves that there is an exact sequence

$$0 \longrightarrow X \longrightarrow \tau^s X \longrightarrow I \longrightarrow 0$$

with I indecomposable injective. This I corresponds to a vertex which is 3 in our case. Kerner assumes that this vertex is a source. To do this he applies Bernstein-Gelfand-Ponomarev reflections. In the case of the component in our proof this changes the quiver to:



The representation X does not change and its support is $\mathbb{T}^{4,3,2}$. This quiver is representation finite. Kerner asks on page 201 in [Ker92], whether this is possible. With the example above we now have shown:

Lemma 3.16. *The Auslander-Reiten quiver of the path algebra of* $\mathbb{T}^{7,3,2}$ *has an exceptional component which contains a quasi-simple module whose support is representation finite.*

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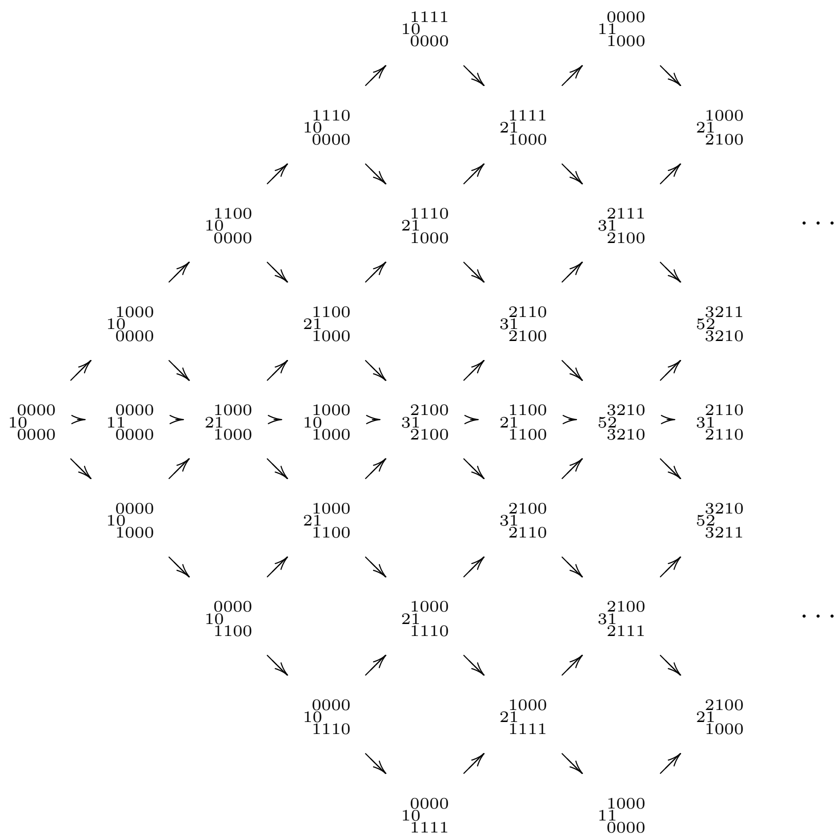
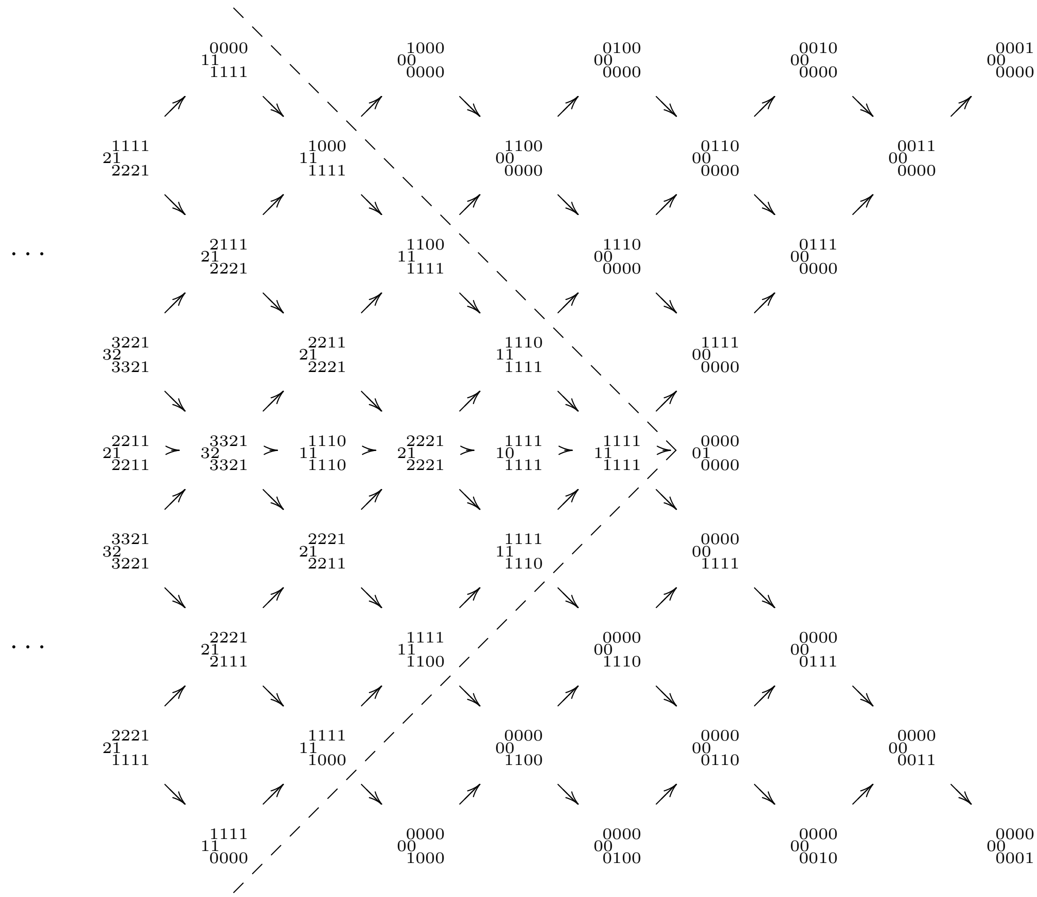
4 Appendix

The preprojective component of the Auslander-Reiten quiver of $\mathbb{T}^{5,2,5}$:



The preinjective component of the Auslander-Reiten quiver of $\mathbb{T}^{5,2,5}$:



The preprojective τ-orbits of $\mathbb{T}^{6,3,2}$ start with the following dimension vectors:

- $\begin{pmatrix} 00000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 10000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21000 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32100 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 00000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 10000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 11000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 21100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22110 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32211 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 00000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 10000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 21000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32210 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 00000 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 10000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 11000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 11100 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 21110 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22111 \\ 321 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 21100 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 32110 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 33211 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 10000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 11000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32110 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 11000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 11100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21110 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32111 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 11100 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 11110 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21111 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 21000 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 32100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 33210 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 11110 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 11111 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 10000 \\ 221 \\ 1 \end{pmatrix} \begin{pmatrix} 21000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 22100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22210 \\ 321 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 32221 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 11111 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 00000 \\ 110 \\ 1 \end{pmatrix} \begin{pmatrix} 10000 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 11000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 11100 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 11110 \\ 211 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 21111 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 11000 \\ 221 \\ 1 \end{pmatrix} \begin{pmatrix} 21100 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 22110 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22211 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 21110 \\ 321 \\ 2 \end{pmatrix}$
$\begin{pmatrix} 32111 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 22100 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 32210 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 33221 \\ 421 \\ 2 \end{pmatrix} \dots$

The preprojective τ-orbits of $\mathbb{T}^{7,3,2}$ start with:

- $\begin{pmatrix} 000000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 100000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 210000 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 321000 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 000000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 100000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 110000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 211000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 221100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 322110 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 000000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 100000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 210000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 221000 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 322100 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 000000 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 100000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 110000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 111000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 211100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 221110 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 322111 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 100000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 110000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 211000 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 321100 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 110000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 111000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 211100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 321110 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 111000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 111100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 211110 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 321111 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 111100 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 111110 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 211111 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 210000 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 321000 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 332100 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 111110 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 111111 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 100000 \\ 221 \\ 1 \end{pmatrix} \begin{pmatrix} 210000 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 221000 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 222100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 322210 \\ 421 \\ 2 \end{pmatrix} \dots$

- $\begin{pmatrix} 111111 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 000000 \\ 110 \\ 1 \end{pmatrix} \begin{pmatrix} 100000 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 110000 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 111000 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 111100 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 211110 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 221111 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 211000 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 321100 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 332110 \\ 421 \\ 2 \end{pmatrix} \dots$

The preprojective τ-orbits of $\mathbb{T}^{n,3,2}$ for $n > 7$ start with:

- $\begin{pmatrix} 0\ldots0 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 210\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 3210\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 0\ldots0 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 110\ldots0 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 2110\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22110\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 322110\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 0\ldots0 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 210\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 2210\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32210\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 0\ldots0 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 110\ldots0 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 1110\ldots0 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 21110\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 221110\ldots0 \\ 321 \\ 1 \end{pmatrix}$
  $\begin{pmatrix} 3221110\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 10\ldots0 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 110\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 2110\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32110\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$

$\vdots$

- $\begin{pmatrix} 1\ldots1000 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 1\ldots100 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21\ldots10 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 321\ldots1 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 1\ldots100 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 1\ldots10 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 21\ldots1 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 210\ldots0 \\ 321 \\ 2 \end{pmatrix} \begin{pmatrix} 3210\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 33210\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 1\ldots10 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 1\ldots1 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 221 \\ 1 \end{pmatrix} \begin{pmatrix} 210\ldots0 \\ 211 \\ 1 \end{pmatrix} \begin{pmatrix} 2210\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 22210\ldots0 \\ 321 \\ 1 \end{pmatrix}$
  $\begin{pmatrix} 322210\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$
- $\begin{pmatrix} 111111 \\ 100 \\ 0 \end{pmatrix} \begin{pmatrix} 0\ldots0 \\ 110 \\ 1 \end{pmatrix} \begin{pmatrix} 10\ldots0 \\ 111 \\ 0 \end{pmatrix} \begin{pmatrix} 110\ldots0 \\ 100 \\ 1 \end{pmatrix} \begin{pmatrix} 1110\ldots0 \\ 110 \\ 0 \end{pmatrix} \begin{pmatrix} 11110\ldots0 \\ 211 \\ 1 \end{pmatrix}$
  $\begin{pmatrix} 211110\ldots0 \\ 210 \\ 1 \end{pmatrix} \begin{pmatrix} 2211110\ldots0 \\ 321 \\ 1 \end{pmatrix} \begin{pmatrix} 32211110\ldots0 \\ 421 \\ 2 \end{pmatrix} \ldots$