Harnack Inequalities and Applications for Stochastic Equations

Shun-Xiang Ouyang

(欧阳顺湘)

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Shun-Xiang Ouyang
(欧阳顺湘)

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Preface

In this thesis we mainly study Harnack inequalities (in the sense of Wang [Wan97]) and their applications to transition semigroups associated with stochastic equations.

Among the stochastic equations we aim at, are finite dimensional stochastic ordinary differential equations with irregular drifts (Chapter 4), infinite dimensional (semi-) linear stochastic partial differential equations with Gaussian or Lévy noise (Chapters 5, 6 and 7); multivalued stochastic differential equations in finite dimension and multivalued stochastic evolution equations in Banach spaces (Chapter 8). The applications of Harnack inequalities include the study of the regularizing property (for instance, the strong Feller property), heat kernel estimates, hyperboundedness etc. of the transition semigroups associated with the stochastic equations.

The main method we used to establish Harnack inequalities is applying Hölder's inequality after a measure transformation. There are two aspects: transformation of measures on state spaces and measures on sample probability spaces of the processes. The method of measure transformation on the probability spaces is due to Arnaudon, Thalmaier and Wang[ATW06] in which they used a coupling argument and a Girsanov transformation to study Harnack inequalities.

Two crucial ingredients of the method of Arnaudon et al. are the absolute continuity and successful coupling of processes. To apply their method to establish Harnack inequalities for Ornstein-Uhlenbeck processes with Lévy noise, we investigate the absolute continuity of Lévy processes in infinite dimension in Chapter 2; to establish Harnack inequalities for stochastic differential equations with general drift, we study the gluing of martingale solutions and its applications to the coupling of stochastic differential equations in Chapter 3.

As a complement to Harnack inequalities, we study entropy cost and HWI inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes in Chapter 9.

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I registered as a PhD student in Beijing Normal University (BNU) from September of 2004, the eighth year of my stay in BNU, but with a break of one year when I taught at Zhuhai Campus of BNU. After my return I got the chance to come to Bielefeld. I am particularly grateful for the opportunity offered by Professor Michael Röckner and the encouragement of Professor Feng-Yu Wang. I appreciate very much the three years' financial support from the DFG through the International Graduate College (IGK) in Bielefeld University since March of 2006. It is hard to imagine how could I have sat down to write down this thesis without it. I am also grateful for the support and understanding of my parents for my study abroad. I really owe a lot to them.

I would like to thank Professors Philippe Blanchard, Friedrich Götze, Yuri Kondratiev, Michael Röckner, Ludwig Streit and Walter Trockel etc. who gave wonderful modularized courses and the visiting researchers of IGK who gave short courses or talks in IGK which opened to me the beautiful worlds of mathematics, physics and economics. I acknowledge the inspiration of the questions from Professors Mu-Fa Chen, Yong-Hua Mao and Doctor Cheng-Gui Yuan. I also appreciate the encouragement of Professor Ken-iti Sato and the discussion with Professors Zeng-Hu Li, Xi-Cheng Zhang and Doctors Kai He, Jing-Hai Shao etc.. It is my pleasure to thank Mrs. Hannelore Litschewsky, Mrs. Nicole Walcker and Mrs. Gaby Windhorst for their kind help during my stay in Bielefeld. I thank my colleagues Peng Jin, Sven Struckmeier and Sven Wiesinger etc. in the IGK group for the communication and help.

I benefited from the summer courses I attended during my study, for example, the Madeira math encounters XXXIII in Portugal and the satellite summer school of the conference on Lévy processes in Denmark in 2007. I am also thankful for the trips to Trento and CIME in Italy, HCM at Bonn, and Swansea in United Kindom in 2007–2008 when parts of the ideas of this thesis were figured out. I am glad to thank the partial supports from the IGK and the local organizers.

Bielefeld, February 12, 2009 Shun-Xiang Ouyang

(欧阳顺湘)

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Chapter 0

Introduction

In this thesis, we mainly devote our study to the dimension free Harnack inequalities (in the sense of Wang [Wan97]) for the transition semigroups of solution processes to some stochastic equations. For various other Harnack inequalities we refer to [LY86, CZ97, BL02, BK05, BBK06, SV07, Kas07, CK09] etc. and references therein.

Wang's Harnack inequality has been extensively studied, see [Wan04b, Wan06] etc.. Here we first shortly review the related literatures.

Wang [Wan97] used a semigroup calculus to establish Harnack inequalities for diffusion processes on Riemannian manifolds with curvature bounded below by a constant. Aida and Kawabi [AK01, Kaw04, Kaw05] obtained Harnack inequalities for some infinite dimensional diffusion processes by adding an ingredient called martingale expansion. Röckner and Wang [RW03a] used the semigroup calculus and also used the relative densities of shifted infinite divisible measures to set up Harnack inequalities for generalized Mehler semigroup.

Recently, Arnaudon, Thalmaier and Wang [ATW06] introduced a new method to establish Harnack inequalities. This method is a combination of a coupling argument and the Girsanov transformation. It has been applied to establish the Harnack inequalities for diffusion processes on Riemannian manifolds with curvature unbounded below in [ATW06] and stochastic porous media equations in infinite dimensional spaces in [Wan07] and singular stochastic equations on Hilbert spaces in Da Prato et al. [DPRW09].

The main technique we will use to establish Harnack inequalities for the transition semigroups is applying Hölder's inequality after a measure transformation. There are two levels for the measure transformation: the measure transformation on the state spaces (image measure transformation) and the measure transformation on the sample spaces.

To explain the idea of measure transformation, let $(E, \mathscr{E})$ be a Polish space and consider a stochastic process X_t starting from $x \in E$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and taking values in E. Denote by μ_t the distribution of X_t on E. We have two representations for the transition semigroup of X_t:

$$P_t f(x) = \int_\Omega f(X_t)\, d\mathbb{P} = \int_E f(y)\, d\mu_t.$$

We can make a measure transformation for $\mathbb{P}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ or take the image measure transformation for μ_t on the state space $(E, \mathscr{E})$. The details will become clear later.

We mention that the idea of the image measure transformation already appeared in [RW03a], but the authors of [RW03a] didn't apply the idea to simple cases to obtain better results than what they proved in [RW03a]. The use of the coupling technique and Girsanov transformation in [ATW06] realized the idea of the measure transformation on probability spaces.

We use the image measure transformation to establish Harnack inequalities for Gaussian Ornstein-Uhlenbeck semigroups: transition semigroups of Ornstein-Uhlenbeck processes driven by Wiener processes in Chapter 5; fractional Ornstein-Uhlenbeck semigroups and Ornstein-Uhlenbeck semigroups on Gaussian probability spaces in Chapter 6.

We also use the image measure transformation to consider Harnack inequalities for the transition semigroups of Ornstein-Uhlenbeck processes driven by Lévy processes in Chapter 7. But the idea can only be applied well to some special cases (e.g. α-stable Ornstein-Uhlenbeck semigroups) when some estimates of the relative densities are available. However, by considering measure transformations on the (sample) probability spaces (via coupling method and Girsanov's transformation), we obtain Harnack inequalities for Lévy Ornstein-Uhlenbeck semigroups which are the same with results for the Gaussian case.

We apply the method of measure transformation on probability spaces to study Harnack inequalities for finite dimensional stochastic differential equations with general drift in Chpater 4; for multivalued stochastic ordinary differential equations and multivalued stochastic evolution equations in Chapter 8.

To deal with the coupling problems for stochastic differential equations with general drifts, we proved a gluing lemma in Chapter 3. We construct a martingale solution for the sum of two second order differential operators separated by a stopping time. This makes it possible for us to study Harnack inequalities for stochastic differential equations without the assumptions of strong solutions which are usually supposed to exist by the other authors.

To apply measure transformation on probability spaces for the study of Harnack inequalities for Lévy Ornstein-Uhlenbeck semigroups, we prove a Girsanov theorem for Lévy processes in infinite dimensional spaces in Chapter 2. But we go further to study the general problem of absolute continuity for Lévy processes in Chapter 2 which is an infinite dimensional version of the lectures notes by Sato [Sat00]. The results may be known to some experts.

The applications of Harnack inequalities are standard now. See [RW03a, RW03b, Wan99, Wan01] for contractivity properties and functional inequalities; [AK01, AZ02, Kaw05] for short time heat kernel estimates of infinite dimensional diffusions; [DPRW09] for regularizing properties; [BGL01] for the transportation-cost inequality; [BLQ97, GW01] for heat kernel estimates etc..

In this thesis, by applying the Harnack inequalities we proved, we correspondingly obtain heat kernel estimates, regularizing properties and contractivity properties of the transition semigroups.

In the last Chapter (Chapter 9), as a complement to Harnack inequalities, we study entropy cost and HWI inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes. We do not claim that the contents of this chapter is really new. But it is interesting to regard this chapter as a first step to consider the corresponding functional inequalities for Lévy case.

In the following, we explain the main contents and main results of this thesis in more detail. For a simple introduction to Harnack inequalities we refer to Section 4.1, where we especially calculate a Harnack inequality for the classical Ornstein-Uhlenbeck processes on Euclidean space.

I Harnack Inequalities for Ornstein-Uhlenbeck Processes Driven by Wiener Processes: Measure Transformation on the State Spaces

For simplicity, we first introduce Harnack inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes. This is the topic of Chapter 5.

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Consider the following linear stochastic partial differential equation

$$dX_t = AX_t + B\,dW_t, \quad X_0 = x \in \mathbb{H}, \tag{1}$$

where A is the generator of some strongly continuous contraction semigroup $(S_t)_{t\geq 0}$ on $\mathbb{H}$, B is a bounded linear operator on $\mathbb{H}$, and $(W_t)_{t\geq 0}$ is a cylindrical Wiener process on $\mathbb{H}$.

Set $R = BB^*$ and

$$Q_t = \int_0^t S_u R S_u^* \, du, \quad 0 \leq t < \infty. \tag{2}$$

Fix $T > 0$. Assume that Q_T is of trace class. Then the solution of Equation (1) exists on $[0, T]$. Denote the transition semigroup of X_t by P_t. For every bounded measurable function f on $\mathbb{H}$, we have

$$P_t f(x) = \int_{\mathbb{H}} f(S_t x + z) \, \mu_t(dz), \tag{3}$$

where $x \in \mathbb{H}$, $t \in [0, T]$, and $\mu_t = N(0, Q_t)$.

For each $y \in \mathbb{H}$, by a change of variables we have

$$\int_{\mathbb{H}} f(S_T x + z) \, \mu_T(dz) = \int_{\mathbb{H}} f(S_T y + z) \frac{dN(S_T(x-y), Q_T)}{dN(0, Q_T)}(z) \, \mu_T(dz),$$

where we suppose that

$$S_T(\mathbb{H}) \subset Q_T^{1/2}(\mathbb{H}). \tag{4}$$

Define

$$\Gamma_T = Q_T^{-1/2} S_T.$$

Then Γ_T is bounded. By applying the Cameron-Martin formula for Gaussian measures on $\mathbb{H}$ and Hölder's inequality, we can prove the following Harnack inequality

$$(P_T f)^\alpha(x) \leq \exp\left(\frac{\beta}{2} |\Gamma_T(x-y)|^2\right) P_T f^\alpha(y), \tag{5}$$

where $x, y \in \mathbb{H}$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

The constant $\exp\left(\frac{\beta}{2}|\Gamma_T(x-y)|^2\right)$ in the Harnack inequality (5) is optimal and new.

Let us look at a diagonal Ornstein-Uhlenbeck process.

Let $\{e_n\}_{n\geq 1}$ be a complete orthonormal basis of the real separable Hilbert space $\mathbb{H}$. Assume that A and R commute. Then there exist two sequences of positive numbers δ_n, γ_n for $n \geq 1$ such that

$$Ae_n = -\delta_n e_n, \quad Re_n = \gamma_n e_n,$$

where $\delta_n \uparrow \infty$ as $n \uparrow \infty$. Under some conditions on δ_n and γ_n (see Subsection 5.4.2 for details), from the inequality (5) we obtain

$$(P_t f)^\alpha(x) \leq \exp\left(\sum_{n=1}^{\infty} \frac{\beta\delta_n \langle x-y, e_n\rangle^2}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)}\right) P_t f^\alpha(y).$$

This inequality is stronger than the result in [RW03a] which states (suppose that $\gamma_n \equiv 1$)

$$(P_t f)^\alpha(x) \leq \exp\left(\frac{\beta\delta_1 |x-y|^2}{\mathrm{e}^{2t\delta_1} - 1}\right) P_t f^\alpha(y).$$

For more examples, we refer the reader to Section 5.4.

Now we explain how to estimate the quantity $|\Gamma_T(x-y)|$ in (5).

Consider the following deterministic linear control system on $\mathbb{H}$ for $t \in [0,T]$,

$$dx_t = Ax_t\, dt + Bu_t\, dt \tag{6}$$

with initial data $x_0 = y - x$, where u is an $\mathbb{H}$-valued square integrable function on $[0,T]$.

By Theorem A.0.2 in Appendix A, under Assumption (4), there exists a control u_t for the system (6) such that $x_T = 0$. Moreover, we know $|\Gamma_T(x-y)|^2$ is the minimal energy for driving the initial state $x_0 = y - x$ to 0 in time T (see (A.4)):

$$|\Gamma_T(x-y)|^2 = \inf\left\{\int_0^T |u_s|^2 \colon u \in L^2([0,T],\mathbb{H}), x_0 = y-x, x_T = 0\right\}. \tag{7}$$

Hence by choosing any concrete control function u such that $x_T = 0$, we can obtain an upper estimate of the constant in the Harnack inequality. See Subsection 5.2.2 for details. Especially, we can obtain the following inequality (8) proved by Röckner and Wang [RW03a].

Let $\langle\cdot,\cdot\rangle_0$, $|\cdot|_0$ be the natural inner product and norm on $R^{1/2}(\mathbb{H})$ respectively defined through $|x|_0 = |R^{-1/2}x|$ for every $x \in \mathbb{H}$. We further assume that

$$|S_t x|_0 \leq \sqrt{\xi(t)^{-1}}\, |x|_0, \quad x \in \mathbb{H},\ t \in [0,T]$$

for some function ξ_t satisfying a certain integrable condition. Then we have

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta|x-y|_0^2}{2\int_0^T \xi(t)\,dt}\right) P_T f^\alpha(y). \tag{8}$$

We can apply these Harnack inequalities to study regularizing properties (such as the strong Feller property), heat kernel bound and hyperboundedness etc. of the Ornstein-Uhlenbeck semigroups. See Section 5.3 for details. Especially, we can prove that the Harnack inequality (5) holds if and only if the semigroup is strongly Feller.

In Chapter 5 (see Section 5.5), we also consider Harnack inequalities for the perturbed Ornstein-Uhlenbeck processes driven by Wiener processes. There we first consider Lipschitz perturbations, then perturbations by gradient of convex potentials using the Moreau-Yosida approximation.

For simplicity we just introduce Lipschitz perturbations. Consider the following semi-linear stochastic partial differential equation

$$dX_t = AX_t dt + F(X_t)dt + dW_t, \quad X_0 = x \in \mathbb{H}, \tag{9}$$

where $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ is self-adjoint such that A^{-1} is of trace class and there exists $\omega > 0$ such that

$$\langle Ax, x\rangle \le -\omega|x|^2, \quad x \in D(A);$$

F is Lipschitz continuous and dissipative

$$\langle F(x) - F(y), x - y\rangle \le 0, \quad x, y \in \mathbb{H}.$$

Then for every $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, and for any $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$, we have the following Harnack inequality for the transition semigroup P_t associated with the solution process X_t of the equation (9):

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\omega\beta|x-y|^2}{\mathrm{e}^{2t\omega} - 1}\right) P_t f^\alpha(y). \tag{10}$$

This result can be proved by approximation using the Harnack inequality (16) for the finite dimensional stochastic differential equations. We refer to the work by Da Prato et al. [DPRW09] for details of the approximation procedure. In this thesis, aiming to show another strategy as a methodological complement, we provide a semigroup calculus to "prove" (10) which is not strictly justified.

II The Method of Measure Transformation on State Spaces for Other Cases

Similarly, in Chapter 6, by using measure transformation on the state space (image measure transformation), we establish Harnack inequalities for fractional Ornstein-Uhlenbeck processes (see Section 6.1) and Ornstein-Uhlenbeck semigroups on Gaussian probability spaces (see Section 6.2).

We also use the image measure transformation to study Harnack inequalities for Lévy driven Ornstein-Uhlenbeck processes (see Section 7.3), especially for α-stable Ornstein-Uhlenbeck processes (see Subsection 7.3.2). By this method, we also show an (implicit) Harnack inequalities hold for irreducible Markov Chains (see Subsection 7.3.3).

The key point of the image measure transformation method for Harnack inequalities is the Radon-Nikodým derivative of a shift of the measure μ with respect to the measure μ. For Gaussian Ornstein-Uhlenbeck semigroups, the measure μ is Gaussian and we can apply the Cameron-Martin formula. For some other cases, the Randon-Nikodým derivatives are not known. For instance, for Lévy Ornstein-Uhlenbeck semigroups, the measure μ we need to deal with is an infinite divisible measure. In this case, nothing is known about the Radon-Nikodým derivative of μ with respect to its shift except some estimates for certain special cases. See Section 7.3 for a more detailed discussion.

III Measure Transformations on Probability Spaces

As mentioned at the begging of Part I of this introduction, we can consider measure transformations on the underlying probability spaces. We first introduce the general idea shortly.

Let E be a polish space and $x, y \in E$. Fix $T > 0$. Consider two E-valued stochastic processes $\widetilde{X}_t^x$ and X_t^y on a filtered probability space $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\leq t\leq T})$ starting from x, y respectively. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\leq t\leq T})$.

We suppose that

(1) The transition law of X_T^y under $\mathbb{P}$ is the same as the transition law of $\widetilde{X}_t^x$ under $\mathbb{Q}$. That is, they have the same transition semigroup P_t:

$$P_t f(y) = \mathbb{E}_{\mathbb{P}} f(X_t^y) \quad \text{and} \quad P_t f(x) = \mathbb{E}_{\mathbb{Q}} f(\widetilde{X}_t^x)$$

for every bounded continuous function f on E.

(2) $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$:

$$\mathbb{Q} \ll \mathbb{P}.$$

(3) The processes $\widetilde{X}_t^x$ and X_t^y meet each other at the fixed time $T > 0$:

$$\widetilde{X}_T^x = X_T^y \quad \mathbb{Q}\text{–a.s.}$$

With the assumptions above, for every $f \in \mathscr{C}_b^+(\mathbb{H})$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, by applying Hölder's inequality, we have

$$\begin{aligned} P_T f(x) &= \mathbb{E}_{\mathbb{Q}} f(\widetilde{X}_T^x) = \mathbb{E}_{\mathbb{Q}} f(X_T^y) = \mathbb{E}_{\mathbb{P}} \frac{d\mathbb{Q}}{d\mathbb{P}} f(X_T^y) \\ &\leq \left[\mathbb{E}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{\beta} \right]^{1/\beta} \left[\mathbb{E}_{\mathbb{P}} f^{\alpha}(X_T^y) \right]^{1/\alpha} \\ &= \left[\mathbb{E}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{\beta} \right]^{1/\beta} \left[P_T f^{\alpha}(y) \right]^{1/\alpha}. \end{aligned}$$

Hence we obtain the following Harnack inequality

$$(P_T f)^{\alpha}(x) \leq \left[\mathbb{E}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{\beta} \right]^{\alpha/\beta} P_T f^{\alpha}(y).$$

In applications, we can take $\widetilde{X}_t^x$ as a drift transformation of X_t^y and choose the drift properly to force the two processes $\widetilde{X}_t^x$ and X_t^y to meet at time T. Under some conditions, we can construct the measure $\mathbb{Q}$ from $\mathbb{P}$ by using the Girsanov theorem and keep the transition law of the processes. This is the idea of the method of the coupling and Girsanov's transformation introduced by Arnaudon et al. [ATW06] for Harnack inequalities. Hence, for preparation (and independent interest), we investigate the absolute continuity of Lévy processes and the existence of coupling in Chapter 2 and Chapter 3 respectively.

IV Absolute Continuity of Lévy Processes

In Chapter 2, we study the absolute continuity of Lévy processes in infinite dimensional spaces.

Denote by $\mathbb{D}$ the Skorokhod space over the Hilbert space $\mathbb{H}$. Let X_t be the canonical process on $\mathbb{D}$ and $\mathscr{F}$, $(\mathscr{F}_t)_{0 \leq t < \infty}$ be the canonical filtrations on $\mathbb{D}$.

Consider two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(\mathbb{D}, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t < \infty})$. We assume that X_t is a Lévy process with characteristic triplet (b_j, R_j, ν_j) under $\mathbb{P}_j$ for $j = 1, 2$.

Fix any $t \geq 0$. We denote the restriction of $\mathbb{P}_j$ to $\mathscr{F}_t$ by $\mathbb{P}_j^t$. We are interested in the absolute continuity of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$ and in the Radon-Nikodým derivative etc..

Sato [Sat99, Sat00] studied such questions extensively for the case $\mathbb{H} = \mathbb{R}^d$. We aim to write down an infinite dimensional version of the main results in [Sat00]. One of the main results for the infinite dimensional case is presented below.

Suppose that for some $0 < r < 1$ we have

$$k_r(\nu_1, \nu_2) < \infty, \quad R := R_1 = R_2, \quad b_{21} \in \mathbb{H}_0 := R^{1/2}(\mathbb{H}),$$

where $k_r(\nu_1, \nu_2)$ is the Hellinger-Kakutani distance between ν_1 and ν_2, and

$$b_{21} = b_2 - b_1 - \int_{\{|x| \leq 1\}} x \, d(\nu_2 - \nu_1).$$

Then we prove that $\mathbb{P}_2^t$ is absolutely continuous with respect to $\mathbb{P}_1^t$.

We also study the Radon-Nikodým derivative of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$ (see Section 2.7 for details). A special case is the following Girsanov transformation for Lévy process (see also [RR07]). We will use this result to study Harnack inequalities for Ornstein-Uhlenbeck processes driven by Lévy processes.

Fix $T > 0$. Let $(X(t))_{0 \leq t \leq T}$ be an $\mathbb{H}$-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with characteristic triplet (b, R, ν). Denote by $X'(t)$ the Gaussian part of $X(t)$. Then $X'(t)$ is an R-Wiener process. Let $(\psi(t))_{0 \leq t \leq T}$ be an $\mathbb{H}_0$-valued $\mathscr{F}_t$-predictable process, independent of the jump part $X - X'$, and such that $\mathbb{E}\, \rho^{X'}(T) = 1$ with

$$\rho^{X'}(T) = \exp\left(\int_0^T \langle \psi(s), dX'(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi(s)|_0^2 \, ds \right).$$

Then

$$\widetilde{X}(t) := X(t) - \int_0^t \psi(s) \, ds, \quad 0 \leq t \leq T$$

is also a Lévy process on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T})$ with the same characteristic triplet (b, R, ν) under the new probability measure $\widetilde{\mathbb{P}}$ defined by

$$\widetilde{\mathbb{P}} = \rho^{X'}(T) \mathbb{P}.$$

This result is easy to show. We only need to use the Girsanov theorem for Wiener processes and use the independence of the three parts in the Lévy-Itô decomposition. In fact, we first motivate and prove this result in Chapter 2. The idea of the proof for the general case is the same.

V Gluing and Coupling

In Chapter 3 we prove a gluing lemma for martingales and studied its applications for couplings.

Set

$$L(a,b) := L(a(t,x), b(t,x)) := \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i}, \tag{11}$$

where $a(t,x)$ is a symmetric nonnegative definite real matrix and $b(t,x) \in \mathbb{R}^n$ defined on $[0,\infty) \times \mathbb{R}^n$.

Let L_1 and L_2 be two second order differential operators of the form (11) on $\mathbb{R}^n$. Let τ be a stopping time on Ω.

Assume that

(1) There exists a solution $\mathbb{P}_1^x$ to the martingale problem for L_1 up to τ;
(2) For each $\omega \in \Omega$, there exists a solution $\mathbb{P}_2^{\tau(\omega), X_{\tau(\omega)}(\omega)}$ to the martingale problem for L_2 starting from $(\tau(\omega), X_{\tau(\omega)}(\omega))$;
(3) and some other conditions (see Theorem 3.1.5).

Define

$$\mathbb{Q}_\omega := \delta_\omega \otimes \mathbb{P}_2^{\tau(\omega), X_{\tau(\omega)}(\omega)} \mathbb{1}_{\{\tau<\infty\}} + \delta_\omega \mathbb{1}_{\{\tau=\infty\}}$$

for every $\omega \in \Omega$. Then $\mathbb{P}_1^x \otimes_\tau \mathbb{Q}$ is a solution to the martingale problem for

$$L = L_1 \mathbb{1}_{\{t<\tau\}} + L_2 \mathbb{1}_{\{t\geq\tau\}}.$$

The proof is based on a result by Stroock and Varadhan [SV79, Theorem 6.1.2]. This gluing lemma generalizes a lemma by Chen and Li [CL89, Lemma 3.4]. Lemma 3.4 in [CL89] studies the gluing of martingale generators via the diffusion coefficients. By the general gluing lemma we proved we can study the gluing of martingale generators via the drifts.

We can use this result to study the existence of coupling and the weak existence of solutions to coupled stochastic differential equations. For example, we have the following result.

Consider the following stochastic differential equations on $\mathbb{R}^{2d}$:

$$\begin{cases} dX_t = \sigma(t, X_t)dW_t + b(X_t)dt, & X_0 = x \in \mathbb{R}^d, \\ dY_t = \sigma(t, Y_t)dW_t + b(Y_t)dt + \xi(t, X_t, Y_t)\mathbb{1}_{\{t<\tau\}}dt, & Y_0 = y \in \mathbb{R}^d, \end{cases} \tag{12}$$

where W_t is an $\mathbb{R}^d$-valued Brownian motion. Suppose that there exists a weak solution to (12) up to τ and there is a weak solution of the following equation

$$\begin{cases} dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt, & X_s = x \in \mathbb{R}^d, \\ dY_t = \sigma(t, Y_t)dW_t + b(t, Y_t)dt, & Y_s = y \in \mathbb{R}^d \end{cases}$$

for every fixed $(s, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Then there exists a weak solution to the equation (12) for all $t \in [0, \infty)$.

With the coupling results, we can study Harnack inequalities for stochastic differential equations with unique weak solutions. This is natural but it is new.

VI Harnack Inequalities for Ornstein-Uhlenbeck Processes Driven by Lévy Processes

Now we turn to the introduction of Harnack inequalities for Lévy driven Ornstein-Uhlenbeck processes studied in Chapter 7. We use the measure transformation on probability space.

Let $(Z_t)_{0\le t\le T}$ be an $\mathbb{H}$-valued Lévy process with characteristic triplet (b, R, ν) on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$.

Consider the following generalized Langevin stochastic differential equation

$$dX_t = AX_t\, dt + dZ_t, \quad X_0 = x. \tag{13}$$

We denote the transition semigroup of X_t by P_t.

Fix $T > 0$ and $x, y \in \mathbb{H}$. Assume (4) holds. Consider

$$\begin{cases} d\widetilde{X}_t = A\widetilde{X}_t\, dt + dZ_t - R^{1/2}u_t\, dt, & \widetilde{X}_0 = x. \\ dY_t = AY_t\, dt + dZ_t, \quad Y_0 = y. \end{cases}$$

Here $u \in L^2([0, T]; \mathbb{H})$ is a control of the system (6) such that $x_T = 0$ and hence it follows that $\widetilde{X_T} = Y_T$. By results from control theory (see Appendix A), the control function u exist.

By Girsanov's theorem, we construct a new probability measure $\mathbb{Q}$ on the space $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T})$ such that $\mathbb{Q}$ is absolute continuous with respect to $\mathbb{P}$

and $\widetilde{Z}_t := Z_t - u_t$ is also a Lévy process with characteristic triplet (b, R, ν), the same characteristic triplet as that of Z_t under $\mathbb{P}$.

Now by using the procedure introduced in (III), we obtain

$$(P_T f)^\alpha(y) \le \exp\left(\frac{\beta}{2}\int_0^T |u_t|^2\,dt\right) P_T f^\alpha(x) \tag{14}$$

for every $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

By optimization Inequality (14) over all possible null control function u of the system (6), and by noting the representation (7), we obtain the Harnack inequality (5) for the Lévy Ornstein-Uhlenbeck semigroup associated with the solution process of the equation (13).

Similar to the Gaussian case, by choosing any control function u (or applying (14)), we also obtain an upper estimate of the coefficient in the Harnack inequality. Especially, we also have (8) which generalizes the Harnack inequalities in Röckner and Wang [RW03a] for the Lévy Ornstein-Uhlenbeck semigroups from $\alpha = 2$ to general order $\alpha > 1$.

We refer to Section 7.5 for the applications of Harnack inequalities for Lévy Ornstein-Uhlenbeck semigroups.

VII Harnack Inequalities for Stochastic Differential Equations

In Chapter 4 we consider Harnack inequalities for the following distorted Brownian motion on $\mathbb{R}^d$:

$$dX_t = b(t, X_t)\,dt + dW_t, \tag{15}$$

where W_t is a Wiener process on $\mathbb{R}^d$, $b(t,x)$ is a measurable function from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}^d$.

Let b be a continuous function satisfying the following linear growth condition

$$|b(t,x)| \le C(1+|x|), \quad x \in \mathbb{R}^d,\ 0 \le t \le T$$

for some constant $C > 0$ and the following classical global monotonicity condition

$$\langle x-y, b(t,x)-b(t,y)\rangle \le K|x-y|^2, \quad x, y \in \mathbb{R}^d,\ 0 \le t \le T$$

for some $K \in \mathbb{R}$. then for the transition semigroup P_t associated with X_t, we have the following well known result

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta K|x-y|^2}{1-\mathrm{e}^{-2KT}}\right) P_T f^\alpha(y). \tag{16}$$

where $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$ and $T > 0$.

The inequality (16) is essentially from [Wan97]. We can reproduce it by coupling and Girsanov's transformation.

We aim to go further and prove Harnack inequalities for P_t under general conditions on b. Especially, we have the following result.

Assume that $b(t,x)$ is of linear growth, the solution to equation (15) is weakly unique and there exists a nonnegative increasing function g on $[0,\infty)$ such that

$$\sup_{|x-y|=r} \frac{1}{r}\langle b(x)-b(y), x-y\rangle \le g(r).$$

Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2}\int_0^T \left[g(|x-y|) + \frac{\xi_t|x-y|}{\int_0^T \xi_u\,du}\right]^2 dt\right) P_T f^\alpha(y),$$

where ξ_t is any positive continuous function on $[0,T]$.

In Chapter 4 we also consider estimates of heat kernels of the transition semigroup by applying Harnack inequalities.

VIII Harnack Inequalities for Multivalued Stochastic Equations

Now we introduce the contents of Chapter 8 which is devoted to the study the Harnack inequalities for multivalued stochastic differential equations and multivalued stochastic evolution equations.

The motivation comes from the study of Harnack inequalities for perturbations of Ornstein-Uhlenbeck processes driven by Wiener processes in Section 5.5. If the perturbation is given by the sub-differential of some convex function, then we come to the multivalued stochastic equations. Below we introduce Harnack inequalities for general multivalued stochastic evolution equations and their applications.

Let $\mathbb{V} \subset \mathbb{H} = \mathbb{H}^* \subset \mathbb{V}^*$ be an evolution triplet, where $\mathbb{V}$ is a real separable and reflexive Banach space which is continuously and densely embedded into a separable Hilbert space $\mathbb{H}$.

Consider the following multivalued stochastic evolution equation

$$\begin{cases} dX_t \in -AX_t\,dt + BX_t\,dt + \sigma(t)\,dW_t, \\ X_0 = x \in \overline{D(A)}, \end{cases} \tag{17}$$

where A is a multivalued maximal monotone operator* on $\mathbb{H}$, $B\colon \mathbb{V} \to \mathbb{V}^*$ is a single valued operator from $\mathbb{V}$ to $\mathbb{V}^*$, $\sigma\colon \mathbb{R}^+ \times \Omega \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$, and W_t is a cylindrical Wiener process on $\mathbb{H}$ with respect to a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

Assume (H1)–(H5) stated in Theorem 8.3.2. We mention here that (H4) says that there exists $\gamma > 0$, $\omega \in \mathbb{R}$ and $q > 1$ such that for every $x, y \in \mathbb{V}$,

$$_{\mathbb{V}}\langle x-y, Bx-By\rangle_{\mathbb{V}^*} \leq -\gamma|x-y|_{\mathbb{V}}^q + \omega|x-y|_{\mathbb{H}}^2.$$

For simplicity, we assume in this introduction that σ is constant and q is strictly greater than 2. We refer to Section 8.3 for details about these conditions.

With these conditions, Zhang [Zha07] proved that Equation (17) has a unique solution X_t. Define $P_tf(x) = \mathbb{E}_{\mathbb{P}}f(X_t)$ for every $f \in \mathscr{B}_b(\overline{D(A)})$. Then P_t is a Markov semigroup. Zhang [Zha07, Theorem 5.8] studied the existence, uniqueness, and finiteness of the second moment of the the invariant measures μ associated with the semigroup P_t. We can prove the following stronger concentration property:

$$\int_{\overline{D(A)}} \left(e^{\theta|x|_{\mathbb{H}}^q} + |x|_{\mathbb{V}}^q \right) \mu(dx) < \infty \tag{18}$$

for some $\theta > 0$.

Assume in addition (i.e. Condition (8.33))

$$\zeta^2|x|_{\sigma}^{2+r} \cdot |x|_{\mathbb{H}}^{q-2-r} \leq |x|_{\mathbb{V}}^q, \quad \text{for all } x \in \mathbb{V},\ t \geq 0$$

for some $\zeta > 0$ and $r \geq q-4$. Then

$$(P_Tf^\alpha)(x) \leq \exp\left(\frac{\beta}{2}\widetilde{\Theta}_T|x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}} \right) P_Tf^\alpha(y),$$

for every $T > 0$, $x, y \in \overline{D(A)}$ and $f \in \mathscr{B}_b^+(\overline{D(A)})$, where $\widetilde{\Theta}_T$ is some constant only dependent on $T, \delta, \gamma, \omega$ and ζ (see (8.37)).

An immediate consequence of the Harnack inequality (with $r > q-4$) is that

*We used the same symbol A which has been used to represent the generator of S_t and hope that this will not cause any confusion.

for every f in $L^p(\overline{D(A)}, \mu)$ with $p > 1$, $P_t f$ is continuous on $\overline{D(A)}$.

We also apply the Harnack inequalities to get the following results.

(1) μ is fully supported by $\overline{D(A)}$.

(2) For every $x \in \overline{D(A)}$, $t > 0$, the transition densities $p_t(x, \cdot)$ of P_t (with respect to μ) exist and for every $\alpha > 1$

$$\|p_t(x,\cdot)\|_{L^\alpha(\mathbb{H},\mu)} \le \left[\int_{\overline{D(A)}} \exp\left(-\frac{\alpha}{2}\widetilde{\Theta}_t |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right)\mu(dy)\right]^{-(\alpha-1)/\alpha}.$$

(3) Suppose $K \le 0$. Then P_t is ultrabounded. More precisely, we have

$$\|P_t\|_{2\to\infty} \le \exp\big(c(1 + t^{-\frac{q}{q-2}})\big)$$

for some constant $c > 0$. Hence, P_t is compact for large $t > 0$.

IX Entropy Cost and HWI Inequalities for Ornstein-Uhlenbeck Processes Driven by Wiener Processes

We are also interested in other functional inequalities for stochastic equations, especially for stochastic equations with Lévy noise. But it seems that this is not easy. As a first step, we write down entropy cost inequalities and HWI inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes in Chapter 9. We also write it as a complement to this thesis. The results maybe be well known for experts.

We go back to consider Equation (1). We assume that A is a bounded self-adjoint and nonnegative definite operator. Suppose that Q_∞ is of trace class and $S_t R = R S_t$ for all $t \ge 0$. Denote by P_t the associated transition semigroup. Then P_t is symmetric. We also assume that there exist $M, \omega > 0$ such that for all $t \ge 0$,

$$\|S_t\| \le M\,\mathrm{e}^{-\omega t}.$$

Then we can prove the following entropy cost inequality

$$\mathrm{Ent}_\mu(P_t f) \le \frac{M\omega}{2(\mathrm{e}^{2\omega t}-1)} W_2(f\mu,\mu)^2,$$

and HWI inequality

$$\mathrm{Ent}_\mu(f) \le M\sqrt{I(f)}W_2(f\mu,\mu) - \frac{M\omega}{2}W_2(f\mu,\mu)^2$$

for every $t \ge 0$ and nonnegative bounded measurable function f on $\mathbb{H}$ with

$\mu(f) = 1$. For the definition of the entropy (“H”) Ent_μ, Wasserstein distance $W_2(\cdot, \cdot)$, and Fisher information $I(\cdot)$, we refer to Section 9.1.

Chapter 1

Preliminaries

This chapter is devoted to some preliminary material which will be used in this thesis. In Section 1.1, we introduce some basic notations. In Section 1.2 we recall Gaussian measures and the Cameron-Martin formula. In Section 1.3 we introduce shortly Wiener processes and stochastic integrals with respect to Wiener processes. In Section 1.4 we introduce Lévy processes and infinite divisible distributions, the Lévy-Itô decomposition, stochastic integral with respect to Lévy processes and symmetric α-stable processes.

1.1 Some Basic Notations

Operators on Hilbert Spaces Let $\mathbb{H}$ be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$. Let T be a bounded linear operator on $\mathbb{H}$. We denote by T^* the adjoint operator of T.

T is called symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every x and y in $\mathbb{H}$ and positive definite if $\langle Tx, x\rangle \geq 0$ for every $x \in \mathbb{H}$.

Let $\{e_k\}_{k\geq 1}$ be a complete orthonormal basis of $\mathbb{H}$. Let T be a bounded symmetric and positive operator on $\mathbb{H}$. We call T a *Hilbert-Schmidt* operator if

$$\sum_{k=1}^{\infty} |Te_k|^2 < +\infty.$$

We call T a *trace class* operator if

$$\operatorname{Tr} T := \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle < +\infty.$$

It is clear that every trace class operator is a Hilbert-Shimidt operator. We call $\mathrm{Tr}\, T$ the *trace* of T.

Spaces of functions We introduce some basic spaces of functions.

As usual, we denote by $\mathscr{C}(\mathbb{H})$ the space of continuous functions on $\mathbb{H}$, $\mathscr{C}_b(\mathbb{H})$ the space of bounded continuous on $\mathbb{H}$, $\mathscr{C}_b^+(\mathbb{H})$ the space of positive bounded continuous functions on $\mathbb{H}$. Similarly, we denote by $\mathscr{B}(\mathbb{H})$, $\mathscr{B}_b(\mathbb{H})$, $\mathscr{B}_b^+(\mathbb{H})$ the spaces of measurable, bounded measurable, positive bounded measurable functions on $\mathbb{H}$.

We denote by $\mathcal{B}(\mathbb{H})$ the Borel σ-algebra on $\mathbb{H}$. Let μ be a probability measure on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$. For every $p > 1$, we denote by $L^p(\mathbb{H}, \mu)$ the space of all measurable functions f on $\mathbb{H}$ such that $|f|^p$ is integrable with respect to μ. For every $f \in L^p(\mathbb{H}, \mu)$, we denote $\|f\|_{L^p(\mathbb{H},\mu)} = \mu(|f|^p)^{1/p}$. If there is no confusion, we also write $\|f\|_p$ for $\|f\|_{L^p(\mathbb{H},\mu)}$.

Denote $\mathscr{E}(\mathbb{H})$ for the space of all exponential functions, that is

$$\mathscr{E}(\mathbb{H}) := \text{Linear Span}\{\mathrm{Re}\varphi_h, \mathrm{Im}\varphi_h :\ h \in \mathbb{H}, \varphi_h(x) := \mathrm{e}^{i\langle h,x\rangle}, x \in \mathbb{H}\}.$$

For a self-adjoint operator $(A, D(A))$ on $\mathbb{H}$, we denote $\mathscr{E}_A(\mathbb{H})$ for the space of all real parts of the functions $\left(\mathrm{e}^{i\langle h,x\rangle}\right)_{h\in D(A^*)}$.

Derivatives For every $\varphi \in \mathscr{E}(\mathbb{H})$ and $h \in \mathbb{H}$ we denote the (weak-) derivative of φ in the direction of h by $D_h\varphi$. Recall that it is defined as usual by

$$D_h\varphi(x) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\varphi(x+\varepsilon h) - \varphi(x)], \quad x \in \mathbb{H}.$$

For $h = e_k$ we simply denote it $D_{e_k}\varphi$ by $D_k\varphi(x)$ as the derivative of φ in the direction of e_k.

The gradient $D\varphi$ is defined by

$$\langle D\varphi, h\rangle = D_h\varphi, \quad \varphi \in \mathscr{E}(\mathbb{H}),\ h \in \mathbb{H}.$$

It can be shown that the linear mappings D_k and

$$D : \mathscr{E}(\mathbb{H}) \subset L^2(\mathbb{H}, \mu) \to L^2(\mathbb{H}, \mu, \mathbb{H}), \quad \varphi \mapsto D\varphi,$$

are closable and we shall still denote the closures respectively by D and D.

Cameron-Martin Spaces Let Q be a trace class operator. It is well known that there exists a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ on $\mathbb{H}$ and a sequence of numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0,\ k \in \mathbb{N}.$$

See Reed and Simon [RS80, Theorem VI.16 and Theorem VI.21], or Dunford and Schwartz [DS88].

We define the square root of Q by

$$Q^{1/2}x = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle x, e_k\rangle e_k, \quad x \in \mathbb{H}.$$

It is obvious that $Q^{1/2}$ is a Hilbert-Schmidt operator. We call the range $\mathbb{H}_0 := Q^{1/2}(\mathbb{H})$ of $Q^{1/2}$ the *Cameron-Martin space* of $\mathbb{H}$. This subspace is also called the *reproducing kernel space* for the measure μ. We know $\mathbb{H}_0$ is densely embedded in $\mathbb{H}$ but with measure zero: $\mu(\mathbb{H}_0) = 0$ (see [DP06, Proposition 1.27]).

We are going to introduce a scalar product on $\mathbb{H}_0$. With this product, the Cameron-Martin space become an Hilbert space. To do so, we first recall an useful concept – *pseudo inverse* of linear operator. See [PR07, Appendix C] for more details.

Pseudo inverse Let T be a linear bounded operator on $\mathbb{H}$. It is not necessary to be one-to-one and onto. Then $\operatorname{Ker} T := \{x\colon Tx = 0\}$ is a closed subspace of $\mathbb{H}$. Denote by $\mathbb{H}_1$ the orthogonal complement of $\operatorname{Ker}(T)$: $\mathbb{H}_1 = \operatorname{Ker}(T)^{\perp}$. The subspace $\mathbb{H}_1$ of $\mathbb{H}$ is also closed. Denote the restriction of T on $\mathbb{H}_1$ by T_1:

$$T_1 = T|_{\mathbb{H}_1}\colon \mathbb{H}_1 \to T_1(\mathbb{H}).$$

Then T_1 is one-to-one. What is more, $T(\mathbb{H}) = T_1(\mathbb{H})$. So we can define the *pseudo inverse* T^{-1} of T by

$$T^{-1}\colon T(\mathbb{H}) \to \mathbb{H}_1, \quad x \mapsto T_1^{-1}x$$

for every $x \in T(\mathbb{H})$.

Remark 1.1.1. Equivalently, for every $x \in T(\mathbb{H})$, the pseudo inverse $T^{-1}x$ can be defined as the element in the hyperplane $\{y \in \mathbb{H} : Ty = x\}$ with minimal norm.

Intrinsic Distance Let Q be a trace class operator on $\mathbb{H}$. Denote the pseudo inverse of $Q^{1/2}$ by $Q^{-1/2}$. We define a scalar product on $\mathbb{H}_0$ by

$$\langle x, y\rangle_{\mathbb{H}_0} := \left\langle Q^{-1/2}x, Q^{-1/2}y\right\rangle, \quad x, y \in \mathbb{H}_0.$$

Denote by $|\cdot|_{\mathbb{H}_0}$ the norm corresponding to the inner product. When there is no confusion, we will simply write $\langle\cdot,\cdot\rangle_0$ and $|\cdot|_0$ for $\langle\cdot,\cdot\rangle_{\mathbb{H}_0}$ and $|\cdot|_{\mathbb{H}_0}$.

Since $\mathbb{H}_0$ is dense in $\mathbb{H}$, we can extend the mapping $\langle\cdot,y\rangle_0$ to the whole space $\mathbb{H}$ for every fixed $y\in\mathbb{H}$.

Remark 1.1.2. Equivalently, we also have

$$\langle x,y\rangle_{\mathbb{H}_0}=\sum_{k=1}^{\infty}\frac{\langle x,e_k\rangle\langle y,e_k\rangle}{\lambda_k}\mathbb{1}_{\{\lambda_k>0\}},$$

where $\{e_k\}_{k\in\mathbb{N}}$ are the eigenvectors of Q with eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$.

Now we can introduce an *intrinsic distance* on $\mathbb{H}$ by

$$\rho(x,y)=\begin{cases}|x-y|_0\,, & \text{if } x-y\in\mathbb{H}_0;\\ \infty, & \text{otherwise.}\end{cases}\tag{1.1}$$

1.2 Absolute Continuity of Gaussian Measures

There are also lots of monographs on Gaussian measures. See, for example, [Xia72, Kuo75, Bog98] etc.. What we will introduce is basic and can also be found in the books by Da Prato and Zabczyk [DPZ92, DPZ02].

Definition 1.2.1. Let $m\in\mathbb{H}$ and Q be a trace class operator on $\mathbb{H}$. A probability measure μ on $(\mathbb{H},\mathcal{B}(\mathbb{H}))$ is called a *Gaussian measure* with mean m and covariance Q if the *Fourier transformation* (*characteristic function*) $\hat{\mu}$ of μ

$$\hat{\mu}(u)=\mathbb{E}_\mu\exp\left(i\langle x,u\rangle\right)=\int_{\mathbb{H}}\exp\left(i\langle x,u\rangle\right)\mu(dx),\quad u\in\mathbb{H}$$

is given by

$$\hat{\mu}(u)=\exp\left(\langle m,u\rangle-\frac{1}{2}\langle Qu,u\rangle\right),\quad u\in\mathbb{H}.$$

In this case, we will write $\mu=N(m,Q)$ or $N_{m,Q}$. When $m=0$ we shall write N_Q instead of $N_{0,Q}$ for short.

It can be shown (see for example [PR07, Definition 2.1.1 and Theorem 2.1.2]) that μ is a Gaussian measure on $(\mathbb{H},\mathcal{B}(\mathbb{H}))$ if and only if every random variable

$$x'\colon\mathbb{H}\to\mathbb{R},\quad u\mapsto\langle x,u\rangle,\quad h\in\mathbb{H}$$

has Gaussian law for every $x \in \mathbb{H}$ under μ.

We are interested at the absolute continuity of Gaussian measures. We first recall some basic notations related to the absolute continuity of measures.

Let σ_1, σ_2 be two σ-finite measures on a general measure space $(E, \mathscr{E})$. If $\sigma_2(A) = 0$ for each $A \in \mathscr{E}$ with $\sigma_1(A) = 0$, then we say σ_2 is *absolutely continuous* with respect to σ_1. And we write it as $\sigma_2 \ll \sigma_1$.

If both $\sigma_1 \ll \sigma_2$ and $\sigma_2 \ll \sigma_1$, then we say σ_1, σ_2 are *mutually absolutely continuous* or *equivalent*. And we denote it by $\sigma_1 \approx \sigma_2$.

If there exists some $A \in \mathscr{E}$ such that $\sigma_1(A) = \sigma_2(E \setminus A) = 0$, then we say σ_1 is *orthogonal* or *singular* to σ_2 and denoted it by $\sigma_1 \perp \sigma_2$. Obviously $\sigma_1 \perp \sigma_2$ if and only if $\sigma_2 \perp \sigma_1$. So we shall also say σ_1 and σ_2 are orthogonal to each other.

When the measure σ_2 is absolutely continuous with respect to the measure σ_1, the famous Radon-Nikodým theorem (see the book Halmos [Hal50]) asserts that there exists a σ_1-a.s determined $\mathscr{E}$-measurable function $f(x)$ such that

$$\sigma_2(A) = \int_A f(x)\, \sigma_1(dx) \quad \text{for every } A \in \mathscr{E}. \tag{1.2}$$

We denote the relation (1.2) by $f(x) = \frac{d\sigma_2}{d\sigma_1}(x)$, and termed it as *Radon-Nikodým derivative* or *relative density* of the measure σ_2 with respect to the measure σ_1.

The equivalence and perpendicularity of two Gaussian measures on a separable Hilbert space $\mathbb{H}$ have been studied for a long time. The first result concerning the equivalence or singularity of two Gaussian measures is essentially due to Kakutani [Kak48]. We have the so called Feldman-Hájek [Fel58, Fel59, Háj58] theorem which states that any two Gaussian measures on a Hilbert space are either equivalent or orthogonal. There are some ways to prove it. For example, it can be proved by using the "Hellinger-Kakutani distance" due to Hellinger [Hel07] and Kakutani [Kak48]; the "method of entropy" due to Hájek [Háj58] and Rozanov [Roz62]; and the "method of reproducing kernel" due to Kallianpur and Oodaira [KO63].

Now one can find the introduction of the absolute continuity of two Gaussian measures in many monographs, see for example, [Xia72, Var68, GS74, Kuo75, DPZ92, Bog98] etc..

In this thesis, we will only need to use the following simple case: the absolute continuity of Gaussian measures $N_{m,Q}$ with respect to N_Q.

To be intuitive, let us first show the formula in one-dimension. Let $N_{b,q}$ and $N_{0,q}$ be two the normal distributions on $\mathbb{R}$ with the same variance q and means b

and 0 respectively. If $q = 0$, then $N_{b,q}$ and $N_{0,q}$ are equivalent only when $b = 0$. If $q \neq 0$, then $N_{b,q}$ and $N_{0,q}$ are always equivalent and the derivative is given by

$$\begin{aligned}\frac{dN_{b,q}}{dN_{0,q}}(x) &= \frac{(2\pi q)^{-1/2}\exp\left[-\frac{(x-b)^2}{2q}\right]}{(2\pi q)^{-1/2}\exp\left[-\frac{x^2}{2q}\right]} \\ &= \exp\left[\frac{2bx-b^2}{2q}\right] = \exp\left[\left\langle q^{-1/2}b, q^{-1/2}x\right\rangle - \frac{1}{2}\left|q^{-1/2}b\right|^2\right].\end{aligned} \tag{1.3}$$

Formally, the infinite dimensional version of the Cameron-Martin formula is the formula (1.3) if we replace $q^{-1/2}$ by $Q^{-1/2}$.

The proof of the following Cameron-Martin formula can be found, for instance, in [GS74, Chapter VII, Section 4, Theorem 1], or [Kuo75, Theorem 3.1] or [DPZ92, Theorem 2.21].

Theorem 1.2.2. *Let $\mu = N(m, Q)$ and $\nu = N(0, Q)$ on $\mathbb{H}$ be two Gaussian measures on $\mathbb{H}$.*

(1) If $m \notin Q^{1/2}(H)$, then μ and ν are singular.

(2) If $m \in Q^{1/2}(H)$, then μ and ν are equivalent and the Radon-Nikodým derivative of μ with respect to ν is given by

$$\frac{d\mu}{d\nu}(x) = \exp\left(\langle x, m\rangle_0 - \frac{1}{2}|m|_0^2\right), \quad \text{for all } x \in \mathbb{H}.$$

Remark 1.2.3. One of the powerful tools in the study of absolute continuity of Gaussian measures is the so called *Kakutani distance* of two measures which is introduced in Kakutani [Kak48]. (As pointed out in Kakutani's paper, Kakutani distance is the same with the *Hellinger integral* introduced by Hellinger in his thesis [Hel07]. So we shall call the distance by Hellinger-Kakutani distance.) We will introduce these concepts in detail in Section 2.3 for the study of absolute continuity of Lévy processes.

1.3 Wiener Processes and Stochastic Integrals

We will first recall the definition of *standard Q-Wiener processes* and stochastic integrals with respect to standard Q-Wiener processes. Then we will introduce *cylindrical Q-Wiener processes* and the stochastic integrals with respect to cylindrical Wiener process. We refer to [DPZ92, PR07] for more details.

Definition 1.3.1. Suppose that Q is a trace class operator on a real separable Hilbert space $\mathbb{H}$. A family of $\mathbb{H}$-valued random variables $W = (W_t)_{0\le t\le T}$ is called a *standard Q-Wiener process* on $[0, T]$ if

(1) $W_0 = 0$;
(2) W has continuous trajectories;
(3) W has independent increments, that is, the random variables

$$W_{t_1}, W_{t_2-t_1}, \dots, W_{t_n} - W_{t_{n-1}}$$

are independent for all $0 \le t_1 < t_2 < \cdots < t_n \le T$ and all $n \in \mathbb{N}$;
(4) The increments of W have Gaussian laws: $W_t - W_s$ is Gaussian distributed as $N_{(t-s)Q}$ for every $0 \le s \le t \le T$.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{H}$ consisting of eigenvectors of Q, and $\{\lambda_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of eigenvalues:

$$Qe_k = \lambda_k e_k, \quad \text{for all } k \in \mathbb{N}.$$

Then $(W_t)_{0\le t\le T}$ is a Q-Wiener process if and only if

$$W_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \quad 0 \le t \le T, \tag{1.4}$$

where $\{\beta_k\}_{k\in\mathbb{N}}$ are independent real valued standard Brownian motions.

The proof of the representation (1.4) of Q-Wiener processes can be found in [DPZ92, Proposition 4.1] or [PR07, Proposition 2.1.10] etc..

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered probability space. We call $(W_t)_{0\le t\le T}$ a *Q-Wiener process* with respect to the filtration $(\mathscr{F}_t)_{0\le t\le T}$, if $(W_t)_{0\le t\le T}$ is a Q-Wiener process satisfying the following conditions

(1) W_t is adapted to $\mathscr{F}_t$ for all $0 \le t \le T$;
(2) $W_t - W_s$ is independent of $\mathscr{F}_s$ for all $0 \le s \le t \le T$.

We say that a filtration $(\mathscr{F}_t)_{0\le t\le T}$, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a *normal filtration* if the following two usual conditions are satisfied.

(1) $\mathscr{F}_0$ (and hence every $\mathscr{F}_t$ for $0 \le t \le T$) contains all elements $A \in \mathscr{F}$ with $\mathbb{P}(A) = 0$.
(2) The filtration is right continuous: $\mathscr{F}_t = \mathscr{F}_{t+} := \cap_{s>t}\mathscr{F}_s$ for every $0 \le t \le T$.

It is easy to show that any Q-Wiener process is a Q-Wiener process with respect to some normal filtration (see [PR07, Proposition 2.1.13]). In this thesis, if it is not explicitly stated, we will always assume that the filtration is normal.

Stochastic Integral

In this thesis we mainly concern the stochastic integrals of deterministic functions with respect to Wiener processes. So we only introduce shortly this simple case. For more on stochastic integrals, we refer to the book by Da Prato and Zabczyk [DPZ92] or the lecture notes by Prévôt and Röckner [PR07].

Let $(W_t)_{0\le t\le T}$, be an $\mathbb{H}$-valued Q-Wiener process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$. Let (1.4) be a representation of $(W_t)_{0\le t\le T}$.

Let U be another real separable Hilbert space. Denote by $L(H,U)$ the space of all linear bounded operators from $\mathbb{H}$ to U. Let $\Phi\colon [0,T] \to L(U,H)$, $t \mapsto \Phi(t)$ be a deterministic function.

For every $0 \le t \le T$, we define the stochastic integral of $\Phi(\cdot)$ with respect to the Wiener process $W(\cdot)$ by

$$\int_0^t \Phi_s \, dW_s := \sum_{k=1}^{\infty} \int_0^t \sqrt{\lambda_k} \Phi_s e_k \, d\beta_k(s) = \sum_{k=1}^{\infty} \int_0^t \Phi_s Q^{1/2} e_k \, d\beta_k(s). \tag{1.5}$$

The generic term

$$\int_0^t \Phi_s Q^{1/2} e_k \, d\beta_k(s)$$

in the series (1.5) is a U-valued Wiener integral defined by

$$\int_0^t \Phi_s Q^{1/2} e_k \, d\beta_k(s) = \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} \int_0^t \langle \Phi_s Q^{1/2} e_k, f_l \rangle \, d\beta_k(s) \right) f_l$$

where $\{f_l\}_{l\in\mathbb{N}}$ is a complete orthogonal normal basis of the Hilbert space U.

It is easy to check

$$\begin{aligned}
\mathbb{E} \left| \int_0^t \Phi_s dW_s \right| &= \sum_{k=1}^{\infty} \int_0^t |\Phi(s) Q^{1/2} e_k|^2 \, ds \\
&= \sum_{k=1}^{\infty} \int_0^t \langle (\Phi_s Q^{1/2})^* \Phi(s) Q^{1/2} e_k, e_k \rangle \, ds \\
&= \int_0^t \mathrm{Tr}\big[(\Phi_s Q^{1/2})^* \Phi(s) Q^{1/2} \big] \, ds.
\end{aligned}$$

Hence, the series (1.5) converges in $L^2(\Omega, \mathscr{F}, U)$ if and only if

$$\int_0^t \operatorname{Tr}\big(\Phi_s Q \Phi_s^*\big)\, ds < \infty. \tag{1.6}$$

In this case, we can show that the stochastic integral Gaussian random variable with covariance (see [DPZ92, Theorem 5.2])

$$Q_t := \int_0^t \Phi_s Q \Phi_s^*\, ds.$$

Cylindrical Q-Wiener Processes and Stochastic Integrals

Formally, in the definition of the stochastic integrals (1.5), the operator Q is not necessary to be of trace class since we only need the condition (1.6) hold. This leads us to introduce *cylindrical Q-Wiener processes.*

Let Q be a bounded, self-adjoint and nonnegative operator on U. Set $U_0 = Q^{1/2}(U)$. Let $\widetilde{U_0}$ be an arbitrary Hilbert space such that U_0 is embedded continuously into $\widetilde{U_0}$

$$J\colon (U_0, |\cdot|_{U_0}) \to (\widetilde{U_0}, |\cdot|_{\widetilde{U_0}})$$

and the embedding J is a Hilbert-Schmidt operator.

Now we define

$$W_t = \sum_{k=1}^{\infty} \beta_k(t) J g_k, \quad 0 \le t \le T,$$

where $\{g_k\}_{k\in\mathbb{N}}$ is an orthogonal normal basis of U_0 and $\{\beta_k\}_{k\in\mathbb{N}}$ is a family of independent real Brownian motions.

Therefore $(W_t)_{0\le t\le T}$ defines a $\widetilde{Q} := JJ^*$-Wiener process on $\widetilde{U_0}$ with $\operatorname{Tr}\widetilde{Q} < \infty$.

If $\operatorname{Tr} Q = \infty$, then we will call the constructed process $(W_t)_{0\le t\le T}$ a *cylindrical Q-Wiener process* on U. If $Q = I$, then we simply call it a *cylindrical Wiener process.*

When there is no confusion, we will simply say *Q-Wiener processes* without distinguishing whether it is the standard one and cylindrical one.

The cylindrical Q-Wiener processes are not uniquely defined but the stochastic integrals with respect to cylindrical Q-Wiener processes are independent of the choices of U_1 and hence are well-defined.

1.4 Lévy Processes

In this section, we introduce some preliminaries on Hilbert space valued Lévy processes. See the books Bertoin [Ber96], K. Sato [Sat99], Applebaum [App04], and Cont and Tankov [CT04] etc. for general introductions to Lévy processes in finite dimension spaces. We refer to the monograph Peszat and Zabczyk [PZ07] and bibliographies therein for the introduction of Lévy processes in infinite dimension.

1.4.1 Lévy Processes and Infinite Divisible Distributions

Let $\mathbb{H}$ be a real separable Hilbert space. A *Lévy process* is a time-homogeneous Markov process with space homogeneity. We give the precise definition of Lévy processes in the following.

Definition 1.4.1. Let $(X_t)_{0\le t<\infty}$ be an $\mathbb{H}$-valued stochastic process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t<\infty}, \mathbb{P})$. We call X_t a Lévy process if it is an $\mathscr{F}_t$-adapted, stochastically continuous process with independent stationary increments. That is, X_t satisfies the following conditions.

(1) $X_0 = 0$ $\mathbb{P}$-a.s.
(2) X_t is adapted: $X_t \in \mathscr{F}_t$ for every $t \ge 0$;
(3) Independent increments: $X_t - X_s$ is independent of $\mathscr{F}_s$ for every $0 \le s \le t$;
(4) Stationary increments: the distribution of $X_t - X_s$ only depends on the time interval $t - s$ for every $0 \le s \le t$;
(5) Stochastically continuous: for every $s \ge 0$ and $\varepsilon > 0$,

$$\lim_{t\to s} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0.$$

It can be proved (see [Sat99, Chap. 1] or [Pro90, Theorem 30]) that there is a unique modification of each Lévy process such that every path of the process is right continuous with left limits. Without loss of generality, we will assume in this thesis that every Lévy process is right continuous with left limits.

Lévy processes are closely related to *infinite divisible distributions (or measures).*

Let μ and ν be two σ-finite measures on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$, their *convolution* $\mu * \nu$ is defined by

$$\mu * \nu(A) = \int_{\mathbb{H}} \mu(A - x)\, \nu(dx), \quad A \in \mathcal{B}(\mathbb{H}).$$

A measure μ (or a random variable with distribution μ) is called *infinitely divisible* if for every $n \in \mathbb{N}$, there exists a measure μ_n such that

$$\mu = \mu_n^{*n} = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

We will introduce a representation of the characteristic functions of infinitely divisible distributions. We first recall the definition of *Lévy measure.*

Definition 1.4.2. We call a σ-finite measure ν on $\mathbb{H}$ a *Lévy measure* if it is concentrated on $\mathbb{H} \setminus \{0\}$ and satisfies

$$\int_{\mathbb{H}\setminus\{0\}} (1 \wedge |z|^2)\, \nu(dx) < \infty.$$

The following Lévy-Kintchine formula characterize the structure of infinite divisible distributions.

Theorem 1.4.3. *If μ is an infinite divisible distribution on $\mathbb{H}$, then the characteristic function of μ is given by*

$$\mu[\exp(i\langle u, x\rangle)] = \exp[-\lambda(u)], \quad \text{for all } u \in \mathbb{H}$$

with

$$\begin{aligned}\lambda(u) = -i\langle u, b\rangle &+ \frac{1}{2}\langle Ru, u\rangle \\ &+ \int_{\mathbb{H}} \left[1 - \exp(i\langle x, u\rangle) + i\langle x, u\rangle \mathbb{1}_{\{|x|\le 1\}}(x)\right] \nu(dx),\end{aligned} \tag{1.7}$$

where $b \in \mathbb{H}$, R is a trace class operator on $\mathbb{H}$, and ν is a Lévy measure on $\mathbb{H}$.

Definition 1.4.4. We call λ the *symbol* or the *characteristic exponent* of the infinite divisible distribution μ and (b, R, ν) the *characteristic triplet* (or *generating triplet*) of μ.

Denote the law of X_t by μ_t. It is easy to see that $\mu_{t+s} = \mu_t * \mu_s$ for all $t, s \ge 0$. This follows that μ_t is an infinite divisible measure for every $t > 0$.

Remark 1.4.5. The distribution $(\mu_t)_{t\ge 0}$ is called a *convolution semigroup*, or an *infinite divisible family.* Comparing with the concepts of *skew convolution semigroup* in Remark 7.1.1.

Define the transition semigroup by

$$P_t f(x) = \int_{\mathbb{H}} f(x+y)\,\mu_t(dy), \quad f \in \mathscr{C}_b^+(\mathbb{H}).$$

By Kolmogorov's theorem, there is a one-to-one correspondence between Lévy processes and infinite divisible distributions. And it is easy to see that for each $t \geq 0$, the random variable X_t is infinitely divisible and

$$\mathbb{E}\exp(i\langle X_t, u\rangle) = \exp(-t\eta(u)), \quad \text{for all } u \in \mathbb{H}.$$

Here $\eta : \mathbb{H} \mapsto \mathbb{C}$ is the characteristic exponent of X_1, that is,

$$\eta(u) = -\log \mathbb{E}\exp(i\langle X_1, u\rangle), \quad \text{for all } u \in \mathbb{H}.$$

By the representation for infinite divisible distribution (see Theorem 1.4.3), we have the following Lévy-Kintchine formula for lévy processes, which is a characterization of Lévy processes.

Theorem 1.4.6. *There exists a triplet (b, R, ν) such that for every $u \in \mathbb{H}$,*

$$\begin{aligned}\eta(u) = -i\langle u, b\rangle + \frac{1}{2}\langle Ru, u\rangle \\ + \int_{\mathbb{H}} \left[1 - \exp(i\langle x, u\rangle) + i\langle x, u\rangle \mathbb{1}_{\{|x|\leq 1\}}(x)\right] \nu(dx),\end{aligned} \tag{1.8}$$

where b is element in $\mathbb{H}$, R is a trace class operator on $\mathbb{H}$, and ν is a Lévy measure on $\mathbb{H}$.

Definition 1.4.7. We call η the *Lévy symbol* or the *characteristic exponent* of the Lévy process X_t and (b, R, ν) the *characteristic triplet* (or *generating triplet*) of X_t.

1.4.2 The Lévy-Itô Decomposition and Stochastic Integrals

Corresponding to the three terms in the Lévy-Kintchine formula (1.8), every Lévy process can be decomposed into three independent processes.

We first introduce Poisson random measure.

Definition 1.4.8. A *Poisson random measure* on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$ with intensity measure ν is a family of random variables $\{N(B), B \in \mathcal{B}(\mathbb{H})\}$ on some probability space Ω

$$N\colon \Omega\times \mathbb{H} \to \mathbb{Z}_+ := \{0,1,2,\dots\},$$
$$(\omega, B) \mapsto N(\omega, B)$$

such that

(1) For almost all $\omega\in\Omega$, $N(\omega,\cdot)$ is an non-negative integer valued measure on $\mathbb{H}$;
(2) For each measurable set $B\in\mathcal{B}(\mathbb{H})$, $N(\cdot,B) = N(B)$ is a Poisson random variable with parameter $\nu(B)$;
(3) For all disjoint measurable sets $B_1, B_2, \dots, B_n$ in $\mathcal{B}(\mathbb{H})$ with $n\in\mathbb{N}$, the random variables $N(B_1), N(B_2), \dots, N(B_n)$ are independent.

We define the corresponding *compensated Poisson random measure* by

$$\widetilde{N}(B) := N(B) - \nu(B)$$

for all $B\in\mathcal{B}(\mathbb{H})$.

Let X_t be a Lévy process on $\mathbb{H}$. The jump of X_t at time t is defined as $\varDelta X_t = X_t - X_{t-}$, where $X_{t-} = \lim_{s\uparrow t} X_s$.

For every $B\in\mathcal{B}(\mathbb{H}\setminus\{0\})$, we count the number of jumps of X_t in B before time t by

$$N(t,B) := \sharp\left\{s\in[0,t)\colon \varDelta X_s\in B\right\}.$$

It can be shown that $N(dt,dx)$ is a Poisson random measure on $[0,+\infty)\times\mathbb{H}$ with intensity measure $dt\times\nu(dx)$. And the associated compensated Poisson random measure of $N(dt,dx)$ is given by

$$\widetilde{N}(dt,dx) = N(dt,dx) - dt\times\nu(dx).$$

Now we can state the famous Lévy-Itô decomposition of Lévy processes.

Proposition 1.4.9. *Let X_t be a Lévy process on $\mathbb{H}$ with characteristic triplet (b,R,ν). Then we have*

$$\begin{aligned} X_t &= bt + W_t + \int_0^t\int_{\{|x|\le 1\}} x\,\widetilde{N}(ds,dx) + \int_0^t\int_{\{|x|>1\}} z\,N(ds,dx) \\ &= bt + W_t + \int_{\{|x|\le 1\}} z\,\widetilde{N}(t,dx) + \int_{\{|x|>1\}} z\,N(t,dx), \end{aligned} \tag{1.9}$$

where W_t is a R-Wiener process on $\mathbb{H}$ and N is a Poisson random measure on

$[0,\infty) \times (\mathbb{H} \setminus \{0\})$ *with intensity measure* $dt \times \nu$.

For a proof, we refer to [AR05] for a general introduction to the Lévy-Itô decomposition on separable Banach spaces.

1.4.3 Stochastic integral with respect to Lévy noise

We will only consider the stochastic integrals with respect to Lévy processes when the integrand is just time-dependent. We refer to [App04, AR05, App06, MR06, App07a] etc. for more details of stochastic integrals with respect to Lévy processes.

Let F be a measurable function from $[0,\infty)$ to the space of all linear bounded operators on $\mathbb{H}$ such that $t \mapsto |F(t)|$ is locally square integrable. Then the integral of F with respect to X_t could be defined in the following via (1.9):

$$\begin{aligned}\int_0^t F(s)\,dX_s := &\int_0^t F(s)b\,ds + \int_0^t F(s)\,dW_s \\ &+ \int_0^t \int_{\{|x|\le 1\}} F(s)x\,\widetilde{N}(ds,dx) + \int_0^t \int_{\{|x|>1\}} F(s)x\,N(ds,dx).\end{aligned}$$

Here, in the right hand side of the definition of integral above, the first and fourth integrals are defined by the standard Bochner integrals; the second integrals are stochastic integral with respect to Wiener process; and third integral is stochastic integral with respect to Poisson random measure (refer to the literatures mentioned above).

We have the following assertion about the stochastic integrals.

Proposition 1.4.10. *For every* $t \ge 0$, *the integral* $\int_0^t F(s)dX_s$ *is infinitely divisible and its characteristic exponent is given by*

$$\lambda_{t,F}(u) := \int_0^t \eta\big(F(s)^* u\big)\,ds, \quad u \in \mathbb{H}.$$

See Chojnowska-Michalik [Cho87, Corollary 2.1], Applebaum [App07b, Proposition 2.1], or [PZ07, Corollary 4.1] for a proof.

The following corollary is an immediate consequence of Equation (1.8) and Proposition 1.4.10.

Corollary 1.4.11. *For every $t \ge 0$, the characteristic triplet (b_t, R_t, ν_t) of the integral $\int_0^t F(s)\, dX_s$ is given by*

$$b_t = \int_0^t F(s) b\, ds + \int_0^t \int_{\mathbb{H}\setminus\{0\}} F(s)x\big[\mathbb{1}_B\big(F(s)x\big) - \mathbb{1}_B(x)\big]\, \nu(dx)\, ds;$$

$$R_t = \int_0^t F(s)RF(s)^*\, ds;$$

$$\nu_t(A) = \int_0^t v(F(s)^{-1}A)\, ds, \quad A \in \mathcal{B}(\mathbb{H}\setminus\{0\}),$$

where $B = \{x \in \mathbb{H} \colon |x| \le 1\}$.

Remark 1.4.12. The conclusion of Corollary 1.4.11 is also stated in [App07b, Corollary 2.1]. But note that there is a misprint therein: $\big[\mathbb{1}_B\big(F(s)x\big) - \mathbb{1}_B(x)\big]$ is written as $\big[\mathbb{1}_B(x) - \mathbb{1}_B\big(F(s)x\big)\big]$ in [App07b, Corollary 2.1].

Proof of Corollary 1.4.11. By applying Proposition 1.4.10 and note the Lévy-Kintchine formula (1.8), for every $u \in \mathbb{H}$, we have

$$\begin{aligned}
&\int_0^t \eta(F(s)^*u)\, ds \\
=&-i\int_0^t \langle b, F(s)^*u\rangle\, ds + \frac{1}{2}\int_0^t \langle F(s)^*u, RF(s)^*u\rangle\, ds \\
&+\int_0^t\int_{\mathbb{H}\setminus\{0\}} \big[1 - e^{i\langle x, F(s)^*u\rangle} + i\,\langle x, F(s)^*u\rangle\, \mathbb{1}_B(x)\big]\, \nu(dx)\, ds \\
=&-i\left\langle u, \int_0^t F(s)b\, ds\right\rangle + \frac{1}{2}\left\langle u, \left(\int_0^t F(s)RF(s)^*\, ds\right) u\right\rangle \\
&+\int_0^t\int_{\mathbb{H}\setminus\{0\}} \big[1 - e^{i\langle F(s)x, u\rangle} + i\,\langle F(s)x, u\rangle\, \mathbb{1}_B(x)\big]\, \nu(dx)\, ds.
\end{aligned}$$

Now we rewrite the last term of the equation above in the following way.

$$\begin{aligned}
&i\int_0^t\int_{\mathbb{H}\setminus\{0\}} [\langle F(s)x, u\rangle\, \mathbb{1}_B(x) - \langle F(s)x, u\rangle\, \mathbb{1}_B(F(s)x)]\, \nu(dx)\, ds \\
&\quad+\int_0^t\int_{\mathbb{H}\setminus\{0\}} \big[1 - e^{i\langle F(s)x, u\rangle} + i\,\langle F(s)x, u\rangle\, \mathbb{1}_B(F(s)x)\big]\, \nu(dx)\, ds \\
=&i\left\langle \int_0^t\int_{\mathbb{H}\setminus\{0\}} F(s)x\, [\mathbb{1}_B(x) - \mathbb{1}_B(F(s)x)]\, \nu(dx)\, ds, u\right\rangle \\
&\quad+\int_0^t\int_{\mathbb{H}\setminus\{0\}} \big[1 - e^{i\langle x, u\rangle} + i\,\langle x, u\rangle\, \mathbb{1}_B(x)\big]\, \nu(F(s)^{-1}dx)\, ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^t \eta(F(s)^*u)\,ds \\
=&-i\left\langle u, \int_0^t F(s)b\,ds + \int_0^t \int_{\mathbb{H}\setminus\{0\}} F(s)x\left[\mathbb{1}_B(F(s)x) - \mathbb{1}_B(x)\right]\nu(dx)\,ds\right\rangle \\
&+\frac{1}{2}\left\langle u, \left(\int_0^t F(s)RF(s)^*\,ds\right)u\right\rangle \\
&+\int_0^t \int_{\mathbb{H}\setminus\{0\}} \left[1 - e^{i\langle x,u\rangle} + i\langle x,u\rangle\,\mathbb{1}_B(x)\right]\nu(F(s)^{-1}dx)\,ds.
\end{aligned}$$

This completes the proof. □

1.4.4 Symmetric α-Stable Processes

We will shortly introduce a special Lévy processes – symmetric α-stable processes on $\mathbb{R}^d$. We refer to the monograph Samorodnitsky and Taqqu [ST94] for more details.

We start with the introduction with *symmetric α-stable random variable.*

Definition 1.4.13. An $\mathbb{R}^d$-valued random variable ζ is called a *symmetric α-stable random variable* if

$$\mathbb{E}\exp(i\langle u,\zeta\rangle) = \exp(-|u|^\alpha), \quad u \in \mathbb{R}^d$$

for some $\alpha \in (0,2]$.

All α-stable random variables have densities. We denote the relative density of the stable random variable ζ with respect to the Lebesgue measure on $\mathbb{R}^d$ by $\hat{p}_\alpha(\cdot)$:

$$\mathbb{P}(\zeta \in A) = \int_A \hat{p}_\alpha(x)\,dx, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

But only for a few stable random variables the densities have closed forms. For $\alpha \neq 2$, the stable random variable have polynomial decay ([BG60, Theorem 2.1]):

$$\hat{p}_\alpha(x) \sim \frac{1}{|x|^{d+\alpha}} \quad (|x| \to +\infty). \tag{1.10}$$

Definition 1.4.14. An $\mathbb{R}^d$-valued process $(X_t)_{t\geq 0}$ is called a *symmetric α-stable (SαS) process* if it is a Lévy process such that for all $t \geq 0$, X_t is a symmetric

α-stable random variable with

$$\mathbb{E}\exp(\langle u, X_t\rangle) = \exp(-t|u|^\alpha), \quad u \in \mathbb{R}^d.$$

Let us denote the transition density of X_t by $\hat{p}_\alpha(t,\cdot)$, and the transition density of X_t starting from x by $\hat{p}_\alpha(t,x,\cdot)$. That is,

$$\mathbb{P}(X_t \in A) = \int_A \hat{p}_\alpha(t,x)\,dx, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

and

$$\mathbb{P}(X_t \in A|X_0 = x) = \int_A \hat{p}_\alpha(t,x,y)\,dy, \quad x \in \mathbb{R}^d,\ A \in \mathcal{B}(\mathbb{R}^d).$$

Their connection is given by

$$\hat{p}_\alpha(t,x,y) = \hat{p}_\alpha(t,x-y), \quad \text{for all } x,y \in \mathbb{R}^d.$$

The transition density $\hat{p}_\alpha(t,x)$ of the process X_t has the following scaling property.

Lemma 1.4.15. *For every $a > 0$, we have*

$$\hat{p}_\alpha(t,x) = a^d \hat{p}_\alpha(a^\alpha t, ax) \tag{1.11}$$

Proof. For each $u \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{i\langle x,u\rangle}\,\hat{p}_\alpha(t,x)\,dx = \exp(-t|u|^\alpha) = \exp(-a^\alpha t|a^{-1}u|^\alpha)\\
=&\int_{\mathbb{R}^d} \mathrm{e}^{i\langle x,a^{-1}u\rangle}\,\hat{p}_\alpha(a^\alpha t,x)\,dx = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle a^{-1}x,u\rangle}\,\hat{p}_\alpha(a^\alpha t,x)\,dx\\
=&\int_{\mathbb{R}^d} \mathrm{e}^{i\langle x,u\rangle}\,\hat{p}_\alpha(a^\alpha t,ax)a^d\,dx.
\end{aligned}$$

□

There is a natural relationship $\hat{p}_\alpha(1,x) = \hat{p}_\alpha(x)$ between the transition density of the process at time 1 and the density of stable random variable. Therefore, by taking $a = t^{-1/\alpha}$ in the scaling property (1.11), we can get

$$\hat{p}_\alpha(t,x) = t^{-d/\alpha}\hat{p}_\alpha(t^{-1/\alpha}x), \quad x \in \mathbb{R}^d,\ t \geq 0.$$

Based on (1.10), Bogdan et al. [BSS03, Theorem 3.1] proved the following estimates of the transition density.

Lemma 1.4.16. *For every $x, y \in \mathbb{R}^d$ with $x \neq y$ and $t > 0$, there exists some constant $K > 0$ such that*

$$K^{-1}\left(\frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \leq \hat{p}_\alpha(t,x,y) \leq K\left(\frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/\alpha}\right)$$

Chapter 2

Absolute Continuity of Lévy Processes in Infinite Dimensional Spaces

In this chapter we aim to generalize the main results in the lecture notes by Sato [Sat00] to the infinite dimensional case. In Section 2.1 we introduce some basic notations and the main problems. We refer to the summary of the structure of this chapter at the end of Section 2.1.

The Girsanov theorem for Lévy processes studied in Section 2.2 will be used in Subsection 7.4.1 to establish Harnack inequalities for Ornstein-Uhlenbeck processes driven by Lévy processes.

2.1 Introduction

Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Denote by $\mathbb{D}$ the *Skorokhod space* $D([0,\infty), \mathbb{H})$ over $\mathbb{H}$. Recall that $\mathbb{D}$ consists of all right continuous with left limits functions from $[0,\infty)$ to $\mathbb{H}$. Denote by X_t the canonical process on $\mathbb{D}$ defined by $X_t(\omega) = \omega(t)$ for every path $\omega \in \mathbb{D}$ and $t \geq 0$.

Set

$$\mathscr{F}_t = \sigma(X_s \colon 0 \leq s \leq t), \quad t \geq 0$$

and

$$\mathscr{F} = \sigma(X_s \colon 0 \leq s < \infty).$$

Note that every Lévy process can be realized as a canonical process on the filtered Skorokhod space $(\mathbb{D}, \mathscr{F}, (\mathscr{F}_t)_{0\le t<\infty})$ with some probability measure $\mathbb{P}$. So we will regard each Lévy process as a probability measure on the Skorokhod space $\mathbb{D}$ and vise versa.

Now we consider two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(\mathbb{D}, \mathscr{F}, (\mathscr{F}_t)_{0\le t<\infty})$. Assume that the characteristic triplet of the canonical Lévy process X_t on $\mathbb{D}$ is (b_j, R_j, ν_j) under $\mathbb{P}_j$ for $j = 1, 2$. In other words, for $j = 1, 2$,

$$\mathbb{E}_{\mathbb{P}_j} \exp(i\langle u, X_t\rangle) = \exp(-t\eta_j(u)), \quad u \in \mathbb{H},$$

where the characteristic symbol η_j is given by

$$\begin{aligned}\eta_j(u) = -i\langle u, b_j\rangle + \frac{1}{2}\langle R_j u, u\rangle \\ + \int_{\mathbb{H}} \left[1 - \exp(i\langle u, x\rangle) + i\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}}(x)\right] \nu_j(dx).\end{aligned}$$

For every $t \ge 0$, we denote the restriction of $\mathbb{P}_j$ on $\mathscr{F}_t$ by $\mathbb{P}_j^t$:

$$\mathbb{P}_j^t = \mathbb{P}_j|\mathscr{F}_t, \qquad j = 1, 2.$$

We are interested at the following problems.

(1) The necessary and sufficient conditions for the absolute continuity and orthogonality of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$;
(2) The Radon-Nikodým derivative of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$ in the case $\mathbb{P}_2^t \ll \mathbb{P}_1^t$;
(3) The Lebesgue decomposition of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$;
(4) The Radon-Nikodým derivative of the absolute continuous part of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$.

For the finite dimensional case, the first two problems were solved by Skorokhod [Sko57, Sko60], Kunita and Watanabe [KW67], Newman [New72, New73] and treated and reformulated by Sato [Sat99, Chapter 6, Section 33]. And the last two problems were treated explicitly first in [Sat00] for the finite dimensional case. We refer to [Sat00] for more notes.

In this chapter, we are going to follow the line in Sato's lecture notes [Sat00] and formulate the main results therein for the infinite dimensional case.

The generalization may sounds trivial to some experts. For example, Jacod and Shiryaev [JS87] studied the absolute continuity of general semi-martingales. But it seems that maybe it is interesting to write down the results for Lévy processes directly. Moreover, as mentioned in [Sat00], the problems (3) and (4) are new.

Now we describe the organization of this chapter.

We start with a Girsanov theorem for Lévy process in Section 2.2. Here we consider a drift transforamtion of a Lévy process with characteristic triplet (b_1, R, ν_1). We obtain another Lévy process with characteristic triplet (b_2, R, ν_2) for the case $\nu_2 = \nu_1$ under a new probability measure. We just apply Girsanov's theorem for the Gaussian part of the Lévy process by using the independence of the Gaussian part and the jump part of the Lévy process. We will generalize this Girsanov theorem from the case $\nu_1 = \nu_2$ to the case when the Hellinger-Kakutani distance of ν_1 and ν_2 is finite. It will turn out that the main idea of this chapter is using the independence.

We introduce *Hellinger-Kakutani inner product* and *distance* of r-order ($r \in (0, 1)$) for any two σ-finite measures in Section 2.3. These concepts have first been introduced by Hellinger [Hel07] and Kakutani [Kak48] for the order $r = 1/2$. They are powerful tools in the study of absolute continuity of measures (see Remark 1.2.3 for bibliographic notes). We list here some related references: Brody [Bro71], Newman [New72, New73], Memin and Shiryayev [MS85] etc.. We also refer to [Sat00] and the references therein.

In Section 2.4 we introduce the non-singularity condition (2.2) for the absolute continuity of two Lévy processes. See Theorem 2.4.1 and Corollary 2.4.5.

To prove Theorem 2.4.1, we need the corresponding results for the Gaussian case which are proved in Section 2.5. The generalization of the non-singularity condition from the finite dimensional case to the infinite dimensional case stems from the Gaussian case.

Then we sketch the proof of Theorem 2.4.1 in Seciton 2.6. In Section 2.7, we study the density of one Lévy process with respect to another.

We mention that some applications which is omitted on the density transformation can be done similar to [Sat00, Section 7].

2.2 Girsanov's Theorem for Lévy Processes

We will prove a special Girsanov theorem for Lévy processes. It says that a drift transformed Lévy process is still a Lévy process with the same distribution under a new probability measure.

Let $\mathbb{H}$ be a separable Hilbert space and R a trace class operator on $\mathbb{H}$. We denote the Cameron-Martin space of $\mathbb{H}$ by $\mathbb{H}_0 = R^{1/2}(\mathbb{H})$ and the inner product of $\mathbb{H}_0$ by $\langle \cdot, \cdot \rangle_0$:

$$\langle x, y \rangle_0 = \left\langle R^{-1/2}x, R^{-1/2}y \right\rangle, \quad x, y \in \mathbb{H}_0.$$

We denote by $|\cdot|_0$ for the norm on $\mathbb{H}_0$ corresponding to the inner product $\langle \cdot, \cdot \rangle_0$.

The following Girsanov's theorem for Wiener processes in infinite dimensional spaces is due to Bensoussan [Ben71] and Kozlov [Koz78] (see also [DPZ92, Theorem 10.14] for a proof).

Theorem 2.2.1. *Let $T > 0$. Suppose that $(W(t))_{0 \le t \le T}$ is an $\mathbb{H}$-valued R-Wiener process on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$. Let $\psi(\cdot)$ be an $\mathbb{H}_0$-valued $\mathscr{F}_t$-predictable process such that*

$$\mathbb{E}\, \rho^W(T) = 1$$

with

$$\rho^W(T) = \exp\left(\int_0^T \langle \psi(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi(s)|_0^2 \, ds \right).$$

Then

$$\widetilde{W}(t) := W(t) - \int_0^t \psi(s)\, ds, \quad 0 \le t \le T$$

is an R-Wiener process on $(\Omega, \mathscr{F}, \mathscr{F}_t, 0 \le t \le T)$ under a new probability measure $\widetilde{\mathbb{P}}$ defined by

$$\widetilde{\mathbb{P}}|_{\mathscr{F}_T} = \rho^W(T)\mathbb{P}.$$

With Theorem 2.2.1, we can prove the following Girsanov theorem for Lévy process.

Theorem 2.2.2. *Let $T > 0$. Suppose that $(X(t))_{0 \le t \le T}$ is an $\mathbb{H}$-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$ with characteristic triplet (b, R, ν). Denote by $X'(\cdot)$ the Gaussian part of $X(\cdot)$. Let $\psi(\cdot)$ be an $\mathbb{H}_0$-valued $\mathscr{F}_t$-predictable process, independent of $X(t) - X'(t)$ such that*

$$\mathbb{E}\, \rho^{X'}(T) = 1$$

with

$$\rho^{X'}(T) = \exp\left(\int_0^T \langle \psi(s), dX'(s) \rangle_0 - \frac{1}{2} \int_0^t |\psi(s)|_0^2 \, ds \right).$$

Then

$$\widetilde{X}(t) := X(t) - \int_0^t \psi(s)\, ds, \quad 0 \le t \le T$$

is also a Lévy process on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T})$ *with the same characteristic triplet* (b, R, ν) *under a new probability measure* $\widetilde{\mathbb{P}}$ *defined by*

$$\widetilde{\mathbb{P}} = \rho^{X'}(T)\mathbb{P}.$$

Proof. Since $(X(t))_{0\le t\le T}$ is a Lévy process with characteristic triplet (b, R, ν) on $\mathbb{H}$, the Fourier transformation of X_t is given by

$$\mathbb{E}_{\mathbb{P}} \exp(i\langle X(t), u\rangle) = \exp\left[-t\vartheta_1(u) - t\vartheta_2(u)\right], \quad u \in \mathbb{H},$$

where for every $u \in \mathbb{H}$,

$$\vartheta_1(u) = \frac{1}{2}\langle Ru, u\rangle$$

and

$$\vartheta_2(u) = -i\langle u, b\rangle + \int_{\mathbb{H}} \left[1 - \exp(i\langle u, x\rangle) + i\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}}(x)\right] \nu(dx).$$

In other words, we have the following (Lévy-Itô) decomposition

$$X(t) = X'(t) + X''(t) = X'(t) + (X(t) - X'(t)).$$

Here $X'(t)$ is the Gaussian part of $X(t)$ with symbol ϑ_1; while $X''(t)$ is a drifted jump process with symbol ϑ_2. These two processes, $X'(t)$ and $X''(t)$, are independent to each other.

For every $0 \le t \le T$, we define

$$\rho^{X'}(t) = \exp\left(\int_0^t \langle\psi(s), dX'(s)\rangle_0 - \frac{1}{2}\int_0^t |\psi(s)|_0^2\, ds\right).$$

Then we have

$$\widetilde{\mathbb{P}}|_{\mathscr{F}_t} = \rho^{X'}(t)\mathbb{P}, \quad 0 \le t \le T.$$

By the Girsanov theorem for Wiener processes on Hilbert space (see Theorem 2.2.1), we know

$$\widetilde{X}'(t) = X'(t) - \int_0^t \psi(s)\, ds$$

is a still an R-Wiener process with respect to the new probability measure $\widetilde{\mathbb{P}}$.

Consequently, for all $0 \le t \le T$ and all $u \in \mathbb{H}$, we have

$$\mathbb{E}_{\mathbb{P}}\rho^{X'}(t) \exp(i\langle u, \widetilde{X}'(t)\rangle) = \mathbb{E}_{\widetilde{\mathbb{P}}} \exp(i\langle u, \widetilde{X}'(t)\rangle) = \exp\left[-t\vartheta_1(u)\right]. \tag{2.1}$$

Therefore, by the independence of X' and X'' and the equation (2.1) above, we get

$$\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{P}}} \exp\big(i\langle u, \widetilde{X}(t)\rangle\big) \\
=&\mathbb{E}_{\mathbb{P}} \rho^{X'}(t) \exp\big(i\langle u, \widetilde{X}(t)\rangle\big) \\
=&\mathbb{E}_{\mathbb{P}} \rho^{X'}(t) \exp\left(i\left\langle u, X(t) - \int_0^t \psi(s)\,ds\right\rangle\right) \\
=&\mathbb{E}_{\mathbb{P}} \rho^{X'}(t) \exp\left(i\left\langle u, X'(t) - \int_0^t \psi(s)\,ds + X''(t)\right\rangle\right) \\
=&\mathbb{E}_{\mathbb{P}} \rho^{X'}(t) \exp\left[i\left\langle u, \widetilde{X}'(t)\right\rangle\right] \cdot \mathbb{E}_{\mathbb{P}} \exp\big(i\langle u, X''(t)\rangle\big) \\
=&\exp\left[-t\vartheta_1(u)\right] \cdot \exp\left[-t\vartheta_2(u)\right] \\
=&\exp\left[-t\vartheta_1(u) - t\vartheta_2(u)\right].
\end{aligned}$$

It follows that the characteristic symbol of $\widetilde{X}$ with respect to $\widetilde{\mathbb{P}}$ is $\vartheta_1 + \vartheta_2$, which is the same with the characteristic symbol of X with respect to $\mathbb{P}$. This fact implies that $\widetilde{X}$ is also a Lévy process with characteristic triplet (b, R, ν) under the new probability measure $\widetilde{\mathbb{P}}$. □

Remark 2.2.3. Ren and Röckner [RR07] considered also a similar Girsanov theorem by martingale methods.

2.3 Hellinger-Kakutani Theory

Let σ_1, σ_2 be two σ-finite measures on a general measurable space $(E, \mathscr{E})$. Consider a σ-finite measure σ on $(E, \mathscr{E})$ such that both σ_1 and σ_2 are absolute continuous with respect to σ:

$$\sigma_1 \ll \sigma \quad \text{and} \quad \sigma_2 \ll \sigma.$$

Note that the measure σ does exist. For example, we can simply take $\sigma = \sigma_1 + \sigma_2$.

For $i = 1, 2$, we denote by $f_i = d\sigma_i / d\sigma$ for the Radon-Nikodým derivative of σ_i with respect to σ. We will fix one version of the derivative f_i for $i = 1, 2$.

Definition 2.3.1. The *Hellinger-Kakutani inner product* $\mathbb{H}_r(\sigma_1, \sigma_2)$ of σ_1 and σ_2

of order $r \in (0,1)$ is defined by

$$\mathbb{H}_r(\sigma_1, \sigma_2)(A) = \int_A f_1^r f_2^{1-r} d\sigma, \quad A \in \mathscr{E}.$$

The *Hellinger-Kakutani integral* $h_r(\sigma_1, \sigma_2)$ is defined by as the total mass of $\mathbb{H}_r(\sigma_1, \sigma_2)$ on E:

$$h_r(\sigma_1, \sigma_2) = \mathbb{H}_r(\sigma_1, \sigma_2)(E).$$

Remark 2.3.2. It is easy to verify (see [Sat00, Remark 2.3, 2.4]) that the definition of $\mathbb{H}_r(\sigma_1, \sigma_2)$ is independent of the choice of σ. Therefore, $\mathbb{H}_r$ is well-defined. And it is also easy to verify that for all $r \in (0,1)$,

$$\mathbb{H}_r(\sigma_1, \sigma_2) \leq r\sigma_1 + (1-r)\sigma_2.$$

The following proposition shows that we can use Hellinger-Kakutani integral and inner product to characterize the orthogonality of two measures. This explains why these two notions are useful.

Proposition 2.3.3. *Two σ-finite measures are orthogonal to each other if and only if their Hellinger-Kakutani inner product (equivalently, Hellinger-Kakutani integral) of some (and hence all) order in (0,1) is zero. That is, for any two σ-finite measures σ_1 and σ_2,*

$$\sigma_1 \perp \sigma_2 \Longleftrightarrow h_r(\sigma_1, \sigma_2) = 0 \Longleftrightarrow \mathbb{H}_r(\sigma_1, \sigma_2) = 0$$

for some (and hence all) order $r \in (0,1)$.

See [Sat00, Remark 2.5] for a proof of Proposition 2.3.3. See also [DPZ92, Proposition 2.19] for the case $r = 1/2$.

Now we continue to introduce the *Hellinger-Kakutani distance* between two σ-finite measures.

Definition 2.3.4. For every $r \in (0,1)$, define

$$\mathbb{K}_r(\sigma_1, \sigma_2)(A) = \int_A \left[r f_1 + (1-r) f_2 - f_1^r f_2^{1-r} \right] d\sigma, \quad A \in \mathscr{E}.$$

The total mass of $\mathbb{K}_r(\sigma_1, \sigma_2)$ on E

$$k_r(\sigma_1, \sigma_2) = \mathbb{K}_r(\sigma_1, \sigma_2)(E)$$

is called the *Hellinger-Kakutani distance* between σ_1 and σ_2.

Remark 2.3.5. As the definition of $\mathbb{H}_r(\sigma_1, \sigma_2)$ (see Remark 2.3.2), the definition of $\mathbb{K}_r(\sigma_1, \sigma_2)$ is also independent of the choice of σ. Moreover, we know $\mathbb{K}_r(\sigma_1, \sigma_2)$ is a σ-finite measure.

We denote the weak convergence of measures by $\mu_n \to \mu$. The following assertions are useful. See [Sat00, Lemma 2.21] for a proof.

Lemma 2.3.6 (Newman, 1973). *Let μ_n, μ, ν_n, ν and π be* finite *measures on $(E, \mathscr{E})$. Fix $r \in (0,1)$. If $\mu_n \to \mu$, $\nu_n \to \nu$, $\mathbb{H}_r(\mu_n, \nu_n) \to \pi$ and* $\inf_n h_r(\mu_n, \nu_n) \geq h_r(\mu, \nu)$*, then $\mathbb{H}_r(\mu, \nu) = \pi$.*

The Kakutani distance $k_r(\sigma_1, \sigma_2)$ of two σ-finite measures σ_1, σ_2 may be infinite. The finiteness of $k_r(\sigma_1, \sigma_2)$ ensures the existence of some integrals we will use.

Lemma 2.3.7. *Let ν_1, ν_2 be two Lévy measures on a separable Hilbert space $\mathbb{H}$. If $k_r(\nu_1, \nu_2) < \infty$ for some $r \in (0,1)$, then*

$$\int_{\{|x| \leq 1\}} |x| \, d\, |\nu_1 - \nu_2| < \infty,$$

and

$$\int_{\{|x| \leq 1\}} |x| \, d\, |\nu_j - \mathbb{H}_r(\nu_1, \nu_2)| < \infty, \quad j = 1, 2.$$

Proof. It is similar to the proof in [New73, Proposition 4] (or [Sat00, Lemma 2.18]). We only need to extend the proof into the infinite dimension case which is easy. □

Remark 2.3.8. By [Sat00, Lemma 2.15], we know if $k_r(\sigma_1, \sigma_2) < \infty$ for some $r \in (0,1)$, then it holds for every $r \in (0,1)$.

Remark 2.3.9. Let $\sigma_2 = \exp(g)\sigma_1$ for some measurable function $g(x)$ satisfying $-\infty \leq g(x) < \infty$ on E. Then $k_r(\sigma_1, \sigma_2) < \infty$ for some $r \in (0,1)$ if and only if

$$\int_{\{|g| \leq 1\}} g^2 d\sigma_1 + \int_{\{g > 1\}} \exp(g) d\sigma_1 + \int_{\{g < -1\}} d\sigma < \infty.$$

See [Sat00, Remark 2.16] and the proof of [Sat00, Lemma 2.15]

We will need the following concept.

Definition 2.3.10. Define

$$C_\sigma(\sigma_1) = \left\{ x \in E \colon \frac{d\sigma_1}{d\sigma} > 0 \right\}, \quad C_\sigma(\sigma_2) = \left\{ x \in E \colon \frac{d\sigma_2}{d\sigma} > 0 \right\}.$$

We call $C_\sigma(\sigma_1)$ (resp. $C_\sigma(\sigma_2)$) the *carrier* of σ_1 (resp. σ_2) relative to σ. Sometimes we simply write $C(\sigma_j)$ for $C_\sigma(\sigma_j)$ for $j = 1, 2$.

2.4 Conditions for Absolute Continuity of Lévy Processes

The following theorem is an infinite dimensional version of Sato [Sat00, Theorem A].

Theorem 2.4.1. *Let $(X_t, \mathbb{P}_1)$ and $(X_t, \mathbb{P}_2)$ be two $\mathbb{H}$-valued Lévy processes on $(\mathbb{D}, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ with characteristic triplets (b_1, R_1, ν_1) and (b_2, R_2, ν_2) respectively.*

(1) Suppose that the following non-singularity conditions *are satisfied for some $r \in (0, 1)$*

$$k_r(\nu_1, \nu_2) < \infty, \quad R := R_1 = R_2, \quad b_{21} \in \mathbb{H}_0 := R^{1/2}(\mathbb{H}), \tag{2.2}$$

where

$$b_{21} = b_2 - b_1 - \int_{\{|x|\leq 1\}} x \, d(\nu_2 - \nu_1). \tag{2.3}$$

Then for every $t > 0$ and $r \in (0, 1)$,

$$\mathbb{H}_r(\mathbb{P}_1^t, \mathbb{P}_2^t) = \exp(-t\Phi_r)\, \mathbb{P}_r^t, \tag{2.4}$$

where

$$\Phi_r = \frac{1}{2} r(1-r)|b_{21}|_0^2 + k_r(\nu_1, \nu_2),$$

and $\mathbb{P}_r$ is the probability measure under which X_t is a Lévy process with characteristic triplet $(b_r, R, \mathbb{H}_r(\nu_1, \nu_2))$. Here

$$b_r = r b_1 + (1-r) b_2 - \int_{\{|x|\leq 1\}} x \, d\, \mathbb{K}_r(\nu_1, \nu_2).$$

(2) If (2.2) is not satisfied, then we have

$$\mathbb{H}_r(\mathbb{P}_1^t, \mathbb{P}_2^t) = 0$$

for all $t > 0$ and $r \in (0, 1)$.

Remark 2.4.2. (1) By Remark 2.3.8, the finiteness of $k_r(\nu_1, \nu_2)$ does not depend on the choice of $r \in (0, 1)$.

(2) By Lemma 2.3.7, the integral in (2.3) is well-defined, and hence b_{21} is well-defined.

(3) By Remark 2.3.2, $\mathbb{H}_r(\nu_1, \nu_2)$ is a Lévy measure.

Remark 2.4.3. To go from the finite dimensional case to the infinite dimensional case, we use the Cameron-Martin space $R^{1/2}(\mathbb{H})$ in the non-singularity condition (2.2) instead of the range $R(\mathbb{H})$ used in [Sat00, Theorem A] for the finite dimensional case. In Section 2.8, we show that if $\mathbb{H}$ is infinite dimensional, then $R(\mathbb{H}) \neq R^{1/2}(\mathbb{H})$.

Remark 2.4.4. Suppose that

$$\int_{\mathbb{H}} x\,\nu_j(dx) < \infty, \quad j = 1, 2.$$

Then by the Lévy-Itô decomposition, we can write for $j = 1, 2$,

$$\begin{aligned} X_t &= tb_j + W_t^j + \int_{\{|x|\le 1\}} x\widetilde{N}(t, dx) + \int_{\{|x|>1\}} x\,N(t, dx) \\ &= t\left[b_j - \int_{\{|x|\le 1\}} x\,\nu_j(dx)\right] + W_t^j + \int_{\{|x|\le 1\}} x\widetilde{N}(t, dx) + \int_{\mathbb{H}} x\,N(t, dx). \end{aligned}$$

Here $N(t, dx)$ is the Poisson random measure associated with X_t and $\widetilde{N}(t, dx)$ is the compensated random measure of $N(t, dx)$.

Then we see b_{21} is the difference of the "drifts":

$$b_{21} = \left[b_2 - \int_{\{|x|\le 1\}} x\,\nu_2(dx)\right] - \left[b_1 - \int_{\{|x|\le 1\}} x\,\nu_1(dx)\right].$$

From Theorem 2.4.1, we have the following corollaries which correspond to [Sat00, Corollaries 3.6-3.15] for the finite dimensional case.

Corollary 2.4.5. *(1) Fix $t > 0$. $\mathbb{P}_1^t$ and $\mathbb{P}_2^t$ are not mutually singular iff condition (2.2) is satisfied. In other words, $\mathbb{P}_1^t \perp \mathbb{P}_2^t$ iff condition (2.2) is not satisfied.*

(2) If $\mathbb{P}_1^t \perp \mathbb{P}_2^t$ for some $t > 0$, then $\mathbb{P}_1^t \perp \mathbb{P}_2^t$ for all $t > 0$.

(3) Fix $t > 0$. If $\mathbb{P}_1^t$ and $\mathbb{P}_2^t$ are not mutually singular, then

$$\nu_2\big[C(\nu_1)^c\big] < \infty, \quad \nu_1\big[C(\nu_2)^c\big] < \infty$$

and

$$\mathbb{P}_2^t\big[C(\mathbb{P}_1^t)\big] = \exp\big[-t\nu_2(C(\nu_1)^c)\big], \quad \mathbb{P}_1^t\big[C(\mathbb{P}_2^t)\big] = \exp\big[-t\nu_1(C(\nu_2)^c)\big].$$

(4) *Fix* $t > 0$. $\mathbb{P}_2^t \ll \mathbb{P}_1^t$ *iff* $\nu_2 \ll \nu_1$ *and (2.2) are both satisfied.*
(5) *If* $\mathbb{P}_2^t \ll \mathbb{P}_1^t$ *for some* $t > 0$, *then* $\mathbb{P}_2^t \ll \mathbb{P}_1^t$ *for all* $t > 0$.
(6) *Fix* $t > 0$. $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ *iff* $\nu_1 \approx \nu_2$ *and (2.2) are both satisfied.*
(7) *If* $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ *for some* $t > 0$, *then* $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ *for all* $t > 0$.
(8) *If* $\nu_1 \approx \nu_2$, *then either* $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ *for all* $t > 0$ *or* $\mathbb{P}_1^t \perp \mathbb{P}_2^t$ *for all* $t > 0$.
(9) *If* $\mathbb{P}_1 \neq \mathbb{P}_2$, *then* $\mathbb{P}_1^t \perp \mathbb{P}_2^t$.
(10) *Suppose that* $\mathbb{P}_1^t$ *and* $\mathbb{P}_2^t$ *are not mutually singular for some* $t > 0$. *Then the following are true.*
  (a) *If* $\nu_1(\mathbb{H}) < \infty$, *then* $\nu_2(\mathbb{H}) < \infty$;
  (b) *If* $\int_{\{|x|\le 1\}} |x|\, \nu_1(dx) < \infty$ *and* $\nu_1(\mathbb{H}) = \infty$, *then* $\int_{\{|x|\le 1\}} |x|\, \nu_2(dx) < \infty$ *and* $\nu_2(\mathbb{H}) = \infty$;
  (c) *If* $\int_{\{|x|\le 1\}} |x|\, \nu_1(dx) = \infty$, *then* $\int_{\{|x|\le 1\}} |x|\, \nu_2(dx) = \infty$.

2.5 Gaussian Case

In this section we prove Theorem 2.4.1 first for the Gaussian case. This section corresponds to Sato [Sat00, Section 5] where finite dimensional Gaussian case is treated. We will use Theorem 2.5.1 to prove Theorem 2.4.1 in the next section. We utilize Girsanov's theorem for Wiener processes in infinite dimensional space and the Cameron-Martin formula for Gaussian measures.

Theorem 2.5.1. *Suppose that* $(X_t, \mathbb{P}_1)$ *and* $(X_t, \mathbb{P}_2)$ *are two Lévy processes on* $(\mathbb{D}, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0})$ *with characteristic triplets* $(b_1, R_1, 0)$ *and* $(b_2, R_2, 0)$ *respectively. For any fixed* $t > 0$, *we have the following statements.*

(1) *The dichotomy holds: either* $\mathbb{P}_2^t \approx \mathbb{P}_1^t$ *or* $\mathbb{P}_1^t \perp \mathbb{P}_2^t$;
(2) $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ *if and only if the following* non-singularity conditions *are satisfied*

$$R := R_1 = R_2, \quad b_{21} := b_2 - b_1 \in \mathbb{H}_0 := R^{1/2}(\mathbb{H}). \tag{2.5}$$

(3) *If* $\mathbb{P}_1^t \approx \mathbb{P}_2^t$, *then for any* $0 < r < 1$,

$$\mathbb{H}_r(\mathbb{P}_1^t, \mathbb{P}_2^t) = \exp(-t\Psi_r)\mathbb{P}_r^t, \tag{2.6}$$

where

$$\Psi_r = \frac{1}{2} r(1-r) |b_{21}|_0^2 ,$$

and $\mathbb{P}_r$ *is the probability measure under which* $(X_t)_{t\geq 0}$ *is a Lévy process with characteristic* $(b_r, R, 0)$. *Here* b_r *is given by*

$$b_r = rb_1 + (1-r)b_2. \tag{2.7}$$

(4) If $\mathbb{P}_1^t \approx \mathbb{P}_2^t$, *then*

$$\frac{d\mathbb{P}_2^t}{d\mathbb{P}_1^t} = \exp(U_t) \tag{2.8}$$

with

$$U_t = \langle b_{21}, X_t - tb_1 \rangle_0 - \frac{t}{2} |b_{21}|_0^2 .$$

Proof. (1) We prove that if (2.5) holds, then $\mathbb{P}_1^t \approx \mathbb{P}_2^t$ and (2.8) holds.

Let $W_s^1 = X_s - sb_1$ for $0 \leq s \leq t$. Obviously $(W_s^1)_{0\leq s\leq t}$ is a R-Wiener process on $(\mathbb{D}, \mathscr{F}_t, (\mathscr{F}_s)_{0\leq s\leq t}, \mathbb{P}_1^t)$. Define a new probability measure $\mathbb{Q}_t$ on $\mathscr{F}_t$ by setting

$$\mathbb{Q}_t = \exp(U_t)\mathbb{P}_1^t|_{\mathscr{F}_t}. \tag{2.9}$$

Then by Girsanov's theorem (refer to Theorem 2.2.1), we see

$$W_s^1 - sb_{21} = X_s - sb_1 - sb_{21} = X_s - sb_2, \quad 0 \leq s \leq t$$

is a R-Wiener process on $(\mathbb{D}, \mathscr{F}_t, (\mathscr{F}_s)_{0\leq s\leq t}, \mathbb{Q}_t)$. That is, X_t is a $(b_2, R, 0)$-Lévy process under $\mathbb{Q}_t$. So, $\mathbb{Q}_t$ coincides with $\mathbb{P}_2^t$. Hence from (2.9), we see $\mathbb{P}_2^t \ll \mathbb{P}_1^t$ and (2.8) holds. Now $\mathbb{P}_1^t \ll \mathbb{P}_2^t$ also follows immediately from (2.9). Therefore we have $\mathbb{P}_1^t \approx \mathbb{P}_2^t$.

(2) We prove that if the non-singularity condition (2.5) is not satisfied, then $\mathbb{P}_1^t \perp \mathbb{P}_2^t$. Condition (2.5) does not hold if (a) $R_1 \neq R_2$ or (b) $R_1 = R_2$ but $b_{21} \notin \mathbb{H}_0$.

(2.a) To prove the implication from $R_1 \neq R_2$ to $\mathbb{P}_1^t \perp \mathbb{P}_2^t$, one method is to use the arguments in [Sat00, Step 2 of the Proof Theorem 4.1].

Since $R_1 \neq R_2$, there exist $z_0 \in \mathbb{H}$ such that $\langle z_0, R_1 z_0 \rangle \neq \langle z_0, R_2 z_0 \rangle$. Let $X_t^{z_0} = \langle z_0, X_t \rangle$. Then $(X_t^{z_0}, \mathbb{P}_j^t)$ is a Lévy process on $\mathbb{R}$ with characteristic triplet $(b_j^{z_0}, R_j^{z_0}, 0)$ for $j = 1, 2$, where $b_j^{z_0} = \langle z_0, b_j \rangle$, $R_j^{z_0} = \langle z_0, R_j z_0 \rangle$. The idea of the proof is to show that $\mathbb{P}_j$ concentrate on the paths with quadratic variation $b_j^{z_0}$ for $j = 1, 2$.

It can be verified that

$$\sum_{k=1}^{n}\left(X_{kt/n}^{z_0}-X_{(k-1)t/n}^{z_0}\right)^2 \to R_j^{z_0}\,t$$

in probability $\mathbb{P}_j$ for each $j=1,2$ as $n\to\infty$. Define for $j=1,2$,

$$\Lambda_j=\left\{\omega\in\Omega\colon\ \sum_{k=1}^{n'}\left(X_{kt/n'}^{z_0}-X_{(k-1)t/n'}^{z_0}\right)^2 \to R_j^{z_0}\,t, \quad \text{as } n\to\infty\right\}.$$

Then $\mathbb{P}_1(\Lambda_1)=1$ and $\mathbb{P}_2(\Lambda_2)=1$. But obviously, Λ_1 is disjoint with Λ_2, hence we have $\mathbb{P}_1(\Lambda_2)=0$. This proves that $\mathbb{P}_1^t\perp\mathbb{P}_2^t$.

Another method is to use the finite dimensional result directly. Indeed, if $R_1\neq R_2$, then there exists some finite dimensional subspace $\mathbb{H}_n$ of $\mathbb{H}$ such that $R_1|_{\mathbb{H}_n}\neq R_2|_{\mathbb{H}_n}$. Therefore, by [Sat00, Theorem A], $\mathbb{P}_1^t$ and $\mathbb{P}_2^t$ are orthogonal when they are confined on $D([0,\infty),\mathbb{H}_n)$. This implies $\mathbb{P}_1^t\perp\mathbb{P}_2^t$ on the whole space $\mathbb{D}$.

(2.b) Suppose $R_1=R_2$ but $b_{21}\notin\mathbb{H}_0$. Note that for $j=1,2$, X_t-tb_j is a R-Wiener process under $\mathbb{P}_j^t$. Hence the random variable X_t is Gaussian distributed with mean tb_j and variance R under $\mathbb{P}_j^t$ for $j=1,2$: $\mathbb{P}_j^t\circ X_t^{-1}=N(tb_j,R)$. By Theorem 1.2.2, the Gaussian measures $N(tb_1,R)$ and $N(tb_2,R)$ are orthogonal to each other since $b_{21}\notin\mathbb{H}_0$. Therefore, there exists a set $A\in\mathcal{B}(\mathbb{H})$ such that

$$\mathbb{P}_1^t\circ X_t^{-1}(A)=0,\quad \mathbb{P}_2^t\circ X_t^{-1}(A)=1.$$

Denote $\widetilde{A}=X_t^{-1}(A)\in\mathscr{F}_t$. Then we have $\mathbb{P}_1^t(\widetilde{A})=0$, $\mathbb{P}_2^t(\widetilde{A})=1$. This proves $\mathbb{P}_1^t\perp\mathbb{P}_2^t$.

(3) Suppose $\mathbb{P}_1^t\approx\mathbb{P}_2^t$. Then the conditions (2.5) are satisfied. By Item (2) of Theorem 2.5.1, we know $b_{21}\in\mathbb{H}_0$. Therefore

$$b_r-b_1=rb_1+(1-r)b_2-b_1=(1-r)b_{21}\in\mathbb{H}_0.$$

By Item (2) of Theorem 2.5.1 again, we get $\mathbb{P}_1^t\approx\mathbb{P}_r^t$. Then the Radon-Nikodým derivative of $\mathbb{P}_r^t$ with respect to $\mathbb{P}_1^t$ is given by

$$\begin{aligned}\frac{d\mathbb{P}_r^t}{d\mathbb{P}_1^t}&=\exp\left(\langle b_r-b_1,X_t-tb_1\rangle_0-\frac{t}{2}\left|b_r-b_1\right|_0^2\right)\\&=\exp\left((1-r)\langle b_{21},X_t-tb_1\rangle_0-\frac{t}{2}(1-r)^2\left|b_{21}\right|_0^2\right).\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\mathbb{H}_r(\mathbb{P}_1^t, \mathbb{P}_2^t) &= \left(\frac{d\mathbb{P}_2^t}{d\mathbb{P}_1^t}\right)^{1-r} \mathbb{P}_1^t = \exp\big((1-r)U(t)\big)\mathbb{P}_1^t \\
&= \exp\left((1-r)\langle b_{21}, X_t - tb_1\rangle_0 - \frac{t}{2}(1-r)\,|b_{21}|_0^2\right)\mathbb{P}_1^t \\
&= \exp\left(\frac{t}{2}\big[(1-r)^2 - (1-r)\big]\,|b_{21}|_0^2\right)\mathbb{P}_r^t \\
&= \exp\left(-\frac{t}{2}r(1-r)\,|b_{21}|_0^2\right)\mathbb{P}_r^t.
\end{aligned}$$

□

2.6 Proof of Theorem 2.4.1

We follow the proof in [Sat00, Section 5] (see also [New73]) to prove Theorem 2.4.1.

For every fixed $t > 0$, let $\mathbb{D}_t = D([0,t], \mathbb{H})$ be the space of all right continuous with left limits functions from $[0,t]$ to $\mathbb{H}$. We still denote

$$\mathscr{F}_s := \sigma(X_u : 0 \le u \le s), \quad s \in [0,t].$$

By $N(du, dx)$ we denote the Poisson random measure on $[0,t] \times \mathbb{H}$ associated with X_t. That is, $N(G)$ is the number of $s \in (0,t]$ such that $(s, \Delta X_s) \in G$ for each $G \in \mathcal{B}\big((0,t] \times \mathbb{H}\big)$. Here we use ΔX_s to denote the jump of X_s at time s:

$$\Delta X_s(\omega) := X_s(\omega) - X_{s-}(\omega), \quad \omega \in \mathbb{D}_t.$$

For every $0 < r < 1$, let $\nu_r = \mathbb{H}_r(\nu_1, \nu_2)$ and for every $0 \le s \le t$, $0 < \varepsilon < 1$ and $0 < r < 1$, we define

$$Y_{\varepsilon,s} = \int_{(0,s]\times\{\varepsilon<|x|\le 1\}} x\,\big(N(du,dx) - \nu_r(dx)du\big) + \int_{(0,s]\times\{|x|>1\}} x\,N(du,dx)$$

and

$$Z_{\varepsilon,s} = X_s - Y_{\varepsilon,s}.$$

If $k_r(\nu_1,\nu_2) < \infty$, then by Lemma 2.3.7, we can take the limit $\varepsilon \to 0$ to obtain

$$Y_{0,s} := \lim_{\varepsilon \downarrow 0} Y_{\varepsilon,s} = \widehat{X}_{j,s} + s \int_{\mathbb{H}} x \, d(\nu_j - \nu_r), \quad j = 1,2,$$

where $\widehat{X}_{j,s}$ is the pure jump part of X_t under $\mathbb{P}_j$:

$$\widehat{X}_{j,s} = \lim_{\varepsilon \downarrow 0} \int_{(0,s]\times\{\varepsilon<|x|\le 1\}} x \left(N(du,dx) - \nu_j(dx)du\right) + \int_{(0,s]\times\{|x|>1\}} x \, N(du,dx).$$

Now we define

$$Z_{0,s} = X_s - Y_{0,s}, \quad \overline{Y}_{\varepsilon,s} = Z_{\varepsilon,s} - Z_{0,s} = Y_{0,s} - Y_{\varepsilon,s}.$$

We denote by

$$\mathbb{Q}^t_{\varepsilon,j},\ \mathbb{Q}^t_{0,j},\ R^t_{\varepsilon,j},\ R^t_{0,j},\ \overline{\mathbb{Q}}^t_{\varepsilon,j}$$

the distribution of

$$Y_{\varepsilon,s},\ Y_{0,s}, Z_{\varepsilon,s},\ Z_{0,s},\ \overline{Y}_{\varepsilon,s}$$

under $\mathbb{P}_j$ on $(\mathbb{D}_t, \mathscr{F}_t)$ respectively. We also denote by $\mathbb{Q}^t_{\varepsilon,r}$ the distribution of $Y_{\varepsilon,s}$ under $\mathbb{P}^t_r$ given in Theorem 2.4.1.

The following lemma is an infinite dimensional version of the lemmas in [Sat00, Section 5].

Lemma 2.6.1. *For every $r \in (0,1)$, $\varepsilon \in (0,1)$ and $j = 1,2$, the following equalities hold.*

(1) $\mathbb{P}^t_j = \mathbb{Q}^t_{\varepsilon,j} * R^t_{\varepsilon,j}$.

(2) $\mathbb{H}_r(\mathbb{P}^t_1, \mathbb{P}^t_2) = \mathbb{H}_r(\mathbb{Q}^t_{\varepsilon,1}, \mathbb{Q}^t_{\varepsilon,2}) * \mathbb{H}_r(R^t_{\varepsilon,1}, R^t_{\varepsilon,2})$.

(3)

$$\mathbb{H}_r(\mathbb{Q}^t_{\varepsilon,1}, \mathbb{Q}^t_{\varepsilon,2}) = \exp\left(-t \int_{\{|x|>\varepsilon\}} d\,\mathbb{K}_r(\nu_1,\nu_2)\right) \mathbb{Q}^t_{\varepsilon,r}. \tag{2.10}$$

(4) Assume that $k_r(\nu_1,\nu_2) < \infty$.

(a) $\mathbb{P}^t_j = \mathbb{Q}^t_{0,j} * R^t_{0,j} = \mathbb{Q}^t_{\varepsilon,j} * \overline{\mathbb{Q}}^t_{\varepsilon,j} * R^t_{0,j}$.

(b)

$$\begin{aligned} \mathbb{H}_r(\mathbb{P}^t_1, \mathbb{P}^t_2) &= \mathbb{H}_r(\mathbb{Q}^t_{0,1}, \mathbb{Q}^t_{0,2}) * \mathbb{H}_r(R^t_{0,1}, R^t_{0,2}) \\ &= \mathbb{H}_r(\mathbb{Q}^t_{\varepsilon,1}, \mathbb{Q}^t_{\varepsilon,2}) * \mathbb{H}_r(\overline{\mathbb{Q}}^t_{\varepsilon,1}, \overline{\mathbb{Q}}^t_{\varepsilon,2}) * \mathbb{H}_r(R^t_{0,1}, R^t_{0,2}) \\ &= \mathbb{H}_r(\overline{\mathbb{Q}}^t_{\varepsilon,1}, \overline{\mathbb{Q}}^t_{\varepsilon,2}) * \mathbb{H}_r(\mathbb{Q}^t_{\varepsilon,1} * R^t_{0,1}, \mathbb{Q}^t_{\varepsilon,2} * R^t_{0,2}) \end{aligned} \tag{2.11}$$

Proof of Theorem 2.4.1. (1) We first show $\mathbb{P}^t_2 \perp \mathbb{P}^t_1$ if condition (2.2) is not satisfied, that is, if one of the following conditions holds

(i) $k_r(\nu_1, \nu_2) = \infty$.

(ii) $k_r(\nu_1, \nu_2) < \infty$ and $R_1 \neq R_2$.

(iii) $k_r(\nu_1, \nu_2) < \infty$ and $R_1 = R_2$ and $b_{21} \notin \mathbb{H}_0$.

Assume that (i) holds. Then the proof of $\mathbb{P}_2^t \perp \mathbb{P}_1^t$ is the same with Step 1 of the proof of Theorem A in [Sat00].

Assume (ii) or (iii) holds. From $k_r(\nu_1, \nu_2) < \infty$, we know the characteristic triplet of the process X_t under $R_{0,j}^t$ for $j = 1, 2$ is given by $(\tilde{b}_{jr}, R_j, 0)$ with

$$\tilde{b}_{jr} := b_j - \int_{\{|x| \leq 1\}} x \, d(\nu_j - \nu_r).$$

If (b) or (c) holds, then we can obtain $h_r(R_{0,1}^t, R_{0,2}^t) = 0$ by applying Theorem 2.5.1. Now $h_r(\mathbb{P}_1^t, \mathbb{P}_2^t) = 0$ follows from the first identity in (2.11) of Lemma 2.6.1.

(2) Suppose that the condition (2.2) holds, we prove (2.4). We can just follow the line in Step 4 of the proof of Theorem A in [Sat00, Section 5]. Similar to the proof in [Sat00] (apply Lemma 2.3.6), we need to show $\mathbb{H}_r(\mathbb{Q}_{\varepsilon,1}^t * R_{0,1}^t, \mathbb{Q}_{\varepsilon,2}^t * R_{0,2}^t)$ tends to $\exp(-t\Phi_r)$ as ε goes to 0. By (2.10) of Lemma 2.6.1 and Theorem 2.5.1 (see (2.6)), we have

$$\begin{aligned}
&\mathbb{H}_r(\mathbb{Q}_{\varepsilon,1}^t * R_{0,1}^t, \mathbb{Q}_{\varepsilon,2}^t * R_{0,2}^t) \\
=&\mathbb{H}_r(\mathbb{Q}_{\varepsilon,1}^t, \mathbb{Q}_{\varepsilon,2}^t) * \mathbb{H}_r(R_{0,1}^t, R_{0,2}^t) \\
=&\left[\exp\left(-t \int_{\{|x|>\varepsilon\}} d\mathbb{K}_r(\nu_1, \nu_2)\right) \mathbb{Q}_{\varepsilon,r}^t\right] * \left[\exp\left(-\frac{1}{2} tr(1-r)|b_{21}|_0^2\right) R_r^t\right],
\end{aligned}$$

where R_r^t corresponds to the Lévy process with characteristic triplet $(b_r, R, 0)$ (i.e. a Gaussian process). Here by (2.7),

$$\begin{aligned}
b_r &= r\tilde{b}_{1r} + (1-r)\tilde{b}_{2r} \\
&= r\left(b_1 - \int_{\{|x|\leq 1\}} x \, d(\nu_1 - \nu_r)\right) + (1-r)\left(b_2 - \int_{\{|x|\leq 1\}} x \, d(\nu_2 - \nu_r)\right) \\
&= rb_1 + (1-r)b_2 - \int_{\{|x|\leq 1\}} x \, d\mathbb{K}_r(\nu_1, \nu_1).
\end{aligned}$$

Here we have used the following fact

$$\begin{aligned}
r(\nu_1 - \nu_r) + (1-r)(\nu_2 - \nu_r) &= r\nu_1 + (1-r)\nu_2 - \nu_r \\
&= r\nu_1 + (1-r)\nu_2 - \mathbb{H}_r(\nu_1, \nu_2) = \mathbb{K}_r.
\end{aligned}$$

As $\varepsilon \downarrow 0$, the measure $\mathbb{Q}^t_{\varepsilon,r}$ goes to $\mathbb{Q}^t_{0,r}$ with triplet $(0, \nu_r, 0)$. The proof is completed by noting that $\mathbb{Q}^t_{0,r} * R^t_r = \mathbb{P}^t_r$ with triplet (b_r, R, ν_r). □

2.7 Density of Lévy Processes

For any two σ-finite measures σ_1 and σ_2, we denote the continuous part and the singular part in the Lebesgue decomposition of σ_2 with respect to σ_1 by σ_2^{ac} and σ_2^{s} respectively.

Take $\nu = \nu_1 + \nu_2$. For $j = 1, 2$, choose the version $f_j := \dfrac{d\nu_j}{d\nu}$ satisfying

$$f_j \geq 0 \quad \text{and} \quad f_1 + f_2 = 1 \quad \nu\text{–a.s. on } \mathbb{H}.$$

Set

$$\begin{aligned} C &= \{f_1 > 0 \text{ and } f_2 > 0\}, \qquad & C_1 &= \{f_1 = 1 \text{ and } f_2 = 0\}, \\ C_2 &= \{f_1 = 0 \text{ and } f_2 = 1\}, \qquad & C_3 &= C_1 \cup C_2. \end{aligned}$$

Then

$$\nu_2^{\mathrm{ac}} = \mathbb{1}_C \nu_2, \quad \nu_2^{\mathrm{s}} = \mathbb{1}_{C_2} \nu_2 = \mathbb{1}_{C_3} \nu_2.$$

and $\dfrac{d\nu_2^{\mathrm{ac}}}{d\nu_1}$ has the following version

$$\frac{d\nu_2^{\mathrm{ac}}}{d\nu_1} = \begin{cases} f_2/f_1 & \text{on } C; \\ 0 & \text{on } C_3. \end{cases}$$

Define

$$g(x) = \begin{cases} \log(f_2/f_1) & \text{on } C; \\ -\infty & \text{on } C_3, \end{cases} \quad \text{and} \quad \tilde{g}(x) = \begin{cases} g(x) & \text{on } C; \\ 0 & \text{on } C_3. \end{cases}$$

As in Section 2.6, we denote by $N(du, dx)$ the random measure associated with X_t. For every $t > 0$, set

$$\Lambda_t = \{N\big((0,t] \times C_3\big) = 0\} = \{\mathit{\Delta} X_s \notin C_3 \text{ for all s} \in (0,\mathrm{t}]\}.$$

The following theorem is an infinite dimensional version of [Sat00, Theorem B].

Theorem 2.7.1. *Suppose that $\mathbb{P}_1^t$ and $\mathbb{P}_2^t$ are not mutually singular. Then*

(1) For every $0 < t < \infty$, the Lebesgue decomposition of $\mathbb{P}_2^t$ with respect to $\mathbb{P}_1^t$ is given by

$$(\mathbb{P}_2^t)^{\mathrm{ac}} = \mathbb{1}_{\Lambda_t}\mathbb{P}_2^t, \quad (\mathbb{P}_2^t)^{\mathrm{s}} = \mathbb{1}_{\mathbb{D}\setminus\Lambda_t}\mathbb{P}_2^t.$$

(2) Consider

$$V_t := \lim_{\varepsilon\to 0}\left(\sum_{(s,\Delta X_s)\in(0,t]\times\{|x|>\varepsilon\}} \tilde{g}(\Delta X_s) - t\int_{\{|x|>\varepsilon\}} (\mathrm{e}^{g(x)} - 1)\nu_1(dx)\right). \quad (2.12)$$

Then the right hand side of (2.12) exists $\mathbb{P}_1$–a.s. and the convergence is uniform on any bounded time interval $\mathbb{P}_1$–a.s.

(3) Let $b \in \mathbb{H}_0$. Define

$$U_t = \langle b, X_t'\rangle_0 - \frac{t}{2}|b|_0^2 + V_t,$$

where X_t' is the Gauss component of the process $(X_t, \mathbb{P}_1)$. It is a Wiener process with covariance R. Then U_t is, under $\mathbb{P}_1$, a real valued Lévy process with characteristic triplet (b_U, R_U, ν_U) given by

$$\begin{aligned} b_U &= -\frac{1}{2}|b|_0^2 + \int_{\mathbb{H}} \left[1 + g(x)\mathbb{1}_{\{|g(x)|\le 1\}} - \mathrm{e}^{g(x)}\right]\nu_1(dx), \\ R_U &= |b|_0^2, \\ \nu_U(A) &= \int_{\mathbb{H}} \mathbb{1}_A\left[g(x)\right]\nu_1(dx), \quad A \in \mathcal{B}(\mathbb{H}\setminus\{0\}). \end{aligned}$$

The processes $(U_t)_{t\ge 0}$ and $(N((0,t]\times C_3))_{t\ge 0}$ are independent under $\mathbb{P}_1$. Moreover,

$$\mathbb{P}_1(\Lambda_t) = \exp\left[-t\nu_1(C_1)\right] \quad \text{and} \quad \mathbb{P}_2(\Lambda_t) = \exp\left[-t\nu_2(C_2)\right].$$

(4) Choose $b = b_{21}$. Then the Radon-Nikodým derivative of $(\mathbb{P}_2^t)^{\mathrm{ac}}$ with respect to $\mathbb{P}_1^t$ is given by

$$\frac{d(\mathbb{P}_2^t)^{\mathrm{ac}}}{d\mathbb{P}_1^t} = \exp\left[-t\nu_2(C_2) + U_t\right]\mathbb{1}_{\Lambda_t}.$$

Let $\mathbb{Q}$ be the probability measure on $(\mathbb{D}, \mathscr{F})$ for which $(X_t, \mathbb{Q})$ is the Lévy process with characteristic triplet $(b_2 - \int_{\{|x|<1\}} x\, d\nu_2^{\mathrm{s}}, R, \nu_2^{\mathrm{ac}})$. Then

$$(\mathbb{P}_2^t)^{\mathrm{ac}} = \exp\left[-t\nu_2(C_2)\right]\mathbb{Q}^t.$$

Proof. We only need to follow the proof in [Sat00, Section 6] with some slight modifications. □

2.8 Appendix: $R(\mathbb{H}) \neq R^{1/2}(\mathbb{H})$

This section is a continuation of Remark 2.4.3. For the finite dimensional case, we have $R^{1/2}(\mathbb{R}^d) = R(\mathbb{R}^d)$. But for infinite dimensional case, we shall show

$$R(\mathbb{H}) \subset R^{1/2}(\mathbb{H}) \quad \text{but} \quad R(\mathbb{H}) \neq R^{1/2}(\mathbb{H}).$$

Obviously $R(\mathbb{H}) = R^{1/2}(R^{1/2}(\mathbb{H})) \subset R^{1/2}(\mathbb{H})$ holds. Let $\{e_k\}_{k\geq 1}$ be a series of eigenvectors which consists of an complete orthogonal normal basis of $\mathbb{H}$ with corresponding eigenvalues $\{\lambda_k\}_{k\geq 1}$.

We first show $R^{1/2}(\mathbb{R}^d) \subset R(\mathbb{R}^d)$. For any $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, take $y = (y_1, y_2, \dots, y_d)$ with

$$y_k = \begin{cases} \dfrac{x_k}{\sqrt{\lambda_k}}, & \text{if } \lambda_k > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$R^{1/2}x = \sum_{k=1}^{d} \sqrt{\lambda_k} x_k e_k = \sum_{k=1}^{d} \mathbb{1}_{\lambda_k > 0} \lambda_k \frac{x_k}{\sqrt{\lambda_k}} e_k = \sum_{k=1}^{d} \lambda_k y_k e_k = Ry.$$

Now we assume $\mathbb{H} = l^2$. We show that $R^{1/2}(\mathbb{H})$ is a real subset of $R(\mathbb{H})$. Take $x = \sum_k \sqrt{\lambda_k} e_k$. Since the operator R is of trace class, we see $x \in \mathbb{H}$. But

$$R^{1/2}x = \sum_k \sqrt{\lambda_k} \langle x, e_k \rangle e_k = \sum_k \lambda_k e_k \notin R(\mathbb{H}).$$

Otherwise if there exist some $y \in \mathbb{H}$ such that $Ry = R^{1/2}x$, then it must be $y \equiv 1 \notin \mathbb{H}$. This is contradict with the fact $y \in \mathbb{H}$.

Chapter 3

Gluing and Coupling

In this chapter we prove a gluing lemma (Lemma 3.1.5) and study its applications. In this lemma we show a martingale solution for operators of the form $L_1\mathbb{1}_{\{t<\tau\}} + L_2\mathbb{1}_{\{t\geq\tau\}}$, where L_1, L_2 are second order differential operators and τ is a stopping time.

The organization is as follows. In Section 3.1, we first recall some basic notations and [SV79, Lemma 6.1.1] and [SV79, Theorem 6.1.2] on which the proof of the gluing lemma is based. Then we state the gluing lemma. The proof of the lemma is given in Section 3.2. In Section 3.3, we apply the gluing lemma to study the existence of coupling and the existence of weak solutions to coupled stochastic differential equations.

Chen and Li [CL89, Lemma 3.4] (see Corollary 3.3.2) studies the gluing the martingale generators via the diffusion coefficients. Our study is stimulated by their statement and hints about the proof noted there. By the general gluing lemma, it is possible to study the gluing of martingale generators via drifts.

3.1 Gluing Lemma

Let $\Omega = C([0,\infty), \mathbb{R}^n)$ be the space of all continuous trajectories from $[0,\infty)$ into $\mathbb{R}^n$. For each $\omega \in \Omega$ and $t \in [0,\infty)$, denote the position of ω at time t by $X_t(\omega) = X(t,\omega) = \omega_t \in \mathbb{R}^n$. For any $0 \leq t_1 < t_2 \leq \infty$, set

$$\mathscr{M}_{t_2}^{t_1} = \sigma(X_s \colon t_1 \leq s \leq t_2).$$

Here we use the convention that we understand $s \le t_2$ as $s < t_2$ if $t_2 = \infty$. We will also use the following simplified notation:

$$\mathscr{M}_t := \mathscr{M}_t^0, \quad \mathscr{M}^t := \mathscr{M}_\infty^t, \quad \text{and } \mathscr{M} := \mathscr{M}_\infty^0.$$

Let S_n represent the space of all $n \times n$ nonnegative definite real matrix. For any measurable functions $a(t,x) \in S_n$ and $b(t,x) \in \mathbb{R}^n$ defined on $[0,\infty) \times \mathbb{R}^n$, let

$$L(a,b) := L(a(t,x), b(t,x)) := \frac{1}{2} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t,x) \frac{\partial}{\partial x_i}. \tag{3.1}$$

Definition 3.1.1. Fix any $(s,x) \in [0,\infty) \times \mathbb{R}^n$. A solution to the martingale problem for $L := L(a,b)$ starting from (s,x) is a probability measure $\mathbb{P}^{s,x}$ on $(\Omega, \mathscr{M})$ such that

$$\mathbb{P}^{s,x}(X_t = x, 0 \le t \le s) = 1 \tag{3.2}$$

and for every $f \in \mathscr{C}_0^\infty(\mathbb{R}^d)$, a compact supported smooth function on $\mathbb{R}^d$,

$$M_t^f := f(X_t) - \int_0^t Lf(X_u)\, du$$

is a $\mathbb{P}^{s,x}$-martingale after time s.

Sometimes we have not the solution for all time. So the following solution concept is useful.

Definition 3.1.2. Fix any $(s,x) \in [0,\infty) \times \mathbb{R}^n$. A solution to the martingale problem for L up to a stopping time τ starting from (s,x) is a probability measure $\mathbb{P}^{s,x}$ on $(\Omega, \mathscr{M})$ such that (3.2) holds and there exist some stopping time sequence $\tau_n \uparrow \tau$ such that for each $n \ge 1$, the stopped process $M_{t \wedge \tau_n}^f$ is a $\mathbb{P}^{s,x}$-martingale.

For convenience, we will denote simply $\mathbb{P}^x$ for $\mathbb{P}^{0,x}$.

The following lemma and theorem are from Stroock and Varadhan [SV79, Lemma 6.1.1] and [SV79, Theorem 6.1.2] respectively.

Lemma 3.1.3. *Let $s \ge 0$ be given and suppose that $\mathbb{P}$ is a probability measure on $(\Omega, \mathscr{M}^s)$. If $\eta \in C([0,s], \mathbb{R}^d)$ and $\mathbb{P}(x_s = \eta_s) = 1$, then there is a unique probability measure $\delta_\eta \otimes_s \mathbb{P}$ on $(\Omega, \mathscr{M})$ such that*

$$\delta_\eta \otimes_s \mathbb{P}(x_t = \eta_t, 0 \le t \le s) = 1$$

and

$$\delta_\eta \otimes_s \mathbb{P}(A) = \mathbb{P}(A), \quad \text{for all } A \in \mathscr{M}^s.$$

Theorem 3.1.4. *Let τ be a finite stopping time on Ω. Suppose that $\omega \to \mathbb{Q}_\omega$ is a mapping of Ω into probability measures on $(\Omega, \mathscr{M})$ such that*

(1) $\omega \to \mathbb{Q}_\omega(A)$ is $\mathscr{M}_\tau$-measurable for all $A \in \mathscr{M}$,
(2) $\mathbb{Q}_\omega(x(\tau(\omega), \cdot) = x(\tau(\omega), \omega)) = 1$ for all $\omega \in \Omega$.

Given a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{M})$, there is a unique probability measure $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.$ on $(\Omega, \mathscr{M})$ such that $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.$ equals $\mathbb{P}$ on $(\Omega, \mathscr{M}_\tau)$ and $\{\delta_\omega \otimes_{\tau(\omega)} \mathbb{Q}_\omega\}$ is a r.c.p.d. (regular conditional probability distribution) of $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.|\mathscr{M}_\tau$.

*In particular, suppose that $\tau \geq s$ and that $\theta : [s, \infty) \times \Omega \to \mathbb{C}$ is a right-continuous, progressively measurable function after time s such that $\theta(t)$ is $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.$-integrable for all $t \geq s$, $(\theta(t \wedge \tau), \mathscr{M}_t, \mathbb{P})$ is a martingale after time s, and $(\theta(t) - \theta(t \wedge \tau), \mathscr{M}_t, \mathbb{Q}_\omega)$ * is a martingale after time s for each $\omega \in \Omega$. Then $(\theta(t), \mathscr{M}_t, \mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.)$ is a martingale after time s.*

By applying Theorem 3.1.4 ([SV79, Theorem 6.1.2]), we will prove the following *gluing lemma* in Section 3.2.

Lemma 3.1.5 (Gluing Lemma)**.** *Let L_1 and L_2 be two second order differential operators as (3.1) on $\mathbb{R}^n$. Let τ be a stopping time on Ω and define*

$$L = L_1 \mathbb{1}_{\{t<\tau\}} + L_2 \mathbb{1}_{\{t \geq \tau\}}.$$

Assume

(1) There exists a solution $\mathbb{P}_1^x$ to the martingale problem for L_1 up to τ;
(2) For each $\omega \in \Omega$, there exists a solution $\mathbb{P}_2^{\tau(\omega), X_{\tau(\omega)}(\omega)}$ to the martingale problem for L_2 starting from $(\tau(\omega), X_{\tau(\omega)}(\omega))$;
(3) There exists a sequence of stopping time τ_n such that $\tau_n \uparrow \tau$ as $n \to \infty$, the following two conditions are satisfied for each $\omega \in \Omega$.
(a)

$$\lim_{n\to\infty} \int_{\tau_n}^{\tau} Lf(X_s)\, ds = 0. \tag{3.3}$$

(b) For every $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, $\displaystyle\int_0^{t\wedge\tau_n} L_1 f(X_s)\, ds$ is bounded and

$$\lim_{n\to\infty} \int_0^{t\wedge\tau_n} L_1 f(X_s)\, ds = \int_0^{t\wedge\tau} L_1 f(X_s)\, ds. \tag{3.4}$$

*In [SV79, Theorem 6.1.2], it is written as $(\theta(t) - \theta(t \wedge \tau(\omega)), \mathscr{M}_t, \mathbb{Q}_\omega)$. This is not true. We shouldn't fix the ω in $\tau(\cdot)$. See [SV79, Theorem 1.2.10].

Define

$$\mathbb{Q}_\omega := \delta_\omega \otimes \mathbb{P}_2^{\tau(\omega), X_{\tau(\omega)}(\omega)} \mathbb{1}_{\{\tau<\infty\}} + \delta_\omega \mathbb{1}_{\{\tau=\infty\}}, \quad \text{for every } \omega \in \Omega.$$

Then $\mathbb{P}_1^x \otimes_\tau \mathbb{Q}$ *is a solution to the martingale problem for* L.

In Section 3.3 we apply this lemma to the existence of couplings and weak solutions of stochastic differential equations.

Remark 3.1.6. It might be possible to consider the gluing of martingales corresponding to Lévy operators. To this aim, we only need to consider the generalization of [SV79, Theorem 6.1.2] to the Lévy case.

3.2 Proof of the Gluing Lemma

For each $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and $t \geq 0$, define

$$\theta_t = f(X_t) - \int_0^t Lf(X_s)\,ds,$$
$$\phi_t = f(X_t) - \int_0^t L_1 f(X_s)\,ds,$$
$$\psi_t = f(X_t) - \int_0^t L_2 f(X_s)\,ds.$$

We first prepare three lemmas. The first lemma show the relationship of θ_t with ϕ_t and ψ_t respectively. The theorem will follow the last two lemmas directly by applying Theorem 3.1.4.

Lemma 3.2.1. *For every* $t \geq 0$ *and* $\omega \in \Omega$, *we have*

$$\theta_{t\wedge\tau} = \phi_{t\wedge\tau} \tag{3.5}$$

and

$$\theta_t - \theta_{t\wedge\tau} = \psi_t - \psi_{t\wedge\tau}. \tag{3.6}$$

Proof. For every $t \geq 0$ and each $n \in \mathbb{N}$, we know

$$\theta_{t\wedge\tau_n} = \phi_{t\wedge\tau_n}. \tag{3.7}$$

So, to prove (3.5) we only need to show that as n goes to infinity, the limits of the left and right hand sides of the equation (3.7) are $\theta_{t\wedge\tau}$ and $\phi_{t\wedge\tau}$ respectively.

By the continuity of the path X_t and the fact $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and the assumption (3.4), we have

$$\begin{aligned}\lim_{n\to\infty} \phi(t\wedge\tau_n) &= \lim_{n\to\infty} f(X_{t\wedge\tau_n}) - \lim_{n\to\infty}\int_0^{t\wedge\tau_n} L_1 f(X_s)\,ds \\ &= f(X_{t\wedge\tau}) - \int_0^{t\wedge\tau} L_1 f(X_s)\,ds \\ &= \phi(t\wedge\tau).\end{aligned} \tag{3.8}$$

On the other hand, by (3.3), we know

$$\begin{aligned}&\lim_{n\to\infty}[\theta_{t\wedge\tau} - \theta_{t\wedge\tau_n}] \\ =&\lim_{n\to\infty}\left[f(X_{t\wedge\tau}) - f(X_{t\wedge\tau_n}) + \int_{t\wedge\tau_n}^{t\wedge\tau} Lf(X_s)\,ds\right] \\ =&0.\end{aligned}$$

That is,

$$\lim_{n\to\infty}\theta_{t\wedge\tau_n} = \theta_{t\wedge\tau} \tag{3.9}$$

(3.5) follows from (3.8) and (3.9).

Now we come to the proof of (3.6). First, it is easy to see

$$(\theta_t - \theta_{t\wedge\tau})\mathbb{1}_{\{t<\tau\}} = 0 = (\psi_t - \psi_{t\wedge\tau})\mathbb{1}_{\{t<\tau\}}$$

and

$$\begin{aligned}(\theta_t - \theta_{t\wedge\tau})\mathbb{1}_{\{t\geq\tau\}} &= \int_\tau^t Lf(X_s)\,ds\mathbb{1}_{\{t\geq\tau\}} \\ &= \int_\tau^t L_2 f(X_s)\,ds\mathbb{1}_{\{t\geq\tau\}} = (\psi_t - \psi_{t\wedge\tau})\mathbb{1}_{\{t\geq\tau\}}.\end{aligned}$$

Hence

$$\begin{aligned}\theta_t - \theta_{t\wedge\tau} &= (\theta_t - \theta_{t\wedge\tau})\mathbb{1}_{\{t<\tau\}} + (\theta_t - \theta_{t\wedge\tau})\mathbb{1}_{\{t\geq\tau\}} \\ &= (\psi_t - \psi_{t\wedge\tau})\mathbb{1}_{\{t<\tau\}} + (\psi_t - \psi_{t\wedge\tau})\mathbb{1}_{\{t\geq\tau\}} = \psi_t - \psi_{t\wedge\tau}.\end{aligned}$$

□

Lemma 3.2.2. $(\theta_{t\wedge\tau}, \mathscr{M}_t, \mathbb{P}_1^x)$ *is a martingale.*

Proof. By (3.5), to prove $\theta_{t\wedge\tau}$ is a martingale, we only need to show that $\phi_{t\wedge\tau}$ is a martingale.

Since $\mathbb{P}_1^x$ is a solution to the martingale problem for L_1 up to τ, we know $(\phi_{t\wedge\tau_n}, \mathscr{M}_t, \mathbb{P}_1^x)$ is a martingale. Therefore, for any $0 \le s < t$, the following equality holds

$$\mathbb{E}(\phi_{t\wedge\tau_n}|\mathscr{M}_s) = \phi_{s\wedge\tau_n}.$$

By assumption (3.3), let $n \to \infty$ and apply the bounded convergence of conditional expectation, we obtain

$$\mathbb{E}(\phi_{t\wedge\tau}|\mathscr{M}_s) = \phi(s \wedge \tau).$$

This proves that $\phi_{t\wedge\tau}$ is a martingale. □

Lemma 3.2.3. *$(\theta_t - \theta_{t\wedge\tau}, \mathscr{M}_t, \mathbb{Q}_\omega)$ is a martingale for each $\omega \in \Omega$.*

Proof. Fix an arbitrary path $\omega_0 \in \Omega$. We set $t_0 := \tau(\omega_0)$ and $X_{t_0}(\omega_0) = x_0$. For any fixed constants $0 \le t_1 \le t_2$, we need to prove

$$\mathbb{E}^{\mathbb{Q}_{\omega_0}}(\theta_{t_2} - \theta_{t_2\wedge\tau}|\mathscr{M}_{t_1}) = \theta_{t_1} - \theta_{t_1\wedge\tau}. \tag{3.10}$$

If $t_0 = \infty$, then $Q_{\omega_0} = \delta_{\omega_0}$. That is, the measure is concentrated on the path ω_0. In this case, Equality (3.10) is trivial since we have

$$\theta_t - \theta_{t\wedge\tau} = \theta_t - \theta_{t\wedge t_0} = \theta_t - \theta_{t\wedge\infty} = 0.$$

In the following we shall assume $t_0 < \infty$ and we will prove (3.10) in the following three cases: (Case 1) $t_0 \le t_1 < t_2$; (Case 2) $t_1 < t_0 \le t_2$; (Case 3) $t_1 < t_2 < t_0$.

Case 1. Assume $t_0 \le t_1 < t_2$. By (3.6), we only need to show

$$\mathbb{E}^{\mathbb{Q}_{\omega_0}}(\psi_{t_2} - \psi_{t_2\wedge\tau}|\mathscr{M}_{t_1}) = \psi_{t_1} - \psi_{t_1\wedge\tau}.$$

In other words, we need to show for any $A \in \mathscr{M}_{t_1}$,

$$\mathbb{Q}_{\omega_0}(\psi_{t_2} - \psi_{t_2\wedge\tau}, A) = \mathbb{Q}_{\omega_0}(\psi_{t_1} - \psi_{t_1\wedge\tau}, A) \tag{3.11}$$

Since $\mathbb{P}_2^{t_0,x_0}$ is a solution to the martingale problem for L_2, we know ψ_t is a $\mathbb{P}_2^{t_0,x_0}$ martingale. Therefore we see $\psi_{t\wedge\tau}$ is a martingale by [SV79, Corollary 1.2.7]. Hence, $\psi_t - \psi_{t\wedge\tau}$ is also a martingale.

The martingale property of $\psi_t - \psi_{t\wedge\tau}$ implies that for any $t_0 \le t_1 < t_2$ and $A \in \mathscr{M}^{t_0}$

$$\mathbb{P}_2^{t_0,x_0}(\psi_{t_2} - \psi_{t_2\wedge\tau}, A) = \mathbb{P}_2^{t_0,x_0}(\psi_{t_1} - \psi_{t_1\wedge\tau}, A). \tag{3.12}$$

Note that it is enough to prove (3.11) for the case when $A = A_1 \times A_2$ with $A_1 \in \mathscr{M}_{t_0}$ and $A_2 \in \mathscr{M}_{t_1}^{t_0}$.

$$\begin{aligned}
\mathbb{Q}_{\omega_0}(\psi_{t_2} - \psi_{t_2\wedge\tau}, A) &= \delta_{\omega_0} \otimes \mathbb{P}_2^{t_0,x_0}(\psi_{t_2} - \psi_{t_2\wedge\tau}, A_1 \times A_2) \\
&= \delta_{\omega_0}(A_1) \times \mathbb{P}_2^{t_0,x_0}(\psi_{t_2} - \psi_{t_2\wedge\tau}, A_2) \\
&= \delta_{\omega_0}(A_1) \times \mathbb{P}_2^{t_0,x_0}(\psi_{t_1} - \psi_{t_1\wedge\tau}, A_2) \\
&= \delta_{\omega_0} \otimes \mathbb{P}_2^{t_0,x_0}(\psi_{t_1} - \psi_{t_1\wedge\tau}, A_1 \times A_2) \\
&= \mathbb{Q}_{\omega_0}(\psi_{t_1} - \psi_{t_1\wedge\tau}, A).
\end{aligned}$$

This proves (3.11).

Case 2. Assume $t_1 < t_0 \leq t_2$. By the fact $\mathscr{M}_{t_1} \subset \mathscr{M}_{t_0}$ and a property of conditional expectation, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_{\omega_0}}(\theta_{t_2} - \theta_{t_2\wedge\tau}|\mathscr{M}_{t_1}) &= \mathbb{E}^{\mathbb{Q}_{\omega_0}}\left(\mathbb{E}^{\mathbb{Q}_{\omega_0}}\left(\psi_{t_2} - \psi_{t_2\wedge\tau}|\mathscr{M}_{t_0}\right)|\mathscr{M}_{t_1}\right) \\
&= \mathbb{E}^{\mathbb{Q}_{\omega_0}}\left(\psi_{t_0} - \psi_{t_0\wedge\tau}|\mathscr{M}_{t_1}\right) \\
&= \mathbb{E}^{\mathbb{Q}_{\omega_0}}\left(\theta_{t_0} - \theta_{t_0\wedge\tau}|\mathscr{M}_{t_1}\right).
\end{aligned}$$

Hence, (3.10) is reduced to prove

$$\mathbb{E}^{\mathbb{Q}_{\omega_0}}\left(\theta_{t_0} - \theta_{t_0\wedge\tau}|\mathscr{M}_{t_1}\right) = \theta_{t_1} - \theta_{t_1\wedge\tau}. \tag{3.13}$$

This is true since we have

$$\mathbb{Q}_{\omega_0}(\theta_{t_0} - \theta_{t_0\wedge\tau}, A) = 0 = \mathbb{Q}_{\omega_0}(\theta_{t_1} - \theta_{t_1\wedge\tau}, A).$$

for any $A \in \mathscr{M}_{t_1}$. In fact, for any $t \leq t_0$, we know

$$\begin{aligned}
\mathbb{Q}_{\omega_0}(\theta_t - \theta_{t\wedge\tau}, A) &= \delta_{\omega_0} \otimes \mathbb{P}_2^{t_0,x_0}(\theta_t - \theta_{t\wedge\tau}, A) \\
&= \delta_{\omega_0}(A) \cdot (\theta_t - \theta_{t\wedge\tau(\omega_0)}) \\
&= \delta_{\omega_0}(A) \cdot (\theta_t - \theta_{t\wedge t_0}) \\
&= \delta_{\omega_0}(A) \cdot (\theta_t - \theta_t) \\
&= 0.
\end{aligned}$$

Here we have used the fact that, $\tau = \tau(\omega_0) = t_0$-$\mathbb{P}_2^{t_0,x_0}$-a.s. when confined on A.

Case 3. Assume $t_1 < t_2 < t_0$. As in the proof in Case 2, for any $A \in \mathscr{M}_{t_1}$

and $t \geq 0$, we have

$$\mathbb{Q}_{\omega_0}(\theta_{t_2} - \theta_{t_2 \wedge \tau}, A) = 0 = \mathbb{Q}_{\omega_0}(\theta_{t_1} - \theta_{t_1 \wedge \tau}, A).$$

□

Proof of Lemma 3.1.5. We need to show that θ_t is a martingale on $(\Omega, \mathscr{M}, \mathbb{P}_1^x \otimes_\tau \mathbb{Q})$. According to Theorem 3.1.4 (i.e. [SV79, Theorem 6.1.2]), it suffices to prove the following two statements.

(a) $(\theta_{t\wedge\tau}, \mathscr{M}_t, \mathbb{P}_1^x)$ is a martingale.

(b) $(\theta_t - \theta_{t\wedge\tau}, \mathscr{M}_t, \mathbb{Q}_\omega)$ is a martingale for each $\omega \in \Omega$.

But they are the conclusions of Lemma 3.2.2 and Lemma 3.2.3 respectively. □

3.3 Coupling

In this section, we apply the Gluing Lemma 3.1.5 to the existence of couplings and the weak existence of coupled stochastic differential equations.

Now we suppose $\Omega = C([0,\infty), \mathbb{R}^{2d})$. Denote $Z_t(\omega) = \omega_t = (X_t(\omega), Y_t(\omega)) \in \mathbb{R}^d \times \mathbb{R}^d$ for each $\omega \in \Omega$. For $i = 1, 2$, let $a_i(t,x)\colon [0,\infty) \times \mathbb{R}^d \to S_d$ and $b_i(t,x)\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable functions. Let $c(t,x,y)$ be a $d \times d$ matrix valued measurable function defined on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Set

$$a(t,x,y) = \begin{pmatrix} a_1(t,x) & c(t,x,y) \\ c^*(t,x,y) & a_2(t,y) \end{pmatrix}, \quad b(t,x,y) = \begin{pmatrix} b_1(t,x) \\ b_2(t,y) \end{pmatrix}.$$

Suppose the martingale problems for $L(a_1(t,x), b_1(t,x))$ and $L(a_2(t,x),b_2(t,x))$ are well-posed. We denote the solutions respectively by $\mathbb{P}_1^x$ and $\mathbb{P}_2^y$.

If $\mathbb{P}^{x,y}$ is a solution of the martingale problem for $L(a(t,x,y), b(t,x,y))$, then $\mathbb{P}^{x,y}$ is a coupling of $\mathbb{P}_1^x$ and $\mathbb{P}_2^y$. That is, the marginal distribution of $\mathbb{P}^{x,y}$ are exactly $\mathbb{P}_1^x$ and $\mathbb{P}_2^y$.

Besides function $c(t,x,y)$ introduced above, we will consider the following functions. Let $\sigma(t,x) \in S_d$ be measurable real matrix defined on $[0,\infty) \times \mathbb{R}^d$. Let $b(t,x)$, $\xi(t,x,y)$ be $\mathbb{R}^d$ valued measurable functions defined on $[0,\infty) \times \mathbb{R}^d$ and $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ respectively. We assume that a, b, ξ all are locally bounded.

The following lemma is proved in [CL89, Theorem 3.1].

Lemma 3.3.1. *Suppose that the martingale problem for the basic coupling $L(a, b)$ with*

$$a(t,x,y) = \begin{pmatrix} \sigma(t,x)\sigma(t,x)^* & \sigma(t,x)\sigma(t,y)^* \\ \sigma(t,y)\sigma(t,x)^* & \sigma(t,y)\sigma(t,y)^* \end{pmatrix}, \quad b(t,x,y) = \begin{pmatrix} b(t,x) \\ b(t,y) \end{pmatrix},$$

is locally well-posed. If we denote the solution by $\mathbb{P}^{x,y}$, then we have

$$X_t = Y_t, \quad t \geq \tau, \quad \mathbb{P}^{x,y}\text{–a.s. on } \{\tau < \infty\}.$$

Here τ is the coupling time of the marginal processes X_t, Y_t of Z_t, i.e.

$$\tau := \inf\{t\colon X_t = Y_t\}.$$

The lemma above describes a fundamental property of basic coupling. Intuitively, basic coupling ensures the marginal processes move together after the coupling time. Refer to Figure 3.1. For this reason, basic coupling is also called march coupling. For more details we refer to the books by Chen [Che04, Che05].

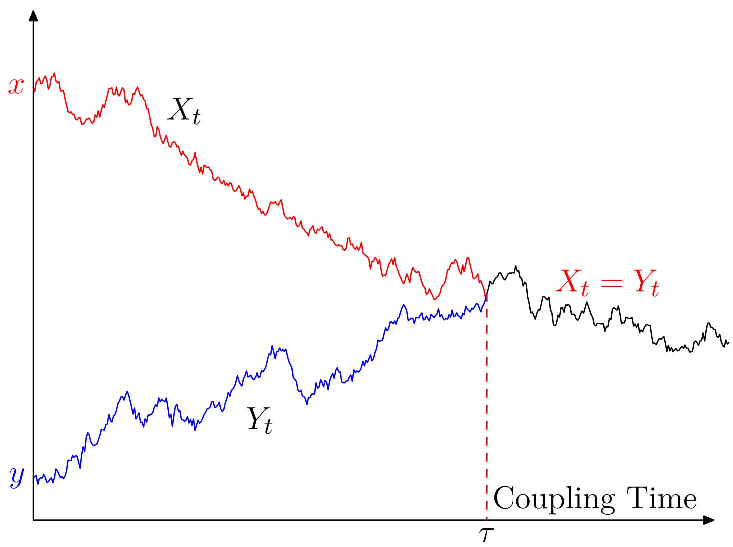

Figure 3.1: March Coupling

Applying Lemma 3.1.5, we get the following corollary which is stated by Chen and Li [CL89, Lemma 3.4] with hints for the proof.

Corollary 3.3.2. *Let $\mathbb{P}_1^{x,y}$ be a solution to the martingale problem for $L(a_1, b)$*

with

$$a_1(t,x,y) = \begin{pmatrix} \sigma(t,x)\sigma(t,x)^* & c(t,x,y) \\ c(t,x,y)^* & \sigma(t,y)\sigma(t,y)^* \end{pmatrix}, \quad b(t,x,y) = \begin{pmatrix} b(t,x) \\ b(t,y) \end{pmatrix}. \tag{3.14}$$

up to some stopping time τ. *For every* $\omega \in \Omega$, *let* $\mathbb{P}_2^{\tau(\omega),Z(\tau(\omega))}$, *be a solution to the martingale problem for the basic coupling operator in Lemma 3.3.1 starting from* $(\tau(\omega), Z(\tau(\omega)))$. *Define* $\mathbb{Q}_\omega$ *for each* $\omega \in \Omega$ *as in Lemma 3.1.5. Then*

$$R = \mathbb{P}_1^{x,y} \otimes_\tau \mathbb{Q}$$

is a solution to the martingale problem for $L(a_2, b)$ *with*

$$a_2(t,x,y) = \begin{pmatrix} \sigma(t,x)\sigma(t,x)^* & \begin{pmatrix} c(t,x,y)\mathbb{1}_{[0,\tau)} \\ +\,\sigma(t,x)\sigma(t,y)^*\mathbb{1}_{[\tau,\infty)} \end{pmatrix} \\ \begin{pmatrix} c(t,x,y)^*\mathbb{1}_{[0,\tau)} \\ +\,\sigma(t,y)\sigma(t,x)^*\mathbb{1}_{[\tau,\infty)} \end{pmatrix} & \sigma(t,y)\sigma(t,y)^* \end{pmatrix}$$

and the drift b *unchanged as in* (3.14).

Remark 3.3.3. A typical use of this fact is the following. First we obtain successful coupling (the marginal processes meet) by choosing $c(t,x,y)$ properly. Then the marginal processes will move together after the coupling time.

Similar to Corollary 3.3.2, we can can obtain coupling by choosing proper drift.

Corollary 3.3.4. *Let* $\mathbb{P}_1^{x,y}$ *be a solution to the martingale problem for* $L(a, b_1)$ *with*

$$a(t,x,y) = \begin{pmatrix} \sigma(t,x)\sigma(t,x)^* & \sigma(t,x)\sigma(t,y)^* \\ \sigma(t,y)\sigma(t,x)^* & \sigma(t,y)\sigma(t,y)^* \end{pmatrix},$$

$$b_1(t,x,y) = \begin{pmatrix} b(t,x) \\ b(t,y) + \xi(t,x,y) \end{pmatrix}.$$

up to some stopping time τ. *For every* $\omega \in \Omega$, *let* $\mathbb{P}_2^{\tau(\omega),Z(\tau(\omega))}$ *be a solution to the martingale problem for the basic coupling operator in Lemma 3.3.1 starting from* $(\tau(\omega), Z(\tau(\omega)))$. *Define* $\mathbb{Q}_\omega$ *for each* $\omega \in \Omega$ *as in Lemma 3.1.5. Then*

$$R = \mathbb{P}_1^{x,y} \otimes_\tau \mathbb{Q}$$

is a solution to the martingale problem for $L(a, b_2)$ *with*

$$b_2(t,x,y) = \begin{pmatrix} b(t,x) \\ b(t,y) + \xi(t,x,y)\mathbb{1}_{\{\tau<t\}} \end{pmatrix}$$

and the diffusion coefficient a *unchanged.*

By the relationship between martingale solution and weak solution of stochastic differential equation (see [KS91]), we can restate Corollary 3.3.4 in the following ways.

Corollary 3.3.5. *Consider the following stochastic differential equations on* $\mathbb{R}^{2d}$

$$\begin{cases} dX_t = \sigma(t,X_t)\,dW_t + b(t,X_t)dt, & X_0 = x \in \mathbb{R}^d, \\ dY_t = \sigma(t,Y_t)\,dW_t + b(t,Y_t)dt + \xi(t,X_t,Y_t)\mathbb{1}_{\{t<\tau\}}dt, & Y_0 = y \in \mathbb{R}^d, \end{cases} \tag{3.15}$$

where W_t *is an* $\mathbb{R}^d$*-valued Brownian motion. Suppose that there exists a weak solution to (3.15) up to* τ*. Assume further that there is a weak solution for all* $t \geq s$ *to the following equation*

$$\begin{cases} dX_t = \sigma(t,X_t)\,dW_t + b(t,X_t)dt, & X_s = \tilde{x} \in \mathbb{R}^d, \\ dY_t = \sigma(t,Y_t)\,dW_t + b(t,Y_t)dt, & Y_s = \tilde{y} \in \mathbb{R}^d, \end{cases} \tag{3.16}$$

for every fixed $(s,(\tilde{x},\tilde{y})) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$*. Then there exists a weak solution to the equation (3.15) for all time.*

Corollary 3.3.6. *Consider the following stochastic differential equation on* $\mathbb{R}^{2d}$

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \sigma(t,X_t,Y_t)\,dW_t + \begin{pmatrix} b(t,X_t) \\ b(t,Y_t) + \xi(t,X_t,Y_t)\mathbb{1}_{\{t<\tau\}} \end{pmatrix} dt, \tag{3.17}$$

with $X_0 = x, Y_0 = y$*, where* W_t *is an* $\mathbb{R}^{2d}$*-valued Wiener process,* $\sigma(t,x,y)$ *is a* $2d \times 2d$ *measurable nonnegative definite matrix. Suppose there exists a weak solution to (3.17) up to* τ *and there is a weak solution for all* $t \geq s$ *for the following equation*

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \sigma(t,X_t,Y_t)\,dW_t + \begin{pmatrix} b(t,X_t) \\ b(t,Y_t) \end{pmatrix} dt,$$

with $X_s = x, Y_s = y$ *for any* $(s,\tilde{x},\tilde{y})) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$*. Then there exists a weak solution to the equation (3.17) for all time.*

Chapter 4

Harnack Inequalities for Stochastic Differential Equations

In this chapter, we show Harnack inequalities for stochastic differential equations and their applications.

In Section 4.1, we introduce Wang's Harnack inequalities ([Wan97]) in which we are interested in this thesis by a simple example. We also refer to a survey paper Wang [Wan06]. In Section 4.2, we recall some known results concerning Harnack inequalities for stochastic differential equations on Euclidean spaces by applying the known results for diffusions on manifolds from [Wan97, ATW06].

We aim to consider Harnack inequalities for stochastic differential equations with more general drifts by the method of coupling and Girsanov's transformation. This method has been introduced by Arnaudon et al. [ATW06] to establish Harnack inequalities for diffusions on manifolds with curvature unbounded below. Coupling methods and Girsanov's transformations are classical tools. For the introduction of coupling methods, see [Lin02, Tho00, Che05] et al.; for Girsanov's theorem, see [KS91, IW81, SV79, RY99] etc..

We establish Harnack inequalities for stochastic differential equations in two frameworks in Section 4.3 and Section 4.4 respectively. We first prove Harnack inequalities with two kinds of abstract assumptions in these two frameworks. The first framework is easier to understand and the second framework involves an approximation procedure. Then we apply these two frameworks to study some concrete examples. In Section 4.5, we consider the classical monotonicity condition under the first framework. In Section 4.6, we assume the stochastic differential equation has linear growth drift and satisfies some regular condition in the second framework.

The method of coupling and Girsanov's transformation doesn't work well for stochastic differential equations driven by general continuous martingales or pure jump processes. We explain the reasons in Section 4.8.

4.1 Introduction to Harnack Inequalities

We demonstrate Harnack inequalities in the sense of Wang [Wan97] for simple Ornstein-Uhlenbeck processes on Euclidean space $\mathbb{R}^d$ by direct computations. For the classical Harnack inequalities, we refer to a survey paper by Kassmann[Kas07] and references therein.

Denote by $\mathscr{C}_b^+(\mathbb{R}^d)$ the set of all nonnegative bounded and continuous function on $\mathbb{R}^d$, and $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$.

Example 4.1.1. Consider the following Ornstein-Uhlenbeck process

$$dX_t = -\kappa X_t dt + dW_t, \tag{4.1}$$

where $\kappa \in \mathbb{R}$ is a constant and W_t is a standard Brownian motion on $\mathbb{R}^d$.

For every initial condition $X_0 = x$, the solution of the stochastic differential equation (4.1) can be written down explicitly as (see for example, [IW81, KS91])

$$X_t = x\,\mathrm{e}^{-\kappa t} + \int_0^t \mathrm{e}^{-\kappa(t-s)}\, dW_s. \tag{4.2}$$

Let

$$\mu_t = N(0, \sigma_t^2),$$

where

$$\sigma_t^2 = \frac{1-\exp(-2\kappa t)}{2\kappa}, \quad t \geq 0.$$

The formula (4.2) allows us to read that for each $t \geq 0$, X_t is Gaussian distributed with mean $x\,\mathrm{e}^{-\kappa t}$ and variance σ_t^2, i.e.

$$X_t \sim N\left(x\,\mathrm{e}^{-\kappa t}, \sigma_t^2\right).$$

Let P_t be the transition semigroup associated with X_t. It can be expressed as

$$P_t f(x) = \int_{\mathbb{R}^d} f(x\mathrm{e}^{-\kappa t} + z)\, d\mu_t(z), \quad x \in \mathbb{R}^d,\ f \in \mathscr{C}_b(\mathbb{R}^d).$$

We shall prove that for every $t > 0, \alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$ and $f \in \mathscr{C}_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d$, we have

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta\kappa|x-y|^2}{\mathrm{e}^{2\kappa t}-1}\right) P_t f^\alpha(y). \tag{4.3}$$

Proof. Note that

$$\mu_t(dz) = (2\pi\sigma_t^2)^{-d/2} \exp\left(-\frac{|z|^2}{2\sigma_t^2}\right) dz.$$

By using a change of variable then applying Hölder's inequality, we can get

$$\begin{aligned}
& P_t f(x) \\
=& \int_{\mathbb{R}^d} f(x\mathrm{e}^{-\kappa t} + z)\, d\mu_t(z) \\
=& \big(2\pi\sigma_t^2\big)^{-d/2} \int_{\mathbb{R}^d} f(x\mathrm{e}^{-\kappa t} + z) \exp\left(-\frac{|z|^2}{2\sigma_t^2}\right) dz \\
=& \big(2\pi\sigma_t^2\big)^{-d/2} \int_{\mathbb{R}^d} f(y\mathrm{e}^{-\kappa t} + z) \exp\left(-\frac{|(y-x)\mathrm{e}^{-\kappa t} + z|^2}{2\sigma_t^2}\right) dz \\
=& \big(2\pi\sigma_t^2\big)^{-d/2} \int_{\mathbb{R}^d} f(y\mathrm{e}^{-\kappa t} + z) \cdot \\
& \qquad \exp\left(-\frac{\mathrm{e}^{-2\kappa t}|x-y|^2 - 2\mathrm{e}^{-\kappa t}\langle x-y, z\rangle + |z|^2}{2\sigma_t^2}\right) dz \\
=& \int_{\mathbb{R}^d} f(y\mathrm{e}^{-\kappa t} + z) \exp\left(-\frac{\mathrm{e}^{-2\kappa t}|x-y|^2 - 2\mathrm{e}^{-\kappa t}\langle x-y, z\rangle}{2\sigma_t^2}\right) d\mu_t(z) \\
=& \exp\left(-\frac{\mathrm{e}^{-2\kappa t}|x-y|^2}{2\sigma_t^2}\right) \int_{\mathbb{R}^d} f(y\mathrm{e}^{-\kappa t} + z) \exp\left(\frac{\mathrm{e}^{-\kappa t}\langle x-y, z\rangle}{\sigma_t^2}\right) d\mu_t(z) \\
\le& \exp\left(-\frac{\mathrm{e}^{-2\kappa t}|x-y|^2}{2\sigma_t^2}\right) \left(\int_{\mathbb{R}^d} f^\alpha(y\mathrm{e}^{-\kappa t} + z) d\mu_t(z)\right)^{1/\alpha} \\
& \qquad \cdot \left(\int_{\mathbb{R}^d} \exp\left[\frac{\beta\mathrm{e}^{-\kappa t}\langle z, x-y\rangle}{\sigma_t^2}\right] d\mu_t(z)\right)^{1/\beta} \\
=& \exp\left(-\frac{\mathrm{e}^{-2\kappa t}|x-y|^2}{2\sigma_t^2}\right) \big(P_t f^\alpha(y)\big)^{1/\alpha} \exp\left(\frac{\beta\mathrm{e}^{-2\kappa t}|x-y|^2}{2\sigma_t^2}\right) \\
=& \exp\left(\frac{(\beta-1)\mathrm{e}^{-2\kappa t}|x-y|^2}{2\sigma_t^2}\right) \big(P_t f^\alpha(y)\big)^{1/\alpha} \\
=& \exp\left(\frac{\mathrm{e}^{-2\kappa t}|x-y|^2}{2(\alpha-1)\sigma_t^2}\right) \big(P_t f^\alpha(y)\big)^{1/\alpha} \\
=& \exp\left(\frac{\kappa|x-y|^2}{(\alpha-1)(\mathrm{e}^{2\kappa t}-1)}\right) \big(P_t f^\alpha(y)\big)^{1/\alpha}.
\end{aligned}$$

□

Remark 4.1.2. We refer to Chapter 5 for more discussions on Harnack inequalities for Ornstein-Uhlenbeck processes.

From the Harnack inequality (4.3), we see what kinds of inequality we are interested at. Let X_t be a general diffusion process on $\mathbb{R}^d$ and denote the corresponding semigroup by P_t, we are looking for inequalities of the following form

$$(P_t f)^\alpha(x) \leq C P_t f^\alpha(y) \tag{4.4}$$

for all $t > 0$, $\alpha > 1$, $x, y \in \mathbb{R}^d$, and $f \in \mathscr{C}_b^+(\mathbb{R}^d)$, where C is some constant depending on t, α, x, y but independent of function f.

One point of this inequality is that we communicate the action of the power α with the action of the semigroup on the function f. To be clear, we can compare the Wang-type Harnack inequality (4.4) with the following celebrated Li-Yau type Harnack inequality (see Li and Yau [LY86]) which communicate the time:

$$P_t f(x) \leq \big(P_{t+s} f(y)\big) \left(\frac{t+s}{t}\right)^{\alpha s/2} \exp\left(\frac{\alpha|x-y|^2}{4s} + \frac{\alpha \kappa s}{4(\alpha-1)}\right)$$

for any $s, t > 0$, $\alpha > 1$ and $f \in \mathscr{C}_b^1(\mathbb{R}^d)$.

Another remarkable feature of this inequality is that it is dimension-free. It is important since dimension is no longer available if the state space is infinite dimensional.

4.2 Harnack Inequalities: Known Results

Harnack inequalities were studied for diffusion processes on Riemannian manifold with curvature bounded and unbounded below in [Wan97] and [ATW06] respectively. We introduce their results and show what is specially known for diffusions on Euclidean spaces.

Let M be a d-dimensional connected complete Riemannian manifold with convex (or empty) boundary. Consider $L = \Delta + Z$ for some C^1-vector field Z on M such that the curvature is bounded below, i.e.

$$\mathrm{Ric}(X, X) - \langle \nabla_X Z, X\rangle \geq -K|X|^2, \quad X \in TM \tag{4.5}$$

for some constant $K \in \mathbb{R}$. Then the diffusion process generated by L is non-explosive and the corresponding semigroup P_t satisfies the following gradient estimation

$$|\nabla P_t f| \le e^{Kt} P_t |\nabla f|, \quad t > 0, f \in \mathscr{C}_b^1(M). \tag{4.6}$$

Wang [Wan97] (see also [Wan04b]) was able to integrate (4.6) along a geodesic and establish the following Harnack inequality: for every $t > 0, \alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, $x, y \in M$, and $f \in \mathscr{C}_b^+(\mathbb{R}^d)$, the following inequality holds

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta K \rho(x,y)^2}{2(1 - e^{-2Kt})}\right) P_t f^\alpha(y), \tag{4.7}$$

where ρ is the distance function on M. When $K = 0$ the right hand side of (4.7) is understood as the limit and (4.7) becomes

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta |x-y|^2}{4t}\right) P_t f^\alpha(y). \tag{4.8}$$

Inequality (4.7) is the best case one can expect under the curvature condition (4.5). Indeed, it is proved in [Wan04a] (see more equivalent statements there) that the curvature condtion (4.5) and the Harnack inequality (4.7) are equivalent.

Since we concentrate on the stochastic differential equations on $\mathbb{R}^d$, we need to understand what we have shown for the special case when the manifold M is reduced to an Euclidean space.

Consider the case when $M = \mathbb{R}^d$ and $Z = b \cdot \nabla$ with $b \in \mathscr{C}^1(\mathbb{R}^d)$. For $i = 1, \dots, d$, we denote $\partial_i = \dfrac{\partial}{\partial x_i}$, and the Jacobian matrix of b by $J = (\partial_j b_i)_{d \times d}$,

Proposition 4.2.1. *Let $M = \mathbb{R}^d$ and $Z = b \cdot \nabla$ with $b \in \mathscr{C}^1(\mathbb{R}^d)$. Then the curvature condition (4.5) is equivalent with each of the following conditions*

(1) For all $\xi \in \mathscr{C}^1(\mathbb{R}^d)$ with $|\xi| = 1$, we have $\langle \xi, J\xi \rangle \le K$. That is,

$$\sum_{i,j=1}^{d} \xi_i \xi_j \partial_j b_i \le K. \tag{4.9}$$

(2) Global weak monotonicity condition:

$$\langle x - y, b(x) - b(y) \rangle \le K|x-y|^2, \quad \text{for all } x, y \in \mathbb{R}^d. \tag{4.10}$$

Proof. Since the Euclidean space is flat, the curvature condition (4.5) is simply

$$\langle \nabla_X Z, X \rangle \le K, \quad |X| = 1.$$

This is true since for every $X = \sum_{i=1}^d \xi_i \partial_i$, we have

$$\begin{aligned}\langle \nabla_X Z, X\rangle &= \left\langle \nabla_X \left(\sum_{i=1}^d b_i\partial_i\right), X\right\rangle = \sum_{i=1}^d \langle \nabla_X(b_i\partial_i), X\rangle \\ &= \sum_{i=1}^d \langle X(b_i)\partial_i + b_i\nabla_X\partial_i, X\rangle = \sum_{i=1}^d Xb_i\langle\partial_i, X\rangle \\ &= \sum_{i=1}^d Xb_i\left\langle \partial_i, \sum_{i=1}^d \xi_j\partial_j\right\rangle = \sum_{i=1}^d \xi_i X b_i = \sum_{i,j=1}^d \xi_i\xi_j\partial_j b_i.\end{aligned}$$

This proves the equivalence between (4.5) and (4.9).

Now we prove that (4.9) implies (4.10).

We will need to use the following Hadamard's formula: for any $x, y \in \mathbb{R}^d$, $i = 1, \ldots, d$, we have

$$b_i(x) - b_i(y) = \sum_{j=1}^d (x_j - y_j)\int_0^1 \partial_j b_i(rx + (1-r)y)\,dr. \tag{4.11}$$

The proof of the formula (4.11) is obvious. We only need to apply the fundamental theorem of calculus to the function $r \mapsto b_i(rx + (1-r)y)$ for $r \in [0, 1]$.

By (4.9) and the Hardamard's formula (4.11), we have

$$\begin{aligned}\langle x - y, b(x) - b(y)\rangle =& \sum_{i=1}^d (x_i - y_i)\big(b_i(x) - b_i(y)\big) \\ =& \sum_{i,j=1}^d (x_i - y_i)(x_j - y_j)\int_0^1 \partial_j b_i\big(rx + (1-r)y\big)dr \\ \le& K|x-y|^2.\end{aligned}$$

Hence the global monotonicity condition (4.10) holds.

It remains to show that the monotonicity condition (4.10) implies (4.9).

By the very definition of derivative, for any $\varepsilon > 0$, there exists a $\delta := \delta(\varepsilon)$ such that for all η satisfying $|\eta| < \delta$, we have

$$\frac{|b(x+\eta) - b(x) - J\eta|}{|\eta|} \le \varepsilon.$$

Hence

$$\langle \eta, J\eta - (b(x+\eta) - b(x))\rangle \le \varepsilon |\eta|^2. \tag{4.12}$$

Now for every $\xi \in \mathbb{R}^d$, $|\xi| = 1$, choosing $\eta \in \mathbb{R}^d$ such that $|\eta| < \delta$ and $\eta = |\eta|\xi$. Take $y = x + \eta$, then deduce from (4.10) we get

$$\langle \eta, b(x+\eta) - b(x)\rangle \le K|\eta|^2,$$

Therefore, we have

$$\frac{\langle \eta, J\eta\rangle}{|\eta|^2} \le K + \frac{\langle \eta, J\eta - (b(x+\eta) - b(x))\rangle}{|\eta|^2}. \tag{4.13}$$

Substitute (4.12) into (4.13) we get

$$\langle \xi, J\xi\rangle \le K + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, (4.9) follows immediately. □

Now we introduce Harnack inequalities proved in [ATW06] for diffusions on manifolds with curvature unbounded below.

Theorem 4.2.2. *Denote by $\rho_0(x) := \rho(o, x)$ the distance of x from a fixed point o. Suppose that*

$$\inf\{\mathrm{Ric}(X,X) : X \in T_xM, |X| = 1\} \ge -C(1 + \rho_0(x)^2), \tag{4.14a}$$

$$\sup\{\langle \nabla_X Z, X\rangle : X \in T_xM, |X| = 1\} \le C(1 + \rho_0(x)), \tag{4.14b}$$

$$\langle Z, \nabla \rho_o(x)\rangle \le C(1 + \rho_0(x)). \tag{4.14c}$$

Assume additionally the corresponding process is non-explosive. Then for any $\varepsilon \in (0,1]$ there exists a constant $c(\varepsilon) > 0$ such that

$$(P_t f)^\alpha(x) \le \exp(N(t, \alpha, x, y, \varepsilon)) P_t f^\alpha(y).$$

where $N(t, \alpha, x, y, \varepsilon)$ is

$$\frac{c(\varepsilon)\alpha^2(\alpha+1)^2}{(\alpha-1)^3}(1 + \rho(x,y)^2)\rho(x,y)^2 + \frac{\alpha(\varepsilon\alpha + 1)\rho(x,y)^2}{2(2-\varepsilon)(\alpha-1)t} + \frac{\alpha - 1}{2}\big(1 + \rho_o(x)^2\big).$$

Again, consider the case $M = \mathbb{R}^d$ and $Z = b\cdot\nabla$ for some $b \in \mathscr{C}^1(\mathbb{R}^d)$. Then the curvature and growth conditions (4.14b) and (4.14c) are reduced to the following conditions

$$\langle x-y, b(t,x)-b(t,y)\rangle \le C(1+|x|+|y|)|x-y|^2,$$
$$\langle x, b(t,x)\rangle \le C(1+|x|^2)$$

for some constant $C > 0$.

4.3 Harnack Inequality: Framework I

Fix $T > 0$. Let $b(t,x)$ be an $\mathbb{R}^d$-valued Borel measurable function defined on $[0,T]\times\mathbb{R}^d$. We aim to study Harnack inequality for the transition semigroup P_t associated with the following stochastic differential equation:

$$dX_t = dW_t + b(t,X_t)dt \tag{4.15}$$

for $t\in[0,T]$, where $(W_t)_{0\le t\le T}$ is standard Brownian motion on $\mathbb{R}^d$,

We turn to consider the following coupled stochastic differential equations on $\mathbb{R}^d$

$$\begin{cases} dX_t = dW_t + b(t,X_t)dt - U_t(X_t,Y_t)dt, & X_0 = x, \qquad (4.16a)\\ dY_t = dW_t + b(t,Y_t)dt, & Y_0 = y. \qquad (4.16b)\end{cases}$$

for $t\in[0,T]$, where $U_t(x,y)$ is an $\mathbb{R}^d$-valued Borel measurable functions defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

For every $0\le t\le T$, set

$$N_t = \int_0^t \langle U_s, dW_s\rangle \quad \text{and} \quad R_t = \exp\left(N_t - \frac{1}{2}[N]_t\right).$$

We will need the following assumption.

Assumption 4.3.1. We assume

(1) The equation (4.16) have a weak solution. That is, there exist processes $(X_t,Y_t,W_t)_{0\le t\le T}$ on some filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{0\le t\le T},\mathbb{P})$ satisfying equation (4.16).

(2) For every starting point, the solution to the equation (4.15) is unique in law.

(3) $(R_t)_{0\le t\le T}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$.

(4) $X_T = Y_T$, $\mathbb{P}$-a.s..

Remark 4.3.2. (1) In applications, we can use the results in Section 3.3 for existence of the weak solution of (4.16).

(2) Item (3) of Assumption 4.3.1 holds if the following Novikov's condition

$$\mathbb{E}\exp\left(\frac{1}{2}[N]_T\right) < \infty \tag{4.17}$$

hold.

With Assumption 4.3.1, we have the following result.

Lemma 4.3.3. *Let Assumption 4.3.1 hold. Then*

$$\begin{aligned} &(P_T f)^\alpha(x) \\ &\le \left[\mathbb{E}_{\mathbb{P}}\exp\left(\frac{1}{2}\beta q(\beta p - 1)\int_0^T |U_s(X_s, Y_s)|^2\,ds\right)\right]^{(\alpha-1)/q} P_T f^\alpha(y) \end{aligned} \tag{4.18}$$

holds for every $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$, *and* $\alpha, \beta, p, q > 1$ *satisfying* $1/\alpha + 1/\beta = 1$ *and* $1/p + 1/q = 1$.

Proof. Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}_T)$ by setting $\mathbb{Q} = R_T\mathbb{P}$. By Girsanov's theorem,

$$\widetilde{W}_t := W_t - \int_0^t U_s\,ds, \quad 0 \le t \le T$$

is also a standard Brownian motion on $(\Omega, \mathscr{F}_T, \mathbb{Q})$. In terms of this new Brownian motion $\widetilde{W}_t$, we can rewrite the equation (4.16) into the following form

$$\begin{cases} dX_t = d\widetilde{W}_t + b(t, X_t)\,dt, & X_0 = x, \quad (4.19a) \\ dY_t = d\widetilde{W}_t + b(t, Y_t)\,dt + U_t(X_t, Y_t), & Y_0 = y. \quad (4.19b) \end{cases}$$

Therefore, $(X_t, \widetilde{W}_t)$ is also a weak solution to stochastic differential equation (4.15) with starting point x. By the uniqueness assumption, we have $P_T f(x) = \mathbb{E}_{\mathbb{Q}} f(X_T)$.

Note the fact that $P_T f(y) = \mathbb{E}_{\mathbb{P}} f(Y_T)$ and $X_T = Y_T$ almost surely, by applying Hölder's inequality, we get

$$\begin{aligned} P_T f(x) &= \mathbb{E}_{\mathbb{Q}} f(X_T) = \mathbb{E}_{\mathbb{Q}} f(Y_T) = \mathbb{E}_{\mathbb{P}} R_T f(Y_T) \\ &\le \left(\mathbb{E}_{\mathbb{P}} R_T^\beta\right)^{1/\beta}\left(\mathbb{E}_{\mathbb{P}} f^\alpha(Y_T)\right)^{1/\alpha} = \left(\mathbb{E}_{\mathbb{P}} R_T^\beta\right)^{1/\beta}\left(P_T f^\alpha(y)\right)^{1/\alpha}. \end{aligned} \tag{4.20}$$

Since $(R_t)_{0\le t\le T}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}} R_T^{\beta}\\
=&\mathbb{E}_{\mathbb{P}} \exp\left(\beta N_T - \frac{1}{2}\beta[N]_T\right)\\
=&\mathbb{E}_{\mathbb{P}} \exp\left(\beta N_T - \frac{1}{2}p\beta^2[N]_T\right)\exp\left(\frac{1}{2}\beta(\beta p-1)[N]_T\right)\\
\le&\left[\mathbb{E}_{\mathbb{P}} \exp\left(p\beta N_T - \frac{1}{2}p^2\beta^2[N]_T\right)\right]^{1/p}\cdot\left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p-1)[N]_T\right)\right]^{1/q}\\
=&\left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p-1)[N]_T\right)\right]^{1/q}.
\end{aligned}\tag{4.21}$$

By substituting the estimate (4.21) above into (4.20), we can get (4.3.3) and finish the proof. □

4.4 Harnack Inequality: Framework II

As in Section 4.3, we fix a constant $T>0$, and let $b(t,x)$ be an $\mathbb{R}^d$-valued Borel measurable function defined on $[0,T]\times\mathbb{R}^d$. We aim to study Harnack inequality for the transition semigroup P_t corresponding to the stochastic differential equation (4.15) with irregular drift.

This time, we turn to consider a approximation of the stochastic differential equation (4.16).

Denote by I the d-dimensional identity matrix. For every $\varepsilon>0$, set

$$\sigma^{\varepsilon} = \frac{1}{2}\begin{pmatrix}(\sqrt{2-\varepsilon}+\sqrt{\varepsilon})\mathrm{I} & (\sqrt{2-\varepsilon}-\sqrt{\varepsilon})\mathrm{I}\\ (\sqrt{2-\varepsilon}-\sqrt{\varepsilon})\mathrm{I} & (\sqrt{2-\varepsilon}+\sqrt{\varepsilon})\mathrm{I}\end{pmatrix}. \tag{4.22}$$

Let us consider the following stochastic differential equation on $\mathbb{R}^{2d}$:

$$dZ_t = \sigma^{\varepsilon} d\overline{W}_t + \overline{b}(t, Z_t)\,dt, \quad Z_0 = z\in\mathbb{R}^{2d}, \tag{4.23}$$

where $\overline{W_t}\in\mathbb{R}^d\times\mathbb{R}^d$ is a $2d$-dimensional Brownian motion, and

$$\overline{b}(t,z) = \overline{b}(t,x,y) = \begin{pmatrix} b(t,x)-\gamma(t,x,y)\\ b(t,y)\end{pmatrix}, \qquad Z_0 = z = \begin{pmatrix}x\\ y\end{pmatrix}\in\mathbb{R}^d\times\mathbb{R}^d.$$

Here the drift γ is an $\mathbb{R}^d$-valued measurable function which maybe dependent on ε.

For every $0 \le t \le T$, let $\overline{W}_{1,t}$ and $\overline{W}_{2,t}$ be the two d-dimensional marginal processes of $\overline{W}_t$:

$$\overline{W}_t = \begin{pmatrix} \overline{W}_{1,t} \\ \overline{W}_{2,t} \end{pmatrix} \in \mathbb{R}^d \times \mathbb{R}^d.$$

Denote

$$W_t = \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} = \sigma^\varepsilon \overline{W}_t.$$

Then

$$\begin{aligned} W_{1,t} &= \frac{1}{2}\left((\sqrt{2-\varepsilon}+\sqrt{\varepsilon})\overline{W}_{1,t} + (\sqrt{2-\varepsilon}-\sqrt{\varepsilon})\overline{W}_{2,t}\right), \\ W_{2,t} &= \frac{1}{2}\left((\sqrt{2-\varepsilon}-\sqrt{\varepsilon})\overline{W}_{1,t} + (\sqrt{2-\varepsilon}+\sqrt{\varepsilon})\overline{W}_{2,t}\right). \end{aligned}$$

Let $(X_t, Y_t) \in \mathbb{R}^d \times \mathbb{R}^d$ be the marginal processes of Z_t.

Now we can rewrite (4.23) as

$$\begin{cases} dX_t = dW_{1,t} + b(t, X_t)dt - \gamma(t, X_t, Y_t)dt, & X_0 = x, \qquad (4.24\text{a}) \\ dY_t = dW_{2,t} + b(t, Y_t)dt, & Y_0 = y, \qquad (4.24\text{b}) \end{cases}$$

for $0 \le t \le T$.

For every $t \in [0, T]$, set

$$N_t = \int_0^t \langle \gamma_s, dW_{2,s} \rangle, \quad R_t = \exp\left(N_t - \frac{1}{2}[N]_t\right).$$

We will work under the following abstract assumption.

Assumption 4.4.1. We assume

(1) For every starting point, the equation (4.23) has a weak solution. That is, there exist couple processes $(Z_t, \overline{W}_t)_{0\le t\le T}$ on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the equation (4.23).
(2) For every starting point, the weak solution to equation (4.15) is weak unique.
(3) $(R_t)_{0\le t\le T}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$.
(4) For a distance ρ on $\mathbb{R}^d$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\rho(X_T, Y_T) \ge \frac{1}{n}\right) = 0. \qquad (4.25)$$

With these assumptions, we can prove the following lemma.

Lemma 4.4.2. *Suppose Assumption 4.4.1 holds. Then*

$$(P_T f)^\alpha(x) \le \lim_{n\uparrow\infty} \lim_{\varepsilon\downarrow 0} \left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p - 1) \int_0^T |\gamma_s|^2 \, ds \right) \right]^{(\alpha-1)/q} P_T f^\alpha(y). \tag{4.26}$$

holds for every $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$ *and* $\alpha, \beta, p, q > 1$ *satisfying* $1/\alpha + 1/\beta = 1$ *and* $1/p + 1/q = 1$.

Proof. Define a new probability measure on $(\Omega, \mathscr{F}_T)$ by setting $\mathbb{Q} = R_T \mathbb{P}$. Then by Girsanov's theorem,

$$\widetilde{W}_{1,t} := W_{1,t} - \int_0^t \gamma_t dt, \quad t \in [0, T]$$

is a Wiener process on $(\Omega, \mathscr{F}_T, \mathbb{Q})$. Now we can rewrite the equation (4.24a) into the following form

$$dX_t = d\widetilde{W}_{1,t} + b(t, X_t)dt, \quad X_0 = x. \tag{4.27}$$

We see $(X_t, \widetilde{W}_{1,t})$ on $(\Omega, \mathscr{F}_T, \mathbb{Q})$ is also a weak solution to the stochastic differential equation (4.15). Therefore, by the weak uniqueness, we have $P_T f(x) = \mathbb{E}_{\mathbb{Q}} f(X_T)$.

Without loss of generality, we can assume that f is a bounded nonnegative function on $\mathbb{R}^d$ such that

$$|f(x) - f(y)| \le \mathrm{Lip}(f)\rho(x, y).$$

for some constant $\mathrm{Lip}(f)$. With this, we have

$$\begin{aligned} P_T f(x) &= \mathbb{E}_{\mathbb{Q}} f(X_T) \\ &= \mathbb{E}_{\mathbb{Q}} f(X_T) \mathbb{1}_{\{\rho(X_T, Y_T) \le \frac{1}{n}\}} + \mathbb{E}_{\mathbb{Q}} f(X_T) \mathbb{1}_{\{\rho(X_T, Y_T) > \frac{1}{n}\}} \\ &\le \mathbb{E}_{\mathbb{Q}} \big[f(Y_T) + \big(f(X_T) - f(Y_T) \big) \big] \mathbb{1}_{\{\rho(X_T, Y_T) \le \frac{1}{n}\}} \\ &\qquad + \|f\|_\infty \mathbb{P}\left(\left\{ \rho(X_T, Y_T) > \frac{1}{n} \right\} \right) \\ &\le \mathbb{E}_{\mathbb{Q}} f(Y_T) + \frac{1}{n} \mathrm{Lip}(f) + \|f\|_\infty \mathbb{P}\left(\left\{ \rho(X_T, Y_T) > \frac{1}{n} \right\} \right). \end{aligned} \tag{4.28}$$

Note that we also have $P_T f(y) = \mathbb{E}_{\mathbb{P}} f(Y_T)$. Therefore, by applying Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}} f(Y_T) = \mathbb{E}_{\mathbb{P}} R_T f(Y_T) &\le \mathbb{E}_{\mathbb{P}}\big(R_T^\beta\big)^{1/\beta}\mathbb{E}_{\mathbb{P}}\big(f^\alpha(Y_T)\big)^{1/\alpha} \\ &= \mathbb{E}_{\mathbb{P}}\big(R_T^\beta\big)^{1/\beta}\big(P_T f^\alpha(y)\big)^{1/\alpha}.\end{aligned} \tag{4.29}$$

Since $(R_t)_{t\in[0,T]}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$, we have (similar to (4.21))

$$\mathbb{E}_{\mathbb{P}} R_T^\beta \le \left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p - 1)\int_0^T |\gamma_s|^2\, ds\right)\right]^{1/q}. \tag{4.30}$$

Now substitute (4.29) and (4.30) into (4.28), we get

$$\begin{aligned}P_T f(x) \le \left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p - 1)\int_0^T |\gamma_s|^2\, ds\right)\right]^{1/(q\beta)} &\big(P_T f^\alpha(y)\big)^{1/\alpha} \\ + \frac{1}{n}\mathrm{Lip}(f) + \|f\|_\infty \mathbb{P}&\left(\left\{\rho(X_T, Y_T) > \frac{1}{n}\right\}\right).\end{aligned} \tag{4.31}$$

We can get (4.26) and finish the proof by letting $\varepsilon \to 0$ and $n \to \infty$ in (4.31) and noting the assumption (4.25). □

4.5 Global Monotonicity Condition

In this section, we apply the framework in Section 4.3 to the classical Global monotonicity condition. In this way we see that we can obtain the known Harnack inequalities for stochastic differential equations on Euclidean spaces (see Section 4.2).

Let b be a continuous function satisfying the following linear growth condition

$$|b(t,x)| \le C(1+|x|), \quad x \in \mathbb{R}^d,\ 0 \le t \le T$$

for some constant $C > 0$ and the following global monotonicity condition

$$\langle x - y, b(t,x) - b(t,y)\rangle \le K|x-y|^2, \quad x, y \in \mathbb{R}^d,\ 0 \le t \le T. \tag{4.32}$$

for some $K \in \mathbb{R}$. We take

$$U_t(X_t, Y_t) := \xi_t |x - y| \frac{X_t - Y_t}{|X_t - Y_t|} \mathbb{1}_{\{t<\tau\}},$$

where $\xi_t > 0$ is a deterministic continuous positive function on $[0, T]$ satisfying

$$\int_0^T \xi_s \, \mathrm{e}^{-Ks} \, ds \geq 1. \tag{4.33}$$

We first prepare two lemmas to check Assumption 4.3.1.

Lemma 4.5.1. *Items* (1) *and* (2) *of Assumption* (4.3.1) *hold.*

Proof. With the global monotonicity condition (4.32), we see Equation (4.16a) has a unique weak solution for all time. On the other hand, for every fixed $s \in [0, T]$, $\tilde{x}, \tilde{y} \in \mathbb{R}^d$, the following equation

$$\begin{cases} dX_t = dW_t + b(t, X_t)\, dt, & X_s = \tilde{x} \in \mathbb{R}^d, \\ dY_t = dW_t + b(t, Y_t)\, dt, & Y_s = \tilde{y} \in \mathbb{R}^d. \end{cases}$$

has a weak solution for $t \in [s, T]$.

Since b is continuous and U_t is also continuous before the coupling time τ, by [IW81, Chapter IV, Theorem 2.3] we know Equation (4.16) has a weak solution up to τ.

Now, by the results in Section 3.3, we know Equation (4.16) has a weak solution on $[0, T]$. □

In the following, we prove that the the two marginal processes meet at time T. Figure 4.5 explains the idea.

Lemma 4.5.2. *We have $X_T = Y_T$ $\mathbb{P}$-a.s.*

Proof. We only need to prove that X_t and Y_t shall meet each other before the fixed time T since X_t and Y_t will move together after the coupling time. That is we need to show $\tau \leq T$.

It is easy to see that

$$d(X_t - Y_t) = \left(b(t, X_t) - b(t, Y_t) - \xi_t |x - y| \frac{X_t - Y_t}{|X_t - Y_t|} \mathbb{1}_{\{t<\tau\}} \right) dt, \quad t < \tau.$$

By (4.32), we have

$$d|X_t - Y_t| \leq K|X_t - Y_t|\, dt - \xi_t |x - y|\, dt, \quad t < \tau.$$

Hence

$$d|X_t - Y_t|\, \mathrm{e}^{-Kt} \leq \xi_t |x - y|\, \mathrm{e}^{-Kt}, \quad t < \tau.$$

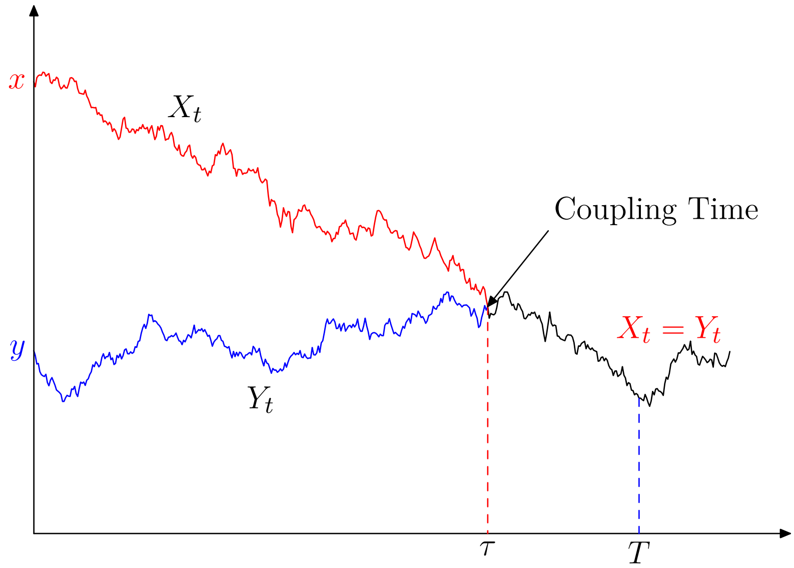

Figure 4.1: Coupling Before Fixed Time

Thus, if $T < \tau$ then

$$0 < |X_T - Y_T|\,\mathrm{e}^{-KT} \le |x-y|\left(1 - \int_0^T \xi_s\,\mathrm{e}^{-Ks}\,ds\right) \le 0.$$

This contradiction implies that $T \ge \tau$ and hence $X_T = Y_T$. □

By we can apply Lemma 4.3.3 to prove the following theorem.

Theorem 4.5.3. *Let* (4.32) *holds. Then*

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta K|x-y|^2}{1-\mathrm{e}^{-2KT}}\right) P_T f^\alpha(y) \tag{4.34}$$

holds for every $T > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$ *and* $\alpha, \beta > 1$ *satisfying* $1/\alpha + 1/\beta = 1$.

Proof. Since U_t is bounded, Item (3) of Assumption 4.3.1 holds automatically by applying Novikov's condition (see (4.17))

(2)). With Lemma 4.5.1 and 4.5.2, we know Assumption 4.3.1 hold.

By (4.18) we get

$$(P_T f)^\alpha(x) \le \left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta(\beta p - 1)\int_0^T |x-y|^2 \xi_s^2\,ds\right)\right]^{\alpha-1} P_T f^\alpha(y), \tag{4.35}$$

where $p > 1$.

Let $p \to 1$ in (4.35) and note that $(\alpha-1)(\beta-1)=1$ we can obtain

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta|x-y|^2}{2}\int_0^T \xi_s^2\, ds\right) P_T f^\alpha(y), \tag{4.36}$$

Now take

$$\xi_t = \frac{\mathrm{e}^{-Kt}}{\int_0^T \mathrm{e}^{-2Ks}\, ds}, \quad 0 \le t \le T. \tag{4.37}$$

Then

$$\int_0^T \xi_t^2\, dt = \frac{\int_0^T \left(\mathrm{e}^{Ks}\,\mathrm{e}^{-2Ks}\right)^2\, ds}{\left(\int_0^T \mathrm{e}^{-2Ks}\, ds\right)^2} = \frac{1}{\int_0^T \mathrm{e}^{-2Ks}\, ds} = \frac{2K}{1-\mathrm{e}^{-2KT}}. \tag{4.38}$$

Substitute (4.38) into (4.36), we can get (4.34) and complete the proof. □

Remark 4.5.4. Let ξ_t be any function satisfy (4.33). Then by Hölder's inequality we have

$$1 \le \left(\int_0^T \xi_s\,\mathrm{e}^{-Ks}\, ds\right)^2 \le \int_0^T \xi_s^2\, ds \int_0^T \mathrm{e}^{-2Ks}\, ds.$$

Therefore,

$$\int_0^T \xi_s^2\, ds \ge \frac{1}{\int_0^T \mathrm{e}^{-2Ks}\, ds} = \frac{2K}{1-\mathrm{e}^{-2KT}}.$$

This means that $\dfrac{2K}{1-\mathrm{e}^{-2KT}}$ is the minimum of $\int_0^T \xi_s^2\, ds$ over all choices of ξ_t satisfying (4.33). Hence the function ξ_t in (4.37) is optimal under all possible choices of ξ_t.

4.6 Linear Growth Condition

We work in the framework introduced in Section 4.4. We take the distance function $\rho(x,y)$ to be the Euclidean distance: $\rho(x,y) = |x-y|$ for every $x, y \in \mathbb{R}^d$.

Fix $T > 0$. Let $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function satisfying the following conditions.

Assumption 4.6.1. (1) There exist some constant $C > 0$ such that

$$|b(t,x)| \le C(1+|x|), \quad 0 \le t \le T,\ x \in \mathbb{R}^d. \tag{4.39}$$

(2) There exists a nonnegative function g on $[0,\infty)$ such that

$$\sup_{|x-y|=r} \frac{1}{r}\langle b(t,x)-b(t,y), x-y\rangle \le g(r). \tag{4.40}$$

Remark 4.6.2. (1) The linear growth condition (4.39) is used to ensure the existence of weak solution the equation. We do not need the continuous of the drift.

(2) We use Condition (4.40) to get a better estimate of the constant in the Harnack inequality we will prove. Condition (4.40) is also used in [PW06, Hypothesis 3.1 iv.]. This condition generalizes substantially the standard condition that $g(r) = cr$ for some $c > 0$, which implies the uniqueness and regularity of strong solutions of the associated stochastic differential equations. If b is uniformly continuous on $\mathbb{R}^d$, we can take g as the modulus of continuity of b, i.e. $g(r) = \sup_{|x-y|\le r} \sup |b(x)-b(y)|$.

We will apply Lemma 4.4.2 to prove the following result.

Theorem 4.6.3. *Suppose that* (4.39), (4.40) *hold and the solution to Equation* (4.15) *is weak unique. For every* $T>0$, $x,y\in\mathbb{R}^d$, $f\in\mathscr{C}_b^+(\mathbb{R}^d)$ *and* $\alpha,\beta>1$ *satisfying* $1/\alpha+1/\beta=1$, *the following inequality holds*

$$\left(P_T f\right)^\alpha(x) \le \exp\left(\frac{\beta}{2}\int_0^T \left[g(|x-y|)+\frac{\xi_t|x-y|}{\int_0^T \xi_u\, du}\right]^2 dt\right) P_T f^\alpha(y). \tag{4.41}$$

Remark 4.6.4. The weak uniqueness of solution to equation (4.15) can be implied from proper choice of g in (4.40). See for example [FZ05].

To prove Theorem 4.6.3, we first take a concrete γ in Lemma 4.4.2 and then check Assumption 4.4.1.

Let $n\in\mathbb{N}$. We take $\varepsilon=\frac{1}{n^4}$ for (4.22) and set

$$\begin{aligned}\gamma(t,X_t,Y_t) = \Bigg[&\left\langle \frac{X_t-Y_t}{|X_t-Y_t|}, b(t,X_t)-b(t,Y_t)\right\rangle^+ \\ &- \frac{\varepsilon(d-1)}{|X_t-Y_t|}\mathbb{1}_{\{|X_t-Y_t|\ge \frac{1}{n^2}\}} - \frac{\xi_s|x-y|}{\int_0^T \xi_s\, ds}\Bigg]\cdot \frac{X_t-Y_t}{|X_t-Y_t|}\mathbb{1}_{\{|X_t-Y_t|>0\}}\end{aligned} \tag{4.42}$$

for the drift in the stochastic differential equation (4.23).

Lemma 4.6.5. *Items* (1) *and* (2) *of Assumption 4.4.1 hold.*

Proof. Since γ is of linear growth, by [KS91, Proposition 3.6], we know Equation (4.23) have a weak solution. Hence Item (1) of Assumption 4.4.1 holds.

Item (2) of Assumption 4.4.1 follows from [KS91, Corollary 3.5.16]. □

We will use the formulae (2.8)–(2.10) in [CL89]. For convenience, we summarize them into the following lemma.

Lemma 4.6.6. *Let* $L = L(a(t,x,y), b(t,x,y))$ *be a second order differential operator of the form* (3.1) *with*

$$a(t,x,y) = \begin{pmatrix} a_1(t,x) & c(t,x,y) \\ c^*(t,x,y) & a_2(t,y) \end{pmatrix}, \quad b(t,x,y) = \begin{pmatrix} b_1(t,x) \\ b_2(t,y) \end{pmatrix}.$$

Denote $\rho(x,y) = |x-y|$ *and*

$$\begin{aligned} A(t,x,y) &= a_1(t,x) + a_2(t,y) - 2c(t,x,y), \\ B(t,x,y) &= b_1(t,x) - b_2(t,y), \\ \widehat{A}(t,x,y) &= \langle x-y, A(t,x,y)(x-y)\rangle, \quad x \neq y, \\ \overline{A}(t,x,y) &= \frac{\widehat{A}(t,x,y)}{|x-y|^2}. \\ \widehat{B}(t,x,y) &= \langle x-y, B(x,y)\rangle. \end{aligned}$$

Then for every $x \neq y$,

$$L\rho(x,y) = \frac{\mathrm{Tr} A(t,x,y) - \overline{A}(t,x,y) + 2\widehat{B}(t,x,y)}{2\rho(x,y)}.$$

Lemma 4.6.7. *For all* $t \in [0,T]$,

$$|X_t - Y_t| \le \sqrt{\varepsilon}(\omega_t - \omega_{\delta_n(t)}) + \frac{1}{n^2} \vee \left(|x-y| \frac{\int_t^T \xi_s\, ds}{\int_0^T \xi_s\, ds} \right),$$

where

$$\delta_n(t) = \sup\{s \in [0,t] \colon |X_s - Y_s| < \frac{1}{n^2}\}, \quad t \in [0,T].$$

Here we use the convention $\sup \emptyset = 0$.

Proof. Note that for every $t \in [0,T]$,

$$W_{1,t} - W_{2,t} = \sqrt{\varepsilon}(\overline{W}_{1,t} - \overline{W}_{2,t}).$$

Let's denote

$$\omega_t := \left\langle \frac{X_t - Y_t}{|X_t - Y_t|}, \overline{W}_{1,t} - \overline{W}_{2,t} \right\rangle.$$

Note that

$$A^\varepsilon := \sigma^\varepsilon (\sigma^\varepsilon)^* = \begin{pmatrix} \mathrm{I} & \mathrm{I} - \varepsilon \\ \mathrm{I} - \varepsilon & \mathrm{I} \end{pmatrix}.$$

Applying Itô's formula and Lemma 4.6.6 directly, we have

$$d|X_t - Y_t| \leq \sqrt{\varepsilon} d\omega_t - \frac{|x-y|\xi_t}{\int_0^T \xi_s ds} + \frac{\varepsilon(d-1)}{|X_t - Y_t|} \mathbb{1}_{\{|X_t - Y_t| < \frac{1}{n^2}\}} dt. \tag{4.43}$$

By integrating Equation (4.43) over $[\delta_n(t), t]$, we obtain (note that the last term disappears)

$$|X_t - Y_t| \leq \sqrt{\varepsilon}(\omega_t - \omega_{\delta_n(t)}) + \Theta(t, n). \tag{4.44}$$

where

$$\Theta(t, n) := |X_{\delta_n(t)} - Y_{\delta_n(t)}| - |x - y| \frac{\int_{\delta_n(t)}^t \xi_s ds}{\int_0^T \xi_s ds}.$$

If $\delta_n(t) = 0$, then

$$\Theta(t, n) = |X_0 - Y_0| - |x - y| \frac{\int_0^t \xi_s ds}{\int_0^T \xi_s ds} = |x - y| \frac{\int_t^T \xi_s ds}{\int_0^T \xi_s ds}. \tag{4.45}$$

If $\delta_n(t) > 0$, then $|X_{\delta_n(t)} - Y_{\delta_n(t)}| = \dfrac{1}{n^2}$. Therefore,

$$\Theta(t, n) = \frac{1}{n^2} - |x - y| \frac{\int_{\delta_n(t)}^t \xi_s ds}{\int_0^T \xi_s ds} \leq \frac{1}{n^2}. \tag{4.46}$$

Hence from (4.45) and (4.46) we see

$$\Theta(t, n) \leq \frac{1}{n^2} \vee \left(|x - y| \frac{\int_t^T \xi_s ds}{\int_0^T \xi_s ds} \right).$$

□

Lemma 4.6.8.

$$\lim_{n \to \infty} \mathbb{P}\left(|X_T - Y_T| > \frac{1}{n} \right) = 0.$$

Proof. Applying Lemma 4.6.7 to the case $t = T$, we have

$$|X_T - Y_T| \le \sqrt{\varepsilon}(\omega_T - \omega_{\delta_n(T)}) + \frac{1}{n^2}.$$

Since $\varepsilon = \frac{1}{n^4}$, $|X_T - Y_T| > \frac{1}{n}$ implies

$$\omega_T - \omega_{\delta_n(T)} \ge n - 1.$$

Hence

$$\begin{aligned}\mathbb{P}\left(|X_T - Y_T| > \frac{1}{n}\right) \le& \mathbb{P}\left(\omega_T - \omega_{\delta_n}(T) \ge n-1\right)\\ \le& \frac{4\mathbb{E}\sup_{s\in[0,T]}|\omega_s|^2}{(n-1)^2} \le \frac{16\mathbb{E}|\omega_T|^2}{(n-1)^2} \to 0.\end{aligned}$$

as $n \to \infty$. The last inequality follows from Doob's martingale inequality. □

Lemma 4.6.9. *For all $t \in [0, T]$,*

$$\lim_{n\to\infty} |X_t - Y_t| \le |x - y|.$$

Proof. By Lemma 4.6.7, we see for all $t \in [0, T]$,

$$|X_t - Y_t| \le \frac{1}{n^2}\omega_t + |x - y| + \frac{1}{n^2}.$$

□

Now Theorem 4.6.3 is easy to show.

Proof of Theorem 4.6.3. With Lemma 4.6.5, Lemma 4.6.8 and Lemma 4.6.9, we can apply Lemma 4.4.2 and Inequality (4.26) to get (4.41). □

4.7 Heat Kernel Estimates

Consider the stochastic differential equation (4.15) with time independent drift b. Suppose that $b(x) = DV(x)$ for some C^2-function V on $\mathbb{R}^d$. Set $\mu(dx) = \exp(V(x))\,dx$. Let P_t be the transition semigroup associated with (4.15). Then P_t is symmetric with respect to μ. Denote by $p_t(x, y)$ the transition kernel of P_t with respect to μ. We aim to estimate p_t by using the measures of balls.

Harnack inequalities is an important tool in the study of heat kernel estimates. In the following we apply Wang's Harnack inequality to estimate the heat kernel.

This kinds of applications has been used in [BLQ97, GW01, ATW06] etc. for diffusion semigroups on manifolds.

We first summarize the application of (Wang's) Harnack inequality to heat kernel estimate by the following lemma. We remind the reader that the generator of the semigroup we consider has the form $\frac{1}{2}\Delta+DV$, while the operator considered in [BLQ97, GW01, ATW06] etc. is of the form $\Delta + Z$ for some Laplace operator Δ and C^1-operator Z on manifolds. Hence there is a slight change.

Lemma 4.7.1. *Suppose that for every $t > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$ and $1 < \alpha < 2$ the following inequality holds*

$$(P_t f)^\alpha(x) \le \exp\big(C(t,\alpha,|x-y|)\big) P_t f^\alpha(y), \tag{4.47}$$

where $C(t,\alpha,\cdot)$ is a positive increasing function. Then

$$p_t(x,y) \le \frac{\exp\Big(\frac{1+\delta}{\delta}C\big(t,\frac{2\delta}{1+\delta},\sqrt{2t}\big)+\frac{4}{\delta-1}\Big)}{\sqrt{\mu\big(B_x(\sqrt{2t})\big)\mu\big(B_y(\sqrt{2t})\big)}}\exp\left(-\frac{|x-y|^2}{2\delta t}\right)$$

holds for every $\delta > 1$, $t > 0$, and $x, y \in \mathbb{R}^d$.

Proof. Let $p = \frac{2}{\alpha} > 1$. Taking power p to both sides of (4.47) we get

$$(P_t f)^2(x) \le (P_t f^\alpha)^p(y)\exp\big(pC(t,\alpha,|x-y|)\big) \tag{4.48}$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$.

Let $T > 0$, $x \in \mathbb{R}^d$, $q = \frac{p}{4(p-1)}$. Set

$$\eta(t,y) = -\frac{|x-y|^2}{2(T-qt)}, \quad \text{for } qt < T.$$

Multiplying both sides of (4.48) by $\exp\big(\eta(t,y)\big)$ and then taking integral with respect to $\mu(dy)$ we can obtain

$$\begin{aligned}&(P_t f)^2(x)\int_{\mathbb{R}^d}\exp\big(-pC(t,\alpha,|x-y|)\big)\exp\big(\eta(t,y)\big)\,\mu(dy)\\ &\le \int_{\mathbb{R}^d}(P_t f^\alpha)^p(y)\exp\big(\eta(t,y)\big)\,\mu(dy).\end{aligned} \tag{4.49}$$

By an integral-maximum principle (see [BLQ97, Proposition 13] or [GW01, (2.9)]), and note the invariance of P_t with respect to μ, we can get the following estimate of the right hand side of (4.49):

$$\begin{aligned}\int_{\mathbb{R}^d}(P_t f^\alpha)^p(y)\exp\big(\eta(t,y)\big)\,\mu(dy) &\le \int_{\mathbb{R}^d} P_t f^2(y)\exp\big(\eta(0,y)\big)\,\mu(dy)\\ &= \int_{\mathbb{R}^d} f^2(y)\exp\left(-\frac{|x-y|^2}{2T}\right)\mu(dy).\end{aligned} \tag{4.50}$$

For the left side of (4.49), confine the integral on the ball $B_x(r) := \{y \in \mathbb{R}^d\colon |x-y| \le r\}$, we can get

$$\int_{\mathbb{R}^d}\exp\big(-pC(t,\alpha,|x-y|)\big)\exp\big(\eta(t,y)\big)\,\mu(dy) \tag{4.51}$$

$$\ge \int_{B_x(r)}\exp\left(-pC(t,\alpha,r)-\frac{r^2}{2(T-qt)}\right)\mu(dy) \tag{4.52}$$

$$= \exp\left(-pC(t,\alpha,r)-\frac{r^2}{2(T-qt)}\right)\mu\big(B_x(r)\big). \tag{4.53}$$

Hence, by combining (4.49), (4.50) and (4.51) we obtain

$$\begin{aligned}(P_t f)^2(x) \le &\frac{\exp\left(pC(t,\alpha,r)+\frac{r^2}{2(T-qt)}\right)}{\mu\big(B_x(r)\big)}\times\\ &\int_{\mathbb{R}^d} f^2(y)\exp\left(-\frac{|x-y|^2}{2T}\right)\mu(dy).\end{aligned} \tag{4.54}$$

For any bounded positive continuous function g on $\mathbb{H}$, we take

$$f(x) = g(y)\exp\left(\frac{|x-y|^2}{4T}\right)$$

in (4.54), we get

$$\begin{aligned}&\left[\int_{\mathbb{R}^d}\exp\left(\frac{|x-y|^2}{4T}\right)g(y)p_t(x,y)\,\mu(dy)\right]^2\\ \le&\frac{\exp\left(pC(t,\alpha,r)+\frac{r^2}{2(T-qt)}\right)}{\mu\big(B_x(r)\big)}\int_{\mathbb{R}^d} g^2(y)\,\mu(dy).\end{aligned} \tag{4.55}$$

Let $\delta > 1$, $T = \frac{\delta t}{2}$, $q = \frac{1+\delta}{4}$, $r = \sqrt{2t}$. Then $p = \frac{1+\delta}{\delta}$, $\alpha = \frac{2\delta}{1+\delta}$ and $T - qt =$

$\frac{\delta-1}{4}t$. Now we can deduce from (4.55) to get

$$
\begin{aligned}
E_\delta(x,t) &:= \int_{\mathbb{R}^d} p_t^2(x,y)\exp\left(\frac{|x-y|^2}{\delta t}\right)\mu(dy)\\
&\le \frac{\exp\Big(pC(t,\alpha,r)+\frac{r^2}{2(T-qt)}\Big)}{\mu\big(B_x(r)\big)}\\
&= \frac{\exp\Big(\frac{1+\delta}{\delta}C\big(t,\frac{2\delta}{1+\delta},\sqrt{2t}\big)+\frac{4}{\delta-1}\Big)}{\mu\big(B_x(r)\big)}.
\end{aligned}
\tag{4.56}
$$

By the estimate (4.56) and the following universal bound (see Grigor′yan [Gri97, (3.4)])

$$
p_t(x,y) \le \sqrt{E_\delta\left(x,\frac{t}{2}\right)E_\delta\left(y,\frac{t}{2}\right)}\exp\left(-\frac{|x-y|^2}{2\delta t}\right),
$$

we can finish the proof. □

Assume the assumptions in Theorem 4.6.3 holds. Then we have Harnack inequality (4.41). Hence we can get the following corollary immediately by Lemma 4.7.1.

Corollary 4.7.2. *Assume for the drift b the conditions in Theorem 4.6.3 and the assumptions at the begging of this section . Then for every $\delta > 1$, $t > 0$, $x, y \in \mathbb{R}^d$ we have*

$$
p_t(x,y) \le \frac{\exp\Big(\frac{1+\delta}{\delta}\widetilde{C}\big(t,\frac{2\delta}{1+\delta},\sqrt{2t}\big)+\frac{4}{\delta-1}\Big)}{\sqrt{\mu\big(B_x(\sqrt{2t})\big)\mu\big(B_y(\sqrt{2t})\big)}}\exp\left(-\frac{|x-y|^2}{2\delta t}\right),
$$

where

$$
\widetilde{C}(t,\alpha,r) = \frac{\alpha}{2(\alpha-1)}\int_0^t\left[g(r)+\frac{r\xi_s}{\int_0^s \xi_u\,du}\right]^2 ds.
$$

4.8 Some Problems in Applying Girsanov's Theorem

Maybe one want to try the coupling and Girsanov transformation method to study Harnack inequalities for stochastic differential equations driven by general continuous martingale or pure jump Lévy processes. Unfortunately, this does not work in general. One of the essential point of the Girsanov transformation we used is that the distribution of the drift transformed process under the new probability measure must be the same with the original process under the original probability measure. In the following, we explain the reasons.

Continuous Martingale Case

The following Girsanov theorem for continuous martingale is well known. See, for example, [RY99].

Theorem 4.8.1. *Let M be a continuous martingale, and*

$$Z_t = \exp\left(M_t - \frac{1}{2}[M]_t\right), 0 \le t < \infty$$

be a positive uniform integrable martingale. Let $\mathbb{Q} = \int Z_\infty d\mathbb{P}$. If N is a continuous $\mathbb{P}$ martingale, then $\widetilde{N}_t \equiv N_t - [N, M]_t$, $0 \le t < \infty$ is a continuous $\mathbb{Q}$ martingale, and $[\widetilde{N}]_t^{\mathbb{Q}} = [N]_t$, for $0 \le t < \infty$.

The Girsanov theorem 4.8.1 states that $\widetilde{N}_t$, the drift transformed N_t, is still a martingale (under the new probability measure $\mathbb{Q}$), and the quadratic variations of N_t and $\widetilde{N}_t$ are the same. But it does not ensure that the distribution of $\widetilde{N}_t$ under $\mathbb{Q}$ is the same with the distribution of N_t under $\mathbb{P}$. It is the case only in some special situation. For example, if N_t is a Brownian motion, then by applying Lévy's characterization of Brownian motions, we know $\widetilde{N}_t$ is still a Brownian motion. And hence their distributions coincide.

Pure Jump Lévy Processes

We first recall a Girsanov theorem for pure jump processes.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t < \infty}, \mathbb{P})$ be a filtered probability space and $N(dt, dz)$ a Poisson random measure on $\Omega \times \mathbb{R}$ with Lévy measure ν. Suppose that the Lévy measure satisfies

$$\int_{\{|z|>1\}} |z|\, |\nu|(dz) < \infty.$$

The compensated measure of $N(dt, dx)$ is given by

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt.$$

The following result is from [ØS05, Lemma 1.33]. See also [JS87, Chap III, Theorems 3.24 and 5.19], [Cha99, Lemma 3.1 and Theorem 3.2] and [Sit05] etc..

Theorem 4.8.2. *Let $\theta(s,x) \le 1$ be a process such that*

$$\rho(t) = \exp\left\{\int_0^t \int_{\mathbb{R}} \log\bigl(1-\theta(s,z)\bigr)\,\widetilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}} \bigl[\log\bigl(1-\theta(s,z)\bigr) + \theta(s,z)\bigr]\,\widetilde{N}(ds,dz)\right\}$$

exists for $0 \le t \le T$. Define a measure $\mathbb{Q}$ on $\mathscr{F}_T$ by $\mathbb{Q} = \rho(T)\mathbb{P}$. Assume that $\mathbb{E}_{\mathbb{P}}\bigl(\rho(T)\bigr) = 1$. Then $\mathbb{Q}$ is a probability measure on $\mathscr{F}_T$ and if we define the random measure $\widetilde{N}^{\mathbb{Q}}(dt,dz)$ by

$$\widetilde{N}^{\mathbb{Q}}(dt,dz) = \widetilde{N}(dt,dz) + \theta(t,z)\,\nu(dz)dt,$$

then

$$\int_0^t \int_A \widetilde{N}(ds,dz) + \int_0^t \int_A \theta(s,z)\,\nu(dz)ds$$

is a $\mathbb{Q}$-local martingale for all $A \in \mathcal{B}(\mathbb{R}\setminus\{0\})$

We claim that the distribution of $\widetilde{N}(dt,dz)$ under $\mathbb{P}$ is not the same with the distribution of $\widetilde{N}^{\mathbb{Q}}(dt,dz) = \widetilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt$. under $\mathbb{Q}$.

The explain follows. First we note that

$$\begin{aligned}\widetilde{N}^{\mathbb{Q}}(dt,dz) + \nu(dz)dt &= [\widetilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt] + \nu(dz)dt \\ &= N(dt,dz) + \theta(t,z)\nu(dz)dt.\end{aligned}$$

Suppose that our claim is not true. Then the distribution of $N(dt,dz) = \widetilde{N}(dt,dz) + \nu(dz)dt$ under $\mathbb{P}$ is the same with the distribution of $\widetilde{N}^{\mathbb{Q}}(dt,dz) + \nu(dz)dt$ under Q. This will not happen. We know $N(dt,dz)$ is integer valued. But $\widetilde{N}^{\mathbb{Q}}(dt,dz) + \nu(dz)dt = N(dt,dz) + \theta(t,z)\nu(dz)dt$ will not take integer value in general.

Remark 4.8.3. With some special transformation (not drift transformation), we could get process with the same distribution. See [BGJ87, Bic02]

Chapter 5

Harnack Inequalities for Ornstein-Uhlenbeck Processes Driven by Wiener Processes

We first give a general introduction to Ornstein-Uhlenbeck processes in Section 5.1. Then we show Harnack inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes in Section 5.2.

In Section 5.3 we consider some properties equivalent to Harnack inequalities. For example, we show that the Harnack inequality for the Gaussian Ornstein-Uhlenbeck semigroup P_t holds if and only if the semigroup P_t is strongly Feller.

In Section 5.4, we show some examples of Harnack inequalities, especially the Harnack inequalities for diagonal Ornstein-Uhlenbeck processes from which we can see clearly why our result is better than the one in [RW03a].

In Section 5.5, we consider Harnack inequalities for Ornstein-Uhlenbeck processes with perturbations driven by Wiener processes. We first consider Lipschitz perturbations. Then we consider gradient systems by approximation. We mention here that there is an independent work by Da Prato et al. [DPRW09]. They considered the perturbation of Ornstein-Uhlenbeck processes with singular drifts. But the spirit is similar.

Section 5.6 is an appendix. We show another proof of the main Harnack inequality by finite dimensional approximation in Subsection 5.6.1. It is especially interesting for readers who only care for the finite dimension case. In Subsection 5.6.2 we show a Mehler formula. It is introduced partially for the motivation of the generalized Mehler semigroups which will be introduced in Section 7.1.

5.1 Ornstein-Uhlenbeck Processes

The story start from Brownian motion. In 1827, the England botanist Robert Brown observed the zigzag path of pollen grains suspended in water under the lens of the microscope. In 1905, Einstein explained the mechanics of the movement. Roughly speaking, if at time t the Brownian particle is at position x, then after arbitrary time Δt, the particle will appear at $x+\varepsilon$, where ε is a Gaussian random variable and independent of the starting position x and time t.

But this theory neglects the viscosity of the medium. Langevin initiated the study and Ornstein and Uhlenbeck [OU30] developed a new theory for Brownian motion. In the following, we just simply introduce it. We refer to the lovely book by Nelson [Nel01] for the dynamical theory of Brownian motion.

Let X_t denote the velocity of a Brownian particle at time t. Let $(W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion and $\kappa > 0$ measures the viscosity. By the second law of Newton and by choosing appropriate units, $\frac{dX_t}{dt}$ means the acceleration of the particle which may be interpreted as the force experienced by the particle. This force is the sum of a systematic viscous force and a stochastic force. Since the viscous force is proportional to the particle's velocity X_t and directed opposite to its velocity, so we can suppose the viscous force is given by $-\kappa X_t$. The stochastic force is modeled by the white noise $\frac{dW_t}{dt}$. Therefore, we have

$$\frac{dX_t}{dt} = -\kappa X_t + \frac{dW_t}{dt}. \tag{5.1}$$

We rewrite it into the following Langevin equation

$$dX_t = -\kappa X_t \, dt + dW_t. \tag{5.2}$$

Let $X_0 = x \in \mathbb{R}$ be the initial data. Then the solution to (5.2) is given by (see the books [IW81, KS91, DPZ92] etc.)

$$X_t = \mathrm{e}^{-\kappa t}\, x + \int_0^t \mathrm{e}^{-\kappa(t-s)} \; dW_s \tag{5.3}$$

Clearly, X_t is random perturbation of the exponential function. The process (5.3) is called *Ornstein-Uhlenbeck process* or simply *OU processes*.

Figure 5.1 in the following indicates the composition of the process X_t.

We can consider more general form of Ornstein-Uhlenbeck processes. The drift maybe general linear function, and the noise $\frac{dW_t}{dt}$ can be fractional Brownian

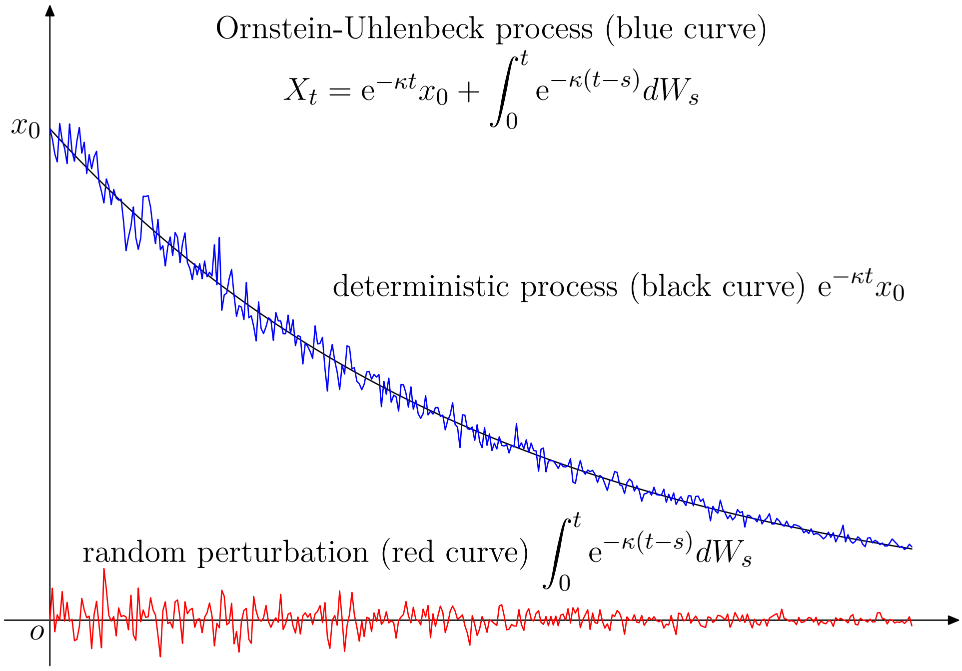

Figure 5.1: Ornstein-Uhlenbeck Process

motion noise, Lévy noise etc..

One of the main general Ornstein-Uhlenbeck type processes which we will consider in this thesis is the *generalized Langevin equation*

$$dX_t = AX_t + dZ_t, \quad X_t = x \tag{5.4}$$

on some Hilbert space $\mathbb{H}$. Here $(Z_t)_{t\geq 0}$ is a Lévy process, and A is the infinitesimal generator of some strong continuous contraction semigroup $(S_t)_{t\geq 0}$.

The *mild solution* of (5.4) can be written down in terms of *stochastic convolution* as

$$X_t = S_t x + \int_0^t S_{t-s}\, dZ_s. \tag{5.5}$$

See [PZ07, Section 9.2] or [App06, Section 4].

The Ornstein-Uhlenbeck process defined in (5.5) generalize the classical one in the following two ways: Firstly, we are working in a infinite dimensional space; and secondly, the noise is a general Lévy process.

Ornstein-Uhlenbeck processes are better reference processes in infinite dimensional analysis than infinite dimensional Brownian motions (or Lévy processes). One of the main reason is that Ornstein-Uhlenbeck processes, in contrast to an infinite dimensional Brownian motion (or more generally Lévy process), can have invariant measures. Another point is that the presence of the linear drifts can have smoothing effects.

Bibliographic Notes on Ornstein-Uhlenbeck Processes The topic related to Ornstein-Uhlenbeck type processes has attracted many people to study for a long time. See Ornstein-Uhlenbeck [OU30] and Kolmogorov [Kol34] etc. for the finite dimensional Gaussian case. See Ito [Itô84b, Itô84a] (or [Itô87, Pages 589-616]), Dawson [Daw75], Da Prato et al. [DPIT82], Chow [Cho87], and the books by Da Prato and Zabczyk [DPZ92, DPZ02], Zabczyk [Zab99] and Da Prato [DP04, DP06] for the infinite dimensional Gaussian case.

The case driven by general Lévy processes were first studied by Wolfe [Wol82] in the scalar case: where A is a positive constant. Sato and Yamazoto [SY83, SY84] generalized this to the multidimensional case where A is a matrix all of whose eigenvalues have positive real parts. Chojnowska-Michalik [CM85, Cho87] considered the generalization to infinite dimension. We also mention a series of papers by Applebaum [App06, App07b, App07a] etc., the monograph by Zabczyk and Peszat [PZ07] for the study of Ornstein-Uhlenbeck type processes in infinite dimensional space with Lévy noise.

We refer also to Page 139 for the bibliographic notes on generalized Mehler semigroup which is closely related to the Ornstein-Uhlenbeck processes driven by Lévy processes.

5.2 Harnack Inequalities

In this section, we first show a main theorem directly in Subsection 5.2.1 by transformation of measures on the state spaces. Then we turn to estimate $\|\Gamma_T\|$ in Subsection 5.2.2 by a result from control theory. By the estimates we can get some corollaries from the main theorem on Harnack inequalities. Especially we can get the Harnack inequalities for Gaussian Ornstein-Uhlenbeck semigroup proved by Röckner and Wang [RW03a].

5.2.1 Main Theorem

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$. Let A be the generator of a strongly continuous contraction semigroup $(S_t)_{t\geq 0}$ on $\mathbb{H}$, and B a linear bounded operator on $\mathbb{H}$.

Consider the following linear stochastic partial differential equation on $\mathbb{H}$

$$dX_t = AX_t + BdW_t, \quad X_0 = x \in \mathbb{H}, \tag{5.6}$$

where $(W_t)_{t\geq 0}$ is a cylindrical Wiener process on $\mathbb{H}$.

Set

$$R = BB^*.$$

Fix $T > 0$. For any $0 \leq t \leq T$, set

$$Q_t = \int_0^t S_u R S_u^* \, du.$$

In control theory, this operator Q_t is called *controllability operator*. See [Zab08] or Appendix A.

We will need the following assumption.

Assumption 5.2.1. We assume that the operator Q_T is of trace-class. That is,

$$\int_0^T \mathrm{Tr}(S_u R S_u^*) \, du < \infty. \tag{5.7}$$

With (5.7), the *mild solution* to the stochastic equation (5.6) on time interval $[0, T]$ is given by (see [DPZ92, Theorem 5.4])

$$X_t = S_t x + \int_0^t S_{t-s} B \, dW_s, \quad 0 \leq t \leq T.$$

This solution is also the unique weak solution of (5.6). See [DPZ92] for details.

Remark 5.2.2. (1) The operator $R = BB^*$ is not necessary to be of trace class for (5.7). For instance, the choice of $B = I$ is allowed if A^{-1} is of trace class.

(2) If Q_T is of trace class for some $T > 0$, then obviously Q_t is of trace class for every $0 \leq t \leq T$.

The stochastic integral

$$W_A(t) = \int_0^t S_{t-s} B \, dW_s, \quad 0 \leq t \leq T$$

is called *stochastic convolution*. By the introduction in Section 1.3, we know $W_A(t)$ is Gaussian distributed mean 0 and covariance Q_t. See also [DPZ92, Theorem 5.2].

Hence the Ornstein-Uhlenbeck process X_t is also Gaussian distributed with mean S_tx and covariance Q_t, i.e. $X_t \sim N(S_tx, Q_t)$.

For every $0 \le t \le T$, we denote

$$\mu_t = N(0, Q_t).$$

Then the transition semigroup associated with the Ornstein-Uhlenbeck process X_t is given by

$$P_tf(x) = \mathbb{E}f(X_t) = \int_{\mathbb{H}} f(S_tx + z)\, \mu_t(dz), \quad f \in \mathscr{C}_b^+(\mathbb{H}). \tag{5.8}$$

We call the semigroup P_t as *Ornstein-Uhlenbeck semigroup.* If $A = 0$, then the semigroup is the classical *heat semigroup.* See [DPZ02] for the detailed discussions of heat semigroup and Ornstein-Uhlenbeck semigroup.

The central result of this chapter is the following Harnack inequality for the Gaussian Ornstein-Uhlenbeck semigroup P_t defined in (5.8). The proof of this result is in the same spirit of the proof of the Harnack inequality (4.3) for the simple Ornstein-Uhlenbeck process (4.1) in Example 4.1.1

Theorem 5.2.3. *Let $T > 0$ and $x, y \in \mathbb{H}$. Assume that the operator Q_T is of trace class and*

$$S_T(x - y) \in Q_T^{1/2}(\mathbb{H}). \tag{5.9}$$

Then for every $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, we have

$$(P_Tf)^\alpha(x) \le \exp\left(\frac{\beta}{2}|\Gamma_T(x - y)|^2\right) P_Tf^\alpha(y), \tag{5.10}$$

for every $f \in \mathscr{C}_b^+(\mathbb{H})$. Here $\Gamma_T = Q_T^{-1/2}S_T$.

Proof. Since (5.9) holds, by Theorem 1.2.2 we know $N(S_Tx - y), Q_T)$ is absolute continuous with respect to $N(0, Q_T)$. Moreover, we have

$$\begin{aligned} &\frac{d\, N\big(S_T(x - y), Q_T\big)}{d\, N(0, Q_T)}(z) \\ =&\exp\left(\langle Q_T^{-1/2}S_T(x - y), Q_T^{-1/2}z\rangle - \frac{1}{2}|Q_T^{-1/2}S_T(x - y)|^2\right) \end{aligned}$$

according the Cameron-Martin formula.

Using a change of variable and the formula above, and applying Hölder's inequality we have

$$
\begin{aligned}
&P_T f(x)\\
&= \int_{\mathbb{H}} f(S_T x + z)\mu_T(dz)\\
&= \int_{\mathbb{H}} f(S_T y + z)\frac{d\,N(S_T(x-y), Q_T)}{d\,N(0,Q_T)}\mu_T(dz)\\
&= \int_{\mathbb{H}} f(S_T y + z)\cdot\\
&\qquad \exp\left(\left\langle Q_T^{-1/2}S_T(x-y), Q_T^{-1/2}z\right\rangle - \frac{1}{2}|Q_T^{-1/2}S_T(x-y)|^2\right)\mu_T(dz)\\
&\le \exp\left(-\frac{1}{2}|\Gamma_T(x-y)|^2\right)\left(\int_{\mathbb{H}} f^\alpha(S_T y + z)\mu_T(dz)\right)^{1/\alpha}\cdot\\
&\qquad \left(\int_{\mathbb{H}} \exp(\beta\langle Q_T^{-1/2}S_T(x-y), Q_T^{-1/2}z\rangle)\mu_T(dz)\right)^{1/\beta}\\
&= \exp\left(\frac{\beta-1}{2}|\Gamma_T(x-y)|^2\right)\left(P_T f^\alpha(y)\right)^{1/\alpha}.
\end{aligned}
$$

□

The following theorem is an immediate consequence of the theorem above.

Theorem 5.2.4. *Suppose that the following null controllability condition*

$$S_T(\mathbb{H}) \subset Q_T^{1/2}(\mathbb{H}) \tag{5.11}$$

hold. Then the Harnack inequality (5.10) *holds for all* $x, y \in \mathbb{H}$*. If we further assume* (5.11) *holds for all* $T > 0$*, then the Harnack inequality* (5.10) *holds also for all* $T \geq 0$*.*

Remark 5.2.5. (1) With the assumption (5.11), the operator $\Gamma_T = Q_T^{-1/2}S_T$ is defined on the whole space $\mathbb{H}$. Hence Γ_T is a bounded operator on $\mathbb{H}$ by the closed graph theorem (see [Yos80]).

(2) By [DPZ02, Theorem B.2.2], if (5.11) holds for all $T > 0$, then $Q_T^{1/2}(\mathbb{H})$ is invariant with respect to $T > 0$. Especially if Q_∞ exists, then

$$Q_T^{1/2}(\mathbb{H}) = Q_\infty^{1/2}(\mathbb{H}), \quad T > 0.$$

Remark 5.2.6. In Section 7.4, we will prove a Harnack inequality for Lévy driven Ornstein-Uhlenbeck processes (See (7.32)). It has the same form as (5.10) prove in this section. For the Lévy case, the covariance operator R of the Gaussian part of the Lévy process is supposed to be of trace class in this thesis. But it is not necessary since we can also consider the "cylindrical Lévy processes". Refer to Theorem 7.4.11 for a Harnack inequality for the stochastic heat equation.

5.2.2 Estimates of $\|\Gamma_T\|$

We can estimate the quantity $|\Gamma_T(x-y)|^2$ in (5.10) according its physical meaning in control theory which we will describe in the following (see details in Appendix A).

Consider the following deterministic linear control system

$$dx_t = Ax_t\,dt + Bu_t\,dt, \quad x_0 = x,\ t \in [0,T] \tag{5.12}$$

on $\mathbb{H}$, where u_t is an $\mathbb{H}$-valued square integrable function on $[0,T]$.

By Theorem A.0.2, if $S_T x \in Q_T^{1/2}(\mathbb{H})$, then there exists a control function u_t for the system (5.12) such that $x_T = 0$. What is more, $|\Gamma_T x|^2$ is the minimal energy for driving x to 0 (see Equation(A.4)). That is,

$$|\Gamma_T x|^2 = \inf\left\{\int_0^T |u_s|^2\,ds\colon u \in L^2([0,t],\mathbb{H}), x_T = 0\right\}. \tag{5.13}$$

From (5.13), we can get an upper estimate of $|\Gamma_t x|^2$ by choosing any concrete control function u_t.

We will use the following simple fact for some explicit controls.

Lemma 5.2.7. *Fix $T > 0$. If $S_T(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$, then*

$$S_t(\mathbb{H}) \subset R^{1/2}(\mathbb{H}) \tag{5.14}$$

holds for every $t \in [0,T]$.

Proof. By Theorem A.0.1, we know the inclusion of $S_T(\mathbb{H})$ in $R^{1/2}(\mathbb{H})$ implies that there exist some constant $C > 0$ such that

$$|R^{1/2}x| \le C\,|S_T x|$$

Therefore for every $t \in [0,T]$ and $x \in \mathbb{H}$, we have

$$|R^{1/2}x| \le C\,|S_T x| = C\,|S_{T-t}S_t x| \le C\,\|S_{T-t}\| \cdot |S_t x|$$

Hence the inclusion (5.14) holds by Theorem A.0.1. □

In what follows, we simply suppose that B is symmetric and $B = R^{1/2}$. As usual, we denote by $|\cdot|_0$ for the canonical norm on the Cameron-Martin space $R^{1/2}(\mathbb{H})$ defined by $|x|_0 = |R^{-1/2}x|$ for every $x \in R^{12/}(\mathbb{H})$.

Proposition 5.2.8. *Fix $T > 0$. Assume that Q_T is of trace class and*

$$S_T(\mathbb{H}) \subset R^{1/2}(\mathbb{H}). \tag{5.15}$$

Let ξ_t be a positive continuous function on $[0, T]$ satisfying

$$\int_0^T |S_t x|_0^2 \, \xi_t^2 \, dt < \infty, \quad \text{for all } x \in \mathbb{H}.$$

Then the null controllability condition

$$S_T \mathbb{H} \subset Q_T^{1/2}(\mathbb{H}) \tag{5.16}$$

are satisfied and the following estimate holds

$$\|\Gamma_T\| \leq \frac{1}{\int_0^T \xi_t \, dt} \left(\int_0^T |S_t x|_0^2 \, \xi_t^2 \, dt \right)^{1/2}. \tag{5.17}$$

Proof. Consider the following control system

$$\begin{cases} dx_t = Ax_t \, dt + R^{1/2} u_t \, dt, \\ x_0 = x, \end{cases} \tag{5.18}$$

for $t \in [0, T]$. The solution of (5.18) is given by

$$x_t = S_t x + \int_0^t S_{t-s} R^{1/2} u_s \, ds, \quad t \in [0, T]. \tag{5.19}$$

By the formula (5.19), it is easy to see that the control

$$u(t) = -\frac{\xi_t}{\int_0^T \xi_t \, dt} R^{-1/2} S_t x, \quad t \in [0, T]$$

transfers the system from $x_0 = x$ to $x_T = 0$. Hence the system (5.18) is null controllable. This implies that the null controllability condition (5.16) holds by Theorem A.0.3.

Moreover, by (5.13), we know

$$\|\Gamma_T\| \le \int_0^T |u_s|^2\, ds.$$

Hence we have the estimate (5.17). □

From Proposition 5.2.8, we have the following corollary.

Corollary 5.2.9. *Assume the assumptions in Proposition 5.2.8 and*

$$|S_t x|_0 \le \sqrt{\xi(t)^{-1}}\, |x|_0, \quad x \in \mathbb{H},\ t \in [0,T].$$

Then

$$\|\Gamma_T\| \le \frac{1}{\int_0^T \xi(t)\, dt}. \tag{5.20}$$

Remark 5.2.10. (1) For the special case $R = I$, the condition (5.15) automatically hold.

(2) If we take $\xi_t \equiv 1$ in Proposition 5.2.8, then we can get [DPZ92, Corollary 9.22]. If we take $R = I$ additionally, we get [DPZ92, Corollary 9.23].

(3) These estimates of $\|\Gamma_T\|$ are also useful in the study of regularizing properties of the transition semigroup corresponding to semi-linear stochastic equations. See [DPZ92, Section 9.4] and [CMG95] etc.. Indeed, in Subsection 7.5 we will use $\|\Gamma_t\|$ to study of the strong Feller property of the Ornstein-Uhlenbeck transition semigroup .

5.2.3 Estimates of Harnack Inequality

From Theorem 5.2.3 and Proposition 5.2.8 we can get the following corollary immediately.

Corollary 5.2.11. *Fix $T > 0$. Let ξ_t be a positive continuous function on $[0,T]$. Suppose that Q_T is of trace class and $S_T(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$. Assume further that*

$$\int_0^T |S_t x|_0^2\, \xi_t^2\, dt < \infty, \quad \text{for all } x \in \mathbb{H}.$$

Then the following inequality

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta \int_0^T |S_t(x-y)|_0^2\, \xi_t^2\, dt}{2\left(\int_0^T \xi_t\, dt\right)^2}\right) P_T f^\alpha(y) \tag{5.21}$$

holds for every $x, y \in \mathbb{H}$*,* $\alpha, \beta > 1$ *with* $1/\alpha + 1/\beta = 1$*, and* $f \in \mathscr{C}_b^+(\mathbb{H})$*.*

From Theorem 5.2.3 and Corollary 5.2.9 we can get the following corollary.

Corollary 5.2.12. *Assume the assumptions in Corollary 5.2.11 and*

$$|S_t x|_0 \le \sqrt{\xi(t)^{-1}}\,|x|_0, \quad x \in \mathbb{H},\ t \in [0, T].$$

Then the following inequality

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta|x-y|_0^2}{2\int_0^T \xi(t)\,dt}\right) P_T f^\alpha(y) \tag{5.22}$$

holds for every $x, y \in \mathbb{H}$*,* $\alpha, \beta > 1$ *with* $1/\alpha + 1/\beta = 1$*, and* $f \in \mathscr{C}_b^+(\mathbb{H})$*.*

Remark 5.2.13.

(1) Corollary 5.2.12 covers (with a slight difference of the conditions) a result in [RW03a] which is recalled in the following for the convenience of comparison. Let $B = R^{1/2}$. Suppose that the following assumptions holds.

  (a) $S_t R(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$ holds for every $t > 0$.

  (b) There is a strictly positive $h_1 \in C[0, \infty)$ such that

$$|S_t R x|_0 \le \sqrt{h_1(t)}|Rx|_0, \quad x \in \mathbb{H},\ t \ge 0,$$

  (c) Other conditions: Item (1) and Item (2) of Assumption 7.2.1.

With these assumptions, Röckner and Wang [RW03a] proved the following Harnack inequality for (5.8)(See Theorem 7.2.2 for details):

$$(P_t f)^\alpha(x) \le \exp\left[\frac{\alpha\rho(x,y)}{2(\alpha-1)\int_0^t h_1(s)^{-1}\,ds}\right] P_t f^\alpha(y)$$

for every $\alpha > 0$, $x, y \in \mathbb{H}$, $t > 0$ and each $f \in \mathscr{C}_b^+(\mathbb{H})$.

(2) But we assume a slightly stronger condition: $S_T(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$ than the condition $S_T R(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$ assumed in [RW03a]. However, we do not assume Item (1) and Item (2) of Assumption 7.2.1 which are required in [RW03a].

Remark 5.2.14. There are three methods in our hands to establish Harnack inequality: semigroup calculus, image measure transformation and Girsanov's transformation methods. In Chapter 7 we shall introduce these three methods.

Röckner and Wang [RW03a] mainly used the semigroup calculus method. The image measure transformation is also used in [RW03a]. But the authors of [RW03a] didn't apply the method to the Gaussian case and didn't realize that this method can obtain optimal inequality.

5.3 Properties Equivalent to Harnack Inequalities

We first recall the strong Feller property of a transition semigroup.

Definition 5.3.1. Let P_t, $t \geq 0$, be a transition semigroup. P_t is called *strongly Feller* if for every bounded measurable function φ on $\mathbb{H}$, $P_t\varphi$ is continuous for every $t > 0$. That is, $P_t\big(\mathscr{B}_b(\mathbb{H})\big) \subset \mathscr{C}_b(\mathbb{H})$.

Da Prato, Röckner and Wang [DPRW09, Proposition 4.1] proved that every Markov transition semigroup has strong Feller property automatically if the Harnack inequality hold. We include the result in the following for convenience.

Proposition 5.3.2. *Let E be a topological space and P a Markov operator on $\mathscr{B}_b(E)$. Assume that for every $\alpha > 1$ there exists a continuous function η_α on $E \times E$ satisfying $\eta_\alpha(x,x) = 0$ for all $x \in E$ and*

$$(Pf)^\alpha(x) \leq \mathrm{e}^{\eta_\alpha(x,y)}\, Pf^\alpha(y)$$

for all $x, y \in E$, $f \in \mathscr{B}_b^+(E)$. Then P is strongly Feller.

Furthermore, for any σ-finite measure μ on $(E, \mathcal{B}(E))$ such that

$$\int_E Pf\, d\mu \leq C \int_E f\, d\mu$$

for all $f \in \mathscr{B}_b^+(E)$ and some fixed constant $C > 0$, P uniquely extends to $L^p(E,\mu)$ with $PL^p(E,\mu) \subset C(E)$ for any $p > 1$.

Now we can prove the following result.

Theorem 5.3.3. *Assume that for every $t \geq 0$, Q_t is of trace class. Then the following statements are equivalent to each other.*

(1) The null controllability condition holds: $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ for all $t \geq 0$.

(2) The system (5.12) is null controllable over each time interval $[0,t]$, $t \geq 0$.

(3) *The Harnack inequality* (5.10) *with bounded* Γ_t *holds for all* $\alpha > 1$, $t \geq 0$, $x, y \in \mathbb{H}$ *and* $f \in \mathscr{C}_b(\mathbb{H})$.

(4) *There exist some constant* $C(t,\alpha) > 0$ *such that*

$$(P_t f)^\alpha(x) \leq \exp(C(t,\alpha)|x-y|^2) P_t f^\alpha(y)$$

holds for all $\alpha > 1$, $t \geq 0$, $x, y \in \mathbb{H}$ *and* $f \in \mathscr{C}_b(\mathbb{H})$.

(5) *The Ornstein-Uhlenbeck transition semigroup* P_t *for* $t \geq 0$ *is strongly Feller. That is* $P_t\big(\mathscr{B}_b(\mathbb{H})\big) \subset \mathscr{C}_b(\mathbb{H})$ *for every* $t \geq 0$.

We assume further that Q_∞ *is of trace class. Hence there exists an invariant measure* μ. *Then the statements* (1)–(5) *above are also equivalent with the following two statements*

(6) *For every* $p > 1$, $P_t\big(L^p(\mathbb{H},\mu)\big) \subset \mathscr{C}(\mathbb{H})$.

(7) *For every* $p > 1$, $P_t\big(L^p(\mathbb{H},\mu)\big) \subset \mathscr{C}^\infty(\mathbb{H})$.

Proof. The equivalence (1)⇔(2) follows from Theorem A.0.2. By Theorem 5.2.3, we know (1)⇒(3). From (3) we obtain (4) immediately since the operator Γ_t is bounded. (4)⇒(5) follows from Proposition 5.3.2 (see also Subsection 7.5). The implication (5)⇒(1) is well known (See Da Prato and Zabzyck [DPZ92, Subsection 9.4.1] or Zabzyck [Zab81]. In fact, it is proved there that (1)⇔(5)).

Suppose that Q_∞ is of trace class. (4)⇒(6) follows from Proposition 5.3.2. (1)⇒(7) come from [DPFZ02, Theorem 2.1] (See also the book Da Prato and Zabczyk [DPZ02]). On the other hand, it is clear that (6)⇒(5) and (7)⇒(5). □

Remark 5.3.4. We refer to [MS02, Theorem 2.1] for the following two more equivalent statements of the strong Feller property of the Ornstein-Uhlenbeck transition semigroup P_t:

(1) P_t is bw strongly Feller.

(2) P_t is bw ultra strongly Feller.

Here "bw" refers to "bounded weak". See [MS02, Section 1] for the definitions of bw and bw-ultra strongly Feller.

Remark 5.3.5. (1) We get a new proof of the well known fact that (1)⇒(5) via Harnack inequality.

(2) Da Prato et al. [DPFZ02, Theorem 2.4] (see also [DPZ02, Theorem 10.3.6]) states that for every $f \in L^1(\mathbb{H},\mu)$, $P_t f$ may fail to be continuous in infinite dimension. (But the author are not clear about their proof.)

Remark 5.3.6. The strong Feller property means that the Ornstein-Uhlenbeck semigroup has a smoothing property. For the heat semigroup, the condition

(5.11) does not hold if the Hilbert space $\mathbb{H}$ is infinite dimensional. In fact, the heat semigroup is regular only in the directions of the Cameron-Martin space $Q^{1/2}(\mathbb{H})$. See [DP06, Proposition 8.4]. This explains one reason for why we prefer to use Ornstein-Uhlenbeck processes in infinite dimensional spaces as reference processes.

Remark 5.3.7. In Section 7.5, we will consider the estimates concerning the strong Feller property.

5.4 Examples of Harnack Inequalities

We work in the framework of the previous section. We show Harnack inequalities for some quite simple examples and general diagonal Ornstein-Uhlenbeck processes in the first and second subsections respectively.

5.4.1 Simple Cases

In the first example we deal with a degenerate finite dimensional stochastic differential equation. In the second and third examples, we deal with two special cases: $B = I$ and $A = -1/2I$ respectively.

Example 5.4.1. Let $\mathbb{H} = \mathbb{R}^2$, and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have $R = BB^* = B$ and for every $t \geq 0$,

$$S_t = \mathrm{e}^{tA} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad S_t R S_t^* = \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix},$$

and hence

$$Q_t = \int_0^t S_u R S_u^* \, du = \int_0^t \begin{pmatrix} 1 & u \\ u & u^2 \end{pmatrix} du = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}.$$

It is obvious that for every $t > 0$, the determinant of Q_t $\det Q_t > 0$. Hence Q_t is non-degenerate and the null controllability condition (5.11) is fulfilled. Let $x = (x_1, x_2)^{\mathrm{tr}}$, $y = (y_1, y_2)^{\mathrm{tr}} \in \mathbb{R}^2$. We have (for example, with the help of mathematical software like MAPLE)

$$|\Gamma_t(x-y)|^2 = |Q_t^{-1/2} S_t(x-y)|^2$$
$$= 4 \cdot [t^{-1}(x_1-y_1)^2 + 3t^{-2}(x_1-y_1)(x_2-y_2) + 3t^{-3}(x_2-y_2)^2].$$

Now we have the following Harnack inequality by Theorem 5.2.3

$$(P_t f)^\alpha(x) \le \exp(C_t) P_t f^\alpha(y)$$

with

$$C_t := 2\beta[t^{-1}(x_1-y_1)^2 + 3t^{-2}(x_1-y_1)(x_2-y_2) + 3t^{-3}(x_2-y_2)^2],$$

for every $f \in \mathscr{C}_b^+(\mathbb{R}^2)$, $x, y \in \mathbb{R}^2$, $t > 0$, and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Example 5.4.2. Assume that $B = I$ and $(-A)^{-1}$ is of trace class. Then $R^{1/2}(\mathbb{H}) = \mathbb{H}$ and hence $S_t(\mathbb{H}) \subset \mathbb{H}$ for every $t \ge 0$. Moreover, it is easy to see that Q_t is of trace class for every $t \ge 0$.

Let $\{e_k\}_{k\in\mathbb{N}}$ be the system of eigenvectors of $(-A)^{-1}$ corresponding with eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Set $\omega = \inf_{k\in\mathbb{N}}(1/\lambda_k)$. For every $x \in \mathbb{H}$ and $t \ge 0$ we have

$$|S_t x|^2 = \sum_{k\in\mathbb{N}} \mathrm{e}^{-2/\lambda_k t}\langle x, \mathrm{e}_k\rangle \le \mathrm{e}^{-2\omega t} |x|^2.$$

Applying Corollary 5.2.11 with $\xi(t) = \mathrm{e}^{-\omega t}$, we get

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta\omega|x-y|^2}{\mathrm{e}^{2\omega t} - 1}\right) P_t f^\alpha(y)$$

for every $f \in \mathscr{C}_b^+(\mathbb{H})$, $x, y \in \mathbb{H}$, $t \ge 0$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Example 5.4.3. Let $A = -1/2I$. Suppose that $R = BB^*$ is of trace class. For every $t \ge 0$, we have

$$S_t = \mathrm{e}^{-t/2} \quad \text{and} \quad Q_t = (1-\mathrm{e}^{-t})R.$$

Hence

$$\Gamma_t = Q_t^{-1/2} S_t = (1-\mathrm{e}^{-t})^{-1/2}\, \mathrm{e}^{-t/2}\, R^{-1/2}, \quad t \ge 0.$$

It is clear that the operator Q_t is of trace class for every $t > 0$.

The transition semigroup is the classical Ornstein-Uhlenbeck semigroup given by

$$P_t f(x) = \int_{\mathbb{H}} f(\mathrm{e}^{-1/2}\, x + z)\, \mu_t(dz),$$

for $x \in \mathbb{H}$, $t \geq 0$ and $f \in \mathscr{C}_b^+(\mathbb{H})$, where $\mu_t = N(0, Q_t)$.

Let $x, y \in \mathbb{H}$ such that $x - y \in \mathbb{H}_0 = R^{1/2}(\mathbb{H})$. Then obviously we have $S_t(x-y) \in Q_t^{1/2}(H)$ for every $t > 0$. Now by Theorem 5.2.3 we have the following Harnack inequality

$$\begin{aligned}(P_t f)^\alpha(x) &\leq \exp\left(\frac{\beta}{2}\left|(1-\mathrm{e}^{-t})^{-1/2}\,\mathrm{e}^{-t/2}\, R^{-1/2}(x-y)\right|^2\right) P_t f^\alpha(y) \\ &= \exp\left(\frac{\beta|x-y|_0^2}{2(\mathrm{e}^t - 1)}\right) P_t f^\alpha(y),\end{aligned} \tag{5.23}$$

for every $f \in \mathscr{C}_b^+(\mathbb{H})$, $t > 0$, and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Remark 5.4.4. By the notation of the intrinsic distance ρ on $\mathbb{H}$ (see (1.1)), we can rewrite (5.23) for every $x, y \in \mathbb{H}$:

$$(P_t f)^\alpha(x) \leq \exp\left(\frac{\beta\,\mathrm{e}^{-t}\,\rho(x,y)^2}{2(1-\mathrm{e}^{-t})}\right) P_t f^\alpha(y).$$

Remark 5.4.5. See also Kusuoka [Kus92, the proof of Lemma 6.12, page 270] for the case $\alpha = \beta = 2$.

5.4.2 Diagonal Ornstein-Uhlenbeck Processes

In the following we consider general diagonal Ornstein-Uhlenbeck processes. It is the important case when the operator A is self-adjoint and commutes with R. The last two examples in the previous subsection are also diagonal Ornstein-Uhlenbeck processes.

Let $\{e_n\}_{n\geq 1}$ be a complete orthonormal basis of the real separable Hilbert space $\mathbb{H}$. Assume that there exist sequences of positive numbers δ_n, γ_n for $n \in \mathbb{N}$, such that

$$Ae_n = -\delta_n e_n \quad \text{and} \quad Re_n = \gamma_n e_n, \tag{5.24}$$

where $\delta_n \uparrow \infty$ as $n \uparrow \infty$. By direct calculation, we can get the following proposition. See also [DPZ92, Section 9.5] or [DPZ02, Example 6.2.11].

Proposition 5.4.6. *Suppose that* (5.24) *hold. Then*

(1) The operator Q_t, $t > 0$, is of trace class if and only if

$$\sum_{n=1}^{\infty} \frac{\delta_n}{\gamma_n} < \infty. \tag{5.25}$$

(2) The null controllability condition $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ holds for every $t > 0$ if and only if

$$\sup_{n\in\mathbb{N}} \frac{2\delta_n}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)} < \infty, \quad t > 0,\ n \in \mathbb{N}. \tag{5.26}$$

Proof. (1) For every $t > 0$, $n \in \mathbb{N}$, we have

$$\begin{aligned} Q_t e_n = \int_0^t S_{2u} R e_n\, du &= \gamma_n \int_0^t \mathrm{e}^{2uA}\, e_n\, du \\ &= \gamma_n \int_0^t \mathrm{e}^{-2u\delta_n}\, e_n\, du = \frac{\gamma_n}{2\delta_n}(1 - \mathrm{e}^{-2\delta_n t}) e_n. \end{aligned}$$

That is,

$$Q_t e_n = \frac{\gamma_n}{2\delta_n}(1 - \mathrm{e}^{-2\delta_n t}) e_n. \tag{5.27}$$

Therefore, Q_t is of trace class for every $t > 0$ if and only the condition (5.25) holds.

(2) From (5.27) we know

$$Q_t^{-1} e_n = \frac{2\delta_n}{\gamma_n(1 - \mathrm{e}^{-2\delta_n t})} e_n$$

for each $t > 0$ and $n \in \mathbb{N}$. Therefore

$$\Gamma_t^2 e_n = Q_t^{-1}\, \mathrm{e}^{2tA}\, e_n = Q_t^{-1}\, \mathrm{e}^{-2t\delta_n}\, e_n = \frac{2\delta_n}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)} e_n.$$

Consequently, for every $z \in \mathbb{H}$,

$$|\Gamma_t z|^2 = \langle \Gamma_t^2 z, z\rangle = \sum_{n=1}^{\infty} \langle \Gamma_t^2 e_n, e_n\rangle \langle z, e_n\rangle^2 = \sum_{n=1}^{\infty} \frac{2\delta_n \langle z, e_n\rangle^2}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)}.$$

By Theorem A.0.3, the null controllability condition is equivalent to

$$\|\Gamma_t\|^2 = \sup_{n\in\mathbb{N}} \frac{2\delta_n}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)} < \infty, \quad t \geq 0,\ n \in \mathbb{N}.$$

□

Example 5.4.7. Assume Conditions (5.24), (5.25) and (5.26). By Theorem 5.2.3, the following inequality holds

$$(P_t f)^\alpha(x) \le \exp\left(\sum_{n=1}^{\infty} \frac{\beta\delta_n \langle x-y, e_n\rangle^2}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)}\right) P_t f^\alpha(y) \tag{5.28}$$

for every $f \in \mathscr{C}_b^+(\mathbb{H})$, $t > 0$, $x, y \in \mathbb{H}$, and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Remark 5.4.8. Suppose that $\gamma_n \equiv 1$. The result in [RW03a] shows

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta\delta_1 |x-y|^2}{\mathrm{e}^{2t\delta_1} - 1}\right) P_t f^\alpha(y). \tag{5.29}$$

Comparing Inequalities (5.28) with (5.29), we see (5.29) is a "first-order" approximation of (5.10).

Now we consider a special case of Example 5.4.7.

Example 5.4.9. Suppose that for each $n \in \mathbb{N}$, $\delta_n = n^\delta$ and $\gamma_n = n^{-\gamma}$ with some $\delta, \gamma > 0$ satisfying

$$\delta + \gamma > 1. \tag{5.30}$$

Then it is obvious that Condition (5.25) holds. Moreover, with (5.30), the null controllability condition (5.11) also holds since

$$\|\Gamma_t\|^2 = \sup_{n\in\mathbb{N}} \frac{2\delta_n}{\gamma_n(\mathrm{e}^{2t\delta_n} - 1)} < \infty$$

for every $t > 0$. Therefore, by (5.28), the following Harnack inequality

$$(P_t f)^\alpha(x) \le \exp\left(\sum_{n=1}^{\infty} \frac{\beta n^{\delta+\gamma}}{\mathrm{e}^{2tn^\delta} - 1} \langle x-y, e_n\rangle^2\right) P_t f^\alpha(y) \tag{5.31}$$

holds for every $t > 0$, $x, y \in \mathbb{H}$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$ and $f \in \mathscr{C}_b^+(\mathbb{H})$.

Example 5.4.10. Consider $\mathbb{H} = L^2(0, \pi)$. Let A be the Laplace operator on $(0, \pi)$ with Dirichlet boundary and $R = I$. Then we have $\delta = 2$ and $\gamma = 0$. Therefore, by (5.31), the following Harnack inequality

$$(P_t f)^\alpha(x) \le \exp\left(\sum_{n=1}^{\infty} \frac{\beta n^2}{\mathrm{e}^{2tn^2} - 1} \langle x-y, e_n\rangle^2\right) P_t f^\alpha(y)$$

holds for every $t > 0$, $x, y \in \mathbb{H}$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$ and $f \in \mathscr{C}_b^+(\mathbb{H})$.

5.5 Perturbations

5.5.1 Lipschitz Perturbation

We assume in this subsection the following assumptions.

Assumption 5.5.1. We assume

(1) $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ is self-adjoint and there exists $\omega > 0$ such that

$$\langle Ax, x\rangle \le -\omega |x|^2, \quad x \in D(A);$$

(2) A^{-1} is of trace class;

(3) F is Lipschitz continuous and dissipative

$$\langle F(x) - F(y), x - y\rangle \le 0, \quad x, y \in \mathbb{H}.$$

We consider the following semi-linear stochastic partial differential equation

$$dX_t = AX_t dt + F(X_t)dt + dW_t, \quad X_0 = x \in \mathbb{H}. \tag{5.32}$$

With Assumption (5.5.1), the equation (5.32) has a unique mild solution (see [DPZ02, Theorem 7.3.5]) given by

$$X_t = S_t x + \int_0^t S_{t-s} F(X_s)\, ds + \int_0^t S_{t-s}\, dW_s,$$

where $S_t = \mathrm{e}^{tA}$, $t \ge 0$ is the semigroup generated by A.

Set

$$P_t f(x) = \mathbb{E} f(X_t), \quad f \in \mathscr{C}_b(\mathbb{H}).$$

It can be proved that there is a unique invariant measure ν for P_t (see [DPZ02, Theorem 11.2.3].) Therefore, the semigroup P_t can be extended to be a strongly continuous semigroup of contraction on $L^p(\mathbb{H}, \nu)$ for $p > 1$ (similar to the proof of [DPZ02, Theorem 10.1.5]).

With Assumption 5.5.1, we have the following Harnack inequality for P_t.

Theorem 5.5.2. *Let Assumption 5.5.1 holds. Then*

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\omega\beta |x-y|^2}{\mathrm{e}^{2t\omega} - 1}\right) P_t f^\alpha(y) \tag{5.33}$$

for every $x, y \in H$, $f \in \mathscr{C}_b^+(\mathbb{H})$, *and* $\alpha, \beta > 1$ *satisfying* $1/\alpha + 1/\beta = 1$.

There are several methods to prove this theorem.

Note that the function $x \mapsto G(x) := Ax + F(x)$ satisfies the following monotonicity condition

$$\langle G(x) - G(y), x - y\rangle \leq -\omega|x-y|^2, \quad x, y \in \mathbb{H},$$

therefore, we can prove Theorem 5.5.2 using the coupling and Girsanov transformation similar to the proof of (4.34) for the finite dimensional case.

We can also consider the finite dimensional approximation. The procedure follows. We project the equation (5.32) on the finite dimensional space. Then the drift of the corresponding finite dimensional stochastic differential equation is also monotone. Hence we get a Harnack inequality for finite dimensional stochastic differential equation (see (4.34)). By the dimension free property of the Harnack inequality, we can get (5.33) by taking limit. We refer the interested author to a recent paper by Da Prato et al. [DPRW09] for details of the approximation.

In the following, we carry out another "proof" by using semigroup calculus. This method was first introduced in [Wan07]. However, we are not able to justify the strictness of the proof. The difficulties come from the domain problem of the semigroup. The reason that we insist to show this method is that we want to present another strategy as a complement of this thesis.

We first introduce some notations and a gradient estimate.

Denote by L the infinitesimal generator of P_t on $L_p(\mathbb{H}, \nu)$. By [DPZ02, Theorem 11.2.14], we know L is the closure of the following differential operator (we still denote it by L)

$$Lf(x) := \frac{1}{2}\mathrm{Tr}[D^2 f(x)] + \langle x, ADf(x)\rangle + \langle F(x), Df\rangle, \quad x \in \mathbb{H}$$

for every $f \in \mathscr{E}_A(\mathbb{H})$. That is, $\mathscr{E}_A(\mathbb{H})$ is a core for L.

We will use the following gradient estimation.

Lemma 5.5.3. *For every* $f \in D(L)$, *we have*

$$|DP_t f(x)| \leq \mathrm{e}^{-\omega t}\, P_t|Df(x)|, \quad x \in \mathbb{H}. \tag{5.34}$$

Proof. First, we consider the approximation of F by

$$F_\varepsilon(x) = \int_{\mathbb{H}} \mathrm{e}^{\varepsilon A^*} F(\mathrm{e}^{\varepsilon A} x + y) N_{\frac{1}{2}A^{-1}(\mathrm{e}^{2\varepsilon A}-1)}(dy),$$

for every $\varepsilon > 0$. Then we have

$$\langle DP_t^\varepsilon f(x), h\rangle = \mathbb{E}\langle Df(X^\varepsilon(t,x), X_x^\varepsilon(t,x))h\rangle, \quad x, h \in \mathbb{H}, \tag{5.35}$$

where $X^\varepsilon(t,x)$ and P_t^ε are defined as X and P_t with F replaced by F_ε.

Note that (refer to the proof of [DPZ02, Proposition 11.2.13])

$$|X_x^\varepsilon(t,x)| \le \mathrm{e}^{-\omega t}, \quad t \ge 0.$$

Therefore, by (5.35), we see

$$|\langle DP_t^\varepsilon f(x), h\rangle| \le \mathrm{e}^{-\omega t} P_t^\varepsilon |Df(X^\varepsilon(t,x))| \cdot |h|.$$

Hence, since the inequality above holds for arbitrary h, we get

$$|DP_t^\varepsilon f(x)| \le \mathrm{e}^{-\omega t} P_t^\varepsilon |Df(X^\varepsilon(t,x))|.$$

Therefore, letting ε tend to 0 we can get (5.34) since it is easy to see that $P_t^\varepsilon f(x) \to P_t f(x)$ as $\varepsilon \to 0$. □

Now we come to show a "proof" of Theorem 5.5.2 by semigroup calculus.

"Proof" of Theorem 5.5.2. We suppose there is a dense subset $\mathscr{E}$ of $\mathscr{C}_b(\mathbb{H})$ such that $\mathscr{E}$ is stable under the action of L and P_t. *

For any $f \in \mathscr{E}$, $\Phi \in \mathscr{C}^2(\mathbb{R})$, we have

$$L\Phi(f) = \frac{1}{2}\Phi''(f)|Df|^2 + \Phi'(f)Lf.$$

Let $\gamma : [0,t] \to \mathbb{H}$ be defined by

$$\gamma(s) = x + \frac{s}{t}(y-x) = (1-\frac{s}{t})x + \frac{s}{t}y.$$

It is the minimal geodesic connecting x and y.

*The existence of this subset is the only reason that we are not able to justify the proof.

Let h be a positive continuous function on $[0,t]$ Taking

$$g(s) := \frac{t\int_0^s h(r)\,\mathrm{e}^{-\omega r}\,dr}{\int_0^t h(r)\,\mathrm{e}^{-\omega r}\,dr}, \quad s\in[0,t].$$

It is the speed function satifying $g(0)=0$, $g(t)=1$. Define for any $s\in[0,t]$,

$$\eta(s)=\gamma(g_s),$$

and

$$\phi(s)=\log P_{t-s}(P_s f)^\alpha(\eta_s).$$

Note that we have

$$\eta(0)=x, \quad \eta(t)=y,$$

then we get

$$\phi(0)=P_t f^\alpha(x), \quad \phi(t)=(P_t f)^\alpha(y).$$

By using the gradient estimate (5.34), we have

$$\begin{aligned}
&P_{t-s}(P_s f)^\alpha(\eta_s)\frac{d}{ds}\phi(s)\\
=&-P_{t-s}L(P_s f)^\alpha(\eta_s)+P_{t-s}\left[\alpha(P_s f)^{\alpha-1}LP_s f(\eta_s)\right]+\left\langle DP_{t-s}(P_s f)^\alpha(\eta_s),\eta'(s)\right\rangle\\
=&-P_{t-s}\left[\alpha(P_s f)^{\alpha-1}LP_s f(\eta_s)+\frac{\alpha(\alpha-1)}{2}(P_s f)^{\alpha-2}|DP_s f|^2(\eta_s)\right]\\
&+P_{t-s}\left[\alpha(P_s f)^{\alpha-1}LP_s f(\eta_s)\right]+\left\langle DP_{t-s}(P_s f)^\alpha(\eta_s),\eta'(s)\right\rangle\\
=&-\frac{\alpha(\alpha-1)}{2}P_{t-s}\left[(P_s f)^{\alpha-2}|DP_s f|^2(\eta_s)\right]+\left\langle DP_{t-s}(P_s f)^\alpha(\eta_s),\eta'(s)\right\rangle\\
\le&-\frac{\alpha(\alpha-1)}{2}P_{t-s}\left[(P_s f)^{\alpha-2}|DP_s f|^2(\eta_s)\right]+\mathrm{e}^{(s-t)\omega}\,|\eta'(s)|\cdot P_{t-s}|D(P_s f)^\alpha|\\
=&P_{t-s}\left\{|P_s f|^\alpha\left[-\frac{\alpha(\alpha-1)}{2}\left(\frac{|DP_s f|}{|P_s f|}\right)^2+\alpha\,\mathrm{e}^{(s-t)\omega}\,|\eta'(s)|\cdot\frac{|DP_s f|}{|P_s f|}\right]\right\}
\end{aligned}$$

Note the following simple facts: for any number $a,b\in\mathbb{R}$ with $a<0$,

$$ax^2+bx=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}\le-\frac{b^2}{4a},$$

we have

$$\frac{d}{ds}\phi(s)\le\frac{\alpha\,\mathrm{e}^{2(s-t)\omega}\,|\eta'(s)|^2}{2(\alpha-1)}.$$

Now integrate both sides of the inequality above from 0 to t, we have

$$\log(P_t f)^\alpha(y) - \log P_t f^\alpha(x) = \phi(t) - \phi(0) \leq \frac{\alpha}{2(\alpha-1)} \int_0^t \mathrm{e}^{2(s-t)\omega} \, |\eta'(s)|^2 ds$$

Inserting

$$\begin{aligned}\eta'(s) = \gamma'(g(s))g'(s) &= \frac{y-x}{t} \cdot \frac{t \cdot h(s)\,\mathrm{e}^{-\omega s}}{\int_0^t h(s)\,\mathrm{e}^{-\omega s}\,ds} \\ &= \frac{h(s)\,\mathrm{e}^{-\omega s}}{\int_0^t h(s)\,\mathrm{e}^{-\omega s}\,ds}(y-x),\end{aligned}$$

we see

$$\log \frac{(P_t f)^\alpha(y)}{P_t f^\alpha(x)} \leq \frac{\alpha|y-x|^2}{2(\alpha-1)} \frac{\mathrm{e}^{-2\omega t}\int_0^t h^2(s)ds}{\left(\int_0^t h(s)\,\mathrm{e}^{-\omega s}\,ds\right)^2}.$$

Hence we have

$$(P_t f)^\alpha(x) \leq P_t f^\alpha(y) \exp\left(\frac{\beta|x-y|^2}{2} \frac{\int_0^t h^2(s)ds}{\left(\int_0^t h(s)\,\mathrm{e}^{(t-s)\omega}\,ds\right)^2}\right). \qquad (5.36)$$

Take $h(s) = \mathrm{e}^{(t-s)\omega}$ in (5.36), we can get (5.33). □

5.5.2 Gradient Systems

We assume in this subsection the following conditions.

Assumption 5.5.4. We assume

(1) $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a self-adjoint operator and there exists a $\omega > 0$ such that

$$\langle Ax, x\rangle \leq -\omega|x|^2, \quad \text{for all } x \in \mathbb{D}(A).$$

(2) A^{-1} is of trace class on $\mathbb{H}$.

(3) $U\colon \mathbb{H} \to (-\infty, +\infty]$ is a convex lower semi-continuous function such that

$$Z := \int_{\mathbb{H}} \mathrm{e}^{-2U(x)} \, \mu(dx) < \infty.$$

where $\mu = N(0, -\frac{1}{2}A^{-1})$.

We consider a perturbation of Ornstein-Uhlenbeck processes as in the previous section but with F replaced by the sub-differential of some convex function $-U$ on $\mathbb{H}$. That is, we consider a stochastic differential inclusion of the form:

$$dX_t \in AX_t\,dt - \partial U(X_t)\,dt + dW_t, \quad X_0 = x \in \mathbb{H}, \tag{5.37}$$

where $(W_t)_{0\le t\le T}$ is a cylindrical Wiener process in $\mathbb{H}$ and ∂U is the *sub-differential* of U defined for every $x \in \mathbb{H}$ as

$$\partial U(x) = \{y \in H\colon U(x+h) \ge U(x) + \langle y, h\rangle, \text{for all } h \in \mathbb{H}\}.$$

Set $K = \{U < +\infty\}$. Note that $\partial U(x)$ is a non-empty closed convex set for every $x \in K$. If U is Fréchet differentiable on $\mathbb{H}$, then ∂U is the gradient DU.

Consider the following *Moreau-Yosida approximation* U_ε of U

$$U_\varepsilon(x) = \inf\left\{U(y) + \frac{1}{2\varepsilon}|x-y|^2 :\ y \in \mathbb{H}\right\}, \quad x \in \mathbb{H},\ \varepsilon > 0.$$

For every $\varepsilon > 0$, U_ε enjoys the following properties:

(1) For every $x \in \mathbb{H}$,

$$\lim_{\varepsilon\to 0} U_\varepsilon(x) = U(x), \quad \lim_{\varepsilon\to 0} DU_\varepsilon(x) = DU(x).$$

(2) U_ε is convex, differentiable and DU_ε is Lipschitz-continuous.

Since DU_ε is Lipschitz-continuous, there exists a unique strong solution $X_t^\varepsilon(x)$ of

$$dX_t^\varepsilon = AX_t^\varepsilon\,dt + DU_\varepsilon(X_t^\varepsilon)\,dt, \quad X_0^\varepsilon = x.$$

Denote the transition semigroup of X_t^ε by P_t^ε. Zambotti [Zam06, Theorem 2.1 and Proposition 3.2] proved the following results on convergence (we include here only parts of the original result).

Theorem 5.5.5. *Assume* $\mu(K) > 0$. *Then*

(1) There exists a semigroup P_t, $t \ge 0$ *on* $\mathscr{C}_b(\mathbb{H})$ *such that for every* $f \in \mathscr{C}_b^+(\mathbb{H})$, $x \in K$, *and* $t \ge 0$,

$$\lim_{\varepsilon\to 0} P_t^\varepsilon f(x) = P_t f(x).$$

(2) For all $x \in K$ *there is a Markov process* X_t, $t \ge 0$, *defined on a probability space* $(\Omega, \mathbb{P}_x)$ *with state space* K *and transition semigroup* P_t, $t \ge 0$, *such that* $\mathbb{P}_x(X_0 = x) = 1$.

(3) For all $f_1, \ldots, f_m \in \mathscr{C}_b(\mathbb{H})$, $0 \le t_1 \le \ldots \le t_m$ *and* $x \in K$,

$$\lim_{\varepsilon \to 0} \mathbb{E}_x \big[f_1(X_{t_1}^\varepsilon) \cdots f_m(X_{t_m}^\varepsilon) = \mathbb{E}_x \big[f_1(X_{t_1}) \cdots f_m(X_{t_m}) \big].$$

Remark 5.5.6. It is not known whether the limiting process of X_t^ε solves the equation (5.37). It is the case only with additional information on U. For example, if the following condition hold

$$\int_{\mathbb{H}} (1 + |x|^2)(1 + |\partial_0 U(x)|)\, \tilde{\mu}(dx) < \infty,$$

where $\tilde{\mu} = 1/Z \cdot \exp(-2U)\, d\mu$. We refer to [DPR02, Section 9] for details (note that there is an erratum [DPR09] to [DPR02]).

For the semigroup P_t defined in Theorem 5.5.5, we have the following Harnack inequality.

Theorem 5.5.7. *For every* $x, y \in H$, $f \in \mathscr{C}_b^+(\mathbb{H})$, *and* $\alpha, \beta > 1$ *satisfying* $1/\alpha + 1/\beta = 1$, *the following inequality holds*

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\omega\beta |x-y|^2}{\mathrm{e}^{2t\omega} - 1} \right) P_t f^\alpha(y). \tag{5.38}$$

Proof. Note that DU_ε is dissipative (see Lemma 5.5.9 which is appended at the end of this section), we can apply Theorem 5.5.2 to P_t^ε and get the following Harnack inequality

$$(P_t^\varepsilon f)^\alpha(x) \le \exp\left(\frac{\omega\beta |x-y|^2}{\mathrm{e}^{2t\omega} - 1} \right) P_t^\varepsilon f^\alpha(y). \tag{5.39}$$

Therefore, we can finish the proof by letting ε tend to 0 in (5.39). □

Remark 5.5.8. We need to point out here that our work here is independent of the recent work by Da Prato et al. [DPRW09]. The spirit is the same. We use Yosida approximation and the result for Ornstein-Uhlenbeck process with Lipschitz perturbation. To some extend, our work is covered by theirs. The singular equations considered in [DPRW09] is a direct generalization of the sub-differential inclusion. We will consider Harnack inequalities for general multivalued stochastic differential equations and stochastic evolution equations in Chapter 8.

Appendix We append here the following simple fact which is used in the proof above.

Lemma 5.5.9. *Let U be a $\mathscr{C}^1$ convex function on a real separable Hilbert space $\mathbb{H}$. Then $-DU$ is dissipative, that is,*

$$\langle DU(x) - DU(y), x - y\rangle \geq 0, \quad x, y \in \mathbb{H}.$$

Proof. Since U is convex, for every $\lambda \in (0,1)$ and $x, y \in \mathbb{H}$, we have

$$U(\lambda x + (1-\lambda)y) \leq \lambda U(x) + (1-\lambda)U(y).$$

Hence we get

$$\frac{U(\lambda(x-y)+y) - U(y)}{\lambda} \leq U(x) - U(y). \tag{5.40}$$

Let λ goes to 0 in (5.40) we obtain

$$\langle DU(y), x - y\rangle \leq U(x) - U(y). \tag{5.41}$$

Similarly, we have

$$\langle DU(x), y - x\rangle \leq U(y) - U(x). \tag{5.42}$$

Now we can complete the proof by adding (5.41) and (5.42). □

5.6 Appendix

5.6.1 Finite Dimensional Approximation

This section is a complement mainly for the readers who are interested at finite dimensional case and who want to have a look of the proof by simple calculus.

We aim to re-prove the Harnack inequality (5.10) for the Gaussian Ornstein-Uhlenbeck transition semigroup (with some additional condition) in Hilbert space by finite dimension approximation.

Finite Dimensional Case Consider the following stochastic differential equation on $\mathbb{R}^n$

$$dX_t = AX\,dt + B\,dW_t, \quad X_0 = x, \tag{5.43}$$

where A and R are $n \times n$ matrices on $\mathbb{R}^n$ and W_t is a Wiener process on $\mathbb{R}^d$. Let $S_t = \exp(tA)$ for each $t \geq 0$. Then the adjoint operator of S_t is $S_t^* = \exp(tA^*)$.

The solution to the equation (5.43) is given by

$$X_t = S_t x + \int_0^t S_{t-s} B \, dW_s.$$

Set $R = BB^*$ and

$$Q_t = \int_0^t S_u R S_u^* \, du, \quad t \geq 0.$$

Denote $\mu_t = N(0, Q_t)$. It is clear that the distribution of $\int_0^t S_{t-s} dW_s$ is μ_t and hence the distribution of X_t is $N(S_t x, Q_t)$. Hence the associated transition semigroup of X_t is given by

$$P_t f(x) = \int_{\mathbb{R}^n} f(S_t x + y) \, \mu_t(dy), \quad x \in \mathbb{R}^d, \ f \in \mathscr{C}_b^+(\mathbb{R}^n).$$

For simplicity we assume the following assumption.

Assumption 5.6.1. The covariance matrix Q_t is non-degenerate.

With Assumption 5.6.1, the determinant of Q_t is positive and hence for any $a \in \mathbb{R}^n$, the Gaussian measure $N(a, Q_t)$ is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover, we have

$$\frac{dN(a, Q_t)}{dx}(x) = \frac{1}{(2\pi)^{n/2} (\det Q_t)^{1/2}} \exp\left(-\frac{1}{2} \left\langle Q_t^{-1}(x-a), x-a \right\rangle \right). \tag{5.44}$$

Now we have the following explicit formula for the transition semigroup P_t which is due to Kolmogorov ([Kol34])

$$P_t f(x) = \frac{1}{(2\pi)^{n/2} (\det Q_t)^{1/2}} \int_{\mathbb{R}^n} f(S_t x + z) \exp\left(-\frac{1}{2} \left\langle Q_t^{-1} z, z \right\rangle \right) dz.$$

Thanks for this explicit formula, we can prove the following Harnack inequality easily.

Proposition 5.6.2. *Assume that Q_t is non-degenerate for every $t > 0$. Let $\Gamma_t = Q_t^{-1/2} S_t$. Then*

$$(P_t f)^\alpha(x) \leq \exp\left(\frac{\beta}{2} \left| \Gamma_t(x-y) \right|^2 \right) P_t f^\alpha(y) \tag{5.45}$$

holds for every $t > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathscr{C}_b^+(\mathbb{R}^d)$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Proof. From (5.44) we know $N(S_t(x-y), Q_t)$ is absolute continuous with respect to $N(0, Q_t)$ and the Radon-Nykodým derivative is given by

$$\frac{dN(S_t(x-y), Q_t)}{dN(0, Q_t)}(z) = \exp\left(-\frac{1}{2}\left|Q_t^{-1/2}S_t(x-y)\right|^2 + \left\langle Q_t^{-1}z, S_t(x-y)\right\rangle\right).$$

By a change of variable and the formula above we have

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}^d} f(S_t x + z)\, \mu_t(dz) \\ &= \int_{\mathbb{R}^d} f(S_t y + z) \frac{d\,N(S_t(x-y), Q_t)}{d\,N(0, Q_t)}\, \mu_t(dz) \\ &= \int_{\mathbb{R}^d} f(S_t y + z) \exp\left(\left\langle Q_t^{-1}z, S_t(x-y)\right\rangle - \frac{1}{2}\left|Q_t^{-1/2}S_t(x-y)\right|^2\right) \mu_t(dz). \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned} P_t f(x) \le & \exp\left(-\frac{1}{2}|\Gamma_t(x-y)|^2\right)\left(\int_{\mathbb{R}^d} f^\alpha(S_t y + z)\, \mu_t(dz)\right)^{1/\alpha} \\ & \left(\int_{\mathbb{R}^d} \exp\left[\beta\left\langle Q_t^{-1}z, S_t(x-y)\right\rangle\right]\right)^{1/\beta} \\ = & \exp\left(\frac{\beta-1}{2}|\Gamma_t(x-y)|^2\right)\left(P_t f^\alpha(y)\right)^{1/\alpha} \end{aligned}$$

□

Infinite Dimensional Case Now we come back to the infinite dimensional settings. Consider the following linear stochastic partial differential equation on a real separable Hilbert space $\mathbb{H}$,

$$dX_t = AX_t + BdW_t, \quad X_0 = x \in \mathbb{H}. \tag{5.46}$$

where A is the generator of some strongly continuous contraction semigroup $(S_t)_{t\ge 0}$ on $\mathbb{H}$, B is a bounded linear operator on $\mathbb{H}$, and $(W_t)_{t\ge0}$ is a cylindrical Wiener process on $\mathbb{H}$.

Set $R = BB^*$. Fix $T > 0$. For any $0 \le t \le T$, set

$$Q_t = \int_0^t S_u R S_u^*\, du.$$

We suppose that Q_T is of trace-class and non-degenerate. Now we prove Theorem

5.10 by using Proposition 5.6.2.

Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal normal basis of $\mathbb{H}$. For each $n \in \mathbb{N}$, denote by $\mathcal{P}_n$ the orthogonal projector on the span of $\{e_1, e_2, \ldots, e_n\}$. In other words, the projection mapping $\mathcal{P}$ is defined by

$$\mathcal{P}_n x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k \quad x \in \mathbb{H}.$$

Consider the following finite dimensional stochastic differential equation

$$dX_t^{(n)} = A^{(n)} X_t + B^{(n)}\, dW_t^{(n)}$$

on $\mathcal{P}_n(\mathbb{H})$, where $A^{(n)} = \mathcal{P}_n A$, $B^{(n)} = \mathcal{P}_n B$, and $W_t^{(n)} = \mathcal{P}_n W_t$ for every $0 \le t \le T$.

Let $S_T^{(n)} := \exp(tA^{(n)})$, $R^{(n)} = B^{(n)}(B^{(n)})^*$ and

$$Q_T^{(n)} := \int_0^T S_u^{(n)} R^{(n)} S_u^{(n)}\, du$$

for every $t \ge 0$. Then we have $S_T^{(n)} = \mathcal{P}_n S_T$ and $Q_T^{(n)} = \mathcal{P} Q_T$.

We know $Q_T^{(n)}$ is non-degenerate since Q_T is so. Therefore, by Proposition 5.6.2, we have the following Harnack inequality for the transition semigroup of $X^{(n)}$ which is equal to $P_T^{(n)} := \mathcal{P} P_T$:

$$(P_T^{(n)} f^{(n)})^\alpha (x^{(n)}) \le \exp(\frac{\beta}{2} |\Gamma_T^{(n)}(x-y)|^2) P_T^{(n)} (f^{(n)})^\alpha (y^{(n)}),$$

where $f^{(n)} = \mathcal{P}_n f$, $x^{(n)} = \mathcal{P}_n x$, $y^{(n)} = \mathcal{P}_n y$ and $\Gamma_T^{(n)} = (Q_T^{(n)})^{-1/2} S_T^{(n)}$.

Let n goes to infinity, we can finish the proof.

5.6.2 Representations of Ornstein-Uhlenbeck Semigroups

We consider the Ornstein-Uhlenbeck transition semigroup introduced in Section 5.2. Assume that Q_t is of trace class for all $t \ge 0$. Recall that the semigroup is given by

$$P_t f(x) = \int_{\mathbb{H}} f(S_t x + y)\, \mu_t(dy), \quad x \in \mathbb{H},\ t \ge 0,\ f \in \mathscr{C}_b(\mathbb{H}).$$

Here $\mu_t = N(0, Q_t)$.

If $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ for all $t > 0$, then $N_{S_t x, Q_t} \ll N_{Q_t}$. By Cameron-Martin formula, we get

$$P_t f(x) = \int_{\mathbb{H}} f(y) d(t,x,y)\, \mu_t(dy)$$

where

$$d(t,x,y) = \frac{dN_{S_t x, Q_t}}{dN_{Q_t}}(y) = \exp\left(\langle \Gamma_t x, Q_t^{-1/2} y\rangle - \frac{1}{2}\|\Gamma_t x\|^2 \right).$$

If Q_∞ is of trace class, then the transition semigroup P_t has an invariant measure $\mu = N_{Q_\infty}$. Assume that $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ for all t. Then we also have $S_t(\mathbb{H}) \subset Q_\infty^{1/2}(\mathbb{H})$ since $Q_t^{1/2}(\mathbb{H}) = Q_\infty^{1/2}(\mathbb{H})$ for all $t > 0$ (see [DPZ02, Appendix B.1]). This leads us to the following formula

$$P_t f(x) = \int_{\mathbb{H}} f(y) k(t,x,y)\, \mu(dy)$$

where

$$k(t,x,y) = \frac{dN_{S_t x, Q_t}}{dN_{Q_\infty}}(y) = d(t,x,y)\frac{dN_{Q_t}}{dNQ_\infty}.$$

See [DPZ02, Equation (10.3.7)] for the tedious formula for $k(t,x,y)$.

Chojnowska-Michalik and Goldys [CMG96, Section 3, Theorem 1] obtained the following Mehler formula by using the second quantization,

$$P_t = \int_{\mathbb{H}} f\left(S_t x + Q_\infty^{1/2}\sqrt{1 - Q_\infty^{1/2} S_t Q_\infty S_t^* Q_\infty^{-1/2}}\, Q_\infty^{-1/2} y \right) \mu(dy). \tag{5.47}$$

See also in the book Da Prato and Zabczyk [DPZ02, Section 10.4].

Remark 5.6.3. Formula (5.47) is a generalization of the classical Mehler formula. Recall that for one dimensional Ornstein-Uhlenbeck process

$$X_t = x\,\mathrm{e}^{-\kappa t} + \mathrm{e}^{-\kappa t}\int_0^t \mathrm{e}^{-\kappa s}\, dW_s,$$

we have the following classical *Mehler semigroup*

$$P_t f(x) = \int_{\mathbb{R}} f(\mathrm{e}^{-\kappa t}\, x + \sqrt{1 - \mathrm{e}^{-2\kappa t}} y)\, \gamma(dy). \tag{5.48}$$

Here γ is the standard Gaussian measure on $\mathbb{R}$.

Here we aim to show another Mehler formula, from which we can also get (5.47) for the Ornstein-Uhlenbeck semigroup by direct calculation.

First we show a proposition.

Proposition 5.6.4. *Let μ be a Gaussian measure with covariance Q. Then for every bounded operator S, the covariance of $\mu \circ S^{-1}$ is SQS^*.*

Proof. For every $h, k \in \mathbb{H}$, we have

$$\int_{\mathbb{H}} \langle h, x\rangle\langle k, x\rangle\, \mu\circ S^{-1}(dx) = \int_{\mathbb{H}} \langle h, Sx\rangle\langle k, Sx\rangle\, \mu(dx)$$
$$= \int_{\mathbb{H}} \langle S^*h, x\rangle\langle S^*k, x\rangle\, \mu(dx) = \langle QS^*h, S^*k\rangle = \langle SQS^*h, k\rangle.$$

□

Now we come to a representation of the Ornstein-Uhlenbeck transition semigroup.

Proposition 5.6.5. *For every $f \in \mathscr{C}_b(\mathbb{H})$, $t \geq 0$, $x \in \mathbb{H}$*

$$P_t f(x) = \int_{\mathbb{H}} f(S_t x + Q_t^{1/2} Q_\infty^{-1/2} y)\, \mu(dy). \tag{5.49}$$

Proof. Let $T = Q_t^{1/2} Q_\infty^{-1/2}$. By Proposition 5.6.4, we have

$$\begin{aligned}\int_{\mathbb{H}} f(S_t x + Ty)\, \mu(dy) &= \int_{\mathbb{H}} f(S_t x + y)\, \mu\circ T^{-1}(dy)\\ &= \int_{\mathbb{H}} f(S_t x + y)\, N_{TQ_\infty T^*}(dy)\\ &= \int_{\mathbb{H}} f(S_t x + y)\, N_{Q_t}(dy)\\ &= P_t f(x).\end{aligned}$$

□

Remark 5.6.6. Note the following relation

$$Q_t = Q_\infty - S_t Q_\infty S_t^* \tag{5.50}$$

for all $t \geq 0$, we can verify that $Q_\infty^{1/2}\sqrt{1 - Q_\infty^{1/2} S_t Q_\infty^{1/2} S_t^*}$, in the Mehler formula (5.47), is a square root of $\mathbb{Q}_t$. Therefore, we can reproduce (5.47) by applying Proposition 5.6.5.

Remark 5.6.7. The formula (5.50) can be obtained from the following relationship between Q_t and S_t by letting $s \to \infty$

$$Q_t + S_t Q_s S_t^* = Q_{t+s}, \quad s, t \geq 0. \tag{5.51}$$

We can compute (5.51) directly as shown in the following. For every $s, t \geq 0$,

$$\begin{aligned} Q_{t+s} &= \int_0^{t+s} S_u R S_u^* \, du = \int_0^t S_u R S_u^* \, du + \int_t^{t+s} S_u R S_u^* \, du \\ &= Q_t + \int_t^{t+s} S_u R S_u^* \, du = Q_t + \int_0^s S_{t+u} R S_{t+u}^* \, du \\ &= Q_t + S_t \int_0^s S_{t+u} R S_u^* \, du S_t^* = Q_t + S_t Q_s S_t^*. \end{aligned}$$

Chapter 6

Harnack Inequalities for Ornstein-Uhlenbeck Semigroups: Two Other Gaussian Cases

In the previous chapter, we have considered Harnack inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes. In this chapter, we study Harnack inequalities for other Gaussian Ornstein-Uhlenbeck semigroups. We still use the Cameron-Martin formula to establish Harnack inequalities.

In Section 6.1, we show Harnack inequalities for the "formal" transition semigroup of fractional Orntein-Uhlenbeck processes. In Section 6.2, we establish Harnack inequalities for Ornstein-Uhlenbeck semigroups on general Gaussian probability spaces.

6.1 Harnack Inequalities for Fractional Orntein-Uhlenbeck Processes

Stochastic (partial) differential equations driven by fractional Brownian motions have met great interest during the last years. We refer to the monograph by Biagini et al. [BØSW04] for topics related to fractional Brownian motion. For linear stochastic equations in Hilbert spaces with a fractional Brownian motion, we refer to Pasik-Ducan et al. [PDDM06] and references therein.

We first shortly introduce fractional Brownian motions and stochastic integrals with respect to fractional Brownian motions. Then we introduce fractional Ornstein-Uhlenbeck semigroups and Harnack inequalities for the semigroups.

6.1.1 Fractional Brownian Motions and Stochastic Integrals

Real Fractional Brownian Motions Case

Definition 6.1.1. A real *fractional Brownian motion* $(\beta^H(t))_{0\le t\le T}$ with *Hurst parameter* $H \in (0,1)$ is a centered Gaussian process with covariance

$$\phi(t,s) = \frac{V_H}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

For every $H \in (0,1)$ and $0 \le s \le t \le T$, define

$$K_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right),$$

where $F(\cdot,\cdot,\cdot,\cdot)$ is the Gauss hyper-geometric function.

Then we have the following representation for fractional Brownian motion

$$\beta^H(t) = \int_0^t K_H(t,u)\, d\beta(u), \quad 0 \le s \le T,$$

where $(\beta(u))_{0\le t\le T}$, is a standard real valued Brownian motion.

Let $\mathbb{H}$ be a real separable Hilbert space. We shall describe the stochastic integral of deterministic $\mathbb{H}$-valued function with respect to a real valued fractional Brownian motion. It is similar to the integral with respect to Wiener process introduced in Section 1.3.

Denote by $\mathscr{E}$ the space of all $\mathbb{H}$-valued step functions on $[0,T]$. Let $\phi \in \mathscr{E}$ with

$$\phi(t) = \sum_{i=1}^{n-1} x_i \mathbb{1}_{[t_i,t_{i+1})}(t),$$

where $x_i \in \mathbb{H}$, $i = 1,2,\ldots,n-1$ and $0 = t_1 < \ldots < t_n = T$ with $n \in \mathbb{N}$.

Define

$$\int_0^T \phi(t)\, d\beta^H(t) = \sum_{i=1}^{n-1} x_i\big(\beta_i^H(t_{i+1}) - \beta^H(t_i)\big). \tag{6.1}$$

It follows that

$$\mathbb{E}\left|\int_0^T \phi^H(t)\, d\beta^H(t)\right|^2 = |\, \mathscr{K}_{\mathrm{H}}^* \phi|^2_{L^2([0,T],\mathbb{H})}, \tag{6.2}$$

where

$$\mathscr{K}_{\mathrm{H}}^* \phi(t) = \phi(t)K_H(T,t) + \int_t^T \big(\phi(u) - \phi(t)\big)\frac{\partial K_H}{\partial u}(u,t)\, du, \quad 0 \le t \le T.$$

By the isometry (6.2), we can extend the stochastic integral (6.1) from the function $\phi \in \mathscr{E}$ to the function $\phi \in \bar{\mathscr{E}}$. Here $\bar{\mathscr{E}}$ is the completion of $\mathscr{E}$ with the inner product

$$\langle \phi, \psi \rangle_{\mathbb{H}} := \langle \mathscr{K}_{\mathrm{H}}^* \phi, \mathscr{K}_{\mathrm{H}}^* \psi \rangle_{L^2([0,T],\mathbb{H})}$$

for every $\phi, \psi \in \mathscr{E}$.

Cylindrical Fractional Brownian motions Case

Now we are going to introduce the cylindrical fractional Brownian motions on a Hilbert space. A usual way is to define it similarly as the definition of cylindrical Wiener processes. We use the following method which is short.

Definition 6.1.2. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. A cylindrical process $\langle W^H, \cdot \rangle : \Omega \times [0,T] \times \mathbb{H} \to \mathbb{R}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called a (standard) *cylindrical fractional Brownian motion* with Hurst parameter $H \in (0,1)$ if

(1) For every $x \in \mathbb{H}$, $x \ne 0$, $\langle W^H(\cdot), \frac{x}{|x|} \rangle$ is a real valued fractional Brownian motion with Hurst parameter H.

(2) For every $0 \le t \le T$, $\langle W^H(t), \cdot \rangle$ is linear. That is,

$$\langle W^H(t), px + qy \rangle = p\langle W^H(t), x \rangle + q\langle W^H(t), y \rangle, \quad \mathbb{P}\text{–a.s.}$$

for every $p, q \in \mathbb{R}$ and $x, y \in \mathbb{H}$.

Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthogonal normal basis of $\mathbb{H}$. For every $n \in \mathbb{N}$, let $\beta_n^H(\cdot) = \langle W^H(\cdot), e_n \rangle$. Then $\{\beta_n\}_{n\in\mathbb{N}}$ is an independent sequence of real fractional Brownian motion with Hurst parameter H.

Definition 6.1.3. For every $0 \le t \le T$, let $\Phi(t)$ be a linear bounded operator on $\mathbb{H}$ and define $\phi_n(t) = \Phi(t)e_n$ for $n \in \mathbb{N}$. Suppose that $\phi_n \in \bar{\mathscr{E}}$. Then we define

$$\int_0^T \Phi(t)\, dW^H(t) = \sum_{n=1}^{\infty} \int_0^T \phi_n(t)\, d\beta_n^H(t) \tag{6.3}$$

provided the infinite series on the right hand side of (6.3) converges in $L^2(\Omega)$.

The following proposition is from [PDDM06, Proposition 11.3, Remark 11.4].

Proposition 6.1.4. *Let*

$$Q^{\mathbb{H}}_{T,\Phi} = \int_0^T \mathscr{K}_H^* \Phi(t) \big(\mathscr{K}_H^* \Phi(t)\big)^* \, dt.$$

If $Q^{\mathbb{H}}_{T,\Phi}$ is a trace class operator on $\mathbb{H}$, then the stochastic integral (6.3) is a well-defined centered Gaussian $\mathbb{H}$-valued random variable with covariance $Q^{\mathbb{H}}_{T,\Phi}$.

6.1.2 Fractional Ornstein-Uhlenbeck Processes and Harnack Inequalities

Consider the following linear stochastic partial differential equation on $\mathbb{H}$

$$dX_t = AX_t dt + B dW_t^H \tag{6.4}$$

with $X_0 = x$ and $t \in [0, T]$, where B is a bounded linear operator on $\mathbb{H}$, A is the generator of some strongly continuous semigroup $(S_t)_{0 \le t \le T}$ on $\mathbb{H}$, $(W_t^H)_{0 \le t \le T}$ is an $\mathbb{H}$-valued cylindrical fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Define

$$Q_T^H = \int_0^T \mathscr{K}_H^* S_u B \big(\mathscr{K}_H^* S_u B\big)^* \, du.$$

By Proposition 6.1.4, if Q_T^H is of trace class, i.e., if

$$\operatorname{Tr} Q_T^H = \int_0^T \operatorname{Tr}\big(\mathscr{K}_H^* S_u B B^* S_u^* \mathscr{K}_H\big) \, du < \infty.$$

then the stochastic equation (6.4) has a mild solution

$$X_t = S_t x + \int_0^t S_{t-s} B \, dW_s^H, \quad 0 \le t \le T.$$

We call it *fractional Ornstein-Uhlenbeck process.*

Note that X_t is Gaussian distributed as $\mu_t^H := N(0, Q_T^H)$. Similar to (5.8), the representation of the transition semigroups for Ornstein-Uhlenbeck processes driven by Wiener processes, we can formally define the "transition semigroup" of fractional Ornstein-Uhlenbeck process by

$$P_t f(x) = \int_0^t f(S_t x + y)\, d\mu_t^H(y),$$

for every $f \in \mathscr{C}_b(\mathbb{H})$.

By applying the Cameron-Martin formula (see Theorem 1.2.2), we have the following Harnack inequality which is similar to (5.10) for Ornstein-Uhlenbeck processes driven by Wiener processes. The proof is also similar to the proof of Theorem 5.10.

Theorem 6.1.5. *Let $T > 0$. Assume that the operator Q_T^H is of trace class. Let $x, y \in \mathbb{H}$ such that $S_T(x-y) \in (Q_T^H)^{1/2}(\mathbb{H})$. Then*

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2}|\Gamma_T^H(x-y)|^2\right) P_T f^\alpha(y), \tag{6.5}$$

for every $f \in \mathscr{C}_b^+(\mathbb{H})$, and every $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$. Here $\Gamma_T^H = \left(Q_T^H\right)^{-1/2} S_T$.

Remark 6.1.6. Similar to the treatment in Chapter 5 on the Harnack inequalities for Ornstein-Uhlenbeck processes driven by Wiener processes, we can consider the estimates of $\|\Gamma_T^H\|$ as in Subsection 5.2.2; and hence we can get estimates of the Harnack inequalities for fractional Ornstein-Uhlenbeck processes as in Subsection 5.2.2. We can also study examples as in Section 5.4.

6.2 Harnack Inequalities for Ornstein-Uhlenbeck Semigroups on Gaussian Probability Spaces

We first recall the definition of Gaussian probability spaces, numerical model for Gaussian probability spaces and the Cameron-Martin theorem on Gaussian probability spaces in Subsection 6.2.1; Then we introduce Ornstein-Uhlenbeck semigroups and Harnack inequalities for them in Section 6.2.2.

We refer to the books [Mal97, HY97, HY00, Nua06] for more detailed background on Gaussian probability spaces and Ornstein-Uhlenbeck semigroups.

6.2.1 Gaussian Probability Spaces and Numerical Models

Gaussian Probability Spaces

Definition 6.2.1. Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space and $\mathbb{H}$ a real separable Hilbert space. Suppose that $\mathscr{H} = \{W_h : h \in \mathbb{H}\}$ is a family of Gaussian random variables such that for all $h, g \in \mathbb{H}$

$$\mathbb{E}(W_h) = 0 \quad \text{and} \quad \mathbb{E}(W_h W_g) = \langle h, g \rangle_{\mathbb{H}}. \tag{6.6}$$

Then we call $(\Omega, \mathscr{F}, \mu; \mathbb{H})$ a *Gaussian probability space.*

There are some typical Gaussian probability spaces. For example, abstract Wiener spaces, white noise spaces. In the following we just recall the classical Wiener space.

Example 6.2.2. Let $\mathbb{W} = C_0([0,1], \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0,1]$ with initial value 0. Equipped with the uniform norm $\|w\|_\infty := \sup_{t\in[0,1]} |w(t)|_{\mathbb{R}^d}$ for every $w \in \mathbb{W}$, the path space $\mathbb{W}$ turns into a separable Banach space.

Let μ be a Wiener measure on $\mathbb{W}$ and $\mathscr{F}$ the μ-completion of the Borel σ-algebra of $\mathbb{W}$. Set $\mathbb{H} := L^2[0,1]$. For every $h \in \mathbb{H}$, define a mapping W_h on $\mathbb{W}$ by

$$W_h(w) = \int_0^1 h(t)\, dw(t), \quad w \in \mathbb{W}$$

according the usual stochastic integrals with respect to Wiener processes. Then $\mathscr{H} = \{W_h : h \in \mathbb{H}\}$ is a family of Gaussian random variables satisfying the conditions in (6.6). Hence $(\Omega, \mathscr{F}, \mu; \mathbb{H})$ is Gaussian probability space.

The mapping $J : \mathbb{H} \to \mathbb{W}$, $h \mapsto \tilde{h}(\cdot) = \int_0^\cdot h(s)\, ds$ for $0 \le t \le 1$ is a continuous linear injective. Then $\widetilde{\mathbb{H}} = J(\mathbb{H})$ is the Cameron-Martin space of $\mathbb{W}$. We call $(\mathbb{W}, \mathscr{F}, \mu; \widetilde{\mathbb{H}})$ the *classical Wiener space.* .

Assumption 6.2.3 (Irreducible Assumption)**.** In this section, we will always assume that every Gaussian probability space is irreducible. That is, $\mathscr{F} = \sigma(\mathscr{F}^0 \cup \mathscr{N})$, where $\mathscr{F}^0 = \sigma\{W_h, h \in \mathbb{H}\}$ and $\mathscr{N}$ is the set of all μ-zero sets.

Numerical Models and The Cameron-Martin Theorem

Denote by

$$\gamma(du) = (2\pi)^{-1/2} \exp(-u^2/2)\, du$$

the standard Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$.

For every $h \in \mathbb{R}$, define $W_h(x) = hx$ for all $x \in \mathbb{R}$. It is obvious that $\{W_h,\ h \in \mathbb{R}\}$ is a family of Gaussian random variables satisfying (6.6). So $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma; \mathbb{R})$ is a one-dimensional Gaussian probability space.

Let H_n be the Hermite polynomial on $\mathbb{R}$ defined as

$$H_n(u) = (-1)^n \, \mathrm{e}^{u^2/2} \frac{d^n}{du^n} \mathrm{e}^{-u^2/2}, \quad u \in \mathbb{R},\ n \in \mathbb{N}.$$

It is well known that $\{(n!)^{-1/2} H_n \colon n \in \mathbb{N}\}$ consists of an orthogonal normal basis of $L^2(\mathbb{R}, \gamma)$.

Now let us consider $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$, which is the infinite product space of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$. For any nonnegative integer sequence $\lambda = \{\lambda_j\}_{j \in \mathbb{N}}$, denote

$$|\lambda| = \sum_j \lambda_j, \quad \lambda! = \prod_j (\lambda_j!),$$
$$\Lambda_n = \{\lambda \in \mathbb{R}^\infty \colon |\lambda| = n\}, \quad \Lambda = \{\lambda \in \mathbb{R}^\infty \colon |\lambda| < \infty\}.$$

For any $\lambda \in \Lambda$, define

$$H_\lambda(x) = \prod_j H_{\lambda_j}(x_j), \quad x = \{x_j\} \in \mathbb{R}^\infty.$$

Then $\{(\lambda!)^{-1/2} H_\lambda \colon \lambda \in \Lambda\}$ is an orthogonal normal basis of $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$.

Let $\mathbb{L}_n = \mathrm{Span}\{H_\lambda \colon \lambda \in \Lambda_n\}$. Then we see

$$L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) = \bigoplus_{n=0}^{\infty} \mathbb{L}_n.$$

Fixing an orthogonal normal basis $\{h_j\}_{j \in \mathbb{N}}$ of $\mathbb{H}$, we see $\mathbb{H} \cong l^2$, where

$$l^2 := \Big\{\{l_j\} \colon \sum_j l_j^2 < \infty\Big\}.$$

Define

$$T \colon \Omega \to \mathbb{R}^\infty, \quad T\omega = \{W_{h_j}(\omega)\}_{j \in \mathbb{N}}.$$

Then T is $\mathscr{F}/\mathcal{B}^\infty$ measurable and $\gamma^\infty = \mu \circ T^{-1}$.

We call the Gaussian probability space $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ a *numerical model* of

$(\Omega, \mathscr{F}, \mu)$.

Define

$$L^{\infty-} = \bigcap_{1<p<\infty} L^p(\Omega, \mathscr{F}, \mu), \quad \text{and} \quad L^{1+} = \bigcup_{1<p<\infty} L^p(\Omega, \mathscr{F}, \mu)$$

respectively by projective limit and inductive limit.

For any $1 \leq p \leq \infty$, $\phi \in L^p(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$, set $T_*\phi(\omega) = \phi(T\omega)$.

6.2.2 Ornstein-Uhlenbeck Semigroups

Let $(\Omega, \mathscr{F}, \mu; \mathbb{H})$ be a Gaussian probability space. Fix an orthogonal normal basis $\{h_j\}_{j\in\mathbb{N}}$ on $\mathbb{H}$. For any $\lambda \in \Lambda$, set

$$\widetilde{H}_\lambda(\omega) = \prod_{j=1}^{\infty} H_{\lambda_j}(W_{h_j}(\omega)).$$

Then $\{(\lambda!)^{-1/2}\widetilde{H}_\lambda \colon \lambda \in \Lambda\}$ consists of an orthogonal normal basis of $L^2(\Omega, \mathscr{F}, \mu)$.

Denote $\widetilde{\mathbb{L}}_0 = \mathbb{R}$ and $\widetilde{\mathbb{L}}_n$ the space spanned by $\{\widetilde{H}_\lambda \colon \lambda \in \Lambda_n\}$. Then we have

$$L^2(\Omega, \mathscr{F}, \mu) = \bigoplus_{n=0}^{\infty} \widetilde{\mathbb{L}}_n.$$

Denote by J_n the orthogonal projection from $L^2(\Omega, \mathscr{F}, \mu)$ to $\widetilde{\mathbb{L}}_n$.

Definition 6.2.4. We call

$$P_t := \sum_{n=0}^{\infty} \mathrm{e}^{-nt} J_n, \quad t \geq 0$$

the Ornstein-Uhlenbeck semigroup on $L^2(\Omega, \mathscr{F}, \mu)$.

Remark 6.2.5. It is possible to study the following semigroup which is a slight generalization of the Ornstein-Uhlenbeck semigroup defined by $P_t = \sum_{n=0}^{\infty} \rho_n J_n$ for $t \geq 0$ and some reasonable real sequence $\rho = \{\rho_n\}$.

The following proposition is from [HY00, Chapter 2, Proposition 3.4].

Proposition 6.2.6. *Let $P^{\mathbb{R}^\infty}$ be the Ornstein-Uhlenbeck semigroup on the nu-*

merical model $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$*. Then*

$$(P_t^{\mathbb{R}^\infty} f)(x) = \int_{\mathbb{R}^\infty} f(\mathrm{e}^{-t} x + \sqrt{1-\mathrm{e}^{-2t}}\, z)\, \gamma^\infty(dz) \tag{6.7}$$

for every $t > 0$*,* $x \in \mathbb{R}^\infty$ *and* $f \in L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$*.*

Since the definition of Ornstein-Uhlenbeck semigroup does not dependent on the choice of the basis of $\mathbb{H}$ (it is called *intrinsic property* in the literatures, see for instance [Mal97]) , the Mehler formula (6.7) for Ornstein-Uhlenbeck semigroup on the numerical model can be regarded as an equivalent definition of Ornstein-Uhlenbeck semigroup on the general Gaussian probability space.

For Ornstein-Uhlenbeck semigroup on general Gaussian probability space, we have the following Mehler formula. We refer to a proof in Nualart [Nua06, Section 1.4].

Proposition 6.2.7. *For each* $f \in L^2(\Omega, \mathscr{F}, \mu)$*, there exist some* $\mathcal{B}^\infty/\mathcal{B}$*-measurable function* $\psi_f : \mathbb{R}^\infty \to \mathbb{R}$ *such that* $f = \psi_f \circ T := (T_*^{-1} f) \circ T$*. Then for every* $\omega \in \Omega$ *and* $t \geq 0$*, the Ornstein-Uhlenbeck semigroup on* $L^2(\Omega, \mathbb{R}, \mu)$ *can be written as*

$$\begin{aligned} P_t f(\omega) &= \int_\Omega T_*^{-1} f(\mathrm{e}^{-t}\, T\omega + \sqrt{1-\mathrm{e}^{-2t}} T\omega')\, \mu(d\omega') \\ &= \int_{\mathbb{R}^\infty} \psi_f(\mathrm{e}^{-t}\, T\omega + \sqrt{1-\mathrm{e}^{-2t}} z)\, \gamma^\infty(dz). \end{aligned}$$

Now we have the following connection between the Ornstein-Uhlenbeck semigroups on general Gaussian probability spaces and the corresponding semigroups on numerical models.

Proposition 6.2.8. *Let* P_t *be the Ornstein-Uhlenbeck semigroup on the Gaussian probability space* $(\Omega, \mathscr{F}, \mu; \mathbb{H})$ *and let* $P_t^{\mathbb{R}^\infty}$ *be corresponding Ornstein-Uhlenbeck semigroup on the numerical model* $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$*. Then for every* $t \geq 0$

$$P_t = T_* \circ \mathbb{P}_t^{\mathbb{R}^\infty} \circ T_*^{-1}.$$

Proof. For any $f \in L^2(\Omega)$, we have

$$\begin{aligned} T_* \circ \mathbb{P}_t^{\mathbb{R}^\infty} \circ T_*^{-1} f(\omega) &= \mathbb{P}_t^{\mathbb{R}^\infty} \circ (T_*^{-1} f)(T\omega) \\ &= \int_{\mathbb{R}^\infty} T_*^{-1} f(\mathrm{e}^{-t}\, T\omega + \sqrt{1-\mathrm{e}^{-2t}} z)\, \mu(dz) = P_t f(\omega). \end{aligned}$$

□

Remark 6.2.9. Proposition 6.2.7 is in a slight different form with the one in Nualart [Nua06, Section 1.4]. Proposition 6.2.8 is summarized by the author.

6.2.3 Harnack Inequalities and Examples

We first recall the Cameron-Martin theorem on Gaussian probability spaces.

By using the mapping

$$T_* : L^{1+}(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) \to L^{1+}(\Omega, \mathscr{F}, \mu)$$

we can define a shift operator on $L^{1+}(\Omega, \mathscr{F}, \mu)$ by

$$\varrho_h = T_* \circ \tau_{J(h)} \circ T_*^{-1}, \quad h \in \mathbb{H}.$$

Here τ is the shift operator of functionals on $\mathbb{R}^\infty$.

We have the following the Cameron-Martin theorem. We refer to [HY00, Theorem 2.5] for a proof.

Theorem 6.2.10. *Define an exponential functional by*

$$\mathscr{E}(h) = \exp(W_h - \frac{1}{2}|h|_{\mathbb{H}}^2), \quad \text{for every } h \in \mathbb{H}.$$

Then $\mathscr{E}(h) \in L^{\infty-}$ *and*

$$\|\mathscr{E}(h)\|_p \le \exp\left(\frac{p-1}{2}|h|^2\right), \quad 1 < p < \infty.$$

Moreover, for every $f \in L^{1+}$, *we have*

$$\mathbb{E}(\varrho_h f) = \mathbb{E}(\mathscr{E}(h) f), \quad h \in \mathbb{H}.$$

Now we can prove a Harnack inequality for the Ornstein-Uhlenbeck semigroups on the numerical model.

Theorem 6.2.11. *Let* $\mathbb{P}_t^{\mathbb{R}^\infty}$ *be the Ornstein-Uhlenbeck semigroup for numerical model. For any* $x, y \in \mathbb{R}^\infty$, *we define*

$$\rho(x, y) = \begin{cases} |x - y|_{l^2}, & \text{if } x - y \in l^2; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for any $x, y \in \mathbb{R}^\infty$, $t \geq 0$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, and for any nonnegative $f \in L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ we have

$$(\mathbb{P}_t^{\mathbb{R}^\infty} f)^\alpha(x) \leq \exp\left(\frac{\beta\rho(x,y)^2}{2(\mathrm{e}^{2t} - 1)}\right) P_t^{\mathbb{R}^\infty} f^\alpha(y).$$

Proof. We only need to consider the case $x - y \in l^2$. For simplicity, we set

$$\sigma_t = \frac{\mathrm{e}^{-t}}{\sqrt{1 - \mathrm{e}^{-2t}}} = \frac{1}{\sqrt{\mathrm{e}^{2t} - 1}}.$$

$$\begin{aligned}
\mathbb{P}_t^{\mathbb{R}^\infty} f(x) &= \int_{\mathbb{R}^\infty} f(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}} y)\, \gamma^\infty(dz) \\
&= \int_{\mathbb{R}^\infty} f\left(\mathrm{e}^{-t} y + \sqrt{1 - \mathrm{e}^{-2t}}\left[z + \frac{\mathrm{e}^{-t}(x - y)}{\sqrt{1 - \mathrm{e}^{-2t}}}\right]\right) \gamma^\infty(dz) \\
&= \int_{\mathbb{R}^\infty} f(\mathrm{e}^{-t} y + \sqrt{1 - \mathrm{e}^{-2t}} z) \cdot \\
&\qquad \exp\left(\sigma_t \langle z, x - y\rangle_{l^2} - \frac{\sigma_t^2}{2}|x - y|_{l^2}\right) \gamma^\infty(dz) \\
&\leq \left[\int_{\mathbb{R}^\infty} f^\alpha(\mathrm{e}^{-t} y + \sqrt{1 - \mathrm{e}^{-2t}} z)\, \gamma^\infty(dz)\right]^{1/\alpha} \cdot \\
&\qquad \left[\int_{\mathbb{R}^\infty} \exp\left(\beta\sigma_t \langle z, x - y\rangle_{l^2} - \frac{\beta\sigma_t^2}{2}|x - y|_{l^2}\right) \gamma^\infty(dz)\right]^{1/\beta} \\
&= \exp\left(\frac{(\beta - 1)\sigma_t^2 |x - y|_{l^2}}{2}\right) (\mathbb{P}_t^{\mathbb{R}^\infty} f^\alpha)^{1/\alpha}(y).
\end{aligned}$$

□

Following from Theorem 6.2.11, we have the following Harnack inequality for the Ornstein-Uhlenbeck semigroup on a general Gaussian probability space.

Theorem 6.2.12. *Let P_t be the Ornstein-Uhlenbeck semigroup on $L^2(\Omega, \mathscr{F}, \mu)$. For any $\omega_1, \omega_2 \in \Omega$, we define*

$$\rho(\omega_1, \omega_2) := \begin{cases} |T\omega_1 - T\omega_2|_{l^2}, & \text{if } T\omega_1 - T\omega_2 \in l^2; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for any $\omega_1, \omega_2 \in \Omega$, $t \geq 0$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, and for any nonnegative $f \in L^2(\Omega, \mathscr{F}, \mu)$ we have

$$(P_t f)^\alpha(\omega_1) \leq \exp\left(\frac{\beta\rho(\omega_1, \omega_2)^2}{2(\mathrm{e}^{2t} - 1)}\right) P_t f^\alpha(\omega_2).$$

Example 6.2.13 (Harnack Inequality for Ornstein-Uhlenbeck Semigroup on Classical Wiener Space)**.** We continue with Example 6.2.2. Consider the Ornstein-Uhlenbeck semigroup P_t on the classical Wiener space $(\mathbb{W}, \mathscr{F}, \mu; \widetilde{\mathbb{H}})$.

Recall that the Cameron-Martin space $\widetilde{\mathbb{H}}$ consists of all absolutely continuous function $h : [0,1] \to \mathbb{R}^d$ with a square integrable derivative. The inner product of $\widetilde{\mathbb{H}}$ is defined by

$$\langle h_1, h_2 \rangle_{\widetilde{\mathbb{H}}} := \int_0^1 \dot{h}_1(s)\dot{h}_2(s)\, ds, \quad h_1, h_2 \in \widetilde{\mathbb{H}}.$$

The intrinsic distance ρ on $(\mathbb{W}, \mathscr{F}, \mu; \widetilde{\mathbb{H}})$ is defined by

$$\rho(w_1, w_2) := \begin{cases} \langle w_1 - w_2, w_1 - w_2 \rangle^{1/2}, & \text{if } w_1 - w_2 \in \widetilde{\mathbb{H}} \\ \infty, & \text{otherwise} \end{cases}$$

for all $w_1, w_2 \in \mathbb{W}$.

Then for any $w_1, w_2 \in \mathbb{W}$, $t \geq 0$, $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$, and for any nonnegative $f \in \mathscr{C}_b^+(\mathbb{W})$ we have

$$(P_t f)^\alpha(w_1) \leq \exp\left(\frac{\beta\rho(w_1, w_2)^2}{2(\mathrm{e}^{2t} - 1)}\right) P_t f^\alpha(w_2).$$

By using semigroup calculus, Shao [Sha07] also studied the Harnack inequalities for the Ornstein-Uhlenbeck semigroups on Wiener spaces.

Chapter 7

Harnack Inequalities for Ornstein-Uhlenbeck Processes Driven by Lévy Processes

In this chapter, we devote our studies to Harnack inequalities for Ornstein-Uhlenbeck processes with Lévy noise. There are three methods available to prove Harnack inequalities: semigroup calculus, measure transformations on the state spaces and measure transformations on the probability spaces. Röckner and Wang [RW03a] used the first two methods and obtained some Harnack inequalities for generalized Mehler semigroups which are naturally associated with Lévy driven Ornstein-Uhlenbeck processes. We present their methods and results in Sections 7.2 and 7.3 respectively. By the first method, only second order Harnack inequalities were able to be obtained. While Harnack inequalities established by the second method are not explicit in general.

In Section 7.3, we apply the measure transformation on the state space for Lévy Ornstein-Uhlenbeck semigroups more concretely. We also use this method to establish Harnack inequalities for α-stable Ornstein-Uhlenbeck process and Markov Chains.

In Section 7.4, by using coupling and Girsanov's transformation, we show Harnack inequalities for Ornstein-Uhlenbeck processes with Lévy noise. The inequalities we prove are more general and sharper than the ones proved in [RW03a]

In Section 7.5 we consider the applications of Harnack inequalities. We mainly study regularizing properties, heat kernel bounds and hyperboundedness of the Lévy Ornstein-Uhlenbeck semigroup.

7.1 Lévy Driven Ornstein-Uhlenbeck Processes

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let R be a trace class operator on $\mathbb{H}$. In this chapter, as before, we always use the notation $\mathbb{H}_0 = R^{1/2}(\mathbb{H})$ for the Cameron-Martin space of $\mathbb{H}$, $\langle \cdot, \cdot \rangle_0$ and $|\cdot|_0$ for the natural inner product and norm on $\mathbb{H}_0$ respectively (refer to Section 1.1).

Fix $T > 0$. Consider the following generalized Langevin equation on $[0, T]$

$$dX_t = AX_t + dZ_t, \quad X_0 = x \in \mathbb{H}, \tag{7.1}$$

where Z_t is a Lévy process with characteristic triplet (b, R, ν), and A is the infinitesimal generator of a strongly continuous contraction semigroup $(S_t)_{t \geq 0}$ on $\mathbb{H}$.

The mild (unique weak) solution of the stochastic differential equation (7.1) is given by (see for example [Cho87, App06] etc.)

$$X_t = S_t x + \int_0^t S_{t-s}\, dZ_s. \tag{7.2}$$

We call this process the *Lévy driven Ornstein-Uhlenbeck process.*

The associated transition semigroup of X_t is given by

$$P_t f(x) = \int_{\mathbb{H}} f(S_t x + y)\, \mu_t(dy) \tag{7.3}$$

for every $x \in \mathbb{H}$ and $f \in \mathscr{C}_b(\mathbb{H})$, where μ_t is the law of $\int_0^t S_{t-s}\, dZ_s$.

By Propositions 1.4.10, we know μ_t is infinitely divisible. Let λ be the characteristic symbol of Z_t. Then the Fourier transform of μ_t is given by

$$\hat{\mu}_t(u) = \exp\left\{ - \int_0^t \lambda(S_r^* u)\, dr \right\}, \quad u \in \mathbb{H}, \tag{7.4}$$

Denote by (b_t, Q_t, ν_t) for the characteristic triplet of μ_t. By Corollary 1.4.11, we have

$$b_t = \int_0^t S_r b\, dr + \int_0^t dr \int_{\mathbb{H}\setminus\{0\}} S_r x \left\{ \mathbb{1}_{\{|x| \leq 1\}}(S_r x) - \mathbb{1}_{\{|x| \leq 1\}}(x) \right\} dx, \tag{7.5}$$

$$Q_t = \int_0^t S_r R S_r^*\, dr, \tag{7.6}$$

$$\nu_t = \int_0^t \nu \circ S_r^{-1}\, dr. \tag{7.7}$$

For a family of probability measures $(\mu_t)_{t\geq 0}$ on $\mathbb{H}$, the semigroup of the form (7.3) is called a *generalized Mehler semigroup* associated with S_t and μ_t. It can be proved that (see for instance [Les01]) the generalized Mehler semigroup become a Markov semigroup if and only if

$$\mu_{s+t}(\xi) = \mu_t * (\mu_s \circ S_t^{-1}\xi), \quad s,t \geq 0,\ \xi \in \mathbb{H}.$$

Remark 7.1.1. The semigroup with the following property $\mu_{s+t} = \mu_t * (\mu_s \circ S_t^{-1})$, for every $s,t \geq 0$ is called *skew convolution semigroup*. This is a generalization the so called *convolution semigroup* (the case when $T_t \equiv I$) corresponding with Lévy process. See the details of convolution semigroup in Page 27.

Under some slight condition, there is a natural one to one corresponding between generalized Mehler semigroup with Markov property and the transition semigroup of Lévy driven Ornstein-Uhlenbeck processes. See [Les01] and references therein for details.

Bibliographic Notes on Generalized Mehler Semigroup Mehler semigroup is named after Mehler [Meh66]. Generalized mehler semigroup has been studied extensively by Röckner and his collaborators in a series papers [BR95, BRS96, FR00, Les01, LR02, Meh66]. See also the papers by Dawson and/or Li et al. [DLSS04, DL06, Li06] etc. for the relation of generalized Mehler semigroup with branching processes. We refer also to Page 96 for the bibliographic notes on Ornstein-Uhlenbeck processes with Gaussian or Lévy noise.

7.2 Semigroup Calculus Approach

We work under the framework in Section 7.1 and consider the Lévy Ornstein-Uhlenbeck semigroup P_t defined in (7.3).

In [RW03a], the following assumptions are used to establish Harnack inequalities.

Assumption 7.2.1. (1) P_t has an invariant probability measure;

(2) There exists $\{x_n\}_{n\geq 1} \subset \mathbb{H}$ consisting of eigenvectors of A^* and separating the points of $\mathbb{H}$;

(3) For every $t \geq 0$, $S_t R(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$ holds and there is a strictly positive

$h_1 \in C[0,\infty)$ such that

$$|S_t Rx|_0 \leq \sqrt{h_1(t)}|Rx|_0, \quad x \in \mathbb{H},\ t \geq 0.$$

Under Assumptions 7.2.1, Röckner and Wang [RW03a] proved the following theorem on Harnack inequalities. Recall that we denote by ρ the intrinsic distance on $\mathbb{H}$ induced by R.

Theorem 7.2.2. *Assume Assumption 7.2.1 holds. Then*

$$(P_t f)^2(x) \leq \exp\left(\frac{\rho(x,y)^2}{\int_0^t h_1(s)^{-1}\,ds}\right) P_t f^2(y) \tag{7.8}$$

holds for every $x, y \in \mathbb{H}$, $t > 0$ and $f \in \mathscr{C}_b^+(\mathbb{H})$. In particular, for the diffusion case, i.e., when $\nu = 0$,

$$(P_t f)^\alpha(x) \leq \exp\left(\frac{\beta\rho^2(x,y)}{2\int_0^t h_1(s)^{-1}\,ds}\right) P_t f^\alpha(y). \tag{7.9}$$

holds for every $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$.

The approach used in [RW03a] to prove Theorem 7.2.2 is semigroup calculus. In the following we just sketch the idea of the proof.

When $\nu = 0$, the generator is a diffusion operator. Then by chain rule, we can take derivative of the following function

$$s \mapsto \log P_{t-s}(P_s f)^\alpha(x_s) \tag{7.10}$$

with respect to s. Here x_s for $s \in [0,t]$ is a geodesic connecting x and y on $\mathbb{H}$.

When $\nu \neq 0$, the generator of the semigroup P_t has the following form

$$\begin{aligned}\mathscr{A} f(x) &= \langle Df(x), b + Ax\rangle + \frac{1}{2}\mathrm{Tr}(RD^2 f(x)) \\ &\quad + \int_{\mathbb{H}\setminus\{0\}} \left[f(x+y) - f(x) - \langle Df(x), y\rangle \mathbb{1}_{\{|x|>1\}}(y)\right] \nu(dy).\end{aligned}$$

for regular enough function f.

This generator $\mathscr{A}$ is not a diffusion operator and hence the chain rule doesn't work. But Röckner and Wang still can get an estimate of the derivative of the function (7.10) for the special case $\alpha = 2$. They used the following explicit formula for the square field operator

$$\Gamma(f,f) := \frac{1}{2}(\mathscr{A}f^2 - 2f\mathscr{A}f) = \frac{1}{2}\bigg(\langle RDf, Df\rangle + \int_{\mathbb{H}}[f(\cdot) - f(\cdot + y)]^2\nu(dy)\bigg) \quad (7.11)$$

for regular enough function f (see [LR02, Propostion 4.1] for details).

From (7.11), it is obvious to see that $\Gamma(f,f) \geq \frac{1}{2}\langle RDf, Df\rangle$. Therefore, the Lévy case is reduced to the Gaussian case. So it is also clear why the exponent of the coefficient in the Harnack inequality (7.8) is independent of the Lévy measure ν.

In the spirit of semigroup calculus, one may try to use the martingale expansion method to calculate the derivatives. This method is used by Kawabi [Kaw04, Kaw05] for diffusions. In the following we point out the difficulty of this method for the jump case.

Let f be a function in some nice class. Consider

$$\begin{aligned} H(r_1, r_2, r_3): \quad & (0,t) \times (0,t) \times (0,t) \to C_b(\mathbb{H}), \\ & (r_1, r_2, r_3) \mapsto P_{r_1}(P_{t-r_2}f)^\alpha(x_{r_3}). \end{aligned}$$

By the martingale expansion method, it is not hard to calculate

$$\frac{\partial H}{\partial r_1}, \quad \frac{\partial H}{\partial r_2}, \quad \frac{\partial H}{\partial r_3}.$$

Let $G(s) = H(s,s,s)$ for $s \in [0,t]$. We want to calculate $G'(s)$. For the diffusion case we have

$$G'(s) = \sum_{i=1}^{3} \frac{\partial H}{\partial r_i}(r_1, r_2, r_3)|_{r_1=r_2=r_3=s}.$$

But it is hard to prove the chain rule above for the jump case.

7.3 Approach by Using Measure Transformation on State Spaces

7.3.1 Main Theorem for Harnack Inequality

We still work under the framework introduced in Section 7.1 and consider the Lévy Ornstein-Uhlenbeck semigroup (7.3).

For the Gaussian case (i.e. $\nu = 0$), we know μ_t is a Gaussian measure. By using the Cameron-Martin formula for Gaussian measures, we proved a Harnack inequality (5.10).

For the Lévy case (i.e. $\nu \neq 0$), we still can use this method of measure transformation on state spaces if we know the Radon-Nikodým derivative of the infinite divisible measure with respect to its shifts. But unfortunately, there are only a few results on the densities. There are some sufficient conditions for the absolute continuity, "but formulae for the densities are not given, because none have been found (Gikhman and Skorokhod [GS66, Section 6, Page 121])."

For convenience, let us denote by $D(m, R, \nu)$ the infinite divisible measure with characteristic triplet (m, R, ν) on $(\mathbb{H}, \mathcal{B})$. That is,

$$\widehat{D(m, R, \nu)} = \exp\Big\{ i\langle u, m\rangle - \frac{1}{2}\langle Ru, u\rangle \\ - \int_{\mathbb{H}} \big[1 - \exp(i\langle x, u\rangle) + i\langle x, u\rangle \mathbb{1}_{\{|x|\le 1\}}(x)\big]\, \nu(dx). \Big\}$$

If $D(\gamma, R, \nu)$ is absolute continuous with respect to $D(0, R, \nu)$, then we will denote the Radon-Nikodým derivative of $D(\gamma, R, \nu)$ with respect to $D(0, R, \nu)$ by $p(\gamma, R, \nu, \cdot)$:

$$\frac{dD(\gamma, R, \nu)}{dD(0, R, \nu)}(x) = p(\gamma, R, \nu, x).$$

In terms of $p(\gamma, R, \nu, \cdot)$, we have the following results on the absolute continuity and Radon-Nikodým derivative of $D(b + \gamma, R, \nu)$ with respect to $D(b, R, \nu)$ for every $b \in \mathbb{H}$.

Proposition 7.3.1. *Suppose that $D(\gamma, R, \nu)$ is absolutely continuous with respect to $D(0, R, \nu)$. Then $D(b + \gamma, R, \nu)$ is absolutely continuous with respect to $D(b, R, \nu)$ and the Radon-Nikodým derivative is given by $p(\gamma, R, \nu, \cdot - b)$.*

Proof. For every $A \in \mathcal{B}(\mathbb{H})$, we have

$$\begin{aligned} D(b+\gamma, R, \nu)(A) &= \int_{\mathbb{H}} \mathbb{1}_A(x)\, D(b+\gamma, R, \nu)(dx) \\ &= \int_{\mathbb{H}} \mathbb{1}_A(x)\, \big(D(\gamma, R, \nu) * D(b, 0, 0)\big)(dx) \\ &= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_A(x+y)\, D(\gamma, R, \nu)(dx)\, D(b, 0, 0)(dy) \\ &= \int_{\mathbb{H}} \mathbb{1}_A(x+b)\, D(\gamma, R, \nu)(dx) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{H}} \mathbb{1}_A(x+b) p(\gamma, R, \nu, x)\, D(0, R, \nu)(dx) \\
&= \int_{\mathbb{H}} \mathbb{1}_A(x) p(\gamma, R, \nu, x-b)\, D(b, R, \nu)(dx).
\end{aligned}
$$

□

Now we can state the following Harnack inequality.

Theorem 7.3.2. *Let $x, y \in \mathbb{H}$. Suppose that the infinite divisible measure $D(S_t(x-y), Q_t, \nu_t)$ is absolute continuous with respect to the infinite divisible measure $D(0, Q_t, \nu_t)$. If there exists a $\beta > 1$ and $t > 0$ such that*

$$
\Phi_{t,\beta}(x-y) := \|p(S_t(x-y), Q_t, \nu_t, \cdot - b_t)\|_{L^\beta(\mathbb{H}, \mu_t)} < \infty,
$$

then

$$
(P_t f)^\alpha(x) \le \Phi_{t,\beta}(x-y)^\alpha P_t f^\alpha(y), \quad f \in \mathscr{C}_b^+(\mathbb{H}), \tag{7.12}
$$

where $\alpha = \frac{\beta}{\beta-1}$.

Proof. By the representation (7.3) of the Ornstein-Uhlenbeck semigroup, we have

$$
\begin{aligned}
P_t f(x) &= \int_{\mathbb{H}} f(S_t x + z)\, \mu_t(dz) \\
&= \int_{\mathbb{H}} f(S_t y + z) p(S_t(x-y), Q_t, \nu_t, z - b_t)\, \mu_t(dz) \\
&\le \left(\int_{\mathbb{H}} f^\alpha(S_t y + z)\, \mu_t(dz) \right)^{1/\alpha} \left(\int_{\mathbb{H}} p(S_t(x-y), Q_t, \nu_t, z - b_t)^\beta \mu_t(dz) \right)^{1/\beta} \\
&= \big(P_t f^\alpha(y)\big)^{1/\alpha} \Phi_{t,\beta}(x-y).
\end{aligned}
$$

This proves (7.12). □

Remark 7.3.3. [RW03a, Theorem 1.5] which is in terms of the Radon-Nikodým derivative $\eta_t(x, \cdot)$:

$$
\eta_t(x, z) := \frac{d\mu_t \circ \theta_{S_t x}^{-1}}{d\mu_t}(z).
$$

where for any $x \in \mathbb{H}$, θ_x is the shift operator $y \mapsto x + y$ for any $y \in \mathbb{H}$. In Theorem 7.3.2, we note the fact that μ_t is an infinite divisible measure and we base our theorem on the Radon-Nikodým derivative $p(\gamma, R, \nu, \cdot)$.

7.3.2 Harnack Inequalities for α-Stable Ornstein-Uhlenbeck Processes

Finite Dimensional α-Stable Ornstein-Uhlenbeck Processes

Let us consider the following stochastic differential equation

$$\begin{cases} dX_t = -\lambda X_t\,dt + dZ_t, \\ X_0 = x \in \mathbb{R}^d, \end{cases} \tag{7.13}$$

where Z_t is a symmetric α-stable process with index $\alpha \in (0,2)$, and $\lambda > 0$ is a constant.

The *mild solution* of (7.13) is given by

$$X_t = \mathrm{e}^{-\lambda t}\,x + \int_0^t \mathrm{e}^{\lambda(u-t)}\,dZ_u, \quad t \geq 0.$$

Denote the transition density of X_t by $p_\alpha(t,x,y)$. It has the following connection with $\hat{p}_\alpha(\cdot,\cdot,\cdot)$ (the transition density of $S\alpha S$ process) and $\hat{p}_\alpha(\cdot)$ (the density of α-stable random variable).

Proposition 7.3.4. *For every $\alpha \in (0,2)$ and $t > 0$, the transition density of X_t is given by*

$$\begin{aligned} p_\alpha(t,x,y) =& \hat{p}_\alpha\left(\frac{1-\mathrm{e}^{\alpha\lambda t}}{\alpha\lambda}, \mathrm{e}^{-\lambda t}\,x, y\right) \\ =& \hat{p}_\alpha\left(\frac{y-\mathrm{e}^{-\lambda t}\,x}{\left(\frac{1-\mathrm{e}^{\alpha\lambda t}}{\alpha\lambda}\right)^{1/\alpha}}\right). \end{aligned}$$

Proof. We only need to note that by Proposition 1.4.10, the characteristic exponent of $\int_0^t \mathrm{e}^{\lambda(u-t)}\,dZ_u$ is given by

$$\begin{aligned} &-\log \mathbb{E}\exp\left\{i\left\langle \xi, \int_0^t \mathrm{e}^{\lambda(u-t)}\ dZ_u\right\rangle\right\} \\ =& \int_0^t |\,\mathrm{e}^{\lambda(u-t)}\,\xi|^\alpha\,du = |\xi|^\alpha \cdot \frac{1-\mathrm{e}^{-\alpha\lambda t}}{\alpha\lambda} \end{aligned}$$

for every $\xi \in \mathbb{R}^d$ □

In terms of the transition density $p_\alpha(t,x,y)$, we can get the following Harnack inequality for the transition semigroup P_t of the α-stable Ornstein-Uhlenbeck process X_t.

Theorem 7.3.5. *For all $t > 0$, $p, q > 1$ with $1/p + 1/q = 1$, and $f \in \mathscr{C}_b^+(\mathbb{R}^d)$, the following inequality holds*

$$(P_t f)^p(x) \le \left[\int_{\mathbb{R}^d} \left(\frac{p_\alpha(t,x,z)}{p_\alpha(t,y,z)}\right)^q p_\alpha(t,y,z)\, dz\right]^{p/q} P_t f^p(y).$$

Proof.

$$\begin{aligned} P_t f(x) &= \int_{\mathbb{R}^d} f(z) p_\alpha(t,x,z)\, dz \\ &= \int_{\mathbb{R}^d} f(z) \frac{p_\alpha(t,x,z)}{p_\alpha(t,y,z)} p_\alpha(t,y,z)\, dz \\ &\le \left[\int_{\mathbb{R}^d} f^p(z) p_\alpha(t,y,z)\, dz\right]^{1/p} \left[\int_{\mathbb{R}^d} \left(\frac{p_\alpha(t,x,z)}{p_\alpha(t,y,z)}\right)^q p_\alpha(t,y,z)\, dz\right]^{1/q} \\ &= [P_t f^p(y)]^{1/p} \left[\int_{\mathbb{R}^d} \left(\frac{p_\alpha(t,x,z)}{p_\alpha(t,y,z)}\right)^q p_\alpha(t,y,z)\, dz\right]^{1/q}. \end{aligned}$$

□

By using the estimate (1.4.16) of $\hat{p}_\alpha(t,x,y)$ and the relation (1.4.15) between $p_\alpha(t,x,y)$ and $\hat{p}_\alpha(t,x,y)$ we have the following corollary.

Corollary 7.3.6. *For every $t > 0$, $p, q > 0$ with $1/p + 1/q = 1$. Let*

$$\begin{aligned} C_1 &= K^{2q+1} \int_{\mathbb{R}^d} \left(\frac{|\,\mathrm{e}^{-\lambda t} y - z|}{|\,\mathrm{e}^{-\lambda t} x - z|}\right)^{q(d+\alpha)} t_*^{-d/\alpha}\, dz, \\ C_2 &= K^{2q+1} \int_{\mathbb{R}^d} \left(\frac{|\,\mathrm{e}^{-\lambda t} y - z|}{|\,\mathrm{e}^{-\lambda t} x - z|}\right)^{q(d+\alpha)} \frac{t_*}{|\,\mathrm{e}^{-\lambda t} y - z|^{d/\alpha}}\, dz, \\ C_3 &= K^{2q+1} \int_{\mathbb{R}^d} \frac{t_*}{|\,\mathrm{e}^{-\lambda t} y - z|^{d/\alpha}}\, dz, \end{aligned}$$

where

$$t_* := \frac{1 - \mathrm{e}^{-\alpha\lambda t}}{\alpha\lambda}.$$

Let $C^{\alpha/q} = \min\{C_1, C_2, C_3\}$. Then

$$(P_t f)^p(x) \le C P_t f^p(y)$$

holds for every $f \in \mathscr{C}_b^+(\mathbb{R}^d)$.

Infinite Diagonal α-stable Ornstein-Uhlenbeck Processes

Let $\{e_n\}_{n\geq 1}$ be an orthogonal normal basis on $\mathbb{H}$. Let A be an self-adjoint operator on $\mathbb{H}$ with eigenvalue $-\lambda_j$ ($j \geq 1$) and associated eigenvector e_j. That is, we assume

$$Ae_j = -\lambda_j e_j, \quad j \geq 1.$$

Suppose that Z_t is an α-Stable process on $\mathbb{H}$. For every $j \geq 1$, set $Z_t^j = \langle Z_t, e_j\rangle$. We suppose that there exist some $\theta_j > 0$ such that

$$\mathbb{E}\, \mathrm{e}^{i\xi Z_t^j} = \mathrm{e}^{-t|\theta_j \xi|^\alpha}$$

for all $\xi \in \mathbb{R}$ and $j \geq 1$.

Consider the following equation

$$\begin{cases} dX_t = AX_t dt + dZ_t, \\ X_0 = x \in \mathbb{H}. \end{cases} \tag{7.14}$$

For any $j \geq 1$, denote by $X_t^j = \langle X_t, e_j\rangle$ and $x_t^j = \langle x, e_j\rangle$. Then the equation (7.14) is equivalent with the following system of equations on $\mathbb{R}$:

$$\begin{cases} dX_t^j = AX_t^j dt + dZ_t^j \\ X_0^j = x^j \in \mathbb{H}. \end{cases} \tag{7.15}$$

The transition density of X_t^j is given by

$$p_\alpha^j(t, x^j, y^j) = \hat{p}_\alpha(t_*^j, \mathrm{e}^{-\lambda_j t}\, x^j, y^j), \tag{7.16}$$

where

$$t_*^j := \frac{\theta_j^\alpha(1 - \mathrm{e}^{-\alpha\lambda_j t})}{\alpha\lambda_j}.$$

Denote by $\mathbb{H}_n := \mathrm{Span}\{e_1, e_2, \cdots, e_n\}$ the n-dimensional subspace of $\mathbb{H}$. Let the projection of X_t, x, Z_t to $\mathbb{H}_n$ be $X_t^{(n)}, x^{(n)}, Z_t^{(n)}$ respectively. Then the transition density of $X_t^{(n)}$ on $\mathbb{H}_n$ is

$$p_\alpha^{(n)}(t, x^{(n)}, y^{(n)}) = \prod_{j=1}^n p_\alpha^j(t, x^j, y^j) = \prod_{j=1}^n \hat{p}_\alpha(t_*^j, \mathrm{e}^{-\lambda_j t}\, x^j, y^j).$$

By Theorem 7.3.5, we have the following result for the transition semigroup

$P_t^{(n)}$ of $X_t^{(n)}$ on the subspace $\mathbb{H}_n$.

Lemma 7.3.7. *For every $t > 0$, $x^{(n)}, y^{(n)} \in \mathbb{H}_n$, $f \in \mathscr{C}_b^+(\mathbb{H}_n)$, and $p, q > 0$ satisfying $1/p + 1/q = 1$, the following inequality holds*

$$(P_t^{(n)} f)^p(x^{(n)}) \leq \left[\int_{\mathbb{R}^d} \left(\frac{p_\alpha^{(n)}(t, x^{(n)}, z)}{p_\alpha^{(n)}(t, y^{(n)}, z)}\right)^q p_\alpha^{(n)}(t, y^{(n)}, z)\, dz\right]^{p/q} P_t^{(n)} f^p(y^{(n)}).$$

By taking limit $n \to \infty$ in Lemma 7.3.7, we can obtain the following Harnack inequality for the transition semigroup P_t of X_t.

Theorem 7.3.8. *For every $t > 0$, $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$, and $p, q > 0$ satisfying $1/p + 1/q = 1$, the following inequality holds*

$$(P_t f)^p(x) \leq \lim_{n \to \infty} \left[\int_{\mathbb{R}^d} \left(\frac{p_\alpha^{(n)}(t, x^{(n)}, z)}{p_\alpha^{(n)}(t, y^{(n)}, z)}\right)^q p_\alpha^{(n)}(t, y^{(n)}, z)\, dz\right]^{p/q} P_t f^p(y).$$

7.3.3 Harnack Inequalities for Markov Chains

Let $(X_t)_{t \geq 0}$ be a homogeneous Markov chain (see, for instance, Norris [Nor98] for the background) with discrete state space $\mathbb{N}$. For every $t \geq 0$, $i, j \in \mathbb{N}$, denote by

$$p_t(i, j) = \mathbb{P}(X_{s+t} = j \mid X_s = i), \quad s \geq 0$$

the transition probability from state i to state j in time t. The transition semigroup of X_t is defined by

$$P_t f(i) = \sum_{k \in \mathbb{N}} p_t(i, k) f(k), \quad i \in \mathbb{N},$$

for every bounded measurable function f defined on $\mathbb{N}$.

We have the following result on Harnack inequality for Markov chain.

Theorem 7.3.9. *Assume the Markov chain is* irreducible. *That is, there exists some $t_0 \geq 0$ such that for all $t \geq t_0$ and every states $i, j \in \mathbb{N}$, $p_t(i, j) > 0$. Then*

$$\big(P_t f\big)^\alpha(i) \leq \left(\sum_{k \in \mathbb{N}} \left(\frac{p_t(i, k)}{p_t(j, k)}\right)^\beta p_t(j, k)\right)^{\alpha/\beta} P_t f^\alpha(j), \quad i, j \in \mathbb{N} \tag{7.17}$$

for every $t \geq t_0$, $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$, and every positive function f defined on $\mathbb{N}$.

Proof.

$$\begin{aligned}
P_t f(i) &= \sum_{k\in\mathbb{N}} p_t(i,k) f(k) = \sum_{k\in\mathbb{N}} \frac{p_t(i,k)}{p_t(j,k)} p_t(j,k) f(k) \\
&\leq \left(\sum_{k\in\mathbb{N}} \left(\frac{p_t(i,k)}{p_t(j,k)} \right)^{\beta} p_t(j,k) \right)^{1/\beta} \left(\sum_{k\in\mathbb{N}} f^{\alpha}(k) p_t(j,k) \right)^{1/\alpha} \\
&= \left(\sum_{k\in\mathbb{N}} \left(\frac{p_t(i,k)}{p_t(j,k)} \right)^{\beta} p_t(j,k) \right)^{1/\beta} \big(P_t f^{\alpha}(j)\big)^{1/\alpha}.
\end{aligned}$$

□

Especially, let $(X_n)_{n\in\mathbb{N}}$ be a discrete time Markov chain on a finite state space $S = \{1, 2, \ldots, l\}$. Denote by $P = \big(p_{ij}\big)_{l\times l}$ for the one-step transition matrix, where

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i), \quad n \in \mathbb{N}.$$

Denote the (i,j)-element of P^n by $p_{ij}^{(n)}$. We know $p_{ij}^{(n)}$ is the n-step transition probability from state i to state j:

$$p_{ij}^{(n)} = \mathbb{P}(X_{n+m} = j \mid X_m = i)$$

for every $m \in \mathbb{N}$.

By Theorem 7.3.9, we have the following Harnack inequality for X_n.

Corollary 7.3.10. *Suppose that for every $i, j \in S$, we have $p_{ij} > 0$ Then*

$$\big(P_n f\big)^{\alpha}(i) \leq \left(\sum_{k=1}^{l} \left(\frac{P_{i,k}^n}{P_{j,k}^n} \right)^{\beta} p_{j,k} \right)^{\alpha/\beta} P_n f^{\alpha}(j), \quad n \in \mathbb{N}, i, j \in S \tag{7.18}$$

holds for every $n \in \mathbb{N}$, $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$, and every positive function f defined on S.

Example 7.3.11. Let $P = \big(p_{ij}\big)_{l\times l}$ with $p_{ij} = 1/l$. Then by Corollary 7.3.10, for every $n \in \mathbb{N}$, $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$, and every function f defined on S, we have,

$$(P_n f)^{\alpha}(i) \leq P_n f^{\alpha}(j), \quad i, j = 1, 2, \ldots l.$$

7.4 Method of Coupling and Girsanov's Transformation

7.4.1 Harnack Inequalities: Using a Control Drift

Recall that our object is the following Ornstein-Uhlenbeck processes

$$dY_t = AY_t dt + dZ_t, \tag{7.19}$$

where A is the generator of a strongly continuous semigroup $(S_t)_{0\le t\le T}$, $(Z_t)_{0\le t\le T}$ is an $\mathbb{H}$-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ with characteristic triplet (b, R, ν).

Denote by P_t for the transition semigroup of the solution process associated with the stochastic equation (7.19). We first prove the following Harnack inequality for P_t.

Lemma 7.4.1. *Let $T > 0$ and $x, y \in \mathbb{H}$. Suppose that there is a control $\gamma \in L^2([0,T], \mathbb{H})$ of the following deterministic control system*

$$\begin{cases} dx_t = Ax_t\, dt + R^{1/2}\gamma_t\, dt, \\ x_0 = y - x, \end{cases} \tag{7.20}$$

such that $x_T = 0$. Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2}\int_0^T |\gamma_u|^2\, du\right) P_T f^\alpha(y). \tag{7.21}$$

for every function $f \in \mathscr{C}_b^+(\mathbb{H})$ and $\alpha, \beta > 0$ satisfying $1/\alpha + 1/\beta = 1$.

Proof. Denote by $(W_t)_{0\le t\le T}$ the Gaussian part of the Lévy process Z_t. It is known that $(W_t)_{0\le t\le T}$ is an R-Wiener process. Set

$$\tilde\gamma_t = R^{1/2}\gamma_t \in \mathbb{H}_0, \quad \text{for every } 0 \le t \le T.$$

Define

$$\rho_t = \exp\left(\int_0^t \langle\tilde\gamma_u, dW_u\rangle_0 - \frac{1}{2}\int_0^t |\tilde\gamma_u|_0^2\, du\right), \quad 0 \le t \le T.$$

It is clear that $(\rho_t)_{0\le t\le T}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$ since γ is square integrable. Hence, we can define a new probability measure $\widetilde{\mathbb{Q}}$ on $\mathscr{F}_T$ by

$$\widetilde{\mathbb{Q}} = \rho_T \mathbb{P}.$$

Now by the Girsanov theorem for Lévy processes (Theorem 2.2.2), we know the following drifted transformed process

$$\widetilde{Z}_t := Z_t - \int_0^t \tilde{\gamma}_u \, du, \quad 0 \le t \le T \tag{7.22}$$

is also a Lévy process with characteristic triplet (b, R, ν) with respect to $\widetilde{\mathbb{Q}}$.

We know the Ornstein-Uhlenbeck process

$$Y_t^y = S_t y + \int_0^t S_{t-u} \, dZ_u$$

solves Equation (7.19) with initial data $Y_0^y = y \in \mathbb{H}$. Hence for $f \in \mathscr{C}_b^+(\mathbb{H})$, we have

$$P_t f(y) = \mathbb{E}_{\mathbb{P}} f(Y_t^y). \tag{7.23}$$

Now we are going to make a drift transformation of

$$Y_t^x = S_t x + \int_0^t S_{t-u} \, dZ_u,$$

which solves Equation (7.19) with initial data $Y_0^x = x \in \mathbb{H}$.

Let us consider another Ornstein-Uhlenbeck process

$$X_t^x = S_t x + \int_0^t S_{t-u} \, d\widetilde{Z}_u \tag{7.24}$$

on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, \widetilde{\mathbb{Q}})$. Obviously we have

$$P_t f(x) = \mathbb{E}_{\widetilde{\mathbb{Q}}} f(X_t^x). \tag{7.25}$$

It is easy to see that X_t^x is a drift transformation of Y_t^x:

$$X_t^x = Y_t^x - \int_0^t S_{t-u} \tilde{\gamma}_u \, du, \quad , \quad 0 \le t \le T. \tag{7.26}$$

From this fact we get the the following relation between the processes X_t^x and

Y_t^y:

$$X_t^x = Y_t^y - x_t, \quad 0 \le t \le T. \tag{7.27}$$

The proof of (7.27) is easy. First, we solve the equation (7.20) and get

$$x_t = S_t(y-x) + \int_0^t S_{t-u}\tilde{\gamma}_u \, du. \tag{7.28}$$

Then we substitute (7.22) into (7.24) and use the fact (7.28), we see

$$\begin{aligned} X_t^x &= S_t x + \int_0^t S_{t-u}\, dZ_u - \int_0^t S_{t-u}\tilde{\gamma}_u \, du \\ &= S_t x + \int_0^t S_{t-u}\, dZ_u - x_t + S_t(y-x) \\ &= S_t y + \int_0^t S_{t-u}\, dZ_u - x_t \\ &= Y_t^y - x_t. \end{aligned}$$

It follows from (7.27) and the fact that $x_T = 0$, we know $X_T^x = Y_T^y$.

Intuitively, the procedure above means that by pulling down each trajectory of Y_t^x with quantity $\int_0^t S_{t-u}\tilde{\gamma}_u \, du$, we get X_t^x and it meets Y_t^y at time T. The main idea is shown in Figure 7.1.

By Hölder's inequality we get

$$\begin{aligned} \mathbb{P}_T f(x) = \mathbb{E}_{\widetilde{\mathbb{Q}}} f(X_T^x) &= \mathbb{E}_{\mathbb{P}} \rho_T f(Y_T^y) \\ &\le \big(\mathbb{E}_{\mathbb{P}} \rho_T^\beta\big)^{1/\beta} \big(\mathbb{E}_{\mathbb{P}} f^\alpha(Y_T^y)\big)^{1/\alpha} \\ &= \big(\mathbb{E}_{\mathbb{P}} \rho_T^\beta\big)^{1/\beta} \big(P_T f(y)\big)^{1/\alpha}. \end{aligned} \tag{7.29}$$

We can calculate the moment of ρ_T^β explicitly.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \rho_T^\beta &= \mathbb{E}_{\mathbb{P}} \exp\left\{\beta \int_0^T \langle \tilde{\gamma}_u, dW_u\rangle_0 - \frac{\beta}{2}\int_0^T |\tilde{\gamma}_u|_0^2 \, du\right\} \\ &= \exp\left\{\frac{\beta^2}{2}\int_0^T |\tilde{\gamma}_u|_0^2 \, du - \frac{\beta}{2}\int_0^T |\tilde{\gamma}_u|_0^2 \, du\right\} \\ &= \exp\left\{\frac{\beta(\beta-1)}{2}\int_0^T |\tilde{\gamma}_u|_0^2 \, du\right\}. \end{aligned} \tag{7.30}$$

Substitute the moment of ρ_T^β worked out in (7.30) into (7.29), we obtain the

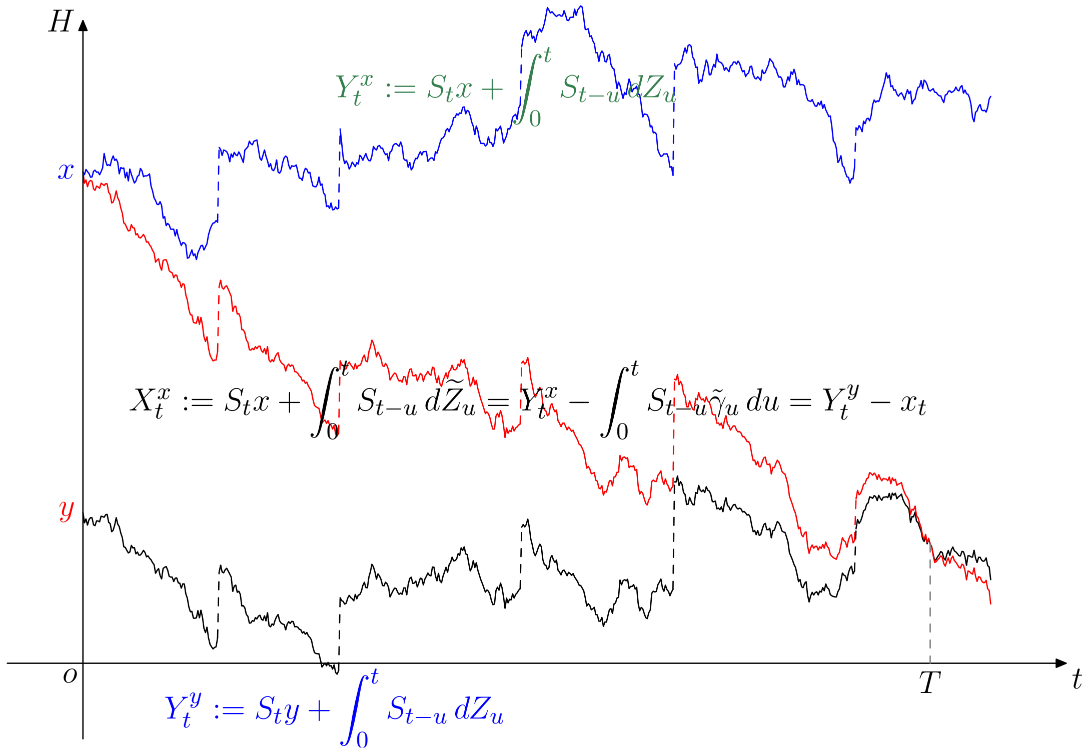

Figure 7.1: Coupling by Drift Transformation

following inequality

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2}\int_0^T |\tilde{\gamma}_u|_0^2\, du\right) P_T f^\alpha(y).$$

The proof is finished by noting that $|\tilde{\gamma}_t|_0 = |\gamma_t|$ for every $t \in [0, T]$. □

Remark 7.4.2. In the proof we use an explicit deterministic drift transformation for the process Y_t to obtain coupling. This is due to the linearity of the stochastic partial differential equation. Therefore we can use Girsanov's transformation which only involves the Gaussian part. If the drift is dependent on the jump part, we cannot apply this method. The reason is explained in Section 4.8. For nonlinear stochastic equations with jumps and Gaussian part, even for one dimensional stochastic differential equations, we are not able to find a proper drift which is independent of the jumps. So we are not able to prove Harnack inequalities use this method.

Remark 7.4.3. It is possible to consider Harnack inequalities for time-dependent Ornstein-Uhlenbeck processes (see [Knä09] and references therein) similarly.

7.4.2 Harnack Inequalities: Optimization Over All Drifts

We will deduce from Lemma 7.4.1 a theorem by taking infimum over all null control drifts.

As in Chapter 5, we use the following notation

$$Q_t := \int_0^t S_u R S_u^* \, du, \quad \Gamma_t := Q_t^{-1/2} S_t$$

for $0 \le t \le T$.

Theorem 7.4.4. *Let $T > 0$ and $x, y \in \mathbb{H}$. Suppose that*

$$S_T(y - x) \in Q_T^{1/2}(\mathbb{H}). \tag{7.31}$$

Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2} |\Gamma_T(x - y)|^2\right) P_T f^\alpha(y). \tag{7.32}$$

holds for every $f \in \mathscr{C}_b^+(\mathbb{H})$ and $\alpha, \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$.

Proof. We first note that (see Appendix A or [Zab08], [DPZ92, Appendix B] etc.) that condition (7.31) hold if and only if the control system (7.20) is null controllable. That is, there exists an $\mathbb{H}$-valued square integrable function $\gamma_t$on $[0, T]$ such that $y_T = 0$.

Moreover, $|\Gamma_t(x - y)|^2$ is the minimal energy for driving $x - y$ to 0:

$$|\Gamma_T(x - y)|^2 = \inf\left\{\int_0^T |\gamma_s|^2 \colon \gamma \in L^2([0, T], \mathbb{H}), y_T = 0\right\}. \tag{7.33}$$

By Lemma 7.4.1 we have inequality (7.21). The proof is completed by taking infimum over all possible choices of the control γ for (7.21) and using the expression (7.33). □

Remark 7.4.5. We have proved the Harnack inequality (7.32) for the Gaussian case (See Proposition 5.2.3) by using the Cameron-Martin formula. For the Gauss Ornstein-Uhlenbeck semigroup, the inequality (7.32) is optimal.

We have the following corollary.

Corollary 7.4.6. *Let $T > 0$. Suppose that*

$$S_T(\mathbb{H}) \subset Q_T^{1/2}(\mathbb{H}). \tag{7.34}$$

Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2}|\Gamma_T(x-y)|^2\right) P_T f^\alpha(y). \tag{7.35}$$

holds for every $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$ *and* $\alpha, \beta > 0$ *satisfying* $1/\alpha + 1/\beta = 1$.

7.4.3 Estimates of the Harnack Inequalities

The coefficient of the Harnack inequality (7.32) is simple but not so direct to compute. However, by taking any explicit choice of the control γ_t for the control system (7.20), we can get an upper bound estimation of $|\Gamma_t(x-y)|$ via the minimal energy representation (7.33). We refer to Subsection 5.2.2 for some estimates on $\|\Gamma_T\|$. In this way (or using Lemma 7.4.1 directly) we can get an explicit Harnack inequality.

The control γ_t naturally determines the behavior of the system x_t. In the following we are going to consider the controls such that the system behave in the following ways:

(1) There is some positive continuous function ξ_t on $[0, T]$ such that

$$x_t = \left(1 - \frac{\int_0^t \xi_u \, du}{\int_0^T \xi_u \, du}\right) S_t(y-x), \quad t \in [0, T]. \tag{7.36}$$

(2)

$$y_t = \left(1 - \frac{t}{T}\right)(y-x), \quad t \in [0, T]. \tag{7.37}$$

Note that the first choice (7.36) is a time-scaling of the following simple case

$$y_t = \left(1 - \frac{t}{T}\right) S_t(y-x), \quad t \in [0, T].$$

By taking the first choice we can obtain the following corollary.

Corollary 7.4.7. *Let* $T > 0$ *be a fixed constant. Suppose that* $S_T(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$. *Let* ξ *be a continuous positive function on* $[0, T]$. *Assume*

$$\int_0^T |S_u x|_0^2 \, \xi_u^2 \, du < \infty \quad \text{for all } x \in \mathbb{H}.$$

Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta \int_0^T |S_u(x-y)|_0^2 \, \xi_u^2 \, du}{2\left(\int_0^T \xi_u \, du\right)^2}\right) P_T f^\alpha(y) \tag{7.38}$$

holds for every $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$ *and* $\alpha, \beta > 1$ *with* $1/\alpha + 1/\beta = 1$.

Proof. Take

$$\gamma_u = R^{-1/2} S_u(x-y) \cdot \frac{\xi_u}{\int_0^T \xi_u \, du}, \quad u \in [0,T].$$

It is obvious that γ_u is a null control of the system (7.20) by noting the formula (7.28). Then we can finish the proof by applying Lemma 7.4.1. □

From Corollary 7.4.7 we have the following assertions if we assume further the estimates on $\|S_t\|$ for $t \in [0,T]$.

Corollary 7.4.8. *Assume the assumptions in Corollary 7.4.7 and*

$$|S_u z|_0 \le \sqrt{\xi(u)^{-1}}\, |x|_0, \quad x \in \mathbb{H},\ u \in [0,T].$$

Then

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta |x-y|_0^2}{2 \int_0^T \xi(u)\, du} \right) P_T f^\alpha(y). \tag{7.39}$$

holds for every $x, y \in \mathbb{H}$, $f \in \mathscr{C}_b^+(\mathbb{H})$, *and* $\alpha, \beta > 1$ *with* $1/\alpha + 1/\beta = 1$.

Remark 7.4.9. (1) Similar to the corollaries on Harnack inequality in Subsection 5.2.2, we can get more inequalities by using the estimates on $\|\Gamma_T\|$.
(2) The inequality (7.39) generalizes a Harnack inequality in [RW03a], where merely the case $\alpha = 2$ was proved. See Theorem 7.2.2 for the result in [RW03a]. Note that we used a condition which is slightly stronger. We refer to Item (2) of Remark 5.2.13 for the explanation of the difference.

By taking the second choice for the control which corresponds to (7.37) we have the following corollary. The point of this corollary is that the coefficient in the Harnack inequality is direct in terms of the operator A instead of the semigroup S_t.

Corollary 7.4.10. *Let* $x, y \in \mathbb{H}$. *Assume that* $x - y \in \mathbb{H}_0$ *and* $A(x-y) \in \mathbb{H}_0$. *For any* $T > 0$, $\alpha > 0, \beta > 0$ *satisfying* $1/\alpha + 1/\beta = 1$, $f \in \mathscr{C}_b^+(\mathbb{H})$, *we have*

$$(P_T f)^\alpha(x) \le \exp\left(\frac{\beta}{2} \int_0^T \left| \left[\left(\frac{t}{T} - 1 \right) A - \frac{1}{T} I \right] (x-y) \right|_0^2 dt \right) P_T f^\alpha(y). \tag{7.40}$$

Proof. Let

$$x_t = \left(1 - \frac{t}{T}\right)(y - x), \quad t \in [0, T].$$

Then $x_T = 0$ and y_t solves the null controllable problem (7.20) by setting

$$\int_0^t R^{1/2}\gamma_s \, ds = x_t - (y - x) - \int_0^t Ax_s \, ds$$

for all $t \in [0, T]$.

Now

$$\gamma_t = R^{-1/2}\left[\left(\frac{t}{T} - 1\right) A - \frac{1}{T} I\right](y - x).$$

By applying Lemma 7.4.1 we can prove the inequality (7.40). □

7.4.4 Examples

Consider the following *stochastic heat equation with Lévy noise*

$$dX_t = \Delta X_t dt + dZ_t \tag{7.41}$$

where Δ is the Laplacian on $(0, 1)$ with Dirichlet boundary condition, and Z_t is a Lévy process on (an extension of) $L^2((0, 1); dx)$ with symbol

$$\lambda(\xi) = |\xi|^\delta + |\xi|^2, \quad \xi \in \mathbb{H}$$

where $\delta \in (0, 2)$ is fixed.

The stochastic heat equation (7.41) was studied in [LR04, Section 8]. It was shown there that the equation (7.41) has a solution in the sense of [LR04, Theorem 7.3] (see [LR04, Corollary 8.2]).

Denote the transition semigroup of X_t by P_t. We have the following theorem on Harnack inequality for P_t. It is a generalization of [RW03a, Theorem 4.1] (see also [Wan04b, Section 7.3.3]) which only stated the case for $\alpha = 2$.

Theorem 7.4.11. *For all $t > 0$, $\alpha > 0, \beta > 0$ satisfying $1/\alpha + 1/\beta = 1$, $x, y \in \mathbb{H}$ and $f \in \mathscr{C}_b^+(\mathbb{H})$, we have*

$$(P_t f)^\alpha(x) \leq \exp\left(\frac{\beta\pi^2|x - y|^2}{2(e^{\pi^2 t} - 1)}\right) P_t f^\alpha(y). \tag{7.42}$$

Proof. The eigenvalues of Δ on $L^2((0,1))$ are $\lambda_k = -k^2\pi^2$, $k \geq 1$, each with multiplicity one. Hence we know the eigenvalues of $e^{t\Delta}$ are $e^{-tk^2\pi^2}$, $k \geq 1$. Therefore, we have $\| e^{t\Delta} \| \leq e^{-\pi^2 t}$ for every $t \geq 0$.

Note that we can modify the proof of Lemma 7.4.1 such that it also works for the (cylindrical) Lévy process Z_t. Hence we can use Corollary 7.4.8 for the stochastic heat equation. □

7.5 Applications of the Harnack Inequalities

7.5.1 Regularizing Property

Let P_t be the transition semigroup (7.3) of the Lévy driven Ornstein-Uhlenbeck processes (7.2) introduced in Section 7.1.

Recall that a transition semigroup P_t is called strongly Feller if for every $t \geq 0$ and every bounded measurable function f on $\mathbb{H}$, $P_t f$ is a continuous function on $\mathbb{H}$.

For the Gaussian case (i.e. $\nu = 0$), we have shown several equivalent statements for the strong Feller property of the semigroup in Theorem 5.3.3. In particular, we know the following statements are equivalent: (i) The semigroup P_t, $t \geq 0$, is strongly Feller. (ii) The Harnack inequality (5.10) holds for all $t \geq 0$, $x, y \in \mathbb{H}$ and $f \in \mathscr{C}_b(\mathbb{H})$. (iii) The following null controllability condition holds

$$S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H}), \quad t \geq 0. \tag{7.43}$$

For the Lévy case, the null controllability condition (7.43) still implies the strong Feller property of the transition semigroup. This result was proved by Röckner and Wang [RW03a, Corollary 1.2] (see also [Wan04b, Corollary 7.3.14]). They used the above mentioned result on strong Feller property for Gaussian Ornstein-Uhlenbeck semigroup.

Now according Da Prato et al. [DPRW09, Proposition 4.1] (see Proposition 5.3.2 in this thesis), we can apply Harnack inequality to prove a property stronger than the strong Feller property.

Theorem 7.5.1. *Let μ be the invariant measure of P_t. Suppose $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$ for some $t > 0$. Then for every $p > 1$, $P_t(L^p(H, \mu)) \subset \mathscr{C}(\mathbb{H})$.*

In the following, we will prove some estimate on the strong Feller property by using coupling method and Girsanov's theorem.

In stead of the strong Feller property, we would like to state the results for a generalization of the concept of strongly Feller property. This concept is also studied in [DPZ92, Subsection 11.2.3] for Gaussian Ornstein-Uhlenbeck semigroup.

Definition 7.5.2. A transition semigroup P_t is *strongly Feller at a moment* $t_0 > 0$, if for every $t \geq t_0$ and bounded measurable function f on $\mathbb{H}$, $P_t f$ is continuous.

We first prove a Lemma.

Lemma 7.5.3. *Assume that for some* $T \geq 0$ *and fixed* $x, y \in \mathbb{H}$,

$$S_T(x-y) \in Q_T^{1/2}(\mathbb{H}).$$

Then for every $f \in \mathscr{B}_b(\mathbb{H})$, *we have*

(1)

$$|P_T f(x) - P_T f(y)| \leq \|f\|_\infty \sqrt{C_{T,\gamma,x-y}} \exp(C_{T,\gamma,x-y}/2), \tag{7.44}$$

where $\|f\|_\infty$ *is the supremum norm of the function* f, *and*

$$C_{T,\gamma,x-y} = \int_0^T |\gamma_u|^2 \, du,$$

and γ_t *is a null control of the system* (7.20) *such that* $y_T = 0$.

(2)

$$|P_T f(x) - P_T f(y)| \leq \|f\|_\infty \, |\Gamma_T(x-y)| \exp(|\Gamma_T(x-y)|^2/2). \tag{7.45}$$

Proof. The second statement is a direct consequence of the first one by taking infimum over all choice of null control γ of the system (7.20) and using the representation (7.33). So we only need to prove (7.44).

Following the line in the proof of Lemma 7.4.1, we know

$$P_T f(x) = \mathbb{E}_{\mathbb{P}} \rho_T f(X_T^x) \quad \text{and} \quad P_T f(y) = \mathbb{E}_{\mathbb{P}} f(Y_T^y)$$

where

$$\rho_T = \exp\left(\int_0^T \langle \tilde{\gamma}_u, dW_u \rangle_0 - \frac{1}{2} \int_0^T |\tilde{\gamma}_u|_0^2 \, du \right)$$

with $\tilde{\gamma} = R^{1/2}\gamma$.

Then we have

$$\begin{aligned} |P_T f(x) - P_T f(y)| &= |\mathbb{E}_{\mathbb{P}} f(\rho_T X_T^x) - \mathbb{E}_{\mathbb{P}} f(Y_T^y)| \\ &= \mathbb{E}_{\mathbb{P}} |(\rho_T - 1) f(Y_T^y)| \leq \|f\|_\infty \mathbb{E}_{\mathbb{P}} |\rho_T - 1|. \end{aligned} \tag{7.46}$$

By (7.30), we know $\mathbb{E}_{\mathbb{P}}\rho_T^2 = \exp(C_{T,\gamma,x-y})$. Using the elementary inequality $\mathrm{e}^r - 1 \le r\,\mathrm{e}^r$ for all $r \ge 0$, we have

$$\begin{aligned}\big(\mathbb{E}_{\mathbb{P}}|\rho_T - 1|\big)^2 \le& \mathbb{E}_{\mathbb{P}}(\rho_T - 1)^2 = \mathbb{E}_{\mathbb{P}}\rho_T^2 - 1\\ =& \exp\big(C_{T,\gamma,x-y}\big) - 1 \le C_{T,\gamma,x-y}\exp(C_{T,\gamma,x-y}).\end{aligned}$$

The proof is completed by substitute the estimate above into (7.46). □

Now we can state the following theorem on the strong Feller property at a moment.

Theorem 7.5.4. *Assume that there exist a $t_0 > 0$ such that*

$$S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H}), \quad \text{for all } t \ge t_0.$$

Then P_t is strongly Feller at moment t_0. Moreover, we have the following estimate

$$|P_t f(x) - P_t f(y)| \le \|f\|_\infty\, |\Gamma_t(x-y)| \exp(|\Gamma_t(x-y)|^2/2).$$

for all $x, y \in \mathbb{H}$, $t \ge t_0$ and $f \in \mathscr{B}_b(\mathbb{H})$.

Remark 7.5.5. By (7.44) we can get explicit estimates of $|P_t f(x) - P_t f(y)|$ by choosing explicit null controls for the system (7.20).

7.5.2 Heat Kernel Bounds

We assume in this section that the Ornstein-Uhlenbeck processes are defined on $[0,\infty)$. We will apply the Harnack inequalities obtained in the previous sections to study norm bounds of the transition density.

We will need the following assumption.

Assumption 7.5.6. P_t has an invariant probability measure μ.

This assumption holds if the following conditions are satisfied (see [FR00, Theorem 3.1]):

(1) $\sup_{t>0} \operatorname{Tr} Q_t < \infty$;

(2)
$$\int_0^\infty dr \int_{\mathbb{H}} (1 \wedge |S_r x|^2)\,\nu(dx) < \infty,$$

(3) $b_\infty := \lim_{t\to\infty}(b_t^{(1)} + b_t^{(2)})$ exists in $\mathbb{H}$, where for any $t \geq 0$,

$$b_t^{(1)} = \int_0^t S_r b \, dr,$$
$$b_t^{(2)} = \int_0^t dr \int_{\mathbb{H}} S_r x \left(\mathbb{1}_B(S_r x) - \mathbb{1}_B(x)\right) \nu(dx),$$

where $B = \{x \in \mathbb{H} \colon |x| \leq 1\}$.

In this case the invariant measure is an infinite divisible measure with characteristic triplet $(b_\infty, Q_\infty, \nu_\infty)$. Here ν_∞ is given by

$$\nu_\infty = \int_0^\infty \nu \circ S_r^{-1} \, dr.$$

The following lemma is from [RW03a, Lemma 2.2]. We include the proof here for completeness.

Lemma 7.5.7. *Let E be a Polish space and $\mathscr{E}$ the Borel σ-algebra of E. Let P_t be a transition semigroup on $(E, \mathscr{E})$ with invariant measure μ. If there exists a constant $\alpha > 1$ and a measurable function $\Phi(x, y) : E \times E \to (0, \infty)$ such that*

$$|P_t f|^\alpha(x) \leq \Phi(x, y) P_t |f|^\alpha(y) \tag{7.47}$$

for every $x, y \in E$ and bounded measurable function f on E. Then the semigroup P_t has a transition density $p_t(x, y)$ with respect to μ.

Moreover, the transition density $p_t(x, y)$ satisfies the following estimate

$$\|p_t(x, \cdot)\|_{L^\beta(E,\mu)} \leq \left(\int_{\mathbb{H}} \frac{\mu(dy)}{\Phi(x, y)}\right)^{-1/\alpha} \tag{7.48}$$

for any $x \in E$, where $\beta = \dfrac{\alpha}{\alpha - 1}$.

Proof. Denote by $P_t(x, \cdot)$, $x \in E$, the transition probability measure corresponding to P_t. That is,

$$P_t(x, A) = P_t \mathbb{1}_A(x), \quad x \in E, \ A \in \mathscr{B}_b(E).$$

We first show that $P_t(x, \cdot)$ is absolutely continuous with respect to μ. Let

$A \in \mathcal{B}_b(E)$ with $\mu(A) = 0$. Inequality (7.47) implies

$$(P_t \mathbb{1}_A)^\alpha(x) \leq P_t \mathbb{1}_A(y) \Phi(x, y).$$

Hence by integrating both sides of the inequality above with respect to μ we can get

$$(P_t \mathbb{1}_A)^\alpha(x) \int_E \frac{\mu(dy)}{\Phi(x, y)} \leq \int_E P_t \mathbb{1}_A(y)\, \mu(dy) = \mu(A) = 0.$$

Therefore $P_t(x, A) = P_t \mathbb{1}_A(x) = 0$. This proves that $P_t(x, \cdot)$ is absolutely continuous with respect to μ.

By (7.47), for every bounded measurable function f on E, we have

$$|P_t f|^\alpha(x) \frac{1}{\Phi(x, y)} \leq P_t |f|^\alpha(y)$$

Integrate the inequality above with respect to $\mu(dy)$, we obtain

$$|P_t f|^\alpha(x) \int_{\mathbb{H}} \frac{\mu(dy)}{\Phi(x, y)} \leq \|f\|^\alpha_{L^\alpha(E,\mu)}.$$

Hence we have

$$\langle p_t(x, \cdot), f\rangle_{L^2(E,\mu)} = \int_E f(y) P_t(x, dy) = P_t f(x) \leq \|f\|_\alpha \left(\int_E \frac{\mu(dy)}{\Phi(x, y)} \right)^{-1/\alpha}.$$

Then the estimate (7.48) follows from the inequality above. □

Let P_t denotes the Lévy driven Ornstein-Uhlenbeck transition semigroup. By applying Theorem 7.4.4 and Lemma 7.5.7 we can obtain the following norm bound for the transition density of the Lévy driven Ornstein-Uhlenbeck transition semigroup.

Corollary 7.5.8. *Assume that $S_t(H) \subset Q_t^{1/2}(H)$ holds for every $t \geq 0$. Then P_t is strongly Feller. Hence $P_t(x, dy)$ has a density $p_t(x, y)$ with respect to μ. Moreover,*

$$\|p_t(x, \cdot)\|_{L^\beta(\mathbb{H},\mu)} \leq \left[\int_{\mathbb{H}} \exp\left(-\frac{\beta}{2} |\Gamma_t(x - y)|^2 \right) \mu(dy) \right]^{-1/\alpha}$$

holds for every $x \in \mathbb{H}$, $\alpha, \beta > 0$ satisfying $1/\alpha + 1/\beta = 1$.

By Corollary 7.4.7 and Lemma 7.5.7, we have the following result.

Corollary 7.5.9. *Suppose that $S_t(\mathbb{H}) \subset R^{1/2}(\mathbb{H})$ holds for all $t \in [0, +\infty)$. Then P_t is strongly Feller. Hence $P_t(x, dy)$ has a density $\rho(x, y)$ with respect to μ.*

Moreover,

$$\|p_t(x,\cdot)\|_{L^\beta(\mathbb{H},\mu)} \le \left[\int_{\mathbb{H}} \exp\left(-\frac{\beta}{2}\frac{\int_0^T |S_u(x-y)|_0^2\, \xi_u^2\, du}{\left(\int_0^T \xi_u\, du\right)^2}\right)\mu(dy)\right]^{-1/\alpha} \tag{7.49}$$

holds for every $x \in \mathbb{H}$, $\alpha, \beta > 0$ satisfying $1/\alpha + 1/\beta = 1$, and for every positive continuous function ξ on $[0, t]$.

Especially, if

$$|S_u x|_0 \le \sqrt{\xi(u)^{-1}}|x|_0, \quad u \in [0, T],$$

then

$$\|p_t(x,\cdot)\|_{L^\beta(\mathbb{H},\mu)} \le \left[\int_{\mathbb{H}} \exp\left(-\frac{\beta|x-y|_0^2}{2\int_0^T \xi(u)\, du}\right)\mu(dy)\right]^{-1/\alpha}.$$

Remark 7.5.10. In [RW03a, Corallary 1.2], only the case $\alpha = \beta = 2$ was studied.

7.5.3 Hyperboundedness

Let μ be a probabilistic measure on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$. Let $p, q \in (0, \infty)$. The operator norm of a linear bounded operator T from $L^p(\mathbb{H}, \mu)$ to $L^q(\mathbb{H}, \mu)$ is defined by

$$\|T\|_{p\to q} = \sup\{\|Tf\|_q \colon \|f\|_p = 1\}.$$

We say that T is hyperbounded if $\|T\|_{p\to q} < \infty$ for some $1 < p < q < \infty$.

If the operator T is contractive on $L^1(\mathbb{H}, \mu)$, then by the Riesz-Thorin interpolation theorem (see for example, [Dav89, Page 3]), T is hyperbounded if and only if $\|T\|_{2\to 4} < \infty$.

Hyperboundedness is a useful concept. For example, for a strongly continuous symmetric diffusion semigroup of contraction, we can deduce a defective Logarithmic Sobolev inequality from the hyperboundedness of the semigroup. We refer to [Wan04b] and references therein for more information.

In the following, we are going to consider the hyperboundedness of the Lévy driven Ornstein-Uhlenbeck semigroup defined by (7.3). We assume that the invariant measure of P_t exist and denote it by μ.

For completeness, we first recall two assertions from [RW03a]. The following proposition is essentially from [RW03a, Theorem 1.5].

Proposition 7.5.11. *Consider the situation of Theorem 7.3.2. If there exists a* $\varepsilon > 0$ *such that*

$$C(t,\beta,\varepsilon) := \int_{\mathbb{H}} \left(\int_{\mathbb{H}} \Phi_{t,\beta}(x-y)^{-\alpha}\mu(dy) \right)^{-(1+\varepsilon)} \mu(dx),$$

then

$$\|P_t\|_{\alpha \mapsto (1+\varepsilon)\alpha} \le C(t,\beta,\varepsilon)^{1/(1+\varepsilon)\alpha}. \tag{7.50}$$

Proof. The proof of (7.50) is the same as in the proof of [RW03a, Theorem 1.5]. We include it in the following. For every $f \in L^\alpha(\mathbb{H},\mu)$, by (7.12), we have

$$(P_t f)^\alpha(x) \le \Phi_{t,\beta}(x-y)^\alpha P_t f^\alpha(y), \tag{7.51}$$

Suppose $\mu(|f|^\alpha) = 1$ and integrate both sides of (7.51) with respect to $\mu(dy)$, we have

$$|P_t|^\alpha f(x) \int_{\mathbb{H}} \Phi_{t,\beta}(x-y)^{-\alpha}\, \mu(dy) \le 1.$$

Hence

$$|P_t f|^{\alpha(1+\varepsilon)}(x) \le \left(\int_{\mathbb{H}} \Phi_{t,\beta}(x-y)^{-\alpha}\, \mu(dy) \right)^{-(1+\varepsilon)}.$$

By integrating the inequality above with respect to $\mu(dx)$, (7.50) follows immediately. □

We will work with the following null controllability condition

$$S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H}) \tag{7.52}$$

for some $t > 0$. The following proposition is from [RW03a, Proposition 1.6].

Proposition 7.5.12. *Consider the situation of Theorem 7.3.2. If* (7.52) *and* $C(t,\beta,0) < \infty$ *hold for some* $t > 0$ *and* $\beta \in (1,\infty]$*, then* P_s *is compact on* $L^\alpha(\mu)$ *for every* $s > t$.

Proof. By (7.12), for every f with $\|f\|_\alpha = 1$, we have

$$|P_t f|^{\alpha(1+\varepsilon)}(x) \le \left(\int_{\mathbb{H}} \Phi_{t,\beta}(x-y)^{-\alpha}\, \mu(dy) \right)^{-(1+\varepsilon)}.$$

If $C(t,\beta,0) < \infty$, then $\{P_t f\colon \|f\|_\alpha \le 1\}$ is uniformly integrable in $L^\alpha(\mathbb{H},\mu)$. Moreover, by Corollary 7.5.8, we know P_t has a density with respect to μ, hence by [GW02, Lemma 3.1], it follows that P_s is compact in $L^p(\mathbb{H},\mu)$ for $s > t$. □

Similar to Propositions 7.5.11 and 7.5.12, by using Harnack inequality (7.35), we can state the following assertion.

Proposition 7.5.13. *Assume* (7.52) *hold for some* $t > 0$. *Let* $\alpha, \beta > 1$ *satisfying* $1/\alpha + 1/\beta = 1$. *If there exist some* $\varepsilon > 0$ *such that*

$$\widetilde{C}(t,\beta,\varepsilon) := \int_{\mathbb{H}} \left[\int_{\mathbb{H}} \exp\left(-\frac{\beta}{2}|\Gamma_t(x-y)|^2\right)\mu(dy)\right]^{-(1+\varepsilon)} \mu(dx) < \infty, \qquad (7.53)$$

then

$$\|P_t\|_{\alpha\to(1+\varepsilon)\alpha} \le \widetilde{C}(t,\beta,\varepsilon)^{\frac{1}{\alpha(1+\varepsilon)}}.$$

Especially if $\widetilde{C}(t,\beta,0) < \infty$ *for some* $t > 0$, *then* P_s *is compact on* $L^\alpha(\mathbb{H},\mu)$ *for every* $s > t$.

By (7.52), we know $\Gamma_t = Q_t^{-1/2} S_t$ is a bounded operator on $\mathbb{H}$ and there exist some $C(t) > 0$ such that (refer to Subsection 5.2.2)

$$\|\Gamma_t\| \le \sqrt{C(t)}, \quad t \ge 0. \qquad (7.54)$$

We will use the assumption (7.54) to study the integrability condition (7.53).

Proposition 7.5.14. *Assume* (7.54). *Let* $r(x)$ *be a positive measurable function on* $\mathbb{H}$. *Suppose that for some* $\varepsilon > 0$ *and* $t > 0$,

$$\int_{\mathbb{H}} \frac{1}{\left[\mu\big(B_{r(x)}(x)\big)\right]^{(1+\varepsilon)}} \exp\left(\frac{\beta(1+\varepsilon)}{2} C(t) r(x)^2\right)\mu(dx) < \infty,$$

where $B_r(x) = \{x \in \mathbb{H}\colon |x| \le r\}$ *for any* $x \in \mathbb{H}$ *and* $r > 0$. *Then* (7.53) *hold. Especially, it is the case if*

$$\int_{\mathbb{H}} \frac{\mu(dx)}{\left[\mu\big(B_r(x)\big)\right]^{(1+\varepsilon)}} < \infty$$

for some $r > 0$.

Proof.

$$\begin{aligned}
&\int_{\mathbb{H}} \left[\int_{\mathbb{H}} \exp\left(-\frac{\beta}{2}|\Gamma_T(x-y)|^2\right)\mu(dy)\right]^{-(1+\varepsilon)} \mu(dx) \\
\leq &\int_{\mathbb{H}} \left[\int_{B_{r(x)}(x)} \exp\left(-\frac{\beta}{2}C(t)r(x)^2\right)\mu(dy)\right]^{-(1+\varepsilon)} \mu(dx) \\
= &\int_{\mathbb{H}} \frac{1}{\left[\mu\big(B_{r(x)}(x)\big)\right]^{(1+\varepsilon)}} \exp\left(\frac{\beta(1+\varepsilon)}{2}C(t)r(x)^2\right)\mu(dx) \\
< &\infty.
\end{aligned}$$

□

In the following, we intend to look at the integrability condition (7.53) by using the structure of μ. Recall that μ is an infinite divisible measure with characteristic triplet $(b_\infty, R_\infty, \nu_\infty)$. Refer to Subsection 7.5.2 for the structure of μ.

We denote by $\mu^{(1)}$, $\mu^{(2)}$ for the infinite divisible measure with characteristic triplet $(b_\infty, R_\infty, 0)$ and $(0, 0, \nu_\infty)$ respectively. That is, $\mu^{(1)} = N(b_\infty, R_\infty)$ is a Gaussian measure and $\mu^{(2)} = D(0, 0, \nu_\infty)$ is an infinite divisible measure with

$$\widehat{\mu^{(2)}}(u) = \int_{\mathbb{H}} \left[1 - \exp(i\langle z, u\rangle) + i\langle z, u\rangle \mathbb{1}_{\{|z|\leq 1\}}(z)\right] \nu_\infty(dz).$$

By the well known Fernique's Theorem (see for example, [DPZ92, Proposition 2.16]), there exist some $\delta > 0$ such that

$$\int_{\mathbb{H}} \exp(\delta|x|^2)\,\mu^{(1)}(dx) < \infty. \tag{7.55}$$

In fact we can take any $\delta \in (0, \delta_{R_\infty})$ with

$$\delta_{R_\infty} = \inf_{\lambda\in\sigma(R_\infty)} \frac{1}{2\lambda} = \frac{1}{\|R_\infty\|}.$$

Here $\sigma(R_\infty)$ is the spectrum of R_∞.

From the integrability of $\mu^{(1)}$ and $\mu^{(2)}$ we can get the integrability of μ easily.

Lemma 7.5.15. *Assume that*

$$\int_{\mathbb{H}} \exp\left(\delta|x|^2\right)\,\mu^{(2)}(dx) < \infty. \tag{7.56}$$

Then

$$\int_{\mathbb{H}} \exp\left(\frac{\delta}{2}|x|^2\right) \mu(dx) < \infty.$$

Proof.

$$\begin{aligned}\int_{\mathbb{H}} \exp\left(\frac{\delta}{2}|x|^2\right) \mu(dx) &= \int_{\mathbb{H}}\int_{\mathbb{H}} \exp\left(\frac{\delta}{2}|x+y|^2\right) \mu^{(1)}(dx)\, \mu^{(2)}(dy) \\ &\le \int_{\mathbb{H}}\int_{\mathbb{H}} \exp\big(\delta(|x|^2+|y|^2)\big)\, \mu^{(1)}(dx)\, \mu^{(2)}(dy) \\ &= \int_{\mathbb{H}} \exp(\delta|x|^2)\, \mu^{(1)}(dx) \int_{\mathbb{H}} \exp(\delta|y|^2)\, \mu^{(2)}(dy) \\ &< \infty.\end{aligned}$$

□

Now we can prove the following theorem on the hyperboundedness of the semigroup P_t.

Theorem 7.5.16. *Assume* (7.52), (7.54) *and* (7.56). *Then* $\|P_t\|_{p\to q(t)} < \infty$ *hold for every* $p > 1$, $t > 0$ *and* $q(t) = \dfrac{\delta(p-1)}{2C(t)}$.

Proof. Let $f \in L^p(\mathbb{H}, \mu)$ with $\|f\|_p = 1$. By Theorem 7.4.4, we have

$$(P_t f)^p(x) \exp\left(-\frac{p}{2(p-1)}|\Gamma_t(x-y)|^2\right) \le P_t f^p(y) \tag{7.57}$$

for every $x, y \in \mathbb{H}$ and $t \ge 0$.

By using (7.54) we see

$$|\Gamma_t(x-y)|^2 \le C(t)|x-y|^2 \le 2C(t)(|x|^2+|y|^2).$$

Therefore, we can deduce from the inequality (7.57) to get

$$(P_t f)^p(x) \exp\left(-\frac{pC(t)}{p-1}\big(|x|^2+|y|^2\big)\right) \le P_t f^p(y).$$

By integrating both sides of the inequality above with respect to $\mu(dy)$ over the ball $B_1(0) := \{x \in \mathbb{H} \colon |x| \le 1\}$, we can obtain

$$(P_t f)^p(x) \exp\left(-\frac{pC(t)}{p-1}\big(1+|x|^2\big)\right) \mu\big(B_1(0)\big) \le \mu(P_t f^p).$$

Note that μ is an invariant measure of P_t, we have

$$(P_t f)^p(x) \le \big[\mu\big(B_1(0)\big)\big]^{-1} \exp\left(\frac{pC(t)}{p-1}\big(1+|x|^2\big)\right).$$

Taking power $\frac{q(t)}{p}$ and then integrating with respect to $\mu(dx)$ for both sides of the inequality above, we get

$$\begin{aligned}\|P_t\|_{q(t)}^{q(t)} &\le \big[\mu\big(B_1(0)\big)\big]^{-1/p} \int_{\mathbb{H}} \exp\left(\frac{q(t)C(t)}{p-1}\big(1+|x|^2\big)\right) \mu(dx)\\ &= \big[\mu\big(B_1(0)\big)\big]^{-1/p} \int_{\mathbb{H}} \exp\left(\frac{\delta}{2}\big(1+|x|^2\big)\right) \mu(dx)\\ &< \infty.\end{aligned}$$

This finishes the proof. □

Chapter 8

Harnack Inequalities for Multivalued Stochastic Equations

The gradient system considered in Subsection 5.5.2 inspires us to consider Harnack inequalities for the transition semigroups associated with general multivalued stochastic equations.

Recently, multivalued stochastic equations have attracted the interest of many researchers. For historic notes and more information about multivalued stochastic differential equations we refer to Krée [Kré82], Cépa [Cép94, Cép95, Cép98], Bensoussan and Rascanu [BR97], Cépa and Lépingle [CL97] and [Zha07] etc..

In this chapter, we first give a general introduction to multivalued maximal monotone operators in Section 8.1. Then we study Harnack inequalities for multivalued stochastic differential equations in finite dimension in Section 8.2.

We devote the remaining sections to multivalued stochastic evolution equations in Banach spaces. In Section 8.3 we recall the existence and uniqueness theorem for the evolution equations due to Zhang [Zha07]. Zhang [Zha07, Theorem 5.8] has proved finiteness second moment of the invariant measure of the transition semigroup associated with evolution equations. In Section 8.4 we prove stronger concentration properties of the invariant measure.

We study Harnack inequalities in Section 8.5 and their applications in Section 8.6 for the transition semigroups associated with evolution equation. In particular, we prove the invariant measure is fully supported on the domain of the underlying multivalued maximal monotone operator; and we also prove the strong Feller property, the hyperboundedness, ultraboundedness and compactness for the transition semigroup.

8.1 Multivalued Maximal Monotone Operator

Denote by $2^{\mathbb{H}}$ for the set of all subsets of $\mathbb{H}$. Let $A\colon \mathbb{H} \to 2^{\mathbb{H}}$ be a set-valued operator*. Define the *domain* of A by

$$D(A) = \{x \in \mathbb{H}\colon Ax \neq \emptyset\}.$$

The multivalued operator A can be characterized by its *graph* defined by

$$\mathrm{Gr}(A) = \{(x, y) \in \mathbb{H} \times \mathbb{H}\colon x \in \mathbb{H}, y \in Ax\}.$$

Definition 8.1.1. (1) A multivalued operator A on $\mathbb{H}$ is called *monotone* if

$$\langle x_1 - y_1, x_2 - y_2\rangle \geq 0, \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathrm{Gr}(A).$$

(2) A monotone operator A is called *maximal monotone* if it must be $(x_1, y_1) \in \mathrm{Gr}(A)$ for any $(x_1, y_1) \in \mathbb{H} \times \mathbb{H}$ satisfying the following property:

$$\langle x_1 - x_2, y_1 - y_2\rangle \geq 0, \quad \text{for all } (x_2, y_2) \in \mathrm{Gr}(A).$$

That is, A is maximal monotone if $\mathrm{Gr}(A)$ is not contained in the graph of any other monotone operator.

The following is a fundamental example of a maximal monotone operator.

Example 8.1.2. Let $U\colon \mathbb{H} \to (-\infty, \infty]$ be a lower semi-continuous convex function on $\mathbb{H}$ such that its domain

$$D(U) = \{x \in \mathbb{H}\colon U(x) < \infty\}$$

is not empty.

We define the *sub-differential* of U by

$$\partial U(x) = \{y \in \mathbb{H}\colon U(x) \leq U(z) + \langle y, x - z\rangle,\ z \in \mathbb{H}\}.$$

Then we see ∂U is a maximal monotone operator on $\mathbb{H}$.

We refer the reader to Brézis [Bré73] for more details of maximal monotone operators. We only note in the following some properties about $\overline{D(A)}$ since it

*We used A to denote the generator of S_t in the previous chapters. We use A here for convenience.

will be the state space of solutions of the multivalued stochastic equations we will consider. We know $\overline{D(A)}$ is a closed and convex subset of $\mathbb{H}$. It is a complete and separable metric space under the norm $|\cdot|_{\mathbb{H}}$.

8.2 Harnack Inequalities for Multivalued Stochastic Differential Equations

Consider the following multivalued stochastic differential equation

$$dX_t + AX_t dt \ni b(X_t)dt + \sigma(X_t)\, dW_t, \quad X_0 = x \in \overline{D(A)}, \tag{8.1}$$

where A is a maximal monotone operator on $\mathbb{R}^d$ with $D(A)^o \neq \emptyset$, W_t is a Wiener process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous.

We first formulate the definition of the solution of the multivalued stochastic differential equation (8.1)

Definition 8.2.1. A pair of continuous $\mathscr{F}_t$-adapted process (X, K) is called a solution of (8.1) if

(1) $X_0 = x \in \overline{D(A)}$, $X_t \in \overline{D(A)}$, $\mathbb{P}$-a.s.;
(2) $X_0 = 0$ and K is of locally finite variation;
(3) X_t is a solution of the following stochastic differential equation

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t - dK_t, \quad 0 \leq t < \infty$$

with initial condition $X_0 = x$;
(4) For every continuous $\mathscr{F}_t$-adapted function (p, q) with $(p_t, q_t) \in \mathrm{Gr}(A)$ for all $t \geq 0$, the measure

$$\langle X_t - p_t, dK_t - q_t\, dt\rangle \geq 0, \quad \mathbb{P}\text{-a.s.}.$$

The following proposition will play a basic role.

Proposition 8.2.2. *Let A be a multivalued maximal monotone operator and (X, K), (X', K') be continous functions with $X, X' \in \overline{D(A)}$, K, K' be of finite variation. Let (p, q) be contiuous functions satisfying*

$$(p_t, q_t) \in \mathrm{Gr}(A) \quad \text{for all } t \geq 0.$$

If

$$\langle X_t - p_t, dK_t - q_t\,dt\rangle \geq 0 \quad \text{and} \quad \langle X_t' - p_t, dK_t' - q_t\,dt\rangle \geq 0,$$

then

$$\langle X_t - X_t', dK_t - dK_t'\rangle \geq 0.$$

We consider the multivalued stochastic differential equation (8.1) with the following assumption.

Assumption 8.2.3. (1) $\sigma \equiv I$ is the unit operator on $\mathbb{R}^d$;
(2) There exist some $K \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$,

$$\langle x - y, b(x) - b(y)\rangle \leq \omega |x - y|^2. \tag{8.2}$$

Then by [RWZ08, Thorem 2.8], the solution X_t exists. The associated transition semigroup is given by

$$P_t f(x) = \mathbb{E}_{\mathbb{P}} f(X_t), \quad t \geq 0,\ f \in \mathscr{B}_b(\mathbb{R}^d).$$

We will use the coupling method and Girsanov transformation to study the Harnack inequalities for P_t. We note that Ren et. al. [RWZ08] used this method and apply it to study the ergodicity of multivalued stochastic differential equations.

Theorem 8.2.4. *Suppose Assumption 8.2.3 hold. Then*

$$(P_T f)^\alpha(x) \leq \exp\left(\frac{\beta\omega|x-y|^2}{1 - \mathrm{e}^{-2\omega T}}\right) P_T f^\alpha(y). \tag{8.3}$$

holds for every $x, y \in \overline{D(A)}$, $T > 0$, $f \in \mathscr{C}_b^+(\overline{D(A)})$ *and* $\alpha, \beta > 1$ *with* $1/\alpha + 1/\beta = 1$.

Proof. We turn to consider the following coupled multivalued stochastic differential equation

$$\begin{cases} dX_t + AX_t\,dt \ni dW_t + b(X_t)\,dt - \xi_t|x-y|\dfrac{X_t - Y_t}{|X_t - Y_t|}\mathbb{1}_{\{t<\tau\}}\,dt, & \text{(8.4a)} \\ dY_t + AY_t\,dt \ni dW_t + b(Y_t)\,dt, & \text{(8.4b)} \end{cases}$$

with initial data $X_0 = x$ and $Y_0 = y$, where W_t is a Wiener process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, τ is the coupling time of X_t and Y_t defined by $\tau = \inf\{t > 0\colon |X_t - Y_t| = 0\}$, and ξ_t is a deterministic positive continuous function satisfying

$$\int_0^T \xi_s \,\mathrm{e}^{-\omega s}\, ds \geq 1.$$

Note that for any $u, v \in \mathbb{R}^d \setminus \{0\}$ we have

$$\begin{aligned}\left|\frac{u}{|u|} - \frac{v}{|v|}\right| &= \frac{1}{|u||v|}\big||v|u - |v|v + |v|v - |u|v\big| \\ &\leq \frac{1}{|u||v|} \cdot 2|v|\,|u - v| = \frac{2}{|u|}|u - v|.\end{aligned}$$

We see the function

$$\mathbb{R}^d \times \mathbb{R}^d \ni (u, v) \mapsto \frac{u - v}{|u - v|}$$

is bounded and locally Lipschitz off the diagonal $\mathbb{R}^d \times \mathbb{R}^d$.

Hence by [RWZ08, Thorem 2.8], we know the equation (8.4) has a solution up to the coupling time τ. That is, there exist continuous processes (X, K) and $(Y, \widetilde{K})$ up to τ satisfying

$$\begin{cases} dX_t = dW_t + b(X_t)\, dt - dK_t - \xi_t |x - y| \dfrac{X_t - Y_t}{|X_t - Y_t|} \mathbb{1}_{\{t<\tau\}}\, dt, & \text{(8.5a)} \\ dY_t = dW_t + b(Y_t)\, dt - d\widetilde{K}_t, & \text{(8.5b)} \end{cases}$$

for $t < \tau$ with initial values $X_0 = x$ and $Y_0 = y$. But it is clear that the solution to Equation (8.4b) (or (8.5b) equivalently) can be extended to all time $t \geq 0$ which is still denoted by $(Y, \widetilde{K})$. Then the solution of (8.4a) (or (8.5a)) can be defined in the following way:

$$X_t := Y_t, \quad K_t := \widetilde{K}_t$$

for all $t \geq \tau$.

First applying Itô's formula to $\sqrt{|X_t - Y_t|^2 + \varepsilon}$ and then letting $\varepsilon \downarrow 0$, by using the assumption (8.2) and Proposition 8.2.2 we can obtain for all $t < \tau$

$$\begin{aligned} d|X_t - Y_t| \leq & \left\langle \frac{X_t - Y_t}{|X_t - Y_t|}, b(X_t) - b(Y_t) - \xi_t |x - y| \frac{X_t - Y_t}{|X_t - Y_t|} \right\rangle dt \\ & - \left\langle \frac{X_t - Y_t}{|X_t - Y_t|}, dK_t - d\widetilde{K}_t \right\rangle dt \\ \leq & \omega |X_t - Y_t|\, dt - \xi_t |x - y|\, dt. \end{aligned}$$

Therefore,

$$d\left(|X_t - Y_t|\, \mathrm{e}^{-\omega t}\right) \le -\xi_t |x-y|\, \mathrm{e}^{-\omega t}\, dt, \quad t < \tau.$$

If $T < \tau$, then by integrating both sides of the inequality above from 0 to T we get

$$0 < |X_T - Y_T|\, \mathrm{e}^{-\omega T} \le |x-y| \left(1 - \int_0^T \xi_t\, \mathrm{e}^{-\omega t}\, dt\right) \le 0.$$

This contradiction implies that $T \ge \tau$ and hence we must have

$$X_T = Y_T. \tag{8.6}$$

As in Section 4.3, for every $0 \le t \le T$, set

$$N_t = \int_0^{t\wedge\tau} \left\langle \xi_t |x-y| \frac{X_t - Y_t}{|X_t - Y_t|}, dW_s \right\rangle,$$

and

$$R_t = \exp\left(N_t - \frac{1}{2}[N]_t\right).$$

It is obvious that $\mathbb{E}(R_T) = 1$. So we can define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}_T)$ by setting $\mathbb{Q}|_{\mathscr{F}_T} = R_T \mathbb{P}$.

By Girsanov's theorem, we know

$$\widetilde{W}_t := W_t - \int_0^t \xi_s |x-y| \frac{X_s - Y_s}{|X_s - Y_s|} \mathbb{1}_{\{t<\tau\}}\, ds$$

for $t \in [0,T]$ is still a Wiener process on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{Q})$. Therefore, on the new probability space $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{Q})$, the process X_t also solves the equation

$$dX_t + AX_t dt \ni d\widetilde{W}_t + b(X_t)\, dt, \quad X_0 = x.$$

By the uniqueness of the solution of the equation, we obtain $P_T f(x) = \mathbb{E}_{\mathbb{Q}} f(X_T)$. Combining (8.6) with the obvious fact $P_T f(y) = \mathbb{E}_{\mathbb{P}} f(Y_T)$, and using Hölder's inequality, we have

$$\begin{aligned} P_T f(x) &= \mathbb{E}_{\mathbb{Q}} f(X_T) = \mathbb{E}_{\mathbb{Q}} f(Y_T) = \mathbb{E}_{\mathbb{P}} R_T f(Y_T) \\ &\le \left(\mathbb{E}_{\mathbb{P}} R_T^\beta\right)^{1/\beta} \left(\mathbb{E}_{\mathbb{P}} f^\alpha(Y_T)\right)^{1/\alpha} = \left(\mathbb{E}_{\mathbb{P}} R_T^\beta\right)^{1/\beta} \left(P_T f^\alpha(y)\right)^{1/\alpha}. \end{aligned} \tag{8.7}$$

Since $(R_t)_{t\in[0,T]}$ is a $\mathscr{F}_t$-martingale with respect to $\mathbb{P}$, we have (refer to (4.21))

$$\mathbb{E}_{\mathbb{P}} R_T^{\beta} \leq \left[\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2}\beta q(\beta p - 1)[N]_T \right) \right]^{1/q}. \tag{8.8}$$

where $p, q > 1$ satisfying $1/p + 1/q = 1$.

Note that

$$[N]_T = |x-y|^2 \int_0^T \xi_t^2 \, dt,$$

we can deduce from (8.8) by by letting p go to 1 to obtain

$$\mathbb{E}_{\mathbb{P}} R_T^{\beta} \leq \exp\left(\frac{1}{2}\beta(\beta - 1)|x-y|^2 \int_0^T \xi_t^2 \, dt \right). \tag{8.9}$$

Substitute (8.9) into (8.7) we have

$$(P_T f)^{\alpha}(x) \leq \exp\left(\frac{\beta |x-y|^2}{2} \int_0^T \xi_s^2 \, ds \right) P_T f^{\alpha}(y). \tag{8.10}$$

Now we can get (8.3) by taking

$$\xi_t = \frac{\mathrm{e}^{-\omega t}}{\int_0^T \mathrm{e}^{-2\omega s} \, ds}, \quad 0 \leq t \leq T. \tag{8.11}$$

□

Remark 8.2.5. From the calculation in Remark 4.5.4 we see the choice of (8.11) is optimal.

Remark 8.2.6. (1) We can study Harnack inequalities for multivalued stochastic differential equations with more general drift as we have done in Chapter 4.

(2) We can also apply the Harnack inequalities we obtained for multivalued stochastic differential equations to study the strong Feller property, hyper-boundedness etc. of the transition semigroup associated with the multivalued stochastic differential equations. Refer to the procedure in Subsection 8.6.

8.3 Multivalued Stochastic Evolution Equations

Let $\mathbb{V}$ be a separable and reflexive Banach space which is continuously and densely embedded in a separable Hilbert space $\mathbb{H}$. Then we have an evolution triplet

$(\mathbb{V}, \mathbb{H}, \mathbb{V}^*)$ satisfying

$$\mathbb{V} \subset \mathbb{H} = \mathbb{H}^* \subset \mathbb{V}^*,$$

where $\mathbb{V}^*$ is the dual space of $\mathbb{V}$ and we identify $\mathbb{H}$ with its own dual $\mathbb{H}^*$.

Denote by $|\cdot|_{\mathbb{V}}$, $|\cdot|_{\mathbb{H}}$, $|\cdot|_{\mathbb{V}^*}$ the norms in $\mathbb{V}$, $\mathbb{H}$ and $\mathbb{V}^*$ respectively; by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ the inner product in $\mathbb{H}$, and ${}_{\mathbb{V}}\langle\cdot,\cdot\rangle_{\mathbb{V}^*}$ the dual relation between $\mathbb{V}$ and $\mathbb{V}^*$. In particular, if $v \in \mathbb{V}$ and $h \in \mathbb{H}$, then

$${}_{\mathbb{V}}\langle v, h\rangle_{\mathbb{V}^*} = \langle v, h\rangle_{\mathbb{H}}.$$

Let W_t be a cylindrical Wiener process on $\mathbb{H}$ with respect to a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

Let A be a multivalued maximal monotone operator on $\mathbb{H}$ and B a single valued operator from $\mathbb{V}$ to $\mathbb{V}^*$; and σ a operator from $\mathbb{R}_+ \times \Omega \times \mathbb{H}$ to $\mathbb{H} \otimes \mathbb{H}$.

We consider the following multivalued stochastic evolution equation

$$\begin{cases} dX_t \in -AX_t\,dt + BX_t\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = x \in \overline{D(A)}. \end{cases} \tag{8.12}$$

Before we explain the meaning of a solution to the equation (8.12), we introduce two sets:

(1) $\mathscr{V}_T(\mathbb{H})$: the set of all $\mathbb{H}$-valued functions of finite variation on $[0, T]$.

(2) $\mathscr{A}_T$: the space of all $[u, K]$ such that $u \in C([0, T]; \overline{D(A)})$, $K \in \mathscr{V}_T(\mathbb{H})$ with $K(0) = 0$, and for all $x, y \in C([0, T], \mathbb{H})$ satisfying $[x(t), y(t)] \in \mathrm{Gr}(A)$, the measure

$$\langle u(t) - x(t), dK(t) - y(t)\,dt\rangle_{\mathbb{H}} \geq 0.$$

Definition 8.3.1. A pair of $\mathscr{F}_t$-adapted random processes (X_t, K_t) is called a solution of Equation (8.12) if

(1) $[X(\cdot, \omega), K(\cdot, \omega)] \in \mathscr{A}_T$ for almost all $\omega \in \Omega$;

(2) For some $q > 1$, $X(\cdot, \omega) \in L^q([0, T]; \mathbb{V})$ for almost all $\omega \in \Omega$;

(3) It holds that

$$X_t = X_0 - K_t + \int_0^t BX_s\,ds + \int_0^t \sigma(s, X_s)\,dW_s,$$

for all $t \in [0, T]$ almost surely.

For the existence and uniqueness of the equation (8.12), we have the following theorem which is due to Zhang [Zha07, Theorem 4.6].

Theorem 8.3.2. *Assume the following conditions.*

(H1) $0 \in D(A)^o$*, where* $D(A)^o$ *denotes the interior of* $D(A)$*;*

(H2) B *is hemicontinuous: for every* $x, y, z \in \mathbb{V}$*,*

$$[0,1] \ni \varepsilon \mapsto {}_{\mathbb{V}}\langle x, B(y+\varepsilon z)\rangle_{\mathbb{V}^*} \text{ is continuous;}$$

(H3) For every $x, y \in \mathbb{V}$*,*

$${}_{\mathbb{V}}\langle x-y, Bx-By\rangle_{\mathbb{V}^*} \leq 0;$$

(H4) There exists $\gamma > 0$*,* $\omega \in \mathbb{R}$ *and* $q > 1$ *such that for every* $x, y \in \mathbb{V}$*,*

$${}_{\mathbb{V}}\langle x-y, Bx-By\rangle_{\mathbb{V}^*} \leq -\gamma|x-y|_{\mathbb{V}}^q + \omega|x-y|_{\mathbb{H}}^2; \tag{8.13}$$

(H5) There exists a $C > 0$ *such that for every* $x \in \mathbb{V}$*,*

$$|Bx|_{\mathbb{V}^*} \leq C(1+|x|_{\mathbb{V}}^{q-1}),$$

where q *is the same as in* (8.13)*;*

(H6) Let $\mathcal{M}$ *be the set of all progressively measurable sets with respect to* $\mathscr{F}_t$*. Assume* σ *is* $\mathcal{M} \times \mathcal{B}(\mathbb{H})/\mathcal{B}(\mathbb{H}\otimes\mathbb{H})$ *measurable and there exists a positive constant* C_σ *such that for all* $(t,\omega) \in \mathbb{R}_+ \times \Omega$ *and* $x, y \in \mathbb{H}$*,*

$$\|\sigma(t,\omega,x) - \sigma(t,\omega,y)\|_{\mathbb{H}\otimes\mathbb{H}} \leq C_\sigma|x-y|_{\mathbb{H}},$$
$$\|\sigma(t,\omega,x)\|_{\mathbb{H}\otimes\mathbb{H}} \leq C_\sigma(1+|x|_{\mathbb{H}}).$$

Then there exists a unique solution to equation (8.12) *in the sense of Definition 8.3.1.*

Remark 8.3.3. The following notes are remarked in [Zha07, Remark 3.1].

(1) Condition (H1) can be replaced by $D(A)^o \neq \emptyset$.

(2) Condition (H2) and (H3) implies that B is demicontinuous [Zei90, Proposition 2.6.4]. That is, if v_n converges strongly to v in $\mathbb{V}$, then Bv_n converges to Bv in $\mathbb{V}^*$. In particular, B is strongly measurable.

By Zhang [Zha07, Theorem 5.5], the process X_t is a Markov process.

We recall here the following proposition which will play an important role. We refer to Zhang [Zha07, Proposition 3.3] for a proof.

Proposition 8.3.4. *Let* $[u, K], [\tilde{u}, \widetilde{K}] \in \mathscr{A}_T$*. Then the measure*

$$\langle u(t) - \tilde{u}(t), dK(t) - d\widetilde{K}(t)\rangle_{\mathbb{H}} \geq 0.$$

8.4 Concentration of Invariant Measures

Suppose that Conditions (H1)-(H6) hold. By Theorem 8.3.2, the equation (8.12) has a unique solution X_t. Define

$$P_t f(x) = \mathbb{E}_{\mathbb{P}} f(X_t)$$

for every $f \in \mathscr{B}_b(\overline{D(A)})$. Let σ is deterministic and time independent. Then P_t is a Markov semigroup (see [Zha07, Theorem 5.5]).

Zhang [Zha07, Theorem 5.8] has studied the the existence, uniqueness of the invariant measures associated with P_t. He also proved that the invariant measure μ satisfies

$$\mu(|x|_{\mathbb{H}}^2) < \infty.$$

Here we aim to study stronger concentration property for the invariant measures.

Theorem 8.4.1. *Assume that (H1)–(H6) holds with $q \geq 2$ and σ deterministic and independent of time. Assume further that $\mathbb{V}$ is compactly embedded in $\mathbb{H}$. If $q = 2$, then suppose in addition that σ is uniformly bounded and $\lambda\omega < \gamma$, where λ is the constant such that $|\cdot|_{\mathbb{H}} \leq \lambda |\cdot|_{\mathbb{V}}$. Then there exist an invariant measure associated with P_t in the sense that*

$$\int_{\overline{D(A)}} P_t f(x)\, \mu(dx) = \int_{\overline{D(A)}} f(x)\, \mu(dx), \quad f \in \mathscr{B}_b(\overline{D(A)}).$$

Moreover,

$$\int_{\overline{D(A)}} |x|_{\mathbb{V}}^q\, \mu(dx) < \infty. \tag{8.14}$$

If σ is always uniformly bounded, then for every $q \geq 2$, there exist some $\theta > 0$ such that

$$\int_{\overline{D(A)}} \mathrm{e}^{\theta |x|_{\mathbb{H}}^q}\, \mu(dx) < \infty. \tag{8.15}$$

Proof. (1) The existence of the invariant measures has been proved in [Zha07, Theorem 5.8 (i)] for the case $q > 2$. The extension to the case $q = 2$ is not hard. We skip the proof here since the main technical can be found below.

(2) From (H3) we know for all $x \in \mathbb{V}$,

$${}_{\mathbb{V}}\langle x, Bx\rangle_{\mathbb{V}^*} \leq -\gamma |x|_{\mathbb{V}}^q + \omega |x|_{\mathbb{H}}^2 + {}_{\mathbb{V}}\langle x, B0\rangle_{\mathbb{V}^*}. \tag{8.16}$$

If $q = 2$, then

$$\omega|x|_{\mathbb{H}}^2 \le \lambda\omega|x|_{\mathbb{V}}^2 < \gamma|x|_{\mathbb{V}}^2. \tag{8.17}$$

If $q > 2$, then by Young's inequality,

$$\omega|x|_{\mathbb{H}}^2 \le \lambda\omega|x|_{\mathbb{V}}^2 \le \frac{2\varepsilon^q}{q}|x|_{\mathbb{V}}^q + \frac{(\lambda\omega)^{p'}}{p'\varepsilon^{p'}}, \tag{8.18}$$

hold for every $\varepsilon > 0$, where p' satisfying $1/p' + 2/q = 1$.

Use the estimate (8.17) and (8.18) in (8.16) for $q = 2$ and $q > 2$ (by taking ε small enough in this case) respectively, we know there are constants $C_1, \gamma' > 0$ such that

$$_{\mathbb{V}}\langle x, Bx\rangle_{\mathbb{V}^*} \le C_1 - 2\gamma'|x|_{\mathbb{V}}^q + {}_{\mathbb{V}}\langle x, B0\rangle_{\mathbb{V}^*}. \tag{8.19}$$

By Young's inequality again, we know for any $\tilde{\varepsilon} > 0$,

$$_{\mathbb{V}}\langle x, B0\rangle_{\mathbb{V}^*} \le |x|_{\mathbb{V}} \cdot |B0|_{\mathbb{V}^*} \le \frac{\tilde{\varepsilon}^q}{q}|x|_{\mathbb{V}}^q + \frac{1}{p\tilde{\varepsilon}^p}|B0|_{\mathbb{V}^*}^p, \tag{8.20}$$

where $p = \frac{q}{q-1}$.

Therefore, we can deduce from (8.19) and (8.20) by taking $\tilde{\varepsilon}$ small enough to get

$$_{\mathbb{V}}\langle x, Bx\rangle_{\mathbb{V}^*} \le C_2 - \gamma'|x|_{\mathbb{V}}^q \tag{8.21}$$

for some constant $C_2, \gamma' > 0$.

Now we fix a y in the set $A0$. Let (X_t, Y_t) be the solution to the multivalued stochastic evolution equation (8.12). By definition, we have

$$\langle X(t) - 0, dK(t) - y\,dt\rangle \ge 0, \tag{8.22}$$

By Itô's formula, using (8.21) and (8.22) and Young's inequality again, we can obtain

$$\begin{aligned}
&\frac{1}{2}d|X_t|_{\mathbb{H}}^2 \\
\le& -{}_{\mathbb{V}}\langle X_t, BX_t\rangle_{\mathbb{V}^*}\,dt - \langle X_t, dK_t\rangle_{\mathbb{H}}\,dt + \frac{1}{2}\|\sigma\|_{\mathbb{H}\otimes\mathbb{H}}^2\,dt + \langle X_t, \sigma dW_t\rangle \\
\le& (C_3 - \gamma'|X(t)|_{\mathbb{V}}^q)\,dt + |y|_{\mathbb{H}} \cdot |X_t|_{\mathbb{H}}\,dt + \langle X_t, \sigma dW_t\rangle \\
\le& (C_4 - \frac{\gamma'}{2}|X(t)|_{\mathbb{V}}^q)\,dt + \langle X_t, \sigma dW_t\rangle,
\end{aligned} \tag{8.23}$$

where $C_3, C_4 > 0$ are some constants.

In the calculation of (8.23) we also used Young's inequality to get control for $\|\sigma(x)\|_{\mathbb{H}\times\mathbb{H}} \leq C_\sigma(1+|x|_{\mathbb{H}})$ if q is strictly greater than 2. If $q=2$, we use the assumption that σ uniformly bounded.

Therefore, by (8.23), we get

$$\int_0^1 \frac{\gamma'}{2}\mathbb{E}^x|X_s|_{\mathbb{V}}^q\, ds \leq C_4 + \frac{1}{2}\left(|x|_{\mathbb{H}}^2 - \mathbb{E}^x|X_1|_{\mathbb{H}}^2\right). \tag{8.24}$$

Consequently we have

$$\int_0^1 P_s|\cdot|_{\mathbb{V}}^q(x)\, ds \leq \frac{1}{\gamma'}(2C_4 + |x|_{\mathbb{H}}^2).$$

Hence we have $\mu(|\cdot|_{\mathbb{V}}^q) < \infty$. This proves (8.14).

(3) For every $\theta > 0$, by (8.23) we have

$$\begin{aligned}
& d\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q} \\
=& \frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{q-2}\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\; d|X_t|_{\mathbb{H}}^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{2}\theta q\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\right)\left(\frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{2(q-2)} + \frac{q-2}{2}|X_t|_{\mathbb{H}}^{q-4}\right) d\langle|X_t|_{\mathbb{H}}^2, |X_t|_{\mathbb{H}}^2\rangle \\
=& \frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{q-2}\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\left(d|X_t|_{\mathbb{H}}^2 + 2\theta q|\sigma|_{\mathbb{H}\otimes\mathbb{H}}^2|X_t|_{\mathbb{H}}^q\, dt + (q-2)|\sigma|_{\mathbb{H}\otimes\mathbb{H}}^2\, dt\right) \\
\leq& \frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{q-2}\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\left(C_5 - \gamma'|X(t)|_{\mathbb{V}}^q + 2\theta q|\sigma|_{\mathbb{H}\otimes\mathbb{H}}^2|X_t|_{\mathbb{H}}^q\right) dt + dM_t
\end{aligned} \tag{8.25}$$

for some constant $C_5 > 0$ and some local martingale M_t.

Since $|\cdot|_{\mathbb{H}} \leq \lambda|\cdot|_{\mathbb{V}}$, for small enough θ, we have

$$d\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q} \leq \frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{q-2}\,\mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\left(C_5 - \frac{\gamma'}{2}|X(t)|_{\mathbb{V}}^q\right) dt + dM_t. \tag{8.26}$$

For convenience, let us focus at the drift of the right hand of (8.26).

By the fact $|\cdot|_{\mathbb{H}} \leq \lambda|\cdot|_{\mathbb{V}}$ and Young's inequality,

$$\begin{aligned}
& \frac{1}{2}\theta q|X_t|_{\mathbb{H}}^{q-2}\left(C_5 - \frac{\gamma'}{2}|X(t)|_{\mathbb{V}}^q\right) \\
\leq& \frac{1}{2}\theta q C_5|X_t|_{\mathbb{H}}^{q-2} - \frac{1}{2}\theta q\cdot\frac{\gamma'}{2}\lambda^{-q}|X(t)|_{\mathbb{H}}^q\cdot|X_t|_{\mathbb{H}}^{q-2} \\
\leq& C_6 - \gamma''|X_t|_{\mathbb{H}}^{2(q-1)}
\end{aligned} \tag{8.27}$$

for some constant $C_6, \gamma'' > 0$.

Now let

$$G = \left\{ |X_t|_{\mathbb{H}}^{2(q-1)} \geq 1 + \frac{C_6}{\gamma''} \right\}.$$

Note that on G^c, both $|X_t|_{\mathbb{H}}^{2(q-1)}$ and $\mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q}$ are bounded. Therefore

$$\begin{aligned}
&\left(C_6 - \gamma'' |X_t|_{\mathbb{H}}^{2(q-1)}\right) \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \\
=& - \gamma'' \left(|X_t|_{\mathbb{H}}^{2(q-1)} - \frac{C_6}{\gamma''} \right) \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \\
\leq& - \gamma'' \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \mathbb{1}_G - \gamma'' \left(|X_t|_{\mathbb{H}}^{2(q-1)} - \frac{C_6}{\gamma''} \right) \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \mathbb{1}_{G^c} \\
\leq& - \gamma'' \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} + \gamma'' \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \mathbb{1}_{G^c} - \gamma'' \left(|X_t|_{\mathbb{H}}^{2(q-1)} - \frac{C_6}{\gamma''} \right) \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \mathbb{1}_{G^c} \\
\leq& C_7 - \gamma'' \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q}
\end{aligned} \tag{8.28}$$

for some constant $C_7 > 0$.

Therefore, from (8.27) and (8.28), we can get an estimate of the drift of the right hand side of (8.26). Consequently, from (8.26), we see

$$d \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \leq \left(C_7 - \gamma'' \, \mathrm{e}^{\theta |X_t|_{\mathbb{H}}^q} \right) dt + dM_t \tag{8.29}$$

By integrating the inequality (8.29) from 0 to n, we get

$$\mathrm{e}^{\theta |X_n|_{\mathbb{H}}^q} \leq \mathrm{e}^{\theta |X_0|_{\mathbb{H}}^q} + C_7 n - \gamma'' \int_0^n \mathrm{e}^{\theta |X_s|_{\mathbb{H}}^q} \, ds + M_n. \tag{8.30}$$

Then we take expectation for both side of (8.30) with respect to $\mathbb{P}^0$, we get

$$\mathbb{E} \, \mathrm{e}^{\theta |X_n|_{\mathbb{H}}^q} \leq 1 + C_7 n - \gamma'' \int_0^n \delta_0 P_s \, \mathrm{e}^{\theta |\cdot|_{\mathbb{H}}^q} \, ds. \tag{8.31}$$

It follows that

$$\mu_n(\mathrm{e}^{\theta |\cdot|_{\mathbb{H}}^q}) \leq \frac{C_7}{\gamma''} + \frac{1}{n\gamma''}, \quad n \geq 1, \tag{8.32}$$

where

$$\mu_n = \frac{1}{n} \int_0^n \delta_0 P_s \, ds, \quad n \geq 1.$$

Note that μ is the weak limit of μ_n (refer to the proof of [Zha07, 5.8]), we can deduce from (8.32) to get $\mu(\mathrm{e}^{\theta |\cdot|_{\mathbb{H}}^q}) < \infty$. This proves (8.15). □

8.5 Harnack Inequalities

In the following we assume conditions (H1)–(H5) in Theorem 8.3.2 and instead of (H6) we suppose that

(H6′) $\sigma\colon [0,\infty)\times\Omega\to\mathbb{H}\otimes\mathbb{H}$ be a nondegenerate Hilbert-Schmidt operator uniformly bounded in time $t\in[0,\infty)$ and $\omega\in\Omega$.

For every $x\in\mathbb{H}$, define

$$|x|_{\sigma_t}=\begin{cases}|y|_{\mathbb{H}} & \text{if } x=\sigma_t y \text{ for some } y\in\mathbb{H},\\ \infty, & \text{otherwise.}\end{cases}$$

The distance associated with $|\cdot|_{\sigma_t}$ is called the *intrinsic distance* induced by σ_t. We refer to Page 19 for more details.

By Theorem 8.3.2, the equation (8.12) has a unique solution and we define $P_tf(x)=\mathbb{E}_{\mathbb{P}}f(X_t)$ for every $f\in\mathscr{B}_b(\overline{D(A)})$. We are going to prove the following Harnack inequality for the semigroup P_t.

Theorem 8.5.1. *Assume (H1)–(H5) and (H6′). Suppose that there exists some nonnegative constant $r\geq q-4$, and some strictly positive continuous function ζ_t on $[0,\infty)$ such that*

$$\zeta_t^2|x|_{\sigma_t}^{2+r}\cdot|x|_{\mathbb{H}}^{q-2-r}\leq|x|_{\mathbb{V}}^q,\quad\text{for all } x\in\mathbb{V},\ t\geq 0 \tag{8.33}$$

holds on Ω. Then for every $T>0$, $x,y\in\overline{D(A)}$, $\alpha,\beta>1$ satisfying $1/\alpha+1/\beta=1$ and $f\in\mathscr{C}_b^+(\overline{D(A)})$, the following inequality holds

$$(P_Tf^\alpha)(x)\leq\exp\left(\frac{\beta}{2}\Theta_T|x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right)P_Tf^\alpha(y), \tag{8.34}$$

where

$$\Theta_T=\Theta(T,\delta,\gamma,\omega,\zeta_t)=4\delta^{-\frac{2(3+r)}{2+r}}\gamma^{-\frac{2}{2+r}}\frac{\left(\int_0^T\zeta_t^2\,\mathrm{e}^{-\delta\omega t}\,dt\right)^{\frac{r}{2+r}}}{\left(\int_0^T\zeta_t\,\mathrm{e}^{-\delta\omega t}\,dt\right)^2} \tag{8.35}$$

with

$$\delta=1-\frac{q}{4+r}. \tag{8.36}$$

Assume the diffusion coefficient σ is independent of (t,ω) and the function ζ_t

in (8.33) *is taken as constant* ζ. *Then* Θ_T *is simplified as*

$$\widetilde{\Theta}_T = 4\delta^{-1}\gamma^{-\frac{2}{2+r}}\zeta^{-\frac{4}{2+r}}\left[\omega^{-1}(1-\mathrm{e}^{-\delta\omega T})\right]^{-\frac{4+r}{2+r}}. \tag{8.37}$$

Proof. We divide the proof into six steps since it is quite long. We outline the main procedure of the proof of (8.34) in the first step and then realize the idea in the next four steps. The simplification from (8.35) to (8.37) is obtained in the last step.

(1) *Main Idea.*

Consider the following coupled multivalued stochastic evolution equation

$$\begin{cases} dX_t \in -AX_t\,dt + BX_t\,dt + \sigma(t)\,dW_t - U_t\,dt, & (8.38\text{a}) \\ dY_t \in -AY_t\,dt + BY_t\,dt + \sigma(t)\,dW_t & (8.38\text{b}) \end{cases}$$

with initial conditions $X_0 = x \in \overline{D(A)}$, $Y_0 = y \in \overline{D(A)}$, and the drift U_t in (8.38a) is of the following form

$$U_t = \frac{\eta_t(X_t - Y_t)}{|X_t - Y_t|_{\mathbb{H}}^{\delta}}\mathbb{1}_{\{t<\tau\}}, \tag{8.39}$$

where the stopping time τ in (8.39) is the coupling time of X_t and Y_t defined by

$$\tau = \inf\{t \geq 0\colon X_t = Y_t\},$$

the power δ in (8.39) is a constant in $(0,1)$ (see (8.36)) and η_t is a deterministic function on $[0,\infty)$. Both δ and η_t in (8.39) will be specified later such that the following two crucial conditions

$$X_T = Y_T \quad \text{a.s.} \tag{8.40}$$

and

$$\mathbb{E}_{\mathbb{P}} \exp\left(\int_0^T \frac{\eta_t^2}{2}\frac{|X_t - Y_t|_{\sigma_t}^2}{|X_t - Y_t|_{\mathbb{H}}^{2\delta}}\mathbb{1}_{\{t<\tau\}}\,dt\right) < \infty. \tag{8.41}$$

are satisfied.

By (8.41) we know

$$R_t = \exp\left(\int_0^t \langle\sigma_s^{-1}U_s, dW_s\rangle - \frac{1}{2}\int_0^t |\sigma_s^{-1}U_s|_{\mathbb{H}}^2\,ds\right), \quad t \in [0,T]$$

is a martingale on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$. Therefore, we can define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T})$ by setting $\mathbb{Q}|_{\mathscr{F}_T} = R_T\mathbb{P}$.

By Girsanov's theorem,

$$\widetilde{W}_t := W_t - \int_0^t \sigma_s^{-1} U_s\, ds$$

is still a cylindrical Wiener process on $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{Q})$. Hence Equation (8.38a) can be rewritten in the following way

$$dX_t \in -AX_t\, dt + BX_t\, dt + \sigma(t)\, d\widetilde{W}_t$$

with initial condition $X_0 = x$.

By the uniqueness of the solution, the transition law of $(X_t)_{t\in[0,T]}$ under $\mathbb{Q}$ is the same with the transition law of $(Y_t)_{t\in[0,T]}$ under $\mathbb{P}$. So by the fact (8.40) which will be verified, we have

$$P_T f(x) = \mathbb{E}_{\mathbb{Q}} f(X_T) = \mathbb{E}_{\mathbb{Q}} f(Y_T) = \mathbb{E}_{\mathbb{P}} R f(Y_T). \tag{8.42}$$

Note that we also have $P_T f(y) = \mathbb{E}_{\mathbb{P}} f(Y_T)$, therefore by applying Hölder's inequality to (8.42), we can get

$$(P_T f)^\alpha(x) \le \left(\mathbb{E}_{\mathbb{P}} R_T^\beta\right)^{\alpha/\beta} P_T f^\alpha(y). \tag{8.43}$$

Then we can finish the proof by an additional estimate of $\mathbb{E}_{\mathbb{P}} R_T^\beta$.

(2) *Existence of the solution of the coupled equation* (8.38).

Note that the function

$$(u, v) \mapsto \frac{u - v}{|u - v|_{\mathbb{H}}^\delta}$$

satisfies the monotone condition off the diagonal (see Appendix of [Wan07]).

Therefore we can apply Theorem 8.3.2 and see that the coupled equation (8.38) has a solution up to the coupling time τ. So there exists continuous processes $(X, K) \in \mathscr{A}_{T\wedge\tau}$ and $(Y, \widetilde{K}) \in \mathscr{A}_{T\wedge\tau}$ such that for all $t < \tau$,

$$\begin{cases} X_t = x - K_t + \int_0^t BX_s\, ds + \int_0^t \sigma(s)\, dW_s - \int_0^t U_s\, ds, & (8.44\text{a}) \\ Y_t = y - \widetilde{K}_t + \int_0^t BY_s\, ds + \int_0^t \sigma(s)\, dW_s. & (8.44\text{b}) \end{cases}$$

On the other hand, it is obvious that the solution of Equation (8.38b) (or equivalently, Equation (8.44b)) can be extended to be a solution for all time $[0, \infty)$. Let $(Y_t, \widetilde{K})_{t\geq 0}$ solves Equation (8.38b). Now we can also solve Equation (8.38a) (or (8.44a)) by defining $X_t = Y_t, \quad K_t = \widetilde{K}_t$ for all $t \geq \tau$.

(3) *Verify* (8.40).

Apply Itô's formula (see [KR79] (or [KR07]), [PR07], or Zhang [Zha07, Theorem A.1] etc.) to $\sqrt{|X_t - Y_t|_{\mathbb{H}}^2 + \varepsilon}$ and then let $\varepsilon \downarrow 0$, by using condition (H4) we have for $t < \tau$

$$\begin{aligned} d|X_t - Y_t|_{\mathbb{H}}^2 \leq &- \langle X_t - Y_t, dK_t - d\widetilde{K}_t\rangle_{\mathbb{H}}\, dt \\ &+ (-\gamma|X_t - Y_t|_{\mathbb{V}}^q + \omega|X_t - Y_t|_{\mathbb{H}}^2 - \eta_t|X_t - Y_t|_{\mathbb{H}}^{2-\delta})\, dt. \end{aligned}$$

By Proposition 8.3.4, for all $t < \tau$ we have

$$d|X_t - Y_t|_{\mathbb{H}}^2 \leq (-\gamma|X_t - Y_t|_{\mathbb{V}}^q + \omega|X_t - Y_t|_{\mathbb{H}}^2 - \eta_t|X_t - Y_t|_{\mathbb{H}}^{2-\delta})\, dt.$$

Then

$$d\left(|X_t - Y_t|_{\mathbb{H}}^2\, \mathrm{e}^{-\omega t}\right) \leq -\,\mathrm{e}^{-\omega t}\left(\gamma|X_t - Y_t|_{\mathbb{V}}^q + \eta_t|X_t - Y_t|_{\mathbb{H}}^{2-\delta}\right)\, dt. \tag{8.45}$$

Hence by (8.45) we get

$$\begin{aligned} &d\left(|X_t - Y_t|_{\mathbb{H}}^2\, \mathrm{e}^{-\omega t}\right)^{\delta/2} \\ \leq &\frac{\delta}{2}\left(|X_t - Y_t|_{\mathbb{H}}^2\, \mathrm{e}^{-\omega t}\right)^{\delta/2-1} \cdot \left(-\,\mathrm{e}^{-\omega t}\,\eta_t|X_t - Y_t|_{\mathbb{H}}^{2-\delta}\right)\, dt \\ = &-\frac{\delta}{2}\,\mathrm{e}^{-\frac{\delta}{2}\omega t}\,\eta_t\, dt. \end{aligned} \tag{8.46}$$

We take

$$\eta_t = \vartheta_T \zeta_t\, \mathrm{e}^{-\frac{\delta}{2}\omega t} \tag{8.47}$$

with

$$\vartheta_T = \frac{2\delta^{-1}|x - y|_{\mathbb{H}}^{\delta}}{\int_0^T \zeta_t\, \mathrm{e}^{-\delta\omega t}\, dt}.$$

Then it must be $T \geq \tau$. Otherwise, if $T < \tau$, then by taking integral from 0 to T for both sides of the inequality (8.46), we can obtain

$$|X_T - Y_T|_{\mathbb{H}}^{\delta}\, \mathrm{e}^{-\frac{\delta}{2}\omega T} \leq |x - y|_{\mathbb{H}}^{\delta} - \frac{\delta}{2}\int_0^T \mathrm{e}^{-\frac{\delta}{2}\omega t}\,\eta_t\, dt. \tag{8.48}$$

By (8.47) the right hand side of (8.48) equals to 0. So we can conclude $X_T = Y_T$

from (8.48). But this is contradict with the assumption that $T < \tau$. Therefore, it must be $T \geq \tau$ and hence $X_T = Y_T$.

(4) *Verify* (8.41).

From (8.45) and the assumption (8.33) we can get for all $t \leq \tau$

$$\begin{aligned}
& d\left(|X_t - Y_t|_{\mathbb{H}}^2 \,\mathrm{e}^{-\omega t}\right)^{\delta} \\
=& \delta\left(|X_t - Y_t|_{\mathbb{H}}^2 \,\mathrm{e}^{-\omega t}\right)^{\delta-1} d\left(|X_t - Y_t|_{\mathbb{H}}^2 \,\mathrm{e}^{-\omega t}\right) \\
\leq & -\delta\gamma \,\mathrm{e}^{-\delta\omega t} |X_t - Y_t|_{\mathbb{H}}^{2(\delta-1)} \cdot |X_t - Y_t|_{\mathbb{V}}^{q} \, dt \\
\leq & -\delta\gamma\zeta_t^2 \,\mathrm{e}^{-\delta\omega t} \frac{|X_t - Y_t|_{\sigma_t}^{2+r}}{|X_t - Y_t|_{\mathbb{H}}^{2+r-2(\delta-1)-q}} \, dt.
\end{aligned} \tag{8.49}$$

Let

$$\delta = 1 - \frac{q}{4+r}.$$

Then

$$2 + r - 2(\delta - 1) - q = \delta(2 + r).$$

Hence, from (8.49) we see

$$d\left(|X_t - Y_t|_{\mathbb{H}}^2 \,\mathrm{e}^{-\omega t}\right)^{\delta} \leq -\delta\gamma\zeta_t^2 \,\mathrm{e}^{-\delta\omega t} \frac{|X_t - Y_t|_{\sigma_t}^{2+r}}{|X_t - Y_t|_{\mathbb{H}}^{\delta(2+r)}} \, dt. \tag{8.50}$$

According to (8.47), we have $\zeta_t^2 = \frac{\eta_t^2}{\vartheta_T^2} \,\mathrm{e}^{-\delta\omega t}$.

By integrating both sides of the inequality (8.50) from 0 to T, we get (note that $X_T = Y_T$)

$$\frac{\delta\gamma}{\vartheta_T^2} \int_0^T \frac{\eta_t^2 |X_t - Y_t|_{\sigma_t}^{2+r}}{|X_t - Y_t|_{\mathbb{H}}^{\delta(2+r)}} \, dt \leq |x - y|_{\mathbb{H}}^{2\delta}.$$

By Hölder's inequality, we have

$$\begin{aligned}
& \int_0^T \frac{\eta_t^2 |X_t - Y_t|_{\sigma_t}^2}{|X_t - Y_t|_{\mathbb{H}}^{2\delta}} \, dt \\
\leq & \left(\int_0^T \frac{\eta_t^2 |X_t - Y_t|_{\sigma_t}^{2+r}}{|X_t - Y_t|_{\mathbb{H}}^{\delta(2+r)}} \, dt \right)^{\frac{2}{2+r}} \left(\int_0^T \eta_t^2 \, dt \right)^{\frac{r}{2+r}} \\
\leq & \left(\frac{\vartheta_T^2}{\delta\gamma} |x - y|_{\mathbb{H}}^{2\delta} \right)^{\frac{2}{2+r}} \cdot \vartheta_T^{\frac{2r}{2+r}} \left(\int_0^T \zeta_t^2 \,\mathrm{e}^{-\delta\omega t} \, dt \right)^{\frac{r}{2+r}} \\
= & (\delta\gamma)^{-\frac{2}{2+r}} \vartheta_T^2 |x - y|_{\mathbb{H}}^{\frac{4\delta}{2+r}} \cdot \left(\int_0^T \zeta_t^2 \,\mathrm{e}^{-\delta\omega t} \, dt \right)^{\frac{r}{2+r}}.
\end{aligned}$$

Note that

$$\vartheta_T^2 = \frac{4\delta^{-2}|x-y|_{\mathbb{H}}^{2\delta}}{\left(\int_0^T \zeta_t \, \mathrm{e}^{-\delta\omega t}\, dt\right)^2},$$

we obtain

$$\begin{aligned}
&\int_0^T \frac{\eta_t^2 |X_t - Y_t|_{\sigma_t}^2}{|X_t - Y_t|_{\mathbb{H}}^{2\delta}}\, dt \\
\leq& 4\delta^{-2}(\delta\gamma)^{-\frac{2}{2+r}} \frac{\left(\int_0^T \zeta_t^2 \, \mathrm{e}^{-\delta\omega t}\, dt\right)^{\frac{r}{2+r}}}{\left(\int_0^T \zeta_t \, \mathrm{e}^{-\delta\omega t}\, dt\right)^2} |x-y|_{\mathbb{H}}^{2\delta+\frac{4\delta}{2+r}} \\
=& 4\delta^{-\frac{2(3+r)}{2+r}} \gamma^{-\frac{2}{2+r}} \frac{\left(\int_0^T \zeta_t^2 \, \mathrm{e}^{-\delta\omega t}\, dt\right)^{\frac{r}{2+r}}}{\left(\int_0^T \zeta_t \, \mathrm{e}^{-\delta\omega t}\, dt\right)^2} |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}.
\end{aligned} \tag{8.51}$$

It is clear now that (8.41) holds.

(5) *Estimate of* $\mathbb{E}R_T^\beta$.

By the martingale property of R_T (with respect to $\mathbb{P}$), we see (refer to (4.21))

$$(\mathbb{E}_{\mathbb{P}} R_T^\beta)^{\alpha/\beta} = \mathbb{E}_{\mathbb{P}} \exp\left(\frac{\beta}{2}\int_0^T \frac{\eta_t^2 |X_t - Y_t|_{\sigma_t}^2}{|X_t - Y_t|_{\mathbb{H}}^{2\delta}}\, dt\right).$$

We can get (8.34) by using the estimate (8.51).

(6) Suppose that σ is independent of (t,ω). And we take ζ_t in (8.33) as a constant ζ. Then we can simplify Θ_T as follows.

$$\begin{aligned}
\Theta_T =& 4\delta^{-\frac{2(3+r)}{2+r}} \gamma^{-\frac{2}{2+r}} \frac{\left(\zeta^2 \int_0^T \mathrm{e}^{-\delta\omega t}\, dt\right)^{\frac{r}{2+r}}}{\left(\zeta \int_0^T \mathrm{e}^{-\delta\omega t}\, dt\right)^2} \\
=& 4\delta^{-\frac{2(3+r)}{2+r}} \gamma^{-\frac{2}{2+r}} \zeta^{-\frac{4}{2+r}} \left[(\delta\omega)^{-1}(1-\mathrm{e}^{-\delta\omega T})\right]^{-\frac{4+r}{2+r}} \\
=& 4\delta^{-1}\gamma^{-\frac{2}{2+r}} \zeta^{-\frac{4}{2+r}} \left[\omega^{-1}(1-\mathrm{e}^{-\delta\omega T})\right]^{-\frac{4+r}{2+r}} = \widetilde{\Theta}_T
\end{aligned}$$

□

Remark 8.5.2. Our proof is similar to the proof of [Wan07, Theorem 1.1].

Remark 8.5.3. We refer to [Wan07, Corollary 1.3] for sufficient conditions for (8.33).

8.6 Applications of Harnack Inequalities

We apply (8.34) to study the strong Feller property, full support of the invariant measure, heat kernel bound and hyperboundedness etc. properties of the semigroup P_t.

Zhang [Zha07, Corollary 5.3] studied Feller property of P_t. We can prove strong Feller property (and even more) for P_t under additional conditions.

Theorem 8.6.1. *Assume (H1)–(H5), (H6′) and* (8.33) *with* $q < 4+r$. *Then for every* f *in* $L^p(\overline{D(A)}, \mu)$ *with* $p > 1$, $P_t f$ *is continuous on* $\overline{D(A)}$. *In particular, the semigroup* P_t *is strongly Feller. Moreover, the following estimate holds*

$$|P_t f(x) - P_t f(y)| \le \|f\|_\infty \Theta_t^{1/2} |x-y|_{\mathbb{H}}^{\frac{4+r-q}{2+r}} \cdot \exp\left(\frac{1}{2}\Theta_t |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right). \quad (8.52)$$

for every $t > 0$, $x, y \in \overline{D(A)}$ *and* $f \in \mathscr{B}_b(\overline{D(A)})$.

Proof. The first statement follows directly from the Harnack inequality (8.34) and Proposition 5.3.2. Now we prove the estimate (8.52).

Use the notation in the proof of Theorem 8.5.1 and we prove (8.52) for fixed $T > 0$. By (8.42), we see

$$\begin{aligned} |P_T f(x) - P_T f(y)| &= |\mathbb{E}_{\mathbb{Q}} f(X_T) - \mathbb{E}_{\mathbb{P}} f(Y_T)| = |\mathbb{E}_{\mathbb{P}} R_T f(X_T) - \mathbb{E}_{\mathbb{P}} f(X_T)| \\ &= \mathbb{E}_{\mathbb{P}} |f(X_T)(1-R_T)| \le \|f\|_\infty \mathbb{E}_{\mathbb{P}} |1-R_T|. \end{aligned} \quad (8.53)$$

It is clear

$$\big(\mathbb{E}_{\mathbb{P}}|1-R_T|\big)^2 \le \mathbb{E}_{\mathbb{P}}(1-R_T)^2 = \mathbb{E}_{\mathbb{P}} R_T^2 - 1. \quad (8.54)$$

By (7.30), we know

$$\mathbb{E}_{\mathbb{P}} R_T^2 = \exp\left(\Theta_T |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right). \quad (8.55)$$

Using the elementary inequality

$$\mathrm{e}^r - 1 \le r\,\mathrm{e}^r \quad \text{for all } r \ge 0,$$

we can deduce from (8.54) and (8.55) to get

$$\big(\mathbb{E}_{\mathbb{P}}|1-R_T|\big)^2 \le \exp\left(\Theta_T |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right) - 1$$

$$\leq \Theta_T |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}} \cdot \exp\left(\Theta_T |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right).$$

Substitute the estimate above into (8.53) we can obtain (8.52). □

Remark 8.6.2. The strong Feller property can also be observed immediately by the dominated convergence theorem:

$$\lim_{y\to x} \mathbb{E}_{\mathbb{P}}|R_T - 1| = \mathbb{E}_{\mathbb{P}} \lim_{y\to x} |R_T - 1| = 0.$$

From now on, we assume that σ is independent of (t, ω). In this case, P_t is a Markov semigroup (see [Zha07, Theorem 5.5]) and the Harnack inequality (8.34) holds for $\widetilde{\Theta}_T$ in place of Θ_T.

We also assume that the invariant measure, denote by μ, of the semigroup P_t exist. See Subsection 8.4 for the study of the concentration property the invariant measure.

Theorem 8.6.3. *Assume (H1)–(H5), (H6′) and* (8.33)*. Then*

(1) The invariant measure μ is fully supported on $\overline{D(A)}$.

(2) For every $x \in \overline{D(A)}$, $t > 0$, the transition density $p_t(x,\cdot)$ (with respect to μ) exist and for every $\alpha > 1$

$$\|p_t(x,\cdot)\|_{L^\alpha(\overline{D(A)},\mu)} \leq \left[\int_{\overline{D(A)}} \exp\left(-\frac{\alpha}{2}\widetilde{\Theta}_t |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right) \mu(dy)\right]^{-(\alpha-1)/\alpha}.$$

(3) Suppose $K \leq 0$.

(i) If $q = 2$ and $\lambda\omega < \gamma$, where λ is the constant such that $|\cdot|_{\mathbb{H}} \leq \lambda|\cdot|_{\mathbb{V}}$, then P_t is hyperbounded .

(ii) If $q > 2$, then P_t is ultrabounded. More precisely, there exist some constant $c > 0$ such that

$$\|P_t\|_{2\to\infty} \leq \exp\big(c(1 + t^{-\frac{q}{q-2}})\big). \tag{8.56}$$

Consequently, P_t is compact for large $t > 0$ for both cases.

Proof. (1) If $\operatorname{supp}\mu \neq \overline{D(A)}$, then there exists some $x_0 \in \overline{D(A)}$, $r > 0$, such that $\mu(B_r(x_0)) = 0$, where $B_r(x_0) = \{y \in \overline{D(A)} \colon |y - x_0| \leq r\}$.

Applying (8.34) to the function $\mathbb{1}_{B_r(x_0)}$ for $\alpha = 2$ and $t \geq 0$, we have

$$(P_t \mathbb{1}_{B_r(x_0)})^2(x) \exp\left(-\widetilde{\Theta}_t |x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right) \leq P_t \mathbb{1}_{B_r(x_0)}(y). \tag{8.57}$$

Hence, by integrating both sides of (8.57) with respect to $\mu(dy)$, we can obtain

$$\begin{aligned}&(P_t\mathbb{1}_{B_r(x_0)})^2(x)\int_{\overline{D(A)}}\exp\left(-\tilde{\Theta}_t|x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right)\mu(dy)\\ \le&\mu(P_t\mathbb{1}_{B_r(x_0)})=\mu(\mathbb{1}_{B_r(x_0)})=0.\end{aligned}$$

This implies $P_t(x_0, B_r(x_0)) = 0$ for all $t \geq 0$. Therefore,

$$\mathbb{P}(\|X_t(x_0)-x_0\|_{\mathbb{H}}\le r)=0,\quad t>0, \tag{8.58}$$

where $X_t(x_0)$ denotes the solution to (8.12) with $X_0(x_0) = x_0$.

Since X_t is continuous on $\mathbb{H}$, by letting $t \to 0$ in (8.58), we have

$$\mathbb{P}(\|X_0(x_0)-x_0\|_{\mathbb{H}}\le r)=0.$$

But obviously this is impossible. So it must be $\operatorname{supp}\mu = \overline{D(A)}$.

(2) It follows immediately from the Harnack inequality (8.34) and Lemma 7.5.7.

(3) Since $K \leq 0$, for any $t > 0$, we know

$$\frac{K}{1-\mathrm{e}^{-\delta Kt}}\le\frac{1}{\delta t}.$$

Therefore, by Theorem 8.5.1, there exist some constant C_8 such that for every $x, y \in \overline{D(A)}$ and $t > 0$,

$$(P_tf)^2(x)\exp\left(-\frac{C_8|x-y|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}}{t^{\frac{4+r}{2+r}}}\right)\le P_tf^2(y), \tag{8.59}$$

where $f \in L^2(\overline{D(A)}, \mu)$ with $\mu(f^2) = 1$.

By integrating the both sides of (8.59) with respect to $\mu(dy)$ over $B_1(0) = \{x \in \overline{D(A)}\colon |x|_{\mathbb{H}} \leq 1\}$, we obtain for every $x \in \overline{D(A)}$ and $t > 0$,

$$(P_tf)^2(x)\le\frac{1}{\mu\big(B_1(0)\big)}\exp\left(\frac{C_8\big(1+|x|_{\mathbb{H}}\big)^{\frac{2(4+r-q)}{2+r}}}{t^{\frac{4+r}{2+r}}}\right). \tag{8.60}$$

(i) If $q = 2$, then by taking square and integration with respect to $\mu(dx)$ for both sides of the equation (8.60), and using Theorem 8.4.1, we have

$$\int_{\overline{D(A)}} (P_t f)^4(x)\, \mu(dx) \le \frac{1}{\mu\big(B_1(0)\big)} \int_{\mathbb{H}} \exp\left(\frac{C_8\big(1+|x|_{\mathbb{H}}\big)^2}{t^{\frac{4+r}{2+r}}}\right) \mu(dx) < \infty$$

for $t > 0$ big enough. This proves $\|P_t\|_{2\to4} < \infty$ for sufficiently big $t > 0$. That is, P_t is hyperbounded.

(ii) Assume $q > 2$. Then the inequality (8.26) implies

$$d\, \mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q} \le \left(C_9 - \gamma'''|X(t)|_{\mathbb{H}}^{2(q-1)}\, \mathrm{e}^{\theta|X_t|_{\mathbb{H}}^q}\right) dt + dM_t \tag{8.61}$$

for some constant $C_9, \theta, \gamma''' > 0$.

Let $g(t)$ be the solution to the following equation

$$dg(t) = \left(C_9 - \gamma'''\theta^{-\frac{2(q-1)}{q}} g(t)\big[\log g(t)\big]^{\frac{2(q-1)}{q}}\right) dt$$

with $g(0) = \mathrm{e}^{\theta|x|_{\mathbb{H}}^q}$.

By the comparison theorem, we have

$$\mathbb{E}\, \mathrm{e}^{\theta|X_t(x)|_{\mathbb{H}}^q} \le g(t) \le \exp\left(C_9(1 + t^{-\frac{q}{q-2}})\right) \tag{8.62}$$

for some constant $C_9 > 0$. By inequality (8.60) we have

$$\|P_t f\|_\infty = \|P_{t/2}P_{t/2}f\|_\infty \le C_{10} \sup_{x\in\overline{D(A)}} \mathbb{E}\exp\left[\frac{C_{11}}{t^{\frac{4+r}{2+r}}}\big|X_{t/2}(x)\big|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right] \tag{8.63}$$

for some constants $C_{10}, C_{11} > 0$.

By using Young's inequality, we see

$$C_{10} \sup_{x\in\overline{D(A)}} \mathbb{E}\exp\left[\frac{C_{11}}{t^{\frac{4+r}{2+r}}}\big|X_{t/2}(x)\big|_{\mathbb{H}}^{\frac{2(4+r-q)}{2+r}}\right] \le \varepsilon\cdot|X_{t/2}^x|_{\mathbb{H}}^q + \varepsilon' t^{-\frac{q}{q-2}} \tag{8.64}$$

for arbitrary $\varepsilon > 0$. By considering small enough $\varepsilon > 0$, it follows from the inequality (8.64) above and (8.62) and (8.63) we can obtain (8.56). This proves that P_t is ultrabounded.

Since P_t has transition density with respect to μ, we know P_t is compact in $L^2(\overline{D(A)}, \mu)$ for large $t > 0$ for these two cases ($q = 2$ or $q > 2$) by [GW02, Lemma 3.1]. □

Remark 8.6.4. We refer to [Zha07, Section 6.2] for an example of multivalued stochastic evolution equation satisfying the conditions we used.

Chapter 9

Functional Inequalities for Ornstein-Uhlenbeck Processes

In the previous chapters, we have concentrated on dimension free Harnack inequalities. It is also interesting to look at other functional inequalities.

Various functional inequalities, for instance, Poincaré and log-Sobolev inequalities (see [CM87, DPZ02, RW03a] etc.), have been well studied for Ornstein-Uhlenbeck processes driven by Wiener processes. For stochastic processes related to Lévy noise, only a few functional inequalities are known. We only know, for example, Poincaré inequalities were obtained under a strong condition on the Lévy measure in [RW03a]; and (modified) log-Sobolev inequalities etc. were considered in [Wu00, CM02, GI08, GI09] etc..

This chapter is on entropy cost and HWI inequalities. These functional inequalities have attracted the interest of many people recently. See the monograph Villani [Vil09] and the bibliography therein for more details.

We prove entropy cost and HWI inequalities for Gaussian Ornstein-Uhlenbeck semigroups. They are not new for experts. But it may be considered as a complement of this thesis and a first step to these functional inequalities for Lévy driven Ornstein-Uhlenbeck processes.

9.1 Entropy Cost and HWI Inequalities

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We assume that R is a bounded self-adjoint and nonnegative definite operator on $\mathbb{H}$ and A generates on $\mathbb{H}$ a strongly continuous semigroup S_t. Consider the following linear partial differential equation

$$dX_t = AX_t dt + R^{1/2} dW_t, \quad X_0 = x, \tag{9.1}$$

where W_t is a standard cylindrical Wiener process on $\mathbb{H}$.

Suppose that

Assumption 9.1.1. (1)

$$Q_\infty = \int_0^\infty S_u R S^* u \, du$$

is of trace class.

(2) For all $t \geq 0$, $S_t R = R S_t$.

(3) There exist $M, \omega > 0$ such that for all $t \geq 0$,

$$|S_t| \leq M \, \mathrm{e}^{-\omega t}. \tag{9.2}$$

We will denote

$$\alpha(t) = M^2 \, \mathrm{e}^{-2\omega t}, \quad t \geq 0.$$

By item (1) of Assumption 9.1.1, the equation (9.1) has a mild solution

$$X_t = S_t x + \int_0^t S_{t-u} R^{1/2} dW_u, \quad t \geq 0.$$

The process X_t is Gaussian and Markov with transition semigroup

$$P_t f(x) = \mathbb{E} f(X_t),$$

where f is a bounded measurable function on $\mathbb{H}$. Moreover, by item (1) of Assumption 9.1.1, we know the semigroup P_t has an invariant measure $\mu = N(0, Q_\infty)$. See [DPZ92].

By item (2) of Assumption 9.1.1, the semigroup P_t is symmetric. We refer to [CMG02] for more characterization of the symmetry of P_t.

We will establish entropy cost and HWI inequalities for P_t. First let us introduce these concepts.

Let ν_1 and ν_2 be two probability measures on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$. A *coupling* of ν_1 and ν_2 is a probability measure π on $(\mathbb{H} \times \mathbb{H}, \mathcal{B}(\mathbb{H}) \times \mathcal{B}(\mathbb{H}))$ such that the marginal distributions of π are ν_1 and ν_2 respectively. That is,

$$\pi(A \times \mathbb{H}) = \nu_1(A), \quad \pi(\mathbb{H} \times B) = \nu_2(B).$$

for every A, $B \in \mathcal{B}(\mathbb{H})$.

Now we can introduce the following useful Wasserstein distance between the probability measures ν_1 and ν_2.

Definition 9.1.2. The *Wasserstein distance* between two probability measures ν_1 and ν_2 on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$ is defined by

$$W_2(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left\{ \int_{\mathbb{H}\times\mathbb{H}} \rho(x, y)^2 \pi(dx, dy) \right\}^{1/2},$$

where $\mathscr{C}(\mu, \nu)$ is the space of all couplings of ν_1 and ν_2, and ρ is the intrinsic distance on $\mathbb{H}$ defined by (see (1.1)):

$$\rho(x, y) = \begin{cases} |x - y|_0 = |R^{-1/2}(x - y)|, & \text{if } x - y \in \mathbb{H}_0 = R^{1/2}(\mathbb{H}); \\ \infty, & \text{otherwise.} \end{cases}$$

Wasserstein distance describes the cost of transporting ν_1-distributed mass to ν_2-distributed mass. Hence this distance is also called *transportation cost.*

Definition 9.1.3. Let ν_1, ν_2 be two probability measures on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$ with $\nu_2 = f\nu_1$.

(1) The *entropy* of f with respect to ν_1 is defined by

$$H(\nu_2|\nu_1) = \mathrm{Ent}_{\nu_1}(f) = \nu_1(f \log f) = \nu_1(f \log f) - \nu_1(f) \log \nu_1(f).$$

(2) The *Fisher information* (or *Fisher-Donsker-Varadhan information*) of f is defined as

$$I(f) = 4\nu_1(\langle RD\sqrt{f}, D\sqrt{f}\rangle).$$

The main results of this chapter are the following theorem. The entropy cost inequality (9.3) deals with the connection between entropy and transportation cost. The HWI inequality inequality (9.4) relates three quantities, i.e. entropy, transportation cost and Fisher information. Here "H" stands for the entropy, "W" for the Wasserstein distance, and "I" for the Fisher information.We need to mention that Shao [Sha07] also considered these two inequalities for classical Ornstein-Uhlenbeck semigroups on Wiener spaces.

Theorem 9.1.4. *Suppose Assumption 9.1.1 holds and $\mu = N(0, Q_\infty)$. Then for every $t \geq 0$, and nonnegative $f \in \mathcal{B}(\mathbb{H})$ with $\mu(f) = 1$, we have*

(1) entropy cost inequality

$$\mathrm{Ent}_\mu(P_t f) \le \frac{M\omega}{2(\mathrm{e}^{2\omega t} - 1)} W_2(f\mu, \mu)^2. \tag{9.3}$$

(2) HWI inequality

$$\mathrm{Ent}_\mu(f) \le M\sqrt{I(f)}W_2(f\mu, \mu) - \frac{M\omega}{2} W_2(f\mu, \mu)^2. \tag{9.4}$$

We will prove these two inequalities in the next two sections respectively.

9.2 Proof of Entropy Cost Inequality

We will need to use the following estimation by assumption (9.2)

Lemma 9.2.1. *For every $\varphi \in \mathscr{E}_A(\mathbb{H})$ and $x, y \in \mathbb{H}$, we have*

$$|(R^{1/2}DP_t\varphi)(x)|^2 \le \alpha(t)P_t(|R^{1/2}D\varphi|^2)(x) \tag{9.5}$$

and for every $s \in [0, t]$

$$\langle DP_s \log P_{t-s}\varphi, y - x\rangle \le \rho(x, y)\alpha^{1/2}(s) \left[P_s|R^{1/2}D(\log P_{t-s}\phi)|^2\right]^{\frac{1}{2}}. \tag{9.6}$$

Proof. The estimate (9.5) is from [DPZ02, Equation (10.5.18)]. The proof of (9.6) is similar to [RW03a, (2.5)]):

$$\begin{aligned}
&\langle DP_s \log P_{t-s}\varphi, y - x\rangle \\
=&\inf_{z \in (R^{1/2})^{-1}(x-y)} \langle DP_s \log P_{t-s}\varphi, R^{1/2}z\rangle \\
\le&\rho(x, y)\left[|R^{1/2}DP_s(\log P_{t-s}\phi)|^2\right]^{1/2} \\
\le&\rho(x, y)\alpha^{1/2}(s)\left[P_s|R^{1/2}D(\log P_{t-s}\phi)|^2\right]^{1/2}.
\end{aligned}$$

□

We introduce some facts related to the transition semigroup P_t. We refer to [DP04] for the proof.

The transition semigroup P_t can be uniquely extended to be strongly continuous semigroup of contractions on $L^p(\mathbb{H}, \mu)$ for any $p \ge 1$. We denote by L_p the infinitesimal generator of P_t in $L^p(\mathbb{H}, \mu)$ and $D(L_p)$ its domain. It is easy to verify that $\mathscr{E}_A(\mathbb{H})$ is stable under the action of P_t and L. Moreover, we know is $\mathscr{E}_A(\mathbb{H})$ dense in $L^p(\mathbb{H},\mu)$. Hence $\mathscr{E}_A(\mathbb{H})$ is a core for L_p.

Fix $p=2$. For every $\varphi \in \mathscr{E}_A(\mathbb{H})$, we have

$$\Gamma(\varphi,\varphi) = \frac{1}{2}\big(L_2\varphi^2 - 2\varphi L_2\varphi)\big) = \langle RD\varphi, D\varphi\rangle.$$

We call $\Gamma(\cdot,\cdot)$ the *square field operator.*

For any function Φ with continuous second order derivatives, we have

$$L_2\Phi(\varphi) = \Phi'(\varphi)L_2\varphi + \Phi''(\varphi)\Gamma(\varphi,\varphi). \tag{9.7}$$

For fixed $t>0$ and $f \in \mathscr{E}_A(\mathbb{H})$, consider the function

$$s \mapsto \Psi(s) = P_s\big(\Phi(P_{t-s}f)\big), \quad s\in[0,t]$$

By the chain rule formula (9.7), for any $s\in[0,t]$ we see

$$\begin{aligned}\Psi'(s) &= P_s[L_2\Phi(P_{t-s}f) - \Phi'(P_{t-s}f)L_2P_{t-s}f]\\ &= P_s[\Phi''(P_{t-s})\Gamma(P_{t-s}f)].\end{aligned} \tag{9.8}$$

To prove (9.3), we first prove the following lemma.

Lemma 9.2.2. *For any $t\ge 0$, $x,y\in\mathbb{H}$, and nonnegative bounded measurable function f on $\mathbb{H}$, we have*

$$P_t\log f(y) \le \log P_tf(x) + \frac{\rho(x,y)^2}{4\int_0^t \alpha(r)^{-1}\,dr}.$$

Proof. We prove it following the line of [BGL01]. Define

$$\gamma(s) = x + \frac{s}{t}(y-x) = \left(1-\frac{s}{t}\right)x + \frac{s}{t}y, \quad s\in[0,t].$$

Take

$$g(s) = \frac{t\int_0^s \alpha(r)^{-1}dr}{\int_0^t \alpha(r)^{-1}dr}, \quad s\in[0,t].$$

Then $g(s)$ is a speed function such that $g(0)=0$ and $g(t)=t$.

Without loss of generality, we assume $f\in\mathscr{E}_A(\mathbb{H})$ and f is strictly positive. Set

$$\phi(s) = P_s\log P_{t-s}f\big(\gamma\left(g(s)\right)\big) \quad \text{for every } s\in[0,t]..$$

By (9.8) and using the estimate in (9.6) we have

$$\begin{aligned}\phi'(s) =& - P_s \left|R^{1/2} D \log P_{t-s} f\right|^2 \left(\gamma\left(g(s)\right)\right) + \frac{g'(s)}{t} \langle DP_s \log P_{t-s} f(\gamma(g(s))), y - x\rangle \\ \leq& - P_s \left|R^{1/2} D \log P_{t-s} f\right|^2 \left(\gamma\left(g(s)\right)\right) \\ &+ \frac{g'(s)}{t} \alpha^{1/2}(s) \left[P_s \left|R^{1/2} D \log P_{t-s} f\right|^2 \left(\gamma\left(g(s)\right)\right)\right]^{1/2} \rho(x, y) \\ \leq& \frac{g'(s)^2 \alpha(s) \rho(x, y)^2}{4t^2} = \frac{\rho(x, y)^2}{4\alpha(s)\left(\int_0^t \alpha(r)^{-1} dr\right)^2}.\end{aligned}$$

The proof is completed by integrating the inequality above with respect to s over $[0, t]$. □

Now we can prove the entropy cost inequality.

Proof of (9.3). Replacing f by $P_t f$ in Lemma 9.2.2 we obtain

$$P_t \log P_t f(y) \leq \log P_{2t} f(x) + \frac{\rho(x, y)^2}{4 \int_0^t \alpha(r)^{-1}\, dr}. \tag{9.9}$$

First integrate the inequality (9.9) above with respect to a coupling measure of $\mu(dx)$ and $f(y)\mu(dy)$, by using the invariance of P_t with respect to μ and then making infimum over all couplings of $\mu(dx)$ and $f(y)\mu(dy)$, we get the following inequality

$$\mu(f P_t \log P_t f) \leq \mu(\log P_{2t} f) + \frac{W_2(f\mu, \mu)}{4 \int_0^t \alpha(r)^{-1} dr}. \tag{9.10}$$

By Jensen's inequality and the invariance of the measure μ with respect to P_t, we see

$$\mu(\log P_{2t} f) \leq \log \mu(P_{2t} f) = \log \mu(f) = 0. \tag{9.11}$$

Moreover, by using the symmetry of P_t with respect to the invariant measure μ, we know

$$\mu(f P_t \log P_t f) = \mu(P_t f \log P_t f) = \mathrm{Ent}_\mu(P_t f). \tag{9.12}$$

Hence from the facts (9.11) and (9.12), we deduce from (9.10) to get

$$\mathrm{Ent}_\mu(P_t f) \leq \frac{W_2(f\mu, \mu)}{4 \int_0^t \alpha(r)^{-1} dr}.$$

□

9.3 Proof of HWI Inequality

Proof of Theorem (9.4). Without loss of generality, we can assume $f \in \mathscr{E}_A(\mathbb{H})$ and that f is bounded below by a strictly positive constant.

Since the semigroup P_t is symmetric with respect to μ, we know

$$\int_{\mathbb{H}} L_2 F \, d\mu = 0, \quad \int_{\mathbb{H}} F L_2 G \, d\mu = \int_{\mathbb{H}} G L_2 F \, d\mu$$

for every $F, G \in D(L_2)$. Using these facts, we have

$$\begin{aligned}
&- \frac{d}{ds} \left(\int_{\mathbb{H}} (P_s f)(\log P_s f) \, d\mu \right) \\
=& - \int_{\mathbb{H}} \left[(L_2 P_s f)(\log P_s f) + P_s f \cdot \frac{L_2 P_s f}{P_s f} \right] d\mu \\
=& \frac{1}{2} \int_{\mathbb{H}} \left[L_2 (P_s f \log P_s f) - (L_2 P_s f)(\log P_s f) - (P_s f)(L_2 \log P_s f) \right] d\mu \\
=& \int_{\mathbb{H}} \Gamma(P_s f, \log P_s f) \, d\mu \\
=& \int_{\mathbb{H}} \left\langle R^{1/2} D P_s f, R^{1/2} D \log P_s f \right\rangle d\mu.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{Ent}_\mu(f) &= \int_{\mathbb{H}} f \log f \, d\mu \\
&= - \int_0^t \frac{d}{ds} \left(\int_{\mathbb{H}} (P_s f)(\log P_s f) \, d\mu \right) ds + \mathrm{Ent}_\mu(P_t f) \\
&= \int_0^t \int_{\mathbb{H}} \left\langle R^{1/2} D P_s f, R^{1/2} D \log P_s f \right\rangle d\mu \, ds + \mathrm{Ent}_\mu(P_t f) \\
&= \int_0^t \int_{\mathbb{H}} \frac{|R^{1/2} D P_s f|^2}{P_s f} \, d\mu \, ds + \mathrm{Ent}_\mu(P_t f).
\end{aligned} \tag{9.13}$$

By the fact that P_s is a Markov kernel, we have, for any measurable function F and G,

$$(P_s G)^2 \leq P_s \left(\frac{G^2}{G} \right) P_s F. \tag{9.14}$$

Applying the inequality (9.14) and using the estimate (9.5), we can obtain

$$\left| R^{1/2} D P_s f \right|^2 \leq \alpha(s) \left(P_s |R^{1/2} D f| \right)^2 \leq \alpha(s) \left(P_s \frac{|R^{1/2} D f|^2}{f} \right) P_s f. \tag{9.15}$$

Therefore, by substituting the estimate (9.15) into (9.13), and applying the entropy cost inequality (9.3), we have

$$\begin{aligned}\mathrm{Ent}_\mu(f) &\le \left(\int_0^t \alpha(r)\,dr\right) I(f) + \mathrm{Ent}_\mu(P_t f)\\ &\le \left(\int_0^t \alpha(r)\,dr\right) I(f) + \frac{1}{4\int_0^t \alpha(r)^{-1}\,dr} W_2(f\mu,\mu)^2 \\ &= \frac{M(1-\mathrm{e}^{-2\omega t})}{2\omega} I(f) + \frac{\omega}{2M(\mathrm{e}^{2\omega t}-1)} W_2(f\mu,\mu)^2.\end{aligned} \tag{9.16}$$

The proof will be completed by minimizing the right side of the above inequality. The minimizing procedure is explained in the following.

Denote $a = I(f),\ b = [W_2(f\mu,\mu)]^2$,

$$\eta(t) = \frac{M(1-\mathrm{e}^{-2\omega t)}}{2\omega} \quad \text{and} \quad \xi(t) = \frac{\omega}{2M(\mathrm{e}^{2\omega t}-1)}.$$

We need to minimize $h(t) := a\eta(t) + b\xi(t)^{-1}$. Solve the equation $h'(t) = 0$ we obtain

$$\sqrt{\frac{a}{b}}[1-\mathrm{e}^{-2\omega t}] = \omega.$$

Consequently, we have

$$\eta(t) = \frac{1}{2}\sqrt{\frac{a}{b}}M \quad \text{and} \quad \xi(t)^{-1} = \frac{1}{2}M\left(\sqrt{\frac{a}{b}} - \omega\right).$$

Therefore

$$h(t) = ah(t) + b\xi(t)^{-1} = M\left(\sqrt{ab} - \frac{\omega}{2}b\right).$$

□

Appendix A

Controllability of Infinite Dimensional Linear System

This appendix is based on the book by Zabczyk [Zab08, Part IV, Chapter 2]. See also the books by Da Prato and Zabczyk [DPZ92, Appendix B] or [DPZ02, Appendix B].

We briefly introduce the comparison of the images of linear operators, and some basic results on null controllability of linear control system.

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let T_1 and T_2 be two linear and bounded operators on$\mathbb{H}$.

The following theorem can be found, for example, [Zab08, Part IV, Theorem 2.2] or [DPZ92, Proposition B.1].

Theorem A.0.1. *The inclusion $T_1(\mathbb{H}) \subset T_2(\mathbb{H})$ holds if and only if there exists a constant $c > 0$ such that $|T_1^* x| \le c|T_2^* x|$ for every $x \in \mathbb{H}$.*

Consider the following linear control system on $\mathbb{H}$

$$\begin{cases} dx_t = Ay_t \, dt + Bu_t \, dt, \\ x_0 = x \in \mathbb{H}, \end{cases} \tag{A.1}$$

where A is the generator of a semigroup of operators S_t for $t \ge 0$ on $\mathbb{H}$, B is a linear bounded operator on $\mathbb{H}$, and $u(\cdot)$ is a $\mathbb{H}$-valued Bochner integrable (see [PR07, Appendix A]) function on $[0, t]$ for every $t \ge 0$.

A weak solution of the control equation (A.1) is given by

$$x_t = S_t x + \int_0^t S_{t-s} B u_s \, ds, \quad t \ge 0. \tag{A.2}$$

For every $t \geq 0$, the variable x_t denotes the *state of the system* in $\mathbb{H}$, the variable u_t represent a *control, strategy or input of the system* .

We say that a *control* u *transfers a state* x *to a state* y at the time $T > 0$ if $x_T = y$ with initial condition $x_0 = x$. We also say that the state x can be *steered* to state y at time T or that the state y is *reachable* or *attainable* from x at time T.

We are especially interested at the case when the state is transferred to state 0 at some fixed time $T \geq 0$.

Define

$$Q_T x = \int_0^T S_t B B^* S_t^* x \, dt, \quad x \in \mathbb{H}. \tag{A.3}$$

We call Q_T as *controllability operator*.

For every $x \in \mathbb{H}$, the function $u\colon u \mapsto S_u B B^* S_u\, x$ is continuous on $[0, T]$ and the Bochner integral in (A.3) is well defined. Moreover, it is obvious that the operator Q_T is linear, continuous, self-adjoint and positive definite.

The following theorem is a special case of [Zab08, Part IV, Theorem 2.3].

Theorem A.0.2. *(1) There exists a strategy $u(\cdot)$ which is square (Bochner) integrable on $[0, T]$ transferring state x to 0 in time T if and only if*

$$S_T x \in Q_T^{1/2}(\mathbb{H}).$$

(2) Among the strategies transferring state x to 0 in time T, there exists exactly one strategy $\hat{u}(\cdot)$ which minimizes the functional $J_T(u) = \int_0^T |u(s)|^2 \, ds$. Moreover,

$$J_T(\hat{u}) = |\Gamma_T x|^2, \tag{A.4}$$

where $\Gamma_T = Q_T^{-1/2} S_T$.

(3) If $S_T x \in Q_T(\mathbb{H})$, then the strategy $\hat{u}(\cdot)$ is given by

$$\hat{u}_t = -B^* S_{T-t}^* Q_T^{-1} S_T \, x \quad t \in [0, T].$$

We say that the system (A.1) is *null controllable* in time T if arbitrary state $x \in \mathbb{H}$ can be transferred to 0 in time T. $J_T(u)$ is the energy for driving x to 0 under the control u.

By Theorem A.0.2, we have the following characterizations ([Zab08, Part IV, Theorem 2.6]).

Theorem A.0.3. *The following conditions are equivalent to each other.*

(1) Control system (A.1) *is null controllable in time* $T > 0$.

(2) There exists a constant $c > 0$ *such that for all* $x \in \mathbb{H}$

$$\int_0^T |B^* S_t^* x|^2 \, dt \geq c |S_T^* x|^2.$$

(3) The image of S_T *is included in the image of* $Q_T^{1/2}$*:*

$$S_T(\mathbb{H}) \subset Q_T^{1/2}(\mathbb{H}). \tag{A.5}$$

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Notations

$\mathbb{N}$	set of positive integer numbers $\{1, 2, 3, \dots\}$.
$a \vee b$	the larger of numbers a and b in $\mathbb{R}$.
f^+	$= f \vee 0$, the positive part of f.
α, β	are conjugate number $1/\alpha + 1/\beta = 1$.
S_n	all $n \times n$ symmetric nonnegative definite real matrix.
$\mathbb{R}^d$	d-dimensional Euclidean space.
$\mathbb{H}$	real separable Hilbert space.
$\langle \cdot, \cdot \rangle$	the usual inner product on $\mathbb{R}^d$ or $\mathbb{H}$.
$\lvert \cdot \rvert$	the norm on $\mathbb{H}$ corresponding with respect to $\langle \cdot, \cdot \rangle$.
$\mathbb{H}_0$	the Cameron-Martin space
$\langle \cdot, \cdot \rangle_0$	an inner product on $\mathbb{H}_0$ (See Page 19).
$\lvert \cdot \rvert_0$	the norm corresponding with $\langle \cdot, \cdot \rangle_0$ on $\mathbb{H}_0$.
$B_x(r)$	$= \{y \in \mathbb{H} \colon \lvert x - y \rvert \leq r\}$, ball with radius r and center x.
$\mathcal{B}(\mathbb{H})$	the Borel σ-algebra on $\mathbb{H}$.
$\mathscr{B}(\mathbb{H})$	the space of bounded measurable functions on $\mathbb{H}$.
$\mathscr{B}_b(\mathbb{H})$	the space of bounded measurable functions on $\mathbb{H}$.
$\mathscr{B}_b^+(\mathbb{H})$	the space of positive bounded measurable functions on $\mathbb{H}$.
$\mathscr{C}(\mathbb{H})$	the space of continuous functions on $\mathbb{H}$.
$\mathscr{C}_b(\mathbb{H})$	the space of bounded continuous functions on $\mathbb{H}$.
$\mathscr{C}^n(\mathbb{H})$	the space of n-th continuously differentiable functions on $\mathbb{H}$.
$\mathscr{C}_b^+(\mathbb{H})$	the space of positive bounded and continuous functions on $\mathbb{H}$.
$\mathscr{C}^\infty(\mathbb{H})$	the space of smooth functions on $\mathbb{H}$.
$\mathscr{C}_0^\infty(\mathbb{H})$	the space of compact supported smooth functions on $\mathbb{H}$.
$\lVert f \rVert_p$	$= \big(\int_{\mathbb{H}} \lvert f \rvert^p \, d\mu \big)^{1/p}$ for $p \geq 1$.
$\lVert f \rVert_\infty$	$= \inf\{C > 0 \colon \lvert f(x) \rvert \leq C \text{almost every}\}$.
$L^p(\mathbb{H}, \mu)$	the space of measurable functions on $\mathbb{H}$ such that $\lVert f \rVert_p < \infty$.
$(\mathbb{V}, \mathbb{H}, \mathbb{V}^*)$	evolution triple.
T^*	adjoint operator of linear bounded operator T.

$Q^{1/2}$	square root of Q.
T^{-1}	pseudo inverse of T.
A	generator of S_t (a multivalued operator in Chapter 8)
S_t	C_0-semigroup on $\mathbb{H}$.
Q_t	$= \int_0^t S_u R S_u^* \, du$, controllability operator.
Γ_t	$= Q_t^{-1/2} S_t$.
$\sigma_2 \ll \sigma_1$	σ_2 is absolutely continuous with respect to σ_1.
$\sigma_2 \perp \sigma_1$	σ_2 is singular (orthogonal) to σ_1.
$\sigma_2 \approx \sigma_1$	σ_2 is equivalent with σ_1.
$d\sigma_2 / d\sigma_1$	the Radon-Nikodým derivative of σ_2 with respect to σ_1.
σ^{ac}	the absolute continuous part of measure σ.
σ^{s}	the singular part of measure σ.
$\hat{\mu}$	Fourier transformation of measure μ.
$\mathbb{E}_{\mathbb{P}}$	expectation with respect to measure $\mathbb{P}$.
$N(m, Q)$	Gaussian measure with mean m and covariance Q.
(b, R, ν)	characteristic triplet of some Lévy process or infinite divisible measure with drift b, covariance R and Lévy measure ν.
$D(b, R, \nu)$	infinite divisible measure with characteristic triplet (b, R, ν).
W_t	standard or cylindrical Wiener processes.