Implementation in Environments with Limited or Delegative Enforcement Power

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1 Introduction and Motivation

Implementation theory is concerned with the question of which (*social*) *choice correspondences* can be *implemented* by the use of certain *mechanisms* in a certain *environment*. Although there were some earlier contributions to what is today known as the theory of implementation, Hurwicz [18] is widely seen as the classic and seminal paper which started the implementation literature.

Standard implementation theory considers situations that can be described (in the sense that all relevant aspects are captured) by an *environment* consisting of

(i) a set of entities and their corresponding interpretations: a designer, a set of agents, a set of feasible outcomes, a set of types for each agent, a set of possible type profiles / states, a type-contingent preference relation or utility function over the set of feasible outcomes for each agent, and a set of *mechanisms* available to the designer, each specifying the agents' possible actions and their respective outcomes in the form of an outcome function (which must be independent of states as a consequence of the assumptions outlined in (ii) and (iii)),

(ii) assumptions on the information structure: in particular, the designer does not know/observe the actual state,

(iii) assumptions on the enforcement structure: for any available mechanism, the designer is able to force the agents to participate in this mechanism according to its rules and is able to enforce the elements of the outcome space as prescribed by its outcome function as a consequence of all agents' actions, and

(iv) assumptions on the behaviour of the agents: the agents play according to a certain noncooperative solution concept,

and a (*social*) *choice correspondence* for the environment, which specifies, for each possible type profile (reflecting the agents' preferences over the set of feasible outcomes), the (socially) desirable outcomes for this state of the environment.

In the remainder of this paper, we will call such an environment, i.e., an environment consisting of (i) to (iv), a *classical environment*.

Given a classical environment and a (social) choice correspondence for this environment, the theory of implementation is concerned with the question of whether or not there exists a mechanism available to the designer that *implements* the (social) choice correspondence with respect to the noncooperative solution concept under consideration, i.e., whether or not there exists a mechanism such that, for each possible type profile, the agents' actions that conform with the noncooperative solution concept result in desirable outcomes for this state of the environment.

Throughout this paper, we will always assume that there is *complete information* and, in particular, *no asymmetric information* between the agents, i.e., all agents are informed about the actual state of the environment. We briefly discuss the *incomplete information* case in Chapter 6, which is concerned with concluding remarks and ideas for future research.

In many situations, however, there are additional relevant aspects which are not captured by the assumptions of the standard theory. In other words, the assumptions are too restrictive to make the theory applicable. The situation in question may, for example, inherit commitment or credibility issues, or may not comply with the assumptions on the enforcement structure outlined in (iii) above.
The latter issue is raised in Hurwicz's [19] "Implementation and Enforcement in Institutional Modeling". Hurwicz [20] uses the following words: "... in general, there is nothing in a specific game form, prescribing particular strategy domains and outcome functions that would prevent players from resorting to 'illegal' strategies, nor is there automatic assurance that outcomes specified by the outcome function will occur unless the required apparatus is in place." Hurwicz suggests "to embed the 'desired' game form in" what he calls "the 'natural' game form, including all feasible behaviors (and not merely those that are 'legal' according to the desired game form) and their natural consequences as the 'natural' outcome function." He uses "the term 'genuine implementation' to refer to the procedures to make ... an institutional arrangement effective."
The idea that something has to be enforced in the absence of an external enforcement institution/mechanism/authority (such as a court) is often referred to by the term "self-enforcement". Self-enforcement issues are addressed, for example, in the literature on contracting, constitutional (rules) economics, international (negotiated) agreements (in particular, international environmental agreements), and on decision-making in international organizations. The usual approach in this literature, to analyse the decision possibilities within the appropriate time horizon, includes the consideration of expected future payoffs, or the problem embedded into an infinitely repeated game (e.g., cooperative behaviour sustained as an equilibrium of an infinitely repeated Prisoner's Dilemma type of game).
Within his discussion of "Ex Post Individual Rationality, Renegotiation, and Credibility", Jackson [22] uses the following words: "At several points I have mentioned that various forms of implementation rely on the belief that the outcomes of the mechanism will be enforced, even if they are 'bad' from society's point of view ex-post. This

can be problematic, to the extent that the positive results depend on such outcomes being used by the mechanism and such beliefs holding. If, for example, a mechanism is constructed to assist bargainers in reaching an efficient agreement, then it is questionable to assume that highly inefficient outcomes will be allowed to stand off (or on) the equilibrium path." Two references mentioned by Jackson as being concerned with credibility or commitment issues are Baliga, Corchon and Sjöström [3], and Baliga and Sjöström [4]. Both articles consider "interactive implementation", i.e., implementation "when the planner is a player" (quoting [3]).

Commitment issues are recently addressed, for example, in the context of auctions, contracting/principal-agent analysis, and mechanism design. Vartiainen [55], for example, analyses "auction design under the hypothesis that parties do not have any commitment power: the seller is allowed to change rules of the auction mechanism at any stage of the game without any cost, and the buyers cannot ever be forced to participate (the value of their outside option is fixed)." Skreta [49] "characterizes the optimal auction in a two-period model under non-commitment. In the first period, a risk-neutral seller designs a mechanism to sell an indivisible object. If no trade takes place, the seller cannot commit not to try to sell the object in the second period." Mitusch and Strausz's [30] "paper studies the role of mediators in a principal-agent problem with ex ante hidden information when the commitment power of the principal as contract designer is limited." Bester and Strausz's [8] "paper provides a modified version of the revelation principle for environments in which the party in the role of the mechanism designer cannot fully commit to the outcome induced by the mechanism." Their "results apply to contracting problems between a principal and a single agent." Bester and Strausz [7] consider the multi-agent case.

In this paper, we analyse the implementation of (social) choice correspondences in *environments with limited enforcement power*, i.e., in environments in which the outcome space is not fully enforceable by the designer, and in which enforcement capabilities on outcomes can be expressed as a function of (and only of) all coalitions of individuals, thereby making the outcome function of a mechanism dependent upon the environment. In such environments, the designer may not be able to fully enforce the outcome functions of those mechanisms he can enforce the agents to participate in (according to its rules). It is in this respect that we deviate from the assumptions of the standard theory, and that we extend the applicability of the theory of implementation.

Enforcement limitations on the side of the designer may be due to non-verifiability. See, for example, our comments concerning Trockel's [52] approach to the implementation of cooperative solution concepts in the remainder of this chapter.[1] They also may be due to a non-existing or non-effective official legal system (combined with non-available private intermediaries). This may be the case, for example, in transition economies, and holds true for many agreements between sovereign states and for international contracts. A third reason for enforcement limitations may be property rights. In the words of Jackson and Palfrey [24a], "a ... source of difficulty with enforcement relates to property rights that are exogenous to a mechanism and impose state-contingent constraints on a social choice rule. In many settings individuals have inalienable rights that guarantee them some outcomes in some states of the world." Similarly to property rights, decisions of many legislative bodies and international organizations are based on some voting system (which may depend on the type of issue under consideration).

Consider, for example, the case in which two or more parties find themselves in a situation that could be represented by a bargaining game (or a cooperative game), and in which these parties consult a 'specialist' or 'mediator' in order to help them solving their decision problem. Then, it might be reasonable to assume that they agree to participate in some mechanism designed to assist them in their decision problem. However, they might not be willing to commit themselves to actually implement the outcome suggested by the mechanism. Or, there might not be an institution that could enforce such a commitment (maybe due to non-verifiability).

These parties could be the different member states of the European Union and their respective representatives within the Council, facing a decision on how to divide the benefits and costs of a public project. Different (sub-)divisions within a firm with a decentralized decision structure could face a decision on how to divide the benefits and costs of a certain project. A married couple could face a divorce and thus a decision on how to divide their belongings. Further examples include international environmental agreements (e.g., the Kyoto Protocol), bilateral conflicts between countries throughout the world, wage negotiations between a trade union and employers (of a certain branch), and the (re-)consideration of a specific reform package which has to be approved by the German Bundestag and Bundesrat (in which different parties may have different majorities).

The idea to express enforcement capabilities as a function of the coalitions can be found, for example, (indirectly) in Gardenfors [13] and (directly in) Moulin and Peleg [35]. Gardenfors defines a "rights-system" as a set of "rights" (satisfying certain conditions) and a right "as a possibility for a group G of individuals to restrict the set of

[1] See also Hahmeier [15], Chapter 7 (on remarks and ideas for future research).

social states to a subset X of S." Similarly, Moulin and Peleg's "effectivity function", a correspondence E (satisfying certain conditions) from the set of coalitions to the set of subsets of the outcome space , "specifies for every coalition T of agents and subset B of outcomes whether or not T is effective for B, i.e., can force the final decision within B".[2] In other words, $B \in E(T)$ allows for the interpretation that coalition T can force the outcome to be an element of B. An effectivity function (and also a rights-system), however, differs conceptionally from our *enforcement structure*, which is a correspondence from the set of coalitions to the outcome space, specifying, for each coalition, the set of outcomes that this coalition is able to enforce. In particular, the concept of an effectivity function is 'richer' than our concept of an enforcement structure in the following sense. For every enforcement structure e, there exists an effectivity function E which completely 'reflects' e. The converse is not true in general.[3]

In environments in which the outcome of a mechanism is not necessarily enforceable by the designer, agents' beliefs (conjectures/perceptions) about the future become important aspects of their decision-making. Our analysis is based on the assumption that each agent has beliefs about what will happen if an outcome suggested by a mechanism is not being implemented. As we will explain in more detail in Section 3.1, these beliefs are in terms of preferences. In other words, our analysis is based on each agent's ordinal

[2]Peleg [38] offers "an axiomatic approach for the investigation of rights by means of game forms." He "introduces a definition of constitution which is a generalization of Gardenfors's definition of rights system", and shows "how a constitution leads in a natural way to an effectivity function which describes the 'distribution of power' in a given society as a result of the assignment of rights ...". Peleg analyses mechanisms ("game forms") that "represent" a constitution in the sense that the effectivity function associated with the mechanism (as introduced by Moulin and Peleg [35]) coincides with the effectivity function corresponding to the constitution. Peleg and Winter [40] and Peleg, Peters and Storcken [39] analyse "constitutional implementation", i.e., the implementation of a social choice correspondence by a mechanism ("game form") in Nash Equilibrium such that the effectivity functions, the one associated with the social choice correspondence and the one associated with the mechanism (as introduced by Moulin and Peleg [35]), coincide. The effectivity function associated with the social choice correspondence is interpreted as specifying or representing a constitution, e.g., in the sense of Peleg [38]. In the words of Peleg and Winter, "... constitutional implementation roughly requires that the implementing game form will induce the same distribution of power as that of the implemented SCC, which we assume to be compatible with some pre-specified constitution."

[3]Consider an outcome space X and a set of agents $N \equiv \{1, \dots, n\}$, $n \in \mathbb{N}$. Given an enforcement structure $e : \mathcal{P}(N) \backslash \{\emptyset\} \Rightarrow X$, the effectivity function $E_e : \mathcal{P}(N) \backslash \{\emptyset\} \Rightarrow \mathcal{P}(X) \backslash \{\emptyset\}$ defined by $E_e(S) := \{X\} \cup \{\{x\} \mid x \in e(S)\} \; \forall \; S \in \mathcal{P}(N) \backslash \{\emptyset, N\}$, $E_e(N) := \mathcal{P}(X) \backslash \{\emptyset\}$, completely 'reflects' enforcement structure e. On the other hand, an enforcement structure cannot 'reflect' an effectivity function E satisfying $\{x_1, x_2\} \in E(S)$ for some $(x_1, x_2, S) \in X \times X \times \mathcal{P}(N) \backslash \{\emptyset\}$.

evaluation of the (unknown) future. Beliefs may differ from one agent to the other, and even if agents have common beliefs on outcomes, these beliefs are solely beliefs and are respected for in a different way than outcomes actually implemented by a mechanism. Reconsidering our bargaining example, the two parties (individuals/firms/countries) might share the (pessimistic) belief that, independent of the outcome suggested by the mechanism, they will end up at the status quo if the suggested outcome is not being implemented. Our assumption is that the decision of whether or not to implement the suggested outcome is based on these beliefs once and for all. This implies, in particular, that beliefs remain constant until an implementation decision is realized or contracted upon in a binding manner.[4]

A main part of this paper, Chapter 3, is devoted to an analysis of the implementability of (social) choice correspondences in environments with limited enforcement power, our focus being on sufficient and necessary conditions for the implementation in Nash Equilibrium, on an extension of the Gibbard-Satterthwaite Theorem, and on a comparison of these environments to their *corresponding classical environments*. In particular, we show that no general implication on the implementability of a (social) choice correspondence between a limited enforcement environment and its corresponding classical environment can be drawn. Our discussion at the end of Chapter 3 will indicate to what extent the implementation decision of the agents following a Nash Equilibrium of a strategic mechanism can be 'copied' by a Subgame Perfect Nash Equilibrium analysis of this mechanism followed by an appropriate extensive decision procedure. Note, however, that these two alternatives require different assumptions, in particular, on the behaviour of the agents.

Chapter 4 is concerned with the notion of implementation that arises for classical environments in which the designer is able to impose an enforcement structure on the agents and to influence their beliefs by specifying the outcome that will be realized in case that the suggested outcome is not being implemented (by a coalition that is able to do so). In *environments with delegative enforcement power* the designer can, in line with classical implementation theory, force the agents to participate in one of a certain set of mechanisms, and is able to enforce each of the feasible outcomes. In addition, and in contrast to standard implementation theory, we now assume that the designer is able to impose one of a certain set of *enforcement and default structure assignments (EDS assignments)* on the agents, thereby capturing applications in which the standard analysis of strategic and extensive game forms does not reflect the enforcement

[4] Nonverifiable outcomes may have an equivalent which can be contracted upon in a binding manner.

capabilities of the designer and the behaviour of the agents.
Consider, for example, a firm (with a centralized decision structure) in which several (sub-)divisions find themselves in a situation that could be represented by a bargaining (or a cooperative) game, e.g., resulting from a decision on how to divide the costs and benefits of a certain project. If relevant information is dispersed among these divisions and not known to the management, a mechanism designed to assist the management in its decision process could impose an enforcement structure on the divisions.[5]
Besides briefly stating the simple counterparts of our sufficient and necessary conditions from Chapter 3 to environments with delegative enforcement power, specific attention is paid to (what we will refer to as) *replica environments*.[6] If all possible EDS assignments are available to the designer, the number of available assignments increases in the number of agents at an increasing rate. Our necessary condition for the implementability of (social) choice correspondences in these environments is independent of any *replica agent*, thereby reducing the maximum number of assignments that have to be checked.
Chapter 4's final section compares environments with delegative enforcement power to their *corresponding classical environments* with respect to the Nash-implementability of (social) choice correspondences. In particular, we show that delegative enforcement power can positively affect the Nash-implementability, and that even the availability of all EDS assignments might not be sufficient for the Nash-implementability of a (social) choice correspondence in environments in which all mechanisms arising from strategic game forms are available to the designer.

In Chapter 5, as an application (referring to our example above), we discuss implications of limited enforcement power on the implementation of cooperative solution concepts, i.e., on the question of whether or not there exists a mechanism that 'implements' a certain cooperative solution concept in a cooperative game situation the exact characteristics of which are not known to the designer. Approaches to the implementation of cooperative solution concepts from the literature can be divided according to whether they are based on a purely welfaristic outcome space or whether they require some addititional structure. We concentrate our analysis on one approach of

[5] Within the literature on contract theory/principal-agent analysis, several papers address the delegation/decentralization of decision-making authority (or contracting rights) in organizations, e.g., articles concerning the design of jobs in firms. See, for example, Aghion and Tirole [1], who develop "a theory of the allocation of formal authority (the right to decide) and real authority (the effective control over decisions) within organizations".

[6] The idea to consider replicas, in the context of general equilibrium theory or cooperative games, can be found, for example, in Debreu and Scarf [10], Shapley and Shubik [47], and in Wooders [57].

each group. Trockel's [52] approach is based on traditional cooperative games that specify the utility profiles available to each coalition, and belongs to the first group. Dagan and Serrano [9] consider games explicitly specifying "physical outcomes" that each coalition can achieve and that agents can evaluate according to some rational preference relation (over these "physical outcomes").[7]
Whereas Trockel's approach leads to a rather positive result, which, in particular, has positive implications for the implementability of the Nash Bargaining Solution concept and (as we will show) of the Core concept, Dagan and Serrano come to a rather negative result in the form of a necessary condition, which, in particular, and in contrast to the Core concept, affects the implementability of the Nash Bargaining Solution concept. Defining a set of single-valued solution concepts as the outcome space, Trockel's approach placed in classical environments implies that the designer can enforce agents to realize a single-valued solution concept without knowing the actual cooperative game — an assumption which might not be an adequate description of many real-world situations. During our analysis in this part of the paper, we will approach the question to what extent, i.e., for what assumptions on the structure of beliefs, Trockel's positive result and its implications extend to environments with limited enforcement power, in which the designer has no enforcement power on solution concepts. The final section of Chapter 5 is devoted to a discussion and an extension of Dagan and Serrano's result to environments with limited enforcement power.

The most recent research related to our paper is by Jackson and Palfrey [24a], who "focus on remedying a specific, but critical, weakness of implementation theory: its use of implausible outcomes off the equilibrium path to enforce equilibrium behaviour and/or to 'break' undesirable equilibria (i.e., assure that undesired strategy combinations are not equilibria). The implausibility stems from the assumption that the outcome function is fully enforceable, which is not the case in many applications."
Jackson and Palfrey (Section 2 and 3) extend Maskin's results on sufficient and necessary conditions for the implementation in Nash Equilibrium to an environment in which the outcome of a mechanism is converted in a state-contingent way via some commonly known "generalized reversion function" G, formally a function from states and outcomes into outcomes. They also present examples showing that a generalized

[7] For a discussion of these two approaches and a third approach by Bergin and Duggan [6], who "suppose that each coalition has some non-empty set ... of conceivable joint plans of action" and allow for the "feasibility" of a coalition's joint plan to depend upon the joint plan of the remaining players, see Hahmeier [15].

reversion function can affect the Nash-implementability of a (social) choice correspondence in both directions. Our concept of implementation is based on Jackson and Palfrey's [24a] notion of "G-Nash implementation". As part of our analysis, we show how their results extend to environments with limited enforcement power.

As a special case of a generalized reversion function, Jackson and Palfrey consider (the consequences of) "voluntary implementation", i.e., implementation in environments in which each individual, after the mechanism has been played, is allowed to veto the outcome suggested by the mechanism. In case of a veto, an exogeneous and commonly known state-contingent "reversion function" then determines the final outcome. Jackson and Palfrey mention two possible generalizations of voluntary implementation "within the framework of the G-function": an outcome-contingent reversion function, and (more general and) state-contingent "blocking coalitions" (e.g. "majority rule approval of the outcome of the mechanism"). The idea that each agent or, more general, blocking coalitions (although not state-contingent blocking coalitions) are allowed to veto the outcome suggested by the mechanism, finds its counterpart in our concept of an enforcement structure (cf., in particular, our comments in Section 4.1). Jackson and Palfrey suggest "natural applications" of voluntary implementation "to problems in which there is a fixed status quo outcome that any agent can revert to." An example mentioned is that of exchange economies, where "it is often natural to assume that each individual can protect their initial endowment." However, after the mechanism has suggested an outcome, each players' only choice is to either accept the suggested outcome or to veto and change to the outcome determined by the reversion function. Their concept of voluntary implementation does not cover applications in which the agents, at this stage of the mechanism, have more options available. And, since these additional options might have an influence on each agent's decisions in the mechanism, they should be respected for in the analysis. In the case of an exchange economy, for example, in which any exchange requires (and requires only) the agreement of all agents participating in this exchange, the agents might, after the mechanism has suggested an outcome and a veto has occured, still face the same situation as before, with the same exchange possibilities.

In Section 4 of their paper, Jackson and Palfrey address this issue as follows: "If an individual vetoes ... it is unnatural to suppose that the world stops at that moment. For example, in a pure exchange environment, if an agent vetoes ... and the endowment results, the individuals in the economy could simply play the mechanism again." Referring to "how game theorists have modeled bargaining", Jackson and Palfrey point out "that the notion of voluntary trade implies that if there are still gains to trade to be exploited, the agents involved will continue playing some game." Jackson and Palfrey "endogenize the generalized reversion function" by analysing a model that allows each individual to either accept the outcome suggested by a strategic mechanism or to veto and thereby forcing the mechanism to be replayed.[8]

Of course, Jackson and Palfrey's reversion function could, in principle, be interpreted in terms of beliefs about the future along our lines outlined above: the outcome determined by the reversion function evaluated at a certain state represents the outcome that, in this state, all agents believe to end up with, if the outcome suggested by the mechanims is not being implemented by a coalition that is able to do so. However, this interpretation entails three restrictive aspects. First of all, the agents are restricted to have beliefs in terms of outcomes (which, for example, does not allow for probabilistic beliefs in outcomes or discounting). Second, the agents are not allowed to have different beliefs. And third, the notion of implementation does not differentiate between outcomes suggested by the mechanism on the one hand and outcomes interpreted as common beliefs on the other hand.

Other papers that address issues related to our research include those already mentioned by Jackson and Palfrey [24a] (and Jackson [22]): Ma, Moore and Turnbull [25], Maskin and Moore [27], and Jackson and Palfrey [23].

In the words of Jackson and Palfrey [24a], "Ma et al. . . . were the first to point out the importance of imposing an individual rationality constraint both in and out of equilibrium." Ma, Moore and Turnbull analyse a one-principal–two-agents setting which allows each agent to sign an enforceable contract on some production/payment schedule with the principal or to refuse to sign the offered contract, in which case the agent expects a certain reservation utility level.

Maskin and Moore use the following words: "Unfortunately, what happens out of equilibrium can profoundly affect what outcomes can occur in equilibrium. In the absence of renegotiation, we might be able to sustain an outcome as an equilibrium by threatening agents with dire consequences should any of them deviate. But if an agent forecasts that those unfavourable consequences would ultimately be renegotiated, he might no longer have sufficient incentive to conform." Maskin and Moore examine implementation in an environment in which the outcome of a mechanism is converted in a

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[8] Jackson and Palfrey analyse "the game form where in a given period the mechanism is played, then agents are called on to veto sequentially, and the process terminates . . . if there is no veto and starts over in the next period if there is a veto" with respect to Markov Perfect Equilibria "where agents do not veto when indifferent".

state-contingent way via some exogenous "renegotiation process" (formally a function from states and outcomes into outcomes), which is assumed to be common knowledge, Pareto-efficient, and individually rational (with respect to the mechanism's outcome).[9] In their Theorem 5, Maskin and Moore present an extension of Maskin's results on sufficient and necessary conditions for the implementation in Nash Equilibrium to their environment. As Jackson and Palfrey [24a] already point out, Maskin and Moore's renegotiation function is an example of a generalized reversion function.
Jackson and Palfrey [23] examine implementation under endogenous individual rationality constraints by analysing a dynamic environment in which, in the first of a finite number of discrete periods, "buyers and sellers are randomly matched into pairs and then play a bargaining game". Both the buyer and the seller can reject to trade at the price suggested by the mechanism, in which case there is no trade and "each is randomly rematched with a new potential trading partner in the next period", except for the last period. And, in the words of Jackson and Palfrey [23], "this places a natural individual rationality, or voluntary participation, constraint on the process: no buyer or seller will consummate a trade that leaves him or her worse off than the discounted expected value of their future rematching in the market."

The remainder of our paper starts with some definitions and results from the standard implementation literature in Chapter 2, and concludes with some remarks and suggestions for future research in Chapter 6.[10] A graphical illustration of the abstract relationship between the different environments considered throughout this paper can be found in Appendix K.

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[9]Building on the work of Maskin and Moore, Segal and Whinston [46] provide a "first-order characterization" of implementable social choice rules in specific two-agents environments with renegotiation, "paralleling Mirrlees's (1971) first-order analysis of standard mechanism design problems." Segal and Whinston do not constrain renegotiation to the set of possible outcomes of a mechanism. Agents have "induced utilities over the pre-renegotiation outcome prescribed by the mechanism (taking the renegotiation process into account)." It is in this freedom with respect to utility, that our approach is more closely related to that by Segal and Whinston than to that by Maskin and Moore. Similarly in this aspect, an article by Schwartz and Watson [45] "adds contracting and renegotiation costs to the standard 'mechanism design with ex post renegotiation' model (Maskin and Moore, 1999; Segal and Whinston, 2002)." For a critical assessment of Maskin and Moore's model, see Watson [56]. Watson studies "how renegotiation opportunities interact with the productive technology of contractual relationships" and relates his research to Hurwicz [20] who, in the words of Watson, "speaks of the importance of incorporating institutional constraints into design problems".

[10]Note that our presentation of definitions and results in Section 2.1, 2.2, 5.1, 5.4.1, and 5.2.1 is similar to that in Hahmeier [15].

2 Implementation in Classical Environments

2.1 Definitions

Throughout, let $\mathbb{N}_k$ denote the set $\{1, \ldots, k\}$ $\forall$ $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout this chapter, let $n \in \mathbb{N}$, $n \geq 2$. The following definitions are standard.[11]

2.1.1 Games in Strategic Form

Definition An *n-person game in normal form* (or *strategic form*) is a tuple $(N, \{S_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N})$, where

$N := \{1, \ldots, n\}$ is the set of players,

$S_i \neq \emptyset$ is player i's strategy/action set, and

$\tilde{u}_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ is player i's utility function, representing her rational (i.e., complete and transitive) preference relation over the set of possible strategy profiles.[12]

Note that each player i's utility function is defined to have an ordinal interpretation.[13]
An n-person normal form game is said to be finite if its set of strategy profiles $S := S_1 \times \ldots \times S_n$ contains only a finite number of elements.
For a strategic n-person game form $(N, \{S_i\}_{i \in N})$, we denote by $\mathcal{C}^n_{nfg}(\{S_i\}_{i \in N})$ the set of n-person normal form games that share game form $(N, \{S_i\}_{i \in N})$.

2.1.2 The Dominant Strategy Equilibrium Concept

Let $\Gamma \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N})$ be an n-person normal form game.

Definition A strategy profile $s \in S := S_1 \times \ldots \times S_n$ *constitutes a Dominant Strategy Equilibrium (DSE) of game* Γ if, for every player $i \in N$, s_i is a dominant strategy for player i, i.e., $\tilde{u}_i(s_i, \hat{s}_{-i}) \geq \tilde{u}_i(\hat{s})$ $\forall$ $\hat{s} \in S$.

Let DSE^n_{nfg} denote the DSE concept for the class of n-person normal form games, i.e., $DSE^n_{nfg}(\Gamma) := \{s \in S \mid s \text{ constitutes a DSE of game } \Gamma\}$.

[11] See, for example, Mas-Colell, Whinston, and Green [29], Osborne and Rubinstein [37], Jackson [22], Moore [32], and Maskin and Sjöström [28]. Note that our usage of the term 'game form' is not in line with most of the literature, which requires an outcome function to be part of a game form.

[12] Note that, throughout this paper, we restrict our analysis to rational preference relations that can be represented by a utility function. A rational preference relation is representable by a utility function, for example, if it is continuous (e.g., if the relation is defined over some finite space).

[13] Since different utility functions can represent the same preference relation, we can identify two different games in this respect.

2.1.3 The Nash Equilibrium Concept

Definition A strategy profile $s \in S$ *constitutes a Nash Equilibrium (NE) of game* Γ if, for every player $i \in N$, s_i is a best response to s_{-i}, i.e. $\tilde{u}_i(s) \geq \tilde{u}_i(\hat{s}_i, s_{-i}) \; \forall \; \hat{s}_i \in S_i$.[14]

Let NE^n_{nfg} denote the NE concept for the class of n-person normal form games, i.e. $NE^n_{nfg}(\Gamma) := \{s \in S \mid s \text{ constitutes a NE of game } \Gamma\}$.

2.1.4 Games in Extensive Form

In extensive form games, a set of histories H describes the possible sequences of players' actions, satisfying the following properties:

(i) The initial history, denoted by $\emptyset$, is an element of H,

(ii) if $(a^k)_{k=1,\ldots,K} \in H$, then $(a^k)_{k=1,\ldots,L} \in H \; \forall \; L \in \{1,\ldots,K-1\}$,

(iii) if $(a^k)_{k\in\mathbb{N}} \in H$, then $(a^k)_{k=1,\ldots,L} \in H \; \forall \; L \in \mathbb{N}$, and

(iv) if $(a^k)_{k\in\mathbb{N}}$ satisfies $(a^k)_{k=1,\ldots,L} \in H \; \forall \; L \in \mathbb{N}$, then $(a^k)_{k\in\mathbb{N}} \in H$.

A history $h \in H$ is said to be terminal in a set of histories H, if $h \neq \emptyset$ and either it is an infinite sequence or it is a finite sequence $h \equiv (a^k)_{k=1,\ldots,K}$ and there is no history $(b^k)_{k=1,\ldots,K+1}$ in H such that $a^k = b^k \; \forall \; k \in \{1,\ldots,K\}$.

For a set of histories H, we let Z_H denote the set of terminal histories in H and A_H denote the set of all possible actions in H.

Definition An *n-person game in extensive form (with possible simultaneous moves)* is a tuple $(N, H, p, \{\tilde{u}_i\}_{i\in N})$, where

$N := \{1,\ldots,n\}$ is the set of players,

H is the set of histories (satisfying properties (i) to (iv)),

$p : H \backslash Z_H \Rightarrow N$ is the player assignment, $p(h) \neq \emptyset$ denoting the set of players who act simultaneously after history h for every $h \in H\backslash Z_H$, and

$\tilde{u}_i : Z_H \to \mathbb{R}$ is player i's utility function, representing her rational preference relation over the set of terminal histories.

For an n-person extensive form game $\Gamma \equiv (N, H, p, \{\tilde{u}_i\}_{i\in N})$, player i's strategy set is $S^\Gamma_i := \{s_i : H_i \to A_H \mid s_i(h) \in A^i_H(h) \; \forall \; h \in H_i\}$, where $H_i := \{h \in H\backslash Z_H \mid i \in p(h)\}$ denotes the set of nonterminal histories after which player i has to move, and, for each nonterminal history $h \in H\backslash Z_H$ and each player $i \in p(h)$, $A^i_H(h) \subseteq A_H$ denotes the set of possible actions for player i after history h:

$$\{ \; (a_i)_{i\in p(h)} \in \prod_{i\in p(h)} A_H \mid (h, (a_i)_{i\in p(h)}) \in H \; \} = \prod_{i\in p(h)} A^i_H(h).$$

[14] In particular, each DSE of game Γ constitutes a NE of game Γ.

Each strategy profile $s \in S^\Gamma := S_1^\Gamma \times \ldots \times S_n^\Gamma$ determines a terminal history $O(s) \in Z_H$ and a utility level $\tilde{u}_i(O(s))$ for each player i.
Note that each player i's utility function is defined to have an ordinal interpretation.
An n-person extensive form game is said to be finite if its set of histories contains only a finite number of elements. An n-person extensive form game is a *game with perfect information* if its player assignment is single-valued.
For an extensive n-person game form (with possible simultaneous moves) (N, H, p), we denote by $\mathcal{C}^n_{efg}(H,p)$ the set of n-person extensive form games that share game form (N, H, p).

2.1.5 The Subgame Perfect Nash Equilibrium Concept

Let $\Gamma \equiv (N, H, p, \{\tilde{u}_i\}_{i\in N})$ be an n-person extensive form game.

The subgame of game Γ *that follows history* $h \in H\backslash Z_H$ is the extensive form game $\Gamma(h) := (N, H^h, p^h, \{\tilde{u}_i^h\}_{i\in N})$, where

$H^h := \{h' \mid (h,h') \in H\}$,

$p^h : H^h\backslash Z_{H^h} \Rightarrow N$ is defined by $p^h(h') := p((h,h'))\ \forall\ h' \in H^h\backslash Z_{H^h}$, and

$\tilde{u}_i^h : Z_{H^h} \to \mathbb{R}$ is defined by $\tilde{u}_i^h(h') := \tilde{u}_i((h,h'))\ \forall\ h' \in Z_{H^h}$.

Definition A strategy profile $s \in S^\Gamma \equiv S_1^\Gamma \times \ldots \times S_n^\Gamma$ *constitutes a Nash Equilibrium (NE) of game* Γ if s constitutes a Nash Equilibrium of the normal form game $(N, \{S_i^\Gamma\}_{i\in N}, \{u_i\}_{i\in N})$ (the strategic form of Γ), where, for each $i \in N$, $u_i : S^\Gamma \to \mathbb{R}$ is defined by $u_i(s) := \tilde{u}_i(O(s))\ \forall\ s \in S^\Gamma$.

Definition A strategy profile $s \in S^\Gamma$ *constitutes a Subgame Perfect Nash Equilibrium (SPNE) of game* Γ if, for every nonterminal history $h \in H\backslash Z_H$, the strategy profile $(s_1^h, \ldots, s_n^h)$ constitutes a NE of the subgame $\Gamma(h)$, where, for each strategy $s_i \in S_i^\Gamma$, s_i^h denotes the strategy in game $\Gamma(h)$ that is induced by s_i, i.e., that is defined by $s_i^h(h') := s_i((h,h'))\ \forall\ h' \in H_i^h := \{h' \in H^h\backslash Z_{H^h} \mid i \in p^h(h')\}$.

In the following, let $SPNE^n$ denote the SPNE concept for the class of n-person extensive form games, i.e. $SPNE^n(\Gamma) := \{s \in S^\Gamma \mid s$ constitutes a SPNE of game $\Gamma\}$.

2.1.6 Mechanisms

Let $N := \{1, \ldots, n\}$ be a set of agents, and let X be a nonempty set of outcomes.

Definition A *strategic n-person mechanism for* (N, X) is a tuple $(N, \{S_i\}_{i\in N}, g)$, where $(N, \{S_i\}_{i\in N})$ is a strategic n-person game form and $g : S_1 \times \ldots \times S_n \to X$ is the outcome function.

Definition An *extensive n-person mechanism (with posssible simultaneous moves) for* (N, X) is a tuple (N, H, p, g), where (N, H, p) is an extensive n-person game form and $g : Z_H \to X$ is the outcome function.

2.1.7 Classical Environments

Throughout this paper, we concentrate our analysis to environments with complete information: in contrast to the designer, who is not informed about the actual state of the environment, each agent knows the other agents' preferences.[15] The structure of the environment is commonly known, i.e., known to both the designer and the agents. For expositional purposes, we define all environments (with complete information) in this paper via an explicit type structure. Our analysis, however, is solely based on the respective state space.

Definition An *n-person classical environment (with complete information)* is a tuple $(N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i'\}_{i\in N}, \mathcal{G})$, where

$N := \{1, \ldots, n\}$ is the set of agents,

$X \neq \emptyset$ is the set of feasible outcomes,

Θ_i is the set of possible types for agent i,

$\Theta \subseteq \Theta_1 \times \ldots \times \Theta_n$ is the set of possible type profiles / states, $\Theta \neq \emptyset$,

$u_i' : X \times \Theta \to \mathbb{R}$, $u_i'(\cdot, \theta) : X \to \mathbb{R}$ being agent i's utility function over outcome space X when the actual state of the environment is $\theta \in \Theta$, representing her rational preference relation over X, and

$\mathcal{G} \equiv \mathcal{G}_{strat} \cup \mathcal{G}_{ext}$ is a set of strategic and/or extensive mechanisms for (N, X).

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i'\}_{i\in N}, \mathcal{G})$ denote a classical n-person environment.

Definition

The game induced by mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ and state $\theta \in \Theta$ in environment E is the n-person normal form game $\Gamma^{E,G,\theta} := (N, \{S_i\}_{i\in N}, \{u_i'(g(\cdot), \theta)\}_{i\in N})$.

The game induced by mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ and state $\theta \in \Theta$ in environment E is the n-person extensive form game $\Gamma^{E,G,\theta} := (N, H, p, \{u_i'(g(\cdot), \theta)\}_{i\in N})$.

2.1.8 Social Choice Correspondences

A (social) choice correspondence specifies, for each possible state of the environment, the (socially) desirable outcomes for this state.

[15] For a survey on the implementation in environments with complete information, see, for example, Moore [32] or Maskin and Sjöström [28].

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ be a classical n-person environment.

Definition A *(social) choice correspondence (SCC) for environment* E is a correspondence $\alpha : \Theta \Rightarrow X$ satisfying $\alpha(\theta) \neq \emptyset \; \forall \; \theta \in \Theta$.[16]

2.1.9 Implementation

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ be a classical n-person environment, let α be a (social) choice correspondence for E, and let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Definition[17]

Mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ *EC-implements α in environment E*

(i) *strongly* if, $\forall \; \theta \in \Theta$, $EC(\Gamma^{E,G,\theta}) \neq \emptyset$ and $g(EC(\Gamma^{E,G,\theta})) \subseteq \alpha(\theta)$,
i.e., if in every possible state of the environment mechanism G induces the agents to establish one of the desirable outcomes for this state assuming that the agents play the game induced by G and θ in E according to equilibrium concept EC.

(ii) *fully* if, $\forall \; \theta \in \Theta$, $g(EC(\Gamma^{E,G,\theta})) = \alpha(\theta)$,[18]
i.e., if mechanism G strongly EC-implements α in environment E and (if in every state) each of the desirable outcomes is possible.

Mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ *$SPNE^n$-implements α in environment E*

(i) *strongly* if, $\forall \; \theta \in \Theta$, $SPNE^n(\Gamma^{E,G,\theta}) \neq \emptyset$ and $g(O(SPNE^n(\Gamma^{E,G,\theta}))) \subseteq \alpha(\theta)$,

(ii) *fully* if, $\forall \; \theta \in \Theta$, $g(O(SPNE^n(\Gamma^{E,G,\theta}))) = \alpha(\theta)$.

Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$.

Definition (Social) choice correspondence α *is strongly/fully EC-implementable in environment* E if there exists a mechanism $G \in \mathcal{G}$ that strongly/fully EC-implements α in E.[19]

Note that if (social) choice correspondence α is fully EC-implementable in environment E, then $\alpha(\theta) = \alpha(\theta')$ for all two states $(\theta, \theta') \in \Theta \times \Theta$ which correspond to the same preference profile over X, i.e., for each pair of states such that each agent has the same preference relation over the set of feasible outcomes in both states.

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[16] Note that, throughout this paper, we will sometimes treat a single-valued correspondence $f : A \Rightarrow B$ as a function, and $f(a)$ as an element of B, and sometimes a function $f : A \rightarrow B$ as a correspondence, and $f(a)$ as a subset of B.

[17] The notation used in the literature to refer to different grades of implementation is not unique.

[18] Note that (i) is equivalent to (ii) if α is single-valued.

[19] Note that whether or not a mechanism implements a (social) choice function is independent of any change in the agents' utility functions which does not change the agents' (ordinal) preferences.

2.2 Conditions for the Implementation in Nash Equilibrium

Proposition 2.1 and 2.2 outline the necessary and the sufficient condition for full implementation of a (social) choice correspondence in Nash Equilibrium presented by Maskin [26] (who considers preference relations and profiles instead of types, type profiles and utility functions).[20]

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i'\}_{i\in N}, \mathcal{G})$ be a classical n-person environment, and let α be a SCC for environment E.

Definition SCC α *is Maskin-monotonic in environment* E if, for all $(\theta, \theta', x) \in \Theta \times \Theta \times X$ that satisfy $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$, there is some agent $i \in N$ and some outcome $x' \in X$ such that $u_i'(x, \theta) \geq u_i'(x', \theta)$ and $u_i'(x, \theta') < u_i'(x', \theta')$.[21]

Proposition 2.1 (Maskin [26]) If α is fully NE^n_{nfg}-implementable in environment E, then α is Maskin-monotonic in E.

Sketch of the proof[22]

Let $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ be a mechanism that fully NE^n_{nfg}-implements α in environment E, and consider arbitrary $(\theta, \theta', x) \in \Theta \times \Theta \times X$ that satisfy $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$. Since G fully NE^n_{nfg}-implements α in E, there exists a strategy profile $s \in S_1 \times \ldots \times S_n$ satisfying $g(s) = x$ which is a Nash Equilibrium of the game induced by mechanism G and type profile θ in environment E and which is not a Nash Equilibrium of the game induced by G and θ' in E.

Since $s \notin NE^n_{nfg}(\Gamma^{E,G,\theta'})$, there exists an $i \in N$ and a strategy $s_i' \in S_i$ such that $u_i'(g(s_i', s_{-i}), \theta') > u_i'(g(s), \theta')$.

Since $s \in NE^n_{nfg}(\Gamma^{E,G,\theta})$, we have that $u_i'(g(s), \theta) \geq u_i'(g(s_i', s_{-i}), \theta)$.

It remains to define $x' := g(s_i', s_{-i})$.

□

Definition SCC α *satisfies no-veto-power in environment* E if $x \in \alpha(\theta)$ for all $(x, \theta) \in X \times \Theta$ that satisfy $\sharp\{i \in N \mid u_i'(x, \theta) \geq u_i'(y, \theta) \ \forall\ y \in X\} \geq n - 1$.

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[20] Conditions that are both necessary and sufficient for the implementation in Nash Equilibrium have been obtained, for example, by Moore and Repullo [33], Dutta and Sen [11] (only for the case of two agents), and Sjöström [48].

[21] This is equivalent to the condition that $x \in \alpha(\theta')$ for all $(x, \theta, \theta') \in X \times \Theta \times \Theta$ that satisfy $x \in \alpha(\theta)$ and $L_i(x, \theta) \subseteq L_i(x, \theta') \ \forall\ i \in N$, where $L_i(x, \theta) := \{y \in X \mid u_i'(x, \theta) \geq u_i'(y, \theta)\}$ denotes agent i's lower contour set for outcome $x \in X$ when the state of the environment is $\theta \in \Theta$.

[22] This sketch follows that of Osborne and Rubinstein [37] (who consider preference relations and profiles instead of types, type profiles and utility functions).

Proposition 2.2 (Maskin [26]) If $\mathcal{G}$ is the set of all strategic mechanisms for (N, X), $\sharp N \geq 3$, and α is Maskin-monotonic and satisfies no-veto-power in environment E, then α is fully NE^n_{nfg}-implementable in E.

Sketch of the proof[23]

Define the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i \in N}, g)$ as follows:
Define $S_i := \{(t_i, x_i, m_i) \mid t_i \in \Theta, x_i \in X, m_i \in \mathbb{N}_0\} \; \forall \; i \in N$.
For all $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$ that satisfy

$$\exists \; (j, \theta, x, m) \in N \times \Theta \times X \times \mathbb{N}_0 \text{ s.t.}$$
$$x \in \alpha(\theta) \text{ and } (t_i, x_i, m_i) = (\theta, x, m) \; \forall \; i \in N \backslash \{j\},$$

define

$$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := \begin{cases} x_j & \text{if } u'_j(x, \theta) \geq u'_j(x_j, \theta) \\ x & \text{otw.} \end{cases} .$$

For all other $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$, define
$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := x_k$, where $k \in N$ satisfies $m_k \geq m_i \; \forall \; i \in N$.
Then, mechanism G fully NE^n_{nfg}-implements α in environment E, i.e.

$$g(NE^n_{nfg}(\Gamma^{E,G,\theta})) = \alpha(\theta) \; \forall \; \theta \in \Theta.$$

We briefly sketch the details in Appendix B.

□

2.3 The Gibbard-Satterthwaite Theorem

The Gibbard-Satterthwaite Theorem is due to Gibbard [14] and Satterthwaite [44]. Several versions and proofs of the Gibbard-Satterthwaite Theorem can be found in the literature.[24] The following (version and proof) is a mixture of elements from Mas-Colell, Whinston, and Green [29] and Osborne and Rubinstein [37]. Our sketch of the proof is divided into two parts by the use of the following lemma, the proof of which can be found in Appendix A.

Lemma 2.3 Let $n \in \mathbb{N}$, $n \geq 2$, and let N denote the set $N := \{1, \ldots, n\}$. Let X be a set that contains at least three elements, let $\mathcal{R}_X$ denote the set of all rational preference relations over X having the property that no two distinct alternatives are indifferent, and let $\mathcal{P}$ denote the set $\mathcal{P} := (\mathcal{R}_X)^N$. If $f : \mathcal{P} \to X$ satisfies

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[23] In the words of Maskin [26], "this elegant proof is due essentially to Repullo" [41] (who also considers preference relations and profiles instead of types, type profiles and utility functions).

[24] For a discussion of different versions/proofs of the Gibbard-Satterthwaite Theorem, see, for example, Barberà [5].

(a) $\forall\, x \in X\ \exists\ \succsim \in \mathcal{P}$ s.t. $f(\succsim) = x$ and

(b) $\forall\, j \in N,\ f(\succsim_j, \succsim_{-j}) \succsim_j f(\succsim'_j, \succsim_{-j})\ \forall\ (\succsim, \succsim'_j) \in \mathcal{P} \times \mathcal{R}_X$,

then $\exists\ j \in N$ such that $\forall \succsim \in \mathcal{P}$ we have that $f(\succsim) \succsim_j x'\ \forall\ x' \in X$.

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ be a classical n-person environment such that X is finite,[25] and let α be a single-valued SCC for environment E.

Proposition 2.3 (Gibbard [14] and Satterthwaite [44]) For all $(i, \theta) \in N \times \Theta$, let $\succsim_i^{(\theta)}$ denote the rational preference relation over X induced by $u'_i(\cdot, \theta)$. For each $\theta \in \Theta$, let $\succsim^{(\theta)}$ denote the preference profile $(\succsim_1^{(\theta)}, \ldots, \succsim_n^{(\theta)})$, and let $\mathcal{R}_X$ denote the set of all rational preference relations over X having the property that no two distinct alternatives are indifferent. Suppose that

X contains at least three elements,

$\mathcal{P} := \{\succsim^{(\theta)} \mid \theta \in \Theta\} = (\mathcal{R}_X)^N$,

$\forall\ x \in X\ \exists\ \theta \in \Theta$ s.t. $\alpha(\theta) = \{x\}$, and that

α is fully ($\Leftrightarrow$ strongly) DSE^n_{nfg}-implementable in environment E.

Then, α is dictatorial, i.e., there exists an agent $j \in N$ such that, $\forall\ \theta \in \Theta$, $u'_j(\alpha(\theta), \theta) \geq u'_j(x', \theta)\ \forall\ x' \in X$.

Sketch of the proof

Let $\beta : \mathcal{P} \to X$ be defined by $\beta(\succsim) := \alpha(\theta)$ where $\theta \in \Theta$ satisfies $\succsim = \succsim^{(\theta)}$.[26]

(a) Consider an arbitrary $x \in X$. By assumption, there exists a $\theta \in \Theta$ such that $\alpha(\theta) = \{x\}$. Then, $\succsim^{(\theta)} \in \mathcal{P}$ satisfies $\beta(\succsim^{(\theta)}) = \alpha(\theta) = \{x\}$. In other words, for each $x \in X$, there exists a preference profile $\succsim \in \mathcal{P}$ such that $\beta(\succsim) = x$.

(b) By assumption, there exists a mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ which fully DSE^n_{nfg}-implements α in E, i.e., $g(DSE^n_{nfg}(\Gamma^{E,G,\theta})) = \alpha(\theta)\ \forall\ \theta \in \Theta$.

For each $(i, \theta) \in N \times \Theta$, let

$$A_i(\theta) := \{s_i \in S_i \mid u'_i(g(s_i, s'_{-i}), \theta) \geq u'_i(g(s'_i, s'_{-i}), \theta)\ \forall\ s' \in S_1 \times \ldots \times S_n\}$$

denote the set of dominant strategies for agent i in game $\Gamma^{E,G,\theta}$.

Note that $A_i(\theta) = A_i(\theta')\ \forall\ (i, \theta') \in N \times \Theta$ such that $\succsim_i^{(\theta)} = \succsim_i^{(\theta')}$. And, since α is fully DSE^n_{nfg}-implementable in E, we have that $A_i(\theta) \neq \emptyset\ \forall\ (i, \theta) \in N \times \Theta$.

Consider an agent $j \in N$ and $(\succsim, \succsim'_j) \in (\mathcal{R}_X)^N \times \mathcal{R}_X$. Since $\mathcal{P} = (\mathcal{R}_X)^N$, there exists a tuple $(\theta, \theta') \in \Theta \times \Theta$ such that $\succsim = \succsim^{(\theta)}$ and $(\succsim'_j, \succsim_{-j}) = \succsim^{(\theta')}$. Note that, in particular, $\succsim_i^{(\theta)} = \succsim_i^{(\theta')}\ \forall\ i \in N \backslash \{j\}$, and, therefore, $A_i(\theta) = A_i(\theta')\ \forall\ i \in$

[25]Note that a finite outcome space X allows every rational preference relation over X to be representable by a utility function.

[26]Remember that, since α is fully DSE^n_{nfg}-implementable in environment E, we have that $\alpha(\theta) = \alpha(\theta')$ for all $(\theta, \theta') \in \Theta \times \Theta$ which correspond to the same preference profile over X.

$N \backslash \{j\}$.

Let $s_i^* \in A_i(\theta) \; \forall \; i \in N$ and $s_j' \in A_j(\theta')$. Then, since $s_j^* \in A_j(\theta)$ and $s^* \in DSE_{nfg}^n(\Gamma^{E,G,\theta})$ and $(s_j', s_{-j}^*) \in DSE_{nfg}^n(\Gamma^{E,G,\theta'})$, we have that $u_j'(\alpha(\theta), \theta) = u_j'(g(s^*), \theta) \geq u_j'(g(s_j', s_{-j}^*), \theta) = u_j'(\alpha(\theta'), \theta)$, which implies that $\beta(\succsim) = \alpha(\theta) \succsim_j \alpha(\theta') = \beta(\succsim_j', \succsim_{-j})$.

Therefore, $\forall \; j \in N$, we have that $\beta(\succsim) \succsim_j \beta(\succsim_j', \succsim_{-j}) \; \forall \; (\succsim, \succsim_j') \in (\mathcal{R}_X)^N \times \mathcal{R}_X$. Lemma 2.3 now implies that there exists an agent $j \in N$ such that $\forall \succsim \in (\mathcal{R}_X)^N$ we have that $\beta(\succsim) \succsim_j x' \; \forall \; x' \in X$.

Thus, $\forall \; \theta \in \Theta$, we have that $\alpha(\theta) = \beta(\succsim^{(\theta)}) \succsim_j^{(\theta)} x' \; \forall \; x' \in X$, i.e. $u_j'(\alpha(\theta), \theta) \geq u_j'(x', \theta) \; \forall \; x' \in X$. In other words, α is dictatorial.

□

3 Implementation in Environments with Limited Enforcement Power

We consider a model for a setting that is characterized by the presence of $n \in \mathbb{N}$, $n \geq 2$, agents (denoted by the numbers 1 to n), a designer (denoted by the number 0), and a set of feasible outcomes $X \neq \emptyset$. We let $N := \{1, \ldots, n\}$ denote the set of agents, and $N^+ := \{0, \ldots, n\}$ the set that consists of all agents and the designer. Within a certain (possibly infinite) time interval T, the agents and the designer can implement exactly one element of the set of feasible outcomes X by performing in some joint course of action. If no feasible outcome $x \in X$ is implemented within this time interval, a specific element of X prevails, which, in the following, will be denoted by $\bar{x}$.

Consider, for example, the exchange economy (as mentioned in the introduction to this paper) in which any exchange requires (and requires only) the agreement of all agents participating in this exchange. The set of feasible outcomes might consist of all those 'consumption bundles' that result from some possible reallocation or the initial allocation, which will be 'consumed' if no reallocation can be agreed upon.

We assume that the designer can force the agents to participate in one of a certain set of mechanisms, i.e., to behave according to its rules. In contrast to the assumptions of the standard theory, however, we assume that the outcome space (and, therefore, possibly the outcome function of a designated mechanism) is not fully enforceable by the designer: the designer might be able to enforce some of the outcomes, but he is not able to enforce all of the outcomes. Instead, certain groups of agents might be able to enforce certain outcomes.

We assume that the *enforcement structure* of the setting can be described by a correspondence from the set of coalitions $\mathcal{N}^+ := \{S \mid S \subseteq N^+, S \neq \emptyset\}$ to the set of outcomes X, specifying, for each coalition, the set of outcomes that this coalition is able to enforce (e.g., by, in the case of more than one member, signing a binding agreement/contract on the outcome or the corresponding joint course of action).

Definition An *enforcement structure for* (N, X) is a correspondence $e : \mathcal{N}^+ \Rightarrow X$ that satisfies $e(N^+) = X$ and the following two consistency requirements:

(1) If a coalition $S \in \mathcal{N}^+$ can enforce an outcome $x \in X$, then every super-coalition $S' \supseteq S$ can also enforce outcome x:

$$e(S') \supseteq e(S) \ \forall\ S, S' \in \mathcal{N}^+, S' \supseteq S.$$

(2) There are no two disjoint coalitions $S \in \mathcal{N}^+$ and $S' \in \mathcal{N}^+$ such that coalition S can enforce an outcome $x \in X$ and coalition S' can enforce some distinct outcome $x' \in X$:

$$\nexists\ (S, S', x, x') \in \mathcal{N}^+ \times \mathcal{N}^+ \times X \times X \text{ s.t. } S \cap S' = \emptyset,\ x \neq x',\ x \in e(S),\ x' \in e(S').$$

For example, a setting in which the implementation of an outcome requires the consent of a majority of agents could be described by the following enforcement structure:

Example *The Majority Voting Enforcement Structure for* (N, X) is defined by

$$e(S) = \begin{cases} X & \text{if } \sharp(S \cap N) > \frac{n}{2} \\ \emptyset & \text{otw.} \end{cases} \qquad \forall\ S \in \mathcal{N}^+.$$

An enforcement structure describing the necessary consent of a higher percentage of agents would be defined correspondingly.

Settings that can be modeled as a bargaining game could be described by the following enforcement structure:

Example *A Bargaining Game Enforcement Structure for* (N, X) is a correspondence $e : \mathcal{N}^+ \Rightarrow X$ that satisfies

$$e(S) = \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{\hat{x}\} & \text{otw.} \end{cases} \qquad \forall\ S \in \mathcal{N}^+$$

for some $\hat{x} \in X$.

And, settings that can be modeled as a cooperative game, or as an exchange or production economy, in which every singleton coalition's 'possibility set' consists of at least two 'alternatives', could be described by the following enforcement structure:

Example *The Cooperative Game Enforcement Structure for* (N, X) is defined by

$$e(S) = \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{otw.} \end{cases} \qquad \forall\, S \in \mathcal{N}^+.$$

The latter two examples will be re-considered in Chapter 5, which is concerned with the implementation of cooperative solution concepts.

As already mentioned above, we assume that the grand coalition can enforce every feasible outcome, i.e., $e(N^+) = X$, and that *the designer has limited enforcement power*: $e(\{0\}) \neq X$. If there exists at least one element of the outcome space that the designer is able to enforce, i.e., if $e(\{0\}) \neq \emptyset$, then we say that *the designer has active enforcement power.*[27] If $e(\{0\}) = \emptyset$, then there might exist a coalition $S \in \mathcal{N} := \{S \mid S \subseteq N, S \neq \emptyset\}$ such that the group consisting of this coalition and the designer is able to enforce more outcomes than this coalition can enforce by itself, i.e., such that $e(S \cup \{0\}) \neq e(S)$. And, depending on whether such a coalition exists or not, we say that *the designer has passive enforcement power* or *no enforcement power.*[28]

In the following, we consider time interval T as divided into three parts.
In part one, the game induced by the mechanism and the type profile is played. Part one results in an outcome $x \in X$ suggested by the mechanism.
In part two, the agents consider the implementation of x. We assume that the designer is committed to support the implementation of every suggested outcome whenever the underlying enforcement structure allows him to do so. We use the tuples $(x, 1)$ and $(x, 0)$ to denote the results that, 'right after the mechanism has been played', outcome x is or is not being implemented, respectively.
If part two results in $(x, 0)$, i.e., outcome x is suggested by the mechanism but is not being implemented, the time remaining (part three) still offers the possibility for an outcome to be implemented. We use the tuple $(y, 2)$ to denote the result that outcome $y \in X \backslash \{\bar{x}\}$ is implemented in this part of the time interval. And, the tuple $(\bar{x}, 2)$ denotes the result that outcome $\bar{x}$ is implemented in this part of the time interval or that outcome $\bar{x}$ prevails since no other outcome is implemented within time interval T. An illustration of the time schedule in tabular form can be found in Appendix J.

[27]Note that if there exists an outcome $x \in X$ such that $x \in e(\{0\})$, then our definition of an enforcement structure requires $e(S) \subseteq \{x\}\ \forall\, S \in \mathcal{N}$.

[28]Note that the designer to have active or passive enforcement power places verifiability restrictions on the set of outcomes.

In Section 3.1, we define environments with limited enforcement power, and, based on Maskin and Moore's [27] notion of "implementation with renegotiation function h" and Jackson and Palfrey's [24a] notion of "G-Nash implementation", the concept of implementation in these environments (*LE Implementation*).

In Section 3.2, we present an important neccessary condition for the implementation of (social) choice correspondences in environments with limited enforcement power.

Section 3.3 shows how Jackson and Palfrey's results on sufficient and necessary conditions for "G-Nash implementation" extend to our environments.

Our discussion in Section 3.4 extends the Gibbard-Satterthwaite Theorem to environments with limited enforcement power. The assumption of the Gibbard-Satterthwaite Theorem that all preference profiles be possible has a somehow abstract counterpart in our environments.

Section 3.5 contrasts environments with limited enforcement power to their *corresponding classical environments*. In the case of *weak pessimistic beliefs*, i.e., all suggested outcomes are implemented by a coalition that is able to do so, the implementability of a (social) choice correspondence in an environment with limited enforcement power implies the implementability of that correspondence in the corresponding classical environment, and vice versa. Jackson and Palfrey [24a] present examples showing that there are voluntarily implementable (social) choice correspondences that are not Nash-implementable, and vice versa. We consider minor and greater modified and adapted versions of these examples to illustrate that, in the case of no weak pessimistic beliefs, every combination of Nash-implementability/non-Nash-implementability of a (social) choice correspondence in an environment with limited enforcement power compared to its corresponding classical environment is possible.

In Section 3.6, we briefly discuss an extensive procedure for the implementation decision of the agents. This procedure sequentially allows, after a strategic mechanism has been played, each agent to either decide in favour or against the implementation of the outcome suggested by the mechanism. In particular, we show that, if every agent is not indifferent between a suggested outcome being implemented or not, and if the enforcement structure is one of those discussed at the beginning of this chapter, then the implementation decision of the agents following a Nash Equilibrium of a strategic mechanism can be 'copied' by a Subgame Perfect Nash Equilibrium analysis of this mechanism followed by the extensive decision procedure. Note, however, that these two alternatives require different assumptions, in particular, on the behaviour of the agents.

3.1 Definitions

3.1.1 Environments with Limited Enforcement Power

Definition An *n-person environment with limited enforcement power* is a tuple $(N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$, where

$N := \{1, \ldots, n\}$ is the set of agents,

$X \neq \emptyset$ is the set of feasible outcomes,

Θ_i is the set of possible types for agent i,

$\Theta \subseteq \Theta_1 \times \ldots \times \Theta_n$ is the set of possible type profiles / states, $\Theta \neq \emptyset$,

$u_i : (X \times \{1\}) \cup (X \times \{0\}) \times \Theta \to \mathbb{R}$,

$u_i(\cdot, \theta) : (X \times \{1\}) \cup (X \times \{0\}) \to \mathbb{R}$ being agent i's utility function over $(X \times \{1\}) \cup (X \times \{0\})$ when the actual state of the environment is $\theta \in \Theta$, representing her rational preference relation over $(X \times \{1\}) \cup (X \times \{0\})$, and satisfying Assumption 3.0 below,

$\mathcal{G} \equiv \mathcal{G}_{strat} \cup \mathcal{G}_{ext}$ is a set of mechanisms for (N, X),

$e : \mathcal{N}^+ \Rightarrow X$ is an enforcement structure for (N, X), satisfying

$e(\{0\}) \neq X$, and

$R : X \times \Theta \to (X \times \{1\}) \cup (X \times \{0\})$ is the *realization function*, satisfying

$$R(x, \theta) = \begin{cases} (x, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \text{ and} \\ & u_i((x,1),\theta) \geq u_i((x,0),\theta)\ \forall\ i \in S \cap N \\ (x, 0) & \text{otw.} \end{cases} \qquad \text{on } X \times \Theta.$$

In environments with limited enforcement power, the agents' utility functions represent their rational preference relations over the *realization space* $(X \times \{1\}) \cup (X \times \{0\})$ based on their beliefs about what will happen if an outcome suggested by a mechanism is not being implemented by a coalition that is able to do so. To be more precise, we assume that, for each agent $i \in N$ and each state $\theta \in \Theta$, the '$X \times \{0\}$ part of $u_i(\cdot, \theta)$' can be interpreted as reflecting agent $i's$ beliefs about what will happen if the suggested outcome is not being implemented, based on (the knowledge of) his own and all other agents' preferences over $X \times \{1\}$ (represented by $\{u_i(\cdot, \theta)_{|X\times\{1\}}\}_{i\in N}$). Each agent i's type corresponds to her preferences over $X \times \{1\}$:[29]

Assumption 3.0 The preference relation over $X \times \{1\}$ induced by $u_i(\cdot, \theta)$ equals the preference relation over $X \times \{1\}$ induced by $u_i(\cdot, \theta')$ for all $(\theta, \theta') \in \Theta \times \Theta$ s.t. $\theta_i = \theta'_i$.

[29] In our definition above, an explicit type structure is introduced for expositional purposes only. This includes, in particular, Assumption 3.0.

In environments with limited enforcement power, the outcome suggested by a mechanism is 'transformed' in a state-contingent way via realization function R. Although this function determines only whether or not the suggested outcome is realized, and does not describe a possible change from a suggested to a different final outcome, it is similar, in spirit, to Jackson and Palfrey's [24a] "generalized reversion function" and Maskin and Moore's [27] "renegotiation function".
An outcome suggested by a mechanism is implemented, or *realized*, if and only if there exists a coalition S which is able to enforce this outcome and all agents in this coalition weakly prefer its realization to its non-realization, i.e., all agents believe that its non-realization is no better than its realization. Note that, given utility functions $\{u_i\}_{i \in N}$, realization function R is completely determined by enforcement structure e.

Sometimes, in the remainder of this paper, we will restrict our analysis to environments in which agents' utilities on $X \times \{0\}$ can be justified by assuming that they have non-probabilistic beliefs in certain outcomes (and that they do not discount the future):
If $u_i((x,0),\theta) = u_i((b_{i1}((x,0),\theta),1),\theta)\ \forall\ (i,\theta,x) \in N \times \Theta \times X$ for some family of functions $\{b_i\}_{i \in N}$, $b_i : (X \times \{0\}) \times \Theta \to X \times \{2\}\ \forall\ i \in N$,[30] then we say that *agents' beliefs can be justified by prediction functions* $\{b_i\}_{i \in N}$ *for* (X, Θ).
A *prediction function* b_i *for agent* i specifies, for each possible non-realization case, her prediction for the final outcome. Prediction function b_i *is outcome-independent*, if agent i's prediction is independent of the outcome suggested by the mechanism, i.e., if $b_i((x,0),\theta) = b_i((x',0),\theta)\ \forall\ (x,x',\theta) \in X \times X \times \Theta$.

We say that *agents have pessimistic beliefs in environment* E if, $\forall\ i \in N$, $u_i((x,1),\theta) \geq u_i((x,0),\theta)\ \forall\ (x,\theta) \in X \times \Theta$. *Agents have weak pessimistic beliefs in environment* E if $R(x,\theta) = (x,1)\ \forall\ (x,\theta) \in X \times \Theta$, i.e., if, in each possible state, agents are 'sufficiently pessimistic' in the sense that for each feasible outcome there exists a coalition that is able and willing to implement this outcome.

3.1.2 Corresponding Classical Environments

For every n-person environment with limited enforcement power

$$E \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, R, e),$$

there exists exactly one classical n-person environment

$$E^C \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u'_i\}_{i \in N}, \mathcal{G})$$

[30] I.e., if, in each state θ, each agent i's utility level from an outcome x being suggested and not being implemented equals her utility level from the corresponding predicted outcome as being realized in the second part of the time interval.

which shares the same outcome space, state space, and mechanism space, and which satisfies $u_i'(x,\theta) = u_i((x,1),\theta)\ \forall\ (x,\theta) \in X \times \Theta$. We refer to this classical environment as *environment E's corresponding classical environment.*[31]

3.1.3 Abstract Environments with State-contingent Mechanisms

Let $N \equiv \{1,\ldots,n\}$ be a set of agents, let X be a nonempty set of outcomes, and let Θ be a nonempty set of states (that correspond to profiles of preferences over X).

Definition An *abstract strategic n-person state-contingent mechanism for* (N,X,Θ) is a tuple $(N,\{S_i\}_{i\in N},g)$, where $(N,\{S_i\}_{i\in N})$ is a strategic n-person game form and $g: S_1 \times \ldots \times S_n \times \Theta \rightarrow X$ is the state-contingent outcome function.

Definition An *abstract extensive n-person state-contingent mechanism (with possible simultaneous moves) for* (N,X,Θ) is a tuple (N,H,p,g), where (N,H,p) is an extensive n-person game form and $g: Z_H \times \Theta \rightarrow X$ is the state-contingent outcome function.

Definition An *abstract n-person environment with state-contingent mechanisms* is a tuple $(N,X,\ \{\Theta_i\}_{i\in N},\Theta,\{u_i'\}_{i\in N},\mathcal{G})$, where N, X, $\{\Theta_i\}_{i\in N}$, Θ, and $\{u_i'\}_{i\in N}$ are as in classical environments, and $\mathcal{G} \equiv \mathcal{G}_{strat} \cup \mathcal{G}_{ext}$ is a set of strategic and/or extensive state-contingent mechanisms for (N,X,Θ).

Let $E \equiv (N,X,\ \{\Theta_i\}_{i\in N},\Theta,\{u_i'\}_{i\in N},\mathcal{G})$ be an abstract n-person environment with state-contingent mechanisms, let α be a (social) choice correspondence for environment E (as defined for classical environments), and let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Definition

The game induced by mechanism $G \equiv (N,\{S_i\}_{i\in N},g) \in \mathcal{G}_{strat}$ *and state* $\theta \in \Theta$ *in environment* E is the n-person normal form game $\Gamma^{E,G,\theta} := (N,\{S_i\}_{i\in N},\{u_i'(g(\cdot,\theta),\theta)\}_{i\in N})$.
The game induced by mechanism $G \equiv (N,H,p,g) \in \mathcal{G}_{ext}$ *and state* $\theta \in \Theta$ *in environment* E is the n-person extensive form game $\Gamma^{E,G,\theta} := (N,H,p,\{u_i'(g(\cdot,\theta),\theta)\}_{i\in N})$.

Definition

Mechanism $G \equiv (N,\{S_i\}_{i\in N},g) \in \mathcal{G}_{strat}$ *EC-implements* α *in* E

(i) *strongly* if, $\forall\ \theta \in \Theta$, $EC(\Gamma^{E,G,\theta}) \neq \emptyset$ and $g(EC(\Gamma^{E,G,\theta}),\theta) \subseteq \alpha(\theta)$.

(ii) *fully* if, $\forall\ \theta \in \Theta$, $g(EC(\Gamma^{E,G,\theta}),\theta) = \alpha(\theta)$.

Mechanism $G \equiv (N,H,p,g) \in \mathcal{G}_{ext}$ $SPNE^n$*-implements* α *in* E

(i) *strongly* if, $\forall\ \theta \in \Theta$, $SPNE^n(\Gamma^{E,G,\theta}) \neq \emptyset$ and $g(O(SPNE^n(\Gamma^{E,G,\theta})),\theta) \subseteq \alpha(\theta)$.

(ii) *fully* if, $\forall\ \theta \in \Theta$, $g(O(SPNE^n(\Gamma^{E,G,\theta})),\theta) = \alpha(\theta)$.

[31] Remember our graphical illustration of the different environments' relationship in Appendix K.

3.1.4 LE Implementation

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power.

Definition A *(social) choice correspondence (SCC) for environment* E is a correspondence $\alpha : \Theta \Rightarrow X$ satisfying $\alpha(\theta) \neq \emptyset \;\forall\; \theta \in \Theta$.

Let α be a (social) choice correspondence for environment E.

Let * denote the function from the set of mechanims for (N, X) to the set of state-contingent mechanisms for $(N, (X \times \{1\}) \cup (X \times \{0\}), \Theta)$ defined by

$(N, \{S_i\}_{i\in N}, g)^* := (N, \{S_i\}_{i\in N}, g^*)$, $g^*(s, \theta) := R(g(s), \theta) \;\forall\; (s, \theta) \in S \times \Theta$, and

$(N, H, p, g)^* := (N, H, p, g^*)$, $g^*(h, \theta) := R(g(h), \theta) \;\forall\; (h, \theta) \in Z_H \times \Theta$.

Let E^* denote the abstract n-person environment with state-contingent mechanisms

$E^* := (N, (X \times \{1\}) \cup (X \times \{0\}), \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}^*)$,

and let $\alpha^* :\; \Theta \Rightarrow (X \times \{1\}) \cup (X \times \{0\})$ denote the SCC for E^* defined by

$\alpha^*(\theta) := \{(x, 1) \mid x \in \alpha(\theta)\} \;\forall\; \theta \in \Theta$.

Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Definition

Mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ *EC-implements* α *in environment* E

(i) *strongly*, if mechanism G^* strongly EC-implements α^* in E^*, i.e., if, $\forall\; \theta \in \Theta$,

$EC(\Gamma^{E^*,G^*,\theta}) \neq \emptyset$ and

$R(g(EC(\Gamma^{E^*,G^*,\theta})), \theta) = g^*(EC(\Gamma^{E^*,G^*,\theta}), \theta) \subseteq \alpha^*(\theta) = \{(x, 1) \mid x \in \alpha(\theta)\}$,

(ii) *fully*, if mechanism G^* fully EC-implements α^* in E^*, i.e., if, $\forall\; \theta \in \Theta$,

$R(g(EC(\Gamma^{E^*,G^*,\theta})), \theta) = g^*(EC(\Gamma^{E^*,G^*,\theta}), \theta) = \alpha^*(\theta) = \{(x, 1) \mid x \in \alpha(\theta)\}$,

where $\Gamma^{E^*,G^*,\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i(R(g(\cdot), \theta), \theta)\}_{i\in N})$ denotes the game induced by mechanism G^* and type profile θ in environment E^*.

Definition

Mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ $SPNE^n$*-implements* α *in environment* E

(i) *strongly*, if mechanism G^* strongly $SPNE^n$-implements α^* in E^*, i.e. if, $\forall\, \theta \in \Theta$,

$SPNE^n(\Gamma^{E^*,G^*,\theta}) \neq \emptyset$ and

$R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta}))), \theta) = g^*(O(SPNE^n(\Gamma^{E^*,G^*,\theta})), \theta) \subseteq \alpha^*(\theta)$,

(ii) *fully*, if mechanism G^* fully $SPNE^n$-implements α^* in E^*, i.e., if, $\forall\; \theta \in \Theta$,

$R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta}))), \theta) = g^*(O(SPNE^n(\Gamma^{E^*,G^*,\theta})), \theta) = \alpha^*(\theta)$,

where $\Gamma^{E^*,G^*,\theta} \equiv (N, H, p, \{u_i(R(g(\cdot), \theta), \theta)\}_{i\in N})$ denotes the game induced by mechanism G^* and type profile θ in environment E^*.

Note that a strategic mechanism $G \equiv (N, \{S_i\}_{i \in N}, g) \in \mathcal{G}_{strat}$ fully EC-implements α in E's corresponding classical environment E^C, if, $\forall\ \theta \in \Theta$, $g(EC(\Gamma^{E^C,G,\theta})) = \alpha(\theta)$, where $\Gamma^{E^C,G,\theta} \equiv (N, \{S_i\}_{i \in N}, \{u_i((g(\cdot), 1), \theta)\}_{i \in N})$.
An extensive mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ fully $SPNE^n$-implements α in E's corresponding classical environment E^C, if, $\forall\ \theta \in \Theta$, $g(O(SPNE^n(\Gamma^{E^C,G,\theta}))) = \alpha(\theta)$, where $\Gamma^{E^C,G,\theta} \equiv (N, H, p, \{u_i((g(\cdot), 1), \theta)\}_{i \in N})$.

Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$.

Definition *SCC α is strongly/fully EC-implementable in environment E* if there exists a mechanism $G \in \mathcal{G}$ that strongly/fully EC-implements α in E.

Note that, according to our definition of LE implementation, any change in the enforcement structure which does not change the realization function will not affect the implementability of a SCC. In particular, any change in the passive enforcement power of the designer will not affect the implementability of a SCC.

3.2 A Necessary Condition: Consistency

Let $E \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power, and let α be a SCC for environment E.

Definition SCC α *is consistent with (realization function R in) environment E* if, $\forall\ (\theta, x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, we have that $R(x, \theta) = (x, 1)$.

The following proposition formalizes a necessary condition which is an immediate consequence of our notion of implementation in environments with limited enforcement power. Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$.

Proposition 3.1 If α is fully EC-implementable in environment E, then α is consistent with (realization function R in) environment E.

Proof
Let $G \equiv (N, \{S_i\}_{i \in N}, g) \in \mathcal{G}_{strat}$ be a mechanism that fully EC-implements α in environment E ($EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$). The proof for the case $G \in \mathcal{G}_{ext}$ is analogous.
Consider an arbitrary tuple $(\theta, x) \in \Theta \times X$ such that $x \in \alpha(\theta)$.
Since, by definition, mechanism G^* fully EC-implements α^* in environment E^*, there exists an EC-Equilibrium $s \in S_1 \times \ldots \times S_n$ of the game induced by G^* and θ in E^* that satisfies $g^*(s, \theta) = R(g(s), \theta) = (x, 1)$. Since $R(g(s), \theta) \in \{(g(s), 1), (g(s), 0)\}$, the preceding implies that $g(s) = x$ and $R(x, \theta) = (x, 1)$.

□

Remark 3.1 Suppose that e is the cooperative game enforcement structure for (N, X). If α is fully EC-implementable in environment E, then, in each state $\theta \in \Theta$, each agent i's beliefs are such that he weakly prefers the realization of any desirable outcome to its non-realization, i.e., $u_i((x,1),\theta) \geq u_i((x,0),\theta)\ \forall\ (i,x) \in N \times \alpha(\theta)$.

3.3 Conditions for the Implementation in Nash Equilibrium

Our concept of LE Implementation extends to the following abstract environments.

Definition An *abstract n-person environment with an unrestricted realization function* is a tuple $(N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R)$, where N, X, $\{\Theta_i\}_{i\in N}$, Θ, $\{u_i\}_{i\in N}$, and $\mathcal{G}$ are as in environments with limited enforcement power, not necessarily satisfying Assumption 3.0, and R is a function from $X \times \Theta$ to $(X \times \{1\}) \cup (X \times \{0\})$.

Lemma 3.2 and Lemma 3.3 provide a necessary and a sufficient condition for full Nash-implementation in these abstract environments, respectively. They follow from the logic of Maskin's [26] conditions, and are extensions of two theorems established by Jackson and Palfrey [24a, 24b] (Theorem 1 and 2), which cover the (from an interpretative point of view) special case $R(X \times \Theta) \subseteq X \times \{1\}$.[32] Both theorems and their proofs carry over to our abstract environments.
Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R)$ be an abstract n-person environment with an unrestricted realization function, and let α be a SCC for environment E.

Lemma 3.2 If α is fully NE^n_{nfg}-implementable in environment E, then, $\forall\ (\theta, x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, there exists an outcome $x' \in X$ which satisfies $R(x', \theta) = (x, 1)$ and the following condition:

$\forall\ \theta' \in \Theta$ s.t. $R(x', \theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i, y) \in N \times X$ s.t. $u_i(R(y, \theta'), \theta') > u_i(R(x', \theta'), \theta')$ and $u_i(R(y, \theta), \theta) \leq u_i(R(x', \theta), \theta)$.

Proof
Let $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ be a mechanism that fully NE^n_{nfg}-implements α in environment E, and consider arbitrary $(\theta, x) \in \Theta \times X$ that satisfy $x \in \alpha(\theta)$.

---

[32] Remember that Jackson and Palfrey do not consider an 'extended outcome space' as we do in environments with limited enforcement power (and thus in abstract environments with an unrestricted realization function). From a purely mathematical point of view, Lemma 3.2 could be seen as a special case / as an implication of Jackson and Palfrey's Theorem 1, if we consider their set of feasible outcomes A to be the set $(X \times \{0\}) \cup (X \times \{1\})$ and focus on those mechanisms whose image lies in $X \times \{1\}$. This perspective leaves open the question for an interpretation of their generalized reversion function, which in this case is a function $G : (X \times \{0\}) \cup (X \times \{1\}) \times \Theta \rightarrow (X \times \{0\}) \cup (X \times \{1\})$.

Since G fully NE^n_{nfg}-implements α in E, we have that $R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})),\theta) = \{(y,1) \mid y \in \alpha(\theta)\}$. Thus, there exists a Nash Equilibrium $a \in S_1 \times \ldots \times S_n$ of the game induced by G^* and θ in E^* that satisfies $R(g(a),\theta) = (x,1)$.

Defining $x' := g(a) \in X$, we have that $R(x',\theta) = (x,1)$.

Also, since strategy profile a is a Nash Equilibrium of the game induced by G^* and θ in E^*, we have that, $\forall\ i \in N$, $u_i(R(g(a'_i,a_{-i}),\theta),\theta) \leq u_i(R(g(a),\theta),\theta)\ \forall\ a'_i \in S_i$.

Consider now an arbitrary $\theta' \in \Theta$ s.t. $R(x',\theta') \notin \alpha(\theta')\times\{1\}$. Then, strategy profile a is not a Nash Equilibrium of the game induced by G^* and θ' in E^*: $a \in NE^n_{nfg}(\Gamma^{E^*,G^*,\theta'})$ would imply $R(g(a),\theta') = R(x',\theta') \in \alpha(\theta') \times \{1\}$, a contradiction. Thus, there exists an $i \in N$ and a strategy $a'_i \in S_i$ s.t. $u_i(R(g(a'_i,a_{-i}),\theta'),\theta') > u_i(R(g(a),\theta'),\theta')$, and it remains to define $y := g(a'_i,a_{-i})$.

□

Lemma 3.3 If $\mathcal{G}$ is the set of all strategic mechanisms for (N,X), $\sharp N \geq 3$, and α satisfies

(i) $\forall\ (\theta,x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, there exists an outcome $x' \in X$ which satisfies $R(x',\theta) = (x,1)$ and the condition of Lemma 3.2,

(ii) $R(x,\theta) \in \alpha(\theta) \times \{1\}\ \forall\ (x,\theta) \in X \times \Theta$ that satisfy $\sharp\{i \in N \mid u_i(R(x,\theta),\theta) \geq u_i(R(y,\theta),\theta)\ \forall\ y \in X\} \geq n-1$,

then α is fully NE^n_{nfg}-implementable in environment E.

Proof

Consider the strategic n-person mechanism $G \equiv (N,\{S_i\}_{i\in N},g)$ defined as follows.

Define $S_i := \{(t_i,x_i,m_i) \mid t_i \in \Theta, x_i \in X, m_i \in \mathbb{N}_0\}\ \forall\ i \in N$.

For all $((t_1,x_1,m_1),\ldots,(t_n,x_n,m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$ that satisfy

$\exists\ (j,\theta,x,m) \in N \times \Theta \times X \times \mathbb{N}_0$ s.t.

$R(x,\theta) \in \alpha(\theta) \times \{1\}$ and

$(t_i,x_i,m_i) = (\theta,x,m)\ \forall\ i \in N\backslash\{j\}$ and

$\forall\ \theta' \in \Theta$ s.t. $R(x,\theta') \notin \alpha(\theta') \times \{1\}\ \exists\ (i,y) \in N \times X$ s.t.

$u_i(R(x,\theta),\theta) \geq u_i(R(y,\theta),\theta)$ and $u_i(R(x,\theta'),\theta') < u_i(R(y,\theta'),\theta')$,

define $g((t_1,x_1,m_1),\ldots,(t_n,x_n,m_n)) := \begin{cases} x_j & \text{if } u_j(R(x,\theta),\theta) \geq u_j(R(x_j,\theta),\theta) \\ x & \text{otw.} \end{cases}$.[33]

For all other $((t_1,x_1,m_1),\ldots,(t_n,x_n,m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$, define $g((t_1,x_1,m_1),\ldots,(t_n,x_n,m_n)) := x_k$, where $k \in N$ satisfies $m_k \geq m_i\ \forall\ i \in N$.

[33]Jackson and Palfrey require the (stronger) condition $(t_i,x_i,m_i) = (\theta,x,0)\ \forall\ i \in N\backslash\{j\}$. The remainder of our proof covers both alternatives.

Then, mechanism G fully NE^n_{nfg}-implements α in environment E, i.e.,

$$R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})),\theta) = \{(y,1) \mid y \in \alpha(\theta)\} \; \forall \; \theta \in \Theta.$$

The details can be found in Appendix B.

□

Proposition 3.2 and 3.3 summarize the implications of the preceding lemmas for environments with limited enforcement power, i.e., for environments with realization functions that, in particular, satisfy $R(x,\theta) \in \{(x,1),(x,0)\} \; \forall \; (x,\theta) \in X \times \Theta$. We give a different proof for Proposition 3.3 in Appendix B, showing that it can be proven directly by using a mechanism that is less complex in a certain way.[34]

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power, and let α be a SCC for environment E.

Proposition 3.2 (Necessary condition for full implementation in Nash Equilibrium) If α is fully NE^n_{nfg}-implementable in environment E, then $\forall \; (\theta,\theta',x) \in \Theta \times \Theta \times X$ satisfying $x \in \alpha(\theta)$ and $R(x,\theta') \notin \alpha(\theta') \times \{1\}$,[35] there exists a tuple $(i,y) \in N \times X$ s.t.

$$u_i(R(y,\theta'),\theta') > u_i(R(x,\theta'),\theta') \text{ and } u_i(R(y,\theta),\theta) \leq u_i(R(x,\theta),\theta).$$

Proof

By Lemma 3.2, $\forall \; (\theta,x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, there exists an outcome $x' \in X$ which satisfies $R(x',\theta) = (x,1)$ and the condition of Lemma 3.2.

Since $R(y,\theta) \in \{(y,1),(y,0)\} \; \forall \; (y,\theta) \in X \times \Theta$, the only outcome $x' \in X$ which can satisfy $R(x',\theta) = (x,1)$ is outcome x itself.

Thus, $\forall \; (\theta,x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, we have that $R(x,\theta) = (x,1)$ and that $\forall \; \theta' \in \Theta$ s.t. $R(x,\theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i,y) \in N \times X$ s.t. $u_i(R(y,\theta'),\theta') > u_i(R(x,\theta'),\theta')$ and $u_i(R(y,\theta),\theta) \leq u_i(R(x,\theta),\theta)$.

□

Remark 3.2 SCC α satisfies the necessary condition of Proposition 3.2 if and only if α satisfies the following condition:

SCC α is consistent with realization function R in environment E, and $\forall \; (\theta,\theta',x) \in \Theta \times \Theta \times X$ such that $\theta \neq \theta'$ and $x \in \alpha(\theta)$ and $R(x,\theta') \notin \alpha(\theta') \times \{1\}$, there exists a tuple $(i,y) \in N \times X$ s.t.

$$u_i(R(y,\theta'),\theta') > u_i(R(x,\theta'),\theta') \text{ and } u_i(R(y,\theta),\theta) \leq u_i(R(x,\theta),\theta).$$

---

[34]Our proof for Proposition 3.3 is, in fact, similar to Jackson and Palfrey's [24a] first proof of their sufficient condition for "G-Nash implementation" (which is corrected in Jackson and Palfrey [24b]).

[35]And, in particular, $\forall \; (\theta,\theta',x)$ s.t. $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$.

Proposition 3.3 (Sufficient condition for full implementation in Nash Equilibrium)
If $\mathcal{G}$ is the set of all strategic mechanisms for (N, X), $\sharp N \geq 3$, and α satisfies

(i) the necessary condition of Proposition 3.2, and

(ii) $R(x, \theta) \in \alpha(\theta) \times \{1\}$ $\forall$ $(x, \theta) \in X \times \Theta$ that satisfy
$\sharp\{i \in N \mid u_i(R(x, \theta), \theta) \geq u_i(R(y, \theta), \theta)\ \forall\ y \in X\} \geq n - 1$,

then α is fully NE^n_{nfg}-implementable in environment E.

Proof

To see that α satisfies (i) of Lemma 3.3, consider arbitrary $(\theta, x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$.
We show that $x' := x$ satisfies $R(x', \theta) = (x, 1)$ and the condition of Lemma 3.2.
First, assume that $R(x, \theta) \neq (x, 1)$, i.e., $R(x, \theta) = (x, 0)$.
Then, in particular, $R(x, \theta) \notin \alpha(\theta) \times \{1\}$, and, by assumption, $\exists$ $(i, y) \in N \times X$ s.t.
$u_i(R(y, \theta), \theta) > u_i(R(x, \theta), \theta)$ and $u_i(R(y, \theta), \theta) \leq u_i(R(x, \theta), \theta)$, a contradiction.
Also, by assumption, $\forall$ $\theta' \in \Theta$ s.t. $R(x, \theta') \notin \alpha(\theta') \times \{1\}$, $\exists$ $(i, y) \in N \times X$ s.t.

$$u_i(R(y, \theta'), \theta') > u_i(R(x, \theta'), \theta') \text{ and } u_i(R(y, \theta), \theta) \leq u_i(R(x, \theta), \theta).$$

□

Remark 3.3 Proposition 3.3(ii) is equivalent to the following condition

(ii)' (a) $x \in \alpha(\theta)$ $\forall$ $(x, \theta) \in X \times \Theta$ that satisfy $R(x, \theta) = (x, 1)$ and
$\sharp\{i \in N \mid u_i((x, 1), \theta) \geq u_i(R(y, \theta), \theta)\ \forall\ y \in X\} \geq n - 1$,

(b) there does not exist a tuple $(x, \theta) \in X \times \Theta$ such that
$R(x, \theta) = (x, 0)$ and $\sharp\{i \in N \mid u_i((x, 0), \theta) \geq u_i(R(y, \theta), \theta)\ \forall\ y \in X\} \geq n - 1$.

For environments in which agents have weak pessimistic beliefs, the preceding propositions imply that Maskin-monotonicity is a necessary condition and that Maskin-monotonicity together with no-veto-power is a sufficient condition for full implementation in Nash Equilibrium:

Corollary 3.2 Suppose that agents have weak pessimistic beliefs in environment E.
If α is fully NE^n_{nfg}-implementable in environment E, then, $\forall$ $(\theta, \theta', x) \in \Theta \times \Theta \times X$ s.t. $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$, there exists a tuple $(i, y) \in N \times X$ such that

$$u_i((y, 1), \theta') > u_i((x, 1), \theta') \text{ and } u_i((y, 1), \theta) \leq u_i((x, 1), \theta).$$

Corollary 3.3 Suppose that agents have weak pessimistic beliefs in environment E.
If $\mathcal{G}$ is the set of all strategic mechanisms for (N, X), $\sharp N \geq 3$, and α satisfies

(i) the necessary condition of Corollary 3.2, and

(ii) $x \in \alpha(\theta)$ for all $(x, \theta) \in X \times \Theta$ that satisfy
$\sharp\{i \in N \mid u_i((x, 1), \theta) \geq u_i((y, 1), \theta)\ \forall\ y \in X\} \geq n - 1$,

then α is fully NE^n_{nfg}-implementable in environment E.

3.4 Gibbard-Satterthwaite in Environments with Limited Enforcement Power

Proposition 3.4 extends the Gibbard-Satterthwaite Theorem outlined in Section 2.3 to environments with limited enforcement power.
The assumption of the Gibbard-Satterthwaite Theorem that all preference profiles be possible has a somehow abstract counterpart in our environments. To be more precise, the preference relations over outcome space X that have to satisfy the respective assumption in environments with limited enforcement power are those induced by utility functions $u_i(R(\cdot,\theta),\theta) : X \to \mathbb{R}$.
Furthermore, in our extension to environments with limited enforcement power, an additional assumption has to be made explicit. As already mentioned at the end of Paragraph 2.1.9, in a classical environment $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$, a (social) choice correspondence α (for E) which is DSE^n_{nfg}-implementable in environment E has to satisfy the following condition: $\alpha(\theta) = \alpha(\theta')$ for all two states $(\theta, \theta') \in \Theta \times \Theta$ which correspond to the same preference profile over X, i.e., for each pair of states such that each agent has the same preference relation over the set of feasible outcomes in both states. In environments with limited enforcement power, the corresponding implication is not necessarily satisfied. To see this, consider an environment with limited enforcement power $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ and two states $(\theta, \theta') \in \Theta\times\Theta$ such that, for every agent $i \in N$, the preference relation over X induced by $u_i(R(\cdot,\theta),\theta)$ equals that induced by $u_i(R(\cdot,\theta'),\theta')$. If $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ denotes a mechanism that fully DSE^n_{nfg}-implements a SCC α (for E) in environment E, then we have that $S^* := DSE^n_{nfg}(\Gamma^{E^*,G^*,\theta}) = DSE^n_{nfg}(\Gamma^{E^*,G^*,\theta'})$ and $R(g(S^*),\theta) = \alpha(\theta) \times \{1\}$ and $R(g(S^*),\theta') = \alpha(\theta') \times \{1\}$. However, this does not necessarily imply that $R(g(S^*),\theta) = R(g(S^*),\theta')$, which in turn would imply that $\alpha(\theta) = \alpha(\theta')$. In our extension of the Gibbard-Satterthwaite Theorem, we explictly assume that $\alpha(\theta) = \alpha(\theta')$.
Note, however, that in the special case of weak pessimistic beliefs our assumptions become an intuitive reflection of those in the Gibbard-Satterthwaite Theorem. Our proof of Proposition 3.4 follows the lines of Section 2.3.

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power such that X is finite, and let α be a single-valued (social) choice correspondence for environment E. For all $(i,\theta) \in N \times \Theta$, let $\succsim^{(\theta)}_i$ denote the rational preference relation over X induced by $u_i(R(\cdot,\theta),\theta)$. For each $\theta \in \Theta$, let $\succsim^{(\theta)}$ denote the preference profile $(\succsim^{(\theta)}_1, \ldots, \succsim^{(\theta)}_n)$, and let $\mathcal{R}_X$ denote the set of all rational preference relations over X having the property that no two distinct alternatives are indifferent.

Proposition 3.4 Suppose that

X contains at least three elements,

$\mathcal{P} := \{\succsim^{(\theta)} | \ \theta \in \Theta\} = (\mathcal{R}_X)^N$,

$\forall \ x \in X \ \exists \ \theta \in \Theta$ s.t. $\alpha(\theta) = \{x\}$,

α is fully DSE^n_{nfg}-implementable in environment E, and that

$\alpha(\theta) = \alpha(\theta') \ \forall \ (\theta, \theta') \in \Theta \times \Theta$ such that $\succsim^{(\theta)} = \succsim^{(\theta')}$.

Then $\exists \ j \in N$ such that, $\forall \ \theta \in \Theta$, $u_j((\alpha(\theta), 1), \theta) \geq u_j(R(x', \theta), \theta) \ \forall \ x' \in X$.

Proof

Let $\beta : \mathcal{P} \to X$ be defined by $\beta(\succsim) := \alpha(\theta)$ where $\theta \in \Theta$ satisfies $\succsim = \succsim^{(\theta)}$.

(a) Consider an arbitrary $x \in X$. By assumption, there exists a $\theta \in \Theta$ such that $\alpha(\theta) = \{x\}$. Then, $\succsim^{(\theta)} \in \mathcal{P}$ satisfies $\beta(\succsim^{(\theta)}) = \alpha(\theta) = \{x\}$.

In other words, $\forall \ x \in X$, there exists a profile $\succsim \in \mathcal{P}$ such that $\beta(\succsim) = x$.

(b) By assumption, there exists a mechanism $G \equiv (N, \{S_i\}_{i \in N}, g) \in \mathcal{G}$ which fully DSE^n_{nfg}-implements α in environment E, i.e.

$$R(g(DSE^n_{nfg}(\Gamma^{E^*, G^*, \theta})), \theta) = (\alpha(\theta), 1) \ \forall \ \theta \in \Theta.$$

In particular, $\forall \ \theta \in \Theta$ and $\forall \ s \in DSE^n_{nfg}(\Gamma^{E^*, G^*, \theta})$, we have that $R(g(s), \theta) = (\alpha(\theta), 1)$, and thus $g(s) = \alpha(\theta)$ and $R(\alpha(\theta), \theta) = (\alpha(\theta), 1)$.

Define $S := S_1 \times \ldots \times S_n$, and, for each $(i, \theta) \in N \times \Theta$, let

$$\begin{aligned} A_i(\theta) &:= \{s_i \in S_i \mid u_i(R(g(s_i, s'_{-i}), \theta), \theta) \geq u_i(R(g(s'_i, s'_{-i}), \theta), \theta) \ \forall \ s' \in S\} \\ &= \{s_i \in S_i \mid g(s_i, s'_{-i}) \succsim_i^{(\theta)} g(s'_i, s'_{-i}) \ \forall \ s' \in S\} \end{aligned}$$

denote the set of dominant strategies for agent i in game $\Gamma^{E^*, G^*, \theta}$.

Note that $A_i(\theta) = A_i(\theta') \ \forall \ \theta' \in \Theta$ such that $\succsim_i^{(\theta)} = \succsim_i^{(\theta')}$. And, since α is fully DSE^n_{nfg}-implementable in E, we have that $A_i(\theta) \neq \emptyset \ \forall \ (i, \theta) \in N \times \Theta$.

Consider an agent $j \in N$ and $(\succsim, \succsim'_j) \in (\mathcal{R}_X)^N \times \mathcal{R}_X$. Since $\mathcal{P} = (\mathcal{R}_X)^N$, there exists a tuple $(\theta, \theta') \in \Theta \times \Theta$ such that $\succsim = \succsim^{(\theta)}$ and $(\succsim'_j, \succsim_{-j}) = \succsim^{(\theta')}$. Note that, in particular, $\succsim_i^{(\theta)} = \succsim_i^{(\theta')} \ \forall \ i \in N \backslash \{j\}$, and, thus, $A_i(\theta) = A_i(\theta') \ \forall \ i \in N \backslash \{j\}$.

Let $s^*_i \in A_i(\theta) \ \forall \ i \in N$ and $s'_j \in A_j(\theta')$. Then, since $s^*_j \in A_j(\theta)$ and $s^* \in DSE^n_{nfg}(\Gamma^{E^*, G^*, \theta})$ and $(s'_j, s^*_{-j}) \in DSE^n_{nfg}(\Gamma^{E^*, G^*, \theta'})$, we have that

$$\beta(\succsim) = \alpha(\theta) = g(s^*) \succsim_j g(s'_j, s^*_{-j}) = \alpha(\theta') = \beta(\succsim'_j, \succsim_{-j}).$$

Hence, $\forall \ j \in N$, we have that $\beta(\succsim) \succsim_j \beta(\succsim'_j, \succsim_{-j}) \ \forall \ (\succsim, \succsim'_j) \in (\mathcal{R}_X)^N \times \mathcal{R}_X$.

Lemma 2.3 now implies that there exists an agent $j \in N$ such that $\forall \succsim \in (\mathcal{R}_X)^N$ we have that $\beta(\succsim) \succsim_j x' \ \forall \ x' \in X$.

Thus, $\forall \ \theta \in \Theta$, we have that $\alpha(\theta) = \beta(\succsim^{(\theta)}) \succsim_j^{(\theta)} x' \ \forall \ x' \in X$, i.e., $u_j((\alpha(\theta), 1), \theta) = u_j(R(\alpha(\theta), \theta), \theta) \geq u_j(R(x', \theta), \theta) \ \forall \ x' \in X$.

□

Corollary 3.4 For all $(i, \theta) \in N \times \Theta$, let $\mathcal{P}_i^{(\theta)}$ denote the rational preference relation over $X \times \{1\}$ induced by $u_i(\cdot, \theta)$. For each $\theta \in \Theta$, let $\mathcal{P}^{(\theta)}$ denote the preference profile $(\mathcal{P}_1^{(\theta)}, \ldots, \mathcal{P}_n^{(\theta)})$, and let $\mathcal{R}_{X\times\{1\}}$ denote the set of all rational preference relations over $X \times \{1\}$ having the property that no two distinct alternatives are indifferent. Suppose that X contains at least three elements,

$\{\mathcal{P}^{(\theta)} \mid \theta \in \Theta\} = (\mathcal{R}_{X\times\{1\}})^N$,

$\forall\, x \in X\ \exists\, \theta \in \Theta$ s.t. $\alpha(\theta) = \{x\}$,

α is fully DSE^n_{nfg}-implementable in environment E, and that

agents have weak pessimistic beliefs in environment E.

Then $\exists\, j \in N$ such that, $\forall\, \theta \in \Theta$, $u_j((\alpha(\theta), 1), \theta) \geq u_j((x', 1), \theta)\ \forall\, x' \in X$.

Proof

For all $(i, \theta) \in N \times \Theta$, let $\succsim_i^{(\theta)}$ denote the rational preference relation over X induced by $u_i(R(\cdot, \theta), \theta)$. For each $\theta \in \Theta$, let $\succsim^{(\theta)}$ denote the preference profile $(\succsim_1^{(\theta)}, \ldots, \succsim_n^{(\theta)})$, and let $\mathcal{R}_X$ denote the set of all rational preference relations over X having the property that no two distinct alternatives are indifferent.

Since $\{\mathcal{P}^{(\theta)} \mid \theta \in \Theta\} = (\mathcal{R}_{X\times\{1\}})^N$ and $\succsim_i^{(\theta)}$ is the rational preference relation over X induced by $u_i(R(\cdot, \theta), \theta) = u_i((\cdot, 1), \theta)$, we have that $\{\succsim^{(\theta)} \mid \theta \in \Theta\} = (\mathcal{R}_X)^N$.

Consider $(\theta, \theta') \in \Theta \times \Theta$ such that $\succsim^{(\theta)} = \succsim^{(\theta')}$. Since $\succsim_i^{(\theta)}$ is the rational preference relation over X induced by $u_i(R(\cdot, \theta), \theta) = u_i((\cdot, 1), \theta)$, we have that $\mathcal{P}^{(\theta)} = \mathcal{P}^{(\theta')}$. And, since α is fully DSE^n_{nfg}-implementable in environment E and agents have weak pessimistic beliefs in E, we have that $\alpha(\theta) = \alpha(\theta')$.

Proposition 3.4 implies that $\exists\, j \in N$ such that, $\forall\, \theta \in \Theta$,

$$u_j((\alpha(\theta), 1), \theta) \geq u_j(R(x', \theta), \theta) = u_j((x', 1), \theta)\ \forall\, x' \in X.$$

□

3.5 Implementability in Corresponding Environments

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power, let α be a (social) choice correspondence for environment E, and let $EC \in \{NE^n_{nfg}, DSE^n_{nfg}, SPNE^n\}$.

Proposition 3.5 If a mechanism $G \in \mathcal{G}$ strongly/fully EC-implements α in environment E, and the image $Y \subseteq X$ of its outcome function satisfies $R(x, \theta) = (x, 1)\ \forall\, (x, \theta) \in Y \times \Theta$, i.e., each outcome in the image is realized, then mechanism G strongly/fully EC-implements α in environment E's corresponding classical environment, and vice versa.

Proof

Consider the case of a strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ and full implementation in $EC \in \{NE^n_{nfg}, DSE^n_{nfg}\}$. The other cases are similar.

Let $E^C \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ denote E's corresponding classical environment, i.e., $u'_i(x,\theta) = u_i((x,1),\theta)\ \forall\ (x,\theta) \in X \times \Theta$.

By definition, mechanism G fully EC-implements α in environment E^C if and only if $g(EC(\Gamma^{E^C,G,\theta})) = \alpha(\theta)\ \forall\ \theta \in \Theta$.

By definition, mechanism G fully EC-implements α in environment E if and only if $R(g(EC(\Gamma^{E^*,G^*,\theta})),\theta) = \{(x,1) \mid x \in \alpha(\theta)\}\ \forall\ \theta \in \Theta$.

Since, by assumption, $R(g(EC(\Gamma^{E^*,G^*,\theta})),\theta) = (g(EC(\Gamma^{E^*,G^*,\theta})),1)$, it is sufficient to show that $EC(\Gamma^{E^C,G,\theta}) = EC(\Gamma^{E^*,G^*,\theta})$.

To see this, note that

$$\Gamma^{E^C,G,\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i((g(\cdot),1),\theta)\}_{i\in N}) = (N, \{S_i\}_{i\in N}, \{u_i(R(g(\cdot),\theta),\theta)\}_{i\in N}) \equiv \Gamma^{E^*,G^*,\theta}.$$

□

In the special case of weak pessimistic beliefs, every outcome in the image of a mechanism's outcome function is realized, and we obtain the following corollary.

Corollary 3.5 If agents have weak pessimistic beliefs in environment E, then full EC-implementability of α in environment E is equivalent to full EC-implementability of α in E's corresponding classical environment.

In the case of no weak pessimistic beliefs, every combination of Nash-implementability/non-Nash-implementability of a SCC in an environment with limited enforcement power compared to its corresponding classical environment is possible:

Remark 3.5 If agents do not have weak pessimistic beliefs in environment E, then, in general, everything is possible:

(a) α is fully NE^n_{nfg}-implementable in environment E and in E's corresponding classical environment, or

(b) α is not fully NE^n_{nfg}-implementable in environment E but is fully NE^n_{nfg}-implementable in E's corresponding classical environment, or

(c) α is fully NE^n_{nfg}-implementable in environment E but is not fully NE^n_{nfg}-implementable in E's corresponding classical environment, or

(d) α is neither fully NE^n_{nfg}-implementable in environment E nor in E's corresponding classical environment.

The following 'Unanimity Voting' examples illustrate each of the four cases.

Jackson and Palfrey [24a] present an example (Section 3, Example 3) showing that

there are Nash-implementable (social) choice correspondences that are not voluntarily implementable. Although, in principle, their example could be adapted into our framework to illustrate case (b), we consider a slightly modified version (with respect to the agents' preferences) which allows us to use Jackson and Palfrey's simple mechanism in order to illustrate both case (a) and case (b) by only changing the agents' identical and outcome-independent predictions.
In their Example 1, Jackson and Palfrey [24a] present a three-agents–two-states–four-outcomes voting example showing that a (social) choice correspondence which is not Nash-implementable may nevertheless satisfy their necessary condition for voluntary implementation. And, a two-agents exchange economy example (Example 2) shows that there are voluntarily implementable (social) choice correspondences that are not Nash-implementable. To illustrate case (c), we consider, for simplicity, a voting example with only three outcomes and three different preference profiles, which again has the merit that we can illustrate both case (c) and case (d) by only changing the agents' identical and outcome-independent predictions. For expositional purposes, we finally adapt Jackson and Palfrey's exchange economy example to our framework in Appendix C (again illustrating case (c)).

Common to all of the following 'Unanimity Voting' examples is that two or three voters can vote for one out of three candidates, i.e., for candidate 0, 1, or candidate 2, to change or to confirm the actual status quo, candidate 0. An unanimous vote for one candidate implies that this candidate is the new (and maybe old) status quo. Only one vote for candidate 0 is sufficient to confirm the actual status quo.

Example 3.5(a)

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be the n-person environment with limited enforcement power, where

$N = \{1, 2\}$,

$X = \{x_0, x_1, x_2\}$,

$\Theta_i = \{\hat{\theta}_i, \tilde{\theta}_i\} \;\forall\; i \in N$,

$\Theta = \{\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2), \tilde{\theta} \equiv (\tilde{\theta}_1, \tilde{\theta}_2)\}$,

$u_1((x_0, 1), \hat{\theta}) = u_2((x_0, 1), \hat{\theta}) = 1, \quad u_1((x_0, 1), \tilde{\theta}) = 2, \quad u_2((x_0, 1), \tilde{\theta}) = 2,$

$u_1((x_1, 1), \hat{\theta}) = u_2((x_1, 1), \hat{\theta}) = 2, \quad u_1((x_1, 1), \tilde{\theta}) = 2, \quad u_2((x_1, 1), \tilde{\theta}) = 3,$

$u_1((x_2, 1), \hat{\theta}) = u_2((x_2, 1), \hat{\theta}) = 3, \quad u_1((x_2, 1), \tilde{\theta}) = 1, \quad u_2((x_2, 1), \tilde{\theta}) = 1,$

$u_i((x_j, 0), \theta) = u_i((x_0, 1), \theta) \;\forall\; (i, j, \theta) \in N \times \{0, 1, 2\} \times \Theta$,

$\mathcal{G}$ is the set of strategic n-person mechanisms for (N, X), and

e is the bargaining game enforcement structure defined by

$$e(S) = \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{x_0\} & \text{otw.} \end{cases} \qquad \forall\ S \in \mathcal{N}^+.$$

Thus, realization function R satisfies

$$R(x,\theta) = \begin{cases} (x_0, 1) & \text{if } x = x_0, \text{ and } u_i((x,1),\theta) \geq u_i((x,0),\theta) \text{ for some } i \in N \\ (x,1) & \text{if } x \neq x_0, \text{ and } u_i((x,1),\theta) \geq u_i((x,0),\theta)\ \forall\ i \in N \\ (x,0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x,0) & \text{if } x = x_2 \text{ and } \theta = \tilde{\theta} \\ (x,1) & \text{otw.} \end{cases} \qquad \forall\ (x,\theta) \in X \times \Theta.$$

In particular, agents do not have weak pessimistic beliefs in environment E.
Let $E^C = (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ denote E's corresponding classical environment, i.e., $u'_i(x,\theta) = u_i((x,1),\theta)\ \forall\ (x,\theta) \in X \times \Theta$, and let α be the (social) choice correspondence for environment E defined by

$$\alpha(\theta) = \begin{cases} \{x_2\} & \text{if } \theta = \hat{\theta} \\ \{x_1\} & \text{if } \theta = \tilde{\theta} \end{cases} \qquad \forall\ \theta \in \Theta.$$

In particular, since $R(x_2,\hat{\theta}) = (x_2,1)$ and $R(x_1,\tilde{\theta}) = (x_1,1)$, α is consistent with (realization function R in) environment E.
Note that agents' beliefs can be justified by outcome-independent prediction functions $\{b_i\}_{i\in N}$ for (X,Θ), where each b_i is defined by $b_i((x,0),\theta) = (x_0,2)\ \forall\ (x,\theta) \in X \times \Theta$.[36]

SCC α is fully NE^n_{nfg}-implementable in environment E:
To see this, consider the strategic mechanism G in which player 2 chooses between x_1 and x_2, which is then the outcome suggested by the mechanism.
The game induced by G^* and $\hat{\theta}$ in E^* has exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses x_2, since $u_2(R(x_2,\hat{\theta}),\hat{\theta}) = u_2((x_2,1),\hat{\theta}) = 3 > 2 = u_2((x_1,1),\hat{\theta}) = u_2(R(x_1,\hat{\theta}),\hat{\theta})$.
The game induced by G^* and $\tilde{\theta}$ in E^* has also exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses x_1, since $u_2(R(x_1,\tilde{\theta}),\tilde{\theta}) = u_2((x_1,1),\tilde{\theta}) = 3 > 2 = u_2((x_0,1),\tilde{\theta}) = u_2((x_2,0),\tilde{\theta}) = u_2(R(x_2,\tilde{\theta}),\tilde{\theta})$.
Thus, $R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\hat{\theta}})),\hat{\theta}) = R(\{x_2\},\hat{\theta}) = \{(x_2,1)\} = \alpha(\hat{\theta}) \times \{1\}$ and
$R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\tilde{\theta}})),\tilde{\theta}) = R(\{x_1\},\tilde{\theta}) = \{(x_1,1)\} = \alpha(\tilde{\theta}) \times \{1\}$.

[36]Since $u_i((x,0),\theta) = u_i((x_0,1),\theta) = u_i((b_{i1}((x,0),\theta),1),\theta)\ \forall\ (i,\theta,x) \in N \times \Theta \times X$.

Also, SCC α is fully NE^n_{nfg}-implementable in environment E^C:
To see this, consider again mechanism G. The game induced by G and $\hat{\theta}$ in E^C has exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses x_2, since $u'_2(x_2,\hat{\theta}) = u_2((x_2,1),\hat{\theta}) = 3 > 2 = u_2((x_1,1),\hat{\theta}) = u'_2(x_1,\hat{\theta})$.
The game induced by G and $\tilde{\theta}$ in E^C has also exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses x_1, since $u'_2(x_1,\tilde{\theta}) = u_2((x_1,1),\tilde{\theta}) = 3 > 1 = u_2((x_2,1),\tilde{\theta}) = u'_2(x_2,\tilde{\theta})$.
Thus, $g(NE^n_{nfg}(\Gamma^{E^C,G,\hat{\theta}})) = \{x_2\} = \alpha(\hat{\theta})$ and $g(NE^n_{nfg}(\Gamma^{E^C,G,\tilde{\theta}})) = \{x_1\} = \alpha(\tilde{\theta})$.

Note that we could as well have considered mechanism G' in which player 2 chooses between x_0, x_1, and x_2, which is then the outcome suggested by the mechanism. To see this, it is sufficient to add the following four equations:

$$u_2(R(x_2,\hat{\theta}),\hat{\theta}) = u_2((x_2,1),\hat{\theta}) = 3 > 1 = u_2((x_0,1),\hat{\theta}) = u_2(R(x_0,\hat{\theta}),\hat{\theta}),$$
$$u_2(R(x_1,\tilde{\theta}),\tilde{\theta}) = u_2((x_1,1),\tilde{\theta}) = 3 > 2 = u_2((x_0,1),\tilde{\theta}) = u_2(R(x_0,\tilde{\theta}),\tilde{\theta}),$$
$$u'_2(x_2,\hat{\theta}) = u_2((x_2,1),\hat{\theta}) = 3 > 1 = u_2((x_0,1),\hat{\theta}) = u'_2(x_0,\hat{\theta}), \text{ and}$$
$$u'_2(x_1,\tilde{\theta}) = u_2((x_1,1),\tilde{\theta}) = 3 > 2 = u_2((x_0,1),\tilde{\theta}) = u'_2(x_0,\tilde{\theta}).$$

Example 3.5(b)

Consider the following modification of Example 3.5(a):

$$u_i((x_j,0),\theta) = u_i((x_1,1),\theta) \;\forall\; (i,j,\theta) \in N \times \{0,1,2\} \times \Theta.$$

Now, agents' beliefs can be justified by outcome-independent prediction functions $\{b_i\}_{i\in N}$, where each b_i is defined by $b_i((x,0),\theta) = (x_1,2) \;\forall\; (x,\theta) \in X \times \Theta$, and realization function R satisfies

$$R(x,\theta) = \begin{cases} (x_0,0) & \text{if } x = x_0 \text{ and } \theta = \hat{\theta} \\ (x_2,0) & \text{if } x = x_2 \text{ and } \theta = \tilde{\theta} \\ (x,1) & \text{otw.} \end{cases} \qquad \forall\; (x,\theta) \in X \times \Theta.$$

In particular, agents do not have weak pessimistic beliefs in environment E, and α is still consistent with environment E.

Since each agent's preferences over $X \times \{1\}$ are the same as before, α is still fully NE^n_{nfg}-implementable in E's corresponding classical environment E^C.

However, the change in each agent's beliefs implies that now α does not satisfy the necessary conditon of Proposition 3.2, and is therefore not fully NE^n_{nfg}-implementable in environment E. To see this, note that $x_2 \in \alpha(\hat{\theta})$, $R(x_2,\tilde{\theta}) = (x_2,0) \notin \alpha(\tilde{\theta}) \times \{1\}$, and that there does not exist a tuple $(i,x') \in N \times X$ such that

$$u_i(R(x',\tilde{\theta}),\tilde{\theta}) > u_i(R(x_2,\tilde{\theta}),\tilde{\theta}) \text{ and } u_i(R(x',\hat{\theta}),\hat{\theta}) \leq u_i(R(x_2,\hat{\theta}),\hat{\theta}) :$$

$$
\begin{array}{ll}
(1,x_0) & u_1(R(x_0,\tilde\theta),\tilde\theta)=u_1((x_0,1),\tilde\theta)=2\not>2=u_1((x_2,0),\tilde\theta)=u_1(R(x_2,\tilde\theta),\tilde\theta),\\
(1,x_1) & u_1(R(x_1,\tilde\theta),\tilde\theta)=u_1((x_1,1),\tilde\theta)=2\not>2,\\
(1,x_2) & u_1(R(x_2,\tilde\theta),\tilde\theta)=u_1((x_2,0),\tilde\theta)=2\not>2,\\
(2,x_0) & u_2(R(x_0,\tilde\theta),\tilde\theta)=u_2((x_0,1),\tilde\theta)=2\not>3=u_2((x_2,0),\tilde\theta)=u_2(R(x_2,\tilde\theta),\tilde\theta),\\
(2,x_1) & u_2(R(x_1,\tilde\theta),\tilde\theta)=u_2((x_1,1),\tilde\theta)=3\not>3,\text{ and}\\
(2,x_2) & u_2(R(x_2,\tilde\theta),\tilde\theta)=u_2((x_2,0),\tilde\theta)=3\not>3.
\end{array}
$$

Finally, note that the alternative modification

$$u_i((x_j,0),\theta)=u_i((x_2,1),\theta)\ \forall\ (i,j,\theta)\in N\times\{0,1,2\}\times\Theta$$

leads to the same result. In this case, realization function R satisfies

$$
R(x,\theta)=\begin{cases}(x_0,0) & \text{if } x=x_0 \text{ and } \theta=\hat\theta\\ (x_1,0) & \text{if } x=x_1 \text{ and } \theta=\hat\theta\\ (x,1) & \text{otw.}\end{cases}\qquad \forall\ (x,\theta)\in X\times\Theta,
$$

$x_1\in\alpha(\tilde\theta)$, $R(x_1,\hat\theta)=(x_1,0)\notin\alpha(\hat\theta)\times\{1\}$,

and there does not exist a tuple $(i,x')\in N\times X$ such that

$$u_i(R(x',\hat\theta),\hat\theta)>u_i(R(x_1,\hat\theta),\hat\theta)\text{ and } u_i(R(x',\tilde\theta),\tilde\theta)\le u_i(R(x_1,\tilde\theta),\tilde\theta):$$

$$
\begin{array}{ll}
(1,x_0) & u_1(R(x_0,\hat\theta),\hat\theta)=u_1((x_0,0),\hat\theta)=3\not>3=u_1((x_1,0),\hat\theta)=u_1(R(x_1,\hat\theta),\hat\theta),\\
(1,x_1) & u_1(R(x_1,\hat\theta),\hat\theta)=u_1((x_1,0),\hat\theta)=3\not>3,\\
(1,x_2) & u_1(R(x_2,\hat\theta),\hat\theta)=u_1((x_2,1),\hat\theta)=3\not>3,\\
(2,x_0) & u_2(R(x_0,\hat\theta),\hat\theta)=u_2((x_0,0),\hat\theta)=3\not>3=u_2((x_1,0),\hat\theta)=u_2(R(x_1,\hat\theta),\hat\theta),\\
(2,x_1) & u_2(R(x_1,\hat\theta),\hat\theta)=u_2((x_1,0),\hat\theta)=3\not>3,\text{ and}\\
(2,x_2) & u_2(R(x_2,\hat\theta),\hat\theta)=u_2((x_2,1),\hat\theta)=3\not>3.
\end{array}
$$

Example 3.5(c)

Let $E\equiv(N,X,\{\Theta_i\}_{i\in N},\Theta,\{u_i\}_{i\in N},\mathcal{G},R,e)$ be the n-person environment with limited enforcement power, where

$N=\{1,2,3\}$,

$X=\{x_0,x_1,x_2\}$,

$\Theta_i=\{\hat\theta_i,\tilde\theta_i\}\ \forall\ i\in N$,

$\Theta=\{\hat\theta\equiv(\hat\theta_1,\hat\theta_2,\hat\theta_3),\tilde\theta\equiv(\tilde\theta_1,\tilde\theta_2,\tilde\theta_3)\}$,

$u_2((x_0,1),\theta)=u_3((x_0,1),\theta)=2\ \forall\ \theta\in\Theta,\ u_1((x_0,1),\hat\theta)=1,\ u_1((x_0,1),\tilde\theta)=3,$

$u_2((x_1,1),\theta)=u_3((x_1,1),\theta)=3\ \forall\ \theta\in\Theta,\ u_1((x_1,1),\hat\theta)=1,\ u_1((x_1,1),\tilde\theta)=2,$

$u_2((x_2,1),\theta)=u_3((x_2,1),\theta)=1\ \forall\ \theta\in\Theta,\ u_1((x_2,1),\hat\theta)=2,\ u_1((x_2,1),\tilde\theta)=1,$

$u_i((x_j,0),\theta)=u_i((x_2,1),\theta)\ \forall\ (i,j,\theta)\in N\times\{0,1,2\}\times\Theta,$

$\mathcal{G}$ is the set of strategic n-person mechanisms for (N,X), and

e is the bargaining game enforcement structure defined by

$$e(S) = \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{x_0\} & \text{otw.} \end{cases} \qquad \forall\ S \in \mathcal{N}^+.$$

Thus, realization function R satisfies

$$R(x, \theta) = \begin{cases} (x_1, 0) & \text{if } x = x_1 \text{ and } \theta = \hat{\theta} \\ (x, 1) & \text{otw.} \end{cases} \qquad \forall\ (x, \theta) \in X \times \Theta.$$

In particular, agents do not have weak pessimistic beliefs in environment E.
Let $E^C = (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i'\}_{i\in N}, \mathcal{G})$ denote E's corresponding classical environment, i.e., $u_i'(x, \theta) = u_i((x, 1), \theta)\ \forall\ (x, \theta) \in X \times \Theta$, and let α be the (social) choice correspondence for environment E defined by

$$\alpha(\theta) = \begin{cases} \{x_0\} & \text{if } \theta = \hat{\theta} \\ \{x_1\} & \text{if } \theta = \tilde{\theta} \end{cases} \qquad \forall\ \theta \in \Theta.$$

In particular, since $R(x_0, \hat{\theta}) = (x_0, 1)$ and $R(x_1, \tilde{\theta}) = (x_1, 1)$, α is consistent with (realization function R in) environment E.
Note that agents' beliefs can be justified by outcome-independent prediction functions $\{b_i\}_{i\in N}$, where each b_i is defined by $b_i((x, 0), \theta) = (x_2, 2)\ \forall\ (x, \theta) \in X \times \Theta$.

SCC α is fully NE^n_{nfg}-implementable in environment E, since both conditions of Proposition 3.3 are satisfied:

(i) For all $(\theta, \theta', x) \in \Theta \times \Theta \times X$ satisfying $x \in \alpha(\theta)$ and $R(x, \theta') \notin \alpha(\theta') \times \{1\}$, there exists a tuple $(i, y) \in N \times X$ s.t.

$$u_i(R(y, \theta'), \theta') > u_i(R(x, \theta'), \theta') \text{ and } u_i(R(y, \theta), \theta) \leq u_i(R(x, \theta), \theta):$$

(i.1) For $(\theta, \theta', x) = (\hat{\theta}, \hat{\theta}, x_0)$, we have that $R(x_0, \hat{\theta}) = (x_0, 1) \in \alpha(\hat{\theta}) \times \{1\}$.

(i.2) For $(\theta, \theta', x) = (\hat{\theta}, \tilde{\theta}, x_0)$, we have that $R(x_0, \tilde{\theta}) \notin \alpha(\tilde{\theta}) \times \{1\}$, and the tuple $(i, y) := (2, x_1)$ satisfies the required inequalities, since

$$u_2(R(x_1, \tilde{\theta}), \tilde{\theta}) = u_2((x_1, 1), \tilde{\theta}) = 3 > 2 = u_2((x_0, 1), \tilde{\theta}) = u_2(R(x_0, \tilde{\theta}), \tilde{\theta}) \text{ and}$$
$$u_2(R(x_1, \hat{\theta}), \hat{\theta}) = u_2((x_1, 0), \hat{\theta}) = u_2((x_2, 1), \hat{\theta}) = 1$$
$$\leq 2 = u_2((x_0, 1), \hat{\theta}) = u_2(R(x_0, \hat{\theta}), \hat{\theta}).$$

(i.3) For $(\theta, \theta', x) = (\tilde{\theta}, \hat{\theta}, x_1)$, we have that $R(x_1, \hat{\theta}) = (x_1, 0) \notin \alpha(\hat{\theta}) \times \{1\}$, and the tuple $(i, y) := (2, x_0)$ satisfies the required inequalities, since

$$u_2(R(x_0, \hat{\theta}), \hat{\theta}) = u_2((x_0, 1), \hat{\theta}) = 2$$
$$> 1 = u_2((x_2, 1), \hat{\theta}) = u_2((x_1, 0), \hat{\theta}) = u_2(R(x_1, \hat{\theta}), \hat{\theta}) \text{ and}$$
$$u_2(R(x_0, \tilde{\theta}), \tilde{\theta}) = u_2((x_0, 1), \tilde{\theta}) = 2 \leq 3 = u_2((x_1, 1), \tilde{\theta}) = u_2(R(x_1, \tilde{\theta}), \tilde{\theta}).$$

(i.4) For $(\theta, \theta', x) = (\tilde{\theta}, \tilde{\theta}, x_1)$, we have that $R(x_1, \tilde{\theta}) = (x_1, 1) \in \alpha(\tilde{\theta}) \times \{1\}$.

(ii) $R(x, \theta) \in \alpha(\theta) \times \{1\} \ \forall \ (x, \theta) \in X \times \Theta$ that satisfy

$$A(x, \theta) := \sharp\{i \in N \mid u_i(R(x, \theta), \theta) \geq u_i(R(y, \theta), \theta) \ \forall \ y \in X\} \geq n - 1:$$

(ii.1) For $(x, \theta) = (x_0, \hat{\theta})$ and

(ii.2) for $(x, \theta) = (x_1, \tilde{\theta})$, we have that $R(x, \theta) \in \alpha(\theta) \times \{1\}$.

(ii.3) For $(x, \theta) = (x_0, \tilde{\theta})$, we have that $A(x, \theta) < n - 1$, since
$u_2(R(x_0, \tilde{\theta}), \tilde{\theta}) = 2 \not\geq 3 = u_2(R(x_1, \tilde{\theta}), \tilde{\theta})$ and
$u_3(R(x_0, \tilde{\theta}), \tilde{\theta}) = 2 \not\geq 3 = u_3(R(x_1, \tilde{\theta}), \tilde{\theta})$.

(ii.4) For $(x, \theta) = (x_1, \hat{\theta})$, we have that $A(x, \theta) < n - 1$, since
$u_2(R(x_1, \hat{\theta}), \hat{\theta}) = u_2((x_1, 0), \hat{\theta}) = 1 \not\geq 2 = u_2(R(x_0, \hat{\theta}), \hat{\theta})$ and
$u_3(R(x_1, \hat{\theta}), \hat{\theta}) = u_3((x_1, 0), \hat{\theta}) = 1 \not\geq 2 = u_3(R(x_0, \hat{\theta}), \hat{\theta})$.

(ii.5) For $(x, \theta) = (x_2, \hat{\theta})$, we have that $A(x, \theta) < n - 1$, since
$u_2(R(x_2, \hat{\theta}), \hat{\theta}) = 1 \not\geq 2 = u_2(R(x_0, \hat{\theta}), \hat{\theta})$ and
$u_3(R(x_2, \hat{\theta}), \hat{\theta}) = 1 \not\geq 2 = u_3(R(x_0, \hat{\theta}), \hat{\theta})$.

(ii.6) For $(x, \theta) = (x_2, \tilde{\theta})$, we have that $A(x, \theta) < n - 1$, since
$u_2(R(x_2, \tilde{\theta}), \tilde{\theta}) = 1 \not\geq 3 = u_2(R(x_1, \tilde{\theta}), \tilde{\theta})$ and
$u_3(R(x_2, \tilde{\theta}), \tilde{\theta}) = 1 \not\geq 3 = u_3(R(x_1, \tilde{\theta}), \tilde{\theta})$.

However, SCC α is not fully NE^n_{nfg}-implementable in environment E's corresponding classical environment, since α is not Maskin-monotonic in E^C. To see this, note that $x_0 \in \alpha(\hat{\theta})$, $x_0 \notin \alpha(\tilde{\theta})$, and that there does not exist a tuple $(i, x') \in N \times X$ such that

$$u'_i(x_0, \hat{\theta}) \geq u'_i(x', \hat{\theta}) \text{ and } u'_i(x_0, \tilde{\theta}) < u'_i(x', \tilde{\theta}) :$$

$(1, x_0)$ $u'_1(x_0, \tilde{\theta}) \not< u'_1(x_0, \tilde{\theta})$,

$(1, x_1)$ $u'_1(x_0, \tilde{\theta}) = 3 \not< 2 = u'_1(x_1, \tilde{\theta})$,

$(1, x_2)$ $u'_1(x_0, \tilde{\theta}) = 3 \not< 1 = u'_1(x_2, \tilde{\theta})$,

$(2, x_0)$ $u'_2(x_0, \tilde{\theta}) \not< u'_2(x_0, \tilde{\theta})$,

$(2, x_1)$ $u'_2(x_0, \hat{\theta}) = 2 \not\geq 3 = u'_2(x_1, \hat{\theta})$,

$(2, x_2)$ $u'_2(x_0, \tilde{\theta}) = 2 \not< 1 = u'_2(x_2, \tilde{\theta})$,

$(3, x_0)$ $u'_3(x_0, \tilde{\theta}) \not< u'_3(x_0, \tilde{\theta})$,

$(3, x_1)$ $u'_3(x_0, \hat{\theta}) = 2 \not\geq 3 = u'_3(x_1, \hat{\theta})$, and

$(3, x_2)$ $u'_3(x_0, \tilde{\theta}) = 2 \not< 1 = u'_3(x_2, \tilde{\theta})$.

Example 3.5(d)

Consider the following modification of Example 3.5(c):

$$u_i((x_j, 0), \theta) = u_i((x_0, 1), \theta) \ \forall \ (i, j, \theta) \in N \times \{0, 1, 2\} \times \Theta.$$

Now, agents' beliefs can be justified by outcome-independent prediction functions

$\{b_i\}_{i\in N}$, where each b_i is defined by $b_i((x,0),\theta) = (x_0,2)\ \forall\ (x,\theta) \in X \times \Theta$, and realization function R satisfies

$$R(x,\theta) = \begin{cases} (x_2,0) & \text{if } x = x_2 \\ (x_1,0) & \text{if } x = x_1 \text{ and } \theta = \tilde{\theta} \\ (x,1) & \text{otw.} \end{cases} \qquad \forall\ (x,\theta) \in X \times \Theta.$$

In particular, agents do not have weak pessimistic beliefs in environment E.

Since each agent's preferences over $X \times \{1\}$ are the same as before, α is still not fully NE^n_{nfg}-implementable in E's corresponding classical environment E^C.
However, the change in each agent's beliefs implies that now α is not consistent with (realization function R in) environment E (since $R(x_1,\tilde{\theta}) = (x_1,0)$ and $\alpha(\tilde{\theta}) = \{x_1\}$), and is therefore not fully NE^n_{nfg}-implementable in environment E.

Finally, note that the alternative modification

$$u_i((x_j,0),\theta) = u_i((x_1,1),\theta)\ \forall\ (i,j,\theta) \in N \times \{0,1,2\} \times \Theta$$

leads to the same result. In this case, realization function R satisfies

$$R(x,\theta) = \begin{cases} (x_2,0) & \text{if } x = x_2 \\ (x,1) & \text{otw.} \end{cases} \qquad \forall\ (x,\theta) \in X \times \Theta,$$

$x_0 \in \alpha(\hat{\theta})$, $R(x_0,\tilde{\theta}) = (x_0,1) \notin \alpha(\tilde{\theta}) \times \{1\}$,
and there does not exist a tuple $(i,x') \in N \times X$ such that

$$u_i(R(x',\tilde{\theta}),\tilde{\theta}) > u_i(R(x_0,\tilde{\theta}),\tilde{\theta}) \text{ and } u_i(R(x',\hat{\theta}),\hat{\theta}) \leq u_i(R(x_0,\hat{\theta}),\hat{\theta}):$$

$$\begin{array}{ll}
(1,x_0) & u_1(R(x_0,\tilde{\theta}),\tilde{\theta}) = u_1((x_0,1),\tilde{\theta}) = 3 \not> 3 = u_1((x_0,1),\tilde{\theta}) = u_1(R(x_0,\tilde{\theta}),\tilde{\theta}), \\
(1,x_1) & u_1(R(x_1,\tilde{\theta}),\tilde{\theta}) = u_1((x_1,1),\tilde{\theta}) = 2 \not> 3 = u_1((x_0,1),\tilde{\theta}) = u_1(R(x_0,\tilde{\theta}),\tilde{\theta}), \\
(1,x_2) & u_1(R(x_2,\tilde{\theta}),\tilde{\theta}) = u_1((x_1,1),\tilde{\theta}) = 2 \not> 3 = u_1((x_0,1),\tilde{\theta}) = u_1(R(x_0,\tilde{\theta}),\tilde{\theta}), \\
(2,x_0) & u_2(R(x_0,\tilde{\theta}),\tilde{\theta}) = u_2((x_0,1),\tilde{\theta}) = 2 \not> 2 = u_2((x_0,1),\tilde{\theta}) = u_2(R(x_0,\tilde{\theta}),\tilde{\theta}), \\
(2,x_1) & u_2(R(x_1,\hat{\theta}),\hat{\theta}) = u_2((x_1,1),\hat{\theta}) = 3 \not\leq 2 = u_2((x_0,1),\hat{\theta}) = u_2(R(x_0,\hat{\theta}),\hat{\theta}), \\
(2,x_2) & u_2(R(x_2,\hat{\theta}),\hat{\theta}) = u_2((x_1,1),\hat{\theta}) = 3 \not\leq 2 = u_2((x_0,1),\hat{\theta}) = u_2(R(x_0,\hat{\theta}),\hat{\theta}), \\
(3,x_0) & u_3(R(x_0,\tilde{\theta}),\tilde{\theta}) = u_3((x_0,1),\tilde{\theta}) = 2 \not> 2 = u_3((x_0,1),\tilde{\theta}) = u_3(R(x_0,\tilde{\theta}),\tilde{\theta}), \\
(3,x_1) & u_3(R(x_1,\hat{\theta}),\hat{\theta}) = u_3((x_1,1),\hat{\theta}) = 3 \not\leq 2 = u_3((x_0,1),\hat{\theta}) = u_3(R(x_0,\hat{\theta}),\hat{\theta}), \\
(3,x_2) & u_3(R(x_2,\hat{\theta}),\hat{\theta}) = u_3((x_1,1),\hat{\theta}) = 3 \not\leq 2 = u_3((x_0,1),\hat{\theta}) = u_3(R(x_0,\hat{\theta}),\hat{\theta}).
\end{array}$$

3.6 An Extensive Procedure for the Implementation Decision

We now define an extensive procedure which, under certain conditions, is able to 'copy' the implementation decision of the agents. An assumption implicit in our definition of realization function R (in Paragraph 3.1.1) is that an agent who is indifferent between a suggested outcome being implemented or not, supports the implementation of this outcome whenever he is able to do so. Since this assumption is 'incompatible' with the SPNE concept, we assume in the following that no agent is indifferent between an outcome being implemented or not.

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an n-person environment with limited enforcement power such that

(i) each utility function u_i satisfies $u_i((x,1),\theta) \neq u_i((x,0),\theta)\ \forall\ (x,\theta) \in X \times \Theta$,

(ii) $\mathcal{G}$ is the set of all strategic and extensive mechanisms for (N, X), and

(iii) enforcement structure e is one of the enforcement structures discussed at the beginning of this chapter, i.e. e is the Majority Voting, Cooperative Game, or a Bargaining Game Enforcement Structure for (N, X).

Let $E^+ \equiv (N, (X \times \{1\}) \cup (X \times \{0\}), \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}^+)$ denote the classical n-person environment where $\mathcal{G}^+$ is the set of all strategic and extensive n-person mechanisms for $(N, (X \times \{1\} \cup (X \times \{0\}))$.

For each strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$, let $G^+ \equiv (N, H, p, g^+) \in \mathcal{G}^+_{ext}$ denote the extensive mechanism for $(N, (X \times \{1\}) \cup \{X \times \{0\}))$ defined by

$$\begin{aligned}
H &:= \{\emptyset\} \cup S_1 \times \ldots \times S_n \\
&\qquad \cup \{(s, a_1, \ldots, a_k) \mid k \in N, s \in S_1 \times \ldots \times S_n, a_i \in \{0,1\}\ \forall\ i \in \mathbb{N}, i \leq k\},\\
Z_H &:= S_1 \times \ldots S_n \times \{0,1\}^n,\\
p(\emptyset) &:= N,\\
p(s) &:= \{1\}\ \forall\ s \in S_1 \times \ldots \times S_n,\\
p(s, a_1, \ldots, a_k) &:= \{k+1\}\ \forall\ k \in N\backslash\{n\}, \forall\ (s, a_1, \ldots, a_k) \in H\backslash Z_H, \text{ and}
\end{aligned}$$

$$g^+(s, a_1 \ldots, a_n) := \begin{cases} (g(s), 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } g(s) \in e(S) \text{ and} \\ & a_i = 1\ \forall\ i \in S \cap N \\ (g(s), 0) & \text{otw.} \end{cases} \qquad \text{on } Z_h.$$

Lemma 3.6 For each strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ and every state $\theta \in \Theta$, the set of 'reduced mechanisms' that result from the backward induction procedure after n steps applied to the game induced by mechanism G^+ and state θ in environment E^+ contains exactly one element: $(N, \{S_i\}_{i\in N}, g^{red}) \in \mathcal{G}^+_{strat}$, $g^{red}(s) := R(g(s),\theta)\ \forall\ s \in S_1 \times \ldots \times S_n$.

Proof

Consider an arbitrary strategy profile $s \in S_1 \times \ldots \times S_n$. We analyse the game tree following strategy profile s in the game induced by mechanism G^+ and state θ in environment E^+ according to the two cases in the defintion of realization function R (applied to $(g(s),\theta)$).

(1) First, suppose that there exists a coalition $S \in \mathcal{N}^+$ such that outcome $g(s) \in e(S)$ and $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta) \;\forall\, i \in N \cap S$.

If $S = \{0\}$, then, by definition of G^+, $g^+(s, a_1, \ldots, a_n) = (g(s), 1) \;\forall\, (a_1, \ldots, a_n) \in \{0,1\}^n$. And, therefore, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s),1) = R(g(s),\theta)$.

If $S \neq \{0\}$, then consider player $i_1 := \max\{\, j \mid j \in S \cap N\}$ at all those of his decision nodes in the game tree (corresponding to strategy profile s) at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s),1)$. Since, by assumption, $u_{i_1}((g(s),1),\theta) > u_{i_1}((g(s),0),\theta)$, player i_1 will choose this alternative.
If $S\backslash\{i_1\} \cap N \neq \emptyset$, then consider player $i_2 := \max\{\, j \mid j \in S\backslash\{i_1\} \cap N\}$ at all those of his decision nodes in the game tree at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s),1)$. Since, by assumption, $u_{i_2}((g(s),1),\theta) > u_{i_2}((g(s),0),\theta)$, player i_2 will choose this alternative.
We can proceed along these lines up to player $i_{\sharp(S\cap N)} := \min\{\, j \mid j \in S \cap N\}$, who also will choose $(g(s),1)$ at all those of his decision nodes in the game tree at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s),1)$. And, in addition, at every node at which he is (possibly) asked for a decision, $(g(s),1)$ is one of player $i_{\sharp(S\cap N)}$'s alternatives.
To verify the latter assertion, assume that there exists a node d at which this player is asked for a decision but at which $(g(s),1)$ is not one of his alternatives, and consider player $i_{\sharp(S\cap N)-1}$'s decision nodes that are possibly reached in the game tree if player $i_{\sharp(S\cap N)}$ chooses action '1' at node d. If at each of these decision nodes at least one of player $i_{\sharp(S\cap N)-1}$'s alternatives (within the backward induction procedure) is $(g(s),1)$, then (as shown above) player $i_{\sharp(S\cap N)-1}$ will choose this alternative (at each of these decision nodes) and we obtain a contradiction. If, on the other hand, there exists a decision node at which $(g(s),1)$ is not one of player $i_{\sharp(S\cap N)-1}$'s alternatives, then consider player $i_{\sharp(S\cap N)-2}$'s decision nodes that are possibly reached in the game tree if player $i_{\sharp(S\cap N)-1}$ chooses action '1' at this node in the game tree. If at each of these decision nodes at least one of player $i_{\sharp(S\cap N)-2}$'s alternatives is $(g(s),1)$, we again obtain a contradiction. If not, we can proceed along these lines, if necessary, up to player i_1. If at each of player i_1's decision nodes at least one of his alternatives is $(g(s),1)$, then (as shown above) player i_1 will choose this alternative and we obtain a contradic-

tion. And, since $g(s) \in e(S)$, and by definition of mechanism G^+, $(g(s), 1)$ has to be one of player i_1's alternatives: By choosing action '1' at this point in the game tree, player i_1 can always enforce $g(s)$.

Since, at every node at which he is (possibly) asked for a decision, $(g(s), 1)$ is one of player $i_{\sharp(S\cap N)}$'s alternatives, and since player $i_{\sharp(S\cap N)}$ will choose this alternative whenever asked for a decision, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s), 1) = R(g(s), \theta)$.

(2) Now, suppose that there does not exist a coalition $S \in \mathcal{N}^+$ such that outcome $g(s) \in e(S)$ and $u_i((g(s), 1), \theta) \geq u_i((g(s), 0), \theta)\ \forall\, i \in N \cap S$.

If e is the cooperative game enforcement structure, then, by defintion of G^+,

$$g^+(s, a_1 \ldots, a_n) = \begin{cases} (g(s), 1) & \text{if } (a_1, \ldots, a_n) = (1, \ldots, 1) \\ (g(s), 0) & \text{otw.} \end{cases} \quad \text{on } Z_h,$$

and, since $g(s) \in X = e(N)$, there exists an agent $i \in N$ such that $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)$. Consider this player at every node at which he is asked for a decision. Outcome function g^+ implies that, within the backward induction procedure, at least one of his alternatives is outcome $(g(s), 0)$. And, since $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)$, player i will choose this alternative. Thus, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s), 0) = R(g(s), \theta)$.

If e is a bargaining enforcement structure, then, either

$$g^+(s, a_1 \ldots, a_n) = \begin{cases} (g(s), 1) & \text{if } (a_1, \ldots, a_n) = (1, \ldots, 1) \\ (g(s), 0) & \text{otw.} \end{cases} \quad \text{on } Z_h$$

and there exists an agent $i \in N$ such that $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)$, or

$$g^+(s, a_1 \ldots, a_n) = \begin{cases} (g(s), 1) & \text{if } (a_1, \ldots, a_n) \neq (0, \ldots, 0) \\ (g(s), 0) & \text{otw.} \end{cases} \quad \text{on } Z_h$$

and $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)\ \forall\, i \in N$.

In the first case, consider player i (satisfying $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)$) at every node at which he is asked for a decision. Outcome function g^+ implies that, within the backward induction procedure, at least one of his alternatives is outcome $(g(s), 0)$. And, since $u_i((g(s), 1), \theta) < u_i((g(s), 0), \theta)$, player i will choose this alternative. Thus, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s), 0) = R(g(s), \theta)$.

In the second case, outcome function g^+ implies that outcome $(g(s), 0)$ is one of player n's alternatives at that node in the mechanism tree which is reached if all previous players have chosen action '0'. Since $u_n((g(s),1),\theta) < u_n((g(s),0),\theta)$, player n will choose this alternative (by choosing action '0'). Thus, within the backward induction procedure, outcome $(g(s),0)$ is one of player $n-1$'s alternatives at that node in the mechanism tree which is reached if all previous players have chosen action '0'. In addition, outcome function g^+ implies that a choice of action '1' by player $n-1$ can only result in outcome $(g(s),1)$. Thus, since $u_{n-1}((g(s),1),\theta) < u_{n-1}((g(s),0),\theta)$, player $n-1$ will choose action '0'. If we proceed along these lines up to player 1, then, within the backward induction procedure, outcome $(g(s),0)$ is one of player 1's alternatives and can only be chosen by deciding for action '0'. Since $u_1((g(s),1),\theta) < u_1((g(s),0),\theta)$, player 1 will choose action '0'. Thus, again, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s),0) = R(g(s),\theta)$.

Finally, if e is the majority voting enforcement structure, then

$$g^+(s,a_1\ldots,a_n) = \begin{cases} (g(s),1) & \text{if } \sum_{i\in N} a_i > \frac{n}{2} \\ (g(s),0) & \text{otw.} \end{cases} \qquad \text{on } Z_h,$$

and there exists a coalition $S \in \mathcal{N}$ of at least $\frac{n}{2}$ agents such that $u_i((g(s),1),\theta) < u_i((g(s),0),\theta)\ \forall\ i \in S$.

To verify the latter assertion, assume, to the contrary, that for every coalition $S \in \mathcal{N}$ of at least $\frac{n}{2}$ agents there exists an agent $i \in S$ satisfying $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta)$.

If n is even, then consider a coalition $S^* \in \mathcal{N}$ consisting of $\frac{n}{2}-1$ agents satisfying $u_i((g(s),1),\theta) < u_i((g(s),0),\theta)\ \forall\ i \in S^*$ (If such a coalition does not exist, then there exists a coalition $S \in \mathcal{N}$ of at least $\frac{n}{2}+2$ agents satisfying $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta)\ \forall\ i \in S$, contradicting that we are in case 2). By assumption and construction, every coalition $S \in \mathcal{N}$ consisting of the members of coalition S^* and one further agent $i \in N\backslash S^*$ satisfies $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta)$. Thus, the coalition $S \in \mathcal{N}$ consisting of all $\frac{n}{2}+1$ potential 'additional agents' satisfies $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta)\ \forall\ i \in S$, contradicting that we are in case 2.

If n is odd, then consider a coalition $S^* \in \mathcal{N}$ consisting of $\frac{n-1}{2}$ agents satisfying $u_i((g(s),1),\theta) < u_i((g(s),0),\theta)\ \forall\ i \in S^*$ (If such a coalition does not exist, then there exists a coalition $S \in \mathcal{N}$ of at least $\frac{n-1}{2}+2$ agents satisfying $u_i((g(s),1),\theta) \geq u_i((g(s),0),\theta)\ \forall\ i \in S$, contradicting that we are in case 2). By assumption and construction, every coalition $S \in \mathcal{N}$ consisting of the members of coalition S^* and one further agent $i \in N\backslash S^*$ satisfies $u_i((g(s),1),\theta) \geq$

$u_i((g(s), 0), \theta)$. Thus, the coalition $S \in \mathcal{N}$ consisting of all $\frac{n-1}{2}+1$ potential 'additional agents' satisfies $u_i((g(s), 1), \theta) \geq u_i((g(s), 0), \theta)\ \forall\ i \in S$, contradicting that we are in case 2, and verifying our assertion.

Now, consider player $i_1 := \max\{\, j \mid j \in S\}$ at all those of his decision nodes in the game tree (corresponding to strategy profile s) at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s), 0)$. Since, by assumption, $u_{i_1}((g(s), 1), \theta) < u_{i_1}((g(s), 0), \theta)$, player i_1 will choose this alternative.

If $S \backslash \{i_1\} \neq \emptyset$, then consider player $i_2 := \max\{\, j \mid j \in S \backslash \{i_1\}\}$ at all those of his decision nodes in the game tree at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s), 0)$. Since $u_{i_2}((g(s), 1), \theta) < u_{i_2}((g(s), 0), \theta)$, player i_2 will choose this alternative.

We can proceed along these lines up to player $i_{\sharp S} := \min\{\, j \mid j \in S\}$, who also will choose $(g(s), 0)$ at all those of his decision nodes in the game tree at which, within the backward induction procedure, at least one of his alternatives is outcome $(g(s), 0)$. And, in addition, at every node at which player $i_{\sharp S}$ is (possibly) asked for a decision, $(g(s), 0)$ is one of his alternatives.

To verify the latter assertion, assume that there exists a node d at which this player is asked for a decision but $(g(s), 0)$ is not one of his alternatives, and consider player $i_{\sharp S-1}$'s decision nodes that are possibly reached in the game tree if player $i_{\sharp S}$ chooses action '0' at node d. If at each of these decision nodes at least one of player $i_{\sharp S-1}$'s alternatives (within the backward induction procedure) is $(g(s), 0)$, then (as shown above) player $i_{\sharp S-1}$ will choose this alternative (at each of these decision nodes) and we obtain a contradiction. If, on the other hand, there exists a decision node at which $(g(s), 0)$ is not one of player $i_{\sharp S-1}$'s alternatives, then consider player $i_{\sharp S-2}$'s decision nodes that are possibly reached in the game tree if player $i_{\sharp S-1}$ chooses action '0' at this node in the game tree. If at each of these decision nodes at least one of player $i_{\sharp S-2}$'s alternatives is $(g(s), 0)$, we again obtain a contradiction. If not, we can proceed along these lines, if necessary, up to player i_1. If at each of player i_1's decision nodes at least one of his alternatives is $(g(s), 0)$, then (as shown above) player i_1 will choose this alternative and we obtain a contradiction. And, by definition of mechanism G^+, $(g(s), 0)$ has to be one of player i_1's alternatives: every terminal node possibly reached after a choice of action '0' by player i_1 at this point in the game tree has to result in outcome $(g(s), 0)$, since only a maximum number $\frac{n}{2}$ players could have chosen action '1'.

Since, at every node at which he is (possibly) asked for a decision, $(g(s), 0)$ is one of player $i_{\sharp S}$'s alternatives, and since player $i_{\sharp S}$ will choose this alternative whenever asked for a decision, every 'reduced mechanism' satisfies $g^{red}(s) = (g(s), 0) = R(g(s), \theta)$.

□

Proposition 3.6 For each strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ and every state $\theta \in \Theta$, we have that $R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})), \theta) = g^+(O(SPNE^n(\Gamma^{E^+,G^+,\theta})))$.

Proof

The set of all SPNE of the game induced by mechanism G^+ and state θ in environment E^+, $\Gamma^{E^+,G^+,\theta}$, can be derived by means of the backward induction procedure.

If $\{(N, \{S_i\}_{i\in N}, g^{red}_j) \mid j \in J\}$ denotes the set of 'reduced mechanisms' that result from the backward induction procedure after n steps applied to game $\Gamma^{E^+,G^+,\theta}$, then $g^+(O(SPNE^n(\Gamma^{E^+,G^+,\theta}))) = \bigcup_{j\in J} g^{red}_j(NE^n_{nfg}(N, \{S_i\}_{i\in N}, \{u_i(g^{red}_j(\cdot), \theta)\}_{i\in N}))$.

Lemma 3.6 implies that there exists only one 'reduced mechanism' $(N, \{S_i\}_{i\in N}, g^{red})$, and that $g^{red} : S_1 \times \ldots \times S_n \to (X \times \{1\}) \cup (X \times \{0\})$ satisfies $g^{red}(s) = R(g(s), \theta)\ \forall\ s \in S_1 \times \ldots \times S_n$.

Thus,
$$\begin{aligned} g^+(O(SPNE^n(\Gamma^{E^+,G^+,\theta}))) &= g^{red}(NE^n_{nfg}(N, \{S_i\}_{i\in N}, \{u_i(g^{red}(\cdot), \theta)\}_{i\in N})) \\ &= R(g(NE^n_{nfg}(N, \{S_i\}_{i\in N}, \{u_i(R(g(\cdot), \theta), \theta)\}_{i\in N})), \theta) \\ &= R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})), \theta). \end{aligned}$$

□

Let α be a SCC for environment E, and let $\alpha^+ : \Theta \rightrightarrows (X \times \{1\}) \cup (X \times \{0\})$ denote the SCC for classical environment E^+ defined by $\alpha^+(\theta) := \{(x, 1) \mid x \in \alpha(\theta)\}\ \forall\ \theta \in \Theta$.

Corollary 3.6 If there exists a strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ that strongly/fully NE^n_{nfg}-implements α in environment E, then mechanism $G^+ \equiv (N, H, p, g^+)$ strongly/fully $SPNE^n$-implements α^+ in classical environment E^+.

Proof

Suppose that mechanism G fully NE^n_{nfg}-implements SCC α in environment E. The other case is analogous.

Since G fully NE^n_{nfg}-implements α in E, we have that
$$R(g(NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})), \theta) = \{(x, 1) \mid x \in \alpha(\theta)\}\ \forall\ \theta \in \Theta.$$
By Proposition 3.6, this implies that
$$g^+(O(SPNE^n(\Gamma^{E^+,G^+,\theta}))) = \{(x, 1) \mid x \in \alpha(\theta)\}\ \forall\ \theta \in \Theta.$$
Thus, mechanism G^+ strongly $SPNE^n$-implements α^+ in environment E^+.

□

4 Implementation in Environments with Delegative Enforcement Power

As before, we consider a model for a setting that is characterized by the presence of $n \in \mathbb{N}$, $n \geq 2$, agents (denoted by the numbers 1 to n), a designer (denoted by the number 0), and a set of feasible outcomes $X \neq \emptyset$. Within a certain time interval T, exactly one element of the set of feasible outcomes X is to be implemented.

And, as before, we assume that the designer can force the agents to participate in one of a certain set of mechanisms. However, in line with classical implementation theory and in contrast to our framework with limited enforcement power, we now assume that the designer can enforce each of the feasible outcomes. In addition, and this aspect is new, we assume that the designer is able to impose one of a certain set of *enforcement and default structure assignments* on the agents. Such an assignment specifies

(1) for each coalition of agents $S \in \mathcal{N}$, the outcomes that, if suggested by the mechanism, this coalition is able to enforce 'right after the mechanism has been played',

(2) the outcomes that, if suggested by the mechanism, the designer commits to implement, and

(3) for each feasible outcome $x \in X$ which is enforceable by a coalition $S \in \mathcal{N}$, the *default outcome* that will be enforced by the designer in case that outcome x is suggested by the mechanism but is not being implemented by a coalition that is able to do so.

In Section 4.1, we define environments with delegative enforcement power and the concept of implementation in these environments (*DE Implementation*). Our concept of implementation is again based on Maskin and Moore's [27] notion of "implementation with renegotiation function h" and Jackson and Palfrey's [24a] notion of "G-Nash implementation" (although, of course, each of these two articles considers a different interpretational context). Subsequently to briefly stating the simple counterparts of our sufficient and necessary conditions for LE implementation in Nash Equilibrium to environments with delegative enforcement power in Section 4.2, specific attention is paid to (what we will refer to as) *replica environments* in Section 4.3. In Section 4.4, two examples show that delegative enforcement power can positively affect the Nash-implementability of a (social) choice correspondence, and that even the availability of all EDS assignments might not be sufficient for the Nash-implementability of a (social) choice correspondence in environments in which all mechanisms arising from strategic game forms are available to the designer.

4.1 Definitions

In the following, we consider time interval T as divided into three parts.

In part one, the game induced by the mechanism and the type profile is played. Part one results in an outcome $x \in X$ suggested by the mechanism.

In part two, the agents and the designer consider the implementation of x. We use the tuples $(x, 1)$ and $(x, 0)$ to denote the results that, 'right after the mechanism has been played', outcome x is or is not being implemented, respectively.

If part two results in $(x, 0)$, the designer implements the default outcome as determined by the *default structure*. We use the tuple $(y, 2)$ to denote the result that outcome $y \in X$ is implemented in this part of the time interval.

Definition An *enforcement and default structure assignment (EDS assignment) for* (N, X) is a tuple $A \equiv (e, d)$, where

$e : \mathcal{N}^+ \Rightarrow X$ satisfies $e(S') \supseteq e(S) \ \forall \ S, S' \in \mathcal{N}^+, S' \supseteq S$,

$e(N) \cap e(\{0\}) = \emptyset$,

$e(N) \cup e(\{0\}) = X$, and

$e(S \cup \{0\}) = e(S) \cup e(\{0\}) \ \forall \ S \in \mathcal{N}$, and

$d : X \times \{0\} \to X \times \{2\}$ satisfies $d(x, 0) = (x, 2) \ \forall \ x \in e(\{0\})$.[37]

In their discussion of "voluntary implementation", Jackson and Palfrey [23] briefly mention a generalization of voluntary implementation to (state-contingent) "blocking coalitions". Note that in our context of delegative enforcement power (and in contrast to that of Chapter 3) we could 'equivalently' define a *coalitional veto and default structure assignment (CVDS assignment)* for (N, X), i.e., a tuple (v, d), where

$v : \mathcal{N} \Rightarrow X$ specifies, for each coalition $S \in \mathcal{N}$, the outcomes that, if suggested by the mechanism, coalition S can veto,
satisfying $v(S') \supseteq v(S) \ \forall \ S, S' \in \mathcal{N}, S' \supseteq S$, and

$d : X \times \{0\} \to X \times \{2\}$ specifies, for each feasible outcome $x \in X$ which can be vetoed by a coalition $S \in \mathcal{N}$, the default outcome that will be enforced by the designer in case that outcome x is suggested by the mechanism but is being vetoed by a coalition that is able to do so,
satisfying $d(x, 0) = (x, 2) \ \forall \ x \in X \backslash v(N)$.

[37]Note that the domain of each of the two mappings e and d is chosen for mathematical reasons. The economic interpretation and effect of an EDS assignment (e, d) depends solely on correspondence e restricted to coalitions in $\mathcal{N} \cup \{\{0\}\}$ and on mapping d restricted to outcomes in $e(N) \times \{0\}$. The degree of freedom arising from the domain as chosen for correspondence e is resolved by the latter two requirements in our definition of an EDS assignment.

Then, for each EDS assignment $A \equiv (e, d)$ (for (N, X)) there exists a CVDS assignment (v^A, d) (for (N, X)) which 'reflects' assignment A in the sense that

$\forall\ (x, S) \in e(\{0\}) \times \mathcal{N}\ : x \notin v^A(S)$, and

$\forall\ (x, S) \in X \backslash e(\{0\}) \times \mathcal{N}\ :\ x \in v^A(S) \Leftrightarrow S \cap S' \neq \emptyset\ \forall\ S' \in \mathcal{N}$ s.t. $x \in e(S')$,

i.e., coalition S can veto x if and only if every coalition S' that can enforce x has at least one member in coaliton S.

And, for each CVDS assignment $A \equiv (v, d)$ (for (N, X)) there exists an EDS assignment (e^A, d) (for (N, X)) which 'reflects' A in the sense that

$x \in e^A(\{0\})\ \forall\ x \in X \backslash v(N)$ and $x \notin e^A(\{0\})\ \forall\ x \in v(N)$,

$x \notin e^A(S)\ \forall\ (x, S) \in X \backslash v(N) \times \mathcal{N}$, and

$\forall\ (x, S) \in v(N) \times \mathcal{N}\ :\ x \notin e^A(S) \Leftrightarrow \exists\ S' \in \mathcal{N}$ s.t. $S \cap S' = \emptyset$ and $x \in v(S')$,

i.e., coalition S can enforce x if and only if there does not exist a disjoint coalition S' that can veto x.

We briefly sketch this 'equivalence' between the two assignments as well as all possible assignments for the case of $N = 3$ agents and $\sharp X = 1$ outcome in Appendix D.

4.1.1 Environments with Delegative Enforcement Power

Definition An *n-person environment with delegative enforcement power* is a tuple $(N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$, where

$N := \{1, \ldots, n\}$ is the set of agents,

$X \neq \emptyset$ is the set of feasible outcomes,

Θ_i is the set of possible types for agent i,

$\Theta \subseteq \Theta_1 \times \ldots \times \Theta_n$ is the set of possible type profiles / states, $\Theta \neq \emptyset$,

$u_i : (X \times \{1\}) \cup (X \times \{2\}) \times \Theta \to \mathbb{R}$,

$u_i(\cdot, \theta)\ :\ (X \times \{1\}) \cup (X \times \{2\}) \to \mathbb{R}$ being agent i's utility function over $(X \times \{1\}) \cup (X \times \{2\})$ when the actual state of the environment is $\theta \in \Theta$, representing her rational preference relation over $(X \times \{1\}) \cup (X \times \{2\})$,

$\mathcal{G} \equiv \mathcal{G}_{strat} \cup \mathcal{G}_{ext}$ is a set of mechanisms for (N, X),

$\mathcal{A} \neq \emptyset$ is a set of EDS assignments for (N, X), satisfying Assumption 4.0 below, and

$\mathcal{R} \equiv \{R^A : X \times \Theta \to (X \times \{1\}) \cup (X \times \{2\}) \mid A \in \mathcal{A}\}$ is the set of *realization functions* corresponding to $\mathcal{A}$, each satisfying

$$R^{(e,d)}(x, \theta) = \begin{cases} (x, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \text{ and} \\ & u_i((x, 1), \theta) \geq u_i(d(x, 0), \theta)\ \forall\ i \in S \cap N \\ d(x, 0) & \text{otw.} \end{cases} \quad \text{on } X \times \Theta.$$

In environments with delegative enforcement power, the outcome suggested by a mechanism is 'transformed' in a state-contingent way via realization function R. This function determines whether the suggested outcome or a different outcome is realized, and at which point in time it is realized. In this respect, it is the counterpart of Jackson and Palfrey's [24a] "generalized reversion function" and Maskin and Moore's [27] "renegotiation function".

Given an EDS assignment, an outcome suggested by a mechanism is implemented 'right after the mechanism has been played' if and only if there exists a coalition S which is able to enforce this outcome (according to the EDS assignment) and all agents in this coalition weakly prefer its realization to its non-realization, i.e., prefer its realization to the implementation of the corresponding default outcome.

There exists exactly one EDS assignment (e, d) for (N, X) which satisfies $e(\{0\}) = X$, i.e., which allocates no enforcement power on the agents and therefore reflects standard implementation theory. We assume that this is one of the designer's options:

Assumption 4.0 The uniquely determined EDS assignment (e, d) for (N, X) which satisfies $e(\{0\}) = X$ is an element of $\mathcal{A}$.

4.1.2 Corresponding Classical Environments

For every n-person environment with delegative enforcement power

$$E \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, \mathcal{R}, \mathcal{A}),$$

there exists exactly one classical n-person environment

$$E^C \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i'\}_{i \in N}, \mathcal{G})$$

which shares the same outcome space, state space, and mechanism space, and which satisfies $u_i'(x, \theta) = u_i((x, 1), \theta) \ \forall \ (x, \theta) \in X \times \Theta$. This classical environment is (as before) referred to as *environment E's corresponding classical environment.*

4.1.3 DE Implementation

Let $E \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$ be an environment with delegative enforcement power.

Definition A *(social) choice correspondence (SCC) for environment E* is a correspondence $\alpha : \Theta \Rightarrow X$ satisfying $\alpha(\theta) \neq \emptyset \ \forall \ \theta \in \Theta$.

Let α be a SCC for environment E, and let $(e, d) \in \mathcal{A}$ be an EDS assignment.

Let $^{*(e,d)}$ denote the function from the set of mechanims for (N, X) to the set of state-contingent mechanisms for $(N, (X \times \{1\}) \cup (X \times \{2\}), \Theta)$ defined by

$$(N, \{S_i\}_{i\in N}, g)^{*(e,d)} := (N, \{S_i\}_{i\in N}, g^{*(e,d)}),$$
$$g^{*(e,d)}(s,\theta) := R^{(e,d)}(g(s),\theta) \ \forall\ (s,\theta) \in S \times \Theta, \text{ and}$$
$$(N, H, p, g)^{*(e,d)} := (N, H, p, g^{*(e,d)}),$$
$$g^{*(e,d)}(h,\theta) := R^{(e,d)}(g(h),\theta) \ \forall\ (h,\theta) \in Z_H \times \Theta.$$

Let $E^{*(e,d)}$ denote the abstract n-person environment with state-contingent mechanisms

$$E^{*(e,d)} := (N, (X \times \{1\}) \cup (X \times \{2\}), \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}^{*(e,d)}),$$

and let $\alpha^{*(e,d)} :\ \Theta \Rightarrow (X \times \{1\}) \cup (X \times \{2\})$ denote the SCC for $E^{*(e,d)}$ defined by

$$\alpha^{*(e,d)}(\theta) := \{(x,1) \mid x \in \alpha(\theta)\} \ \forall\ \theta \in \Theta.$$

Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Definition Mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ *EC-implements α in environment E under EDS assignment* (e,d)

(i) *strongly,* if mechanism $G^{*(e,d)}$ strongly EC-implements $\alpha^{*(e,d)}$ in $E^{*(e,d)}$, i.e. if, $\forall\ \theta \in \Theta$, $EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}) \neq \emptyset$ and

$$R^{(e,d)}(g(EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta})),\theta) = g^{*(e,d)}(EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}),\theta) \subseteq \alpha^{*(e,d)}(\theta) = \{(x,1) \mid x \in \alpha(\theta)\},$$

(ii) *fully,* if mechanism $G^{*(e,d)}$ fully EC-implements $\alpha^{*(e,d)}$ in $E^{*(e,d)}$, i.e. if, $\forall\ \theta \in \Theta$, $R^{(e,d)}(g(EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta})),\theta) = g^{*(e,d)}(EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}),\theta)$

$$= \alpha^{*(e,d)}(\theta) = \{(x,1) \mid x \in \alpha(\theta)\},$$

where $\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta} \ \equiv\ (N, \{S_i\}_{i\in N}, \{u_i(R^{(e,d)}(g(\cdot),\theta),\theta)\}_{i\in N})$ denotes the game induced by mechanism $G^{*(e,d)}$ and type profile θ in environment $E^{*(e,d)}$.

Definition Mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ *$SPNE^n$-implements α in environment E under EDS assignment* (e,d)

(i) *strongly,* if mechanism $G^{*(e,d)}$ strongly $SPNE^n$-implements $\alpha^{*(e,d)}$ in $E^{*(e,d)}$, i.e. if, $\forall\ \theta \in \Theta$, $SPNE^n(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}) \neq \emptyset$ and

$$R^{(e,d)}(g(O(SPNE^n(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}))),\theta) \subseteq \{(x,1) \mid x \in \alpha(\theta)\},$$

(ii) *fully,* if mechanism $G^{*(e,d)}$ fully $SPNE^n$-implements $\alpha^{*(e,d)}$ in $E^{*(e,d)}$, i.e. if $\forall\ \theta \in \Theta$, $R^{(e,d)}(g(O(SPNE^n(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta}))),\theta) = \{(x,1) \mid x \in \alpha(\theta)\}$,

where $\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta} \ \equiv\ (N, H, p, \{u_i(R^{(e,d)}(g(\cdot),\theta),\theta)\}_{i\in N})$ denotes the game induced by mechanism $G^{*(e,d)}$ and type profile θ in environment $E^{*(e,d)}$.

Note that a strategic mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ fully EC-implements α in E's corresponding classical environment E^C if, $\forall\ \theta \in \Theta$, $g(EC(\Gamma^{E^C,G,\theta})) = \alpha(\theta)$, where $\Gamma^{E^C,G,\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i((g(\cdot),1),\theta)\}_{i\in N})$.

An extensive mechanism $G \equiv (N, H, p, g) \in \mathcal{G}_{ext}$ fully $SPNE^n$-implements α in E's corresponding classical environment E^C if, $\forall\ \theta \in \Theta$, $g(O(SPNE^n(\Gamma^{E^C,G,\theta}))) = \alpha(\theta)$, where $\Gamma^{E^C,G,\theta} \equiv (N, H, p, \{u_i((g(\cdot),1),\theta)\}_{i\in N})$.

Let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$.

Definition *SCC α is strongly/fully EC-implementable in environment E* if there exists a mechanism $G \in \mathcal{G}$ and an EDS assignment $(e,d) \in \mathcal{A}$ such that G strongly/fully EC-implements α in E under assignment (e,d).

4.2 Conditions for the Implementation in Nash Equilibrium

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$ be an environment with delegative enforcement power, and let α be a SCC for environment E.

Definition SCC α *is consistent with environment* E if there exists an EDS assignment $(e,d) \in \mathcal{A}$ such that, $\forall\ (x,\theta) \in X \times \Theta$ satisfying $x \in \alpha(\theta)$, we have that $R^{(e,d)}(x,\theta) = (x,1)$.

Corollary 4.1, 4.2 and Proposition 4.3 are the counterparts of Proposition 3.1, 3.2 and 3.3, respectively. Our proofs for Proposition 4.2 and 4.3 are again based on Lemma 3.2 and 3.3. Proposition 4.1 and Corollary 4.1 hold true for all three equilibrium concepts considered throughout this paper, i.e., for each $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$.

Proposition 4.1 If a mechanism $G \in \mathcal{G}$ fully EC-implements α in environment E under an assignment $(e,d) \in \mathcal{A}$, then $R^{(e,d)}(x,\theta) = (x,1)\ \forall\ (x,\theta) \in X \times \Theta$ s.t. $x \in \alpha(\theta)$.

Proof

Let $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ be the mechanism that fully EC-implements α in environment E ($EC \in \{NE^n_{nfg}, DSE^n_{nfg}\}$). The proof for $G \in \mathcal{G}_{strat}$ is analogous.

Consider an arbitrary tuple $(\theta, x) \in \Theta \times X$ such that $x \in \alpha(\theta)$.

Since, by definition, mechanism $G^{*(e,d)}$ fully EC-implements $\alpha^{*(e,d)}$ in environment $E^{*(e,d)}$, there exists an EC-Equilibrium $s \in S_1 \times \ldots \times S_n$ of the game induced by $G^{*(e,d)}$ and θ in $E^{*(e,d)}$ that satisfies $g^{*(e,d)}(s,\theta) = R^{(e,d)}(g(s),\theta) = (x,1)$.

Since $R^{(e,d)}(g(s),\theta) \in \{(g(s),1),(g(s),2)\}$, the preceding implies that $g(s) = x$ and $R^{(e,d)}(x,\theta) = (x,1)$.

□

Corollary 4.1 If α is fully EC-implementable in environment E, then α is consistent with environment E.

Proposition 4.2 If a mechanism $G \in \mathcal{G}_{strat}$ fully NE^n_{nfg}-implements α in environment E under an assignment $(e,d) \in \mathcal{A}$, then, $\forall\ (\theta,\theta',x) \in \Theta \times \Theta \times X$ satisfying $x \in \alpha(\theta)$ and $R^{(e,d)}(x,\theta') \notin \alpha(\theta') \times \{1\}$, there exists a tuple $(i,y) \in N \times X$ s.t.

$$u_i(R^{(e,d)}(y,\theta'),\theta') > u_i(R^{(e,d)}(x,\theta'),\theta') \text{ and } u_i(R^{(e,d)}(y,\theta),\theta) \leq u_i(R^{(e,d)}(x,\theta),\theta).$$

Proof

By defintion of LE and DE Implementation, mechanism G fully NE^n_{nfg}-implements α in the abstract n-person environment with an unrestricted realization function $E^{(e,d)} \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R^{(e,d)})$ (identifying space $X \times \{2\}$ with $X \times \{0\}$).

By Lemma 3.2, $\forall\ (\theta, x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, there exists an outcome $x' \in X$ which satisfies $R^{(e,d)}(x', \theta) = (x, 1)$ and the following condition:

$\forall\ \theta' \in \Theta$ s.t. $R^{(e,d)}(x', \theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i, y) \in N \times X$ s.t. $u_i(R^{(e,d)}(y, \theta'), \theta') > u_i(R^{(e,d)}(x', \theta'), \theta')$ and $u_i(R^{(e,d)}(y, \theta), \theta) \leq u_i(R^{(e,d)}(x', \theta), \theta)$.

Since $R^{(e,d)}(y, \theta) \in \{(y, 1), (y, 2)\}\ \forall\ (y, \theta) \in X \times \Theta$, the only outcome $x' \in X$ which can satisfy $R^{(e,d)}(x', \theta) = (x, 1)$ is outcome x itself.

Thus, $\forall\ (\theta, x) \in \Theta \times X$ s.t. $x \in \alpha(\theta)$, we have that $R^{(e,d)}(x, \theta) = (x, 1)$ and that $\forall\ \theta' \in \Theta$ s.t. $R^{(e,d)}(x, \theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i, y) \in N \times X$ s.t. $u_i(R^{(e,d)}(y, \theta'), \theta') > u_i(R^{(e,d)}(x, \theta'), \theta')$ and $u_i(R^{(e,d)}(y, \theta), \theta) \leq u_i(R^{(e,d)}(x, \theta), \theta)$.

□

Corollary 4.2 (Necessary condition for full implementation in Nash Equilibrium)
If α is fully NE^n_{nfg}-implementable in environment E, then there exists an EDS assignment $(e, d) \in \mathcal{A}$ such that the condition of Proposition 4.2 is satisfied.

Proposition 4.3 (Sufficient condition for full implementation in Nash Equilibrium)
If $\mathcal{G}$ is the set of all strategic mechanisms for (N, X), $\sharp N \geq 3$, and there exists an EDS assignment $(e, d) \in \mathcal{A}$ such that α satisfies

(i) the necessary condition of Proposition 4.2, and

(ii) $R^{(e,d)}(x, \theta) \in \alpha(\theta) \times \{1\}\ \forall\ (x, \theta) \in X \times \Theta$ that satisfy
$\sharp\{i \in N \mid u_i(R^{(e,d)}(x, \theta), \theta) \geq u_i(R^{(e,d)}(y, \theta), \theta)\ \forall\ y \in X\} \geq n - 1$,

then α is fully NE^n_{nfg}-implementable in environment E.

Proof

We only have to show that, $\forall\ (\theta, x) \in \Theta \times X$ such that $x \in \alpha(\theta)$, there exists an outcome $x' \in X$ which satisfies $R^{(e,d)}(x', \theta) = (x, 1)$ and the following condition:

$\forall\ \theta' \in \Theta$ s.t. $R^{(e,d)}(x', \theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i, y) \in N \times X$ s.t. $u_i(R^{(e,d)}(y, \theta'), \theta') > u_i(R^{(e,d)}(x', \theta'), \theta')$ and $u_i(R^{(e,d)}(y, \theta), \theta) \leq u_i(R^{(e,d)}(x', \theta), \theta)$.

Lemma 3.3 then implies that α is fully NE^n_{nfg}-implementable in the abstract n-person environment with an unrestricted realization function $E^{(e,d)} \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R^{(e,d)})$ (identifying space $X \times \{2\}$ with $X \times \{0\}$), and it remains to note that, by definition of LE and DE Implementation, the mechanism $G \in \mathcal{G}_{strat}$ which fully NE^n_{nfg}-implements α in $E^{(e,d)}$ also fully NE^n_{nfg}-implements α in E under (e, d).

Consider arbitrary $(\theta, x) \in \Theta \times X$ satisfying $x \in \alpha(\theta)$.
If we can proof that $R^{(e,d)}(x, \theta) = (x, 1)$, then assumption (i) directly implies that $x' := x$ is an outcome as required.
Assume, to the contrary, that $R^{(e,d)}(x, \theta) = (x, 2)$. Then, in particular, $R^{(e,d)}(x, \theta) \notin \alpha(\theta) \times \{1\}$, and, by assumption (i), there exists a tuple (i, y) such that

$$u_i(R^{(e,d)}(y, \theta), \theta) > u_i(R^{(e,d)}(x, \theta), \theta) \text{ and } u_i(R^{(e,d)}(y, \theta), \theta) \leq u_i(R^{(e,d)}(x, \theta), \theta),$$

a contradiction.

□

4.3 Nash-Implementability in Replica Environments

Let $\hat{E} \equiv (\hat{N}, X, \{\Theta_i\}_{i \in \hat{N}}, \hat{\Theta}, \{\hat{u}_i\}_{i \in N}, \hat{\mathcal{G}}, \hat{\mathcal{R}}, \hat{\mathcal{A}})$ be an $\hat{n}$-person environment with delegative enforcement power, and let $\hat{\alpha}$ be a SCC for environment $\hat{E}$. Furthermore, suppose that $\tilde{E}$ is a (possibly asymmetric) 'replica of environment $\hat{E}$' and that $\tilde{\alpha}$ is 'the corresponding extension of $\hat{\alpha}$':
Let $\tilde{E} \equiv (\tilde{N}, X, \{\Theta_i\}_{i \in \tilde{N}}, \tilde{\Theta}, \{\tilde{u}_i\}_{i \in N}, \tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{A}})$ be an $\tilde{n}$-person environment with delegative enforcement power, let $\tilde{\alpha}$ be a SCC for environment $\tilde{E}$, and let $\iota : \tilde{N} \to \hat{N}$ be a mapping, such that

$$\iota(i) = i \;\forall\; i \in \hat{N},$$
$$\tilde{n} > \hat{n},$$
$$\Theta_i = \Theta_{\iota(i)} \;\forall\; i \in \tilde{N},$$
$$\tilde{\Theta} = T(\hat{\Theta}),$$
$$\tilde{u}_i(\cdot, T(\hat{\theta})) \equiv \hat{u}_{\iota(i)}(\cdot, \hat{\theta}) \;\forall\; (i, \theta) \in \tilde{N} \times \hat{\Theta}, \text{ and}$$
$$\tilde{\alpha}(T(\hat{\theta})) = \hat{\alpha}(\hat{\theta}) \;\forall\; \hat{\theta} \in \hat{\Theta},$$

where $T : \hat{\Theta} \to \tilde{\Theta}$ is defined by $T(\hat{\theta}) := (\hat{\theta}_1, \ldots, \hat{\theta}_{\hat{n}}, \hat{\theta}_{\iota(\hat{n}+1)}, \ldots, \hat{\theta}_{\iota(\tilde{n})}) \;\forall\; \hat{\theta} \in \hat{\Theta}$.

According to Corollary 4.2, if $\tilde{\alpha}$ is fully NE^n_{nfg}-implementable in environment $\tilde{E}$, then there exists an EDS assignment $(e, d) \in \tilde{\mathcal{A}}$ such that the condition of Proposition 4.2 is satisfied. Now, suppose that all possible EDS assignments are available to the designer both in environment $\hat{E}$ and $\tilde{E}$:

$\hat{\mathcal{A}}$ is the set of all EDS assignments for $(\hat{N}, X)$, and
$\tilde{\mathcal{A}}$ is the set of all EDS assignments for $(\tilde{N}, X)$.

If all possible EDS assignments are available to the designer, the number of available assignments increases in the number of agents at an increasing rate. Analysing the Nash-implementability of a (social) choice correspondence by the use of Corollary 4.2, the number of available assignments, in turn, increases the maximum number of EDS assignments that have to be checked with respect to the condition of Proposition 4.2.

If $\hat{E}$ is a classical n-person environment and $\hat{\alpha}$ is a SCC for environment E, and if environment $\tilde{E}$ is a replica of E (in the sense described above) and $\tilde{\alpha}$ is the corresponding extension of α, then it is easy to see that either Maskin-monotonicity is satisfied by both $\hat{\alpha}$ in $\hat{E}$ and $\tilde{\alpha}$ in $\tilde{E}$ or it is not satisfied by both $\hat{\alpha}$ in $\hat{E}$ and $\tilde{\alpha}$ in $\tilde{E}$. In the following, we will show that this equivalence extends to environments with delegative enforcement power.

Lemma 4.4

(i) For each realization function $\hat{R} \in \hat{\mathcal{R}}$, there exists a realization function $\tilde{R} \in \tilde{\mathcal{R}}$ such that $\hat{R}(x, \hat{\theta}) = \tilde{R}(x, T(\hat{\theta}))\ \forall\ (x, \hat{\theta}) \in X \times \hat{\Theta}$.

(ii) For each realization function $\tilde{R} \in \tilde{\mathcal{R}}$, there exists a realization function $\hat{R} \in \hat{\mathcal{R}}$ such that $\hat{R}(x, \hat{\theta}) = \tilde{R}(x, T(\hat{\theta}))\ \forall\ (x, \hat{\theta}) \in X \times \hat{\Theta}$.

Proof

(i) Let $\hat{R} \in \hat{\mathcal{R}}$. Then, there exists an EDS assignment $(\hat{e}, d) \in \hat{\mathcal{A}}$ such that

$$\hat{R}(x, \hat{\theta}) = \hat{R}^{(\hat{e},d)}(x, \hat{\theta}) \equiv \begin{cases} (x,1) & \text{if } \exists\ S \in \hat{\mathcal{N}}^+ \text{ s.t. } x \in \hat{e}(S) \text{ and} \\ & \hat{u}_i((x,1), \hat{\theta}) \geq \hat{u}_i(d(x,0), \hat{\theta})\ \forall\ i \in S \cap \hat{N} \\ d(x,0) & \text{otw.} \end{cases}$$

on $X \times \hat{\Theta}$, where $\hat{\mathcal{N}}^+ := \{S \mid S \subseteq \hat{N}^+, S \neq \emptyset\}$.

For each $S \in \tilde{\mathcal{N}}^+ := \{S \mid S \subseteq \tilde{N}^+, S \neq \emptyset\}$, there exists a unique decomposition $S = \hat{S} \,\dot{\cup}\, \tilde{S}$ such that $\hat{S} \in \hat{\mathcal{N}}^+$ and $\tilde{S} \in \tilde{\mathcal{N}}^+ \backslash \hat{\mathcal{N}}^+$. Throughout the proof of (i), we let, for each $S \in \tilde{\mathcal{N}}^+$, $\hat{S}$ and $\tilde{S}$ denote the respective parts of this decomposition. Let $\tilde{e} : \tilde{\mathcal{N}}^+ \Rightarrow X$ be defined by $\tilde{e}(S) := \hat{e}(\hat{S})\ \forall\ S \in \tilde{\mathcal{N}}^+$. Then,

(a) $\tilde{e}(S) \supseteq \tilde{e}(S')\ \forall\ (S, S') \in \tilde{\mathcal{N}}^+ \times \tilde{\mathcal{N}}^+$ satisfying $S \supseteq S'$:
Let $(S, S') \in \tilde{\mathcal{N}}^+ \times \tilde{\mathcal{N}}^+$ satisfying $S \supseteq S'$. Then, $\tilde{e}(S) = \tilde{e}(\hat{S} \,\dot{\cup}\, \tilde{S}) = \hat{e}(\hat{S})$, and $\tilde{e}(S') = \tilde{e}(\hat{S}' \,\dot{\cup}\, \tilde{S}') = \hat{e}(\hat{S}')$.
Since $(\hat{e}, d)$ is an EDS assignment for $(\hat{N}, X)$ and ($S \supseteq S'$ implies that) $\hat{S} \supseteq \hat{S}'$, we have that $\hat{e}(\hat{S}) \supseteq \hat{e}(\hat{S}')$. It follows that $\tilde{e}(S) = \hat{e}(\hat{S}) \supseteq \hat{e}(\hat{S}') = \tilde{e}(S')$.

(b) $\tilde{e}(S \cup \{0\}) = \tilde{e}(S) \cup \tilde{e}(\{0\})\ \forall\ S \in \tilde{\mathcal{N}}$:
Let $S \in \tilde{\mathcal{N}}$. Then, $\tilde{e}(S) = \tilde{e}(\hat{S} \,\dot{\cup}\, \tilde{S}) = \hat{e}(\hat{S})$, $\tilde{e}(S \cup \{0\}) = \tilde{e}((\hat{S} \,\dot{\cup}\, \{0\}) \cup \tilde{S}) = \hat{e}(\hat{S} \cup \{0\})$, and $\tilde{e}(\{0\}) = \hat{e}(\{0\})$.
Since $(\hat{e}, d)$ is an EDS assignment for $(\hat{N}, X)$, we have that $\hat{e}(\hat{S} \cup \{0\}) = \hat{e}(\hat{S}) \cup \hat{e}(\{0\})$. It follows that $\tilde{e}(S \cup \{0\}) = \hat{e}(\hat{S} \cup \{0\}) = \hat{e}(\hat{S}) \cup \hat{e}(\{0\}) = \tilde{e}(S) \cup \tilde{e}(\{0\})$.

(c) $d(x, 0) = (x, 2)\ \forall\ x \in \tilde{e}(\{0\})$:
Since $(\hat{e}, d)$ is an EDS assignment for $(\hat{N}, X)$, we have that $d(x, 0) =$

$(x,2)\ \forall\ x \in \hat{e}(\{0\})$. It follows that $d(x,0) = (x,2)\ \forall\ x \in \tilde{e}(\{0\}) = \hat{e}(\{0\})$.

(d) $\tilde{e}(\tilde{N}) \cup \tilde{e}(\{0\}) = X$ and $\tilde{e}(\tilde{N}) \cap \tilde{e}(\{0\}) = \emptyset$:

By definition of $\tilde{e}$, we have that $\tilde{e}(\tilde{N}) = \tilde{e}(\hat{N} \cup (\tilde{N}\backslash\hat{N})) = \hat{e}(\hat{N})$, and $\tilde{e}(\{0\}) = \hat{e}(\{0\})$.

Since $(\hat{e}, d)$ is an EDS assignment for $(\hat{N}, X)$, we have that $\hat{e}(\hat{N}) \cup \hat{e}(\{0\}) = X$ and $\hat{e}(\hat{N}) \cap \hat{e}(\{0\}) = \emptyset$. It follows that $\tilde{e}(\tilde{N}) \cup \tilde{e}(\{0\}) = \hat{e}(\hat{N}) \cup \hat{e}(\{0\}) = X$ and $\tilde{e}(\tilde{N}) \cap \tilde{e}(\{0\}) = \hat{e}(\hat{N}) \cap \hat{e}(\{0\}) = \emptyset$.

Properties (a) to (d) imply that $(\tilde{e}, d)$ is an EDS assignment for $(\tilde{N}, X)$, and therefore, by assumption, an element of $\tilde{A}$.

Let $\tilde{R} : X \times T(\hat{\Theta}) \to (X \times \{1\}) \cup (X \times \{2\})$ be defined by

$$\tilde{R}(x, T(\hat{\theta})) := \tilde{R}^{(\tilde{e},d)}(x, T(\hat{\theta}))$$
$$\equiv \begin{cases} (x,1) & \text{if } \exists\ S \in \tilde{\mathcal{N}}^+ \text{ s.t. } x \in \tilde{e}(S) \text{ and} \\ & \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\ \forall\ i \in S \cap \tilde{N} \\ d(x,0) & \text{otw.} \end{cases}$$

on $X \times \hat{\Theta}$, and note that, in particular, $\tilde{R} \in \tilde{\mathcal{R}}$.

If the condition

$$\exists\ S \in \tilde{\mathcal{N}}^+ \text{ s.t. } x \in \tilde{e}(S) \text{ and } \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\ \forall\ i \in S \cap \tilde{N}$$

is equivalent to the condition

$$\exists\ S \in \hat{\mathcal{N}}^+ \text{ s.t. } x \in \hat{e}(S) \text{ and } \hat{u}_i((x,1), \hat{\theta}) \geq \hat{u}_i(d(x,0), \hat{\theta})\ \forall\ i \in S \cap \hat{N}$$

for each $(x, \hat{\theta}) \in X \times \hat{\Theta}$, then we have that $\tilde{R}(x, T(\hat{\theta})) = \hat{R}(x, \hat{\theta})$ on $X \times \hat{\Theta}$, which proves part (i). To see this equivalence, consider the two directions:

'⇒' Let $S \in \tilde{\mathcal{N}}^+$ satisfy the first condition. Then, $\hat{S}$ satisfies the second condition: $x \in \tilde{e}(S) = \tilde{e}(\hat{S} \dot{\cup} \tilde{S}) = \hat{e}(\hat{S})$, and, for each $i \in \hat{S} \cap \hat{N}$, we have that $\hat{u}_i((x,1), \hat{\theta}) = \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta})) = \hat{u}_i(d(x,0), \hat{\theta})$.

'⇐' Let $S \in \hat{\mathcal{N}}^+$ satisfy the second condition. Then, $S = \hat{S}$ also satisfies the first condition: $x \in \hat{e}(S) = \tilde{e}(\hat{S} \dot{\cup} \emptyset) = \tilde{e}(\hat{S}) = \tilde{e}(S)$, and for each $i \in S \cap \tilde{N} = S \cap \hat{N}$ we have that $\tilde{u}_i((x,1), T(\hat{\theta})) = \hat{u}_i((x,1), \hat{\theta}) \geq \hat{u}_i(d(x,0), \hat{\theta}) = \tilde{u}_i(d(x,0), T(\hat{\theta}))$.

(ii) Let $\tilde{R} \in \tilde{\mathcal{R}}$. Then, there exists an EDS assignment $(\tilde{e}, d) \in \tilde{\mathcal{A}}$ such that

$$\tilde{R}(x, T(\hat{\theta})) = \tilde{R}^{(\tilde{e},d)}(x, T(\hat{\theta}))$$
$$\equiv \begin{cases} (x,1) & \text{if } \exists\ S \in \tilde{\mathcal{N}}^+ \text{ s.t. } x \in \tilde{e}(S) \text{ and} \\ & \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\ \forall\ i \in S \cap \tilde{N} \\ d(x,0) & \text{otw.} \end{cases}$$

on $X \times \hat{\Theta}$.

Let $\hat{e} : \hat{\mathcal{N}}^+ \Rightarrow X$ be constructed according to the following instructions:

(1) $\hat{e}(\{0\}) := \tilde{e}(\{0\})$.

(2) $\forall\ (x, \hat{\theta}) \in X \backslash \hat{e}(\{0\}) \times \hat{\Theta}$ satisfying $\tilde{R}^{(\tilde{e},d)}(x, T(\hat{\theta})) = (x, 1)$, let outcome x be an element of $\hat{e}(\{i \in \hat{N} \mid \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\})$.

(3) $\forall\ x \in X \backslash \hat{e}(\{0\})$, let outcome x be an element of $\hat{e}(\hat{N})$.

(4) $\forall\ S \in \hat{\mathcal{N}}$, let all outcomes in $\bigcup_{S' \in \hat{\mathcal{N}}, S' \subseteq S, S' \neq S} \hat{e}(S')$ be an element of $\hat{e}(S)$.

(5) $\hat{e}(S \cup \{0\}) := \hat{e}(S) \,\dot{\cup}\, \hat{e}(\{0\})\ \forall\ S \in \hat{\mathcal{N}}$.

Then,

(a) $\hat{e}(S) \supseteq \hat{e}(S')\ \forall\ (S, S') \in \hat{\mathcal{N}}^+ \times \tilde{\mathcal{N}}^+$ satisfying $S \supseteq S'$,

(b) $\hat{e}(S \cup \{0\}) = \hat{e}(S) \cup \hat{e}(\{0\})\ \forall\ S \in \tilde{\mathcal{N}}$,

(c) $d(x, 0) = (x, 2)\ \forall\ x \in \hat{e}(\{0\})$,

(d) $\hat{e}(\hat{N}) \cup \hat{e}(\{0\}) = X$ and $\hat{e}(\hat{N}) \cap \hat{e}(\{0\}) = \emptyset$.

Conditions (a) to (d) are a direct consequence of $\hat{e}$'s construction,[38] and imply that $(\hat{e}, d)$ is an EDS assignment for $(\hat{N}, X)$. Therefore, by assumption, $(\hat{e}, d)$ is an element of $\hat{A}$.

Now, let $\hat{R} : X \times \hat{\Theta} \to (X \times \{1\}) \cup (X \times \{2\})$ be defined by

$$\hat{R}(x, \hat{\theta}) := \hat{R}^{(\hat{e},d)}(x, \hat{\theta})$$
$$\equiv \begin{cases} (x,1) & \text{if } \exists\ S \in \hat{\mathcal{N}}^+ \text{ s.t. } x \in \hat{e}(S) \text{ and} \\ & \hat{u}_i((x,1), \hat{\theta}) \geq \hat{u}_i(d(x,0), \hat{\theta})\ \forall\ i \in S \cap \hat{N} \\ d(x,0) & \text{otw.} \end{cases}$$

on $X \times \hat{\Theta}$, and note that, in particular, $\hat{R} \in \hat{\mathcal{R}}$.

If the condition

$$\exists\ S \in \tilde{\mathcal{N}}^+ \text{ s.t. } x \in \tilde{e}(S) \text{ and } \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\ \forall\ i \in S \cap \tilde{N}$$

is equivalent to the condition

$$\exists\ S \in \hat{\mathcal{N}}^+ \text{ s.t. } x \in \hat{e}(S) \text{ and } \hat{u}_i((x,1), \hat{\theta}) \geq \hat{u}_i(d(x,0), \hat{\theta})\ \forall\ i \in S \cap \hat{N}$$

for each $(x, \hat{\theta}) \in X \times \hat{\Theta}$, then we have that $\hat{R}(x, \hat{\theta}) = \tilde{R}(x, T(\hat{\theta}))$ on $X \times \hat{\Theta}$, which proves part (ii). To see this equivalence, consider the two directions:

'$\Rightarrow$' First, note that, if there exits a coalition $S \in \tilde{\mathcal{N}}^+$ that satisfies the first condition, then either there exists a coalition $S \in \tilde{\mathcal{N}}$ that satisfies the first condition or the coalition $S = \{0\}$ satisfies the first condition.[39]

If a coalition $S \in \tilde{\mathcal{N}}$ satisfies the first condition, then

$$\hat{S} := \{i \in \hat{N} \mid \tilde{u}_i((x,1), T(\hat{\theta})) \geq \tilde{u}_i(d(x,0), T(\hat{\theta}))\}$$

is nonempty and thus an element of $\hat{\mathcal{N}}^+$. And, since

[38] In particular, since $d(x, 0) = (x, 2)\ \forall\ x \in \tilde{e}(\{0\})$, condition (c) follows from step (1) of $\hat{e}$'s construction.

[39] Remember that $\tilde{e}(\tilde{N}) \cup \tilde{e}(\{0\}) = X$ and $\tilde{e}(\tilde{N}) \cap \tilde{e}(\{0\}) = \emptyset$.

- $x \in \hat{e}(\hat{S})$ by construction of $\hat{e}$, and
- $\hat{u}_i((x,1),\hat{\theta}) = \tilde{u}_i((x,1),T(\hat{\theta})) \geq \tilde{u}_i(d(x,0),T(\hat{\theta})) = \hat{u}_i(d(x,0),\hat{\theta})$ $\forall\ i \in \hat{S} \cap \hat{N} = \hat{S}$,

$\hat{S}$ satisfies the second condition.

If $S = \{0\}$ satisfies the first condition, then $\hat{S} := \{0\} \in \hat{\mathcal{N}}^+$ satisfies the second condition.

'⇐' First, note that, if there exits a coalition $S \in \hat{\mathcal{N}}^+$ that satisfies the second condition, then either there exists a coalition $S \in \hat{\mathcal{N}}$ that satisfies the second condition or the coalition $S = \{0\}$ satisfies the second condition.

If $S = \{0\}$ satisfies the second condition, then $\tilde{S} := \{0\} \in \tilde{\mathcal{N}}^+$ satisfies the first condition.

If $S = \hat{N}$ satisfies the second condition, then $\tilde{S} := \tilde{N} \in \tilde{\mathcal{N}}^+$ satisfies the first condition, since

- ($x \in \hat{e}(\hat{N})$ implies that $x \notin \hat{e}(\{0\}) = \tilde{e}(\{0\})$ and thus) $x \in \tilde{e}(\tilde{N})$, and
- $\hat{u}_i((x,1),\hat{\theta}) \geq \hat{u}_i(d(x,0),\hat{\theta})\ \forall\ i \in \hat{N}$ implies that $\tilde{u}_i((x,1),T(\hat{\theta})) = \hat{u}_{\iota(i)}((x,1),\hat{\theta}) \geq \hat{u}_{\iota(i)}(d(x,0),\hat{\theta}) = \tilde{u}_i(d(x,0),T(\hat{\theta}))\ \forall\ i \in \tilde{N}$.

If a coalition $S \in \hat{\mathcal{N}}$, $S \neq \hat{N}$, satisfies the second condition, then, by construction of $\hat{e}$, there exists a subset $\hat{S} \subseteq S$, $\hat{S} \in \hat{\mathcal{N}}$, satisfying

$\exists\ (\hat{\theta}', \tilde{S}) \in \hat{\Theta} \times \tilde{\mathcal{N}}^+$, $\tilde{S} \neq \{0\}$, such that

$x \in \tilde{e}(\tilde{S})$,

$x \in \hat{e}(\hat{S})$,

$\tilde{u}_i((x,1),T(\hat{\theta}')) \geq \tilde{u}_i(d(x,0),T(\hat{\theta}'))\ \forall\ i \in \tilde{S} \cap \tilde{N}$, and

$\hat{S} = \{i \in \hat{N} \mid \tilde{u}_i((x,1),T(\hat{\theta}')) \geq \tilde{u}_i(d(x,0),T(\hat{\theta}'))\}$.

Coalition $\tilde{S} \cap \tilde{N}$ consists of members of $\hat{N}$ and replicas of members of $\hat{N}$ (only). Let $\hat{M}$ denote the set of all such members, i.e. $\hat{M} := \iota(\tilde{S} \cap \tilde{N})$. Since $\tilde{u}_i((x,1),T(\hat{\theta}')) \geq \tilde{u}_i(d(x,0),T(\hat{\theta}'))\ \forall\ i \in \tilde{S} \cap \tilde{N}$, all $i \in \hat{M}$ must satisfy $\hat{u}_i((x,1),\hat{\theta}') \geq \hat{u}_i(d(x,0),\hat{\theta}')$, i.e., must be an element of $\hat{S} = \{i \in \hat{N} \mid \hat{u}_i((x,1),\hat{\theta}') \geq \hat{u}_i(d(x,0),\hat{\theta}')\}$. In other words, $\hat{M} \subseteq \hat{S}$.
By assumption, all $i \in S$ satisfy $\hat{u}_i((x,1),\hat{\theta}) \geq \hat{u}_i(d(x,0),\hat{\theta})$. Since $\hat{M} \subseteq \hat{S} \subseteq S$, all $i \in \hat{M}$ must satisfy this inequality as well. This implies that all $i \in \tilde{S} \cap \tilde{N}$ must satisfy $\tilde{u}_i((x,1),T(\hat{\theta})) = \hat{u}_{\iota(i)}((x,1),\hat{\theta}) \geq \hat{u}_{\iota(i)}(d(x,0),\hat{\theta}) = \tilde{u}_i(d(x,0),T(\hat{\theta}))$. Therefore, coalition $\tilde{S} \in \tilde{\mathcal{N}}^+$ satisfies the first condition.

□

Proposition 4.4 The respective necessary condition for full implementation in Nash Equilibrium as outlined in Corollary 4.2 is either satisfied in both environment $\hat{E}$ and $\tilde{E}$ or it is not satisfied in both $\hat{E}$ and $\tilde{E}$:
Condition

(i) There exists an EDS assignment $(\hat{e}, \hat{d}) \in \hat{\mathcal{A}}$ such that,
$\forall\ (\hat{\theta}, \hat{\theta}', x) \in \hat{\Theta} \times \hat{\Theta} \times X$ satisfying $x \in \hat{\alpha}(\hat{\theta})$ and $\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}') \notin \hat{\alpha}(\hat{\theta}') \times \{1\}$,
there exists a tuple $(i, y) \in \hat{N} \times X$ s.t.
$\hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}'), \hat{\theta}') > \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}'), \hat{\theta}')$ and $\hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}), \hat{\theta}) \leq \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}), \hat{\theta})$.

is equivalent to condition

(ii) There exists an EDS assignment $(\tilde{e}, \tilde{d}) \in \tilde{\mathcal{A}}$ such that,
$\forall\ (\tilde{\theta}, \tilde{\theta}', x) \in \tilde{\Theta} \times \tilde{\Theta} \times X$ satisfying $x \in \tilde{\alpha}(\tilde{\theta})$ and $\tilde{R}^{(\tilde{e},\tilde{d})}(x, \tilde{\theta}') \notin \tilde{\alpha}(\tilde{\theta}') \times \{1\}$,
there exists a tuple $(i, y) \in \tilde{N} \times X$ s.t.
$\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, \tilde{\theta}'), \tilde{\theta}') > \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, \tilde{\theta}'), \tilde{\theta}')$ and $\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, \tilde{\theta}), \tilde{\theta}) \leq \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, \tilde{\theta}), \tilde{\theta})$.

Proof

'$\Rightarrow$' According to Lemma 4.4(i), there exists an EDS assignment $(\tilde{e}, \tilde{d}) \in \tilde{\mathcal{A}}$ such that $\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}) = \tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}))\ \forall\ (x, \hat{\theta}) \in X \times \hat{\Theta}$.
Let $(\hat{\theta}, \hat{\theta}', x) \in \hat{\Theta} \times \hat{\Theta} \times X$ satisfy

$x \in \tilde{\alpha}(T(\hat{\theta}))$ and $\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')) \notin \tilde{\alpha}(T(\hat{\theta}')) \times \{1\}$.

Then, $x \in \tilde{\alpha}(T(\hat{\theta})) = \hat{\alpha}(\hat{\theta})$ and $\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}') = \tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')) \notin \tilde{\alpha}(T(\hat{\theta}')) \times \{1\} = \hat{\alpha}(\hat{\theta}') \times \{1\}$, and (i) implies that there exists a tuple $(i, y) \in \hat{N} \times X$ s.t.
$\hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}'), \hat{\theta}') > \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}'), \hat{\theta}')$ and $\hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}), \hat{\theta}) \leq \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}), \hat{\theta})$.
Thus, $(i, y) \in \hat{N} \times X \subseteq \tilde{N} \times X$ satisfies

$$\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta}')), T(\hat{\theta}')) = \hat{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta}')), \hat{\theta}') = \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}'), \hat{\theta}')$$
$$> \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}'), \hat{\theta}') = \hat{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')), \hat{\theta}') = \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')), T(\hat{\theta}'))$$

and

$$\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta})), T(\hat{\theta})) = \hat{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta})), \hat{\theta}) = \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(y, \hat{\theta}), \hat{\theta})$$
$$\leq \hat{u}_i(\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}), \hat{\theta}) = \hat{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta})), \hat{\theta}) = \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta})), T(\hat{\theta})).$$

'$\Leftarrow$' According to Lemma 4.4(ii), there exists an EDS assignment $(\hat{e}, \hat{d}) \in \hat{\mathcal{A}}$ such that $\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}) = \tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}))\ \forall\ (x, \hat{\theta}) \in X \times \hat{\Theta}$.
Let $(\hat{\theta}, \hat{\theta}', x) \in \hat{\Theta} \times \hat{\Theta} \times X$ satisfy

$x \in \hat{\alpha}(\hat{\theta})$ and $\hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}') \notin \hat{\alpha}(\hat{\theta}') \times \{1\}$.

Then, $x \in \hat{\alpha}(\hat{\theta}) = \tilde{\alpha}(T(\hat{\theta}))$ and $\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')) = \hat{R}^{(\hat{e},\hat{d})}(x, \hat{\theta}') \notin \hat{\alpha}(\hat{\theta}') \times \{1\} = \tilde{\alpha}(T(\hat{\theta}')) \times \{1\}$, and (ii) implies that there exists a tuple $(i, y) \in \tilde{N} \times X$ s.t.

$\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta}')), T(\hat{\theta}')) > \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta}')), T(\hat{\theta}'))$ and
$\tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y, T(\hat{\theta})), T(\hat{\theta})) \leq \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x, T(\hat{\theta})), T(\hat{\theta}))$.

Thus, $(i, y) \in \tilde{N} \times X$ satisfies $(\iota(i) \in \hat{N}$ and$)$

$$\hat{u}_{\iota(i)}(\hat{R}^{(\hat{e},\hat{d})}(y,\hat{\theta}'),\hat{\theta}') = \hat{u}_{\iota(i)}(\tilde{R}^{(\tilde{e},\tilde{d})}(y,T(\hat{\theta}')),\hat{\theta}') = \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y,T(\hat{\theta}')),T(\hat{\theta}'))$$
$$> \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x,T(\hat{\theta}')),T(\hat{\theta}')) = \hat{u}_{\iota(i)}(\tilde{R}^{(\tilde{e},\tilde{d})}(x,T(\hat{\theta}')),\hat{\theta}') = \hat{u}_{\iota(i)}(\hat{R}^{(\hat{e},\hat{d})}(x,\hat{\theta}'),\hat{\theta}')$$

and

$$\hat{u}_{\iota(i)}(\hat{R}^{(\hat{e},\hat{d})}(y,\hat{\theta}),\hat{\theta}) = \hat{u}_{\iota(i)}(\tilde{R}^{(\tilde{e},\tilde{d})}(y,T(\hat{\theta})),\hat{\theta}) = \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(y,T(\hat{\theta})),T(\hat{\theta}))$$
$$\leq \tilde{u}_i(\tilde{R}^{(\tilde{e},\tilde{d})}(x,T(\hat{\theta})),T(\hat{\theta})) = \hat{u}_{\iota(i)}(\tilde{R}^{(\tilde{e},\tilde{d})}(x,T(\hat{\theta})),\hat{\theta}) = \hat{u}_{\iota(i)}(\hat{R}^{(\hat{e},\hat{d})}(x,\hat{\theta}),\hat{\theta}).$$

□

Corollary 4.4 SCC $\tilde{\alpha}$ satisfies the necessary condition for full Nash-implementation in environment $\tilde{E}$ as outlined by Corollary 4.2 if and only if $\hat{\alpha}$ satisfies the respective condition for full Nash-implementation in environment $\hat{E}$.

In particular, if E is an environment with delegative enforcement power in which all possible EDS assignments are available to the designer, and α is a SCC for E which does not satisfy the necessary condition of Proposition 4.2 (and therefore is not fully NE^n_{nfg}-implementable in E), then the corresponding extension of α is not fully NE^n_{nfg}-implementable in any replica of E.

4.4 Implementability in Corresponding Environments

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$ be an environment with delegative enforcement power, let $E^C \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ denote its corresponding classical environment, and let α be a SCC for environment E.
Let $EC \in \{NE^n_{nfg}, DSE^n_{nfg}, SPNE^n\}$.
Proposition 4.5 is a consequence of Assumption 4.0.

Proposition 4.5 If α is strongly/fully EC-implementable in environment E's corresponding classical environment E^C, then α is strongly/fully EC-implementable in E. If $\mathcal{A}$ contains only the single EDS assignment (e, d) which satisfies $e(\{0\}) = X$, then α is strongly/fully EC-implementable in E^C if and only if α is strongly/fully EC-implementable in E.

Proof

Let $(e, d) \in \mathcal{A}$ denote the EDS assignment which satisfies $e(\{0\}) = X$.
Let $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ be a mechanism that fully EC-implements α in E^C ($EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$). The other cases are similar.
Since mechanism G fully EC-implements α in E^C, we have that

$$g(EC(\Gamma^{E^C,G,\theta})) = \alpha(\theta)\ \forall\ \theta \in \Theta,$$

where $\Gamma^{E^C,G,\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i((g(\cdot), 1), \theta)\}_{i\in N})$.
Since $R^{(e,d)}(x, \theta) = (x, 1)\ \forall\ (x, \theta) \in X \times \Theta$, this implies that

$R^{(e,d)}(g(EC(\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta})),\theta) = \{(x,1) \mid x \in \alpha(\theta)\} \; \forall \; \theta \in \Theta,$

where $\Gamma^{E^{*(e,d)},G^{*(e,d)},\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i(R^{(e,d)}(g(\cdot),\theta),\theta)\}_{i\in N})$.

Thus, α is fully EC-implementable in E.

If $\mathcal{A} = \{(e,d)\}$, and α is fully EC-implementable in E, then there exists a mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}_{strat}$ that fully EC-implements α in E under (e,d). Since $R^{(e,d)}(x,\theta) = (x,1) \; \forall \; (x,\theta) \in X \times \Theta$, we obtain that α is fully EC-implementable in environment E's corresponding classical environment E^C.

□

Remark 4.5 If α is not fully NE^n_{nfg}-implementable in environment E's corresponding classical environment E^C, and $\mathcal{A}$ contains more EDS assignments than just the single EDS assignment (e,d) which satisfies $e(\{0\}) = X$, then, in general, α may be fully NE^n_{nfg}-implementable in environment E. However, in general, α may not be fully NE^n_{nfg}-implementable in environment E even though $\mathcal{A}$ is the set of all possible EDS assignments for (N,X).

Example 4.5 and 4.6 illustrate each of the two cases. In particular, the former one provides an example of a replica environment.

Example 4.5

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$ be an n-person environment with delegative enforcement power, where

$N = \{1,2,3\}$,

$X = \{x_0, x_1, x_2\}$,

$\Theta_i = \{\hat{\theta}_i, \tilde{\theta}_i\} \; \forall \; i \in N$,

$\Theta = \{\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3), \tilde{\theta} \equiv (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)\}$,

$u_2((x_0,1),\theta) = u_3((x_0,1),\theta) = 2 \; \forall \; \theta \in \Theta, \;\; u_1((x_0,1),\hat{\theta}) = 1, \;\; u_1((x_0,1),\tilde{\theta}) = 3,$

$u_2((x_1,1),\theta) = u_3((x_1,1),\theta) = 3 \; \forall \; \theta \in \Theta, \;\; u_1((x_1,1),\hat{\theta}) = 1, \;\; u_1((x_1,1),\tilde{\theta}) = 2,$

$u_2((x_2,1),\theta) = u_3((x_2,1),\theta) = 1 \; \forall \; \theta \in \Theta, \;\; u_1((x_2,1),\hat{\theta}) = 2, \;\; u_1((x_2,1),\tilde{\theta}) = 1,$

$u_i((x_j,2),\theta) = u_i((x_j,1),\theta) \; \forall \; (i,j,\theta) \in N \times \{0,1,2\} \times \Theta,$[40]

$\mathcal{G}$ is the set of strategic n-person mechanisms for (N,X), and

$\mathcal{A}$ is a set of EDS assignments containing the assignment (e^*,d^*) defined by

$$e^*(S) := \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{x_0\} & \text{otw.} \end{cases} \qquad \forall \; S \in \mathcal{N}^+ \text{ and}$$

[40] I.e., each agent is indifferent between an outcome being implemented ‘right after the mechanism has been played’ and the same outcome being implemented as the default outcome.

$d^*(x,0) := (x_2, 2)\ \forall\ x \in X$.[41]

Thus, realization function $R^{(e^*,d^*)}$ satisfies

$$R^{(e^*,d^*)}(x,\theta)$$

$$= \begin{cases} (x,1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e^*(S) \text{ and} \\ & u_i((x,1),\theta) \geq u_i((d^*(x,0),\theta)\ \forall\ i \in S \cap N \\ d^*(x,0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x_0,1) & \text{if } x = x_0 \text{ and } u_i((x,1),\theta) \geq u_i((d^*(x,0),\theta) \text{ for some } i \in N \\ (x,1) & \text{if } x \neq x_0 \text{ and } u_i((x,1),\theta) \geq u_i((d^*(x,0),\theta)\ \forall\ i \in N \\ d^*(x,0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x_0,1) & \text{if } x = x_0 \text{ and } u_i((x,1),\theta) \geq u_i((x_2,1),\theta) \text{ for some } i \in N \\ (x,1) & \text{if } x \neq x_0 \text{ and } u_i((x,1),\theta) \geq u_i((x_2,1),\theta)\ \forall\ i \in N \\ (x_2,2) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x_0,1) & \text{if } x = x_0 \\ (x_1,1) & \text{if } x = x_1 \text{ and } \theta = \hat{\theta} \\ (x_2,2) & \text{if } x = x_1 \text{ and } \theta = \tilde{\theta} \\ (x_2,1) & \text{if } x = x_2 \end{cases}$$

$$= \begin{cases} (x_2,2) & \text{if } x = x_1 \text{ and } \theta = \hat{\theta} \\ (x,1) & \text{otw.} \end{cases}$$

on $X \times \Theta$.

Let $E^C \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i'\}_{i\in N}, \mathcal{G})$ denote E's corresponding classical environment, i.e., $u_i'(x,\theta) = u_i((x,1),\theta)\ \forall\ (x,\theta) \in X\times\Theta$, and let α be the SCC for environment E defined by

$$\alpha(\theta) := \begin{cases} \{x_0\} & \text{if } \theta = \hat{\theta} \\ \{x_1\} & \text{if } \theta = \tilde{\theta} \end{cases} \quad \forall\ \theta \in \Theta.$$

In particular, since $R^{(e^*,d^*)}(x_0,\hat{\theta}) = (x_0,1)$ and $R^{(e^*,d^*)}(x_1,\tilde{\theta}) = (x_1,1)$, α is consistent with environment E.

SCC α is fully NE^n_{nfg}-implementable in environment E, since both conditions of Proposition 4.3 are satisfied with respect to EDS assignment (e^*,d^*). See the corresponding part in Example 3.5(c), replacing R by $R^{(e^*,d^*)}$ and every tuple $(x_1,0)$ by $(x_2,2)$.

However, SCC α is not fully NE^n_{nfg}-implementable in environment E^C, since α is not Maskin-monotonic in E^C. Again, see the corresponding part in Example 3.5(c).

[41] Note that e^* is an enforcement structure for (N,X) (as defined in Chapter 3).

4.4.1 Two-Agents–Two-Outcomes Environments

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, \mathcal{R}, \mathcal{A})$ be a two-person environment with delegative enforcement power such that $X \equiv \{x_1, x_2\}$ contains only two outcomes and $\mathcal{A}$ is the set of all EDS assignments for (N, X). Let α be a SCC for environment E. Assume, without loss of generality,[42] that $u_i((x,1),\theta) \in \{1,2\}\ \forall\ (i,x,\theta) \in N \times X \times \Theta$, and that, $\forall\ (i,\theta) \in N \times \Theta$, there exists an outcome $x \in X$ such that $u_i((x,1),\theta) = 1$.[43] Furthermore, assume that $u_i((x,2),\theta) = u_i((x,1),\theta) - l\ \forall\ (i,x,\theta) \in N \times X \times \Theta$, where $l \in \mathbb{R}_+ := \{r \in \mathbb{R} \mid r \geq 0\}$. If $l = 0$, then all agents are indifferent between an outcome being implemented 'right after the mechanism has been played' or the same outcome being implemented as the default outcome by the designer. If $l > 0$, then an outcome being implemented 'right after the mechanism has been played' is strictly preferred to the same outcome being implemented as the default outcome by the designer.
We proof the following lemma in Appendix E.

Lemma 4.6 Suppose that α is not Maskin-monotonic in environment E's corresponding classical environment E^C, i.e. that $\exists\ (x,\theta,\theta') \in X \times \Theta \times \Theta$ such that

(i) $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$,

(ii) $u_1((y,1),\theta') \leq u_1((x,1),\theta')$ or $u_1((y,1),\theta) > u_1((x,1),\theta)$, and

(iii) $u_2((y,1),\theta') \leq u_2((x,1),\theta')$ or $u_2((y,1),\theta) > u_2((x,1),\theta)$,

where $y \in X$ denotes the uniquely determined outcome $y \neq x$. Then, in particular, outcome y is an element of $\alpha(\theta')$ (since $\alpha(\theta') \subseteq \{x,y\}$, $\alpha(\theta') \neq \emptyset$, and $x \notin \alpha(\theta')$).
If $l \geq 1$, then each EDS assignment $(e,d) \in \mathcal{A}$ such that $R^{(e,d)}(x,\theta) = (x,1)$ and $R^{(e,d)}(y,\theta') = (y,1)$ satisfies condition (iv) below, i.e., every suggested outcome is implemented 'right after the mechanism has been played'.
If either $l \in (0,1)$ and $y \in \alpha(\theta)$, or if $l = 0$, then, for each EDS assignment $(e,d) \in \mathcal{A}$ satisfying $R^{(e,d)}(x,\theta) = (x,1)$ and $R^{(e,d)}(y,\theta') = (y,1)$, at least one of the following four conditions is satisfied:

(iv) $R^{(e,d)}(x,\theta') = (x,1)$ and $R^{(e,d)}(y,\theta) = (y,1)$.

(v) $y \in \alpha(\theta)$ and $R^{(e,d)}(y,\theta) = d(y,0)$.

(vi) $u_1(R^{(e,d)}(y,\theta'),\theta') \leq u_1(R^{(e,d)}(x,\theta'),\theta')$ or $u_1(R^{(e,d)}(y,\theta),\theta) > u_1(R^{(e,d)}(x,\theta),\theta)$,
$u_2(R^{(e,d)}(y,\theta'),\theta') \leq u_2(R^{(e,d)}(x,\theta'),\theta')$ or $u_2(R^{(e,d)}(y,\theta),\theta) > u_2(R^{(e,d)}(x,\theta),\theta)$.

(vii) $R^{(e,d)}(y,\theta) \notin \alpha(\theta) \times \{1\}$,
$u_1(R^{(e,d)}(x,\theta),\theta) \leq u_1(R^{(e,d)}(y,\theta),\theta)$ or $u_1(R^{(e,d)}(x,\theta'),\theta') > u_1(R^{(e,d)}(y,\theta'),\theta')$,
$u_2(R^{(e,d)}(x,\theta),\theta) \leq u_2(R^{(e,d)}(y,\theta),\theta)$ or $u_2(R^{(e,d)}(x,\theta'),\theta') > u_2(R^{(e,d)}(y,\theta'),\theta')$.

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[42] Note that this assumption does not restrict the set of possible preference profiles over $X \times \{1\}$.

[43] In other words, $(u_i((x_1,1),\theta), u_i((x_2,1),\theta)) \in \{(1,1),(2,1),(1,2)\}\ \forall\ (i,\theta) \in N \times \Theta$.

Proposition 4.6 Suppose that α is not Maskin-monotonic in environment E's corresponding classical environment E^C, i.e. that $\exists\ (x,\theta,\theta') \in X \times \Theta \times \Theta$ such that

(i) $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$,

(ii) $u_1((y,1),\theta') \leq u_1((x,1),\theta')$ or $u_1((y,1),\theta) > u_1((x,1),\theta)$, and

(iii) $u_2((y,1),\theta') \leq u_2((x,1),\theta')$ or $u_2((y,1),\theta) > u_2((x,1),\theta)$,

where $y \in X$ denotes the uniquely determined outcome $y \neq x$. If $y \in \alpha(\theta)$ or $l \notin (0,1)$, then α is not fully NE^n_{nfg}-implementable in environment E.

Proof

Assume, to the contrary, that α is fully NE^n_{nfg}-implementable in environment E, i.e. that there exists a mechanism $G \in \mathcal{G}_{strat}$ and an EDS assignment $(e,d) \in \mathcal{A}$ such that G fully NE^n_{nfg}-implements α in E under assignment (e,d). Then, Proposition 4.1 and 4.2 imply that the following four conditions are satisfied:

(viii) $R^{(e,d)}(x,\theta) = (x,1)$ and $R^{(e,d)}(y,\theta') = (y,1)$ (since $x \in \alpha(\theta)$ and $y \in \alpha(\theta')$).

(ix) If $y \in \alpha(\theta)$, then $R^{(e,d)}(y,\theta) = (y,1)$.

(x) There exists an $i \in N \equiv \{1,2\}$ satisfying
$u_i(R^{(e,d)}(y,\theta'),\theta') > u_i(R^{(e,d)}(x,\theta'),\theta')$ and $u_i(R^{(e,d)}(y,\theta),\theta) \leq u_i(R^{(e,d)}(x,\theta),\theta)$
(since, by condition (i), $x \in \alpha(\theta)$ and $R^{(e,d)}(x,\theta') \notin \alpha(\theta') \times \{1\}$).

(xi) If $R^{(e,d)}(y,\theta) \notin \alpha(\theta) \times \{1\}$, then $\exists\ i \in N \equiv \{1,2\}$ satisfying
$u_i(R^{(e,d)}(x,\theta),\theta) > u_i(R^{(e,d)}(y,\theta),\theta)$ and $u_i(R^{(e,d)}(x,\theta'),\theta') \leq u_i(R^{(e,d)}(y,\theta'),\theta')$
(since $y \in \alpha(\theta')$).

Since condition (viii) is satisfied, the preceding lemma implies that at least one of the following four conditions is satisfied:[44]

(iv) $R^{(e,d)}(x,\theta') = (x,1)$ and $R^{(e,d)}(y,\theta) = (y,1)$.

(v) $y \in \alpha(\theta)$ and $R^{(e,d)}(y,\theta) = d(y,0)$.

(vi) $u_1(R^{(e,d)}(y,\theta'),\theta') \leq u_1(R^{(e,d)}(x,\theta'),\theta')$ or $u_1(R^{(e,d)}(y,\theta),\theta) > u_1(R^{(e,d)}(x,\theta),\theta)$,
$u_2(R^{(e,d)}(y,\theta'),\theta') \leq u_2(R^{(e,d)}(x,\theta'),\theta')$ or $u_2(R^{(e,d)}(y,\theta),\theta) > u_2(R^{(e,d)}(x,\theta),\theta)$.

(vii) $R^{(e,d)}(y,\theta) \notin \alpha(\theta) \times \{1\}$,
$u_1(R^{(e,d)}(x,\theta),\theta) \leq u_1(R^{(e,d)}(y,\theta),\theta)$ or $u_1(R^{(e,d)}(x,\theta'),\theta') > u_1(R^{(e,d)}(y,\theta'),\theta')$,
$u_2(R^{(e,d)}(x,\theta),\theta) \leq u_2(R^{(e,d)}(y,\theta),\theta)$ or $u_2(R^{(e,d)}(x,\theta'),\theta') > u_2(R^{(e,d)}(y,\theta'),\theta')$.

If condition (iv) is satisfied, then conditions (ii),(iii), and (viii) imply that condition (vi) is satisfied. Condition (v) contradicts condition (ix). Condition (vi) contradicts condition (x). And, condition (vii) contradicts condition (xi).

□

[44] If $y \in \alpha(\theta)$, then consider the three cases '$l = 0$', '$l \geq 1$', and '$l \in (0,1)$'. If $l \notin (0,1)$, then consider the two cases '$l = 0$' and '$l \geq 1$'.

Remark 4.6 Suppose that α is not Maskin-monotonic in environment E's corresponding classical environment E^C, i.e. that $\exists\ (x, \theta, \theta') \in X \times \Theta \times \Theta$ such that

(i) $x \in \alpha(\theta)$ and $x \notin \alpha(\theta')$,

(ii) $u_1((y,1),\theta') \leq u_1((x,1),\theta')$ or $u_1((y,1),\theta) > u_1((x,1),\theta)$, and

(iii) $u_2((y,1),\theta') \leq u_2((x,1),\theta')$ or $u_2((y,1),\theta) > u_2((x,1),\theta)$,

where $y \in X$ denotes the uniquely determined outcome $y \neq x$. If $y \notin \alpha(\theta)$ and $l \in (0,1)$, then there may exist an EDS assignment $(e,d) \in \mathcal{A}$ such that the necessary conditions (for full implementation in Nash Equilibrium) as outlined in Proposition 4.1 and 4.2 are satisfied. See Example 4.6.

Example 4.6

For each $l \in \mathbb{R}_+$, let $E^{(l)} \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i^{(l)}\}_{i\in N}, \mathcal{G}, \mathcal{R}_{(l)}, \mathcal{A})$ be the n-person environment with delegative enforcement power, where

$N = \{1,2\}$,

$X = \{x_1, x_2\}$,

$\Theta_i = \{\hat{\theta}_i, \tilde{\theta}_i\}\ \forall\ i \in N$,

$\Theta = \{\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2), \tilde{\theta} \equiv (\tilde{\theta}_1, \tilde{\theta}_2)\}$,

$u_1^{(l)}((x_1,1),\hat{\theta}) = 1,\ u_1^{(l)}((x_2,1),\hat{\theta}) = 2,\ u_2^{(l)}((x_1,1),\hat{\theta}) = 2,\ u_2^{(l)}((x_2,1),\hat{\theta}) = 1$,

$u_1^{(l)}((x_1,1),\tilde{\theta}) = 1,\ u_1^{(l)}((x_2,1),\tilde{\theta}) = 1,\ u_2^{(l)}((x_1,1),\tilde{\theta}) = 2,\ u_2^{(l)}((x_2,1),\tilde{\theta}) = 1$,

$u_i^{(l)}((x_j,2),\theta) = u_i^{(l)}((x_j,1),\theta) - l\ \forall\ (i,j,\theta) \in N \times \{1,2\} \times \Theta$,

$\mathcal{G}$ is the set of strategic n-person mechanisms for (N, X), and

$\mathcal{A}$ is the set of all EDS assignments for (N, X).

Let $E^C = (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ denote each environment $E^{(l)}$'s corresponding classical environment, i.e., $u'_i(x,\theta) = u_i^{(l)}((x,1),\theta)\ \forall\ (x,\theta) \in X \times \Theta$ (for some $l \in \mathbb{R}_+$). Let α be the SCC for each environment $E^{(l)}$ defined by

$$\alpha(\theta) := \begin{cases} \{x_2\} & \text{if } \theta = \hat{\theta} \\ \{x_1\} & \text{if } \theta = \tilde{\theta} \end{cases} \qquad \forall\ \theta \in \Theta.$$

SCC α is not fully NE^n_{nfg}-implementable in environment E^C, since α is not Maskin-monotonic in E^C: $x_2 \in \alpha(\hat{\theta})$, $x_2 \notin \alpha(\tilde{\theta})$,

$u'_1(x_1,\tilde{\theta}) = 1 \ngtr 1 = u'_1(x_2,\tilde{\theta})$, and

$u'_2(x_1,\hat{\theta}) = 2 \nleq 1 = u'_2(x_2,\hat{\theta})$.

And, by Proposition 4.6, this implies that, for each $l \notin (0,1)$, SCC α is not fully NE^n_{nfg}-implementable in environment $E^{(l)}$. In particular, $E^{(l)}$, $l \notin (0,1)$, provides an example of an environment in which a (social) choice correspondence is not fully Nash-implementable even though all possible EDS assignments are available to the designer.

However, for each $l \in (0,1)$ and for each EDS assignment $(e,d) \in \mathcal{A}$ which satisfies

$e(\{1,2\}) = \{x_1, x_2\}$, $e(\{0\}) = e(\{2\}) = \emptyset$, $d(x_1, 0) = d(x_2, 0) = (x_2, 2)$,[45]

the necessary conditions as outlined in Proposition 4.1 and 4.2 are satisfied:
First,

$$R^{(e,d)}_{(l)}(x,\theta) = \begin{cases} (x,1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \text{ and} \\ & u^{(l)}_i((x,1),\theta) \geq u^{(l)}_i(d(x,0),\theta)\ \forall\ i \in S \cap N \\ d(x,0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x,1) & \text{if } u^{(l)}_i((x,1),\theta) \geq u^{(l)}_i((x_2,1),\theta) - l\ \forall\ i \in N \\ (x,1) & \text{if } x \in e(\{1\}) \text{ and } u^{(l)}_1((x,1),\theta) \geq u^{(l)}_1((x_2,1),\theta) - l \\ (x_2,2) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x,1) & \text{if } x = x_1 \text{ and } \theta = \tilde{\theta} \\ (x,1) & \text{if } x = x_2 \\ (x,1) & \text{if } x = x_1 \text{ and} \\ & \quad x \in e(\{1\}) \text{ and } u^{(l)}_1((x,1),\theta) \geq u^{(l)}_1((x_2,1),\theta) - l \\ (x_2,2) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x_2,2) & \text{if } x = x_1 \text{ and } \theta = \hat{\theta} \\ (x,1) & \text{otw.} \end{cases} \qquad (\text{on } X \times \Theta)$$

$$= (x,1)\ \forall\ (x,\theta) \in X \times \Theta \text{ satisfying } x \in \alpha(\theta).$$

Second, $\forall\ (\theta, \theta', x) \in \Theta \times \Theta \times X$ satisfying $x \in \alpha(\theta)$ and $R^{(e,d)}_{(l)}(x,\theta') \notin \alpha(\theta') \times \{1\}$, there exists a tuple $(i,y) \in N \times X$ such that $u^{(l)}_i(R^{(e,d)}_{(l)}(y,\theta'),\theta') > u^{(l)}_i(R^{(e,d)}_{(l)}(x,\theta'),\theta')$ and $u^{(l)}_i(R^{(e,d)}_{(l)}(y,\theta),\theta) \leq u^{(l)}_i(R^{(e,d)}_{(l)}(x,\theta),\theta)$:

$$\begin{aligned} & x_1 \in \alpha(\tilde{\theta}),\ R^{(e,d)}_{(l)}(x_1,\hat{\theta}) = (x_2,2) \notin \alpha(\hat{\theta}) \times \{1\}, \\ & u^{(l)}_1(R^{(e,d)}_{(l)}(x_2,\hat{\theta}),\hat{\theta}) = u^{(l)}_1((x_2,1),\hat{\theta}) = 2 \\ & \qquad > 2 - l = u^{(l)}_1((x_2,2),\hat{\theta}) = u^{(l)}_1(R^{(e,d)}_{(l)}(x_1,\hat{\theta}),\hat{\theta}), \\ & u^{(l)}_1(R^{(e,d)}_{(l)}(x_2,\tilde{\theta}),\tilde{\theta}) = u^{(l)}_1((x_2,1),\tilde{\theta}) = 1 \\ & \qquad \leq 1 = u^{(l)}_1((x_1,1),\tilde{\theta}) = u^{(l)}_1(R^{(e,d)}_{(l)}(x_1,\tilde{\theta}),\tilde{\theta}), \end{aligned}$$

and

$$\begin{aligned} & x_2 \in \alpha(\hat{\theta}),\ R^{(e,d)}_{(l)}(x_2,\tilde{\theta}) = (x_2,1) \notin \alpha(\tilde{\theta}) \times \{1\}, \\ & u^{(l)}_2(R^{(e,d)}_{(l)}(x_1,\tilde{\theta}),\tilde{\theta}) = u^{(l)}_2((x_1,1),\tilde{\theta}) = 2 \\ & \qquad > 1 = u^{(l)}_2((x_2,1),\tilde{\theta}) = u^{(l)}_2(R^{(e,d)}_{(l)}(x_2,\tilde{\theta}),\tilde{\theta}), \\ & u^{(l)}_2(R^{(e,d)}_{(l)}(x_1,\hat{\theta}),\hat{\theta}) = u^{(l)}_2((x_2,2),\hat{\theta}) = 1 - l \\ & \qquad \leq 1 = u^{(l)}_2((x_2,1),\hat{\theta}) = u^{(l)}_2(R^{(e,d)}_{(l)}(x_2,\hat{\theta}),\hat{\theta}). \end{aligned}$$

[45]Note that this allows e to be an enforcement structure (as defined in Chapter 3).

5 Classical and LE Implementation of Cooperative Solution Concepts

As an application, we now discuss implications of limited enforcement power on two different approaches to the implementation of cooperative solution concepts, with particular emphasis on the Core concept and the Nash Bargaining Solution concept.

One approach, by Trockel [52], is based on a "purely welfaristic" outcome space, and leads to a rather positive result, an "Embedding Principle": "... I propose a general procedure of embedding the Nash program into the theory of implementation. That procedure enables us in our framework to transform any support result from the Nash program into an implementation result in mechanism theory." Trockel's approach has positive implications on the implementability of the Nash Bargaining Solution concept and, as we will show by presenting an appropriate support result, the Core concept.[46] However, defining a set of single-valued solution concepts as the outcome space, his approach placed in classical environments is bound to the assumption that the designer can enforce agents to realize a single-valued solution concept without knowing the actual cooperative game — an assumption which might not be an adequate description of many real-world situations. In the words of Trockel, "it may ... be questioned whether the outcome space and the mechanism employed for our ... embedding lemma are very reasonable from a practical point of view. Such considerations, however, lead us immediately back to the question to what extent the presently established modeling of implementation via game forms is an adequate one, a question that led Hurwicz (1994) to suggest 'genuine implementation'."

In Section 5.3, we approach the question to what extent, i.e., for what assumptions on the structure of beliefs, Trockel's positive result and its implications concerning the Nash Bargaining Solution concept and the Core concept extend to environments with limited enforcement power, in which the designer has no enforcement power on single-valued solution concepts.

Another approach to the implementation of cooperative solution concepts, by Dagan and Serrano [9], is based on "coalitional games" specifying "physical outcomes" that each coalition can achieve and that agents can evaluate according to some rational

[46] Note that the implementation of the Core as well as non-cooperative characterizations (or "foundations" as termed by Bergin and Duggan [6]) of the Core, with respect to cooperative games (with transferable as well as with non-transferable utility), specific exchange/production economies, and with respect to different matching problems, are addressed by several articles. See, for example (and also for further references), Bergin and Duggan [6] and Okada and Winter [36].

preference relation (over these "physical outcomes"). Dagan and Serrano come to a rather negative result in the form of a necessary condition, which, in particular (and in contrast to the Core concept), affects the implementability of the Nash Bargaining Solution concept: "... major solution concepts in coalitional games (e.g., the Nash bargaining solution, the NTU-Shapley value) can be derived strategically only by considering the possibility of random outcomes: either chance moves, mixed strategies, or pure strategy equilibrium refinements based on trembles must be part of the analysis." Our extension of Dagan and Serrano's result in Section 5.4 indicates that in environments with limited enforcement power, and in contrast to their result for classical environments, not every solution concept which is fully implementable by an ordinally invariant equilibrium concept must be ordinally invariant.
We begin our discussion with definitions and notation in Section 5.1. Section 5.2 continues with a more detailed description of Trockel's [52] approach to the implementation of cooperative solution concepts.

5.1 Definitions

Throughout this chapter, let $n \in \mathbb{N}$, $n \geq 2$.
For any partition $\mathcal{R}$ of $\mathbb{N}_n$ (i.e., for any collection of disjoint nonempty subsets of $\mathbb{N}_n$ whose union is $\mathbb{N}_n$) and each $i \in \mathbb{N}_n$, let $\mathcal{R}(i)$ denote the element of $\mathcal{R}$ that contains i.
The definitions in Paragraph 5.1.1 and 5.1.2 are standard.[47] The definitions in Paragraph 5.1.3 and 5.1.4 are based on Dagan and Serrano [9].[48]

5.1.1 NTU Games and the Core

Definition An *n-person game in characteristic form with nontransferable utility* (an *n-person NTU game*) is a tuple (N, V), where $N \equiv \{1, \ldots, n\}$ is the set of players, and V is a correspondence that assigns to each coalition $S \in \mathcal{N} := \{S \mid S \subseteq N, S \neq \emptyset\}$ a (possibly empty) *utility possibility set* $V(S) \subseteq \mathbb{R}^S$.

Note that we often identify an n-person NTU-game (N, V) with its correspondence V. For every pair of coalitions $(S, S') \in \mathcal{N} \times \mathcal{N}$, $S' \subseteq S$, and each element $u \in \mathbb{R}^S$, we let $u_{S'}$ denote the projection of u to the coordinates corresponding to coalition S'.

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47 See, for example, Mas-Colell, Whinston, and Green [29] (Appendix A to Chapter 18 and Section 22.E), Rosenmüller [42] (Chapter 4) and [43] (Chapter 8).

48 These kind of games are also considered by several other authors. See, for example, Osborne and Rubinstein [37], pp. 268-269, 274-275, 299-301, and 312. (Osborne and Rubinstein use the term "consequences" instead of "physical outcomes".)

Let $\mathcal{V}^n$ denote the set of all n-person NTU games V satisfying $F_{up}(V) \neq \emptyset$, where $F_{up}(V) := \{u \in \mathbb{R}^N \mid \exists$ a partition $\mathcal{R}$ of N s.t. $u_S \in V(S) \; \forall \; S \in \mathcal{R}\}$ denotes the set of *feasible utility profiles* in game (N, V).
Let $\mathcal{C}^n_{ntu} \subseteq \mathcal{V}^n$ be nonempty. The following definition is implicit in several sources, and explicitly stated, for example, in Trockel [52].[49]

Definition A *solution concept for* $\mathcal{C}^n_{ntu}$ is a correspondence L that assigns to each game $(N, V) \in \mathcal{C}^n_{ntu}$ a (possibly empty) subset $L(V)$ of its feasible utility profiles $F_{up}(V)$.

Defintion *The Core of* $(N, V) \in \mathcal{V}^n$ is the set of utility allocations in the utility possibility set of the grand coalition with the property that no coalition could on its own make all of its members better off, i.e. $Core(V) := \{u \in V(N) \mid \nexists$ (a blocking coalition) $S \in \mathcal{N}$ for which $\exists \; u' \in V(S)$ s.t. $u'_i > u_i \; \forall \; i \in S\}$.

5.1.2 Bargaining Games and the Nash Bargaining Solution

An *n-person bargaining game* is a tuple (N, U, u^*), where $N \equiv \{1, \ldots, n\}$ is the set of players, $U \subseteq \mathbb{R}^N$ is the *utility possibility set*, and $u^* \in U$ is the *status-quo* utility allocation. In the words of Mas-Colell et al. [29], "the set U represents the allocations of utility that can be settled on if there is cooperation among the different agents. The point u^* is the outcome that will occur if there is a breakdown of cooperation." Formally, a bargaining game can be considered as a specific NTU game.

Definition An *n-person bargaining game in NTU form (with status quo* $0 \in \mathbb{R}^N$*)* is an n-person NTU game (N, V) satisfying $V(S) = \{0\} \; \forall \; S \in \mathcal{N} \setminus \{N\}$ and $0 \in V(N)$.

Let $\mathcal{B}^n$ denote the set of all n-person bargaining games in NTU form (N, V) satisfying

$V(N) \subseteq \mathbb{R}^N_+$ is compact, convex, and comprehensive with respect to $\mathbb{R}^N_+$,[50]
and $\max_{u \in V(N)} u_i = 1 \; \forall \; i \in N$.

Note that as far as we restrict our analysis of bargaining games to the Nash Bargaining Solution concept, which is "independent of utility origins" and "independent of utility units" as termed by Mas-Colell et al. [29] (Rosenmüller [43] uses the terminology "covariant with affine transformation of utility"), the formal restriction to bargaining games with a status quo of 0 and a maximum value of 1 for each player places in fact only a 'normalization' on the set of games under consideration (satisfying $\exists \; u \in V(N)$ s.t. $u_i > u^*_i \; \forall \; i \in N$ instead).

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[49] Trockel requires a "solution" to be non-empty-valued.

[50] A set $U \subseteq \mathbb{R}^N_+$ is comprehensive with respect to $\mathbb{R}^N_+$ if $(u - \mathbb{R}^N_+) \cap \mathbb{R}^N_+ \subseteq U \; \forall \; u \in U$.

Definition *The Nash Bargaining Solution of* $(N, V) \in \mathcal{B}^n$ is the set consisting of the unique utility allocation in the utility possibility set of the grand coalition which maximizes the product of its coordinates on $V(N)$, i.e.

$Nash(V) := \arg\max_{u \in V(N)} u_1 \cdot \ldots \cdot u_n$.[51]

5.1.3 Cooperative and Bargaining Games with Physical Outcomes

Definition An *n-person cooperative game with physical outcomes* is a tuple $(N, \bar{X}, \{u_i\}_{i \in N})$, where $N \equiv \{1, \ldots, n\}$ is the set of players, $\bar{X}$ is a correspondence that assigns to each coalition $S \in \mathcal{N}$ a *physical outcome possibility set* $\bar{X}(S)$, satisfying

$\exists\ Q \in \mathcal{N}$ such that $\bar{X}(Q) \neq \emptyset$, and

$\forall\ S \in \mathcal{N} \backslash \{N\}$ satisfying $\bar{X}(S) \neq \emptyset$, there exists a partition $\mathcal{R}$ of $N \backslash S$ such that $\bar{X}(S) \neq \emptyset\ \forall\ S \in \mathcal{R}$,

and $u_i : \bigcup_{S \in \mathcal{N} :\, i \in S} \bar{X}(S) \to \mathbb{R}$ is player i's utility function.[52]

Let $\mathcal{C}^n_{po}$ be a nonempty set of n-person cooperative games with physical outcomes all sharing the same game form $(N, \bar{X})$.

Definition A *solution concept for* $\mathcal{C}^n_{po}$ is a correspondence ψ that assigns to each game $\Gamma \equiv (N, \bar{X}, \{u_i\}_{i \in N}) \in \mathcal{C}^n_{po}$ a (possibly empty) subset $\psi(\Gamma)$ of its *feasible outcomes* $F_o(\bar{X}) := \{(S, x^S)_{S \in \mathcal{R}} \mid \mathcal{R}$ is a partition of N and $x^S \in \bar{X}(S)\ \forall\ S \in \mathcal{R}\}$.[53]

For each game $(N, \bar{X}, \{u_i\}_{i \in N}) \in \mathcal{C}^n_{po}$, let each u_i also denote player i's utility function over $F_o(\bar{X})$ induced by $u_i : \bigcup_{S \in \mathcal{N} :\, i \in S} \bar{X}(S) \to \mathbb{R}$, i.e., let $u : F_o(\bar{X}) \to \mathbb{R}^N$ be defined by $u_i((S, x^S)_{S \in \mathcal{R}}) := u_i(x^{\mathcal{R}(i)})\ \forall\ (i, (S, x^S)_{S \in \mathcal{R}}) \in N \times F_o(\bar{X})$.

Let ψ and L be a solution concept for $\mathcal{C}^n_{po}$ and $\mathcal{C}^n_{ntu} \subseteq \mathcal{V}^n$, respectively.
The following definition (as already present in Hahmeier [15]) interprets Dagan and Serrano: "Solution concepts which are defined for characteristic function games can be adapted into our framework by assigning to each outcome of the characteristic function game a nonempty set of outcomes of the coalitional game."

Definition Solution concept ψ *is induced by* L if, for each $\Gamma \equiv (N, \bar{X}, \{u_i\}_{i \in N}) \in \mathcal{C}^n_{po}$, $\psi(\Gamma) = \{(S, x^S)_{S \in \mathcal{R}} \in F_o(\bar{X}) \mid u((S, x^S)_{S \in \mathcal{R}}) \in L(V^\Gamma)\}$, where game $V^\Gamma \in \mathcal{C}^n_{ntu} \subseteq \mathcal{V}^n$ is defined by $V^\Gamma(S) := \{y \in \mathbb{R}^S \mid \exists\ x \in \bar{X}(S)$ s.t. $y_i = u_i(x)\ \forall\ i \in S\}\ \forall\ S \in \mathcal{N}$.

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[51]For an axiomatic foundation of the Nash Bargaining Solution concept, see, for example, Rosenmüller [43] (Chapter 8). As for the uniqueness on $V(N)$, note that the mapping $u \mapsto \prod_{i=1}^n u_i$ is strictly quasi-concave on $\{u \in \mathbb{R}^n \mid u_1 > 0, \ldots, u_n > 0\}$.

[52]Dagan and Serrano (and Osborne and Rubinstein [37]) use the term "coalitional game".

[53]Dagan and Serrano use the term "pure solution".

Let $\Gamma^{(1)} \equiv (N, \bar{X}, \{u_i^{(1)}\}_{i \in N})$ and $\Gamma^{(2)} \equiv (N, \bar{X}, \{u_i^{(2)}\}_{i \in N})$ be in $\mathcal{C}_{po}^n$.

Definition Game $\Gamma^{(2)}$ *is an order preserving transformation of* $\Gamma^{(1)}$ if, for all $i \in N$ and each pair x, x' of physical outcomes in $\bigcup_{S \in \mathcal{N}:\, i \in S} \bar{X}(S)$, $u_i^{(2)}(x) > u_i^{(2)}(x')$ if and only if $u_i^{(1)}(x) > u_i^{(1)}(x')$ (or, equivalently, if, $\forall\, (i, q, q') \in N \times F_o\,(\bar{X}) \times F_o\,(\bar{X})$, $u_i^{(2)}(q) > u_i^{(2)}(q')$ if and only if $u_i^{(1)}(q) > u_i^{(1)}(q')$).

Definition Solution concept ψ *is ordinally invariant on* $\mathcal{C}_{po}^n$ if $\psi(\Gamma^{(1)}) = \psi(\Gamma^{(2)})$ for each pair $\Gamma^{(1)} \equiv (N, \bar{X}, \{u_i^{(1)}\}_{i \in N})$, $\Gamma^{(2)} \equiv (N, \bar{X}, \{u_i^{(2)}\}_{i \in N})$ of games in $\mathcal{C}_{po}^n$ such that $\Gamma^{(2)}$ is an order preserving transformation of $\Gamma^{(1)}$.

Dagan and Serrano [9] do not explicitly define bargaining games. The following definition is in accordance with the preceding paragraph (and similar to Osborne and Rubinstein's [37] definition of a "bargaining problem").

Definition An *n-person bargaining game with physical outcomes* is an n-person cooperative game with physical outcomes $\Gamma \equiv (N, \bar{X}, \{u_i\}_{i \in N})$ that satisfies $V^\Gamma \in \mathcal{B}^n$ and

$$\exists\; x_0 \in \bar{X}(N) \text{ s.t. } u_i(x_0) = 0 \;\forall\; i \in N \text{ and } \bar{X}(S) = \{x_0\} \;\forall\; S \in \mathcal{N} \setminus \{N\}.$$

In the following, let ψ^{Core} denote the solution concept that is induced by the Core concept for $\mathcal{V}^n$, i.e., $\psi^{Core}(\Gamma) := \{(S, x^S)_{S \in \mathcal{R}} \in F_o\,(\bar{X}) \mid \exists\; x \in \bar{X}(N) \text{ s.t. } u_i((S, x^S)_{S \in \mathcal{R}}) = u_i(x) \;\forall\; i \in N, \text{ and } \nexists\; S \in \mathcal{N} \text{ for which } \exists\; u' \in V^\Gamma(S) \text{ s.t. } u'_i > u_i((S, x^S)_{S \in \mathcal{R}}) \;\forall\; i \in S\}$ for every cooperative game with physical outcomes $\Gamma \equiv (N, \bar{X}, \{u_i\}_{i \in N})$. Correspondingly, let ψ^{Nash} denote the solution concept that is induced by the Nash Bargaining Solution concept for $\mathcal{B}^n$, i.e., $\psi^{Nash}(\Gamma) \equiv \{(N, x^*) \mid x^* \in \bar{X}(N), (u_1(x^*), \ldots, u_n(x^*)) \in \arg\max_{u \in V^\Gamma(N)} u_1 \cdot \ldots \cdot u_n\}$ for every bargaining game with physical outcomes $\Gamma \equiv (N, \bar{X}, \{u_i\}_{i \in N})$. Dagan and Serrano [9] assert that ψ^{Core} is ordinally invariant on every set of cooperative games with physical outcomes sharing the same game form, and that ψ^{Nash} in general is not ordinally invariant. For a proof of their first assertion, see Hahmeier [15] (pp. 33-34). In Appendix F, adapting two examples in Hahmeier [15] (pp. 37-39) to our definitions and notation, we define a set of bargaining games with physical outcomes sharing the same game form on which ψ^{Nash} is not ordinally invariant as well as a set on which every solution concept (and, in particular, ψ^{Nash}) is ordinally invariant (since there are no two distinct games in this set such that one is an order preserving transformation of the other).[54]

[54] Howard [17] considers a class of games similar to the latter set and shows that the Nash Bargaining Solution concept is $SPNE^n$-implementable on this class of games (cf. Hahmeier [15], pp. 71-74). Moulin [34] $SPNE^n$-implements the Kalai-Smorodinsky Solution concept in this kind of setting. A recent reference on the "Subgame-Perfect Implementation of Bargaining Solutions" is Miyagawa [31].

5.1.4 Ordinally Invariant Noncooperative Equilibrium Concepts

Let $\mathcal{C}^n_{nfg}$ be a set of n-person normal form games that share the same game form, and let $EC \in \{NE^n_{nfg}, DSE^n_{nfg}\}$.
Let $\Gamma \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i\}_{i\in N})$ and $\Gamma' \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}'_i\}_{i\in N})$ be in $\mathcal{C}^n_{nfg}$, and define $S := S_1 \times \ldots \times S_n$.

Definition Game Γ' *is an order preserving transformation of* Γ if, $\forall\ (i, s, \hat{s}) \in N \times S \times S$, $\tilde{u}'_i(s) > \tilde{u}'_i(\hat{s})$ if and only if $\tilde{u}_i(s) > \tilde{u}_i(\hat{s})$.

Definition Equilibrium concept EC *is ordinally invariant on* $\mathcal{C}^n_{nfg}$ if $EC(\Gamma) = EC(\Gamma')$ for each pair $\Gamma \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i\}_{i\in N})$, $\Gamma' \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}'_i\}_{i\in N})$ of games in $\mathcal{C}^n_{nfg}$ such that Γ' is an order preserving transformation of Γ.

We add the analogous definition for extensive form games and $SPNE^n$ (as already present in Hahmeier [15]). Let $\mathcal{C}^n_{efg}$ be a set of n-person extensive form games with perfect information that share the same game form.

Definition Equilibrium concept $SPNE^n$ *is ordinally invariant on* $\mathcal{C}^n_{efg}$ if $SPNE^n(\Gamma) = SPNE^n(\Gamma')$ for each pair $\Gamma \equiv (N, H, p, \{\tilde{u}_i\}_{i\in N})$, $\Gamma' \equiv (N, H, p, \{\tilde{u}'_i\}_{i\in N})$ of games in $\mathcal{C}^n_{efg}$ such that Γ' *is an order preserving transformation of* Γ, i.e., such that, $\forall\ (i, h, \hat{h}) \in N \times Z_H \times Z_H$, we have that $\tilde{u}'_i(h) > \tilde{u}'_i(\hat{h})$ if and only if $\tilde{u}_i(h) > \tilde{u}_i(\hat{h})$.

Dagan and Serrano [9] assert that "the class of ordinally invariant equilibrium concepts includes pure strategy Nash equilibrium, pure strategy subgame perfect equilibrium (and its stationary refinements), pure undominated strategies, iterative elimination of dominated actions, among others." For a proof that NE^n_{nfg} and DSE^n_{nfg} are both ordinally invariant on every set of n-person normal form games that share the same game form, and that $SPNE^n$ is ordinally invariant on every set of n-person extensive form games with perfect information that share the same game form, see Hahmeier [15], pp. 10-13.

5.2 Implementation of Solution Concepts for NTU Games

5.2.1 Trockel's "Embedding Principle"

For the implementation via strategic mechanisms, Trockel [52] introduces an "Embedding Principle" which transforms specific "support results" into strong implementation results. As shown in Hahmeier [15], an 'equivalent' principle holds for the implementation via extensive mechanisms. We outline these results in Proposition 5.1(a) and 5.1(b), and briefly sketch their proofs in Appendix G.

Let $\mathcal{C}^n_{ntu} \subseteq \mathcal{V}^n$ be a nonempty set of n-person NTU games, and let $T : \mathcal{C}^n_{ntu} \to (\mathcal{C}^n_{ntu})^n$ be defined by $T(V) := (V, \ldots, V)\ \forall\ V \in \mathcal{C}^n_{ntu}$.

Let $\mathcal{L}$ denote the set of single-valued solution concepts for $\mathcal{C}^n_{ntu}$, i.e.

$$\mathcal{L} := \{\ l : \mathcal{C}^n_{ntu} \to \bigcup_{V \in \mathcal{C}^n_{ntu}} F_{up}(V) \mid l(V) \in F_{up}(V)\ \forall\ V \in \mathcal{C}^n_{ntu}\}.$$

Define the classical n-person environment $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ by

$X := \mathcal{L}$,

$\Theta_i := \mathcal{C}^n_{ntu}\ \forall\ i \in N$,

$\Theta := T(\mathcal{C}^n_{ntu}) \equiv \{T(V) \mid V \in \mathcal{C}^n_{ntu}\}$,

$u'_i(l, T(V)) := (l(V))_i\ \forall\ (i, l, V) \in N \times \mathcal{L} \times \mathcal{C}^n_{ntu}$,[55] and

$\mathcal{G}$ is the set of all strategic and extensive mechanisms for $(N, \mathcal{L})$.

For each nonempty-valued solution concept L for $\mathcal{C}^n_{ntu}$, let $\alpha_L : \Theta \rightrightarrows \mathcal{L}$ denote the (social) choice correspondence for environment E defined by

$$\alpha_L(T(V)) := \bigcup_{l \in \mathcal{S}_L} [l]_V \quad \forall\ V \in \mathcal{C}^n_{ntu}\ ,^{56}$$

where $[l]_V := \{l' \in \mathcal{L} \mid l'(V) = l(V)\}\ \forall\ (l, V) \in \mathcal{L} \times \mathcal{C}^n_{ntu}$, and

$\mathcal{S}_L := \{l : \mathcal{C}^n_{ntu} \to \mathbb{R}^N \mid l(V) \in L(V)\ \forall\ V \in \mathcal{C}^n_{ntu}\}$ denotes the set of selections of L.[57]

In other words, $\alpha_L(T(V))$ is the set of all single-valued solution concepts l' for $\mathcal{C}^n_{ntu}$ for which there exists a selection l of L that takes the same value on V.

Let L be a nonempty-valued solution concept for $\mathcal{C}^n_{ntu}$, and let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Proposition 5.1(a) (Trockel's [52] "Embedding Principle")

Suppose that there exists a collection of n-person normal form games (all of which share the same game form) $\{\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})\}_{V \in \mathcal{C}^n_{ntu}}$ such that

(i) $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ *(strongly) EC-supports solution concept L on* $\mathcal{C}^n_{ntu}$, i.e. $\forall\ V \in \mathcal{C}^n_{ntu}$, $EC(\tilde{\Gamma}^V) \neq \emptyset$ and $\tilde{u}^V(EC(\tilde{\Gamma}^V)) \subseteq L(V)\ (\subseteq \mathbb{R}^N)$,[58] and

(ii) $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{u}^V(s) \in F_{up}(V)\ (\subseteq \mathbb{R}^N)\ \forall\ s \in S := S_1 \times \ldots \times S_n$.

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[55]In the words of Trockel [52], "this definition of utility functions reflects the fact that the players' subjective evaluations are determined by what they get in the actual game independently of what players would receive in a different game V'."

[56]In the words of Trockel [52], "this social choice rule reflects the idea that any population of n players as characterized by V evaluates a solution concept only on the basis of what that solution allocates to them in the game V. This population does not care about what a solution might give to other populations' players characterized by some $V' \neq V$."

[57]Note that $\mathcal{S}_L \neq \emptyset$ (since $L(V) \neq \emptyset\ \forall\ V \in \mathcal{C}^n_{ntu}$), and that, $\forall\ V \in \mathcal{C}^n_{ntu}$, $[l]_V \neq \emptyset\ \forall\ l \in \mathcal{S}_L$ (since $l \in [l]_V$). Therefore, $\alpha_L(\theta) \neq \emptyset\ \forall\ \theta \in \Theta$.

[58]Trockel requires $\tilde{u}^V(EC(\tilde{\Gamma}^V)) = L(V)\ \forall\ V \in \mathcal{C}^n_{ntu}$.

Then, the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g)$ for $(N, \mathcal{L})$ defined by

$$g : S \to \mathcal{L},\ g(s)(V) := \tilde{u}^V(s)\ (\in F_{up}(V))\ \forall\ s \in S$$

strongly EC-implements α_L in environment E, i.e.

$$EC(\Gamma^{E,G,T(V)}) \neq \emptyset \text{ and } g(EC(\Gamma^{E,G,T(V)})) \subseteq \alpha_L(T(V))\ \forall\ V \in \mathcal{C}^n_{ntu}.$$

Under the assumptions of Proposition 5.1(a), there exists, for every game $V \in \mathcal{C}^n_{ntu}$, at least one equilibrium of the game induced by mechanism G and type profile $T(V)$ in environment E, and in each such equilibrium $s^* \in EC(\Gamma^{E,G,T(V)})$ the players receive payoffs $(u'_1(g(s^*), T(V)), \ldots, u'_n(g(s^*), T(V))) = ((g(s^*)(V))_1, \ldots, (g(s^*)(V))_n)$ as if they had applied a single-valued solution concept to their game V that takes the same value on V as some selection of L $(g(s^*) \in \alpha_L(T(V)))$.

Proposition 5.1(b) (Hahmeier [15])

Suppose that there exists a collection of n-person extensive form games with perfect information (all of which share the same game form) $\{\tilde{\Gamma}^V \equiv (N, H, p, \{\tilde{u}^V_i\}_{i\in N})\}_{V\in\mathcal{C}^n_{ntu}}$ such that

(i) $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ *(strongly)* $SPNE^n$*-supports solution concept* L *on* $\mathcal{C}^n_{ntu}$, i.e.
$\forall\ V \in \mathcal{C}^n_{ntu},\ SPNE^n(\tilde{\Gamma}^V) \neq \emptyset$ and $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}^V))) \subseteq L(V)$, and

(ii) $\forall\ V \in \mathcal{C}^n_{ntu},\ \tilde{u}^V(x) \in F_{up}(V)\ \forall\ x \in Z_H$.

Then, the extensive n-person mechanism with perfect information $G \equiv (N, H, p, g)$ for $(N, \mathcal{L})$ defined by

$$g : Z_H \to \mathcal{L},\ g(x)(V) := \tilde{u}^V(x) \in (F_{up}(V))\ \forall\ x \in Z_H$$

strongly $SPNE^n$-implements α_L in environment E, i.e.

$$SPNE^n(\Gamma^{E,G,T(V)}) \neq \emptyset \text{ and } g(O(SPNE^n(\Gamma^{E,G,T(V)}))) \subseteq \alpha_L(T(V))\ \forall\ V \in \mathcal{C}^n_{ntu}.$$

5.2.2 Implementation of the Core Concept

Suppose that, for each $V \in \mathcal{C}^n_{ntu}$,

(i) $V(N)$ is comprehensive (i.e., $(u - \mathbb{R}^N_+) \subseteq V(N)\ \forall\ u \in V(N)$),

(ii) $V(\{i\}) \neq \emptyset$ and $\sup V(\{i\}) \in (0, \infty)\ \forall\ i \in N$, and
$V(\{i\})$ is comprehensive $\forall\ i \in N$, and

(iii) the set of efficient Core elements is nonempty, i.e.

$$EfCore(V) := \{u \in Core(V) \mid \nexists\ u' \in V(N) \text{ s.t. } u' \geq u \text{ and } u' \neq u\} \neq \emptyset,$$

and that solution concept L satisfies $L(V) = Core(V) \equiv \{u \in V(N) \mid \nexists\ S \in \mathcal{N}$ for which $\exists\ u' \in V(S)$ s.t. $u'_i > u_i\ \forall\ i \in S\}\ \forall\ V \in \mathcal{C}^n_{ntu}$.

We now introduce a collection of n-person normal form games satisfying the assumptions of Proposition 5.1(a) for $EC = NE^n_{nfg}$, thereby inducing a result on the Nash-implementability of the Core concept on $\mathcal{C}^n_{ntu}$.

Consider the collection of n-person normal form games $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}_{ntu}}$, where each $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i^V\}_{i\in N})$ is defined by $S_i := [0,\infty)\ \forall\ i \in N$ and

$$\tilde{u}_i^V(s) := \begin{cases} s_i \text{ if } s \in Core(V) \text{ or } s_i < \sup V(\{i\}) \\ 0 \text{ otherwise} \end{cases} \quad \forall\ (i,s) \in N \times [0,\infty)^n .^{59}$$

Then, $\forall\ V \in \mathcal{C}_{ntu}^n$, we have that $s \in EfCore(V) \Leftrightarrow s \in NE_{nfg}^n(\tilde{\Gamma}^V)\ \forall\ s \in [0,\infty)^n$:

'$\Rightarrow$' Let $s \in EfCore(V)$, and assume that there exists an $i \in N$ and a strategy $s_i' \in S_i$ such that $\tilde{u}_i^V(s_i', s_{-i}) > \tilde{u}_i^V(s)$. Since $s \in Core(V)$, we have that $\tilde{u}_i^V(s) = s_i \geq \sup V(\{i\})$. Therefore, $\tilde{u}_i^V(s_i', s_{-i}) > s_i \geq \sup V(\{i\}) \geq 0$. By definition of $\tilde{u}_i^V$, $\tilde{u}_i^V(s_i', s_{-i}) > s_i \geq 0$ implies that

$s_i' > s_i$, and

$(s_i', s_{-i}) \in Core(V)$ or $s_i' < \sup V(\{i\})$.

Since $s_i' > s_i \geq \sup V(\{i\})$, we must have $(s_i', s_{-i}) \in Core(V)\ (\subseteq V(N))$.

Now, $s_i' > s_i$ implies a contradiction to $s \in EfCore(V)$.

'$\Leftarrow$' Let $s \in [0,\infty)^n \setminus EfCore(V)$.

First, suppose that $s \in Core(V)$:

Since $s \notin EfCore(V)$, there exists an $u \in V(N)$ s.t. $u \geq s$ and $u \neq s$. Since $V(N)$ is comprehensive, it follows that there exists an $i \in N$ and an $u' \in V(N)$ s.t. $u_i' > s_i$ and $u_j' = s_j\ \forall\ j \in N\setminus\{i\}$. And, $s \in Core(V)$ implies that $u' \in Core(V)$.[60]

Thus, $\tilde{u}_i^V(u_i', s_{-i}) = \tilde{u}_i^V(u') = u_i' > s_i = \tilde{u}_i^V(s)$, i.e., $s \notin NE_{nfg}^n(\tilde{\Gamma}^V)$.

Next, suppose that $s \notin Core(V)$.

Since $s \notin Core(V)$, we have that $\tilde{u}_i^V(s) < \sup V(\{i\})\ \forall\ i \in N$. Thus, for each $i \in N$, strategy $s_i' := \tilde{u}_i^V(s) + \frac{1}{2} \cdot (\sup V(\{i\}) - \tilde{u}_i^V(s))$ satisfies $\tilde{u}_i^V(s_i', s_{-i}) = s_i' > \tilde{u}_i^V(s)$. In other words, $s \notin NE_{nfg}^n(\tilde{\Gamma}^V)$.

And, collection $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}_{ntu}^n}$ satisfies the assumptions of Proposition 5.1(a) for $EC = NE_{nfg}^n$. For each $V \in \mathcal{C}_{ntu}^n$, we have that

$NE_{nfg}^n(\tilde{\Gamma}^V) = EfCore(V) \neq \emptyset$,

$\tilde{u}^V(NE_{nfg}^n(\tilde{\Gamma}^V)) = \tilde{u}^V(EfCore(V)) = EfCore(V) \subseteq Core(V)$, and

$\tilde{u}^V(s) \in Core(V) \cup [0, \sup V(\{1\})) \times \ldots \times [0, \sup V(\{n\})) \subseteq F_{up}(V)\ \forall\ s \in [0,\infty)^n$,

where the last inclusion follows, in particular, from assumption (ii).

[59] Note that our definition of each player's utility function entails a feature that may well be subject to the same kind of criticism often expressed against 'integer games' or 'modulo games' or another, in the words of Jackson [21], "questionable feature of a mechanism".

[60] Assume that $\exists\ S \in \mathcal{N}$ for which $\exists\ u'' \in V(S)$ s.t. $u_i'' > u_i'\ \forall\ i \in S$. Then, $u_i'' > u_i' \geq s_i\ \forall\ i \in S$, contradicting $s \in Core(V)$.

5.2.3 Implementation of the Nash Bargaining Solution Concept

Suppose that $\mathcal{C}^n_{ntu} \subseteq \mathcal{B}^n$, and that solution concept L satisfies $L(V) = Nash(V)\ \forall\ V \in \mathcal{C}^n_{ntu}$. Trockel [51] mentions the following collection of normal form games (for the specific case of $n = 2$), in the following denoted by $\{\tilde{\Gamma}^V_{(a)}\}_{V \in \mathcal{C}^n_{ntu}}$, which satisfies the assumptions of Proposition 5.1(a) for $EC = DSE^n_{nfg}$.

(a) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V_{(a)} \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by

$$S_i := \mathbb{R}_+\ \forall\ i \in N \text{ and } \tilde{u}^V_i(s) := \begin{cases} s_i & \text{if } s = Nash(V) \\ 0 & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times \mathbb{R}^n_+.$$

According to Trockel [51], "this game provides support in a dominant strategy equilibrium for the Nash solution. It fails, however, to be a 'sensitive strategic model' as Osborne and Rubinstein (1990) require it for the Nash program. And it does not supplement the cooperative bargaining game such that, in the words of Nash (1953) 'each helps to justify and clarify' the other. In fact, any arbitrary bargaining solution could be supported in the same way."[61] The same is true for the following two collections of normal form games, each again satisfying the assumptions of Proposition 5.1(a) for $EC = DSE^n_{nfg}$.

(b) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V_{(b)} \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by $S_i := [0,1]\ \forall\ i \in N$

$$\text{and } \tilde{u}^V_i(s) := \begin{cases} s_i & \text{if } s_i \leq (Nash(V))_i \\ 0 & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times [0,1]^n,$$

(c) and define $\tilde{\Gamma}^V_{(c)} \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by

$$S_i := \{1\}\ \forall\ i \in N \text{ and } \tilde{u}^V_i(s) := (Nash(V))_i\ \forall\ (i,s) \in N \times \{1\}^n.$$ [62]

In "A Walrasian Approach to Bargaining Games", Trockel [50] introduces an "alternative characterization" of the Nash Bargaining Solution concept,[63] resulting in three different support results (also satisfying assumption (ii) in the respective part of Proposition 5.1) presented in Trockel [51]:[64]

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[61] As Trockel points out, "a similar point of view is represented by Proposition 1 of Bergin and Duggan (1999)."

[62] Note that every game $\tilde{\Gamma}^V$ in each of the three preceding collections has a unique DSE $\hat{s}^V$ and this unique DSE satisfies $\hat{s}^V = Nash(V)$.

[63] In the words of Trockel [51], "there, the Nash solution of any bargaining game is shown to coincide with the unique Walrasian equilibrium of a naturally induced economy with production and private ownership. The equilibrium price system evaluates the allocated utilities of players (interpreted as commodities) such that each player gets the same part of the total utility allocation in terms of value."

[64] Trockel [51], Proposition 1, 2, and 3, in Section 3, 4, and 5, respectively. A main part of Section 6 is on Trockel's [52] "Embedding Principle".

(d) A collection of n-person extensive form games with perfect information, in the following denoted by $\{\tilde{\Gamma}^V_{(d)}\}_{V \in \mathcal{C}^n_{ntu}}$, $SPNE^n$-supports solution concept $L \equiv Nash$ on every $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$, where $\bar{\mathcal{B}}^n$ denotes the set of all bargaining games $(N, V) \in \mathcal{B}^n$ that satisfy
   (1) $V(N)$ is strictly convex, and
   (2) the mapping g that associates with every vector $x \in \mathbb{R}^n$ in the efficient boundary $\partial(V(N))$ of $V(N)$ the normal vector at x to the efficient boundary $\partial(V(N))$, normalized by $\|g(x)\|_2 = 1$, is well defined as a continuously differentiable mapping $g : \partial(V(N)) \to \mathbb{R}^n_{++}$.

(e) A collection of n-person normal form games DSE^n_{nfg}-supports solution concept $L \equiv Nash$ on every $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$. And, as is true for collection $\{\tilde{\Gamma}^V_{(b)}\}_{V \in \mathcal{C}^n_{ntu}}$, every game $\tilde{\Gamma}^V$ in this collection allows each utility profile in $\{u \in \mathbb{R}^N_+ \mid u \leq Nash(V)\}$ to be realized.

(f) And, a third collection, again of n-person normal form games, NE^n_{nfg}-supports solution concept $L \equiv Nash$ on every $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$. In contrast to the preceding collection, however, every game $\tilde{\Gamma}^V$ in this collection allows each utility profile in $V(N)$ to be realized. And, according to Trockel [51], "this property which allows it to realize via coordinated strategic actions any feasible utility allocation of the cooperative game also in the non-cooperative game provides a good justification for the implicit assumption made in the support and implementation literature that players voluntarily participate in the non-cooperative game."

We denote the latter two collections by $\{\tilde{\Gamma}^V_{(e)}\}_{V \in \mathcal{C}^n_{ntu}}$ and $\{\tilde{\Gamma}^V_{(f)}\}_{V \in \mathcal{C}^n_{ntu}}$, respectively. Both are based on a specific collection of functions $\{m^V_i\}_{(i,V) \in N \times \mathcal{C}^n_{ntu}}$, each $m^V_i : [0,1] \to \mathbb{R}$ being a continuous function that equals the identity on $[0, (Nash(V))_i]$ and is strictly decreasing on $[(Nash(V))_i, 1]$:[65]

For each $V \in \mathcal{C}^n_{ntu}$, $\tilde{\Gamma}^V_{(e)} \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})$ satisfies

$$S_i = [0,1] \; \forall \; i \in N \text{ and } \tilde{u}^V_i(s) = m^V_i(s_i) \; \forall \; (i,s) \in N \times [0,1]^n,$$

and $\tilde{\Gamma}^V_{(f)} \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})$ satisfies

$$S_i = [0,1] \; \forall \; i \in N \text{ and } \tilde{u}^V_i(s) = \begin{cases} s_i & \text{if } s \in V(N) \\ m^V_i(s_i) & \text{otherwise} \end{cases} \quad \forall \; (i,s) \in N \times [0,1]^n.$$

Due to the properties of the functions m^V_i, every game $\tilde{\Gamma}^V$ in the former collection has a unique DSE $\hat{s}^V$ and this DSE satisfies $\hat{s}^V = Nash(V)$. Similarly, every game $\tilde{\Gamma}^V$ in the latter collection has a unique NE $\hat{s}^V$ and this NE satisfies $\hat{s}^V = Nash(V)$.

[65] Note that, since $V(N)$ is strictly convex, $(Nash(V))_i < 1 \; \forall \; i \in N$.

The details of these two collections are summarized in Appendix H, together with a brief description of collection $\{\tilde{\Gamma}^V_{(d)}\}_{V\in\mathcal{C}^n_{ntu}}$ and some modifications leading to a fourth collection, in the following denoted by $\{\tilde{\Gamma}^V_{(d')}\}_{V\in\mathcal{C}^n_{ntu}}$. Collection $\{\tilde{\Gamma}^V_{(d')}\}_{V\in\mathcal{C}^n_{ntu}}$ also satisfies the assumptions of Proposition 5.1(b) and is, in parts, similar to an earlier working paper version of collection $\{\tilde{\Gamma}^V_{(d)}\}_{V\in\mathcal{C}^n_{ntu}}$ in Trockel [53].

5.3 LE Implementation of Solution Concepts for NTU Games

5.3.1 The "Embedding Principle" in Environments with Limited Enforcement Power

Proposition 5.2 extends Trockel's [52] "Embedding Principle" to environments with limited enforcement power, in which the designer has no enforcement power on single-valued solution concepts.

Let $\mathcal{C}^n_{ntu} \subseteq \mathcal{V}^n$ be a nonempty set of n-person NTU games such that either
(i) $\sharp\{V(\{i\})\} \geq 2 \;\forall\; (i,V) \in N \times \mathcal{C}^n_{ntu}$, or
(ii) each $V \in \mathcal{C}^n_{ntu}$ is a bargaining game in NTU form.
Let $T : \mathcal{C}^n_{ntu} \to (\mathcal{C}^n_{ntu})^n$ be defined by $T(V) := (V,\ldots,V) \;\forall\; V \in \mathcal{C}^n_{ntu}$, and let $\mathcal{L}$ denote the set of single-valued solution concepts for $\mathcal{C}^n_{ntu}$, i.e.

$$\mathcal{L} := \{\; l : \mathcal{C}^n_{ntu} \to \bigcup_{V\in\mathcal{C}^n_{ntu}} F_{up}(V) \mid l(V) \in F_{up}(V) \;\forall\; V \in \mathcal{C}^n_{ntu}\}.$$

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an an n-person environment with limited enforcement power such that

$X = \mathcal{L}$,
$\Theta_i = \mathcal{C}^n_{ntu} \;\forall\; i \in N$,
$\Theta = T(\mathcal{C}^n_{ntu}) \equiv \{T(V) \mid V \in \mathcal{C}^n_{ntu}\}$,
$u_i((l,1), T(V)) = (l(V))_i \;\forall\; (i,l,V) \in N \times \mathcal{L} \times \mathcal{C}^n_{ntu}$,
$\mathcal{G}$ is the set of all strategic and extensive mechanisms for $(N,\mathcal{L})$, and
$e \equiv e^c$ in case (i) and $e \equiv e^b$ in case (ii),

where e^c and e^b denote the two enforcement structures for $(N,\mathcal{L})$ that satisfy

$$e^c(S) = \begin{cases} \mathcal{L} & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{otw.} \end{cases} \quad \forall\; S \in \mathcal{N}^+ \text{ and}$$

$$e^b(S) = \begin{cases} \mathcal{L} & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{l^{(0)}\} & \text{otw.} \end{cases} \quad \forall\; S \in \mathcal{N}^+,$$

and $l^{(0)} \in \mathcal{L}$ denotes the solution concept that satisfies $l^{(0)}(V) = (0, \ldots, 0) \; \forall \; V \in \mathcal{C}^n_{ntu}$. For each nonempty-valued solution concept L for $\mathcal{C}^n_{ntu}$, let $\alpha_L : \Theta \Rightarrow \mathcal{L}$ denote the (social) choice correspondence for environment E defined by

$$\alpha_L(T(V)) := \bigcup_{l \in \mathcal{S}_L} [l]_V \quad \forall \; V \in \mathcal{C}^n_{ntu} \; ,$$

where $[l]_V := \{l' \in \mathcal{L} \mid l'(V) = l(V)\} \; \forall \; (l, V) \in \mathcal{L} \times \mathcal{C}^n_{ntu}$, and
$\mathcal{S}_L := \{l : \mathcal{C}^n_{ntu} \to \mathbb{R}^N \mid l(V) \in L(V) \; \forall \; V \in \mathcal{C}^n_{ntu}\}$ denotes the set of selections of L.

Let L be a nonempty-valued solution concept for $\mathcal{C}^n_{ntu}$, and let $EC \in \{DSE^n_{nfg}, NE^n_{nfg}\}$.

Proposition 5.2

(a) Suppose that there exists a collection of n-person normal form games (sharing the same game form) $\{\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})\}_{V \in \mathcal{C}^n_{ntu}}$ such that

  (i) $\{\hat{\Gamma}^V\}_{V \in \mathcal{C}^n_{ntu}}$ *(strongly) EC-supports solution concept L on* $\mathcal{C}^n_{ntu}$, i.e. $\forall \; V \in \mathcal{C}^n_{ntu}$, $EC(\hat{\Gamma}^V) \neq \emptyset$ and $\hat{u}^V(EC(\hat{\Gamma}^V)) \subseteq L(V)$ $(\subseteq \mathbb{R}^N)$,

  (ii) $\forall \; V \in \mathcal{C}^n_{ntu}$, $\tilde{u}^V(s) \in F_{up}(V)$ $(\subseteq \mathbb{R}^N) \; \forall \; s \in S := S_1 \times \ldots \times S_n$, and

  (iii) $\forall \; V \in \mathcal{C}^n_{ntu}$, $R(g(s), T(V)) = (g(s), 1) \; \forall \; s \in EC(\hat{\Gamma}^V)$,

where, $\forall \; V \in \mathcal{C}^n_{ntu}$, game $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i \in N}, \{\hat{u}^V_i\}_{i \in N})$ is defined by

$\hat{u}^V_i(s) := u_i(R(g(s), T(V)), T(V)) \; \forall \; (i, s) \in N \times S$,

and $g : S \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s)$ $(\in F_{up}(V)) \; \forall \; (s, V) \in S \times \mathcal{C}^n_{ntu}$. Then, the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i \in N}, g)$ strongly EC-implements α_L in environment E, i.e., $\forall \; V \in \mathcal{C}^n_{ntu}$, we have that

$EC(\Gamma^{E^*, G^*, T(V)}) \neq \emptyset$ and

$R(g(EC(\Gamma^{E^*, G^*, T(V)})), T(V)) \subseteq \{(x, 1) \mid x \in \alpha_L(T(V))\}$.

(b) Suppose that there exists a collection of n-person extensive form games with perfect information (sharing the same game form) $\{\tilde{\Gamma}^V \equiv (N, H, p, \{\tilde{u}^V_i\}_{i \in N})\}_{V \in \mathcal{C}^n_{ntu}}$ such that

  (i) $\{\hat{\Gamma}^V\}_{V \in \mathcal{C}^n_{ntu}}$ *(strongly) $SPNE^n$-supports solution concept L on* $\mathcal{C}^n_{ntu}$, i.e. $\forall \; V \in \mathcal{C}^n_{ntu}$, $SPNE^n(\hat{\Gamma}^V) \neq \emptyset$ and $\hat{u}^V(O(SPNE^n(\hat{\Gamma}^V))) \subseteq L(V)$,

  (ii) $\forall \; V \in \mathcal{C}^n_{ntu}$, $\tilde{u}^V(x) \in F_{up}(V) \; \forall \; x \in Z_H$, and

  (iii) $\forall \; V \in \mathcal{C}^n_{ntu}$, $R(g(x), T(V)) = (g(x), 1) \; \forall \; x \in O(SPNE^n(\hat{\Gamma}^V))$,

where, $\forall \; V \in \mathcal{C}^n_{ntu}$, game $\hat{\Gamma}^V \equiv (N, H, p, \{\hat{u}^V_i\}_{i \in N})$ is defined by

$\hat{u}^V_i(x) := u_i(R(g(x), T(V)), T(V)) \; \forall \; (i, x) \in N \times Z_H$,

and $g : Z_H \to \mathcal{L}$ is defined by $g(x)(V) := \tilde{u}^V(x)$ $(\in F_{up}(V)) \; \forall \; (x, V) \in Z_H \times \mathcal{C}^n_{ntu}$. Then, the extensive n-person mechanism with perfect information $G \equiv (N, H, p, g)$ strongly $SPNE^n$-implements α_L in environment E, i.e., $\forall \; V \in \mathcal{C}^n_{ntu}$, we have that

$SPNE^n(\Gamma^{E^*,G^*,T(V)}) \neq \emptyset$ and
$R(g(O(SPNE^n(\Gamma^{E^*,G^*,T(V)}))), T(V)) \subseteq \{(x,1) \mid x \in \alpha_L(T(V))\}$.

Proof

(a) Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.
Since $u_i(R(g(s), T(V)), T(V)) = \hat{u}^V_i(s)\ \forall\ (i,s) \in N \times S$, we have that
$\Gamma^{E^*,G^*,T(V)} \equiv (N, \{S_i\}_{i\in N}, \{u_i(R(g(\cdot), T(V)), T(V))\}_{i\in N}) = \hat{\Gamma}^V$.
Thus, $EC(\Gamma^{E^*,G^*,T(V)}) = EC(\hat{\Gamma}^V) \neq \emptyset$, and it remains to show that
$R(g(EC(\hat{\Gamma}^V)), T(V)) \subseteq \{(x,1) \mid x \in \alpha_L(T(V))\}$.
Consider an arbitrary $s \in EC(\hat{\Gamma}^V)$.
By assumption (iii), it is sufficient to show that $g(s) \in \alpha_L(T(V))$.
Since $\hat{u}^V(EC(\hat{\Gamma}^V)) \subseteq L(V)$, we have that $\hat{u}^V(s) \in L(V)$.
Therefore, there exists a selection $l^* \in \mathcal{S}_L$ of L such that $l^*(V) = \hat{u}^V(s)$.
By definition of $\hat{u}$ and by assumption (iii), it follows that $l^*(V) = \hat{u}^V(s) = u(R(g(s), T(V)), T(V)) = u((g(s), 1), T(V)) = g(s)(V)$.
In other words, $g(s) \in [l^*]_V\ (= \{l' \in \mathcal{L} \mid l'(V) = l^*(V)\}\ )$.
Since $l^* \in \mathcal{S}_L$, it follows that $g(s) \in [l^*]_V \subseteq \bigcup_{l\in\mathcal{S}_L}[l]_V = \alpha_L(T(V))$.

(b) Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.
Since $u_i(R(g(x), T(V)), T(V)) = \hat{u}^V_i(x)\ \forall\ (i,x) \in N \times Z_H$, we have that
$\Gamma^{E^*,G^*,T(V)} \equiv (N, H, p, \{u_i(R(g(\cdot), T(V)), T(V))\}_{i\in N}) = \hat{\Gamma}^V$.
Thus, $SPNE^n(\Gamma^{E^*,G^*,T(V)}) = SPNE^n(\hat{\Gamma}^V) \neq \emptyset$, and it remains to show that
$R(g(O(SPNE^n(\hat{\Gamma}^V))), T(V)) \subseteq \{(x,1) \mid x \in \alpha_L(T(V))\}$.
Consider an arbitrary $x \in O(SPNE^n(\hat{\Gamma}^V))$.
By assumption (iii), it is sufficient to show that $g(x) \in \alpha_L(T(V))$.
Since $\hat{u}^V(O(SPNE^n(\hat{\Gamma}^V))) \subseteq L(V)$, we have that $\hat{u}^V(x) \in L(V)$.
Therefore, there exists a selection $l^* \in \mathcal{S}_L$ of L such that $l^*(V) = \hat{u}^V(x)$.
By definition of $\hat{u}$ and by assumption (iii), it follows that $l^*(V) = \hat{u}^V(x) = u(R(g(x), T(V)), T(V)) = u((g(x), 1), T(V)) = g(x)(V)$.
In other words, $g(x) \in [l^*]_V\ (= \{l' \in \mathcal{L} \mid l'(V) = l^*(V)\}\ )$.
Since $l^* \in \mathcal{S}_L$, it follows that $g(x) \in [l^*]_V \subseteq \bigcup_{l\in\mathcal{S}_L}[l]_V = \alpha_L(T(V))$.

□

The following corollary covers the case that the supporting collection allows every outcome in the image of the corresponding mechanism's outcome function to be realized.

Corollary 5.2

(a) Suppose that there exists a collection of n-person normal form games (sharing the same game form) $\{\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})\}_{V\in\mathcal{C}^n_{ntu}}$ such that

(i) $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ *(strongly) EC-supports solution concept L on* $\mathcal{C}^n_{ntu}$, i.e.
$\forall\ V \in \mathcal{C}^n_{ntu}$, $EC(\tilde{\Gamma}^V) \neq \emptyset$ and $\tilde{u}^V(EC(\tilde{\Gamma}^V)) \subseteq L(V)$ $(\subseteq \mathbb{R}^N)$,

(ii) $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{u}^V(s) \in F_{up}(V)$ $(\subseteq \mathbb{R}^N)$ $\forall\ s \in S := S_1 \times \ldots \times S_n$, and

(iii) $\forall\ V \in \mathcal{C}^n_{ntu}$, $R(g(s), T(V)) = (g(s), 1)$ $\forall\ s \in S$,

where $g : S \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s)$ $\forall\ (s, V) \in S \times \mathcal{C}^n_{ntu}$.

Then, the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g)$ strongly *EC*-implements α_L in environment E.

(b) Suppose that there exists a collection of n-person extensive form games with perfect information (sharing the same game form) $\{\tilde{\Gamma}^V \equiv (N, H, p, \{\tilde{u}^V_i\}_{i\in N})\}_{V\in\mathcal{C}^n_{ntu}}$ such that

(i) $\{\tilde{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ *(strongly)* $SPNE^n$*-supports solution concept L on* $\mathcal{C}^n_{ntu}$, i.e.
$\forall\ V \in \mathcal{C}^n_{ntu}$, $SPNE^n(\tilde{\Gamma}^V) \neq \emptyset$ and $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}^V))) \subseteq L(V)$,

(ii) $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{u}^V(x) \in F_{up}(V)$ $\forall\ x \in Z_H$, and

(iii) $\forall\ V \in \mathcal{C}^n_{ntu}$, $R(g(x), T(V)) = (g(x), 1)$ $\forall\ x \in Z_H$,

where $g : Z_H \to \mathcal{L}$ is defined by $g(x)(V) := \tilde{u}^V(x)$ $(\in F_{up}(V))$ $\forall\ (x, V) \in Z_H \times \mathcal{C}^n_{ntu}$. Then, the extensive n-person mechanism with perfect information $G \equiv (N, H, p, g)$ strongly $SPNE^n$-implements α_L in environment E.

Proof

(a) For each $V \in \mathcal{C}^n_{ntu}$, let $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\hat{u}^V_i\}_{i\in N})$ be defined by
$$\hat{u}^V_i(s) := u_i(R(g(s), T(V)), T(V))\ \forall\ (i, s) \in N \times S.$$
Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.
Then, $\hat{u}^V_i(s) = u_i((g(s), 1), T(V)) = (g(s)(V))_i = \tilde{u}^V_i(s)$ $\forall\ (i, s) \in N \times S$, and thus $\hat{\Gamma}^V = \tilde{\Gamma}^V$.
It follows that $EC(\hat{\Gamma}^V) = EC(\tilde{\Gamma}^V) \neq \emptyset$, and that
$\hat{u}^V(EC(\hat{\Gamma}^V)) = \tilde{u}^V(EC(\tilde{\Gamma}^V)) \subseteq L(V)$.

(b) For each $V \in \mathcal{C}^n_{ntu}$, let $\hat{\Gamma}^V \equiv (N, H, p, \{\hat{u}^V_i\}_{i\in N})$ be defined by
$$\hat{u}^V_i(x) := u_i(R(g(x), T(V)), T(V))\ \forall\ (i, x) \in N \times Z_H.$$
Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.
Then, $\hat{u}^V_i(s) = u_i((g(x), 1), T(V)) = (g(x)(V))_i = \tilde{u}^V_i(x)$ $\forall\ (i, x) \in N \times Z_H$, and thus $\hat{\Gamma}^V = \tilde{\Gamma}^V$.
It follows that $SPNE^n(\hat{\Gamma}^V) = SPNE^n(\tilde{\Gamma}^V) \neq \emptyset$, and that
$\hat{u}^V(O(SPNE^n(\hat{\Gamma}^V))) = \tilde{u}^V(O(SPNE^n(\tilde{\Gamma}^V))) \subseteq L(V)$.

□

Remark 5.2 Note that condition (iii) of Corollary 5.2 is satisfied, in particular, if agents have weak pessimistic beliefs in environment E.

In the following two paragraphs, we re-consider the support results for the Nash Bargaining Solution concept and the Core concept outlined in the preceding section. Assuming that agents' beliefs can be justified by outcome-independent prediction functions, we approach the question to what extent these support results can be adjusted in order to imply implementation results by the application of Proposition 5.2.

5.3.2 LE Implementation of the Nash Bargaining Solution Concept

Suppose that $\mathcal{C}^n_{ntu} \subseteq \mathcal{B}^n$, that $e \equiv e^b$, and that solution concept L satisfies $L(V) = Nash(V)\ \forall\ V \in \mathcal{C}^n_{ntu}$.[66] Furthermore, suppose that agents' beliefs can be justified by prediction functions $\{b_i\}_{i\in N}$ for $(\mathcal{L}, T(V))$, i.e.

$$u_i((l,0),T(V)) = u_i((b_{i1}((l,0),T(V)),1),T(V))\ \forall\ (i,V,l) \in N \times \mathcal{C}^n_{ntu} \times \mathcal{L},$$

and that these prediction functions are outcome-independent,i.e.

$$b_i((l,0),T(V)) = b_i((l',0),T(V))\ \forall\ (i,l,l',V) \in N \times \mathcal{L} \times \mathcal{L} \times \mathcal{C}_{ntu}.$$

Define, $\forall\ (i,V) \in N \times \mathcal{C}_{ntu}$,

$b^{i,V} \in \mathbb{R}^N$ by $b^{i,V} := (b_{i1}((l,0),T(V)))(V)$ for some $l \in \mathcal{L}$,

i.e., $b^{i,V}$ reflects agent $i's$ prediction for the final utility allocation in state $T(V)$ if the outcome suggested by the mechanism is not being implemented by a coalition that is able to do so,

$b_i^V \in \mathbb{R}$ by $b_i^V := (b^{i,V})_i$, and

$n_i^V \in \mathbb{R}$ by $n_i^V := (Nash(V))_i$,

and note that, in particular,

$$\begin{aligned} u_i((l,0),T(V)) &= u_i((b_{i1}((l,0),T(V)),1),T(V)) \\ &= (b_{i1}((l,0),T(V))(V))_i \\ &= b_i^V\ \forall\ (i,V,l) \in N \times \mathcal{C}^n_{ntu} \times \mathcal{L}, \text{ and} \end{aligned}$$

$$R(l,T(V)) = \begin{cases} (l,1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } l \in e(S) \text{ and} \\ & u_i((l,1),T(V)) \geq u_i((l,0),T(V))\ \forall\ i \in S \cap N \\ (l,0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (l,1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } l \in e(S) \text{ and} \\ & (l(V))_i \geq b_i^V\ \forall\ i \in S \cap N \\ (l,0) & \text{otherwise} \end{cases} \quad \text{on } \mathcal{L} \times \Theta.$$

Remember, that both collection $\{\tilde{\Gamma}^V_{(e)}\}_{V\in\mathcal{C}^n_{ntu}}$ and collection $\{\tilde{\Gamma}^V_{(f)}\}_{V\in\mathcal{C}^n_{ntu}}$ are based on a collection of functions $\{m_i^V\}_{(i,V)\in N\times\mathcal{C}^n_{ntu}}$, each $m_i^V : [0,1] \to \mathbb{R}$ being a continuous function that equals the identity on $[0, n_i^V]$ and is strictly decreasing on $[n_i^V, 1]$.[67] Now,

[66] Note that, since $\alpha_L(T(V)) \equiv \bigcup_{l\in\mathcal{S}_L}[l]_V = [Nash]_V = \{Nash\}\ \forall\ V \in \mathcal{C}^n_{ntu}$, strong implementation of α_L is equivalent to full implementation of α_L in environment E.

[67] See Paragraph 5.2.3 and Appendix H.

for each $V \in \mathcal{C}^n_{ntu}$ satisfying $b^V \leq n^V$, define

$$q_i^V := \begin{cases} \sup\{u_i \in (n_i^V, 1] \mid m_i^V(u_i) > b_i^V\} & \text{if } b_i^V < n_i^V \\ b_i^V & \text{if } b_i^V = n_i^V \end{cases} \quad \forall\, i \in N.$$

Figure 5.1 illustrates the notation.

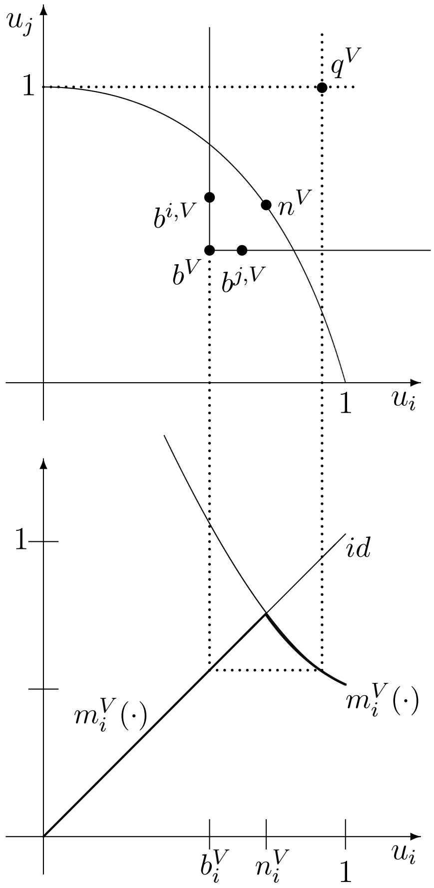

Figure 5.1

If agents' beliefs are such that $0 \leq b^V \leq n^V \;\forall\; V \in \mathcal{C}^n_{ntu}$, i.e., each agent i's prediction for his final utility level (in case that the outcome suggested by the mechanism is not being implemented) is no higher than his Nash coordinate, then collection $\{\tilde{\Gamma}^V_{(c)}\}_{V \in \mathcal{C}^n_{ntu}}$ satisfies the assumptions of Corollary 5.2(a) for $EC = DSE^n_{nfg}$.

If agents' beliefs are such that $0 \leq b_i^V < n_i^V \;\forall\; (i, V) \in N \times \mathcal{C}^n_{ntu}$, then the following modifications of collection $\{\tilde{\Gamma}^V_{(a)}\}_{V \in \mathcal{C}^n_{ntu}}$ and $\{\tilde{\Gamma}^V_{(b)}\}_{V \in \mathcal{C}^n_{ntu}}$ also satisfy the assumptions of Corollary 5.2(a) for $EC = DSE^n_{nfg}$.[68]

(a) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}_i^V\}_{i \in N})$ by

$$S_i := \mathbb{R}_+ \;\forall\; i \in N \text{ and } \tilde{u}_i^V(s) := \begin{cases} s_i & \text{if } s = Nash(V) \\ b_i^V & \text{otw.} \end{cases} \quad \forall\, (i, s) \in N \times \mathbb{R}^n_+.$$

[68] In our modification of collection $\{\tilde{\Gamma}^V_{(a)}\}_{V \in \mathcal{C}^n_{ntu}}$, we merely substitute 0 by b_i^V.

(b) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by
$S_i := [0,1]\ \forall\ i \in N$ and

$$\tilde{u}^V_i(s) := \begin{cases} s_i & \text{if } b^V_i \leq s_i \leq (Nash(V))_i \\ b^V_i & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times [0,1]^n.$$ [69]

If, as before, agents' beliefs are such that $0 \leq b^V_i < n^V_i\ \forall\ (i,V) \in N \times \mathcal{C}^n_{ntu}$, and if, in addition, $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$, then the following two modifications of $\{\tilde{\Gamma}^V_{(e)}\}_{V\in\mathcal{C}^n_{ntu}}$ and $\{\tilde{\Gamma}^V_{(f)}\}_{V\in\mathcal{C}^n_{ntu}}$ satisfy the assumptions of Corollary 5.2(a) for $EC = DSE^n_{nfg}$ and $EC = NE^n_{nfg}$, respectively. In Appendix H, we provide a modification of collection $\{\tilde{\Gamma}^V_{(d)}\}_{V\in\mathcal{C}^n_{ntu}}$ (and $\{\tilde{\Gamma}^V_{(d')}\}_{V\in\mathcal{C}^n_{ntu}}$) which satisfies the assumptions of Corollary 5.2(b).

(e) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by

$$S_i := [0,1]\ \forall\ i \in N \text{ and } \tilde{u}^V_i(s) := m^V_i(f^V_i(s_i))\ \forall\ (i,s) \in N \times [0,1]^n,$$

where $f^V_i : [0,1] \to [0,1]$ is defined by $f^V_i(s_i) := b^V_i + s_i \cdot (q^V_i - b^V_i)\ \forall\ s_i \in [0,1]$.[70]
In other words, the argument $s_i \in [0,1]$ in the definition of collection $\{\tilde{\Gamma}^V_{(e)}\}_{V\in\mathcal{C}^n_{ntu}}$ is substituted by its projection on the interval $[b^V_i, q^V_i]$.

(f) For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N})$ by
$S_i := [0,1]\ \forall\ i \in N$ and

$$\tilde{u}^V_i(s) := \begin{cases} f^V_i(s_i) & \text{if } f^V(s) \in V(N) \\ m^V_i(f^V_i(s_i)) & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times [0,1]^n,$$

where $f^V_i : [0,1] \to [0,1]$ is defined by $f^V_i(s_i) := b^V_i + s_i \cdot (q^V_i - b^V_i)\ \forall\ s_i \in [0,1]$.

If agents' beliefs are such that $0 < b^V_i < n^V_i\ \forall\ (i,V) \in N \times \mathcal{C}^n_{ntu}$, i.e., each agent i's prediction lies strictly between 0 and his Nash coordinate, then each of the following four collections from Paragraph 5.2.3 satisfies the assumptions of Proposition 5.2(a) for $EC = DSE^n_{nfg}$, where again $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$ is required in case (e):

(a) If, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}^V_i\}_{i\in N}) = \tilde{\Gamma}^V_{(a)}$, i.e.

$$S_i = \mathbb{R}_+\ \forall\ i \in N \text{ and } \tilde{u}^V_i(s) = \begin{cases} s_i & \text{if } s = Nash(V) \\ 0 & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times \mathbb{R}^n_+,$$

then, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\hat{u}^V_i\}_{i\in N})$ defined by

$$\hat{u}^V_i(s) := \begin{cases} s_i & \text{if } s = Nash(V) \\ b^V_i & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times \mathbb{R}^n_+$$

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[69] Note that in both case (a) and (b), $\forall\ (s,V) \in S_1 \times \ldots \times S_n \times \mathcal{C}^n_{ntu}$,

$$R(g(s), T(V)) = \begin{cases} (g(s), 1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and } \tilde{u}^V_i(s) \geq b^V_i\ \forall\ i \in S \\ (g(s), 0) & \text{otherwise} \end{cases} = (g(s), 1).$$

[70] Note that $\tilde{u}^V([0,1]^n) \subseteq \{u \in \mathbb{R}^N_+ \mid u \leq n^V\}$, and thus $\tilde{u}^V(s) \in V(N)\ \forall\ (s,V) \in [0,1]^n \times \mathcal{C}^n_{ntu}$.

satisfies

$$R(g(s),T(V)) = \begin{cases} (g(s),1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and} \\ & \tilde{u}_i^V(s)\,(= (g(s)(V))_i) \ \geq b_i^V\ \forall\ i \in S \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } s = Nash(V) \\ (g(s),0) & \text{otherwise} \end{cases} \qquad \forall\ s \in \mathbb{R}_+^n,$$

and thus

$$u_i(R(g(s),T(V)),T(V)) = \begin{cases} u_i((g(s),1),T(V)) & \text{if } s = Nash(V) \\ u_i((g(s),0),T(V)) & \text{otw.} \end{cases}$$

$$= \begin{cases} (g(s)(V))_i & \text{if } s = Nash(V) \\ b_i^V & \text{otw.} \end{cases}$$

$$= \begin{cases} \tilde{u}_i^V(s) & \text{if } s = Nash(V) \\ b_i^V & \text{otw.} \end{cases}$$

$$= \begin{cases} s_i & \text{if } s = Nash(V) \\ b_i^V & \text{otw.} \end{cases}$$

$$= \hat{u}_i^V(s)\ \forall\ (i,s) \in N \times \mathbb{R}_+^n,$$

where $g : \mathbb{R}_+^n \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s)\ \forall\ (s,V) \in \mathbb{R}_+^n \times \mathcal{C}_{ntu}^n$. And, collection $\{\hat{\Gamma}^V\}_{V \in \mathcal{C}_{ntu}^n}$ satisfies assumption (i) and (iii) of Proposition 5.2(a).

(b) If, $\forall\ V \in \mathcal{C}_{ntu}^n$, $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i^V\}_{i\in N}) = \tilde{\Gamma}_{(b)}^V$, i.e., $S_i = [0,1]\ \forall\ i \in N$ and

$$\tilde{u}_i^V(s) = \begin{cases} s_i & \text{if } s_i \leq (Nash(V))_i \\ 0 & \text{otw.} \end{cases} \qquad \forall\ (i,s) \in N \times [0,1]^n,$$

then, $\forall\ V \in \mathcal{C}_{ntu}^n$, $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\hat{u}_i^V\}_{i\in N})$ defined by

$$\hat{u}_i^V(s) := \begin{cases} s_i & \text{if } b^V \leq s \leq Nash(V) \\ b_i^V & \text{otw.} \end{cases} \qquad \forall\ (i,s) \in N \times [0,1]^n$$

satisfies

$$R(g(s),T(V)) = \begin{cases} (g(s),1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and} \\ & \tilde{u}_i^V(s)\,(= (g(s)(V))_i) \ \geq b_i^V\ \forall\ i \in S \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } g(s) \neq l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V\ \forall\ i \in N \\ (g(s),1) & \text{if } g(s) = l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V \text{ for some } i \in N \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } g(s) \neq l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V \ \forall\ i \in N \\ (g(s),1) & \text{if } g(s) = l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V \ \forall\ i \in N \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } g(s) \neq l^{(0)} \text{ and } b^V \leq s \leq n^V \\ (g(s),1) & \text{if } g(s) = l^{(0)} \text{ and } b^V \leq s \leq n^V \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } b^V \leq s \leq n^V \\ (g(s),0) & \text{otherwise} \end{cases} \quad \forall\ s \in [0,1]^n,^{71}$$

and thus

$$u_i(R(g(s),T(V)),T(V)) = \begin{cases} u_i((g(s),1),T(V)) & \text{if } b^V \leq s \leq n^V \\ u_i((g(s),0),T(V)) & \text{otw.} \end{cases}$$

$$= \begin{cases} \tilde{u}_i^V(s) & \text{if } b^V \leq s \leq n^V \\ b_i^V & \text{otw.} \end{cases}$$

$$= \begin{cases} s_i & \text{if } b^V \leq s \leq n^V \\ b_i^V & \text{otw.} \end{cases}$$

$$= \hat{u}_i^V(s) \ \forall\ (i,s) \in N \times [0,1]^n,$$

where $g : [0,1]^n \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s) \ \forall\ (s,V) \in [0,1]^n \times \mathcal{C}^n_{ntu}$.
And, collection $\{\hat{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ satisfies assumption (i) and (iii) of Proposition 5.2(a).

(c) If, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i^V\}_{i\in N}) = \tilde{\Gamma}^V_{(c)}$, i.e.

$$S_i = \{1\} \ \forall\ i \in N \text{ and } \tilde{u}_i^V(s) = (Nash(V))_i \ \forall\ (i,s) \in N \times \{1\}^n,$$

then, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\hat{u}_i^V\}_{i\in N})$ defined by

$$\hat{u}_i^V(s) := (Nash(V))_i \ \forall\ (i,s) \in N \times \{1\}^n \text{ satisfies}$$

$$R(g(s),T(V)) = \begin{cases} (g(s),1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and} \\ & \tilde{u}_i^V(s)\,(= (g(s)(V))_i) \ \geq b_i^V \ \forall\ i \in S \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(s),1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and} \\ & n_i^V \geq b_i^V \ \forall\ i \in S \\ (g(s),0) & \text{otherwise} \end{cases}$$

$$= (g(s),1) \ \forall\ s \in \{1\}^n,$$

and thus

[71] As for the third equality, note that, if $g(s) = l^{(0)}$, then $g(s)(V) = (0,\ldots,0) = \tilde{u}^V(s)$ (by definition of $l^{(0)}$ and $g : [0,1]^n \to \mathcal{L}$ below), and thus $\tilde{u}_i^V(s) = 0 < b_i^V \ \forall\ i \in N$.

$$u_i(R(g(s),T(V)),T(V)) = \tilde{u}_i^V(s) = n_i^V = \hat{u}_i^V(s)\ \forall\ (i,s) \in N \times \{1\}^n,$$

where $g : [0,1]^n \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s)\ \forall\ (s,V) \in [0,1]^n \times \mathcal{C}^n_{ntu}$. And, collection $\{\hat{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ satisfies assumption (i) and (iii) of Proposition 5.2(a).[72]

(e) If, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\tilde{u}_i^V\}_{i\in N}) = \tilde{\Gamma}^V_{(e)}$, i.e.

$$S_i = [0,1]\ \forall\ i \in N \text{ and } \tilde{u}_i^V(s) = m_i^V(s_i)\ \forall\ (i,s) \in N \times [0,1]^n,$$

then, $\forall\ V \in \mathcal{C}^n_{ntu}$, $\hat{\Gamma}^V \equiv (N, \{S_i\}_{i\in N}, \{\hat{u}_i^V\}_{i\in N})$ defined by

$$\hat{u}_i^V(s) := \begin{cases} m_i^V(s_i) & \text{if } m_j^V(s_j) \geq b_j^V\ \forall\ j \in N \\ b_i^V & \text{otw.} \end{cases} \quad \forall\ (i,s) \in N \times [0,1]^n$$

satisfies

$$\begin{aligned} R(g(s),T(V)) &= \begin{cases} (g(s),1) & \text{if } \exists\ S \in \mathcal{N} \text{ s.t. } g(s) \in e(S) \text{ and} \\ & \quad \tilde{u}_i^V(s)\,(= (g(s)(V))_i) \geq b_i^V\ \forall\ i \in S \\ (g(s),0) & \text{otherwise} \end{cases} \\ &= \begin{cases} (g(s),1) & \text{if } g(s) \neq l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V\ \forall\ i \in N \\ (g(s),1) & \text{if } g(s) = l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V \text{ for some } i \in N \\ (g(s),0) & \text{otherwise} \end{cases} \\ &= \begin{cases} (g(s),1) & \text{if } g(s) \neq l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V\ \forall\ i \in N \\ (g(s),1) & \text{if } g(s) = l^{(0)} \text{ and } \tilde{u}_i^V(s) \geq b_i^V\ \forall\ i \in N \\ (g(s),0) & \text{otherwise} \end{cases} \\ &= \begin{cases} (g(s),1) & \text{if } m_i^V(s_i) \geq b_i^V\ \forall\ i \in N \\ (g(s),0) & \text{otherwise} \end{cases} \quad \forall\ s \in [0,1]^n,^{73} \end{aligned}$$

and thus

$$\begin{aligned} u_i(R(g(s),T(V)),T(V)) &= \begin{cases} u_i((g(s),1),T(V)) & \text{if } m_j^V(s_j) \geq b_j^V\ \forall\ j \in N \\ u_i((g(s),0),T(V)) & \text{otw.} \end{cases} \\ &= \begin{cases} \tilde{u}_i^V(s) & \text{if } m_j^V(s_j) \geq b_j^V\ \forall\ j \in N \\ b_i^V & \text{otw.} \end{cases} \\ &= \begin{cases} m_i^V(s_i) & \text{if } m_j^V(s_j) \geq b_j^V\ \forall\ j \in N \\ b_i^V & \text{otw.} \end{cases} \\ &= \hat{u}_i^V(s)\ \forall\ (i,s) \in N \times [0,1]^n, \end{aligned}$$

where $g : [0,1]^n \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s)\ \forall\ (s,V) \in [0,1]^n \times \mathcal{C}^n_{ntu}$. And, collection $\{\hat{\Gamma}^V\}_{V\in\mathcal{C}^n_{ntu}}$ satisfies assumption (i) and (iii) of Proposition 5.2(a).

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[72] Note that, for collection $\{\tilde{\Gamma}^V_{(c)}\}_{i\in N}$, it is sufficient to require $0 \leq b^V \leq n^V\ \forall\ V \in \mathcal{C}^n_{ntu}$.

[73] As for the third equality, note that, if $g(s) = l^{(0)}$, then $g(s)(V) = (0,\ldots,0) = \tilde{u}^V(s)$ (by definition of $l^{(0)}$ and $g : [0,1]^n \to \mathcal{L}$ below), and thus $\tilde{u}_i^V(s) = 0 < b_i^V\ \forall\ i \in N$.

Up to this point, we have assumed that $0 \leq b^V \leq n^V \; \forall \; V \in \mathcal{C}^n_{ntu}$ (and, sometimes, even $0 \leq b^V_i < n^V_i \; \forall \; (i,V) \in N \times \mathcal{C}^n_{ntu}$ or $0 < b^V_i < n^V_i \; \forall \; (i,V) \in N \times \mathcal{C}^n_{ntu}$).
A point made by Maskin and Moore [27], to justify "the 'point expectation' assumption" implicit in their definition of a "renegotiation process" as a function into outcomes, is the following: "... uncertainty about the realization ... may actually facilitate implementation rather than impede it. This is because even though ... each realization ... is Pareto optimal ... the expected utilities ... (which correspond to a convex combination of the utilities from each realization) may lie in the interior of the utility possibility set." Applied to our context, if each agent's beliefs reflect the expected value of some non-trivial probability distribution over the efficient utility allocations, and if each agent's probability distribution is 'sufficiently pessimistic', then we may well expect each b^V_i to lie between 0 and n^V_i.

If, however, there exists a tuple $(i,V) \in N \times \mathcal{C}^n_{ntu}$ such that $b^V_i > n^V_i$, then, for each $EC \in \{DSE^n_{nfg}, NE^n_{nfg}, SPNE^n\}$, we have that α_L is not strongly ($\Leftrightarrow$ fully) EC-implementable in environment E.
Assume, to the contrary, that α_L is EC-implementable in E.
Since $Nash \neq l^{(0)}$ (on $\mathcal{C}^n_{ntu} \subseteq \mathcal{B}^n$), and, $\forall \; (l,V) \in \mathcal{L} \times \mathcal{C}^n_{ntu}$,

$$R(l,T(V)) = \begin{cases} (l,1) \text{ if } l \neq l^{(0)} \text{ and } u_i((l,1),T(V)) \geq u_i((l,0),T(V)) \; \forall \; i \in N \\ (l,1) \text{ if } l = l^{(0)} \text{ and } \exists \; i \in N \text{ s.t. } u_i((l,1),T(V)) \geq u_i((l,0),T(V)) \\ (l,0) \text{ otw.} \end{cases},$$

consistency of α_L in environment E requires

$$u((Nash,1),T(V)) \geq u((Nash,0),T(V)) \; \forall \; V \in \mathcal{C}^n_{ntu}.$$

However, since $u((Nash,1),T(V)) = Nash(V) = n^V$ and $u((Nash,0),T(V)) = b^V$ for all $V \in \mathcal{C}^n_{ntu}$, this implies that $n^V \geq b^V \; \forall \; V \in \mathcal{C}^n_{ntu}$, contradicting that there exists a tuple $(i,V) \in N \times \mathcal{C}^n_{ntu}$ such that $b^V_i > n^V_i$.

5.3.3 LE Implementation of the Core Concept

Suppose that $\mathcal{C}^n_{ntu}$ satisfies

$$V(N) \text{ is comprehensive } \forall \; V \in \mathcal{C}^n_{ntu} \text{ and}$$
$$\sharp\{V(\{i\})\} \geq 2 \; \forall \; (i,V) \in N \times \mathcal{C}^n_{ntu},$$

that $e \equiv e^c$, and that, $\forall \; V \in \mathcal{C}^n_{ntu}$, solution concept L satisfies
$L(V) = Core(V) \equiv \{u \in V(N) \mid \nexists \; S \in \mathcal{N} \text{ for which } \exists \; u' \in V(S) \text{ s.t. } u'_i > u_i \; \forall \; i \in S\}$.
Furthermore, suppose that agents' beliefs can be justified by prediction functions $\{b_i\}_{i \in N}$ for $(\mathcal{L}, T(V))$, i.e.,

$$u_i((l,0),T(V)) = u_i((b_{i1}((l,0),T(V)),1),T(V)) \; \forall \; (i,V,l) \in N \times \mathcal{C}^n_{ntu} \times \mathcal{L},$$

and that these prediction functions are outcome-independent, i.e.,

$$b_i((l,0),T(V)) = b_i((l',0),T(V)) \;\forall\; (i,l,l',V) \in N \times \mathcal{L} \times \mathcal{L} \times \mathcal{C}^n_{ntu}.$$

Define, $\forall\; (i,V) \in N \times \mathcal{C}^n_{ntu}$, $b^{i,V} \in \mathbb{R}^N$ and $b^V_i \in \mathbb{R}$ as in the preceding paragraph. And, suppose that, $\forall\; V \in \mathcal{C}^n_{ntu}$, the set of efficient Core elements above $b^V$is nonempty, i.e.

$$\widehat{EfCore}(V,b^V) := \{u \in Core(V) \mid u \geq b^V, \nexists\; u' \in V(N) \text{ s.t. } u' \geq u \text{ and } u' \neq u\} \neq \emptyset,$$

and satisfies $\widehat{EfCore}(V,b^V) \cap \{u \in \mathbb{R}^N \mid u_i > b^V_i \;\forall\; i \in N\} \neq \emptyset$.

We now introduce a collection of n-person normal form games $\{\tilde{\Gamma}^V\}_{V \in \mathcal{C}^n_{ntu}}$ satisfying the assumptions of Corollary 5.2(a) for $EC = \widehat{NE}^n_{nfg}$, where $\widehat{NE}^n_{nfg}(\Gamma)$ denotes the *payoff-dominant* Nash Equilibria of game Γ, i.e.

$$\widehat{NE}^n_{nfg}(\Gamma) := \{s \in S_1 \times \ldots \times S_n \mid s \in NE^n_{nfg}(\Gamma) \text{ and } \nexists\; s' \in NE^n_{nfg}(\Gamma) \text{ such that } \tilde{u}_i(s') > \tilde{u}_i(s) \;\forall\; i \in N\}$$

for all n-person normal form games $\Gamma \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}_i\}_{i \in N})$.

The concept of "payoff-dominance" as an equilibrium selection criterion was introduced by Harsanyi and Selten [16], and is discussed, for example, by Van Huyck, Battalio, and Beil [54a, 54b, 54c], and Fudenberg and Tirole [12] (Paragraph 1.2.4).

For each $V \in \mathcal{C}^n_{ntu}$, define $\tilde{\Gamma}^V \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})$ by $S_i := [0,\infty) \;\forall\; i \in N$ and

$$\tilde{u}^V_i(s) := \begin{cases} s_i \;\text{ if } s \in Core(V) \text{ and } s \geq b^V \\ b^V_i \;\text{ otw.} \end{cases} \quad \forall\; (i,s) \in N \times [0,\infty)^n.$$

Then, $\forall\; V \in \mathcal{C}^n_{ntu}$, $s \in \widehat{EfCore}(V,b^V) \Leftrightarrow s \in \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V) \;\forall\; s \in [0,\infty)^n$:

'$\Rightarrow$' Let $s \in \widehat{EfCore}(V,b^V)$, and assume that there exists an $i \in N$ and a strategy $s'_i \in S_i$ s.t. $\tilde{u}^V_i(s'_i, s_{-i}) > \tilde{u}^V_i(s)$.
Since $s \in Core(V)$ and $s \geq b^V$, we have that $\tilde{u}^V_i(s) = s_i \geq b^V_i$.
By definition of $\tilde{u}^V_i$, $\tilde{u}^V_i(s'_i, s_{-i}) > s_i \geq b^V_i$ implies that

$$(s'_i, s_{-i}) \in Core(V),\; (s'_i, s_{-i}) \geq b^V, \text{ and } s'_i > s_i,$$

contradicting $s \in \widehat{EfCore}(V,b^V)$.
Thus, $s \in NE^n_{nfg}(\tilde{\Gamma}^V)$.
Now, assume $\exists\; s' \in NE^n_{nfg}(\tilde{\Gamma}^V)$ such that $\tilde{u}^V_i(s') > \tilde{u}^V_i(s) \;\forall\; i \in N$.
Since $s \in Core(V)$ and $s \geq b^V$, we have that $\tilde{u}^V_i(s) = s_i \geq b^V_i \;\forall\; i \in N$. Therefore, $\tilde{u}^V_i(s') > s_i \geq b^V_i \;\forall\; i \in N$, and, by definition of $\tilde{u}^V_i$, we have that $s' \in Core(V)$, $s' \geq b^V$, and $\tilde{u}^V_i(s') = s'_i \;\forall\; i \in N$.
Thus, $s' \in Core(V) \subseteq V(N)$ and $s'_i = \tilde{u}^V_i(s') > \tilde{u}^V_i(s) = s_i \;\forall\; i \in N$, contradicting $s \in Core(V)$.
It follows that $s \in \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V)$.

'$\Leftarrow$' Let $s \in [0,\infty)^n \backslash \widehat{EfCore}(V, b^V)$.

First, suppose that $s \in Core(V)$ and $s \geq b^V$:

Since $s \notin \widehat{EfCore}(V, b^V)$, there exists an $u \in V(N)$ s.t. $u \geq s$ and $u \neq s$.

Since $V(N)$ is comprehensive, it follows that there exists an $i \in N$ and an $u' \in V(N)$ s.t. $u'_i > s_i$ and $u'_j = s_j \ \forall\ j \in N\backslash\{i\}$.

And, $s \in Core(V)$ implies that $u' \in Core(V)$.[74]

Thus, $\tilde{u}_i^V(u'_i, s_{-i}) = \tilde{u}_i^V(u') = u'_i > s_i = \tilde{u}_i^V(s)$, i.e., $s \notin NE^n_{nfg}(\tilde{\Gamma}^V)$.

In particular, $s \notin \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V)$.

Next, suppose that $s \notin Core(V)$ or $s \ngeq b^V$:

By definiton of $\tilde{u}_i^V$, we have that $\tilde{u}_i^V(s) = b_i^V \ \forall\ i \in N$.

Let $s^* \in \widehat{EfCore}(V, b^V) \cap \{u \in \mathbb{R}^N \mid u_i > b_i^V \ \forall\ i \in N\}$ ($\neq \emptyset$ by assumption). According to '$\Rightarrow$', we have that $s^* \in \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V)$.

Since $\tilde{u}_i^V(s^*) = s_i^* > b_i^V = \tilde{u}_i^V(s) \ \forall\ i \in N$, we have that $s \notin \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V)$.

And, collection $\{\tilde{\Gamma}^V\}_{V \in \mathcal{C}^n_{ntu}}$ satisfies the assumptions of Corollary 5.2(a) for $EC = \widehat{NE}^n_{nfg}$:

For each $V \in \mathcal{C}^n_{ntu}$, we have that

$$\widehat{NE}^n_{nfg}(\tilde{\Gamma}^V) \neq \emptyset \text{ (since } \widehat{NE}^n_{nfg}(\tilde{\Gamma}^V) = \widehat{EfCore}(V, b^V) \neq \emptyset),$$

$$\tilde{u}^V(\widehat{NE}^n_{nfg}(\tilde{\Gamma}^V)) = \tilde{u}^V(\widehat{EfCore}(V, b^V)) = \widehat{EfCore}(V, b^V) \subseteq Core(V) = L(V),$$

$$\tilde{u}^V(s) = \begin{cases} s & \text{if } s \in Core(V) \text{ and } s \geq b^V \\ b^V & \text{otw.} \end{cases} \in \begin{cases} Core(V) \\ V(N) \end{cases}$$

$$\subseteq V(N) \subseteq F_{up}(V) \ \forall\ s \in [0,\infty)^n,$$[75] and

$$R(g(s), T(V)) = \begin{cases} (g(s), 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } g(s) \in e(s) \text{ and} \\ & u_i((g(s),1), T(V)) \geq u_i((g(s),0), T(V)) \\ & \forall\ i \in S \cap N \\ (g(s), 0) & \text{otw.} \end{cases}$$

$$= (g(s), 1) \ \forall\ s \in [0,\infty)^n$$

$$\text{(since, } \forall\ i \in N,\ u_i((g(s),1), T(V)) = (g(s)(V))_i = \tilde{u}_i^V(s) \geq b_i^V = u_i((g(s),0), T(V))\text{)},$$

where $g : [0,\infty)^n \to \mathcal{L}$ is defined by $g(s)(V) := \tilde{u}^V(s) \ \forall\ s \in [0,\infty)^n$.

---

[74]Assume that $\exists\ S \in \mathcal{N}$ for which $\exists\ u'' \in V(S)$ s.t. $u''_i > u'_i \ \forall\ i \in S$. Then, $u''_i > u'_i \geq s_i \ \forall\ i \in S$, contradicting $s \in Core(V)$.

[75]Note that, since $\widehat{EfCore}(V, b^V) \neq \emptyset$, there exists an $u \in Core(V) \subseteq V(N)$ such that $u \geq b^V$. Since $V(N)$ is comprehensive, we have that $b^V \in V(N)$.

5.4 Classical and LE Implementation of Solution Concepts for Cooperative Games with Physical Outcomes

5.4.1 Classical Implementation

The following approach is due to Dagan and Serrano [9]. Proposition 5.3 reflects their assertion that every solution concept which is fully implementable by an ordinally invariant equilibrium concept must be ordinally invariant (Dagan and Serrano [9], Result 2). Adapting the proof in Hahmeier [15] to our definitions and notation, we briefly sketch a proof in Appendix G.

Let $\mathcal{C}^n_{po} \equiv \{\Gamma^\theta \equiv (N, \bar{X}, \{u_i^{\theta_i}\}_{i\in N})\}_{\theta\in\Theta}$ be a nonempty set of n-person cooperative games with physical outcomes (all sharing the same game form $(N, \bar{X})$), where,

$\forall\ i \in N$, Θ_i is the set of possible types for agent i, and

$\Theta \subseteq \Theta_1 \times \ldots \times \Theta_n$ is the set of possible type profiles / states.

Define the classical n-person environment $E \equiv E(\mathcal{C}^n_{po}) \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ by $X := F_o(\bar{X})$,

$u'_i(q, \theta) := u_i^{\theta_i}(q)\ \forall\ (i, q, \theta) \in N \times F_o(\bar{X}) \times \Theta$, and

$\mathcal{G}$ is the set of all strategic and extensive mechanisms for $(N, F_o(\bar{X}))$.

For each nonempty-valued solution concept ψ for $\mathcal{C}^n_{po}$, let $\alpha_\psi : \Theta \Rightarrow F_o(\bar{X})$ denote the (social) choice correspondence for environment E defined by $\alpha_\psi(\theta) := \psi(\Gamma^\theta)\ \forall\ \theta \in \Theta$.

Let ψ be a nonempty-valued solution concept for $\mathcal{C}^n_{po}$, and let $EC \in \{NE^n_{nfg}, DSE^n_{nfg}\}$.

Proposition 5.3 (Dagan and Serrano [9])

(i) If there exists a strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ that fully EC-implements α_ψ in environment E, then ψ is ordinally invariant on $\mathcal{C}^n_{po}$, i.e., $\psi(\Gamma^{\theta'}) = \psi(\Gamma^\theta)\ \forall\ (\theta, \theta') \in \Theta \times \Theta$ such that $\Gamma^{\theta'}$ is an order preserving transformation of Γ^θ.[76]

(ii) If there exists an n-person extensive form mechanism with perfect information $G \equiv (N, H, p, g) \in \mathcal{G}$ that fully $SPNE^n$-implements α_ψ in environment E, then ψ is ordinally invariant on $\mathcal{C}^n_{po}$, i.e., $\psi(\Gamma^{\theta'}) = \psi(\Gamma^\theta)\ \forall\ (\theta, \theta') \in \Theta \times \Theta$ such that $\Gamma^{\theta'}$ is an order preserving transformation of Γ^θ.

In contrast to solution concept ψ^{Core}, which is ordinally invariant on every set of cooperative games with physical outcomes (sharing the same game form), ψ^{Nash} might not be ordinally invariant on $\mathcal{C}^n_{po}$.[77] Thus, Proposition 5.3 represents a necessary condition

[76] I.e. $\psi(\Gamma^{\theta'}) = \psi(\Gamma^\theta)\ \forall\ (\theta, \theta') \in \Theta\times\Theta$ satisfying $u'_i(q, \theta') > u'_i(\hat{q}, \theta') \Leftrightarrow u'_i(q, \theta) > u'_i(\hat{q}, \theta)\ \forall\ (i, q, \hat{q}) \in N \times F_o(\bar{X}) \times F_o(\bar{X})$.

[77] Remember our discussion in Paragraph 5.1.3.

for the implementation of $\alpha_{\psi^{Nash}}$.[78]
Remark 5.3 follows from the proof of the preceding proposition. However, this 'version' of Proposition 5.3 allows for an extension to environments with limited enforcement power, which will be presented in the following paragraph.

Remark 5.3

(i) If there exists a strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ that fully EC-implements α_ψ in environment E, then $\psi(\Gamma^\theta) = \psi(\Gamma^{\theta'})\ \forall\ (\theta, \theta') \in \Theta\times\Theta$ such that $\Gamma^{E,G,\theta'}$ is an order preserving transformation of $\Gamma^{E,G,\theta}$, i.e. $\forall\ (\theta,\theta') \in \Theta\times\Theta$ satisfying $u_i'(g(s),\theta') > u_i'(g(\hat{s}),\theta') \Leftrightarrow u_i'(g(s),\theta) > u_i'(g(\hat{s}),\theta)\ \forall\ (i,s,\hat{s}) \in N \times S \times S$, where $S := S_1 \times \ldots \times S_n$.

(ii) If there exists an n-person extensive form mechanism with perfect information $G \equiv (N, H, p, g) \in \mathcal{G}$ that fully $SPNE^n$-implements α_ψ in environment E, then $\psi(\Gamma^\theta) = \psi(\Gamma^{\theta'})\ \forall\ (\theta,\theta') \in \Theta \times \Theta$ such that $\Gamma^{E,G,\theta'}$ is an order preserving transformation of $\Gamma^{E,G,\theta}$, i.e. $\forall\ (\theta,\theta') \in \Theta \times \Theta$ satisfying $u_i'(g(h),\theta') > u_i'(g(\hat{h}),\theta') \Leftrightarrow u_i'(g(h),\theta) > u_i'(g(\hat{h}),\theta)\ \forall\ (i,h,\hat{h}) \in N \times Z_H \times Z_H$.

5.4.2 LE Implementation

Let $\mathcal{C}^n_{po} \equiv \{\Gamma^\theta \equiv (N, \bar{X}, \{u_i^{\theta_i}\}_{i\in N})\}_{\theta\in\Theta}$ be a nonempty set of n-person cooperative games with physical outcomes (all sharing the same game form $(N, \bar{X})$), where,

$\forall\ i \in N$, Θ_i is the set of possible types for agent i, and

$\Theta \subseteq \Theta_1 \times \ldots \times \Theta_n$ is the set of possible type profiles / states,

such that either

(i) $\sharp\{\bar{X}(\{i\})\} \geq 2\ \forall\ i \in N$, or

(ii) each $\Gamma^\theta \in \mathcal{C}^n_{po}$ is a bargaining game with physical outcomes.

Let $E \equiv (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u_i\}_{i\in N}, \mathcal{G}, R, e)$ be an an n-person environment with limited enforcement power such that

$X = F_o(\bar{X})$,

$u_i((q,1),\theta) = u_i^{\theta_i}(q)\ \forall\ (i,q,\theta) \in N \times F_o(\bar{X}) \times \Theta$,

$\mathcal{G}$ is the set of all strategic and extensive mechanisms for $(N, F_o(\bar{X}))$, and

$e \equiv e^c$ in case (i) and $e \equiv e^b$ in case (ii),

---

[78]Note that ordinally invariance of ψ^{Nash} does not necessarily imply Maskin-monotonicity of $\alpha_{\psi^{Nash}}$. In Appendix F, adapting an example in Hahmeier [15] to our definitions and notation, we define a set $\mathcal{C}^n_{po}$ of bargaining games with physical outcomes (sharing the same game form) on which ψ^{Nash} is ordinally invariant but $\alpha_{\psi^{Nash}}$ is not Maskin-monotonic in environment $E(\mathcal{C}^n_{po})$. Our example is based on a similar example / comparable result by Howard [17].

where e^c and e^b denote the two enforcement structures for $(N, F_o(\bar{X}))$ that satisfy

$$e^c(S) = \begin{cases} F_o(\bar{X}) & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{otw.} \end{cases} \quad \forall\, S \in \mathcal{N}^+ \quad \text{and}$$

$$e^b(S) = \begin{cases} F_o(\bar{X}) & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{(N, x_0)\} & \text{otw.} \end{cases} \quad \forall\, S \in \mathcal{N}^+,$$

and $x_0 \in \bar{X}(N)$ denotes the outcome that satisfies $\bar{X}(S) = \{x_0\}\ \forall\, S \in \mathcal{N}\backslash\{N\}$.
For each nonempty-valued solution concept ψ for $\mathcal{C}^n_{po}$, let $\alpha_\psi : \Theta \Rightarrow F_o(\bar{X})$ denote the (social) choice correspondence for environment E defined by $\alpha_\psi(\theta) := \psi(\Gamma^\theta)\ \forall\, \theta \in \Theta$. Proposition 5.4 indicates that a (social) choice correspondence α_ψ might be implementable in an environment with limited enforcement power without ψ being ordinally invariant. In contrast to Proposition 5.3, our extension to environments with limited enforcement power represents a necessary condition for both $\alpha_{\psi^{Nash}}$ and $\alpha_{\psi^{Core}}$.[79]
Let ψ be a nonempty-valued solution concept for $\mathcal{C}^n_{po}$, and let $EC \in \{NE^n_{nfg}, DSE^n_{nfg}\}$.

Proposition 5.4

(i) If there exists a strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g) \in \mathcal{G}$ that fully EC-implements α_ψ in environment E, then $\psi(\Gamma^\theta) = \psi(\Gamma^{\theta'})\ \forall\, (\theta, \theta') \in \Theta\times\Theta$ such that $\Gamma^{E^*,G^*,\theta'}$ is an order preserving transformation of $\Gamma^{E^*,G^*,\theta}$, i.e., $\forall\, (\theta,\theta') \in \Theta \times \Theta$ satisfying $u_i(R(g(s),\theta'),\theta') > u_i(R(g(\hat{s}),\theta'),\theta') \Leftrightarrow u_i(R(g(s),\theta),\theta) > u_i(R(g(\hat{s}),\theta),\theta)\ \forall\, (i,s,\hat{s}) \in N \times S \times S$, where $S := S_1 \times \ldots \times S_n$.[80]

(ii) If there exists an n-person extensive form mechanism with perfect information $G \equiv (N, H, p, g) \in \mathcal{G}$ that fully $SPNE^n$-implements α_ψ in environment E, then $\psi(\Gamma^\theta) = \psi(\Gamma^{\theta'})\ \forall\, (\theta,\theta') \in \Theta \times \Theta$ such that $\Gamma^{E^*,G^*,\theta'}$ is an order preserving transformation of $\Gamma^{E^*,G^*,\theta}$, i.e., $\forall\, (\theta,\theta') \in \Theta \times \Theta$ satisfying $u_i(R(g(h),\theta'),\theta') > u_i(R(g(\hat{h}),\theta'),\theta') \Leftrightarrow u_i(R(g(h),\theta),\theta) > u_i(R(g(\hat{h}),\theta),\theta)\ \forall\, (i,h,\hat{h}) \in N \times (Z_H)^2$.

Proof

(i) Consider a tuple $(\theta,\theta') \in \Theta \times \Theta$ such that $\Gamma^{E^*,G^*,\theta'}$ is an order preserving transformation of $\Gamma^{E^*,G^*,\theta}$.

Since EC is ordinally invariant (on every set of n-person normal form games that share the same game form), we have that $EC(\Gamma^{E^*,G^*,\theta'}) = EC(\Gamma^{E^*,G^*,\theta})$.

[79] Note that, in general, our necessary condition depends on the environment via its realization function and also on the mechanism's outcome function.

[80] Remember that, $\forall\, \theta \in \Theta$, $\Gamma^{E^*,G^*,\theta} \equiv (N, \{S_i\}_{i\in N}, \{u_i(R(g(\cdot),\theta),\theta)\}_{i\in N})$ denotes the game induced by mechanism G^* and state θ in environment E^*.

Thus, $g(EC(\Gamma^{E^*,G^*,\theta'})) = g(EC(\Gamma^{E^*,G^*,\theta}))$.

Since mechanism G fully EC-implements α_ψ in environment E, we have that $\{(q,1) \mid q \in \alpha_\psi(\theta')\} = R(g(EC(\Gamma^{E^*,G^*,\theta'})),\theta')$, and $\{(q,1) \mid q \in \alpha_\psi(\theta)\} = R(g(EC(\Gamma^{E^*,G^*,\theta})),\theta)$.

In particular, $R(g(EC(\Gamma^{E^*,G^*,\theta'})),\theta') = \{(q,1) \mid q \in g(EC(\Gamma^{E^*,G^*,\theta'}))\}$, and $R(g(EC(\Gamma^{E^*,G^*,\theta})),\theta) = \{(q,1) \mid q \in g(EC(\Gamma^{E^*,G^*,\theta}))\}$.

It follows from the preceding that

$$\begin{aligned}
\{(q,1) \mid q \in \psi(\Gamma^{\theta'})\} &= \{(q,1) \mid q \in \alpha_\psi(\theta')\} \\
&= R(g(EC(\Gamma^{E^*,G^*,\theta'})),\theta') \\
&= \{(q,1) \mid q \in g(EC(\Gamma^{E^*,G^*,\theta'}))\} \\
&= \{(q,1) \mid q \in g(EC(\Gamma^{E^*,G^*,\theta}))\} \\
&= R(g(EC(\Gamma^{E^*,G^*,\theta})),\theta) \\
&= \{(q,1) \mid q \in \alpha_\psi(\theta)\} \\
&= \{(q,1) \mid q \in \psi(\Gamma^{\theta})\}.
\end{aligned}$$

(ii) Consider a tuple $(\theta,\theta') \in \Theta \times \Theta$ such that $\Gamma^{E^*,G^*,\theta'}$ is an order preserving transformation of $\Gamma^{E^*,G^*,\theta}$.

Since $SPNE^n$ is ordinally invariant (on every set of n-person extensive form games with perfect information that share the same game form), we have that $SPNE^n(\Gamma^{E^*,G^*,\theta'}) = SPNE^n(\Gamma^{E^*,G^*,\theta})$.

Thus, $g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'}))) = g(O(SPNE^n(\Gamma^{E^*,G^*,\theta})))$.

Since mechanism G fully $SPNE^n$-implements α_ψ in environment E, we have that $\{(q,1) \mid q \in \alpha_\psi(\theta')\} = R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'}))),\theta')$, and

$$\{(q,1) \mid q \in \alpha_\psi(\theta)\} = R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta}))),\theta).$$

In particular,

$R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'}))),\theta') = \{(q,1) \mid q \in g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'})))\}$, and $R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta}))),\theta) = \{(q,1) \mid q \in g(O(SPNE^n(\Gamma^{E^*,G^*,\theta})))\}$.

It follows from the preceding that

$$\begin{aligned}
\{(q,1) \mid q \in \psi(\Gamma^{\theta'})\} &= \{(q,1) \mid q \in \alpha_\psi(\theta')\} \\
&= R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'}))),\theta') \\
&= \{(q,1) \mid q \in g(O(SPNE^n(\Gamma^{E^*,G^*,\theta'})))\} \\
&= \{(q,1) \mid q \in g(O(SPNE^n(\Gamma^{E^*,G^*,\theta})))\} \\
&= R(g(O(SPNE^n(\Gamma^{E^*,G^*,\theta}))),\theta) \\
&= \{(q,1) \mid q \in \alpha_\psi(\theta)\} \\
&= \{(q,1) \mid q \in \psi(\Gamma^{\theta})\}.
\end{aligned}$$

□

6 Concluding Remarks

As stated at the beginning of this paper, implementation theory is concerned with the question of which (social) choice correspondences can be implemented by the use of certain mechanisms in certain environments. The standard theory's assumptions on the enforcement structure, however, are too restrictive for many applications. Our approach (of implementation in environments with limited enforcement power) accounts for this by explicitly introducing a 'variable' enforcement structure describing the enforcement capabilities on outcomes as a function of all coalitions of individuals. The future is thereby not explicitly modeled but implicitly summarized in each agent's beliefs about what will happen if an outcome suggested by a mechanism is not being implemented.

Throughout the preceding chapters, we have assumed that the designer knows these beliefs in dependence upon the state of the environment, that is, the designer knows the realization function. And, although this might be a reasonable assumption for a variety of situations, it might not cover others.

Amorós [2] "studies Nash implementation when the outcomes of the mechanism can be renegotiated among the agents but the planner does not know the renegotiation function that they will use." Amorós assumes "that there exists a set of admissible renegotiation functions" (such that "(1) renegotiated outcomes are always Pareto-efficient and, (2) no agent ends up worse off after renegotiating") and proposes "a new form of implementation where the same mechanism must work for every admissible renegotiation function."[81] In particular, he extends Jackson and Palfrey's [24a] results on sufficient and necessary conditions for the implementation in Nash Equilibrium to his setting.

Analogously, in our approach, instead of assuming that the designer knows the realization function, a set of *possible beliefs* for each player in each state of the environment would imply a set of *possible realization functions* and would allow for an extension of Amorós's results to environments with limited enforcement power.[82] An analysis into

[81]As Amorós notes, "alternatively, this could be interpreted as an enlargement of the set of possible states. As Maskin and Moore (1999) argue, two states s and s' might be identical in preferences and differ only in terms of how renegotiation would proceed. We prefer to model the set of admissible renegotiation functions seperately in order to illustrate its effect on the set of implementable social choice rules."

[82]This is in line with Amorós's concluding remark: "Another line of research could involve to extend our analysis to the case in which the no enforcement of the mechanism is due to individual rationality constraints."

this direction could result in a classification of mechanisms/support results according to the set of agents' beliefs for which they work.
Amorós sees "some scope for further development and extension" of his model: "One line of research could involve to study the case in which the true renegotiation function is unknown not only to the planner, but also to the agents ...". Correspondingly, we could extend our approach by deviating from the assumption that all agents are completely informed about the actual state of the environment. Whereas the complete information (between the agents) assumption might be a good starting point for research in this area, in specific applications it may be unrealistic, for example, to assume that every agent knows all other agents' beliefs.[83]

Another important question not answered in the present paper is the following: what are *reasonable beliefs*? Probably, an answer to this question can only be based on more information about the specific environment under consideration. An interesting aspect, however, is present in Jackson and Palfrey [24a], who "endogenize the generalized reversion function" by analysing a model that allows each individual to either accept the outcome suggested by a strategic mechanism or to veto and thereby forcing the mechanism to be replayed. Similarly, in our model (of LE implementation), agents' beliefs could be endogenized by analysing such a mechanism with respect to certain assumptions on the behaviour of the agents (e.g., in the form of an equilibrium concept).
Jackson and Palfrey [24a] note that, "more generally, a veto might trigger an alternative mechanism which is played." Following this direction, one could analyse mechanisms that allow agents to actually implement an outcome and that always continue or that might continue if this opportunity is not taken. Such an analysis could extend our implementation results in both environments with limited and with delegative enforcement power, thereby covering applications in which the enforcement capabilities of the designer allow for such *active mechanisms*.[84] And, in the words of Jackson and Palfrey, "there is a rich array of applications where dynamics is a crucial element, ranging from the operation of continuous trading institutions to the rules governing electoral and legislative institutions."

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[83]There is an extensive literature on the implementation of (social) choice correspondences in classical environments with incomplete information. Cf. Jackson [22], pp. 691-693.

[84]Note that this kind of mechanism requires outcomes to be verifiable.

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Appendix

A Proof for Lemma 2.3

For each $(i, x) \in N \times X$ and every $\succsim_i \in \mathcal{R}_X$, define $L(x, \succsim_i) := \{y \in X \mid x \succsim_i y\}$.

(i) We have that $f(\succsim) = f(\succsim') \; \forall \; (\succsim, \succsim') \in \mathcal{P} \times \mathcal{P}$ satisfying

$$\exists \; h \in N \text{ s.t. } \succsim_{-h} = \succsim'_{-h} \text{ and } L(f(\succsim), \succsim_h) \subseteq L(f(\succsim), \succsim'_h):$$[85]

Suppose not, i.e., suppose that there are $(\succsim, \succsim') \in \mathcal{P} \times \mathcal{P}$ such that

$f(\succsim) \neq f(\succsim')$, and

$\exists \; h \in N$ s.t. $\succsim_{-h} = \succsim'_{-h}$ and $L(f(\succsim), \succsim_h) \subseteq L(f(\succsim), \succsim'_h)$.

Define $x := f(\succsim)$ and $y := f(\succsim')$.

Since $x \neq y$, and since no two distinct alternatives are indifferent, we have that either $y \succ_h x$ or $x \succsim_h y$.

Assumption (b) implies that $x = f(\succsim_h, \succsim_{-h}) \succsim_h f(\succsim'_h, \succsim_{-h}) = f(\succsim') = y$, contradicting the first alternative.

If $x \succsim_h y$, then, since $L(f(\succsim), \succsim_h) \subseteq L(f(\succsim), \succsim'_h)$, we have that $x \succsim'_h y$. Furthermore, since $x \neq y$, and since no two distinct alternatives are indifferent, it follows that $x \succ'_h y$. However, assumption (b) implies that $y = f(\succsim'_h, \succsim'_{-h}) \succsim'_h f(\succsim_h, \succsim'_{-h}) = f(\succsim_h, \succsim_{-h}) = x$, a contradiction.

(ii) We have that f is "monotonic" (Mas-Colell et al. [29], Definition 21.E.4), i.e., $f(\succsim) = f(\succsim') \; \forall \; (\succsim, \succsim') \in \mathcal{P} \times \mathcal{P}$ satisfying $L(f(\succsim), \succsim_i) \subseteq L(f(\succsim), \succsim'_i) \; \forall \; i \in N$:[86]

Let $(\succsim, \succsim') \in \mathcal{P} \times \mathcal{P}$ satisfy $L(f(\succsim), \succsim_i) \subseteq L(f(\succsim), \succsim'_i) \; \forall \; i \in N$.

Since $L(f(\succsim), \succsim_1) \subseteq L(f(\succsim), \succsim'_1)$, (i) implies that $f(\succsim) = f(\succsim'_1, \succsim_2, \ldots, \succsim_N)$.

It follows that

$$\begin{aligned} L(f(\succsim'_1, \succsim_{-1}), \succsim_2) &= L(f(\succsim), \succsim_2) \\ &\subseteq L(f(\succsim), \succsim'_2) = L(f(\succsim'_1, \succsim_{-1}), \succsim'_2), \end{aligned}$$

and, again, (i) implies that $f(\succsim'_1, \succsim_2, \ldots, \succsim_N) = f(\succsim'_1, \succsim'_2, \succsim_3, \ldots, \succsim_N)$.

An iteration of this process leads to the result that

$$f(\succsim) = f(\succsim'_1, \succsim_2, \ldots, \succsim_N) = \ldots = f(\succsim'_1, \ldots, \succsim'_{N-1}, \succsim_N) = f(\succsim').$$

(iii) We have that f is "weakly Paretian" (Mas-Colell et al. [29], Definition 21.E.2), i.e. $\forall \; \succsim \in \mathcal{P} \; \nexists \; x \in X$ s.t. $x \succ_i f(\succsim) \; \forall \; i \in N$:[87]

Suppose not, i.e., suppose that there exists a tuple $(\succsim, x) \in \mathcal{P} \times X$ such that $x \succ_i f(\succsim) \; \forall \; i \in N$.

Assumption (a) implies that there exists a profile $\succsim' \in \mathcal{P}$ such that $f(\succsim') = x$.

[85] Cf. Mas-Colell, Whinston, and Green [29], proof of Proposition 21.E.2

[86] Cf. Mas-Colell, Whinston, and Green [29], proof of Proposition 23.C.3 (Step 1).

[87] Cf. Mas-Colell, Whinston, and Green [29], proof of Proposition 23.C.3 (Step 2).

Since $\mathcal{P} = (\mathcal{R}_X)^N$, there exists a profile $\succsim'' \in \mathcal{P}$ such that, for every $i \in N$, $x \succ''_i f(\succsim) \succ''_i z \ \forall \ z \in X \backslash \{f(\succsim), x\}$.

Since $L(x, \succsim'_i) \subseteq X = L(x, \succsim''_i) \ \forall \ i \in N$, monotonicity of f implies that $f(\succsim') = f(\succsim'')$, i.e., that $x = f(\succsim'')$.

Since $L(f(\succsim), \succsim_i) \subseteq X = L(f(\succsim), \succsim''_i) \ \forall \ i \in N$, monotonicity of f implies that $f(\succsim) = f(\succsim'')$.

Thus, $f(\succsim) = f(\succsim'') = x$ — a contradiction to $x \succ_i f(\succsim) \ \forall \ i \in N$.

(iv) Mas-Colell et al. [29], Proposition 21.E.1, implies that f is "dictatorial", i.e. $\exists \ j \in N$ such that $\forall \succsim \in \mathcal{P}$ we have that $f(\succsim) \succsim_j x' \ \forall \ x' \in X$.

B Proofs for Proposition 2.2 and 3.3, and Lemma 3.3

B.1 Proof for Proposition 2.2

Consider the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i \in N}, g)$ defined as follows.
Define $S_i := \{(t_i, x_i, m_i) \mid t_i \in \Theta, x_i \in X, m_i \in \mathbb{N}_0\} \ \forall \ i \in N$.
For all $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$ that satisfy

$\exists \ (j, \theta, x, m) \in N \times \Theta \times X \times \mathbb{N}_0$ s.t.

$x \in \alpha(\theta)$ and $(t_i, x_i, m_i) = (\theta, x, m) \ \forall \ i \in N \backslash \{j\}$,

define

$$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := \begin{cases} x_j & \text{if } u'_j(x, \theta) \geq u'_j(x_j, \theta) \\ x & \text{otw.} \end{cases} .$$

For all other $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$, define $g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := x_k$, where $k \in N$ satisfies $m_k \geq m_i \ \forall \ i \in N$.
To see that mechanism G fully NE^n_{nfg}-implements α in environment E, i.e., to see that $g(NE^n_{nfg}(\Gamma^{E,G,\theta})) = \alpha(\theta) \ \forall \ \theta \in \Theta$, consider an arbitrary $\theta \in \Theta$.

'$\supseteq$'

Let $x \in \alpha(\theta)$, let $s_i := (\theta, x, 0) \ \forall \ i \in N$, and $s := (s_1, \ldots, s_n)$. Then, $g(s) = x$.
To see that s constitutes a Nash Equilibrium of the game induced by mechanism G and type profile θ in environment E (i.e., that $s \in NE^n_{nfg}(\Gamma^{E,G,\theta})$), consider an arbitrary deviation by an arbitrary player $j \in N$, say to $s'_j := (\theta', x', m')$:
Since

$$u'_j(g(s'_j, s_{-j}), \theta) = \begin{cases} u'_j(x', \theta) & \text{if } u'_j(x, \theta) \geq u'_j(x', \theta) \\ u'_j(x, \theta) & \text{otw.} \end{cases} \leq u'_j(x, \theta) = u'_j(g(s), \theta),$$

player j's deviation is not profitable.

'$\subseteq$'

Let $s^* \equiv (s_1^*, \ldots, s_n^*) \in NE^n_{nfg}(\Gamma^{E,G,\theta})$, and define $x^* := g(s^*)$. To see that $x^* \in \alpha(\theta)$, consider the following four cases.

(a) Suppose that $\exists\ (\theta', x', m') \in \Theta \times X \times \mathbb{N}_0$ such that $s_i^* = (\theta', x', m')\ \forall\ i \in N$. This implies that $x^* \equiv g(s^*) = x'$, i.e., that $s_i^* = (\theta', x^*, m')$.

(a.1) Suppose that $\theta' = \theta$.

Assume that $x^* \notin \alpha(\theta)$ and that there exists a tuple $(i, x'') \in N \times X$ such that $u_i'(x'', \theta) > u_i'(x^*, \theta)$.

To see that s^* does not constitute a NE of the game induced by mechanism G and type profile θ in environment E (implying a contradiction), consider player i's deviation to $s_i := (\theta, x'', m'')$ for some $m'' > m'$:

Since $g(s_i, s_{-i}^*) = x''$, we have that

$u_i'(g(s^*), \theta) = u_i'(x^*, \theta) < u_i'(x'', \theta) = u_i'(g(s_i, s_{-i}^*), \theta)$,

i.e., player i can profitably deviate.

Thus, $x^* \in \alpha(\theta)$ or $u_i'(x^*, \theta) \geq u_i'(x'', \theta)\ \forall\ (i, x'') \in N \times X$. Since α satisfies no-veto-power in environment E, also the latter condition implies that $x^* \in \alpha(\theta)$.

(a.2) Suppose that $\theta' \neq \theta$ and that $x^* \in \alpha(\theta')$.

Assume that $x^* \notin \alpha(\theta)$. Maskin-monotonicity of α implies that there exists a tuple $(i, y) \in N \times X$ such that $u_i'(x^*, \theta') \geq u_i'(y, \theta')$ and $u_i'(x^*, \theta) < u_i'(y, \theta)$.

To see that s^* does not constitute a NE of the game induced by mechanism G and type profile θ in environment E (implying a contradiction), consider player i's deviation to $s_i := (\theta, y, 0)$:

Since

$$g(s_i, s_{-i}^*) = \begin{cases} y & \text{if } u_i'(x^*, \theta') \geq u_i'(y, \theta') \\ x^* & \text{otw.} \end{cases} = y,$$

we have that $u_i'(g(s^*), \theta) = u_i'(x^*, \theta) < u_i'(y, \theta) = u_i'(g(s_i, s_{-i}^*), \theta)$,

i.e., player i can profitably deviate.

(a.3) Suppose that $\theta' \neq \theta$ and that $x^* \notin \alpha(\theta')$.

Assume that $\exists\ (i, x'') \in N \times X$ such that $u_i'(x'', \theta) > u_i'(x^*, \theta)$.

To see that s^* does not constitute a NE of the game induced by mechanism G and type profile θ in environment E (implying a contradiction), consider player i's deviation to $s_i := (\theta', x'', m'')$ for some $m'' > m'$:

Since $g(s_i, s_{-i}^*) = x''$, we have that

$u_i'(g(s^*), \theta) = u_i'(x^*, \theta) < u_i'(x'', \theta) = u_i'(g(s_i, s_{-i}^*), \theta)$,

i.e., player i can profitably deviate.

Thus, $u'_i(x^*, \theta) \geq u'_i(x'', \theta)\ \forall\ (i, x'') \in N \times X$.

Since α satisfies no-veto-power in E, this implies that $x^* \in \alpha(\theta)$.

(b) Suppose that $s^*_i \neq s^*_j$ for some $i, j \in N$.

Since $\sharp N \geq 3$, $\exists\ h \in N\backslash\{i, j\}$. Since $s^*_i \neq s^*_j$, we have that $s^*_h \neq s^*_i$ or $s^*_h \neq s^*_j$.

Without loss of generality, suppose that $s^*_h \neq s^*_i$.

Assume that $\exists\ (k, x') \in N\backslash\{i\} \times X$ such that $u'_k(x', \theta) > u'_k(x^*, \theta)$.

To see that s^* does not constitute a NE of the game induced by mechanism G and type profile θ in environment E (implying a contradiction), consider player k's deviation to $s_k := (\theta', x', m')$ for some m' that satisfies $m' > m_l\ \forall\ l \in N\backslash\{k\}$ (where $s^*_i = (t_i, x_i, m_i)\ \forall\ i \in N$) and for some $\theta' \in \Theta$:

Since $g(s_k, s^*_{-k}) = x'$,[88] we have that

$u'_k(g(s^*), \theta) = u'_k(x^*, \theta) < u'_k(x', \theta) = u'_k(g(s_k, s^*_{-k}), \theta)$,

i.e., player k can profitably deviate.

Thus, $u'_k(x^*, \theta) \geq u'_k(x', \theta)\ \forall\ (k, x') \in N\backslash\{i\} \times X$.

Since α satisfies no-veto-power in E, this implies that $x^* \in \alpha(\theta)$.

B.2 Proof for Lemma 3.3

Consider the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i\in N}, g)$ defined as follows.

Define $S_i := \{(t_i, x_i, m_i) \mid t_i \in \Theta, x_i \in X, m_i \in \mathbb{N}_0\}\ \forall\ i \in N$.

For all $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$ that satisfy

$\exists\ (j, \theta, x, m) \in N \times \Theta \times X \times \mathbb{N}_0$ s.t.

$R(x, \theta) \in \alpha(\theta) \times \{1\}$ and

$(t_i, x_i, m_i) = (\theta, x, m)\ \forall\ i \in N\backslash\{j\}$ and

$\forall\ \theta' \in \Theta$ s.t. $R(x, \theta') \notin \alpha(\theta') \times \{1\}\ \exists\ (i, y) \in N \times X$ s.t.

$u_i(R(x, \theta), \theta) \geq u_i(R(y, \theta), \theta)$ and $u_i(R(x, \theta'), \theta') < u_i(R(y, \theta'), \theta')$,

define

$$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := \begin{cases} x_j & \text{if } u_j(R(x, \theta), \theta) \geq u_j(R(x_j, \theta), \theta) \\ x & \text{otw.} \end{cases}.$$

For all other $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$, define $g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := x_k$, where $k \in N$ satisfies $m_k \geq m_i\ \forall\ i \in N$.

To see that mechanism G fully NE^n_{nfg}-implements α in environment E, i.e., to see that $R(g(NE^n_{nfg}(\Gamma^{E^*, G^*, \theta})), \theta) = \{(y, 1) \mid y \in \alpha(\theta)\}\ \forall\ \theta \in \Theta$, consider an arbitrary $\theta \in \Theta$.

[88] Note that if $k \neq j$, then $i \neq j \neq k \neq i$ and $s^*_i \neq s^*_j \neq s_k \neq s^*_i$. If $k = j$, then $i \neq j = k \neq h \neq i$ and $s^*_i \neq s_k \neq s^*_h \neq s^*_i$.

'$\supseteq$'

Let $x \in \alpha(\theta)$. Then, by assumption (i), there exists an $x' \in X$ such that

$R(x', \theta) = (x, 1)$ and

$\forall\ \theta' \in \Theta$ s.t. $R(x', \theta') \notin \alpha(\theta') \times \{1\}$ there exists a tuple $(i, y) \in N \times X$ s.t.

$u_i(R(y, \theta'), \theta') > u_i(R(x', \theta'), \theta')$ and $u_i(R(y, \theta), \theta) \leq u_i(R(x', \theta), \theta)$.

Let $a_i := (\theta, x', 0)\ \forall\ i \in N$, and $a := (a_1, \ldots, a_n)$. Then, $g(a) = x'$ and $R(g(a), \theta) = R(x', \theta) = (x, 1)$.

To see that a constitutes a NE of the game induced by mechanism G^* and type profile θ in environment E^* (i.e., that $a \in NE^n_{nfg}(\Gamma^{E^*, G^*, \theta})$), consider an arbitrary deviation by an arbitrary player $j \in N$, say to $a'_j := (\theta', x'', m')$:

Since

$$u_j(R(g(a'_j, a_{-j}), \theta), \theta) = \begin{cases} u_j(R(x'', \theta), \theta) & \text{if } u_j(R(x', \theta), \theta) \geq u_j(R(x'', \theta), \theta) \\ u_j(R(x', \theta), \theta) & \text{otw.} \end{cases}$$

$$\leq u_j(R(x', \theta), \theta) = u_j(R(g(a), \theta), \theta),$$

player j's deviation is not profitable.

'$\subseteq$'

Let $a^* \equiv (a^*_1, \ldots, a^*_n) \in NE^n_{nfg}(\Gamma^{E^*, G^*, \theta})$, define $x := g(a^*)$ and $x^* := R(g(a^*), \theta) = R(x, \theta)$. To see that $x^* \in \alpha(\theta) \times \{1\}$, consider the following four cases.

(a) Suppose that $\exists\ (\theta', x', m') \in \Theta \times X \times \mathbb{N}_0$ such that $a^*_i = (\theta', x', m')\ \forall\ i \in N$. This implies that $x \equiv g(a^*) = x'$, i.e., that $a^*_i = (\theta', x, m')$.

(a.1) Suppose that $\theta' = \theta$.

Assume that $x^* = R(x, \theta) \notin \alpha(\theta) \times \{1\}$ and that there exists a tuple $(i, x'') \in N \times X$ such that $u_i(R(x'', \theta), \theta) > u_i(R(x, \theta), \theta)$.

To see that a^* does not constitute a NE of the game induced by mechanism G^* and type profile θ in environment E^* (implying a contradiction), consider player i's deviation to $a_i := (\theta, x'', m'')$ for some $m'' > m'$:

Since $R(g(a_i, a^*_{-i}), \theta) = R(x'', \theta)$ and $R(g(a^*), \theta) = R(x, \theta)$, we have that $u_i(R(g(a^*), \theta), \theta) = u_i(R(x, \theta), \theta) < u_i(R(x'', \theta), \theta) = u_i(R(g(a_i, a^*_{-i}), \theta), \theta)$, i.e., player i can profitably deviate.

Thus, $R(x, \theta) \in \alpha(\theta) \times \{1\}$ or $u_i(R(x, \theta), \theta) \geq u_i(R(x'', \theta), \theta)\ \forall\ (i, x'') \in N \times X$. By assumption (ii), also the latter condition implies that $R(x, \theta) \in \alpha(\theta) \times \{1\}$.

(a.2) Suppose that $\theta' \neq \theta$ and that

$R(x, \theta') \in \alpha(\theta') \times \{1\}$ and

$\forall\ \theta'' \in \Theta$ s.t. $R(x, \theta'') \notin \alpha(\theta'') \times \{1\}\ \exists\ (i, y) \in N \times X$ s.t.

$u_i(R(x, \theta'), \theta') \geq u_i(R(y, \theta'), \theta')$ and $u_i(R(x, \theta''), \theta'') < u_i(R(y, \theta''), \theta'')$.

Assume that $x^* = R(x,\theta) \notin \alpha(\theta) \times \{1\}$. Then, there exists a tuple $(i, y) \in N \times X$ s.t. $u_i(R(y,\theta),\theta) > u_i(R(x,\theta),\theta)$ and $u_i(R(y,\theta'),\theta') \leq u_i(R(x,\theta'),\theta')$.

To see that a^* does not constitute a NE of the game induced by mechanism G^* and type profile θ in environment E^* (implying a contradiction), consider player i's deviation to $a_i := (\theta, y, m'')$ for some $m'' > m$:

Since $R(g(a_i, a^*_{-i}), \theta) = R(y,\theta)$ and $R(g(a^*),\theta) = R(x,\theta)$, we have that $u_i(R(g(a^*),\theta),\theta) = u_i(R(x,\theta),\theta) < u_i(R(y,\theta),\theta) = u_i(R(g(a_i,a^*_{-i}),\theta),\theta)$, i.e., player i can profitably deviate.

(a.3) Suppose that $\theta' \neq \theta$ and that

$R(x,\theta') \notin \alpha(\theta') \times \{1\}$ or

$\exists\ \theta'' \in \Theta$ s.t. $R(x,\theta'') \notin \alpha(\theta'') \times \{1\}$ and $\nexists\ (i,y) \in N \times X$ s.t.

$u_i(R(x,\theta'),\theta') \geq u_i(R(y,\theta'),\theta')$ and $u_i(R(x,\theta''),\theta'') < u_i(R(y,\theta''),\theta'')$.

Assume that $\exists\ (i, x'') \in N \times X$ such that $u_i(R(x'',\theta),\theta) > u_i(R(x,\theta),\theta)$.

To see that a^* does not constitute a NE of the game induced by mechanism G^* and type profile θ in environment E^* (implying a contradiction), consider player i's deviation to $a_i := (\theta', x'', m'')$ for some $m'' > m'$:

Since $R(g(a_i, a^*_{-i}), \theta) = R(x'',\theta)$ and $R(g(a^*),\theta) = R(x,\theta)$, we have that $u_i(R(g(a^*),\theta),\theta) = u_i(R(x,\theta),\theta) < u_i(R(x'',\theta),\theta) = u_i(R(g(a_i,a^*_{-i}),\theta),\theta)$, i.e., player i can profitably deviate.

Thus, $u_i(R(x,\theta),\theta) \geq u_i(R(x'',\theta),\theta)\ \forall\ (i,x'') \in N \times X$.

By assumption (ii), this implies that $R(x,\theta) \in \alpha(\theta) \times \{1\}$.

(b) Suppose that $a^*_i \neq a^*_j$ for some $i, j \in N$.

Since $\sharp N \geq 3$, $\exists\ h \in N \backslash \{i,j\}$. Since $a^*_i \neq a^*_j$, we have that $a^*_h \neq a^*_i$ or $a^*_h \neq a^*_j$.

Without loss of generality, suppose that $a^*_h \neq a^*_i$.

Assume that $\exists\ (k, x') \in N\backslash\{i\} \times X$ such that $u_k(R(x',\theta),\theta) > u_k(R(x,\theta),\theta)$.

To see that $a^* \notin NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})$ (implying a contradiction), consider player k's deviation to $a_k := (\theta', x', m')$ for some m' that satisfies $m' > m_l\ \forall\ l \in N\backslash\{k\}$ (where $a^*_i = (t_i, x_i, m_i)\ \forall\ i \in N$) and for some $\theta' \in \Theta$:

Since $R(g(a_k, a^*_{-k}),\theta) = R(x',\theta)$,[89] and $R(g(a^*),\theta) = R(x,\theta)$, we have that $u_k(R(g(a^*),\theta),\theta) = u_k(R(x,\theta),\theta) < u_k(R(x',\theta),\theta) = u_k(R(g(a_k,a^*_{-k}),\theta),\theta)$, i.e., player k can profitably deviate.

Thus, $u_k(R(x,\theta),\theta) \geq u_k(R(x',\theta),\theta)\ \forall\ (k,x') \in N\backslash\{i\} \times X$.

By assumption (ii), this implies that $R(x,\theta) \in \alpha(\theta) \times \{1\}$.

---

[89]Note that if $k \neq j$, then $i \neq j \neq k \neq i$ and $a^*_i \neq a^*_j \neq a_k \neq a^*_i$. If $k = j$, then $i \neq j = k \neq h \neq i$ and $a^*_i \neq a_k \neq a^*_h \neq a^*_i$.

B.3 Proof for Proposition 3.3

Consider the strategic n-person mechanism $G \equiv (N, \{S_i\}_{i \in N}, g)$ defined as follows.
Define $S_i := \{(t_i, x_i, m_i) \mid t_i \in \Theta, x_i \in X, m_i \in \mathbb{N}_0\} \ \forall\, i \in N$.
For all $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$ that satisfy

$$\exists\, (j, \theta, x, m) \in N \times \Theta \times X \times \mathbb{N}_0 \text{ s.t.}$$
$$x \in \alpha(\theta) \text{ and } (t_i, x_i, m_i) = (\theta, x, m) \ \forall\, i \in N \backslash \{j\},$$

define

$$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := \begin{cases} x_j & \text{if } u_j(R(x, \theta), \theta) \geq u_j(R(x_j, \theta), \theta) \\ x & \text{otw.} \end{cases} .$$

For all other $((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) \in (\Theta \times X \times \mathbb{N}_0)^n$, define
$g((t_1, x_1, m_1), \ldots, (t_n, x_n, m_n)) := x_k$, where $k \in N$ satisfies $m_k \geq m_i \ \forall\, i \in N$.

To see that mechanism G fully NE^n_{nfg}-implements α in environment E, i.e., to see that $R(g(NE^n_{nfg}(\Gamma^{E^*, G^*, \theta})), \theta) = \{(y, 1) \mid y \in \alpha(\theta)\} \ \forall\, \theta \in \Theta$, consider an arbitrary $\theta \in \Theta$.

'$\subseteq$'

Let $a^* \equiv (a^*_1, \ldots, a^*_n) \in NE^n_{nfg}(\Gamma^{E^*, G^*, \theta})$, define $x := g(a^*)$ and $x^* := R(g(a^*), \theta) = R(x, \theta) \in \{(x, 1), (x, 0)\}$. To see that $x^* \in \alpha(\theta) \times \{1\}$, consider the following four cases.

(a) Suppose that $\exists\, (\theta', x', m') \in \Theta \times X \times \mathbb{N}_0$ such that $a^*_i = (\theta', x', m') \ \forall\, i \in N$. This implies that $x \equiv g(a^*) = x'$, i.e., that $a^*_i = (\theta', x, m')$.

(a.1) Suppose that $\theta' = \theta$.
Assume that $x \notin \alpha(\theta)$ and that there exists a tuple $(i, x'') \in N \times X$ s.t. $u_i(R(x'', \theta), \theta) > u_i(R(x, \theta), \theta)$.
To see that a^* does not constitute a NE of the game induced by mechanism G^* and type profile θ in environment E^* (implying a contradiction), consider player i's deviation to $a_i := (\theta, x'', m'')$ for some $m'' > m'$:
Since $R(g(a_i, a^*_{-i}), \theta) = R(x'', \theta)$ and $R(g(a^*), \theta) = R(x, \theta)$, we have that
$u_i(R(g(a^*), \theta), \theta) = u_i(R(x, \theta), \theta) < u_i(R(x'', \theta), \theta) = u_i(R(g(a_i, a^*_{-i}), \theta), \theta)$,
i.e., player i can profitably deviate.
Thus, $x \in \alpha(\theta)$ or $u_i(R(x, \theta), \theta) \geq u_i(R(x'', \theta), \theta) \ \forall\, (i, x'') \in N \times X$. By assumptions (i) and (ii), respectively, both alternatives imply that $R(x, \theta) \in \alpha(\theta) \times \{1\}$.

(a.2) Suppose that $\theta' \neq \theta$ and $x \in \alpha(\theta')$. See part (a.2) in the proof for Lemma 3.3.

(a.3) Suppose that $\theta' \neq \theta$ and $x \notin \alpha(\theta')$. See part (a.3) in the proof for Lemma 3.3.

(b) Suppose that $a^*_i \neq a^*_j$ for some $i, j \in N$. See part (b) in the proof for Lemma 3.3.

'$\supseteq$'

Let $x \in \alpha(\theta)$, let $a_i := (\theta, x, 0)\ \forall\ i \in N$, and $a := (a_1, \ldots, a_n)$. Then, $g(a) = x$.

To see that a constitutes a Nash Equilibrium of the game induced by mechanism G^* and type profile θ in environment E^* (i.e., that $a \in NE^n_{nfg}(\Gamma^{E^*,G^*,\theta})$), consider an arbitrary deviation by an arbitrary player $j \in N$, say to $a'_j := (\theta', x', m')$:

Since

$$u_j(R(g(a'_j, a_{-j}), \theta), \theta) = \begin{cases} u_j(R(x', \theta), \theta) & \text{if } u_j(R(x, \theta), \theta) \geq u_j(R(x', \theta), \theta) \\ u_j(R(x, \theta), \theta) & \text{otw.} \end{cases}$$

$$\leq u_j(R(x, \theta), \theta) = u_j(R(g(a), \theta), \theta),$$

player j's deviation is not profitable.

It follows, as we have already shown (by '$\subseteq$'), that $R(g(a), \theta) \in \alpha(\theta) \times \{1\} \subseteq X \times \{1\}$.

Since $R(g(a), \theta) \in \{(g(a), 1), (g(a), 0)\}$, it follows that $R(g(a), \theta) = (g(a), 1) = (x, 1)$.

C An Exchange Economy Example [90]

Let $E \equiv (N, X, \{\Theta_i\}_{i \in N}, \Theta, \{u_i\}_{i \in N}, \mathcal{G}, R, e)$ be the n-person environment with limited enforcement power, where

$N = \{1, 2\}$,

$X = \{((x_{11}, x_{12}), (x_{21}, x_{22})) \in (\mathbb{R}^2)^2 \mid x_{11} + x_{21} = 6, x_{12} + x_{22} = 6\}$,

$\bar{x} = ((1, 5), (5, 1)) \in X$ (is the initial endowment),

$\Theta_i = \{\hat{\theta}_i, \tilde{\theta}_i\}\ \forall\ i \in N$,

$\Theta = \{\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2), \tilde{\theta} \equiv (\tilde{\theta}_1, \tilde{\theta}_2)\}$,

$\mathcal{G}$ is the set of strategic n-person mechanisms for (N, X),

each agent i's utility function satisfies

$$u_i((x, 1), \theta) = \begin{cases} x_{i1} \cdot x_{i2} & \text{if } \theta = \hat{\theta} \\ \min\{x_{i1}, x_{i2}\} & \text{if } \theta = \tilde{\theta} \end{cases} \quad \forall\ (i, x, \theta) \in N \times X \times \Theta,$$ [91] and

$u_i((x, 0), \theta) = u_i((\bar{x}, 1), \theta)\ \forall\ (i, x, \theta) \in N \times X \times \Theta$, and

e is the bargaining game enforcement structure defined by

$$e(S) = \begin{cases} X & \text{if } S = N \text{ or } S = N^+ \\ \emptyset & \text{if } S = \{0\} \\ \{\bar{x}\} & \text{otw.} \end{cases} \qquad \forall\, S \in \mathcal{N}^+.$$

---

[90] This example is adapted from Jackson and Palfrey [24a]. Cf. Mas-Colell, Whinston and Green [29], Example 23.BB.1.

[91] I.e., in state $\hat{\theta}$ both agents have Cobb-Douglas preferences, and in state $\tilde{\theta}$ both agents have Leontief preferences.

Thus, realization function R satisfies

$$R(x,\theta) = \begin{cases} (\bar{x},1) & \text{if } x=\bar{x}, \text{ and } u_i((x,1),\theta) \geq u_i((x,0),\theta) \text{ for some } i \in N \\ (x,1) & \text{if } x \neq \bar{x}, \text{ and } u_i((x,1),\theta) \geq u_i((x,0),\theta)\ \forall\ i \in N \\ (x,0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x,1) & \text{if } u_i((x,1),\theta) \geq u_i((\bar{x},1),\theta)\ \forall\ i \in N \\ (x,0) & \text{otw.} \end{cases}.$$

Note that agents do not have weak pessimistic beliefs in environment E. For example, $x' := ((0,0),(6,6))$ satisfies $R(x',\hat{\theta}) = (x',0)$.

Let $E^C = (N, X, \{\Theta_i\}_{i\in N}, \Theta, \{u'_i\}_{i\in N}, \mathcal{G})$ denote E's corresponding classical environment, i.e., $u'_i(x,\theta) = u_i((x,1),\theta)\ \forall\ (x,\theta) \in X \times \Theta$, and let α be the SCC for environment E defined by

$$\alpha(\theta) = \begin{cases} \{\hat{x} := ((3,3),(3,3))\} & \text{if } \theta = \hat{\theta} \\ \{\tilde{x} := ((2,2),(4,4))\} & \text{if } \theta = \tilde{\theta} \end{cases}.$$

In particular, since $R(\hat{x},\hat{\theta}) = (\hat{x},1)$ and $R(\tilde{x},\tilde{\theta}) = (\tilde{x},1)$, (social) choice correspondence α is consistent with environment E.

Note that agents' beliefs can be justified by outcome-independent prediction functions $\{b_i\}_{i\in N}$, where each b_i is defined by $b_i((x,0),\theta) = (\bar{x},2)\ \forall\ (x,\theta) \in X \times \Theta$.

SCC α is fully NE^n_{nfg}-implementable in environment E.

To see this, consider the strategic mechanism G in which player 2 chooses between $\hat{x}$ and $\tilde{x}$, which is then the outcome suggested by the mechanism. The game induced by G^* and $\hat{\theta}$ in E^* has exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses $\hat{x}$, since $u_2(R(\hat{x},\hat{\theta}),\hat{\theta}) = u_2((\hat{x},1),\hat{\theta}) = 9 > 5 = u_2((\bar{x},1),\hat{\theta}) = u_2((\tilde{x},0),\hat{\theta}) = u_2(R(\tilde{x},\hat{\theta}),\hat{\theta})$. The game induced by G^* and $\tilde{\theta}$ in E^* has also exactly one Nash Equilibrium. In this Nash Equilibrium, player 2 chooses $\tilde{x}$, since $u_2(R(\tilde{x},\tilde{\theta}),\tilde{\theta}) = u_2((\tilde{x},1),\tilde{\theta}) = 4 > 3 = u_2((\hat{x},1),\tilde{\theta}) = u_2(R(\hat{x},\tilde{\theta}),\tilde{\theta})$.

However, (social) choice correspondence α is not fully NE^n_{nfg}-implementable in environment E^C, since α is not Maskin-monotonic in E^C. To see this, note that $\hat{x} \in \alpha(\hat{\theta})$, $\hat{x} \notin \alpha(\tilde{\theta})$, and that there does not exist a tuple $(i,x') \in N \times X$ s.t. $u'_i(\hat{x},\hat{\theta}) \geq u'_i(x',\hat{\theta})$ and $u'_i(\hat{x},\tilde{\theta}) < u'_i(x',\tilde{\theta})$: Assume that there exists such a tuple (i,x'). Then, $3 = u'_i(\hat{x},\tilde{\theta}) < u'_i(x',\tilde{\theta}) = \min\{x'_{i1}, x'_{i2}\}$ implies that $x'_{i1} > 3$ and $x'_{i2} > 3$, which contradicts $9 = u'_i(\hat{x},\hat{\theta}) \geq u'_i(x',\hat{\theta}) = x'_{i1} \cdot x'_{i2}$.

Figure C.1 illustrates the setting (Cf. Jackson and Palfrey [24a], Figure 1).

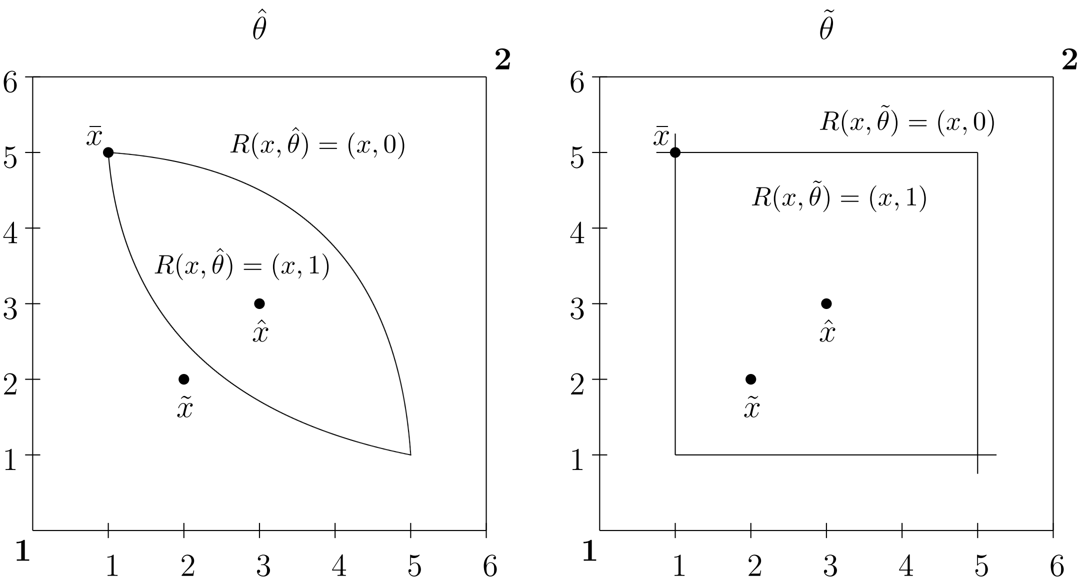

Figure C.1

D 'Equivalence' between EDS and CVDS Assignments

Let $A \equiv (e, d)$ be an EDS assignment , and define $v^A : \mathcal{N} \Rightarrow X$ by

$\forall\ (x, S) \in e(\{0\}) \times \mathcal{N} : x \notin v^A(S)$, and

$\forall\ (x, S) \in X \backslash e(\{0\}) \times \mathcal{N}\ :\ x \in v^A(S) \Leftrightarrow S \cap S' \neq \emptyset\ \forall\ S' \in \mathcal{N}$ s.t. $x \in e(S')$.

Then, $X \backslash v^A(N) = e(\{0\})$, and $v^A(S') \supseteq v^A(S)\ \forall\ (S, S') \in \mathcal{N} \times \mathcal{N}$ satisfying $S' \supseteq S$ (implying that (v^A, d) is a CVDS assignment):

Let $(x, S, S') \in X \backslash e(\{0\}) \times \mathcal{N} \times \mathcal{N}$ s.t. $x \in v^A(S)$ and $S' \supseteq S$. Since $x \in v^A(S)$, we have that $S \cap S'' \neq \emptyset\ \forall\ S'' \in \mathcal{N}$ s.t. $x \in e(S'')$. Since $S' \supseteq S$, this implies that $S' \cap S'' \neq \emptyset\ \forall\ S'' \in \mathcal{N}$ s.t. $x \in e(S'')$. Thus, $x \in v^A(S')$.

Let $A \equiv (v, d)$ be a CVDS assignment, and define $e^A : \mathcal{N}^+ \Rightarrow X$ by

$\forall\ (x, S) \in v(N) \times \mathcal{N}\ :\ x \notin e^A(S) \Leftrightarrow \exists\ S' \in \mathcal{N}$ s.t. $S \cap S' = \emptyset$ and $x \in v(S')$,

$x \notin e^A(S)\ \forall\ (x, S) \in X \backslash v(N) \times \mathcal{N}$,

$x \in e^A(\{0\})\ \forall\ x \in X \backslash v(N)$, $x \notin e^A(\{0\})\ \forall\ x \in v(N)$, and

$e^A(S \cup \{0\}) = e^A(S) \cup e^A(\{0\})\ \forall\ S \in \mathcal{N}$.

Then, $X \backslash v(N) = e^A(\{0\})$, and

(i) $e^A(N) \cup e^A(\{0\}) = X$ and $e^A(N) \cap e^A(\{0\}) = \emptyset$:

Let $x \in X$. If $x \in v(N)$, then $x \in e^A(N)$ (since $\nexists\ S' \in \mathcal{N}$ s.t. $N \cap S' = \emptyset$) and $x \notin e^A(\{0\})$. If $x \notin v(N)$, then $x \in e^A(\{0\})$ and $x \notin e^A(N)$.

Thus, $e^A(N) \cap e^A(\{0\}) = \emptyset$, and $X = X \backslash v(N) \cup v(N) \subseteq e^A(\{0\}) \cup e^A(N) \subseteq X$.

(ii) $e^A(S') \supseteq e^A(S)\ \forall\ (S,S') \in \mathcal{N} \times \mathcal{N}$ s.t. $S' \supseteq S$:
Let $(x,S,S') \in X \times \mathcal{N} \times \mathcal{N}$ s.t. $x \in e^A(S)$ and $S' \supseteq S$. Then, in particular, $x \in v(N)$.
Since $x \in e^A(S)$, we have that $\nexists\ S'' \in \mathcal{N}$ s.t. $S \cap S'' = \emptyset$ and $x \in v(S'')$.
Since $S' \supseteq S$, this implies that $\nexists\ S'' \in \mathcal{N}$ s.t. $S' \cap S'' = \emptyset$ and $x \in v(S'')$.
Thus, $x \in e^A(S')$.

(ii) $e^A(S') \supseteq e^A(S)\ \forall\ (S,S') \in \mathcal{N}^+ \times \mathcal{N}^+$ s.t. $S' \supseteq S$:
If $S' \in \mathcal{N}$, then $e^A(S') \supseteq e^A(S)$ follows from (ii).
If $S' = \{0\}$, then $S = \{0\}$ and $e^A(S') = e^A(S)$.
If $S' = S^* \cup \{0\}$ for some $S^* \in \mathcal{N}$, and if $S \subseteq S^*$, then (iii) implies $e^A(S^*) \supseteq e^A(S)$, and thus $e^A(S') = e^A(S^*) \cup e^A(\{0\}) \supseteq e^A(S) \cup e^A(\{0\}) \supseteq e^A(S)$.
If $S' = S^* \cup \{0\}$ for some $S^* \in \mathcal{N}$, and if $S = \{0\}$, then
$e^A(S') = e^A(S^*) \cup e^A(\{0\}) \supseteq e^A(\{0\}) = e^A(S)$.
If $S' = S^* \cup \{0\}$ for some $S^* \in \mathcal{N}$, and if $S = \tilde{S} \cup \{0\}$ for some $\tilde{S} \in \mathcal{N}$, $\tilde{S} \subseteq S^*$, then $e^A(S') = e^A(S^*) \cup e^A(\{0\}) \supseteq e^A(\tilde{S}) \cup e^A(\{0\}) = e^A(S)$.

Conditions (i) and (iii) imply that (e^A, d) is an EDS assignment.

The following table lists all possible CVDS and EDS assignments for the case of $N = 3$ agents and $\sharp X = 1$ outcome. The letters to the left enumerate the assignments, the letters to the right point to the respective corresponding assignment as considered above.

	$v(\{1\})$	$v(\{2\})$	$v(\{3\})$	$v(\{1,2\})$	$v(\{2,3\})$	$v(\{1,3\})$	$v(\{1,2,3\})$		
	$e(\{1\})$	$e(\{2\})$	$e(\{3\})$	$e(\{1,2\})$	$e(\{2,3\})$	$e(\{1,3\})$	$e(\{1,2,3\})$	$e(\{0\})$	
(a)								x	(a)
(b)	x			x		x	x		(b)
(c)		x		x	x		x		(c)
(d)	x	x		x	x	x	x		(i)
(e)			x		x	x	x		(e)
(f)	x		x	x	x	x	x		(n)
(g)		x	x	x	x	x	x		(k)
(h)	x	x	x	x	x	x	x		(s)
(i)				x			x		(d)
(j)			x	x	x	x	x		(q)
(k)					x		x		(g)
(l)				x	x		x		(o)
(m)	x			x	x	x	x		(p)
(n)						x	x		(f)

	$v(\{1\})$	$v(\{2\})$	$v(\{3\})$	$v(\{1,2\})$	$v(\{2,3\})$	$v(\{1,3\})$	$v(\{1,2,3\})$		
	$e(\{1\})$	$e(\{2\})$	$e(\{3\})$	$e(\{1,2\})$	$e(\{2,3\})$	$e(\{1,3\})$	$e(\{1,2,3\})$	$e(\{0\})$	
(o)		x		x	x	x	x		(l)
(p)				x		x	x		(m)
(q)					x	x	x		(j)
(r)				x	x	x	x		(r)
(s)							x		(h)

E Proof for Lemma 4.6

Assignment (e, d) has to match one of the following 400 cases (I.1.A to IV.25.D):

(I) $e(\{0\}) = \emptyset$
(II) $e(\{0\}) = \{x\}$
(III) $e(\{0\}) = \{y\}$
(IV) $e(\{0\}) = \{x, y\}$

(1) $e(\{1\}) = \emptyset$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \emptyset$
(2) $e(\{1\}) = \emptyset$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{x\}$
(3) $e(\{1\}) = \emptyset$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{y\}$
(4) $e(\{1\}) = \emptyset$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{x, y\}$
(5) $e(\{1\}) = \emptyset$, $e(\{2\}) = \{x\}$, and $e(\{1,2\}) = \{x\}$
(6) $e(\{1\}) = \emptyset$, $e(\{2\}) = \{x\}$, and $e(\{1,2\}) = \{x, y\}$
(7) $e(\{1\}) = \emptyset$, $e(\{2\}) = \{y\}$, and $e(\{1,2\}) = \{y\}$
(8) $e(\{1\}) = \emptyset$, $e(\{2\}) = \{y\}$, and $e(\{1,2\}) = \{x, y\}$
(9) $e(\{1\}) = \emptyset$, $e(\{2\}) = \{x, y\}$, and $e(\{1,2\}) = \{x, y\}$
(10) $e(\{1\}) = \{x\}$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{x\}$
(11) $e(\{1\}) = \{x\}$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{x, y\}$
(12) $e(\{1\}) = \{x\}$, $e(\{2\}) = \{x\}$, and $e(\{1,2\}) = \{x\}$
(13) $e(\{1\}) = \{x\}$, $e(\{2\}) = \{x\}$, and $e(\{1,2\}) = \{x, y\}$
(14) $e(\{1\}) = \{x\}$, $e(\{2\}) = \{y\}$, and $e(\{1,2\}) = \{x, y\}$
(15) $e(\{1\}) = \{x\}$, $e(\{2\}) = \{x, y\}$, and $e(\{1,2\}) = \{x, y\}$
(16) $e(\{1\}) = \{y\}$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{y\}$
(17) $e(\{1\}) = \{y\}$, $e(\{2\}) = \emptyset$, and $e(\{1,2\}) = \{x, y\}$
(18) $e(\{1\}) = \{y\}$, $e(\{2\}) = \{x\}$, and $e(\{1,2\}) = \{x, y\}$
(19) $e(\{1\}) = \{y\}$, $e(\{2\}) = \{y\}$, and $e(\{1,2\}) = \{y\}$
(20) $e(\{1\}) = \{y\}$, $e(\{2\}) = \{y\}$, and $e(\{1,2\}) = \{x, y\}$
(21) $e(\{1\}) = \{y\}$, $e(\{2\}) = \{x, y\}$, and $e(\{1,2\}) = \{x, y\}$

(22) $e(\{1\}) = \{x, y\}$, $e(\{2\}) = \emptyset$, and $e(\{1, 2\}) = \{x, y\}$

(23) $e(\{1\}) = \{x, y\}$, $e(\{2\}) = \{x\}$, and $e(\{1, 2\}) = \{x, y\}$

(24) $e(\{1\}) = \{x, y\}$, $e(\{2\}) = \{y\}$, and $e(\{1, 2\}) = \{x, y\}$

(25) $e(\{1\}) = \{x, y\}$, $e(\{2\}) = \{x, y\}$, and $e(\{1, 2\}) = \{x, y\}$

(A) $d(x, 0) = (x, 2)$ and $d(y, 0) = (y, 2)$

(B) $d(x, 0) = (x, 2)$ and $d(y, 0) = (x, 2)$

(C) $d(x, 0) = (y, 2)$ and $d(y, 0) = (y, 2)$

(D) $d(x, 0) = (y, 2)$ and $d(y, 0) = (x, 2)$

Cases IV.·.·

Full enforcement power on the side of the designer implies condition (iv).

Cases I.·.A, II.·.A, and III.·.A

$$R^{(e,d)}(y, \theta) = \begin{cases} (y, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } y \in e(S) \text{ and} \\ & u_i((y, 1), \theta) \geq u_i(d(y, 0), \theta)\ \forall\ i \in S \cap N \\ d(y, 0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (y, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } y \in e(S) \text{ and} \\ & u_i((y, 1), \theta) \geq u_i((y, 2), \theta) = u_i((y, 1), \theta) - l\ \forall\ i \in S \cap N \\ d(y, 0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (y, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } y \in e(S) \\ d(y, 0) & \text{otw.} \end{cases}$$

$$= (y, 1),$$

since $R^{(e,d)}(y, \theta') = (y, 1)$ implies that $\exists\ S \in \mathcal{N}^+$ s.t. $y \in e(S)$.

$$R^{(e,d)}(x, \theta') = \begin{cases} (x, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \text{ and} \\ & u_i((x, 1), \theta') \geq u_i(d(x, 0), \theta')\ \forall\ i \in S \cap N \\ d(x, 0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \text{ and} \\ & u_i((x, 1), \theta') \geq u_i((x, 2), \theta') = u_i((x, 1), \theta') - l\ \forall\ i \in S \cap N \\ d(x, 0) & \text{otw.} \end{cases}$$

$$= \begin{cases} (x, 1) & \text{if } \exists\ S \in \mathcal{N}^+ \text{ s.t. } x \in e(S) \\ d(x, 0) & \text{otw.} \end{cases}$$

$$= (x, 1),$$

since $R^{(e,d)}(x, \theta) = (x, 1)$ implies that $\exists\ S \in \mathcal{N}^+$ s.t. $x \in e(S)$.

Thus, condition (iv) is satisfied.

Cases I.·.B

Following the same steps as in cases I.·.A, we obtain $R^{(e,d)}(x,\theta') = (x,1)$.

Assume that condition (iv) is not satisfied, i.e., that $R^{(e,d)}(y,\theta) = (x,2)$.

Since $e(\{0\}) = \emptyset$, we have that $e(\{1,2\}) = \{x,y\}$.

Since $e(\{1,2\}) = \{x,y\}$ and $R^{(e,d)}(y,\theta) = (x,2)$, we have that

$$u_1((y,1),\theta) < u_1(d(y,0),\theta) = u_1((x,2),\theta) = u_1((x,1),\theta) - l \text{ or}$$
$$u_2((y,1),\theta) < u_2(d(y,0),\theta) = u_2((x,2),\theta) = u_2((x,1),\theta) - l.$$

If $l \geq 1$, then this implies that

$$1 \leq u_1((y,1),\theta) < u_1((x,1),\theta) - l \leq 2 - 1 = 1 \text{ or}$$
$$1 \leq u_2((y,1),\theta) < u_2((x,1),\theta) - l \leq 2 - 1 = 1,$$

a contradiction.

If $l = 0$ and $y \in \alpha(\theta)$, then condition (v) is satisfied.

If $l = 0$ and $y \notin \alpha(\theta)$, then $R^{(e,d)}(y,\theta) \notin \alpha(\theta) \times \{1\}$,

$$u_1(R^{(e,d)}(x,\theta),\theta) = u_1((x,1),\theta) \leq u_1((x,1),\theta) - l = u_1(R^{(e,d)}(y,\theta),\theta), \text{ and}$$
$$u_2(R^{(e,d)}(x,\theta),\theta) = u_2((x,1),\theta) \leq u_2((x,1),\theta) - l = u_2(R^{(e,d)}(y,\theta),\theta),$$

i.e., condition (vii) is satisfied.

If $l \in (0,1)$ and $y \in \alpha(\theta)$, then condition (v) is satisfied.

Cases I.·.C

Following the same steps as in cases I.·.A, we obtain $R^{(e,d)}(y,\theta) = (y,1)$.

Assume that condition (iv) is not satisfied, i.e., that $R^{(e,d)}(x,\theta') = (y,2)$.

Since $e(\{0\}) = \emptyset$, we have that $e(\{1,2\}) = \{x,y\}$.

Since $e(\{1,2\}) = \{x,y\}$ and $R^{(e,d)}(x,\theta') = (y,2)$, we have that

$$u_1((x,1),\theta') < u_1(d(x,0),\theta') = u_1((y,2),\theta') = u_1((y,1),\theta') - l \text{ or}$$
$$u_2((x,1),\theta') < u_2(d(x,0),\theta') = u_2((y,2),\theta') = u_2((y,1),\theta') - l.$$

If $l \geq 1$, then this implies that

$$1 \leq u_1((x,1),\theta') < u_1((y,1),\theta') - l \leq 2 - 1 = 1 \text{ or}$$
$$1 \leq u_2((x,1),\theta') < u_2((y,1),\theta') - l \leq 2 - 1 = 1,$$

a contradiction.

If $l = 0$, then condition (vi) is satisfied:

$$u_1(R^{(e,d)}(y,\theta'),\theta') = u_1((y,1),\theta') \leq u_1((y,1),\theta') - l = u_1(R^{(e,d)}(x,\theta'),\theta'), \text{ and}$$
$$u_2(R^{(e,d)}(y,\theta'),\theta') = u_2((y,1),\theta') \leq u_2((y,1),\theta') - l = u_2(R^{(e,d)}(x,\theta'),\theta').$$

If $l \in (0,1)$ and $y \in \alpha(\theta)$, then,

- in cases I.4.C, I.8.C, I.17.C, and I.20.C ($x \notin e(\{1\})$, $x \notin e(\{2\})$):

  $R^{(e,d)}(x,\theta) = (x,1)$ implies that

  $$u_1((x,1),\theta) \geq u_1(d(x,0),\theta) = u_1((y,2),\theta) = u_1((y,1),\theta) - l \text{ and}$$

$u_2((x,1),\theta) \geq u_2(d(x,0),\theta) = u_2((y,2),\theta) = u_2((y,1),\theta) - l.$

Since all utility levels are by assumption integer values and since $l \in (0,1)$, this implies that $u_1((x,1),\theta) \geq u_1((y,1),\theta)$ and $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.

By assumption (ii) and (iii), it follows that

$u_1((y,1),\theta') \leq u_1((x,1),\theta')$ and $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.

On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that

$u_1((x,1),\theta') < u_1(d(x,0),\theta') = u_1((y,2),\theta') = u_1((y,1),\theta') - l$ or

$u_2((x,1),\theta') < u_2(d(x,0),\theta') = u_2((y,2),\theta') = u_2((y,1),\theta') - l,$

i.e., $u_1((x,1),\theta') < u_1((y,1),\theta')$ or $u_2((x,1),\theta') < u_2((y,1),\theta')$,

a contradiction.

- in cases I.6.C, I.9.C, I.18.C, and I.21.C ($x \notin e(\{1\})$, $x \in e(\{2\})$):
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.
  By assumption (iii), it follows that $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that $u_2((x,1),\theta') < u_2((y,1),\theta')$,
  a contradiction.
- in cases I.11.C, I.14.C, I.22.C, and I.24.C ($x \in e(\{1\})$, $x \notin e(\{2\})$):
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that $u_1((x,1),\theta) \geq u_1((y,1),\theta)$.
  By assumption (ii), it follows that $u_1((y,1),\theta') \leq u_1((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that $u_1((x,1),\theta') < u_1((y,1),\theta')$,
  a contradiction.
- in cases I.13.C, I.15.C, I.23.C, and I.25.C ($x \in e(\{1\})$, $x \in e(\{2\})$):
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that
  $u_1((x,1),\theta) \geq u_1((y,1),\theta)$ or $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.
  By assumption (ii) and (iii), it follows that
  $u_1((y,1),\theta') \leq u_1((x,1),\theta')$ or $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that
  $u_1((x,1),\theta') < u_1((y,1),\theta')$ and $u_2((x,1),\theta') < u_2((y,1),\theta')$,
  a contradiction.

Cases I.·.D

If we assume that $R^{(e,d)}(y,\theta) = (x,2)$, then, following the same steps as in cases I.·.B, we obtain our assertion for the cases '$l = 0$' and '$l \in (0,1)$ and $y \in \alpha(\theta)$', and obtain a contradiction for the case '$l \geq 1$'.

If we assume that $R^{(e,d)}(x,\theta') = (y,2)$, then, following the same steps as in cases I.·.C, we obtain our assertion for the case '$l = 0$', and obtain a contradiction for the cases '$l \geq 1$' and '$l \in (0,1)$ and $y \in \alpha(\theta)$'.

Thus, for the case '$l = 0$', we obtain our assertion.

And, for the case '$l \geq 1$', condition (iv) is satisfied.

Finally, consider the case '$l \in (0,1)$ and $y \in \alpha(\theta)$'.

First, we obtain $R^{(e,d)}(x, \theta') = (x, 1)$.

Second, if $R^{(e,d)}(y, \theta) = (x, 2)$, then we obtain our assertion, otherwise condition (iv) is satisfied.

Cases II.·.B and II.·.D

Since $x \in e(\{0\})$, we have that $R^{(e,d)}(x, \theta') = (x, 1)$.

Assume that condition (iv) is not satisfied, i.e., that $R^{(e,d)}(y, \theta) = (x, 2)$.

Since $y \notin e(\{0\})$ and $R^{(e,d)}(y, \theta') = (y, 1)$, we have that $y \in e(\{1, 2\})$.

Since $y \in e(\{1, 2\})$ and $R^{(e,d)}(y, \theta) = (x, 2)$, we have that

$$u_1((y,1),\theta) < u_1(d(y,0),\theta) = u_1((x,2),\theta) = u_1((x,1),\theta) - l \text{ or}$$

$$u_2((y,1),\theta) < u_2(d(y,0),\theta) = u_2((x,2),\theta) = u_2((x,1),\theta) - l.$$

If $l \geq 1$, then this implies that

$$1 \leq u_1((y,1),\theta) < u_1((x,1),\theta) - l \leq 2 - 1 = 1 \text{ or}$$

$$1 \leq u_2((y,1),\theta) < u_2((x,1),\theta) - l \leq 2 - 1 = 1,$$

a contradiction.

If $l = 0$ and $y \in \alpha(\theta)$, then condition (v) is satisfied.

If $l = 0$ and $y \notin \alpha(\theta)$, then $R^{(e,d)}(y, \theta) \notin \alpha(\theta) \times \{1\}$,

$$u_1(R^{(e,d)}(x,\theta),\theta) = u_1((x,1),\theta) \leq u_1((x,1),\theta) - l = u_1(R^{(e,d)}(y,\theta),\theta), \text{ and}$$

$$u_2(R^{(e,d)}(x,\theta),\theta) = u_2((x,1),\theta) \leq u_2((x,1),\theta) - l = u_2(R^{(e,d)}(y,\theta),\theta),$$

i.e., condition (vii) is satisfied.

If $l \in (0,1)$ and $y \in \alpha(\theta)$, then condition (v) is satisfied.

Cases II.·.C

Following the same steps as in cases I.·.A, we obtain $R^{(e,d)}(y, \theta) = (y, 1)$. Since $x \in e(\{0\})$, we have that $R^{(e,d)}(x, \theta') = (x, 1)$. Thus, condition (iv) is satisfied.

Cases III.·.B

Following the same steps as in cases I.·.A, we obtain $R^{(e,d)}(x, \theta') = (x, 1)$. Since $y \in e(\{0\})$, we have that $R^{(e,d)}(y, \theta) = (y, 1)$. Thus, condition (iv) is satisfied.

Cases III.·.C and III.·.D

Since $y \in e(\{0\})$, we have that $R^{(e,d)}(y, \theta) = (y, 1)$.

Assume that condition (iv) is not satisfied, i.e., that $R^{(e,d)}(x, \theta') = (y, 2)$.

Since $x \notin e(\{0\})$ and $R^{(e,d)}(x, \theta) = (x, 1)$, we have that $x \in e(\{1, 2\})$.

Since $x \in e(\{1, 2\})$ and $R^{(e,d)}(x, \theta') = (y, 2)$, we have that

$u_1((x,1),\theta') < u_1(d(x,0),\theta') = u_1((y,2),\theta') = u_1((y,1),\theta') - l$ or
$u_2((x,1),\theta') < u_2(d(x,0),\theta') = u_2((y,2),\theta') = u_2((y,1),\theta') - l$.

If $l \geq 1$, then this implies that
$1 \leq u_1((x,1),\theta') < u_1((y,1),\theta') - l \leq 2 - 1 = 1$ or
$1 \leq u_2((x,1),\theta') < u_2((y,1),\theta') - l \leq 2 - 1 = 1$,
a contradiction.

If $l = 0$, then condition (vi) is satisfied:
$u_1(R^{(e,d)}(y,\theta'),\theta') = u_1((y,1),\theta') \leq u_1((y,1),\theta') - l = u_1(R^{(e,d)}(x,\theta'),\theta')$, and
$u_2(R^{(e,d)}(y,\theta'),\theta') = u_2((y,1),\theta') \leq u_2((y,1),\theta') - l = u_2(R^{(e,d)}(x,\theta'),\theta')$.

If $l \in (0,1)$ and $y \in \alpha(\theta)$, then,

- in case I.2.C:
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that
  $u_1((x,1),\theta) \geq u_1(d(x,0),\theta) = u_1((y,2),\theta) = u_1((y,1),\theta) - l$ and
  $u_2((x,1),\theta) \geq u_2(d(x,0),\theta) = u_2((y,2),\theta) = u_2((y,1),\theta) - l$.
  Since all utility levels are by assumption integer values and since $l \in (0,1)$, this implies that $u_1((x,1),\theta) \geq u_1((y,1),\theta)$ and $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.
  By assumption (ii) and (iii), it follows that
  $u_1((y,1),\theta') \leq u_1((x,1),\theta')$ and $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that
  $u_1((x,1),\theta') < u_1(d(x,0),\theta') = u_1((y,2),\theta') = u_1((y,1),\theta') - l$ or
  $u_2((x,1),\theta') < u_2(d(x,0),\theta') = u_2((y,2),\theta') = u_2((y,1),\theta') - l$,
  i.e., $u_1((x,1),\theta') < u_1((y,1),\theta')$ or $u_2((x,1),\theta') < u_2((y,1),\theta')$,
  a contradiction.
- in case I.5.C:
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.
  By assumption (iii), it follows that $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that $u_2((x,1),\theta') < u_2((y,1),\theta')$,
  a contradiction.
- in case I.10.C:
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that $u_1((x,1),\theta) \geq u_1((y,1),\theta)$.
  By assumption (ii), it follows that $u_1((y,1),\theta') \leq u_1((x,1),\theta')$.
  On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that $u_1((x,1),\theta') < u_1((y,1),\theta')$,
  a contradiction.
- in case I.12.C:
  $R^{(e,d)}(x,\theta) = (x,1)$ implies that
  $u_1((x,1),\theta) \geq u_1((y,1),\theta)$ or $u_2((x,1),\theta) \geq u_2((y,1),\theta)$.

By assumption (ii) and (iii), it follows that

$u_1((y,1),\theta') \leq u_1((x,1),\theta')$ or $u_2((y,1),\theta') \leq u_2((x,1),\theta')$.

On the other hand, $R^{(e,d)}(x,\theta') = (y,2)$ implies that

$u_1((x,1),\theta') < u_1((y,1),\theta')$ and $u_2((x,1),\theta') < u_2((y,1),\theta')$,

a contradiction.

F Three Sets of Bargaining Games with Physical Outcomes

F.1 A set on which ψ^{Nash} is not ordinally invariant

Define $X := \{x \in \mathbb{R}^2 \mid \exists\, \alpha \in \mathbb{R}^4_+$ such that $\sum_{i=1}^4 \alpha_i = 1$ and

$$x = \alpha_2 \cdot (1,0) + \alpha_3 \cdot (0,1) + \alpha_4 \cdot (\tfrac{1}{5}, 1 - \tfrac{1}{25})\},\text{ and}$$

$D := 0 \in X$.

Let $\mathcal{C}^n_{po}$ be the set of all two-person bargaining games with physical outcomes $\Gamma \equiv (N, \bar{X}, \{u_i^\Gamma\}_{i\in N})$ such that $\bar{X}(\{1,2\}) = X$ and $\bar{X}(\{i\}) = \{D\}\ \forall\, i \in N \equiv \{1,2\}$.

Let $\Gamma \equiv (N, \bar{X}, \{u_i^\Gamma\}_{i\in N}) \in \mathcal{C}^n_{po}$ and $\Gamma' \equiv (N, \bar{X}, \{u_i^{\Gamma'}\}_{i\in N}) \in \mathcal{C}^n_{po}$ be defined by

$u_i^\Gamma : X \to \mathbb{R}$, $u_i^\Gamma(x) := x_i\ \forall\, (i,x) \in N \times X$, and

$u_i^{\Gamma'} : X \to \mathbb{R}$, $u_2^{\Gamma'}(x) := x_2\ \forall\, x \in X$ and

$$u_1^{\Gamma'}(x) := \begin{cases} 4 \cdot x_1 & \forall\, x \in X \text{ s.t. } x_1 \leq \frac{1}{5} \\ \frac{1}{4} \cdot (x_1 - \frac{1}{5}) + \frac{4}{5} & \forall\, x \in X \text{ s.t. } x_1 > \frac{1}{5} \end{cases},$$

and note that Γ' is an order preserving transformation of game Γ.

Figure F.1 illustrates the two games.

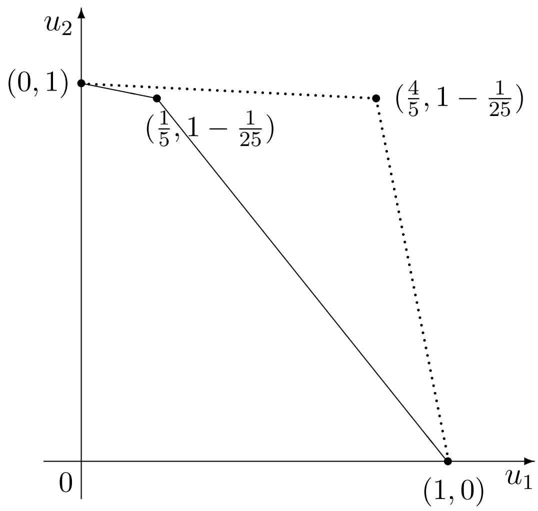

Figure F.1

Let $\bar{x}$ denote the unique element of X that satisfies $u_1^\Gamma(\bar{x}) \cdot u_2^\Gamma(\bar{x}) = \bar{x}_1 \cdot \bar{x}_2 \geq x_1 \cdot x_2 = u_1^\Gamma(x) \cdot u_2^\Gamma(x)\ \forall\, x \in X$, and note that $u_1^\Gamma(\bar{x}) = \bar{x}_1 > \frac{1}{5}$.

Let $\bar{x}'$ denote the unique element of X that satisfies $u_1^{\Gamma'}(\bar{x}')\cdot u_2^{\Gamma'}(\bar{x}') \geq u_1^{\Gamma'}(x)\cdot u_2^{\Gamma'}(x)\ \forall\, x \in X$, and note that $(u_1^{\Gamma'}(\bar{x}'), u_2^{\Gamma'}(\bar{x}')) = (\frac{4}{5}, 1-\frac{1}{25})$ and $\bar{x}' = (\frac{1}{5}, 1-\frac{1}{25})$.
In particular, $\bar{x}' \neq \bar{x}$, and thus $\psi^{Nash}(\Gamma) = \{(N,\bar{x})\} \neq \{(N,\bar{x}')\} = \psi^{Nash}(\Gamma')$.
It follows that ψ^{Nash} is not ordinally invariant on $\mathcal{C}_{po}^n$.

F.2 A set on which every solution concept is ordinally invariant

Let $m \in \mathbb{N}$, $m \geq 2$, and define game form $(N, \bar{X})$ by $N := \{1,2\}$ and

$\bar{X}(\{1,2\}) := X := \{(x_0,\ldots,x_m) \in \mathbb{R}_+^{m+1} \mid \sum_{k=0}^m x_k = 1\}$ and

$\bar{X}(\{i\}) := \{D\}\ \forall\, i \in N$, where $D := e_0 := (1,0,\ldots,0) \in X$.

Let $\mathcal{C}_{po}^n$ be the set of all two-person bargaining games with physical outcomes $\Gamma \equiv (N,\bar{X},\{u_i^\Gamma\}_{i\in N})$ such that each $u_i^\Gamma : X \to \mathbb{R}$ satisfies

$\exists\ \alpha^{(\Gamma,i)} \in \mathbb{R}_+^m$ such that $u_i^\Gamma(x_0,\ldots,x_m) = \sum_{k=1}^m \alpha_k^{(\Gamma,i)} \cdot x_k\ \forall\, (x_0,\ldots,x_m) \in X$.[92]

Let $\Gamma \equiv (N,\bar{X},\{u_i^\Gamma\}_{i\in N})$ and $\Gamma' \equiv (N,\bar{X},\{u_i^{\Gamma'}\}_{i\in N})$ be bargaining games in $\mathcal{C}_{po}^n$ such that Γ' is an order preserving transformation of game Γ.
Then, $\forall\, (i,x,x') \in N \times X \times X$, $u_i^\Gamma(x) > u_i^\Gamma(x') \Leftrightarrow u_i^{\Gamma'}(x) > u_i^{\Gamma'}(x')$.
This implies that, $\forall\, (i,x,x') \in N \times X \times X$, $u_i^\Gamma(x) = u_i^\Gamma(x') \Leftrightarrow u_i^{\Gamma'}(x) = u_i^{\Gamma'}(x')$, and thus, $u_i^\Gamma(x) \geq u_i^\Gamma(x') \Leftrightarrow u_i^{\Gamma'}(x) \geq u_i^{\Gamma'}(x')$.
Without loss of generality, let $e_1 := (0,1,0,\ldots,0) \in X$ and $e_2 := (0,0,1,0,\ldots,0) \in X$ satisfy $(u_1^\Gamma(e_1), u_2^\Gamma(e_1)) = (1,0)$ and $(u_1^\Gamma(e_2), u_2^\Gamma(e_2)) = (0,1)$.
Let $(i,j) \in N \times \mathbb{N}_m$, and define $e_l := (\delta_{0l},\ldots,\delta_{ml}) \in X$, where $\delta_{ij} := 1$ for $i = j$ and $\delta_{ij} := 0$ for $i \neq j$.
If $u_i^\Gamma(e_l) = 0$, then $u_i^\Gamma(e_l) \leq u_i^\Gamma(x)\ \forall\, x \in X$. Since Γ' is an order preserving transformation of Γ, this implies that $u_i^{\Gamma'}(e_l) \leq u_i^{\Gamma'}(x)\ \forall\, x \in X$. Thus, $u_i^{\Gamma'}(e_l) = 0$.
If $u_i^\Gamma(e_l) = 1$, then $u_i^\Gamma(e_l) \geq u_i^\Gamma(x)\ \forall\, x \in X$. Since Γ' is an order preserving transformation of Γ, this implies that $u_i^{\Gamma'}(e_l) \geq u_i^{\Gamma'}(x)\ \forall\, x \in X$. Thus, $u_i^{\Gamma'}(e_l) = 1$.
In particular, $(u_1^{\Gamma'}(e_1), u_2^{\Gamma'}(e_1)) = (1,0)$ and $(u_1^{\Gamma'}(e_2), u_2^{\Gamma'}(e_2)) = (0,1)$.
We now proof that $u_i^\Gamma(e_l) = u_i^{\Gamma'}(e_l)\ \forall\, (i,j) \in N \times \{3,\ldots,m\}$ such that $u_i^\Gamma(e_l) \in (0,1)$.
Without loss of generality, suppose that $i = 2$.
Define $\hat{X} := \{x \in X \mid u_2^{\Gamma'}(x) = \hat{c} := \alpha_l^{(\Gamma,2)}\ (= u_2^\Gamma(e_l))\}$

$= \{x \in X \mid x_2 + \alpha_3^{(\Gamma',2)} \cdot x_3 + \ldots + \alpha_m^{(\Gamma',2)} \cdot x_m = \hat{c}\}$.

Let $x' \in X$ satisfy $u_2^{\Gamma'}(x') = \hat{c} \in (0,1)$, and
define $\tilde{X} := \{x \in X \mid u_2^\Gamma(x) = \tilde{c} := u_2^\Gamma(x')\}$

$= \{x \in X \mid x_2 + \alpha_3^{(\Gamma,2)} \cdot x_3 + \ldots + \alpha_m^{(\Gamma,2)} \cdot x_m = \tilde{c}\}$.

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[92] If, for example, each $u_i : X \to \mathbb{R}$ satisfies $u_i(x_0,\ldots,x_m) = \sum_{k=1}^m \alpha_k^i \cdot x_k$ on X, $\alpha_k^i \in [0,1]\ \forall\, (i,k) \in N \times \mathbb{N}_m$, $(\alpha_1^1, \alpha_1^2) = (1,0)$, and $(\alpha_2^1, \alpha_2^2) = (0,1)$, then $(N,\bar{X},\{u_i\}_{i\in N})$ is an element of $\mathcal{C}_{po}^n$.

Since Γ' is an order preserving transformation of Γ, we have that $\hat{X} = \tilde{X}$.
Therefore, $(1 - u_2^\Gamma(e_l), 0, u_2^\Gamma(e_l), 0 \ldots, 0) \in \hat{X}$ has to be an element of $\tilde{X}$, which implies that $\tilde{c} = u_2^\Gamma(e_l)$ $(= \alpha_l^{(\Gamma,2)})$.
This implies, in particular, that e_l is an element of $\tilde{X}$, and thus has to be an element of $\hat{X}$. The latter property implies that $\alpha_l^{(\Gamma',2)} = \hat{c}$, and we obtain

$$u_2^{\Gamma'}(e_l) = \alpha_l^{(\Gamma',2)} = \hat{c} \equiv \alpha_l^{(\Gamma,2)} = u_2^\Gamma(e_l).$$

By now, we have shown that $u_i^{\Gamma'}(e_l) = \alpha_l^{(\Gamma',i)} = \alpha_l^{(\Gamma,i)} = u_i^\Gamma(e_l)$ $\forall$ $(i,l) \in N \times \mathbb{N}_m$, which implies that $u_i^\Gamma \equiv u_i^{\Gamma'}$. It follows that $\Gamma = \Gamma'$, and thus every solution concept is ordinally invariant on $\mathcal{C}_{po}^n$.

F.3 A set $\mathcal{C}_{po}^n$ of bargaining games on which ψ^{Nash} is ordinally invariant but $\alpha_{\psi^{Nash}}$ is not Maskin-monotonic in environment $E(\mathcal{C}_{po}^n)$

Consider the set $\mathcal{C}_{po}^n$ as defined in the preceding paragraph for $m = 4$.
Let $\hat{\mathcal{C}}_{po}^n \equiv \{(N, \bar{X}, \{u_i^{\theta_i}\}_{i\in N})\}_{\theta\in\Theta}$ be a subset of $\mathcal{C}_{po}^n$ that contains the two games $\Gamma^\theta \equiv (N, \bar{X}, \{u_i^{\theta_i}\}_{i\in N})$ and $\Gamma^{\theta'} \equiv (N, \bar{X}, \{u_i^{\theta_i'}\}_{i\in N})$ defined by

$$\begin{aligned}
&u_1^{\theta_1}(e_1) = 1, \quad u_1^{\theta_1'}(e_1) = 1, \; u_2^{\theta_2}(e_1) = 0, \; u_2^{\theta_2'}(e_1) = 0,\\
&u_1^{\theta_1}(e_2) = 0, \quad u_1^{\theta_1'}(e_2) = 0, \; u_2^{\theta_2}(e_2) = 1, \; u_2^{\theta_2'}(e_2) = 1,\\
&u_1^{\theta_1}(e_3) = \tfrac{1}{2}, \quad u_1^{\theta_1'}(e_3) = \tfrac{7}{8}, \; u_2^{\theta_2}(e_3) = 1, \; u_2^{\theta_2'}(e_3) = 1,\\
&u_1^{\theta_1}(e_4) = 1, \quad u_1^{\theta_1'}(e_4) = 1, \; u_2^{\theta_2}(e_4) = \tfrac{3}{4}, \; u_2^{\theta_2'}(e_4) = \tfrac{3}{4}.
\end{aligned}$$

Then, $\alpha_{\psi^{Nash}}(\theta) = \{(N, e_4)\}$, $\alpha_{\psi^{Nash}}(\theta') = \{(N, e_3)\}$, and $\alpha_{\psi^{Nash}}$ is not Maskin-monotonic in environment $E(\hat{\mathcal{C}}_{po}^n)$.[93] Figure F.2 illustrates the two games.

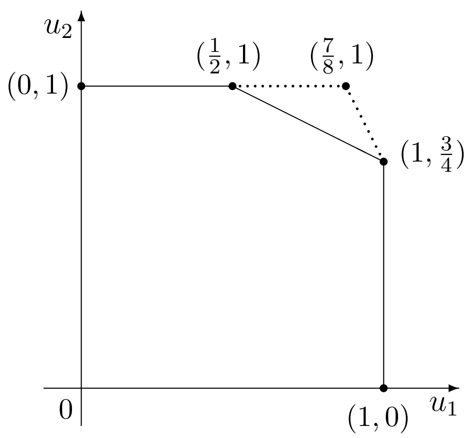

Figure F.2

[93] For $\bar{x} := (N, e_4)$ we have that $\bar{x} \in \alpha_{\psi^{Nash}}(\theta)$, $\bar{x} \notin \alpha_{\psi^{Nash}}(\theta')$, and there does not exist a tuple $(i, x') \in N \times X$ such that $u_i^{\theta_i}(e_4) \geq u_i^{\theta_i}(x')$ and $u_i^{\theta_i'}(e_4) < u_i^{\theta_i'}(x')$. Assume, to the contrary, that there exists such a tuple. If $i = 1$, then $u_i^{\theta_i'}(e_4) = 1 < u_i^{\theta_i'}(x') \leq 1$ provides a contradiction. If $i = 2$, then $u_i^{\theta_i} \equiv u_i^{\theta_i'}$ provides a contradiction.

G Proofs for Proposition 5.1(a), 5.1(b), and Proposition 5.3

G.1 Proof for Proposition 5.1(a)

Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.

Since $u'_i(g(s), T(V)) = (g(s)(V))_i = (\tilde{u}^V(s))_i = \tilde{u}^V_i(s) \; \forall \; (i, s) \in N \times S$, we have that $\Gamma^{E,G,T(V)} = (N, \{S_i\}_{i \in N}, \{u'_i(g(\cdot), T(V))\}_{i \in N}) = (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N}) = \tilde{\Gamma}^V$. Thus, $EC(\Gamma^{E,G,T(V)}) = EC(\tilde{\Gamma}^V) \neq \emptyset$, and it remains to show that $g(EC(\tilde{\Gamma}^V)) \subseteq \alpha_L(T(V))$.

Consider an arbitrary $s \in EC(\tilde{\Gamma}^V)$.

Since $\tilde{u}^V(EC(\tilde{\Gamma}^V)) \subseteq L(V)$, we have that $\tilde{u}^V(s) \in L(V)$. Therefore, there exists a selection $l^* \in \mathcal{S}_L$ of L such that $l^*(V) = \tilde{u}^V(s)$. By definition of g, it follows that $l^*(V) = \tilde{u}^V(s) = g(s)(V)$. In other words, $g(s) \in [l^*]_V \; (= \{l' \in \mathcal{L} \mid l'(V) = l^*(V)\})$. Since $l^* \in \mathcal{S}_L$, it follows that $g(s) \in [l^*]_V \subseteq \bigcup_{l \in \mathcal{S}_L} [l]_V = \alpha_L(T(V))$.

G.2 Proof for Proposition 5.1(b)

Consider an arbitrary $V \in \mathcal{C}^n_{ntu}$.

Since $u'_i(g(x), T(V)) = (g(x)(V))_i = (\tilde{u}^V(x))_i = \tilde{u}^V_i(x) \; \forall \; (i, x) \in N \times Z_H$, we have that $\Gamma^{E,G,T(V)} = (N, \{S_i\}_{i \in N}, \{u'_i(g(\cdot), T(V))\}_{i \in N}) = (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N}) = \tilde{\Gamma}^V$.

Thus, $SPNE^n(\Gamma^{E,G,T(V)}) = SPNE^n(\tilde{\Gamma}^V) \neq \emptyset$,

and it remains to show that $g(O(SPNE^n(\tilde{\Gamma}^V))) \subseteq \alpha_L(T(V))$.

Consider an arbitrary $x \in O(SPNE^n(\tilde{\Gamma}^V))$.

Since $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}^V))) \subseteq L(V)$, we have that $\tilde{u}^V(x) \in L(V)$. Therefore, there exists a selection $l^* \in \mathcal{S}_L$ of L such that $l^*(V) = \tilde{u}^V(x)$. By definition of g, it follows that $l^*(V) = \tilde{u}^V(x) = g(x)(V)$. In other words, $g(x) \in [l^*]_V \; (= \{l' \in \mathcal{L} \mid l'(V) = l^*(V)\})$. Since $l^* \in \mathcal{S}_L$, it follows that $g(x) \in [l^*]_V \subseteq \bigcup_{l \in \mathcal{S}_L} [l]_V = \alpha_L(T(V))$.

G.3 Proof for Proposition 5.3

(i) Consider a tuple $(\theta, \theta') \in \Theta \times \Theta$ such that $\Gamma^{\theta'}$ is an order preserving transformation of Γ^{θ}. Then, $\forall \; (i, q, q') \in N \times F_o(\bar{X}) \times F_o(\bar{X})$,

$$u'_i(q, \theta') > u'_i(q', \theta') \text{ if and only if } u'_i(q, \theta) > u'_i(q', \theta).$$

It follows that, $\forall \; (i, s, \hat{s}) \in N \times S \times S$, where $S := S_1 \times \ldots \times S_n$,

$$u'_i(g(s), \theta') > u'_i(g(\hat{s}), \theta') \text{ if and only if } u'_i(g(s), \theta) > u'_i(g(\hat{s}), \theta).$$

In other words, $\Gamma^{E,G,\theta'}$ is an order preserving transformation of $\Gamma^{E,G,\theta}$.

Since EC is ordinally invariant, we have that $EC(\Gamma^{E,G,\theta'}) = EC(\Gamma^{E,G,\theta})$.

Thus, $g(EC(\Gamma^{E,G,\theta'})) = g(EC(\Gamma^{E,G,\theta}))$.

Since mechanism G fully EC-implements α_ψ in environment E, we have that $g(EC(\Gamma^{E,G,\theta'})) = \alpha_\psi(\theta')$ and $g(EC(\Gamma^{E,G,\theta'})) = \alpha_\psi(\theta)$.

It follows from the preceding that $\psi(\Gamma^{\theta'}) \equiv \alpha_\psi(\theta') = \alpha_\psi(\theta) \equiv \psi(\Gamma^\theta)$.

(ii) Consider a tuple $(\theta, \theta') \in \Theta \times \Theta$ such that $\Gamma^{\theta'}$ is an order preserving transformation of Γ^θ. Then, $\forall\ (i, q, q') \in N \times F_o(\bar{X}) \times F_o(\bar{X})$,

$$u_i'(q, \theta') > u_i'(q', \theta') \text{ if and only if } u_i'(q, \theta) > u_i'(q', \theta).$$

It follows that, $\forall\ (i, h, \hat{h}) \in N \times Z_H \times Z_H$,

$$u_i'(g(h), \theta') > u_i'(g(\hat{h}), \theta') \text{ if and only if } u_i'(g(h), \theta) > u_i'(g(\hat{h}), \theta).$$

In other words, $\Gamma^{E,G,\theta'}$ is an order preserving transformation of $\Gamma^{E,G,\theta}$.

Since $SPNE^n$ is ordinally invariant, we have that

$$SPNE^n(\Gamma^{E,G,\theta'}) = SPNE^n(\Gamma^{E,G,\theta}).$$

Thus, $g(O(SPNE^n(\Gamma^{E,G,\theta'}))) = g(O(SPNE^n(\Gamma^{E,G,\theta})))$.

Since mechanism G fully $SPNE^n$-implements α_ψ in environment E, we have that $g(O(SPNE^n(\Gamma^{E,G,\theta'}))) = \alpha_\psi(\theta')$ and $g(O(SPNE^n(\Gamma^{E,G,\theta}))) = \alpha_\psi(\theta)$.

It follows from the preceding that $\psi(\Gamma^{\theta'}) \equiv \alpha_\psi(\theta') = \alpha_\psi(\theta) \equiv \psi(\Gamma^\theta)$.

H Collections of Games Supporting the Nash Bargaining Solution Concept

Collections $\{\tilde{\Gamma}^V_{(f)}\}_{V \in \mathcal{C}^n_{ntu}}$, $\{\tilde{\Gamma}^V_{(e)}\}_{V \in \mathcal{C}^n_{ntu}}$, and $\{\tilde{\Gamma}^V_{(d)}\}_{V \in \mathcal{C}^n_{ntu}}$ are due to Trockel [51].[94] Collection $\{\tilde{\Gamma}^V_{(d')}\}_{V \in \mathcal{C}^n_{ntu}}$ is a slight modification of the latter one, and is, in parts, similar to an earlier working paper version of collection $\{\tilde{\Gamma}^V_{(d)}\}_{V \in \mathcal{C}^n_{ntu}}$ in Trockel [53].

H.1 Definition of $\{\tilde{\Gamma}^V_{(f)}\}_{V \in \mathcal{C}^n_{ntu}}$, $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$

Consider a game $V \in \mathcal{C}^n_{ntu}$, and let U denote the set $V(N)$ of utility allocations feasible for the grand coalition.

For each $k \in \mathbb{N}_n$, define

$$M_k(U) := \{\, (i_1, \ldots, i_k, \bar{u}_{i_1}, \ldots, \bar{u}_{i_k}) \mid \{i_1, \ldots, i_k\} \subseteq N,\ \sharp\{i_1, \ldots, i_k\} = k, \text{ and } \exists\ u \in U \text{ s.t. } u_{i_l} = \bar{u}_{i_l}\ \forall\ l \in \mathbb{N}_k\}.$$

For each $k \in \{2, \ldots, n\}$, $\forall\ m_{k-1} \equiv (i_1, \ldots, i_{k-1}, \bar{u}_{i_1}, \ldots, \bar{u}_{i_{k-1}}) \in M_{k-1}(U)$, $\forall\ i_k \in N \backslash \{i_1, \ldots, i_{k-1}\}$, define

$$D_{i_k}(U, m_{k-1}) := \{u_{i_k} \in \mathbb{R} \mid (i_1, \ldots, i_k, \bar{u}_{i_1}, \ldots, \bar{u}_{i_{k-1}}, u_{i_k}) \in M_k(U)\}.$$

If $n \geq 3$, then, for each $k \in \mathbb{N}_{n-2}$, bargaining game $(N, U, 0)$ combined with a tuple $m_k \equiv (i_1, \ldots, i_k, \bar{u}_{i_1}, \ldots, \bar{u}_{i_k}) \in M_k(U)$ induces an $(n-k)$-person bargaining game $(N_{m_k}, U(U, m_k), 0)$ via

$$N_{m_k} := N \backslash \{i_1, \ldots, i_k\} \text{ and}$$

[94] Our presentation of collection $\{\tilde{\Gamma}^V_{(d)}\}_{V \in \mathcal{C}^n_{ntu}}$ to $\{\tilde{\Gamma}^V_{(f)}\}_{V \in \mathcal{C}^n_{ntu}}$ is similar to that in Hahmeier [15].

$$U(U, m_k) := \{y \in \mathbb{R}^{N_{m_k}} \mid \exists\, u \in U \text{ s.t. } u_{i_l} = \bar{u}_{i_l}\ \forall\, l \in \mathbb{N}_k \text{ and } y_i = u_i\ \forall\, i \in N_{m_k}\}.$$

Trockel [50] shows that, for each $k \in \{0, \ldots, n-2\}$, and, if $k \neq 0$, for each $m_k \in M_k(U)$ such that $U(U, m_k) \neq \{0\}$, the *bargaining economy induced by the* $(n-k)$*-person bargaining game*

$$G \equiv (N_G, U_G, 0) := \begin{cases} (N, U, 0) & \text{if } k = 0 \\ (N_{m_k}, U(U, m_k), 0) & \text{if } k > 0 \end{cases},$$ [95]

which is defined as the tuple $\mathcal{E}_G \equiv (N_G, (\succeq_i, e_i, \theta_i)_{i \in N_G}, Y)$, where

$Y := U_G$ is the production possibility set,

$\theta_i := \frac{1}{n-k}$ is agent i's share in Y,

$e_i := 0 \in \mathbb{R}^{N_G}$ is agent i's initial endowment, and

$\succeq_i$ is agent i's preference relation over the consumption set $\mathbb{R}_+^{N_G}$, which is assumed to be representable by the function $u_i : \mathbb{R}_+^{N_G} \to \mathbb{R}$ defined by $u_i(x) := x_i\ \forall\, x \in \mathbb{R}_+^{N_G}$,

has a unique Walrasian equilibrium, and that the equilibrium allocation $x^*(G)$ coincides with the Nash Bargaining Solution of bargaining game G.

For each $k \in \{0, \ldots, n-2\}$, and, if $k \neq 0$, for each $m_k \in M_k(U)$, consider the $(n-k)$-person bargaining game $G \equiv (N_G, U_G, 0)$ as defined above, and, $\forall\, i_{k+1} \in N_G$, define $d^G_{i_{k+1}} : D_{i_{k+1}} \to \mathbb{R}$ as follows:

- If $U_G = \{0\}$, define $D_{i_{k+1}} := \{0\} \subseteq \mathbb{R}$ and $d^G_{i_{k+1}}(\{0\}) := 0$.
- If $U_G \neq \{0\}$, define

$$D_{i_{k+1}} := \begin{cases} [0, 1] & \text{if } k = 0 \\ D_{i_{k+1}}(U, m_k) & \text{if } k > 0 \end{cases},$$

and, for each $\bar{u}_{i_{k+1}} \in D_{i_{k+1}}$, define $d^G_{i_{k+1}}(\bar{u}_{i_{k+1}}) \in \mathbb{R}$ as agent i_{k+1}'s demand for commodity i_{k+1} in bargaining economy $\mathcal{E}_G$ at the (normalized) price system $g'(s^G(i_{k+1}, \bar{u}_{i_{k+1}}))$ and income

$$\frac{1}{n-k} \cdot \sum_{i \in N_G} (g'(s^G(i_{k+1}, \bar{u}_{i_{k+1}})))_i \cdot (s^G(i_{k+1}, \bar{u}_{i_{k+1}}))_i\ ,$$

[95] Trockel [50] uses the terminology "associated bargaining economy". In the words of Trockel [51], "the set of feasible utility allocations of the players in a bargaining game is interpreted as a production possibility set describing all technologically possible ways of producing joint utility vectors. All players have equal shares in this technology set and, hence, in any resulting profit from production. The different players' utilities are the commodities. Each player, as an agent of the economy, is only interested in 'his' commodity, namely his utility. Endowments are zero for each player to guarantee that the only source of income is the profit earned from production of joint utilities."

where g' denotes the mapping that associates with every vector $x \in \partial U_G$ the normal vector at x to ∂U_G normalized by $\|g'(x)\|_2 = 1$,[96]

and $s^G(i_{k+1}, \bar{u}_{i_{k+1}}) \in \partial U_G$ denotes the utility (commodity) allocation of $\mathcal{E}_G$ in which agent i_{k+1} gets $\bar{u}_{i_{k+1}}$ and each agent $i \in N_G \backslash \{i_{k+1}\}$ gets

$$\max\{y \in \mathbb{R} \mid \exists\ u \in U_G \text{ s.t. } (u_{i_{k+1}}, u_i) = (\bar{u}_{i_{k+1}}, y)\} \text{ if } n-k=2$$

$$\text{and } \begin{cases} 0 & \text{if } U_{G'} = \{0\} \\ (x^*(G'))_i & \text{if } U_{G'} \neq \{0\} \end{cases} \text{ if } n-k \geq 3,$$

$G' \equiv (N_{G'}, U_{G'}, 0)$ denoting the $(n-(k+1))$-person bargaining game that is induced by game $(N, U, 0)$ and $m_{k+1} \equiv (i_1, \ldots, i_{k+1}, \bar{u}_{i_1}, \ldots, \bar{u}_{i_{k+1}})$.

The (so defined) function $d^G_{i_{k+1}} : D_{i_{k+1}} \rightarrow \mathbb{R}$ is strictly decreasing and continuous on $D_{i_{k+1}}$, has a unique fixed point, and this unique fixed point is at $(x^*(G))_{i_{k+1}}$. The function $\min\{\cdot, d^G_{i_{k+1}}(\cdot)\} : D_{i_{k+1}} \rightarrow \mathbb{R}$ is continuous and attains its maximum $(x^*(G))_{i_{k+1}}$ (which coincides with the Nash Bargaining Solution of G evaluated at i_{k+1}) in and only in $(x^*(G))_{i_{k+1}}$ (which lies in the interior of $D_{i_{k+1}}$).

For each $V \in \mathcal{C}^n_{ntu}$, define the normal form game $\tilde{\Gamma}^V_{(f)} \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})$ by

$S_i := [0,1]\ \forall\ i \in N$ and

$$\tilde{u}^V_i(s) := \begin{cases} s_i & \text{if } s \in V(N) \\ \min\{s_i, d_i^{(N,V(N),0)}(s_i)\} & \text{otherwise} \end{cases} \quad \forall\ (i,s) \in N \times [0,1]^n.$$ [97]

Then, for each $V \in \mathcal{C}^n_{ntu}$, game $\tilde{\Gamma}^V_{(f)}$ has a unique NE $\hat{s}^V$ and this unique NE satisfies $\tilde{u}^V(\hat{s}^V) = \hat{s}^V = Nash(V)$ (Trockel [51], Proposition 1).

H.2 Definition of $\{\tilde{\Gamma}^V_{(e)}\}_{V \in \mathcal{C}^n_{ntu}}$, $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$

For each $V \in \mathcal{C}^n_{ntu}$, define the normal form game $\tilde{\Gamma}^V_{(e)} \equiv (N, \{S_i\}_{i \in N}, \{\tilde{u}^V_i\}_{i \in N})$ by

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[96]Note that, due to the assumptions on bargaining games in $\bar{\mathcal{B}}^n$, U_G is strictly convex and mapping g' is a continuously differentiable mapping $g' : \partial U_G \rightarrow \mathbb{R}^{N_G}_{++}$. Since $(g'(s^G(i_{k+1}, \bar{u}_{i_{k+1}})))_i \neq 0\ \forall\ i \in N_G$, agent i_{k+1}'s demand $d^G_{i_{k+1}}(\bar{u}_{i_{k+1}})$ is well defined as an element of $\mathbb{R}$. Note that, due to her preferences in $\mathcal{E}_G$, agent i_{k+1}'s demand for commodity $j \neq i_{k+1}$ is 0.

[97]In the words of Trockel [51], "the effect of this payoff rule, reflecting a Walrasian evaluation of utility allocations, is an 'adequate' claim of each player. A very modest utility claim of player 1 results in ... a high level of his demand for commodity one, which turns out to be in excess to x_1. So he gets only his modest claim. A very high utility claim ... results ... in a low level of his resulting demand. Then he receives only this small demand. For all players together it is strategically optimal to claim 'adequate' utility levels, i.e. those which coincide with the derived demands. If all players act accordingly, there is no need for the hypothetical market system, as the resulting utility allocation is feasible in V."

$S_i := [0,1] \ \forall\ i \in N$ and
$\tilde{u}_i^V(s) := \min\{s_i, d_i^{(N,V(N),0)}(s_i)\} \ \forall\ (i,s) \in N \times [0,1]^n.$

For each $V \in \mathcal{C}^n_{ntu}$, game $\tilde{\Gamma}^V_{(e)}$ has a unique DSE $\hat{s}^V$ and this unique DSE satisfies $\tilde{u}^V(\hat{s}^V) = \hat{s}^V = Nash(V)$ (Trockel [51], Proposition 2).

H.3 Definition of $\{\tilde{\Gamma}^V_{(d)}\}_{V \in \mathcal{C}^n_{ntu}}$, $\mathcal{C}^n_{ntu} \subseteq \bar{\mathcal{B}}^n$

For each game $V \in \mathcal{C}^n_{ntu}$, define the n-person extensive form game with perfect information $\tilde{\Gamma}^V_{(d)} \equiv (N, H, p, \{\tilde{u}_i^V\}_{i \in N})$ as follows.

In stage 1, player 1 chooses an element from the set $[0,1]$.
If, in stage $i \in \{1,\ldots,n-2\}$, player i has chosen an element from the set $[0,1)$, then, in stage $i+1$, player $i+1$ chooses an element from the set $[0,1]$.
If, in stage $n-2$, player $n-2$ has choosen an element from the set $[0,1)$,then, in stage $n-1$, player $n-1$ chooses an element from the set $[0,1]$, and, in stage n, player n chooses an element from the set { 'left','right'}.
In other words, if $n \geq 3$, then

$$Z_H = [0,1)^{n-2} \times [0,1] \times \{\text{'left','right'}\} \cup \{(1)\}$$
$$\cup \{(a_1,\ldots,a_{k-1},1) \in [0,1)^{k-1} \times \{1\} \mid k \in \{2,\ldots,n-2\}\}, \text{ and}$$
$$H = \{\emptyset\} \ \cup \ Z_H \ \cup \ \{(a_1,\ldots,a_k) \in [0,1)^k \mid k \in \mathbb{N}_{n-2}\} \ \cup \ [0,1)^{n-2} \times [0,1],$$

and, if $n = 2$, then

$$Z_H = [0,1] \times \{\text{'left','right'}\}, \text{ and}$$
$$H = \{\emptyset\} \ \cup \ Z_H \ \cup \ [0,1].$$

In both cases,

$p : H \backslash Z_H \Rightarrow N$ is defined by $p(\emptyset) := \{1\}$ and
$$p(a_1,\ldots,a_k) := \{k+1\} \ \forall\ k \in \mathbb{N}_{n-1}, \forall\ (a_1,\ldots,a_k) \in H \backslash Z_H.$$

Let U denote the set $V(N)$.
For all $i \in \{2,\ldots,n-1\}$, $\forall\ (a_1,\ldots,a_{i-1}) \in H \backslash Z_H$, $\forall\ a_i \in [0,1]$, define (recursively)

$$A_i(a_1,\ldots,a_i) := a_i \cdot \max\{\ y \in [0,1] \mid y \in D_i(U, m_{i-1})\},$$

where $m_{i-1} \equiv (1,\ldots,i-1,a_1,A_2(a_1,a_2),\ldots,A_{i-1}(a_1,\ldots,a_{i-1}))$.

If $n \geq 3$, then, $\forall\ i \in \mathbb{N}_n$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by

$$\tilde{u}_1^V(a_1,\ldots,a_n) := \min\{a_1, d_1^{(N,U,0)}(a_1)\},$$
$$\tilde{u}_i^V(a_1,\ldots,a_n) := \min\{A_i(a_1,\ldots,a_i), d_i^{(N\backslash\{1,\ldots,i-1\},U(U,m_{i-1}),0)}(A_i(a_1,\ldots,a_i))\} \quad \forall\ i \in \{2,\ldots,n-2\},$$

$$\tilde{u}_i^V(a_1,\ldots,a_n) := \begin{cases} l_i^{(a_1,\ldots,a_{n-1})} & \text{if } a_n = \text{'left'} \\ r_i^{(a_1,\ldots,a_{n-1})} & \text{if } a_n = \text{'right'} \end{cases} \quad \forall\, i \in \{n-1, n\},$$

$\tilde{u}_1^V(a_1, \ldots, a_{k-1}, 1) := \min\{a_1, d_1^{(N,U,0)}(a_1)\} \; \forall \; k \in \{2, \ldots, n-2\}$,
$\tilde{u}_i^V(a_1, \ldots, a_{k-1}, 1) := \min\{A_i(a_1, \ldots, a_i), d_i^{(N\setminus\{1,\ldots,i-1\}, U(U,m_{i-1}),0)}(A_i(a_1, \ldots, a_i))\}$
$\forall \; k \in \{2, \ldots, n-2\}$ and $\forall \; i \in \{2, \ldots, n-3\}$ such that $i < k$,
$\tilde{u}_i^V(a_1, \ldots, a_{i-1}, 1) := 0 \; \forall \; i \in \{2, \ldots, n-2\}$,[98]
$\tilde{u}_i^V(a_1, \ldots, a_{k-1}, 1) := 0 \; \forall \; k \in \{2, \ldots, n-2\}$ and $\forall \; i \in \{3, \ldots, n\}$ s.t. $i > k$,
$\tilde{u}_1^V(1) := 0$,[99] and $\tilde{u}_i^V(1) := 0 \; \forall \; i \in \{2, \ldots, n\}$,
where $l^{(a_1,\ldots,a_{n-1})}$ denotes the efficient element of $U(U, m_{n-2}) \subseteq \mathbb{R}^{\{n-1,n\}}$ that satisfies
$l_{n-1}^{(a_1,\ldots,a_{n-1})} = A_{n-1}(a_1, \ldots, a_{n-1})$,
$r^{(a_1,\ldots,a_{n-1})}$ denotes the efficient element of $U(U, m_{n-2}) \subseteq \mathbb{R}^{\{n-1,n\}}$ that satisfies

$$r_{n-1}^{(a_1,\ldots,a_{n-1})} = \min\{d_{n-1}^{(N\setminus\{1,\ldots,n-2\}, U(U,m_{n-2}),0)}(A_{n-1}(a_1, \ldots, a_{n-1})), \max_{y \in D_{n-1}(U,m_{n-2})} y\},$$

and m_{i-1} denotes the respective $(1, \ldots, i-1, a_1, A_2(a_1, a_2), \ldots, A_{i-1}(a_1, \ldots, a_{i-1}))$.

If $n = 2$, then, $\forall \; i \in \{1, 2\}$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by

$$\tilde{u}_i^V(a_1, a_2) := \begin{cases} l_i^{(a_1)} & \text{if } a_2 = \text{'left'} \\ r_i^{(a_1)} & \text{if } a_2 = \text{'right'} \end{cases} \quad \forall \, i \in \{1, 2\},$$

where $l^{(a_1)}$ denotes the efficient element of U that satisfies $l_1^{(a_1)} = a_1$, and $r^{(a_1)}$ denotes the efficient element of U that satisfies $r_1^{(a_1)} = \min\{d_1^{(N,U,0)}(a_1), 1\}$.

Then, for each $V \in \mathcal{C}_{ntu}^n$, the extensive form game $\tilde{\Gamma}_{(d)}^V \equiv (N, H, p, \{\tilde{u}_i^V\}_{i \in N})$ has a SPNE, and $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}_{(d)}^V))) = Nash(V)$ (Trockel [51], Proposition 3).[100]

H.4 Definition of $\{\tilde{\Gamma}_{(d')}^V\}_{V \in \mathcal{C}_{ntu}^n}$, $\mathcal{C}_{ntu}^n \subseteq \bar{\mathcal{B}}^n$

For each game $V \in \mathcal{C}_{ntu}^n$, consider the following modification $\tilde{\Gamma}_{(d')}^V \equiv (N, H, p, \{\tilde{u}_i^V\}_{i \in N})$ of game $\tilde{\Gamma}_{(d)}^V$.
In stage 1, player 1 chooses an element from the set $[0, 1]$. If, in stage $i \in \{1, \ldots, n-2\}$, player i has chosen an element from the set $[0, 1)$, then, in stage $i + 1$, player $i + 1$

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[98] Trockel [51] defines $\tilde{u}_i^V(a_1, \ldots, a_{i-1}, 1) := \frac{1}{n-i} \cdot \max\{y \mid y \in D_i(U, m_{i-1})\}$.

[99] Trockel [51] defines $\tilde{u}_1^V(1) := \frac{1}{n-1} \cdot \max\{y \mid y \in [0, 1]\} = \frac{1}{n-1}$.

[100] The idea is the following: if player $n-1$ is able, given the choices of player 1 to $n-2$, to choose his Nash coordinate (by 'playing' an appropriate action a_{n-1}), he will do so in every SPNE, taking into account player n's optimal response to his choice. If player $i \in \{2, \ldots, n-2\}$ is able, given the choices of player 1 to $i-1$, to choose his Nash coordinate, he will do so in every SPNE, taking into account his preferences. And, player 1 can and will choose his Nash coordinate in every SPNE, taking into account his preferences. Player 1's choice (of his Nash coordinate) now allows player 2 to choose his Nash coordinate. Player 2's choice allows player 3 to choose his Nash coordinate, and so on.

chooses an element from the set $[0,1]$.

In other words, if $n \geq 3$, then

$$Z_H = \{(a_1,\dots,a_{n-1}) \mid a_i \in [0,1)\ \forall\ i \in \mathbb{N}_{n-2}, a_{n-1} \in [0,1]\}$$
$$\cup\ \{(1)\}\ \cup\ \{(a_1,\dots,a_{k-1},1) \in [0,1)^{k-1} \times \{1\} \mid k \in \{2,\dots,n-2\}\}, \text{ and}$$
$$H = \{\emptyset\}\ \cup\ Z_H\ \cup\ \{(a_1,\dots,a_k) \in [0,1)^k \mid k \in \mathbb{N}_{n-2}\},$$

and, if $n = 2$, then

$$Z_H = [0,1], \text{ and}$$
$$H = \{\emptyset\}\ \cup\ Z_H.$$

In both cases,

$p : H\backslash Z_H \rightrightarrows N$ is defined by $p(\emptyset) := \{1\}$ and

$$p(a_1,\dots,a_k) := \{k+1\}\ \forall\ k \in \mathbb{N}_{n-2}, \forall\ (a_1,\dots,a_k) \in H\backslash Z_H.$$

Let U denote the set $V(N)$.

Define $A_1(a_1) := a_1\ \forall\ a_1 \in [0,1]$, and,

$\forall\ i \in \{2,\dots,n-1\}$, $\forall\ (a_1,\dots,a_{i-1}) \in H\backslash Z_H$, $\forall\ a_i \in [0,1]$, define (recursively)

$$A_i(a_1,\dots,a_i) := a_i \cdot \max\{\ y \in [0,1] \mid y \in D_i(U, m_{i-1})\},$$

where $m_{i-1} \equiv (1,\dots,i-1, A_1(a_1), A_2(a_1,a_2),\dots,A_{i-1}(a_1,\dots,a_{i-1}))$.

If $n \geq 3$, then, $\forall\ i \in \mathbb{N}_n$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by

$$\tilde{u}_1^V(a_1,\dots,a_{n-1}) := \min\{A_1(a_1), d_1^{(N,U,0)}(A_1(a_1))\},$$
$$\tilde{u}_i^V(a_1,\dots,a_{n-1}) := \min\{A_i(a_1,\dots,a_i), d_i^{(N\backslash\{1,\dots,i-1\},U(U,m_{i-1}),0)}(A_i(a_1,\dots,a_i))\}$$
$$\forall\ i \in \{2,\dots,n-1\},$$

$$\tilde{u}_n^V(a_1,\dots,a_{n-1}) := \begin{cases} f_n^{(a_1,\dots,a_{n-1})} & \text{if } a_{n-1} \neq 1 \\ 0 & \text{if } a_{n-1} = 1 \end{cases},$$

$$\tilde{u}_1^V(a_1,\dots,a_{k-1},1) := \min\{A_1(a_1), d_1^{(N,U,0)}(A_1(a_1))\}\ \forall\ k \in \{2,\dots,n-2\},$$
$$\tilde{u}_i^V(a_1,\dots,a_{k-1},1) := \min\{A_i(a_1,\dots,a_i), d_i^{(N\backslash\{1,\dots,i-1\},U(U,m_{i-1}),0)}(A_i(a_1,\dots,a_i))\}$$
$$\forall\ k \in \{2,\dots,n-2\} \text{ and } \forall\ i \in \{2,\dots,n-3\} \text{ such that } i < k,$$
$$\tilde{u}_i^V(a_1,\dots,a_{i-1},1) :=$$
$$\min\{A_i(a_1,\dots,a_{i-1},1), d_i^{(N\backslash\{1,\dots,i-1\},U(U,m_{i-1}),0)}(A_i(a_1,\dots,a_{i-1},1))\}$$
$$\forall\ i \in \{2,\dots,n-2\},$$
$$\tilde{u}_i^V(a_1,\dots,a_{k-1},1) := 0\ \forall\ k \in \{2,\dots,n-2\} \text{ and } \forall\ i \in \{2,\dots,n\} \text{ s.t. } i > k,$$
$$\tilde{u}_1^V(1) := \min\{A_1(1), d_1^{(N,U,0)}(A_1(1))\}, \text{ and}$$
$$\tilde{u}_i^V(1) := 0\ \forall\ i \in \{2,\dots,n\},$$

where $f^{(a_1,\dots,a_{n-1})}$ denotes the efficient element of $U(U, m_{n-2}) \subseteq \mathbb{R}^{\{n-1,n\}}$ that satisfies

$$f_{n-1}^{(a_1,\dots,a_{n-1})} = \min\{A_{n-1}(a_1,\dots,a_{n-1}), d_{n-1}^{(N\backslash\{1,\dots,n-2\},U(U,m_{n-2}),0)}(A_{n-1}(a_1,\dots,a_{n-1}))\},$$

and m_{i-1} denotes the respective $(1,\dots,i-1, A_1(a_1), A_2(a_1,a_2),\dots,A_{i-1}(a_1,\dots,a_{i-1}))$.

If $n = 2$, then, $\forall\, i \in \{1, 2\}$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by
$\tilde{u}_1^V(a_1) := \min\{A_1(a_1), d_1^{(N,U,0)}(A_1(a_1))\}$, and

$$\tilde{u}_2^V(a_1) := \begin{cases} f_2^{(a_1)} & \text{if } a_1 \in [0,1) \\ 0 & \text{if } a_1 = 1 \end{cases},$$

where $f^{(a_1)}$ denotes the efficient element of U that satisfies

$$f_1^{(a_1)} = \min\{A_1(a_1), d_1^{(N,U,0)}(A_1(a_1))\}.$$

Then, for each $V \in \mathcal{C}_{ntu}^n$, the extensive form game $\tilde{\Gamma}_{(d')}^V \equiv (N, H, p, \{\tilde{u}_i^V\}_{i\in N})$ has a SPNE, and $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}_{(d')}^V))) = Nash(V)$.[101]

H.5 A Collection of Games Satisfying the Assumptions of Corollary 5.2(b)

For each game $V \in \mathcal{C}_{ntu}^n$, define the n-person extensive form game with perfect information $\tilde{\Gamma}^V \equiv (N, H, p, \{\tilde{u}_i^V\}_{i\in N})$ as follows.
In stage $i \in \{1, \ldots, n-1\}$, player i chooses an element from the set $[0, 1]$.
In other words,

$$Z_H = \{(a_1, \ldots, a_{n-1}) \mid a_i \in [0,1]\ \forall\, i \in \mathbb{N}_{n-1}\},$$
$$H = \{\emptyset\} \cup Z_H \cup \{(a_1, \ldots, a_k) \in [0,1]^k \mid k \in \mathbb{N}_{n-2}\}, \text{ and}$$
$$p : H \backslash Z_H \Rightarrow N \text{ is defined by } p(\emptyset) := \{1\} \text{ and}$$
$$p(a_1, \ldots, a_k) := \{k+1\}\ \forall\, k \in \mathbb{N}_{n-2}, \forall\, (a_1, \ldots, a_k) \in H\backslash Z_H.$$

Let U denote the set $V(N)$, and define

$$\tilde{A}_1(a_1) := b_1^V + a_1 \cdot (q_1^V - b_1^V),$$
$$A_1(a_1) := \min\{\tilde{A}_1(a_1), d_1^{(N,U,0)}(\tilde{A}_1(a_1))\}\ \forall\, a_1 \in [0,1],$$

and, $\forall\, i \in \{2, \ldots, n-1\}$, $\forall\, (a_1, \ldots, a_{i-1}) \in H\backslash Z_H$, $\forall\, a_i \in [0,1]$, define (recursively)

$$\tilde{A}_i(a_1, \ldots, a_i) := b_i^V + a_i \cdot (\min\{\max\{\, y \in [0,1] \mid y \in D_i(U, m_{i-1})\}, q_i^V\} - b_i^V),$$
$$A_i(a_1, \ldots, a_i) := \min\{\tilde{A}_i(a_1, \ldots, a_i), d_i^{(N\backslash\{1,\ldots,i-1\}, U(U,m_{i-1}),0)}(\tilde{A}_i(a_1, \ldots, a_i))\},$$

where $m_{i-1} \equiv (1, \ldots, i-1, a_1, A_2(a_1, a_2), \ldots, A_{i-1}(a_1, \ldots, a_{i-1}))$.

If $n \geq 3$, then, $\forall\, i \in \mathbb{N}_n$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by
$\tilde{u}_i^V(a_1, \ldots, a_{n-1}) := A_i(a_1, \ldots, a_i)\ \forall\, i \in \{1, \ldots, n-1\}$,
$\tilde{u}_n^V(a_1, \ldots, a_{n-1}) := f_n^{(a_1,\ldots,a_{n-1})}$,

[101] The idea is almost the same as for the preceding collection: if player $i \in \{2, \ldots, n-1\}$ is able, given the choices of player 1 to $i-1$, to choose his Nash coordinate (by 'playing' an appropriate action a_i), he will do so in every SPNE, taking into account his preferences. And, player 1 can and will choose his Nash coordinate in every SPNE, taking into account his preferences. Player 1's choice (of his Nash coordinate) now allows player 2 to choose his Nash coordinate. Player 2's choice allows player 3 to choose his Nash coordinate, and so on.

where $f^{(a_1,\ldots,a_{n-1})}$ denotes the efficient element of $U(U, m_{n-2}) \subseteq \mathbb{R}^{\{n-1,n\}}$ that satisfies $f_{n-1}^{(a_1,\ldots,a_{n-1})} = A_{n-1}(a_1, \ldots, a_{n-1})$, where m_{n-2} denotes
$m_{n-2} \equiv (1, \ldots, n-2, A_1(a_1), A_2(a_1, a_2), \ldots, A_{n-2}(a_1, \ldots, a_{n-2}))$.

If $n = 2$, then, $\forall\, i \in \{1, 2\}$, player i's utility function $\tilde{u}_i^V : Z_H \to \mathbb{R}$ is defined by
$\tilde{u}_1^V(a_1) := A_1(a_1)$ and $\tilde{u}_2^V(a_1) := f_2^{(a_1)}$,
where $f^{(a_1)}$ denotes the efficient element of U that satisfies $f_1^{(a_1)} = A_1(a_1)$.

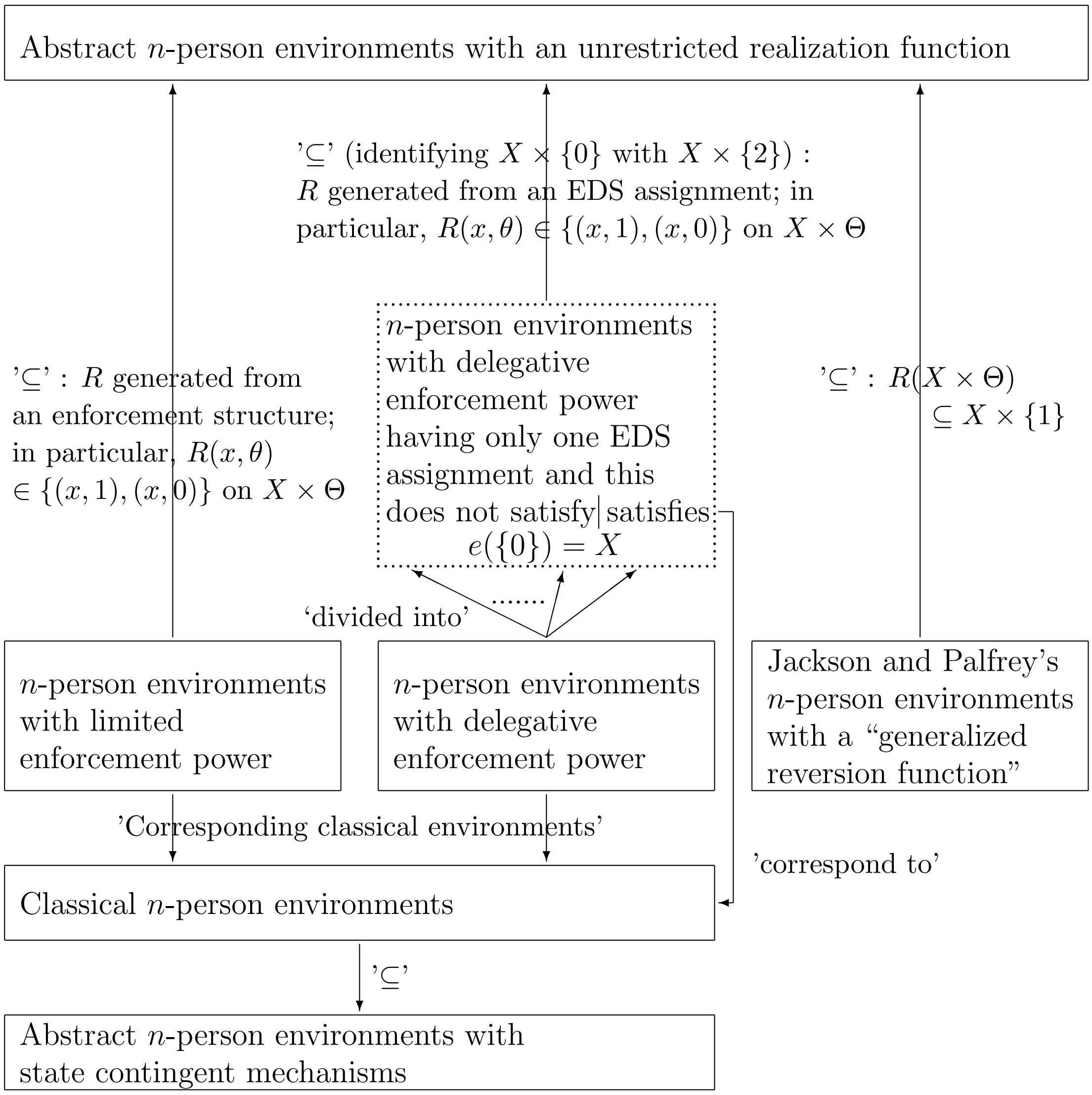
Then, for each $V \in \mathcal{C}_{ntu}^n$, game $\tilde{\Gamma}^V$ has a SPNE, $\tilde{u}^V(O(SPNE^n(\tilde{\Gamma}^V))) = Nash(V)$,[102] $\tilde{u}^V(x) \in V(N) \subseteq F_{up}(V)\ \forall\, x \in Z_H$, and $R(g(x), T(V)) = (g(x), 1)\ \forall\, x \in Z_H$ (since $\tilde{u}_i^V(x) \geq b_i^V\ \forall\, i \in N$).

J Time Schedule

Time Period	Entry Conditions and Parameters	Actions	Possible Results and Interpretation	
T_1	—	mechanism is 'played'	$x \in X$	outcome x is suggested by mechanism
T_2	$x \in X$	implementation of x is considered and possibly realized	$(x, 0)$	outcome x is not implemented in T_2 ('right after the mechanism has been played')
			$(x, 1)$	outcome x is implemented in T_2 ('right after the mechanism has been played')
T_3	T_2 results in $(x, 0)$, $x \in X$	implementation problem is reconsidered; an implementation is possibly realized	$(y, 2)$	outcome $y \in X \setminus \{\bar{x}\}$ is implemented in T_3
			$(\bar{x}, 2)$	outcome $\bar{x}$ is implemented in T_3 or outcome $\bar{x}$ prevails since no other outcome is implemented in T_3

[102] The idea is the following: each player $i \in \{1, \ldots, n-1\}$ is able, given the choices of player 1 to $i-1$, to choose his Nash coordinate (by 'playing' $a_i \in [0, 1]$ such that $\tilde{A}_i(a_1, \ldots, a_i) = (Nash(V))_i$). And, taking into account his preferences, he will do so in every SPNE.

K Abstract Relationship between the Environments



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