Recent Results in Infinite Dimensional Analysis and Applications to Feynman Integrals

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Chapter 1

Introduction

In recent years Gaussian analysis and in particular white noise analysis have developed to a useful tool in applied mathematics and mathematical physics. White noise analysis is a mathematical framework which offers various generalizations of concepts known from finite dimensional analysis, among them are differential operators and Fourier transform. For a detailed exposition of the theory and for many examples of applications we refer the reader to the recent monographs [BeKo88, HKPS93, Ob94, Hi80] and the introductory articles [Kuo92, Po91, S94, W93].

This work consists of three different main parts:

- The generalization of the theory to an infinite dimensional analysis with underlying *non-Gaussian* measure.
- Further development of Gaussian analysis.
- Applications to the theory of path-integrals.

Some of the results presented here have already been published as joint works, see [KLPSW94, LLSW94a, LLSW94b, CDLSW95, KoSW95, KSWY95]. We present here a systematic exposition of this circle of ideas.

Non-Gaussian infinite dimensional analysis

An approach to such a theory was recently proposed by [AKS93]. For smooth probability measures on infinite dimensional linear spaces a biorthogonal decomposition is a natural extension of the orthogonal one that is well known in Gaussian analysis. This biorthogonal "Appell" system has been constructed for smooth measures by Yu.L. Daletskii [Da91]. For a detailed description of its use in infinite dimensional analysis we refer to [ADKS94].

Aim of the present work (Chapter 3). We consider the case of non–degenerate measures on co-nuclear spaces with analytic characteristic functionals. It is worth emphasizing that no further condition such as quasi–invariance of the measure or smoothness of logarithmic derivatives are required. The point here is that the important example of Poisson noise is now accessible.
For any such measure μ we construct an Appell system $\mathbb{A}^\mu$ as a pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ of Appell polynomials $\mathbb{P}^\mu$ and a canonical system of generalized functions $\mathbb{Q}^\mu$, properly associated to the measure μ.

Central results. Within the above framework

- we obtain an explicit description of the test function space introduced in [ADKS94] (Theorem 28)
- in particular this space is in fact identical for all the measures that we consider
- characterization theorems for generalized as well as test functions are obtained analogously as in Gaussian analysis [KLS94] for more references see [KLPSW94] (Theorems 33 and 35)
- the well known Wick product and the corresponding Wick calculus [KLS94] extends rather directly (Section 3.7)
- similarly, a full description of positive distributions (as measures) will be given (Section 3.8).

Finally we should like to underline here the important conceptual role of holomorphy here as well as in earlier studies of Gaussian analysis (see e.g., [PS91, Ou91, KLPSW94, KLS94] as well as the references cited therein).

All these results are collected in Chapter 3 as well as [KSWY95].

Gaussian analysis

In recent years there was an increasing interest in white noise analysis, due to its rapid developments in mathematical structure and applications in various domains. Especially, the circle of ideas going under the heading 'characterization theorems' has played quite an important role in the last few years. These results [Ko80a, Lee89, PS91], and their variations and refinements (see, e.g., [KPS91, MY90, Ob91, SW93, Yan90, Zh92], and references quoted there), provide a deep insight into the structure of spaces of smooth and generalized random variables over the white noise space or – more generally – Gaussian spaces. Also, they allow for rather straightforward applications of these notions to a number of fields: for example, Feynman integration [FPS91, HS83, KaS92, LLSW94a], representation of quantum field theory [AHPS89, PS93], stochastic equations [CLP93, Po91, Po92, Po94], intersection local times [FHSW94, Wa91], Dirichlet forms [AHPRS90a, AHPRS90b, HPS88], infinite dimensional harmonic analysis [Hi89] and so forth. Moreover, characterization theorems have been at the basis of new methods for the construction of smooth and generalized random variables [KoS93, MY90] which seem to be useful in applications untractable by existing methods (e.g., [HLØUZ93a, HLØUZ93b]).

One of the basic technical ideas in the development of the theory is the use of dual pairs of spaces of test and generalized functionals. Since the usefulness of a particular test function space depends on the application one has in mind various dual pairs appear in the literature. In this work we are particulary interested in the following spaces:

- **The Hida spaces**
  We construct a nuclear rigging

$$(\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})'$$

We give the construction of the second quantized space $(\mathcal{N})$ solely in terms of the topology of $\mathcal{N}$, independent of the particular representation as a projective limit. The purpose of Section 4.1 is four-fold: We wish 1. to clarify and generalize the structure of the existing characterization theorems, and at the same time, 2. to review and unify recent developments in this direction, 3. to establish the connection to rich, related mathematical literature [AR73, Co82, Di81, Si69, Za76], which might be helpful in future developments, and – last but not least – 4. to fill a gap in the article [PS91]. In the course of doing this, we also establish some new results, for instance an analytic extension property of U–functionals, and the topological invariance of certain spaces of generalized random variables with respect to different construction schemes.
The material presented in Section 4.1 is the central part of [KLPSW94]

- **The test function spaces $\mathcal{G}$ and $\mathcal{M}$**
  In section 4.3 we discuss the space $\mathcal{G}$ introduced in [PT94]. This space and is dual are interesting because all terms in the chaos expansion are given by Hilbert space kernels. So also the distributions have an expansion in a series of n-fold stochastic integrals.
  A second useful property is that $\mathcal{G}$ is an algebra under pointwise multiplication [PT94] which is larger than $(\mathcal{N})$. Since we are interested in more general pointwise products, we introduce a second test function space $\mathcal{M}$ which again is bigger than $\mathcal{G}$. One can not expect that $\mathcal{M}$ is closed under multiplication but we will show that pointwise multiplication is a separately continuous bilinear map $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$. (Corollary 66).
  We will see that the shift operator $\tau_\eta : \varphi \mapsto \varphi(\cdot + \eta)$, $\quad \eta \in \mathcal{H}$ is well defined from $\mathcal{G}$ into $\mathcal{G}$ and $\mathcal{M}$ into $\mathcal{M}$ and that we can extend τ_η to complex $\eta \in \mathcal{H}_\mathbb{C}$ (Theorem 67) Using this it is easy to see that $\mathcal{G}$ and $\mathcal{M}$ are closed under Gâteaux differentiation.
  In section 4.3.3.2 we consider the composition of test functions with projection operations on $\mathcal{N}'$. This is of particular interest to understand the action of Donsker's delta on test functions. Note that Donsker's delta is in $\mathcal{M}'$ (Theorem 90). We will understand what it means to integrate out this delta distribution (Proposition 72).

- **The Meyer–Yan triple**
  We sketch the well-known construction of the triple [MY90] and state a convenient form of the characterization theorem for generalized functions [KoS93]. For later use we add a corollary controlling the convergence of a sequence of generalized functions. Furthermore the integration of a family of distributions is discussed and controlled in terms of S-transform.

Besides the discussion of various spaces of test and generalized functions Chapter 4 also contains a discussion of the scaling operator σ_z which suggests one of the possible approaches to path integrals in a white noise framework. We will collect some properties of σ_z and specify its domain and range where it acts continuously (similar to [HKPS93]). But applications to path integrals require extended domains of σ_z. This naturally leads to the study of traces. In Proposition 84 we give sufficient conditions on $\varphi \in L^2(\mu)$ to ensure that $\sigma_z\varphi$ exists in some useful sense.

We close Chapter 4 by a detailed discussion of Donsker's delta function. In particular we study its behavior under σ_z. Most of this results have already been published in [LLSW94b].

Applications to Feynman integrals

Path integrals are a useful tool in many branches of theoretical physics including quantum mechanics, quantum field theory and polymer physics. We are interested in a rigorous treatment of such path integrals. As our basic example we think of a quantum mechanical particle.

On one hand it is possible to represent solutions of the heat equation by a path integral representation, based on the Wiener measure in a mathematically rigorous way. This is stated by the famous Feynman Kac formula. On the other hand there have been a lot of attempts to write solutions of the Schrödinger equation as a Feynman (path) integral in a useful mathematical sense.

Unfortunately, however there can be no hope of extending the theory of invariant measure from finite to infinite dimensional spaces. For example, one may easily prove that no reasonably well-behaved translation invariant measure exists on any infinite-dimensional Hilbert space. More specifically, for any translation invariant measure on a infinite dimensional Hilbert space such that all balls are measurable sets there must be many balls whose measure is either zero or ∞. This is the reason why the formal expression $\mathcal{D}^\infty x$ used in some physical textbooks is problematic and misleading.

One may have some hope that the ill defined term $\mathcal{D}^\infty x$ combines with the kinetic energy term to produce a well defined complex measure with imaginary variance $\sigma^2 = i$, or that this combination is the limit of Gaussian measures. But this causes problems if we assume that cylinder functions are integrated in the obvious way, see [Ex85, p.217].

Theorem 1 (Cameron [C60])
Any finite (complex or real) measure with N-dimensional densities

$$\rho_{t_N > \cdots > t_0}(x_N, \ldots, x_0) = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi i \gamma (t_j - t_{j-1})}} \ \exp\left(i \frac{(x_j - x_{j-1})^2}{2\gamma\,(t_j - t_{j-1})} \right)$$

must have $i\gamma \in \mathbb{R}_+$.

So there is no hope for measure theory to solve the problems with path integration.

The successful methods are always more involved and less direct than in the Euclidean (i.e., Feynman Kac) case. Among them are analytic continuation, limits of finite dimensional approximations and Fourier transform. We are not interested in giving full reference on various theories of Feynman integrals (brief surveys can be found in [Ex85, Ta75]) but we like to mention the method in [AHK76] using Fresnel integrals. Here we have chosen a white noise approach.

The idea of realizing Feynman integrals within the white noise framework goes back to [HS83]. The “average over all paths” is performed with a Hida distribution as the weight (instead of a measure). The existence of such Hida distributions corresponding to Feynman integrands has been established in [FPS91].

In this white noise framework we define the Feynman *integrand* as a white noise distribution. Its expectation reduces to the Feynman *integral*, which had to be defined. But the Feynman *integrand* defined before may also be very useful. The point is that at least the pairing with the corresponding test functions is well defined. In this sense the Feynman integrand serves as an integrator. We will illustrate this by two examples.

1. Since all of our test function spaces contain smooth polynomials, all "moments" of the Feynman integrand are well defined. Furthermore the singularity of the Feynman integrand gives some information on the growth of the moments (w.r.t. n) i.e., if we have
$$\left| \left\langle\!\!\left\langle \mathrm{I}, \prod_{j=1}^{n} \langle \cdot, \xi_j \rangle \right\rangle\!\!\right\rangle \right| \leq (n!)^x C^n \prod_{j=1}^{n} |\xi_j|_p \,, \qquad \xi_j \in \mathcal{S}(\mathbb{R})$$
for some $C > 0$ and some continuous norm on $\mathcal{S}(\mathbb{R})$, then
$x = 1 \Leftrightarrow \mathrm{I} \in (\mathcal{S})^{-1}$,
$x = 1/2 \Leftrightarrow \mathrm{I} \in (\mathcal{S})'$.

2. Exponential functions are test functions (at least of finite order). So if I is a generalized white noise functional we can study the pairing
$$\langle\!\langle \mathrm{I}, e^{i\langle \cdot, \xi \rangle} \rangle\!\rangle \,, \qquad \xi \in \mathcal{S}(\mathbb{R}) \,,$$
which is the Fourier-Gauss transform of I. From the singularity of the distribution I we get some additional information of the analyticity of the above pairing with respect to ξ.

A second advantage is that the general white noise mathematics allows some manipulations with the Feynman integrand I, since there are many operators acting continuously on the corresponding distribution space. This allows well defined calculations. We will illustrate this advantage in Chapter 9. There we will *prove* Ehrenfest's theorem and derive a functional form of the canonical commutation relations for one particular class of potentials. The first argument is essentially been done by applying the adjoint of a differentiation operator to I. After calculating this expression we take expectation and obtain Ehrenfest's theorem.

We also want to stress that the white noise setting gives a good "conceptual background" to discuss some of the numerous independent definitions of path integrals in a "common language". In some sense this has been done in Chapters 7–8:

- We present an analytic continuation approach related to the work of Doss [D80] based on our discussion of the scaling operator in section 4.5. As a by-product we will also see the relation to the definition of Hu and Meyer [HM88]. We will also use our discussion in section 4.3.3 and integrate out Donsker's delta (introduced to fix the endpoints of the paths). This gives a convenient form of the propagator.

- In the white noise framework the first attempt to include interaction with a potential was done in [KaS92]. Khandekar and Streit constructed the Feynman integrand for a large class of potentials including singular ones. Basically they constructed a strong Dyson series converging in the space of Hida distributions. This approach only works for one space dimension. We will generalize this construction to (one dimensional) time-dependent potentials of non-compact support (Theorem 103). In Section 7.2 we will show that the expectation of the constructed Feynman integrand is indeed the physical propagator, i.e., it solves the Schrödinger equation. This results can also be found in [LLSW94a].

- The above construction is not restricted to perturbations of the free Feynman integrand. For example we may expand around the Feynman integrand of the harmonic oscillator. This construction works for small times for the same large class of potentials (Section 7.3 or [CDLSW95]).

- Modifications and generalizations of the Khandekar Streit construction as above suffer from the restriction to one dimensional quantum systems. In the work [AHK76] Feynman integrals for potentials which are Fourier transforms of bounded complex measures are discussed (with independent methods). This class of potentials can also be considered in the white noise framework, without restriction to the space dimension d. We need some integrability condition of the measure associated to the potential to ensure that the expansion the Feynman integrand converges in $(\mathcal{S}_d)^{-1}$ (Theorem 110).
  A smaller distribution space to control the convergence of the perturbative expansion may be obtained by sharpening the integrability condition on the measure.
  We will also allow time dependent potentials which surprisingly may be more singular than in the previous construction (Theorem 111).
  For this class of Feynman integrands we will show that our mathematical background allows to prove some relations which were based before on some heuristic arguments, (see Feynman and Hibbs [FH65, p.175]). In Chapter 9 we will do this for Ehrenfest's theorem and for the functional form of the canonical commutation relations.

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Chapter 2

Preliminaries

2.1 Some facts on nuclear triples

We start with a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. For a given separable nuclear space $\mathcal{N}$ (in the sense of Grothendieck) densely topologically embedded in $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

The dual pairing $\langle\cdot,\cdot\rangle$ of $\mathcal{N}'$ and $\mathcal{N}$ then is realized as an extension of the inner product in $\mathcal{H}$

$$\langle f,\, \xi\rangle = (f,\xi) \quad f \in \mathcal{H},\ \xi \in \mathcal{N}$$

Instead of reproducing the abstract definition of nuclear spaces (see e.g., [Sch71]) we give a complete (and convenient) characterization in terms of projective limits of Hilbert spaces.

Theorem 2 *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{p\in\mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p,\ p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ and $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The topology of $\mathcal{N}$ is given by the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

The Hilbertian norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$. Without loss of generality we always suppose that $\forall p \in \mathbb{N}, \forall \xi \in \mathcal{N} : |\xi| \leq |\xi|_p$ and that the system of norms is ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ if $p < q$. By general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p}.$$

with inductive limit topology τ_{ind} by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}_p',\ p \in \mathbb{N}\}$. The inductive limit topology (w.r.t. this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$. Let us mention that in our setting the topology τ_{ind} coincides with the Mackey topology $\tau(\mathcal{N}',\mathcal{N})$ and the strong topology $\beta(\mathcal{N}',\mathcal{N})$. Further note that the dual pair $\langle\mathcal{N}',\mathcal{N}\rangle$ is reflexive if $\mathcal{N}'$ is equipped with $\beta(\mathcal{N}',\mathcal{N})$. In addition we have that convergence of sequences is equivalent in $\beta(\mathcal{N}',\mathcal{N})$ and the weak topology $\sigma(\mathcal{N}',\mathcal{N})$, see e.g., [HKPS93, Appendix 5].

Further we want to introduce the notion of tensor power of a nuclear space. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$ of Hilbert spaces. Since there is no danger of confusion we will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$ respectively. Using the definition

$$\mathcal{N}^{\otimes n} := \operatorname{pr}\lim_{p\in\mathbb{N}} \mathcal{H}_p^{\otimes n}$$

one can prove [Sch71] that $\mathcal{N}^{\otimes n}$ is a nuclear space which is called the n^{th} tensor power of $\mathcal{N}$. The dual space of $\mathcal{N}^{\otimes n}$ can be written

$$\left(\mathcal{N}^{\otimes n}\right)' = \operatorname{ind}\lim_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\otimes n}$$

Most important for the applications we have in mind is the following 'kernel theorem', see e.g., [BeKo88].

Theorem 3 *Let $\xi_1, ..., \xi_n \mapsto F_n(\xi_1, ..., \xi_n)$ be an n-linear form on $\mathcal{N}^{\otimes n}$ which is $\mathcal{H}_p$-continuous , i.e.,*

$$|F_n(\xi_1, ..., \xi_n)| \le C \prod_{k=1}^{n} |\xi_k|_p$$

for some $p \in \mathbb{N}$ and $C > 0$.
Then for all $p' > p$ such that the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt there exist a unique $\Phi^{(n)} \in \mathcal{H}_{-p'}^{\otimes n}$ such that

$$F_n(\xi_1, \ldots, \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \cdots \otimes \xi_n\rangle \ , \qquad \xi_1, ..., \xi_n \in \mathcal{N}$$

and the following norm estimate holds

$$\left|\Phi^{(n)}\right|_{-p'} \le C \ \|i_{p',p}\|_{HS}^n$$

using the Hilbert-Schmidt norm of $i_{p',p}$.

Corollary 4 *Let $\xi_1, ..., \xi_n \mapsto F(\xi_1, ..., \xi_n)$ be an n-linear form on $\mathcal{N}^{\otimes n}$ which is $\mathcal{H}_{-p}$-continuous, i.e.,*

$$|F_n(\xi_1, \ldots, \xi_n)| \le C \prod_{k=1}^{n} |\xi_k|_{-p}$$

for some $p \in \mathbb{N}$ and $C > 0$.
Then for all $p' < p$ such that the embedding $i_{p,p'} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is Hilbert-Schmidt there exist a unique $\Phi^{(n)} \in \mathcal{H}_{p'}^{\otimes n}$ such that

$$F_n(\xi_1, ..., \xi_n) = \langle \Phi^{(n)}, \xi_1 \otimes \cdots \otimes \xi_n\rangle, \qquad \xi_1, ..., \xi_n \in \mathcal{N}$$

and the following norm estimate holds

$$\left|\Phi^{(n)}\right|_{p'} \le C \ \|i_{p,p'}\|_{HS}^n \ .$$

If in Theorem 3 (and in Corollary 4 respectively) we start from a symmetric n-linear form F_n on $\mathcal{N}^{\otimes n}$ i.e., $F_n(\xi_{\pi_1}, \ldots, \xi_{\pi_n}) = F_n(\xi_1, \ldots, \xi_n)$ for any permutation π, then the corresponding kernel $\Phi^{(n)}$ can be localized in $\mathcal{H}_{p'}^{\hat{\otimes} n} \subset \mathcal{H}_{p'}^{\otimes n}$ (the nth symmetric tensor power of the Hilbert space $\mathcal{H}_{p'}$). For $f_1, \ldots, f_n \in \mathcal{H}$ let $\hat{\otimes}$ also denote the symmetrization of the tensor product

$$f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := \frac{1}{n!} \sum_{\pi} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n} \ ,$$

where the sum extends over all permutations of n letters. All the above quoted theorems also hold for complex spaces, in particular the complexified space $\mathcal{N}_{\mathbb{C}}$. By definition an element $\theta \in \mathcal{N}_{\mathbb{C}}$ decomposes into $\theta = \xi + i\eta$, $\xi, \eta \in \mathcal{N}$. If we also introduce the corresponding complexified Hilbert spaces $\mathcal{H}_{p,\mathbb{C}}$ the inner product becomes

$$(\theta_1, \theta_2)_{\mathcal{H}_{p,\mathbb{C}}} = (\theta_1, \bar{\theta}_2)_{\mathcal{H}_p} = (\xi_1, \xi_2)_{\mathcal{H}_p} + (\eta_1, \eta_2)_{\mathcal{H}_p} + i(\eta_1, \xi_2)_{\mathcal{H}_p} - i(\xi_1, \eta_2)_{\mathcal{H}_p}$$

for $\theta_1, \theta_2 \in \mathcal{H}_{p,\mathbb{C}}$, $\theta_1 = \xi_1 + i\eta_1$, $\theta_2 = \xi_2 + i\eta_2$, $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}_p$. Thus we have introduced a nuclear triple

$$\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \subset \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n} \subset \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \right)'$$

We also want to introduce the (Boson or symmetric) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$$

with the convention $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} 0} := \mathbb{C}$ and the Hilbertian norm

$$\|\vec{\varphi}\|_{\Gamma(\mathcal{H})}^2 = \sum_{n=0}^{\infty} n! \ \left|\varphi^{(n)}\right|^2 \ , \quad \vec{\varphi} = \left\{ \varphi^{(n)} \ \middle| \ n \in \mathbb{N}_0 \right\} \in \Gamma(\mathcal{H}) \ .$$

2.2 Holomorphy on locally convex spaces

We shall collect some facts from the theory of holomorphic functions in locally convex topological vector spaces $\mathcal{E}$ (over the complex field $\mathbb{C}$), see e.g., [Di81]. Let $\mathcal{L}(\mathcal{E}^n)$ be the space of n-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric n-linear forms. Also let $P^n(\mathcal{E})$ denote the n-homogeneous polynomials on $\mathcal{E}$. There is a linear bijection $\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \widehat{A} \in P^n(\mathcal{E})$. Now let $\mathcal{U} \subset \mathcal{E}$ be open and consider a function $G : \mathcal{U} \to \mathbb{C}$.

G is said to be **G-holomorphic** if for all $\theta_0 \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C}$: $\lambda \to G(\theta_0 + \lambda\theta)$ is holomorphic in some neighborhood of zero in $\mathbb{C}$. If G is G-holomorphic then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{\mathrm{d}^n G(\eta)}$ such that

$$G(\theta + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n G(\eta)}(\theta)$$

for all θ from some open set $\mathcal{V} \subset \mathcal{U}$. G is said to be **holomorphic**, if for all η in $\mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that $\sum_{n=0}^{\infty} \frac{1}{n!}\widehat{\mathrm{d}^n G(\eta)}(\theta)$ converges uniformly on

$\mathcal{V}$ to a continuous function. We say that G is holomorphic at θ_0 if there is an open set $\mathcal{U}$ containing θ_0 such that G is holomorphic on $\mathcal{U}$. The following proposition can be found e.g., in [Di81].

Proposition 5 *G is holomorphic if and only if it is G-holomorphic and locally bounded.*

Let us explicitly consider a function holomorphic at the point $0 \in \mathcal{E} = \mathcal{N}_{\mathbb{C}}$, then

1) there exist p and $\varepsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \leq \varepsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function of one complex variable $\lambda \to G(\xi_0 + \lambda\xi)$ is analytic at $0 \in \mathbb{C}$, and

2) there exists $c > 0$ such that for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq \varepsilon : |G(\xi)| \leq c$.

As we do not want to discern between different restrictions of one function, we consider germs of holomorphic functions, i.e., we identify F and G if there exists an open neighborhood $\mathcal{U} : 0 \in \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ such that $F(\xi) = G(\xi)$ for all $\xi \in \mathcal{U}$. Thus we define $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ as the algebra of germs of functions holomorphic at zero equipped with the inductive topology given by the following family of norms

$$\mathrm{n}_{p,l,\infty}(G) = \sup_{|\theta|_p \leq 2^{-l}} |G(\theta)|, \quad p, l \in \mathbb{N}.$$

Let use now introduce spaces of entire functions which will be useful later. Let $\mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ denote the set of all entire functions on $\mathcal{H}_{-p,\mathbb{C}}$ of growth $k \in [1,2]$ and type 2^{-l}, $p, l \in \mathbb{Z}$. This is a linear space with norm

$$\mathrm{n}_{p,l,k}(\varphi) = \sup_{z \in \mathcal{H}_{-p,\mathbb{C}}} |\varphi(z)| \exp\left(-2^{-l}|z|^k_{-p}\right), \qquad \varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$$

The space of entire functions on $\mathcal{N}'_{\mathbb{C}}$ of growth k and minimal type is naturally introduced by

$$\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}}) := \underset{p,l \in \mathbb{N}}{\mathrm{pr\,lim}}\ \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}),$$

see e.g., [Ou91]. We will also need the space of entire functions on $\mathcal{N}_{\mathbb{C}}$ of growth k and finite type:

$$\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}}) := \underset{p,l \in \mathbb{N}}{\mathrm{ind\,lim}}\ \mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}}).$$

In the following we will give an equivalent description of $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}})$. Cauchy's inequality and Corollary 4 allow to write the Taylor coefficients in a convenient form. Let $\varphi \in \mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$ and $z \in \mathcal{N}'_{\mathbb{C}}$, then there exist kernels $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$ such that

$$\langle z^{\otimes n}, \varphi^{(n)} \rangle = \frac{1}{n!}\widehat{\mathrm{d}^n\varphi(0)}(z)$$

i.e.,

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle. \tag{2.1}$$

This representation allows to introduce a nuclear topology on $\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}})$, see [Ou91] for details. Let $\mathrm{E}^{\beta}_{p,q}$ denote the space of all functions of the form (2.1) such that the following Hilbertian norm

$$|||\varphi|||^2_{p,q,\beta} := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left|\varphi^{(n)}\right|^2_p, \quad p, q \in \mathbb{N} \tag{2.2}$$

is finite for $\beta \in [0,1]$. (By $\left|\varphi^{(0)}\right|_p$ we simply mean the complex modulus for all p.) The space $\mathrm{E}^{-\beta}_{-p-,q}$ with the norm $\|\varphi\|_{-p,-q,-\beta}$ is defined analogously.

Theorem 6 *The following topological identity holds:*

$$\mathrm{pr}\lim_{p,q\in\mathbb{N}} \mathrm{E}^{\beta}_{p,q} = \mathcal{E}^{\frac{2}{1+\beta}}_{\min}(\mathcal{N}'_{\mathbb{C}}) \quad .$$

The proof is an immediate consequence of the following two lemmata which show that the two systems of norms are in fact equivalent.

Lemma 7 *Let $\varphi \in \mathrm{E}^{\beta}_{p,q}$ then $\varphi \in \mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ for $l = \frac{q}{1+\beta}$. Moreover*

$$\mathrm{n}_{p,l,k}(\varphi) \le \|\varphi\|_{p,q,\beta} \ , \quad k = \tfrac{2}{1+\beta} \ . \tag{2.3}$$

Proof. We look at the convergence of the series $\varphi(z) = \sum_{n=0}^{\infty}\langle z^{\otimes n}, \varphi^{(n)}\ \rangle$, $z \in \mathcal{H}_{-p,\mathbb{C}}$, $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}$ if $\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|\varphi^{(n)}|_p^2 = \|\varphi\|^2_{p,q,\beta}$ is finite. The following estimate holds:

$$\begin{aligned}
\sum_{n=0}^{\infty}|\langle z^{\otimes n}, \varphi^{(n)}\ \rangle| &\le \left(\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|\varphi^{(n)}|_p^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{(n!)^{1+\beta}}2^{-nq}|z|_{-p}^{2n}\right)^{1/2} \\
&\le \|\varphi\|_{p,q,\beta}\cdot\left(\sum_{n=0}^{\infty}\left\{\frac{1}{n!}2^{-\frac{nq}{1+\beta}}|z|_{-p}^{\frac{2n}{1+\beta}}\right\}^{1+\beta}\right)^{1/2} \\
&\le \|\varphi\|_{p,q,\beta}\left(\sum_{n=0}^{\infty}\frac{1}{n!}2^{-\frac{nq}{1+\beta}}|z|_{-p}^{\frac{2n}{1+\beta}}\right)^{(1+\beta)/2} \\
&\le \|\varphi\|_{p,q,\beta}\exp\left(2^{-\frac{q}{1+\beta}}|z|_{-p}^{\frac{2}{1+\beta}}\right).
\end{aligned}$$

□

Lemma 8 *For any $p', q \in \mathbb{N}$ there exist $p, l \in \mathbb{N}$ such that*

$$\mathcal{E}^{\frac{2}{1+\beta}}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}) \subset \mathrm{E}^{\beta}_{p',q}$$

i.e., there exists a constant $C > 0$ such that

$$\|\varphi\|_{p',q,\beta} \le C\ \mathrm{n}_{p,l,k}(\varphi), \quad \varphi \in \mathcal{E}^{k}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}), \quad k = \tfrac{2}{1+\beta}.$$

Remark. More precisely we will prove the following: If $\varphi \in \mathcal{E}^{k}_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ then $\varphi \in \mathrm{E}^{\beta}_{p',q}$ for $k = \frac{2}{1+\beta}$ and $\rho := 2^{q-2l/k}k^{2/k}e^2\,\|i_{p',p}\|^2_{HS} < 1$ (in particular this requires $p' > p$ to be such that the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt).
Moreover the following bound holds

$$\|\varphi\|_{p',q,\beta} \le \mathrm{n}_{p,l,k}(\varphi)\cdot(1-\rho)^{-1/2} \ . \tag{2.4}$$

Proof. The assumption $\varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ implies a bound of the growth of φ :

$$|\varphi(z)| \leq \mathrm{n}_{p,l,k}(\varphi) \exp(2^{-l}|z|^k_{-p}) \ .$$

For each $\rho > 0$, $z \in \mathcal{H}_{-p,\mathbb{C}}$ the Cauchy inequality from complex analysis [Di81] gives

$$\left| \frac{1}{n!} \widehat{\mathrm{d}^n \varphi(0)}(z) \right| \leq \mathrm{n}_{p,l,k}(\varphi) \rho^{-n} \exp(\rho^k 2^{-l}) \ |z|^n_{-p} \ .$$

By polarization [Di81] it follows for $z_1, \ldots, z_n \in \mathcal{H}_{-p,\mathbb{C}}$

$$\left| \frac{1}{n!} \mathrm{d}^n \varphi(0)(z_1, \ldots, z_n) \right| \leq \mathrm{n}_{p,l,k}(\varphi) \frac{1}{n!} \left(\frac{n}{\rho} \right)^n \exp(\rho^k 2^{-l}) \prod_{k=1}^n |z_k|_{-p} \ .$$

For $p' > p$ such that $\|i_{p',p}\|_{HS}$ is finite, an application of the kernel theorem guarantees the existence of kernels $\varphi^{(n)} \in \mathcal{H}^{\hat{\otimes} n}_{p',\mathbb{C}}$ such that

$$\varphi(z) = \sum_{n=0}^\infty \langle z^{\hat{\otimes} n}, \varphi^{(n)} \ \rangle$$

with the bound

$$\left|\varphi^{(n)}\right|_{p'} \leq \mathrm{n}_{p,l,k}(\varphi) \frac{1}{n!} \left(\frac{n}{\rho} \|i_{p',p}\|_{HS} \right)^n \exp(\rho^k \cdot 2^{-l}) \ .$$

We can optimize the bound with the choice of an n-dependent ρ. Setting $\rho^k = 2^l n/k$ we obtain

$$\begin{aligned} \left|\varphi^{(n)}\right|_{p'} &\leq \mathrm{n}_{p,l,k}(\varphi) \frac{1}{n!} n^{n(1-1/k)} \left(\tfrac{1}{k} 2^l \right)^{-n/k} \|i_{p',p}\|^n_{HS} \, e^{n/k} \\ &\leq \mathrm{n}_{p,l,k}(\varphi) \ (n!)^{-1/k} \left\{ (k 2^{-l})^{1/k} e \, \|i_{p',p}\|_{HS} \right\}^n \ , \end{aligned}$$

where we used $n^n \leq n! \, e^n$ in the last estimate. Now choose $\beta \in [0,1]$ such that $k = \frac{2}{1+\beta}$ to estimate the following norm:

$$\begin{aligned} \|\varphi\|^2_{p',q,\beta} &\leq \mathrm{n}^2_{p,l,k}(\varphi) \sum_{n=0}^\infty (n!)^{1+\beta-\frac{2}{k}} 2^{qn} \left\{ (k 2^{-l})^{1/k} e \, \|i_{p',p}\|_{HS} \right\}^{2n} \\ &\leq \mathrm{n}^2_{p,l,k}(\varphi) \left(1 - 2^q \left\{ (k 2^{-l})^{1/k} e \, \|i_{p',p}\|_{HS} \right\}^2 \right)^{-1} \end{aligned}$$

for sufficiently large l. This completes the proof. □

Analogous estimates for these systems of norms also hold if β, p, q, l become negative. This implies the following theorem. For related results see e.g., [Ou91, Prop.8.6].

Theorem 9
If $\beta \in [0,1)$ then the following topological identity holds:

$$\operatorname*{ind\,lim}_{p,q \in \mathbb{N}} \ \mathrm{E}^{-\beta}_{-p,-q} = \mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}}).$$

If $\beta = 1$ we have

$$\operatorname*{ind\,lim}_{p,q \in \mathbb{N}} \ \mathrm{E}^{-1}_{-p,-q} = \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}) \ .$$

This theorem and its proof will appear in the context of section 3.6. The characterization of distributions in infinite dimensional analysis is strongly related to this theorem. From this point of view it is natural to postpone its proof to section 3.6.

Chapter 3

Generalized functions in infinite dimensional analysis

3.1 Measures on linear topological spaces

To introduce probability measures on the vector space $\mathcal{N}'$, we consider $\mathcal{C}_\sigma(\mathcal{N}')$ the σ-algebra generated by cylinder sets on $\mathcal{N}'$, which coincides with the Borel σ-algebras $\mathcal{B}_\sigma(\mathcal{N}')$ and $\mathcal{B}_\beta(\mathcal{N}')$ generated by the weak and strong topology on $\mathcal{N}'$ respectively. Thus we will consider this σ-algebra as the *natural* σ-algebra on $\mathcal{N}'$. Detailed definitions of the above notions and proofs of the mentioned relations can be found in e.g., [BeKo88].

We will restrict our investigations to a special class of measures μ on $\mathcal{C}_\sigma(\mathcal{N}')$, which satisfy two additional assumptions. The first one concerns some analyticity of the Laplace transformation

$$l_\mu(\theta) = \int_{\mathcal{N}'} \exp\langle x, \theta\rangle \; \mathrm{d}\mu(x) =: \mathbb{E}_\mu(\exp\langle \cdot, \theta\rangle) \ , \ \theta \in \mathcal{N}_\mathbb{C}.$$

Here we also have introduced the convenient notion of expectation $\mathbb{E}_\mu$ of a μ-integrable function.

Assumption 1 The measure μ has an analytic Laplace transform in a neighborhood of zero. That means there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}_\mathbb{C}$ of zero, such that l_μ is holomorphic on $\mathcal{U}$, i.e., $l_\mu \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. This class of *analytic measures* is denoted by $\mathcal{M}_a(\mathcal{N}')$.

An equivalent description of analytic measures is given by the following lemma.

Lemma 10 *The following statements are equivalent*

1) $\mu \in \mathcal{M}_a(\mathcal{N}')$

2) $\exists p_\mu \in \mathbb{N}, \quad \exists C > 0 : \qquad \left| \int_{\mathcal{N}'} \langle x, \theta\rangle^n \, \mathrm{d}\mu(x) \right| \leq n!\, C^n \, |\theta|_{p_\mu}^n \ , \quad \theta \in \mathcal{H}_{p_\mu,\mathbb{C}}$

3) $\exists p'_\mu \in \mathbb{N}, \quad \exists \varepsilon_\mu > 0 : \qquad \int_{\mathcal{N}'} \exp(\varepsilon_\mu \, |x|_{-p'_\mu})\, \mathrm{d}\mu(x) < \infty$

Proof. The proof can be found in [KoSW95]. We give its outline in the following. The only non-trivial step is the proof of 2)⇒3).
By polarization [Di81] 2) implies

$$\left| \int_{\mathcal{N}'} \langle x^{\otimes n}, \bigotimes_{j=1}^{n} \xi_j \rangle \ \mathrm{d}\mu(x) \right| \leq n!\ C^n \prod_{j=1}^{n} |\xi_j|_{p_\mu} \quad , \quad \xi_j \in \mathcal{H}_{p'} \tag{3.1}$$

for a (new) constant $C > 0$. Choose $p' > p_\mu$ such that the embedding $i_{p',p_\mu} : \mathcal{H}_{p'} \to \mathcal{H}_{p_\mu}$ is of Hilbert-Schmidt type. Let $\{e_k,\ k \in \mathbb{N}\} \subset \mathcal{N}$ be an orthonormal basis in $\mathcal{H}_{p'}$. Then $|x|^2_{-p'} = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2$, $x \in \mathcal{H}_{-p'}$. We will first estimate the moments of even order

$$\int_{\mathcal{N}'} |x|^{2n}_{-p'} \ \mathrm{d}\mu(x) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \int_{\mathcal{N}'} \langle x, e_{k_1} \rangle^2 \ \cdots \langle x, e_{k_n} \rangle^2 \ \mathrm{d}\mu(x) \ ,$$

where we changed the order of summation and integration by a monotone convergence argument. Using the bound (3.1) we have

$$\begin{aligned} \int_{\mathcal{N}'} |x|^{2n}_{-p'} \ \mathrm{d}\mu(x) &\leq C^{2n} \ (2n)! \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} |e_{k_1}|^2_{p_\mu} \cdots |e_{k_n}|^2_{p_\mu} \\ &= C^{2n} \ (2n)! \left(\sum_{k=1}^{\infty} |e_k|^2_{p_\mu} \right)^n \\ &= \left(C \cdot \left\| i_{p',p_\mu} \right\|_{HS} \right)^{2n} (2n)! \end{aligned}$$

because

$$\sum_{k=1}^{\infty} |e_k|^2_{p_\mu} = \left\| i_{p',p_\mu} \right\|^2_{HS} \ .$$

The moments of arbitrary order can now be estimated by the Schwarz inequality

$$\begin{aligned} \int |x|^n_{-p'} \ \mathrm{d}\mu(x) &\leq \sqrt{\mu(\mathcal{N}')} \left(\int |x|^{2n}_{-p} \ \mathrm{d}\mu(x) \right)^{\frac{1}{2}} \\ &\leq \sqrt{\mu(\mathcal{N}')} \left(C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n \sqrt{(2n)!} \\ &\leq \sqrt{\mu(\mathcal{N}')} \left(2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n n! \end{aligned}$$

since $(2n)! \leq 4^n (n!)^2$.
Choose $\varepsilon < \left(2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^{-1}$ then

$$\begin{aligned} \int e^{\varepsilon |x|_{-p'}} \mathrm{d}\mu(x) &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int |x|^n_{-p'} \ \mathrm{d}\mu(x) \\ &\leq \sqrt{\mu(\mathcal{N}')} \sum_{n=0}^{\infty} \left(\varepsilon \ 2C \left\| i_{p',p_\mu} \right\|_{HS} \right)^n < \infty \end{aligned} \tag{3.2}$$

Hence the lemma is proven. □

For $\mu \in \mathcal{M}_a(\mathcal{N}')$ the estimate in statement 2 of the above lemma allows to define the moment kernels $\mathrm{M}_n^\mu \in (\mathcal{N}^{\hat{\otimes} n})'$. This is done by extending the above estimate by a simple polarization argument and applying the kernel theorem. The kernels are determined by

$$l_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathrm{M}_n^\mu, \theta^{\otimes n} \rangle$$

or equivalently

$$\langle \mathrm{M}_n^\mu, \theta_1 \hat{\otimes} \cdots \hat{\otimes} \theta_n \rangle = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} l_\mu(t_1\theta_1 + \cdots + t_n\theta_n)\bigg|_{t_1 = \cdots = t_n = 0} .$$

Moreover, if $p > p_\mu$ is such that embedding $i_{p,p_\mu} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_\mu}$ is Hilbert-Schmidt then

$$|\mathrm{M}_n^\mu|_{-p} \leq \left(nC \left\|i_{p,p_\mu}\right\|_{HS}\right)^n \leq n! \ \left(eC \left\|i_{p,p_\mu}\right\|_{HS}\right)^n . \tag{3.3}$$

Definition 11 *A function $\varphi : \mathcal{N}' \to \mathbb{C}$ of the form $\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle$, $x \in \mathcal{N}'$, $N \in \mathbb{N}$, is called a continuous polynomial (short $\varphi \in \mathcal{P}(\mathcal{N}')$) iff $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $\forall n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.*

Now we are ready to formulate the second assumption:

Assumption 2 For all $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\varphi = 0$ μ-almost everywhere we have $\varphi \equiv 0$. In the following a measure with this property will be called *non-degenerate*.

Note. Assumption 2 is equivalent to:
Let $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\int_A \varphi \, d\mu = 0$ for all $A \in \mathcal{C}_\sigma(\mathcal{N}')$ then $\varphi \equiv 0$.
A sufficient condition can be obtained by regarding admissible shifts of the measure μ. If $\mu(\cdot + \xi)$ is absolutely continuous with respect to μ for all $\xi \in \mathcal{N}$, i.e., there exists the Radon-Nikodym derivative

$$\rho_\mu(\xi, x) = \frac{d\mu(x+\xi)}{d\mu(x)} , \quad x \in \mathcal{N}' ,$$

Then we say that μ is $\mathcal{N}$*–quasi-invariant* see e.g., [GV68, Sk74]. This is sufficient to ensure Assumption 2, see e.g., [KoTs91, BeKo88].

Example 1 In Gaussian Analysis (especially White Noise Analysis) the Gaussian measure $\gamma_{\mathcal{H}}$ corresponding to the Hilbert space $\mathcal{H}$ is considered. Its Laplace transform is given by

$$l_{\gamma_{\mathcal{H}}}(\theta) = e^{\frac{1}{2}\langle \theta, \theta \rangle} , \qquad \theta \in \mathcal{N}_{\mathbb{C}} ,$$

hence $\gamma_{\mathcal{H}} \in \mathcal{M}_a(\mathcal{N}')$. It is well known that $\gamma_{\mathcal{H}}$ is $\mathcal{N}$–quasi-invariant (moreover $\mathcal{H}$–quasi-invariant) see e.g., [Sk74, BeKo88]. Due to the previous note $\gamma_{\mathcal{H}}$ satisfies also Assumption 2.

Example 2 *(Poisson measures)*
Let use consider the classical (real) Schwartz triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) .$$

The Poisson white noise measure μ_p is defined as a probability measure on $\mathcal{C}_\sigma(\mathcal{S}'(\mathbb{R}))$ with the Laplace transform

$$l_{\mu_p}(\theta) = \exp\left\{\int_{\mathbb{R}} (e^{\theta(t)} - 1)\,\mathrm{d}t\right\} = \exp\left\{\langle e^\theta - 1, 1\rangle\right\}, \qquad \theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}),$$

see e.g., [GV68]. It is not hard to see that l_{μ_p} is a holomorphic function on $\mathcal{S}_{\mathbb{C}}(\mathbb{R})$, so Assumption 1 is satisfied. But to check Assumption 2, we need additional considerations.

First of all we remark that for any $\xi \in \mathcal{S}(\mathbb{R})$, $\xi \neq 0$ the measures μ_p and $\mu_p(\cdot + \xi)$ are orthogonal (see [VGG75] for a detailed analysis). It means that μ_p is not $\mathcal{S}(\mathbb{R})$-quasi-invariant and the note after Assumption 2 is not applicable now.

Let some $\varphi \in \mathcal{P}(\mathcal{S}'(\mathbb{R}))$, $\varphi = 0$ μ_p-a.s. be given. We need to show that then $\varphi \equiv 0$. To this end we will introduce a system of orthogonal polynomials in the space $L^2(\mu_p)$ which can be constructed in the following way. The mapping

$$\theta(\cdot) \mapsto \alpha(\theta)(\cdot) = \log(1 + \theta(\cdot)) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}), \quad \theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$$

is holomorphic on a neighborhood $\mathcal{U} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R})$, $0 \in \mathcal{U}$. Then

$$e^\alpha_{\mu_p}(\theta; x) = \frac{e^{\langle \alpha(\theta), x\rangle}}{l_{\mu_p}(\alpha(\theta))} = \exp\{\langle \alpha(\theta), x\rangle - \langle \theta, 1\rangle\}, \quad \theta \in \mathcal{U},\ x \in \mathcal{S}'(\mathbb{R})$$

is a holomorphic function on $\mathcal{U}$ for any $x \in \mathcal{S}'(\mathbb{R})$. The Taylor decomposition and the kernel theorem (just as in subsection 3.2.1 below) give

$$e^\alpha_{\mu_p}(\theta; x) = \sum_{n=0}^\infty \frac{1}{n!}\langle \theta^{\otimes n}, C_n(x)\rangle,$$

where $C_n : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})^{\hat{\otimes} n}$ are polynomial mappings. For $\varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, we define Charlier polynomials

$$x \mapsto C_n(\varphi^{(n)}; x) = \langle \varphi^{(n)}, C_n(x)\rangle \in \mathbb{C},\ \ x \in \mathcal{S}'(\mathbb{R}).$$

Due to [Ito88, IK88] we have the following orthogonality property:

$$\forall \varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n},\ \forall \psi^{(m)} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$$

$$\int C_n(\varphi^{(n)}) C_m(\psi^{(m)})\,\mathrm{d}\mu_p = \delta_{nm} n! \langle \varphi^{(n)}, \psi^{(n)}\rangle.$$

Now the rest is simple. Any continuous polynomial φ has a uniquely defined decomposition

$$\varphi(x) = \sum_{n=0}^N \langle \varphi^{(n)}, C_n(x)\rangle, \quad x \in \mathcal{S}'(\mathbb{R}),$$

where $\varphi^{(n)} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$. If $\varphi = 0$ μ_p-a.e. then

$$\|\varphi\|^2_{L^2(\mu_p)} = \sum_{n=0}^N n!\,\langle \varphi^{(n)}, \overline{\varphi^{(n)}}\rangle = 0.$$

Hence $\varphi^{(n)} = 0$, $n = 0, \ldots, N$, i.e., $\varphi \equiv 0$. So Assumption 2 is satisfied.

3.2 Concept of distributions in infinite dimensional analysis

In this section we will introduce a preliminary distribution theory in infinite dimensional non-Gaussian analysis. We want to point out in advance that the distribution space constructed here is in some sense too big for practical purposes. In this sense section 3.2 may be viewed as a stepping stone to introduce the more useful structures in §3.3 and §3.4.

We will choose $\mathcal{P}(\mathcal{N}')$ as our (minimal) test function space. (The idea to use spaces of this type as appropriate spaces of test functions is rather old see [KMP65]. They also discussed in which sense this space is "minimal".) First we have to ensure that $\mathcal{P}(\mathcal{N}')$ is densely embedded in $L^2(\mu)$. This is fulfilled because of our assumption 1 [Sk74, Sec.ğ10 Th.1]. The space $\mathcal{P}(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes a nuclear space [BeKo88]. The topology on $\mathcal{P}(\mathcal{N}')$ is chosen such that is becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ see e.g., [Sch71, Ch II 6.1, Ch III 7.4]

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} .$$

via

$$\varphi(x) = \sum_{n=0}^{\infty} \left\langle x^{\otimes n}, \varphi^{(n)} \right\rangle \longleftrightarrow \vec{\varphi} = \left\{ \varphi^{(n)} \,\middle|\, n \in \mathbb{N}_0 \right\} .$$

Note that only a finite number of $\varphi^{(n)}$ is non-zero. We will not reproduce the full construction here, but we will describe the notion of convergence of sequences this topology on $\mathcal{P}(\mathcal{N}')$. For $\varphi \in \mathcal{P}(\mathcal{N}')$, $\varphi(x) = \sum_{n=0}^{N(\varphi)} \left\langle x^{\otimes n}, \varphi^{(n)} \right\rangle$ let $p_n : \mathcal{P}(\mathcal{N}') \to \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ denote the mapping p_n is defined by $p_n\varphi := \varphi^{(n)}$. A sequence $\{\varphi_j,\ j \in \mathbb{N}\}$ of smooth polynomials converges to $\varphi \in \mathcal{P}(\mathcal{N}')$ iff the $N(\varphi_j)$ are bounded and $p_n\varphi_j \underset{n\to\infty}{\longrightarrow} p_n\varphi$ in $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}$.

Now we can introduce the dual space $\mathcal{P}'_\mu(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to $L^2(\mu)$. As a result we have constructed the triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}')$$

The (bilinear) dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\mu$ between $\mathcal{P}'_\mu(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by

$$\langle\!\langle \varphi,\ \psi \rangle\!\rangle_\mu = (\varphi,\ \overline{\psi})_{L^2(\mu)} \ , \quad \varphi \in L^2(\mu),\ \psi \in \mathcal{P}(\mathcal{N}') \ .$$

Since the constant function 1 is in $\mathcal{P}(\mathcal{N}')$ we may extend the concept of expectation from random variables to distributions; for $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$

$$\mathbb{E}_\mu(\Phi) := \langle\!\langle \Phi, 1 \rangle\!\rangle_\mu \ .$$

The main goal of this section is to provide a description of $\mathcal{P}'_\mu(\mathcal{N}')$, see Theorem 19 below. The simplest approach to this problem seems to be the use of so called μ-Appell polynomials.

3.2.1 Appell polynomials associated to the measure μ

Because of the holomorphy of l_μ and $l_\mu(0) = 1$ there exists a neighborhood of zero

$$\mathcal{U}_0 = \left\{ \theta \in \mathcal{N}_{\mathbb{C}} \;\middle|\; 2^{q_0} \left|\theta\right|_{p_0} < 1 \right\}$$

$p_0, q_0 \in \mathbb{N}$, $p_0 \geq p'_\mu$, $2^{-q_0} \leq \varepsilon_\mu$ (p'_μ, ε_μ from Lemma 10) such that $l_\mu(\theta) \neq 0$ for $\theta \in \mathcal{U}_0$ and the normalized exponential

$$e_\mu(\theta; z) = \frac{e^{\langle z,\theta\rangle}}{l_\mu(\theta)} \quad \text{for } \theta \in \mathcal{U}_0, \quad z \in \mathcal{N}'_{\mathbb{C}} \, , \tag{3.4}$$

is well defined. We use the holomorphy of $\theta \mapsto e_\mu(\theta; z)$ to expand it in a power series in θ similar to the case corresponding to the construction of one dimensional Appell polynomials [Bo76]. We have in analogy to [AKS93, ADKS94]

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n e_\mu(0, z)}(\theta)$$

where $\widehat{\mathrm{d}^n e_\mu(0; z)}$ is an n-homogeneous continuous polynomial. But since $e_\mu(\theta; z)$ is not only G-holomorphic but holomorphic we know that $\theta \to e_\mu(\theta; z)$ is also locally bounded. Thus Cauchy's inequality for Taylor series [Di81] may be applied, $\rho \leq 2^{-q_0}$, $p \geq p_0$

$$\left| \frac{1}{n!} \widehat{\mathrm{d}^n e_\mu(0; z)}(\theta) \right| \leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} |e_\mu(\theta; z)| \, |\theta|_p^n \leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} \frac{1}{l_\mu(\theta)} e^{\rho |z|_{-p}} \, |\theta|_p^n \tag{3.5}$$

if $z \in \mathcal{H}_{-p,\mathbb{C}}$. This inequality extends by polarization [Di81] to an estimate sufficient for the kernel theorem. Thus we have a representation $\widehat{\mathrm{d}^n e_\mu(0; z)}(\theta) = \langle P_n^\mu(z), \theta^{\otimes n} \rangle$ where $P_n^\mu(z) \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$. The kernel theorem really gives a little more: $P_n^\mu(z) \in \mathcal{H}_{-p'}^{\hat{\otimes} n}$ for any $p' (> p \geq p_0)$ such that the embedding operator $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt. Thus we have

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^\mu(z), \theta^{\otimes n} \right\rangle \quad \text{for } \theta \in \mathcal{U}_0, \; z \in \mathcal{N}'_{\mathbb{C}} \, . \tag{3.6}$$

We will also use the notation

$$P_n^\mu(\varphi^{(n)})(z) := \left\langle P_n^\mu(z), \varphi^{(n)} \right\rangle , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \quad n \in \mathbb{N}.$$

Thus for any measure satisfying Assumption 1 we have defined the $\mathbb{P}^\mu$-system

$$\mathbb{P}^\mu = \left\{ \left\langle P_n^\mu(\cdot), \varphi^{(n)} \right\rangle \;\middle|\; \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \; n \in \mathbb{N} \right\}.$$

Let us collect some properties of the polynomials $P_n^\mu(z)$.

Proposition 12 *For $x, y \in \mathcal{N}'$, $n \in \mathbb{N}$ the following holds*

(P1)

$$P_n^\mu(x) = \sum_{k=0}^{n} \binom{n}{k} x^{\otimes k} \hat{\otimes} P_{n-k}^\mu(0), \tag{3.7}$$

(P2)

$$x^{\otimes n} = \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(x) \hat{\otimes} \mathrm{M}_{n-k}^\mu \tag{3.8}$$

(P3)

$$\begin{aligned} P_n^\mu(x+y) &= \sum_{k+l+m=n} \frac{n!}{k!\,l!\,m!} P_k^\mu(x) \hat{\otimes} P_l^\mu(y) \hat{\otimes} \mathrm{M}_m^\mu \\ &= \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(x) \hat{\otimes} y^{\otimes(n-k)} \end{aligned} \tag{3.9}$$

(P4) Further we observe

$$\mathbb{E}_\mu(\langle P_m^\mu(\cdot), \varphi^{(m)} \rangle) = 0 \qquad \text{for } m \neq 0 \text{ }, \varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m} \text{ }. \tag{3.10}$$

(P5) For all $p > p_0$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0}$ is Hilbert–Schmidt and for all $\varepsilon > 0$ small enough $\left(\varepsilon \leq \frac{2^{-q_0}}{e \left\| i_{p,p_0} \right\|_{HS}} \right)$ there exists a constant $C_{p,\varepsilon} > 0$ with

$$|P_n^\mu(z)|_{-p} \leq C_{p,\varepsilon}\, n!\, \varepsilon^{-n}\, e^{\varepsilon |z|_{-p}}, \quad z \in \mathcal{H}_{-p,\mathbb{C}} \tag{3.11}$$

Proof. We restrict ourselves to a sketch of proof, details can be found in [ADKS94].
(P1) This formula can be obtained simply by substituting

$$\frac{1}{l_\mu(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^\mu(0), \theta^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}}, |\theta|_q < \delta \tag{3.12}$$

and

$$e^{\langle x,\theta \rangle} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle x^{\otimes n}, \theta^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}}, x \in \mathcal{N}'$$

in the equality $e_\mu(\theta; x) = e^{\langle x,\theta \rangle} l_\mu^{-1}(\theta)$. A comparison with (3.6) proves (P1). The proof of (P2) is completely analogous to the proof of (P1).
(P3) We start from the following obvious equation of the generating functions

$$e_\mu(\theta; x+y) = e_\mu(\theta; x)\, e_\mu(\theta; y)\, l_\mu(\theta)$$

This implies

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\mu(x+y), \theta^{\otimes n} \rangle = \sum_{k,l,m=0}^{\infty} \frac{1}{k!\,l!\,m!} \langle P_k(x) \hat{\otimes} P_l(y) \hat{\otimes} \mathrm{M}_m,\ \theta^{\otimes(k+l+m)} \rangle$$

from this (P3) follows immediately.
(P4) To see this we use, $\theta \in \mathcal{N}_{\mathbb{C}}$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_\mu(\langle P_m^\mu(\cdot), \theta^{\otimes n} \rangle) = \mathbb{E}_\mu(e_\mu(\theta; \cdot)) = \frac{\mathbb{E}_\mu(e^{\langle \cdot, \theta \rangle})}{l_\mu(\theta)} = 1 \text{ }.$$

Then a comparison of coefficients and the polarization identity gives the above result.

(P5) We can use

$$|P_n^\mu(z)|_{-p'} \leq n! \left(\sup_{|\theta|_p=\rho} \frac{1}{l_\mu(\theta)} \right) e^{\rho|z|_{-p}} \left(\frac{e}{\rho} \, \|i_{p',p}\|_{HS} \right)^n , \qquad z \in \mathcal{H}_{-p,\mathbb{C}} \tag{3.13}$$

$p > p_0, p', \rho$ defined above. (3.13) is a simple consequence of the kernel theorem by (3.5). In particular we have

$$|P_n^\mu(0)|_{-p} \leq n! \left(\sup_{|\theta|_{p_0}=\rho} \frac{1}{l_\mu(\theta)} \right) \left(\frac{e}{\rho} \, \|i_{p,p_0}\|_{HS} \right)^n$$

If $p > p_0$ such that $\|i_{p,p_0}\|_{HS}$ is finite. For $0 < \varepsilon \leq 2^{-q_0}/e \, \|i_{p,p_0}\|_{HS}$ we can fix $\rho = \varepsilon \, e \, \|i_{p,p_0}\|_{HS} \leq 2^{-q_0}$. With

$$C_{p,\varepsilon} := \sup_{|\theta|_{p_0}=\rho} \frac{1}{l_\mu(\theta)}$$

we have

$$|P_n^\mu(0)|_{-p} \leq C_{p,\varepsilon} \, n! \, \varepsilon^{-n}.$$

Using (3.7) the following estimates hold

$$\begin{aligned}
|P_n^\mu(z)|_{-p} &\leq \sum_{k=0}^n \binom{n}{k} |P_k^\mu(0)|_{-p} \, |z|_{-p}^{n-k} , \qquad z \in \mathcal{H}_{-p,\mathbb{C}} \\
&\leq C_{p,\varepsilon} \sum_{k=0}^n \tbinom{n}{k} k! \, \varepsilon^{-k} \, |z|_{-p}^{n-k} \\
&= C_{p,\varepsilon} \, n! \, \varepsilon^{-n} \sum_{k=0}^n \tfrac{1}{(n-k)!} (\varepsilon \, |z|_{-p})^{n-k} \\
&\leq C_{p,\varepsilon} \, n! \, \varepsilon^{-n} \, e^{\varepsilon |z|_{-p}} .
\end{aligned}$$

This completes the proof. □

Note. The formulae (3.7) and (3.12) can also be used as an alternative definition of the polynomials $P_n^\mu(x)$.

Example 3 Let us compare to the case of Gaussian Analysis. Here one has

$$l_{\gamma_{\mathcal{H}}}(\theta) = e^{\frac{1}{2}\langle\theta,\theta\rangle} , \qquad \theta \in \mathcal{N}_{\mathbb{C}}$$

Then it follows

$$\mathrm{M}_{2n}^\mu = (-1)^n P_{2n}^\mu(0) = \frac{(2n)!}{n! \, 2^n} \mathrm{Tr}^{\hat{\otimes} n} , \qquad n \in \mathbb{N}$$

and $\mathrm{M}_n^\mu = P_n^\mu(0) = 0$ if n is odd. Here $\mathrm{Tr} \in \mathcal{N}'^{\otimes 2}$ denotes the trace kernel defined by

$$\langle \mathrm{Tr}, \eta \otimes \xi \rangle = (\eta, \xi) , \qquad \eta, \xi \in \mathcal{N} \tag{3.14}$$

A simple comparison shows that

$$P_n^\mu(x) =: x^{\otimes n} :$$

and

$$e_\mu(\theta;x) =: e^{\langle x,\theta\rangle} :$$

where the r.h.s. denotes usual Wick ordering see e.g., [BeKo88, HKPS93]. This procedure is uniquely defined by

$$\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle = 2^{-\frac{n}{2}} |\xi|^n \, H_n\left(\tfrac{1}{\sqrt{2}|\xi|}\langle x,\xi\rangle\right) \ , \qquad \xi \in \mathcal{N}$$

where H_n denotes the Hermite polynomial of order n (see e.g., [HKPS93] for the normalization we use).

Now we are ready to give the announced description of $\mathcal{P}(\mathcal{N}')$.

Lemma 13 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$\varphi(x) = \sum_{n=0}^{N} \left\langle P_n^\mu(x), \varphi^{(n)}\right\rangle \ , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \tag{3.15}$$

and vice versa, any functional of the form (3.15) is a smooth polynomial.

Proof. The representations from Definition 11 and equation (3.15) can be transformed into one another using (3.7) and (3.8). □

3.2.2 The dual Appell system and the representation theorem for $\mathcal{P}'_\mu(\mathcal{N}')$

To give an internal description of the type (3.15) for $\mathcal{P}'_\mu(\mathcal{N}')$ we have to construct an appropriate system of generalized functions, the $\mathbb{Q}^\mu$-system. The construction we propose here is different from that of [ADKS94] where smoothness of the logarithmic derivative of μ was demanded and used for the construction of the $\mathbb{Q}^\mu$-system. To avoid this additional assumption (which excludes e.g., Poisson measures) we propose to construct the $\mathbb{Q}^\mu$-system using differential operators.

Define a differential operator of order n with constant coefficient $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$

$$D(\Phi^{(n)})\langle x^{\otimes m}, \varphi^{(m)}\rangle = \begin{cases} \dfrac{m!}{(m-n)!}\langle x^{\otimes(m-n)}\hat{\otimes}\Phi^{(n)}, \varphi^{(m)}\rangle & \text{for } m \ge n \\ 0 & \text{for } m < n \end{cases}$$

($\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}, m \in \mathbb{N}$) and extend by linearity from the monomials to $\mathcal{P}(\mathcal{N}')$.

Lemma 14 *$D(\Phi^{(n)})$is a continuous linear operator from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$.*

Remark. For $\Phi^{(1)} \in \mathcal{N}'$ we have the usual Gâteaux derivative as e.g., in white noise analysis [HKPS93]

$$D(\Phi^{(1)})\varphi = D_{\Phi^{(1)}}\varphi := \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\cdot + t\Phi^{(1)})|_{t=0}$$

for $\varphi \in \mathcal{P}(\mathcal{N})$ and we have $D((\Phi^{(1)})^{\otimes n}) = (D_{\Phi^{(1)}})^n$ thus $D((\Phi^{(1)})^{\otimes n})$ is in fact a differential operator of order n.

Proof. By definition $\mathcal{P}(\mathcal{N}')$ is isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n} .$$

Via this isomorphism $D(\Phi^{(n)})$ transforms each component $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m}$, $m \geq n$ by

$$\varphi^{(m)} \mapsto \frac{n!}{(m-n)!} (\Phi^{(n)},\ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}}$$

where the contraction $(\Phi^{(n)},\ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}(m-n)}$ is defined by

$$\langle x^{\otimes(m-n)},\ (\Phi^{(n)},\ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}} \rangle := \langle x^{\otimes(m-n)} \hat{\otimes} \Phi^{(n)}, \varphi^{(m)} \rangle \tag{3.16}$$

for all $x \in \mathcal{N}'$. It is easy to verify that

$$|(\Phi^{(n)},\ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}}|_q \leq |\Phi^{(n)}|_{-q} |\varphi^{(m)}|_q , \qquad q \in \mathbb{N}$$

which guarantees that $(\Phi^{(n)},\ \varphi^{(m)})_{\mathcal{H}^{\hat{\otimes}n}} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}(m-n)}$ and shows at the same time that $D(\Phi^{(n)})$ is continuous on each component. This is sufficient to ensure the stated continuity of $D(\Phi^{(n)})$ on $\mathcal{P}(\mathcal{N}')$. □

Lemma 15 *For $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes}n}_{\mathbb{C}}$, $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m}$ we have*

$$\text{(P6)} \quad D(\Phi^{(n)}) \langle P_m^{\mu}(x), \varphi^{(m)} \rangle = \begin{cases} \dfrac{m!}{(m-n)!} \left\langle P_{m-n}^{\mu}(x) \hat{\otimes} \Phi^{(n)},\ \varphi^{(m)} \right\rangle & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} \tag{3.17}$$

Proof. This follows from the general property of Appell polynomials which behave like ordinary powers under differentiation. More precisely, by using

$$\langle P_m^{\mu}, \theta^{\otimes m} \rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^m e_{\mu}(t\theta; \cdot) \bigg|_{t=0} , \qquad \theta \in \mathcal{N}_{\mathbb{C}}$$

we have

$$\begin{aligned} D(\Phi^{(1)}) \langle P_m^{\mu}(x), \theta^{\otimes m} \rangle &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \langle P_m^{\mu}(x + \lambda \Phi^{(1)}), \theta^{\otimes m} \rangle \bigg|_{\lambda=0} \\ &= \left(\frac{\partial}{\partial t} \right)^m \frac{\partial}{\partial \lambda} e_{\mu}(t\theta; x + \lambda \Phi^{(1)}) \bigg|_{\substack{t=0 \\ \lambda=0}} \\ &= \langle \Phi^{(1)}, \theta \rangle \left(\tfrac{\partial}{\partial t} \right)^m t\, e_{\mu}(t\theta; x) \big|_{t=0} \\ &= \langle \Phi^{(1)}, \theta \rangle \sum_{k=0}^{m} \tbinom{m}{k} \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^k t \right) \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^{m-k} e_{\mu}(t\theta; x) \bigg|_{t=0} \\ &= m \langle \Phi^{(1)}, \theta \rangle \left(\tfrac{\mathrm{d}}{\mathrm{d}t} \right)^{m-1} e_{\mu}(t\theta; x) \Big|_{t=0} \\ &= m \langle \Phi^{(1)}, \theta \rangle \left\langle P_{m-1}^{\mu}(x), \theta^{\otimes(m-1)} \right\rangle . \end{aligned}$$

This proves

$$D(\Phi^{(1)})\langle P^\mu_m, \varphi^{(m)}\rangle = m\,\left\langle P^\mu_{m-1}\hat{\otimes}\Phi^{(1)},\ \varphi^{(m)}\right\rangle .$$

The property (3.17), then follows by induction. □

In view of Lemma 21 it is possible to define the adjoint operator $D(\Phi^{(n)})^* : \mathcal{P}'_\mu(\mathcal{N}') \to \mathcal{P}'_\mu(\mathcal{N}')$ for $\Phi^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}$. Further we can introduce the constant function $1\!\!1 \in \mathcal{P}'_\mu(\mathcal{N}')$ such that $1\!\!1(x) \equiv 1$ for all $x \in \mathcal{N}'$, so

$$\langle\!\langle 1\!\!1,\ \varphi\rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x)\,\mathrm{d}\mu(x) = \mathbb{E}_\mu(\varphi).$$

Now we are ready to define our $\mathbb{Q}$-system.

Definition 16 *For any* $\Phi^{(n)} \in \left(\mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}\right)'$ *we define* $Q^\mu_n(\Phi^{(n)}) \in \mathcal{P}'_\mu(\mathcal{N}')$ *by*

$$Q^\mu_n(\Phi^{(n)}) = D(\Phi^{(n)})^* 1\!\!1 \ .$$

We want to introduce an additional formal notation $Q^\mu_n(x)$ which stresses the linearity of $\Phi^{(n)} \mapsto Q^\mu_n(\Phi^{(n)}) \in P'_\mu(\mathcal{N}')$:

$$\langle Q^\mu_n, \Phi^{(n)}\rangle := Q^\mu_n(\Phi^{(n)}) \ .$$

Example 4 It is possible to put further assumptions on the measure μ to ensure that the expression is more than formal. Let us assume a smooth measure (i.e., the logarithmic derivative of μ is infinitely differentiable, see [ADKS94] for details) with the property

$$\exists q \in \mathbb{N}\ ,\ \exists\{C_n \geq 0,\ n \in \mathbb{N}\} : \forall \xi \in \mathcal{N}$$

$$\left|\int D^n_\xi \varphi \ \mathrm{d}\mu(x)\right| \leq C_n \left\|\varphi\right\|_{L^2(\mu)} |\xi|^n_q$$

where φ is any finitely based bounded $\mathcal{C}^\infty$-function on $\mathcal{N}'$. This obviously establishes a bound of the type

$$\|Q^\mu_n(\xi_1 \otimes \cdots \otimes \xi_n)\|_{L^2(\mu)} \leq C'_n \prod_{j=1}^n |\xi_j|_q \ , \qquad \xi_1, \ldots, \xi_n \in \mathcal{N}\ ,\ n \in \mathbb{N}$$

which is sufficient to show (by means of kernel theorem) that there exists $Q^\mu_n(x) \in \left(\mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}\right)'$ for almost all $x \in \mathcal{N}'$ such that we have the representation

$$Q^\mu_n(\varphi^{(n)})(x) = \langle Q^\mu_n(x), \varphi^{(n)}\rangle \ , \qquad \varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$$

for almost all $x \in \mathcal{N}'$. For any smooth kernel $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$ we have then that the function

$$x \mapsto \langle Q^\mu_n(x), \varphi^{(n)}\rangle \ = Q^\mu_n\left(\varphi^{(n)}\right)(x)$$

belongs to $L^2(\mu)$.

Example 5 The simplest non trivial case can be studied using finite dimensional real analysis. We consider $\mathbb{R}$ as our basic Hilbert space and as our nuclear space $\mathcal{N}$. Thus the nuclear "triple" is simply

$$\mathbb{R} \subseteq \mathbb{R} \subseteq \mathbb{R}$$

and the dual pairing between a "test function" and a "distribution" degenerates to multiplication. On $\mathbb{R}$ we consider a measure $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$ where ρ is a positive $\mathcal{C}^\infty$–function on $\mathbb{R}$ such that Assumptions 1 and 2 are fulfilled. In this setting the adjoint of the differentiation operator is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^* f(x) = -\left(\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right) f(x)\ , \qquad f \in \mathcal{C}^1(\mathbb{R})$$

where the logarithmic derivative β of the measure μ is given by

$$\beta = \frac{\rho'}{\rho}$$

This enables us to calculate the $\mathbb{Q}^\mu$-system. One has

$$Q_n^\mu(x) = \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^*\right)^n 1\!\!1 = (-1)^n \left(\tfrac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right)^n 1\!\!1$$

$$= (-1)^n \frac{\rho^{(n)}(x)}{\rho(x)}\ .$$

The last equality can be seen by simple induction.
If $\rho = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the Gaussian density Q_n^μ is related to the nth Hermite polynomial:

$$Q_n^\mu(x) = 2^{-n/2} H_n\left(\tfrac{x}{\sqrt{2}}\right)\ .$$

Definition 17 *We define the $\mathbb{Q}^\mu$-system in $\mathcal{P}'_\mu(\mathcal{N}')$ by*

$$\mathbb{Q}^\mu = \left\{ Q_n^\mu(\Phi^{(n)}) \;\middle|\; \quad \Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)',\ n \in \mathbb{N}_0 \right\}\ ,$$

and the pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ will be called the Appell system $\mathbb{A}^\mu$ generated by the measure μ.

Now we are going to discuss the central property of the Appell system $\mathbb{A}^\mu$.

Theorem 18 (Biorthogonality w.r.t. μ)

$$\left\langle\!\left\langle \langle Q_n^\mu(\Phi^{(n)}),\ \langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\right\rangle_\mu = \delta_{m,n}\, n!\, \langle \Phi^{(n)}, \varphi^{(n)}\rangle \tag{3.18}$$

for $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ and $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} m}$.

Proof. It follows from (3.10) and (3.17) that

$$\begin{aligned}\left\langle\!\left\langle Q_n^\mu(\Phi^{(n)}),\ \langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\right\rangle_\mu &= \left\langle\!\left\langle 1\!\!1, D(\Phi^{(n)})\langle P_m^\mu, \varphi^{(m)}\rangle \right\rangle\!\right\rangle_\mu \\ &= \frac{m!}{(m-n)!}\mathbb{E}_\mu\left(\langle P_{(m-n)}^\mu \hat{\otimes}\Phi^{(n)},\ \varphi^{(m)}\rangle\right) \\ &= m!\ \delta_{m,n}\ \langle \Phi^{(m)}, \varphi^{(m)}\rangle\ .\end{aligned}$$

□

Now we are going to characterize the space $\mathcal{P}'_\mu(\mathcal{N}')$

Theorem 19 *For all $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ there exists a unique sequence $\{\Phi^{(n)} |\ n \in \mathbb{N}_0\}$, $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ such that*

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}) \equiv \sum_{n=0}^{\infty} \langle Q_n^\mu, \Phi^{(n)}\rangle \tag{3.19}$$

and vice versa, every series of the form (3.19) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

Proof. For $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ we can uniquely define $\Phi^{(n)} \in \left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}\right)'$ by

$$\langle \Phi^{(n)}, \varphi^{(n)}\rangle = \frac{1}{n!}\langle\!\langle \Phi,\ \langle P_n^\mu, \varphi^{(n)}\rangle\rangle\!\rangle_\mu\ , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$$

This definition is possible because $\langle P_n^\mu, \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$. The continuity of $\varphi^{(n)} \mapsto \langle \Phi^{(n)}, \varphi^{(n)}\rangle$ follows from the continuity of $\varphi \mapsto \langle\!\langle \Phi, \varphi\rangle\!\rangle$, $\varphi \in \mathcal{P}(\mathcal{N}')$. This implies that $\varphi \mapsto \sum_{n=0}^{\infty} n!\ \langle \Phi^{(n)}, \varphi^{(n)}\rangle$ is continuous on $\mathcal{P}(\mathcal{N}')$. This defines a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$, which we denote by $\sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)})$. In view of Theorem 18 it is obvious that

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)})\ .$$

To see the converse consider a series of the form (3.19) and $\varphi \in \mathcal{P}(\mathcal{N}')$. Then there exist $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that we have the representation

$$\varphi = \sum_{n=0}^{N} P_n^\mu(\varphi^{(n)})\ .$$

So we have

$$\left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}), \varphi \right\rangle\!\!\right\rangle_\mu := \sum_{n=0}^{N} n!\ \langle \Phi^{(n)}, \varphi^{(n)}\rangle$$

because of Theorem 18. The continuity of $\varphi \mapsto \langle\!\langle \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}), \varphi\rangle\!\rangle_\mu$ follows because $\varphi^{(n)} \mapsto \langle \Phi^{(n)}, \varphi^{(n)}\rangle$ is continuous for all $n \in \mathbb{N}$. □

3.3 Test functions on a linear space with measure

In this section we will construct the test function space $(\mathcal{N})^1$ and study its properties. On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the representation from Lemma 13. Let

$$\varphi = \sum_{n=0}^{N} \langle P_n^\mu,\ \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$$

be given, then $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}$ for each $p \geq 0$ $(n \in \mathbb{N})$. Thus we may define for any $p, q \in \mathbb{N}$ a Hilbertian norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|_{p,q,\mu}^2 = \sum_{n=0}^{N} (n!)^2\ 2^{nq}\ |\varphi^{(n)}|_p^2$$

The completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|_{p,q,\mu}$ is called $(\mathcal{H}_p)_{q,\mu}^1$.

Definition 20 *We define*

$$(\mathcal{N})_\mu^1 := \underset{p,q\in\mathbb{N}}{\mathrm{pr\ lim}}\ (\mathcal{H}_p)_{q,\mu}^1\ .$$

This space has the following properties

Theorem 21 *$(\mathcal{N})_\mu^1$ is a nuclear space. The topology $(\mathcal{N})_\mu^1$ is uniquely defined by the topology on $\mathcal{N}$: It does not depend on the choice of the family of norms $\{|\cdot|_p\}$.*

Proof. Nuclearity of $(\mathcal{N})_\mu^1$ follows essentially from that of $\mathcal{N}$. For fixed p, q consider the embedding

$$I_{p',q',p,q} : (\mathcal{H}_{p'})_{q',\mu}^1 \to (\mathcal{H}_p)_{q,\mu}^1$$

where p' is chosen such that the embedding

$$i_{p',p} : \mathcal{H}_{p'} \to \mathcal{H}_p$$

is Hilbert–Schmidt. Then $I_{p',q',p,q}$ is induced by

$$I_{p',q',p,q}\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, i_{p',p}^{\otimes n}\varphi^{(n)}\rangle \quad \text{for} \quad \varphi = \sum_{n=0}^{\infty}\langle P_n^\mu, \varphi^{(n)}\rangle \in (\mathcal{H}_{p'})_{q',\mu}^1\ .$$

Its Hilbert–Schmidt norm is easily estimated by using an orthonormal basis of $(\mathcal{H}_{p'})_{q',\mu}^1$. The result is the bound

$$\|I_{p',q',p,q}\|_{HS}^2 \leq \sum_{n=0}^{\infty} 2^{n(q-q')}\ \|i_{p',p}\|_{HS}^{2n}$$

which is finite for suitably chosen q'.

Let us assume that we are given two different systems of Hilbertian norms $|\cdot|_p$ and $|\cdot|_k'$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|_{k,l,\mu}'$ by $\|\cdot\|_{p,q,\mu}$ for some p, q (and vice versa which is completely analogous). Since $|\cdot|_k'$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$, there exists p and a constant C such that $|f|'_k \le C\,|f|_p$, for all $f \in \mathcal{N}$, i.e., the injection i from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by i also its linear extension from $\mathcal{H}_{p,\mathbb{C}}$ into $\mathcal{K}_{k,\mathbb{C}}$. It follows that $i^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$ into $\mathcal{K}_{k,\mathbb{C}}^{\otimes n}$. Now we choose q such that $2^{\frac{q-l}{2}} \ge C$. Then

$$\begin{aligned}
\|\cdot\|'^2_{k,l,\mu} &= \sum_{n=0}^{\infty} (n!)^2\, 2^{nl}\, |\cdot|'^2_k \\
&\le \sum_{n=0}^{\infty} (n!)^2\, 2^{nl} C^{2n}\, |\cdot|^2_p \\
&\le \|\cdot\|^2_{p,q,\mu} \ ,
\end{aligned}$$

which had to be proved. □

Lemma 22 *There exist $p, C, K > 0$ such that for all n*

$$\int |P_n^\mu(x)|^2_{-p}\, \mathrm{d}\mu(x) \le (n!)^2\, C^n\, K \tag{3.20}$$

Proof. The estimate (3.13) may be used for $\rho \le 2^{-q_0}$ and $\rho \le 2\varepsilon_\mu$ (ε_μ from Lemma 10). This gives

$$\int |P_n^\mu(x)|^2_{-p}\, \mathrm{d}\mu(x) \le (n!)^2 \left(\frac{e}{\rho}\, \|i_{p,p_0}\|_{HS}\right)^{2n} \int e^{2\rho|x|_{-p_0}}\, \mathrm{d}\mu(x)$$

which is finite because of Lemma 10. □

Theorem 23 *There exist $p', q' > 0$ such that for all $p \ge p',\ q \ge q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu} \subset L^2(\mu)$ holds.*

Proof. Elements of the space $(\mathcal{N})^1_\mu$ are defined as series convergent in the given topology. Now we need to study the convergence of these series in $L^2(\mu)$. Choose q' such that $C > 2^{q'}$ (C from estimate (3.20)). Let us take an arbitrary

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$$

For $p > p'$ (p' as in Lemma 22) and $q > q'$ the following estimates hold

$$\begin{aligned}
\|\varphi\|_{L^2(\mu)} &\le \sum_{n=0}^{\infty} \left\| \langle P_n^\mu, \varphi^{(n)} \rangle \right\|_{L^2(\mu)} \\
&\le \sum_{n=0}^{\infty} |\varphi^{(n)}|_{-p}\, \|\, |P_n^\mu|_{-p} \|_{L^2(\mu)} \\
&\le K \sum_{n=0}^{\infty} n!\, 2^{nq/2} \left|\varphi^{(n)}\right|_{-p} (C 2^{-q})^{n/2} \\
&\le K \left(\sum_{n=0}^{\infty} (C\, 2^{-q})^n \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} (n!)^2\, 2^{qn} \left|\varphi^{(n)}\right|^2_{-p} \right)^{\frac{1}{2}} \\
&= K \left(1 - C\, 2^{-q}\right)^{-1/2} \|\varphi\|_{p,q,\mu} \,.
\end{aligned}$$

Taking the closure the inequality extends to the whole space $(\mathcal{H}_p)^1_q$. □

Corollary 24 *$(\mathcal{N})^1_\mu$ is continuously and densely embedded in $L^2(\mu)$.*

Example 6 *(μ-exponentials as test functions)*
The μ-exponential given in (3.6) has the following norm

$$||e_\mu(\theta;\cdot)||^2_{p,q,\mu} = \sum_{n=0}^{\infty} 2^{nq}\,|\theta|_p^{2n}\ , \qquad \theta \in \mathcal{N}_{\mathbb{C}}$$

This expression is finite if and only if $2^q|\theta|_p^2 < 1$. Thus we have $e_\mu(\theta;\cdot) \notin (\mathcal{N})^1_\mu$ if $\theta \neq 0$. But we have that $e_\mu(\theta;\cdot)$ is a test function of finite order i.e., $e_\mu(\theta;\cdot) \in (\mathcal{H}_p)^1_q$ if $2^q|\theta|_p^2 < 1$. This is in contrast to some useful spaces of test functions in Gaussian Analysis, see e.g., [BeKo88, HKPS93].

The set of all μ–exponentials $\{e_\mu(\theta;\cdot) \mid 2^q|\theta|_p^2 < 1,\ \theta \in \mathcal{N}_{\mathbb{C}}\}$ is a total set in $(\mathcal{H}_p)^1_q$. This can been shown using the relation $\mathrm{d}^n e_\mu(0;\cdot)(\theta_1, ..., \theta_n) = \langle P_n^\mu, \theta_1 \hat{\otimes} \cdots \hat{\otimes} \theta_n \rangle$.

Proposition 25 *Any test function φ in $(\mathcal{N})^1_\mu$ has a uniquely defined extension to $\mathcal{N}'_{\mathbb{C}}$ as an element of $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$*

Proof. Any element φ in $(\mathcal{N})^1_\mu$ is defined as a series of the following type

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle\ , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$$

such that

$$\|\varphi\|^2_{p,q,\mu} = \sum_{n=0}^{\infty} (n!)^2\, 2^{nq}\, |\varphi^{(n)}|_p^2$$

is finite for each $p, q \in \mathbb{N}$. In this proof we will show the convergence of the series

$$\sum_{n=0}^{\infty} \langle P_n^\mu(z), \varphi^{(n)} \rangle, \quad z \in \mathcal{H}_{-p,\mathbb{C}}$$

to an entire function in z.

Let $p > p_0$ such that the embedding $i_{p,p_0} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0}$ is Hilbert-Schmidt. Then for all $0 < \varepsilon \leq 2^{-q_0}/e\, \|i_{p,p_0}\|_{HS}$ we can use (3.11) and estimate as follows

$$\begin{aligned}
\sum_{n=0}^{\infty} |\langle P_n^\mu(z), \varphi^{(n)} \rangle| &\leq \sum_{n=0}^{\infty} |P_n^\mu(z)|_{-p} |\varphi^{(n)}|_p \\
&\leq C_{p,\varepsilon}\, e^{\varepsilon |z|_{-p}} \sum_{n=0}^{\infty} n!\, |\varphi^{(n)}|_p\, \varepsilon^{-n} \\
&\leq C_{p,\varepsilon}\ e^{\varepsilon|z|_{-p}} \left(\sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|_p^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} 2^{-nq} \varepsilon^{-2n} \right)^{1/2} \\
&= C_{p,\varepsilon}\ \left(1 - 2^{-q}\varepsilon^{-2}\right)^{-1/2}\ \|\varphi\|_{p,q,\mu}\ \ e^{\varepsilon|z|_{-p}}
\end{aligned}$$

if $2^q > \varepsilon^{-2}$. That means the series $\sum_{n=0}^{\infty} \langle P_n^\mu(z), \varphi^{(n)} \rangle$ converges uniformly and absolutely in any neighborhood of zero of any space $\mathcal{H}_{-p,\mathbb{C}}$. Since each term $\langle P_n^\mu(z), \varphi^{(n)} \rangle$ is entire in z the uniform convergence implies that $z \mapsto \sum_{n=0}^{\infty} \langle P_n^\mu(z), \varphi^{(n)} \rangle$ is entire on each $\mathcal{H}_{-p,\mathbb{C}}$ and hence on $\mathcal{N}'_{\mathbb{C}}$. This completes the proof. □

The following corollary is an immediate consequence of the above proof and gives an explicit estimate on the growth of the test functions.

Corollary 26 *For all $p > p_0$ such that the norm $\|i_{p,p_0}\|_{HS}$ of the embedding is finite and for all $0 < \varepsilon \leq 2^{-q_0}/e \, \|i_{p,p_0}\|_{HS}$ we can choose $q \in \mathbb{N}$ such that $2^q > \varepsilon^{-2}$ to obtain the following bound.*

$$|\varphi(z)| \leq C \; \|\varphi\|_{p,q,\mu} \; e^{\varepsilon |z|_{-p}} \; , \qquad \varphi \in (\mathcal{N})^1_\mu, \; z \in \mathcal{H}_{-p,\mathbb{C}} \; ,$$

where

$$C = C_{p,\varepsilon} \; \left(1 - 2^{-q} \varepsilon^{-2}\right)^{-1/2} .$$

Let us look at Proposition 25 again. On one hand any function $\varphi \in (\mathcal{N})^1_\mu$ can be written in the form

$$\varphi(z) = \sum_{n=0}^{\infty} \langle P_n^\mu(x), \varphi^{(n)} \rangle \; , \qquad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \; , \tag{3.21}$$

on the other hand it is entire, i.e., it has the representation

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \tilde{\varphi}^{(n)} \rangle \; , \qquad \tilde{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \; , \tag{3.22}$$

To proceed further we need the explicit correspondence $\left\{\varphi^{(n)}, n \in \mathbb{N}\right\} \longleftrightarrow \left\{\tilde{\varphi}^{(n)}, n \in \mathbb{N}\right\}$ which is given in the next lemma.

Lemma 27 (Reordering)
Equations (3.21) and (3.22) hold iff

$$\tilde{\varphi}^{(k)} = \sum_{n=0}^{\infty} \binom{n+k}{k} \left(P_n^\mu(0), \varphi^{(n+k)}\right)_{\mathcal{H}^{\hat{\otimes} n}}$$

or equivalently

$$\varphi^{(k)} = \sum_{n=0}^{\infty} \binom{n+k}{k} \left(\mathrm{M}_n^\mu, \tilde{\varphi}^{(n+k)}\right)_{\mathcal{H}^{\hat{\otimes} n}}$$

where $\left(P_n^\mu(0), \varphi^{(n+k)}\right)_{\mathcal{H}^{\hat{\otimes} n}}$ and $\left(\mathrm{M}_n^\mu, \tilde{\varphi}^{(n+k)}\right)_{\mathcal{H}^{\hat{\otimes} n}}$ denote contractions defined by (3.16).

This is a consequence of (3.7) and (3.8). We omit the simple proof.

Proposition 25 states

$$(\mathcal{N})^1_\mu \subseteq \mathcal{E}^1_{\min}(\mathcal{N}')$$

as sets, where

$$\mathcal{E}^1_{\min}(\mathcal{N}') = \left\{ \varphi|_{\mathcal{N}'} \;\middle|\; \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}) \right\} \; .$$

Corollary 26 then implies that the embedding is also continuous. Now we are going to show that the converse also holds.

Theorem 28 *For all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$ we have the topological identity*

$$(\mathcal{N})^1_\mu = \mathcal{E}^1_{\min}(\mathcal{N}') \ .$$

To prove the missing topological inclusion it is convenient to use the nuclear topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ (given by the norms $\|\!|\cdot|\!\|_{p,q,1}$) introduced in section 2. Theorem 6 ensures that this topology is equivalent to the projective topology induced by the norms $\mathrm{n}_{p,l,k}$. Then the above theorem is an immediate consequence of the following norm estimate.

Proposition 29 *Let $p > p_\mu$ (p_μ as in Lemma 10) such that $\left\|i_{p,p_\mu}\right\|_{HS}$ is finite and $q \in \mathbb{N}$ such that $2^{q/2} > K_p$ ($K_p := eC\left\|i_{p,p_\mu}\right\|_{HS}$ as in (3.3)). For any $\varphi \in \mathrm{E}^1_{p,q}$ the restriction $\varphi|_{\mathcal{N}'}$ is a function from $(\mathcal{H}_p)^1_{q',\mu}$, $q' < q$. Moreover the following estimate holds*

$$||\varphi||_{p,q',\mu} \leq \|\!|\varphi|\!\|_{p,q,1}\,(1-2^{-q/2}K_p)^{-1}(1-2^{q'-q})^{-1/2} \ .$$

Proof. Let $p, q \in \mathbb{N}$, K_p be defined as above. A function $\varphi \in \mathrm{E}^1_{p,q}$ has the representation (3.22). Using the Reordering lemma combined with (3.3) and

$$\left|\tilde{\varphi}^{(n)}\right|_p \leq \frac{1}{n!}\, 2^{-nq/2}\, \|\!|\varphi|\!\|_{p,q,1}$$

we obtain a representation of the form (3.21) where

$$\begin{aligned}
\left|\varphi^{(n)}\right|_p &\leq \sum_{k=0}^{\infty}\binom{n+k}{k}\,|\mathrm{M}^\mu_k|_{-p}\,\left|\tilde{\varphi}^{(n+k)}\right|_p \\
&\leq \|\!|\varphi|\!\|_{p,q,1}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{k!}{(n+k)!}K_p^k\,2^{-(n+k)q/2} \\
&\leq \|\!|\varphi|\!\|_{p,q,1}\,\frac{1}{n!}2^{-nq/2}\sum_{k=0}^{\infty}(2^{-q/2}K_p)^k \\
&\leq \|\!|\varphi|\!\|_{p,q,1}\,\frac{1}{n!}2^{-nq/2}(1-2^{-q/2}K_p)^{-1}.
\end{aligned}$$

For $q' < q$ this allows the following estimate

$$\begin{aligned}
||\varphi||^2_{p,q',\mu} &= \sum_{n=0}^{\infty}(n!)^2\,2^{q'n}\,|\varphi^{(n)}|^2_p \\
&\leq \|\!|\varphi|\!\|^2_{p,q,1}\,(1-2^{-q/2}K_p)^{-2}\sum_{k=0}^{\infty}2^{n(q'-q)} < \infty
\end{aligned}$$

This completes the proof. □

Since we now have proved that the space of test functions $(\mathcal{N})^1_\mu$ is isomorphic to $\mathcal{E}^1_{\min}(\mathcal{N}')$ for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$, we will now drop the subscript μ. The test function space $(\mathcal{N})^1$ is the same for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

Corollary 30 *$(\mathcal{N})^1$ is an algebra under pointwise multiplication.*

Corollary 31 *$(\mathcal{N})^1$ admits 'scaling' i.e., for $\lambda \in \mathbb{C}$ the scaling operator $\sigma_\lambda : (\mathcal{N})^1 \to (\mathcal{N})^1$ defined by $\sigma_\lambda\varphi(x) := \varphi(\lambda x)$, $\varphi \in (\mathcal{N})^1$, $x \in \mathcal{N}'$ is well–defined.*

Corollary 32 *For all $z \in \mathcal{N}'_{\mathbb{C}}$ the space $(\mathcal{N})^1$ is invariant under the shift operator $\tau_z : \varphi \mapsto \varphi(\cdot + z)$.*

3.4 Distributions

In this section we will introduce and study the space $(\mathcal{N})^{-1}_{\mu}$ of distributions corresponding to the space of test functions $(\mathcal{N})^1$. Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ the space $(\mathcal{N})^{-1}_{\mu}$ can be viewed as a subspace of $\mathcal{P}'_{\mu}(\mathcal{N}')$

$$(\mathcal{N})^{-1}_{\mu} \subset \mathcal{P}'_{\mu}(\mathcal{N}')$$

Let us now introduce the Hilbertian subspace $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ of $\mathcal{P}'_{\mu}(\mathcal{N}')$ for which the norm

$$\|\Phi\|^2_{-p,-q,\mu} := \sum_{n=0}^{\infty} 2^{-qn} \left|\Phi^{(n)}\right|^2_{-p}$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu}_n(\Phi^{(n)}) \in \mathcal{P}'_{\mu}(\mathcal{N}')$$

from Theorem 19. The space $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ is the dual space of $(\mathcal{H}_p)^1_q$ with respect to $L^2(\mu)$ (because of the biorthogonality of $\mathbb{P}-$and $\mathbb{Q}-$systems). By general duality theory

$$(\mathcal{N})^{-1}_{\mu} := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu}$$

is the dual space of $(\mathcal{N})^1$ with respect to $L^2(\mu)$. As we noted in section 2 there exists a natural topology on co-nuclear spaces (which coincides with the inductive limit topology). We will consider $(\mathcal{N})^{-1}_{\mu}$ as a topological vector space with this topology. So we have the nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu} \ .$$

The action of $\Phi = \sum_{n=0}^{\infty} Q^{\mu}_n(\Phi^{(n)}) \in (\mathcal{N})^{-1}_{\mu}$ on a test function $\varphi = \sum_{n=0}^{\infty}\langle P^{\mu}_n, \varphi^{(n)}\rangle \in (\mathcal{N})^1$ is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle \ .$$

For a more detailed characterization of the singularity of distributions in $(\mathcal{N})^{-1}_{\mu}$ we will introduce some subspaces in this distribution space. For $\beta \in [0,1]$ we define

$$(\mathcal{H}_{-p})^{-\beta}_{-q,\mu} = \left\{ \Phi \in \mathcal{P}'_{\mu}(\mathcal{N}') \;\middle|\; \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} \left|\Phi^{(n)}\right|^2_{-p} < \infty \text{ for } \Phi = \sum_{n=0}^{\infty} Q^{\mu}_n(\Phi^{(n)}) \right\}$$

and

$$(\mathcal{N})^{-\beta}_{\mu} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-\beta}_{-q,\mu} \ ,$$

It is clear that the singularity increases with increasing β:

$$(\mathcal{N})^{-0} \subset (\mathcal{N})^{-\beta_1} \subset (\mathcal{N})^{-\beta_2} \subset (\mathcal{N})^{-1}$$

if $\beta_1 \leq \beta_2$.We will also consider $(\mathcal{N})^{\beta}_{\mu}$ as equipped with the natural topology.

Example 7 *(Generalized Radon–Nikodym derivative)*
We want to define a generalized function $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property

$$\langle\!\langle \rho_\mu(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\, \mathrm{d}\mu(x) \ , \qquad \varphi \in (\mathcal{N})^1 \ .$$

That means we have to establish the continuity of $\rho_\mu(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$. If $p' \geq p$ is sufficiently large and $\varepsilon > 0$ small enough, Corollary 26 applies i.e., $\exists q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varphi(x-z)\mathrm{d}\mu(x) \right| &\leq C \,\|\varphi\|_{p',q,\mu} \int_{\mathcal{N}'} e^{\varepsilon|x-z|_{-p'}} \mathrm{d}\mu(x) \\ &\leq C \,\|\varphi\|_{p',q,\mu}\, e^{\varepsilon|z|_{-p'}} \int_{\mathcal{N}'} e^{\varepsilon|x|_{-p'}} \mathrm{d}\mu(x) \end{aligned}$$

If ε is chosen sufficiently small the last integral exists. Thus we have in fact $\rho(z,\cdot) \in (\mathcal{N})^{-1}_\mu$. It is clear that whenever the Radon–Nikodym derivative $\frac{\mathrm{d}\mu(x+\xi)}{\mathrm{d}\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu(\xi,\cdot)$ defined above. We will now show that in $(\mathcal{N})^{-1}_\mu$ we have the canonical expansion

$$\rho_\mu(z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n Q^\mu_n(z^{\otimes n}).$$

It is easy to see that the r.h.s. defines an element in $(\mathcal{N})^{-1}_\mu$. Since both sides are in $(\mathcal{N})^{-1}_\mu$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}_{\mathbb{C}}$ we have

$$\begin{aligned} \left\langle\!\!\left\langle \rho_\mu(z,\cdot), \langle P^\mu_n, \varphi^{(n)} \rangle \right\rangle\!\!\right\rangle_\mu &= \int_{\mathcal{N}'} \langle P^\mu_n(x-z), \varphi^{(n)} \rangle \, \mathrm{d}\mu(x) \\ &= \sum_{k=0}^{\infty} \binom{n}{k} (-1)^{n-k} \int_{\mathcal{N}'} \langle P^\mu_k(x) \hat{\otimes} z^{\otimes n-k}, \varphi^{(n)} \rangle \, \mathrm{d}\mu(x) \\ &= (-1)^n \langle z^{\otimes n}, \varphi^{(n)} \rangle \\ &= \left\langle\!\!\left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k Q^\mu_k(z^{\otimes k}), \langle P^\mu_n, \varphi^{(n)} \rangle \right\rangle\!\!\right\rangle_\mu \ , \end{aligned}$$

where we have used (3.9), (3.10) and the biorthogonality of $\mathbb{P}$- and $\mathbb{Q}$-systems. This had to be shown. In other words, we have proven that $\rho_\mu(-z,\cdot)$ is the generating function of the $\mathbb{Q}$-functions

$$\rho_\mu(-z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^\mu_n(z^{\otimes n}) \ . \tag{3.23}$$

Let use finally remark that the above expansion allows for more detailed estimates. It is easy to see that $\rho_\mu \in (\mathcal{N})^{-0}_\mu$.

Example 8 *(Delta distribution)*
For $z \in \mathcal{N}'_{\mathbb{C}}$ we define a distribution by the following $\mathbb{Q}$-decomposition:

$$\delta_z = \sum_{n=0}^{\infty} \frac{1}{n!} Q^\mu_n(P^\mu_n(z))$$

If $p \in \mathbb{N}$ is large enough and $\varepsilon > 0$ sufficiently small there exists $C_{p,\varepsilon} > 0$ according to (3.11) such that

$$\begin{aligned} \|\delta_z\|^2_{-p,-q,\mu} &= \sum_{n=0}^{\infty} (n!)^{-2} 2^{-nq} \left|P_n^\mu(z)\right|^2_{-p} \\ &\leq C^2_{p,\varepsilon}\, e^{2\varepsilon|z|_{-p}} \sum_{n=0}^{\infty} 2^{-nq} \varepsilon^{-2n} \,, \qquad z \in \mathcal{H}_{-p,\mathbb{C}} \,, \end{aligned}$$

which is finite for sufficiently large $q \in \mathbb{N}$. Thus $\delta_z \in (\mathcal{N})^{-1}_\mu$.

For $\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in (\mathcal{N})^1$ the action of δ_z is given by

$$\langle\!\langle \delta_z, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} \langle P_n^\mu(z), \varphi^{(n)} \rangle = \varphi(z)$$

because of (3.18). This means that δ_z (in particular for z real) plays the role of a "δ-function" (evaluation map) in the calculus we discuss.

3.5 Integral transformations

We will first introduce the Laplace transform of a function $\varphi \in L^2(\mu)$. The global assumption $\mu \in \mathcal{M}_a(\mathcal{N}')$ guarantees the existence of $p'_\mu \in \mathbb{N}$, $\varepsilon_\mu > 0$ such that $\int_{\mathcal{N}'} \exp(\varepsilon_\mu |x|_{-p'_\mu})\, \mathrm{d}\mu(x) < \infty$ by Lemma 10. Thus $\exp(\langle x, \theta \rangle) \in L^2(\mu)$ if $2|\theta|_{p'_\mu} \leq \varepsilon_\mu$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$. Then by Cauchy–Schwarz inequality the Laplace transform defined by

$$L_\mu \varphi(\theta) := \int_{\mathcal{N}'} \varphi(x) \exp\langle x, \theta \rangle \, \mathrm{d}\mu(x)$$

is well defined for $\varphi \in L^2(\mu)$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$ with $2|\theta|_{p'_\mu} \leq \varepsilon_\mu$. Now we are interested to extend this integral transform from $L^2(\mu)$ to the space of distributions $(\mathcal{N})^{-1}_\mu$.

Since our construction of test function and distribution spaces is closely related to $\mathbb{P}$- and $\mathbb{Q}$-systems it is useful to introduce the so called S_μ-transform

$$S_\mu \varphi(\theta) := \frac{L_\mu \varphi(\theta)}{l_\mu(\theta)} \,.$$

Since $e_\mu(\theta; x) = e^{\langle x, \theta \rangle}/l_\mu(\theta)$ we may also write

$$S_\mu \varphi(\theta) = \int_{\mathcal{N}'} \varphi(x)\, e_\mu(\theta; x) \, \mathrm{d}\mu(x) \,.$$

The μ-exponential $e_\mu(\theta, \cdot)$ is not a test function in $(\mathcal{N})^1$, see Example 6 . So the definition of the S_μ-transform of a distribution $\Phi \in (\mathcal{N})^{-1}_\mu$ must be more careful. Every such Φ is of finite order i.e., $\exists p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu}$. As shown in Example 6 $e_\mu(\theta, \cdot)$ is in the corresponding dual space $(\mathcal{H}_p)^1_{q,\mu}$ if $\theta \in \mathcal{H}_{p,\mathbb{C}}$ is such that $2^q |\theta|^2_p < 1$. Then we can define a consistent extension of S_μ-transform.

$$S_\mu \Phi(\theta) := \langle\!\langle \Phi,\ e_\mu(\theta, \cdot) \rangle\!\rangle_\mu$$

if θ is chosen in the above way. The biorthogonality of $\mathbb{P}$- and $\mathbb{Q}$-system implies

$$S_\mu\Phi(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle \ .$$

It is easy to see that the series converges uniformly and absolutely on any closed ball $\left\{ \theta \in \mathcal{H}_{p,\mathbb{C}} \middle| \ |\theta|_p^2 \leq r, \ r < 2^{-q} \right\}$, see the proof of Theorem 35. Thus $S_\mu\Phi$ is holomorphic a neighborhood of zero, i.e., $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. In the next section we will discuss this relation to the theory of holomorphic functions in more detail.

The third integral transform we are going to introduce is more appropriate for the test function space $(\mathcal{N})^1$. We introduce the convolution of a function $\varphi \in (\mathcal{N})^1$ with the measure μ by

$$C_\mu\varphi(y) := \int_{\mathcal{N}'} \varphi(x+y) \, \mathrm{d}\mu(x), \quad y \in \mathcal{N}'.$$

From Example 7 the existence of a generalized Radon–Nikodym derivative $\rho_\mu(z,\cdot)$, $z \in \mathcal{N}'_\mathbb{C}$ in $(\mathcal{N})_\mu^{-1}$ is guaranteed. So for any $\varphi \in (\mathcal{N})^1$, $z \in \mathcal{N}'_\mathbb{C}$ the convolution has the representation

$$C_\mu\varphi(z) = \langle\!\langle \rho_\mu(-z,\cdot), \ \varphi \rangle\!\rangle_\mu \ .$$

If $\varphi \in (\mathcal{N})^1$ has the canonical representation

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \ \varphi^{(n)} \rangle$$

we have by equation (3.23)

$$C_\mu\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle \ .$$

In Gaussian Analysis C_μ- and S_μ-transform coincide. It is a typical non-Gaussian effect that these two transformations differ from each other.

3.6 Characterization theorems

Gaussian Analysis has shown that for applications it is very useful to characterize test and distribution spaces by the integral transforms introduced in the previous section. In the non-Gaussian setting first results in this direction have been obtained by [AKS93, ADKS94].

We will start to characterize the space $(\mathcal{N})^1$ in terms of the convolution C_μ.

Theorem 33 *The convolution C_μ is a topological isomorphism from $(\mathcal{N})^1$ on $\mathcal{E}^1_{\min}(\mathcal{N}'_\mathbb{C})$.*

Remark. Since we have identified $(\mathcal{N})^1$ and $\mathcal{E}^1_{\min}(\mathcal{N}')$ by Theorem 28 the above assertion can be restated as follows. We have

$$C_\mu : \mathcal{E}^1_{\min}(\mathcal{N}') \to \mathcal{E}^1_{\min}(\mathcal{N}'_\mathbb{C})$$

as a topological isomorphism.

Proof. The proof has been well prepared by Theorem 6, because the nuclear topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ is the most natural one from the point of view of the above theorem. Let $\varphi \in (\mathcal{N})^1$ with the representation

$$\varphi = \sum_{n=0}^{\infty} \langle P^{\mu}_n, \varphi^{(n)} \rangle \ .$$

From the previous section it follows

$$C_\mu \varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle$$

It is obvious from (2.2) that

$$\| C_\mu \varphi \|_{p,q,1} = \| \varphi \|_{p,q,\mu}$$

for all $p, q \in \mathbb{N}_0$, which proves the continuity of

$$C_\mu : (\mathcal{N})^1 \to \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}) \ .$$

Conversely let $F \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. Then Theorem 6 ensures the existence of a sequence of generalized kernels $\left\{ \varphi^{(n)} \in \mathcal{N}'_{\mathbb{C}} \mid n \in \mathbb{N}_0 \right\}$ such that

$$F(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle \ .$$

Moreover for all $p, q \in \mathbb{N}_0$

$$\| F \|^2_{p,q,1} = \sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left| \varphi^{(n)} \right|^2_p$$

is finite. Choosing

$$\varphi = \sum_{n=0}^{\infty} \langle P^{\mu}_n, \varphi^{(n)} \rangle$$

we have $\| \varphi \|_{p,q,\mu} = \| F \|_{p,q,1}$. Thus $\varphi \in (\mathcal{N})^1$. Since $C_\mu \varphi = F$ we have shown the existence and continuity of the inverse of C_μ. □

To illustrate the above theorem in terms of the natural topology on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ we will reformulate the above theorem and add some useful estimates which relate growth in $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ to norms on $(\mathcal{N})^1$.

Corollary 34
1) Let $\varphi \in (\mathcal{N})^1$ then for all $p, l \in \mathbb{N}_0$ and $z \in \mathcal{H}_{-p,\mathbb{C}}$ the following estimate holds

$$|C_\mu \varphi(z)| \leq \| \varphi \|_{p,2l,\mu} \exp(2^{-l} |z|_{-p})$$

i.e., $C_\mu \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.
2) Let $F \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. Then there exists $\varphi \in (\mathcal{N})^1$ with $C_\mu \varphi = F$. The estimate

$$|F(z)| \leq C \exp(2^{-l} |z|_{-p})$$

for $C > 0, \ p, q \in \mathbb{N}_0$ implies

$$\| \varphi \|_{p',q,\mu} \leq C \left(1 - 2^{q-2l} e^2 \, \| i_{p',p} \|^2_{HS} \right)^{-1/2}$$

if the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert-Schmidt and $2^{l-q/2} > e \, \| i_{p',p} \|_{HS}$.

Proof. The first statement follows from

$$|C_\mu\varphi(z)| \le \mathrm{n}_{p,l,1}(C_\mu\varphi)\cdot\exp(2^{-l}|z|_{-p})$$

which follows from the definition of $\mathrm{n}_{p,l,1}$ and estimate (2.3). The second statement is an immediate consequence of Lemma 8. □

The next theorem characterizes distributions from $(\mathcal{N})_\mu^{-1}$ in terms of S_μ-transform.

Theorem 35 *The S_μ-transform is a topological isomorphism from $(\mathcal{N})_\mu^{-1}$ on* $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$.

Remark. The above theorem is closely related to the second part of Theorem 9. Since we left the proof open we will give a detailed proof here.

Proof. Let $\Phi \in (\mathcal{N})_\mu^{-1}$. Then there exists $p, q \in \mathbb{N}$ such that

$$\|\Phi\|^2_{-p,-q,\mu} = \sum_{n=0}^\infty 2^{-nq}|\Phi^{(n)}|^2_{-p}$$

is finite. From the previous section we have

$$S_\mu\Phi(\theta) = \sum_{n=0}^\infty \langle\Phi^{(n)}, \theta^{\otimes n}\rangle \ . \tag{3.24}$$

For $\theta \in \mathcal{N}_\mathbb{C}$ such that $2^q|\theta|_p^2 < 1$ we have by definition (Formula (2.2))

$$\|S_\mu\Phi\|_{-p,-q,-1} = \|\Phi\|_{-p,-q,\mu} \ .$$

By Cauchy–Schwarz inequality

$$\begin{aligned} |S_\mu\Phi(\theta)| &\le \sum_{n=0}^\infty |\Phi^{(n)}|_{-p}|\theta|_p^n \\ &\le \left(\sum_{n=0}^\infty 2^{-nq}|\Phi^{(n)}|^2_{-p}\right)^{1/2}\left(\sum_{n=0}^\infty 2^{nq}|\theta|_p^{2n}\right)^{1/2} \\ &= \|\Phi\|_{-p,-q,\mu}\left(1-2^q|\theta|_p^2\right)^{-1/2} \ . \end{aligned}$$

Thus the series (3.24) converges uniformly on any closed ball $\left\{\theta \in \mathcal{H}_{p,\mathbb{C}} \middle| \ |\theta|_p^2 \le r, \ r < 2^{-q}\right\}$. Hence $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ and

$$\mathrm{n}_{p,l,\infty}(S_\mu\Phi) \le \|\Phi\|_{-p,-q,\mu}\,(1-2^{q-2l})^{-1/2}$$

if $2l > q$. This proves that S_μ is a continuous mapping from $(\mathcal{N})_\mu^{-1}$ to $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. In the language of section 2.2 this reads

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \ \mathrm{E}^{-1}_{-p,-q} \subset \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$$

topologically.

Conversely, let $F \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ be given, i.e., there exist $p, l \in \mathbb{N}$ such that $\mathrm{n}_{p,l,\infty}(F) < \infty$. The first step is to show that there exists $p', q \in \mathbb{N}$ such that

$$\|F\|_{-p',-q,-1} < \mathrm{n}_{p,l,\infty}(F) \cdot C \ ,$$

for sufficiently large $C > 0$. This implies immediately

$$\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}) \subset \operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \mathrm{E}^{-1}_{-p,-q}$$

topologically, which is the missing part in the proof of the second statement in Theorem 9.

By assumption the Taylor expansion

$$F(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n F(0)}(\theta)$$

converges uniformly on any closed ball $\left\{\theta \in \mathcal{H}_{p,\mathbb{C}} | \ |\theta|_p^2 \le r, \ r < 2^{-l}\right\}$ and

$$|F(\theta)| \le \mathrm{n}_{p,l,\infty}(F) \ .$$

Proceeding analogously to Lemma 7, an application of Cauchy's inequality gives

$$\begin{aligned} \frac{1}{n!}\widehat{\mathrm{d}^n F(0)}(\theta) &\le 2^l |\theta|_p^n \sup_{|\theta|_p \le 2^{-l}} |F(\theta)| \\ &\le \mathrm{n}_{p,l,\infty}(F) \ \cdot 2^{nl} \cdot |\theta|_p^n \end{aligned}$$

The polarization identity gives

$$\left|\frac{1}{n!}\mathrm{d}^n F(0)(\theta_1, \ldots, \theta_n)\right| \le \mathrm{n}_{p,l,\infty}(F) \ \cdot e^n \cdot 2^{nl} \prod_{j=1}^{n} |\theta_j|_p$$

Then by kernel theorem (Theorem 3) there exist kernels $\Phi^{(n)} \in \mathcal{H}^{\hat{\otimes} n}_{-p',\mathbb{C}}$ for $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ such that

$$F(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle \ .$$

Moreover we have the following norm estimate

$$\left|\Phi^{(n)}\right|_{-p'} \le \mathrm{n}_{p,l,\infty}(F) \ \left(2^l e \, \|i_{p',p}\|_{HS}\right)^n$$

Thus

$$\begin{aligned} \|F\|^2_{-p',-q,-1} &= \sum_{n=0}^{\infty} 2^{-nq} \left|\Phi^{(n)}\right|^2_{-p'} \\ &\le \mathrm{n}^2_{p,l,\infty}(F) \sum_{n=0}^{\infty} \left(2^{2l-q} e^2 \, \|i_{p',p}\|^2_{HS}\right)^n \\ &= \mathrm{n}^2_{p,l,\infty}(F) \left(1 - 2^{2l-q} e^2 \, \|i_{p',p}\|^2_{HS}\right)^{-1} \end{aligned}$$

if $q \in \mathbb{N}$ is such that $\rho := 2^{2l-q}e^2 \left\|i_{p',p}\right\|_{HS}^2 < 1$. So we have in fact

$$\|F\|_{-p',-q,-1} \leq \mathrm{n}_{p,l,\infty}(F)(1-\rho)^{-1/2}.$$

Now the rest is simple. Define $\Phi \in (\mathcal{N})_\mu^{-1}$ by

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)})$$

then $S_\mu \Phi = F$ and

$$\|\Phi\|_{-p',-q,\mu} = \|F\|_{-p',-q,-1}$$

This proves the existence of a continuous inverse of the S_μ–transform. Uniqueness of Φ follows from the fact that μ-exponentials are total in any $(\mathcal{H}_p)_q^1$. □

We can extract some useful estimates from the above proof which describe the degree of singularity of a distribution.

Corollary 36 *Let $F \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ be holomorphic for all $\theta \in \mathcal{N}_\mathbb{C}$ with $|\theta|_p \leq 2^{-l}$. If $p' > p$ with $\|i_{p',p}\|_{HS} < \infty$ and $q \in \mathbb{N}$ is such that $\rho := 2^{2l-q}e^2 \left\|i_{p',p}\right\|_{HS}^2 < 1$. Then $\Phi \in (\mathcal{H}_{-p'})_{-q}^{-1}$ and*

$$\|\Phi\|_{-p',-q,\mu} \leq \mathrm{n}_{p,l,\infty}(F) \cdot (1-\rho)^{-1/2}.$$

For a more detailed discussion of the degree of singularity the spaces $(\mathcal{N})^{-\beta}$, $\beta \in [0,1)$ are useful. In the following theorem we will characterize these spaces by means of S_μ-transform.

Theorem 37 *The S_μ-transform is a topological isomorphism from $(\mathcal{N})_\mu^{-\beta}$, $\beta \in [0,1)$ on $\mathcal{E}_{\max}^{2/(1-\beta)}(\mathcal{N}_\mathbb{C})$.*

Remark. The proof will also complete the proof of Theorem 9.

Proof. Let $\Phi \in (\mathcal{H}_{-p})_{-q,\mu}^{-\beta}$ with the canonical representation $\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)})$ be given. The S_μ-transform of Φ is given by

$$S_\mu \Phi(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle.$$

Hence

$$\|S_\mu \Phi\|_{-p,-q,-\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta}\, 2^{-nq} |\Phi^{(n)}|_{-p}^2$$

is finite. We will show that there exist $l \in \mathbb{N}$ and $C < 0$ such that

$$\mathrm{n}_{-p,-l,2/(1-\beta)}(S_\mu \Phi) \leq C\ \|S_\mu \Phi\|_{-p,-q,-\beta}\ .$$

We can estimate as follows

$$\begin{aligned}|S_\mu\Phi(\theta)| &\leq \sum_{n=0}^{\infty}\left|\Phi^{(n)}\right|_{-p}|\theta|_p^n\\ &\leq \left(\sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-nq}|\Phi^{(n)}|_{-p}^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{(n!)^{1-\beta}}2^{nq}\,|\theta|_p^{2n}\right)^{1/2}\\ &= \|S_\mu\Phi\|_{-p,-q,-\beta}\left(\sum_{n=0}^{\infty}\rho^{n\beta}\cdot\frac{1}{(n!)^{1-\beta}}2^{nq}\rho^{-n\beta}\,|\theta|_p^{2n}\cdot\right)^{1/2},\end{aligned}$$

where we have introduced a parameter $\rho \in (0,1)$. An application of Hölder's inequality for the conjugate indices $\frac{1}{\beta}$ and $\frac{1}{1-\beta}$ gives

$$\begin{aligned}|S_\mu\Phi(\theta)| &\leq \|S_\mu\Phi\|_{-p,-q,-\beta}\left(\sum_{n=0}^{\infty}\rho^n\right)^{\beta/2}\cdot\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(2^q\rho^{-\beta}|\theta|_p^2\right)^{\frac{n}{1-\beta}}\right)^{\frac{1-\beta}{2}}\\ &= \|S_\mu\Phi\|_{-p,-q,-\beta}\,(1-\rho)^{-\beta/2}\exp\left(\tfrac{1-\beta}{2}\,2^{\frac{q}{1-\beta}}\,\rho^{-\frac{\beta}{1-\beta}}\,|\theta|_p^{\frac{2}{1-\beta}}\right)\end{aligned}$$

If $l \in \mathbb{N}$ is such that

$$2^{l-\frac{q}{1-\beta}} > \tfrac{1-\beta}{2}\rho^{-\frac{\beta}{1-\beta}}$$

we have

$$\begin{aligned}\mathrm{n}_{-p,-l,2/(1-\beta)}(S_\mu\Phi) &= \sup_{\theta\in\mathcal{H}_{p,\mathbb{C}}}|S_\mu\Phi(\theta)|\,\exp\left(-2^l|\theta|_p^{2/(1-\beta)}\right)\\ &\leq (1-\rho)^{-\beta/2}\,\|S_\mu\Phi\|_{-p,-q,-\beta}\end{aligned}$$

This shows that S_μ is continuous from $(\mathcal{N})_\mu^{-\beta}$ to $\mathcal{E}_{\max}^{2/(1-\beta)}(\mathcal{N}_\mathbb{C})$. Or in the language of Theorem 9

$$\operatorname*{ind\,lim}_{p,q\in\mathbb{N}}\ \mathrm{E}_{-p,-q}^{-\beta} \subset \mathcal{E}_{\max}^{2/(1-\beta)}(\mathcal{N}_\mathbb{C})$$

topologically.

The proof of the inverse direction is closely related to the proof of Lemma 8. So we will be more sketchy in the following.
Let $F \in \mathcal{E}_{\max}^k(\mathcal{N}_\mathbb{C})$, $k = \frac{2}{1-\beta}$. Hence there exist $p, l \in \mathbb{N}_0$ such that

$$|F(\theta)| \leq \mathrm{n}_{-p,-l,k}(F)\exp(2^l|\theta|_p^k)\ , \qquad \theta \in \mathcal{N}_\mathbb{C}$$

From this we have completely analogous to the proof of Lemma 8 by Cauchy inequality and kernel theorem the representation

$$F(\theta) = \sum_{n=0}^{\infty}\langle\Phi^{(n)},\theta^{\otimes n}\rangle$$

and the bound

$$\left|\Phi^{(n)}\right|_{-p'} \leq \mathrm{n}_{-p,-l,k}(F)\ (n!)^{-1/k}\left\{(k2^l)^{1/k}e\,\|i_{p',p}\|_{HS}\right\}^n\ ,$$

where $p' > p$ is such that $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert–Schmidt. Using this we have

$$\begin{aligned} \|F\|^2_{-p',-q,-\beta} &= \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} \left|\Phi^{(n)}\right|^2_{-p'} \\ &\leq \mathrm{n}^2_{-p,-l,k}(F) \sum_{n=0}^{\infty} (n!)^{1-\beta-2/k} 2^{-qn} \left\{(k2^l)^{1/k} e \left\|i_{p',p}\right\|_{HS}\right\}^{2n} \\ &\leq \mathrm{n}^2_{-p,-l,k}(F) \sum_{n=0}^{\infty} \rho^n \end{aligned}$$

where we have set $\rho := 2^{-q+2l/k} k^{2/k} e^2 \left\|i_{p',p}\right\|^2_{HS}$. If $q \in \mathbb{N}$ is chosen large enough such that $\rho < 1$ the sum on the right hand side is convergent and we have

$$\|F\|_{-p',-q,-\beta} \leq \mathrm{n}_{-p,-l,2/(1-\beta)}(F) \cdot (1-\rho)^{-1/2} \ . \tag{3.25}$$

That means

$$\mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}}) \subset \operatorname*{ind\,lim}_{p,q \in \mathbb{N}_0} \mathrm{E}^{-\beta}_{-p,-q}$$

topologically.

If we set

$$\Phi := \sum_{n=0}^{\infty} Q^{\mu}_n(\Phi^{(n)})$$

then $S_\mu \Phi = F$ and $\Phi \in (\mathcal{H}_{-p'})^{-\beta}_{-q}$ since

$$\sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |\Phi^{(n)}|^2_{-p'}$$

is finite. Hence

$$S_\mu : (\mathcal{N})^{-\beta}_\mu \to \mathcal{E}^{2/(1-\beta)}_{\max}(\mathcal{N}_{\mathbb{C}})$$

is one to one. The continuity of the inverse mapping follows from the norm estimate (3.25). □

3.7 The Wick product

In Gaussian Analysis it has been shown that $(\mathcal{N})^{-1}_{\gamma_{\mathcal{H}}}$ (and other distribution spaces) is closed under so called Wick multiplication (see [KLS94] and [BeS95, Øk94, Va95] for applications). This concept has a natural generalization to the present setting.

Definition 38 *Let $\Phi, \Psi \in (\mathcal{N})^{-1}_\mu$. Then we define the Wick product $\Phi \diamond \Psi$ by*

$$S_\mu(\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi \ .$$

This is well defined because $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra and thus by the characterization Theorem 35 there exists an element $\Phi \diamond \Psi \in (\mathcal{N})^{-1}_\mu$ such that $S_\mu(\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi$.

By definition we have

$$Q^{\mu}_n(\Phi^{(n)}) \diamond Q^{\mu}_m(\Psi^{(m)}) = Q^{\mu}_{n+m}(\Phi^{(n)} \hat{\otimes} \Psi^{(m)}) \ ,$$

$\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n})'$ and $\Psi^{(m)} \in (\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}m})'$. So in terms of $\mathbb{Q}$–decompositions $\Phi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Phi^{(n)})$ and $\Psi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Psi^{(n)})$ the Wick product is given by

$$\Phi \diamond \Psi = \sum_{n=0}^{\infty} Q_n^{\mu}(\Xi^{(n)})$$

where

$$\Xi^{(n)} = \sum_{k=0}^{n} \Phi^{(k)} \hat{\otimes} \Psi^{(n-k)}$$

This allows for concrete norm estimates.

Proposition 39 *The Wick product is continuous on $(\mathcal{N})_{\mu}^{-1}$. In particular the following estimate holds for $\Phi \in (\mathcal{H}_{-p_1})_{-q_1,\mu}^{-1}$, $\Psi \in (\mathcal{H}_{-q_2})_{-q_2}^{-1}$ and $p = \max(p_1, p_2)$, $q = q_1 + q_2 + 1$*

$$\|\Phi \diamond \Psi\|_{-p,-q,\mu} = \|\Phi\|_{-p_1,-q_1,\mu} \|\Psi\|_{-p_2,-q_2,\mu} \ .$$

Proof. We can estimate as follows

$$\begin{aligned}
\|\Phi \diamond \Psi\|_{-p,-q,\mu}^2 &= \sum_{n=0}^{\infty} 2^{-nq} \left|\Xi^{(n)}\right|_{-p}^2 \\
&= \sum_{n=0}^{\infty} 2^{-nq} \left(\sum_{k=0}^{n} \left|\Phi^{(k)}\right|_{-p} \left|\Psi^{(n-k)}\right|_{-p} \right)^2 \\
&\leq \sum_{n=0}^{\infty} 2^{-nq} (n+1) \sum_{k=0}^{n} \left|\Phi^{(k)}\right|_{-p}^2 \left|\Psi^{(n-k)}\right|_{-p}^2 \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-nq_1} \left|\Phi^{(n)}\right|_{-p}^2 2^{-nq_2} \left|\Psi^{(n-k)}\right|_{-p}^2 \\
&\leq \left(\sum_{n=0}^{\infty} 2^{-nq_1} \left|\Phi^{(k)}\right|_{-p_1}^2 \right) \left(\sum_{n=0}^{\infty} 2^{-nq_2} \left|\Psi^{(n)}\right|_{-p_2}^2 \right) \\
&= \|\Phi\|_{-p_1,-q_1,\mu}^2 \|\Psi\|_{-p_2,-q_2,\mu}^2 \ .
\end{aligned}$$

□

Similar to the Gaussian case the special properties of the space $(\mathcal{N})_{\mu}^{-1}$ allow the definition of *Wick analytic functions* under very general assumptions. This has proven to be of some relevance to solve equations e.g., of the type $\Phi \diamond X = \Psi$ for $X \in (\mathcal{N})_{\mu}^{-1}$. See [KLS94] for the Gaussian case.

Theorem 40 *Let $F : \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of the point $z_0 = \mathbb{E}_{\mu}(\Phi)$, $\Phi \in (\mathcal{N})_{\mu}^{-1}$. Then $F^{\diamond}(\Phi)$ defined by $S_{\mu}(F^{\diamond}(\Phi)) = F(S_{\mu}\Phi)$ exists in $(\mathcal{N})^{-1}$.*

Proof. By the characterization Theorem 35 $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. Then $F(S_\mu\Phi) \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ since the composition of two analytic functions is also analytic. Again by characterization Theorem we find $F^\diamond(\Phi) \in (\mathcal{N})^{-1}_\mu$. □

Remark. If $F(z) = \sum_{n=0}^\infty a_k(z-z_0)^n$ then the *Wick series* $\sum_{n=0}^\infty a_k(\Phi - z_0)^{\diamond n}$ (where $\Psi^{\diamond n} = \Psi \diamond \ldots \diamond \Psi$ n-times converges in $(\mathcal{N})^{-1}$ and $F^\diamond(\Phi) = \sum_{n=0}^\infty a_k(\Phi - z_0)^{\diamond n}$ holds.

Example 9 The above mentioned equation $\Phi \diamond X = \Psi$ can be solved if $\mathbb{E}_\mu(\Phi) = S_\mu\Phi(0) \neq 0$. That implies $(S_\mu\Phi)^{-1} \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. Thus $\Phi^{\diamond(-1)} = S_\mu^{-1}\left((S_\mu\Phi)^{-1}\right) \in (\mathcal{N})^{-1}_\mu$. Then $X = \Phi^{\diamond(-1)} \diamond \Psi$ is the solution in $(\mathcal{N})^{-1}_\mu$. For more instructive examples we refer the reader to [KLS94].

3.8 Positive distributions

In this section we will characterize the positive distributions in $(\mathcal{N})^{-1}_\mu$. We will prove that the positive distributions can be represented by measures in $\mathcal{M}_a(\mathcal{N}')$. In the case of the Gaussian Hida distribution space $(\mathcal{S})'$ similar statements can be found in works of Kondratiev [Ko80a, b] and Yokoi [Yok90, Yok93], see also [Po87] and [Lee91]. In the Gaussian setting also the positive distributions in $(\mathcal{N})^{-1}$ have been discussed, see [KoSW95].

Since $(\mathcal{N})^1 = \mathcal{E}^1_{\min}(\mathcal{N}')$ we say that $\varphi \in (\mathcal{N})^1$ is positive ($\varphi \geq 0$) if and only if $\varphi(x) \geq 0$ for all $x \in \mathcal{N}'$.

Definition 41 *An element $\Phi \in (\mathcal{N})^{-1}_\mu$ is positive if for any positive $\varphi \in (\mathcal{N})^1$ we have $\langle\langle \Phi, \varphi\rangle\rangle_\mu \geq 0$. The cone of positive elements in $(\mathcal{N})^{-1}_\mu$ is denoted by $(\mathcal{N})^{-1}_{\mu,+}$.*

Theorem 42 *Let $\Phi \in (\mathcal{N})^{-1}_{\mu,+}$. Then there exists a unique measure $\nu \in \mathcal{M}_a(\mathcal{N}')$ such that $\forall \varphi \in (\mathcal{N})^1$*

$$\langle\langle \Phi, \varphi\rangle\rangle_\mu = \int_{\mathcal{N}'} \varphi(x)\, \mathrm{d}\nu(x) \; . \tag{3.26}$$

Vice versa, any (positive) measure $\nu \in \mathcal{M}_a(\mathcal{N}')$ defines a positive distribution $\Phi \in (\mathcal{N})^{-1}_{\mu,+}$ by (3.26).

Remarks.

1. For a given measure ν the distribution Φ may be viewed as the generalized Radon-Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ of ν with respect to μ. In fact if ν is absolutely continuous with respect to μ then the usual Radon-Nikodym derivative coincides with Φ.

2. Note that the cone of positive distributions generates the same set of measures $\mathcal{M}_a(\mathcal{N}')$ for all initial measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

Proof. To prove the first part we define moments of a distribution Φ and give bounds on their growth. Using this we construct a measure ν which is uniquely defined by given moments*. The next step is to show that any test functional $\varphi \in (\mathcal{N})^1$ is integrable with respect to ν.

*Since the algebra of exponential functions is not contained in $(\mathcal{N})^1_\mu$ we cannot use Minlos' theorem to construct the measure. This was the method used in Yokoi's work [Yok90].

Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ we may define moments of a positive distribution $\Phi \in (\mathcal{N})_\mu^{-1}$ by

$$\mathrm{M}_n(\xi_1, ..., \xi_n) = \left\langle\!\!\left\langle \Phi, \prod_{j=1}^n \langle \cdot, \xi_j \rangle \right\rangle\!\!\right\rangle_\mu , \quad n \in \mathbb{N}, \quad \xi_j \in \mathcal{N}, 1 \le j \le n$$

$$\mathrm{M}_0 = \langle\!\langle \Phi, \mathbb{1} \rangle\!\rangle .$$

We want to get estimates on the moments. Since $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu}$ for some $p, q > 0$ we may estimate as follows

$$\left| \left\langle\!\!\left\langle \Phi, \left\langle x^{\otimes n}, \bigotimes_{j=1}^n \xi_j \right\rangle \right\rangle\!\!\right\rangle_\mu \right| \le \|\Phi\|_{-p,-q,\mu} \left\| \left\langle x^{\otimes n}, \bigotimes_{j=1}^n \xi_j \right\rangle \right\|_{p,q,\mu} .$$

To proceed we use the property (3.8) and the estimate (3.3) to obtain

$$\begin{aligned}
\left\| \left\langle x^{\otimes n}, \bigotimes_{j=1}^n \xi_j \right\rangle \right\|^2_{p,q,\mu} &= \sum_{k=0}^n \binom{n}{k}^2 \left\| \left\langle P_k^\mu \hat{\otimes} \mathrm{M}^\mu_{n-k}, \bigotimes_{j=1}^n \xi_j \right\rangle \right\|^2_{p,q,\mu} \\
&\le \sum_{k=0}^n \binom{n}{k}^2 (k!)^2\, 2^{kq}\, |\mathrm{M}^\mu_{n-k}|^2_{-p} \prod_{j=1}^n |\xi_j|^2_p \\
&= \prod_{j=1}^n |\xi_j|^2_p \sum_{k=0}^n \binom{n}{k}^2 (k!)^2 \left((n-k)!\right)^2 K^{2(n-k)} 2^{kq} \\
&\le \prod_{j=1}^n |\xi_j|^2_p\, (n!)^2\, 2^{nq} \sum_{k=0}^n 2^{-(n-k)q} K^{2(n-k)} \\
&\le \prod_{j=1}^n |\xi_j|^2_p\, (n!)^2\, 2^{nq} \sum_{k=0}^\infty 2^{-kq} K^{2k}
\end{aligned}$$

which is finite for p, q large enough. Here K is determined by equation (3.3).
Then we arrive at

$$\Big| \mathrm{M}_n(\xi_1, ...\xi_n) \Big| \le K\ C^n\ n! \prod_{j=1}^n |\xi_j|_p \tag{3.27}$$

for some $K, C > 0$.

Due to the kernel theorem 3 we then have the representation

$$\mathrm{M}_n(\xi_1, ...\xi_n) = \left\langle \mathrm{M}^{(n)}, \xi_1 \otimes ... \otimes \xi_n \right\rangle ,$$

where $\mathrm{M}^{(n)} \in (\mathcal{N}^{\hat{\otimes} n})'$. The sequence $\left\{\mathrm{M}^{(n)},\ n \in \mathbb{N}_0\right\}$ has the following property of positivity: for any finite sequence of smooth kernels $\left\{g^{(n)}, n \in \mathbb{N}\right\}$ (i.e., $g^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$ and $g^{(n)} = 0$ $\forall\ n \ge n_0$ for some $n_0 \in \mathbb{N}$) the following inequality is valid

$$\sum_{k,j}^{n_0} \left\langle \mathrm{M}^{(k+j)}\ , g^{(k)} \otimes \overline{g^{(j)}} \right\rangle \ge 0 . \tag{3.28}$$

This follows from the fact that the left hand side can be written as $\langle\!\langle \Phi, |\varphi|^2 \rangle\!\rangle$ with

$$\varphi(x) = \sum_{n=0}^{n_0} \left\langle x^{\otimes n}, g^{(n)} \right\rangle, \quad x \in \mathcal{N}' ,$$

which is a smooth polynomial. Following [BS71, BeKo88] inequalities (3.27) and (3.28) are sufficient to ensure the existence of a uniquely defined measure ν on $(\mathcal{N}',\mathcal{C}_\sigma(\mathcal{N}'))$, such that for any $\varphi \in \mathcal{P}(\mathcal{N}')$ we have

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x)\ \mathrm{d}\nu(x)\ .$$

From estimate (3.27) we know that $\nu \in \mathcal{M}_a(\mathcal{N}')$. Then Lemma 10 shows that there exists $\varepsilon > 0$, $p \in \mathbb{N}$ such that $\exp(\varepsilon |x|_{-p})$ is $\nu-$integrable. Corollary 26 then implies that each $\varphi \in (\mathcal{N})^1$ is ν-integrable.

Conversely let $\nu \in \mathcal{M}_a(\mathcal{N}')$ be given. Then the same argument shows that each $\varphi \in (\mathcal{N})^1$ is ν-integrable and from Corollary 26 we know that

$$\left| \int_{\mathcal{N}'} \varphi(x) \mathrm{d}\nu(x) \right| \leq C\ \|\varphi\|_{p,q,\mu} \int_{\mathcal{N}'} \exp(\varepsilon |x|_{-p})\ \mathrm{d}\nu(x)$$

for some $p, q \in \mathbb{N}$, $C > 0$. Thus the continuity of $\varphi \mapsto \int_{\mathcal{N}'} \varphi\ \mathrm{d}\nu$ is established, showing that Φ defined by equation (3.26) is in $(\mathcal{N})^{-1}_{\mu,+}$. □

3.9 Change of measure

Suppose we are given two measures $\mu, \hat{\mu} \in \mathcal{M}_a(\mathcal{N}')$ both satisfying Assumption 2. Let a distribution $\hat{\Phi} \in (\mathcal{N})^{-1}_{\hat{\mu}}$ be given. Since the test function space $(\mathcal{N})^1$ is invariant under changes of measures in view of Theorem 28, the continuous mapping

$$\varphi \mapsto \langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}}\ , \qquad \varphi \in (\mathcal{N})^1$$

can also be represented as a distribution $\Phi \in (\mathcal{N})^{-1}_\mu$. So we have the implicit relation $\Phi \in (\mathcal{N})^{-1}_\mu \leftrightarrow \hat{\Phi} \in (\mathcal{N})^{-1}_{\hat{\mu}}$ defined by

$$\langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}} = \langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu\ .$$

This section will provide formulae which make this relation more explicit in terms of re-decomposition of the $\mathbb{Q}$-series. First we need an explicit relation of the corresponding $\mathbb{P}$-systems.

Lemma 43 *Let $\mu, \hat{\mu} \in \mathcal{M}_a(\mathcal{N}')$ then*

$$P^\mu_n(x) = \sum_{k+l+m=n} \frac{n!}{k!\,l!\,m!} P^{\hat{\mu}}_k(x) \hat{\otimes} P^\mu_l(0) \hat{\otimes} \mathrm{M}^\mu_m\ .$$

Proof. Expanding each factor in the formula

$$e_\mu(\theta, x) = e_{\hat{\mu}}(\theta, x) l^{-1}_\mu(\theta) l_{\hat{\mu}}(\theta)\ ,$$

we obtain

$$\sum_{n=0}^\infty \frac{1}{n!} \langle P^\mu_n(x), \theta^{\otimes n} \rangle = \sum_{k,l,m=0}^\infty \frac{1}{k!\,l!\,m!} \langle P^\mu_k(x) \otimes P^{\hat{\mu}}_l(0) \otimes \mathrm{M}^\mu_m, \theta^{\otimes(k+l+m)} \rangle\ .$$

A comparison of coefficients gives the above result. □

An immediate consequence is the next reordering lemma.

Lemma 44 *Let $\varphi \in (\mathcal{N})^1$ be given. Then φ has representations in $\mathbb{P}^\mu$-series as well as $\mathbb{P}^{\hat{\mu}}$-series:*

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle = \sum_{n=0}^{\infty} \langle P_n^{\hat{\mu}}, \hat{\varphi}^{(n)} \rangle$$

where $\varphi^{(n)}, \hat{\varphi}^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}_0$, and the following formula holds:

$$\hat{\varphi}^{(n)} = \sum_{l,m=0}^{\infty} \frac{(l+m+n)!}{l!\, m!\, n!} \left(P_l^\mu(0) \hat{\otimes} \mathrm{M}_m^{\hat{\mu}}, \varphi^{(l+m+n)} \right)_{\mathcal{H}^{\otimes(l+m)}} . \tag{3.29}$$

Now we may prove the announced theorem.

Theorem 45 *Let $\hat{\Phi} = \sum_{n=0}^{\infty} \langle Q_n^{\hat{\mu}}, \hat{\Phi}^{(n)} \rangle \in (\mathcal{N})_{\hat{\mu}}^{-1}$. Then $\Phi = \sum_{n=0}^{\infty} \langle Q_n^\mu, \Phi^{(n)} \rangle$ defined by*

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \langle\!\langle \hat{\Phi}, \varphi \rangle\!\rangle_{\hat{\mu}} , \qquad \varphi \in (\mathcal{N})^1$$

is in $(\mathcal{N})_\mu^{-1}$ and the following relation holds

$$\Phi^{(n)} = \sum_{k+l+m=n} \frac{1}{l!\, m!} \hat{\Phi}^{(k)} \hat{\otimes} P_l^\mu(0) \hat{\otimes} \mathrm{M}_m^{\hat{\mu}}$$

Proof. We can insert formula (3.29) in the formula

$$\sum_{n=0}^{\infty} n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle = \sum_{n=0}^{\infty} n!\, \langle \hat{\Phi}^{(n)}, \hat{\varphi}^{(n)} \rangle$$

and compare coefficients again. □

Chapter 4

Gaussian analysis

The primordial object of Gaussian analysis (e.g., [BeKo88, HKPS93, Ko80a, KT80],) is a real separable Hilbert space $\mathcal{H}$. One then considers a rigging of $\mathcal{H}$, $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$, where $\mathcal{N}$ is a real nuclear space (see below and [GV68]), densely and continuously embedded into $\mathcal{H}$, and $\mathcal{N}'$ is its dual ($\mathcal{H}$ being identified with its dual). A typical example (which appears for instance in white noise analysis) is the rigging $\mathcal{S}(I\!R) \subset L^2(I\!R) \subset \mathcal{S}'(I\!R)$ of $L^2(I\!R)$ (with Lebesgue measure) by the Schwartz spaces of test functions and tempered distributions. Via Minlos' theorem the canonical Gaussian measure μ on $\mathcal{N}'$ is introduced by giving its characteristic function

$$C(\xi) = \int_{\mathcal{N}'} e^{i\langle\omega,\xi\rangle}\,\mathrm{d}\mu(\omega) = e^{-\frac{1}{2}|\xi|^2_{\mathcal{H}}}, \quad \xi \in \mathcal{N}.$$

Of course the basic setting has already been introduced in the previous chapter, see e.g., Example 3 on page 20. For traditional reasons we have some changes in the notation at this point. The Gaussian measure connected with the Hilbert space $\mathcal{H}$ previously denoted by $\gamma_{\mathcal{H}}$ is called μ from now on. Since it is fixed throughout the rest of the work, we will drop some subscripts μ. Since S_μ-transform and convolution C_μ coincide both will be denoted by S. The letter C will be reserved for the characteristic function of the Gaussian measure, which will be used more frequently than its Laplace transform. The basic variable of integration, in the previous chapter called $x \in \mathcal{N}'$ will now be called $\omega \in \mathcal{N}'$.

The space $L^2(\mathcal{N}', \mu) \equiv L^2(\mu)$ of (equivalence classes of) complex valued functions on $\mathcal{N}'$ which are square-integrable with respect to μ has the well-known Wiener–Itô–Segal chaos decomposition [Ne73, Si74, Se56], and one has the familiar Segal isomorphism $\mathcal{I}$ between $L^2(\mu)$ and the complex Fock space $\Gamma(\mathcal{H})$ over the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$. Spaces of smooth functions on $\mathcal{N}'$ can be constructed by mapping appropriate subspaces of $\Gamma(\mathcal{H})$ into $L^2(\mu)$ via the unitary mapping $\mathcal{I}^{-1} : \Gamma(\mathcal{H}) \to L^2(\mu)$, see, e.g., the construction using second quantized operators in [BeKo88, HKPS93]. So $\varphi \in L^2(\mu)$ has a representation

$$\varphi = \sum_{n=0}^{\infty} \langle :\omega^{\otimes n}:, \varphi^{(n)}\rangle\ , \qquad \varphi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$$

with norm

$$\|\varphi\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n!\,\left|\varphi^{(n)}\right|^2 \qquad .$$

4.1 The Hida spaces $(\mathcal{N})$ and $(\mathcal{N})'$

4.1.1 Construction and properties

In the Gaussian setting it is of course also possible to study the triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1} \qquad .$$

This will be done in the next section. But for Gaussian measures one also has the very important possibility to construct the space of Hida test functionals $(\mathcal{N}) \supset (\mathcal{N})^1$ which is much bigger and was historically considered first. We will only sketch the well known construction of $(\mathcal{N})$.

Consider the space of smooth polynomials $\mathcal{P}(\mathcal{N}')$. For $\varphi = \sum_{n=0}^{\mu} \langle :\omega^{\otimes n}:, \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$, $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ we introduce the Hilbertian norm, $p, q, \in \mathbb{N}_0$

$$\|\varphi\|_{p,q}^2 = \sum_{n=0}^{\infty} n!\, 2^{nq} \left|\varphi^{(n)}\right|_p^2 \ . \tag{4.1}$$

Note. Of course the notation is now in some sense misleading, since $\|\cdot\|_{p,q}$ is different from $\|\cdot\|_{p,q,\mu}$ from the previous chapter. Despite of this the notation (4.1) will be used to stay consistent with the literature. The norm $\|\cdot\|_{p,q,\mu}$ will be called $\|\cdot\|_{p,q,1}$ in the Gaussian setting.

Denote the closure of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q}$ by $(\mathcal{H}_p)_q$. Finally we set

$$(\mathcal{N}) = \operatorname*{pr\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_p)_q \ .$$

Remark. Evidently substitution of the value 2 in equation (4.1) by any other number strictly larger than 1 produces the same space $(\mathcal{N})$.

It is worthwhile to note that $(\mathcal{N})$ is continuously embedded in $L^2(\mu)$ in the Gaussian case. This is due to the fact that our $\mathbb{P}$-system used here coincides with the orthonormal basis of Hermite functions.

Lemma 46 *$(\mathcal{N})$ is nuclear.*

Proof. Nuclearity of $(\mathcal{N})$ follows essentially from that of $\mathcal{N}$. For fixed p, q consider the embedding

$$I : (\mathcal{H}_{p'})_{q'} \to (\mathcal{H}_p)_{q'}$$

where p' is chosen such that the embedding

$$i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$$

is Hilbert–Schmidt. Then I is given by

$$\mathcal{I} I \mathcal{I}^{-1} = \bigoplus_n i_{p',p}^{\otimes n}.$$

where $\mathcal{I}$ is the Segal isomorphism. Its Hilbert–Schmidt norm is easily estimated by using an orthonormal basis (cf., e.g., [HKPS93, Appendix A. 2]) of $(\mathcal{H}_{p'})_{q'}$. The result is the bound

$$\|I\|_{HS}^2 \leq \sum_{n=0}^{\infty} 2^{n(q-q')} \, \|i_{p',p}\|_{HS}^{2n}$$

which is finite for suitably chosen q'. □

Theorem 47 *The topology on $(\mathcal{N})$ is uniquely determined by the topology on $\mathcal{N}$.*

Proof. Let us assume that we are given two different systems of Hilbertian norms $|\cdot|_p$ and $|\cdot|'_k$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|'_{k,l}$ by $\|\cdot\|_{p,q}$ for some p, q (and vice versa which is completely analogous). Since $|\cdot|'_k$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$, there exists p and a constant C such that $|f|'_k \leq C\,|f|_p$, for all $f \in \mathcal{N}$, i.e., the injection ι from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by ι also its linear extension from $\mathcal{H}_{\mathbb{C},p}$ into $\mathcal{K}_{\mathbb{C},k}$. It follows from a straightforward modification of the proof of the Proposition on p. 299 in [ReSi72], that $\iota^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{\mathbb{C},p}^{\otimes n}$ into $\mathcal{K}_{\mathbb{C},k}^{\otimes n}$. Now we choose q such that $2^{\frac{q-l}{2}} \geq C$. Then

$$\|\cdot\|'^2_{k,l} = \sum_{n=0}^{\infty} n!\, 2^{nl}\, |\cdot|'^2_k \leq \sum_{n=0}^{\infty} n!\, 2^{nl} C^{2n}\, |\cdot|_p^2 \leq \|\cdot\|_{p,q}^2 \,,$$

which had to be proved. □

From general duality theory on nuclear spaces we know that the dual of $(\mathcal{N})$ is given by

$$(\mathcal{N})' = \operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \, (\mathcal{H}_{-p})_{-q} \ \ ,$$

where

$$(\mathcal{H}_{-p})_{-q} = (\mathcal{H}_p)'_q \ .$$

We shall denote the bilinear dual pairing on $(\mathcal{N})' \times (\mathcal{N})$ by $\langle\!\langle \cdot,\cdot \rangle\!\rangle$:

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle,$$

where $\Phi \in (\mathcal{H}_{-p})_{-q}$ corresponds to the sequence $(\Phi^{(n)}, n \in \mathbb{N})$ with $\Phi^{(0)} \in \mathbb{C}$, and $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C},-p}^{\hat{\otimes} n}$, $n \in \mathbb{N}$.

Remark. Consider the particular choice $\mathcal{N} = \mathcal{S}(I\!R)$. Then $(\mathcal{N})^{(\prime)}$ coincide with the well-known spaces $(\mathcal{S})^{(\prime)}$ of white noise functionals, see, e.g., [HKPS93, PS91]. For the norms $\|\varphi\|_p \equiv \|\Gamma(A^p)\varphi\|_0$ introduced there, we have $\|\cdot\|_p = \|\cdot\|_{p,0}$, and $\|\cdot\|_{p,q} \leq \|\cdot\|_{p+\frac{q}{2}}$. More generally, if the norms on $\mathcal{N}$ satisfy the additional assumption that for all $p \geq 0$ and all $\varepsilon > 0$ there exists $p' \geq 0$ such that $|\cdot|_p \leq \varepsilon |\cdot|_{p'}$, then the construction of Kubo and Takenaka [KT80] (and other authors) leads to the same space $(\mathcal{N})$. The construction presented here has the advantage of being manifestly independent of the choice of any concrete system of Hilbertian norms topologizing $\mathcal{N}$.

For Wick exponentials

$$:\exp\langle\cdot,\xi\rangle := e^{\langle\cdot,\xi\rangle-\frac{1}{2}\langle\xi,\xi\rangle} = \sum_{n=0}^{\infty}\langle :\omega^{\otimes n}:, \frac{1}{n!}\xi^{\otimes n}\rangle$$

one calculates the norms

$$\| :\exp\langle\cdot,\xi\rangle: \|_{p,q}^2 = e^{2^q|\xi|_p^2},$$

and hence for all $\xi \in \mathcal{N}$ they are in $(\mathcal{N})$. This then allows for the following

Definition 48 *Let $\Phi \in (\mathcal{N})'$. The* **S–transform of** Φ *is the mapping from $\mathcal{N}$ into $\mathbb{C}$ given by*

$$S\Phi(\xi) := \langle\langle \Phi, :\exp\langle\cdot,\xi\rangle: \rangle\rangle, \quad \xi \in \mathcal{N}.$$

We note that the exponential vectors $\{:\exp\langle\cdot,\xi\rangle:, \xi \in \mathcal{N}\}$, are a total set $\mathcal{E}$ in $(\mathcal{N})$, and hence elements of $(\mathcal{N})'$ are characterized by their S–transforms. Furthermore, it is obvious that the S–transform of $\Phi \in (\mathcal{N})'$ extends to $\mathcal{N}_{\mathbb{C}}$: for $\theta \in \mathcal{N}_{\mathbb{C}}$ set $S\Phi(\theta) = \langle\langle \Phi, :\exp\langle\cdot,\theta\rangle: \rangle\rangle$, where $:\exp\langle\cdot,\theta\rangle: \in (\mathcal{N})$ has complex kernels.

4.1.2 U–functionals and the characterization theorems

We begin with a definition.

Definition 49 *Let $F : \mathcal{N} \to \mathbb{C}$ be such that*

C.1 for all $\xi, \eta \in \mathcal{N}$, the mapping $l \longmapsto F(\eta + l\xi)$ from $\mathbb{R}$ into $\mathbb{C}$ has an entire extension to $l \in \mathbb{C}$,
C.2 for some continuous quadratic form B on $\mathcal{N}$ there exists constants $C, K > 0$ such that for all $f \in \mathcal{N}$, $z \in \mathbb{C}$,

$$|F(z\xi)| \le C \exp(K\,|z|^2 |B(\xi)|).$$

Then F is called a **U–functional**.

Remark. Condition C.2 is actually equivalent to the more conventional

C.2′ there exists constants $C, K > 0$ and $p \in \mathbb{N}_0$, so that for all $\xi \in \mathcal{N}$, $z \in \mathbb{C}$,

$$|F(z\xi)| \le C \exp(K\,|z|^2 |\xi|_p^2). \tag{4.2}$$

To proceed we need a result which is related to the celebrated "cross theorem" of Bernstein. For a review of such results we refer the interested reader also to [AR73]. The following is a special case of a result by Siciak: if we make use of the fact that any segment of the real line in the complex plane has strictly positive transfinite diameter, then Corollary 7.3 in [Si69] implies

Proposition 50 *Let $n \in \mathbb{N}$, $n \ge 2$, and f be a complex valued function on $\mathbb{R}^n$. Assume that for all $k = 1, 2, \ldots, n$, and $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, the mapping*

$$x_k \longmapsto f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n),$$

from $\mathbb{R}$ into $\mathbb{C}$ has an entire extension. Then f has an entire extension to $\mathbb{C}^n$.

Lemma 51 *Every U–functional F has a unique extension to an entire function on $\mathcal{N}_{\mathbb{C}}$. Moreover, if the bound on F holds in the form (4.2) then for all $\rho \in (0,1)$,*

$$|F(\theta)| \le C' \exp(K' \, |\theta|_p^2), \quad \theta \in \mathcal{N}_{\mathbb{C}},$$

with $C' = C(1-\rho)^{-\frac{1}{2}}$, $K' = 2\rho^{-1}e^2 K$.

Proof. First we show that a U–functional F has a G–entire extension. The extension of F (denoted by the same symbol) is given by $F(\theta) = F(\xi_0 + z\xi_1)$, $\theta = \xi_0 + z\xi_1 \in \mathcal{N}_{\mathbb{C}}$, $\xi_0, \xi_1 \in \mathcal{N}$, $z \in \mathbb{C}$. Let $\theta \in \mathcal{N}_{\mathbb{C}}$ be of the form $\theta = \xi_2 + i\xi_3, \xi_2, \xi_3 \in \mathcal{N}$. Consider the mapping

$$(\lambda_1, \lambda_2, \lambda_3) \longmapsto F(\xi_0 + \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3),$$

from $\mathbb{R}^3$ into $\mathbb{C}$. Condition C.1 and Proposition 50 imply that this function has an entire extension to $\mathbb{C}^3$. In particular, F is G–entire on $\mathcal{N}_{\mathbb{C}}$. Let $\theta \in \mathcal{N}_{\mathbb{C}}$, and consider the Taylor expansion of $F(\theta)$ at the origin :

$$F(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n F(0)}(\theta). \tag{4.3}$$

For all $\xi \in \mathcal{N}$, $n \in \mathbb{N}$, $R > 0$, we obtain from C.2$'$ and Cauchy's inequality the estimate

$$|\widehat{\mathrm{d}^n F(0)}(\xi)| \le C \, n! \, R^{-n} e^{R^2 K |\xi|_p^2}.$$

We choose $R = (\frac{n}{2K})^{\frac{1}{2}}$, and get for $\xi \in \mathcal{N}$ with $|\xi|_p = 1$ the inequality

$$|\widehat{\mathrm{d}^n F(0)}(\xi)| \le C \, n! \left(\frac{2eK}{n}\right)^{n/2}.$$

A standard polarization argument (see, e.g., [Na69, sec.3]) and homogeneity of $\widehat{\mathrm{d}^n F(0)}$ yield the following bound for the n–linear form $\mathrm{d}^n F(0)$:

$$|\mathrm{d}^n F(0)(\xi_1, \ldots, \xi_n)| \le C(n! \, (2e^2 K)^n)^{\frac{1}{2}} \prod_{k=1}^{n} |\xi_k|_p, \tag{4.4}$$

where $\xi_1, \ldots, \xi_n \in \mathcal{N}$ (and we used $\frac{n^n}{n!} \le e^n$). Since $\mathrm{d}^n F(0)$ is n–linear on $\mathcal{N}_{\mathbb{C}}$, the last inequality gives the estimate

$$|\mathrm{d}^n F(0)(\theta_1, \ldots, \theta_n)| \le C \, (n! \, (4e^2 K)^n)^{\frac{1}{2}} \prod_{k=1}^{n} |\theta_k|_p, \tag{4.5}$$

for $\theta_1, \ldots, \theta_n \in \mathcal{N}_{\mathbb{C}}$. In particular, the Taylor coefficients in (4.3) have absolute value bounded by

$$C \left(\frac{(4e^2 K |\theta|_p^2)^n}{n!}\right)^{\frac{1}{2}},$$

and we get (by Schwarz' inequality) the following estimate for all $\rho \in (0,1)$,

$$|F(\theta)| \le C(1-\rho)^{-\frac{1}{2}} e^{2\rho^{-1} e^2 K |\theta|_p^2}, \quad \theta \in \mathcal{N}_{\mathbb{C}}.$$

Hence F is locally bounded on $\mathcal{N}_{\mathbb{C}}$, and therefore Proposition 5 implies that F is entire. □

Now we are ready to prove the following generalization of the main result in [PS91] which characterizes the space $(\mathcal{N})'$ in terms of its S–transform.

Theorem 52 *A mapping $F : \mathcal{N} \to \mathbb{C}$ is the S–transform of an element in $(\mathcal{N})'$ if and only if it is a U–functional.*

Proof. Let $\Phi \in (\mathcal{N})'$. Then $\Phi \in (\mathcal{H}_{-p})_{-q}$ for some p, $q \in \mathbb{N}_0$. As we have remarked after Definition 48, the S–transform of Φ extends to $\mathcal{N}_{\mathbb{C}}$, and therefore it makes sense to consider the mapping $\theta \mapsto S\Phi(\theta)$ from $\mathcal{N}_{\mathbb{C}}$ into $\mathbb{C}$. We shall show that this mapping is entire. We have

$$S\Phi(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}}.$$

We estimate as follows:

$$\begin{aligned}
|S\Phi(\theta)| &\leq \sum_{n=0}^{\infty} |\Phi^{(n)}|_{-p} |\theta|_p^n \\
&\leq \left(\sum_{n=0}^{\infty} n!\, 2^{-qn} |\Phi^{(n)}|_{-q}^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{n!} 2^{qn} |\theta|_p^{2n} \right)^{1/2} \\
&= \|\Phi\|_{-p,-q}\, e^{2^{q-1}|\theta|_p^2}.
\end{aligned}$$

The last estimation shows that the power series for $S\Phi$ on $\mathcal{N}_{\mathbb{C}}$ converges uniformly on every bounded neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$, and therefore it defines an entire function on this space [Di81]. In particular, C.1 holds for $S\Phi$. Moreover, the choice $\theta = z\xi$, $z \in \mathbb{C}$, $\xi \in \mathcal{N}$, shows that also C.2′ is fulfilled. Hence $S\Phi$ is a U–functional.

Conversely let F be a U–functional. We may assume the bound in the form (4.2). Consider the n–linear form $\mathrm{d}^n F(0)$ on $\mathcal{N}_{\mathbb{C}}$ constructed in the proof of Lemma 51. The estimate (4.5) shows that $\mathrm{d}^n F(0)$ is separately continuous on $\mathcal{N}_{\mathbb{C}}$ in its n variables. Hence by the Kernel Theorem 3 there exists $\Phi^{(n)} \in (\mathcal{N}'_{\mathbb{C}})^{\widehat{\otimes} n}$ so that

$$\langle \Phi^{(n)}, \theta_1 \widehat{\otimes} \cdots \widehat{\otimes} \theta_n \rangle = \frac{1}{n!} \mathrm{d}^n F(0)(\theta_1, \ldots, \theta_n), \quad \theta_1, \ldots, \theta_n \in \mathcal{N}_{\mathbb{C}}$$

and from (4.4) we have the norm estimate

$$\left|\Phi^{(n)}\right|_{-p'} \leq C (n!)^{\frac{1}{2}} \left(2e^2 K \, \|i_{p',p}\|_{HS}^2 \right)^{n/2} \tag{4.6}$$

if $p' > p$ is such that the embedding $i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is Hilbert Schmidt. For Φ given by the sequence $(\Phi^{(n)}, n \in \mathbb{N}_0)$ $(\Phi^{(0)} \equiv F(0))$ we have

$$\begin{aligned}
\|\Phi\|_{-p',-q}^2 &= \sum_{n=0}^{\infty} n!\, 2^{-nq} \left|\Phi^{(n)}\right|_{-p'}^2 \\
&\leq C^2 \sum_{n=0}^{\infty} (2^{1-q} e^2 K \, \|i_{p',p}\|_{\mathrm{HS}}^2)^n \\
&= C^2 (1 - 2^{1-q} e^2 K \, \|i_{p',p}\|_{\mathrm{HS}}^2)^{-1} \\
&< +\infty,
\end{aligned}$$

if we choose q large enough so that $2^{1-q} e^2 K \, \|i_{p',p}\|_{HS}^2 < 1$. In particular, $\Phi \in (\mathcal{N})'$, and for $f \in \mathcal{N}$ we have by (4.3),

$$\begin{aligned}
S\Phi(\xi) &= \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \xi^{\otimes n} \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n F(0)}(\xi) \\
&= F(\xi).
\end{aligned}$$

Uniqueness of $\Phi = S^{-1}F$ follows from the fact that the exponential vectors are total in $(\mathcal{N})$. □

As a by-product of the above proof we obtain the following localization result for generalized functionals.

Corollary 53 *Given a U–functional F satisfying C.2′. Let $p\prime > p$ be such that the embedding $i_{p',p} : \mathcal{H}_{p\prime} \to \mathcal{H}_p$ is Hilbert–Schmidt, and $q \in \mathbb{N}_0$ so that $\rho := 2^{1-q}e^2K \|i_{p',p}\|_{HS}^2 < 1$. Then $\Phi := S^{-1}F \in (\mathcal{H}_{-p\prime})_{-q}$, and*

$$\|\Phi\|_{-p\prime,-q} \leq C(1-\rho)^{-1/2}. \tag{4.7}$$

For analogous results in white noise analysis see, e.g., [KoS92, Ob91, Yan90].

We close this section by the corresponding characterization theorem for $(\mathcal{N})$. This result is independently due to [Ko80a, KPS91, Lee89], and has been generalized and modified in various ways, e.g., [Ob91, Yan90, Zh92].

Theorem 54 *A mapping $F : \mathcal{N} \to \mathbb{C}$ is the S–transform of an element in $(\mathcal{N})$ if and only if it admits C.1 and the following condition*

C.3 there exists a system of norms $(|\cdot|_{-p}\,,\, p \in \mathbb{N}_0)$, which yields the inductive limit topology on $\mathcal{N}'$, and such that for all $p \geq 0$ and $\varepsilon > 0$ there exists $C_{p,\varepsilon} > 0$ so that

$$|F(z\xi)| \leq C_{p,\varepsilon} \exp\left(\varepsilon |z|^2 \, |\xi|_{-p}^2\right), \quad \xi \in \mathcal{N},\, z \in \mathbb{C}. \tag{4.8}$$

If for F conditions C.1 and C.3 are satisfied we say that F is of order 2 and minimal type, i.e., $F \in \mathcal{E}^2_{\min}(\mathcal{N}')$.

Proof. If $\varphi \in (\mathcal{N})$ then condition C.1 is satisfied as a consequence of Theorem 52. For any $p, q \geq 0$ we estimate as follows

$$\begin{aligned}
|S\varphi(zf)| &= |\sum_{n=0}^{\infty} \langle \varphi^{(n)}, (z\xi)^{\otimes n}\rangle| \\
&\leq \sum_{n=0}^{\infty} |z|^n |\varphi^{(n)}|_p \, |\xi|_{-p}^n \\
&\leq (\sum_{n=0}^{\infty} n!\, 2^{nq}\, |\varphi^{(n)}|_p^2)^{1/2} (\sum_{n=0}^{\infty} \frac{1}{n!}\, (2^{-q}|z|^2 |\xi|_{-p}^2)^n)^{1/2} \\
&= \|\varphi\|_{p,q} \, \exp(2^{1-q}|z|^2|\xi|_{-p}^2).
\end{aligned}$$

Hence condition C.3, too, is necessary.

Conversely, let F be a U–functional of order 2 and minimal type. From F, construct a sequence $\varphi = (\varphi^{(n)},\, n \in I\!N_0)$ of continuous linear forms $\varphi^{(n)}$ on $\mathcal{N}^{\hat{\otimes} n}$ as in the proof of Lemma 51. We have to show that φ belongs to $(\mathcal{H}_r)_q$ for all $r, q \in I\!N_0$. Let $r, q \in I\!N_0$ be given. Choose $p > r$ such that the injection $i_{p,r} : \mathcal{H}_p \to \mathcal{H}_r$ is Hilbert–Schmidt. Then so is the injection $i^*_{p,r} : \mathcal{H}_{-r} \to \mathcal{H}_{-p}$. $\varepsilon > 0$ in (4.8) is chosen so that $\rho := \varepsilon 2^{1+q} e^2 \, \|i^*_{p,r}\|^2_{HS} < 1$. Then the analogue of (4.6) reads

$$\begin{aligned} \|\varphi^{(n)}\|^2_{r,q} &\leq C^2_{p,\varepsilon} (2^{q+1} e^2 \varepsilon \, \|i^*_{p,r}\|^2_{\mathrm{HS}})^n \\ &= C^2_{p,\varepsilon} \rho^n, \end{aligned}$$

and we get

$$\begin{aligned} \|\varphi\|_{r,q} &= (\sum_{n=0}^{\infty} \|\varphi^{(n)}\|^2_{r,q})^{1/2} \\ &\leq C_{p,\varepsilon} (1-\rho)^{-1/2}. \end{aligned}$$

Thus $\varphi \in (\mathcal{N})$, and the proof is complete. □

Within the framework established here one can treat the following and numerous other examples in a unified way.

Example 10 We choose the triplet

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

and equip $\mathcal{S}'(\mathbb{R}^n)$ with the Gaussian measure with characteristic functional

$$C(\xi) = e^{-\frac{1}{2}\int \xi^2(t)\,\mathrm{d}^n t}, \quad \xi \in \mathcal{S}\left(\mathbb{R}^n\right).$$

Then the framework allows to discuss functionals of white noise with n–dimensional time parameter [SW93].

Example 11 *(Vector valued white noise)*

The starting point is the real separable Hilbert space $L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d$, $d \in \mathbb{N}$ which is isomorphic to a direct sum of d identical copies of $L^2(\mathbb{R})$. In this space we choose a densely imbedded nuclear space. Here we fix this space to be $\mathcal{S}_d := \mathcal{S}(\mathbb{R}) \otimes \mathbb{R}^d$. A typical element $f \in \mathcal{S}_d$ is a d-dimensional vector where each component $f_j \quad 1 \leq j \leq d$ is a Schwartz test function. The topology on $\mathcal{S}_d$ may be represented by a system of Hilbertian norms

$$\left|\vec{f}\right|^2_p = \sum_{j=1}^{d} |f_j|^2_p \qquad , \quad p \geq 0, \ \vec{f} \in \mathcal{S}_d$$

where $|\cdot|_p$ on the r.h.s. is an increasing system of Hilbertian norms topologizing $\mathcal{S}(\mathbb{R})$. For notational simplicity we identify $|\cdot|_0$ with the norm on L^2_d. Together with the dual space $\mathcal{S}'_d \equiv \mathcal{S}'(\mathbb{R}) \otimes \mathbb{R}^d$ of $\mathcal{S}_d$ we obtain the basic nuclear triple

$$\mathcal{S}_d \subset L^2_d \subset \mathcal{S}'_d \, .$$

On $\mathcal{S}'_d$ the canonical Gaussian measure is introduced by the characteristic function

$$C(\vec{\xi}) = e^{-\frac{1}{2}\langle \vec{\xi}, \vec{\xi} \rangle}, \quad \vec{\xi} \in \mathcal{S}_d.$$

If we introduce the vector valued random variable

$$\vec{B}(t, \vec{\omega}) := \langle \vec{\omega}, 1\!\!1_{[0,t)} \rangle = \left(\langle \omega_j, 1\!\!1_{[0,t)} \rangle, \ j = 1..d \right), \quad \vec{\omega} \in \mathcal{S}'_d$$

a representation of an d–dimensional Brownian motion is obtained. In this setting the above theorem gives the characterization of the space of Hida distributions of the noise of an d–dimensional Brownian motion [SW93].

Example 12 For later use we are interested in the formal expression

$$\Phi = \exp\left(\frac{1}{2}(1 - z^{-2})\langle\omega, \omega\rangle\right) , \qquad z \in \mathbb{C}/\{0\} .$$

Using finite dimensional approximations to calculate its S–transform, we see that the sequence factorizes in a convergent sequence of U-functionals and a divergent pre-factor. So instead of constructing the ill defined expression Φ, we consider its multiplicative renormalization (see [HKPS93] for more details) $\mathrm{J}_z = \Phi/\mathbb{E}(\Phi)$. So the divergent pre-factor cancels in each step of approximation. For J_z we also use the suggestive notation of *normalized exponential*

$$\mathrm{J}_z = \mathrm{Nexp}\left(\frac{1}{2}(1 - z^{-2})\langle\omega, \omega\rangle\right)$$

The resulting S–transform is given by

$$S\mathrm{J}_z(\theta) = \exp\left(-\frac{1}{2}(1 - z^{2})\langle\theta, \theta\rangle\right) , \qquad \theta \in \mathcal{H}_{\mathbb{C}} .$$

The right hand side is obviously a U-functional and thus by characterization $\mathrm{J}_z \in (\mathcal{N})'$.

Let us now choose $z = \lambda \in \mathbb{R}_+$ and consider the T-transform:

$$T\mathrm{J}_\lambda(\theta) = C(\theta) \cdot S\mathrm{J}_\lambda(i\theta) = \exp\left(-\frac{\lambda^2}{2}\langle\theta, \theta\rangle\right) = C_{\lambda^2}(\theta)$$

which is the characteristic function of the Gaussian measure μ_{λ^2} with variance λ^2. This implies

$$\mathrm{J}_\lambda = \frac{\mathrm{d}\mu_{\lambda^2}}{\mathrm{d}\mu}$$

where the right hand side is the generalized Radon Nikodym derivative (see Example 7 for this concept). The fact that $\mathrm{J}_\lambda \notin L^2(\mu)$ for $\lambda \neq 1$ is in agreement with the fact that μ_{λ^2} and μ are singular measures if $\lambda \neq 1$.

Example 13 *(A simple second quantized operator)*

Let $z \in \mathbb{C}$ and $\Phi \in (\mathcal{N})'$. Then $S\Phi$ has an entire analytic extension and we may consider the function

$$\theta \mapsto S\Phi(z\theta) , \qquad \theta \in \mathcal{N}'_{\mathbb{C}} .$$

This function is also an element of $\mathcal{E}^2_{\max}(\mathcal{N}'_{\mathbb{C}})$. Thus we may define $\Gamma_z\Phi$ by

$$S\left(\Gamma_z\Phi\right)(\theta) = S\Phi(z\theta) .$$

Moreover Γ_z is continuous from $(\mathcal{N})'$ into $(\mathcal{N})'$. Γ_z is an extension of $\Gamma(z1\!\!1)$ where Γ is the usual second quantization, see e.g., [Si74].

Example 14 *(Wick product)*

The characterization theorems give simple arguments why the spaces $(\mathcal{N})'$ and $(\mathcal{N})$ are closed under the so called Wick product (already discussed in the non-Gaussian setting). Besides the defining equation

$$S(\Phi \diamond \Psi) = S\Phi \cdot S\Psi \ , \qquad \Phi, \Psi \in (\mathcal{N})^{(\prime)}$$

we only need to mention that $\mathcal{E}^2_{\max}(\mathcal{N}_{\mathbb{C}})$ and $\mathcal{E}^2_{\min}(\mathcal{N}'_{\mathbb{C}})$ are both algebras under pointwise multiplication.

4.1.3 Corollaries

One useful application of Theorem 52 is the discussion of convergence of a sequence of generalized functionals. A first version of this theorem is worked out in [PS91]. Here we use our more general setting to state

Theorem 55 *Let $(F_n,\, n \in \mathbb{N})$ denote a sequence of U–functionals such that*

1. $(F_n(\xi),\, n \in \mathbb{N})$ is a Cauchy sequence for all $\xi \in \mathcal{N}$,

2. there exists a continuous norm $|\cdot|$ on $\mathcal{N}$ and $C,\, K > 0$ such that $|F_n(z\xi)| \le Ce^{K|z|^2|\xi|^2}$ for all $\xi \in \mathcal{N}$, $z \in \mathbb{C}$, and for almost all $n \in \mathbb{N}$. Then $(S^{-1}F_n,\, n \in \mathbb{N})$ converges strongly in $(\mathcal{N})'$.

Proof. The assumptions and inequality (4.7) imply that there exist $p, q \ge 0$ and $\rho \in (0,1)$ such that for all $n \in I\!N$,

$$\|\Phi_n\|_{-p,-q} \le C(1-\rho)^{-\frac{1}{2}}$$

where $\Phi_n = S^{-1}F_n$. Since $\mathcal{E}$ is total in $(\mathcal{H}_{-p})_{-q}$, assumption 1 implies that $(\langle\!\langle \Phi_n, \varphi \rangle\!\rangle, n \in \mathbb{N})$ is a Cauchy sequence for all $\varphi \in (\mathcal{N})$. Since $(\mathcal{N})'$ is the dual of the countable Hilbert space $(\mathcal{N})$, which is in particular Fréchet, it follows from the Banach–Steinhaus theorem that $(\mathcal{N})'$ is weakly sequentially complete. Thus there exists $\Phi \in (\mathcal{N})'$ such that Φ is the weak limit of $(\Phi_n,\, n \in \mathbb{N})$. The proof is concluded by the remark that weak and strong convergence of sequences coincide in the duals of nuclear spaces (e.g., [GV68]). □

As a second application we consider a theorem which concerns the integration of a family of generalized functionals.

Theorem 56 *Let $(\Lambda, \mathcal{A}, \nu)$ be a measure space, and $\lambda \mapsto \Phi_\lambda$ a mapping from Λ to $(\mathcal{N})'$. We assume that the S–transform $F_\lambda = S\Phi_\lambda$ satisfies the following conditions:*

1. for every $\xi \in \mathcal{N}$ the mapping $\lambda \mapsto F_\lambda(\xi)$ is measurable,

2. there exists a continuous norm $|\cdot|$ on $\mathcal{N}$ so that for all $l \in \Lambda$, F_λ satisfies the bound $|F_\lambda(z\xi)| \le C_\lambda e^{K_\lambda |z|^2 |\xi|^2}$, and such that $\lambda \mapsto K_\lambda$ is bounded ν–a.e., and $\lambda \mapsto C_\lambda$ is integrable with respect to ν.

Then there are $q, p \ge 0$ such that $\Phi_{\cdot}$ is Bochner integrable on $(\mathcal{H}_{-p})_{-q}$. Thus in particular,

$$\int_\Lambda \Phi_\lambda \, \mathrm{d}\nu(\lambda) \in (\mathcal{N})' \, ,$$

and

$$S\left(\int_\Lambda \Phi_\lambda \, \mathrm{d}\nu(\lambda)\right)(\xi) = \int_\Lambda S\Phi_\lambda(\xi) \, \mathrm{d}\nu(\lambda), \quad \xi \in \mathcal{N}.$$

Proof. In inequality (4.2) for $F_l(z\xi)$ we can replace K_l by its bound. With this modified estimate and Corollary 12 we can find $p, q \geq 0$ and $\rho \in (0,1)$ such that for all $\lambda \in \Lambda$,

$$\|\Phi_\lambda\|_{-p,-q} \leq C_\lambda (1-\rho)^{-\frac{1}{2}}. \tag{4.9}$$

Since the right hand side of (4.9) is integrable with respect to ν, we only need to show the weak measurability of $\lambda \mapsto \Phi_\lambda$ (see [Yo80]). But this is obvious because $\lambda \mapsto \langle\langle \Phi_\lambda, \varphi \rangle\rangle$ is measurable for all $\varphi \in \mathcal{E}$ which is total in $(\mathcal{H}_p)_q$. □

Example 15 Let us look at Donsker's delta function (see section 4.6 for the definition)

$$\delta(\langle \omega, \eta \rangle - a) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda(\langle \omega, \eta \rangle - a)} \mathrm{d}\lambda \ , \qquad \eta \in \mathcal{N},\ a \in \mathbb{R} \tag{4.10}$$

in the sense of Bochner integration (see [HKPS93] and compare Theorem 91).

Remark.
For later use we have to define pointwise products of a Hida distribution Φ with a Donsker delta function

$$\delta\left(\langle \omega, \eta \rangle - a\right) \ , \qquad \eta \in L^2(\mathbb{R}) \ , \ a \in \mathbb{R} \ .$$

If $T\Phi$ has an extension to $L^2_{\mathbb{C}}(\mathbb{R})$ and the mapping $\lambda \longmapsto T\Phi(\theta + \lambda\eta)$, $\theta \in \mathcal{N}_{\mathbb{C}}$ is integrable on $\mathbb{R}$ the following formula may be used to define the product $\Phi \cdot \delta$

$$T\left(\Phi \cdot \delta(\langle \omega, \eta \rangle - a)\right)(\theta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda a}\ T\Phi\left(\theta + \lambda\eta\right)\ \mathrm{d}\lambda, \tag{4.11}$$

in case the right hand integral is indeed a U-functional.

This definition extends the usual definition of pointwise multiplication where one factor is a test function. This is easily seen by use of (4.10) in the following calculation, $\varphi \in (\mathcal{N})$

$$\begin{aligned} T\left(\varphi \cdot \delta(\langle \omega, \eta \rangle - a)\right)(\theta) &= \left\langle\!\left\langle \delta(\langle \omega, \eta \rangle - a),\ \varphi \cdot e^{i\langle \omega, \theta \rangle} \right\rangle\!\right\rangle \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda a} \left\langle\!\left\langle e^{i\lambda\langle \omega, \eta \rangle},\ \varphi \cdot e^{i\langle \omega, \theta \rangle} \right\rangle\!\right\rangle\ \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda a}\ T\varphi\left(\theta + \lambda\eta\right)\ \mathrm{d}\lambda\ . \end{aligned}$$

4.2 The nuclear triple $(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}$

4.2.1 Construction

Consider the space $\mathcal{P}(\mathcal{N}')$ of continuous polynomials on $\mathcal{N}'$, i.e., any $\varphi \in \mathcal{P}(\mathcal{N}')$ has the form $\varphi(\omega) = \sum_{n=0}^{N} \left\langle \omega^{\otimes n}, \tilde{\varphi}^{(n)} \right\rangle$, $\omega \in \mathcal{N}'$, $N \in \mathbb{N}$ for kernels $\tilde{\varphi}^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$. It is well-known that any $\varphi \in \mathcal{P}(\mathcal{N}')$ can be written as a Wick polynomial i.e., $\varphi(\omega) = \sum_{n=0}^{N} \left\langle : \omega^{\otimes n} :, \varphi^{(n)} \right\rangle$, $\varphi^{(n)} \in \mathcal{N}^{\hat{\otimes} n}$, $N \in \mathbb{N}$ (see e.g., equations (3.7) and (3.8)). To construct test functions we define for $p, q \in \mathbb{N}$, $\beta \in [0,1]$ the following Hilbertian norm on $\mathcal{P}(\mathcal{N}')$

$$\|\varphi\|^2_{p,q,\beta} = \sum_{n=0}^{\infty} (n!)^{(1+\beta)} 2^{nq} \left|\varphi^{(n)}\right|_p^2 \quad , \quad \varphi \in \mathcal{P}(\mathcal{N}')\ . \tag{4.12}$$

Then we define $(\mathcal{H}_p)_q^\beta$ to be the completion of $\mathcal{P}(\mathcal{N}')$ with respect to $\|\cdot\|_{p,q,\beta}$. Or equivalently

$$(\mathcal{H}_p)_q^\beta = \left\{ \varphi \in L^2(\mu) \;\middle|\; \|\varphi\|_{p,q,\beta} < \infty \right\} .$$

Finally, the space of test functions $(\mathcal{N})^\beta$ is defined to be the projective limit of the spaces $(\mathcal{H}_p)_q^\beta$:

$$(\mathcal{N})^\beta = \operatorname*{pr\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_p)_q^\beta .$$

For $0 \le \beta < 1$ the corresponding spaces have been studied in [KoS92] and in the special case of Gaussian product measures all the spaces for $0 \le \beta \le 1$ were introduced in [Ko78]. For $\beta = 0$ and $\mathcal{N} = \mathcal{S}(\mathbb{R})$ the well-known space $(\mathcal{S}) = (\mathcal{S})^0$ of Hida test functions is obtained (e.g., [KoSa78, Ko80a, Ko80b, KT80, HKPS93, BeKo88, KLPSW94]), while in this section we concentrate on the smallest space $(\mathcal{N})^1$.

Let $(\mathcal{H}_{-p})_{-q}^{-1}$ be the dual with respect to $L^2(\mu)$ of $(\mathcal{H}_p)_q^1$ and let $(\mathcal{N})^{-1}$ be the dual with respect to $L^2(\mu)$ of $(\mathcal{N})^1$. We denote by $\langle\!\langle\, . \,,\, . \,\rangle\!\rangle$ the corresponding bilinear dual pairing which is given by the extension of the scalar product on $L^2(\mu)$. We know from general duality theory that

$$(\mathcal{N})^{-1} = \operatorname*{ind\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})_{-q}^{-1} \quad .$$

In particular, we know that every distribution is of finite order i.e., for any $\Phi \in (\mathcal{N})^{-1}$ there exist $p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})_{-q}^{-1}$. The chaos decomposition introduces the following natural decomposition of $\Phi \in (\mathcal{N})^{-1}$. Let $\Phi^{(n)} \in (\mathcal{N}'_{\mathbb{C}})^{\widehat{\otimes} n}$ be given. Then there is a distribution $\left\langle :\omega^{\otimes n}:, \Phi^{(n)} \right\rangle$ in $(\mathcal{N})^{-1}$ acting on $\varphi \in (\mathcal{N})^1$ as

$$\left\langle\!\left\langle \left\langle :\omega^{\otimes n}:, \Phi^{(n)} \right\rangle, \varphi \right\rangle\!\right\rangle = n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle .$$

Any $\Phi \in (\mathcal{N})^{-1}$ then has a unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \left\langle :\omega^{\otimes n}:, \Phi^{(n)} \right\rangle \quad ,$$

where the sum converges in $(\mathcal{N})^{-1}$ and we have

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle \quad , \ \varphi \in (\mathcal{N})^1 \quad .$$

From the definition it is not hard to see that $(\mathcal{H}_{-p})_{-q}^{-1}$ is a Hilbert space with norm

$$\|\Phi\|_{-p,-q,-1}^2 = \sum_{n=0}^{\infty} 2^{-nq} \left|\Phi^{(n)}\right|_{-p}^2 \quad .$$

Remark. Considering also the above mentioned spaces $(\mathcal{N})^\beta$ and their duals $(\mathcal{N})^{-\beta}$ we have the following chain of spaces

$$(\mathcal{N})^1 \subset ... \subset (\mathcal{N})^\beta \subset ... \subset (\mathcal{N}) = (\mathcal{N})^0 \subset L^2(\mu) \subset (\mathcal{N})' \subset ... \subset (\mathcal{N})^{-\beta} \subset ... \subset (\mathcal{N})^{-1} \quad .$$

4.2.2 Description of test functions by infinite dimensional holomorphy

We state a theorem proven in [KLS94] which shows that functions from $(\mathcal{N})^1$ have a pointwise meaning on $\mathcal{N}'$ and are even (real) analytic on this space. Since the space $(\mathcal{N})^1$ is discussion in great detail in the previous chapter, we can refer to Theorem 28. But we will also give an independent proof using different methods.

Corollary 57 *Any test function in $(\mathcal{N})^1$ has a pointwise defined version which has an analytic continuation onto the space $\mathcal{N}'_{\mathbb{C}}$ as an element of $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. Vice versa the restriction of any function in $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$ to $\mathcal{N}'$ is in $(\mathcal{N})^1$.*

In the rest of the paper we identify any $\varphi \in (\mathcal{N})^1$ with its version in $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$. In this sense we may write

$$(\mathcal{N})^1 = \mathcal{E}^1_{\min}(\mathcal{N}') = \left\{ u|_{\mathcal{N}'} \;\middle|\; u \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}) \right\} .$$

We will give an independent and short proof of Corollary 26.

Corollary 58 *For all $\varphi \in (\mathcal{N})^1$ and $q \geq 0$ we have the following pointwise bound*

$$|\varphi(\omega)| \leq C_{p,\varepsilon} \;\; \|\varphi\|_{p,q,1} \, e^{\varepsilon|\omega|_{-p}}, \; \omega \in \mathcal{H}_{-p} \; , \tag{4.13}$$

where $\varepsilon = 2^{-\frac{q}{2}}$ and

$$C_{p,\varepsilon} \;\; = \int_{\mathcal{N}'} e^{\varepsilon|\omega|_{-p}} \; \mathrm{d}\mu(\omega) \; .$$

Here $p > 0$ is taken such that the embedding $i_{p,0} : \mathcal{H}_p \hookrightarrow \mathcal{H}_0$ is of Hilbert-Schmidt type.

Proof. Let us introduce the following function

$$w(z) = \sum_{n=0}^{\infty} (-i)^n \left\langle z^{\otimes n}, \varphi^{(n)} \right\rangle \; , \; z \in \mathcal{N}'_{\mathbb{C}}$$

using the chaos decomposition $\varphi(\omega) = \sum_{n=0}^{\infty} \left\langle : \omega^{\otimes n} :, \varphi^{(n)} \right\rangle$ of φ. Using the inequality

$$\left|\varphi^{(n)}\right|_p \leq \frac{1}{n!} \varepsilon^n \|\varphi\|_{p,q,1} \;\; , \; \varepsilon = 2^{-\frac{q}{2}}$$

we may estimate $|w(z)|$ for $z \in \mathcal{H}_{-p}$ as follows

$$\begin{aligned} |w(z)| &\leq \sum_{n=0}^{\infty} \left|\varphi^{(n)}\right|_p |z|^n_{-p} \\ &\leq \|\varphi\|_{p,q,1} \sum_{n=0}^{\infty} \frac{1}{n!} \varepsilon^n \, |z|^n_{-p} \\ &= \|\varphi\|_{p,q,1} \exp\left(\varepsilon \, |z|_{-p} \right) \; . \end{aligned}$$

To achieve a bound of the type (4.13) we use the relation [KLS94, BeKo88]

$$\varphi(\omega) = \int_{\mathcal{N}'} w(y + i\omega) \; \mathrm{d}\mu(y) \; , \; \omega \in \mathcal{N}' \; .$$

This allows to estimate

$$\begin{aligned} |\varphi(\omega)| &\leq \|\varphi\|_{p,q,1} \int_{\mathcal{N}'} \exp\left(\varepsilon\, |y+i\omega|_{-p}\right) \,\mathrm{d}\mu(y) \\ &\leq \|\varphi\|_{p,q,1}\, e^{\varepsilon|\omega|_{-p}} \int_{\mathcal{N}'} e^{\varepsilon|y|_{-p}} \,\mathrm{d}\mu(y) \end{aligned}$$

We conclude the proof with the inequality

$$C_{p,\varepsilon} = \int_{\mathcal{N}'} e^{\varepsilon|\omega|_{-p}} \,\mathrm{d}\mu(\omega) \leq e^{\frac{\varepsilon^2}{4\alpha}} \int_{\mathcal{N}'} e^{\alpha|\omega|^2_{-p}} \,\mathrm{d}\mu(\omega)$$

for $\alpha > 0$. If $p > 0$ is such that the embedding $i_{p,0}$ is of Hilbert-Schmidt type and α is chosen sufficiently small the right hand integral is finite, see e.g., [Kuo75, Fernique's theorem]. □

4.3 The spaces $\mathcal{G}$ and $\mathcal{M}$

4.3.1 Definitions and examples

For applications it is often useful to have distribution spaces with kernels $\Phi^{(n)}$ but not more singular than $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. To this end Potthoff and Timpel [PT94] introduced a triple

$$\mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' .$$

We will introduce a second triple which is embedded in the above chain

$$\mathcal{G} \subset \mathcal{M} \subset L^2(\mu) \subset \mathcal{M}' \subset \mathcal{G}' . \tag{4.14}$$

We compare the properties of the two triples and we will discuss some interesting interplay between these spaces.

Let use first note that there is no need in the definition of $(\mathcal{H})_q := (\mathcal{H}_0)_q$ to choose $q \in \mathbb{N}$. We will denote the new real parameter replacing q by $\alpha \in \mathbb{R}_+$

Definition 59 *We can define*

$$\mathcal{G} := \operatorname*{pr\,lim}_{\alpha>0}\, (\mathcal{H})_\alpha$$

and

$$\mathcal{M} := \operatorname*{ind\,lim}_{\alpha>0}\, (\mathcal{H})_\alpha$$

Both spaces are Fréchet spaces continuously embedded in $L^2(\mu)$. By general duality theory [Sch71] we have the representations

$$\mathcal{G}' = \operatorname*{ind\,lim}_{\alpha>0}\, (\mathcal{H})_{-\alpha}$$

$$\mathcal{M}' = \operatorname*{pr\,lim}_{\alpha>0}\, (\mathcal{H})_{-\alpha} .$$

Obviously (4.14) holds.

Example 16 Let $\eta \in \mathcal{H}$, $c \in \mathbb{R}$, $2c|\eta|^2 < 1$ and

$$\varphi = e^{c\langle\cdot,\eta\rangle^2} .$$

The S-transform is easy to calculate, $\xi \in \mathcal{N}$

$$S\varphi(\xi) = (1-2c|\eta|^2)^{-1/2} \exp\left(\frac{c}{1-2c|\eta|^2}\langle\xi,\eta\rangle^2\right) ,$$

expanding the exponential we obtain the kernels

$$\varphi^{(2n)} = \frac{1}{n!}\left(\frac{c}{1-2c|\eta|^2}\right)^n \eta^{\otimes 2n} , \quad \varphi^{(n)} = 0 \text{ if } n \text{ is odd.}$$

Then

$$\begin{aligned} \|\varphi\|_{0,\alpha}^2 &= \sum_{n=0}^{\infty} n!\, 2^{\alpha n} |\varphi^{(n)}|^2 \\ &\leq \sum_{n=0}^{\infty} 2^{2\alpha n}\left(\frac{2c|\eta|^2}{1-2c|\eta|^2}\right)^{2n} \end{aligned}$$

which is finite if $2^\alpha \frac{2c|\eta|^2}{1-2c|\eta|^2} < 1$.
From this it follows

$\varphi \notin \mathcal{G}$	if	$c \neq 0$	but	$\varphi \in \mathcal{M}$	if	$4c\|\eta\|^2 < 1$
$\varphi \notin L^2(\mu)$	if	$4c\|\eta\|^2 = 1$	but then	$\varphi \in \mathcal{M}'$		
$\varphi \notin \mathcal{M}'$	if	$4c\|\eta\|^2 > 1$	but	$\varphi \in \mathcal{G}'$	if	$2c\|\eta\|^2 < 1$.

In section 4.6 we will prove that we also can define Donsker's delta $\delta(\langle\cdot,\eta\rangle - a) \in \mathcal{M}'$ for $a \in \mathbb{C}$, $\eta \in \mathcal{H}_{\mathbb{C}}$, $\arg\langle\eta,\eta\rangle \neq \pi$. This was in fact one of the main motivations to introduce $\mathcal{M}'$. We wanted to study pointwise multiplication of δ with other functions.

The next proposition will produce whole classes of examples.

Proposition 60
1) Let $\varphi \in L^p(\mu)$ for some $p > 1$ then $\varphi \in \mathcal{G}'$, i.e.,

$$\bigcup_{p>1} L^p(\mu) \subset \mathcal{G}' .$$

2) Let $\varphi \in L^p(\mu)$ for all $1 < p < 2$ then $\varphi \in \mathcal{M}'$, i.e.,

$$\bigcap_{p<2} L^p(\mu) \subset \mathcal{M}' .$$

Proof. The argument is based on Nelson's Hypercontractivity Theorem [Ne73]. In particular we obtain $2^{-\alpha N/2} : L^p(\mu) \to L^2(\mu)$ (here N denotes the well known number operator) is a contraction if $2^{-\alpha} \leq p-1$. Otherwise $2^{-\alpha N/2}$ is unbounded. Hence

$$\|\varphi\|_{0,-\alpha} = \left\|2^{-\alpha N/2}\varphi\right\|_{L^2(\mu)} \leq \|\varphi\|_{L^p(\mu)} \qquad \text{if } 2^{-\alpha} \leq p-1$$

If $\varphi \in L^p(\mu)$ for some $p > 1$, there exists a $\alpha > 0$ such that the above inequality holds. Now we prove the second assertion. For any $\alpha > 0$ we may choose $p \in (1, 2)$ such that the above estimate holds, hence $\varphi \in \mathcal{M}'$. □

Notes.
1. The first assertion is already proved in [PT94].
2. From Example 16 we know that for all $p \in (1, 2)$, there exists $\varphi \in \mathcal{M}'$ such that $\varphi \notin L^p(\mu)$, i.e., $\forall p \in (1, 2)$

$$L^p(\mu) \not\subset \mathcal{M}' \ .$$

Moreover $L^1(\mu) \not\subset \mathcal{G}'$.
3. Let us also mention the trivial consequence that any $L^p(\mu)$-function, $p > 1$, has a chaos expansion with all kernels contained in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$.

Now we state the 'dual result' which may be proved along the same lines.

Proposition 61
1.

$$\varphi \in \mathcal{G} \Rightarrow \varphi \in \bigcap_{p>1} L^p(\mu) \ .$$

2.

$$\varphi \in \mathcal{M} \Rightarrow \exists p > 2 : \varphi \in L^p(\mu) \ .$$

Note. Assertion 1 is related to the inclusion $\mathcal{G} \subset \mathcal{D}$ where $\mathcal{D}$ is the so called Meyer-Watanabe space, see [HKPS93] for a definition and [PT94] for a proof of the inclusion. We can again refer to Example 16. For all $p > 2$ there exists $\varphi \in \mathcal{M}$ such that $\varphi \notin L^p(\mu)$, i.e., $\forall p > 2$

$$\mathcal{M} \not\subset L^p(\mu) \ ,$$

in particular

$$\mathcal{M} \not\subset \mathcal{D} \ .$$

4.3.2 The pointwise product

It is well known that $(\mathcal{N})$ and $(\mathcal{N})^1$ are algebras under pointwise multiplication (sometimes called Wiener Product). In [PT94] it has been shown that also $\mathcal{G}$ has this property. On the other hand it is obvious that $\mathcal{M}$ can not be an algebra. To see this consider $\varphi = e^{\langle \cdot, \eta \rangle^2}$, $|\eta|^2 = 1/8$ then $\varphi \in \mathcal{M}$ but $\varphi^2 = e^{2\langle \cdot, \eta \rangle^2} \notin L^2(\mu)$, see Example 16 on page 60. We will show that the pointwise product can also be defined if one factor is in $\mathcal{M}$ and the other in $\mathcal{G}$. To prove this we found it useful to have a detailed discussion of pointwise products in [Ob94] which we were able to modify to the present setting.

The first question is, how does the pointwise product look like in terms of chaos expansion?

Lemma 62 *Let $\varphi, \psi \in \mathcal{G}$ be given by*

$$\varphi = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, \varphi^{(n)} \rangle \ , \quad \psi = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, \psi^{(n)} \rangle \ .$$

Then the chaos expansion of

$$\varphi\psi = \sum_{l=0}^{\infty} \langle :\omega^{\otimes l}:, f^{(l)} \rangle$$

is given by

$$f^{(l)} = \sum_{m+n=l} \sum_{k=0}^{\infty} k! \binom{m+k}{k} \binom{n+k}{k} \varphi^{(m+k)} \hat{\underset{k}{\otimes}} \psi^{(n+k)} \tag{4.15}$$

where the contraction $\varphi^{(m+k)} \hat{\underset{k}{\otimes}} \psi^{(n+k)}$ *of the kernels* $\varphi^{(m+k)}$ *and* $\psi^{(n+k)}$ *is the symmetrization of the partial scalar product* $\left(\varphi^{(m+k)}, \psi^{(n+k)}\right)_{\mathcal{H}^{\otimes k}} \in \mathcal{H}_{\mathbb{C}}^{\otimes(m+n)}$.

Note that

$$\left| \varphi^{(m+k)} \hat{\underset{k}{\otimes}} \psi^{(n+k)} \right| \leq \left|\varphi^{(m+k)}\right| \left|\psi^{(n+k)}\right|$$

Now we are giving a variant of Lemma 3.5.4 in [Ob94]. The only qualitative change is that we do not need smoother kernels in the estimate.

Lemma 63 *Let* $\alpha, \beta \geq 0$*, then* $f^{(l)}$ *defined by (4.15) can be estimated*

$$l! \left|f^{(l)}\right|^2 \leq (l+1)(2^{-\alpha} + 2^{-\beta})^l \; \|\varphi\|_{0,\beta}^2 \; \|\psi\|_{0,\beta}^2 \sum_{k=0}^{\infty} \binom{l+2k}{2k} 2^{-k(\alpha+\beta)} \; .$$

Following the lines of the proof in [Ob94] a little further we obtain.

Theorem 64 *Let* $\alpha, \beta, \gamma \geq 0$ *satisfy* $2^{-(\alpha+\beta)/2} + 2^{\gamma-\alpha} + 2^{\gamma-\beta} < 1$ *then*

$$\|\varphi\psi\|_{0,\gamma} \leq \frac{\sqrt{1 - 2^{-(\alpha+\beta)/2}}}{1 - 2^{-(\alpha+\beta)/2} - 2^{\gamma-\alpha} - 2^{\gamma-\beta}} \|\varphi\|_{0,\alpha} \, \|\psi\|_{0,\beta} \; .$$

Corollary 65 $\mathcal{G}$ *is closed under pointwise multiplication and multiplication is a separately continuous bilinear map from* $\mathcal{G} \times \mathcal{G}$ *into* G*.*

This result is also shown in [PT94].

Corollary 66 *The pointwise multiplication is a separately continuous bilinear map from* $\mathcal{G} \times \mathcal{M}$ *into* $\mathcal{M}$*.*

Proof. First fix the factor $\varphi \in \mathcal{G}$. To prove that $\psi \mapsto \varphi \cdot \psi$ from $\mathcal{M}$ into itself is continuous we have to show that for all $\beta > 0$ there exists a $\gamma > 0$ such that $\psi \mapsto \varphi \cdot \psi$ is continuous from $(\mathcal{H})_\beta$ into $(\mathcal{H})_\gamma$. But this follows from the above theorem, since we may choose $\gamma < \beta$ and α large enough.
For fixed $\psi \in \mathcal{M}$ we have to show that there exist $\alpha > 0$ and $\gamma > 0$ such that $\varphi \mapsto \varphi \cdot \psi$ is continuous from $(\mathcal{H})_\alpha$ into $(\mathcal{H})_\gamma$. Also this is clear from the above theorem. □

Now we can extend the concept of pointwise multiplication to products where one factor is a distribution.

Let $\Phi \in \mathcal{M}'$, $\varphi \in \mathcal{G}$ then $\Phi \cdot \varphi \in \mathcal{M}'$ defined by

$$\langle\!\langle \Phi \cdot \varphi, \psi \rangle\!\rangle := \langle\!\langle \Phi, \varphi\psi \rangle\!\rangle \, , \qquad \psi \in \mathcal{M}$$

is well defined because of the previous corollary. More useful is the following:

Let $\Phi \in \mathcal{M}'$, $\psi \in \mathcal{M}$ then $\Phi \cdot \psi \in \mathcal{G}'$ given by

$$\langle\!\langle \Phi \cdot \psi, \varphi \rangle\!\rangle := \langle\!\langle \Phi, \varphi\psi \rangle\!\rangle, \quad \varphi \in \mathcal{G} \tag{4.16}$$

is well defined.

4.3.3 Integrating out Donsker's delta

Let $a \in \mathbb{C}$, $\eta \in \mathcal{N}$ such that $|\eta| = 1$ and $\varphi \in (\mathcal{N})$ than a simple calculation yields

$$\langle\!\langle \delta(\langle \cdot, \eta \rangle - a), \varphi \rangle\!\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} \mathbb{E}\left(\varphi(\cdot + (a - \langle \cdot, \eta \rangle)\eta)\right). \tag{4.17}$$

Since φ has a pointwise well defined version which has some analytic continuation, we understand what $\varphi(\omega + a\eta)$ for $a \in \mathbb{C}$ means. So everything is well defined. In this section we want to extend the above formula to $\varphi \in \mathcal{M}$ and $\eta \in \mathcal{H}$. This raises at least the following questions

1. What does $\varphi\left(\omega + (a - \langle \omega, \eta \rangle)\eta\right)$ mean? In particular in what sense do we have an analytic continuation?
2. Is the expectation value at the end of the procedure well defined?

4.3.3.1 Analyticity of shifts

In this section we want to define an operator

$$\tau_\eta : \mathcal{M} \to \mathcal{M}, \;\; \varphi \mapsto \varphi(\cdot + \eta) \;\; \text{for } \eta \in \mathcal{H}_{\mathbb{C}}.$$

(Note that this operation surely has no sense pointwisely. Consider e.g., $\varphi = \langle : \omega^{\otimes 2} :, \varphi^{(2)} \rangle$ then $\tau_\eta \varphi(0) = \langle : \eta^{\otimes 2} :, \varphi^{(2)} \rangle = \langle \eta^{\otimes 2}, \varphi^{(2)} \rangle - \langle \mathrm{Tr}, \varphi^{(2)} \rangle$ which is ill defined if $\varphi^{(2)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} 2}$ allows no trace.)

To give a meaningful definition we use

$$: (\omega + \eta)^{\otimes n} : = \sum_{k=0}^{n} \binom{n}{k} : \omega^{\otimes (n-k)} : \otimes \eta^{\otimes k}$$

and define $\forall \eta \in \mathcal{H}_{\mathbb{C}}$

$$\tau_\eta \varphi = \varphi(\cdot + \eta) := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l}{k} \langle : \omega^{\otimes l} :, \left(\eta^{\otimes k}, \varphi^{(k+l)}\right)_{\mathcal{H}^{\otimes k}} \rangle$$

whenever the series converges.

Theorem 67
1. Let $\varphi \in \mathcal{M}$, $\eta \in \mathcal{H}_{\mathbb{C}}$ then $\tau_\eta \varphi \in \mathcal{M}$. Moreover the mapping $\eta \mapsto \tau_\eta \varphi \in \mathcal{E}^2_{\max}(\mathcal{H}_{\mathbb{C}}, \mathcal{M})$.
2. Let $\varphi \in \mathcal{G}$, $\eta \in \mathcal{H}_{\mathbb{C}}$ then $\tau_\eta \varphi \in \mathcal{G}$ and moreover the mapping $\eta \mapsto \tau_\eta \varphi \in \mathcal{E}^2_{\min}(\mathcal{H}_{\mathbb{C}}, \mathcal{G})$.

Proof. Define the k-homogeneous polynomials on $\mathcal{H}_{\mathbb{C}}$

$$\frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)} := \sum_{l=0}^{\infty} \binom{k+l}{k} \langle :\omega^{\otimes l}:, \left(\eta^{\otimes k}, \varphi^{(k+l)}\right)_{\mathcal{H}^{\otimes k}}\rangle$$

with values in $\mathcal{M}$ (or $\mathcal{G}$ respectively). To show that this is well defined let $\varphi \in (\mathcal{H})_\alpha$, $\alpha > 0$. Such that $\left|\varphi^{(k)}\right|^2 \leq \|\varphi\|_{0,\alpha}^2 \frac{1}{k!} 2^{-\alpha k}$ and choose $\gamma \in (0, \alpha)$ to estimate

$$\begin{aligned}
\left\| \frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)} \right\|_{0,\gamma}^2 &\leq \sum_{l=0}^{\infty} l!\, 2^{\gamma l} \binom{k+l}{k}^2 |\eta|^{2k} |\varphi^{(k+l)}| \\
&\leq \|\varphi\|_{0,\alpha}^2 |\eta|^{2k} \sum_{l=0}^{\infty} \binom{k+l}{k}^2 \frac{l!}{(k+l)!} 2^{l\gamma} 2^{-\alpha(k+l)} \\
&\leq \|\varphi\|_{0,\alpha}^2 \frac{1}{k!} 2^{-\alpha k} |\eta|^{2k} \sum_{l=0}^{\infty} \binom{k+l}{k} 2^{l(\gamma-\alpha)} \\
&= \|\varphi\|_{0,\alpha}^2 \frac{1}{k!} 2^{-\alpha k} |\eta|^{2k} (1 - 2^{\gamma-\alpha})^{-(k+1)} \\
&= (1 - 2^{\gamma-\alpha})^{-1} \|\varphi\|_{0,\alpha}^2 \frac{1}{k!} |\eta|^{2k} (2^\alpha - 2^\gamma)^{-k}.
\end{aligned}$$

This shows that $\frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)}$ is in fact a k-homogeneous continuous polynomial. If $\varphi \in \mathcal{G}$ and if $\varphi \in \mathcal{M}$ then $\frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)} \in \mathcal{M}$.

The next step is to show that $\sum_{k=0}^{\infty} \frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)}$ converges uniformly on any ball in $\mathcal{H}_{\mathbb{C}}$ in the topology of $\mathcal{M}$ (or $\mathcal{G}$ respectively). So we estimate

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\| \frac{1}{k!}\widehat{\mathrm{d}^k\varphi(\eta)} \right\|_{0,\gamma} &\leq \|\varphi\|_{0,\alpha} (1 - 2^{\gamma-\alpha})^{-1/2} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} (2^\alpha - 2^\gamma)^{-k/2} |\eta|^k \\
&\leq \|\varphi\|_{0,\alpha} (1 - 2^{\gamma-\alpha})^{-1/2} \left(\sum_{k=0}^{\infty} \frac{1}{k!} 2^{k(\alpha-\gamma)} (2^\alpha - 2^\gamma)^{-k} |\eta|^{2k} \right)^{1/2} \left(\sum_{k=0}^{\infty} 2^{k(\gamma-\alpha)} \right)^{1/2} \\
&= \|\varphi\|_{0,\alpha} (1 - 2^{\gamma-\alpha})^{-1} \exp\left(\frac{2^{\alpha-\gamma}}{2(2^\alpha - 2^\gamma)} |\eta|^2 \right).
\end{aligned}$$

showing the uniform and absolute convergence of the series.
If $\varphi \in \mathcal{M}$ we have shown that $\eta \mapsto \tau_\eta\varphi \in \mathcal{E}^2_{\max}(\mathcal{H}_{\mathbb{C}}, \mathcal{M})$.
If $\varphi \in \mathcal{G}$ choose e.g., $\alpha = 2\gamma$, then the type of growth is bounded by $(2^{\gamma+1} - 2)^{-1}$ which converges to zero for growing γ. Thus $\eta \mapsto \tau_\eta\varphi \in \mathcal{E}^2_{\min}(\mathcal{H}_{\mathbb{C}}, \mathcal{G})$. □

The second term in the Taylor series coincides with the Gâteaux derivative. So we have the following corollary.

Corollary 68 *Let $\eta \in \mathcal{H}_{\mathbb{C}}$ then the Gâteaux derivative defined by*

$$D_\eta\varphi := \sum_{l=0}^{\infty} l\, \langle :\omega^{\otimes l}:, \left(\eta, \varphi^{(l+1)}\right)_{\mathcal{H}}\rangle, \qquad \varphi \in \mathcal{M}$$

is a well defined operator from $\mathcal{M}$ into itself and from $\mathcal{G}$ into itself.

4.3.3.2 Composition with projection operators

Let $\eta \in \mathcal{N}$ with $|\eta| = 1$. In view of the aim of this section we also want to understand how to define $\varphi(\omega - \langle \omega, \eta \rangle \eta)$ (if $\varphi \notin (\mathcal{N})$ this is non trivial). Note that

$$P_\perp : \mathcal{N}' \to \mathcal{N}' , \quad \omega \mapsto P_\perp \omega = \omega - \langle \omega, \eta \rangle \eta$$

is the projection on the orthogonal complement of the subspace spanned by $\eta \in \mathcal{H}_\mathbb{C}$. For $\varphi \in \mathcal{G}$ (also for $\varphi \in \mathcal{M}$) we want to define $P\varphi = \varphi \circ P_\perp$.

First we have to understand what happens in terms of chaos expansion.

Lemma 69 *For $\eta \in \mathcal{N}$, $|\eta| = 1$ and $\omega \in \mathcal{N}'$ we have $: (P_\perp \omega)^{\otimes n} : \in (\mathcal{N}^{\hat{\otimes} n})'$ and the following relation holds*

$$: (P_\perp \omega)^{\otimes n} := \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!\,(n-2k)!} (-\frac{1}{2})^k \left(: \omega^{\otimes (n-2k)} : \circ P_\perp^{\otimes (n-2k)} \right) \hat{\otimes} \eta^{\otimes 2k}$$

where $: \omega^{\otimes l} : \circ P_\perp^{\otimes l} \in (\mathcal{N}^{\hat{\otimes} l})'$ is defined by

$$\langle : \omega^{\otimes l} : \circ P_\perp^{\otimes l}, \varphi^{(l)} \rangle = \langle : \omega^{\otimes l} :, P_\perp^{\otimes l} \varphi^{(l)} \rangle \,.$$

Proof. Choose $\varphi =: \exp\langle \cdot, \xi \rangle :$ then

$$\begin{aligned} P\varphi &= \exp\left(\langle P_\perp \omega, \xi \rangle - \frac{1}{2} |\xi|^2 \right) \\ &= \; : \exp\langle \omega, P_\perp \xi \rangle : \; \exp\left(-\frac{1}{2} (|\xi|^2 - |P_\perp \xi|^2) \right) \\ &= \; : \exp\langle \omega, P_\perp \xi \rangle : \; \exp\left(-\frac{1}{2} \langle \eta, \xi \rangle^2 \right) \end{aligned}$$

Expanding both sides of this equation we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle : (P_\perp \omega)^{\otimes n} :, \xi^{\otimes n} \rangle = \sum_{l=0}^{\infty} \frac{1}{l!} \langle : \omega^{\otimes l} :, (P_\perp \xi)^{\otimes l} \rangle \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{1}{2})^k \langle \eta^{\otimes 2k}, \xi^{\otimes 2k} \rangle$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{k!\,(n-2k)!} (-\frac{1}{2})^k \left\langle \left(: \omega^{\otimes (n-2k)} : \circ P_\perp^{\otimes (n-2k)} \right) \otimes \eta^{\otimes 2k}, \xi^{\otimes n} \right\rangle \,.$$

A comparison of coefficients proves the lemma. □

An immediate consequence is the following lemma

Lemma 70 *Let $\eta \in \mathcal{N}$, $|\eta| = 1$ and $\varphi = \sum_{n=0}^{N} \langle : \omega^{\otimes n} :, \varphi^{(n)} \rangle \in \mathcal{G}$ be a finite linear combination. Then*

$$P\varphi = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} :, \tilde{\varphi}^{(n)} \rangle \tag{4.18}$$

where

$$\tilde{\varphi}^{(n)} = \sum_{k=0}^{\infty} \frac{(n+2k)!}{k!\,n!} \left(-\frac{1}{2} \right)^k P_\perp^{\otimes n} \left(\eta^{\otimes 2k}, \varphi^{(n+2k)} \right)_{\mathcal{H}^{\otimes 2k}} \,. \tag{4.19}$$

(We have no problems of convergence since all sums are in fact finite.)

Proof. Follows from

$$\begin{aligned}
P\varphi &= \sum_{n=0}^{\infty}\langle :(P_\perp\omega)^{\otimes n}:,\varphi^{(n)}\rangle \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!\,(n-2k)!}(-\frac{1}{2})^k\left\langle\left(:\omega^{\otimes(n-2k)}:\circ P_\perp^{\otimes(n-2k)}\right)\otimes\eta^{\otimes 2k},\ \varphi^{(n)}\right\rangle \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(l+2k)!}{k!\,l!}\left(-\frac{1}{2}\right)^k\left\langle :\omega^{\otimes l}:,\ P_\perp^{\otimes l}\left(\eta^{\otimes 2k},\varphi^{(l+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle .
\end{aligned}$$

□

Now we observe that the kernels $\tilde{\varphi}^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ in (4.19) are well defined if we extend from $\eta\in\mathcal{N}$ to $\eta\in\mathcal{H}$. Also the definition of P is not restricted to finite linear combinations as the following theorem shows.

Theorem 71 *The linear mapping P has the following well defined extensions; for $\eta\in\mathcal{H}$, $|\eta|=1$*

$$P:\mathcal{G}\to\mathcal{G}$$

$$P:\mathcal{M}\to\mathcal{G}'$$

$$P:(\mathcal{H})_\alpha\to\mathcal{M}\,,\qquad \alpha>1,$$

more precisely

$$P:(\mathcal{H})_\alpha\to(\mathcal{H})_\gamma\qquad \text{if}\quad 2^\alpha-1>2^\gamma$$

Proof. We discuss the convergence of (4.18) with (4.19). First note

$$\left|P_\perp^{\otimes l}\left(\eta^{\otimes 2k},\varphi^{(l+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right|\le|\eta|^{2k}\left|\varphi^{(l+2k)}\right|\le\left|\varphi^{(l+2k)}\right|$$

since $|\eta|=1$. If we assume $\varphi\in(\mathcal{H})_\alpha$ then $n!\left|\varphi^{(n)}\right|^2\le 2^{-\alpha n}\|\varphi\|_{0,\alpha}^2$. For $\gamma<\alpha$ we have

$$\begin{aligned}
&\left\|\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!\,n!}\left(-\frac{1}{2}\right)^k\left\langle :\omega^{\otimes n}:,P_\perp^{\otimes n}\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle\right\|_{0,\gamma}^2 \\
&\quad= \sum_{n=0}^{\infty}n!\left(\frac{(n+2k)!}{k!\,n!}\right)^2 2^{n\gamma}\left(\frac{1}{2}\right)^{2k}\left|\varphi^{(n+2k)}\right|^2 \\
&\quad= \sum_{n=0}^{\infty}\binom{n+2k}{2k}\frac{(2k)!}{(k!)^2 2^{2k}}(n+2k)!\left|\varphi^{(n+2k)}\right|^2 2^{n\gamma} \\
&\quad\le \|\varphi\|_{0,\alpha}^2\,2^{-2k\alpha}\sum_{n=0}^{\infty}\binom{n+2k}{2k}2^{n(\gamma-\alpha)} \\
&\quad= \|\varphi\|_{0,\alpha}^2\,2^{-2k\alpha}(1-2^{\gamma-\alpha})^{-(2k+1)} \\
&\quad= (1-2^{\gamma-\alpha})^{-1}\|\varphi\|_{0,\alpha}^2\,(2^\alpha-2^\gamma)^{-2k}
\end{aligned}$$

Hence

$$\|P\varphi\|_{0,\gamma}\le\sum_{k=0}^{\infty}\left\|\sum_{n=0}^{\infty}\frac{(n+2k)!}{k!\,n!}\left(-\frac{1}{2}\right)^k\left\langle :\omega^{\otimes n}:,P_\perp^{\otimes n}\left(\eta^{\otimes 2k},\varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}}\right\rangle\right\|_{0,\gamma}$$

$$\leq (1-2^{\gamma-\alpha})^{-1/2} \, \|\varphi\|_{0,\alpha} \sum_{k=0}^{\infty} (2^\alpha - 2^\gamma)^{-k}$$

which is convergent if $2^\alpha - 2^\gamma > 1$.
If $\varphi \in \mathcal{G}$ then for every $\gamma > 0$ there exists an $\alpha > 0$ (e.g., $\alpha = \gamma + 1$) such that $2^\alpha - 2^\gamma > 1$. Hence $P\varphi \in \mathcal{G}$.

If $\varphi \in \mathcal{M}$ there exists $\alpha > 0$ such that $\|\varphi\|_{0,\alpha} < \infty$. Then we choose $\gamma \in \mathbb{R}$ such that $2^\alpha - 1 > 2^\gamma$ (γ possibly negative), to obtain $\|P\varphi\|_{0,\gamma} < \infty$ i.e., $P\varphi \in \mathcal{G}'$.
If $\varphi \in (\mathcal{H})_\alpha$, $\alpha > 1$ then we can find $\gamma > 0$ such that $2^\alpha - 1 > 2^\gamma$, i.e., $\varphi \in \mathcal{M}$. □

Now we are going back to our motivating example.
Let $\eta \in \mathcal{N}$, $|\eta| = 1$ and $\varphi \in (\mathcal{N})$. Starting from expression (4.17) we calculate

$$\varphi(\omega + (a - \langle \omega, \eta \rangle)\eta) = \varphi(P_\perp \omega + a\eta) = P\varphi(\omega + a\eta) = P\tau_{a\eta}\varphi(\omega) \, .$$

The last expression can be extended to $\eta \in \mathcal{H}$, $|\eta| = 1$, $\varphi \in \mathcal{M}$ in view of Propositions 67 and 71, since

$$\varphi \in \mathcal{M} \Rightarrow \tau_{a\eta}\varphi \in \mathcal{M} \Rightarrow P\tau_{a\eta}\varphi \in \mathcal{G}'.$$

Hence we can take expectation $\mathbb{E}(P\tau_{a\eta}\varphi)$ without problems. So now we can formulate equation (4.17) as a proposition

Proposition 72 *Let $\eta \in \mathcal{H}$, $|\eta| = 1$, $a \in \mathbb{C}$ and $\varphi \in \mathcal{M}$. Then*

$$\langle\!\langle \delta(\langle \cdot, \eta \rangle - a), \varphi \rangle\!\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a^2} \mathbb{E}(P\tau_{a\eta}\varphi) \, .$$

Proof. It is easy to show that both sides coincide if we choose $\varphi =: \exp\langle \cdot, \xi \rangle :$, $\xi \in \mathcal{H}$. Then a continuity argument shows that they agree for all $\varphi \in \mathcal{M}$. □

4.4 The Meyer–Yan triple

If the S-(or T-)transform of a distribution is well defined as an entire function, but of infinite order of growth, then the distributions spaces $(\mathcal{N})^{-\beta}$ for $0 \leq \beta < 1$ are too small. Obviously we can use $(\mathcal{N})^{-1}$ but then we will only require that the S-transform is analytic in a neighborhood of zero. Meyer and Yan [MY90] introduced a triple which helps to close this gap. In [KoS93] this triple was discussed in great detail. So we will only introduce the notation and quote some useful results from the second work. The only new results are stated in the two corollaries.

Definition 73 *The Meyer–Yan space is defined by*

$$\mathcal{Y} := \underset{p \in \mathbb{N}}{\text{pr lim}} \; \underset{q \in \mathbb{N}}{\text{ind lim}} \; (\mathcal{H}_p)^1_{-q} \, .$$

The dual space can be represented as

$$\mathcal{Y}' = \underset{p \in \mathbb{N}}{\text{ind lim}} \; \underset{q \in \mathbb{N}}{\text{pr lim}} \; (\mathcal{H}_{-p})^{-1}_{+q} \, .$$

Then we have

$$(\mathcal{N})^1 \subset \mathcal{Y} \subset (\mathcal{N})^\beta \subset (\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})' \subset (\mathcal{N})^{-\beta} \subset \mathcal{Y}' \subset (\mathcal{N})^{-1}$$

for $\beta \in [0,1)$.

We will only need a characterization of distributions.

Theorem 74 *A mapping $F : \mathcal{N}_\mathbb{C} \to \mathbb{C}$ is the S-transform of a distribution $\Phi \in \mathcal{Y}'$ if and only if*
1. F is entire on $\mathcal{N}_\mathbb{C}$.
2. There exists $p \in \mathbb{N}_0$ such that for any $R > 0$ exists $C > 0$:

$$|F(\theta)| \leq C\,, \quad |\theta|_p \leq R\,, \qquad \theta \in \mathcal{N}_\mathbb{C}\,. \tag{4.20}$$

In particular if inequality (4.20) holds and $p', q \in \mathbb{N}_0$ are such that $\|i_{p',q}\|_{HS} < \infty$ and $2^q < (\|i_{p',q}\|_{HS}\, R/e)^2$ then

$$\|\Phi\|_{-p',q,-1} \leq \sqrt{2}C\,.$$

Of course the same is true for the T-transform.

As consequences of the above theorem we discuss the convergence of a sequence of distributions as well as an integration theorem.

Corollary 75 *Let $(F_n\,,\, n \in \mathbb{N})$ denote a sequence of entire functions $F_n : \mathcal{N}_\mathbb{C} \to \mathbb{C}$ such that*
1. $(F_n\,(\theta)\;,\, n \in \mathbb{N})$ is a Cauchy sequence for all $\theta \in \mathcal{N}_\mathbb{C}$.
2. There exists $p \in \mathbb{N}_0$ such that $\forall R > 0\ \exists C > 0$:

$$|F_n(\theta)| \leq C\,, \quad |\theta|_p \leq R\,,\, \theta \in \mathcal{N}_\mathbb{C}$$

uniformly in $n \in \mathbb{N}$.
Then the sequence $(\Phi_n = S^{-1}F_n\,,\, n \in \mathbb{N})$ converges weakly to a distribution $\Phi \in \mathcal{Y}'$, i.e.,

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \lim_{n\to\infty} \langle\!\langle \Phi_n, \varphi \rangle\!\rangle\,, \qquad \varphi \in \mathcal{Y}\,. \tag{4.21}$$

Proof. Let $\Phi_n := S^{-1}F_n \in \mathcal{Y}'$. Since $\mathcal{E}$ is total in $\mathcal{Y}$ assumption 1 implies that $(\langle\!\langle \Phi_n, \varphi \rangle\!\rangle\;,\, n \in \mathbb{N})$ is a Cauchy sequence for all $\varphi \in \mathcal{Y}$. Theorem 74 implies that there exists $p' \in \mathbb{N}$ such that $\forall q \in \mathbb{N}\ \exists C > 0$:

$$\|\Phi_n\|_{-p',q,-1} \leq \sqrt{2}C\,.$$

Thus

$$\left|\lim_{n\to\infty} \langle\!\langle \Phi_n, \varphi \rangle\!\rangle\right| \leq \sqrt{2}C\, \|\varphi\|_{p',-q,+1}$$

which proves the continuity of the linear functional Φ defined by (4.21). □

Now we are going to prove the analog of Theorem 56. Since the representation of $\mathcal{Y}'$ involves a projective limit, it is more convenient to use Pettis integration instead of Bochner integration.

Corollary 76 *Let $(\Lambda, \mathcal{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ a mapping from Λ to $\mathcal{Y}'$. We assume that the S-transform $F_\lambda = S\Phi_\lambda$ satisfies the following conditions:*
1. for every $\theta \in \mathcal{N}_{\mathbb{C}}$ the mapping $\lambda \mapsto F_\lambda(\theta)$ is measurable,
2. there exists $p \in \mathbb{N}$ such that $\forall R > 0 \; \exists C \in L^1(\nu)$:

$$|F_\lambda(\theta)| \leq C_\lambda \,, \quad |\theta|_p \leq R \,, \, \theta \in \mathcal{N}_{\mathbb{C}}$$

for almost all $\lambda \in \Lambda$.
Then $\lambda \mapsto \Phi_\lambda$ is Pettis integrable i.e., $\exists \Phi \in \mathcal{Y}'$:

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \int_\Lambda \langle\!\langle \Phi_\lambda, \varphi \rangle\!\rangle \, \mathrm{d}\nu(\lambda) \,, \;\; \varphi \in \mathcal{Y} \,.$$

Proof. Let $\varphi \in \mathcal{E}$, then assumption 1 implies that $\lambda \mapsto \langle\!\langle \Phi_\lambda, \varphi \rangle\!\rangle$ is measurable. From Theorem 74 and assumption 2 we know that there exists $p' \in \mathbb{N}$ such that $\forall q \in \mathbb{N} \; \exists C \in L^1(\nu) : \|\Phi_\lambda\|_{-p',q,-1} \leq \sqrt{2} C_\lambda$. Thus

$$\int_\Lambda |\langle\!\langle \Phi_\lambda, \varphi \rangle\!\rangle| \, \mathrm{d}\nu(\lambda) \, \leq \sqrt{2} \, \|\varphi\|_{p',-q,1} \int_\Lambda C_\lambda \, \mathrm{d}\nu(\lambda) \, , \, \varphi \in \mathcal{E} \, .$$

This implies that Φ defined by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \int_\Lambda \langle\!\langle \Phi_\lambda, \varphi \rangle\!\rangle \, \mathrm{d}\nu(\lambda)$$

is well defined for all $\varphi \in \mathcal{E}$. The definition of Φ may now be extended by continuity to $\varphi \in \mathcal{Y}$. □

4.5 The scaling operator

In this section we collect some facts about the so called "scaling operator", which has some interesting applications in the theory of Feynman integrals. We first define this operator on a small domain, collect some properties and afterwards extend the domain to include more interesting examples. For the definition we follow [HKPS93].

Let $\varphi \in (\mathcal{N})$ be given. Without loss of generality we assume that φ coincides with its pointwisely defined, continuous version. Let $z \in \mathbb{C}$ be given and define

$$(\sigma_z \varphi)(\omega) := \varphi(z\omega) \qquad .$$

Theorem 77 *For all $z \in \mathbb{C}$ the mapping $\sigma_z : \varphi \mapsto \sigma_z \varphi$ is continuous from $(\mathcal{N})$ into itself.*

We will give a proof later (which is related to the one in [HKPS93]). Let $\varphi \in (\mathcal{N})$ be given by its chaos expansion $\varphi = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, \varphi^{(n)} \rangle$. It is easy to calculate the expansion

$$\sigma_z \varphi = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, \tilde{\varphi}^{(n)} \rangle \; ,$$

$$\tilde{\varphi}^{(n)} = z^n \sum_{k=0}^\infty \frac{(n+2k)!}{k! \; n!} \left(\frac{z^2 - 1}{2} \right)^k \mathrm{tr}^k \varphi^{(n+2k)} \tag{4.22}$$

where $\mathrm{tr}^k\varphi^{(n+2k)}$ is shorthand for the contraction (def. eq. (3.16)) of iterated traces (def. eq. (3.14)) with $\varphi^{(n+2k)}$:

$$\mathrm{tr}^k\varphi^{(n+2k)} := \left(\mathrm{Tr}^{\otimes k}, \varphi^{(n+2k)}\right)_{\mathcal{H}^{\otimes 2k}} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n} \ .$$

Lemma 78 *Tr$\in \mathcal{H}_{-p}^{\hat{\otimes}2}$ if and only if $i_{p,0} : \mathcal{H}_p \ \to \mathcal{H}$ is of Hilbert–Schmidt type. Moreover*

$$|\mathrm{Tr}|_{-p} = \|i_{p,0}\|_{HS} \ .$$

Proof. Let $\{e_j |\ j \in \mathbb{N}_0\}$ be an orthonormal basis of $\mathcal{H}$. Then the expansion

$$\mathrm{Tr} = \sum_{j=0}^{\infty} e_j \otimes e_j$$

is valid, and we may calculate

$$|\mathrm{Tr}|_{-p}^2 = \left|\sum_{j=0}^{\infty} e_j \otimes e_j\right|_{-p}^2 = \sum_{j=0}^{\infty} |e_j|_{-p}^2 = \|i_{0,-p}\|_{HS}^2 = \|i_{p,0}\|_{HS}^2 \ .$$

□

For $p > 0$ large enough, the estimate

$$\left|\mathrm{tr}^k\varphi^{(n+2k)}\right|_p \le |\mathrm{Tr}|_{-p}^k \left|\varphi^{(n+2k)}\right|_p = \|i_{p,0}\|_{HS}^k \cdot \left|\varphi^{(n+2k)}\right|_p \tag{4.23}$$

shows that smooth kernels $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ allow the action of iterated traces. Now we are ready to prove a statement which is a little more general then Theorem 77.

Theorem 79 *Let $z \in \mathbb{C}$, and $p > 0$ be such that $i_{p,0}$ is of Hilbert–Schmidt type. If q' and $q' - q$ are large enough, σ_z is continuous from $(\mathcal{H}_p)_{q'}$ into $(\mathcal{H}_p)_q$.*

Proof. Note that $\frac{(2k)!}{(k!)^2 2^{2k}} \le 1$ for all $k \in \mathbb{N}_0$. Using this and the estimate (4.23) we can estimate as follows

$$\begin{aligned}
\left|\tilde{\varphi}^{(n)}\right|_p &\le (n!)^{-1/2}|z|^n \sum_{k=0}^{\infty} \binom{n+2k}{2k}^{\frac{1}{2}} \frac{\sqrt{(2k)!}}{k!\ 2^k} |z^2-1|^k \sqrt{(n+2k)!}\, \|i_{p,0}\|_{HS}^k \left|\varphi^{(n+2k)}\right|_p \\
&\le (n!)^{-1/2}|z|^n \sum_{k=0}^{\infty} \binom{n+k}{k}^{\frac{1}{2}} |z^2-1|^{k/2}\, \|i_{p,0}\|_{HS}^{k/2} \cdot \sqrt{(n+k)!}\, \left|\varphi^{(n+k)}\right|_p \\
&\le (n!)^{-1/2} 2^{-nq'/2}|z|^n \left(\sum_{k=0}^{\infty} \binom{n+k}{k} 2^{-q'k} \left(|z^2-1|\, \|i_{p,0}\|_{HS}\right)^k\right)^{\frac{1}{2}} \cdot \\
&\qquad\qquad \cdot \left(\sum_{k=0}^{\infty} (n+k)!\, 2^{q'(n+k)} \left|\varphi^{(n+k)}\right|_p^2\right)^{\frac{1}{2}} \\
&\le \|\varphi\|_{p,q'}\, (n!)^{-1/2}\, 2^{-nq'/2} |z|^n \left(1 - 2^{-q'}|z^2-1|\, \|i_{p,0}\|_{HS}\right)^{-\frac{n+1}{2}}
\end{aligned}$$

if q' is such that $2^{q'} > |z^2 - 1| \, \|i_{p,0}\|_{HS}$. Then we get

$$\|\sigma_z \varphi\|_{p,q}^2 \le \|\varphi\|_{p,q'}^2 \cdot \sum_{n=0}^{\infty} 2^{n(q-q')} |z|^{2n} \left(1 - 2^{-q'} |z^2 - 1| \, \|i_{p,0}\|_{HS}\right)^{-(n+1)} .$$

The sum on the right hand side converges if $q' - q$ is large enough. □

Note. We can also give a completely different proof of Theorem 77 using the powerful theorem describing the space $(\mathcal{N})$. Since $(\mathcal{N}) = \mathcal{E}^2_{\min}(\mathcal{N}')$, every test function $\varphi \in (\mathcal{N})$ has a version which has an extension to a function from $\mathcal{E}^2_{\min}(\mathcal{N}'_{\mathbb{C}})$. The function

$$\omega \mapsto \sigma_z \varphi(\omega) = \varphi(z\omega)$$

is also entire of the same growth. Since φ is of minimal type also $\omega \mapsto \varphi(z\omega)$ is of minimal type. Thus $\sigma_z \varphi \in (\mathcal{N})$. In fact $\sigma_z : \mathcal{E}^2_{\min}(\mathcal{N}'_{\mathbb{C}}) \to \mathcal{E}^2_{\min}(\mathcal{N}'_{\mathbb{C}})$ is continuous. The same argument based on Theorem 28 shows:

Theorem 80 *For all $z \in \mathbb{C}$ the mapping σ_z is continuous from $(\mathcal{N})^1$ into $(\mathcal{N})^1$.*

This kind of argument also shows

Theorem 81 *For $\varphi, \psi \in (\mathcal{N})$ the following equation holds $\sigma_z(\varphi \cdot \psi) = (\sigma_z \varphi) \cdot (\sigma_z \psi)$.*

Note. To prove these relations without referring to the description of $(\mathcal{N})$, only based on chaos expansions, requires much more effort.

Since σ_z is continuous from $(\mathcal{N})$ into $(\mathcal{N})$ it is possible to define its adjoint operator $\sigma_z^\dagger : (\mathcal{N})' \to (\mathcal{N})'$ by

$$\langle\!\langle \sigma_z^\dagger \Phi, \psi \rangle\!\rangle = \langle\!\langle \Phi, \sigma_z \psi \rangle\!\rangle \ , \quad \psi \in (\mathcal{N}) \ . \tag{4.24}$$

Of course there also exists a well defined extension $\sigma_z^\dagger : (\mathcal{N})^{-1} \to (\mathcal{N})^{-1}$.

The next Lemma will be useful later.

Lemma 82 *For $z \in \mathbb{C}$, $\Phi \in (\mathcal{N})^{-1}$ we have*

$$\sigma_z^\dagger \Phi = \mathrm{J}_z \diamond \Gamma_z \Phi$$

in particular

$$\sigma_z^\dagger 1\!\!1 = \mathrm{J}_z$$

where J_z is defined in Example 12 .

Proof. The following calculation is valid

$$\begin{aligned} S(\sigma_z^\dagger \Phi)(\xi) &= \langle\!\langle \Phi, :\exp\langle \cdot, z\xi\rangle :\rangle\!\rangle e^{-\frac{1}{2}(1-z^2)\langle \xi,\xi\rangle} \\ &= S\Phi(z\xi)\cdot S\mathrm{J}_z(\xi) \\ &= S\left(\Gamma_z\Phi \diamond \mathrm{J}_z\right)(\xi)\ . \end{aligned}$$

□

We can also derive some useful formulae concerning the pointwise product of $\mathrm{J}_z \in (\mathcal{N})'$ with a test functional.

Lemma 83 *Let* $\varphi \in (\mathcal{N})$ *then*

$$\mathrm{J}_z\varphi = \sigma_z^\dagger(\sigma_z\varphi) \tag{4.25}$$

or if we prefer to rewrite the r.h.s. as a Wick product

$$\mathrm{J}_z\varphi = \mathrm{J}_z \diamond \Gamma_z(\sigma_z\varphi)\ . \tag{4.26}$$

Proof. Let $\varphi, \psi \in (\mathcal{N})$

$$\langle\!\langle \mathrm{J}_z\varphi, \psi\rangle\!\rangle = \langle\!\langle \sigma_z^\dagger 1\!\!1, \varphi\cdot\psi\rangle\!\rangle = \langle\!\langle \sigma_z\varphi, \sigma_z\psi\rangle\!\rangle = \langle\!\langle \sigma_z^\dagger(\sigma_z\varphi), \psi\rangle\!\rangle\ ,$$

hence (4.25) follows. □

Example 17 Let us discuss the above formula for the concrete choice $\varphi = \langle \cdot, \eta\rangle^n\ ,\ \eta \in \mathcal{N}$. Then

$$\langle \cdot, \eta\rangle^n\, \mathrm{J}_z = z^{2n} \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-2k)!} \left(\frac{1}{2z^2}|\eta|^2\right)^k \langle \cdot, \eta\rangle^{\diamond(n-2k)} \diamond \mathrm{J}_z \tag{4.27}$$

by use of formula (4.26) and the expansion

$$\langle \cdot, \eta\rangle^n \ = \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-2k)!} \left(\frac{1}{2}|\eta|^2\right)^k :\langle \cdot, \eta\rangle^{n-2k}:\ .$$

Formula (4.27) allows to express pointwise products by Wick products which are well defined in more general situations. The right hand side immediately extends to $\eta \in \mathcal{H}_{\mathbb{C}}$. Then formula (4.27) may serve as a definition of the pointwise product on the left hand side. By polarization this is also possible for mixed products. Examples:

1)

$$\langle \cdot, \eta\rangle \mathrm{J}_z = z^2 \langle \omega, \eta\rangle \diamond \mathrm{J}_z.$$

Note that pointwise multiplication has a non-trivial translation to Wick multiptication.

2)

$$\langle \cdot, \eta\rangle\langle \cdot, \xi\rangle \mathrm{J}_z = z^4 \langle \cdot, \eta\rangle \diamond \langle \cdot, \xi\rangle \diamond \mathrm{J}_z + z^2(\eta, \xi)\mathrm{J}_z.$$

This formula allows to read off the "covariance" of J_z

$$\mathbb{E}\left(\langle \cdot, \eta\rangle\langle \cdot, \xi\rangle \mathrm{J}_z\right) = z^2(\eta, \xi)\,.$$

For the applications we have in mind the domain of σ_z given in Theorem 79 is too small. We want to apply σ_z to Donsker's delta and the interaction term in Feynman integrals. Both have kernels in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ but obviously not in $\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}$ for $p > 0$. Thus we need to study extensions of σ_z. Of course this is not trivial, since we may construct elements in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$ where a contraction with iterated traces is *not* well defined. On the other hand kernels consisting of tensor products raise no problems in this context. Let $\xi_j \in \mathcal{H}_{\mathbb{C}},\ 1 \le j \le n+2k$ then

$$\mathrm{tr}^k(\xi_1\hat{\otimes}\cdots\hat{\otimes}\xi_{n+2k}) = \frac{1}{(n+2k)!}\sum_{\pi}(\xi_{\pi_1},\xi_{\pi_2})\cdots\left(\xi_{\pi_{2k-1}},\xi_{\pi_{2k}}\right)\ \xi_{\pi_{2k+1}}\hat{\otimes}\cdots\hat{\otimes}\xi_{\pi_{n+2k}}\ ,$$

where the sum extends over the symmetric group of order $n+2k$. Obviously also finite sums of tensor products are allowed. The next step is to discuss infinite sums of tensor products. We give a sufficient condition discussed in [JK93].

Proposition 84 *Let $\varphi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$. Suppose there exists a complete orthonormal system $\{e_j \mid j \in \mathbb{N}_0\}$ of $\mathcal{H}$ such that the expansion*

$$\varphi^{(n)} = \sum_{j_1,\cdots,j_n=1}^{\infty} a_{j_1,\cdots,j_n}\, e_{j_1}\otimes\cdots\otimes e_{j_n}$$

holds. If the coefficients $(a_{j_1,\cdots,j_n})$ are in l_1, i.e.,

$$C_n := \sum_{j_1,\cdots,j_n=0}^{\infty} |a_{j_1,\cdots,j_n}| \tag{4.28}$$

is finite, then for every $k,\ 0 \le k \le [\frac{n}{2}]$ $\quad \mathrm{tr}^k\varphi^{(n)}$ exists in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}(n-2k)}$ and is given by

$$\mathrm{tr}^k\varphi^{(n)} = \sum_{j_{2k+1},\cdots,j_n=0}^{\infty}\left(\sum_{j_1,\cdots,j_k=0}^{\infty} a_{\underbrace{j_1,j_1}\cdots,\underbrace{j_k,j_k},\underbrace{j_{2k+1},\cdots,j_n}}\right) e_{j_{2k+1}}\otimes\cdots\otimes e_{j_n}\ . \tag{4.29}$$

Moreover

$$\left|\mathrm{tr}^k\varphi^{(n)}\right|_{\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}(n-2k)}} \le C_n\ .$$

Now we are going to use Proposition 84 to extend σ_z. We will take the chaos expansion of $\sigma_z\varphi$ given by equation (4.22) as the fundamental definition, whenever this is well defined. The question arises to find sufficient conditions to ensure that $\sigma_z\varphi \in \mathcal{M}$ (or $L^2(\mu)$ or $\mathcal{G}'$) or at least the existence of the expectation value

$$\mathbb{E}(\sigma_z\varphi) = \tilde{\varphi}^{(0)}\ .$$

We will formulate this type of conditions in terms of the C_n appearing in Proposition 84.

Lemma 85 *Let $\alpha \in \mathbb{R}$ such that $2^\alpha > |z^2-1|$ and assume that K_α defined by*

$$K_\alpha^2 = \sum_{n=0}^{\infty} n!\,2^{\alpha n}\,C_n^2$$

is finite. Then

$$|\tilde{\varphi}^{(n)}|^2 \le K_\alpha^2\frac{1}{n!}|z|^{2n}2^{-\alpha n}\left(1-2^{-\alpha/2}|z^2-1|\right)^{-(n+1)}$$

Proof. The proof is very similar to the proof of Theorem 79. We can use the bound (4.29) and $\frac{1}{k!\,2^k}\sqrt{(2k)!} \leq 1$ in the following estimate:

$$
\begin{aligned}
|\tilde{\varphi}^{(n)}| &\leq |z|^n \frac{1}{\sqrt{n!}} \sum_{k=0}^{\infty} \sqrt{\frac{(n+2k)!}{n!\,(2k)!}} \frac{\sqrt{(2k)!}}{k!\,2^k} |z^2-1|^k \sqrt{(n+2k)} C_{n+2k} \\
&\leq |z|^n (n!\,2^{\alpha n})^{-1/2} \left(\sum_{k=0}^{\infty} \binom{n+2k}{2k} |z^2-1|^{2k} 2^{-2\alpha k} \right)^{1/2} \left(\sum_{k=0}^{\infty} (n+2k)!\, 2^{\alpha(n+2k)} C_{n+2k}^2 \right)^{1/2} \\
&\leq K_\alpha (n!\,2^{\alpha n})^{-1/2} |z|^n \left(\sum_{k=0}^{\infty} \binom{n+k}{k} |z^2-1|^k 2^{-\alpha k} \right)^{1/2} \\
&= K_\alpha (n!\,2^{\alpha n})^{-1/2} |z|^n (1 - 2^\alpha |z^2-1|)^{-\frac{n+1}{2}}
\end{aligned}
$$

if $2^\alpha > |z^2 - 1|$. □

Proposition 86 *Assume all definitions as before.*
If $\alpha \in \mathbb{R}$ is such that $2^\alpha > |z^2-1|$, then $K_\alpha < \infty$ implies $\sigma_z \varphi \in \mathcal{G}'$.
If $\alpha \in \mathbb{R}$ is such that $2^\alpha > |z|^2 + |z^2-1|$, then $K_\alpha < \infty$ implies $\sigma_z \varphi \in \mathcal{M}$.

Note. In the case $z = \sqrt{i}$ we obtain

$$K_\alpha < \infty \ , \ \alpha > 0.5 \ \Rightarrow \ \sigma_z \varphi \in \mathcal{G}'$$

$$K_\alpha < \infty \ , \ \alpha \gtrsim 1.27 \ \Rightarrow \ \sigma_z \varphi \in \mathcal{M} \ .$$

Proof. For $\alpha, \beta \in \mathbb{R}$ with $2^\alpha > |z^2-1|$ we can estimate

$$
\begin{aligned}
\|\sigma_z \varphi\|_{0,\beta}^2 &= \sum_{n=0}^{\infty} n!\, 2^{\beta n} |\tilde{\varphi}^{(n)}|^2 \\
&\leq K_\alpha^2 \sum_{n=0}^{\infty} |z|^{2n} 2^{n(\beta-\alpha)} (1 - |z^2-1|\, 2^{-\alpha})^{-(n+1)}.
\end{aligned}
$$

1) If $\beta < 0$ is chosen small enough the series on the right hand side is convergent, such that $\sigma_z \varphi \in \mathcal{G}'$.
2) If $\alpha \in \mathbb{R}$ is such that $2^\alpha > |z|^2 + |z^2-1|$ then $|z|^2\, 2^{-\alpha}(1 - |z^2-1|\, 2^{-\alpha})^{-1} < 1$. Hence there exists $\beta > 0$ such that $\|\sigma_z \varphi\|_{0,\beta} < \infty$, which proves $\sigma_z \varphi \in \mathcal{M}$. □

Now let us check if we get less restrictive conditions if we only define $\tilde{\varphi}^{(0)}$ (and interpret this as $\mathbb{E}(\sigma_z \varphi)$). We have

$$
\begin{aligned}
|\tilde{\varphi}^{(0)}| &\leq \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,2^k} |z^2-1|^k C_{2k} \\
&\leq \sum_{k=0}^{\infty} \sqrt{(2k)!}\, |z^2-1|^k\, C_{2k} \ .
\end{aligned}
$$

This series is convergent if $\sum_{k=0}^{\infty}(2k)!\,2^{2k\alpha}C_{2k}^2$ is finite for $\alpha \in \mathbb{R}$ such that $2^\alpha > |z^2-1|$, i.e., we get the same type of growth, but conditions are only put on the kernels $\varphi^{(2k)}$ of even order.

One important application of the scaling operator is the following theorem from [S93] (see also [HKPS93]) which gives an explicit relation to pointwise multiplication with J_z:

Theorem 87 *Let φ_n be a sequence of test functionals in $(\mathcal{N})$. Then the following statements are equivalent:*
(i) The sequence $\mathrm{J}_z\varphi_n \to \Psi$ converges in $(\mathcal{N})'$.
(ii) The sequence $\sigma_z\varphi_n$ converges in $(\mathcal{N})'$.
(iii) The sequence $\mathbb{E}\,(\psi\cdot\sigma_z\varphi_n)$ converges for all $\psi \in (\mathcal{N})$.
The action of Ψ is given by

$$\langle\!\langle \Psi, \psi\rangle\!\rangle = \lim_{n\to\infty} \mathbb{E}\,(\sigma_z\,(\varphi_n\psi))\,,$$

if one of the conditions (i) to (iii) holds.

The proof is an immediate consequence of Lemma 83.

One may be tempted to extend σ_z by continuity arguments. This has to be done with great care as the following illustrative example shows.

Example 18 We will construct a sequence $\{\varphi_n|\, n \in \mathbb{N}\} \subset (\mathcal{N})$ converging to zero in the topology of $L^2(\mu)$. But for $z \neq 1$ the sequence $\{\sigma_z\varphi_n|\, n \in \mathbb{N}\}$ converges to a constant different from zero in $L^2(\mu)$.

Let $\{\varphi_n^{(2)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}2}|\ n \in \mathbb{N}\}$ denote a sequence converging to $\varphi^{(2)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}2}$, such that the sequence $\{c_n := \mathrm{tr}\,\varphi_n^{(2)}|\ n \in \mathbb{N}\} \subset \mathbb{C}$ is divergent. We consider the sequence of test functions

$$\varphi_n = \frac{1}{c_n}\langle\omega^{\otimes 2}, \varphi_n^{(2)}\rangle$$

which by construction converges to zero in $L^2(\mu)$.
On the other hand

$$\sigma_z\varphi_n = \frac{1}{c_n}\langle\omega^{\otimes 2}, z^2\varphi_n^{(2)}\rangle + \frac{1}{c_n}(z^2-1)\mathrm{tr}\,\varphi_n^{(2)}$$

$$= \frac{1}{c_n}\langle\omega^{\otimes 2}, z^2\varphi_n^{(2)}\rangle + (z^2-1)$$

such that

$$\lim_{n\to\infty}\sigma_z\varphi_n = (z^2-1)$$

which is different from zero for $z \neq 1$, concluding the example.

4.6 Donsker‘s delta “function”

4.6.1 Complex scaling of Donsker’s delta

Consider again the S-transform of Donsker’s delta function:

$$F(\theta) = \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \exp\left(-\frac{1}{2\langle\eta,\eta\rangle}\left(\langle\theta,\eta\rangle - a\right)^2\right), \qquad \theta \in \mathcal{N}_{\mathbb{C}}\ ,\ \eta \in \mathcal{H}_+ \ .$$

This is clearly analytic in the parameter $a \in \mathbb{R}$. We can thus extend to complex a and the resulting expression is still a U-functional. The same argument holds if we extend to $\eta \in \mathcal{H}_{\mathbb{C}}$. We only have to be careful with regard to the square root. For our purpose it is convenient to cut the complex plane along the negative axis. So we have to exclude $\eta \in \mathcal{H}_{\mathbb{C}}$ with $\langle\eta,\eta\rangle$ negative. Hence by Theorem 52 it is possible to define $\delta\left(\langle\omega,\eta\rangle - a\right)$ for this choice of parameters. First of all we calculate the chaos expansion of Donsker’s delta.

Lemma 88 *Let* $a \in \mathbb{C}\ ,\ \eta \in \mathcal{H}_{\mathbb{C}}\ ,\ \arg\langle\eta,\eta\rangle \neq \pi\ ,$ *then*

$$\delta(\langle\cdot,\eta\rangle - a) = \sum_{n=0}^{\infty}\langle :\omega^{\otimes n}:, f^{(n)}\rangle$$

where

$$f^{(n)} = \frac{e^{-\frac{a^2}{2\langle\eta,\eta\rangle}}}{\sqrt{2\pi\langle\eta,\eta\rangle}} \frac{1}{n!} H_n\left(\frac{a}{\sqrt{2\langle\eta,\eta\rangle}}\right)\left(2\langle\eta,\eta\rangle\right)^{-n/2}\eta^{\otimes n} \tag{4.30}$$

is in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}}$. *Here* H_n *denotes the* n^{th} *Hermite polynomial (in the normalization of* [HKPS93]*).*

Proof. We can expand the S-transform of δ

$$\begin{aligned} S\delta(\langle\cdot,\eta\rangle - a)(\theta) &= \frac{1}{\sqrt{2\pi\langle\eta,\eta\rangle}} \exp\left(-\frac{1}{2\langle\eta,\eta\rangle}(\langle\theta,\eta\rangle - a)^2\right) \\ &= \frac{e^{-\frac{a^2}{2\langle\eta,\eta\rangle}}}{\sqrt{2\pi\langle\eta,\eta\rangle}} \sum_{n=0}^{\infty}\frac{1}{n!}H_n\left(\frac{a}{\sqrt{2\langle\eta,\eta\rangle}}\right)\left(2\langle\eta,\eta\rangle\right)^{-n/2}\langle\theta^{\otimes n},\eta^{\otimes n}\rangle\ . \end{aligned}$$

Then it is easy to read of the kernels $f^{(n)}$ given by equation (4.30). □

To discuss the convergence of the chaos expansion we need estimates on the growth of the sequence $\{H_n(\lambda) \mid n \in \mathbb{N}_0\}$ at a fixed complex point λ. This is a well known fact for $\lambda \in \mathbb{R}$, but for complex λ the sequence grows faster.

Lemma 89 [Sz39, Th. 8.22.7 and eq. (8.23.4)]
Let $\lambda \in \mathbb{C}$, *then*

$$\lim_{n\to\infty}(2n)^{-1/2}\log\left\{\frac{\Gamma(n/2+1)}{\Gamma(n+1)}|H_n(\lambda)|\right\} = |\mathrm{Im}(\lambda)|\ .$$

Note. That means for n large we have the asymptotic behavior

$$\begin{aligned} |H_n(\lambda)| &\sim \frac{\Gamma(n+1)}{\Gamma(n/2+1)} \cdot e^{\sqrt{2n}|\mathrm{Im}(\lambda)|} \\ &\lesssim \sqrt{n!\,2^n} e^{\sqrt{2n}\,|\mathrm{Im}(\lambda)|} . \end{aligned} \tag{4.31}$$

Theorem 90 *Let $a \in \mathbb{C}$, $\eta \in \mathcal{H}_{\mathbb{C}}$, $\arg\langle\eta,\eta\rangle \neq \pi$ be given. Then $\delta(\langle\cdot,\eta\rangle - a)$ defined by the chaos expansion (4.30) (or equivalently by its S-transform) is in $\mathcal{M}'$, i.e., for all $\alpha > 0$, $\|\delta(\langle\cdot,\eta\rangle - a)\|_{0,-\alpha}$ is finite.*

Proof. We have

$$\begin{aligned} \|\delta\|_{0,-\alpha}^2 &= \sum_{n=0}^{\infty} n!\,2^{-n\alpha} |f^{(n)}|^2 \\ &\leq \left| \frac{e^{-\frac{a^2}{2\langle\eta,\eta\rangle}}}{\sqrt{2\pi\langle\eta,\eta\rangle}} \right| \sum_{n=0}^{\infty} \frac{1}{n!} 2^{-n\alpha} \left| H_n\left(\frac{a}{\sqrt{2\langle\eta,\eta\rangle}}\right)\right|^2 |2\langle\eta,\eta\rangle|^{-n} |\eta|^{2n} \\ &\lesssim \frac{e^{\frac{|a|^2}{2|\eta|^2}}}{\sqrt{2\pi}|\eta|} \sum_{n=0}^{\infty} 2^{-n\alpha} \exp\left(2\sqrt{2n}\,|\,\mathrm{Im}(\frac{a}{2\langle\eta,\eta\rangle})|\right) \end{aligned}$$

in view of (4.31). The series is convergent for any $\alpha > 0$. □

Now we intend to study complex scaling of a sequence of test functionals converging to δ. This is done in the spirit of Theorem 87. Let $\eta_n \in \mathcal{N}$ be a sequence of real Schwartz test functions converging to $\eta \in \mathcal{H}$.
Choose $|\alpha| < \frac{\pi}{4}$ and $z \in \mathbf{S}_\alpha \equiv \left\{ z \in \mathbb{C} \mid \arg z \in \left(-\frac{\pi}{4}+\alpha, \frac{\pi}{4}+\alpha\right)\right\}$ and define

$$\varphi_{n,z}(\omega) = \frac{1}{2\pi} \int_{-ne^{-i\alpha}}^{ne^{-i\alpha}} e^{i\lambda(z\langle\omega,\eta_n\rangle - a)} \mathrm{d}\lambda \,. \tag{4.32}$$

To shorten notation we call the basic sequence $\varphi_{n,1}$ simply φ_n. Note that given any z with $|\arg z| < \frac{\pi}{2}$ one can choose α such that z and 1 are in $\mathbf{S}_\alpha$. In this section we will establish the following results:

Theorem 91 [LLSW94b]
For all $z \in \mathbf{S}_\alpha$ we have:

i) $\varphi_{n,z} \in (\mathcal{N})$.

ii) $\sigma_z \varphi_n = \varphi_{n,z}$.

iii) $\varphi_n \to \delta$ *in* $(\mathcal{N})'$.

iv) $\sigma_z\varphi_n$ *converges in* $(\mathcal{N})'$.
The limit element is called $\sigma_z\delta$.

Remark. The limit elements in (iii) and (iv) do not depend on α.

Proof. **i)** First of all we calculate the S-transform of the integrand of equation (4.32), $\theta \in \mathcal{N}_{\mathbb{C}}$:

$$S\left(\exp\left(i\lambda\left(z\left\langle\omega,\eta_n\right\rangle - a\right)\right)\right)(\theta) = \exp\left(-\frac{1}{2}z^2\lambda^2\left|\eta_n\right|_0^2 + i\lambda\left(z\langle\theta,\eta_n\rangle - a\right)\right) .$$

This fulfills the requirements of Theorem 56, thus the integral (4.32) is well-defined. Hence

$$\begin{aligned} S\varphi_{n,z}(\theta) &= \frac{1}{2\pi}\int_{-ne^{-i\alpha}}^{ne^{-i\alpha}} S\left(\exp\left(i\lambda\left(z\left\langle\omega,\eta_n\right\rangle - a\right)\right)\right)(\theta)\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-ne^{-i\alpha}}^{ne^{-i\alpha}} \exp\left(-\frac{1}{2}z^2\lambda^2\left|\eta_n\right|_0^2 + i\lambda\left(z\langle\theta,\eta_n\rangle - a\right)\right)\mathrm{d}\lambda\ . \end{aligned}$$

We substitute $\nu = e^{i\alpha}\lambda$, this leads to

$$S\varphi_{n,z}(\theta) = \frac{e^{-i\alpha}}{2\pi}\int_{-n}^{n} \exp\left(-\frac{1}{2}z^2e^{-2i\alpha}\nu^2\left|\eta_n\right|_0^2 + ie^{-i\alpha}\nu\left(z\langle\theta,\eta_n\rangle - a\right)\right)\mathrm{d}\nu. \tag{4.33}$$

Now take the absolute value

$$\begin{aligned} \left|S\varphi_{n,z}(\theta)\right| &\leq \frac{1}{2\pi}\int_{-n}^{n} \exp\left(\frac{1}{2}\left|z\right|^2\nu^2\left|\eta_n\right|_0^2 + \left|\nu\right|\left|z\right|\left|\theta\right|_{-p}\left|\eta_n\right|_p + \left|\nu\right|\left|a\right|\right)\mathrm{d}\nu \\ &\leq \frac{1}{2\pi}\int_{-n}^{n} \exp\left(\frac{1}{2}\left|z\right|^2\nu^2\left|\eta_n\right|_0^2 + \frac{1}{2s^2}\nu^2\left|z\right|^2\left|\eta_n\right|_p^2 + \frac{1}{2}s^2\left|\theta\right|_{-p}^2 + \left|\nu\right|\left|a\right|\right)\mathrm{d}\nu \\ &\leq \frac{n}{\pi}\exp\left(\frac{n^2}{2}\left|z\right|^2\left(\left|\eta_n\right|_0^2 + \frac{1}{s^2}\left|\eta_n\right|_p^2\right) + n\left|a\right|\right)\exp\left(+\frac{1}{2}s^2\left|\theta\right|_{-p}^2\right) . \end{aligned}$$

This estimate holds for all $s \in \mathbb{R}$ and $p \in \mathbb{N}$. Thus it fulfills the requirements of the characterization Theorem 54 and we arrive at $\varphi_{n,z} \in (\mathcal{N})$.

ii) Now we study the action of σ_z on φ_n. A direct computation yields

$$\begin{aligned} \varphi_n(\omega) &= \frac{1}{2\pi}\int_{-ne^{-i\alpha}}^{ne^{-i\alpha}} e^{i\lambda(\langle\omega,\eta_n\rangle - a)}\mathrm{d}\lambda \\ &= \frac{1}{2\pi i\left(\langle\omega,\eta_n\rangle - a\right)}\left(\exp\left(ine^{-i\alpha}\left(\langle\omega,\eta_n\rangle - a\right)\right) - \exp\left(-ine^{-i\alpha}\left(\langle\omega,\eta_n\rangle - a\right)\right)\right) \\ &= \frac{1}{\pi\left(\langle\omega,\eta_n\rangle - a\right)}\sin\left(ne^{-i\alpha}\left(\langle\omega,\eta_n\rangle - a\right)\right) . \end{aligned}$$

This is defined pointwise and continuous. Thus

$$\sigma_z\varphi_n(\omega) = \frac{1}{\pi\left(z\left\langle\omega,\eta_n\right\rangle - a\right)}\sin\left(ne^{-i\alpha}\left(z\left\langle\omega,\eta_n\right\rangle - a\right)\right) .$$

On the other hand we get

$$\frac{1}{2\pi}\int_{-ne^{-i\alpha}}^{ne^{-i\alpha}} e^{i\lambda(z\langle\omega,\eta_n\rangle-a)}\mathrm{d}\lambda \;=\; \frac{1}{\pi\left(z\left\langle\omega,\eta_n\right\rangle-a\right)}\sin\left(ne^{-i\alpha}\left(z\left\langle\omega,\eta_n\right\rangle-a\right)\right)=\varphi_{n,z}\left(\omega\right).$$

Hence

$$\sigma_z\varphi_n\left(\omega\right)=\varphi_{n,z}\left(\omega\right)\ ,\ z\in\mathbf{S}_\alpha.$$

iii,iv) Let us look at the convergence of (4.33). The following estimate holds:

$$\left|S\sigma_z\varphi_n\left(\theta\right)\right|\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\mathrm{Re}\left(z^2e^{-2i\alpha}\right)\nu^2\left|\eta_n\right|_0^2+\nu\mathrm{Re}\left(ie^{-i\alpha}\left(z\langle\theta,\eta_n\rangle-a\right)\right)\right)\mathrm{d}\nu\ .$$

The integral exists if $\mathrm{Re}\left(z^2e^{-2i\alpha}\right)>0$.
This condition is satisfied for $-\frac{\pi}{4}+\alpha<\arg z<\frac{\pi}{4}+\alpha$. We get

$$\begin{aligned}\left|S\sigma_z\varphi_n\left(\theta\right)\right| &\leq \frac{1}{2\pi}\sqrt{\frac{2\pi}{\mathrm{Re}\left(z^2e^{-2i\alpha}\right)\left|\eta_n\right|_0^2}}\exp\left(\frac{\left[\mathrm{Re}\left(ie^{-i\alpha}\left(z\langle\theta,\eta_n\rangle-a\right)\right)\right]^2}{2\mathrm{Re}\left(z^2e^{-2i\alpha}\right)\left|\eta_n\right|_0^2}\right)\\ &\leq \frac{1}{\sqrt{2\pi\mathrm{Re}\left(z^2e^{-2i\alpha}\right)\frac{1}{2}\left|\eta\right|_0^2}}\exp\left(\frac{\left(\left|z\right|\left|\theta\right|_0 2\left|\eta\right|_0+\left|a\right|\right)^2}{2\mathrm{Re}\left(z^2e^{-2i\alpha}\right)\frac{1}{2}\left|\eta\right|_0^2}\right)\ ,\end{aligned}$$

for n large enough.
Now the convergence Theorem 55 applies:

$$\begin{aligned}\lim_{n\to\infty}S\sigma_z\varphi_n\left(\theta\right) &= \frac{1}{2\pi}e^{-i\alpha}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}z^2e^{-2i\alpha}\nu^2\left|\eta\right|_0^2+i\nu e^{-i\alpha}\left(z\langle\theta,\eta\rangle-a\right)\right)\mathrm{d}\nu\\ &= e^{-i\alpha}\frac{1}{\sqrt{2\pi}ze^{-i\alpha}\left|\eta\right|_0}\exp\left(\frac{-e^{-2i\alpha}\left(z\langle\theta,\eta\rangle-a\right)^2}{2\left|\eta\right|_0^2z^2e^{-2i\alpha}}\right)\\ &= \frac{1}{\sqrt{2\pi}z\left|\eta\right|_0}\exp\left(\frac{-\left(z\langle\theta,\eta\rangle-a\right)^2}{2\left|\eta\right|_0^2z^2}\right)\ .\end{aligned}\tag{4.34}$$

Note that the limit does not depend on α. □

Proposition 92 *δ is homogeneous of degree -1 in $z\in\mathbf{S}_\alpha$:*

$$\sigma_z\delta\left(\langle\omega,\eta\rangle-a\right)=\frac{1}{z}\delta\left(\langle\omega,\eta\rangle-\frac{a}{z}\right).$$

Proof. From formula (4.34) we have

$$S\sigma_z\delta\left(\langle\omega,\eta\rangle-a\right)\left(\theta\right)=\frac{1}{\sqrt{2\pi}z\left|\eta\right|_0}\exp\left(-\frac{1}{2}\frac{\left(a-z\langle\theta,\eta\rangle\right)^2}{z^2\left|\eta\right|_0^2}\right)\tag{4.35}$$

$$= \frac{1}{z}\frac{1}{\sqrt{2\pi}\,|\eta|_0}\exp\left(-\frac{1}{2}\frac{\left(\frac{a}{z}-\langle\theta,\eta\rangle\right)^2}{|\eta|_0^2}\right)$$
$$= S\left(\frac{1}{z}\delta\left(\langle\omega,\eta\rangle-\frac{a}{z}\right)\right)(\theta)\ .$$

□

4.6.2 Products of Donsker's deltas

To define products of scaled Donsker's deltas, we use the following ansatz

$$\Phi = \prod_{j=1}^{n}\sigma_z\delta\left(\langle\cdot,\eta_j\rangle - a_j\right) = \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\int_\gamma \exp\left(i\lambda_j\left(z\langle\cdot,\eta_j\rangle - a_j\right)\right)\mathrm{d}\lambda_j\ , \tag{4.36}$$

here $\gamma = \{e^{-i\alpha}t \mid t\in\mathbb{R}\}$, $z\in\mathbf{S}_\alpha$, where α is chosen such that $|\alpha|<\frac{\pi}{4}$, η_j are real, linear independent elements of $\mathcal{H}$ and $a_j\in\mathbb{C}$. We use the notation

$$\exp\left(\sum_{j=1}^{n} i\lambda_j\left(\langle\cdot,\eta_j\rangle - a_j\right)\right) = \exp\left(i\vec{\lambda}\left(\langle\cdot,\vec{\eta}\rangle - \vec{a}\right)\right).$$

To prove that Φ is well-defined, we calculate it's T-transform:

$$\begin{aligned}
T\Phi(\theta) &= \frac{e^{-i\alpha n}}{(2\pi)^n}\int\int \exp\left(ize^{-i\alpha}\left\langle\omega,\vec{\lambda}\vec{\eta}\right\rangle - ie^{-i\alpha}\vec{\lambda}\vec{a} + i\langle\omega,\theta\rangle\right)\mathrm{d}\mu\,\mathrm{d}^n\lambda\\
&= \frac{1}{(2\pi)^n}e^{-i\alpha n}\int\int \exp\left(i\left\langle\omega, ze^{-i\alpha}\vec{\lambda}\vec{\eta}+\theta\right\rangle - ie^{-i\alpha}\vec{\lambda}\vec{a}\right)\mathrm{d}\mu\,\mathrm{d}^n\lambda\\
&= \frac{1}{(2\pi)^n}e^{-i\alpha n}\int C\left(ze^{-i\alpha}\vec{\lambda}\vec{\eta}+\theta\right)\exp\left(-ie^{-i\alpha}\vec{\lambda}\vec{a}\right)\mathrm{d}^n\lambda\ .
\end{aligned}$$

To calculate $C(ze^{-i\alpha}\vec{\lambda}\vec{\eta}+\theta)$ consider now

$$\langle ze^{-i\alpha}\vec{\lambda}\vec{\eta}, ze^{-i\alpha}\vec{\lambda}\vec{\eta}\rangle = z^2e^{-i2\alpha}\sum_{k,l}\lambda_k\lambda_l\left(\eta_k,\eta_l\right) = z^2e^{-i2\alpha}\vec{\lambda}M\vec{\lambda},$$

where $M \equiv (\eta_k,\eta_l)_{k,l}$. This is a Gram matrix of linear independent vectors and thus positive definite.

$$\begin{aligned}
T\Phi(\theta) &= \frac{1}{(2\pi)^n}\,e^{-i\alpha n}e^{-\frac{1}{2}\langle\theta,\theta\rangle}\\
&\quad\times\int\exp\left[-\frac{1}{2}z^2e^{-i2\alpha}\vec{\lambda}M\vec{\lambda} - ze^{-i\alpha}\vec{\lambda}\langle\vec{\eta},\theta\rangle - ie^{-i\alpha}\vec{\lambda}\vec{a}\right]\mathrm{d}^n\lambda\\
&= \sqrt{\frac{(2\pi)^n}{\left(z^2e^{-i2\alpha}\right)^n\det M}}\,\frac{e^{-i\alpha n}}{(2\pi)^n}\,e^{-\frac{1}{2}\langle\theta,\theta\rangle}\\
&\times\exp\left[\frac{1}{2}\left(ze^{-i\alpha}\langle\vec{\eta},\theta\rangle + ie^{-i\alpha}\vec{a}\right)\left(z^2e^{-i2\alpha}M\right)^{-1}\left(ze^{-i\alpha}\langle\vec{\eta},\theta\rangle + ie^{-i\alpha}\vec{a}\right)\right],
\end{aligned}$$

this Gaussian integral exists if $\mathrm{Re}\left(z^2e^{-i2\alpha}\right)>0$, which is equivalent to $z\in\mathbf{S}_\alpha$. The last expression is a U-functional, so we get:

Theorem 93 [LLSW94b]
Let $a_j \in \mathbb{C}$, $\eta_j \in L^2(\mathbb{R})$ *linear independent and* $M = (\eta_k, \eta_l)_{k,l}$ *the corresponding Gram matrix.*
Then for all $z \in \mathbf{S}_\alpha$ $\Phi = \prod_{j=1}^{n} \sigma_z \delta\left(\langle \cdot, \eta_j \rangle - a_j\right)$ *is a Hida distribution with S-transform*

$$S\Phi\left(\theta\right) = \frac{1}{\sqrt{\left(2\pi z^2\right)^n \det M}} \exp\left[-\frac{1}{2}\left(\langle \vec{\eta}, \theta \rangle - \frac{1}{z}\vec{a}\right) M^{-1} \left(\langle \vec{\eta}, \theta \rangle - \frac{1}{z}\vec{a}\right)\right] . \tag{4.37}$$

4.6.3 Complex scaling of finite dimensional Hida distributions

We can use Theorem 93 to extend the scaling operator. Let $\eta_j \in \mathcal{H}$, $1 \leq j \leq n$ be linear independent and $G : \mathbb{R}^n \to \mathbb{C}$ in $L^p_{\mathbb{C}}\left(\mathbb{R}^n, \exp(-\frac{1}{2}\vec{x} \cdot M^{-1}\vec{x})\,\mathrm{d}^n x\right)$ for some $p > 1$ where $M = (\eta_k, \eta_l)_{k,l}$ is positive definite. These assumptions allow to define

$$\varphi := G\left(\langle \cdot, \eta_1 \rangle, \ldots, \langle \cdot, \eta_n \rangle\right)$$

such that $\varphi \in L^p_{\mathbb{C}}(\mu)$. In view of Proposition 61.1 $\varphi \in \mathcal{G}'$. Since φ depends only on a finite number of "coordinates" $\langle \cdot, \eta_j \rangle$ we call it a finite dimensional Hida distribution (similar to [KK92] where this notion was restricted to smooth $\eta_j \in \mathcal{N}$, $1 \leq j \leq \eta$).

Lemma 94 *In the case of the above assumptions the following representation holds*

$$\varphi = \int_{\mathbb{R}^n} G(\vec{x})\, \delta^n(\langle \cdot, \vec{\eta} \rangle - \vec{x})\, \mathrm{d}^n x$$

where the integral in $(\mathcal{N})'$ *is in the sense of Bochner and*

$$\delta^n(\langle \cdot, \vec{\eta} \rangle - \vec{x}) := \prod_{j=1}^{n} \delta(\langle \cdot, \eta_j \rangle - x_j)$$

is defined in Theorem 93.

The proof is postponed because the existence of the Bochner integral will follow from the more general discussion in the next theorem. Then the equality follows from a comparison on the dense set of exponential functions.

Now it is natural to try the following extension of σ_z :

$$\sigma_z \varphi = \int_{\mathbb{R}^n} G(\vec{x})\, \sigma_z \delta^n(\langle \cdot, \vec{\eta} \rangle - \vec{x})\, \mathrm{d}^n x$$

whenever the right hand side is a well defined Bochner integral in $(\mathcal{N})'$. To do this, stronger assumptions on G are needed. In the next theorem we will give a sufficient condition.

Theorem 95 *Let* $z \in \mathbf{S}_0$ *(i.e.,* $\mathrm{Re}\frac{1}{z^2} > 0$*) and let*

$$G \in L^p_{\mathbb{C}}\left(\mathbb{R}^n, \exp\left(-\frac{1}{2}(\mathrm{Re}\frac{1}{z^2})\, \vec{x} \cdot M^{-1}\vec{x}\right) \mathrm{d}^n x\right) \text{ for some } p > 1.$$

Then

$$\int_{\mathbb{R}^n} G(\vec{x})\, \sigma_z \delta^n(\langle \cdot, \vec{\eta} \rangle - \vec{x})\, \mathrm{d}^n x$$

is a well defined Bochner integral in $(\mathcal{N})'$.

Proof. From equation (4.37) we can estimate

$$|S\sigma_z\delta^n(\langle\cdot,\vec{\eta}\rangle - \vec{x})(\theta)| \leq$$

$$\leq \frac{1}{\sqrt{(2\pi|z|^2)^n \det M}} \exp\left(-\frac{1}{2}\mathrm{Re}\frac{1}{z^2}\,\vec{x}\cdot M^{-1}\vec{x} + \frac{1}{|z|}\left|\langle\theta,\vec{\eta}\rangle\cdot M^{-1}\vec{x}\right| + \frac{1}{2}|\theta|^2|\vec{\eta}||M^{-1}\vec{\eta}|\right).$$

The term linear in $\vec{x}$ can now be estimated using a general estimate for positive quadratic forms; for all $\varepsilon > 0$

$$\frac{1}{|z|}\left|\langle\theta,\vec{\eta}\rangle\cdot M^{-1}\vec{x}\right| \leq \varepsilon\vec{x}\cdot M^{-1}\vec{x} + \frac{1}{4\varepsilon|z|^2}\langle\theta,\vec{\eta}\rangle\cdot M^{-1}\langle\theta,\vec{\eta}\rangle$$

$$\leq \varepsilon\vec{x}\cdot M^{-1}\vec{x} + \frac{1}{4\varepsilon|z|^2}|\theta|^2|\vec{\eta}||M^{-1}\vec{\eta}|\ .$$

Thus

$$|S\sigma_z\delta^n(\langle\cdot,\vec{\eta}\rangle - \vec{x})(\theta)| \leq$$

$$\leq \frac{1}{\sqrt{(2\pi|z|^2)^n \det M}} \exp\left(-\left(\frac{1}{2}\mathrm{Re}\frac{1}{z^2} - \varepsilon\right)\vec{x}\cdot M^{-1}\vec{x} + \frac{1}{2}\left(\frac{1}{2\varepsilon|z|^2} + 1\right)|\vec{\eta}||M^{-1}\vec{\eta}||\theta|^2\right)\ .$$

Now we choose $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$ such that $q\varepsilon < \frac{1}{2}\mathrm{Re}\frac{1}{z^2}$. Then

$$\int_{\mathbb{R}^n} |G(\vec{x})| \exp\left(-\left(\frac{1}{2}\mathrm{Re}\frac{1}{z^2} - \varepsilon\right)\vec{x}\cdot M^{-1}\vec{x}\right) \mathrm{d}^n x \leq$$

$$\left(\int |G(\vec{x})|^p \exp\left(-\frac{1}{2}\mathrm{Re}\frac{1}{z^2}\vec{x}\cdot M^{-1}\vec{x}\right)\mathrm{d}^n x\right)^{1/p} \cdot \left(\int \exp\left(-\left(\frac{1}{2}\mathrm{Re}\frac{1}{z^2} - q\varepsilon\right)\vec{x}\cdot M^{-1}\vec{x}\right)\mathrm{d}^n x\right)^{1/q}$$

is finite because of our assumptions. Hence Theorem 56 applies and proves the theorem.□

Notes.
1. Instead of integration with respect to $G(\vec{x})\cdot\mathrm{d}^n x$ we can use complex measures v on $\mathbb{R}^n$ to define the more general distribution

$$\int_{\mathbb{R}^n} \sigma_z\delta^n(\langle\cdot,\vec{\eta}\rangle - \vec{x})\,\mathrm{d}^n v(x)\ .$$

2. If $z \notin \mathbf{S}_0$ (not negative) it is sill possible to define $\sigma_z\delta^n(\langle\cdot,\vec{\eta}\rangle - \vec{x})$ (if we do not insist on the existence of an integral representation of type (4.36)). Then equation (4.37) defines this object. In this case $\mathrm{Re}\frac{1}{z^2}$ may be negative and the analog of Theorem 95 would require rapid decrease of $|G|$ at infinity, more precisely there has to be an $\varepsilon > 0$ such that

$$\int |G(\vec{x})| \exp\left(\left(\varepsilon - \frac{1}{2}\mathrm{Re}\frac{1}{z^2}\right)\vec{x}\cdot M^{-1}\vec{x}\right)\mathrm{d}^n x$$

is finite.

4.6.4 Series of Donsker's deltas

We set

$$\Phi_N = \sum_{n=-N}^{N} \sigma_z \, \delta\left(\langle \omega, \eta \rangle - a + n\right), \quad a \in \mathbb{C} \, .$$

This is a well-defined Hida distribution and its S-transform is given by

$$S\Phi_N\left(\theta\right) = \frac{1}{\sqrt{2\pi |\eta|^2 z}} \sum_{n=-N}^{N} \exp\left(-\frac{1}{2|\eta|^2 z^2}\left(a - n - z\langle \eta, \theta\rangle\right)^2\right) \, .$$

We now assume $\mathrm{Re}\frac{1}{z^2} > 0$. To study the limit $N \to \infty$ we calculate a uniform bound (in N) for

$$\begin{aligned}
|S\Phi_N\left(\theta\right)| &\leq \frac{1}{|z| \sqrt{2\pi |\eta|^2}} \sum_{n=-N}^{N} \exp\left(-\frac{1}{2|\eta|^2}\mathrm{Re}\left(\frac{1}{z^2}\left(a - n - z\langle \eta, \theta\rangle\right)^2\right)\right) \\
&\leq \frac{1}{|z| \sqrt{2\pi |\eta|^2}} \sum_{n=-N}^{N} \exp\left(\frac{1}{2|\eta|^2}\left(-n^2 \mathrm{Re}\frac{1}{z^2} + 2\left|\frac{n}{z}\right|\left|\frac{a}{z} - \langle \eta, \theta\rangle\right| + \left|\frac{a}{z} - \langle \eta, \theta\rangle\right|^2\right)\right) \\
&\leq \frac{1}{|z| \sqrt{2\pi |\eta|^2}} \sum_{n=-\infty}^{\infty} \exp\left(\frac{1}{2|\eta|^2}\left(-n^2 \frac{1}{2}\mathrm{Re}\frac{1}{z^2} + \left(1 + \frac{2\,|z|^2}{\mathrm{Re}\ z^2}\right)\left|\frac{a}{z} - \langle \eta, \theta\rangle\right|^2\right)\right) \\
&= \frac{1}{|z| \sqrt{2\pi |\eta|^2}} \exp\left(\frac{1}{2|\eta|^2}\left(1 + \frac{2\,|z|^2}{\mathrm{Re}\ z^2}\right)\left(\left|\frac{a}{z}\right| - |\eta|_0 \, |\theta|_0\right)^2\right) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{4|\eta|^2}\mathrm{Re}\frac{1}{z^2} n^2\right) .
\end{aligned}$$

The infinite sum converges if $\mathrm{Re}\frac{1}{z^2} > 0$, i.e., if $z \in \mathbf{S}_0$. The sum can also be expressed as $\vartheta\left(0, \frac{i}{4\pi|\eta|^2}\mathrm{Re}\frac{1}{z^2}\right)$ using the theta function (see [Mu79])

$$\vartheta\left(\rho, \tau\right) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n \rho\right) \, .$$

Now Theorem 55 applies and we get:

$$\begin{aligned}
S\Phi\left(\theta\right) &= \frac{1}{z\sqrt{2\pi |\eta|^2}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{2z^2|\eta|^2}\left(a - n - z\langle \eta, \theta\rangle\right)^2\right) \\
&= \frac{1}{z\sqrt{2\pi |\eta|^2}} \exp\left(-\frac{1}{2|\eta|^2}\left(\langle \eta, \theta\rangle - \frac{a}{z}\right)^2\right) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{2z^2|\eta|^2} - \frac{n}{|\eta|^2}\left(\langle \eta, \theta\rangle - \frac{a}{z}\right)\right) \\
&= \frac{1}{z\sqrt{2\pi |\eta|^2}} \exp\left(-\frac{1}{2|\eta|^2}\left(\langle \eta, \theta\rangle - \frac{a}{z}\right)^2\right) \vartheta\left(\frac{i}{2\pi|\eta|^2}\left(\langle \eta, \theta\rangle - \frac{a}{z}\right), \frac{i}{2\pi z^2 |\eta|^2}\right) \\
&= S\sigma_z \delta\left(\theta\right) \cdot \vartheta\left(\frac{i}{2\pi|\eta|^2}\left(\langle \eta, \theta\rangle - \frac{a}{z}\right), \frac{i}{2\pi z^2 |\eta|^2}\right) \, .
\end{aligned}$$

Thus we have proved:

Theorem 96 [LLSW94b]
For all $a \in \mathbb{C}$ and all $z \in \mathbf{S}_0$ the infinite sum

$$\Phi = \sum_{n=-\infty}^{\infty} \sigma_z \, \delta \left(\langle \omega, \eta \rangle - a + n \right)$$

exists as a Hida distribution with S-transform

$$S\Phi\left(\theta\right) = S\sigma_z \, \delta\left(\theta\right) \cdot \vartheta\left(\frac{i}{2\pi|\eta|^2} \left(\langle \theta, \eta \rangle - \frac{a}{z} \right), \frac{i}{2\pi z^2 |\eta|^2} \right) \quad .$$

4.6.5 Local Time

In the next two sections we choose the nuclear triple

$$\mathcal{S}_d \subset L^2_d \subset \mathcal{S}'_d$$

and $\eta = 1\!\!1_{[0,t)}$ the indicator function of a real interval. As is well known Brownian motion may be represented in the framework of Gaussian analysis as $\vec{B}(t) = \left\langle \vec{\omega}, 1\!\!1_{[0,t]} \right\rangle$. Let us consider the local time, which intuitively should measure the mean time a Brownian particle spends at a given point. Informally the local time is given by "Tanaka's formula"

$$L(\tau, \vec{a}) = \frac{1}{\tau} \int_0^\tau \delta^d(\vec{B}(t) - \vec{a}) \, \mathrm{d}t \quad , \quad \vec{a} \in \mathbb{R}^d, \tau \in \mathbb{R}_+ \quad ,$$

where $\delta^d(\vec{B}(t) - \vec{a})$ is Donsker delta function with S-transform given by

$$S\delta^d(\vec{B}(t) - \vec{a})(\vec{\xi}) = \left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \exp\left(-\frac{1}{2t} \left[\int_0^t \vec{\xi}(s) \mathrm{d}s - \vec{a} \right]^2 \right) \quad .$$

For dimension $d \geq 2$ this expression is usually treated by renormalization, i.e., the cancellation of divergent terms, see e.g., [SW93, FHSW94]. However, for $\vec{a} \neq 0$ — Brownian motion starts in 0, thus one expects a strong divergence for $\vec{a} = 0$ — the local time can be rigorously defined in $(\mathcal{S}_d)^{-1}$ using Bochner integrals. In fact we can estimate as follows

$$\left| S\delta^d(\vec{B}(t) - \vec{a})(\vec{\theta}) \right| \leq \left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \exp\left(\frac{1}{2t} \left| 1\!\!1_{[0,t]} \right|^2 \left| \vec{\theta} \right|^2 + \frac{|\vec{a}|}{t} t \sup_{0 \leq s \leq t} \left| \vec{\theta}(s) \right| - \frac{\vec{a}^2}{2t} \right)$$

$$\leq \left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \exp\left(-\frac{\vec{a}^2}{2t} \right) \exp\left(\left| \vec{\theta} \right|_1^2 \right) \exp\left(\frac{1}{2} \left| \vec{a} \right|^2 \right) \quad .$$

Now

$$C(t) := \left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \exp\left(-\frac{\vec{a}^2}{2t} \right) \exp\left(\frac{1}{2} \left| \vec{a} \right|^2 \right)$$

is integrable on any interval $[0, \tau]$ with respect to Lebesgue measure $\mathrm{d}t$. Hence the conditions of Theorem 56 are satisfied and we have

$$L(\tau, \vec{a}) = \frac{1}{\tau} \int_0^\tau \delta^d(\vec{B}(t) - \vec{a}) \, \mathrm{d}t \in (\mathcal{S}_d)' \quad , \quad 0 \neq \vec{a} \in \mathbb{R}^d, \tau \in \mathbb{R}_+ \quad .$$

In the case $d = 1$ and $a \in \mathbb{C}$ the relevant estimate becomes

$$\left|S\delta\left(B\left(t\right)-a\right)\left(\theta\right)\right| \leq \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2}|a|^2} e^{-\frac{\operatorname{Re}\left(a^2\right)}{2t}} \exp\left(|\theta|_1^2\right) \ .$$

This fulfills the conditions of Theorem 56 if $\operatorname{Re}\left(a^2\right) \geq 0$. Thus we have an analytic extension of $L\left(\tau, a\right) = \frac{1}{\tau} \int\limits_0^\tau \delta\left(B\left(t\right)-a\right) \mathrm{d}t$ to $a \in \mathbb{C}$ with $\operatorname{Re}\left(a^2\right) \geq 0$. That means the point $a = 0$ is allowed in this case (Note that $\int\limits_0^\tau t^{-d/2}\mathrm{d}t$ only exists for $d = 1$.).

Chapter 5

Concept of path integration in a white noise framework

5.1 The free Feynman integrand

The idea of realizing Feynman integrals within the white noise framework goes back to [HS83]. The "average over all paths" is performed with a Hida distribution as the weight. The existence of such Hida distributions corresponding to Feynman integrands has been established in [FPS91]. There the Feynman integrand for the free motion (in one space dimension) reads:

$$\mathrm{I}_{0,old} = \mathrm{Nexp}\left(\frac{i+1}{2}\int_{t_0}^{t} \omega^2(\tau)\ \mathrm{d}\tau\right)\delta\left(x(t)-x\right).$$

However the distribution

$$\mathrm{J} = \mathrm{Nexp}\left(\frac{i+1}{2}\int_{\mathbb{R}} \omega^2(\tau)\ \mathrm{d}\tau\right)$$

has recently been seen to be particularly useful in this context because of its relation to complex scaling (see Theorem 87). It turns out that it is unnecessary to use the time interval $[t_0, t]$ in the kinetic energy factor; the delta function introduces the interval into the resulting distribution $\mathrm{I}_0 := \mathrm{J}\delta$. Indeed it will be shown that I_0 produces the correct physical results. As the choice of I_0 rather than $\mathrm{I}_{0,old}$ as a starting point produces only minor modifications in calculations and formulae, all the pertinent results in [FPS91] can be established in a completely analogous manner.

Let us look at the construction of the free Feynman integrand (in more than one space dimension) in more detail. We are going to use a variant of white noise analysis which allows vector valued white noise and hence the possibility to build up Brownian paths in d-dimensional space, see Example 11 on page 53.

We introduce the heuristic term

$$\exp\left(\frac{1+i}{2}\int_{\mathbb{R}} \vec{\omega}^2(\tau)\ \mathrm{d}\tau\right)$$

where $\vec{\omega} \in \mathcal{S}'_d$ is a d–tuple of independent white noises. Formal speaking we expect this term to consist of one factor representing Feynman's factor introduced for the kinetic part and one factor compensating the Gaussian fall-off of the white noise measure which is used instead of Feynman's ill defined flat measure on path–space. The above term is not well defined because of its infinite expectation. Formally this can be cured by dividing out this infinite constant. This leads to the normalized exponential $\mathrm{J} := \mathrm{J}_{\sqrt{i}} = \mathrm{Nexp}\left(\frac{1+i}{2}\int_{\mathbb{R}} \vec{\omega}^2(\tau)\,\mathrm{d}\tau\right)$ in $(\mathcal{S}_d)'$ studied in Example 12 on page 54. It can be defined rigorously by its T-transform

$$\exp\left(-\frac{i}{2}\int_{\mathbb{R}} \vec{\xi}^2(\tau)\,\mathrm{d}\tau\right), \qquad \vec{\xi} \in \mathcal{S}_d$$

where $\vec{\xi}^2 = \sum_{j=1}^{d} \xi_i^2$ denotes the Euclidean inner product. The version of Brownian motion we are going to use starts in point $\vec{x}_0$ at time t_0:

$$\vec{x}(\tau) = \vec{x}_0 + \left\langle \vec{\omega}, \mathbb{1}_{[t_0,\tau)}\right\rangle \quad , \quad \vec{\omega} \in \mathcal{S}_d' \tag{5.1}$$

here $\mathbb{1}_{[t_0,\tau)}$ denotes the indicator function of the interval $[t_0, \tau)$. Since we will discuss propagators we also have to fix the endpoint $\vec{x}$ of the paths at time t. To this end we introduce Donsker's delta function which is the formal composition of a delta function and a Brownian motion:

$$\delta^d(\vec{x}(t) - \vec{x})\ .$$

This is a well defined distribution in $(\mathcal{S}_d)'$ as can be verified by calculating its T-transform. Here we give instead the slightly more general T-transform where the scaling operator $\sigma_z : \varphi(\vec{\omega}) \longrightarrow \varphi(z\vec{\omega})$ has been applied to Donsker's delta. This can be justified following the lines of section 4.6.1;

$$T(\sigma_z\delta)(\vec{\xi}) = \left(2\pi z^2(t-t_0)\right)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}\left|\vec{\xi}\right|^2 - \frac{1}{2(t-t_0)}\left(i\int_{t_0}^{t} \vec{\xi}(\tau)\,\mathrm{d}\tau - \frac{\vec{x}-\vec{x}_0}{z}\right)^2\right).$$

Now we have to justify the pointwise multiplication of J and δ to get the so called free Feynman integrand I_0. This was initially been done in [FPS91] for $d = 1$ and in [SW93] for higher dimensions. So we use a short way to reproduce this result. Using the relation of Nexp and complex scaling which is condensed in the following formula (Lemma 83)

$$T(\mathrm{I}_0)(\vec{\xi}) = T(\mathrm{J}\delta)(\vec{\xi}) = T(\sigma_{\sqrt{i}}\delta)(\sqrt{i}\vec{\xi})\ , \tag{5.2}$$

we arrive at the T-transform of the free Feynman integrand

$$T(\mathrm{I}_0)(\vec{\xi}) = (2\pi i(t-t_0))^{-\frac{d}{2}} \exp\left(-\frac{i}{2}\left|\vec{\xi}\right|^2 - \frac{1}{2i(t-t_0)}\left(\int_{t_0}^{t} \vec{\xi}(\tau)\,\mathrm{d}\tau + (\vec{x}-\vec{x}_0)\right)^2\right). \tag{5.3}$$

This is clearly a U-functional and can be used to define I_0 in $(\mathcal{S}_d)'$. If it is necessary to be more precise we will also use the notation $\mathrm{I}_0\left(\vec{x}, t|\vec{x}_0, t_0\right)$.

Furthermore the Feynman integral $\mathbb{E}\left(\mathrm{I}_0\right) = T\mathrm{I}_0\left(0\right)$ is indeed the (causal) free particle propagator $(2\pi i(t-t_0))^{-\frac{d}{2}} \exp\left[\frac{i}{2|t-t_0|}\ (\vec{x}-\vec{x}_0)^2\right]$.
Not only the expectation but also the T–transform has a physical meaning. By a formal integration by parts

$$T\mathrm{I}_0\left(\vec{\xi}\right) = \mathbb{E}\left(\mathrm{I}_0\ e^{-i\int_{t_0}^{t} \vec{x}(\tau)\dot{\vec{\xi}}(\tau)\,\mathrm{d}\tau}\right)\ e^{i\vec{x}\vec{\xi}(t) - i\vec{x}_0\vec{\xi}(t_0)}\ e^{-\frac{i}{2}\left|\vec{\xi}_{[t_0,t]^c}\right|^2}.$$

$(\vec{\xi}_{[t_0,t]}c$ denotes the restriction of $\vec{\xi}$ to the complement of $[t_0,t]$). The term $e^{-i\int_{t_0}^{t}\vec{x}(\tau)\dot{\vec{\xi}}(\tau)\,\mathrm{d}\tau}$ would thus arise from a time-dependent potential $W(\vec{x},\tau) = \dot{\vec{\xi}}(\tau)\vec{x}$. And indeed it is straightforward to verify that

$$\Theta(t-t_0)\cdot T\mathrm{I}_0\left(\vec{\xi}\right) = K_0^{\left(\dot{\vec{\xi}}\right)}(\vec{x},t|\vec{x}_0,t_0)\; e^{i\vec{x}\vec{\xi}(t)-i\vec{x}_0\vec{\xi}(t_0)}\; e^{-\frac{i}{2}\left|\vec{\xi}_{[t_0,t]^c}\right|^2}, \tag{5.4}$$

where

$$K_0^{\left(\dot{\vec{\xi}}\right)}(\vec{x},t|\vec{x}_0,t_0) = \frac{\Theta(t-t_0)}{\sqrt{2\pi i\,|t-t_0|}}\times$$
$$\exp\left(i\vec{x}_0\vec{\xi}(t_0)-i\vec{x}\vec{\xi}(t)-\frac{i}{2}\left|\vec{\xi}_{[t_0,t]}\right|^2+\frac{i}{2\,|t-t_0|}\left(\int_{t_0}^{t}\vec{\xi}(\tau)\,\mathrm{d}\tau+\vec{x}-\vec{x}_0\right)^2\right) \tag{5.5}$$

is the Green's function corresponding to the potential W, i.e., $K_0^{\left(\dot{\vec{\xi}}\right)}$ obeys the Schrödinger equation

$$\left(i\partial_t+\frac{1}{2}\triangle-\dot{\vec{\xi}}(t)\,x\right)K_0^{\left(\dot{\vec{\xi}}\right)}(\vec{x},t|\vec{x}_0,t_0) = i\,\delta(t-t_0)\;\delta^d(\vec{x}-\vec{x}_0)\,.$$

More generally one calculates (e.g., using Theorem 93)

$$\begin{aligned} T\left(\mathrm{J}\prod_{j=1}^{n+1}\delta^d(\vec{x}(t_j)-\vec{x}_j)\right)\left(\vec{\xi}\right) &= e^{-\frac{i}{2}\left|\vec{\xi}_{[t_0,t]^c}\right|^2} e^{i\vec{x}\vec{\xi}(t)-i\vec{x}_0\vec{\xi}(t_0)}\prod_{j=1}^{n+1}K_0^{\left(\dot{\vec{\xi}}\right)}(\vec{x}_j,t_j|\vec{x}_{j-1},t_{j-1})\,. \\ &= e^{\frac{in}{2}|\vec{\xi}|^2}\prod_{j=1}^{n+1}T\mathrm{I}_0(\vec{x}_j,t_j|\vec{x}_{j-1},t_{j-1})(\vec{\xi}) \end{aligned} \tag{5.6}$$

Here $t_0<t_1<...<t_n<t_{n+1}\equiv t$ and $\vec{x}_{n+1}\equiv\vec{x}$.

5.2 The unperturbed harmonic oscillator

In this section we first review some results of [FPS91]. Then we prepare a proposition on which we base the perturbative method in section 7.3.

To define the Feynman integrand

$$\mathrm{I}_h = \mathrm{I}_0\exp\left(-i\int_{t_0}^{t}U(x(\tau))\;\mathrm{d}\tau\right),\quad U(x)=\frac{1}{2}k^2x^2$$

of the harmonic oscillator (for space dimension $d=1$), at least two things have to be done.

First we have to justify the pointwise multiplication of I_0 with the interaction term and secondly it has to be shown that $\mathbb{E}(\mathrm{I}_h)$ solves the Schrödinger equation for the harmonic oscillator. Both has been done in [FPS91]. There the T-transform of I_h has been calculated and shown to be a U-functional. Thus $\mathrm{I}_h\in(\mathcal{S})'$. Later we will use the following modified version of their result:

$$T\mathrm{I}_h(\xi) = \sqrt{\frac{k}{2\pi i\sin k\,|\Delta|}}\exp\left(-\frac{i}{2}|\xi|^2\right)\exp\left\{\frac{ik}{2\sin k\,|\Delta|}\Big[\left(x_0^2+x^2\right)\cdot\right.$$

$$\cdot \cos k\,|\Delta| - 2x_0 x + 2x \int_{t_0}^{t} \mathrm{d}t'\, \xi(t') \cos k(t'-t_0) - 2x_0 \int_{t_0}^{t} \mathrm{d}t'\, \xi(t') \cos k(t-t')$$

$$+ 2 \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2\, \xi(s_1)\xi(s_2) \cos k(t-s_1) \cos k(s_2-t_0) \Big] \Big\}, \tag{5.7}$$

with $0 < k\,|\Delta| < \dfrac{\pi}{2}$, $\Delta = [t_0, t]$, $|\Delta| = |t - t_0|$. $T\mathrm{I}_h$ is easily seen to be a U-functional.

For our purposes it is convenient to introduce

$$K_h^{(\dot{\xi})}(x, t \mid x_0, t_0) = \theta(t-t_0)\; T\mathrm{I}_h(\xi) \cdot \exp \tfrac{i}{2} \left|\xi_{\Delta^c}\right|^2 \cdot \exp\left(ix_0 \xi(t_0) - ix\xi(t)\right) \;,$$

which is the propagator of a particle in a time dependent potential $\frac{1}{2}k^2x^2 + x\dot{\xi}(t)$. (Again $\xi_{\Delta}c$ denotes the restriction of ξ to the complement of Δ.) This allows for an independent check on the correctness of the above result. In advanced textbooks of quantum mechanics such as [Ho92] the propagator for a harmonic oscillator coupled to a source j (forced harmonic oscillator) is worked out. Upon setting $j = \dot{\xi}$ their result is easily seen to coincide with the formula given above.

As in equation (5.6) we also need a definition of the (pointwise) product

$$\mathrm{I}_h \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right)$$

in $(\mathcal{S})'$. The expectation of this object can be interpreted as the propagator of a particle in a harmonic potential, where the paths all are "pinned" such that $x(t_j) = x_j$, $1 \le j \le n$. Following the ideas of the remark at the end of the section 4.1.3 we will have to apply (4.11) repeatedly. But due to the form of $T\mathrm{I}_h(\xi)$, which contains ξ only in the exponent up to second order, all these integrals are expected to be Gaussian.

Using this we arrive at the following proposition.

Proposition 97 *For $x_0 < x_j < x$, $1 \le j \le n$, $\quad t_0 < t_j < t_{j+1} < t$, $1 \le j \le n-1$, $\mathrm{I}_h \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right)$ is a Hida distribution and its T-transform is given by*

$$T\Big(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(x(t_j) - x_j\right)\Big)(\xi) = e^{-\frac{i}{2}|\xi_{\Delta^c}|^2} e^{i(x\xi(t) - x_0\xi(t_0))} \prod_{j=1}^{n+1} K_h^{(\dot{\xi})}\left(x_j, t_j | x_{j-1}, t_{j-1}\right) \;.$$

Proof. For $n = 1$ we may check the assertion by direct computation using formula (4.11). To perform induction one needs the following lemma.

Lemma 98 *Let $[t_0, t] \subset [t_0', t']$ then*

$$K_h^{\left(\left(\xi + \lambda \mathbb{1}_{[t_0', t']}\right)^{\cdot}\right)}\left(x, t | x_0, t_0\right) = K_h^{(\dot{\xi})}\left(x, t | x_0, t_0\right), \;\; \forall \lambda \in \mathbb{R} \;.$$

The lemma is also proven by a lengthy but straightforward computation. On a formal level the assertion of the lemma is obvious as both sides of the equation are solutions of the same Schrödinger equation if $[t_0, t] \subset [t_0', t']$. □

The proposition states what one intuitively expects, ordinary propagation from one intermediate position to the next.

5.3 An example: Quantum mechanics on a circle

In this section we study a free quantum system whose one degree of freedom is constrained to a unit circle. Constructing a path integral for such a system, one has to take into account paths with different winding numbers n. Thus the following ansatz for the Feynman integrand seems to be natural:

$$\mathrm{I}\left(\varphi_1, t|\varphi_0, 0\right) \equiv \sum_{n=-\infty}^{\infty} \mathrm{J}\,\delta\left(\varphi\left(t\right)-\varphi_1+2\pi n\right)\ , \quad \mathrm{J}{=}\mathrm{J}_{\sqrt{i}}$$

where $\varphi(t) = \varphi_0 + B(t)$ is the angle of position modulo 2π. (Other quantizations would arise if we summed up the contributions from different winding numbers with a phase factor $e^{i\theta n}$ [Ri87].) However multiplication by J corresponds to complex scaling by $z = \sqrt{i}$ and we have seen in section 4.6.4 that the series does not converge for this value of z. A formal calculation (e.g., using Theorem 96 and equation (5.2)) would lead to the following S-transform:

$$S\mathrm{I}\left(\varphi_1, t|\varphi_0, 0\right)\left(\theta\right) = S\mathrm{I}_0\left(\varphi_1, t|\varphi_0, 0\right)\left(\theta\right)\cdot\vartheta\left(\frac{1}{t}\left(i\int_0^t \theta\left(s\right)\mathrm{d}s - \left(\varphi_1-\varphi_0\right)\right), \frac{2\pi}{t}\right)\ .$$

However the ϑ-function does not converge for these arguments, see [Mu79]. To stay within the ordinary white noise framework we thus consider as final states smeared wave packets F instead of strictly localized states. So let

$$F\left(\varphi\right) = \sum_{l=-\infty}^{\infty} a_l\, e^{il\varphi}\ ,$$

where $\sum\limits_{l=-\infty}^{\infty} |a_l| \exp\left(\frac{1}{2}s^2 l^2\right) < \infty$ for some $s > 0$. This leads to:

$$\begin{aligned}\mathrm{I} &= \mathrm{J}F\left(B\left(t\right)+\varphi_0\right)\\ &= \sum_{l=-\infty}^{\infty} a_l \mathrm{J}\exp\left(il\left(\left\langle\omega, 1\!\!1_{[0,t)}\right\rangle+\varphi_0\right)\right).\end{aligned}$$

It is then easy to calculate

$$\begin{aligned}T\,\mathrm{I}\left(\theta\right) &= \sum_{l=-\infty}^{\infty} a_l\, e^{il\varphi_0} T\left(\mathrm{J}\exp\left(il\left\langle\omega, 1\!\!1_{[0,t)}\right\rangle\right)\right)\left(\theta\right)\\ &= \sum_{l=-\infty}^{\infty} a_l\, e^{il\varphi_0}\exp\left(-\frac{i}{2}\int\left(\theta+l1\!\!1_{[0,t)}\right)^2\mathrm{d}\tau\right)\\ &= e^{-\frac{i}{2}\int\theta^2\mathrm{d}\tau}\sum_{l=-\infty}^{\infty} a_l\,\exp\left(-\frac{i}{2}l^2 t + il\left(-\int_0^t\theta\left(s\right)\mathrm{d}s+\varphi_0\right)\right)\ .\end{aligned}$$

To ensure convergence of the series we estimate:

$$\left|T\,\mathrm{I}\left(\theta\right)\right| \le \sum_{l=-\infty}^{\infty} |a_l|\ e^{|l|\left|\left(1\!\!1_{[0,t)},\theta\right)\right|}\ e^{\frac{1}{2}|\theta|_0^2}$$

$$\leq \sum_{l=-\infty}^{\infty} |a_l| \; e^{|l|\sqrt{t}|\theta|_0} \, e^{\frac{1}{2}|\theta|_0^2}$$
$$\leq \left(\sum_{l=-\infty}^{\infty} |a_l| \, e^{\frac{1}{2}s^2 l^2} \right) e^{\frac{1}{2}\left(1+\frac{t}{s^2}\right)|\theta|_0^2} \; .$$

This is a uniform bound, sufficient for the application of Theorem 55. Thus we have proved $\mathrm{I} \in (\mathcal{S})'$. It is straightforward to check that the Feynman integral

$$\mathbb{E}(\mathrm{I}) = \sum_{t=-\infty}^{\infty} a_l \exp\left(-\frac{i}{2} l^2 t + i l \varphi_0 \right)$$

solves the corresponding Schrödinger equation.

Chapter 6

Feynman integrals and complex scaling

6.1 General remarks

It has been shown in section 5.1 that the kinetic energy term and the factor compensating the Gaussian fall-off of the white noise measure combine to give a well-defined Hida distribution

$$\mathrm{J} := \mathrm{J}_{\sqrt{i}} = \mathrm{Nexp}\left(\tfrac{i+1}{2}|\omega|^2\right)$$

So the central question in realizing Feynman integrals in terms of white noise distributions is the definition of $\mathrm{J} \cdot \varphi$ for most general φ (e.g., $\varphi = \delta(\vec{B}(t) - \vec{x})$ in order to construct the free particle propagator).

A very elegant and general way of defining products of J and other distributions has been suggested in [S93], where the connection between J and complex scaling was noted. One has $\mathrm{J} = \sigma^{\dagger}_{\sqrt{i}} 1\!\!1$ by Lemma 82. In order to define products with J one approximates the other factor by test functionals and then studies the convergence of the scaled sequence according to Theorem 87.

Here we have to remind the reader of Example 18 on page 75. If we want to define the action of σ_z on $\Phi \in (\mathcal{N})'$ by a limiting procedure, the result depends on the choice of the approximating sequence $\varphi_n \to \Phi$. In view of Theorem 87 the same care is necessary if we want to define the pointwise product $\mathrm{J} \cdot \Phi$.

If we choose $\mathcal{N} = \mathcal{S}_d(\mathbb{R})$ and

$$\varphi = \delta(\vec{B}(t) - \vec{x}) \cdot \exp(-i \int_0^t V(\vec{B}(\tau)) \mathrm{d}\tau)$$

then $\mathbb{E}(\mathrm{J} \cdot \varphi)$ is a representation of a Feynman integral, i.e., it should coincide with the particle propagator $K(\vec{x}, t|\vec{0}, 0)$.

In this section we follow the idea suggested by Lemma 83. There we proved

$$\mathrm{J}_z \cdot \varphi = \sigma^{\dagger}_z(\sigma_z \varphi)\,, \quad \varphi \in (\mathcal{N})\,.$$

We will use the right hand side as a definition of the left side for a larger class of φ if this makes sense. Since the functionals φ in question have kernels in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ we discussed extensions of the scaling operator, which led to sufficient conditions (in Proposition 86) on

φ to ensure the existence of $\sigma_z\varphi$ in $\mathcal{G}'$ or more restrictive ones for $\sigma_z\varphi \in \mathcal{M}$. The last case is interesting if we choose $\varphi = \exp(-i\int_0^t V(\vec{B}(\tau))\mathrm{d}\tau)$. Then we can justify the multiplication with $\sigma_z\delta$ afterwards (Theorem 90).

Now assume $\varphi \in \mathcal{G}'$ such that $\sigma_z\varphi$ is well defined in $\mathcal{G}'$. Then we define

$$\mathrm{J}_z\varphi := \sigma_z^\dagger(\sigma_z\varphi) \in (\mathcal{N})' \ ,$$

such that

$$\mathbb{E}(\mathrm{J}_z\varphi) = \mathbb{E}(\sigma_z^\dagger(\sigma_z\varphi)\,) = \mathbb{E}(\sigma_z\varphi) \ .$$

Using equation (4.22) we obtain

$$\mathbb{E}(\sigma_z\varphi) = \tilde{\varphi}^{(0)} = \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,2^k}(z^2-1)^k \,\mathrm{tr}^k\varphi^{(2k)} \, .$$

Considerations of this type have been used by Hu and Meyer [HM88]. They defined the Feynman integral by

$$\mathbb{E}(\mathrm{J}_{\sqrt{i}}\varphi) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,2^k}\,(i-1)^k \,\mathrm{tr}^k\varphi^{(2k)} \, ,$$

whenever the right hand side is well defined. (Note that they used different normalization in the definition of chaos expansion.) One central question in this approach is the existence of iterated traces if $\varphi^{(2k)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}2k}$. This was one important motivation in the work [JK93], see Proposition 84 for a brief account. The second problem in this approach is that it depends on the knowledge of the chaos expansion of φ, which often cannot be calculated explicitly enough. An alternative approach is suggested by the work of Doss [D80].

6.2 Inspection of the Doss approach

Let $V : \mathbb{R}^d \to \mathbb{R}$ denote a potential on d dimensional space. We assume that V has an extension to an analytic function (also denoted by V) defined on the following "strip"

$$\mathbf{S} = \left\{ \vec{x} + \sqrt{i}\vec{y} \;\middle|\; \vec{x} \in \mathbf{D} \text{ and } \vec{y} \in \mathbb{R}^d \right\}$$

where $\mathbf{D} \subset \mathbb{R}^d$ is a connected open set.

Doss studies the expression

$$\psi(t,\vec{x}) = \mathbb{E}\left\{ f\left(\vec{x} + \sqrt{i}\vec{B}(t)\right) \exp\left(-i\int_0^t V\left(\vec{x} + \sqrt{i}\vec{B}(\tau)\right)\mathrm{d}\tau\right)\right\}$$

to obtain Feynman Kac type solutions of the time dependent Schrödinger equation, where $f : \mathbf{S} \to \mathbb{C}$ plays the role of the initial wave function $f(\vec{x}) = \psi(0,\vec{x})$. He introduces conditions on V to define $\exp\left(-i\int_0^t V(\vec{x} + \sqrt{i}\vec{B}(\tau))\,\mathrm{d}\tau\right)$ as a well defined random variable. Nevertheless these conditions are not very transparent, so we will restrict ourselves to sub–classes of potentials where the meaning of the conditions becomes more obvious. But before we need to give the underlying lemma. On the space $\mathcal{C}\left([0,\infty),\mathbb{R}^d\right)$ we introduce the norms

$$\|\vec{g}\|_t = \sup_{j=1,\ldots,d}\ \sup_{\tau\in[0,t)} |\vec{g}_j(\tau)| \ , \qquad \vec{g} \in \mathcal{C}\left([0,t),\mathbb{R}^d\right) .$$

Lemma 99 [D80]
Let $k : \mathbb{R}_+ \to \mathbb{R}_+$ denote a measurable function. Then

$$\mathbb{E}\left(k(\left\|\vec{B}\right\|_t)\right) \leq 2d\sqrt{\frac{2}{\pi t}} \cdot \int_0^\infty k(u)\exp(-\frac{u^2}{2t})\,\mathrm{d}u\,.$$

Now we state our central assumption

Definition 100 *Let $z \in \mathbb{C}$. An analytic function $V : \mathbf{S} \to \mathbb{C}$ is said to be in the Doss class (with parameters z, a, b) if there exist $a, b \geq 0$ such that V obeys the following bound*

$$\operatorname{Im} V(\vec{x} + z\vec{y}) \leq a + b|\vec{y}|^2\,, \qquad \vec{x} \in \mathbf{D}\,,\ \ \vec{y} \in \mathbb{R}^d\ . \tag{6.1}$$

The above definition is interesting in view of the following proposition.

Proposition 101 *Let V be in the Doss class with parameters $z \in \mathbb{C}$, $a, b \geq 0$.*
1. Let $\vec{x} \in \mathbf{D}$, $p \geq 1$ and $b < \frac{3}{2pt^2}$ then

$$\exp\left(-i\int_0^t V\left(\vec{x} + z\vec{B}(\tau)\right)\mathrm{d}\tau\right) \in L^p(\mu)\ .$$

2. Let $\mathbf{D}$ be convex and $\vec{x}, \vec{y} \in \mathbf{D}$, $p \geq 1$ and $b < \frac{3}{14pt^2}$ then

$$\exp\left(-i\int_0^t V\left(\vec{x} + \frac{\tau}{t}(\vec{y} - \vec{x}) + z\left(\vec{B}(\tau) - \frac{\tau}{t}\vec{B}(t)\right)\right)\mathrm{d}\tau\right) \in L^p(\mu)\ \ .$$

Proof. We prove assertion 2. the proof of statement 1. is completely analogous. The following holds

$$\left|\exp -i\int_0^t V\left(\vec{x} + \frac{\tau}{t}(\vec{y} - \vec{x}) + z\left(\vec{B}(\tau) - \frac{\tau}{t}\vec{B}(t)\right)\right)\mathrm{d}\tau\right|$$

$$\begin{aligned}
&= \exp\int_0^t \operatorname{Im} V\left(\vec{x} + \frac{\tau}{t}(\vec{y} - \vec{x}) + z\left(\vec{B}(\tau) - \frac{\tau}{t}\vec{B}(t)\right)\right)\mathrm{d}\tau \\
&\leq \exp\left(ta + b\int_0^t \left\|\vec{B}(\tau) - \frac{\tau}{t}\vec{B}(t)\right\|_t^2 \mathrm{d}\tau\right) \\
&\leq \exp\left(ta + b\int_0^t \left(\left\|\vec{B}(t)\right\|_t^2 + 2\frac{\tau}{t}\left\|\vec{B}(t)\right\|_t^2 + \frac{\tau^2}{t^2}\left\|\vec{B}(t)\right\|_t^2\right)\mathrm{d}\tau\right) \\
&= \exp\left(ta + \frac{7}{3}bt\left\|\vec{B}(t)\right\|_t^2\right)\ .
\end{aligned}$$

Lemma 99 gives

$$\mathbb{E}\left(\left|\exp -i\int_0^t V\left(\vec{x} + \frac{\tau}{t}(\vec{y} - \vec{x}) + z\left(\vec{B}(\tau) - \frac{\tau}{t}\vec{B}(t)\right)\right)\mathrm{d}\tau\right|^p\right)$$

$$\leq \mathbb{E}\left(\exp\left(pta + \frac{7}{3}ptb\left\|\vec{B}(t)\right\|_t^2\right)\right)$$
$$\leq 2d\sqrt{\frac{2}{\pi t}}\int_0^\infty \exp\left(pta - \left(\frac{1}{2t} - \frac{7}{3}ptb\right)u^2\right)\mathrm{d}u$$

which is finite if $b < \frac{3}{14pt^2}$. □

Example 19 Let $d = 1$, $z = \sqrt{i}$, and $\mathbf{D}$ bounded. Then consider the polynomial potential

$$V(x) = g\sum_{k=0}^{n} c_k x^k\,, \qquad g, c_k \in \mathbb{R} \qquad k \leq n \quad \text{and} \quad c_n = 1\,.$$

First we assume the harmonic oscillator potential, i.e., $n = 2$. If $g < b$ then there exist $a = a(\mathbf{D}) > 0$ such that (6.1) is fulfilled. Note in particular that negative values of the coupling constant g are allowed. For positive g the restriction $g < b$ is consistent with the fact that the propagator $K_h(x,t|0,0)$ of the harmonic oscillator is only defined for small times t (compare (5.7) with Proposition 101.2, which is the relevant case for propagators as we will see).

Now let $n = 2 + 8m$, $m \in \mathbb{N}_0$ and $g < 0$ **or** $n = 6 + 8m$, $m \in \mathbb{N}_0$ and $g > 0$, then the dominant behavior of the highest power shows that $\mathrm{Im}V(x + \sqrt{i}y) < 0$ for y large enough. So due to the smoothness of V

$$\mathrm{Im}\,V(x + \sqrt{i}y) < a$$

for some $a = a(\mathbf{D}) \geq 0$.
Note that we have included an interesting class of repulsive potentials i.e., $g < 0$.
Let us also mention that this example allows a comparison with results in the recent monograph [Us94]. He obtained a nice behaviour for sextic oscillators i.e., for some polynomial interactions with leading power x^6. If the coefficients c_k satisfy an additional condition, the Schrödinger equation becomes quasi–exact solvable, i.e., a finite number of energy eigenvalues E_m and eigenfunctions can be calculated explicitly. On the other hand the work of Bender and Wu [BeWu69] demonstrated that potentials with leading power x^4 produce very complicated non–perturbative effects (e.g., rapid growth of $|E_m| \sim m!\,A^m$ and the "horn structure" of the singularities of the function $g \mapsto E(g) := \sum_{m=0}^\infty E_m g^m$ in a neighborhood of zero).
For more examples see [D80].

To shorten notation we define

$$\varphi := \exp -i\int_0^t V(\vec{x}_0 + \vec{B}(\tau))\,\mathrm{d}\tau\,, \quad x_0 \in \mathbf{D}\,.$$

Of course $\varphi \in L^p(\mu)$ for any $p \geq 0$. If V satisfies the conditions of Proposition 101 for some $p \geq 0$ we will write

$$\sigma_z\varphi \overset{\text{def}}{=} \exp -i\int_0^t V(\vec{x}_0 + z\vec{B}(\tau))\,\mathrm{d}\tau \in L^p(\mu)$$

since the right hand side may be viewed as a well defined extension of the scaling operator σ_z. (Any useful extension of the scaling operator is expected to reflect the structure of the original definition $\sigma_z\varphi(\vec{\omega}) = \varphi(z\vec{\omega})$ for $\varphi \in (\mathcal{S}_d)$.)

Smooth final wave function. We want to define $\mathrm{J}_z \cdot \varphi \cdot \psi$ by extension of formula (4.25):

$$\mathrm{J}_z \cdot \varphi \cdot \psi = \sigma_z^\dagger (\sigma_z \varphi \cdot \sigma_z \psi) \ .$$

If we assume V in the Doss class for some a and $b < \frac{3}{2t^2}$ then there exists $p > 1$ such that $\sigma_z \varphi \in L^p(\mu)$, i.e., by Proposition 60 $\sigma_z \varphi \in \mathcal{G}'$. Let $\psi \in \mathcal{G}$ such that $\sigma_z \psi \in \mathcal{G}$ (for example $\psi \in (\mathcal{S}_d)$) then $\sigma_z \varphi \cdot \sigma_z \psi \in \mathcal{G}'$ and $\sigma_z^\dagger(\sigma_z \varphi \cdot \sigma_z \psi) \in (\mathcal{S}_d)'$. For example we can choose ψ to be an approximation of $\delta^d(\vec{B}(t) - (\vec{x} - \vec{x}_0))$ and $z = \sqrt{i}$ then $\mathrm{J}_{\sqrt{i}} \varphi \psi$ defined above is an approximation of the Feynman integrand. Thus

$$\mathbb{E}(\mathrm{J}_{\sqrt{i}} \varphi \psi) = \mathbb{E}\left(\sigma_{\sqrt{i}}^\dagger (\sigma_{\sqrt{i}} \varphi \cdot \sigma_{\sqrt{i}} \psi) \right) = \mathbb{E}(\sigma_{\sqrt{i}} \varphi \cdot \sigma_{\sqrt{i}} \psi)$$

is an approximation of the propagator $K_V(\vec{x}, t | \vec{x}_0, 0)$.

Rewriting the propagator. Now assume V in the Doss class for some a and $b < \frac{3}{14t^2}$ such that $\sigma_z \varphi \in \mathcal{M}$. This condition is not easy to check but nevertheless at the end of this consideration it will be possible to extend the validity of the result to more general V.

We define

$$\mathrm{J}_z \cdot \delta \cdot \varphi := \sigma_z^\dagger (\sigma_z \delta \cdot \sigma_z \varphi)$$

here δ is shorthand for $\delta^d \left(\vec{B}(t) - (\vec{x} - \vec{x}_0) \right)$. This is well defined since $\sigma_z \delta \in \mathcal{M}'$ (Theorem 90) and the pointwise product $\sigma_z \delta \cdot \sigma_z \varphi$ is in $\mathcal{G}'$ in view of (4.16).
The homogeneity property in Proposition 92 writes

$$\sigma_z \delta^d \left(\vec{B}(t) - (\vec{x} - \vec{x}_0) \right) = \left(\frac{1}{z\sqrt{t}} \right)^d \delta^d \left(\langle \vec{\omega}, \frac{1\!\!1_t}{\sqrt{t}} \rangle - \frac{\vec{x} - \vec{x}_0}{z\sqrt{t}} \right) ,$$

since δ^d is no more than the product of d independent Donsker deltas. Using this and Proposition 72 we can calculate:

$$\begin{aligned} \mathbb{E}\left(\sigma_z^\dagger (\sigma_z \varphi \cdot \sigma_z \delta) \right) &= \mathbb{E}(\sigma_z \delta \cdot \sigma_z \varphi) \\ &= \left(\frac{1}{z\sqrt{t}} \right)^d \left\langle\!\!\left\langle \delta^d \left(\langle \cdot, \frac{1\!\!1_t}{\sqrt{t}} \rangle - \frac{\vec{x} - \vec{x}_0}{z\sqrt{t}} \right), \sigma_z \varphi \right\rangle\!\!\right\rangle \\ &= \left(\frac{1}{\sqrt{2\pi t}\ z} \right)^d e^{-\frac{(\vec{x} - \vec{x}_0)^2}{2z^2 t}} \mathbb{E}\left(P \tau_{\frac{\vec{x} - \vec{x}_0}{zt} 1\!\!1_t} (\sigma_z \varphi) \right) \end{aligned}$$

where $P : \mathcal{M} \mapsto \mathcal{G}'$ is defined as in section 4.3.3.2 with $\eta = 1\!\!1_t / \sqrt{t}$. Using the definitions we obtain

$$\begin{aligned} P \tau_{\frac{\vec{x} - \vec{x}_0}{zt} 1\!\!1_t} (\sigma_z \varphi) &= \exp -i \int_0^t V \left(\vec{x}_0 + z \left\langle \vec{\omega} - \langle \vec{\omega}, 1\!\!1_t \rangle 1\!\!1_t / t + \frac{\vec{x} - \vec{x}_0}{zt} 1\!\!1_t, 1\!\!1_\tau \right\rangle \right) \mathrm{d}\tau \\ &= \exp -i \int_0^t V \left(\vec{x}_0 + \frac{\tau}{t} (\vec{x} - \vec{x}_0) + z \left(\vec{B}(\tau) - \frac{\tau}{t} B(t) \right) \right) \mathrm{d}\tau \ . \end{aligned}$$

This term can now be substituted in the above formula. Furthermore it is possible to change the representation of Brownian motion in the expectation. If $\vec{B}(\tau)$ is a Wiener process then also $l\vec{B}(\frac{\tau}{l^2})$, $l > 0$ and they have the same covariance $\min(\tau, \tau')$. This property is called scaling invariance of Brownian motion.

Thus we have derived

$$\begin{aligned}
\mathbb{E}(\mathrm{J}_z\delta\varphi) &= \left(\frac{1}{\sqrt{2\pi z^2 t}}\right)^d \exp\left(-\frac{(\vec{x}-\vec{x}_0)^2}{2z^2t}\right)\cdot \\
&\qquad \cdot\mathbb{E}\left(\exp -i\int_0^t V\left(\vec{x}_0+\frac{\tau}{t}(\vec{x}-\vec{x}_0)+z\left(\vec{B}(\tau)-\frac{\tau}{t}\vec{B}(t)\right)\right)\mathrm{d}\tau\right) \\
&= \left(\frac{1}{\sqrt{2\pi z^2 t}}\right)^d \exp\left(-\frac{(\vec{x}-\vec{x}_0)^2}{2z^2t}\right)\cdot \\
&\qquad \cdot\mathbb{E}\left(\exp -it\int_0^1 V\left(\vec{x}_0+s(\vec{x}-\vec{x}_0)+z\sqrt{t}\left(\vec{B}(s)-s\vec{B}(1)\right)\right)\mathrm{d}s\right)
\end{aligned}$$

Note that $\tau\mapsto\vec{B}(\tau)-\frac{\tau}{t}\vec{B}(t)$ is a representation of the Brownian bridge from zero to t. In the physically relevant case $z=\sqrt{i}$ the factor

$$\left(\frac{1}{\sqrt{2\pi it}}\right)^d \exp\left(-\frac{(\vec{x}-\vec{x}_0)^2}{2it}\right) = K_0(\vec{x},t|\vec{x}_0,0)$$

appearing in the above formulae is the free particle propagator. So we obtained a well defined probabilistic expression. To write down the right hand side of the above equations it is only necessary that V satisfied the Doss condition for some a and $b<\frac{3}{14t^2}$. Then the functional in the expectation is in $L^p(\mu)$ for some $p>1$. Hence the physical relevant quantity is well defined, also if $\sigma_z\varphi\in\mathcal{M}$ is not true. It remains to show that this is in fact the fundamental solution of the corresponding Schrödinger equation. We will only give a partial answer to this question. Expressions of the type discussed above were also studied in the work of Yan [Yan93]. We will state his result in our setting

Proposition 102 [Yan93, Th's 3.9 and 5.2]
Let V be as above. The expression

$$\begin{aligned}
q(\vec{x},\lambda t|\vec{x}_0,0) &= \frac{1}{\sqrt{2\pi\lambda t}}\exp\left(-\frac{(\vec{x}-\vec{x}_0)^2}{2\lambda t}\right)\cdot \\
&\qquad \cdot\mathbb{E}\left(\exp -it\int_0^1 V\left(\vec{x}_0+s(\vec{x}-\vec{x}_0)+z\sqrt{\lambda t}(\vec{B}(s)-s\vec{B}(1))\right)\mathrm{d}s\right)
\end{aligned}$$

has an analytic continuation to all $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda>0$. Moreover $q(\vec{x},\lambda t|\vec{x}_0,0)$ is the fundamental solution of

$$\frac{\partial\psi}{\partial t}=\lambda(\frac{1}{2}\Delta-V)\psi\,,\qquad \lambda\in\mathbb{C}_+\,.$$

So the quantities constructed above solve the right (partial) differential equation if z^2 $(\sim\lambda)$ is such that $\operatorname{Re}z^2>0$. The open question remains if this is also true for $z^2=i$.

Chapter 7

Quantum mechanical propagators in terms of white noise distributions

7.1 An extension of the Khandekar Streit method

In order to pass from the free motion to more general situations, one has to give a rigorous definition of the heuristic expression

$$\mathrm{I} = \mathrm{I}_0 \exp\left(-i\int_{t_0}^{t} V\left(\vec{x}\left(\tau\right)\right)\, \mathrm{d}\tau\right).$$

In [KaS92] Khandekar and Streit accomplished this by perturbative methods in the case $d = 1$ and V is a finite signed Borel measure with compact support. (Note that singular potentials are included in this class.) We generalize the construction by allowing time-dependent potentials and a Gaussian fall–off instead of a bounded support. In section 7.1.2 potentials of exponential fall–off are considered, for the price that we need to use a larger distribution space.

Let $\mathrm{D} \equiv [\mathsf{T}_0, \mathsf{T}] \supset \Delta = [t_0, t]$ and let v be a finite signed Borel measure on $\mathbb{R} \times \mathrm{D}$. Let v_x denote the marginal measure

$$v_x\left(A\right) \equiv v\left(A \times \mathrm{D}\right)\ , \qquad A \in \mathcal{B}(\mathbb{R})$$

similarly

$$v_t\left(B\right) \equiv v\left(\mathbb{R} \times B\right)\ , \qquad B \in \mathcal{B}(\mathrm{D}).$$

7.1.1 The Feynman integrand as a Hida distribution

We assume that $|v|_x$ and $|v|_t$ satisfy:

i) $\exists\, R > 0\ \forall\, r > R : |v|_x\left(\{x :\ |x| > r\}\right) < e^{-\beta r^2}$ for some $\beta > 0$,

ii) $|v|_t$ has a L^∞density.

The essential bound of this density is denoted by C_v.
Let us first describe heuristically the construction by treating v as an ordinary function V before stating the rigorous result Theorem 103. The starting point is a power series expansion of $\exp\left(-i\int_{t_0}^{t} V(x(\tau),\tau)\,\mathrm{d}\tau\right)$ using $V(x(\tau),\tau) = \int \mathrm{d}x\, V(x,\tau)\,\delta\left(x(\tau) - x\right)$:

$$\exp\left(-i\int_{t_0}^{t} V(x(\tau),\tau)\mathrm{d}\tau\right) = \sum_{n=0}^{\infty}(-i)^n\int_{\Lambda_n}\mathrm{d}^n t\prod_{i=1}^{n}\int \mathrm{d}x_i\, V(x_i,t_i)\delta(x(t_i)-x_i)$$

where

$$\Lambda_n = \{(t_1,...,t_n)|\, t_0 < t_1 < ... < t_n < t\}\,. \tag{7.1}$$

If necessary we will also use the notation $\Lambda_n(t,t_0)$.

More generally we can show:

Theorem 103

$$\mathrm{I} = \mathrm{I}_0 + \sum_{n=1}^{\infty}(-i)^n\int_{\mathbb{R}^n}\int_{\Lambda_n}\prod_{i=1}^{n} v\left(\mathrm{d}x_i,\mathrm{d}t_i\right)\, \mathrm{I}_0\prod_{j=1}^{n}\delta\left(x\left(t_j\right)-x_j\right) \tag{7.2}$$

exists as a Hida distribution in case V obeys i) and ii).

Proof. **1)** $\mathrm{I}_n = \int_{\mathbb{R}^n}\int_{\Lambda_n}\prod_{i=1}^{n} v\left(\mathrm{d}x_i,\mathrm{d}t_i\right)\, \mathrm{I}_0\prod_{j=1}^{n}\delta\left(x\left(t_j\right)-x_j\right)$ is a Hida distribution for $n\geq 1$. This is shown by applying Theorem 56.

Hence we have to establish a bound of the required type for the T-transform of the integrand. From formulae (5.5) and (5.6) we find ($\theta\in\mathcal{S}_{\mathbb{C}}$)

$$\begin{aligned}
T\left(\mathrm{I}_0\prod_{j=1}^{n}\delta(x(t_j)-x_j)\right)(\theta) &= \exp(-\frac{i}{2}|\theta|_0^2)\cdot\prod_{j=1}^{n+1}\frac{1}{\sqrt{2\pi i(t_j-t_{j-1})}}\cdot\\
&\cdot\exp\left(\sum_{j=1}^{n+1} i\frac{(x_j-x_{j-1})^2}{2(t_j-t_{j-1})}\right)\\
&\cdot\exp\left(\sum_{j=1}^{n+1} i\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s\right)\\
&\cdot\exp\left(\sum_{j=1}^{n+1}\frac{i}{2(t_j-t_{j-1})}\left[\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s\right]^2\right)\,.
\end{aligned}$$

It is easy to estimate the last term

$$\begin{aligned}
\left|\sum_{j=1}^{n+1}\frac{i}{2(t_j-t_{j-1})}\left[\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s\right]^2\right| &\leq \sum_{j=1}^{n+1}\frac{1}{2(t_j-t_{j-1})}\left[\int_{t_{j-1}}^{t_j}|\theta(s)|^2\mathrm{d}s\cdot\int_{t_{j-1}}^{t_j}\mathbb{1}^2_{[t_{j-1},t_j]}(s)\mathrm{d}s\right]\\
&\leq \sum_{j=1}^{n+1}\frac{1}{2}\int_{t_{j-1}}^{t_j}|\theta(s)|^2\mathrm{d}s \leq \frac{1}{2}|\theta|_0^2\,.
\end{aligned}$$

In order to estimate the term

$$\exp\left(\sum_{j=1}^{n+1} i\frac{x_j-x_{j-1}}{t_j-t_{j-1}}\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s\right)\,, \tag{7.3}$$

we proceed as follows

$$\sum_{j=1}^{n+1} i\frac{x_j - x_{j-1}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \theta(s)\mathrm{d}s = \frac{x}{t-t_n}\int_{t_n}^{t}\theta(s)\mathrm{d}s - \frac{x_0}{t_1-t_0}\int_{t_0}^{t_1}\theta(s)\mathrm{d}s + \sum_{j=1}^{n} x_j\left(\frac{\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s}{t_j-t_{j-1}} - \frac{\int_{t_j}^{t_{j+1}}\theta(s)\mathrm{d}s}{t_{j+1}-t_j}\right) .$$

By the mean value theorem

$$\sum_{j=1}^{n} x_j\left(\frac{\int_{t_{j-1}}^{t_j}\theta(s)\mathrm{d}s}{t_j-t_{j-1}} - \frac{\int_{t_j}^{t_{j+1}}\theta(s)\mathrm{d}s}{t_{j+1}-t_j}\right) = \sum_{j=1}^{n} x_j(\theta(\tau_j)-\theta(\tau_{j+1})) ,$$

where $\tau_k \in (t_{k-1}, t_k)$. Then we can estimate

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j(\theta(\tau_j)-\theta(\tau_{j+1}))\right| &\leq \sum_{j=1}^{n}|x_j||\theta(\tau_j)-\theta(\tau_{j+1})| \\ &\leq \left(\sup_{1\leq j\leq n}|x_j|\right)\cdot\sum_{j=1}^{n}\int_{\tau_j}^{\tau_{j+1}}|\theta'(s)|\mathrm{d}s \\ &\leq \left(\sup_{1\leq j\leq n}|x_j|\right)\cdot\sum_{j=1}^{n}\int_{t_j}^{t_{j+1}}|\theta'(s)|\mathrm{d}s \\ &\leq \left(\sup_{1\leq j\leq n}|x_j|\right)\cdot\int_{t_0}^{t}|\theta'(s)|\mathrm{d}s .\end{aligned}$$

Therefore we have

$$\begin{aligned}&\left|\exp\left(\sum_{j=1}^{n+1} i\frac{x_j - x_{j-1}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \theta(s)\mathrm{d}s\right)\right| \\ &\leq \exp\left(|x|\sup_{t_n\leq s\leq t}|\theta(s)| + |x_0|\sup_{t_0\leq s\leq t_1}|\theta(s)| + \left(\sup_{1\leq j\leq n}|x_j|\right)\cdot\int_{t_0}^{t}|\theta'(s)|\mathrm{d}s\right) \\ &\leq \exp\left(\left(\sup_{0\leq j\leq n+1}|x_j|\right)\cdot\left[\sup_{t_n\leq s\leq t}|\theta(s)| + \sup_{t_0\leq s\leq t_1}|\theta(s)| + \int_{t_0}^{t}|\theta'(s)|\mathrm{d}s\right]\right) .\end{aligned}$$

Let us introduce the following norm on $\mathcal{S}_{\mathbb{C}}(\mathbb{R})$

$$\|\theta\| \equiv \int_{t_0}^{t}|\theta'(s)|\mathrm{d}s + \sup_{t_0\leq s\leq t}|\theta(s)| .$$

Clearly this is a continuous norm on $\mathcal{S}_{\mathbb{C}}(\mathbb{R})$. From the last estimate we obtain

$$\begin{aligned}\left|\exp\left(\sum_{j=1}^{n+1} i\frac{x_j - x_{j-1}}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \theta(s)\mathrm{d}s\right)\right| &\leq \exp\left(\left(\sup_{0\leq j\leq n+1}|x_j|\right)\cdot\|\theta\|\right) \\ &\leq \exp\left[\gamma\left(\sup_{0\leq j\leq n+1}|x_j|\right)^2\right]\cdot\exp\left(\frac{1}{\gamma}\|\theta\|^2\right) ,\end{aligned}$$

where $0 < \gamma$ is to be chosen later. Now we can estimate as follows

$$\left| T\left(\mathrm{I}_0 \prod_{j=1}^{n} \delta(x(t_j) - x_j) \right)(\theta) \right| \leq \exp\left(\frac{1}{2} |\theta|_0^2 \right) \cdot \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}$$

$$\cdot \exp\left[\gamma \left(\sup_{0 \leq j \leq n+1} |x_j| \right)^2 \right] \cdot \exp\left(\frac{1}{\gamma} \|\theta\|^2 \right) \cdot \exp\left(\frac{1}{2} |\theta|_0^2 \right) \; .$$

If we introduce the norm

$$|||\theta||| \equiv \|\theta\| + |\theta|_0 \; ,$$

which is obviously also continuous on $\mathcal{S}_{\mathbb{C}}(\mathbb{R})$, we have the bound

$$\left| T\left(\mathrm{I}_0 \prod_{j=1}^{n} \delta(x(t_j) - x_j) \right)(\theta) \right|$$

$$\leq \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \cdot \exp\left[\gamma \left(\sup_{0 \leq j \leq n+1} |x_j| \right)^2 \right] \cdot \exp\left(\frac{1+\gamma}{\gamma} |||\theta|||^2 \right) \; .$$

In order to apply Theorem 56 we have to show the integrability of the first two factors with respect to v. To this end we will use Hölder's inequality.

Choose $q > 2$ and $0 < \gamma < \beta/q$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. The property i) of v yields that $e^{\gamma x^2} \in L^q(\mathbb{R} \times \mathrm{D}, |v|)$. Let $Q \equiv \left(\int_{\mathbb{R}} |v|_x \, (\mathrm{d}x) \, e^{\gamma q x^2} \right)^{1/q}$, then

$$\left(\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{i=1}^{n} |v| \, (\mathrm{d}x_i, \mathrm{d}t_i) e^{\gamma q \left(\sup_{0 \leq i \leq n+1} |x_i| \right)^2} \right)^{1/q} \leq e^{\gamma |x_0|^2} e^{\gamma |x|^2} Q^n.$$

Using the property ii) of v and the formula

$$\int_{\Lambda_n} \mathrm{d}^n t \prod_{j=1}^{n+1} \frac{1}{(2\pi \, |t_j - t_{j-1}|)^{\alpha}} = \left(\frac{\Gamma(1-\alpha)}{(2\pi)^{\alpha}} \right)^{n+1} \frac{|t - t_0|^{n(1-\alpha)-\alpha}}{\Gamma\left((n+1)(1-\alpha) \right)} \; , \; \alpha < 1$$

we obtain the following estimate:

$$\left(\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{i=1}^{n} |v| \, (\mathrm{d}x_i, \mathrm{d}t_i) \prod_{j=1}^{n+1} \left(\frac{1}{\sqrt{2\pi \, |t_j - t_{j-1}|}} \right)^p \right)^{1/p} \leq C_v^{\frac{n}{p}} \frac{\Gamma(\frac{2-p}{2})^{\frac{n+1}{p}}}{(2\pi)^{\frac{n+1}{2}}} \frac{|\Delta|^{\frac{n}{p} - \frac{1}{2}(n+1)}}{\Gamma\left((n+1)(\frac{2-p}{2}) \right)^{1/p}}$$

Let

$$C_n(x, |\Delta|) \equiv e^{\gamma |x_0|^2} e^{\gamma |x|^2} Q^n C_v^{\frac{n}{p}} \frac{\Gamma(\frac{2-p}{2})^{\frac{n+1}{p}}}{(2\pi)^{\frac{n+1}{2}}} \frac{|\Delta|^{\frac{n}{p} - \frac{1}{2}(n+1)}}{\Gamma\left((n+1)(\frac{2-p}{2}) \right)^{1/p}} \; .$$

Hölder's inequality yields the following estimate:

$$\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{i=1}^{n} |v| \, (\mathrm{d}x_i, \mathrm{d}t_i) \left| T\left(\mathrm{I}_0 \prod_{j=1}^{n} \delta(x(t_j) - x_j) \right)(\theta) \right|$$

$$\leq C_n \exp\left(\frac{1+\gamma}{\gamma} |||\theta|||^2 \right) \tag{7.4}$$

This establishes the bound required for the application of Theorem 56 and hence I_n exists as a Bochner integral in $(\mathcal{S})'$.

2) $\mathrm{I} = \sum_{n=0}^{\infty} \mathrm{I}_n$ exists in $(\mathcal{S})'$.

As the C_n are rapidly decreasing in n the hypotheses of Theorem 55 are fulfilled and hence the convergence in $(\mathcal{S})'$ is established. □

Remark. Conditions i) and ii) allow for some rather singular potentials, e.g., $\tilde{v} = \sum e^{-n^2} \delta_n$. For a cut-off interaction, i.e., compactly supported v_x, condition i) is of course valid. Note also that v is not supposed to be a product measure, hence the time dependence can be more intricate than simple multiplication by a function of time. For example we can take two bounded continuous functions f and g on Δ. Use one to move the potential around and the other one to vary its strength: $v(x,t) = f(t)\,\tilde{v}(x - g(t))$.

7.1.2 The Feynman integrand in $(\mathcal{S})^{-1}$

Instead of $(\mathcal{S})'$ we can also use $(\mathcal{S})^{-1}$ to discuss the convergence of the perturbative expansion (7.2). In this case some of the technical difficulties in estimating the term (7.3) disappear. Furthermore we obtain a larger class of potentials which allows some weaker decrease in the space direction.

Theorem 104 *Let v be a finite signed Borel measure on $\mathbb{R} \times \mathrm{D}$ such that the (absolute) marginal measures satisfy*

i') $\exists\, R > 0\ \forall\, r > R : |v|_x\,(\{x : \ |x| > r\}) < e^{-\beta r}$ *for some* $\beta > 0$,

ii) $|v|_t$ *has a* L^∞ *density.*

Then I *defined by (7.2) exists in* $(\mathcal{S})^{-1}$.

Proof. The term (7.3) may be estimated

$$\left| \exp i \sum_{j=1}^{n+1} \frac{x_j - x_{j-1}}{t_j - t_{j-1}} \int_{t_{i-1}}^{t_j} \theta(s) \mathrm{d}s \right| \leq \exp\left(2|\theta|_\infty \sum_{j=1}^{n+1} |x_j| \right) .$$

Using this we obtain

$$\left| T\left(\mathrm{I}_0 \prod_{j=1}^{n} \delta(x(t_j) - x_j) \right)(\theta) \right| \leq e^{|\theta|_0^2} \cdot \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(2|\theta|_\infty \sum_{j=1}^{n+1} |x_j| \right) .$$

To prove that this bound is integrable w.r.t. the (n–fold) product measure we refer to Hölder's inequality again. Choose $q > 2$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. In this case it is sufficient to do this for all $\theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ in a neighborhood of zero. A possible choice is

$$\theta \in \mathcal{U} := \left\{ \theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R});\, |\theta|_\infty < \frac{\beta}{2q} \right\}.$$

Then

$$\left(\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{i=1}^{n} |v|\,(\mathrm{d}x_i, \mathrm{d}t_i) \exp\left(2q|\theta|_\infty \sum_{j=1}^{n+1} |x_j| \right) \right)^{1/q} \leq e^{\frac{\beta}{q}(|x|+|x_0|)} \left(\int_{\mathbb{R}} |v|_x\,(\mathrm{d}x)\; e^{\beta|x|} \right)^{\frac{n}{q}}$$

for all $\theta \in \mathcal{U}$. The rest of the proof is along the lines of the proof of Theorem 103. The main difference is that the convergence of the integrals and of the series here are controlled by the corresponding theorems for $(\mathcal{S})^{-1}$ (Theorems 5 and 6 in [KLS94]) □

7.2 Verifying the Schrödinger equation

In this section we prove that the expectation of the Feynman integrand constructed in section 7.1.1, i.e., the Feynman integral, does indeed solve the usual integral equation for quantum mechanical propagators, which corresponds to the Schrödinger equation. (In this section we will always assume the situation of section 7.1.1 for simplicity.)

As in the case of the free motion we expect

$$K^{(\dot{\xi})}\left(x,t|x_0,t_0\right) \equiv \; e^{+\frac{i}{2}|\xi_\Delta c|^2}\; e^{-ix\xi(t)+ix_0\xi(t_0)}\,\Theta(t-t_0)T\mathrm{I}\left(x,t|x_0,t_0\right)(\xi) \tag{7.5}$$

to be the propagator corresponding to the potential $W(x,t) = V(x,t) + \dot{\xi}(t)x$. More precisely we have to use the measure $w(\mathrm{d}x,\mathrm{d}t) = v\,(\mathrm{d}x,\mathrm{d}t) + \dot{\xi}(t)x\;\mathrm{d}x\;\mathrm{d}t$. We now proceed to show some properties of $K^{(\dot{\xi})}$. As the propagators $K_0^{(\dot{\xi})}$ are continuous on $\mathbb{R}^2\times\Lambda_2$ (see (5.5)), the product $\prod_{j=1}^{n+1} K_0^{(\dot{\xi})}\left(x_j,t_j|x_{j-1},t_{j-1}\right)$ is continuous on $\mathbb{R}^{n+1}\times\Lambda_{n+1}$. Set

$$K^{(\dot{\xi})}\left(x,t|x_0,t_0\right) = \sum_{n=0}^{\infty} K_n^{(\dot{\xi})}\left(x,t|x_0,t_0\right) \tag{7.6}$$

where

$$K_n^{(\dot{\xi})}\left(x,t\mid x_0,t_0\right) = (-i)^n \int_{\mathbb{R}^n}\int_{\Lambda_n} \prod_{i=1}^{n} v\left(\mathrm{d}x_i,\mathrm{d}t_i\right) \prod_{j=1}^{n+1} K_0^{(\dot{\xi})}\left(x_j,t_j|x_{j-1},t_{j-1}\right).$$

As the test functions ξ are real the explicit formula (5.5) yields

$$|K_0^{(\dot{\xi})}\left(x,t|x_0,t_0\right)| = \frac{\Theta(t-t_0)}{\sqrt{2\pi|t-t_0|}} \equiv M_0 \tag{7.7}$$

and for $n\geq 1$ the bounds

$$\left|K_n^{(\dot{\xi})}\left(x,t|x_0,t_0\right)\right| \leq \int_{\mathbb{R}^n}\int_{\Lambda_n} \prod_{i=1}^{n} |v|\left(\mathrm{d}x_i,\mathrm{d}t_i\right) \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi\left|t_j-t_{j-1}\right|}} \tag{7.8}$$

$$\leq C_v^n \frac{|t-t_0|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} \leq C_v^n \frac{|\Delta|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} \equiv M_n.$$

Hence $\prod_{j=1}^{n+1} K_0^{(\dot{\xi})}\left(x_j,t_j|x_{j-1},t_{j-1}\right)$ is integrable on $\mathbb{R}^n\times\Lambda_n$ with respect to v^n. (This is also established in the course of a detailed proof of Theorem 103 and we have reproduced the argument here for the convenience of the reader.) Thus we can apply Fubini's theorem to change the order of integration in $K_n^{(\dot{\xi})}$ to obtain

$$K_n^{(\dot{\xi})}\left(x,t|x_0,t_0\right) = -i\iint v\left(\mathrm{d}x_n,\mathrm{d}t_n\right) K_0^{(\dot{\xi})}\left(x,t|x_n,t_n\right)\;\times$$

$$(-i)^{n-1}\int_{\mathbb{R}^{n-1}}\int_{\Lambda'_{n-1}} \prod_{i=1}^{n-1} v\left(\mathrm{d}x_i,\mathrm{d}t_i\right) \prod_{j=1}^{n} K_0^{(\dot{\xi})}\left(x_j,t_j|x_{j-1},t_{j-1}\right)$$

$(\Lambda'_{n-1} = \{(t_1, ..., t_{n-1}) \mid t_0 < t_1 < ... < t_{n-1} < t_n\})$. This establishes the following recursion relation for $K_n^{(\dot{\xi})}$

$$K_n^{(\dot{\xi})}(x, t|x_0, t_0) = (-i) \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0^{(\dot{\xi})}(x, t|y, \tau) K_{n-1}^{(\dot{\xi})}(y, \tau|x_0, t_0). \tag{7.9}$$

We now claim that the series $K^{(\dot{\xi})}(y, \tau|x_0, t_0) = \sum_n K_n^{(\dot{\xi})}(y, \tau|x_0, t_0)$ converges uniformly in y, τ on $\mathbb{R} \times (t_0, \mathsf{T})$. To see this recall the above estimate (7.8) which is uniform in y, τ. Because the M_n are rapidly decreasing it follows that

$$\sum_{n=1}^{\infty} \sup \left\{ \left| K_n^{(\dot{\xi})}(y, \tau|x_0, t_0) \right| \Big| (y,\ \tau) \in \mathbb{R} \times (t_0, \mathsf{T}) \right\} \leq \sum_{n=1}^{\infty} M_n < \infty\ .$$

Due to the uniform convergence we may interchange summation and integration in the following expression

$$\begin{aligned} -i \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0^{(\dot{\xi})}(x, t|y, \tau) K^{(\dot{\xi})}(y, \tau|x_0, t_0) \\ = -i \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0^{(\dot{\xi})}(x, t|y, \tau) \sum_n K_n^{(\dot{\xi})}(y, \tau|x_0, t_0) \\ = \sum_n -i \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0^{(\dot{\xi})}(x, t|y, \tau) K_n^{(\dot{\xi})}(y, \tau|x_0, t_0). \end{aligned}$$

By the above recursion relation (7.9) for $K_n^{(\dot{\xi})}$ this equals

$$\sum_n K_{n+1}^{(\dot{\xi})}(x, t|x_0, t_0) = K^{(\dot{\xi})}(x, t|x_0, t_0) - K_0^{(\dot{\xi})}(x, t|x_0, t_0).$$

Hence we obtain the following

Theorem 105 *$K^{(\dot{\xi})}$ as defined in (7.5) obeys the following integral equation:*

$$K^{(\dot{\xi})}(x, t|x_0, t_0) = K_0^{(\dot{\xi})}(x, t|x_0, t_0) - i \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0^{(\dot{\xi})}(x, t|y, \tau) K^{(\dot{\xi})}(y, \tau|x_0, t_0).$$

In particular the Feynman integral $\mathbb{E}(\mathrm{I}) \equiv K$ obeys the well-known propagator equation:

$$K(x, t|x_0, t_0) = K_0(x, t|x_0, t_0) - i \iint v(\mathrm{d}y, \mathrm{d}\tau) K_0(x, t|y, \tau) K(y, \tau|x_0, t_0).$$

We now proceed to show that this corresponds to the Schrödinger equation. To prove this we first prepare the following

Lemma 106 *The mapping $(x, t) \mapsto K^{(\dot{\xi})}(x, t|x_0, t_0)$ is continuous on $\mathbb{R} \times (t_0, \mathsf{T})$.*

Proof. Because the series (7.6) converges uniformly it is sufficient to show the continuity of $K_n^{(\dot{\xi})}$. For $n = 0, 1$ this is straightforward from the explicit formula (5.5). For $n > 1$ we use (7.9) and the estimate (7.8) to obtain

$$\left| K_n^{(\dot{\xi})}(x', t'|x_0, t_0) - K_n^{(\dot{\xi})}(x, t|x_0, t_0) \right|$$
$$\leq M_{n-1} \int_{\mathbb{R}} \int_{\Delta} |v| (\mathrm{d}x_n, \mathrm{d}t_n) \left| K_0^{(\dot{\xi})}(x', t'|x_n, t_n) - K_0^{(\dot{\xi})}(x, t|x_n, t_n) \right| .$$

Using the explicit form (5.5) of $K_0^{(\dot{\xi})}$ it is now straightforward to check that

$$\int_{\mathbb{R}} \int_{\Delta} |v| (\mathrm{d}x_n, \mathrm{d}t_n) \left| K_0^{(\dot{\xi})}(x', t'|x_n, t_n) - K_0^{(\dot{\xi})}(x, t|x_n, t_n) \right|$$
$$\leq \ | x - x' | \, C(x, t) \ + \ | t - t' |^{\alpha} \, C_{\alpha}(x, t)$$

where $0 < \alpha < \frac{1}{2}$ and $x > x'$, $t > t'$. □

An application of Lemma 106 combined with the estimate (7.7) shows that $K^{(\dot{\xi})}(., .|x_0, t_0)$ is locally integrable on $\mathbb{R} \times (\mathsf{T}_0, \mathsf{T})$ with respect to both v and Lebesgue measure. We can thus regard $K^{(\dot{\xi})}$ as a distribution on $\mathcal{D}(\Omega) \equiv \mathcal{D}(\mathbb{R} \times (\mathsf{T}_0, \mathsf{T}))$:

$$\left\langle K^{(\dot{\xi})}, \varphi \right\rangle = \iint \mathrm{d}x \, \mathrm{d}t \, K^{(\dot{\xi})}(x, t|x_0, t_0) \, \varphi(x, t) , \qquad \varphi \in \mathcal{D}(\Omega) .$$

And we can also define a distribution $vK^{(\dot{\xi})}$ by setting

$$\left\langle vK^{(\dot{\xi})}, \varphi \right\rangle = \iint v(\mathrm{d}x, \, \mathrm{d}t) \, K^{(\dot{\xi})}(x, t|x_0, t_0) \, \varphi(x, t) , \qquad \varphi \in \mathcal{D}(\Omega) .$$

($K^{(\dot{\xi})}$ is locally integrable with respect to v, φ is bounded with compact support and v is finite, hence $\varphi K^{(\dot{\xi})}$ is integrable with respect to v).

We now proceed to show that $K^{(\dot{\xi})}$ solves the Schrödinger equation as a distribution. To abbreviate we set $\hat{L} = \left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}(t) \, x \right)$ and let $\hat{L}^*$ denote its adjoint. Let $\varphi \in \mathcal{D}(\Omega)$. By Theorem 105 we have

$$\left\langle \hat{L}K^{(\dot{\xi})}, \varphi \right\rangle = \left\langle K_0^{(\dot{\xi})}(x, t|x_0, t_0) - i \iint v(\mathrm{d}y, \mathrm{d}\tau) \, K_0^{(\dot{\xi})}(x, t|y, \tau) \, K^{(\dot{\xi})}(y, \tau|x_0, t_0) \, , \hat{L}^*\varphi \right\rangle .$$

By Fubini's theorem this equals

$$\left\langle K_0^{(\dot{\xi})}, \hat{L}^*\varphi \right\rangle - i \iint v(\mathrm{d}y, \mathrm{d}\tau) \left[\iint \mathrm{d}x \, \mathrm{d}t \, K_0^{(\dot{\xi})}(x, t|y, \tau) \, \hat{L}^*\varphi(x, t) \right] K^{(\dot{\xi})}(y, \tau|x_0, t_0) .$$

As $K_0^{(\dot{\xi})}$ is a Green's function of $\hat{L}$ we obtain

$$i\varphi(x_0, t_0) + \iint v(\mathrm{d}y, \mathrm{d}\tau) \, \varphi(y, \tau) \, K^{(\dot{\xi})}(y, \tau|x_0, t_0) = \langle i\delta_{x_0}\delta_{t_0}, \varphi \rangle + \left\langle vK^{(\dot{\xi})}, \varphi \right\rangle .$$

Hence we have the following

Theorem 107 $K^{(\dot{\xi})}$ *is a Green's function for the full Schrödinger equation, i.e.,*

$$\left(i\,\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}\,(t)\,x - v\right) K^{(\dot{\xi})}\,(x,t|x_0,t_0) = i\,\delta_{x_0}\,\delta_{t_0}.$$

In particular the Feynman integral $\mathbb{E}\,(\mathrm{I}) = K$ *solves the Schrödinger equation*

$$i\,\partial_t\;K\,(x,t|x_0,t_0) = \left(-\frac{1}{2}\partial_x^2 + v\right) K\,(x,t|x_0,t_0)\,, \quad \text{for } t > t_0.$$

Hence the construction proposed by Khandekar and Streit yields a rigorously defined Feynman integrand whose expectation is the correct quantum mechanical propagator.

7.3 The Feynman integrand for the perturbed harmonic oscillator

In this section we carry the ideas of section 7.1 over to perturbations of the harmonic oscillator. Hence instead of constructing a Dyson series around the free particle Feynman integrand we expand around the Feynman integrand of the harmonic oscillator. The external potentials to which the oscillator is submitted correspond to the wide class of time-dependent singular potentials treated in section 7.1.

In [AHK76, chap 5] the path integral of the unharmonic oscillator is defined within the theory of Fresnel integrals. Compared to our ansatz this procedure has the advantage of being manifestly independent of the space dimension. Despite the lack of a generalization to higher dimensional quantum systems our construction has some interesting features:

- The admissible potentials may be very singular.
- We are not restricted to smooth initial wave functions and may thus study the propagator directly.

In this section we construct the Feynman integrand for the harmonic oscillator in an external potential $V(x,t)$. Thus we have to define

$$\mathrm{I}_V = \mathrm{I}_h \cdot \exp\left(-i\int_{t_0}^{t} V(x(\tau),\tau)\,\mathrm{d}\tau\right) \;.$$

As for the free particle we introduce the perturbation V via the series expansion of the exponential. Hence we have to find conditions for V such that the following object exists in $(\mathcal{S})'$

$$\mathrm{I}_V = \mathrm{I}_h + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \mathrm{d}^n x \int_{\Lambda_n} \mathrm{d}^n t \prod_{j=1}^{n} V\,(x_j,t_j)\,\delta\,(x\,(t_j) - x_j)\,\mathrm{I}_h \;.$$

We are able to treat the same class of singular time-dependent potentials as in section 7.1.1 i.e., we consider ν a finite signed Borel measure on $\mathbb{R}\times\mathrm{D}$, where $\mathrm{D} = [\mathsf{T}_0,\mathsf{T}] \supset \Delta = [t_0,t]$. The following theorem contains conditions under which the Feynman integrand I_V exists as a Hida distribution.

Theorem 108 *Let v be a finite signed Borel measure on $\mathbb{R} \times \mathrm{D}$ satisfying* i) *and* ii) *of section 7.1.1. Then*

$$\mathrm{I}_V = \mathrm{I}_h + \sum_{n=1}^{\infty} (-i)^n \int_{\mathbb{R}^n} \int_{\Lambda_n} \Big(\prod_{j=1}^{n} v(\mathrm{d}x_j, \mathrm{d}t_j) \Big) \mathrm{I}_h \prod_{j=1}^{n} \delta\left(x\left(t_j\right) - x_j\right) \tag{7.10}$$

is a Hida distribution.

Proof.

1. part. In the first part of the proof we have to perform some technicalities which are necessary to establish the central estimate (7.11). We have to use a very careful procedure to achieve that (7.11) survives n-fold integration and summation in the second part of the proof.

Let $\theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. From Proposition 97 and the explicit formula (5.7) we find ($\Delta_j = [t_{j-1}, t_j]$)

$$\left| T\Big(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(B(t_j) - x_j\right)\Big)(\theta) \right| \leq e^{\frac{1}{2}|\theta|_0^2} \left(\prod_{j=1}^{n+1} \sqrt{\tfrac{1}{4|\Delta_j|}} \right) \exp\left(\left(|x_{n+1}| + |x_0|\right) \frac{\pi}{2} \sup_{\Delta} |\theta| \right) \cdot$$
$$\cdot \left| \exp\Big(\Big\{ \sum_{j=1}^{n} i k x_j \Big[\frac{1}{\sin k\,|\Delta_j|} \int_{\Delta_j} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_{j-1}\right) \right.$$
$$\left. - \frac{1}{\sin k\,|\Delta_{j+1}|} \int_{\Delta_{j+1}} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_{j+1}\right) \Big] \Big\} \Big) \right| \cdot$$
$$\cdot \exp\left\{ \sum_{j=1}^{n} \frac{\pi}{2\,|\Delta_j|} \int_{\Delta_j} \mathrm{d}s_1 \int_{\Delta_j} \mathrm{d}s_2\, |\theta\left(s_1\right)|\, |\theta\left(s_2\right)| \right\}$$

We define

$$X = \sup_{0 \leq j \leq n+1} |x_j|$$

and

$$\|\theta\| \equiv \sup_{\Delta} |\theta| + \sup_{\Delta} |\theta'| + |\theta|_0 \ .$$

With these

$$\left| T\Big(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(B(t_j) - x_j\right)\Big)(\theta) \right| \leq e^{\frac{1}{2}\|\theta\|^2} \left(\prod_{j=1}^{n+1} \sqrt{\tfrac{1}{4|\Delta_j|}} \right) \exp\left(X\pi \|\theta\| + \frac{\pi}{2} |\Delta|\, \|\theta\|^2 \right) \cdot$$
$$\cdot \left| \exp\Big(\Big\{ \sum_{j=1}^{n} i k x_j \Big[\frac{1}{\sin k\,|\Delta_j|} \int_{\Delta_j} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_{j-1}\right) \right.$$
$$\left. - \frac{1}{\sin k\,|\Delta_{j+1}|} \int_{\Delta_{j+1}} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_{j+1}\right) \Big] \Big\} \Big) \right| .$$

To estimate the last factor we proceed as follows:

$$\left| \sum_{j=1}^{n} i k x_j \left[\frac{1}{\sin k\,|\Delta_{j+1}|} \int_{\Delta_{j+1}} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_j\right) - \frac{1}{\sin k\,|\Delta_{j+1}|} \int_{\Delta_{j+1}} \mathrm{d}t\, \theta\left(t\right) \cos k\left(t - t_{j+1}\right) \right] \right|$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n} kX \frac{1}{\sin k\left|\Delta_{j+1}\right|}\left|\int_{\Delta_{j+1}} \mathrm{d}t\ \theta(t) \int_{t_j}^{t_{j+1}} k \sin k(t-\tau)\, d\tau\right| \\
&\leq \sum_{j=1}^{n} kX \sup_{\Delta}|\theta| \frac{\pi}{2}\left|\Delta_{j+1}\right| \\
&\leq \frac{\pi}{2} Xk\|\theta\||\Delta|
\end{aligned}
$$

To obtain a bound for the remaining term

$$
\left|\sum_{j=1}^{n} ikx_j\left[\frac{1}{\sin k\left|\Delta_j\right|} \int_{\Delta_j} \mathrm{d}t\, \theta(t) \cos k\left(t-t_{j-1}\right)-\frac{1}{\sin k\left|\Delta_{j+1}\right|} \int_{\Delta_{j+1}} \mathrm{d}t\, \theta(t) \cos k\left(t-t_j\right)\right]\right|
$$

we expand $F(t_{j-1}) = \int_{t_{j-1}}^{t_j} \mathrm{d}t\, \theta(t) \cos k\left(t-t_{j-1}\right)$ and $G(t_{j+1}) = \int_{t_j}^{t_{j+1}} \mathrm{d}t\, \theta(t) \cos k\left(t-t_j\right)$ around t_j. This yields with $\eta_j \in \Delta_j$ and $\eta_{j+1} \in \Delta_{j+1}$

$$
\begin{aligned}
\leq\ & kX\left|\sum_{j=1}^{n} \theta(t_j)\left[\frac{\left|\Delta_j\right|}{\sin k\left|\Delta_j\right|}-\frac{\left|\Delta_{j+1}\right|}{\sin k\left|\Delta_{j+1}\right|}\right]\right|+ \\
& +kX \sum_{j=1}^{n}\left[\frac{\left(t_{j-1}-t_j\right)^2}{2 \sin k\left|\Delta_j\right|}\left(-\theta'(\eta_j)-k^2 \int_{\eta_j}^{t_j} \mathrm{d}t\, \theta(t) \cos k\left(t-\eta_j\right)\right)\right]- \\
& -kX \sum_{j=1}^{n}\left[-\frac{\left(t_{j+1}-t_j\right)^2}{2 \sin k\left|\Delta_{j+1}\right|}\left(\theta'(\eta_{j+1}) \cos k\left(\eta_{j+1}-t_j\right)-k \theta(\eta_{j+1}) \sin k\left(\eta_{j+1}-t_j\right)\right)\right]
\end{aligned}
$$

Since $\sup\limits_{0 \leq x \leq \frac{\pi}{2}}\left(\frac{x}{\sin x}\right)' = 1$ then the first term above is bounded by

$$
2k|\Delta| X\|\theta\|
$$

For the second term we obtain the bound

$$
X \frac{|\Delta|}{4} \sup_{\Delta}\left|\theta'\right|+\frac{k^2|\Delta|^2}{4} \sup_{\Delta}|\theta|+\frac{\pi}{4}|\Delta| \sup_{\Delta}\left|\theta'\right|+\frac{\pi k}{4}|\Delta| \sup_{\Delta}|\xi| \leq
$$

$$
\leq \frac{X|\Delta| \pi}{4}\left(\|\theta\|\left(2+k^2|\Delta|+k\right)\right)
$$

Putting all of this together we finally arrive at

$$
\left|T\left(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(B(t_j)-x_j\right)\right)(\theta)\right| \leq\left(\prod_{j=1}^{n+1} \sqrt{\frac{1}{4\left|\Delta_j\right|}}\right) \exp \left(LX\|\theta\|+\left(\frac{\pi}{2}|\Delta|+\frac{1}{2}\right)\|\theta\|^2\right)
$$

where $L = \pi + \frac{3}{4}\pi k|\Delta| + 2k|\Delta| + \frac{\pi}{4}|\Delta|\left(2+k^2|\Delta|\right)$ is a constant.

Hence for $\theta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ we have the following estimate

$$
\left|T\left(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(B(t_j)-x_j\right)\right)(z\xi)\right| \leq\left(\prod_{j=1}^{n+1} \sqrt{\frac{1}{4\left|\Delta_j\right|}}\right) \exp\left(X^2 \gamma\right) \exp\left[|z|^2\|\xi\|^2\left(\frac{1}{2}+\frac{\pi}{2}|\Delta|+\frac{L^2}{2 \gamma}\right)\right] \tag{7.11}
$$

where $\gamma > 0$ is chosen later.

2. part. In this final step we use the method developed in the proof of Theorem 103 to control the convergence of (7.10). Although the slight modification to our case is easy we give the basic steps for the convenience of the reader.

In order to apply Theorem 56 to perform the integration we need to show that

$$\left(\prod_{j=1}^{n+1} \sqrt{\frac{1}{4|\Delta_j|}}\right) \exp\left(X^2\gamma\right)$$

is integrable with respect to v. To this end we choose $q > 2$ and $0 < \gamma < \frac{\beta}{q}$. With this choice of γ the property i) of v yields that $\exp\left(\gamma X^2\right) \in L^q\left(\mathbb{R}^n \times \Lambda_n, |v|\right)$ and with

$$Q \equiv \left(\int_{\mathbb{R}} \int_{\Delta} |v|\left(\mathrm{d}x, \mathrm{d}t\right) \exp\left(\gamma q x^2\right)\right)^{\frac{1}{q}}$$

we have

$$\left(\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n} |v|\left(\mathrm{d}x_j, \mathrm{d}t_j\right) \exp\left(\gamma q X^2\right)\right)^{\frac{1}{q}} \leq \exp\left(\gamma\left(x_0^2 + x^2\right)\right) Q^n < \infty.$$

Now we choose p such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the property ii) of v and the formula

$$\int_{\Lambda_n} \mathrm{d}^n t \prod_{j=1}^{n+1} \left(\frac{1}{4\left|t_j - t_{j-1}\right|}\right)^{\alpha} = \left(\frac{\Gamma\left(1-\alpha\right)}{4^{\alpha}}\right)^{n+1} \frac{|\Delta|^{n(1-\alpha)-\alpha}}{\Gamma\left(\left(n+1\right)\left(1-\alpha\right)\right)}, \ \alpha < 1$$

we obtain the following bound

$$\left[\int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n} |v|\left(\mathrm{d}x_j, \mathrm{d}t_j\right) \prod_{j=1}^{n+1} \left(\frac{1}{4\left|t_j - t_{j-1}\right|}\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \leq$$

$$\leq C_v^{n/p} \frac{\Gamma\left(\frac{2-p}{2}\right)^{\frac{n+1}{p}} |\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{4^{\frac{n+1}{2}} \Gamma\left(\left(n+1\right)\frac{2-p}{2}\right)^{\frac{1}{p}}} < \infty$$

(remember: C_v is the essential supremum of the L^∞-density of $|v|_t$).

Finally an application of Hölder's inequality gives

$$\left|\left(\prod_{j=1}^{n+1} \sqrt{\frac{1}{4|t_j - t_{j-1}|}}\right) \exp\left(\gamma X^2\right)\right|_{L^1(|v|)} \leq$$

$$\leq \exp\left(\gamma x_0^2 + \gamma x^2\right) Q^n C_v^{n/p} \frac{\Gamma\left(\frac{2-p}{2}\right)^{\frac{n+1}{p}} |\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{2^{n+1} \Gamma\left(\left(n+1\right)\frac{2-p}{2}\right)^{\frac{1}{p}}} \equiv C_n < \infty$$

Hence Theorem 56 yields

$$\mathrm{I}_n \equiv \int_{\mathbb{R}^n} \int_{\Lambda_n} \prod_{j=1}^{n} v\left(\mathrm{d}x_j, \mathrm{d}t_j\right) \left(\mathrm{I}_h \prod_{j=1}^{n} \delta\left(B(t_j) - x_j\right)\right) \in \left(\mathcal{S}\right)'.$$

As the C_n are rapidly decreasing in n the hypotheses of Theorem 55 are fulfilled and hence

$$\mathrm{I}_V = \sum_{n=0}^{\infty} \mathrm{I}_n \in (\mathcal{S})' \ .$$

$\square$

Chapter 8

The Feynman integrand for the Albeverio Høegh-Krohn class

8.1 Introduction

First we will introduce a further class of potentials for which we will define a path integral representation of the Green's function for the Schrödinger equation. Potentials of this type already appeared in earlier (mathematically rigorous) works on Feynman integrals, see e.g., the works [AHK76, Ga74, Ito66]. The most elegant construction of a path integral for this class of potentials has been proposed by Albeverio and Høegh-Krohn [AHK76]. They used the so-called Fresnel integral, an extension of the Lebesgue integral. Thus I will call potentials of that kind the Albeverio Høegh-Krohn class.

Besides the fact that we use completely different methods than [AHK76, Ga74, Ito66] this approach differs from previous ones in two main points.

1. In the works [AHK76, Ga74] smooth initial wave functions were used and their propagation was handled by construction of a path integral. In our white noise framework we are able to introduce delta like initial wave functions. Thus we can go back to Feynman's original idea to treat propagators by path integrals.

2. We wish to give a meaning to the integrand itself. Its expectation yields the desired propagator. We will see later that this seems to be the reason why we will have to put one additional assumption on the class of potentials we are able to handle.

Definition 109 *Let* m *denote a bounded complex measure on the Borel sets of* $\mathbb{R}^d$, $d \geq 1$. *A complex valued function* V *on* $\mathbb{R}^d$ *is called Fresnel integrable (following [AHK76]) if*

$$V(\vec{x}) = \int_{\mathbb{R}^d} e^{i\vec{\alpha}\vec{x}}\, \mathrm{d}^d\mathrm{m}(\alpha) \tag{8.1}$$

Since the bounded complex Borel measures form an algebra under convolution, the Fresnel integrable functions are an algebra $\mathcal{F}(\mathbb{R}^d)$ *under pointwise multiplication.* $\mathcal{F}(\mathbb{R}^d)$ *is called the Albeverio Høegh-Krohn class.*
We will call $V \in \mathcal{F}(\mathbb{R}^d)$ *admissible, if*

$$\int_{\mathbb{R}^d} e^{\varepsilon|\vec{\alpha}|}\mathrm{d}^d|\mathrm{m}|(\alpha) \tag{8.2}$$

is finite for some $\varepsilon > 0$.
For later use we need also the condition

$$\int_{\mathbb{R}^d} \exp(\varepsilon|\vec{\alpha}|^{1+\delta})\, \mathrm{d}^d|\mathrm{m}|(\alpha) \tag{8.3}$$

is finite for some $\varepsilon, \delta > 0$.

Remark. It is clear from the definition that our admissible potentials are $\mathcal{C}^\infty(\mathbb{R}^d)$ since all moments of the corresponding measure m exist. In fact admissible potentials are analytic in the open ball of radius ε. Since formula (8.1) now makes sense for all $\vec{x} \in \mathbb{C}^d : |\mathrm{Im}(\vec{x}\,)| < \varepsilon$, an admissible potential is regular in this strip containing the real axis. A useful reference on (analytic) characteristic functions has proven to be [Lu70]. Condition (8.3) implies that V in fact is an entire function.

In their well known work [AHK76] Albeverio and Høegh-Krohn made extensive use of the fact that a suitable choice of norm makes $\mathcal{F}(\mathbb{R}^d)$ to be a Banach algebra. Since the measures m have to be bounded, $\mathcal{F}(\mathbb{R}^d)$ contains only functions bounded on the real line.

Example 20 One particular example is given by the 2 dimensional (periodic) potential

$$V(\vec{x}) = \{\cos(x_1) \cdot \sin(x_2)\}^\beta$$

which is of special interest in the theory of *antidot (super) lattices,* see [FGKP95] for a recent review. The integer valued even parameter β determines if the potential is "soft or hard". This potential generates chaotic behavior in classical systems and the Hamiltonian has a fractal spectrum of eigenvalues in a quantum mechanical treatment. Since the measure m corresponding to V is a linear combination of (products of) delta measures and thus has compact support, the condition (8.3) is satisfied. Details will be presented in a forthcoming "Diplomarbeit" of M. Grothaus.

8.2 The Feynman integrand as a generalized white noise functional

Now we proceed to introduce these interactions into the Feynman integral. Mathematically speaking this means to give a rigorous definition to the pointwise product

$$\mathrm{I} = \mathrm{I}_0 \cdot \exp\left[-i \int_{t_0}^{t} V(\vec{x}(\tau))\, \mathrm{d}\tau\right] .$$

In Chapter 7 we already saw a method which works in the one dimensional case. There the potential was "expanded in terms of delta functions". In a second step the expansion of the exponential led to a convergent series of Hida distributions. Since in higher dimensions d the delta functions cause problems (in respect to the t-integration) we here use a Fourier decomposition of the potential. Then we will proceed as in Chapter 7 but we have to face the fact that the occurring integrals in the perturbation series are only convergent in some larger distribution space $(\mathcal{S}_d)^{-1}$ (at least $\mathcal{Y}'$ is necessary).

Theorem 110 *Let V be an admissible potential in the Albeverio Høegh-Krohn class, i.e., there exists a bounded complex Borel measure* m *satisfying (8.2). Then*

$$\mathrm{I} = \mathrm{I}_0 + \sum_{n=1}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d \mathrm{m}(\alpha_j)\ \mathrm{I}_0 \cdot \prod_{j=1}^{n} e^{i\vec{\alpha}_j \vec{x}(\tau_j)} \tag{8.4}$$

exists as a generalized white noise functional i.e., $\mathrm{I} \in (\mathcal{S}_d)^{-1}$*. If also (8.3) is satisfied then* I *is a Meyer-Yan distribution i.e.,* $\mathrm{I} \in \mathcal{Y}'$*. The integrals we are using here are in sense of Theorems 56 and 76 respectively.* Λ_n *is defined by eq. (7.1).*

Notes.

1. From physical reasons V may supposed to be real, but in mathematical respect this is irrelevant here.
2. The integral $\int_{\Lambda_n}$ can be replaced by $\frac{1}{n!}\int_{[t_0,t]^n} =: \frac{1}{n!}\int_{\square_n}$.
3. The fact, that $\mathbb{E}(\mathrm{I})$ is physically reasonable i.e., it is the elementary solution of the (time dependent) Schrödinger equation is well known. It coincides with the series developed in [AHK76]. In [Ga74] it is proved explicitly that this series is in fact the physical solution.

Proof. As a first step we have to justify the pointwise product

$$\Phi_n = \mathrm{I}_0 \cdot \prod_{j=1}^{n} e^{i\vec{\alpha}_j \vec{x}(\tau_j)} \tag{8.5}$$

Since the explicit formula (5.3) allows an extension of $T(\mathrm{I}_0)(\vec{\xi})$ to all $\vec{\xi} \in L^2_d$ we may use the following ansatz to calculate

$$\begin{aligned} T\Phi_n(\vec{\xi}) &= T(\mathrm{I}_0)\left(\vec{\xi} + \sum_{j=1}^{n} \vec{\alpha}_j \mathbb{1}_{[t_0,\tau_j)}\right) \cdot \exp\left(i\vec{x}_0 \sum_{j=1}^{n} \vec{\alpha}_j\right) \\ &= (2\pi i(t-t_0))^{-\frac{d}{2}} \exp\left(i\vec{x}_0 \sum_{j=1}^{n} \vec{\alpha}_j\right) \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \left[\vec{\xi}(s) + \sum_{j=1}^{n} \vec{\alpha}_j \mathbb{1}_{[t_0,\tau_j)}(s)\right]^2 \mathrm{d}s\right) \\ &\quad \cdot \exp\left(-\frac{1}{2i(t-t_0)}\left[\int_{t_0}^{t} \vec{\xi}(s)\mathrm{d}s + \left\{\sum_{j=1}^{n} \vec{\alpha}_j(\tau_j - t_0) + \vec{x} - \vec{x}_0\right\}\right]^2\right) \end{aligned}$$

Obviously this has an extension in $\vec{\xi} \in \mathcal{S}_d(\mathbb{R})$ to all $\vec{\theta} \in \mathcal{S}_{d,\mathbb{C}}(\mathbb{R})$ and is locally bounded in a neighborhood of zero (see estimate (8.6) below.) Thus $\Phi_n \in (\mathcal{S}_d)^{-1}$ (in fact $\Phi_n \in (\mathcal{S}_d)'$). Now we want to apply Theorem 56. Since $T\Phi_n(\vec{\theta})$ is a measurable function of $(\tau_1, ..., \tau_n; \vec{\alpha}_1, ..., \vec{\alpha}_n)$ for all $\vec{\theta} \in \mathcal{S}_{d,\mathbb{C}}(\mathbb{R})$, we only have to find an integrable local bound

$$\begin{aligned} |T\Phi_n(\vec{\theta})| &\le (2\pi(t-t_0))^{-\frac{d}{2}} \exp\Big(\frac{1}{2}\left|\vec{\theta}\right|_0^2 + \sum_{j=1}^{n} \left|(\vec{\theta}, \vec{\alpha}_j \mathbb{1}_{[t_0,\tau_j)})\right| + \frac{1}{2(t-t_0)}\Big[(t-t_0)\left|\vec{\theta}\right|_0^2 \\ &\qquad + 2\left|\int_{t_0}^{t} \vec{\theta}(s)\mathrm{d}s \cdot \left\{\sum_{j=1}^{n} \vec{\alpha}_j(\tau_j - t_0) + \vec{x} - \vec{x}_0\right\}\right|\Big]\Big) \\ &\le (2\pi(t-t_0))^{-\frac{d}{2}} \exp\left(\left|\vec{\theta}\right|_0^2 + 2\sqrt{t-t_0}\left|\vec{\theta}\right|_0 \sum_{j=1}^{n} |\vec{\alpha}_j| + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{t-t_0}}\left|\vec{\theta}\right|_0\right) \\ &=: \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \end{aligned} \tag{8.6}$$

Since m has the property (8.2) we can find neighborhood of zero

$$\mathcal{U}_{0,r} = \left\{ \vec{\theta} \in \mathcal{S}_{d,\mathbb{C}} \middle| \left|\vec{\theta}\right|_0 < r \right\} \quad , \; r = \frac{\varepsilon}{2\sqrt{t-t_0}} \tag{8.7}$$

such that

$$\int_{\mathbb{R}^d} \mathrm{d}^d |\mathrm{m}|(\alpha) \; e^{2|\vec{\alpha}|\sqrt{t-t_0}|\vec{\theta}|}$$

is finite for all $\vec{\theta} \in \mathcal{U}_{0,r}$. So we have

$$\int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d|\mathrm{m}|(\alpha_j) \; \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \leq \tag{8.8}$$

$$\leq \frac{1}{n!}(t-t_0)^n (2\pi(t-t_0))^{-\frac{d}{2}} \exp\left(\left|\vec{\theta}\right|_0^2 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{(t-t_0)}} \left|\vec{\theta}\right|_0 \right) \left(\int_{\mathbb{R}^d} \mathrm{d}^d|\mathrm{m}|(\alpha) \; e^{2|\vec{\alpha}|\sqrt{t-t_0}|\vec{\theta}|_0} \right)^n$$

Thus we have proved the existence of an integrable bound. Also the convergence of the series in n is established because the right hand side may be summed up. Thus we have proved I defined by (8.4) is in $(\mathcal{S}_d)^{-1}$ and established the bound

$$\left|T\mathrm{I}(\vec{\theta})\right| \leq \frac{1}{(2\pi(t-t_0))^{\frac{d}{2}}} \exp\left(\left|\vec{\theta}\right|_0 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{(t-t_0)}} \left|\vec{\theta}\right|_0 + (t-t_0) \int_{\mathbb{R}^d} \mathrm{d}^d|\mathrm{m}|(\alpha) \; e^{2|\vec{\alpha}|\sqrt{t-t_0}|\vec{\theta}|_0} \right)$$

for all $\vec{\theta} \in \mathcal{U}$.

Now we assume that also (8.3) is satisfied. Then it is useful to estimate C_n in (8.6). We use the elementary estimate

$$\alpha \cdot \beta \leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q \,, \qquad \alpha, \beta > 0, \; \frac{1}{p} + \frac{1}{q} = 1$$

to show

$$2\sqrt{t-t_0}|\vec{\theta}|_0|\vec{\alpha}_j| \leq \varepsilon|\vec{\alpha}_j|^{1+\delta} + \delta\varepsilon^{-1/\delta} \left(\frac{2\sqrt{t-t_0}|\theta|_0}{1+\delta} \right)^{\frac{1+\delta}{\delta}}$$

then

$$\int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d|\mathrm{m}|(\alpha_j) \; \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \leq$$

$$\leq \frac{1}{n!}(t-t_0)^n (2\pi(t-t_0))^{-\frac{d}{2}} \exp\left(\left|\vec{\theta}\right|_0^2 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{(t-t_0)}} \left|\vec{\theta}\right|_0 + n\delta\varepsilon^{-1/\delta} \left(\frac{(2\sqrt{t-t_0}|\theta|_0}{1+\delta} \right)^{\frac{1+\delta}{\delta}} \right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}^d} \mathrm{d}^d|\mathrm{m}|(\alpha) \; \exp(\varepsilon|\vec{\alpha}|^{1+\delta}) \right)^n$$

This shows that the assumption of Theorem 76 are satisfied so that the integrals in equation (8.4) are well defined Pettis integrals in $\mathcal{Y}'$. The series in n now is no problem in view of Corollary 75 since the right hand side of the above bound can be summed up. Thus $I \in \mathcal{Y}'$ (defined by (8.4)) and we have the bound

$$|T\mathrm{I}(\vec{\theta})| \leq (2\pi(t-t_0))^{-d/2} \cdot \exp\left(|\vec{\theta}|_0^2 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{t-t_0}}|\vec{\theta}|_0\right) \cdot$$

$$\cdot \exp\left((t-t_0) \cdot \exp\left(\delta\varepsilon^{-1/\delta}\left(\frac{2\sqrt{t-t_0}|\theta|_0}{1+\delta}\right)^{\frac{1+\delta}{\delta}}\right) \cdot \int_{\mathbb{R}^d} \mathrm{d}|\mathrm{m}|(\alpha)\exp(\varepsilon|\vec{\alpha}|^{1+\delta})\right) .$$

□

The above construction may be generalized to include also explicitly time dependent potentials.

Theorem 111 *Let* m *denote a complex measure on* $\mathbb{R}^d \times [t_0, t]$ *,* $d \geq 1$ *such that*

$$\int_{\mathbb{R}^d}\int_{t_0}^{t} e^{\varepsilon|\vec{\alpha}|}\ |\mathrm{m}|(\mathrm{d}^d\alpha, \mathrm{d}\tau) < \infty \tag{8.9}$$

Then

$$\mathrm{I} = \mathrm{I}_0 + \sum_{n=1}^{\infty}(-i)^n \int_{\Lambda_n}\int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{m}(\mathrm{d}^d\alpha_j, \mathrm{d}\tau_j)\ \mathrm{I}_0 \cdot \prod_{j=1}^{n} e^{i\vec{\alpha}_j\vec{x}(\tau_j)} \tag{8.10}$$

exists as in $(\mathcal{S}_d)^{-1}$ *i.e., as a generalized white noise functional.*

Proof. The proof can be done in the same way as in the previous theorem. But inequality (8.8) has to be modified

$$\int_{\Lambda_n}\int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} |\mathrm{m}|(\mathrm{d}^d\alpha_j, \mathrm{d}\tau_j)\ \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \leq$$

$$\leq (2\pi(t-t_0))^{-\frac{d}{2}} \exp\left(\left|\vec{\theta}\right|_0^2 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{(t-t_0)}}\left|\vec{\theta}\right|_0\right) \int_{\Lambda_n}\int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} |\mathrm{m}|(\mathrm{d}^d\alpha_j, \mathrm{d}\tau_j)\ e^{2|\vec{\alpha}_j|\sqrt{t-t_0}\left|\vec{\theta}\right|_0}$$

$$\leq \frac{1}{n!}(2\pi(t-t_0))^{-\frac{d}{2}} \exp\left(\left|\vec{\theta}\right|_0^2 + \frac{|\vec{x}-\vec{x}_0|}{\sqrt{(t-t_0)}}\left|\vec{\theta}\right|_0\right) \left(\int_{t_0}^{t}\int_{\mathbb{R}^d} |\mathrm{m}|(\mathrm{d}^d\alpha, \mathrm{d}\tau)\ e^{2|\vec{\alpha}|\sqrt{t-t_0}\left|\vec{\theta}\right|_0}\right)^n .$$

This shows that the integration and the summation in equation (8.10) are well defined in $(\mathcal{S}_d)^{-1}$. □

Remark. Note that the time dependence of our 'potentials' may be very singular here. Let us consider an admissible potential V in $\mathcal{F}(\mathbb{R}^d)$ represented by the measure $\tilde{\mathrm{m}}$

$$V(\vec{x}) = \int_{\mathbb{R}^d} e^{i\vec{\alpha}\vec{x}}\mathrm{d}^d\tilde{\mathrm{m}}(\alpha)$$

We wish to study a quantum mechanical system which is 'kicked' a finite number of times $s_j \in (t_0, t)$. This is done by multiplication with a delta measure in time. Thus we have to introduce

$$\mathrm{m}(\mathrm{d}^d\alpha, \mathrm{d}\tau) := \sum_j \tilde{\mathrm{m}}(\mathrm{d}^d\alpha) \cdot \delta_{s_j}(\mathrm{d}\tau),$$

which clearly fulfills (8.9).

Chapter 9

A new look at Feynman Hibbs

9.1 Transition amplitudes

Since [FH65] it is quite common to discuss so called transition amplitudes which in our framework would read

$$\mathbb{E}(F \cdot \mathrm{I})$$

where $F : \vec{x}(\cdot) \mapsto F(\vec{x}(\cdot))$ is a function of the path. Of course this is well defined (by writing $\mathbb{E}(F \cdot \mathrm{I}) = \langle\langle \mathrm{I}, F \rangle\rangle$) whenever $F \in (\mathcal{S}_d)^1$. Since this is too restrictive for relevant cases we shall discuss some special extensions of this pairing before we start to discuss the physical interpretation of the transition amplitudes we have defined. Throughout this chapter we assume the setting of Theorem 110 for simplicity.

First let us introduce some convenient notations which will help to keep formulas little shorter:

$$\vec{\zeta} := \sum_{j=1}^{n} \vec{\alpha}_j \mathbb{1}_{[t_0,\tau_j)}, \qquad \vec{\chi} := \sum_{j=1}^{n} \vec{\alpha}_j (\tau_j - t_0)$$

and

$$\vec{X} := \vec{x} - \vec{x}_0, \qquad \Delta = [t_0, t]\,, \qquad |\Delta| = t - t_0\,.$$

For later use we collect some useful formulas.

Lemma 112 *Let* $\vec{\eta}, \vec{\eta}_1, \vec{\eta}_2 \in \mathcal{S}_d(\mathbb{R})\,,\, \vec{\theta} \in \mathcal{S}_{d,\mathbb{C}}(\mathbb{R})$ *then*

$$T(\langle \cdot, \vec{\eta} \rangle \Phi_n)(\vec{\theta}) = -\left[\langle \vec{\eta}, \vec{\theta} + \vec{\zeta} \rangle - \tfrac{1}{t-t_0} \int_\Delta \vec{\eta} \left\{ \int_\Delta \vec{\theta} + \vec{\chi} + \vec{X} \right\}\right] T\Phi_n(\vec{\theta}) \tag{9.1}$$

and

$$T(\langle \cdot, \vec{\eta}_1 \rangle \langle \cdot, \vec{\eta}_2 \rangle \Phi_n)(\vec{\theta}) =$$

$$= T\Phi_n(\vec{\theta}) \cdot \left\{ i(\vec{\eta}_1, \vec{\eta}_2) + \frac{i}{|\Delta|} \int_\Delta \vec{\eta}_1 \int_\Delta \vec{\eta}_2 + \left[(\vec{\eta}_1, \vec{\theta} + \vec{\zeta}) - \tfrac{1}{|\Delta|} \int_\Delta \vec{\eta}_1 \left\{ \int_\Delta \vec{\theta} + \vec{\chi} + \vec{X} \right\}\right] \cdot \right.$$

$$\left. \cdot \left[(\vec{\eta}_2, \vec{\theta} + \vec{\zeta}) - \tfrac{1}{|\Delta|} \int_\Delta \vec{\eta}_2 \left\{ \int_\Delta \vec{\theta} + \vec{\chi} + \vec{X} \right\}\right] \right\} \tag{9.2}$$

of course $\langle \cdot, \vec{\eta} \rangle \cdot \Phi_n$ *and* $\langle \cdot, \eta_1 \rangle \langle \cdot, \eta_2 \rangle \cdot \Phi_n \in (\mathcal{S}_d)'$.

Proof. Use the formula

$$T(\langle\cdot,\vec\eta\rangle^k\Phi_n)(\vec\theta) = \left(\frac{\mathrm{d}}{i\,\mathrm{d}\lambda}\right)^k T\Phi_n(\vec\theta+\lambda\vec\eta)\Big|_{\lambda=0}$$

and polarization identity. □

Furthermore we need the possibility for an intermediate pinning of the paths, which is prepared by the next proposition. This is a generalization of formula (5.6).

Proposition 113 *The distribution*

$$\Phi_n(\vec x,t|\vec x_0,t_0)\cdot\prod_{l=1}^m \delta^d(\vec x(t_l)-\vec x_l)\ , \qquad t_0<t_l\le t,\ \ \vec x_l\in\mathbb{R}^d;\ 1\le l\le m$$

is well defined in $(\mathcal{S}_d)'$ with T-transform

$$T\left(\Phi_n(\vec x,t|\vec x_0,t_0)\prod_{l=1}^m \delta^d(\vec x(t_l)-\vec x_l)\right)(\vec\theta) = e^{i\frac{m}{2}|\vec\theta|^2}\prod_{l=1}^{m+1} T\Phi_{n_l}(\vec x_l,t_l|\vec x_{l-1},t_{l-1})(\vec\theta)$$

here $\vec x_{m+1}\equiv\vec x$, $t_{m+1}\equiv t$, $n_l=\#\{j\mid t_{l-1}<\tau_j\le t_l\}$ and Φ_{n_l} depends on the parameters $\{\alpha_j,\tau_j\mid t_{l-1}<\tau_j\le t_l\}$.

Proof. To simplify the calculation we propose the use of

$$\prod_{l=1}^{m+1}\delta^d(\vec x(t_l)-\vec x_l)\prod_{j=1}^n \exp\left(i\vec\alpha_j(\vec x_0+\langle\cdot,1\!\!1_{[t_0,\tau_j)}\rangle)\right) =$$

$$=\prod_{l=1}^{m+1}\left\{\delta^d(\vec x(t_l)-\vec x_l)\prod_{t_{l-1}<\tau_j\le t_l}\exp\left(i\vec\alpha_j(\vec x_{l-1}+\langle\cdot,1\!\!1_{[t_{l-1},\tau_j)}\rangle)\right)\right\}$$

which is simply checked by a comparison T-transforms. In view of formula (5.6) $\mathrm{J}\prod_{l=1}^{m+1}\delta^d(\vec x(t_l)-\vec x_l)$ is well defined due to its explicit T-transform. Then the starting point of the calculation is

$$T\left(\Phi_n\prod_{l=1}^{m+1}\delta^d(\vec x(t_l)-\vec x_l)\right)(\vec\theta) =$$

$$T\left(\mathrm{J}\prod_{l=1}^{m+1}\delta^d(\vec x(t_l)-\vec x_l)\right)\left(\vec\theta+\sum_{l=1}^{m+1}\sum_{\{j|t_{l-1}<\tau_j\le t_l\}}\vec\alpha_j 1\!\!1_{[t_{l-1},\tau_j)}\right)\exp\left(\sum_{l=0}^{m+1}\vec x_{l-1}\sum_{\{j|t_{l-1}<\tau_j\le t_l\}}\vec\alpha_j\right)$$

which can be evaluated without problems. □

Now we continue the discussion of Feynman integrands defined in Theorem 110. Pointwise products of $\mathrm{I}=\mathrm{I}(\vec x,t|\vec x_0,t_0)$ with $\vec x(s)$, $\dot{\vec x}(s)$ and $\ddot{\vec x}(s)$ for $t_0<s<t$ have natural interpretations in usual quantum mechanics as we will see in the next section.

In the white noise framework $\vec x(s)$ is represented as $\vec x_0+\langle\cdot,1\!\!1_{[t_0,s)}\rangle$, $\dot{\vec x}(s)$ as $\langle\cdot,\delta_s\rangle$ and by a formal partial integration we obtain $\ddot{\vec x}(s)=-\langle\cdot,\delta_s'\rangle$. So we have to study products of the form $\langle\cdot,\vec T\rangle\cdot\mathrm{I}$ for suitable distributions $\vec T\in\mathcal{S}_d'$. This can conveniently be done by approximating the first factor by test functions.

Definition 114 *Let* $\vec{T} \in \mathcal{S}_d'$ *and* $\{\vec{\eta}_l \in \mathcal{S}_d\,,\, l \in \mathbb{N}\}$ *a sequence of test functions such that* $\lim_{l\to\infty} \vec{\eta}_l = \vec{T}$. *We define*

$$\langle \cdot, \vec{T}\rangle \cdot \mathrm{I} := \lim_{l\to\infty} \langle \cdot, \vec{\eta}_l\rangle \cdot \mathrm{I}$$

if the limit exists and is independent of the sequence.

That this is fulfilled in the three mentioned cases is shown in the following proposition.

Proposition 115 *In the sense of the above definition we have* $x_k(s)\cdot \mathrm{I}\,,\, \dot{x}_k\,(s)\cdot\mathrm{I}\,,\, \ddot{x}_k(s)\cdot \mathrm{I} \in (\mathcal{S}_d)^{-1}$ *(the index* $1 \le k \le d$ *indicates the* k^{th}*-component) and*

$$x_k\,(s)\cdot \mathrm{I} = \sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j)\; x_k(s)\cdot \Phi_n \tag{9.3}$$

$$\dot{x}_k\,(s)\cdot \mathrm{I} = \sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j)\; \dot{x}_k(s)\cdot \Phi_n\ .$$

Proof. Since $\langle\cdot,\vec{\eta}_l\rangle$ is in $(\mathcal{S}_d)$ pointwise multiplication intertwines with Bochner integration and the infinite sum:

$$\langle\cdot,\vec{\eta}_l\rangle \mathrm{I} = \sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j)\; \langle\cdot,\vec{\eta}_l\rangle \cdot \Phi_n\ .$$

In the case $\vec{\eta}_l \to 1\!\!1_{[t_0,s)}\vec{e}_k$ ($\{\vec{e}_k, 1 \le k \le d\}$ denotes the canonical basis of $\mathbb{R}^d$) equation (9.1) gives the estimate

$$\left|T\left(\langle\cdot,\vec{\eta}_l\rangle\cdot\Phi_n\right)(\vec{\theta})\right| \le 2\,|\vec{\eta}_l| \left(\left|\vec{\theta}\right| + \sqrt{t-t_0}\sum_{j=1}^{n}|\alpha_{j,k}| + \frac{|\vec{X}|}{2\sqrt{t-t_0}}\right)\left|T\Phi_n(\vec{\theta})\right|$$

$$\le 2\,|\vec{\eta}_l| \left(\left|\vec{\theta}\right| + \sqrt{t-t_0}\sum_{j=1}^{n}|\alpha_{j,k}| + \frac{|\vec{X}|}{2\sqrt{t-t_0}}\right) \mathrm{C}_n(\vec{\alpha}_1,\dots,\vec{\alpha}_n,\vec{\theta})$$

where $\alpha_{j,k} = \vec{\alpha}_j\cdot\vec{e}_k$. It is easy to see that for $\vec{\theta} \in \mathcal{U}_{0,r}$ (defined in (8.7) the right hand side of the estimate is integrable on $\Lambda_n \times \mathbb{R}^{nd}$ w.r.t. $\mathrm{d}^n\tau \prod_{j=1}^n \mathrm{d}^d|\mathrm{m}|(\alpha_j)$ such that

$$\left|T(\langle\cdot,\vec{\eta}_l\rangle \mathrm{I})(\vec{\theta})\right| \le |\vec{\eta}_l|\cdot K_n$$

on $\mathcal{U}_{0,r}$ for some rapidly decreasing sequence K_n. This is sufficient to ensure the convergence of the sequence $l \to \langle\cdot,\vec{\eta}_l\rangle\mathrm{I}$. The limit of the sequence $l \to (x_{0,k} + \langle\cdot,\vec{\eta}_l\rangle)\mathrm{I}$ is given by (9.3) with

$$T(\vec{x}(s)\cdot\Phi_n)(\vec{\theta}) = \Big\{\frac{t-s}{t-t_0}\left(\vec{x}_0 - \int_{t_0}^{s}\vec{\theta} - \sum_{j=1}^{k}\vec{\alpha}_j(\tau_j - t_0)\right)$$

$$+\frac{s-t_0}{t-t_0}\left(\vec{x} + \int_{s}^{t}\vec{\theta} - \sum_{j=k+1}^{n}\vec{\alpha}_j(t-\tau_j)\right)\Big\} T\Phi_n(\vec{\theta})\ .$$

By a similar argument we discuss the product with $\dot{x}_k(s)$. Here the basic formula is

$$T(\dot{x}_k(s)\Phi_n)(\vec{\theta}) = \Big(-\theta_k(s) - \sum_{j=1}^{k} \alpha_{j,k} \mathbb{1}_{[t_0,\tau_j)}(s) +$$

$$+\frac{1}{t-t_0}\left(\int_{t_0}^{t} \theta_k + \sum_{j=1}^{n} \alpha_{j,k}(\tau_j - t_0) + x_k - x_{0,k}\right)\Big) T\Phi_n(\vec{\theta}) \ .$$

Again we can find a neighborhood of zero in $\mathcal{S}_d$ such that the resulting estimate

$$\left| T(\dot{x}_k(s)\Phi_n)(\vec{\theta}) \right| \leq 2\left(\left|\vec{\theta}\right|_\infty + \sum_{j=1}^{n} |\alpha_{j,k}| + \frac{|\vec{X}|}{2\cdot|t-t_0|}\right) \mathrm{C}_n(\vec{\alpha}_1, \dots \vec{\alpha}_n, \vec{\theta}) \tag{9.4}$$

is integrable and can be summed up ($|\cdot|_\infty$denotes the sup–norm). In particular this is sufficient to show that the requirement of Definition 114 is fulfilled.

In the third case we have

$$T(\ddot{x}_k(s)\Phi_n)(\vec{\theta}) = \left(-\dot{\theta}_k(s) + \sum_{j=1}^{n} \alpha_{j,k} \cdot \delta_s(\tau_j) \right) T\Phi_n(\vec{\theta}) \ .$$

The term $\dot{\theta}(s) \cdot T\Phi_n(\vec{\theta})$ causes no problem. To ensure integrability one integration (w.r.t. τ_j) has to be regarded as integration with respect to Dirac measure. This is possible because the bound (8.6) is independent of τ_j. The rest of the proof is as before. □

Note. In terms of Wick products we may write

$$\langle \cdot, \vec{\eta} \rangle \cdot \mathrm{I} = i\left(\langle \cdot, \vec{\eta} \rangle - \frac{1}{t-t_0}\int_{t_0}^{t} \vec{\eta}\left(\langle \cdot, \mathbb{1}_{[t_0,t)} \rangle - \vec{x} + \vec{x}_0 \right) \right) \diamond \mathrm{I}$$

$$+\sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j) \left(\sum_{j=1}^{n} \vec{\alpha}_j \left(\frac{\tau_j - t_0}{t-t_0} \int_{t_0}^{t} \vec{\eta} - \int_{t_0}^{\tau_j} \vec{\eta} \right) \right) \Phi_n \ .$$

In fact the bound (9.4) proves that $\dot{x}_k(s) \cdot \Phi_n$ is integrable with respect to the product measure $\mathrm{d}^n\tau \prod_{j=1}^{n} \mathrm{d}^d|\mathrm{m}|(\alpha_j) \cdot \mathrm{d}s$ on the domain $\Lambda_n \times \mathbb{R}^{dn} \times [t_0, t]$. This implies that $\dot{x}_k(s) \cdot \mathrm{I}$ is Bochner integrable w.r.t. $\mathrm{d}s$, i.e.,

$$\int_{t_0}^{s} (\dot{x}_k(s) \cdot \mathrm{I}) \, \mathrm{d}s = x_k(s) \cdot \mathrm{I} - x_{0,k}\mathrm{I} \ ,$$

in particular

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}(\vec{x}(s) \cdot \mathrm{I}) = \mathbb{E}(\dot{\vec{x}}\ (s) \cdot \mathrm{I}) \ . \tag{9.5}$$

Furthermore we need pointwise products of the type $F(\vec{x}(s)) \cdot \mathrm{I}$ for fixed $s \in (t_0, t]$ and appropriate functions F. Since $F(\vec{x}(s))$ is not in $(\mathcal{S}_d)^1$ we have to give an extension of this pointwise product. We will give two alternative definitions which have different advantages such that later we can use the most convenient one.

Definition 116 *Let* $F \in \mathcal{F}(\mathbb{R}^d)$ *such that* $F(\vec{x}) = \int_{\mathbb{R}^d} e^{i\vec{\alpha}\vec{x}} \mathrm{d}^d \mathrm{m}_F(\alpha)$ *and* $\Phi \in (\mathcal{S}_d)^{-1}$*. If the product* $\Phi \cdot e^{i\vec{\alpha}\vec{x}(s)}$ *is well defined in* $(\mathcal{S}_d)^{-1}$ *and the Bochner integral with respect to* $\mathrm{d}^d|\mathrm{m}_F|(\alpha)$ *exists we define*

$$F(\vec{x}(s)) \cdot \Phi := \int_{\mathbb{R}^d} \Phi \cdot e^{i\vec{\alpha}\vec{x}(s)} \mathrm{d}^d \mathrm{m}_F(\alpha) \qquad .$$

Remark. One can show that this definition extends the usual definition of pointwise multiplication. Without loss of generality let $F \in \mathcal{F}(\mathbb{R})$ and $\eta \in \mathcal{S}$. Further we assume that $|\mathrm{m}_F|$ satisfies the following integrability condition: $\forall K > 0$

$$\int e^{K\alpha^2} \mathrm{d}|\mathrm{m}_F|(\alpha) < \infty \ .$$

Then

$$\left| \int e^{i\alpha\langle z,\eta\rangle} \mathrm{dm}_F(\alpha) \right| \leq \left(\int e^{K\alpha^2} \mathrm{d}|\mathrm{m}_F|(\alpha) \right) \exp\left(\frac{1}{4K} |z|^2_{-p} |\eta|^2_p \right)$$

for all $p > 0$. This shows that $F(\langle \cdot, \eta \rangle) = \int e^{i\alpha\langle \cdot,\eta\rangle} \mathrm{dm}_F(\alpha)$ is in $\mathcal{E}^2_{\min}(\mathcal{S}')$ which is the same as $(\mathcal{S})$. For this class of multipliers the coincidence of the two definitions can now easily be seen. Let $\Phi \in (\mathcal{S})'$ be arbitrary and $\varphi \in (\mathcal{S})$

$$\begin{aligned} \left\langle\!\!\left\langle \int \Phi \cdot e^{i\alpha\langle \cdot,\eta\rangle} \mathrm{dm}_F(\alpha), \varphi \right\rangle\!\!\right\rangle &= \int \langle\!\langle \Phi, \varphi \cdot e^{i\alpha\langle \cdot,\eta\rangle} \rangle\!\rangle \, \mathrm{dm}_F(\alpha) \\ &= \left\langle\!\!\left\langle \Phi, \varphi \cdot \int e^{i\alpha\langle \cdot,\eta\rangle} \mathrm{dm}_F(\alpha) \right\rangle\!\!\right\rangle \\ &= \langle\!\langle \Phi, \varphi \cdot F(\langle \cdot, \eta \rangle) \rangle\!\rangle \ . \end{aligned}$$

Lemma 117 *Let* $F \in \mathcal{F}(\mathbb{R}^d)$ *be admissible i.e., there exists* $\varepsilon > 0$ *such that* $\int_{\mathbb{R}^d} e^{\varepsilon|\vec{\alpha}|} \mathrm{d}^d |\mathrm{m}_F|(\alpha)$ *is finite. Then* $F(\vec{x}(s)) \cdot \mathrm{I} \in (\mathcal{S}_d)^{-1}$ *in the sense of the above definition. Moreover*

$$F(\vec{x}(s)) \cdot \mathrm{I} = \sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d \mathrm{m}(\alpha_j) \int_{\mathbb{R}^d} \mathrm{d}^d \mathrm{m}_F(\beta) \ \Phi_n \cdot e^{i\vec{\beta}\vec{x}(s)} \ .$$

The proof is a simple modification of the proof of Theorem 110.

A different class of multipliers is obtained by the following definition.

Definition 118 *Let* $F : \mathbb{R}^d \to \mathbb{C}$ *denote a Borel measurable function and* $\Phi \in (\mathcal{S}_d)^{-1}$*. If the product* $\Phi \cdot \delta^d(\vec{x}(s) - \vec{y})$ *is well defined and Bochner integrable with respect to* $|F(\vec{y})| \, \mathrm{d}^d y$ *then*

$$F(\vec{x}(s)) \cdot \Phi := \int_{\mathbb{R}^d} F(\vec{y}) \, \Phi \cdot \delta^d(\vec{x}(s) - \vec{y}) \, \mathrm{d}^d y \ .$$

Theorem 119 *Let* $F : \mathbb{R}^d \to \mathbb{C}$ *such that*

$$\int e^{\varepsilon|\vec{y}|} \, |F(\vec{y})| \, \mathrm{d}^d y < \infty$$

for some $\varepsilon > 0$*. Then* $F(\vec{x}(s)) \cdot \mathrm{I}$ *is defined in the sense of the above definition. Moreover*

$$T\left(F(\vec{x}(s)) \cdot \mathrm{I}\right)(\vec{\theta}) = e^{\frac{i}{2}|\vec{\theta}|^2} \int_{\mathbb{R}^d} T\mathrm{I}(\vec{x}, t|\vec{y}, s)(\vec{\theta}) \, F(\vec{y}) \, T\mathrm{I}(\vec{y}, s|\vec{x}_0, t_0)(\vec{\theta}) \, \mathrm{d}^d y \ . \qquad (9.6)$$

Remark. If the last formula is evaluated at $\vec{\theta} = \vec{0}$ we obtain

$$\mathbb{E}(F(\vec{x}(s)) \cdot \mathrm{I}) = \int K(\vec{x}, t|\vec{y}, s)\, F(\vec{y})\, K(\vec{y}, s|\vec{x}_0, t_0)\, \mathrm{d}^d y \tag{9.7}$$

which is one of the key formulas in Feynman Hibbs [FH65].

Proof. The expression $\mathrm{I} \cdot \delta^d(\vec{x}(s) - \vec{y})$ has a natural sense in view of Proposition 113. More precisely we have

$$T\left(\Phi_n \cdot \delta(\vec{x}(s) - \vec{y})\right)(\vec{\theta}) = \sum_{k=0}^{n} \mathbb{1}_{(\tau_k, \tau_{k+1}]}(s)\, e^{\frac{i}{2}|\vec{\theta}|^2} T\Phi_{n-k}(\vec{x}, t|\vec{y}, s)(\vec{\theta})\, T\Phi_k(\vec{y}, s|\vec{x}_0, t_0)(\vec{\theta}) \tag{9.8}$$

which can be estimated as follows (similar to (8.6))

$$T\left(\Phi_n \cdot \delta^d(\vec{x}(s) - \vec{y})\right)(\vec{\theta}) \le n \cdot \left(4\pi^2(t-s)(s-t_0)\right)^{-d/2}$$
$$\cdot \exp\left(|\vec{\theta}|^2 + 2\sqrt{t-t_0}|\vec{\theta}| \sum_{j=1}^{n} |\vec{\alpha}_j| + |\vec{\theta}||\vec{y}| \left(\tfrac{1}{\sqrt{t-s}} + \tfrac{1}{\sqrt{s-t_0}}\right) + |\vec{\theta}| \left(\tfrac{|\vec{x}|}{\sqrt{t-s}} + \tfrac{|\vec{y}|}{\sqrt{s-t_0}}\right)\right) .$$

Analogous to the proof of Theorem 110 we can find a neighborhood of zero in $\mathcal{S}_d$ such that this bound is integrable with respect to $\int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d|\mathrm{m}|(\alpha_j) \int_{\mathbb{R}^d} |F(\vec{y})| \mathrm{d}^d y$. In particular

$$\mathrm{I} \cdot \delta^d(\vec{x}(s) - \vec{y}) = \sum_{n=0}^{\infty} (-i)^n \int_{\Lambda_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j)\, \Phi_n \cdot \delta^d(\vec{x}(s) - \vec{y})$$

is well defined and integrable w.r.t. $|F(\vec{y})|\, \mathrm{d}^d y$.

Now we deduce formula (9.6). Note that each term in the sum in equation (9.8) factorizes in one factor depending on $\tau_j, \vec{\alpha}_j$, $1 \le j \le k$ and a second factor depending on the remaining τ_j, α_j , $k+1 \le j \le n$. So it is natural to use the corresponding decomposition of the domain of integration

$$\Lambda_n(t, t_0) = \bigcup_{k=0}^{n} \Lambda'_{n-k}(t, s) \times \Lambda_k(s, t_0)$$

where the prime indicates that $\Lambda'_{n-k}(t, s)$ is a set in the $\tau_{k+1} \times \ldots \times \tau_n$–plane. Then

$$\int_{\Lambda_n} T\left(\Phi_n \delta^d(\vec{x}(s) - \vec{y})\right)(\vec{\theta})\, \mathrm{d}^n\tau =$$
$$= e^{\frac{i}{2}|\theta|^2} \sum_{k=0}^{n} \int_{\Lambda'_{n-k}(t,s)} T\Phi_{n-k}(\vec{x}, t|\vec{y}, s)(\vec{\theta}) \cdot \int_{\Lambda_k(s,t_0)} T\Phi_k(\vec{y}, s|\vec{x}_0, t_0)(\vec{\theta}) .$$

This implies

$$T\left(\mathrm{I} \cdot \delta^d(\vec{x}(s) - \vec{y})\right)(\vec{\theta}) = e^{\frac{i}{2}|\vec{\theta}|^2} T\mathrm{I}(\vec{x}, t|\vec{y}, s)(\vec{\theta}) \cdot T\mathrm{I}(\vec{y}, s|\vec{x}_0, t_0)(\vec{\theta}) . \tag{9.9}$$

Integration with respect to $F(\vec{y}) \cdot \mathrm{d}^d y$ gives (9.6). □

Remark. Since the situation in Proposition 113 is more general, we can show the following generalization of Theorem 119. Let $t_0 < s_1 < s_2 < \ldots < s_n < t$ and $F_j : \mathbb{R}^d \to \mathbb{C}$, $1 \leq j \leq n$ as in Theorem 119. Then $\prod_{j=1}^n F_j(\vec{x}(s_j)) \cdot \mathrm{I}$ is in $(\mathcal{S}_d)^{-1}$. Moreover (9.6) generalizes to

$$T\left(\prod_{j=1}^n F_j(\vec{x}(s_j)) \cdot \mathrm{I}\right)(\vec{\theta}) = \tag{9.10}$$
$$= e^{\frac{in}{2}|\vec{\theta}|^2} \int_{\mathbb{R}^{dn}} T\mathrm{I}(\vec{x},t|\vec{y}_n,s_n)(\vec{\theta}) \prod_{j=1}^n F_j(\vec{y}_j)\, T\mathrm{I}(\vec{y}_j,s_j|\vec{y}_{j-1},s_{j-1})(\vec{\theta})\, \mathrm{d}^d y_j \ .$$

There are two relevant cases which are not covered by the previous theorem. First of all we want to discuss (9.7) for the constant function $F = 1$. In other words we want to apply the identity

$$\int_{\mathbb{R}^d} \delta^d(\vec{x}(s) - \vec{y})\, \mathrm{d}^d y = 1$$

to (9.9). The second case is $F(\vec{y}) = y_k$. Here we want to compare the two definitions 114 and 118. In both cases the integral in (9.7) is not absolutely convergent (easily seen in the free case) but has a sense as a Fresnel integral. Since the notion of a Fresnel integral of a family white noise distributions is not yet developed we will use a regularization procedure.

Proposition 120 *Let $F_\varepsilon(\vec{y}) = e^{-\frac{\varepsilon^2}{2}y^2}$ and $s \in (t_0, t)$. Then the following two limits exist in $(\mathcal{S}_d)^{-1}$ and are given by*
1)

$$\lim_{\varepsilon \to 0} (F_\varepsilon(\vec{x}(s)) \cdot \mathrm{I}) = \mathrm{I}$$

2)

$$\lim_{\varepsilon \to 0} (x_k(s) F_\varepsilon(\vec{x}(s)) \cdot \mathrm{I}) = x_k(s) \cdot \mathrm{I}$$

Proof. Let $g(\vec{y})$ be a real valued function which either is identical 1 or equal to y_k. By definition

$$T(g(\vec{x}(s))F_\varepsilon(\vec{x}(s)) \cdot \mathrm{I})(\theta) = e^{\frac{i}{2}|\vec{\theta}|^2} \int_{\mathbb{R}^d} g(\vec{y})F_\varepsilon(\vec{y})\, T\mathrm{I}(\vec{x},t|\vec{y},s)(\vec{\theta})\, T\mathrm{I}(\vec{y},s|\vec{x}_0 t_0)(\vec{\theta})\, \mathrm{d}^d y$$
$$= \sum_{n=0}^\infty \sum_{k=0}^n (-i)^n \int_{\Lambda'_{n-k}(t,s)\times\Lambda_k(s,t_0)} \mathrm{d}^n \tau \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \mathrm{d}^d \mathrm{m}(\alpha_j)\, G_\varepsilon$$

with

$$G_\varepsilon = \int_{\mathbb{R}^d} g(\vec{y})F_\varepsilon(\vec{y})\, T\Phi_{n-k}(\vec{x},t|\vec{y},s)(\vec{\theta})\, T\Phi_k(\vec{y},s|\vec{x}_0 t_0)(\vec{\theta})\, \mathrm{d}^d y \ .$$

The Gaussian integral G_ε can be evaluated explicitly:

$$G_\varepsilon = \left(\frac{2\pi}{a_\varepsilon}\right)^{d/2} g\left(\frac{i\vec{b}}{a_\varepsilon}\right) T\Phi_{n-k}(\vec{x},t|\vec{0},s)(\vec{\theta})\, T\Phi_k(\vec{0},s|\vec{x}_0,t_0)(\vec{\theta}) \cdot \exp\left(-\frac{\vec{b}^2}{2a_\varepsilon}\right)$$

with

$$a_\varepsilon = \varepsilon^2 - \frac{i(t-t_0)}{(t-s)(s-t_0)}$$

$$\vec{b} = \frac{1}{s-t_0}\left(\int_{t_0}^{s} \vec{\theta} + \sum_{j=1}^{k} \vec{\alpha}_j(\tau_j - t_0) - \vec{x}_0\right) - \frac{1}{t-s}\left(\int_{s}^{t} \vec{\theta} - \sum_{j=k+1}^{n} \vec{\alpha}_j(t-\tau_j) + \vec{x}\right) .$$

From this the following bound can be calculated

$$|G_\varepsilon| \le (2\pi|\Delta|)^{d/2} \mathrm{C}_{n-k}(\vec{x},t|\vec{0},s)\ \mathrm{C}_k(\vec{0},s|\vec{x}_0,t_0) \cdot \left| g\left(\frac{i\vec{b}}{a}\right)\right|$$

$$\cdot \exp \frac{|\Delta|}{2}\left(|\vec{\theta}|_\infty^2 + 2|\vec{\theta}|_\infty \left(\sum_{j=1}^{n} |\vec{\alpha}_j| + \frac{|\vec{x}|}{|\Delta|} + \frac{|\vec{x}_0|}{|\Delta|}\right)\right)$$

where $\left| g\left(\frac{i\vec{b}}{a}\right)\right|$ is either equal to 1 or

$$\left| g\left(\frac{i\vec{b}}{a}\right)\right| \le \left(|\vec{x}_0| + |\vec{x}| + |\Delta||\vec{\theta}|_\infty + |\Delta| \sum_{j=1}^{n} |\vec{\alpha}_j|\right) .$$

It is easy to see that in both cases this bound can be integrated for $\vec{\theta}$ in some neighborhood of zero w.r.t. $\mathrm{d}^n\tau \prod_{j=1}^{n} \mathrm{d}^d|\mathrm{m}|(\alpha_j)$ on $\mathbb{R}^{nd} \times \Lambda'_{n-k}(t,s) \times \Lambda_k(s,t_0)$ and stays finite after $\sum_{n=0}^{\infty}\sum_{k=0}^{n}$. Thus the limit $\varepsilon \to 0$ exists in both cases. The limit itself can be identified by an elementary calculation of $\lim_{\varepsilon\to 0} G_\varepsilon$. □

Consequences. The above proposition together with (9.7) gives

$$\begin{aligned} K(\vec{x},t|\vec{x}_0,t_0) &= \mathbb{E}(\mathrm{I}(\vec{x},t|\vec{x}_0,t_0)) \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbb{E}(\mathrm{I}(\vec{x},t|\vec{y},s))\ F_\varepsilon(\vec{y})\ \mathbb{E}(\mathrm{I}(\vec{y},s|\vec{x}_0,t_0))\ \mathrm{d}^d\vec{y} \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} K(\vec{x},t|\vec{y},s)\ F_\varepsilon(\vec{y})\ K(\vec{y},s|\vec{x}_0,t_0)\ \mathrm{d}^d y \qquad . \end{aligned}$$

We will use this as a substitute of Feynman's

$$\text{“ } K(\vec{x},t|\vec{x}_0,t_0) = \int_{\mathbb{R}^d} K(\vec{x},t|\vec{y},s)\ K(\vec{y},s|\vec{x}_0,t_0)\ \mathrm{d}^d y \text{ ”}$$

which is not absolutely convergent. The second consequence we want to mention is

$$\mathbb{E}(\vec{x}(s)\cdot \mathrm{I}) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} K(\vec{x},t|\vec{y},s)\ \vec{y}\, F_\varepsilon(\vec{y})\ K(\vec{y},s|\vec{x}_0,t_0)\ \mathrm{d}^d y \qquad (9.11)$$

which allows to calculate the transition element directly from the propagator.

9.2 Relation to operator notation

In usual quantum mechanics time evolution can be represented by an unitary group $U(t, t_0)$ acting on a suitable Hilbert space. The infinitesimal generator of $U(t, t_0)$ is assumed to be the Hamiltonian H. In the Schrödinger representation the matrix element of $U(t, t_0)$ is given by the propagator

$$\langle \vec{x} | U(t, t_0) | \vec{x}_0 \rangle = K(\vec{x}, t | \vec{x}_0, t_0) \ .$$

(We have not proved this explicitly since this question is discussed in [Ga74] in great detail.) The above formula may be viewed as the standard connection of path integral techniques to usual quantum mechanics.

For our discussion we will choose the Heisenberg picture where states are time independent and observables evolve in time according to the time evolution operator. Concretely the position operator is given by

$$\vec{q}(t) = U^*(t, t_0)\, \vec{q}\, U(t, t_0)$$

with

$$\vec{q}\, |\vec{x}\rangle = \vec{x}\, |\vec{x}\rangle$$

and the star denotes the adjoint operator. Now we are ready to connect the transition amplitudes from the previous section to quantum mechanical observables. From (9.11) it follows

$$\begin{aligned}
\mathbb{E}(\vec{x}(s) \cdot \mathrm{I}) &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \langle \vec{x} | U(t, s) | \vec{y} \rangle\, F_\varepsilon(\vec{y})\, \vec{y}\, \langle \vec{y} | U(s, t_0) | \vec{x}_0 \rangle \, \mathrm{d}^d y \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \langle \vec{x} | U(t, s) F_\varepsilon(\vec{q})\, \vec{q}\, | \vec{y} \rangle \, \langle \vec{y} | U(s, t_0) | \vec{x}_0 \rangle \, \mathrm{d}^d y \\
&= \lim_{\varepsilon \to 0} \langle \vec{x} | U(t, s)\, F_\varepsilon(\vec{q})\, \vec{q}\, U(s, t_0) | \vec{x}_0 \rangle \\
&= \langle \vec{x} | U(t, t_0)\, U^*(s, t_0) \vec{q} U(s, t_0) | \vec{x}_0 \rangle \\
&= \langle \vec{x} |\, U(t, t_0)\, \vec{q}(s)\, | \vec{x}_0 \rangle \ .
\end{aligned}$$

More generally we can show, based on (9.10)

$$\mathbb{E}(\vec{x}(s_1) \ldots \vec{x}(s_n) \cdot \mathrm{I}) = \langle \vec{x} | U(t, t_0)\, \mathsf{T} \vec{q}(s_1) \ldots \vec{q}(s_n)\, | \vec{x}_0 \rangle \ ,$$

where the usual time ordering of operators appears $\mathsf{T} \vec{q}(s_1) \ldots \vec{q}(s_n) = \vec{q}(s_n) \ldots \vec{q}(s_1)$ if $s_n > s_{n-1} > \ldots > s_1$.

Before $\mathbb{E}(\dot{\vec{x}}(s) \cdot \mathrm{I})$ is discussed we need one more assumption. Assume the Hamiltonian H not to be explicitly time dependent. Then the Heisenberg equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}s} \vec{q}(s) = \frac{i}{\hbar} [H, \vec{q}(s)]$$

holds where the square brackets denote the commutator. If furthermore the Hamiltonian is of the form

$$H = \frac{1}{2m} \vec{p}^{\,2} + V(\vec{q})$$

we can use the relation

$$[p_l^n, q_k] = -n i \hbar p_l^{n-1} \delta_{k,l}$$

to find

$$[H, \vec{q}(s)] = -i\frac{\hbar}{m}\vec{p}(s) \ .$$

Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\vec{q}(s) = \frac{1}{m}\vec{p}(s) \ .$$

The starting point of the following calculation is formula (9.5):

$$\mathbb{E}(\dot{\vec{x}}\ (s)\cdot \mathrm{I}) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}(\vec{x}(s)\cdot \mathrm{I}) = \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{x}|U(t,t_0)\,\vec{q}(s)\,|\vec{x}_0\rangle = \frac{1}{m}\langle \vec{x}|U(t,t_0)\,\vec{p}(s)\,|\vec{x}_0\rangle \ .$$

To give more evidence on these relations between transition elements and operator notation we will try to identify the canonical commutation relations in the language of transition elements.

9.3 A functional form of the canonical commutation relations

A well-known fact from quantum mechanics is the non commutativity of momentum and position operators at equal times. This seems to have no direct translation in a path-integral formulation of quantum mechanics. But on a heuristic level Feynman and Hibbs [FH65] found an argument to show that $\mathbb{E}(\dot{x}(s+\varepsilon)x(s)\mathrm{I}) \neq \mathbb{E}(\dot{x}(s-\varepsilon)x(s)\mathrm{I})$ for infinitesimal small ε and that the difference is given by the commutator. In this section we will prove this fact rigorously for the class of potentials introduced in the previous chapter.

First of all we use the convergence theorem to extend the validity of formula (9.2). We study the two limits $\vec{\eta}_1 \rightarrow 1\!\!1_{[t_0,s)}\vec{e}_k$, $\vec{\eta}_2 \rightarrow \delta_{s\pm\varepsilon}\vec{e}_l$, $\ t_0 < s < t$. To avoid further terms in our formulas we assume without loss of generality $\vec{x}_0 = 0$. Then we have

Lemma 121

$$\dot{x}_l(s\pm\varepsilon)x_k(s)\Phi_n \in (\mathcal{S}_d)'$$

with T-transform

$$T(\dot{x}_l(s\pm\varepsilon)x_k(s)\Phi_n)(\vec{\theta}) = \lim_{\substack{\vec{\eta}_1\rightarrow 1\!\!1_{(t_0,s]}\vec{e}_k \\ \vec{\eta}_2\rightarrow\delta_{s\pm\varepsilon}\vec{e}_l}} T(\langle\cdot,\vec{\eta}_1\rangle\langle\cdot,\vec{\eta}_2\rangle\Phi_n)(\vec{\theta})$$

$$= T\Phi_n(\vec{\theta})\Big\{(\theta_l(s\pm\varepsilon)+\zeta_l(s\pm\varepsilon))\left[\textstyle\int_{t_0}^s\theta_k + \int_{t_0}^s\zeta_k - \frac{s-t_0}{|\Delta|}\left(\int_\Delta\theta_k+\chi_k+X_k\right)\right]$$

$$-\frac{1}{|\Delta|}\left(\textstyle\int_\Delta\theta_l+\chi_l+X_l\right)\left[\int_{t_0}^s\theta_k+\int_{t_0}^s\zeta_k - \frac{s-t_0}{|\Delta|}\left(\int_\Delta\theta_k+\chi_k+X_k\right)\right]$$

$$+i\tfrac{s-t_0}{|\Delta|}\delta_{kl} + i1\!\!1_{[t_0,s)}(s\pm\varepsilon)\delta_{kl}\Big\}$$

where the dependence $\tau_1,...,\tau_n \rightarrow \dot{x}_l(s\pm\varepsilon)x_k(s)\cdot\Phi_n$ is now only defined in $L^2(\mathbb{R}^n)$ sense. (The value at the point $\tau_j = s\pm\varepsilon$ is not uniquely defined, which causes no problems in respect to later integration.)

Proof. Let us look to the terms which may cause problems.

1) The sequence $\{\vec{\eta}_{2,m}, m \in \mathbb{N}\}$ may be chosen such that the support of each $\vec{\eta}_{2,m}$ does not contain the point s where $1\!\!1_{[t_0,s)}$ has its jump. Thus the convergence of (η_1, η_2) causes no problems.

2) We may find uniform bounds in m for $(\vec{\zeta}, \vec{\eta}_{1,m})$. For simplicity suppose $\vec{\eta}_{1,m} \cdot \vec{e}_k \leq 1\!\!1_{[t_0,s)}$, $m \in \mathbb{N}$ then

$$\left|(\vec{\zeta}, \vec{\eta}_{1,m})\right| \leq \left|\sum_{j=1}^{n} \alpha_{j,k} \int_{t_0}^{\tau_j} 1\!\!1_{[t_0,s)}\right| \leq (s-t_0) \sum_{j=1}^{n} |\vec{\alpha}_j| \, .$$

3) The limit $\vec{\eta}_{2,m} \to \delta_{s\pm\varepsilon}\vec{e}_l$ in the term $(\vec{\zeta}, \vec{\eta}_{2,m})$ is more subtle. The form $\vec{\zeta} = \sum_{j=1}^{n} \vec{\alpha}_j 1\!\!1_{[t_0,\tau_j)}$ urges us to study the action of a delta sequence on a step function. If we write

$$\lim_{m\to\infty} (\vec{\eta}_{2,m}, 1\!\!1_{[t_0,\tau_j)}) = 1\!\!1_{[t_0,\tau_j)}(s) = 1\!\!1_{[s,t)}(\tau_j)$$

this formula is only valid (pointwise) for $s \neq \tau_j$ or in respect to τ_j-dependence in $L^2(\mathbb{R})$-sense. (In the point $s = \tau_j$ we may find delta-sequences which produce any value between 0 and 1.) □

Remark. Compare to (9.17) where a similar extension procedure forces us to view τ_j-dependence as a distribution.

Now we are interested to study the difference

$$\begin{aligned} &T((\dot{x}_l(s+\varepsilon)x_k(s) - x_k(s)\dot{x}_l(s-\varepsilon))\Phi_n)(\vec{\theta}) = \\ &= iT\Phi_n(\vec{\theta}) \cdot \delta_{kl} + (\theta_l(s+\varepsilon) - \theta_l(s-\varepsilon) + \zeta_l(s+\varepsilon) - \zeta_l(s-\varepsilon)) \\ &\quad \cdot \left(\int_{t_0}^{s} \theta_k + \int_{t_0}^{s} \zeta_k - \frac{s-t_0}{|\Delta|} \left(\int_\Delta \theta_k + \chi_k + X_k\right)\right) T\Phi_n(\vec{\theta}) \, . \end{aligned} \tag{9.12}$$

Lemma 122 *The series*

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\square_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}^d\mathrm{m}(\alpha_j) \ (\dot{x}_l(s+\varepsilon)x_k(s) - x_k(s)\dot{x}_l(s-\varepsilon))\Phi_n$$

converges in $(\mathcal{S}_d)^{-1}$.

Proof. The convergence of the above stated limits of the first term on the r.h.s. of (9.12) is proven in Theorem 110. The second term may be bounded by

$$\begin{aligned} 2|\Delta|\Big(2\varepsilon|\vec{\theta}'|_\infty &+ \sum_{k=1}^{n} |\vec{\alpha}_k| 1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k)\Big)\Big(|\vec{\theta}|_\infty + \sum_{j=1}^{n} |\vec{\alpha}_j| + \tfrac{|\vec{X}|}{2|\Delta|}\Big) \mathrm{C}_n(\alpha_1, ..., \alpha_n, \vec{\theta}) \leq \\ \leq \ & |\Delta| \sum_{k=1}^{n}\sum_{j=1}^{n} 1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k) \ (|\vec{\alpha}_k|^2 + |\vec{\alpha}_j|^2) \ \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \\ & +2|\Delta| \sum_{k=0}^{n} \Big(2\varepsilon|\vec{\theta}'|_\infty + \Big(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\Big) \ 1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k)\Big) \ |\alpha_k| \ \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \\ & +4|\Delta|\varepsilon|\vec{\theta}'|_\infty \ \Big(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\Big) \ \mathrm{C}_n(\vec{\alpha}_1, ..., \vec{\alpha}_n, \vec{\theta}) \, . \end{aligned} \tag{9.13}$$

For all $\vec{\theta} \in \mathcal{S}_d(\mathbb{R})$ such that $2|\vec{\theta}|_0\sqrt{t-t_0} < \varepsilon$ we know that

$$\int e^{\varepsilon|\vec{\alpha}|}\ |\mathrm{m}|(\mathrm{d}\alpha) := K_1, \quad \int |\vec{\alpha}|\ e^{\varepsilon|\vec{\alpha}|}\ |\mathrm{m}|(\mathrm{d}\alpha) := K_2 \quad \text{and} \quad \int |\vec{\alpha}|^2 e^{\varepsilon|\vec{\alpha}|}\ |\mathrm{m}|(\mathrm{d}\alpha) := K_3$$

all are finite. Thus the above bound is integrable w.r.t. $\prod_{j=1}^n |\mathrm{m}|(\mathrm{d}\alpha_j)$ and w.r.t. $\tau_1 ... \tau_n$. The convergence of the sum causes no problems. □

In the last step we want to prove that the last term of (9.12)

$$\begin{aligned} F_\varepsilon(\vec{\theta}) &:= \left((\theta_l(s+\varepsilon) - \theta_l(s-\varepsilon) + \zeta_l(s+\varepsilon) - \zeta_l(s-\varepsilon)\right)) \\ &\quad \cdot \left(\int_{t_0}^s \theta_k + \int_{t_0}^s \zeta_k - \frac{s-t_0}{|\Delta|}\left(\int_\Delta \theta_k + \chi_k + X_k\right)\right) T\Phi_n(\vec{\theta}) \end{aligned}$$

has the following property $\sum \frac{1}{n!}\int_{\square_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^n |\mathrm{m}|(\mathrm{d}\alpha_j)\ |F_\varepsilon(\vec{\theta})|$ is of order ε, i.e., the part of equation (9.12) connected to $F_\varepsilon(\vec{\theta})$ vanishes in the limit $\varepsilon \to 0$.

Proof. Let us consider (9.13) after $\prod_{j=1}^n |\mathrm{m}|(\mathrm{d}\alpha_j)$-integration is performed

$$\begin{aligned} \int_{\mathbb{R}^{dn}} \prod_{j=1}^n |\mathrm{m}|(\mathrm{d}\alpha_j)\ |F_\varepsilon(\vec{\theta})| \ &\leq \ |\Delta| \sum_{k=1}^n \sum_{j=1}^n 1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k)\ 2K_3K_1^{n-1} \\ &\quad +2|\Delta| \sum_{k=0}^n \left(2\varepsilon|\vec{\theta}'|_\infty + \left(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right) 1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k)\right) K_2K_1^{n-1} \\ &\quad +4|\Delta|\varepsilon|\vec{\theta}'|_\infty \left(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right) K_1^n \end{aligned}$$

the whole estimate will now be integrated w.r.t. $\tau_1, ..., \tau_n$. Here each integration produces a factor of 2ε iff $1\!\!1_{[s-\varepsilon,s+\varepsilon]}(\tau_k)$ appears in the integrand. Thus

$$\begin{aligned} &\int_{\square_n} \mathrm{d}^n\tau \int_{\mathbb{R}^{dn}} \prod_{j=1}^n |\mathrm{m}|(\mathrm{d}\alpha_j)\ |F_\varepsilon(\vec{\theta})| \leq 2|\Delta|K_3K_1^{n-1} \cdot |\Delta|^{n-1} \cdot 2\varepsilon \cdot n^2 \\ &\qquad +2|\Delta| \left(2|\Delta|\varepsilon|\vec{\theta}'|_\infty + 2\varepsilon\left(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right)\right) K_2K_1^{n-1}|\Delta|^{n-1} \cdot n \\ &\qquad +4|\Delta|\varepsilon|\vec{\theta}'|_\infty \left(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right) K_1^n|\Delta|^n \\ &= 4\varepsilon|\Delta|^n K^{n-1} \left\{K_3n^2 + \left(|\vec{\theta}'|_\infty|\Delta| + |\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right) K_2 \cdot n + K_1|\vec{\theta}'|_\infty|\Delta| \left(|\vec{\theta}|_\infty + \tfrac{|\vec{X}|}{2|\Delta|}\right)\right\}. \end{aligned}$$

The sum converges due to the rapidly decreasing factor $\frac{1}{n!}$. The additional quadratic polynomial (in braces) in n does not prevent convergence. □

Thus for $\vec{\theta} \in \mathcal{S}_d(\mathbb{R})$ with $2\sqrt{t-t_0}|\theta|_0 < \varepsilon$ we have

$$\lim_{\varepsilon\to 0} T((\dot{x}_k(s+\varepsilon)x_l(s) - x_l(s)\dot{x}_k(s-\varepsilon))\ \mathrm{I})(\vec{\theta}) = -iT\mathrm{I}(\vec{\theta}) \cdot \delta_{kl}.$$

Using the characterization theorem we have the following result.

Theorem 123 *In the space* $(\mathcal{S}_d)^{-1}$ *we have the following identity*

$$\lim_{\varepsilon\to 0}((\dot{x}_k(s+\varepsilon)x_l(s) - x_l(s)\dot{x}_k(s-\varepsilon))\mathrm{I}) = -i\delta_{kl}\mathrm{I}.$$

In particular we have in terms of expectation values

$$\lim_{\varepsilon\to 0}\mathbb{E}((\dot{x}_k(s+\varepsilon)x_l(s) - x_l(s)\dot{x}_k(s-\varepsilon))\mathrm{I}) = -i\delta_{kl}\mathbb{E}(\mathrm{I}).$$

This reflects the well-known fact that the quantum mechanical observables position and momentum do *not* commute. Thus we have proved a functional integral form of the canonical commutation relations, which was derived by a heuristic argument in [FH65]. The above theorem also shows that the important sample paths in the mean value can not have a continuous derivative. Hence this form of the canonical commutation relations reflects the lack of smoothness of the sample paths.

9.4 Ehrenfest's theorem

This section is intended to demonstrate that it is worthwhile to work in a white noise framework for the discussion of path integrals. The underlying ideas are simple. We exploit identities from general Gaussian analysis like

$$\mathbb{E}(D^*_{\vec{T}}\mathrm{I}) = 0\ , \quad \mathrm{I}\in(\mathcal{S}_d)^{-1},\ \vec{T}\in\mathcal{S}'_d(\mathbb{R})\ .$$

A good choice of $\vec{T}$ and a calculation of the derivative may lead to interesting quantum mechanical relations if I is chosen to be a Feynman integrand. The above formula may be viewed as a partial integration formula in functional integrals.

We start with the Feynman integrand defined in Theorem 110 and choose

$$\vec{T} := \delta'_s\cdot\vec{e}_k\ , \qquad t_0 < s < t \tag{9.14}$$

where $\{\vec{e}_k,\ 1\le k\le d\}$ is the canonical basis of $\mathbb{R}^d$. For convenience of notation we will introduce the following abbreviation $D_{s,k} := D_{\delta'_s\cdot\vec{e}_k}$. This differential operator has the following interesting property, which is the basic motivation for our choice (9.14)

$$D_{s,k}\vec{x}(t) = \delta(t-s)\ \vec{e}_k\ . \tag{9.15}$$

So this represents a kind of "partial derivative sensitive to the paths at given time". In textbooks of theoretical physics (9.15) is often used as a definition of the so called functional derivative.

Now we apply $D^*_{\vec{T}}$ to I and interchange it with limit and integration. This is allowed because

$$\begin{aligned} T(D^*_{\vec{T}}\mathrm{I})(\vec{\theta}) &= i\,\langle\vec{T},\vec{\theta}\rangle\ T\mathrm{I}(\vec{\theta}) \\ &= i\sum(-i)^n\int_{\Lambda_n}\mathrm{d}^n\tau\int_{\mathbb{R}^{dn}}\prod_{j=1}^n\mathrm{d}^d\mathrm{m}(\alpha_j)\ \langle\vec{T},\vec{\theta}\rangle\ T\Phi_n(\vec{\theta}) \\ &= \sum(-i)^n\int_{\Lambda_n}\mathrm{d}^n\tau\int_{\mathbb{R}^{dn}}\prod_{j=1}^n\mathrm{d}^d\mathrm{m}(\alpha_j)\ T(D^*_{\vec{T}}\Phi_n)(\vec{\theta}). \end{aligned}$$

In the next step we apply the (slightly generalized) product rule with $\vec{\eta} \in \mathcal{S}_d(\mathbb{R})$ to Φ_n from (8.5)

$$D^*_{\vec{\eta}}\Phi_n = D^*_{\vec{\eta}}(\mathrm{I}_0)\prod_{j=1}^{n}\exp(i\vec{\alpha}_j\vec{x}(\tau_j)) - \mathrm{I}_0 D_{\vec{\eta}}\exp\Big(i\sum_{j=1}^{n}\vec{\alpha}_j\vec{x}(\tau_j)\Big)$$
$$= -i\,\langle\vec{\omega},\vec{\eta}\rangle\cdot\Phi_n - i\Big(\sum_{j=1}^{n}\vec{\alpha}_j\langle 1\!\!1_{[t_0,\tau_j)},\vec{\eta}\rangle\Big)\Phi_n\ . \tag{9.16}$$

Here we had to be careful because the product rule in this form requires smooth directions $\vec{\eta} \in \mathcal{S}_d(\mathbb{R})$ of differentiation. To ensure that the second equality holds, we used only such $\vec{\eta}$ for which $\int_{t_0}^{t}\vec{\eta}(\tau)\,\mathrm{d}\tau = \vec{0}$ in order to avoid an additional term coming from the differentiation of Donsker's delta.

In the next step a careful discussion of the limit $\vec{\eta} \longrightarrow \delta'_s\vec{e}_k$ is necessary. As we have seen in Proposition 115 the first term in (9.16) becomes $i\ddot{x}_k(s)\cdot\Phi_n$ Thus let us fix our intermediate result

$$D^*_{s,k}\Phi_n = i\ddot{x}_k(s)\Phi_n - i\sum_{j=1}^{n}\alpha_{j,k}\delta_s(\tau_j)\Phi_n\ . \tag{9.17}$$

Remark. In fact the whole argument holds in a more general situation. Let $\vec{T} \in \mathcal{S}'_d(\mathbb{R})$ of order 1 such that there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}$ of 0 and t respectively with

$$T = 0 \text{ on } \mathcal{U}_0 \text{ and } \mathcal{U}.$$

Then there exists $\vec{S} \in \mathcal{S}'_d(\mathbb{R})$ of order 0 (i.e., a Radon measure), such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S} = \vec{T} \text{ and } \vec{S} = 0 \text{ on } \mathcal{U}_0$$

If we assume further $\vec{S} = 0$ on $\mathcal{U}$ then we have

$$D^*_{\vec{T}}\Phi_n = -i\left\langle\vec{\omega},\vec{T}\right\rangle\Phi_n - i\sum_{j=1}^{n}\vec{\alpha}_j\vec{S}(\tau_j)\cdot\Phi_n$$

(We restricted the order of the distributions $\vec{T}$ and $\vec{S}$ to have a τ_j-dependence in the last formula which allows τ_j-integration later in this section).

Let us now consider the second term in (9.17) after integration and summing up

$$\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{l=1}^{n}\int_{\square_n}\mathrm{d}^n\tau\int\Big(\prod_{j=1}^{n}\mathrm{d}^d\mathrm{m}(\alpha_j)\Big)\alpha_{l,k}\,\delta_s(\tau_l)\,\Phi_n =: (*)\ .$$

Φ_n contains a factor $e^{i\vec{\alpha}_l\vec{x}(\tau_l)}$ which is continuous in τ_l μ-a.e.. Thus τ_l– integration amounts to substitution of τ_l by s. (A different argument can easily be produced by considering the integration of the corresponding T-transform). Then one has to do a renumbering of integration variables:

$$\begin{aligned}(*) &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}n\int_{\square_{n-1}}\mathrm{d}^{n-1}\tau\int_{\mathbb{R}^{d(n-1)}}\prod_{j=1}^{n-1}\mathrm{d}^{\mathrm{d}}\mathrm{m}(\alpha_j)\,\Phi_{n-1}\int\mathrm{d}^{\mathrm{d}}\mathrm{m}(\alpha)\,\alpha_{,k}\,e^{i\vec{\alpha}\vec{x}(s)}\\ &= -\Big(\int\mathrm{d}^{\mathrm{d}}\mathrm{m}(\alpha)\,i\,\alpha_{,k}\,e^{i\vec{\alpha}\vec{x}(s)}\Big)\sum_{n=0}^{\infty}\frac{(-i)^{n-1}}{(n-1)!}\int_{\square_{n-1}}\mathrm{d}^{n-1}\tau\int_{\mathbb{R}^{d(n-1)}}\prod_{j=1}^{n-1}\mathrm{d}^{\mathrm{d}}\mathrm{m}(\alpha_j)\,\Phi_{n-1}\\ &= -(\nabla_k V)(\vec{x}(s))\cdot\mathrm{I}\end{aligned}$$

in the sense of Definition 118. Hence together with Proposition 115 we have derived

$$D^*_{s,k}\mathrm{I} = i\ddot{x}_k(s) \cdot \mathrm{I} + i(\nabla_k V)(\vec{x}(s)) \cdot \mathrm{I} \ .$$

Collecting all components and taking expectation we have

$$\mathbb{E}(\ddot{\vec{x}}\ (s) \cdot \mathrm{I}) = -\mathbb{E}(\vec{\nabla} V(\vec{x}(s)) \cdot \mathrm{I}) \ .$$

This is a variant of the well-known Ehrenfest theorem of quantum mechanics. The mean-values of the quantum observables satisfy the classical law of motion.

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