The Gabriel-Roiter Measure For Representation-Finite Hereditary Algebras

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Contents

Introduction

The Gabriel-Roiter measure was first introduced by Gabriel (under the name 'Roiter measure', [15]) in 1973 in order to clarify the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. But it was forgotten for nearly 30 years. Recently, Ringel showed ([28], [27]) in some way the Gabriel-Roiter measure provides a foundation for representation theory of artin algebras. The Gabriel-Roiter filtration and the Gabriel-Roiter submodule play an important role in the topic. So-called Gabriel-Roiter submodules of an indecomposable module are indecomposable submodules with a certain maximality: there do not exist proper indecomposable submodules containing a Gabriel-Roiter submodule. Gabriel-Roiter submodules of an indecomposable module Y always exist in case Y is not simple. One of the most interesting property of Gabriel-Roiter submodules is that if Y is an indecomposable non-simple module and X is a Gabriel-Roiter submodule of Y, then Y/X is indecomposable ([28], [27], also 1.6 below). Therefore, any indecomposable non-simple module Y is an extension of indecomposable modules.

Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k. Schofield's Theorem ([25], [32], also 1.5 below) tells us that the exceptional Λ-modules are extensions of orthogonal exceptional pairs. This shows that there is an inductive procedure in order to construct all the indecomposable modules starting from the simple modules, namely forming extensions of orthogonal bricks.

Schofield's Theorem raises the following problems:

- If Λ is not hereditary, can we find such orthogonal exceptional pairs to an exceptional Λ-module?
- If Λ is hereditary, the existence of orthogonal pairs to an indecomposable exceptional module follows directly from Schofield's Theorem. But how to construct such pairs of indecomposable modules?

To solve the first problem, we have to find, for each indecomposable (exceptional) module M, an indecomposable submodule U of M such that U^u is again a submodule of M for some $u > 0$ and the corresponding factor module M/U^u has, up to isomorphism, only one indecomposable summand. But it seems to be difficult to go further. Now we consider the simplest case: for each indecomposable (exceptional) module, we look for an indecomposable submodule such that the corresponding factor module is indecomposable, again. This motivates us to consider the Gabriel-Roiter measure, study the Gabriel-Roiter submodules and their factors.

There are several reasons which lead us to work mainly on the so-called directed algebras ([23] and 1.3 below). First, all indecomposable modules over a directed algebra are exceptional modules. Second, a factor algebra of a directed algebra is again directed. Thus we may only consider sincere directed algebras, i.e., directed algebras affording a sincere indecomposable module. Recall that the global dimension of a sincere directed algebra is bounded by 2 ([23], 2.4.7 and 1.3 below), and that all representation-finite hereditary algebras are directed with global dimension 1. Third, directed algebras are always representation finite, i.e., they afford only finitely many isomorphism classes of indecomposable modules. On one hand, we can easily calculate the Gabriel-Roiter measure of each indecomposable module. On the other hand, sincere directed algebras are simply connected, and any representation finite algebra admits simply connected coverings, ([4], [17]). Using this technique, Bongartz showed that any indecomposable non-simple module over a representation-finite algebra is an extension of an indecomposable module and a simple one ([7]). So it is interesting to know whether we can write the indecomposable non-simple modules over directed algebras as extensions of orthogonal indecomposable modules.

We now assume Λ is a representation-finite hereditary algebra. Then, Schofield's Theorem implies that for each indecomposable non-simple module Y, there exist exactly $s(Y)-1$ short exact sequences $0\to X\to Y\to Z\to 0$ with X, Z indecomposable and $\mathrm{Hom}(X,Z)=0$, where $s(Y)$ is the number of isomorphism classes of composition factors of Y. There does not yet exist a convenient procedure to determine the possible submodules X (and then Z), when Y is given. One of my aim in this thesis is to provide a method to find at least some of these modules X, namely the Gabriel-Roiter submodules of Y.

In particular, we will show the following theorem:

Theorem A. *Let Λ be a representation-finite hereditary k-algebra.*
(1). If T is a Gabriel-Roiter submodule of M, then $\mathrm{Hom}(T,M/T)=0$.
(2). Each indecomposable module M possesses at most 3 Gabriel-Roiter submodules.

We get immediately the following consequences from the theorem:

(1). $(M/T,T)$ is an orthogonal exceptional pair to M.
(2). $\dim\mathrm{Hom}(T,M)=1$.
(3). $\dim\mathrm{Ext}^1(M/T,T)=1$.
(4). If N is an indecomposable submodule of M which is different from T and M, then $\mathrm{Hom}(T,N)=0$.

As a conjecture, we claim **Theorem A** still holds for directed algebras over algebraically closed fields.

If X is a Gabriel-Roiter submodule of Y, we call the inclusion a Gabriel-Roiter inclusion which is a mono-irreducible map (1.7 below). If X is a Gabriel-Roiter submodule of Y, and X' is a proper submodule of Y which contains X, then X is a direct summand of X'. Recall that the irreducible monomorphisms have the same property. This leads us to consider the connection between the irreducible monomorphisms and the Gabriel-Roiter inclusions.

Let Z be the cokernel of an irreducible monomorphism f which is not a source map. H.Krause ([21]) proved that if Z is not simple and, either the domain or range of f is indecomposable, then the middle term of the almost split sequence ending at Z is indecomposable. This was generalized by S.Brenner ([8]), who only required that Z is not simple.

Assume $T \subset M$ is a Gabriel-Roiter submodule. In view of the formal similarities between Gabriel-Roiter inclusions and irreducible monomorphisms, it is natural to ask if the middle term of the almost split sequence ending at M/T is indecomposable. Unfortunately, this is not always true even we assume M/T is not simple. But we can still formulate the following theorem:

Theorem B. *Let Λ be a representation-finite hereditary k-algebra and T be a Gabriel-Roiter submodule of M. If M/T is not injective, then the AR sequence terminating in M/T has an indecomposable middle term.*

The paper is organized as follows. In chapter 1 we recall and give some basic notions and results which will be needed later on. Chapter 2 is devote to a discussion of the properties of Gabriel-Roiter measure. We will give the proof of Theorem A in chapter 3. Chapter 4 deals with the Auslander-Reiten sequences ending with a Gabriel-Roiter factor module. We will give the proof of Theorem B and some examples which prevent us from extending the theorem to lager classes of algebras.

Chapter 1

Preliminaries

The aim of this chapter is to formulate some notions, definitions and some known results which will be needed later on.

Throughout the paper, we assume k is an algebraically closed field and algebras are finite dimensional k-algebras. By modules, we always mean finite dimensional left modules. For an algebra Λ, we denote by modΛ the category of Λ-modules and by indΛ the category of indecomposable Λ-modules. For the details we refer to [1] and [23].

1.1 Path algebras and representation of quivers

A **quiver** $\Delta = (\Delta_0, \Delta_1)$, or more precisely, $\Delta = (\Delta_0, \Delta_1, s, e)$ is given by two sets Δ_0, Δ_1 and two maps s,e: $\Delta_1 \to \Delta_0$; the set Δ_0 is called the set of vertices, the set Δ_1 is called the set of arrows, and given an arrow $\alpha \in \Delta_1$, then $s(\alpha)$ is called the starting vertex, and $e(\alpha)$ its end vertex; we write $a \xrightarrow{\alpha} b$ where $s(\alpha) = a$, $e(\alpha) = b$. We denote by $\overline{\Delta}$ the underlying graph which is obtained from Δ by forgetting the orientation of the arrows. We say Δ has no multiple arrows in case for any $a, b \in \Delta_0$, there is at most one arrow from a to b.

Given a quiver Δ, we can define the **path algebra** $k\Delta$. For each vertex a of Δ, we define a path denoted by e_a of length 0 from a to a. A path of length $t \geq 1$ from a to b in a quiver is of the form $\alpha_t\alpha_{t-1}\cdots\alpha_1$ where $s(\alpha_i) = e(\alpha_{i-1})$ for $2 \leq i \leq t$, and $s(\alpha_1) = a$, $e(\alpha_t) = b$. We say $s(\alpha_t\alpha_{t-1}\cdots\alpha_1) = s(\alpha_1)$ and $e(\alpha_t\alpha_{t-1}\cdots\alpha_1) = e(\alpha_t)$. A path of length $t \geq 1$ from a to a is called a **cyclic path**. The path algebra $k\Delta$ is defined to be the vector space spanned by all the paths and the multiplication of two paths is defined as follows:

$$\beta \cdot \alpha = \begin{cases} \beta\alpha & \text{if} \quad s(\beta) = e(\alpha); \\ \beta & \text{if} \quad s(\beta) = a, \alpha = e_a; \\ \alpha & \text{if} \quad e(\alpha) = b, \beta = e_b; \\ 0 & \quad \text{otherwise.} \end{cases}$$

Note that the path algebra of Δ is finite dimensional if and only if, first of all, Δ is finite, (i.e., Δ_0, Δ_1 are finite sets,) and, in addition, there is no cyclic path in Δ. In $k\Delta$, we denote by $k\Delta^+$ the ideal generated by all arrows. Note that $(k\Delta^+)^n$ is the ideal generated by all paths of length $\geq n$.

We recall that the radical of an algebra Λ, denoted by radΛ, is the intersection of all maximal ideals. A finite dimensional k-algebra Λ is **basic** provided $\Lambda/\text{rad}\Lambda$ is a product of copies of k. Any finite dimensional k-algebra Λ is Morita equivalent to a basic algebra. There is the following structure theorem for basic algebras:

Theorem 1.1.1 (Gabriel)**.** *Any basic finite dimensional k-algebra is isomorphic to $k\Delta/I$ for some uniquely determined finite quiver Δ and some ideal I with $(k\Delta^+)^n \subseteq I \subseteq (k\Delta^+)^2$, for some $n \geq 2$.*

The associated quiver in the above theorem is call **Gabriel** quiver of the k-algebra.

Given vertices $a, b \in \Delta_0$, and paths $\{\varrho_i | i\}$ from a to b of length ≥ 2. A finite linear combination of these ϱ_i is called a relation on Δ. Any ideal $I \subset (k\Delta^+)^2$ can be generated, as an ideal, by relations. Write $I =< \varrho_i | i >$. For example, a commutativity relation is a relation of the form $\varrho - \varrho'$ where ϱ and ϱ' are paths having the same starting vertex and the same end vertex. A zero relation is given by a single path ϱ.

Given a quiver $\Delta = (\Delta_0, \Delta_1)$, a representation $V = (V_a, h_\alpha)$ of Δ over k is given by a family of finite dimensional vector spaces V_a for all $a \in \Delta$, and linear maps $h_\alpha : V_a \rightarrow V_b$, for any arrow $a \xrightarrow{\alpha} b$. If V and V' are two representations of Δ over k, a map $f = (f_a) : V \rightarrow V'$ is given by maps $f_a : V_a \rightarrow V'_a$ for $a \in \Delta$ such that $h'_\alpha f_a = f_b h_\alpha$ for any arrow $a \xrightarrow{\alpha} b$. In other words, f is given by the following commutative diagram:

$$\begin{array}{ccc} V_a & \xrightarrow{h_\alpha} & V_b \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_b} \\ V'_a & \xrightarrow[h'_\alpha]{} & V'_b \end{array}$$

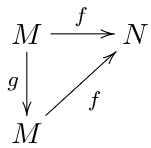
Given a quiver with relations $(\Delta, \{\varrho_i | i\})$, we define the representation to be the representation of quiver such that the compositions of maps corresponding to the paths satisfy the same relations.

Theorem 1.1.2. *Given a quiver with relations $(\Delta, \{\varrho_i | i\})$, its representation category is equivalent to the category of $k\Delta/I$-modules with $I =< \varrho_i >$.*

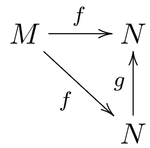
A basic algebra Λ is hereditary if and only if it is given by a quiver with no relations. A theorem of Gabriel says that a basic hereditary algebra Λ is representation-finite if and only if it is isomorphic to a path algebra kQ where the underlying graph $\overline{Q}$ is one of the Dynkin diagrams: A_n, D_n, E_6, E_7, or E_8. One may find an elegant proof in [5].

1.2 Almost split sequences and AR quiver

Fix a finite dimensional k-algebra Λ. A morphism $f : M \to N$ is called **right minimal** provided any morphism g fitting into the following commutative diagram



is an automorphism. A morphism $f : M \to N$ is called **left minimal** provided any morphism g fitting into the following commutative diagram



is an automorphism.

A morphism $g : B \to C$ is **right almost split** if (1) g is not a split epimorphism and (2) any morphism $X \to C$ which is not a split epimorphism factors through g. Dually, a morphism $g : A \to B$ is **left almost split** if (1) g is not a split monomorphism and (2) any morphism $A \to Y$ which is not a split monomorphism factors through g.

A morphism is said to be a **minimal left (right) almost split** morphism or a **source (sink) map** if it is both left (right) minimal and left (right) almost split. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called an **almost split sequence** or **AR-sequence** if f is minimal left almost split and g is minimal right almost split.

It is easy to check that the canonical inclusion $\mathrm{rad}P \to P$ for an indecomposable projective module P is minimal right almost split and dually, the canonical epimorphism $I \to I/\mathrm{soc}I$ for an indecomposable injective module I is minimal left almost split.

Proposition 1.2.1. *(1). If C is an indecomposable non-projective module, then there exists an almost split sequence $0 \to A \to B \to C \to 0$ with A, B are uniquely determined, up to isomorphism, by C. We denote by $A = \tau C$.*
(2). If A is an indecomposable non-injective module, then there exists an almost split sequence $0 \to A \to B \to C \to 0$ with C, B are uniquely determined, up to isomorphism, by A. We denote by $C = \tau^{-1} A$.

Let X, Y be two Λ-modules. A map $f : X \to Y$ is said to be **irreducible** if f is neither a split monomorphism nor a split epimorphism, and h is a split monomorphism or g is a split epimorphism whenever $f = gh$ for $g : M \to Y$, $h : X \to M$ and Λ-module M.

Proposition 1.2.2. *Let $\delta : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence which is not split. Then f is irreducible if and only if for any homomorphism $h : X \to C$ there is either a morphism $t : X \to B$ with $h = gt$ or a morphism $s : B \to X$ with $g = hs$.*

We now assume X and Y are indecomposable modules and $f : X \to Y$ is an irreducible monomorphism. Then $Z = \mathrm{coker} f$ is an indecomposable module. Also $\mathrm{Im} f (\cong X)$ is a direct summand of any proper submodule of Y containing $\mathrm{Im} f$. Therefore any homomorphism to Z which is not an epimorphism factors through the canonical projection $Y \to \mathrm{coker} f$. It follows that all irreducible maps to Z are epimorphisms. Furthermore, if $Z \cong \mathrm{coker} f$ is not simple, and $0 \to \tau Z \to M \to Z \to 0$ is an almost split sequence , then M is indecomposable . Note that the corresponding statements hold for the kernel of an irreducible epimorphism ([1],[8],[21]).

The relationship between the almost split morphisms and the irreducible maps can be formulated as follows.

Proposition 1.2.3. *Let $f : A \to X$ be a morphism with A indecomposable. Then f is irreducible if and only if there is an X' such that $A \overset{\binom{f}{f'}}{\to} X \oplus X'$ is minimal left almost split. Dually, a morphism $g : Y \to C$ with C indecomposable is irreducible if and only if there is a Y' such that $Y \oplus Y' \overset{(g,g')}{\to} C$ is minimal right almost split.*

If X, Y are indecomposable modules, denote by $\mathrm{rad}(X, Y)$ the set of non-invertible morphisms from X to Y. Given direct sums $X = \oplus_{i=1}^{s} X_i$, $Y = \oplus_{j=1}^{t} Y_j$, a map $f : X \to Y$ can be written in the form $f = (f_{ij})$ with $f_{ij} \in \mathrm{Hom}(X_i, Y_j)$. f is said to belong to $\mathrm{rad}(X, Y)$ provided for all i,j, f_{ij} belong to $\mathrm{rad}(X_i, Y_j)$. Define $\mathrm{rad}^2(X, Y)$ to be the set of maps of the form gf with $f \in \mathrm{rad}(X, Z)$, $g \in \mathrm{rad}(Z, Y)$ for some module Z. Note that $\mathrm{rad}^2(X, Y) \subseteq \mathrm{rad}(X, Y) \subseteq \mathrm{Hom}(X, Y)$ are k-spaces and in fact $\mathrm{End}(X)$-$\mathrm{End}(Y)$-subbimodules of $\mathrm{Hom}(X, Y)$. If we denote by

$$\mathrm{Irr}(X, Y) = \mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y),$$

then $\mathrm{End}(X)$-$\mathrm{End}(Y)$-bimodule $\mathrm{Irr}(X, Y)$ is annihilated from the left by $\mathrm{rad}(X, X)$, from the right by $\mathrm{rad}(Y, Y)$. It is easy to see that a map $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}(X, Y) \setminus \mathrm{rad}^2(X, Y)$. $\dim_k \mathrm{Irr}(X, Y)$ gives multiplicity of modules in middle terms of AR sequence. It is called the bimodule of irreducible maps.

The Auslander-Reiten quiver $\Gamma(\Lambda)$ of Λ is defined as follows: its vertices are the isomorphism classes of the indecomposable modules, and we draw $d_{XY} = \dim_k Irr(X, Y)$ arrows from X to Y. Here we X both the indecomposable module and its isomorphism class. For indecomposable Λ-module X and Y, we say X is **before** Y if there is a path from X to Y in the AR quiver.

A **sectional path** in AR quiver is a path $X_1 \to X_2 \to \cdots \to X_n$ such that $X_i \not\cong \tau X_{i+2}$ for each $1 \leq i \leq n-2$.

For each indecomposable module M, there exists a minimal right almost split morphism $\oplus_{i=1}^{n} X_i \to M$ with X_i indecomposable and uniquely determined by M for each i. We denote by $\alpha(M) = n$, the number of the indecomposable summands of the middle term, and by

$\alpha(\Lambda) = \max\{\alpha(M)|M$ is an indecomposable Λ-module$\}$. The following theorem shows that $\alpha(\Lambda)$ has an upper bound if Λ is a representation-finite algebras.

Theorem 1.2.4 ([2]). *Let Λ be a representation-finite algebra and suppose $0 \to A \to \oplus_{i=1}^{n} B_i \to C \to 0$ is an almost split sequence of Λ-modules with B_i non-zero and indecomposable for $1 \leq i \leq n$. Then $n \leq 4$ and, if $n = 4$, then one of the B_i is both projective and injective.*

1.3 Directed algebras

In this section, we will present some known results for directed algebras. One may find all the proofs in [23]. Let Λ be a basic finite dimensional algebra over k. Then Λ is given by a quiver and relations. We denote by modΛ the category of Λ-modules of finite length and by $\underline{\dim} M$ the dimensional vector of the Λ-module M. A **path** from an indecomposable module M to an indecomposable module N in modΛ is a sequence of morphisms $M \stackrel{f_1}{\to} M_1 \stackrel{f_2}{\to} M_2 \stackrel{f_3}{\to} \cdots \stackrel{f_{t-1}}{\to} M_{t-1} \stackrel{f_t}{\to} N$ between indecomposable modules, where $t \geq 1$ and each f_i is not zero and not an isomorphism. A path from M to M is called a **cycle** in modΛ, and the number of morphisms in the path is called the length of the cycle. Note that a path in the Auslander-Reiten quiver $\Gamma(\Lambda)$ of Λ gives rise to a path in modΛ. An indecomposable module M is said to be **directing** if M does not belong to any cycle. An algebra Λ is said to be **directed** provided every indecomposable Λ-module is directing.

Proposition 1.3.1. *Let M be an indecomposable Λ-module.*
(1). If M lies on a cyclic path in the Auslander-Reiten quiver $\Gamma(\Lambda)$, then M lies on a cycle in modΛ.
(2). If Λ is of finite representation type, then M lies on a cycle in modΛ *if and only if M lies on a cyclic path in $\Gamma(\Lambda)$.*

Proposition 1.3.2. *Let M be a directing Λ-module. Then* $\mathrm{End}(M) = k$ *and for all $i \geq 1$,* $\mathrm{Ext}^i(M, M) = 0$. *Also, if N is an indecomposable Λ-module with $\underline{\dim} M = \underline{\dim} N$, then $M \cong N$.*

A Λ-module M is **sincere** if every simple Λ-module occurs as a composition factor of M, or equivalently, $(\underline{\dim} M)_i \geq 1\ \forall i \in \Delta_0$ where $\Delta = (\Delta_0, \Delta_1)$ is the corresponding Gabriel quiver. An algebra Λ is said to be sincere if it has sincere indecomposable modules. M is called **faithful** provided the only element $a \in \Lambda$ satisfying $aM = 0$, is the element a=0. A faithful module is always sincere. An indecomposable module M is said to be a **thin** module if $(\underline{\dim} M)_i$=0 or 1 for each i. Note that M is a thin module if and only if each simple module occurs as a composition factor at most once.

Proposition 1.3.3. *Let M be a directing Λ-module. Then M is sincere if and only if it is faithful.*

Proposition 1.3.4. *Let M be a sincere directing Λ-module. Then the projective dimension* p.d.$M \leq 1$*, the injective dimension* i.d.$M \leq 1$ *and the global dimension* gl.d.$\Lambda \leq 2$.

Proposition 1.3.5. *Let Λ be a directed algebra. Then all the indecomposable projective modules and the indecomposable injective modules are thin modules.*

Given a finite dimensional algebra Λ with finite global dimension, we define the bilinear form $< -, - >$ on the Grothendieck group as follows:

$$< \underline{\dim}X, \underline{\dim}Y >= \dim\mathrm{Hom}(X,Y) + \sum_{i\geq 1}(-1)^i \dim\mathrm{Ext}^i(X,Y).$$

We denote by $\mathcal{X}_\Lambda$ the corresponding quadratic form, thus $\mathcal{X}_\Lambda(z) =< z, z >$. We endow $\mathbb{Z}^n$ a partial ordering defined componentwise: $z = (z_1, \cdots, z_n) \in \mathbb{Z}^n$ is said to be **positive**, written $z > 0$, provided $z \neq 0$ and $z_i \geq 0$. The integral quadratic form $\mathcal{X}$ is said to be **weakly positive** if $\mathcal{X}(z) \geq 0$ for all positive $z \in \mathbb{Z}^n$. And an element $z \in \mathbb{Z}^n$ satisfying $\mathcal{X}(z) = 1$ is called a **root** of $\mathcal{X}$.

Theorem 1.3.6. *Let Λ be a finite dimensional directed algebra. If* gl.d.$\Lambda \leq 2$ *(for example, if Λ is sincere), then $\mathcal{X}$ is weakly positive, and* $\underline{\dim}$ *furnishes a bijection between the indecomposable Λ-modules and the positive roots of $\mathcal{X}$.*

Corollary 1.3.7. *A directed algebra is representation-finite.*

Corollary 1.3.8. *Let Λ be a sincere directed algebra, and M an indecomposable Λ-module. Then the components of* $\underline{\dim}M$ *are bounded by 6.*

The Auslander-Reiten quiver $\Gamma(\Lambda)$ of a directed algebra Λ is preprojective, i.e., it is contains no cyclic paths, and only finitely many τ-orbits and such that any τ-orbit contains a projective module. Hence it is standard (see [23] Lemma 2.3.3), that is indΛ is equivalent to the mesh category of $\Gamma(\Lambda)$. It follows that the integer-valued function $f_M = \dim\mathrm{Hom}(M, -)$: ind$\Lambda \rightarrow \mathbb{Z}$ is an addictive function (Gabriel [16]) for each indecomposable module (vertex in $\Gamma(\Lambda)$. The function f_M satisfies the properties that $f_M(X) = 1$ whenever there is a sectional path from M to X in $\Gamma(\Lambda)$, and if $0 \rightarrow \tau X \rightarrow \oplus Y_i \rightarrow X \rightarrow 0$ is an almost split sequence, then $f_M(\tau X) + f_M(X) = \sum f_M(Y_i)$. Note that there is also an addictive function $\dim\mathrm{Hom}(-, M)$ for each indecomposable module M.

If the AR quiver of Λ is preprojective, we denote by $\mathcal{O}(\Lambda)$ its orbit quiver: the vertices of $\mathcal{O}(\Lambda)$ are the τ orbits of the AR quiver of Λ; or, equivalently, the isomorphism classes of the indecomposable projective modules. Given an indecomposable projective module P in the AR quiver, let $Y_1, Y_2, \cdots, Y_n$ be the direct predecessors of P. For any i, there exist $t_i \geq 0$ and a projective vertex P_i with $\tau^{t_i} Y_i = P_i$. Let $n(Y_i, P)$ be the number of arrows from Y_i to P. In $\mathcal{O}(\Lambda)$, there will be $n(Y_i, P)$ arrows from P_i to P. We also denote by $\overline{\mathcal{O}(\Lambda)}$ the underlying graph of the orbit quiver $\mathcal{O}(\Lambda)$.

Denote by $[[M]]$ the τ orbit of M which corresponds to a point in the orbit graph.

Theorem 1.3.9 (Bautista-Larrion-Salmeron, Bongartz). *Let Λ be a sincere directed algebra. Then the orbit graph $\overline{\mathcal{O}(\Lambda)}$ is a tree with at most 4 endpoints.*

Let Λ be a sincere directed algebra. Then the AR quiver is a preprojective translation quiver and the orbit graph Γ is a tree with at most 4 end points. Now assume the orbit graph of Λ is a star with 3 branches (for example, D_n, $E_{6,7,8}$), and M is indecomposable. M is said to **lie on the center** if in the orbit quiver $[[M]]$ has exactly 3 neighbors. And M is said to **lie on the quasi-center** if $[[M]]$ has two neighbors and one of the neighbors, say $[[N]]$, lies on the center. M is said to **lie on the boundary** if M is either projective or injective, or $[[M]]$ has exactly 1 neighbor. In other words, if M is neither projective nor injective, then M lies on the boundary if and only if $\alpha(M) = 1$. Since the orbit graph is a star, for each indecomposable module M, we may define $sl(M)$ to be the length of $[[M]]$ in the branch containing $[[M]]$. It follows that $sl(M) = 0$ if M lies on the center and $sl(M) = 1$ if M lies on the quasi center.

1.4 Representation-finite hereditary algebras

The most important examples of directed algebras are the path algebras of Dynkin quivers. For a path algebra Λ of a Dynkin quiver, there is a one to one correspondence between the isomorphism classes of indecomposable Λ-modules and the positive roots of the corresponding semisimple Lie algebra.

Let $\mathrm{D} = \mathrm{Hom}(-, k)$ be the dual. The formula in the following theorem is called Auslander-Reiten (AR) formula:

Theorem 1.4.1. *Let Λ be a hereditary algebra. Then*

$$\mathrm{Ext}^1(X, Y) \cong \mathrm{DHom}(Y, \tau X) \cong \mathrm{DHom}(\tau^{-1}Y, X).$$

So by using the additive functions $\mathrm{dimHom}(M, -)$ and $\mathrm{dimHom}(-, M)$ for each indecomposable module M, we can also calculate the dimension of all extension groups $\mathrm{Ext}^1(M, X)$ and $\mathrm{Ext}^1(X, M)$ for every indecomposable module X.

Proposition 1.4.2 ([18]). *Let Λ be a hereditary algebra and X, Y be indecomposable Λ-modules with $\mathrm{Ext}^1(Y, X) = 0$. Then any non-zero map from X to Y is either injective or surjective.*

Suppose Λ is a directed algebra, M and N are two indecomposable Λ-modules. If there is a sectional path from $[M]$ to $[N]$ in the AR quiver, then $\mathrm{dimHom}(M, N) = 1$ and $\mathrm{Ext}^1(N, M) = \mathrm{Ext}^1(M, N) = 0 = \mathrm{Hom}(N, M)$. In particular, if Λ is a representation-finite hereditary algebra, and there is a sectional path from $[M]$ to $[N]$, then up to a scalar factor, the unique non-zero map from M to N is either a monomorphism or an epimorphism.

1.5 Schofield's Theorem

An indecomposable Λ-module M with $\mathrm{End}(M) \cong k$ and $\mathrm{Ext}^i(M, M)$=0 for all $i \geq 1$ is said to be **exceptional**. It follows exceptional modules are indecomposable. By 1.3.2, all indecomposable modules over a directed algebra are exceptional. Two indecomposable modules V and U are said to be **orthogonal** if $\mathrm{Hom}(U, V) = 0 = \mathrm{Hom}(V, U)$. A pair of exceptional modules (V, U) is said to be an **orthogonal exceptional pair** if U and V are orthogonal and $\mathrm{Ext}^1(U, V)$=0. An orthogonal exceptional pair (V, U) is said to be an **orthogonal exceptional pair to** M if there exists a short exact sequence $0 \to U^u \to M \to V^v \to 0$ for some pair of positive integers (u, v).

Now we assume that Λ is a hereditary algebra. We are going to present a theorem of Schofield which yields an inductive way for constructing all exceptional modules in modΛ The theorem asserts that we can find, for each exceptional module, orthogonal exceptional pairs to it, i.e., any exceptional module M is obtained as the middle term of a suitable exact sequence

$$(*) \qquad 0 \longrightarrow U^u \longrightarrow M \longrightarrow V^v \longrightarrow 0$$

where U, V are again exceptional modules and (V, U) is an orthogonal exceptional pair. Given an orthogonal exceptional pair (V, U), we denote by $\mathcal{E}(U, V)$ the full subcategory of all Λ-modules which have a filtration with factors of the form U and V. Note that for any module M in $\mathcal{E}(U, V)$ there exists an exact sequence of the form $(*)$ with non-negative integers u, v.

The reduction problems to be considered is the following: Given an exceptional module, we want to find orthogonal exceptional pair (V, U) such that M belongs to $\mathcal{E}(U, V)$, but M is not one of the two simple modules of $\mathcal{E}(U, V)$. One may ask for all possible pairs of this kind, and it is amazing that there exists an intrinsic characterization of the number of such pairs.

Theorem 1.5.1 (Schofield)**.** *Let Λ be a finite dimensional hereditary k-algebra and M be an exceptional Λ-module. Let $s(M)$ be the number of the isomorphism classes of composition factors of M. Then there are precisely $s(M)$-1 orthogonal exceptional pairs (V_i, U_i) such that M belongs to $\mathcal{E}(U_i, V_i)$ and is not a simple object in $\mathcal{E}(U_i, V_i)$.*

1.6 The Gabriel-Roiter measure

We will give the definition of the Gabriel-Roiter measure for modules of finite length ([28], [27]). We fix a finite dimensional k-algebra Λ.

Let $\mathbb{N}_1$={$1, 2, \cdots$} be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ the set of all subsets $I \subseteq \mathbb{N}_1$. We use the symbol $\subset$ to denote proper inclusion. We consider the set $\mathcal{P}(\mathbb{N}_1)$ as a totally ordered set as follows: If I,J are two different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I\backslash J)\cup(J\backslash I)$ belongs to J. Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J\backslash I$, we have $a < b$. We say that J **starts with** I provided $I = J$ or $I \ll J$. It is easy to check that

(1). If $I \subseteq J \subseteq \mathbb{N}_1$, then $I \leq J$.
(2). If $I_1 \leq I_2 \leq I_3$, and I_3 starts with I_1, then I_2 starts with I_1.

For each Λ-module M, denote by $|M|$ the length of M. Let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \cdots, |M_t|\}$ where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure**(briefly **GR measure**) of M. If M is an indecomposable Λ-module, then a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_t = M$ with $\mu(M) = \{|M_1|, |M_2|, \cdots |M_t|\}$ is called a **Gabriel-Roiter filtration**(briefly **GR filtration**) of M. We call an inclusion $N \subset M$ of indecomposable Λ-modules a **Gabriel-Roiter inclusion**(briefly **GR inclusion**) provided $\mu(M) = \mu(N) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(N)$. Note that a chain $M_1 \subset M_2 \subset \cdots \subset M_t = M$ is a GR filtration if and only if all the inclusions $M_i \subset M_{i+1}$ are GR inclusions. The factor module of a GR inclusion is called **Gabriel-Roiter factor**(briefly **GR factor**). A short exact sequence $0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} X \longrightarrow 0$ is called a **GR sequence** provided the monomorphism f is a GR inclusion.

Lemma 1.6.1. *Let X, Y and Z be indecomposable modules.*
(1). X is a proper submodule of Y, then $\mu(X) < \mu(Y)$.
(2). If $\mu(X) < \mu(Y) < \mu(Z)$ and X is a GR submodule of Z, then $|Y| > |Z|$.

Proof. We only show (2) since (1) follows directly from the definition. Since X is a GR submodule of Z, $\mu(Z)$ starts with $\mu(X)$, and hence $\mu(Y)$ starts with $\mu(X)$. We may assume $\mu(X) = \{l_1 = 1, l_2, \cdots, l_m\}$, $\mu(Z) = \{l_1 = 1, l_2, \cdots, l_m, l_{m+1}\}$ and, $\mu(Y) = \{l_1 = 1, l_2, \cdots, l_m, r_1, r_2, \cdots, r_n\}$. $\mu(Y) < \mu(Z)$ implies $l_{m+1} < r_1$. Then $r_n \geq r_1 > l_{m+1}$, that is $|Y| > |Z|$. □

Example. (1). If P is an indecomposable projective Λ-module, the GR submodules of M are the direct summands of rad$P$ with maximal GR measures.

(2). If I is an indecomposable injective Λ-module (more generally, indecomposable module with simple socle), then the GR measure of I is $\mu(I) = \{1, 2, \cdots, |I|-1, |I|\}$. Thus the corresponding GR factor modules are simple modules.

(3). If M is a local indecomposable module with Loewy length 2, then $\mu(M) = \{1, |M|\}$.

Example. The Kronecker quiver $\widetilde{A}_{11}$. It is the path algebra $k\Delta$ where Δ has two vertices a, b and two arrows from a to b. There are two simple modules, the simple projective module $P(b)$ and the simple injective module $I(a)$. If M is an indecomposable module, then the dimension vector of M is $\underline{\dim}M = (d_a, d_b)$ with $|d_a - d_b| \leq 1$.

(1). The pre-projective modules P_n for $n \in \mathbb{N}_0$, with $\underline{\dim}P_n = (n, n+1)$. Since $P_n \oplus P_n \rightarrow P_{n+1}$ is a sink map, P_n is a (and hence the unique up to iso) GR submodule of P_{n+1} and $\mu(P_n) = \{1, 3, 5, \cdots, 2n+1\}$.

(2). The regular modules $R_\lambda(n)$ for $\lambda \in \mathbb{P}^1(k)$ and $n \in \mathbb{N}_1$, with $\underline{\dim}R_\lambda(n) = (n, n)$. It is easy to see that the GR submodule of $R_\lambda(1)$ is $P_0 = P_b$, the simple projective modules. Hence $\mu(R_\lambda(1)) = \{1, 2\}$. For $R_\lambda(n)$ with $n \geq 2$, the almost split sequences are $0 \rightarrow R_\lambda(n) \rightarrow R_\lambda(n+1) \oplus R_\lambda(n-1) \rightarrow R_\lambda(n) \rightarrow 0$. The GR submodule (unique up to isomorphism) of $R_\lambda(n)$ is $R_\lambda(n-1)$ and $\mu(R_\lambda(n)) = \{1, 2, 4, 6, \cdots, 2n\}$.

(3). The pre-injective modules I_n for $n \in \mathbb{N}_0$, with $\underline{\dim}I_n = (n+1, n)$. The regular modules $R_\lambda(n)$ are GR submodules of I_n and $\mu(I_n) = \{1, 2, 4, 6, \cdots, 2n, 2n+1\}$. Note that there are infinitely many non-isomorphic GR submodules for each indecomposable pre-injective modules.

There is a second possibility for introducing the Gabriel-Roiter measure. Namely, we can define the Gabriel-Roiter measure by induction on the length of modules. It will be a rational number in [0,1]. For the zero module 0, let $\mu(0) = 0$. Given a module of length $m > 0$. we may assume by induction that $\mu(M')$ is already defined for any proper submodule M' of M. Let

$$\mu(M) = \max\mu(M') + \begin{cases} 2^{-m}, & M \quad \text{indecomposable} \\ 0, & M \quad \text{decomposable} \end{cases}$$

Here the maximum is taken over all proper submodules M' of M. Note that the maximum always exists.

Let I, J be two subsets of $\mathcal{P}(\mathbb{N}_1)$. Then we have

$$I < J \Leftrightarrow \sum_{i \in I} 2^{-i} < \sum_{j \in J} 2^{-j}.$$

This shows the order introduced on $\mathcal{P}(\mathbb{N}_1)$ and the usual ordering of rational numbers are compatible. Therefore, we have the two definitions of the Gabriel-Roiter measure are equivalent via the following map: if $M_1 \subset M_2 \subset \cdots \subset M_t = M$ is a GR filtration, then $\{|M_1|, |M_2|, \cdots |M_t| = |M|\}$ is mapped to the rational number $\sum_{i=1}^{t} \frac{1}{2^{|M_i|}}$. In this paper, we will use the first definition.

1.7 Basic properties of the Gabriel-Roiter measure

In this section, we want to present some basic properties of the Gabriel-Roiter measure which will needed later on. We fix a finite dimensional k-algebra Λ.

Main property.(Gabriel) *Let X, Y_1, $\cdots$,Y_t be indecomposable Λ-modules and assume*

that there is a monomorphism $f : X \longrightarrow \oplus_{i=1}^{t} Y_i$. *Then*
(1). $\mu(X) \leq \max\{\mu(Y_i)\}$.
(2). If $\mu(X) = \max\{\mu(Y_i)\}$, *then* f *splits.*
(3). If $\max\{\mu(Y_i)\}$ *starts with* $\mu(X)$, *then there is some* j *such that* $\pi_j f$ *is injective, where* $\pi_j : \oplus_i Y_i \longrightarrow Y_j$ *is the canonical projection.*

In [28], one may find the proof of this main property.

Example. The morphism $\pi_j f$ is not necessarily a monomorphism if $\mu(Y_j) = \max\{\mu(Y_i)\}$. Let $\Lambda = kA_5$ with the following orientation:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

Then, $\dim\mathrm{Hom}(P_4, I_4) = 1$ and the (unique) non-zero map is neither a monomorphism nor an epimorphism. By direct calculation, we get $\mu(P_4) = \{1, 2\}$, $\mu(P_3) = \{1, 2, 3\}$ and $\mu(I_4) = \{1, 2, 3, 4\}$. We get a monomorphism $(f, l) : P_4 \to I_4 \oplus P_3$ where l the inclusion and f is the (unique) map from P_2 to I_4. $\mu(I_4) > \mu(P_3)$ but f is not injective.

Corollary 1.7.1. *Suppose* $M_1, \cdots, M_t$ *are indecomposable* Λ*-modules. Then* $\mu(\oplus M_i) = \max\{\mu(M_i)\}$.

Proposition 1.7.2 ([28]). *Let* $T \subset M$ *be a GR inclusion, and* $f : T \longrightarrow M$ *an injective map. Then for any factorization* $f = f''f'$, *where* $f'' : T' \longrightarrow M$ *is a proper monomorphism, the map* $f' : T \longrightarrow T'$ *is a split monomorphism.*

Proof. First assume that T' is indecomposable. If f' is not an isomorphism, i.e., f' is a proper monomorphism, then $\mu(T) \cup \{|T'|, |M|\} \leq \mu(M)$. However, by assumption $\mu(M) = \mu(T) \cup \{|M|\} \leq \mu(T) \cup \{|T'|, |M|\}$, a contradiction. For the general case: Write $T' = \oplus T_i$ with indecomposable modules T_i. The main property asserts that $\mu(T) \leq \max\mu(T_i)$. On the other hand, we have $\mu(T_i) < \mu(M)$ for each i since that T' is a submodule of M. Therefore, $\max\mu(T_i)$ starts with $\mu(T)$, and it follows there exist j such that $\pi_j f' : T \to T_j$ is monomorphism where $\pi_j : T' \to T_j$ is the canonical projection. There is also a monomorphism $T_j \to T' \to M$. Since N_j is a proper submodule of M and indecomposable, we are in the first case. Thus $\pi_j f'$ is an isomorphism, so that f' is a split monomorphism. □

Definition 1.7.3. *A monomorphism* $f : T \longrightarrow M$ *is called* **mono-irreducible** *provided either* $s : N \longrightarrow M$ *is a split epimorphism or* $t : T \longrightarrow N$ *is a split monomorphism whenever* $f = st$ *with* s, t *monomorphisms.*

Clearly, irreducible injective maps and GR inclusions are mono-irreducible. And if the inclusion $T \subset M$ is mono-irreducible, then T is a direct summand of any proper submodule X of M containing T.

Proposition 1.7.4. *Assume the inclusion* $T \subset M$ *is mono-irreducible with* M *indecomposable. Then* M/T *is indecomposable.*

Proof. Assume M/T is decomposable. Then there exist two proper submodules X_1, X_2 of M containing T such that $M/T \cong X_1/T \oplus X_2/T$. But the mono-irreducibility implies that the inclusions $T \to X_1$ and $T \to X_2$ split. It follows $X_1 = T \oplus X'$ and $X_2 = T \oplus X''$. This implies $M = T \oplus X' \oplus X''$, a contradiction. □

Proposition 1.7.5. *Let $T \subset M$ be a mono-irreducible map with M indecomposable. Then all irreducible maps to M/T are epimorphisms.*

Proof. Note that T is a direct summand of any proper submodule of M containing T. Consider the exact sequence $0 \to T \xrightarrow{f} M \xrightarrow{g} M/T \to 0$, and assume $h : X \to M/T$ is an irreducible monomorphism. Then it follows that the induced short exact sequence $0 \to T \to g^{-1}(\mathrm{Im} h) \to \mathrm{Im} h \to 0$ splits. Hence we have $h = gt$ for some $t : X \to M$. Since g is not a split epimorphism and h is irreducible, we get t is a split monomorphism, and consequently an isomorphism. Thus h is an epimorphism since g is, a contradiction. Therefore any irreducible morphism to M/T is an epimorphism. □

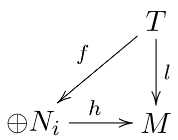
Proposition 1.7.6. *Let $\delta : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence which is not split. Then f is mono-irreducible if and only if for any monomorphism $h : X \to C$ there is either a morphism $t : X \to B$ with $h = gt$ or a morphism $s : B \to X$ with $g = hs$.*

Proof. We may copy the proof for the case of irreducible monomorphisms ([1], Prop.5.6, p.170). □

We now collect some properties of the GR inclusions which will be quite often used later on.

Corollary 1.7.7. *Let $\delta : 0 \longrightarrow T \xrightarrow{l} M \xrightarrow{\pi} M/T \longrightarrow 0$ be a GR sequence. Then the following statements hold:*
(1). T is a direct summand of all proper submodules of M containing T.
(2). M/T is indecomposable.
(3). Any map to M/T which is not an epimorphism factors through π.
(4). All irreducible maps to M/T are epimorphisms.
(5). If all irreducible maps to M are monomorphisms, then l is an irreducible map.
(6). M/T is a factor module of $\tau^{-1}T$ and $M/T \cong \tau^{-1}T$ if and only if δ is an almost split sequence.

Proof. Proofs of (1)–(4) are straightforward. For (5), let $\oplus N_i \xrightarrow{h} M$ be the minimal right almost split map. Then we have the following commutative diagram:



Since l is a monomorphism, f is also a monomorphism , therefore $\mu(T) \leq \max\mu(N_i)$. Thus $\mu(N_i) = \mu(\mathrm{Im}(h_i)) \leq \mu(T) \leq \max\mu(N_i)$ since every irreducible map $N_i \xrightarrow{h_i} M$ is injective. So we have $\max \mu(N_i) = \mu(T)$ and f is split by the main property. Thus, l is irreducible.

For statement (6), we assume $\epsilon : 0 \longrightarrow T \xrightarrow{f} E \xrightarrow{g} \tau^{-1}T \longrightarrow 0$ be an almost split sequence. Consider the following commutative diagram:

$$\begin{array}{llllllllll}
\epsilon: & 0 & \longrightarrow & T & \xrightarrow{f} & E & \xrightarrow{g} & \tau^{-1}T & \longrightarrow & 0 \\
 & & & \| & & \downarrow u & & \downarrow h & & \\
\delta: & 0 & \longrightarrow & T & \xrightarrow{l} & M & \xrightarrow{\pi} & M/T & \longrightarrow & 0
\end{array}$$

u and h exist since ϵ is almost split and l is a GR inclusion which is not a split monomorphism. We claim that h is an epimorphism. If not, h factors through π since M/T is a GR factor module. It follows ϵ is split sequence since E in fact is the pull back. We get a contradiction. Therefore h is an epimorphism and M/T is a factor module of $\tau^{-1}T$. Furthermore, $\tau^{-1}T \cong M/T$ if and only if h is an isomorphism, if and only if u is an isomorphism. Thus, $\tau^{-1}T \cong M/T$ if and only if δ is an almost split sequence. □

Chapter 2

Gabriel-Roiter submodules

We fix a finite dimensional algebra Λ. We will study the interplay of modules defined via GR-properties and the AR quiver.

2.1 Maps between the modules of a GR inclusion

Let X, Y be two indecomposable modules. We denote by $\mathrm{Sing}(X,Y)$ the subset of $\mathrm{Hom}(X,Y)$ which consists of all non-injective maps. If $T \subset M$ is a GR inclusion, then $\mathrm{Sing}(T,M)$ has the following nice property:

Proposition 2.1.1. *Let $T \subset M$ be a GR inclusion. Then $\mathrm{Sing}(T,M)$ is a subgroup of $\mathrm{Hom}(T,M)$.*

Proof. Let $f, g \in \mathrm{Sing}(T,M)$ be two morphisms. Then $f+g$ is the composition of the following maps: $T \overset{\binom{f}{g}}{\to} \mathrm{Im}f \oplus \mathrm{Im}g \overset{(l_f,l_g)}{\to} M$ where l_f and l_g are canonical inclusions. If $f+g$ is a monomorphism, then $\binom{f}{g}$ is a monomorphism. By the main property (1.7), we get $\mu(T) \leq \max\{\mu(\mathrm{Imf}), \mu(\mathrm{Im}g)\} \leq \mu(T)$ since T is a GR submodule of M and $\mathrm{Im}f$, $\mathrm{Im}g$ are both proper submodules of M. Again by the main property, $\binom{f}{g}$ is split. Thus f or g is an isomorphism, a contradiction. □

Proposition 2.1.2. *Let Λ be a directed algebra and $T \subset M$ be a GR inclusion. Then either the inclusion is an irreducible map, or there exists a path of irreducible maps $T \overset{f_n}{\to} X_n \overset{f_{n-1}}{\to} X_{n-1} \overset{f_{n-2}}{\to} \cdots \overset{f_1}{\to} X_1 \overset{f_0}{\to} X_0 = M$, such that the composition $f_i f_{i+1} \cdots f_n$ is injective for each $0 \leq i \leq n$, and the composition $f_0 f_1 \cdots f_j$ is surjective for each $0 \leq j \leq n-1$.*

Proof. Since Λ is directed and hence representation finite, any morphisms from T to M is a sum of compositions of irreducible maps. Assume the GR inclusion l is not irreducible and $g_1, g_2, \cdots g_m$ are all possible compositions of irreducible maps from T to M. Without loss of generality, we may write $l = \sum g_i$. It follows that the map $T \overset{(g_i)}{\to} \oplus \mathrm{Im}g_i$

is a monomorphism. Since T is a GR submodule of M, we get $\mu(T) \leq \max\mu(\mathrm{Im}g_i) \leq \mu(T)$. Thus by the main property (1.7), there exists an index i such that the map g_i is an isomorphism, say $g_i = g$. We may assume $T \overset{f_n}{\to} X_n \overset{f_{n-1}}{\to} X_{n-1} \overset{f_{n-2}}{\to} \cdots \overset{f_1}{\to} X_1 \overset{f_0}{\to} X_0 = M$ is the path corresponding to g, i.e., $g = f_0 f_1 \cdots f_n$. Thus $f_i f_{i+1} \cdots f_n$ are monomorphisms for all i. On the other hand, if there is some $1 \leq j \leq n-1$ such that the composition $f_0 f_1 \cdots f_j$ is not an epimorphism, then the image X is a proper submodule of M and contains $N = \mathrm{Im}(f_0 f_1 \cdots f_n) \cong T$ as a submodule. Thus, N is also GR submodule of M and is isomorphic to a direct summand of X. In any case, we get a path of morphisms $T \to X_n \to \cdots X_j \to X \cong T$, a contradiction. □

Proposition 2.1.3. *Let Λ be a directed algebra and $0 \longrightarrow \tau M \longrightarrow \oplus_{i=1}^{4} X_i \overset{(g_i)}{\longrightarrow} M \longrightarrow 0$ an almost split sequence with 4 indecomposable summands. Then the GR inclusions of M are given by irreducible maps, and M has at most 3 different GR submodules.*

Proof. By Theorem 1.2.4, we know that one of these X_i's is projective and injective, and the remaining X_j are neither projective nor injective and pairwise non-isomorphic. So, we may assume $X_1 = P_a = I_b$ where a, b are in the index set of the simple Λ-modules. Since X_1 is injective, g_1 is an epimorphism. If, say, g_2 is an epimorphism, then there exist non zero map $h : X_1 = P_a \to X_2$ such that $g_1 = g_2 h$. In particular, $(\underline{\dim} X_2)_a = \dim\mathrm{Hom}(P_a, X_2) \neq 0$. As Λ is directed and there is an irreducible map from τM to P_a, it follows that $(\underline{\dim}\tau M)_a = \dim\mathrm{Hom}(P_a, \tau M) = 0$. Using

$$(\underline{\dim}\tau M)_a + (\underline{\dim}M)_a = (\underline{\dim}P_a)_a + \sum_{i=2}^{4}(\underline{\dim}X_i)_a$$

and the fact $(\underline{\dim}P_a)_a = (\underline{\dim}M)_a$, we have $(\underline{\dim}X_i)_a = 0$, which is a contradiction. Thus, g_i is a monomorphism for each $i \neq 1$. Let $I = \max\{\mu(X_i) | i = 2, 3, 4\}$ and $T \overset{l}{\longrightarrow} M$ be a GR inclusion. Since (g_i) is an right almost split morphism, there exists $f = (f_i) : T \longrightarrow \oplus X_i$ such that $\sum g_i f_i = l$. Since $\mathrm{Hom}(I_b, M) \neq 0$, we obtain $\mathrm{Hom}(M, I_b) = 0$. Thus, $\mathrm{Hom}(T, X_1) = \mathrm{Hom}(T, I_b) = \mathrm{imHom}(l, I_b) = 0$. Consequence, $f_1 = 0$ and we have a monomorphism $T \longrightarrow \oplus_{j=2}^{4} X_j$ which implies by the main property (1.7):

$$\mu(T) \leq I = \max\{\mu(X_j) | j \neq 1\} \leq \mu(T).$$

Thus, $T \cong X_j$ for some $j \in \{2, 3, 4\}$. Since there is an irreducible map $X_i \to M$, it follows $\dim\mathrm{Hom}(X_i, M) = 1$. Thus, there are at most 3 different GR submodules. □

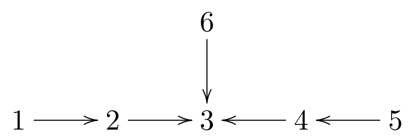
Proposition 2.1.4. *Let M be an indecomposable module over a directed algebra Λ. Then τM is not a GR submodule of M.*

Proof. First recall that if Λ is representation-finite, then any non-zero map can be written as a sum of compositions of irreducible maps and, for directed algebras, if there is an irreducible map $X \to Y$ with X and Y indecomposable, then $\dim \mathrm{Hom}(X, Y) = 1$. We assume M is not projective and $0 \to \tau M \overset{f=(f_i)}{\to} X = \oplus_{i=1}^{n} X_i \overset{g=(g_i)}{\to} M \to 0$ is an almost split sequence. By Theorem 1.2.4, we have $n \leq 4$.

If τM is a GR submodule of M, by 2.1.2, we may assume the irreducible maps $f_1 : \tau M \to X_1$ is a monomorphism and $g_1 : X_1 \to M$ is an epimorphism. Thus, $n \geq 2$. Comparing the length, we get $|\tau M| - \sum_{i \neq 1} |X_i| = |X_1| - |M| > 0$, thus, the irreducible map $(f_i)_{i \neq 1}$ is an epimorphism. On the other hand, since $\sum_i g_i f_i = 0$, we have the GR inclusion $l = \sum a_i g_i f_i = \sum_{i \neq 1} a'_i g_i f_i$ for some $a_i, a'_i \in k$. It follows the map $(f_i)_{i \neq 1} : \tau M \to \oplus_{i \neq 1} X_i$ is a monomorphism. A contradiction. □

The following example shows there exists indecomposable module M such that $\tau^2 M$ is a GR submodule of M.

Example. Let $\Lambda = kE_6$ with the following orientation:



The indecomposable module $M = \begin{matrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{matrix} \cong \tau^{-3} P_6$. Up to isomorphism, it has 3 GR submodules, $\tau^{-1} P_1$, $\tau^{-1} P_3$ and $\tau^2 M \cong \tau^{-1} P_6$.

Proposition 2.1.5. *Let $T \subset M$ be a GR inclusion with T a directing module. Assume f is a non-zero map in Sing(T, M). Then either $T + f(T) = M$ or $T \cap f(T) = 0$.*

Proof. The assertion is a direct result of the following general case: If X is a proper indecomposable submodule of M which is not isomorphic to T and $\mathrm{Hom}(T, X) \neq 0$, then either $T + X = M$ or $T \cap X = 0$.

Now we begin the proof of the general statement. Assume $T + X \neq M$. We claim that $T + X \neq X$: if the equality holds, then $T \subset X$. Thus, T is a direct summand of X since T is a GR submodule of M and X is a proper submodule of M containing T. It follows that $\mathrm{Hom}(X, T) \neq 0$. A contradiction since $\mathrm{Hom}(T, X) \neq 0$ and T is directing . Therefore, X is a proper submodule of $T + X$. On the other hand, $\mathrm{Hom}(T, X) \neq 0$ implies $\mathrm{Hom}(X, T) = 0$ since T is a directing module. In particular, $T \subset T + X$ is a proper inclusion. Thus $T + X = T \oplus Y$ for some submodule Y of M. The inclusion $X \subset T + X$ induces a monomorphism from X to Y since $\mathrm{Hom}(X, T) = 0$. Hence $|X| \leq |Y| = |T + X| - |T| = |X| - |T \cap X| \leq |X|$. Thus we have $T \cap X = 0$. □

2.2 Socle and the GR socle

This section is devoted to a discussion of the socle and the Gabriel-Roiter measure of an indecomposable module. We will give a characterization of a module with simple socle by using the GR measure.

Proposition 2.2.1. *Let M be an indecomposable module and $\mu(M) = \{l_1, l_2, \cdots, l_m = |M|\}$. Then $|\{i : l_{i+1} - l_i > 1\}| + 1 \leq |\mathrm{soc}M| \leq |M| - m + 1$.*

Proof. If M is not simple, then we have a GR sequence $0 \to T \to M \to M/T \to 0$. First assume M/T is simple. Thus T is a maximal submodule of M and if S is a simple submodule of M which is not contained in T, then $M = T + S = T \oplus S$. A contradiction since M is indecomposable. This contradiction implies $S \subset T$ and therefore $\mathrm{soc}M = \mathrm{soc}T$. Now assume M/T is not simple. Then the canonical inclusion $\mathrm{soc}(M/T) \subset M/T$ factors through M. Thus $\mathrm{soc}(M/T)$ is isomorphic to a submodule X of $\mathrm{soc}M$. Conversely, for any simple submodule $S \subset M$, if S is not contained in T then corresponding $(T+S)/T \cong S$ is a simple submodule of M/T and hence, S is contained in X. Therefore, we have $|\mathrm{soc}M| = |\mathrm{soc}T| + |X| = |\mathrm{soc}T| + |\mathrm{soc}(M/T)|$.

(1). For the second inequality, we use induction on the length of M . The assertion is trivial if M is simple. Now assume $|M| > 1$. Let T be a GR submodule of M. If M/T is simple, by induction, $|\mathrm{soc}M| = |\mathrm{soc}T| \leq |T| - (m-1) + 1 \leq |M| - m + 1$. If M/T is not simple, we have $|\mathrm{soc}M| = |\mathrm{soc}T| + |\mathrm{soc}M/T| \leq |T| - (m-1) + 1 + |\mathrm{soc}M/T| \leq |T| - m + 2 + |M| - |T| - 1 = |M| - m + 1$.

(2). We use induction on $r_M := |\{i : l_{i+1} - l_i > 1\}| + 1$ to show the first inequality. Assume $M_1 \subset M_2 \subset \cdots \subset M_m = M$ is a GR filtration. If $r_M = 1$, i.e., $\mu(M) = \{1, 2, 3, \cdots, |M|\}$, then $\mathrm{soc}M$ is simple and hence, $|\mathrm{soc}M| = 1 = r$. Now assume $r_M > 1$. Let j be the largest index with $l_{j+1} - l_j > 1$, Then $r_{M_j} = r_{M_{j+1}} - 1 = r_M - 1$. By induction, we obtain $|\mathrm{soc}M_j| \geq r_{M_j}$. Note that $|\mathrm{soc}M_{j+1}| = |\mathrm{soc}M_{j+1}/M_j| + |\mathrm{soc}M_j|$ since M_{j+1}/M_j is not simple. Therefore $r_{M_{j+1}} = r_{M_j} + 1 \leq |\mathrm{soc}M_j| + 1 \leq |\mathrm{soc}M_{j+1}|$. On the other hand, since M_{s+1}/M_s are simple modules for all $s \geq j+1$, we have $\mathrm{soc}M = \mathrm{soc}M_{m-1} = \cdots = \mathrm{soc}M_{j+1}$. Thus, $r_M = r_{M_{j+1}} \leq |\mathrm{soc}M_{j+1}| = |\mathrm{soc}M|$. □

Now we give a characterization of indecomposable modules with simple socle.

Proposition 2.2.2 ([27]). *Let M be a module of length n. Then the following are equivalent:*
(1). socle of M is simple.
(2). any non-zero submodule of M is indecomposable.
(3). there exist a composition series of M with all terms indecomposable.
(4). $\mu(M) = \{1, 2, \cdots, n\}$.
(5). $\mu(M') < \mu(M)$, for any proper factor module M' of M.
(6). $\mu(M/S) < \mu(M)$ for any simple submodule of M.

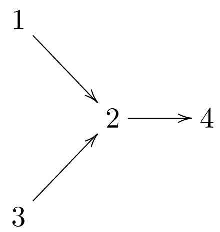
Proof. The equivalences of the first 4 statements are well-known and, the implications (4)⇒(5)⇒(6) are obvious. It remains to show that (6)⇒(1). Assume M has two different simple submodules, say S and S'. Then the canonical maps give rise to an embedding $M \to M/S \oplus M/S'$. The main property in 1.7, yields $\mu(M) \leq \max\{\mu(M/S), \mu(M/S')\}$. On the other hand, $\mu(M/S) < \mu(M)$ and $\mu(M/S') < \mu(M)$ by assumption. A contradiction. □

This proposition tells us if soc$M$ is not simple, then there exist a simple submodule S of M such that $\mu(M/S) > \mu(M)$. The question is: Can we determine the number of simple submodules S of M such that $\mu(M/S) > \mu(M)$? To answer the question, we need the following definition.

Definition 2.2.3. *The* **Gabriel-Roiter socle** (**GR socle**) *of an indecomposable module M, denoted by* $\mathrm{soc}_{GR}M$*, is the sum of all simple submodules of M which can occur as the first term of some GR filtration of M.*

For any indecomposable non-simple module M, we have $0 \subset \mathrm{soc}_{GR}M \subseteq \mathrm{soc}M$. The following example shows that $\mathrm{soc}_{GR}M$ is in general a proper submodule of soc$M$.

Example. Let $\Lambda = kD_4$ with the following orientation:



Consider the indecomposable M of maximal length. Then $\mu(M) = \{1, 2, 3, 5\}$ and M has 2 simple submodules: S_4 and S_2. Since there is an irreducible map from S_2 to M, it follows there are no indecomposable modules lying in between. Therefore $\mathrm{soc}_{GR}M = S_4$ and S_2 is not a summand of $\mathrm{soc}_{GR}M$.

Lemma 2.2.4. *Let X be an indecomposable non-simple module and X' be the intersection of kernels of all maps $X \to N$ with $\mu(N) < \mu(X)$. Then*
(1). $\mathrm{soc}_{GR}X \subseteq X' \subseteq \mathrm{rad}X$. *In particular,* $\mathrm{soc}_{GR}X \subseteq Z$ *for any proper submodule Z of X with $\mu(X/Z) < \mu(X)$.*
(2). $X' = \mathrm{rad}X$ if and only if $\mu(N) > \mu(X)$ for any proper non-semisimple factor module N of X.

Proof. We first consider the following assertion: if $f : X \to Y$ is a non-zero map with $\mu(X) > \mu(Y)$, then $f(X_1) = 0$ for any GR filtration $X_1 \subset X_2 \subset \cdots \subset X_n = X$.

The assertion implies directly the first inclusion $\mathrm{soc}_{GR}X \subseteq X'$. In particular, if Z is a proper submodule of X with $\mu(X/Z) < \mu(X)$, then X' is obviously a submodule of $Z = \ker\pi$ where $\pi : X \to X/Z$ is the canonical projection. Thus, $\mathrm{soc}_{GR}M \subseteq M$. The second inclusion $X' \subseteq \mathrm{rad}X$ holds since all simple factor modules of X have smaller GR measure and the radical of a module is the intersection of all kernels of maps from X to simple modules.

The assertion was proved by Ringel in [27]. We re-write the proof of the above assertion. If f is a monomorphism, then $\mu(X) \leq \mu(Y)$ by the main property, a contradiction. Thus, $\ker f \neq 0$ and we choose a minimal i such that $\ker f \cap X_i \neq 0$. If $i = 1$, then $X_1 \subseteq \ker f$ since X_1 is simple. If $i > 1$, we have $X_i \cap \ker f \neq 0$ and $X_{i-1} \cap \ker f = 0$. Consider the restriction: $f' = f|_{X_i} : X_i \to Y$. It is not zero and the induced map $X_{i-1} \to X_i/\ker f' \cong \mathrm{Im} f' \subseteq Y$ is injective since $X_{i-1} \cap \ker f' \subseteq X_{i-1} \cap \ker f = 0$. Thus $\mu(X_{i-1}) < \mu(\mathrm{Im} f') \leq \mu(Y) < \mu(X)$. Thus $\mu(\mathrm{Im} f')$ starts with $\mu(X_{i-1})$ since $\mu(X)$ starts with $\mu(X_{i-1})$. Since $X_{i-1} \subset X_i$ is a GR inclusion, we get $|\mathrm{Im} f'| > |X_i|$ by Lemma 1.6.1. But on the other hand, $|X_i| > |\mathrm{Im} f'|$ since $\mathrm{Im} f'$ is a factor module of X_i. A contradiction. Thus, the minimal index i with $\ker f \cap X_i \neq 0$ is 1 and $X_1 \subseteq \ker f$.

Now we prove statement (2). If $\mu(N) > \mu(X)$ for any proper non-semisimple factor module N of X, then the intersection of kernels of all maps $X \to Y$ with $\mu(Y) < \mu(X)$ is the intersection of all the maps $X \to S$ with S a simple module, and thus is the radical of X.

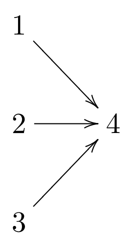
Conversely, assume $X' = \mathrm{rad} X$ and N is a non-semisimple proper factor module of X with $\mu(N) < \mu(X)$. Let $\pi : X \to N$ be the projection, then $X' \subseteq \mathrm{rad} X \cap \ker \pi \subseteq \mathrm{rad} X = X'$. It follows $X' = \mathrm{rad} X = \mathrm{rad} X \cap \ker \pi$ and hence $\mathrm{rad} X \subseteq \ker \pi$. Thus we have $N \cong X/\ker \pi$ is semisimple, a contradiction. □

Proposition 2.2.5. *Let M be an indecomposable module. There exist at most one simple submodule of M such that $\mu(M/S) < \mu(M)$. If such simple submodule S exists, then $S = \mathrm{soc}_{GR} M$.*

Proof. Let S be a simple submodule of M with $\mu(M/S) < \mu(M)$. Consider the canonical projection $\pi : M \to M/S$. Then $\mathrm{soc}_{GR} M \subset S = \ker \pi$ by Lemma 2.2.4. Therefore $\mathrm{soc}_{GR} M = S$ is a simple. If S' is a simple submodule of M with $\mu(M/S') < \mu(M)$. Then we have $S' = \mathrm{soc}_{GR} M = S$. It follows that there exists at most one simple submodule of M such that $\mu(M/S) < \mu(M)$. □

The following example shows that for an indecomposable module M, there may not exist simple submodule such that $\mu(M/S) < \mu(M)$. By proposition 2.2.2, this can only occur when $\mathrm{soc} M$ is not simple.

Example. Let Λ be the hereditary algebra of type D_4 with the following orientation:



Let S be the simple projective module P_4 and M the indecomposable module of maximal length. Then $\mu(M) = \{1, 2, 5\}$ and $\dim \mathrm{Hom}(S, M) = 2$. There is a monomorphism $S \to M$

with an indecomposable cokernel, the indecomposable injective module I_4 of length 4, thus $\mu(M/S) = \mu(I_4) = \{1, 2, 3, 4\}$. There are three different kinds of monomorphisms with decomposable cokernel: the direct sum of a simple injective module and an indecomposable module of length 3 whose GR measure is $\{1, 2, 3\}$. Hence for any simple submodule of M the GR measure of the corresponding factor module is larger than $\mu(M)$.

2.3 Examples on the difference between two GR submodules

We have seen that an indecomposable module M may have, up to isomorphism, more than one (even infinitely many) GR submodules. In some sense, all of these non-isomorphic GR submodules behave totally differently. Except for their length, two GR submodules may have nothing in common. In this section, we want to present more examples to show the possible difference between GR submodules.

Let M be an indecomposable Λ-module. $\mathrm{ann}M = \{\lambda \in \Lambda | \lambda M = 0\}$ is an ideal of Λ. Let Λ' be the quotient $\Lambda/\mathrm{ann}M$. Therefore M is an Λ' module. It follows $\mu_\Lambda(M) = \mu_{\Lambda'}(M)$. By using this assertion, we can show the following proposition which provide a good method for our construction:

Proposition 2.3.1. *Let M be an indecomposable Λ-module and Λ' the one point extension:* $\Lambda' = \begin{bmatrix} \Lambda & M \\ 0 & k \end{bmatrix}$. *Then $\mu_\Lambda(M) = \mu_{\Lambda'}(M)$.*

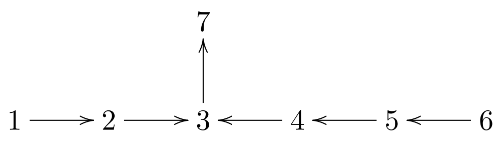
Proof. Clearly, the category $\mathrm{mod}\Lambda$ can be identified with the subcategory of Λ_1:

$$\{X \in \mathrm{mod}\Lambda' | \mathrm{Hom}(P_\omega, X) = 0\} = \{X | e_\omega X = 0\},$$

where $P_\omega = \Lambda_1 e_\omega$ is the indecomposable projective Λ'-module with $\mathrm{rad}P_\omega = M$. □

Example. This example shows there exist indecomposable modules with different GR submodules, and one of the corresponding GR factor module is local module, but the other one is not.

Let $\Lambda = kE_7$ with the following orientation:



We select $M = \begin{array}{cccccc} & & 2 & & & \\ 1 & 2 & 3 & 2 & 1 & 0 \end{array}$. Then $\mu(M) = \{1, 2, 3, 4, 6, 7, 10\}$ and M has 3 non-isomorphic GR submodules:

$$T_1 = \begin{array}{cccccc} & & 1 & & & \\ 0 & 1 & 2 & 2 & 1 & 0 \end{array}, \qquad M/T_1 = \begin{array}{cccccc} & & 0 & & & \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

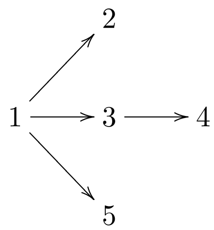
$$T_2 = \begin{array}{cccccc} & & 1 & & & \\ 1 & 2 & 2 & 1 & 0 & 0 \end{array}, \qquad M/T_2 = \begin{array}{cccccc} & & 0 & & & \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

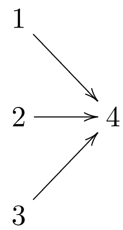
$$T_3 = \begin{array}{cccccc} & & 1 & & & \\ 1 & 1 & 2 & 1 & 1 & 0 \end{array}, \qquad M/T_3 = \begin{array}{cccccc} & & 0 & & & \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}$$

top(M/T_1) and top(M/T_2) are simple modules, but top(M/T_3) is not simple.

Example. This example shows the lengths of the socle of two non-isomorphic GR submodules are not necessary to be the same.

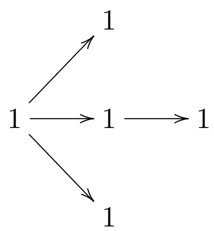
Assume $\Lambda_1 = kD_5$ and $\Lambda_2 = kD_4$ with the following orientations respectively:



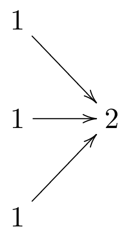


Let M_1 and M_2 are Λ_1 and Λ_2 module:

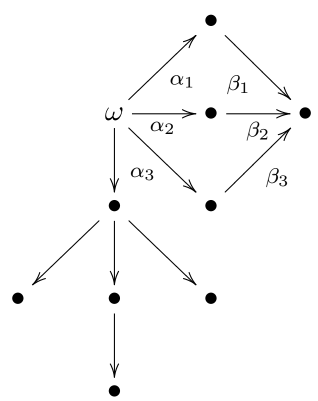
M_1 :



M_2 :



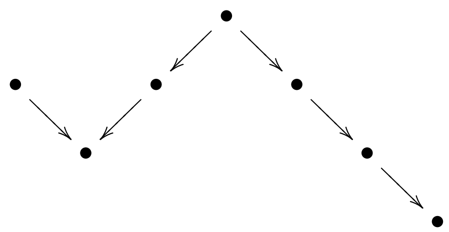
These two modules are both of GR measure $\{1, 2, 5\}$. Then let Λ be the direct sum $\Lambda_1 \oplus \Lambda_2$ and $M = M_1 \oplus M_2$. We construct the one point extension: $\Lambda' = \begin{bmatrix} \Lambda & M \\ 0 & k \end{bmatrix}$. Then Λ' is given by the following together with relation $\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3 = 0$.



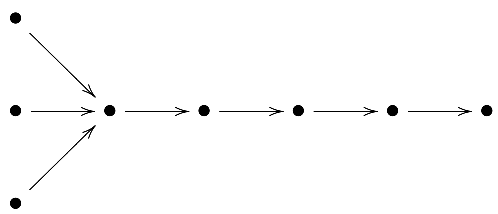
We now consider the indecomposable projective module P_ω whose radical is $M = M_1 \oplus M_2$. Then the GR measure of P_ω is $\mu(P_\omega) = \{1, 2, 5, 11\}$ and it has two non-isomorphic GR submodules M_1 and M_2. But $|\mathrm{soc} M_1| = 3$, $|\mathrm{soc} M_2| = 2$.

Note that if one of the GR submodules has simple socle, then so are all the other ones.

Example. Again from the above example, we can see that two different GR submodules of an indecomposable modules need not have the same length of top. We will construct the more general examples. Let $\Lambda = kA_7$ with the following orientation:



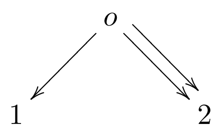
Let M be the unique sincere indecomposable module. Then M has, up to iso, two GR submodules, say T and N. $|\mathrm{top} T| = 2$ and $|\mathrm{top} N| = 1$. In general, we have the following construction: Let $\Lambda_{r,n-r}, r \leq n$ be the following quiver: the underlying graph is a star with $r+1$ branches such that r branches are of length 1 and the other one has length $n - r$. For the orientation, we select as in the following example: : $\Lambda_{3,4}$ is



Fix $n \geq 3$, we obtain n algebras $\Lambda_{r,n-r} = kA_{r,n-r}$. For each algebra, we select the sincere indecomposable module M_r such that the components in the dimension vectors are 1. Since all the modules have simple socle, the GR measure are the same, i.e., $\mu(M_r) = \{1, 2, \cdots, n+1\}$. But $|\mathrm{top}(M_r)| = r$. Again let $\Lambda = \oplus_r \Lambda_{r,n-r}$ and $M = \oplus M_r$ and we get the one point extension $\Lambda' = \begin{bmatrix} \Lambda & M \\ 0 & k \end{bmatrix}$. Let P be the indecomposable projective Λ' module whose radical is $M = \oplus M_r$. Easy to see all these M_r are GR submodules of P and pairwise non-isomorphic. By this way, for any sequence of positive integers $(a_1, a_2, \cdots a_s)$, we can construct indecomposable module M with s non-isomorphic GR submodules T_i such that the $|\mathrm{top} T_i| = a_i$.

Example. If T and N are two non-isomorphic GR submodules of M, then $\dim \mathrm{Hom}(T, M)$ may not be equal to $\dim \mathrm{Hom}(N, M)$.

Let Λ be the wild hereditary algebra kQ with $Q_{1,2}$ is the following quiver:



Let P_o be the indecomposable projective module and S_1 ,S_2 are the two simple modules. Easy to see, $\dim\mathrm{Hom}(S_1, P_o) = 1$ and $\dim\mathrm{Hom}(S_2, P_o) = 2$. In this way, for any pair of integral number (a, b), we consider the algebra $\Lambda = kQ_{a,b}$. The indecomposable projective Λ-module P_o has two non-isomorphic GR submodules such that the corresponding Hom space have dimension a, b respectively. More generally, for any sequence of positive integrals $(a_1, a_2 \cdots, a_n)$, we consider the algebra $\Lambda = kQ_{a_1,a_2,\cdots,a_n}$ and the indecomposable projective Λ-module P_o. Then P_o has n non-isomorphic GR submodules such that the corresponding Hom spaces have dimension $a_1, a_2, \cdots, a_n$ respectively.

2.4 Number of GR submodules

Lemma 2.1.3 tells us that if Λ is a directed algebra and M is an indecomposable module with $\alpha(M) = 4$, then M has at most 3 GR submodules. In this section, we will present another kinds of indecomposable modules which have, up to isomorphism, at most 3 GR submodules. We fix a finite dimensional algebra Λ.

Proposition 2.4.1. *Suppose M is an indecomposable module and T is a GR submodule of M with $|T| = \frac{1}{2}|M|$. Then,*
(1). up to isomorphism, T is the unique GR submodule of M.
(2). $\mu(M) > \mu(M/T)$.

Proof. (1). Assume N is a GR submodule of M which is not isomorphic to T. Note that $|T| = |N|$, and T (N) is a direct summand of any proper submodule of M containing T (N). Consider the submodule $T + N$ which contains both T and N as proper submodules. If $T + N$ is a proper submodule of M, then $T \oplus X = T + N = Y \oplus N$ for some X and Y since T, N are GR submodules of M. $N \ncong T$ implies N is isomorphic to a direct summand of X. It follows $|N| \leq |X|$. But

$$|X| = |T + N| - |T| < |M| - |T| = \frac{1}{2}|M| = |N|.$$

We get a contradiction.

Now we assume $T + N = M$, then

$$|M| = |T + N| = |T| + |N| - |T \cap N| = |M| - |T \cap N|.$$

It follows that $T \cap N = 0$ and hence $M = T + N = T \oplus N$. It is a contradiction since M is indecomposable. Hence, up to isomorphism, T is the unique GR submodule of M.

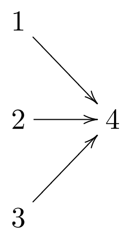
(2). It is trivial if M/T is simple. Assume M/T is not simple and $\mu(M) < \mu(M/T)$. Let X be a GR submodule of M/T. Then the GR inclusion $X \subset M/T$ factors through M since $T \subset M$ is a GR submodule. In particular there is a monomorphism from X to M. Thus, $\mu(X) \leq \mu(T) < \mu(M) < \mu(M/T)$. $|X| < |M/T| = \frac{1}{2}|M| = |T|$ implies $\mu(X) < \mu(T)$. Since X is a GR submodule of M/T, $\mu(T)$ starts with $\mu(X)$. Hence $|T| > |M/T|$ by 1.6.1. A contradiction. □

We can show the following proposition by using the same method:

Proposition 2.4.2. *Suppose M is an indecomposable module and T, N are non-isomorphic GR submodules of M. Then*
(1). $|T| > \frac{1}{2}|M|$ if and only if $T + N = M$.
(2). $|T| < \frac{1}{2}|M|$ if and only if $T \cap N = 0$.
(3). if $|T| = \frac{1}{m}|M|$, then M has, up to isomorphism, at most $m-1$ GR submodules.

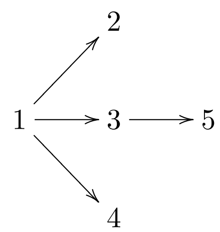
The following example shows that if $|T| \neq \frac{1}{2}|M|$, both $\mu(M) < \mu(M/T)$ and $\mu(M) > \mu(M/T)$ may happen.

Example. First consider hereditary algebra $\Lambda = kD_4$ with the following orientation:



Consider the indecomposable module M with $\underline{\dim} M = (1,1,1,2)$. It is easy to see that $\mu(P_1) = \mu(P_2) = \mu(P_3) = \{1,2\}$ and $\mu(M) = \{1,2,5\}$. Since there is an almost split sequence $0 \to P_4 \to P_1 \oplus P_2 \oplus P_3 \to M \to 0$, P_1, P_2 and P_3 are 3 GR submodules of M. Note that for each GR submodule of M, the corresponding GR factor module has GR measure $\{1,2,3\}$ which is larger than $\mu(M)$.

Now consider hereditary algebra $\Lambda' = kD_5$ with the following orientation:



Note that rad$P_1 = P_2 \oplus P_3 \oplus P_4$, $\mu(P_3) = \{1,2\}$ and P_2, P_4 are both simple projective modules. Hence $\mu(P_1) = \{1,2,5\}$ and, up to isomorphism, P_3 is the unique GR submodule of P_1. $\mu(P_1/P_3) = \{1,3\} < \mu(P_1)$.

We now consider indecomposable modules with simple submodules as GR submodules. The following lemma is straightforward.

Lemma 2.4.3. *Let M be an indecomposable module with $|M| = m$. The following are equivalent:*
(1). $\mu(M) = \{1, m\}$.
(2). every simple submodule of M is a GR submodule.
(3). every proper submodule of M is semisimple.
(4). $\mathrm{soc}M$ *is the unique maximal submodule of M.*
(5). M is a local module with loewy length 2.

Proposition 2.4.4. *Let Λ be a directed algebra and M be an indecomposable Λ-module with $\mu(M) = \{1, m\}$. Then*
(1). M is a thin module. And for any simple submodule S of M, $(M/S, S)$ is an orthogonal exceptional pair to M. Thus, $\mathrm{dimHom}(S, M) = 1$.
(2). M has exactly $m-1$ GR submodules.
(3). $m \leq n$ where n is the number of isomorphism classes of simple modules.
(4). $m \leq 4$.

Proof. (1). By the above lemma, M is a local module. Thus the projective cover of M is indecomposable, and any factor module of M is indecomposable . Thus M is a thin module since all indecomposable projective over a directed algebra are thin modules. For each simple submodule S of M, $\mathrm{Hom}(S, M/S) = 0$ since M is a thin module. By using the long exact sequences induced by $0 \to S \to M \to M/S \to 0$, we obtain $(M/S, S)$ is an orthogonal exceptional pair and $\mathrm{dimHom}(S, M) = 1$.

(2). Given a simple submodule S of M. If M/S is not simple, then any proper submodule of M/S is of the form N/S with N a proper submodule of M. Any proper submodule N of M is semisimple since $\mu(M) = \{1, m\}$. Thus N/S is semisimple and $\mu(M/S) = \{1, m-1\}$. M is a thin module implies any simple module occur at most once as composition factor. If $m = 2$, then M has unique GR submodule. Now assume $m \geq 3$ and S is a simple submodule of M. Thus $\mu(M/S) = \{1, m-1\}$ and by induction, M/S has exactly $m-2$ pairwise non-isomorphic GR (simple) submodules, say, S_2, S_3, $\cdots$ S_{m-1} and $S_i \ncong S$ for $2 \leq i \leq n-2$. Note $\mathrm{soc}(M/S) \subset \mathrm{soc}M$. Then $S_1 = S$, S_2,$\cdots$ S_{m-1} are pairwise non isomorphic simple modules and hence GR submodules of M. On the other hand, $|S| = 1 = \frac{1}{m}$, up to isomorphism, there are at most $m-1$ GR submodules of M. Since $\mathrm{dimHom}(S, M) = 1$, $S \cong S'$ implies $S = S'$. Thus M has exactly $m-1$ different GR submodules.

(3). It follows from (2).

(4). The M has exactly $m-1$ GR submodules, say, S_1, S_2, $\cdots$, S_{m-1}. Without loss of generality, we may assume M is a sincere module and all S_i, $1 \leq i \leq m-1$ are simple projective modules. Then $m-1$ simple projective modules corresponding to the $m-1$ end points in the orbit quiver of Λ. Since $m \leq n$ where n is the number of isomorphism classes of simple Λ-modules. There exists some other indecomposable projective module which corresponds to another end point in the orbit quiver. Hence the orbit quiver has at least m end points. But for a sincere representation directed algebra, the orbit quiver should be a tree with at most 4 end points. Hence we have $m \leq 4$. □

Corollary 2.4.5. *Let M be an indecomposable module over a directed algebra. Assume $\mu(M) = \{1 = l_1, l_2, \cdots, l_m = |M|\}$. Then $l_2 \leq 4$.*

The next example shows if Λ is not directed and M is an indecomposable module with GR measure $\mu(M) = \{1, |M|\}$. Then, up to isomorphism, there may not exist $m - 1$ GR submodules.

Example. Let Λ be a Kronecker algebra and P_0, P_1 the indecomposable projective module, where P_0 is the simple projective module. Clearly $\mu(P_0) = \{1\}$, and $\mu(P_1) = \{1, 3\}$. Up to isomorphism, P_0 is the unique GR submodule of P_1. But we should note that $\text{dimHom}(P_0, P_1) = 2$.

Example. We can easily construct indecomposable module with GR measure $\{1, n\}$ for any $n > 4$. Consider the algebra given by a star quiver with n outgoing arrows from the center vertex v_0. Then easy to see the indecomposable projective non simple module P_{v_0} is with $\mu(P_{v_0}) = \{1, n\}$.

Chapter 3

The Gabriel-Roiter measure and Hom-Orthogonality

This chapter will be devoted to a discussion of the Hom-orthogonality of the Gabriel-Roiter measure. We shall give the proof of the following theorem:

Theorem A. *Let* Λ *be a representation-finite hereditary* k*-algebra.*
(1). If T *is a Gabriel-Roiter submodule of* M*, then* $\mathrm{Hom}(T, M/T) = 0$.
(2). Each indecomposable module M *possesses at most 3 Gabriel-Roiter submodules.*

3.1 Some Lemmas

In this section, we collect a few subsidiary results.

Lemma 3.1.1. *Let* Λ *be a directed algebra and* $\delta : 0 \to T \to M \to M/T \to 0$ *be a short exact sequence of indecomposable* Λ*-modules. Then the following are equivalent:*
(1). $\mathrm{Hom}(T, M/T) = 0$.
(2). $\dim\mathrm{Hom}(T, M) = 1$.
(3). $\dim\mathrm{Hom}(M, M/T) = 1$.
(4). $\dim\mathrm{Ext}^1(M/T, T) = 1$.
(5). $\mathrm{Ext}^1(M, T) = 0$.
(6). $\mathrm{Ext}^1(M/T, M) = 0$.
(7). $(M/T, T)$ *is an orthogonal exceptional pair to* M*.*

Proof. First recall that any indecomposable module M over a directed algebra is a brick without self-extensions, i.e., End(M) $= k$ and $\mathrm{Ext}^1(M, M) = 0$. Applying the functors $\mathrm{Hom}(T, -)$, $\mathrm{Hom}(-, T)$, $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$, $\mathrm{Hom}(M/T, -)$, $\mathrm{Hom}(-, M/T)$ to the short exact sequence δ, we get 6 exact sequences. By comparing the dimensions of these vector spaces, we easily get the first 6 equivalent conditions. Since Λ is directed, conditions (1) and (7) are also equivalent. □

For each indecomposable module M, we denote by $\alpha(M)$ the number of the direct summands of X where $X \to M$ is a minimal right almost split map.

Lemma 3.1.2. *Let Λ be a directed algebra and M be a sincere indecomposable Λ-module.*
(1). Assume $M \to X_1 \to X_2 \cdots \to X_n$ is a sectional path with n maximal and $\alpha(X_i) \leq 2$ for each i. Then the irreducible map $\tau X_1 \to M$ is a monomorphism.
(2). Assume $Y_m \to \cdots \to Y_2 \to Y_1 \to M$ is a sectional path with m maximal and $\alpha(\tau^{-1}Y_j) \leq 2$ for each j. Then the irreducible map $M \to \tau^{-1}Y_1$ is an epimorphism.

Lemma 3.1.3. *Let*

$$0 \to A_1 \overset{\binom{f_1}{g_1}}{\to} B_1 \oplus A_2 \overset{(h_1, f_2)}{\to} B_2 \to 0$$

and

$$0 \to A_2 \overset{\binom{f_2}{g_2}}{\to} B_2 \oplus A_3 \overset{(h_2, f_3)}{\to} B_3 \to 0$$

be two exact sequences. Then the sequence

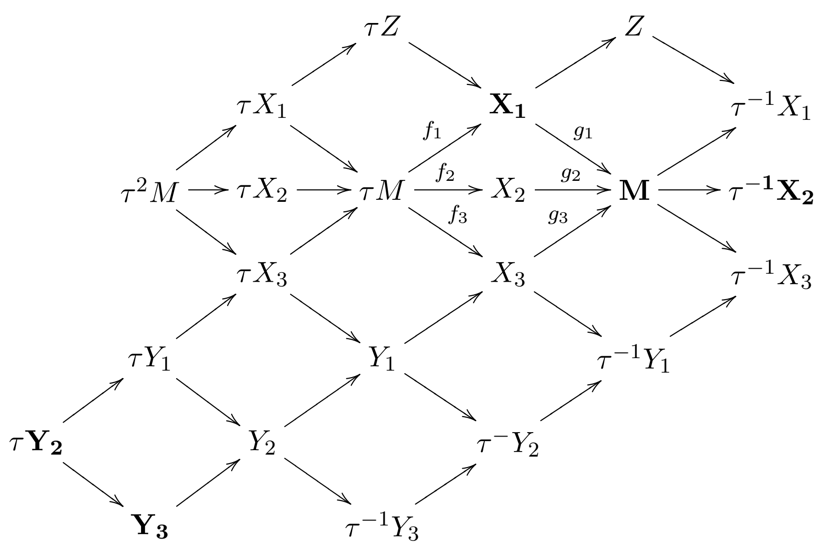
$$0 \to A_1 \overset{\binom{f_1}{g_2 g_1}}{\to} B_1 \oplus A_3 \overset{(h_2 h_1, -f_3)}{\to} B_3 \to 0$$

is exact.

Proof. Straightforward. □

By way of example, we illustrate the use of 3.1.3.

Example*. Consider the following full subquiver of an AR quiver of $\Lambda = kE_8$ for some fixed orientation.



Our aim is to study the composition of the irreducible maps, $\mathbf{X_1} \to \mathbf{M} \to \tau^{-1}\mathbf{X_2}$. Since it is a sectional path, $\mathrm{Ext}^1(\tau^{-1}X_2, X_1) = 0$ and $\mathrm{dimHom}(X_1, \tau^{-1}X_2) = 1$. Thus, the composition is either injective or surjective by 1.4.2. To decide which alternative applies, we only need to compare the length of the two modules. We claim that

$$|X_1| - |\tau^{-1}X_2| = |\tau Y_2| - |Y_3|.$$

We have the following two almost split sequences:

$$0 \to \tau M \to X_1 \oplus X_2 \oplus X_3 \to M \to 0,$$

and

$$0 \to X_2 \to M \to \tau^{-1}X_2 \to 0.$$

By Lemma 3.1.3, we obtain a new short exact sequence:

$$0 \to \tau M \to X_1 \oplus X_3 \to \tau^{-1}X_2 \to 0.$$

Again, by using the almost split sequence $0 \to \tau X_3 \to \tau M \oplus Y_1 \to X_3 \to 0$, we obtain the following new short exact sequence:

$$0 \to \tau X_3 \to Y_1 \oplus X_1 \to \tau^{-1}X_2 \to 0.$$

Thus, we have

$$|X_1| - |\tau^{-1}X_2| = |\tau X_3| - |Y_1| = |\tau Y_1| - |Y_2| = |\tau Y_2| - |Y_3|.$$

The last two identities follow from the fact that the squares involved are push-out and pull-back diagrams, i.e., short exact sequences. If Y_3 is projective, then the irreducible map $\tau Y_2 \to Y_3$ is a monomorphism, and hence, $|X_1| - |\tau^{-1}X_2| < 0$ which means the composition $X_1 \to M \to \tau^{-1}X_2$ is a monomorphism. If Y_3 is not projective, then there is a short exact sequence $0 \to \tau Y_3 \to \tau Y_2 \to Y_3 \to 0$. Thus, the composition $X_1 \to M \to \tau^{-1}X_2$ is an epimorphism.

Definition 3.1.4. *An indecomposable module M is* **Gabriel-Roiter maximal** *(briefly* **GR maximal***), if it is not a GR submodule of any indecomposable module.*

By definition, all indecomposable injective modules are GR maximal. If M is a maximal indecomposable module over a representation-finite algebra Λ, then M is GR maximal. These are trivial GR maximal modules. Our next lemma shows that non-trivial GR maximal modules exist.

Lemma 3.1.5. *Let Λ be an arbitrary finite dimensional algebra. Assume T is an indecomposable Λ-module and $T \stackrel{f=(f_i)}{\to} \oplus_{i=1}^{n} X_i$ is a minimal left almost split map such that each f_i is an epimorphism. Then T is GR maximal.*

Proof. Since any injective module is GR maximal, we may assume that T is not injective. In this case $f = (f_i)$ is injective, and $n \geq 2$ since each f_i is an epimorphism. If $l : T \to M$ is a GR inclusion for some indecomposable module M, we get the following commutative diagram since f is minimal left almost split.

$$\begin{array}{ccccc} 0 & \longrightarrow & T & \xrightarrow{(f_i)} & \oplus_{i=1}^{n} X_i \\ & & \Big\downarrow{\scriptstyle l} & \swarrow{\scriptstyle (g_i)} & \\ & & M & & \end{array}$$

The map $T \overset{(g_i f_i)}{\to} \oplus_{i=1}^{n} \mathrm{Im}(g_i f_i)$ is injective since $l = \sum_{i=1}^{n} g_i f_i$ is injective. Then, by the main property 1.7, we have $\mu(T) \leq \max \mu(\mathrm{Im}(g_i f_i))$. Note for each i, $g_i f_i$ is not injective since f_i is a proper epimorphism. Hence the above inequality is strict. On the other hand, for each i, $\mathrm{Im}(g_i f_i)$ is a proper submodule of M. So we get

$$\mu(T) < \max \mu(\mathrm{Im}(g_i f_i)) < \mu(M),$$

which is a contradiction since T is a GR submodule of M. □

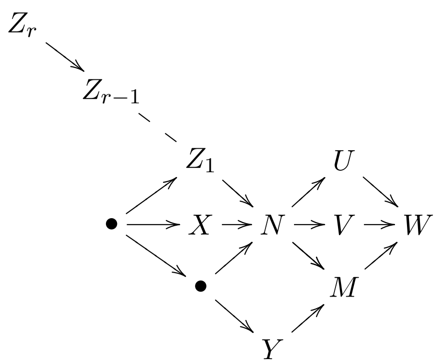
3.2 Reduction

Recall that the orbit quiver of a sincere directed algebra is a tree with at most 4 end points. If the orbit quiver is a star with 3 branches, we say that M lies on the center if $[[M]]$ has exactly 3 neighbors, namely the center vertex of the star; and say that M lies on the quasi center if $[[M]]$ has exactly 2 neighbors such that one of the neighbors is the center vertex of the star.

Proposition 3.2.1. *Let Λ be a directed algebra whose orbit quiver is a star with 3 branches and one of the branch is of length 1 (for example, D_n, $E_{6,7,8}$). If M is a sincere indecomposable Λ-module which lies on the center or the quasi-center, then M has at most 3 GR submodules and for each GR submodule T of M,* $\mathrm{Hom}(T, M/T) = 0$.

Proof. First assume M lies on the center. Then $[[M]]$ is the unique point in $\overline{\mathcal{O}(\Lambda)}$ with 3 neighbors and any other point $[[N]]$ has at most 2 neighbors. Let $g : Y \to M$ be an irreducible epimorphism. By Lemma 3.1.2, we get g is a monomorphism. And hence all irreducible maps to M are monomorphism. Therefore, any GR submodule of M is given by an irreducible map, see (1.7.7). Thus, up to isomorphism , M has at most 3 GR submodules. Note that if $T \to M$ is an irreducible map, then $\dim\mathrm{Hom}(T, M) = 1$. Thus M has at most 3 GR submodules.

Now we assume M lies on the quasi center. Consider the following subquiver of the AR quiver:



M is sincere implies the irreducible map $Y \to M$ is an injective (3.1.2) . If the irreducible map $N \to M$ is also injective, then any GR submodule of M is isomorphic to either N or Y. So we may assume the irreducible map $N \to M$ is an epimorphism.

Stating with the two short exact sequence

$$0 \to X \to N \to V \to 0$$

and

$$0 \to N \to U \oplus V \oplus M \to W \to 0,$$

we get the following short exact sequence

$$0 \to X \to M \oplus U \to W \to 0$$

by using Lemma 3.1.3. Thus we get $|X| - |M| = |U| - |W|$. Let $N \to U = U_1 \to \cdots \to U_s$ be the sectional path with s maximal. M is sincere implies U_s is not injective. It follows that the irreducible map $U \to W$ is a monomorphism. Thus $|X| < |M|$. Since $\dim\mathrm{Hom}(X, M) = 1$, the image X' of the unique map is an indecomposable submodule of M. If $X \ncong X'$, then there is a path from X to X', then to M. Thus $X' \cong N$, a contradiction since the irreducible map $N \to M$ is an epimorphism. Thus, the composition $X \to N \to M$ is a monomorphism.

Assume T is a GR submodule which does not lie on the sectional paths $Z_r \to \cdots \to N \to M$, $X \to N \to M$, or $\cdots \to Y \to M$. Then the GR inclusion factors through $X \oplus Y$. In particular, there is a monomorphism $T \to X \oplus Y$. It follows T is isomorphic to X or Y since T is a GR submodule of M and both X and Y are submodule of M. This contradicts our assumption. So any GR submodule of M lies on one of the 3 sectional paths. In particular, $\dim\mathrm{Hom}(T, M) = 1$. Note that on each sectional path, there exist at most one GR submodule of M. Therefore, M has at most 3 GR submodules and for each GR submodule T of M, $\dim\mathrm{Hom}(T, M) = 1$. Therefore, $\mathrm{Hom}(T, M/T) = 0$ by 3.1.1. $\square$

From now on, we assume Λ is a representation-finite hereditary algebra and M is an indecomposable Λ-module.

Let Λ' be the quotient $\Lambda/\mathrm{ann}M$, where $\mathrm{ann}M = \{\lambda \in \Lambda | \lambda M = 0\}$ is an ideal of Λ. Then M is an indecomposable Λ' module. T, as a Λ-module, is a GR submodule of M if and only if it is, as a Λ'-module, a GR submodule of M. It follows that $\mu_\Lambda(M) = \mu_{\Lambda'}(M)$. It is easy to see $\mathrm{ann}M = \sum_i Ae_iA$, where each e_i is a primitive idempotent such that $\dim\mathrm{Hom}(P_i, M) = (\underline{\dim}M)_i = 0$. It follows that the Gabriel quiver of $\Lambda' = \Lambda/\mathrm{ann}M$ is obtained from the Gabriel quiver of Λ by deleting vertices. Thus, Λ' is again representation-finite and hereditary. This allows us to assume M is a sincere indecomposable Λ-module.

Let T be a GR submodule of M. By Lemma 3.1.1, to show the orthogonal property $\mathrm{Hom}(T, M/T) = 0$, is equivalent to show $\dim\mathrm{Hom}(T, M) = 1$. Note that in this case, if N is also a submodule of M with $N \cong T$, then $N = T$.

If M is projective, then all irreducible maps to M are monomorphisms. Thus, all GR submodules of M are given by irreducible maps and, for each GR submodule T, $\dim\mathrm{Hom}(T, M) = 1$. Since there are at most 3 sectional paths to M, M has at most 3 GR submodules. If M is injective, then M/T is also injective and there is a sectional path from M to M/T since Λ is hereditary. Note that there are at most 3 sectional paths going out from M and on each sectional path, there exists at most one corresponding GR factor module. Thus, M has at most 3 GR submodules. Therefore, **Theorem A** holds for indecomposable projective modules and indecomposable injective modules. This allows us to assume M is neither projective nor injective.

As an upshot of our discussion, we shall henceforth assume:

- M does not lie on the center or the quasi-centers.
- M is a sincere indecomposable module.
- M is neither projective nor injective.

3.3 Proof of Theorem A

This section is devoted to the proof of **Theorem A**.

Recall that $\Lambda = kQ$ with the underlying graph $\overline{Q}$ of Q being of type A_n, D_n, and $E_{6,7,8}$. There is a one to one correspondence between the isomorphism classes of indecomposable Λ-modules and the positive roots of the corresponding semisimple Lie algebras. Precisely, the dimension vectors of simple modules correspond to simple roots.

We assume M is indecomposable, sincere, not projective, not injective, and that does not lie on the center or the quasi-centers. If T is a GR submodule of M, we need to show $\dim\mathrm{Hom}(T, M) = 1$ by Lemma 3.1.1. Recall that if there is a sectional path from X to Y, then $\mathrm{Ext}^1(Y, X) = 0$ and $\dim\mathrm{Hom}(X, Y) = 1$ (1.4.2). It follows that the composition of the irreducible maps from X to Y is either injective or surjective and hence, any two indecomposable modules on the same sectional path have different length. Therefore, on each sectional path, there exists at most one GR submodule of M.

The main idea of the proof is the following:

(1). Find several indecomposable submodules of M. They are said to be **test submodules**

of M. For each test submodule X of M, dimHom$(X, M) = 1$. The direct sum of the test submodules is called a **test module**.
(2). Find an indecomposable module C before (C is before X if there is a path from C to X), the test submodules of M such that any map from C to M factors through the test module we have selected. In particular, if a GR submodule T of M is before C , then the GR inclusion factors through the test module. It follows that there is a monomorphism from T to the test module of M. Thus, T is isomorphic to one of the test submodules by the main property 1.7. This contradiction shows that any GR submodule T of M is not before C.
(3). Check the modules which are before M but not before C.
(4). In some cases, we can not find test module of M. But we may get the possibilities of the orientation of the underlying graph, and hence the dimensional vector of M. We may calculate the GR submodules of M directly.

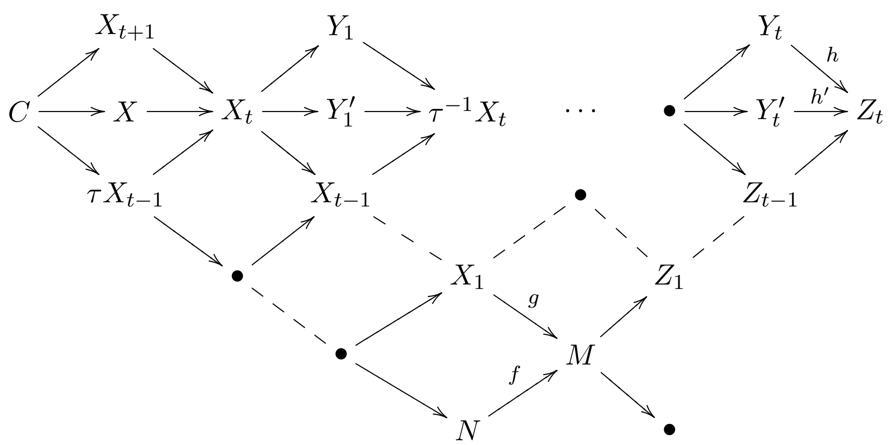
Now we will show this theorem case by case:

(1). A_n **type.**

In this case, there is only one sincere positive root. Thus, M is sincere implies it is the unique sincere indecomposable module and all irreducible maps to M are monomorphisms. Therefore the GR inclusions are given by irreducible maps. Thus, dimHom$(T, M) = 1$ for any GR submodule T and M has at most 2 GR submodules since there exists at most 2 irreducible maps to M.

(2). D_n **type.**

First assume $sl(M) > 1$. (Recall that if the orbit quiver is a star, then $sl(M)$ is defined to be the distance to the center vertex to $[[M]]$. Thus, $sl(M) = 0$ if M lies on the center and $sl(M) = 1$ if M lies on a quasi-center.) Consider the following full subquiver of the AR quiver:



Since M is sincere, $Z_t \neq 0$, and Y_t, Y_t' are not injective. by Lemma 3.1.2, f is injective if $\alpha(M) = 2$ and $f = 0$ ($N = 0$) if $\alpha(M) = 1$. The arguments given in the **Example*** in

3.1 show

$$|M| - |X| = |M| - |X_t| + |Y_1'| = |Z_1| - |Y_1| = \cdots = \begin{cases} |Z_t| - |Y_t'| & \text{if t is even.} \\ |Z_t| - |Y_t| & \text{if t is odd.} \end{cases}$$

Since Y_t and Y_t' are not injective, h and h' are monomorphisms. It follows that $|M| > |X|$. Therefore the composition of the irreducible maps $X \to X_t \to X_{t-1} \to \cdots \to M$ is a monomorphism.

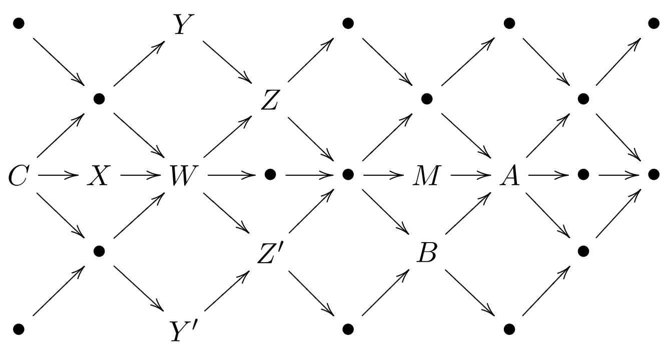
We may select $X \oplus N$ as the test module.

If T is not on the sectional paths $X_{t+1} \to \cdots \to X_1 \to M$ or $X \to \cdots \to X_1 \to M$, then $T \cong N$ since the GR inclusion factors through N. It follow that there is a sectional path from T to M. Thus, we have $\dim\mathrm{Hom}(T, M) = 1$. Therefore, M has at most 3 GR submodules.

If M lies on the boundary with $sl(M) = 1$ (using the above picture, say $M = Y_1'$), The arguments given in the **Example*** in 3.1 show $|M| - |X_{t+1}| = |M| + |Y_1| - |X_t| = |\tau^{-1}X_t| - |X_{t-1}|$. Since $Y_1' = M$ is sincere and not injective, $\tau^{-1}X_t$ is sincere. Lemma 3.1.2 implies the irreducible map $X_{t-1} \to \tau^{-1}X_t$ is a monomorphism. Therefore the composition $X_{t+1} \to X_t \to M$ is a monomorphism. We may select X_{t+1} as the test module. It follows that any GR submodule of M is either isomorphic to X_{t+1} or lies on the sectional path $\cdots \to \tau X_{t-1} \to X_t \to M = Y_1'$. Therefore M has at most 2 GR submodules.

(3). E_6 **type.**

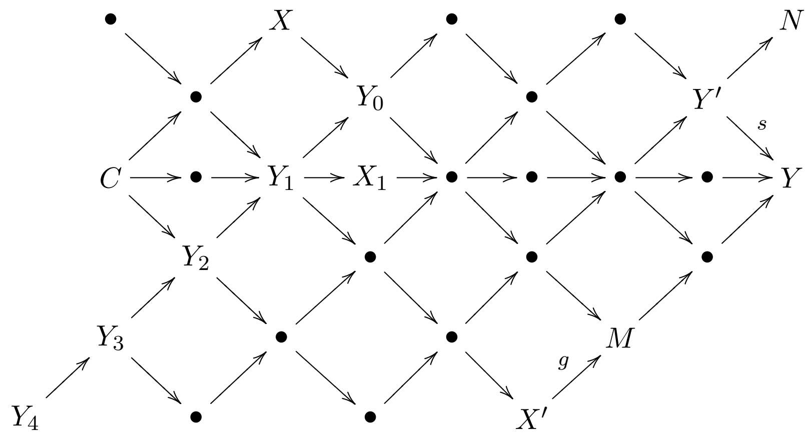
In this case, all sincere indecomposable modules lie either on the center or the quasi-centers, or on the boundary with $sl(M) = 1$. So we need only consider the case $sl(M) = 1$:



Since M is sincere and not injective, A is sincere and not injective. Lemma 3.1.2 implies that the irreducible map $B \to A$ is injective. By using the arguments given in **Example*** in 3.1, we obtain $|M| - |Y| = |M| + |\tau^{-1}Z| - |\tau A| = |A| - |B| > 0$. Thus, the composition of irreducible maps $Y \to Z \to \bullet \to M$ is a monomorphism. For the same reason, the composition of irreducible maps from Y' to M is also a monomorphism. We select $Y \oplus Y'$ as the test module. Thus T is not before C and $\dim\mathrm{Hom}(T, M) = 1$. Examine all the modules lying before M but not before C. Without loss of generality, we may assume the compositions of the irreducible maps $Z \to \bullet \to M'$ and $Z' \to \bullet \to M$ are epimorphisms. It follows Z Z' are sincere and τY, $\tau Y'$ are not zero. Thus, W is GR maximal since all irreducible maps going out from M are epimorphisms, see (3.1.5). It is easy to see dim $\mathrm{Hom}(\tau Z, M) = 1$, and the unique non-zero map from τZ to M factors through Y, thus is neither an epimorphism nor a monomorphism. Finally, $\mathrm{Hom}(\tau Y, M) = 0 = \mathrm{Hom}(\tau Y', M)$. Thus, M has at most 3 GR submodules with X, Y, Y' being the 3 possibilities.

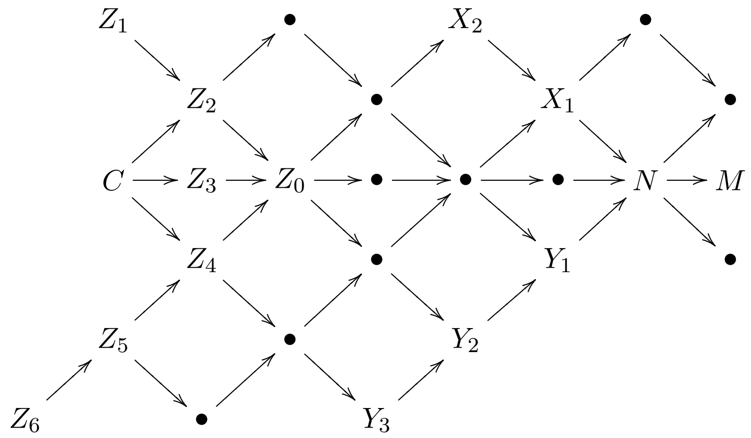
(4). E_7 **type.**

We first assume $\alpha(M) = 2$ and $sl(M) = 2$.



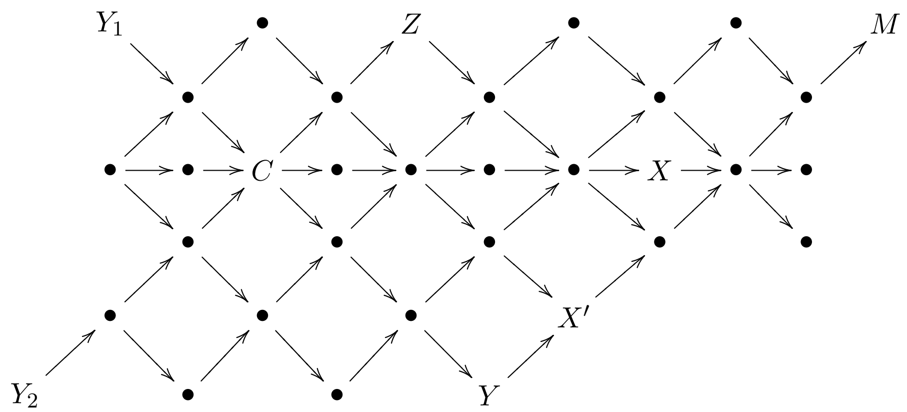
Since M is sincere, g is a monomorphism by Lemma 3.1.2. If the composition of irreducible maps from X_1 (or Y_0) to M is a monomorphism, we may select $X_1 \oplus X'$ ($Y_0 \oplus X'$) as the test module. We now assume both of the compositions are epimorphisms. It follows that X_1 and Y_0 are sincere, non-projective, non-injective modules and τX, Y_4 are not zero. M is sincere implies $N \neq 0$ and not injective. Thus, the irreducible map $s : Y' \to Y$ is a monomorphism. By using the arguments given in **Example*** in 3.1, we obtain $|M| - |X| = |Y| - |Y'| > 0$, and hence, the composition of irreducible maps from X to M is a monomorphism. Let $X \oplus X'$ be the test module. Thus any GR submodule of M is not before C. Note that Y_1 is GR maximal since the outgoing irreducible maps are epimorphisms. For modules τX_1, τY_0, τX, the corresponding Hom spaces are of dimension 1. But the corresponding morphisms are neither epimorphisms nor monomorphisms, thus there are not GR submodules of M. $|X_1| - |Y_2| = |X_1| + |X'| - |Y_1| = |M| - |X_1| < 0$ since we have assumed there is an epimorphism from X_1 to M. Thus $|Y_2| > |X_1| > |M|$. Therefore, if T is a GR submodule of M then T is isomorphic to X', X or one of Y_4, Y_3. Therefore M has at most 3 GR submodules and for each GR submodule T, $\mathrm{dimHom}(T, M) = 1$.

Now we begin to consider the cases $\alpha(M) = 1$. First assume $sl(M) = 1$.



We may assume the compositions $X_1 \to X_0 = N \to M$, $Y_2 \to Y_1 \to Y_0 = N \to M$ are epimorphisms and $C \neq 0$, else we may select X_1, Y_2 as test submodules. Under this assumption, Z_1 and Z_6 are not zero. As before, by using the arguments given in **Example*** in 3.1, we obtain that the compositions of the irreducible maps $X = X_2 \to X_1 \to X_0 = N \to M$ and $Y = Y_3 \to Y_2 \to Y_1 \to Y_0 = N \to M$ are both injective. Select $X_2 \oplus Y_3$ as the test module. Thus any GR submodule of M is not before C and not isomorphic to X_i for $i = 4, 5, 6$. Z_0, $\tau^{-1}Z_0$ are GR maximal since the irreducible maps outgoing are all epimorphisms. Note that we have assumed the composition $Y_2 \to Y_1 \to N \to M$ to be surjective, thus there is epimorphism from τY_1 to M. All maps from Z_3 and τX_1 factors through X_2, hence are neither epimorphism nor monomorphism. Thus if T is a GR submodule of M, then T is isomorphic to X_2, or Y_3, or one of Z_1, Z_2, $\tau^{-1}Z_3$. Thus, M has at most 3 GR submodules and for each GR submodule T, $\dim\mathrm{Hom}(T, M) = 1$.

If $sl(M) = 2$, we consider the following section of the AR quiver:



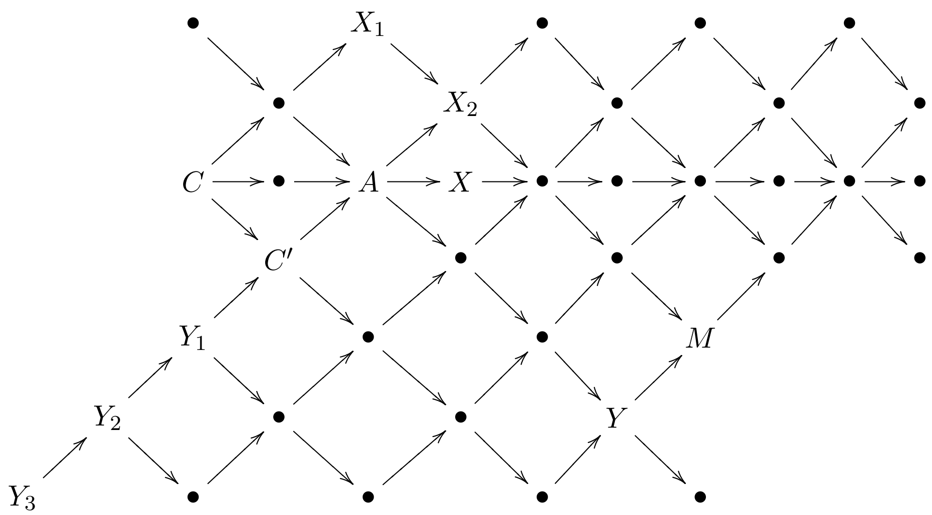
We may assume the compositions of irreducible maps from X to M and from X' to M are both epimorphisms, since other cases are similar. It follows X and X' are both sincere. By calculating the dimensions of the Hom-spaces we can easily get that Y_1 and Y_2 are not zero.

We first note there is a monomorphism from Y to M. Now consider the indecomposable module Z. By the AR-formula (1.4.1), we obtain $\mathrm{Ext}^1(M,Z) \cong \mathrm{DHom}(Z,\tau M) = 0$. Also we have $\mathrm{dimHom}(Z,M) = 1$. It follows that the unique map from Z to M is an epimorphism or a monomorphism (1.4.2) and thus, a monomorphism since M is sincere. We select $Y \oplus Z$ as the test module. All GR submodules of M are not before C. Again, it easily follows that M has at most 3 GR submodules, and for each GR submodule T of $\mathrm{dimHom}(T,M) = 1$.

If $sl(M) = 3$, then $(\underline{\mathrm{dim}}M)_i = 1$, i.e, M is a thin module. Thus, $\mathrm{Hom}(T, M/T) = 0$ since $(\underline{\mathrm{dim}}M)_i = 1$ if and only if $(\underline{\mathrm{dim}}M/T)_i = 0$. Consider the two sectional paths $X_5 \to X_4 \to X_3 \to X_2 \to X_1 \to M$ and $X' \to X_3 \to X_2 \to X_1 \to M$ with $\alpha(X_5) = 1 = \alpha(X')$. Since M is neither projective nor injective, both $\mathrm{Hom}(\tau^i M, M) \neq 0$ implies $i = 4$ or 7, and $\mathrm{Hom}(M, \tau^{-j} M) = 0$ implies $j = 4$ or 7. But for each indecomposable X, $\tau^{10} X = 0$. Thus, we have $\tau^4 M$ is projective. It follows the unique map from X_5 to M is a monomorphism. We may select X_5 to be the test module.

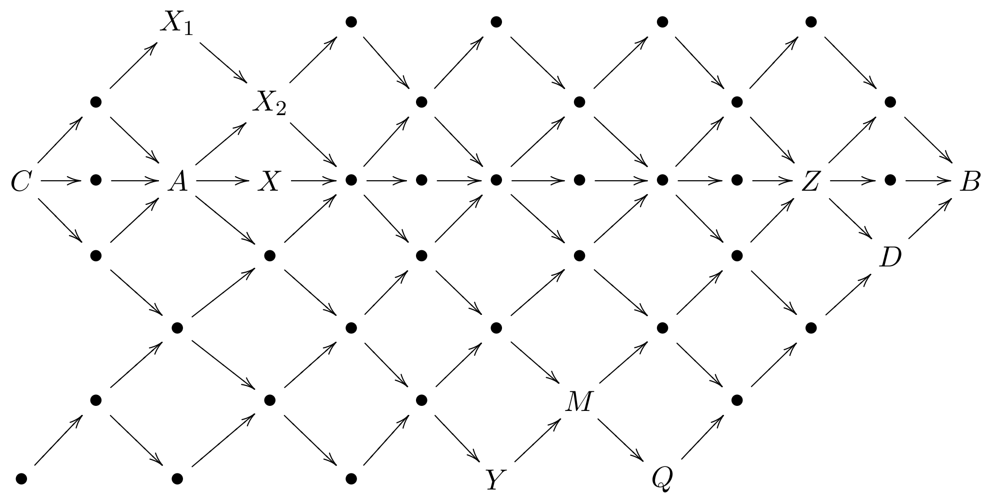
(5). E_8 **type.**

The same method will be used. We outline the proof. First consider the case $\alpha(M) = 2$ and $sl(M) = 2$.



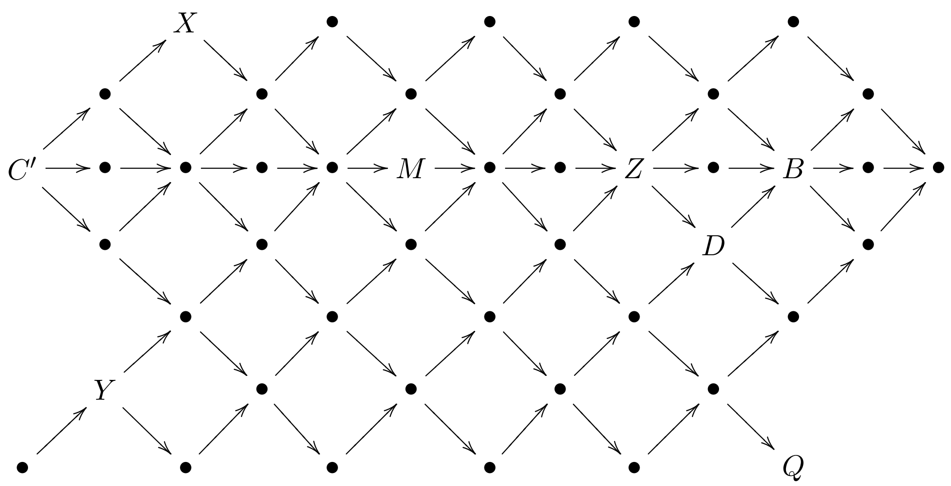
It is obvious that the irreducible map $Y \to M$ is injective. Without loss of generality, we may assume the compositions of the irreducible maps from X to M and from X_2 to M are both epimorphisms. Thus Y_3 and τX_1 are not zero. By comparing the length of X_1 and M, we get a monomorphism from X_1 to M. Thus, we may select $X_1 \oplus Y$ as the test module. The modules with 2 dimensional Hom-space to M are A and C' and τX_2. But A is GR maximal, $|X| - |C'| = |\tau^{-1} Y| - |X_2| \leq |M| - |X_2| \leq 0$ implies $|C'| > |X| > |M|$. Any morphism from τX_2 to M factors through $X_1 \oplus Y$. Hence $\mathrm{dimHom}(T,M) = 1$ and M has at most 3 GR submodules. Namely, a GR submodule of M is isomorphic to X_1, or Y, or one of Y_i, $1 \leq i \leq 3$.

Now assume $\alpha(M) = 2$ and $sl(M) = 3$.



As before, we may assume the morphisms from X_2 and X to M are both epimorphisms. If Z is injective, then $B = 0$. In this case, for any indecomposable injective module I, we have $(\underline{\dim} M)_i = \dim\mathrm{Hom}(M, I) = 1$. Thus, the orthogonality holds. Since Λ is hereditary and M is sincere, neither projective nor injective, only 3 possible orientations of E_8 occur. [Note that Q is not injective implies $\tau^{-i}Q$ are not injective for $i = 1, 2, 3$ since $\mathrm{Hom}(M, \tau^{-i}Q) = 0$ and M is sincere.] We can calculate one by one and get that there is only one GR factor modules, hence only one GR submodules in each case (see Appendix 1, Table-1). Assume Z is not injective. $|M| - |X_1| = |B| - |D| > 0$ if Q is not injective. Here we use that M is sincere and $\dim\mathrm{Hom}(M, \tau^{-i}Y) = 0$ for $1 \le i \le 4$. Select $Y \oplus X_1$ as the test module. If Q is injective, only 6 possible orientations of E_8 can occur. We can check one by one and again get only one GR submodule in each case (see Appendix 1, Table-2).

Now we assume M is on the boundary. We first consider the case $sl(M) = 1$ and the following full subquiver of the AR quiver:



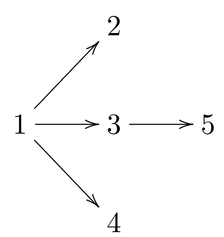
If Z is injective, then for all possibilities of orientations of E_8, $(\underline{\dim}M)_i = 1$ for all $i \in Q_0$. So $\mathrm{Hom}(T, M/T) = 0$. For each orientation, we get easily the GR submodules and corresponding factor modules (see appendix 1, Table-3). Assume that Z is not injective. Then $|M| - |Y| = |B| - |D| > 0$ if Q is not injective. [Note that if Q is not injective and $\mathrm{Hom}(M, \tau^{-1}Q) = 0$ implies $\tau^{-1}Q$ is not injective.] In this case, we select $Y \oplus X$ as the test module and $C = \tau C'$. If Q is injective, by calculating the dimension of the Hom-space, we have 64 possibilities of the orientations of E_8 such that M is sincere and not injective. For each orientation, we can easily get the dimension vector of M and calculate the GR submodules. In each case, we get exactly one GR submodule [similar to the situation in the case $\alpha(M) = 2$ and $sl(M) = 3$)].

Now let us come to the sincere indecomposable modules lying on the boundary. The unique proper sincere indecomposable with $sl(M) = 4$ in this orbit, has dimension vector $(1, 1, 1, 1, 1, 1, 1, 2)$. We assume $X \to M$ is the unique irreducible map. Then there is a unique irreducible map from $\tau^5 M$ to $\tau^4 X$. Note that $\dim\mathrm{Hom}(\tau^4 X, M) = 1$ by using the AR-formula (1.4.1) and direct calculation of dimension vector, $\tau^4 X$ is a submodule of M. So we may select $\tau^4 X$ as the test module. If $sl(M) = 2$, then $(\underline{\dim}M)_i \leq 2$. Except for only several possibilities of orientations of E_8, $\tau^3 M$ is a submodule of M, and we may select $\tau^3 M \oplus Y$ as the test module where Y lies on the boundary with $sl(Y) = 4$ and there is a sectional path from Y to M. If $\tau^3 M$ is not a submodule of M, then we may get the dimension vector of M for each orientation and calculate the GR submodule of M. In each case, we get only one GR submodule. □

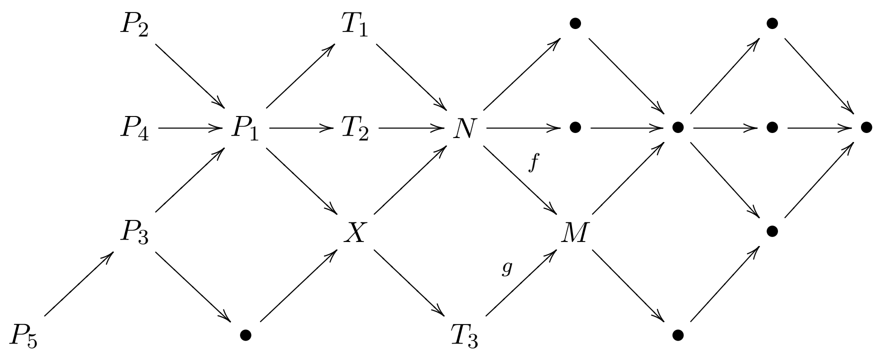
3.4 Examples

In this section, we want to show some examples. The first example shows that the GR inclusions of an indecomposable module are not necessarily given by irreducible maps even there do exist irreducible monomorphisms to it. Also some GR maximal modules are given there.

Example. Consider the hereditary algebra of type D_5 with the following orientation:



The AR quiver is the following:



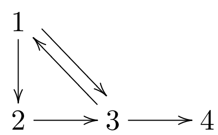
We consider the indecomposable module M where $\underline{\dim}M = (2,1,1,1,1)$. Then, f is an epimorphism and g is a monomorphism. By direct calculation, we get $\mu(T_1)$=$\mu(T_2)$=$\{1,2,4\}$, $\mu(T_3) = \{1,3\}$ and $\mu(N) = \{1,2,4,7\}$, $\mu(M) = \{1,2,4,6\}$. Hence T_1 and T_2 are the only two GR submodules of M and T_3 is not a GR submodule of M although the irreducible map g is monomorphism. Also in this example, M, X with $\underline{\dim}X = (1,1,1,1,0)$ and $\mu(X) = \{1,4\}$, P_1, N, $\tau^{-1}N$ with $\underline{\dim}\tau^{-1}N = (2,1,1,1,0)$ and $\mu(\tau^{-1}N) = \{1,3,5\}$, together with all the indecomposable injective modules are all the GR maximal modules.

Example. Assume $\Lambda = kQ$ where Q is the Kronecker quiver. Up to isomorphism, the pre-projective modules P_n is the unique GR submodule of P_{n+1}. Different embedding gives rise to non-isomorphism GR factor module. These GR factor modules are the regular module $R_\lambda(1)$ for $\lambda \in \mathbb{P}^1(k)$. But $\mathrm{Hom}(P_n, R_\Lambda(1)) \neq 0$ for each all n and $\lambda \in \mathbb{P}^1(k)$. $\mathrm{Hom}(P_n, X) = 0$ for all indecomposable submodule X of P_{n+1} which is not isomorphic to P_n.

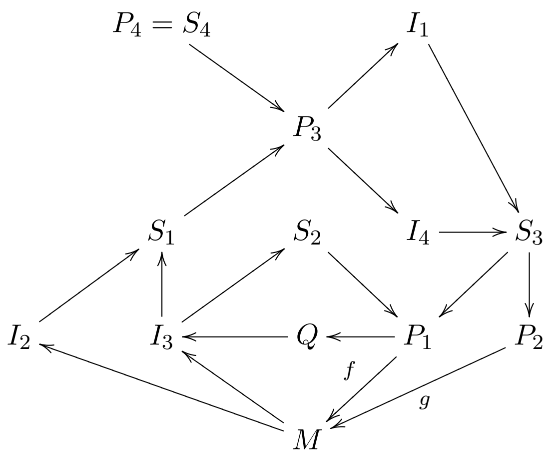
The following two examples show that if Λ is representation finite, we can find a GR inclusion $T \subset M$ such that $\mathrm{Hom}(T, M/T) \neq 0$ and there exists an indecomposable submodule X of M, such that $\mathrm{Hom}(T, X) \neq 0$.

Example. Let $\Lambda = k[x]/(x^n)$. There exist a unique simple module S and each indecomposable Λ-module is of the form $S[i]$ for $1 \leq i \leq n$, where $S = S[1]$, and $S[n]$ is the projective-injective module. Fix an $i \geq 2$, then $S[j]$ is submodule of $S[i]$ for each $j \leq i$ and $S[i-1]$ is a GR submodule of $S[i]$. Thus, in case $i \geq 3$, for any submodule $S[j]$ of $S[i]$ ($j \leq i-2$), $\mathrm{Hom}(S[i-1], S[j]) \neq 0$. Note that $S[1]$ is the GR factor module of the GR inclusion $S[i-1] \subset S[i]$ and $\mathrm{Hom}(S[i-1], S[1]) \neq 0$.

Example. Let $\Lambda = kQ/r^2$ where Q is the following quiver and r is the radical, i.e., the ideal generated by all arrows:



The AR quiver is the following:



Here $\underline{\dim}M = (1,2,1,0)$ and $\underline{\dim}Q = (1,0,1,0)$. Consider the indecomposable M and the almost split sequence $0 \to S_3 \to P_1 \oplus P_2 \overset{(f,g)}{\to} M \to 0$. Since f and g are monomorphisms, any GR submodule of M is isomorphic to P_1 or P_2. By easy calculation, we have $\mu(P_2) = \{1,2\}$ and $\mu(P_3) = \{1,3\}$, hence P_2 is a GR submodule of M and $\mu(M) = \{1,2,4\}$. But $\mathrm{Hom}(P_2, P_1) \neq 0$ since $\dim\mathrm{Hom}(P_2, P_1) = (\underline{\dim}P_1)_2 = 1$.

Chapter 4

The AR-sequences of Gabriel-Roiter factors

Assume $f : X \to Y$ is an irreducible monomorphism. Then $\operatorname{coker} f$ is indecomposable and all irreducible maps to $\operatorname{coker} f$ are epimorphisms. In [21], H.Krause proved that if either X or Y is indecomposable and $\operatorname{coker} f$ is not simple, then $\alpha(\operatorname{coker} f) = 1$ which means that in the almost split sequence $0 \to \tau(\operatorname{coker} f) \to Z \to \operatorname{coker} f \to 0$, the middle term Z is indecomposable. In [8], S.Brenner generalized the situation to irreducible monomorphisms with X and Y not necessarily indecomposable. We have seen some similarities between the mono-irreducibles (in particular, the GR inclusions) and irreducible monomorphisms. In view of the above, it is natural to ask whether an analogous result holds for a mono-irreducible map. In particular, whether $\alpha(M/T) = 1$ holds when $T \subset M$ is a GR inclusion. This section is devoted to a discussion of this problem.

We will give the proof of the following theorem:

Theorem B. *Let Λ be a representation-finite hereditary k-algebra and T be a Gabriel-Roiter submodule of M. If M/T is not injective, then the AR sequence terminating in M/T has an indecomposable middle term.*

We will also give some examples which illustrate that our result can not be generalized to directed algebras.

4.1 Some Lemmas

Lemma 4.1.1. *Let Λ be a hereditary algebra and $0 \to T \to M \to M/T \to 0$ be a GR sequence such that M/T is not injective. Let $0 \to \tau(M/T) \to X \to M/T \to 0$ be an almost split sequence. Then $|\tau^{-1}M| \geq |\tau^{-1}X|$ and equality holds if and only if $X \cong M$.*

Proof. Let $0 \to A \to B \to C \to 0$ be an arbitrary short exact sequence with C indecomposable and non-injective. Applying the functor $D = \operatorname{Hom}(-, k)$, we obtain an exact sequence

$0 \to D(C) \to D(B) \to D(A) \to 0$ of right Λ-modules. The functor $\mathrm{Hom}(-, \Lambda)$ induces a long exact sequence:

$$\to \mathrm{Hom}(D(C), \Lambda) \to \mathrm{Ext}^1(D(A), \Lambda) \to \mathrm{Ext}^1(D(B), \Lambda) \to \mathrm{Ext}^1(D(C), \Lambda) \to 0.$$

Since C is not injective, $D(C)$ is not projective. Then $\mathrm{Hom}(D(C), \Lambda) = 0$ since C is indecomposable, Λ and hence Λ^{op} is hereditary. Using $\tau^{-1}Y \cong \mathrm{Ext}^1(D(Y), \Lambda)$, we get a short exact sequence $0 \to \tau^{-1}A \to \tau^{-1}B \to \tau^{-1}C \to 0$. In particular, we get the following two short exact sequences:

$$0 \to M/T \to \tau^{-1}X \to \tau^{-1}(M/T) \to 0,$$

$$0 \to \tau^{-1}T \to \tau^{-1}M \to \tau^{-1}(M/T) \to 0.$$

Therefore, $|\tau^{-1}M| = |\tau^{-1}X| - |M/T| + |\tau^{-1}T| \geq |\tau^{-1}X|$, and equality holds if and only if $|\tau^{-1}T| = |M/T|$. Recall that if T is a GR submodule of M, then M/T is a factor module of $\tau^{-1}T$ and $\tau^{-1}T \cong M/T$ if and only if $0 \to T \to M \to M/T \to 0$ is an almost split sequence (1.7.7). thus, $|\tau^{-1}T| = |M/T|$ if and only if $\tau^{-1}T \cong M/T$, if and only if $M \cong X$. □

Lemma 4.1.2. *Let Λ be a representation-finite hereditary algebra and X an indecomposable non-injective Λ-module. Suppose $X_n \to X_{n-1} \to \cdots \to X_1 \to X$ is a sectional path such that n is maximal and $\alpha(X_i) \leq 2$ for each i. If there is an irreducible epimorphism $Y \to X$ with $Y \not\cong X_1$, then the composition of the irreducible maps $X_n \to X_{n-1} \to \cdots \to X_1 \to X$ is a monomorphism.*

Proof. Each X_i is not injective since X is not injective and Λ is hereditary. In particular, $\tau^{-1}X_1 \neq 0$. If X_i is projective for some i, then there exists a non-zero morphism from X_i to Y since the irreducible map $Y \to M$ is an epimorphism. Thus we obtain a path in the AR quiver from X_i to Y, then to X, since Λ is a representation-finite algebra. But the sectional path $X_n \to \cdots \to X_1 \to X$ is the unique path from X_i to X, a contradiction. Thus, all X_i's are not projective. In particular, X_n is not projective. Let $0 \to X_n \to X_{n-1} \oplus Z \to \tau^{-1}X_n \to 0$ be an almost split sequence with $Z \neq 0$. if Z is projective, then X_n is projective since Λ is hereditary, a contradiction. If Z is not projective, then there is an irreducible map $\tau Z \to X_n$. Thus, we obtain a sectional path $\tau Z \to X_n \to \cdots \to X_1 \to X$ which contradicts with the maximality of n. Therefore $Z = 0$. Starting with short exact sequences $0 \to X_n \to X_{n-1} \to \tau^{-1}X_n \to 0$ and $0 \to X_{n-1} \to X_{n-2} \oplus \tau^{-1}X_n \to \tau^{-1}X_{n-2} \to 0$, we obtain a short exact sequence $0 \to X_n \to X \to \tau^{-1}X_1 \to 0$ by using Lemma 3.1.3 continuously. In particular, the non-zero composition of irreducible maps $X_n \to X_{n-1} \to \cdots \to X_1 \to X$ is a monomorphism. □

Lemma 4.1.3. *Let $\Lambda = kQ$ with Q a quiver of type $D_n (n \geq 4)$, or $E_n (n = 6, 7, 8)$. Assume N is an indecomposable non-injective module with $\alpha(N) = 3$. Then N is not GR factor module.*

Proof. We may assume N is not projective and consider the AR sequence

$$0 \to \tau N \overset{(f_i)}{\to} \oplus_{i=1}^{3} X_i \overset{(g_i)}{\to} N \to 0.$$

Note that the orbit quiver of Λ is a star and at least one of the $[[X_i]]$ has only one neighbor, say $i = 1$. N is not injective implies X_1 is not injective since Λ is hereditary . We have an AR sequence $0 \to X_1 \to N \to \tau^{-1} X_1 \to 0$ which means the irreducible map g_1 is a monomorphism. N is not a GR factor module since all irreducible maps to a GR factor module are surjective.

□

4.2 Proof of Theorem B

In this section, we will present the proof of **theorem B**. We will proceed case by case.

We always assume $T \subset M$ is a GR submodule and assume $N = M/T$ is the corresponding non-injective GR factor module. Assume for a contradiction that $\alpha(N) \geq 2$. Owing to Lemma 4.1.3, it suffices to consider the case $\alpha(N) = 2$. We should keep in mind that all irreducible maps to M/T are surjective and any homomorphism $X \to N = M/T$ which is not an epimorphism factors through M (1.7.7).

(1). A_n **type.**

In this case, $\alpha(\Lambda) = 2$, i.e., for any indecomposable Λ-module M, $\alpha(M) \leq 2$. Assume there is an AR sequence $0 \to \tau(M/T) \to X \oplus Y \overset{(g_x, g_y)}{\to} M/T \to 0$ with X Y indecomposable and g_x,g_y epimorphisms. There are two sectional paths $Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y \overset{g_y}{\to} M/T$ and $X_m \to X_{m-1} \to \cdots \to X_1 = X \overset{g_x}{\to} M/T$ with n, m maximal respectively. By Lemma 4.1.2, we get two monomorphisms $Y_n \to M/T$ and $X_m \to M/T$ which factor through M. Then there are paths $Y_n \to \cdots \to M \to \cdots \to M/T$ and $X_m \to \cdots \to M \to \cdots \to M/T$. In particular, M lies on both of the two sectional paths. But the unique indecomposable module on both sectional path is M/T. We get a contradiction. Thus, $\alpha(M/T) = 1$ if M/T is not injective.

We also claim that M/T is uniserial. Assume the vertices A_n is indexed as follows

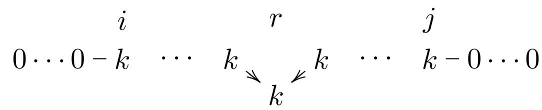
$$1 \text{---} 2 \text{---} 3 \text{---} \bullet \qquad \cdots \qquad \bullet \text{---} n$$

For $i \leq j$, we denote by $[i, j]$ the indecomposable module (for any orientation)

$$\begin{array}{ccccccccc} & & i & & & & j & & \\ 0 \cdots 0 & \text{---} & k & \text{---} & k \quad \cdots \quad k & \text{---} & k & \text{---} & 0 \cdots 0 \end{array}$$

Then, all indecomposable modules are of the form $[a, b]$ with $1 \leq a, b \leq n$. An indecomposable modules is not uniserial if and only if it is of one of the following 2 forms:

$$\begin{array}{ccccccc} & i & & r & & j & \\ & & & k & & & \\ 0 \cdots 0 - k & \cdots & k \swarrow & & \searrow k & \cdots & k - 0 \cdots 0 \end{array}$$

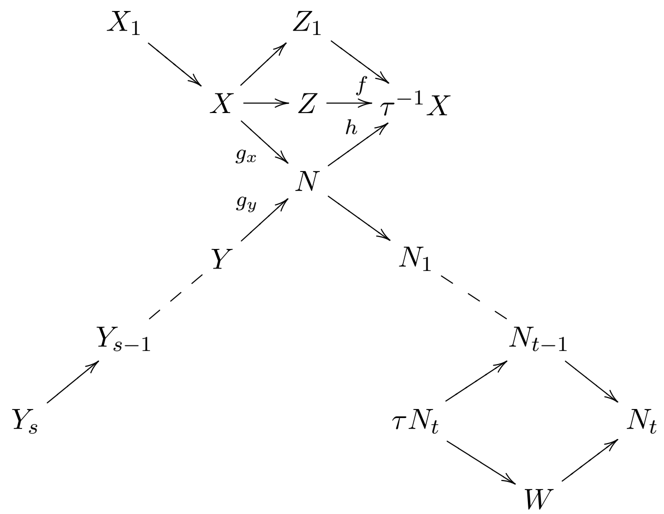


Assume M/T is not uniserial and is of the first form. Then, $T = [a, i-1]$, $M = [a, j]$, for some $a \geq 1$ and $[a, r-1]$ is an indecomposable submodule of M containing T, a contradiction. Or, $T = [j+1, b]$, $M = [i, b]$ for some $b \leq n$, and $[r+1, b]$ is an indecomposable submodule of M containing T, a contradiction.

Assume M/T is of the second form. Then $T = [a, i-1]$, $M = [a, j]$, for some $a \geq 1$ and $[a, r]$ is an indecomposable submodule of M containing T, a contradiction. Or, $T = [j+1, b]$, $M = [i, b]$ for some $b \leq n$, and $[r, b]$ is an indecomposable submodule of M containing T, a contradiction. Thus, M/T is uniserial.

(2). D_n **type.**

Our result is obvious for D_4, so we assume $n \geq 5$. Suppose first that $N = M/T$ lies on the quasi-center. Consider the following full subquiver of the AR quiver:



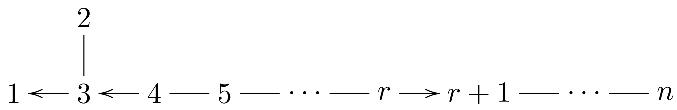
The maps g_x, g_y are both epimorphisms since N is a GR factor. Consider the sectional path $Y_s \to \cdots \to Y_0 = Y \to N$ with s maximal. By Lemma 4.1.2, we get a monomorphism $Y_s \to N$. Hence there is a sectional path $Y_s \to \cdots M \to \cdots \to Y \to N = M/T$.

Case 1. Z is not injective.
In this case, f is a monomorphism since $Z \to \tau^{-1}X$ is a source map. In view of $|N| - |X_1| = |\tau^{-1}X| - |Z| > 0$, the composition $X_1 \to X \to N$ is a monomorphism, thus, factors through M. It follows that there is a path from X to M. But we have shown there is a sectional path from $M \to \cdots \to Y \to N$. A contradiction.

Case 2. Z is injective.
In this case $\tau^{-1}X$ is injective. Let $N = N_0 \to N_1 \to \cdots \to N_t$ be the sectional path with t maximal and $N_1 \cong \tau^{-1}Y$. The irreducible map $\tau N_t \to N_{t-1}$ is an epimorphism since g_y is an epimorphism. Therefore, there exist an indecomposable module W such that

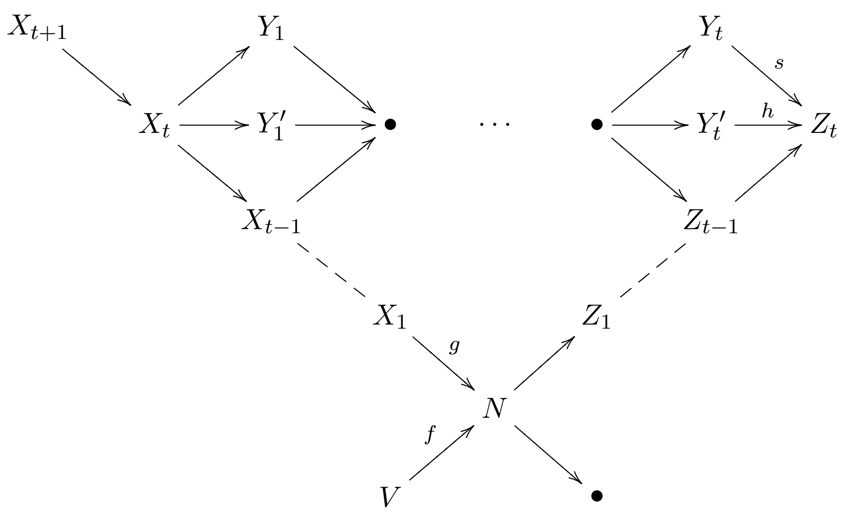
$0 \to \tau N_t \to N_{t-1} \oplus W \to N_t \to 0$ is an almost split sequence and the irreducible map $W \to N_t$ is an epimorphism. It follows that W, hence N_t, is injective since t is maximal.

Since Z, N_t and W are all injective, the ordinary quiver of D_n is of the following form:



Here $t = r - 4$ and the non-oriented edges can be endowed with arbitrary orientation. Also Z, $\tau^{-1}X$ and N_t are the indecomposable injective module I_1, I_3 and I_r respectively. Note that $\dim \mathrm{Hom}(M, I) \leq 1$ for any indecomposable injective module I and $\mathrm{Hom}(M, Z) = 0$ since M lies on the sectional path $Y_s \to \cdots \to Y \to N$. Hence we may consider M as an indecomposable kA_{n-1} module where A_{n-1} is obtained from D_n by deleting the vertex 1. Since $\tau^{-1}X$ and N_t are injective, there exists an unique $0 \leq j \leq t-1$ such that $\tau^{-1}N_j$ is a simple injective module, i.e., there is an integer $4 \leq p \leq r-1$ such that p is a source point in the ordinary quiver D_n. Then $(\underline{\dim} N)_3 = \dim \mathrm{Hom}(N, \tau^{-1}X) = 1 = (\underline{\dim} N)_r$ implies that, as kA_{n-1} module, N is not uniserial module. A contradiction.

We now assume that N does not lie on the quasi-center. Then we get the following full subquiver of the AR quiver:



By Lemma 4.1.2, the composition of irreducible maps $V_n \to \cdots \to V \to N$ is a monomorphism and hence, we get a sectional path $M \to \cdots \to V \to N = M/T$. Note that each X_i is not projective since $f : V \to N$ is an epimorphism. If Z_j is injective for some j, we may reduce the case to some A_m by using the same argument in the case of N lying on the quasi-center.

We may assume the subquiver of the AR quiver is complete, i.e., X_{t+1} and Z_t are not zero and Z_t is not injective and hence s, and h are both monomorphisms. Starting with the two short exact sequences:

$$0 \to X_{t+1} \to Y'_1 \oplus N \to Z_1 \to 0$$

and

$$0 \rightarrow Y_1' \rightarrow Y_2 \oplus Z_1 \rightarrow Z_2 \rightarrow 0,$$

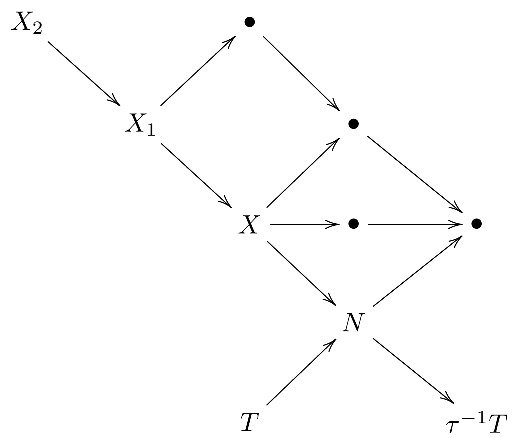
we obtain the following short exact sequence by using Lemma 3.1.3 continuously,

$$\begin{cases} 0 \rightarrow X_{t+1} \rightarrow N \oplus Y_t \rightarrow Z_t \rightarrow 0 & \text{if } n \text{ is even} \\ 0 \rightarrow X_{t+1} \rightarrow N \oplus Y_t' \rightarrow Z_t \rightarrow 0 & \text{if } n \text{ is odd} \end{cases}$$

So $|N|-|X_{t+1}| = |Z_t|-|Y_t| < 0$ or $|N|-|X_{t+1}| = |Z_t|-|Y_t'| < 0$. Therefore the composition of the irreducible maps from X_{t+1} to N is a monomorphism and hence, factors through M. Thus M lies on sectional path $X_{t+1} \rightarrow X_t \rightarrow \cdots \rightarrow X_1 \rightarrow N$. This is a contradiction since M lies on the other sectional path $\cdots \rightarrow V \rightarrow N$.

(3). E_6 **type.**

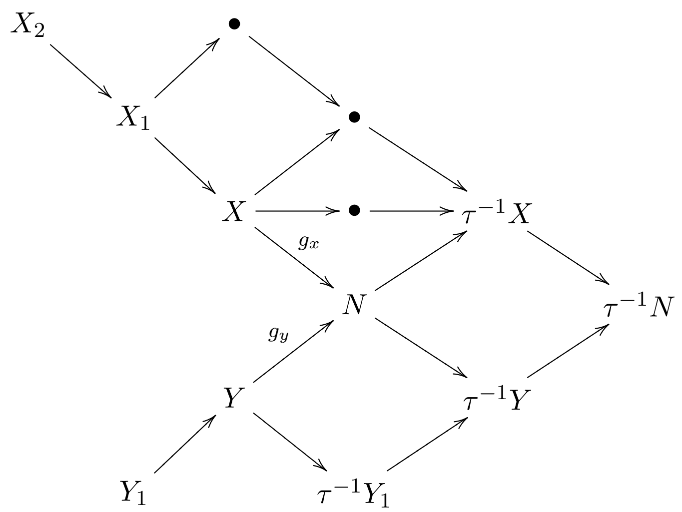
Consider the following subquiver of the AR quiver:



N is not injective implies T is not injective and thus the irreducible map $T \rightarrow N$ is injective. Hence N is not a GR factor module.

(4). E_7 **type.**

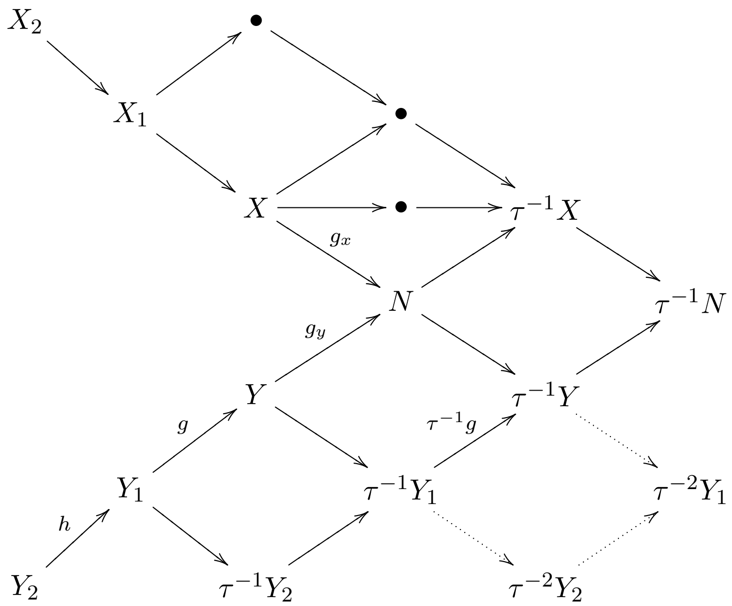
Due to the proof of type E_6 type, we need only to consider the case that N lies on the quasi-center as in the following full subquiver of the AR quiver:



The irreducible maps g_x, g_y are surjective. By Lemma 4.1.2, the composition $Y_1 \to Y \to N$ is injective and hence, M lies on the sectional path $Y_1 \to Y \to N$. Therefore $M \cong Y$. Since X is not injective, we have $\tau^{-1}X \neq 0$ and $|\tau^{-1}M| = |\tau^{-1}Y| < |\tau^{-1}Y| + |\tau^{-1}X|$. This contradicts Lemma 4.1.1.

(5). E_8 **type**

Due to the proof of type E_7, we need only to consider the cases that N lies on the quasi-center as in the following full subquiver of the AR quiver:



Since g_x and g_y are epimorphisms, all Y, Y_1 and Y_2 are not zero and not projective. The composition $g_y g h$ is injective by Lemma 4.1.2 and, M lies on the sectional path $Y_2 \to Y_1 \to Y \to N$. If $M \cong Y$, then $|\tau^{-1}M| = |\tau^{-1}Y_1| < |\tau^{-1}Y| + |\tau^{-1}X|$ which is a contradiction to 4.1.1. So we assume $M \cong Y_1$.

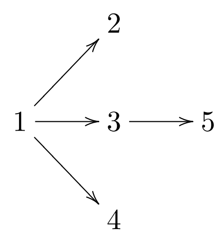
Case 1. $\tau^{-1}Y_1$ and $\tau^{-1}Y_2$ are both injective.
In this case, we have $\tau^{-1}Y$ and $\tau^{-1}N$ are injective modules and $\tau^{-1}Y_1 \overset{\tau^{-1}g}{\to} \tau^{-1}Y$. Then $|\tau^{-1}Y_1| - |\tau^{-1}Y| = 1$ since $\tau^{-1}Y_1$ is injective. X is not injective implies $\tau^{-1}X \neq 0$. Also the irreducible map $\tau^{-1}X \to \tau^{-1}N$ is an epimorphism since $Y_1 \to \tau^{-1}Y_2$ is surjective. Thus $|\tau^{-1}X| > |\tau^{-1}N| \neq 0$ and $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 1 < |\tau^{-1}Y| + |\tau^{-1}X|$.

Case 2. $\tau^{-1}Y_1$ in injective but $\tau^{-1}Y_2$ is not.
In this case, there is a irreducible map from $\tau^{-1}Y_1$ to the simple injective module $\tau^{-2}Y_2$ which means $\tau^{-1}Y_1/\mathrm{soc}\tau^{-1}Y_1$ has two direct summands. So $|\tau^{-1}Y_1| - |\tau^{-1}Y| = |\tau^{-2}Y_2| + 1 = 2$. we have $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 2 \leq |\tau^{-1}Y| + |\tau^{-1}X|$ since $\tau^{-1}X$ is not simple.

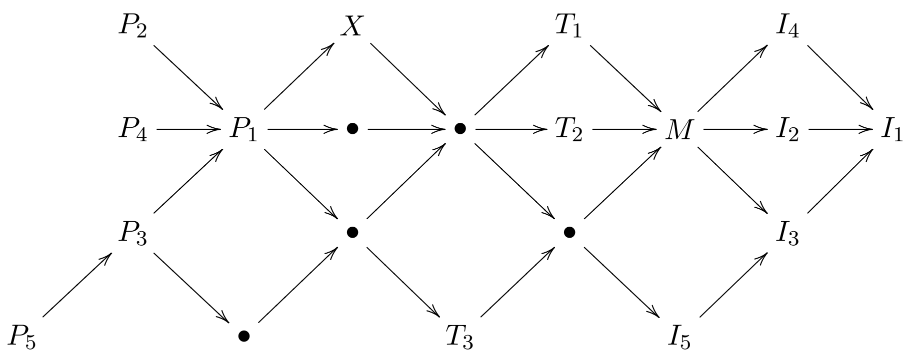
Case 3. $\tau^{-1}Y_1$ is not injective.
In this case, $\tau^{-2}Y_2$ and $\tau^{-2}Y_1$ are not zero. g_y is an epimorphism implies the irreducible map $\tau^{-2}Y_2 \to \tau^{-2}Y_1$ is an epimorphism and hence $\tau^{-2}Y_2$, $\tau^{-2}Y_1$ are injective modules. $|\tau^{-1}Y_1| - |\tau^{-1}Y| = |\tau^{-2}Y_2| - |\tau^{-2}Y_1| = 1$. Therefore $|\tau^{-1}M| = |\tau^{-1}Y_1| = |\tau^{-1}Y| + 1 < |\tau^{-1}Y| + |\tau^{-1}X|$. In all the cases, we get $|\tau^{-1}M| \leq |\tau^{-1}Y| + |\tau^{-1}X|$ which contradicts Lemma 4.1.1. □

4.3 Examples

Example. Let $\Lambda = kD_5$ with the following orientation:



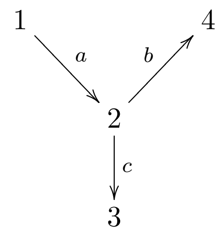
The AR quiver is the following:



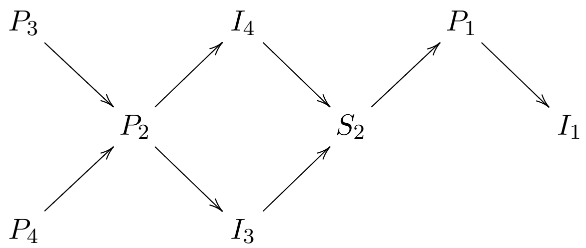
(1). If a GR factor module N is injective, then $\alpha(N) \neq 1$ may happen. In the example, up to isomorphism, M has 3 GR submodules, T_1, T_2 and T_3. And the corresponding GR factor modules are I_4, I_2 and I_3 respectively. $\alpha(I_4) = \alpha(I_2) = 1$, but $\alpha(I_3) = 2$. Also $\alpha(I_1) = 3$ and any non-projective simple module is a GR factor.

(2). The indecomposable module Y with $\alpha(Y) = 1$ may not be a GR factor module. In the example, $\alpha(X) = \alpha(I_5) = 1$, but they are not GR factor modules.

Example. Let $\Lambda = kQ/I$ where Q is the following quiver :

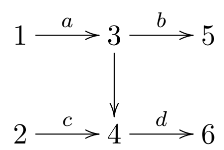


and $I =< ca, ba >$. The AR quiver of Λ is:



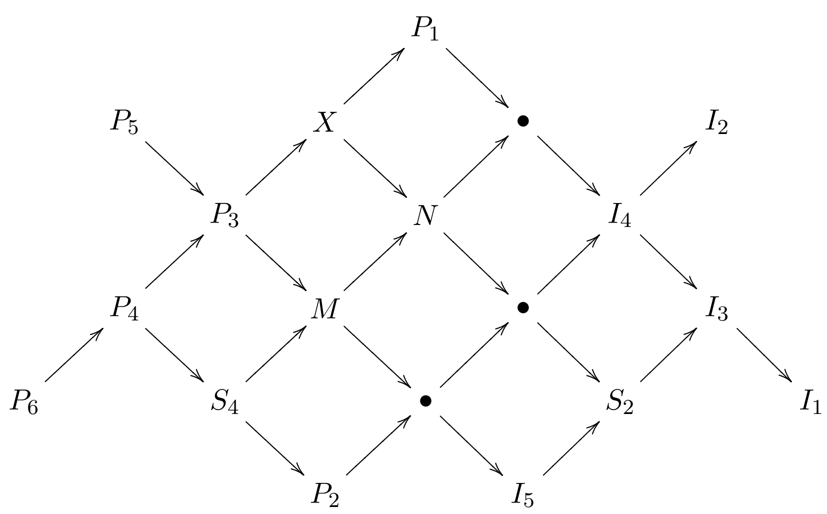
The simple module $S_2 \cong P_2/P_4$ is a GR factor and not injective, but $\alpha(S_2) = 2$.

Example. Let $\Lambda = kQ/I$ with Q the following quiver:



and $I =< ba, dc >$

The AR quiver is of the following shape:

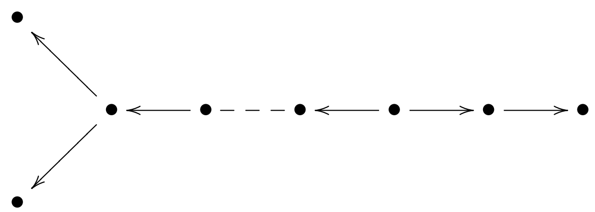


Here

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

and $0 \to P_5 \to M \to N \to 0$ is a GR sequence with $\alpha(N) = 2$. Note that N is not simple.

Example. We add an example here to show the GR factor modules are not necessary to be uniserial. If $\Lambda = kD_n$, where D_n is with the following orientation:



Consider the indecomposable module M with $\underline{\dim} M = (1, 1 \cdots, 1)$. Then it is easy to see the GR measure is $\{1, 2, n\}$ and corresponding GR factor module has length $n - 2$ which is not uniserial.

Appendix 1

Table-1

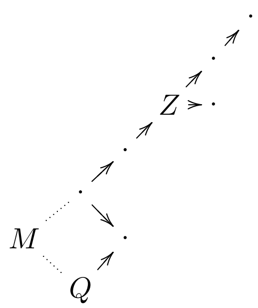
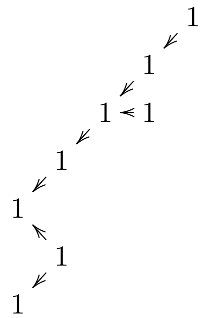
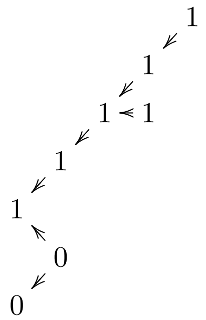
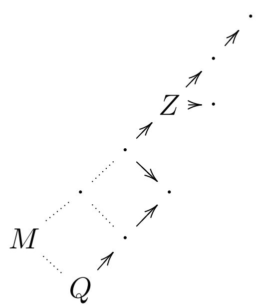
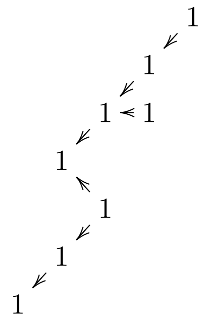
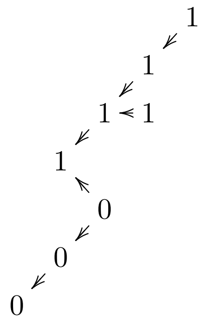
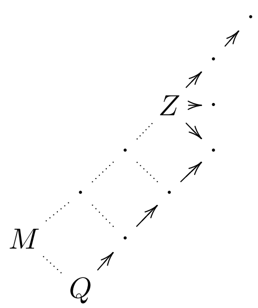
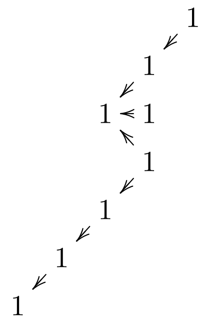
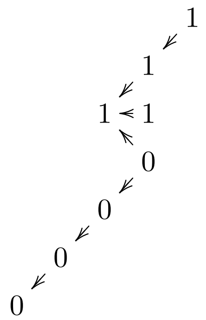
AR quiver	M	T	$\mu(M)$
			{1,2,3,4,5,6,8}
			{1,2,3,4,5,8}
			{1,2,3,4,8}

Table-2

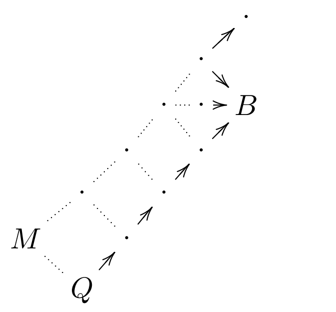
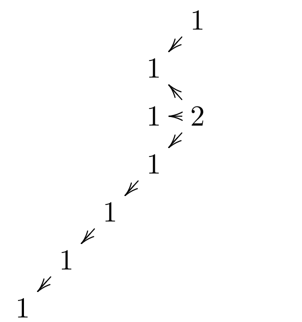
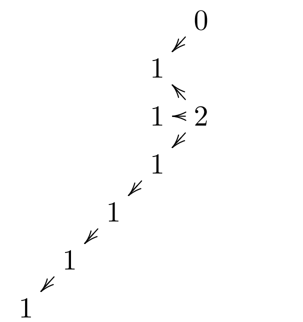
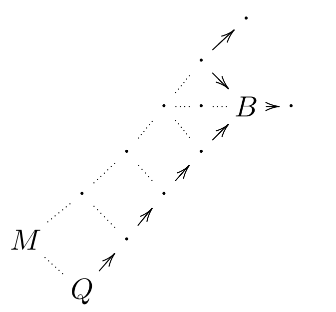
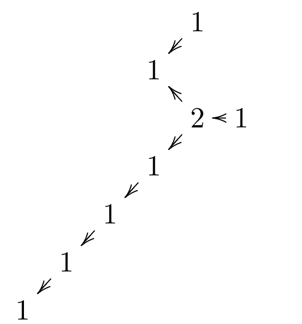
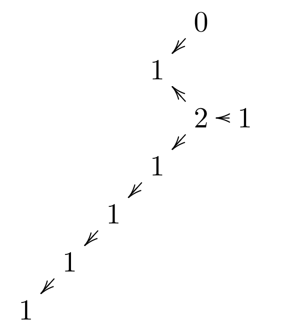
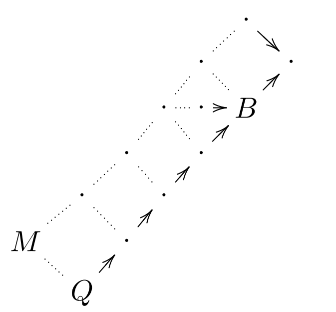
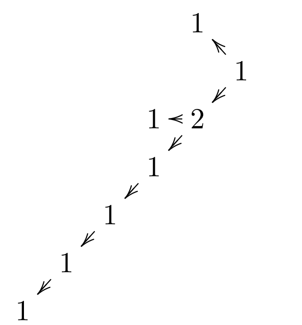
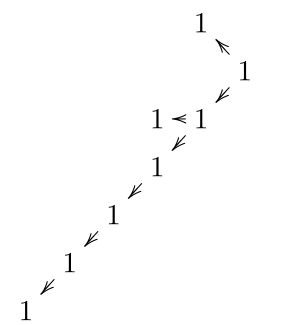
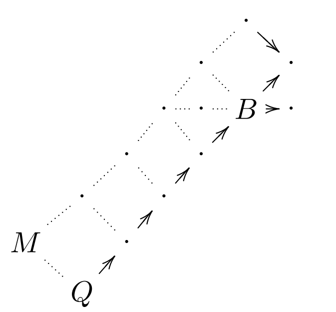
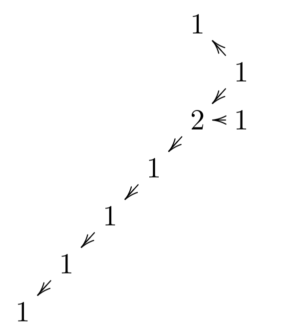
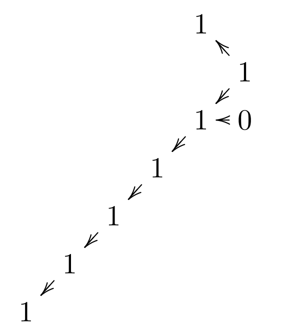
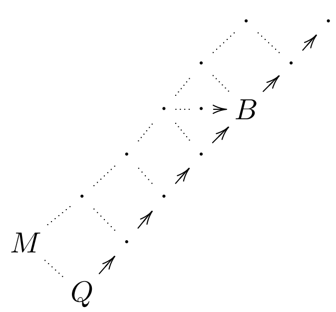
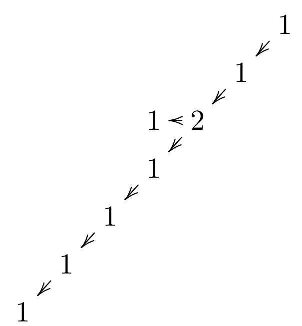
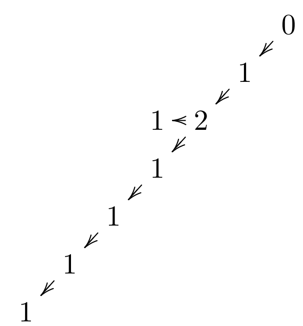
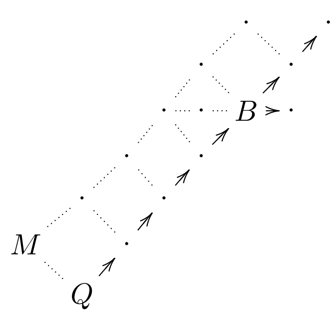
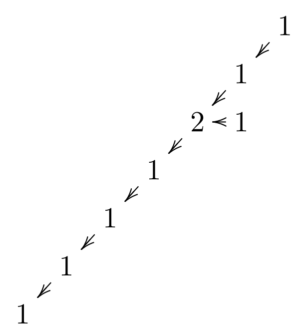
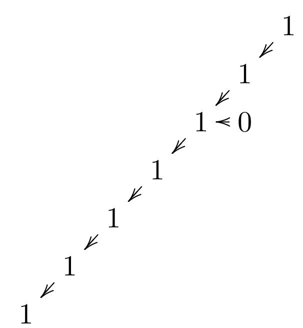
AR quiver	M	T	$\mu(M)$
			{1,2,3,4,6,8,9}
			{1,2,3,4,8,9}
			{1,2,3,4,6,8,9}
			{1,2,3,4,5,7,9}
			{1,2,3,4,5,6,8,9}
			{1,2,3,4,5,6,7,9}

Table-3

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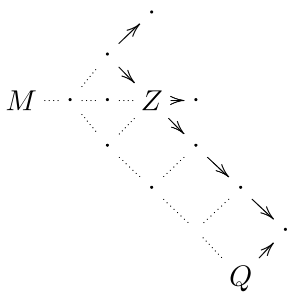
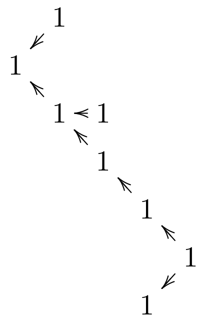
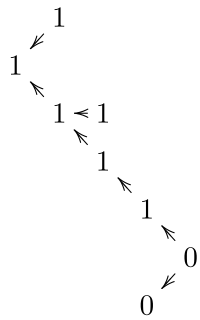
<tr><td>
</td><td>
</td><td>
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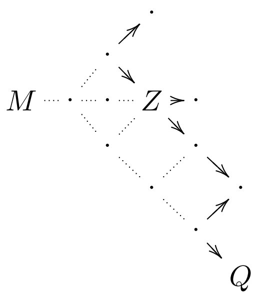
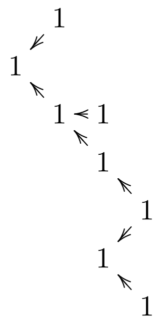
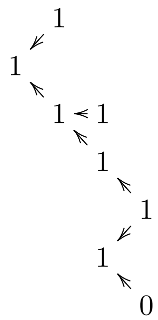
<tr><td>
</td><td>
</td><td>
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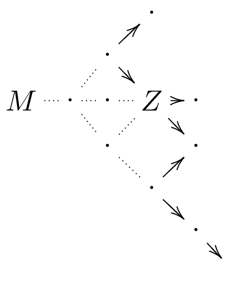
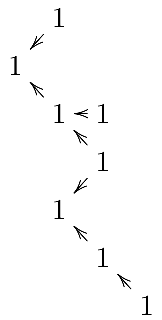
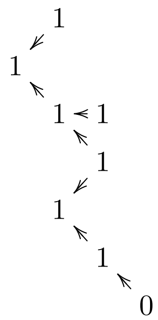
<tr><td>
</td><td>
</td><td>
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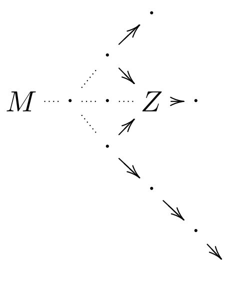
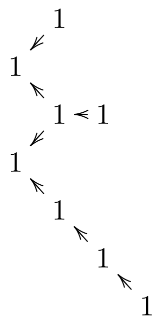
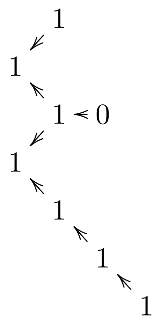
<tr><td>
</td><td>
</td><td>
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Appendix 2: Open questions

We will list some open questions in this section. We still assume that algebras are finite-dimensional k-algebras where k is an algebraically closed field.

1. We conjecture that **Theorem A** holds for any directed algebra . (One may find a proof of part (1) in the preprint [29]. The author first gives the proof for split directed algebra over a finite field by using Hall polynomials for directed algebras and then generalizes to arbitrary fields.)

2. We conjecture that part (2) of **Theorem A** holds for any representation-finite algebra. (Since representation-finite algebras admit simply connected coverings, this problem is related to question 1.)

3. Let Λ be a directed algebra and $0 \to T \stackrel{(f_i)}{\to} \oplus_{i=1}^{3} X_i \stackrel{(g_i)}{\to} Y \to 0$ be an almost split sequence with 3 middle terms. Assume T is not a GR submodule of X_i for each i. Is T GR maximal? (The statement is true if all f_i's are epimorphisms, see **Lemma 3.1.5.**)

4. Does there exist GR factor module N over some directed algebra Λ such that $\alpha(N) \geq 2$ and $|N| \geq 3$? (See the examples at the end of **Chapter 4**.)

5. Suppose Λ is a k-algebra and $T \subset M$ is a GR inclusion. Does $T \subset \text{rad}M$ imply that $\text{top}M$ is simple?

6. Suppose Λ is a k-algebra and M is an indecomposable module. Is τM (isomorphic to) a GR submodule of M. (This is never the case for directed algebras, see **Proposition 2.1.4**.)

7. Let Λ be either a directed algebra, given by a quiver with only commutative relations, or a representation-infinite hereditary algebra. Assume N is a non-injective GR factor. Is $\alpha(N) = 1$? (In [6], the authors showed that representation-finite algebras admit normed multiplicative bases. It follows that representation-finite algebras are given by quivers with zero relations and commutativity relations. Since the examples at the end of

Chapter 4 show that we can not generalize **Theorem B** to algebras [at least for those of representation-finite type] with zero relations, the conditions of this question are natural.)

(8). Let Λ be a tame hereditary algebra. Does each indecomposable preprojective module M have, up to isomorphism, at most 4 GR submodules? (If Λ is a representation-finite hereditary algebras, then $\alpha(M) \leq 3$ for all indecomposable modules. If Λ is a tame hereditary algebra, then for each indecomposable module M, $\alpha(M) \leq 4$.)

(9). Are the following equivalent for a finite dimensional algebra Λ:
(a). Λ is representation-infinite.
(b). There exists an indecomposable module which has, up to isomorphism, infinitely many GR submodules.
(c). There exist infinitely many indecomposable modules which have, up to isomorphism, infinitely many GR submodules.
(d). There exists an indecomposable module which is a GR submodule of infinitely many indecomposable modules.
(e). There exist infinitely many indecomposable modules such that each such indecomposable is a GR submodule of infinitely many indecomposables, up to isomorphism.

(10). Let Λ be a representation-infinite algebra. Are there infinitely many GR maximal modules?

Bibliography

[1] M.Auslander, I.Reiten and S.O.Smalø, Representation theory of Artin algebras. Cambridge university press(1996).

[2] R.Bautista, S.Brenner, *On the number of terms in the middle of an almost split sequence.* Representation of algebras. Lecture Notes in Mathematics **903**(1981), 1–8.

[3] K.Bongartz, *Indecomposable over representation-finite algebras are extensions of an indecomposable and a simple.* Math. Z. **187**(1984), 75–80.

[4] K.Bongartz and P.Gabriel, *Covering space in representation theory.* Invent. Math. **65**(1982), 331–378.

[5] I.Bernstein, I.Gelfand and V.A.Ponomarev, *Coxeter functors and Gabriel's theorem.* Uspechi Mat. Nauk **28**(1973), 19–33.

[6] R.Bautista, P.Gabriel, A.V.Roiter, and L.Salmeron, *Representation-finite algebras and multiplicative bases.* Invent. Math. **81**(1985), 217–285.

[7] S.Brenner, *The Auslander translate of a short exact sequence.* Colloquium Math. **78**(1998), no.1, 49–56.

[8] S.Brenner, *On the kernel of an irreducible map.* Linear Algebra and Its Applications. **365**(2003), 91–97.

[9] P.Brown, *Non-split extensions over representation-finite hereditary algebras.* Comm. Algebra. **26(4)**(1998), 1005–1015.

[10] W.Crawley-Boevey, *Exceptional sequences of representations of quivers.* Proceedings of ICRA VI, Carleton-Ottawa Math. LNS. **14**(1992).

[11] V.Dlab and C.M.Ringel, *Indecomposable representations of graphs and algebras.* Mem. Amer. Math. Soc.**173**(1976).

[12] P.Dräxler, *On indecomposable modules over directed algebras.* Proc. Amer. Math. Soc. **112**(1991), no. 2, 321–327.

[13] P.Dräxler, *Representation-directed diamonds.* LMS journal of computation and Mathematics **4**(2001), 13–20.

[14] P.Dräxler, *Normal forms for representations of representation-finite algebra.* Journal of Symbolic Computation **32**(2001), 491–497.

[15] P.Gabriel, *Indecomposable representations II.* Symposia. Math. Ist. Naz. AltaMat. **6**(1973), 81–104.

[16] P.Gabriel, *Auslander-Reiten sequence and representation-finite algebras.* Lecture Notes in Mathematics **831**(1980), 1–71.

[17] P. Gabriel, *The universal cover of a representation finite algebra.* Representation of algebras. Lecture Notes in Mathematics **903**(1981), 68–105.

[18] D.Happel and C.M.Ringel, *Tilted algebras.* Trans. Amer. Math. Soc. **272**(1982), no.2, 399–443.

[19] J.A. De la Peña, *On omnipresent modules in simply connected algebras.* J. London Math. Soc.(2), **36**(1987), no.3, 385–392.

[20] H.Krause, *On the number of almost split sequence with indecomposable middle term.* Bull. London Math. Soc. **26**(1994), no.5, 422–426.

[21] H.Krause, *The kernel of an irreducible map.* Proc. Amer. Math. Soc.**121**(1994), no.1, 57–66.

[22] C.M.Ringel, *Representations of K species and bimodules.* J. Algebra **41**(1976), 269–302.

[23] C.M.Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Mathematics **1099**. (Spring, Berlin, 1984)

[24] C.M.Ringel, *Hall polynomials for the representation-finite hereditary algebras.* Adv. in Math.**84**(1990), 137–178.

[25] C.M.Ringel, *Exceptional objects in hereditary categories.* Proceedings Constantza Conference. An. St. Univ. Ovidius Constantza Vol.**4**(1998), no.2, 150–158.

[26] C.M.Ringel, *Exceptional modules are tree modules.* Linear Algebra and Its Applications **275-276**(1998), 471–493.

[27] C.M.Ringel, *Foundation of the representation theory of Artin algebras, Using the Gabriel-Roiter measure.* Proceedings of the 36th Symposium on Ring Theory and Representation Theory. **2**, 1–19, Symp. Ring Theory Represent Theory Organ. Comm., Yamanashi, 2004.

[28] C.M.Ringel, *The Gabriel-Roiter measure.* Bull. Sci. math. **129**(2005), 726-748.

[29] C.M.Ringel, *The Theorem of Bo Chen and Hall polynomials.* to appear in Nagoya Journal.

[30] A.Rogat and T.Tesche *The Gabriel Quiver of the sincere simply connected algebras.* preprint, University of Bielefeld.

[31] L.Unger. *On the number of maximal sincere modules over sincere directed algebras.* J. Algebra **133**(1990), 211–231.

[32] A.Schofield, *The internal structure of real Schur representations.* Preprint.

[33] A.Schofield, *Semi-invariants of quivers.* J. London Math. Soc. **43**(1991), 383–395.