Analytical methods in constructive measure theory on configuration spaces

Dissertation

zur
Erlangung des Doktorgrades (Dr. math.)
der
Fakultät für Mathematik
der
Universität Bielefeld

vorgelegt von
Oleksandr Kutoviy
aus
Kyiv, Ukraine
Bielefeld 2003

Analytical methods in constructive measure theory on configuration spaces

Dissertation

zur
Erlangung des Doktorgrades (Dr. math.)
der
Fakultät für Mathematik
der
Universität Bielefeld

vorgelegt von
Oleksandr Kutoviy

Contents

Chapter 1

Introduction

The space of configurations Γ_X over a Riemannian manifold X consists of all locally finite subsets of X. Such spaces play an important role in the topology, the theory of point processes, the mathematical physics and several other areas of the mathematics and its applications. As objects of infinite dimensional analysis configuration spaces form a class of infinite dimensional manifolds which are not in the well-known categories of Banach or Fréchet manifolds. Nevertheless, they can be equipped with a natural differentiable structure (coming from the underlying manifold X) with quite rich analytic and geometrical properties, see [2], [3]. This leads us to the first application in mathematical physics. Directly, the configuration space appears in applications to classical mechanical systems of infinite many particles describing the position of indistinguishable particles. More comprehensive is the knowledge about configuration spaces in the branch of general measure theory, cf. e.g. [36], [63]. One should note that even in the stochastics there are still many open questions related to the general theory of point processes on configuration spaces. The general analysis and stochastics on configuration spaces can be conditionally divided in two parts. One of them, which can be characterized as general analysis, in particular connected with the so-called Poissonian White Noise analysis. This analysis is a special modification of the well-known Gaussian White Noise analysis (see, *e.g.*, the books [10], [33], [34], [52] for a detailed exposition of the theory and examples of applications, and the introductory articles [51], [81], and [82]).

One of the first approaches to non-Gaussian analysis was proposed in [4] and developed in [1]. Gaussian white noise analysis is essentially based on the Wiener-Itô-Segal chaos decomposition of the L^2-space with respect to a

Gaussian measure on an orthogonal system of Hermite polynomials. This has motivated the aforementioned construction. There, the orthogonal system of Hermite polynomials is replaced by a biorthogonal Appell system – a system first constructed by Yu. L. Daletsky [18] for a special class of measures and extending the previous system – and the chaos decomposition is replaced by a biorthogonal decomposition. Through a slight modification on the conditions of the probability measures used in these references, the previous construction of non-Gaussian analysis was enlarged in [49] to a class of measures which, in particular, includes the gamma and the Poisson measures (see also [48], [78], and [44]). This construction is supported on a general concept of generalized Appell systems ([47], [49], [78]). All these generalizations towards a non-Gaussian analysis are clearly based on the theory of Appell systems. Recently, some aspects of this theory were further developed by generalizing the theory of Appell systems to an analysis on hypergroups [7], [11]. An alternative approach to non-Gaussian analysis including Poissonian analysis and based on the spectral representation of a special family of Jacobi fields in the Fock space was developed by Yu. M. Berezansky, see, *e.g.*, [8], [9], [13], [14], [59].

Poisson measures appear in several areas of mathematics and in applications to problems of physics, biology, chemistry, economics, and other fields of modern science. In particular, Poisson measures are related to the study of point processes in probability theory, representation theory for diffeomorphism groups and current algebras, models of non-relativistic quantum fields, classical and quantum statistical mechanics. We only refer to [3] for a reasonable list of applications in mathematical physics and corresponding references. Apart from the diversity of the related topics, Poisson measures are by themselves a subject of interest in infinite dimensional analysis, because these, similar to the Gaussian measures, are defined on infinite dimensional spaces whose analysis is of a constructive character and has a very rich structure.

In the Poissonian White Noise analysis are manifested the general features of infinite dimensional analysis as well as arise new structures, related with the specific of configuration spaces on which Poisson measures are considered (see [2], [3], [38], [39], [40], [42], [43]). The recent dissertation [66] of M. J. Oliveira was devoted to the detailed study of relations between structures of Poissonian White Noise analysis and specific structures peculiar to the analysis on configuration spaces.

Another part can be characterized as constructive infinite dimensional

analysis on the configuration spaces. First of all, it is related with the investigation of some classes of measures on configuration spaces. Among measures, considered on the configuration spaces, one should distinguish the class of measures constructed via potentials of interaction. These measures are known in mathematical physics as Gibbs measures. The rigorous mathematical definition of such objects came back to [61], [19], [20], [22], and [53]. There exist many equivalent description of Gibbs measures, see [27], [69], and [65]. To the detailed study of Gibbs measures were also devoted such papers as [68], [70], [30], [28], [29]. The aim of the following dissertation is a construction and study of such measures, using analytical methods. Methods of infinite dimensional analysis, used for the study of Gibbs measures, were developed, in particular, in works of [2], [3], [27], [38], [39], [41]. This work is a continuation of this direction.

In the sequel we describe the contents of the work chapter by chapter in more details.

General facts and notations

Chapter 2 begins with a description of the configuration spaces used in this work. These spaces are constructed over an Euclidean space $\mathbb{R}^d$, $d \geq 1$, but without loss of generality the most of results can be transferred to the case of a general non-compact Riemannian manifold X. In the Section 2.1 we describe the space of finite configurations, i.e.

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \,|\, |\eta| < \infty\},$$

where $|\eta|$ denotes the number of elements of the set η. Some topological properties of Γ_0 are also considered in Section 2.1.

In Section 2.2 we consider configuration space Γ with it basic topological properties which is defined as

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \big| \, |\gamma \cap \Lambda| < \infty, \text{ for all compact } \Lambda \subset \mathbb{R}^d \right\}.$$

Classes of functions on Γ_0 and Γ are also discussed in this section. Moreover, in Section 2.2 we define and present some properties of the K-transform, a mapping which transforms functions defined on Γ_0 into functions on Γ. The K-transform plays the role of the Fourier transform in configuration space analysis and has purely combinatorial nature. It plays a key-role in the construction of combinatorial harmonic analysis on configuration spaces

introduced and developed in [38], [39], see also [42] , [43] . The operator nature of the K-transform was first recognized and studied by A. Lenard in a series of works, [56], [57]. A last part of Section 2.2, is devoted to the Lebesgue-Poisson measure $\lambda_{z\sigma}$ on Γ_0, a Poisson measure $\pi_{z\sigma}$ on Γ and the dual operator of the K-transform denoted by K^*. The latter operator maps probability measures μ on Γ into correlation measures $K^*\mu$ on Γ_0.

Section 2.3 is devoted to the the space of multiple configuration, its basic topological and measure theoretical properties.

Detailed structure and some topological properties of the configuration space Γ

In the study of problems of stochastics and mathematical physics, topological and metrical structures on configuration spaces play an essential role. Questions related to these structures were mostly studied on the space of multiple configurations $\ddot{\Gamma}$ comparing with the configuration space Γ. First of all, it concerns with the possibility to metrize vague topology on $\ddot{\Gamma}$. This metrization is not proper for the case of simple configurations and demands some modification.

The aim of this chapter is to order our knowledge concerning some topological properties of Γ. We would like to emphasize that the main new results of this chapter are related with a metrical structure of configuration space Γ. We construct a family of metrics on Γ, which makes it complete, separable metric space and such that topologies generated by these metrics are equivalent to the vague topology on Γ. The construction of such a metrics is motivated by observations made by A. Skorokhod in [80]. Such metrical structures on Γ give us a possibility to describe relatively compact sets in Γ. In this chapter, we propose new simple proof of the Holley-Stroock criterion for relatively compact subsets of Γ, see [35]. Using it, we introduce a family of compact functions on Γ. Such functions are a standard tool in the study of many problems of mathematical physics and stochastics.

On relations between a priori bounds for measures on configuration spaces

The measure theory on configuration spaces has several specific aspects comparing with the well developed one in the case of linear spaces. Namely, in

the linear case we have useful relations between such characteristics of measure as moments, the Laplace transform, support and integrability properties for some classes of functions on linear spaces, see e.g. [10] for a review and related historical comments and references. These characteristics need to be modified properly in configuration space analysis. Important instructive ideas in this area are coming from the theory of stochastic processes and statistical physics. In these applications measures on configuration spaces correspond to point processes and states of continuous systems respectively and in both areas we have already many deep results concerning properties of particular classes of such measures.

The point of view developed in Chapter 4 is motivated mainly by results of classical statistical mechanics of continuous systems. In particular, in pioneering works of R. L. Dobrushin [21] and D. Ruelle [75] dedicated to the study of equilibrium states (*Gibbs measures*) in the case of pair potentials were discovered several properties of these measures related with analysis of their characteristics. Namely, the first characteristic of configuration space measures is the system of correlation functions (that is the system of reduced moments or coincidence densities in the point process theory). Correlation functions can be considered as an analog of the moments of measures in the linear space analysis. In the case of superstable pair potentials their satisfy so-called Ruelle bound (RB) [75] which is very useful in applications. Another important bound obtained in the same paper is related with the density of finite volume projections of Gibbs measures (Ruelle probability bound (RPB)) which also became a standard technical tool in the equilibrium statistical physics. In particular, (RPB) gives information about the support of Gibbs measures. R.L.Dobrushin [21] proved exponential integrability w.r.t. Gibbs measures of some local functions on configuration spaces (Dobrushin exponential bound (DEB)) which also gives useful information about these measures.

In Chapter 4 we consider measures on configuration spaces which satisfy (some generalizations of) the mentioned bounds. We have shown that these bounds, in fact, are related among each other and do not need to be restricted to the class of Gibbs measures. This is important, in particular, in applications to non-equilibrium problems. More precisely, in the study of the dynamics (e.g., Hamiltonian or stochastic) of continuous systems we need, typically, to restrict the class of initial states assuming one or another kind of a priori bounds on them. Actually, the necessity to transport the description of the time evolution from the traditional classical mechanics point of view

(in terms of particle trajectories) to the evolution of states is a specific point in the rigorous statistical physics of continuous systems. We refer the reader to the excellent discussion of this concept in the review by R.L.Dobrushin, Ya.G.Sinai and Yu.M.Suhov [24]. In concrete examples we can see that the possibility to construct the time evolution of an initial state depends on the level of the deviation from the equilibrium state (i.e., on the information about "how non-equilibrium is the initial state").

Moreover, even in the case when the initial state is a Gibbs measure, the time evolution usually does not preserve the Gibbs property (at least, in the class of Gibbs measures with interactions of a finite order). But we can expect that the time evolution can be realized in a class on configuration space measures with certain a priori bounds. This hope is supported, in particular, by recent results on the stochastic dynamics of infinite particle systems [46]. One of the aims of this chapter is to clarify which kinds of a priori bounds can be reasonable, in principle, for measures in the configuration space analysis and how modifications of these bounds are reflected in the properties of the measures (e.g., support properties etc.).

Note, that even in the case of Gibbs measures with pair potentials, modifications of classical bounds are useful. For example, a generalization of the Ruelle bound for correlation functions, which we discussed in this chapter, was already used essentially in [3] for the construction of equilibrium gradient stochastic dynamics of continuous systems with pair singular interactions. An additional motivation for the analysis developed in this chapter is related with an important class of so called fermion and boson measures, see e.g. [60] and references therein. Such measures are defined via explicitly given correlation functions and do not admit clear Gibbs type descriptions. Only one way to study the properties of such measures is based on using the bounds on correlation functions and their consequences.

Existence problem for Gibbs measures on configuration spaces

In the first section of Chapter 5 we consider the existence problem for Gibbs states of continuous systems with pair interactions. Such problem was investigated in the fundamental works of R. L. Dobrushin and D. Ruelle, see [21], [76]. Ruelle's approach was based on the concept of superstability, and the existence result was obtained using a priori bounds on correlation func-

tions, which are now known as Ruelle bounds. In turn, these bounds were established with the help of the technically rather complicated use of superstability. Dobrushin's approach was based on the consideration of the associated lattice system with further use of the general Dobrushin criterion for lattice systems, see [21]. It was modified in the work of Pechersky and Zhukov, where the existence problem was analyzed under condition of the finite range interaction (see [67]). In Chapter 5 we develop the approach of Pechersky and Zhukov for the case of interaction with an infinite range.

In the second section of Chapter 5 we use a modified approach to the study the existence problem for Gibbs states of continuous systems with pair interactions, using Dobrushin approach described in [21]. Namely, we use Dobrushin existence criterion, which is proven for the lattice models in $\mathbb{Z}^d$. To apply this criterion to continuous models in $\mathbb{R}^d$, we reduce such a continuous model to an equivalent one on $\mathbb{Z}^d$ by appropriate partition of $\mathbb{R}^d$ into cubes, c.f. [67].

Comparing with [21], [67], we consider a different compact function on the spin space, that gives us more a priori information about the class of Gibbs measures. Similar to Dobrushin, we consider the potentials with infinite radius of the interaction but subject to the conditions close to those used in the papers [67], [76]. For simplicity we consider from the outset of Section 5.2 a concrete class of the potentials of the (DFR)-type (Dobrushin-Fisher-Ruelle type), although, as shown in the Theorem 5.2.2, the existence result holds true for more general potentials. In combination with the statements of Chapter 4 we have more a priori properties of Gibbs measures. Under the conditions of Dobrushin's existence criterion with some compact function, there exist certain a priori bounds of integrability of this function, see (5.31). As shown in the proof of Theorem 5.2.1 this implies integrability of functions $e^{\varrho|\gamma_\Lambda|^2}$ for some $\rho \geq 0$, where $|\gamma_\Lambda|$ denotes the number of particles in a finite volume $\Lambda \subset \mathbb{R}^d$. Starting from relations between bounds on probability measures on configuration spaces (see Chapter 4), we obtain information about the probability of occupation of particles in a finite volume (RPB) (Ruelle probability bound), as well as information about supports for Gibbs measures.

Furthermore, the use of the new compact function (5.25) allows us to avoid the consideration of multiple configurations from the outset of the work. We introduce a new metric (5.24) on the spin space that uses explicitly a compact function. The corresponding metric space is complete and separable, see Chapter 3. In the works of [21], [67] the problem of local-

ization of Gibbs states on the simple configuration spaces was subject of an additional analysis. Note, that comparing with [67] we do not need the assumption of the finite range interaction.

In the third section of the Chapter 5 we consider the case of multibody interaction. It is well known that one of the important (and essentially open) problems of equilibrium statistical mechanics is the construction of Gibbs states for continuous particle systems with many-body interactions. In the pioneering works by W. Greenberg [32] and H. Moral [64] the problem was analyzed via Kirkwood-Salsburg equations(KSE). For sufficiently small activity parameter z they proved existence of the unique solution of KSE, but with rather unnatural assumptions on the potentials which, in fact, take place only for finite range and positive interactions. In [74] the convergence of the Brydges-Federbush type cluster expansion is proved for dilute continuous systems with n-body ($n \leq M$) interaction. The proof requires a stable potential satisfying an integrability condition and exponential decay of the many-body potentials at large distances. In the following paper [71] the authors consider the system of hard-core spheres interacting via infinite group of many body potentials (for all n) which are bounded and integrable. They prove the convergence of the Mayer series for the pressure in thermodynamic limit and establish the region of analyticity in the activity z. In a recent work by V. Belitsky and E. A. Pechersky [6] the problem of existence and uniqueness of Gibbs state in $\mathbb{R}^d$ with finite group of n-body interactions was investigated using the technique of Dobrushin's type [22], [23].

In this section we give a simple proof of the existence of Gibbs state with infinite group of many body potentials. We establish some kind of modified Ruelle's bound for finite volume correlation functions. It gives a possibility to prove existence of at least one Gibbs measure in thermodynamic limit. We consider these results as some further development in solving of the existence problem for general potentials of interaction.

Acknowledgment

Chapter 2

General facts and notations

2.1 The space of finite configurations

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. By $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all open and Borel sets, respectively. $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ denote the system of all sets in $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, respectively, which are bounded.

The space of n-point configuration is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \big| \, |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set A. In the following, the symbol $|\cdot|$ may also represent Lebesgue measure or Euclidean norm in $\mathbb{R}^d$ but the meaning will always be clear from the context. Analogously the space $\Gamma_{0,\Lambda}^{(n)}$ is defined for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, which we denote for short by $\Gamma_\Lambda^{(n)}$.

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping

$$N_\Lambda : \Gamma_0^{(n)} \to \mathbb{N}_0; \quad N_\Lambda(\eta) := |\eta \cap \Lambda|.$$

For short we write $\eta_\Lambda := \eta \cap \Lambda$.

To define topological structure on $\Gamma_0^{(n)}$ we may use the following natural mapping

$$sym^n : \widetilde{(\mathbb{R}^d)^n} \mapsto \Gamma_0^{(n)}, \quad n \in \mathbb{N}, \tag{2.1}$$

$$sym^n((x_1, \ldots, x_n)) := \{x_1, \ldots, x_n\},$$

where

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \big| \, x_k \neq x_l \text{ if } k \neq l \right\}.$$

Using the mapping (2.1) one can identify $\Gamma_0^{(n)}$ with the symmetrization of $\widetilde{(\mathbb{R}^d)^n}$, i.e. $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group over $\{1, \ldots, n\}$. Hence $\Gamma_0^{(n)}$ inherits the structure of an $n \cdot d$ - dimensional manifold. Applying this we can introduce a topology $\mathcal{O}(\Gamma_0^{(n)})$ on $\Gamma_0^{(n)}$. The corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0^{(n)})$ coincides with the σ-algebra generated by the mappings N_Λ, i.e.,

$$\mathcal{B}(\Gamma_0^{(n)}) = \sigma\left(N_\Lambda \,\middle|\, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\right),$$

see e.g. [56]. Moreover, it is well known (see e.g. [63]) that a basis of the topology $\mathcal{O}(\Gamma_0^{(n)})$ is given by the following set

$$U_1 \hat{\times} \cdots \hat{\times} U_n := \left\{\eta \in \Gamma_0^{(n)} \,\middle|\, N_{U_1}(\eta) = 1, \ldots, N_{U_n}(\eta) = 1\right\},$$

where $U_1, \ldots, U_n \in \mathcal{O}(\mathbb{R}^d)$ with $U_i \cap U_j = \emptyset$ for $i \neq j$.

The space of finite configurations

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology $\mathcal{O}(\Gamma_0)$ of disjoint union. The corresponding Borel σ-algebra is denoted by $\mathcal{B}(\Gamma_0)$. A set $K \in \mathcal{B}(\Gamma_0)$ is compact iff there exists an $N \in \mathbb{N}$ such that $K \cap \Gamma_0^{(n)}$ is compact in $\Gamma_0^{(n)}$ for all $n \leq N$. A set $K \subset \Gamma_0^{(n)}$ is compact iff $(sym^n)^{-1}K$ is compact in $\widetilde{(\mathbb{R}^d)^n}$. A set $B \in \mathcal{B}(\Gamma_0)$ is called bounded iff there exists a $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and an $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. Any compact set is bounded in this sense, but not every closed and bounded set is compact.

2.2 Configuration space

The configuration space is defined as

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d \,\middle|\, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\right\}.$$

One can identify any $\gamma \in \Gamma$ with the positive Radon measure

$$\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d),$$

where $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$. Therefore, the configuration space Γ can be endowed with relative topology $\mathcal{O}(\Gamma)$ as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology such that all functions

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

are continuous for all $f \in C_0(\mathbb{R}^d)$ (the set of all continuous functions on $\mathbb{R}^d$ with bounded support). Moreover, well known (see e.g. [56], [63]) that a subbasis of the topology $\mathcal{O}(\Gamma)$ is given by the sets of the form

$$\{\gamma \in \Gamma |\ |\gamma_\Lambda| = n,\ \gamma_{\partial\Lambda} = \emptyset\},$$

where $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, and $\partial\Lambda$ is the topological boundary of Λ. This topology is separable and metrizable, see e.g. [63]. The convergence of the sequence $(\gamma^{(n)})_{n \in \mathbb{N}}$ to γ in the topology $\mathcal{O}(\Gamma)$ can be described in the following way: $(\gamma^{(n)})_{n \in \mathbb{N}}$ converges to γ in $\mathcal{O}(\Gamma)$ iff $N_\Lambda(\gamma^{(n)}) \to N_\Lambda(\gamma)$ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ with $N_{\partial\Lambda}(\gamma) = 0$.

The Borel σ-algebra $\mathcal{B}(\Gamma)$ is equal to the smallest σ-algebra for which all the mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$ are measurable, i.e.,

$$\mathcal{B}(\Gamma) = \sigma(N_\Lambda \,|\Lambda \in \mathcal{B}_c(\mathbb{R}^d))$$

and filtration on Γ given by

$$\mathcal{B}_\Lambda(\Gamma) := \sigma(N_{\Lambda'} \,|\Lambda' \in \mathcal{B}_c(\mathbb{R}^d),\ \Lambda' \subset \Lambda).$$

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the configuration space Γ_Λ is defined as

$$\Gamma_\Lambda = \{\gamma \in \Gamma \,|\, \gamma \subset \Lambda\}.$$

It is equipped with the induced topology $\mathcal{O}(\Gamma_\Lambda)$ of the topology $\mathcal{O}(\Gamma)$. The Borel σ-algebra generated by $\mathcal{O}(\Gamma_\Lambda)$ is denoted by $\mathcal{B}(\Gamma_\Lambda)$. Obviously, the configuration space Γ_Λ can be represented as

$$\Gamma_\Lambda = \bigsqcup_{n \in \mathbb{N}_0} \Gamma_\Lambda^{(n)}. \tag{2.2}$$

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a projection

$$p_\Lambda : \Gamma \to \Gamma_\Lambda; \quad p_\Lambda(\gamma) := \gamma_\Lambda$$

and w.r.t. this projections Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$. The following classes of function are used in the following: $L^0(\Gamma_0)$ is the set of all measurable functions on Γ_0, $L^0_{\mathrm{ls}}(\Gamma_0)$ is the set of functions which have additionally a local support, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ such that $G \restriction_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$. $L^0_{\mathrm{bs}}(\Gamma_0)$ denotes the measurable functions with bounded support, $B(\Gamma_0)$ the set of bounded measurable functions. On Γ we consider the set of a cylinder functions $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable function $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. These functions are characterized by the following relation: $F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda)$.

Next we would like to describe some facts from Harmonic analysis on configuration space based on [38, 39].

The following mapping between functions on Γ_0, e.g. $L^0_{\mathrm{ls}}(\Gamma_0)$, and functions on Γ, e.g. $\mathcal{F}L^0(\Gamma)$, plays a key role in our further considerations:

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad \gamma \in \Gamma,$$

where $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, see e.g. [56, 57]. The summation in the latter expression is extend over all finite subconfigurations of γ, in symbols $\xi \Subset \gamma$. K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.3}$$

Lemma 2.2.1 *For all* $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $F \in \mathcal{F}L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$

$$K^{-1}F(\eta) = \mathbb{1}_{\Gamma_\Lambda}(\eta) K^{-1}F(\eta), \quad \forall \eta \in \Gamma_0.$$

Proof.

$$K^{-1}F(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi) = \sum_{\xi_1 \subset \eta_\Lambda} \sum_{\xi_2 \subset \eta_{\mathbb{R}^d \setminus \Lambda}} (-1)^{|\eta \setminus (\xi_1 \cup \xi_2)|} F(\xi_1 \cup \xi_2) =$$

$$= \sum_{\xi_2 \subset \eta_{\mathbb{R}^d \setminus \Lambda}} (-1)^{|\xi_2|} \sum_{\xi_1 \subset \eta_\Lambda} (-1)^{|\eta \setminus \xi_1|} F(\xi_1) = 0^{|\eta_{\mathbb{R}^d \setminus \Lambda}|} K^{-1}F(\eta_\Lambda) =$$

$$= \mathbb{1}_{\Gamma_\Lambda}(\eta) K^{-1}F(\eta). \qquad \blacksquare$$

One can introduce a convolution

$$
\begin{aligned}
\star : L^0(\Gamma_0) \times L^0(\Gamma_0) &\to L^0(\Gamma_0) \qquad\qquad\qquad (2.4)\\
(G_1, G_2) &\mapsto (G_1 \star G_2)(\eta)\\
&:= \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2)\, G_2(\xi_2 \cup \xi_3),
\end{aligned}
$$

where $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions (ξ_1, ξ_2, ξ_3) of η in 3 parts, i.e., all triples (ξ_1, ξ_2, ξ_3) with $\xi_i \subset \eta$, $\xi_i \cap \xi_j = \emptyset$ if $i \neq j$, and $\xi_1 \cup \xi_2 \cup \xi_3 = \eta$. It has the property that for $G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$ we have $K(G_1 \star G_2) = KG_1 \cdot KG_2$. Due to this convolution we can interpret K transform as Fourier transform in configuration space analysis, see also [12].

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ which have finite local moments of all orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$. A measure ρ on Γ_0 is called locally finite iff $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$, the set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$
\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).
$$

$\rho_\mu := K^*\mu$ we call the correlation measure corresponding to μ.

As shown in [38] for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series

$$
KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \qquad\qquad\qquad (2.5)
$$

is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$
\int_{\Gamma_0} G(\eta)\,\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\,\mu(d\gamma). \qquad\qquad (2.6)
$$

Fix a non-atomic and locally finite measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$ and hence on $\Gamma^{(n)}_0$. The measure on $\Gamma^{(n)}_0$ we denote by $\sigma^{(n)}$.

The *Lebesgue-Poisson measure* $\lambda_{z\sigma}$ on Γ_0 is defined as

$$
\lambda_{z\sigma} := \sum_{n=0}^\infty \frac{z^n}{n!} \sigma^{(n)}.
$$

Here $z > 0$ is the so called activity parameter. The restriction of $\lambda_{z\sigma}$ to Γ_Λ will be also denoted by $\lambda_{z\sigma}$.

The *Poisson measure* $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_{z\sigma}^\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$, where $\pi_{z\sigma}^\Lambda$ is the measure on Γ_Λ defined by

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}.$$

A measure $\mu \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ is called locally absolutely continuous w.r.t. $\pi_{z\sigma}$ iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi_{z\sigma}^\Lambda = \pi_{z\sigma} \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_\Lambda(\mathbb{R}^d)$. In this case $\rho_\mu = K^* \mu$ is absolutely continuous w.r.t $\lambda_{z\sigma}$. We denote by

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_\sigma}(\eta), \ \eta \in \Gamma_0.$$

The functions

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}^+ \tag{2.7}$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are well known correlation functions of statistical physics, see e.g [76], [75].

2.3   The space of multiple configurations

The space of multiple configurations is defined as

$$\ddot{\Gamma} = \left\{ (\gamma, n), \ \gamma \subset \mathbb{R}^d, \ n : \gamma \to \mathbb{N} \ \Bigg| \ \sum_{x \in \gamma_\Lambda} n(x) < \infty \ \text{ for all } \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\},$$

where $\gamma_\Lambda = \gamma \cap \Lambda, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

Multiple configuration can be interpreted in the following way. The set γ is a set of positions from $\mathbb{R}^d$ where particles are located and for every $x \in \gamma$ the number $n(x)$ is the number of particles located at the position x. In the sequel, notation $\gamma \in \ddot{\Gamma}$ will be understood as $(\gamma, n) \in \ddot{\Gamma}$. Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma \in \ddot{\Gamma}$. We use the following notations: $\gamma_\Lambda = \gamma \cap \Lambda$ and $\sigma_\Lambda = (\gamma_\Lambda, n_\Lambda)$, where $n_\Lambda = n|_{\gamma_\Lambda}$. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma \in \ddot{\Gamma}$ we denote by $|\gamma_\Lambda|$ or $|\sigma_\Lambda|$ the number of particles of the configuration (γ, n) in Λ, i.e

$$|\sigma_\Lambda| = |\gamma_\Lambda| := \sum_{x \in \gamma_\Lambda} n(x).$$

We say that $(\gamma_1, n_1) \in \ddot{\Gamma}$ is included in (γ_2, n_2) if $\gamma_1 \subset \gamma_2$ and $n_1(x) \leq n_2(x)$ for $x \in \gamma_1$. If $\gamma_1 \cap \gamma_2 = \emptyset$, then the union $(\gamma_1, n_1) \cup (\gamma_2, n_2)$ is the configuration (γ_3, n_3), where $\gamma_3 = \gamma_1 \cup \gamma_2$ and for $x \in \gamma_3$

$$n_3(x) = \begin{cases} n_1(x), & \text{if } x \in \gamma_1 \\ n_2(x), & \text{if } x \in \gamma_2. \end{cases}$$

The empty configuration θ is the configuration $\theta = (\emptyset, 0)$, i.e. $\gamma = \emptyset$ and $n \equiv 0$. We define $(\gamma_1, n_1) \cap (\gamma_2, n_2) = (\gamma_3, n_3)$, where $\gamma_3 = \gamma_1 \cap \gamma_2$ and $n_3(x) = \min\{n_1(x), n_2(x)\}$ for $x \in \gamma_3$. If $\gamma_1 \cap \gamma_2 = \emptyset$ we write $(\gamma_1, n_1) \cap (\gamma_2, n_2) = \theta$. As in Section 2.2, one can identify any $\gamma \in \ddot{\Gamma}$ with the positive Radon measure

$$\sum_{x \in \gamma} n(x) \varepsilon_x \in \mathcal{M}(\mathbb{R}^d).$$

Therefore, the space of multiple configurations $\ddot{\Gamma}$ can be endowed with relative topology $\mathcal{O}(\ddot{\Gamma})$ as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology. This topology is separable and metrizable, see e.g. [63]. Moreover, since for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\sigma = (\gamma, n)$, the sum $\sum_{x \in \gamma \cap \Lambda} n(x)$ is finite, a Borel σ-algebra $\mathcal{B}(\ddot{\Gamma})$ on $\ddot{\Gamma}$ is generated by

$$\mathcal{A}_\Lambda^{(m)} := \left\{ \gamma \in \ddot{\Gamma} \,\middle|\, |\gamma_\Lambda| = m \right\},$$

which are called cylindrical sets. For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the configuration space $\ddot{\Gamma}_\Lambda$ is defined as

$$\ddot{\Gamma}_\Lambda = \{\gamma \in \ddot{\Gamma} \,|\, \gamma \subset \Lambda\}.$$

It is equipped with the induced topology $\mathcal{O}(\ddot{\Gamma}_\Lambda)$ of the topology $\mathcal{O}(\ddot{\Gamma})$. Obviously, $\Gamma_\Lambda \subset \ddot{\Gamma}_\Lambda$ for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

The space of m-point multiple configuration is

$$\ddot{\Gamma}_{0,\mathbb{R}^d}^{(m)} = \ddot{\Gamma}_0^{(m)} := \left\{ (\eta, n) \in \ddot{\Gamma}, \ \eta \subset \mathbb{R}^d \,\middle|\, |\sigma| = \sum_{x \in \eta} n(x) = m \right\}, \quad m \in \mathbb{N}_0.$$

Analogously, the space $\ddot{\Gamma}_{0,\Lambda}^{(m)}$ is defined for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, which we denote for short by $\ddot{\Gamma}_\Lambda^{(m)}$.

The space of finite multiple configurations

$$\ddot{\Gamma}_0 := \bigsqcup_{m \in \mathbb{N}_0} \ddot{\Gamma}_0^{(m)}.$$

Obviously, the space $\ddot{\Gamma}_\Lambda$ can be represented as

$$\ddot{\Gamma}_\Lambda = \bigsqcup_{m \in \mathbb{N}_0} \ddot{\Gamma}_\Lambda^{(m)}. \tag{2.8}$$

The *Lebesgue-Poisson measure* λ_z for cylindrical sets is defined as

$$\lambda_z(\{\sigma \in \ddot{\Gamma} \mid |\sigma_\Lambda| = m\}) = \frac{z^m}{m!}|\Lambda|^m,$$

where $z > 0$ is the so-called activity parameter and the symbol $|\cdot|$ represents Lebesgue measure. We are able to extend the measure λ_z to the whole σ-algebra $\mathcal{B}(\ddot{\Gamma})$ using the equality

$$\lambda_z(\mathcal{A}_{\Lambda_1}^{(n_1)} \cap \mathcal{A}_{\Lambda_2}^{(n_2)}) = \lambda_z(\mathcal{A}_{\Lambda_1}^{(n_1)})\lambda_z(\mathcal{A}_{\Lambda_2}^{(n_2)}), \quad \Lambda_1 \cap \Lambda_2 = \emptyset.$$

The measure λ_z restricted to $\ddot{\Gamma}_\Lambda$ is also denoted by λ_z.

Chapter 3

Detailed structure and some topological properties of the configuration space Γ

3.1   Metrical structures on configuration space

It is well known from [63] that the space of multiple configurations $\ddot{\Gamma}$ is a Polish space. Let ρ be a metric on $\ddot{\Gamma}$ such that $(\ddot{\Gamma}, \rho)$ is separable and complete.

Lemma 3.1.1 (c.f. [83]) *The configuration space Γ is a G_δ-set in $\ddot{\Gamma}$.*

Proof. Let $\{K_i\}_{i \geq 1}$ be an increasing sequence of compact sets such that

$$\bigcup_{i \geq 1} K_i = \mathbb{R}^d.$$

Then Γ can be represented as

$$\Gamma = \bigcap_{i \geq 1} \left[\ddot{\Gamma} \setminus \Gamma(K_i) \right],$$

where
$$\Gamma(K_i) = \{\gamma \in \ddot{\Gamma} \,|\, \exists\, x \in K_i \,:\, n(x) \geq 2\}.$$

The only thing to show now is that for any $i \geq 1$ the set $\Gamma(K_i)$ is vaguely closed.

Let $i \in \mathbb{N}$ be arbitrary and $\{\gamma_n\}_{n \geq 1}$ be a sequence from $\Gamma(K_i)$, such that $\gamma_n \to \gamma$, $n \to \infty$ vaguely. Let $f \in C_0(\mathbb{R}^d)$ be arbitrary and fixed. Then, for every $n \geq 1$ there exists $x_n \in K_i$ such that the following holds

$$\langle f, \gamma_n \rangle \geq 2f(x_n). \tag{3.1}$$

Because K_i is compact in $\mathbb{R}^d$, there exists a convergent subsequence $\{x_{n_m}\}_{m \geq 1}$ of the sequence $\{x_n\}_{n \geq 1}$. Moreover, the correspondent limit $x \in K_i$. Therefore, using continuity of the function f, inequality (3.1) yields

$$\langle f, \gamma \rangle \geq 2f(x).$$

The function f was fixed to be arbitrary, hence, the latter inequality holds for any $f \in C_0(\mathbb{R}^d)$, which implies $x \in \gamma$. Taking the function $f \in C_0(\mathbb{R}^d)$ such that $f(x) \neq 0$ and $\langle f, \gamma \rangle = n(x)f(x)$ we will have $n(x) \geq 2$, which means that $\Gamma(K_i)$ is vaguely closed. ∎

Remark 3.1.1 *It is well known from [17] that any G_δ-set of the Polish space is a Polish space. Therefore, Γ is a Polish space.*

Consider $\psi : \mathbb{R}^d \mapsto (0, 1]$, $\psi \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and a continuous decreasing $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$, such that

(I_α) $\alpha_0 := \lim_{t \to 0+} \alpha(t) = +\infty$;

(II_α) $\alpha_+ := \lim_{t \to +\infty} \alpha(t) \geq 1$;

The set of all pairs of functions (α, ψ) which satisfy the conditions above will be denoted by $\mathcal{F}$.

Let $I = \{I_k\}_{k \in \mathbb{N}}$ be an arbitrary collection of functions from $C_0(\mathbb{R}^d)$ such that $I_k : \mathbb{R}^d \mapsto [0, 1]$, $\mathrm{supp}\, I_k \subset \Lambda_k$, $k \geq 1$, and for all $x \in \Lambda_k : I_{k+1}(x) \neq 0$, $k \geq 1$. We let $\psi_k := \psi I_k$.

Define

$$\Gamma^{\alpha, \psi} = \left\{ \gamma \in \Gamma \;\middle|\; \sum_{\{x,y\} \subset \gamma} \psi(x)\alpha(|x - y|)\psi(y) < \infty \right\}, \quad (\alpha, \psi) \in \mathcal{F}$$

and

$$E^{\alpha, \psi}(\gamma) = \sum_{\{x,y\} \subset \gamma} \psi(x)\alpha(|x - y|)\psi(y), \; \gamma \in \Gamma^{\alpha, \psi}.$$

For any $k \geq 1$ we define also

$$E_k^{\alpha, \psi}(\gamma) = \sum_{\{x,y\} \subset \gamma} \psi_k(x) \alpha(|x - y|) \psi_k(y), \ \gamma \in \Gamma.$$

Lemma 3.1.2 *Let $\{\gamma^{(n)}\}_{n \geq 1}$ be a sequence from Γ, such that*

$$\gamma^{(n)} \to \gamma, \quad n \to \infty$$

in the vague topology. Then

$$\forall x \in \gamma \ \ \exists \{x_{n_k}^i\}_{k \geq 1}, \ \ 1 \leq i \leq n(x) : \ x_{n_k}^i \in \gamma^{(n_k)}, \ k \geq 1, \ 1 \leq i \leq n(x)$$

and

$$x_{n_k}^i \to x, \ k \to \infty, \ 1 \leq i \leq n(x).$$

Proof. Let $x \in \gamma$ be arbitrary. There exists $N \in \mathbb{N}$ such that

$$\dot{B}_{\frac{1}{k}}(x) \cap \gamma = \emptyset, \quad \forall k \geq N,$$

where

$$\dot{B}_{\frac{1}{k}}(x) = B_{\frac{1}{k}}(x) \setminus \{x\}, \quad B_{\frac{1}{k}}(x) = \left\{ y \in \mathbb{R}^d \ \middle| \ |x - y| \leq \frac{1}{k} \right\}.$$

Let us consider the sequence of functions $f_k \in C_0(\mathbb{R}^d)$, $k \geq N$ such that

$$\mathbb{1}_{B_{\frac{1}{k+1}}(x)} \leq f_k \leq \mathbb{1}_{B_{\frac{1}{k}}(x)}, \quad k \geq N.$$

For any $k \geq N$ there exists $N_1 = N_1(k) \in \mathbb{N}$ such that for any $n \geq N_1$:

$$|\gamma^{(n)}_{B_{\frac{1}{k+1}}(x)}| - n(x) \leq \langle f_k, \gamma^{(n)} \rangle - \langle f_k, \gamma \rangle < \frac{1}{2} \tag{3.2}$$

and $N_2 = N_2(k) \in \mathbb{N}$ such that for any $n \geq N_2$:

$$|\gamma^{(n)}_{B_{\frac{1}{k}}(x)}| - n(x) \geq \langle f_k, \gamma^{(n)} \rangle - \langle f_k, \gamma \rangle > -\frac{1}{2}. \tag{3.3}$$

Inequalities (3.29) and (3.3) give us

$$\forall k \geq N + 1 \ \ \exists N^\star(k) = \max \{N_1(k - 1), N_2(k)\} :$$

$$\forall\, n \geq N^{\star}(k) :\ |\gamma^{(n)}_{B_{\frac{1}{k}}(x)}| = n(x).$$

Set

$$M_1 = N^{\star}(N + 1),\ \ M_l = \max\{M_{l-1} + 1,\ N^{\star}(N + l)\},\ l \geq 2.$$

Let

$$\gamma^{(M_l)}_{B_{\frac{1}{N+l}}(x)} = \{x_l^1, \ldots, x_l^{n(x)}\},\ l \geq 1.$$

Obviously, for any $1 \leq i \leq n(x)$

$$x_l^i \to x,\ l \to \infty.\quad \blacksquare$$

Lemma 3.1.3 *For any $(\alpha,\, \psi) \in \mathcal{F}$ and $k \geq 1$ the function $E_k^{\alpha,\,\psi}(\gamma)$ is continuous on Γ.*

Proof. Let $\gamma^{(n)} \to \gamma \in \Gamma$, $n \to \infty$ vaguely and let $\gamma_{\Lambda_k} = \{x_1, \ldots, x_p\}$. Set

$$l = \min_{1 \leq i < j \leq p} |x_i - x_j|.$$

Suppose that $\gamma_{\Lambda_k^c} \neq \emptyset$, then we define

$$2r = \inf_{x \in \gamma_{\Lambda_k^c},\, y \in \Lambda_k} |x - y| > 0.$$

Let $\varepsilon > 0$ be an arbitrary and fixed. A straightforward arguments insures that for any $\bar{\varepsilon} > 0$ function α is uniformly continuous on $[\bar{\varepsilon},\, \infty)$. Because function ψ_k is continuous with compact support, there exists $\delta > 0$ such that for all $x,\, y\, \in \mathbb{R}^d$, $r_1,\, r_2 \in [l/3,\, \infty)$:

$$|x - y| < \delta,\quad |r_1 - r_2| < 2\delta$$

holds

$$|\psi_k(x) - \psi_k(y)| < \varepsilon\quad \text{and}\quad |\alpha(r_1) - \alpha(r_2)| < \varepsilon. \tag{3.4}$$

Denote $\delta^{\star} = \min\{\delta,\, l/3,\, r\}$. Vague convergence of $\gamma^{(n)}$ to γ, when $n \to \infty$ implies

$$\forall f \in C_0(\mathbb{R}^d)\ \ \exists N \in \mathbb{N} :\ \ \forall n \geq N\ \ |\langle f,\, \gamma^{(n)} \rangle - \langle f,\, \gamma \rangle| < \frac{1}{2}. \tag{3.5}$$

Define

$$\Lambda_{k+r} = \left\{x \in \mathbb{R}^d \,|\, \mathrm{dist}_{\mathbb{R}^d}(x,\, \Lambda_k) <\, r\right\}.$$

Applying (3.5) to the functions $f_i \in C_0(\mathbb{R}^d)$, $1 \le i \le p + 2$ of the form

$$f_i \le \mathbb{1}_{B_{\delta^\star}(x_i)}, \quad x_i \in \operatorname{supp} f_i, \quad 1 \le i \le p$$

and

$$\mathbb{1}_{\Lambda_k} \le f_{p+1} \le \mathbb{1}_{\Lambda_{k+r}}, \quad \mathbb{1}_{\Lambda_{k+r}} \le f_{p+2} \le \mathbb{1}_{\Lambda_{k+2r}},$$

we have

$$\exists N^\star \in \mathbb{N} : \forall n \ge N^\star \quad |\gamma_{\Lambda_{k+r}}| = |\gamma^{(n)}_{\Lambda_{k+r}}| = p.$$

Moreover, for any $1 \le i \le p$ and any $n \ge N^\star$

$$\exists! \, x_i^n \in \gamma^{(n)} \quad \text{such that} \quad |x_i^n - x_i| < \delta^\star. \tag{3.6}$$

Therefore, for any $1 \le i < j \le p$ and any $n \ge N^\star$

$$|\psi_k(x_i)\alpha(|x_i - x_j|)\psi_k(x_j) - \psi_k(x_i^n)\alpha(|x_i^n - x_j^n|)\psi_k(x_j^n)| \le \tag{3.7}$$

$$\le \alpha(l)|\psi_k(x_i) - \psi_k(x_i^n)| + \alpha(l)|\psi_k(x_j) - \psi_k(x_j^n)| + |\alpha(|x_i - x_j|) - \alpha(|x_i^n - x_j^n|)|.$$

Using (3.4), (3.6) and bound

$$|x_i^n - x_j^n| \ge |x_i - x_j| - |x_i - x_i^n| - |x_j - x_j^n| \ge l - l/3 - l/3 = l/3, \quad n \ge N^\star$$

we can estimate (3.7) by

$$\varepsilon(2\alpha(l) + 1).$$

Finally, for any $n \ge N^\star$

$$|E_k^{\alpha, \psi}(\gamma^{(n)}) - E_k^{\alpha, \psi}(\gamma)| \le \frac{\varepsilon p(p - 1)(2\alpha(l) + 1)}{2}.$$

We omit the case $\gamma_{\Lambda_k^c} = \emptyset$ because of its triviality. $\blacksquare$

Consider a function $\rho_{\alpha, \psi} : \Gamma \times \Gamma \to \mathbb{R}_+$ which is defined by

$$\rho_{\alpha, \psi}(\gamma_1, \gamma_2) = \rho(\gamma_1, \gamma_2) + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|E_k^{\alpha, \psi}(\gamma_1) - E_k^{\alpha, \psi}(\gamma_2)|}{1 + |E_k^{\alpha, \psi}(\gamma_1) - E_k^{\alpha, \psi}(\gamma_2)|}. \tag{3.8}$$

Obviously, this function is a metric on configuration space Γ.

Theorem 3.1.1 *For any $(\alpha, \psi) \in \mathcal{F}$ metric space $(\Gamma, \rho_{\alpha, \psi})$ is complete and separable. Moreover topology on Γ generated by metric $\rho_{\alpha, \psi}$ is equivalent to the vague topology on Γ.*

Proof. Let $(\alpha, \psi) \in \mathcal{F}$ be arbitrary and let $\gamma^{(n)} \to \gamma \in \Gamma$, $n \to \infty$ vaguely. Because vague topology and topology generated by metric ρ are equivalent, we have

$$\rho(\gamma^{(n)}, \gamma) \to 0, \quad n \to \infty.$$

Using Lemma 3.1.3, for any $k \in \mathbb{N}$ we have

$$|E_k^{\alpha, \psi}(\gamma^{(n)}) - E_k^{\alpha, \psi}(\gamma)| \to 0, \quad n \to \infty$$

yielding

$$\rho_{\alpha, \psi}(\gamma^{(n)}, \gamma) \to 0, \quad n \to \infty. \tag{3.9}$$

Hence, the collection of all closed sets in the topology generated by $\rho_{\alpha, \psi}$ and the vague topology is the same, which means that corresponding topologies are equivalent.

It is well known from [63] that vague topology on Γ is separable. Therefore, metric space $(\Gamma, \rho_{\alpha, \psi})$ is separable.

Let $\{\gamma^{(n)}\}_{n \geq 1}$ be a Cauchy sequence in $(\Gamma, \rho_{\alpha, \psi})$. Then, $\{\gamma^{(n)}\}_{n \geq 1}$ is also a Cauchy sequence in $(\ddot{\Gamma}, \rho)$. The completeness of $(\ddot{\Gamma}, \rho)$ implies existence of $\gamma \in \ddot{\Gamma}$ such that $\rho(\gamma^{(n)}, \gamma) \to 0$, $n \to \infty$. If $\gamma \in \Gamma$ then as was shown before we have (3.9). Suppose that there exists $x \in \gamma$ such that $n(x) > 1$. Let $x \in \Lambda_{\hat{k}}$ for some $\hat{k} \in \mathbb{N}$. Using Lemma 3.1.2 we obtain

$$\exists \{x_{n_k}^i\}_{k \geq 1}, \ i = 1, 2: \ x_{n_k}^i \in \gamma^{(n_k)}, k \geq 1, i = 1, 2$$

and

$$x_{n_k}^i \to x, \ k \to \infty, \ i = 1, 2.$$

Therefore,

$$|x_{n_k}^1 - x_{n_k}^2| \to 0, \quad k \to \infty. \tag{3.10}$$

Let us denote

$$\psi^\star = \inf_{x \in \Lambda_{\hat{k}}} \psi_{\hat{k}+1}^2(x).$$

As for any $x \in \Lambda_{\hat{k}}: \ \psi_{\hat{k}+1}(x) \neq 0$, the number $\psi^\star > 0$. The fact that $\{\gamma^{(n)}\}_{n \geq 1}$ is a Cauchy sequence implies

$$\exists N_1 \in \mathbb{N}: \ \forall k \geq N_1, m \geq N_1 \quad |E_{\hat{k}+1}^{\alpha, \psi}(\gamma^{(n_k)}) - E_{\hat{k}+1}^{\alpha, \psi}(\gamma^{(m)})| < \psi^\star. \tag{3.11}$$

We fix $m \geq N_1$ and define a number

$$C = (\psi^\star)^{-1} E_{\hat{k}+1}^{\alpha, \psi}(\gamma^{(m)}) < \infty.$$

From the property (I_α) and (3.10) we conclude

$$\exists\, N_2 \in \mathbb{N} : \quad \forall\, k \geq N_2, \quad \alpha(|x_{n_k}^1 - x_{n_k}^2|) > C + 2,$$

which yields the following estimate

$$E_{\hat{k}+1}^{\alpha,\psi}(\gamma^{(n_k)}) > \psi^\star(C+2), \quad k \geq N_2. \tag{3.12}$$

Finally, with the help of (3.11), for any $k \geq \max\{N_1, N_2\}$ we obtain

$$E_{\hat{k}+1}^{\alpha,\psi}(\gamma^{(n_k)}) < \psi^\star + \psi^\star C,$$

which contradicts to (3.12). Hence, for any $x \in \gamma : n(x) = 1$. This concludes the proof. $\blacksquare$

We consider a function $\rho^\Lambda : \Gamma_\Lambda \times \Gamma_\Lambda \to \mathbb{R}_+$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ defined by

$$\rho^\Lambda(\eta_1, \eta_2) = \begin{cases} \frac{1}{2\,diam(\Lambda)|\eta_1|} \min_\pi \sum_{i=1}^{|\eta_1|} |x_i - y_{\pi(i)}|, & \text{if } |\eta_1| = |\eta_2| \\ 1, & \text{otherwise.} \end{cases} \tag{3.13}$$

In (3.13) minimum is taken over the set of all permutations π of the set $\{1, \dots, |\eta_1|\}$, configuration $\eta_1 = \{x_1, \dots, x_{|\eta_1|}\}$ and $\eta_2 = \{y_1, \dots, y_{|\eta_2|}\}$. As shown in [67] for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the function ρ^Λ is a metric on $\ddot{\Gamma}_\Lambda$ and, hence, on Γ_Λ. Moreover, for any compact set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ metric space $(\ddot{\Gamma}_\Lambda, \rho^\Lambda)$ is complete and separable, although, metric space $(\Gamma_\Lambda, \rho^\Lambda)$ is not complete.

Let $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be an arbitrary continuous decreasing function, which satisfies conditions I_α and II_α. One can introduce the Hamiltonian which corresponds to potential $\alpha(|x|)$:

$$E^\alpha(\eta) = \sum_{\{x,y\} \subset \eta} \alpha(|x - y|), \ \eta \in \Gamma_0, \ |\eta| \geq 2.$$

Consider a function $d_\alpha : \Gamma_\Lambda \times \Gamma_\Lambda \to \mathbb{R}_+$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ which is defined by

$$d_\alpha(\eta_1, \eta_2) = \rho^\Lambda(\eta_1, \eta_2) + |E^\alpha(\eta_1) - E^\alpha(\eta_2)|, \quad \eta_1, \eta_2 \in \Gamma_\Lambda. \tag{3.14}$$

Proposition 3.1.1 *For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the function d_α is a metric on Γ_Λ. Moreover, if Λ is a closed set then the metric space $(\Gamma_\Lambda, d_\alpha)$ is complete and separable.*

Proof. Symmetry and triangle inequality of d_α follow straightforward. If $\eta_1 = \eta_2$ then $d_\alpha(\eta_1, \eta_2) = 0$ by the definition of d_α. If $d_\alpha(\eta_1, \eta_2) = 0$ then $\rho^\Lambda(\eta_1, \eta_2) = 0$ and $\eta_1 = \eta_2$. Therefore, $(\Gamma_\Lambda, d_\alpha)$ is a metric space.

Let $\{\eta_n\}_{n\geq 1}$ be arbitrary Cauchy sequence in the metric space $(\Gamma_\Lambda, d_\alpha)$. Then $\{\eta_n\}_{n\geq 1}$ will be also a Cauchy sequence in $(\Gamma_\Lambda, \rho^\Lambda)$. As $\Gamma_\Lambda \subset \ddot{\Gamma}_\Lambda$ for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $(\ddot{\Gamma}_\Lambda, \rho^\Lambda)$ is a complete metric space, there exists $\sigma \in \ddot{\Gamma}_\Lambda$ such that

$$\rho^\Lambda(\eta_n, \sigma) \to 0, \quad n \to 0. \tag{3.15}$$

Moreover, from the definition of ρ^Λ follows

$$\exists\, N_0 \in \mathbb{N} \quad \forall\, n \geq N_0 \; : \; |\eta_n| = |\sigma| =: p.$$

Let $\eta_n = \{x_1^n, \ldots, x_p^n\}$, $n \geq N_0$. Then from (3.15) we have convergence of η_n to σ in the following sense:

$$x_k^n \to x_k, \quad n \to \infty, \quad 1 \leq k \leq p,$$

where x_k, $1 \leq k \leq p$ are all positions of particles of the configuration σ, which may be repeated. We will show that $\sigma \in \Gamma_\Lambda$. Suppose that this is not true, i.e.

$$\exists\, 1 \leq k < j \leq p \; : \; x_k = x_j.$$

This implies

$$|x_k^n - x_j^n| \to 0, \; n \to \infty. \tag{3.16}$$

The sequence $\{\eta_n\}_{n\geq 1}$ is a Cauchy sequence in $(\Gamma_\Lambda, d_\alpha)$. Hence,

$$\exists\, N_1 \in \mathbb{N}, \; N_1 \geq N_0 \quad \forall\, n \geq N_1, \, m \geq N_1 \; :$$

$$\left| \sum_{1 \leq i < r \leq p} \alpha(|x_i^n - x_r^n|) - \sum_{1 \leq i < r \leq p} \alpha(|x_i^m - x_r^m|) \right| < 1.$$

Let us fix $m \geq N_1$ and define number

$$C = \sum_{1 \leq i < r \leq p} \alpha(|x_i^m - x_r^m|) < \infty.$$

From the property $\alpha(0+) = +\infty$ and (3.16) follows

$$\exists\, N_2 \in \mathbb{N} \quad \forall\, n \geq N_2 \; : \; \alpha(|x_k^n - x_j^n|) > C + 2. \tag{3.17}$$

Hence,

$$\sum_{1 \leq i < r \leq p} \alpha(|x_i^n - x_r^n|) > C + 2. \tag{3.18}$$

Denote $N := \max\{N_1 \, N_2\}$. Then for any $n \geq N$

$$\left| \sum_{1 \leq i < r \leq p} \alpha(|x_i^n - x_r^n|) \right| \leq \left| \sum_{1 \leq i < r \leq p} \alpha(|x_i^n - x_r^n|) - \sum_{1 \leq i < r \leq p} \alpha(|x_i^m - x_r^m|) \right| +$$

$$+ \sum_{1 \leq i < r \leq p} \alpha(|x_i^m - x_r^m|) < 1 + C.$$

The latter inequality contradicts to (3.18). Hence, $\sigma \in \Gamma_\Lambda$. From the continuity of the function α follows

$$d_\alpha(\eta_n, \, \sigma) \to \infty, \; n \to \infty.$$

We have proved that $(\Gamma_\Lambda, \, d_\alpha)$ is a complete metric space.

Obviously, the set $\Gamma_{\Lambda_\mathbb{Q}}$, with $\Lambda_\mathbb{Q} := \Lambda \cap \mathbb{Q}^d$, will be a countable dense subset in Γ_Λ. Therefore, $(\Gamma_\Lambda, \, d_\alpha)$ is also separable. ∎

3.2 Relatively compact sets and compact functions on configuration space

The description of relatively compact subsets of the configuration space Γ in the vague topology was obtained in the work [35]. Below, we propose an alternative proof of the corresponding criterion which is based on the metric structures defined before.

Theorem 3.2.1 *A set $S \subset \Gamma$ is relatively compact in the vague topology, iff for any compact set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ holds*

$$\sup_{\gamma \in S} |\gamma_\Lambda| < \infty \;\; \text{and} \;\; \inf_{\gamma \in S} \min_{\{x, y\} \subset \gamma_\Lambda} |x - y| > 0. \tag{3.19}$$

Proof. Let $S \subset \Gamma$ be relatively compact in the vague topology. Then, it is relatively compact in Γ with respect to the metric $\rho_{\alpha, \psi}$. From the Hausdorff criterion it follows that for any $\varepsilon > 0$ there exists a finite ε-net for S in

$(\Gamma, \rho_{\alpha,\phi})$. Hence, S is relatively compact in $(\ddot{\Gamma}, \rho)$. It is well known from [63], that in this case, the condition

$$\sup_{\gamma \in S} |\gamma_\Lambda| < \infty$$

is fulfilled. Assume that there exists a compact set $\Lambda^\star \in \mathcal{B}_c(\mathbb{R}^d)$ such that

$$\inf_{\gamma \in S} \min_{\{x,y\} \subset \gamma_{\Lambda^\star}} |x - y| = 0. \tag{3.20}$$

Let $k^\star \in \mathbb{N}$ be such that $\Lambda^\star \subset \Lambda_{k^\star}$. The condition (3.20) implies

$$\exists \{\gamma^{(k)}\}_{k \geq 1} \subset S \; : \; \forall \, k \geq 1 \quad \exists \, x_k, y_k \in \gamma_{\Lambda^\star}^{(k)} \quad |x_k - y_k| < \frac{1}{k}.$$

Therefore,

$$E_{\Lambda_{k^\star+1}}^{\alpha,\psi}(\gamma^{(k)}) \geq \alpha \left(\frac{1}{k} \right) \psi^\star, \tag{3.21}$$

where

$$\psi^\star = \inf_{x \in \Lambda_{k^\star}} \psi_{k^\star+1}^2(x) > 0.$$

The right hand side of (3.21) tends to infinity, when $k \to \infty$. But Lemma 3.1.3 implies $E_{\Lambda_{k^\star+1}}^{\alpha,\psi}$ is continuous on Γ and hence bounded on S. Therefore, assumption (3.20) does not hold.

Vice versa, suppose that assumptions (3.19) are fulfilled. ¿From the general criterion of the relative compactness on the space of multiple configurations $\ddot{\Gamma}$ (see [63]) it follows that set $S \subset \Gamma$ is relatively compact in $\ddot{\Gamma}$. Therefore, given an arbitrary $\{\gamma^{(n)}\}_{n \geq 1} \subset S$, we may assume that $\gamma^{(n)} \to \gamma \in \ddot{\Gamma}$, $n \to \infty$ vaguely (otherwise we will consider the subsequence of $\{\gamma^{(n)}\}_{n \geq 1}$ which converge to γ due to the relative compactness of S in $\ddot{\Gamma}$). What remains to be shown is that $\gamma \in \Gamma$. Suppose that there exists $x \in \gamma$ such that $n(x) > 1$. Let $x \in \Lambda_{k^\star}$ for some $k^\star \in \mathbb{N}$. Using Lemma 3.1.2 we obtain

$$\exists \{x_{n_k}^i\}_{k \geq 1}, \; i = 1, 2 : \; x_{n_k}^i \in \gamma^{(n_k)}, \, k \geq 1, \, i = 1, 2$$

and

$$x_{n_k}^i \to x, \, k \to \infty, \, i = 1, 2.$$

Therefore,

$$|x_{n_k}^1 - x_{n_k}^2| \to 0, \quad k \to \infty. \tag{3.22}$$

Moreover,

$$\exists\, K \in \mathbb{N} \;\; \forall k \geq K : \;\; x^i_{n_k} \in \Lambda_{k^\star+1}, \; i = 1,\, 2.$$

Finally, for the compact set $\Lambda_{k^\star+1}$ in $\mathbb{R}^d$ we have

$$\inf_{\gamma \in S} \min_{\{x,\,y\} \subset \gamma_{\Lambda_{k^\star+1}}} |x - y| \leq \inf_{k \geq K} |x^1_{n_k} - x^2_{n_k}| = 0.$$

This contradicts the second assumption in (3.19). Consequently, $\gamma \in \Gamma$, which yields relative compactness of S in Γ. ∎

Definition 3.2.1 *The measurable function $F : \Gamma \mapsto \mathbb{R}_+ \cup \{+\infty\}$ is called compact if for any $C > 0$ the set*

$$\{\gamma \in \Gamma \mid F(\gamma) \leq C\}$$

is relatively compact in Γ.

Proposition 3.2.1 *For any $(\alpha,\, \psi) \in \mathcal{F}$ and an arbitrary $D > 0$ the set*

$$\{\gamma \in \Gamma \mid E^{\alpha,\,\psi}(\gamma) \leq D\} \tag{3.23}$$

is a relatively compact in $(\Gamma,\, \rho_{\alpha,\,\psi})$.

Proof. Let $(\alpha,\, \psi) \in \mathcal{F}$ be an arbitrary and fixed. From the definition of function ψ for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ follows

$$\exists\, C_\Lambda > 0 : \;\; \psi(x) \geq C_\Lambda, \;\; \forall x \in \Lambda.$$

Therefore, for any $\gamma \in \Gamma$ such that $E^{\alpha,\,\psi}(\gamma) \leq D$ we have

$$C_\Lambda^2 \sum_{\{x,y\} \subset \gamma} \alpha(|x - y|) \leq \sum_{\{x,y\} \subset \gamma} \psi(x)\alpha(|x - y|)\psi(y) \leq D,$$

which give us the following bounds:

$$|\gamma_\Lambda| \leq \max\left\{1,\, \frac{2\sqrt{D}}{C_\Lambda}\right\}$$

and

$$\forall\, \{x,\, y\} \subset \gamma_\Lambda : \;\; |x - y| > \alpha^{-1}\left(\frac{D}{C_\Lambda^2}\right).$$

Hence, the conditions of Theorem 3.2.1 for the set (3.23) are fulfilled. ∎

Corollary 3.2.1 *Let us extend function $E^{\alpha,\psi}(\gamma)$ on the whole Γ setting*

$$E^{\alpha,\psi}(\gamma) = +\infty, \;\; \gamma \in \Gamma \setminus \Gamma^{\alpha,\psi}.$$

Then, for any $(\alpha, \psi) \in \mathcal{F}$ the function $E^{\alpha,\psi}(\gamma)$ is compact on Γ.

Remark 3.2.1 *Let $\psi_1 : \mathbb{R}^d \mapsto (0, 1]$, $\psi_1 \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. It is not difficult to show that for any $(\alpha, \psi) \in \mathcal{F}$ the function*

$$\mathbb{V}(\gamma) = E^{\alpha,\psi}(\gamma) + \langle \psi_1, \gamma \rangle, \quad \gamma \in \Gamma$$

is compact on Γ.

    In some dynamical models such functions are used as Lyapunov functions, see [45].

Proposition 3.2.2 *For any $C > 0$ and any closed $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the set*

$$\{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\}$$

is a relatively compact in Γ_Λ.

Proof. Let $C > 0$ be fixed. First let us notice that

$$\forall \eta \in \{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\} \; : \; |\eta| \leq \max\{1, 2\sqrt{C}\}. \qquad (3.24)$$

This follows from simple inequality for $\eta \in \{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\} \; : \; |\eta| \geq 2$

$$\frac{|\eta|^2}{4} \leq \frac{|\eta|(|\eta| - 1)}{2} \leq \sum_{\{x,y\} \subset \eta} \alpha(|x - y|) \leq C. \qquad (3.25)$$

Using the Hausdorff criterion, our aim will be to show that for

$$\{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\}$$

there exists a finite ε-net in Γ_Λ. Let $\varepsilon > 0$ be given. A straightforward arguments insures that for any $\bar{\varepsilon} > 0$ function α uniformly continuous on $[\bar{\varepsilon}, \infty)$. Therefore,

$$\exists \delta > 0 \;\; \forall x, y \in [\alpha^{-1}(C)/2, \infty), \;\; |x - y| < \delta \; :$$

$$|\alpha(x) - \alpha(y)| < \frac{\varepsilon}{2C}. \qquad (3.26)$$

Because Λ is a compact, for any $\varepsilon^\star > 0$ there exists a finite $\varepsilon^\star$-net for Λ. In particular, for $\varepsilon^\star = \min\{diam(\Lambda)\varepsilon,\ \alpha^{-1}(C)/4,\ \delta/4\}$ we have

$$\exists\, G,\ |G| < \infty,\quad \forall\, x \in \Lambda\ \exists\, y \in G\ :$$

$$|x - y| < \min\{diam(\Lambda)\varepsilon,\ \alpha^{-1}(C)/4,\ \delta/4\}. \tag{3.27}$$

Now we consider the set Γ_G. Clear, that this set will be finite. Moreover, from (3.27) we have

$$\forall\, \eta = \{x_1, \ldots, x_p\} \in \{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\}\quad \exists\, \{y_k\}_{1 \leq k \leq p} \subset G\ :$$

$$|x_k - y_k| < \min\{diam(\Lambda)\varepsilon,\ \alpha^{-1}(C)/4,\ \delta/4\},\ 1 \leq k \leq p. \tag{3.28}$$

We define $\eta^\star := \{y_1, \ldots, y_p\}$. We will show that $\eta^\star \in \Gamma_G$. To do this we have to show that

$$\forall\, i,\, j\,:\, 1 \leq i < j \leq p\quad y_i \neq y_j.$$

Suppose that $\exists\, m,\, n\,:\, 1 \leq m < n \leq p$ such that $y_m = y_n$. Then

$$|x_m - x_n| \leq |x_m - y_m| + |y_m - y_n| + |y_n - x_n| = |x_m - y_m| + |y_n - x_n| <$$

$$< 2\min\{diam(\Lambda)\varepsilon,\ \alpha^{-1}(C)/4,\ \delta/4\} < \alpha^{-1}(C)/2. \tag{3.29}$$

But

$$\alpha(|x_m - x_n|) \leq \sum_{1 \leq i < j \leq p} \alpha(|x_i - x_j|) \leq C \tag{3.30}$$

and hence $|x_m - x_n| \geq \alpha^{-1}(C)$. We have contradiction with (3.29). Therefore, $\eta^\star \in \Gamma_G$. As the conclusion to (3.29) and (3.30) we have $\forall\, i,\, j\,:\, 1 \leq i < j \leq p$

$$|y_i - y_j| \geq |x_i - x_j| - |x_i - y_i| - |y_j - x_j| \geq$$

$$\geq \alpha^{-1}(C) - 2\min\{diam(\Lambda)\varepsilon,\ \alpha^{-1}(C)/4,\ \delta/4\} \geq \alpha^{-1}(C)/2. \tag{3.31}$$

Eventually, we have only to show that $d(\eta, \eta^\star) < \varepsilon$.

$$d_\alpha(\eta, \eta^\star) = \frac{1}{2diam(\Lambda)|\eta|} \sum_{i=1}^{p} |x_i - y_i| + \left| \sum_{1 \leq i < j \leq p} [\alpha(|x_i - x_j|) - \alpha(|y_i - y_j|)] \right|.$$

Using (3.28) we have

$$d_\alpha(\eta, \eta^\star) < \frac{1}{2diam(\Lambda)p}\, diam(\Lambda)\varepsilon p + \left| \sum_{1 \leq i < j \leq p} [\alpha(|x_i - x_j|) - \alpha(|y_i - y_j|)] \right| =$$

$$= \frac{\varepsilon}{2} + \left| \sum_{1 \leq i < j \leq p} [\alpha(|x_i - x_j|) - \alpha(|y_i - y_j|)] \right|.$$

Inequality (3.30), (3.31) and bound (3.28) give us

$$\forall i, j \; : 1 \leq i < j \leq p \; : \; |y_i - y_j|, \, |x_i - x_j| \in [\alpha^{-1}(C)/2, \, \infty). \qquad (3.32)$$

and

$$||x_i - x_j| - |y_i - y_j|| \leq |(x_i - y_i) - (x_j - y_j)| \leq |x_i - y_i| + |x_j - y_j| \leq$$

$$\leq 2 \min \{ diam(\Lambda)\varepsilon, \, \alpha^{-1}(C)/4, \, \delta/4 \} \leq \delta.$$

Finally, with (3.24), (3.25) and (3.26) we have

$$d_\alpha(\eta, \eta^\star) < \frac{\varepsilon}{2} + \frac{p(p-1)}{2} \frac{\varepsilon}{2C} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for any $C > 0$ the set

$$\{ \eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C \}$$

is relatively compact in Γ_Λ.   ∎

Corollary 3.2.2 *For any closed $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ function*

$$h_\alpha(\eta) = e^{E^\alpha(\eta)} \qquad (3.33)$$

is a compact function on Γ_Λ.

Chapter 4

On relations between a priori bounds for measures on configuration spaces

4.1   A priori bounds

Let σ be Lebesgue measure and $\|x\| = \max_k |x_k|$, $x \in \mathbb{R}^d$. For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, let

$$l_\Lambda = \sup_{x,\,y \in \Lambda} \|x - y\|$$

and $|\Lambda|$ denote the Lebesgue measure of Λ.

Let $V : \Gamma_0^{(2)} \to \mathbb{R}$ be a pair potential.

Definition 4.1.1 *A potential V is called stable (see [76]) iff there exists a constant $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any configuration $\gamma \in \Gamma_\Lambda$ holds*

$$\sum_{\{x,y\} \subset \gamma} V(x,y) \geq -B|\gamma|. \tag{4.1}$$

In the following we assume that all potentials under consideration are stable.

Consider $\mu \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ locally absolutely continuous w.r.t. $\pi_{z\sigma}$ and three type of bounds on it.

We will say that a measure μ satisfies the generalized Ruelle bound with potential V if the following holds:

- $(GRB)_V$: *The correlation function $k_\mu(\eta)$ satisfies the inequality*

$$k_\mu(\eta) \le C^{|\eta|} \exp\left[-\sum_{\{x,y\}\subset\eta} V(x,y) \right], \ \eta \in \Gamma_0, \qquad (4.2)$$

  with some $C > 0$.

We will say that a measure μ satisfies the Ruelle's probability bound if the following holds:

- *(RPB): For any $g > 0$ there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ and $N \in \mathbb{N}_0$*

$$\mu(\{\gamma \mid |\gamma_\Lambda| \ge N\}) \le \exp\left\{ -\alpha \frac{N^2}{l_\Lambda^d} + \delta l_\Lambda^d \right\}. \qquad (4.3)$$

We will say that a measure μ satisfies the Dobrushin's exponential bound of type $\lambda > 0$ and order $p > 0$ if the following holds:

- $(DEB)_{(\lambda,\ p)}$: *For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ there exists a constant $C_\Lambda > 0$ such that*

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p} \mu(d\gamma) < C_\Lambda. \qquad (4.4)$$

Remark 4.1.1 *Obviously, for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ with $l_\Lambda = 0$ the bound (4.4) holds automatically. Therefore, in the sequel we will consider $(DEB)_{(\lambda,\ p)}$ only for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda > 0$.*

Definition 4.1.2 *A potential V is called superstable in the sense of Ginibre (see [31, 58]) iff for any $g > 0$ there exist $A > 0$ and $B \ge 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ and any configuration $\gamma \in \Gamma_\Lambda$ holds*

$$\sum_{\{x,y\}\subset\gamma} V(x,y) \ge A\frac{|\gamma|^2}{l_\Lambda^d} - B|\gamma|. \qquad (4.5)$$

In the sequel, we will write sometimes α_g, δ_g, A_g, B_g, instead of α, δ, A, B, to emphasize that these constants depend on g.

Theorem 4.1.1

1. For any $\lambda > 0$ and $p \in (0, 1]$
$$(GRB)_V \Rightarrow (DEB)_{(\lambda,\ p)}.$$
2. Let V be superstable in the sense of Ginibre. Then

   2.1. $(GRB)_V \Rightarrow (RPB),$

   2.2.   *for any $\lambda > 0$ and $p \in (1, 2)$*
$$(GRB)_V \Rightarrow (DEB)_{(\lambda,\ p)},$$

   2.3.   *for any $\lambda > 0$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $0 < l_\Lambda^d \le A_{l_\Lambda} \lambda^{-1}$*
$$(GRB)_V \Rightarrow (DEB)_{(\lambda,\ 2)}.$$

3. For any $\lambda > 0$ and $p \in (0, 2)$
$$(RPB) \Rightarrow (DEB)_{(\lambda,\ p)}.$$
4. For any $\lambda > 0$ $(DEB)_{(\lambda,\ 2)}$ with $C_\Lambda \le e^{\delta\, l_\Lambda^d}$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\delta > 0$ implies (RPB).

Proof.

   1. Using (2.3), stability of V and according to the bound on the correlation functions we have

$$\int_{\Gamma_\Lambda} \left| K^{-1}\left[e^{\lambda|\eta|^p}\right] \right| \rho_\mu(d\eta) = \int_{\Gamma_\Lambda} \left| \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} e^{\lambda|\xi|^p} \right| \rho_\mu(d\eta) \le$$

$$\le \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1,\dots,x_n\}} e^{\lambda|\xi|^p} C^n e^{-\sum_{\{x,y\} \subset \{x_1,\dots,x_n\}} V(x,y)} dx_1 \dots dx_n \le$$

$$\le \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1,\dots,x_n\}} e^{\lambda|\xi|} C^n e^{Bn} dx_1 \dots dx_n = \sum_{n=0}^{\infty} \frac{[2zC|\Lambda|e^{\lambda+B}]^n}{n!} =$$

$$= \exp\left\{2zC|\Lambda|e^{\lambda+B}\right\}.$$

Because of Lemma 2.2.1 and (2.6) we conclude

$$\int_{\Gamma} e^{\lambda|\gamma_\Lambda|^p} \mu(d\gamma) = \int_{\Gamma_\Lambda} K^{-1}\left[e^{\lambda|\eta|^p}\right] \rho_\mu(d\eta) \le \exp\left\{2zC|\Lambda|e^{\lambda+B}\right\}.$$

   2. Now suppose that V is superstable in the sense of Ginibre.

   2.1. Define $S_\Lambda := \{\gamma \in \Gamma|\ |\gamma_\Lambda| \ge N\}$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Let $g > 0$ be any and given. Then, using (2.3) for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \ge g$ we have

$$\int_{\Gamma_\Lambda} \left| K^{-1}\left[\mathbb{1}_{S_\Lambda}(\eta)\right] \right| \rho_\mu(d\eta) =$$

$$= \int_{\Gamma_\Lambda} \left| \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \mathbb{1}_{S_\Lambda}(\xi) \right| \rho_\mu(d\eta) = \int_{\Gamma_\Lambda} \left| \mathbb{1}_{S_\Lambda}(\eta) \sum_{\xi \subset \eta, |\xi| \geq N} (-1)^{|\eta \setminus \xi|} \right| \rho_\mu(d\eta).$$

According to the bound on the correlation functions and the superstability for given g, the latter expression can be estimated by

$$\int_{\Gamma_\Lambda} \mathbb{1}_{S_\Lambda}(\eta) C^{|\eta|} \sum_{\xi \subset \eta, |\xi| \geq N} e^{-\sum_{\{x,y\} \subset \eta} V(x,y)} \lambda_{z\sigma}(d\eta) \leq$$

$$\leq \int_{\Gamma_\Lambda} \mathbb{1}_{S_\Lambda}(\eta) (2C)^{|\eta|} e^{-A|\eta|^2 l_\Lambda^{-d} + B|\eta|} \lambda_{z\sigma}(d\eta) \leq \exp\left\{ -A\frac{N^2}{l_\Lambda^d} + 2zCe^B l_\Lambda^d \right\}.$$

In the last inequality we have used the fact that integration actually extends only over all $\eta \in \Gamma_\Lambda : |\eta| \geq N$.

Finally, Lemma 2.2.1 and (2.6) give us

$$\mu(\{\gamma | \, |\gamma_\Lambda| \geq N\}) = \int_{\Gamma_\Lambda} K^{-1} \left[\mathbb{1}_{S_\Lambda}(\eta) \right] \rho_\mu(d\eta) \leq \exp\left\{ -A\frac{N^2}{l_\Lambda^d} + 2zCe^B l_\Lambda^d \right\}.$$

2.2. Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda > 0$ be arbitrary and fixed. Using (2.3) we have

$$\int_{\Gamma_\Lambda} \left| K^{-1} \left[e^{\lambda |\eta|^p} \right] \right| \rho_\mu(d\eta) = \int_{\Gamma_\Lambda} \left| \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} e^{\lambda |\xi_\Lambda|^p} \right| \rho_\mu(d\eta) \leq$$

$$\leq \int_{\Gamma_\Lambda} \sum_{\xi \subset \eta} \exp\{\lambda |\xi|^p\} \rho_\mu(d\eta).$$

The estimation for the correlation functions and the superstability of V for $g = l_\Lambda$ imply the following bound for the latter integral

$$\int_{\Gamma_\Lambda} \sum_{\xi \subset \eta} e^{\lambda |\xi|^p - \sum_{\{x,y\} \subset \eta} V(x,y)} C^{|\eta|} \lambda_{z\sigma}(d\eta) \leq$$

$$\leq \int_{\Gamma_\Lambda} \sum_{\xi \subset \eta} e^{\lambda |\xi|^p - A l_\Lambda^{-d} |\eta|^2 + B|\eta|} C^{|\eta|} \lambda_{z\sigma}(d\eta) \leq$$

$$\leq \int_{\Gamma_\Lambda} (2C)^{|\eta|} e^{B|\eta|} e^{\lambda |\eta|^p - A l_\Lambda^{-d} |\eta|^2} \lambda_{z\sigma}(d\eta) \leq e^{2zC|\Lambda| e^B + C_\Lambda^\star}, \tag{4.6}$$

where $C_\Lambda^\star > 0$ is some constant s.t.

$$\lambda |\eta|^p - \frac{A}{l_\Lambda^d} |\eta|^2 \leq C_\Lambda^\star.$$

Such constant exists, because $p \in (1,2)$ and $\lim_{n\to\infty}(\lambda n^p - \frac{A}{|\Lambda|}n^2) = -\infty$. Therefore, using Lemma 2.2.1 and (2.6) we have

$$\int_\Gamma e^{\lambda |\gamma_\Lambda|^p} \mu(d\gamma) = \int_{\Gamma_\Lambda} K^{-1}\left[e^{\lambda |\eta|^p}\right]\rho_\mu(d\eta) \leq e^{2zC|\Lambda|e^B + C_\Lambda^\star}.$$

2.3. Doing the same as in *2.2* for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $0 < l_\Lambda \leq A_{l_\Lambda}\lambda^{-1}$ we obtain

$$\int_{\Gamma_\Lambda} \left| K^{-1}\left[e^{\lambda |\eta|^2}\right]\right|\rho_\mu(d\eta) \leq$$

$$\leq \sum_{n=0}^\infty \frac{(zC)^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1,\dots,x_n\}} e^{\lambda |\xi|^2 - A_{l_\Lambda} l_\Lambda^{-d} n^2 + B_{l_\Lambda} n} dx_1 \dots dx_n. \qquad (4.7)$$

Because $l_\Lambda^d \leq A_{l_\Lambda}\lambda^{-1}$, (4.7) is bounded by

$$\sum_{n=0}^\infty \frac{(zC)^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1 \dots x_n\}} e^{\lambda [|\xi|^2 - n^2] + B_{l_\Lambda} n} dx_1 \dots dx_n \leq$$

$$\leq \sum_{n=0}^\infty \frac{(2zC|\Lambda|e^B)^n}{n!} \leq e^{2zC|\Lambda|e^{B_{l_\Lambda}}}.$$

The statement is now a direct consequence of Lemma 2.2.1 and (2.6).

3. To prove this part of the theorem we need the following lemma which follows directly from the definition of distribution function for a random variable.

Lemma 4.1.1 *For any measurable $\xi : \Gamma \to \mathbb{R}_+$ and differentiable $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$ we have*

$$\int_\Gamma f \circ \xi(\gamma)\mu(d\gamma) = \int_0^\infty f'(x)\mu(\{\gamma \in \Gamma \mid \xi(\gamma) > x\})dx.$$

Proof. As

$$\int_\Gamma f \circ \xi(\gamma)\mu(d\gamma) = \int_0^\infty f(x)\mu_\xi(dx)$$

and

$$\mu(\{\gamma \in \Gamma \mid \xi(\gamma) > x\}) = \mu_\xi((x, \ \infty))$$

where $\mu_\xi(B) = \mu(\{\gamma \in \Gamma \mid \xi(\gamma) \in B\})$, $B \in \mathcal{B}(\mathbb{R})$ we need only to show that

$$\int_0^\infty f(x)\mu_\xi(dx) = \int_0^\infty f'(x)\mu_\xi((x, \ \infty))dx.$$

Using theorem Tonelli we have

$$\int_0^\infty f(x)\mu_\xi(dx) = \int_0^\infty \int_0^x f'(y)dy\mu_\xi(dx) =$$

$$= \int_0^\infty f'(y) \int_y^\infty \mu_\xi(dx)dy = \int_0^\infty f'(y)\left(\int_y^\infty \mu_\xi(dx)\right)dy =$$

$$= \int_0^\infty f'(y)\mu_\xi((y, \ \infty))dy. \quad \blacksquare$$

Using Lemma 4.1.1 for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda > 0$ we have

$$\int_\Gamma e^{\lambda|\gamma_\Lambda|^p}\mu(d\gamma) = \int_0^\infty \mu(\{\gamma \in \Gamma \mid e^{\lambda|\gamma_\Lambda|^p} > y\})dy = \qquad (4.8)$$

$$\int_0^\infty \mu\left(\left\{\gamma \ \middle| \ |\gamma_\Lambda| > \frac{(\ln y)^{\frac{1}{p}}}{\lambda^{\frac{1}{p}}}\right\}\right) dy.$$

Due to (RPB) for $g = l_\Lambda$ we bound (4.8) by

$$\int_0^{\exp\left[(2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}\right]} 1dy + \int_{\exp\left[(2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}\right]}^\infty \exp\left\{-\alpha\frac{(\ln y)^{2/p}}{\lambda^{2/p}l_\Lambda^d} + \delta l_\Lambda^d\right\}dy.$$

$$\leq \exp\left[2\lambda^{2/p}l_\Lambda^d\alpha^{-1}\right]^{\frac{p}{2-p}} + e^{\delta l_\Lambda^d}\int_{\exp[2\lambda^{2/p}l_\Lambda^d\alpha^{-1}]^{\frac{p}{2-p}}}^\infty y^{-\frac{\alpha(\ln y)^{\frac{2}{p}-1}}{\lambda^{2/p}l_\Lambda^d}} dy \leq$$

$$\leq \exp\left[2\lambda^{2/p}l_\Lambda^d\alpha^{-1}\right]^{\frac{p}{2-p}} + e^{\delta l_\Lambda^d}\int_{\exp\left[2\lambda^{2/p}l_\Lambda^d\alpha^{-1}\right]^{\frac{p}{2-p}}}^{\infty} y^{-2}dy \leq$$

$$\leq \exp\left[2\lambda^{2/p}l_\Lambda^d\alpha^{-1}\right]^{\frac{p}{2-p}} + \exp\left[\delta l_\Lambda^d - (2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}\right] \leq$$

$$\leq 2\exp\left[\delta l_\Lambda^d + (2\lambda^{2/p}l_\Lambda^d\alpha^{-1})^{\frac{p}{2-p}}\right].$$

4. Let $(DEB)_{(\lambda,\,2)}$ holds for some $\lambda > 0$ with $C_\Lambda \leq e^{\delta\,l_\Lambda^d}$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\delta > 0$ and $g > 0$ be arbitrary and given.

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ consider a function $g_\Lambda(x) = e^{\alpha l_\Lambda^{-d}x^2}$, $x \geq 0$, $0 \leq \alpha \leq \lambda g^d$. This function is increasing and $\int_\Gamma g_\Lambda(|\gamma_\Lambda|)\mu(d\gamma) \leq C_\Lambda$ (it follows from $(DEB)_{(\lambda,\,2)}$ and inequality $l_\Lambda \geq g$). Thus, the generalized Chebyshev inequality

$$P(\xi \geq \varepsilon^*) \leq \frac{Ef(\xi)}{f(\varepsilon^*)}, \tag{4.9}$$

where f be increasing and positive function, $\varepsilon^* > 0$, shows that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$:

$$\mu(\{\gamma \mid |\gamma_\Lambda| \geq N\}) \leq \frac{\int_\Gamma g_\Lambda(|\gamma_\Lambda|)\mu(d\gamma)}{e^{\alpha N^2 l_\Lambda^{-d}}} \leq C_\Lambda e^{-\alpha N^2 l_\Lambda^{-d}} \leq e^{-\alpha N^2 l_\Lambda^{-d} + \delta\,l_\Lambda^d}.\blacksquare$$

4.2 Support properties

For each $i \in \mathbb{Z}^d$, let

$$Q_i = \{r \in \mathbb{R}^d \mid i_k - 1/2 < r_k \leq i_k + 1/2, k = 1, \ldots, d\}.$$

Define $|\gamma_i| = |\gamma \cap Q_i|$. For $k \in \mathbb{N}$, let Λ_k be the hypercube of the sidelength $2k - 1$ centered at the origin in $\mathbb{R}^d$. Actually, Λ_k is then a union of $(2k - 1)^d$ unit cubes of the form Q_i. Note, that $|\Lambda_k| = l_{\Lambda_k}^d = (2k - 1)^d$, $k \in \mathbb{N}$. We will also sometimes regard Λ_k as a subset of $\mathbb{Z}^d$ by letting Λ_k represent $\Lambda_k \cap \mathbb{Z}^d$. For $i \in \mathbb{Z}^d$ let $\ln_+ \|i\| = \max\{1, \ln\|i\|\}$.

Following Ruelle [76] a measure μ is called tempered if μ is supported by the set

$$R_\infty = \bigcup_{N=1}^{\infty} R_N,$$

where $R_N = \{\gamma \in \Gamma \mid \sum_{i \in \Lambda_k} |\gamma_i|^2 \le N^2 |\Lambda_k|, \ \forall \ k \ge 1\}$.

Consider two subsets of the configuration space:

$$P_\infty = \bigcup_{N=1}^\infty P_N,$$

where $P_N = \{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| \le N|\Lambda_k|, \ \forall \ k \ge 1\}$ and

$$U_\infty = \bigcup_{n=1}^\infty U_n,$$

where $U_n = \{\gamma \in \Gamma \mid |\gamma_i| \le n(\ln_+ \|i\|)^{\frac{1}{2}}, \ \forall i \in \mathbb{Z}^d\}$.

Obviously, $R_\infty \subset P_\infty$ and for any tempered measure μ with (RPB), it is also possible to show that $\mu(U_\infty) = 1$ (see [37, 54]).

Proposition 4.2.1 *(RPB) implies* $\mu(P_\infty) = 1$.

Proof. Obviously,

$$P_\infty = \bigcup_{N=1}^\infty \bigcap_{k \ge 1} \{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| \le N|\Lambda_k|\}$$

and

$$\Gamma \backslash P_\infty = \bigcap_{N \ge 1} \bigcup_{k \ge 1} \{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| > N|\Lambda_k|\}.$$

Note that for any $k \ge 1$

$$\{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| > N|\Lambda_k|\} \supset \{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| > (N+1)|\Lambda_k|\}, \ N \ge 1.$$

Using σ - semi-additivity and monotonicity of the measure μ we have

$$\mu(\Gamma \backslash P_\infty) = \lim_{N \to \infty} \mu \left(\bigcup_{k \ge 1} \{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| > N|\Lambda_k|\} \right) \le$$

$$\le \lim_{N \to \infty} \sum_{k \ge 1} \mu \left(\{\gamma \in \Gamma \mid |\gamma_{\Lambda_k}| > N|\Lambda_k|\} \right). \tag{4.10}$$

Due to *(RPB)* one can show that the right-hand side of (4.10) can be estimated by

$$\lim_{N \to \infty} \sum_{k \ge 1} e^{-(\alpha N^2 - \delta)|\Lambda_k|} = \lim_{N \to \infty} \sum_{k \ge 1} e^{-(\alpha N^2 - \delta)(2k-1)^d} \le$$

$$\leq \lim_{N \to \infty} \frac{e^{-(\alpha N^2 - \delta)}}{1 - e^{-(\alpha N^2 - \delta)}} = 0. \quad \blacksquare$$

Remark 4.2.1 *Proposition 4.2.1 holds, if (RPB) is replaced by the following weaker probability bound:*

there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ such that for any $N \geq N_0$, $N_0 \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\mu(\{\gamma \mid |\gamma_{\Lambda_k}| \geq N|\Lambda_k|\}) \leq \exp\{-(\alpha N - \delta)|\Lambda_k|\}. \tag{4.11}$$

Proposition 4.2.2 *(RPB) implies $\mu(U_\infty) = 1$.*

Proof. Define $U_n^i := \{\gamma \in \Gamma \mid |\gamma_i| \leq n(\ln_+ \|i\|)^{\frac{1}{2}}\}$, $i \in \mathbb{Z}^d$. Then

$$U_\infty = \bigcup_{n=1}^{\infty} \bigcap_{i \in \mathbb{Z}^d} U_n^i,$$

$$\Gamma \backslash U_\infty = \bigcap_{n \geq 1} \bigcup_{i \in \mathbb{Z}^d} \{\gamma \in \Gamma \mid |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\}.$$

Note that

$$\{\gamma \in \Gamma \mid |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\} \supset \{\gamma \in \Gamma \mid |\gamma_i| > (n+1)(\ln_+ \|i\|)^{\frac{1}{2}}\}, \ n \geq 1.$$

Using σ - semi-additivity and monotonicity of the measure μ we have

$$\mu(\Gamma \backslash U_\infty) = \lim_{n \to \infty} \mu\left(\bigcup_{i \in \mathbb{Z}^d} \{\gamma \in \Gamma \mid |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\}\right) \leq$$

$$\leq \lim_{n \to \infty} \sum_{i \in \mathbb{Z}^d} \mu\left(\{\gamma \in \Gamma \mid |\gamma_i| > n(\ln_+ \|i\|)^{\frac{1}{2}}\}\right). \tag{4.12}$$

Due to *(RPB)* we estimate (4.12) by

$$\lim_{n \to \infty} \sum_{i \in \mathbb{Z}^d} e^{-(\alpha n^2 (\ln_+ \|i\|) - \delta)} =$$

$$= \lim_{n \to \infty} \left(e^{-\alpha n^2 + \delta} + \sum_{i=1}^{\infty} [(2i+1)^d - (2i-1)^d] e^{-(\alpha n^2 (\ln_+ i) - \delta)}\right) \leq$$

$$\leq \lim_{n\to\infty} \left(\sum_{i=1}^{2} 2d(2i+1)^{d-1} e^{-(\alpha n^2 - \delta)} + \sum_{i=3}^{\infty} 2d(2i+1)^{d-1} e^{-(\alpha n^2 \ln i - \delta)} \right) \leq$$

$$\leq \lim_{n\to\infty} 2^{2d-1} d e^{\delta} \sum_{i=3}^{\infty} i^{d-1} e^{-\alpha n^2 \ln i} = \lim_{n\to\infty} 2^{2d-1} d e^{\delta} \sum_{i=3}^{\infty} i^{d-1} i^{-\alpha n^2} \leq$$

$$\leq \lim_{n\to\infty} 2^{2d-1} d e^{\delta} \int_{2}^{\infty} x^{d-1-\alpha n^2} dx = \lim_{n\to\infty} 2^{3d-1} d e^{\delta} \frac{2^{-\alpha n^2}}{\alpha n^2 - d} = 0. \quad \blacksquare$$

Remark 4.2.2 *We will say that a measure μ satisfy $(RPB)^p$, $p > 0$ if the following holds:*

- *$(RPB)^p$: For any $g > 0$ there exist constants $\alpha > 0$ and $\delta \in \mathbb{R}$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ and $N \geq N_0$ for some $N_0 \in \mathbb{N}$*

$$\mu(\{\gamma \,|\, |\gamma_\Lambda| \geq N\}) \leq \exp\left\{ -\alpha \frac{N^p}{l_\Lambda^d} + \delta l_\Lambda^d \right\}. \tag{4.13}$$

Similar to the proof of Proposition 4.2.2 one can show that for any $p > 0$ the fulfillment of $(RPB)^p$ on the sets Q_i, $i \in \mathbb{Z}^d$ implies $\mu(U_\infty^p) = 1$. Here

$$U_\infty^p = \bigcup_{n=1}^{\infty} U_n^p,$$

$$U_n^p = \{\gamma \in \Gamma \,|\, |\gamma_i| \leq n(\ln_+ \|i\|)^{\frac{1}{p}}, \ \forall i \in \mathbb{Z}^d\}.$$

4.3   Stronger consequences of generalized Ruelle bound

In this section we describe further conclusions which follow from $(GRB)_V$. As before one can consider the partition of $\mathbb{R}^d$ on cubes, but now with sidelength equal to $g > 0$. Namely, for each $i \in \mathbb{Z}^d$ and any $g > 0$ let

$$Q_i^g = \{r \in \mathbb{R}^d \,|\, g(i_k - 1/2) < r_k \leq g(i_k + 1/2), k = 1, \ldots, d\}$$

and $|\gamma_{i,g}| = |\gamma \cap Q_i^g|$.

By $\mathcal{J}_g(\mathbb{R}^d)$ we denote all finite unions of cubes of the form Q_i^g (such sets are used in the construction of the Jordan measure). Sometimes we will regard $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ as a subset of $\mathbb{Z}^d$ by letting Λ represent $\{i \in \mathbb{Z}^d \,|\, Q_i^g \subset \Lambda\}$.

Let $W : \Gamma_0 \to \mathbb{R}$ be a measurable increasing function, i.e. for $\gamma, \; \gamma' \in \Gamma_0$ s.t. $\gamma \subset \gamma' : W(\gamma) \leq W(\gamma')$.

We will say that a measure μ satisfies the $(RPB)_V^W$ if the following holds:

- $(RPB)_V^W$: *For any $g > 0$ there exist constants $B > 0$ and $\delta \in \mathbb{R}$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$, any configuration $\gamma \in \Gamma_\Lambda$ and $L \in \mathbb{R}_+$*

$$\mu(\{\gamma \,|\, W(\gamma_\Lambda) \geq L\}) \leq \exp\{-L + \delta|\Lambda|\}, \qquad (4.14)$$

  and

$$\sum_{\{x,y\}\subset\gamma} V(x,y) - W(\gamma) \geq -B|\gamma|. \qquad (4.15)$$

Proposition 4.3.1 *Suppose that there exists a measurable increasing function $W : \Gamma_0 \to \mathbb{R}$ which satisfies (4.15). Then $(GRB)_V$ implies $(RPB)_V^W$.*

Proof. Let $g > 0$ and $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ be arbitrary. Define $S := \{\gamma \in \Gamma \,|\, W(\gamma_\Lambda) \geq L\}$. Then using (4.15) we have

$$\mathbb{1}_S(\eta)e^{-\sum_{\{x,y\}\subset\eta} V(x,y)} \leq \mathbb{1}_S(\eta)e^{-W(\eta)+B|\eta|} \leq e^{-L+B|\eta|}, \; \eta \in \Gamma_\Lambda. \qquad (4.16)$$

Therefore, similarly to the proof of the Theorem 4.1.1(*2.1*) we obtain

$$\mu(\{\gamma| \; W(\gamma_\Lambda) \geq L\}) = \int_\Gamma \mathbb{1}_S(\gamma)\mu(d\gamma) =$$

$$= \int_{\Gamma_\Lambda} \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|}\mathbb{1}_S(\xi)\rho_\mu(d\eta) = \int_{\Gamma_\Lambda} \mathbb{1}_S(\eta) \sum_{\xi\subset\eta, \; \xi\in S} (-1)^{|\eta\setminus\xi|}\rho_\mu(d\eta).$$

According to the bound on the correlation function and (4.16) the latter expression can be estimate by

$$\int_{\Gamma_\Lambda} \mathbb{1}_S(\eta)C^{|\eta|} \sum_{\xi\subset\eta, \; \xi\in S} e^{-\sum_{\{x,y\}\subset\eta} V(x,y)}\lambda_{z\sigma}(d\eta) \leq$$

$$\leq e^{-L} \int_{\Gamma_\Lambda} (2C)^{|\eta|}e^{B|\eta|}\lambda_{z\sigma}(d\eta) = e^{-L+2zCe^B|\Lambda|}. \qquad \blacksquare$$

Remark 4.3.1 *For any* $0 \leq \varepsilon < 1$ *inequality (4.14) implies*

$$\int_\Gamma e^{W(\gamma_\Lambda)^{1-\varepsilon}} \mu(d\gamma) < C_\Lambda, \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

with some $C_\Lambda > 0$.

Indeed, let $g > 0$ be given. We increase any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ to a set $\Lambda_{\mathcal{J}} \in \mathcal{J}_g(\mathbb{R}^d)$ which is a union of all cubes $Q_{i,g}$, which have nonempty intersection with Λ. Then using Lemma 4.1.1 and the fact that the function W is increasing we have

$$\int_\Gamma e^{W(\gamma_\Lambda)^{1-\varepsilon}} \mu(d\gamma) \leq \int_0^\infty \mu(\{\gamma \mid e^{W(\gamma_{\Lambda_{\mathcal{J}}})^{1-\varepsilon}} > y\}) dy =$$

$$= \int_0^\infty \mu\left(\{\gamma \mid W(\gamma_{\Lambda_{\mathcal{J}}}) > (\ln y)^{\frac{1}{1-\varepsilon}}\}\right) dy.$$

Inequality (4.14) implies the following bound for the latter integral:

$$\int_0^{\exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right]} 1 dy + \int_{\exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right]}^\infty e^{-(\ln y)^{(1-\varepsilon)^{-1}} + \delta|\Lambda_{\mathcal{J}}|} dy = \qquad (4.17)$$

$$= \exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right] + e^{\delta|\Lambda_{\mathcal{J}}|} \int_{\exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right]}^\infty y^{-(\ln y)^{\varepsilon(1-\varepsilon)^{-1}}} dy \leq$$

$$\leq \exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right] + e^{\delta|\Lambda_{\mathcal{J}}|} \int_{\exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right]}^\infty y^{-2} dy \leq$$

$$\leq \exp\left[2^{(1-\varepsilon)\varepsilon^{-1}}\right] + \exp\left[\delta|\Lambda_{\mathcal{J}}| - \left(2^{(1-\varepsilon)\varepsilon^{-1}}\right)\right] \leq$$

$$\leq 2 \exp\left[\delta|\Lambda_{\mathcal{J}}| + 2^{(1-\varepsilon)\varepsilon^{-1}}\right]. \quad \blacksquare$$

In the literature different non-equivalent versions of the Ruelle's probability bound are known. The definition of (RPB) we used here can be found in [37], [54]. Besides this bound, Ruelle in [76] used also another one. Namely, we will say that a measure μ satisfies the strong Ruelle's probability bound if the following holds:

- *(SRPB): For any* $g > 0$ *there exist constants* $\alpha > 0$ *and* $\delta \in \mathbb{R}$ *(may be* g *dependent) such that for any* $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ *and* $N \in \mathbb{N}$

$$\mu\left(\left\{\gamma \mid \sum_{i \in \Lambda} |\gamma_{i,g}|^2 \geq N^2|\Lambda|\right\}\right) \leq \exp\left\{-(\alpha N^2 - \delta)|\Lambda|\right\}. \qquad (4.18)$$

As shown in [76] $(SRPB)$ implies (RPB).

Definition 4.3.1 *A potential V is called superstable in the sense of Ruelle [76] if for any $g > 0$ there exist $A > 0$, $B \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\gamma \in \Gamma_\Lambda$*

$$\sum_{\{x,y\} \subset \gamma} V(x,y) \geq \sum_{i \in \Lambda} [A|\gamma_{i,g}|^2 - B|\gamma_{i,g}|]$$

Lemma 4.3.1 *Ruelle's supestability implies Ginibre's supestability.*

Proof. Let $g > 0$ be given. We first increase, as before, any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ to a set $\Lambda_{\mathcal{J}} \in \mathcal{J}_g(\mathbb{R}^d)$ which is a union of all cubes $Q_{i,g}$, which have nonempty intersection with Λ. Then for any $\gamma \in \Gamma_\Lambda \subset \Gamma_{\Lambda_{\mathcal{J}}}$, Ruelle's supestability gives

$$\sum_{\{x,y\} \subset \gamma} V(x,y) \geq \sum_{i \in \Lambda_{\mathcal{J}}} A|\gamma_{i,g}|^2 - B|\gamma| \geq A\frac{g^d|\gamma|^2}{|\Lambda_{\mathcal{J}}|} - B|\gamma|.$$

Because $l_\Lambda \geq g$, one can show that for large $\kappa > 1$ the following inequality holds

$$|\Lambda_{\mathcal{J}}| \leq \kappa l_\Lambda^d$$

and the assertion of the lemma is now obvious. ∎

Proposition 4.3.2 *Let V be superstable in the sense of Ruelle. Then*

$$(GRB)_V \Rightarrow (SRPB).$$

Proof. It follows immediately from Proposition 4.3.1 by choosing $W(\gamma) = A\sum_{i \in \Lambda} |\gamma_{i,g}|^2$. ∎

Proposition 4.3.3 $(SRPB)$ *implies* $\mu(R_\infty) = 1$.

Proof. Let, as above, Λ_k denote the hypercube of sidelength $2k - 1$ centered at the origin in $\mathbb{R}^d$. Obviously,

$$R_\infty = \bigcup_{N=1}^{\infty} \bigcap_{k \geq 1} \left\{ \gamma \in \Gamma \ \middle| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 \leq N^2|\Lambda_k| \right\}$$

and

$$\Gamma \backslash R_\infty = \bigcap_{N \geq 1} \bigcup_{k \geq 1} \left\{ \gamma \in \Gamma \ \Big| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 > N^2 |\Lambda_k| \right\}.$$

Note that for any $k \geq 1$ and $N \in \mathbb{N}$

$$\left\{ \gamma \in \Gamma \ \Big| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 > N^2 |\Lambda_k| \right\} \supset \left\{ \gamma \in \Gamma \ \Big| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 > (N+1)^2 |\Lambda_k| \right\}.$$

Using σ - semi-additivity and monotonicity of the measure μ, we have

$$\mu(\Gamma \backslash R_\infty) = \lim_{N \to \infty} \mu \left(\bigcup_{k \geq 1} \left\{ \gamma \in \Gamma \ \Big| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 > N^2 |\Lambda_k| \right\} \right) \leq$$

$$\leq \lim_{N \to \infty} \sum_{k \geq 1} \mu \left(\left\{ \gamma \in \Gamma \ \Big| \ \sum_{i \in \Lambda_k} |\gamma_i|^2 > N^2 |\Lambda_k| \right\} \right). \qquad (4.19)$$

Due to *(SRPB)* for $g = 1$ we bound (4.19) by

$$\lim_{N \to \infty} \sum_{k \geq 1} e^{-(\alpha N^2 - \delta)|\Lambda_k|} = \lim_{N \to \infty} \sum_{k \geq 1} e^{-(\alpha N^2 - \delta)(2k-1)^d} \leq$$

$$\leq \lim_{N \to \infty} \frac{e^{-(\alpha N^2 - \delta)}}{1 - e^{-(\alpha N^2 - \delta)}} = 0. \qquad \blacksquare$$

Remark 4.3.2 *Proposition 4.3.3 holds if in* (SRPB) *we substitute (4.18) by the following weaker probability bound:*
for any $N \geq N_0$, $N_0 \in \mathbb{N}$,

$$\mu \left(\left\{ \gamma \ \Big| \ \sum_{i \in \Lambda} |\gamma_{i,g}|^2 \geq N^2 |\Lambda| \right\} \right) \leq \exp \left\{ -(\alpha N - \delta)|\Lambda| \right\}. \qquad (4.20)$$

Corollary 4.3.1 *Let V be superstable in the sense of Ruelle. Then*

$$(GRB)_V \Rightarrow \mu(R_\infty) = \mu(P_\infty) = \mu(U_\infty) = 1.$$

4.4 Examples

In this section we consider some class of examples known from the statistical physics to which the results of this article can be applied. One of them is related to the so-called Gibbs states (see [27] for more details) and another with states constructed by a given family of correlation functions (see [5]).

Example 1. (Gibbs states with pair potentials).

The Hamiltonian $E^V : \Gamma_0 \to \mathbb{R}$ which corresponds to the potential V (even function on $\mathbb{R}^d$) is defined by

$$E^V(\eta) = \sum_{\{x,y\} \subset \eta} V(x-y), \ \eta \in \Gamma_0, \ |\eta| \geq 2$$

Having in mind applications in mathematical physics, we will always assume positivity of V for small distances. More precisely, we suppose that there exists g, $0 < g < \infty$, such that $V(x) \geq 0$ for $|x| \leq g$.

For fixed V we will write for short $E = E^V$ and for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ we will sometimes write $E_\Lambda(\eta)$ instead of $E(\eta)$.

For a given $\bar{\gamma} \in \Gamma$ define the interaction energy between $\eta \in \Gamma_\Lambda$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \backslash \Lambda$ as

$$W_\Lambda(\eta|\bar{\gamma}) = \sum_{x \in \eta, \ y \in \bar{\gamma} \cap \Lambda^c} V(x-y). \tag{4.21}$$

Define

$$E_\Lambda(\eta|\bar{\gamma}) = E_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma}).$$

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\gamma} \in \Gamma$. The finite volume Gibbs state with boundary configuration $\bar{\gamma}$ for E and $z > 0$ is

$$\mu_\Lambda(d\eta|\ \bar{\gamma}) = \frac{\exp\{-E_\Lambda(\eta|\ \bar{\gamma})\}}{Z_\Lambda(\bar{\gamma})} \lambda_{z\sigma}(d\eta),$$

where

$$Z_\Lambda(\bar{\gamma}) = \int_{\Gamma_\Lambda} \exp\{-E_\Lambda(\eta|\ \bar{\gamma})\} \lambda_{z\sigma}(d\eta).$$

This finite volume Gibbs state is well defined if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma$ the interaction energy $W_\Lambda(\eta|\bar{\gamma})$ does not become $-\infty$ and partition function $Z_\Lambda(\bar{\gamma})$ is finite. The assumptions, under which these conditions hold true will be introduced later.

When $\bar{\gamma} = \emptyset$, let $\mu_\Lambda(d\eta|\emptyset) \equiv \mu_\Lambda(d\eta)$.

Let $\{\pi_\Lambda\}$ denote the specification associated with z and the Hamiltonian E (see [69]), which is defined on Γ by

$$\pi_\Lambda(A|\,\bar{\gamma}) = \int_{A'} \mu_\Lambda(d\eta|\,\bar{\gamma})$$

where $A' = \{\eta \in \Gamma_\Lambda :\; \eta \cup \bar{\gamma}_{\Lambda^c} \subset A\}$, $A \in \mathcal{B}(\Gamma)$.

A probability measure μ on Γ is called a Gibbs state for E and z if

$$\mu(\pi_\Lambda(A|\,\bar{\gamma})) = \mu(A)$$

for every $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is well known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [27] for more details. The class of all Gibbs states we denote by $\mathcal{G}(V, z)$.

About the potential V we will assume:

Assumption 4.4.1

1.Regularity:

$$\int_{\mathbb{R}^d} |1 - e^{-V(x)}|\sigma(dx) < \infty.$$

2. V is superstable in the sense of Ruelle.

3. V is lower regular, e.g. there exists a positive function ψ on the nonnegative integers such that $\psi(m) \leq Km^{-\lambda}$ for $m \geq 1$, and for any Λ_1 and Λ_2 which are each finite unions of unit cubes of the form Q_i, with $\gamma \subset \Lambda_1$ and $\bar{\gamma} \subset \Lambda_2$,

$$W(\gamma|\,\bar{\gamma}) \geq -\sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \psi(\|i - j\|)|\gamma_i||\bar{\gamma}_j|$$

where $K > 0$, $\lambda > d$ are fixed.

Let

$$V^+(x) = \inf_{\widetilde{x}:0<|\widetilde{x}|\leq|x|} V(\widetilde{x}), \quad V^-(x) = \min(0, \inf_{\widetilde{x}:|x-\widetilde{x}|\leq\frac{3}{2}g} V(\widetilde{x})),$$

$$\bar{V}(x) = \max(0, \sup_{\widetilde{x}:|x-\widetilde{x}|\leq\frac{3}{2}g} V(\widetilde{x})),$$

where the symbol $|\cdot|$ represent Euclidean norm in $\mathbb{R}^d$, and let

$$C_1 = \frac{1}{2}(v_d)^{-1} \int_{0 < |x| < g} V^+(x)[1 + g^{-1}|x|]^{-d-1} dx,$$

$$C_2 = -n^{n/2} \int_{\mathbb{R}^d} V^-(x) dx,$$

where v_d is the volume of a d - dimensional sphere of radius 1.

Assumption 4.4.2 ([21])
 1. *The inequalities $C_2 < C_1$, $C_2 < \infty$ hold.*
 2. *For some $D < \infty$:* $\int_{x:|x| \geq D} \bar{V}(x) dx < \infty$.

It is well known from [76] that under Assumption 4.4.1 the set of tempered Gibbs states is nonempty. Let us denote this set by $\mathcal{G}_t(V, z)$.

Analogous existence result for Gibbs states under Assumption 4.4.2 can be found in [21].

The following propositions collect some known results concerning Gibbs measures.

Proposition 4.4.1 ([3]) *Suppose that Assumption 4.4.1 is fulfilled. Then for any $\mu \in \mathcal{G}_t(V, z)$ the correlation functions $k_\mu^{(n)}(x_1, \ldots, x_n)$ satisfy the following inequality*

$$k_\mu^{(n)}(x_1, \ldots, x_n) \leq C^n \exp\left[-\sum_{i<j} V(x_i - x_j)\right]. \qquad (4.22)$$

with some $C > 0$.

Proposition 4.4.2 ([37]) *Suppose that Assumptions 4.4.1.2, 4.4.1.3 hold. Let Λ be a finite union of unit cubes of the form Q_i. Suppose $\widetilde{\Lambda} \supset \Lambda$, $\widetilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)$. For any $\mu \in \mathcal{G}_t(V, z)$ there exist constants $\alpha > 0$ and δ, depending only on z (independent of $\widetilde{\Lambda}$ and Λ), such that for any $N \in \mathbb{N}_0$*

$$\mu_{\widetilde{\Lambda}}(\{\gamma \mid |\gamma_\Lambda| \geq N|\Lambda|\}) \leq \exp\{-(\alpha N^2 - \delta)|\Lambda|\}. \qquad (4.23)$$

Proposition 4.4.3 ([21]) *Suppose that Assumption 4.4.2 holds and let $\varphi(y)$, $0 < y < \infty$, be a positive monotonically increasing convex function is such that for some $h > 0$, $L < \infty$*

$$\varphi(m) \leq L \exp\{m^2(H(m) - g^{-d}C_2 - h)\}, m = 0, 1, \ldots,$$

where

$$H(m) = \frac{1}{2}g^{-d}(v_d)^{-1}\int_{x:g(m^{\frac{1}{d}}-1)^{-1}\leq|x|\leq g} V^{+}(x)[1 + g^{-1}|x|]^{-d-1}dx$$

Then for any $\mu \in \mathcal{G}(V,z)$ *there exists a constant* $C_\Lambda(\varphi)$ *such that for any* $\widetilde{\Lambda} \in \mathcal{B}_c(\mathbb{R}^d)$ *the following inequality holds*

$$\int_{\Gamma_{\widetilde{\Lambda}}} \varphi(|\gamma_\Lambda|)\mu_{\widetilde{\Lambda}}(d\gamma) < C_\Lambda(\varphi), \ \ for \ all \ \Lambda \subset \widetilde{\Lambda}, \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \quad (4.24)$$

The conditions on function φ *are satisfied if*

$$\varphi(m) = \exp\{dm^2\}, \ \ 0 < d < (C_1 - C_2 g^{-d}).$$

Corollary 4.4.1 *Under Assumption 4.4.1 for any* $\mu \in \mathcal{G}_t(V,z)$ *we have:*

- *(RPB) (Ruelle's probability bound (4.23)).*

- *Dobrushin's bound (4.24) for all bounded* $\Lambda \subset \widetilde{\Lambda}$ *such that* $l_\Lambda \leq gd^{-\frac{1}{2}}$,

- *Dobrushin's bound (4.24) for function* $\varphi(x) = e^{\lambda x^p}$, $\lambda > 0$, $p \in (0,2)$.

Under Assumptions 4.4.1.2, 4.4.1.3 for any $\mu \in \mathcal{G}_t(V,z)$ *we have Dobrushin's bound for function* $\varphi(x) = e^{\lambda x^p}$, $\lambda > 0$, $p \in (0,2)$.
The conditions of Proposition 4.4.3 imply (RPB).

   Proof. We will prove only that under Assumption 4.4.1 for any $\mu \in \mathcal{G}_t(V,z)$ we have Dobrushin's bound for every bounded $\Lambda \subset \widetilde{\Lambda}$ such that $l_\Lambda \leq gd^{-\frac{1}{2}}$ and that the conditions of Proposition 4.4.3 imply (RPB). The proof of the remaining statements in this corollary is a direct consequence of Theorem 4.1.1 for measure $\mu = \mu_{\widetilde{\Lambda}}$.
   Using (2.3) and an estimate of the function φ we have

$$\int_{\Gamma_\Lambda} \left|K^{-1}[\varphi(|\eta|)]\right| \rho_\mu(d\eta) \leq \int_{\Gamma_\Lambda}\sum_{\xi\subset\eta}\varphi(|\xi|)\rho_{\mu_\Lambda}(d\eta) \leq$$

$$\leq L \int_{\Gamma_\Lambda}\sum_{\xi\subset\eta}\exp\{|\xi|^2 H(|\xi|) - g^{-d}C_2 - h\}\rho_{\mu_\Lambda}(d\eta). \quad (4.25)$$

The estimate on the correlation functions implies the bound for (4.25)

$$L \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{\xi \subset \{x_1, \ldots, x_n\}} e^{|\xi|^2 H(|\xi|) - \sum_{i<j} V(x_i - x_j)} C^n dx_1 \ldots dx_n. \qquad (4.26)$$

We bound $e^{-\sum_{i<j} V(x_i - x_j)}$ using the following result from [21]: there exists $m_0 \geq 2^d$ s.t. for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \leq gd^{-\frac{1}{2}}$ and $\eta \in \Gamma_\Lambda$, $|\eta| \geq m_0$ holds

$$E^V(\eta) \geq |\eta|^2 H(|\eta|).$$

Therefore, we can estimate (4.26) by

$$Le^{m_0^2 H(|m_0|)} \sum_{n=0}^{m_0} \frac{(2zC|\Lambda|)^n}{n!} + L \sum_{n=m_0+1}^{\infty} \frac{(2zC|\Lambda|)^n}{n!} \leq Le^{2zC|\Lambda| + m_0^2 H(|m_0|)}.$$

The equalities (2.2.1) and (2.6) give

$$\int_{\Gamma_{\widetilde{\Lambda}}} \varphi(|\eta_\Lambda|) \mu_{\widetilde{\Lambda}}(d\eta) = \int_{\Gamma_\Lambda} K^{-1}[\varphi(|\eta_\Lambda|)] \rho_\mu(d\eta) \leq Le^{2zC|\Lambda| + m_0^2 H(|m_0|)}.$$

To show that conditions of Proposition 4.4.3 imply (RPB) one should take in the proof of Theorem 4.1.1(4) the constant $\lambda = C_1 - C_2 g^{-d}$ and use the fact from [21] that $C_\Lambda \leq e^{\delta|\Lambda|}$ for some $\delta > 0$. ∎

Remark 4.4.1 *Let us note that the Poisson measure $\pi_{z\sigma}$ satisfy (4.11). Really, we have*

$$\pi_{z\sigma}(\{\gamma| \; |\gamma_\Lambda| \geq N|\Lambda|\}) =$$

$$= \sum_{n \geq N|\Lambda|} e^{-z|\Lambda|} \frac{(z|\Lambda|)^n}{n!} = e^{-z|\Lambda|} (z|\Lambda|)^{n_0} \sum_{n=0}^{\infty} \frac{(z|\Lambda|)^n}{n!} \frac{n!}{(n+n_0)!}, \qquad (4.27)$$

where n_0 is the smallest integer greater than or equal to $N|\Lambda|$. Using inequality

$$\frac{n!}{(n+n_0)!} \leq \frac{1}{n_0!}, \; n \geq 1,$$

and Stirling formula we can bound (4.27) by

$$\frac{z^{n_0} |\Lambda|^{n_0}}{n_0!} \leq z^{n_0} \frac{|\Lambda|^{n_0}}{e^{-n_0} n_0^{n_0}} \leq \frac{(ze)^{n_0}}{N^{n_0}}.$$

Considering $N \geq e^2 z$ the latter expression can be estimated by $e^{-N|\Lambda|}$.

Moreover, this implies $\pi_{z\sigma}(P_\infty) = \pi_{z\sigma}(U_\infty^1) = 1$. ∎

Remark 4.4.2 *The Poisson measure $\pi_{z\sigma}$ does not satisfy (RPB). Indeed, suppose that (RPB) for $\pi_{z\sigma}$ holds. Then from Theorem 4.1.1 we have that $\pi_{z\sigma}$ satisfy $(DEB)_{(1,\ 2-\varepsilon)}$, where $0 < \varepsilon < 1$. But by the definition of the Poisson measure*

$$\int_\Gamma e^{|\gamma_\Lambda|^{2-\varepsilon}} \pi_{z\sigma}(d\gamma) = e^{-z|\Lambda|} \sum_{n=0}^\infty e^{n^{2-\varepsilon}} \frac{(z|\Lambda|)^n}{n!},$$

where the latter series obviously diverges.

So, our assumption that the Poisson measure satisfies (RPB) is false. ∎

Example 2. Let $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a nonnegative pair potential and the function $k_\alpha^V : \Gamma_0 \to \mathbb{R}$ defined by

$$k_\alpha^V(\eta) = \alpha^{|\eta|} e^{-E^V(\eta)} = \alpha^{|\eta|} e^{-\sum_{\{x,y\} \subset \eta} V(x,y)}, \ \eta \in \Gamma_0, \ |\eta| \geq 2, \qquad (4.28)$$

$$k_\alpha^V(\eta) = \alpha, \ |\eta| = 1,$$

$$k_\alpha^V(\emptyset) = 1.$$

with some constant $\alpha > 0$.

Assume that $c := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (1 - e^{-V(x,y)}) dy < \infty$. As shown in [5] under assumption $\alpha c e < 1$ there exists probability measure μ on $\mathcal{B}(\Gamma)$ s.t.

$$k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda_\sigma}(\eta) = k_\alpha^V(\eta), \ \eta \in \Gamma_0,$$

where σ denotes the Lebesgue measure on $\mathbb{R}^d$. Moreover, the bound $0 \leq k_\mu(\eta) \leq \alpha^{|\eta|}$, $\eta \in \Gamma_0$ implies the uniqueness (c.f. [38]).

The measure μ is not Gibbs state associated with a pair potential. Moreover, it is difficult to show that μ corresponds to a potential in an explicit form. Even if this is true, such a potential should include interactions of all orders. In spite of this, we know that correlation functions of μ satisfy $(GRB)_V$. Therefore, all results of this chapter connected with $(GRB)_V$ are applicable to this measure. In particular, we have information about support properties and probability bounds depending on the behavior of V on the diagonal.

Chapter 5

Existence problem for Gibbs measures on configuration spaces

On $\mathbb{R}^d$ one can consider the following norms

$$\|x\|_p = \left(\sum_{k=1}^{d} |x_k|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_{1 \leq k \leq d} |x_k|, \tag{5.1}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

In the whole Chapter 5 we will use only norm (5.1). Therefore, for brevity we will use notation $|\cdot|$ instead of $\|\cdot\|_\infty$.

For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, let

$$l_\Lambda = \sup_{x, y \in \Lambda} |x - y|.$$

As in Section 4.3, for every $i \in \mathbb{Z}^d$ we define a cube

$$Q_i = \left\{ r \in \mathbb{R}^d \Big| \ g\left(i_k - \frac{1}{2} \right) < r_k \leq g\left(i_k + \frac{1}{2} \right), \ k = 1, \ldots, d \right\}, \tag{5.2}$$

where $g > 0$ will be chosen later. As before, $\mathcal{J}_g(\mathbb{R}^d)$ is denoted all finite unions of cubes of the form Q_i. In the Chapter 5, sometimes we will regard

$\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ as a subset of $\mathbb{Z}^d$ by letting Λ represent $\{i \in \mathbb{Z}^d \,|\, gi \in \Lambda\}$. For a nonnegative integer k, let Λ_k be the hypercube of length $g(2k-1)$ centered at the origin in $\mathbb{R}^d$. Λ_k is then a union of $(2k-1)^d$ unit cubes of the form Q_i. For $x \in \mathbb{R}^d$ let $\ln_+ |x| = \max\{1, \ln|x|\}$.

5.1   Existence of Gibbs states for pair long-range potentials

5.1.1   Potentials and Hamiltonians

A measurable function $\bar{V} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called a *pair potential*. We formulate the conditions on potential V which will be used in the Section 5.1:

(V1) Symmetry: $\bar{V}(x,\, y) = \bar{V}(y,\, x)$ for all $(x,\, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

(V2) Translation invariance: for any $(x,\, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and any $r \in \mathbb{R}^d$

$$\bar{V}(x + r,\, y + r) = \bar{V}(x,\, y).$$

We are able now introduce the function $V(x)$, $x \in \mathbb{R}^d$, by the equality

$$V(x - y) = \bar{V}(x,\, y).$$

(V3) There exist constants $\varepsilon > 0$ and $L > 0$ such that for any $x \in \mathbb{R}^d$, $x \neq 0$

$$V(x) \geq -\frac{L}{|x|^{d+\varepsilon}}.$$

(V4) $V \in C(\mathbb{R}^d \setminus \{0\})$.

The Hamiltonian $H : \ddot{\Gamma}_0 \to \mathbb{R}$ which corresponds to the potential V is defined by

$$H(\sigma) = \sum_{\{x,y\} \subset \eta} n(x)n(y)V(x - y) + V(0) \sum_{x \in \eta} \binom{n(x)}{2}, \qquad (5.3)$$

where $\sigma = (\eta, n) \in \ddot{\Gamma}_0$.

If $n(x) \equiv 1$ for all $x \in \eta$ and $|\sigma| \geq 2$ then

$$H(\sigma) = \sum_{\{x,y\} \subset \eta} V(x-y).$$

For the case $V(0) = +\infty$ and $n(x) > 1$ at least for one $x \in \eta$ it is clear that $H(\sigma) = +\infty$. For simplicity we will write

$$H(\sigma) = \sum_{x,y \in \sigma} V(x-y)$$

instead of (5.3). For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\sigma \in \ddot{\Gamma}_\Lambda$ we will sometimes write $H_\Lambda(\sigma)$ instead of $H(\sigma)$.

Having in mind applications in mathematical physics, we will always assume superstability of V (see [31, 58]). More precisely,

(V5) Superstability: for any $g > 0$ there exist $A > 0$ and $B \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ and any configuration $\sigma \in \ddot{\Gamma}_\Lambda$ holds

$$\sum_{x,\,y \in \sigma} V(x-y) \geq A\frac{|\sigma|^2}{l_\Lambda^d} - B|\sigma|. \tag{5.4}$$

In the sequel, we will write sometimes A_g, B_g, instead of A, B, to emphasize that these constants depend on g.

Remark 5.1.1 *Obviously, conditions* **(V3)** *and* **(V5)** *give us*

$$M_g = \inf_{0 < |x| < 2g} V(x) \geq \min\left\{\frac{4A_g}{g^d} - 2B_g, -\frac{L}{g^{d+\varepsilon}}\right\}.$$

Define $|\sigma_i| = |\sigma \cap Q_i|$.

Lemma 5.1.1 *Let the conditions* **(V3)**, **(V5)** *be fulfilled. Then, there exists a constant $K > 0$ such that for any $i, j \in \mathbb{Z}^d$, $|i - j| > 1$ and for any $x \in Q_i$, $y \in Q_j$*

$$V(x-y) \geq -\frac{K}{(g|i-j|)^{d+\varepsilon}}.$$

Proof. We set

$$K = 2^{d+\varepsilon} L.$$

Let $i, j \in \mathbb{Z}^d$, $|i - j| > 1$ be arbitrary and fixed. Then for any $x \in Q_i$, $y \in Q_j$:

$$0 < g(|i - j| - 1) \leq |x - y| \leq g(|i - j| + 1).$$

If $g|i - j| \leq |x - y| \leq g(|i - j| + 1)$ then

$$V(x - y) \geq -\frac{L}{|x - y|^{d+\varepsilon}} \geq -\frac{L}{(g|i - j|)^{d+\varepsilon}}. \tag{5.5}$$

If $0 < g(|i - j| - 1) \leq |x - y| \leq g|i - j|$ then

$$V(x - y) \geq -\frac{L}{|x - y|^{d+\varepsilon}} \geq -\frac{L}{(g[|i - j| - 1])^{d+\varepsilon}} \geq -\frac{2^{d+\varepsilon} L}{(g|i - j|)^{d+\varepsilon}}. \tag{5.6}$$

In the last inequality we have used the fact that $|i - j| \geq 2$. The claim of the Lemma now follows from the representation of the constant K, bounds (5.5) and (5.6). ∎

We introduce an additional assumption which will be necessary in the following.

(g) There exists $g > 0$ such that

$$a \max \{2^{d+\varepsilon} L, -g^{d+\varepsilon} M_g\} \leq A_g \, g^\varepsilon,$$

where

$$a := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|j|^{d+\varepsilon}} < \infty.$$

In the sequel, in this section we will consider g which satisfies condition **(g)**.

Remark 5.1.2 *It is well-known from Dobrushin-Fisher-Ruelle criterion (see [76]) that V is superstable if $V(x) \sim \frac{C_1}{|x|^{d+\delta_1}}$, $x \to 0$ and $|V(x)| \sim \frac{C_2}{|x|^{d+\delta_2}}$, $x \to \infty$ for some $C_1, C_2, \delta_1, \delta_2 > 0$. Moreover, by choosing the length of the sides of the cubes Q_i, $i \in \mathbb{Z}^d$ appropriately small, assumption **(g)** can be fulfilled automatically.*

*Indeed, one can show that in this case constant $A_g = Ag^{-\delta_1}$ with $A > 0$ independent of g and, hence, assumption **(g)** will have the form*

$$a \max \{2^{d+\delta_2} L, -g^{d+\delta_2} M_g\} \leq A \, g^{\delta_2 - \delta_1}. \tag{5.7}$$

Without lose of generality, one can regard $\delta_1 > \delta_2$. Because $-M_g$ decrease when $g \to 0$, inequality (5.7) holds for small $g > 0$. ∎

Consider a subset in the space of multiple configurations:

$$\ddot{\Gamma}^t = \bigcup_{n=1}^{\infty} \ddot{\Gamma}^t_n,$$

where $\ddot{\Gamma}^t_n = \left\{ \sigma \in \ddot{\Gamma} \ \middle| \ |\sigma_i| \leq n(\ln_+ |i|)^{\frac{1}{2}}, \ \forall i \in \mathbb{Z}^d \right\}.$

Definition 5.1.1 *Configuration $\sigma \in \ddot{\Gamma}$ is said to be tempered if $\sigma \in \ddot{\Gamma}^t$.*

For a given $\bar{\sigma} = (\bar{\gamma}, \bar{n}) \in \ddot{\Gamma}^t$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ define the interaction energy between $\sigma = (\eta, n) \in \ddot{\Gamma}_\Lambda$ and $\bar{\sigma}_{\Lambda_c} = \bar{\sigma} \cap \Lambda^c = (\bar{\gamma} \cap \Lambda^c, n)$, $\Lambda^c = \mathbb{R}^d \setminus \Lambda$ as

$$W_\Lambda(\sigma|\bar{\sigma}) = \sum_{x \in \sigma, \ y \in \bar{\sigma} \cap \Lambda^c} V(x - y),$$

where the sum at the right-hand side is a simplified notation for

$$\sum_{x \in \gamma, \ y \in \bar{\gamma} \cap \Lambda^c} n(x)\bar{n}(y)V(x - y).$$

The interaction energy is said to be well-defined if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\bar{\sigma} \in \ddot{\Gamma}^t$ it does not become $-\infty$.

Define

$$H_\Lambda(\sigma|\bar{\sigma}) = H_\Lambda(\sigma) + W_\Lambda(\sigma|\bar{\sigma})$$

and let

$$Z_\Lambda(\bar{\sigma}) := \int_{\ddot{\Gamma}_\Lambda} \exp\left\{-\beta H_\Lambda(\sigma|\ \bar{\sigma})\right\} \lambda_z(d\sigma)$$

be the so-called partition function.

Lemma 5.1.2 *Let conditions* **(V1)**-**(V5)** *be fulfilled. Then for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\sigma \in \ddot{\Gamma}_\Lambda$ and $\bar{\sigma} = (\bar{\gamma}, \bar{n}) \in \ddot{\Gamma}^t$ the interaction energy $W_\Lambda(\sigma|\bar{\sigma})$ is well defined and partition function $Z_\Lambda(\bar{\sigma})$ is finite.*

Proof. Using representation $\ddot{\Gamma}_\Lambda := \bigsqcup_{N \in \mathbb{N}_0} \ddot{\Gamma}^{(N)}_\Lambda$ we have

$$Z_\Lambda(\bar{\sigma}) = \int_{\ddot{\Gamma}_\Lambda} e^{-\beta H_\Lambda(\sigma|\bar{\sigma})} \lambda_z(d\sigma) = \sum_{N=0}^{\infty} \int_{\ddot{\Gamma}^{(N)}_\Lambda} e^{-\beta H_\Lambda(\sigma|\bar{\sigma})} \lambda_z(d\sigma).$$

With (**V5**) it is not difficult to show that there exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any $\sigma \in \ddot{\Gamma}_\Lambda$, $\sigma \geq 2$ holds

$$\sum_{x,y \in \sigma} V(x-y) \geq -B|\sigma|.$$

In particular, potential V is bounded from below by $-2B$.

Define $\Lambda_{\mathcal{J}}$ as a union of all cubes Q_i which have nonempty intersection with Λ. Without lose of generality we will assume that for any $i \in \Lambda$ and $j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}$ holds $|i-j| > 1$. Otherwise we will add to $\Lambda_{\mathcal{J}}$ all cubes with such a numbers $j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}$. Then, according to Lemma 5.1.1 the interaction energy can be estimated by

$$W_\Lambda(\sigma|\bar{\sigma}) \geq -2B|\sigma|\,|\bar{\sigma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}| - K \sum_{i \in \Lambda} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}} \frac{|\sigma_i|\,|\bar{\sigma}_j|}{(g|i-j|)^{d+\varepsilon}} \geq$$

$$\geq -|\sigma|\left(2B|\bar{\sigma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}| + K \max_{i \in \Lambda} \sum_{j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}} \frac{|\bar{\sigma}_j|}{(g|i-j|)^{d+\varepsilon}}\right). \tag{5.8}$$

Let i_0 maximize the sum in (5.8). Then

$$W_\Lambda(\sigma|\bar{\sigma}) \geq -|\sigma|\left(2B|\bar{\sigma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}| + K \sum_{j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}} \frac{|\bar{\sigma}_j|}{(g|i_0-j|)^{d+\varepsilon}}\right).$$

Since $\bar{\sigma} \in \Gamma^t$, the series

$$S := \sum_{j \in \mathbb{Z}^d \setminus \Lambda_{\mathcal{J}}} \frac{|\bar{\sigma}_j|}{|i_0-j|^{d+\varepsilon}}$$

is finite.

Therefore, the interaction energy is well defined. Moreover, the partition function can be estimated by

$$\exp\left\{z|\Lambda|e^{B\left[1+2|\bar{\sigma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}|\right]+KS}\right\} < \infty. \quad \blacksquare$$

Definition 5.1.2 *A potential V is called stable (see [76]) iff there exists a constant $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any configuration $\sigma \in \ddot{\Gamma}_\Lambda$ it holds*

$$\sum_{x,y \in \sigma} V(x,y) \geq -B|\sigma| \tag{5.9}$$

Remark 5.1.3 *Lemma 5.1.2 holds true if instead of the condition* **(V5)** *we assume stability of the potential V.*

In the following, we will consider only tempered configurations.

5.1.2   Specifications

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\sigma} \in \ddot{\Gamma}^t$. The finite volume Gibbs state with boundary configuration $\bar{\sigma}$ for H, $\beta > 0$ and $z > 0$ on $\ddot{\Gamma}_\Lambda$ is defined by

$$P_{\Lambda, \bar{\sigma}}(d\sigma) = \frac{\exp\left\{-\beta H_\Lambda(\sigma|\ \bar{\sigma})\right\}}{Z_\Lambda(\bar{\sigma})} \lambda_z(d\sigma).$$

When $\bar{\sigma} = \emptyset$, let $P_{\Lambda, \emptyset}(d\sigma) \equiv P_\Lambda(d\sigma)$.

Let $\{\pi_\Lambda\}$ denote the specification associated with z, β and the Hamiltonian H (see [69]) defined on $\ddot{\Gamma}$ by

$$\pi_\Lambda(A|\ \bar{\sigma}) = \int_{A'} P_{\Lambda, \bar{\sigma}}(d\sigma)$$

where $A' = \{\sigma \in \ddot{\Gamma}_\Lambda :\ \sigma \cup \bar{\sigma}_{\Lambda^c} \subset A\}$, $A \in \mathcal{B}(\ddot{\Gamma})$ and $\bar{\sigma} \in \ddot{\Gamma}^t$.

A probability measure μ on $\ddot{\Gamma}$ is called a Gibbs state for H, β and z if

$$\mu(\pi_\Lambda(A|\ \bar{\sigma})) = \mu(A)$$

for every $A \in \mathcal{B}(\ddot{\Gamma})$ and every $\Lambda \in \mathcal{B}(\mathbb{R}^d)$.

This relation is well-known as (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [27] for more details.

Let $\chi > 0$. The class of all Gibbs states μ which satisfy

$$\int_{\ddot{\Gamma}} e^{\chi|\sigma_\Lambda|} \mu(d\sigma) < \infty, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

will be denoted by $\mathcal{G}_\chi(V, z, \beta)$.

5.1.3   Main result

Theorem 5.1.1 *Let conditions* **(V1)**-**(V5)** *and* **(g)** *be satisfied. Then for any $\chi > 0$, $z > 0$ and $\beta > 0$*

$$\mathcal{G}_\chi(V, z, \beta) \neq \emptyset.$$

The proof of this theorem is based on the general Dobrushin's theorem about the existence of Gibbs states for lattice models (see [22]). The formulation of this theorem can be found also in [27], [67], [79]. For reader's convenience we quote the theorem from [22] in the next subsection.

5.1.4   Existence theorem on $\mathbb{Z}^d$

Let X be a complete separable metric space. A configuration $\bar{\mathbf{x}}$ on $\Lambda \subseteq \mathbb{Z}^d$ is a map $\bar{\mathbf{x}} : \Lambda \mapsto X$. X^Λ denote the set of all configurations on Λ. Let $\{P_{t,\bar{\mathbf{x}}}\}$ be the family of probability measures on X (a specification) indexed by parameters $t \in \mathbb{Z}^d$ and $\bar{\mathbf{x}} \in X^{\mathbb{Z}^d \setminus \{t\}}$.

A random field $\xi(t)$, $t \in \mathbb{Z}^d$, taking values in X corresponds to the specification $\{P_{t,\bar{\mathbf{x}}}\}$ if for every $t \in \mathbb{Z}^d$ and every Borel set $\mathcal{A} \subseteq X$

$$\Pr\{\xi(t) \in \mathcal{A} \,|\, \xi(u) = \bar{\mathbf{x}}(u), \, u \neq t\} = P_{t,\bar{\mathbf{x}}}(\mathcal{A}).$$

The compact function is a non-negative measurable function $h : X \mapsto \mathbb{R}_+$ such that for any $d \in \mathbb{R}_+$ the set

$$\{x \,|\, h(x) \leq d, \, x \in X\}$$

is relatively compact in X.

Theorem 5.1.2 *For the existence of the field $\{\xi(t), \, t \in \mathbb{Z}^d\}$ with a prescribed system of specification $\{P_{t,\bar{\mathbf{x}}}\}$, the fulfillment of the following two conditions is sufficient:*

1. There exist a compact function $h(x)$, $x \in X$, and constants C, $0 \leq C < \infty$, and $c_t \geq 0$, $t \in \mathbb{Z}^d \setminus \{0\}$, such that the conditional mathematical expectation

$$\int_X h(x) P_{t_0,\bar{\mathbf{x}}}(dx) \leq C + \sum_{t \in \mathbb{Z}^d \setminus \{t_0\}} c_{t-t_0} h(\bar{\mathbf{x}}(t))$$

for all $t_0 \in \mathbb{Z}^d$ and all $\bar{\mathbf{x}} \in X^{\mathbb{Z}^d \setminus \{t_0\}}$, and

$$\sum_{t \in \mathbb{Z}^d \setminus \{0\}} c_t < 1.$$

2. For any $t_0 \in \mathbb{Z}^d$, there exist a sequence of finite sets $U_{t_0}^1 \subset U_{t_0}^2 \cdots$, whose union is $\mathbb{Z}^d \setminus \{t_0\}$, constants d_t^n, $t \in \mathbb{Z}^d \setminus \{0\}$, $n \in \mathbb{N}$, and constants D_n such that

$$\sum_{t \in \mathbb{Z}^d \setminus \{0\}} d_t^n \leq D_n,$$

where D_n tends to 0 as $n \to \infty$. Moreover, for any continuous function $\varphi(x)$, $x \in X$, with $|\varphi(x)| \leq 1$ there exist functions $f_n(\bar{\mathbf{x}}(t), t \in U_{t_0}^n)$ which are continuous on $X^{U_{t_0}^n}$, $n \in \mathbb{N}$ such that

$$\left| \int_X \varphi(x) P_{t_0, \bar{\mathbf{x}}}(dx) - f_n(\bar{\mathbf{x}}(t), t \in U_{t_0}^n) \right| \leq D_n + \sum_{t \in \mathbb{Z}^d \setminus \{t_0\}} d_{t-t_0}^n h(\bar{\mathbf{x}}(t)).$$

The field $\{\xi_t, t \in \mathbb{Z}^d\}$ with specification $\{P_{t, \bar{\mathbf{x}}}\}$ can be constructed in such a way that the mathematical expectations

$$\sup_{t \in \mathbb{Z}^d} E h(\xi_t) < \infty.$$

5.1.5 Lattice structure associated with continuous system

In this subsection we introduce a lattice structure associated with our continuous system, c.f. [67]. For any $t \in \mathbb{Z}^d$ let us denote by $\mathbf{X}_t$ the configuration space $\ddot{\Gamma}_{\bar{Q}_t}$ in the closure $\bar{Q}_t$ of the cube Q_t. Let

$$\mathcal{X} = \times_{t \in \mathbb{Z}^d} \mathbf{X}_t$$

be the associated lattice configuration space and let $\mathcal{B}(\mathcal{X})$ be the correspondent Borel σ-algebra on it.

As in Section 3.1 we consider function $\rho : \ddot{\Gamma}_{\bar{Q}_t} \times \ddot{\Gamma}_{\bar{Q}_t} \to \mathbb{R}_+$,

$$\rho(\sigma_1, \sigma_2) = \begin{cases} \frac{1}{2g|\sigma_1|} \min_\pi \sum_{i=1}^{|\sigma_1|} |x_i - y_{\pi(i)}|, & \text{if } |\sigma_1| = |\sigma_2| \\ 1, & \text{otherwise.} \end{cases} \qquad (5.10)$$

In (5.10) the minimum is taken over the set of all permutations π of the set $\{1, \ldots, |\sigma_1|\}$, configuration $\sigma_1 = \{x_1, \ldots, x_{|\sigma_1|}\}$ and $\sigma_2 = \{y_1, \ldots, y_{|\sigma_2|}\}$.

As shown in [67], the function ρ is a metric on $\ddot{\Gamma}_{\bar{Q}_t}$. Moreover, metric space $(\ddot{\Gamma}_{\bar{Q}_t}, \rho)$ is a Polish space.

Having a continuous configuration $\sigma = (\gamma, n) \in \ddot{\Gamma}$, we construct the lattice configuration $\xi = (\xi(t), n_t)_{t \in \mathbb{Z}^d} \in \mathcal{X}$ in the following way:

$$\xi(t) = \sigma \cap Q_t, \quad t \in \mathbb{Z}^d$$

and for $x \in \xi(t)$

$$n_t(x) = n(x).$$

Denote this correspondence by $T : \ddot{\Gamma} \to \mathcal{X}$. For $\xi = T\sigma$: $\xi(t) \subset Q_t \subset \bar{Q}_t$, $t \in \mathbb{Z}^d$. Therefore

$$T(\ddot{\Gamma}) \subset \mathcal{X}$$

and T is an injective map.

The inverse map T^{-1} can be constructed as follows. If $\xi = (\xi(t), n_t)_{t \in \mathbb{Z}^d} \in T(\ddot{\Gamma})$ then $T^{-1}\xi := \sigma = (\gamma, n)$ is defined by

$$\gamma = \bigcup_{t \in \mathbb{Z}^d} \xi(t). \tag{5.11}$$

Because $\xi \in T(\ddot{\Gamma})$, configurations $\xi(t)$ and $\xi(s)$ do not intersect for $t \neq s$. Therefore, for any $x \in \gamma$ there exists only one point $t \in \mathbb{Z}^d$ such that $x \in \xi(t)$ and we are able to define

$$n(x) = n_t(x).$$

The map T is a measurable embedding of $\ddot{\Gamma}$ into $\mathcal{X}$. Hence, every measure on $\ddot{\Gamma}$ induces a measure on $\mathcal{X}$. The inverse map T^{-1} can be extended to the whole $\mathcal{X}$. If $\xi = (\xi(t), n_t) \in \mathcal{X} \backslash T(\ddot{\Gamma})$ then there exists $t \in \mathbb{Z}^d$ with $x \in \xi(t)$ on $\bar{Q}_t \backslash Q_t$. To define T^{-1} for this case, we are able to use (5.11) for γ and

$$n(x) = \sum_{t:\, x \in \xi(t)} n_t(x).$$

Thus, the existence of the lattice model implies the existence of the continuous one.

For any $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, $\xi \in T(\ddot{\Gamma}_{\cup_{t \in \Lambda} Q_t})$ and $\bar{\xi} \in T(\ddot{\Gamma}^t)$ the conditional energy $H_\Lambda(\xi \,|\, \bar{\xi})$ is defined as

$$H_\Lambda(\xi \,|\, \bar{\xi}) = H_{\cup_{t \in \Lambda} Q_t}((T^{-1}\xi)_{\cup_{t \in \Lambda} Q_t} \,|\, (T^{-1}\bar{\xi})_{\cup_{t \in \Lambda^c} Q_t}),$$

where $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$ and $(T^{-1}\xi)_G$ is the restriction of the $T^{-1}\xi \in \ddot{\Gamma}$ to the set $G \subset \mathbb{R}^d$.

Using Lemma 5.1.2 we can define finite volume Gibbs states for the lattice counterpart of the continuous model. Namely, for any $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ the

finite volume Gibbs state $P_{\Lambda,\bar{\xi}}$ under condition $\bar{\xi} \in T(\ddot{\Gamma}^t)$ is given on $\times_{t\in\Lambda}\mathbf{X}_t$ by

$$P_{\Lambda,\bar{\xi}}(d\xi_\Lambda) = \frac{\exp\left\{-\beta H_\Lambda(\xi|\,\bar{\xi})\right\}}{Z_\Lambda(\bar{\xi})}d\lambda_z(\xi_\Lambda) = \frac{\exp\left\{-\beta H_{\cup_{t\in\Lambda}Q_t}(\sigma|\,\bar{\sigma})\right\}}{Z_\Lambda(\bar{\xi})}d\lambda_z(\sigma),$$

where

$$Z_\Lambda(\bar{\xi}) = \int_{\Gamma_{\cup_{t\in\Lambda}Q_t}} e^{-\beta H_{\cup_{t\in\Lambda}Q_t}(\sigma\,|\,\bar{\sigma})}\lambda_z(d\sigma) = Z_{\cup_{t\in\Lambda}Q_t}(\bar{\sigma}),$$

$(T^{-1}\xi)_{\cup_{t\in\Lambda}Q_t} = \sigma$ and $T^{-1}\bar{\xi} = \bar{\sigma}$.

The corresponding specifications are defined by

$$\pi_\Lambda(A|\,\bar{\xi}) = \int_{A'} P_{\Lambda,\bar{\xi}}(d\xi)$$

where $A' = \{\xi \in \times_{t\in\Lambda}\mathbf{X}_t : \; \xi \times \bar{\xi}_{\mathbb{Z}^d\setminus\Lambda} \in A\}$, $A \in \mathcal{B}(\mathcal{X})$ and $\bar{\xi}_{\mathbb{Z}^d\setminus\Lambda}$ is projection of $\bar{\xi} \in T(\Gamma^t)$ on $\times_{t\in\mathbb{Z}^d\setminus\Lambda}\mathbf{X}_t$.

A probability measure μ on $\mathcal{X}$ is called a Gibbs state for z and β if

$$\mu(\pi_\Lambda(A|\,\bar{\xi})) = \mu(A)$$

for every $A \in \mathcal{B}(\mathcal{X})$ and every $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$. For more details, see [6], [67].

In the Section 5.1, we will need only single point Gibbs states, i.e. $\{P_{t,\bar{\xi}}|t \in \mathbb{Z}^d, \; \bar{\xi} \in T(\Gamma^t)\}$. Obviously, all spaces $\mathbf{X}_t$, $t \in \mathbb{Z}^d$ are isomorphic to the space $\mathbf{X}_0$, which we will denote for brevity by $\mathbf{X}$. We will consider for simplicity $\{P_{t,\bar{\xi}}|t \in \mathbb{Z}^d, \; \bar{\xi} \in T(\Gamma^t)\}$ on $\mathbf{X}$. For more details about the associated lattice structure, see [6], [67]

5.1.6 Proof

In this subsection we check Dobrushin's conditions for the lattice model with compact function (see [6], [67])

$$h(\xi) = e^{\chi|\xi|},\; \chi > 0$$

on $\mathbf{X}$ under assumptions **(V1)-(V5)**, **(g)**.

Because all spaces $\mathbf{X}_t$, $t \in \mathbb{Z}^d$ are topologically identical to the space $\mathbf{X}_0$, which we have denoted by $\mathbf{X}$, in proofs we drop index t, considering, instead of $\mathbf{X}_t$ and $\xi_t \in \mathbf{X}_t$, $t \in \mathbb{Z}^d$ the space $\mathbf{X}$ and the configuration $\xi \in \mathbf{X}$.

Lemma 5.1.3 *For any $\chi > 0$, $t \in \mathbb{Z}^d$ and $\bar{\xi} \in T(\ddot{\Gamma}^t)$ there exists $C > 0$ and $c_j \geq 0$, $j \in \mathbb{Z}^d \setminus \{0\}$, such that*

$$\sum_{j \in \mathbb{Z}^d \setminus \{0\}} c_j < 1$$

and

$$\int_X e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi) \leq C + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{j-t} e^{\chi|\bar{\xi}(j)|}. \tag{5.12}$$

Proof. For simplicity we will use the notation $\bar{\xi}_j$ instead of $\bar{\xi}(j)$, $j \in \mathbb{Z}^d \setminus \{0\}$. The spin space $\mathbf{X}$ can be represented as

$$\mathbf{X} = \bigcup_{N=0}^{\infty} \mathbf{X}^N, \ \mathbf{X}^N = \{\xi \in \mathbf{X} | \ |\xi| = N\}.$$

Using this representation we have

$$\int_{\mathbf{X}} e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi) = \sum_{N \leq N_{\bar{\xi}}} \int_{\mathbf{X}^N} e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi) + \sum_{N > N_{\bar{\xi}}} \int_{\mathbf{X}^N} e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi). \tag{5.13}$$

$N_{\bar{\xi}} \in \mathbb{N}$ will be chosen later. Let us estimate the second term in (5.13):

$$I_{N_{\bar{\xi}}} := \sum_{N > N_{\bar{\xi}}} \int_{\mathbf{X}^N} e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi) \leq \sum_{N > N_{\bar{\xi}}} e^{\chi N} \int_{\mathbf{X}^N} e^{-\beta H(\xi|\bar{\xi})} \lambda_z(d\xi).$$

We have used fact that the partition function $Z_t(\bar{\xi})$ is greater than 1. We set

$$\partial t = \bigcup_{i \, : \, |t-i|=1} Q_i.$$

Then, using Lemma 5.1.1 and condition (**V5**), $I_{N_{\bar{\xi}}}$ can be estimated by

$$\sum_{N > N_{\bar{\xi}}} \frac{z^N}{N!} e^{(\chi+\beta B)N} \exp \left\{ \beta N \left(-\frac{A}{g^d} N - M_g |\bar{\xi}_{\partial t}| + K \sum_{j \in \mathbb{Z}^d \setminus \partial t} \frac{|\bar{\xi}_j|}{(g|t-j|)^{d+\varepsilon}} \right) \right\}.$$

Denote

$$D_g = \frac{\max\{K, -g^{d+\varepsilon} M_g\}}{Ag^\varepsilon}.$$

Suppose that $\bar{\xi} \in T(\ddot{\Gamma}_n^t)$ for some $n \in \mathbb{N}$. Then

$$D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{|\bar{\xi}_j|}{|t-j|^{d+\varepsilon}} \leq D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{n(\ln_+ |j|)}{|t-j|^{d+\varepsilon}} < \infty. \tag{5.14}$$

Choosing $N_{\bar{\xi}}$ as the largest integer less or equal than

$$D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{|\bar{\xi}_j|}{|t-j|^{d+\varepsilon}},$$

we have for all $N > N_{\bar{\xi}}$

$$\frac{A}{g^d} N \geq K \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{|\bar{\xi}_j|}{(g|t-j|)^{d+\varepsilon}} - M_g |\bar{\xi}_{\partial t}|.$$

This implies that

$$I_{N_{\bar{\xi}}} \leq \sum_{N > N_{\bar{\xi}}} \frac{z^N}{N!} e^{(\chi+\beta B)N} \leq \exp\left\{ z e^{\chi+\beta B} \right\} - 1.$$

To estimate the first term in (5.13) let us observe that

$$J_{N_{\bar{\xi}}} := \sum_{N \leq N_{\bar{\xi}}} \int_{\mathbf{X}^N} e^{\chi|\xi|} P_{t,\bar{\xi}}(d\xi) = \sum_{N \leq N_{\bar{\xi}}} e^{\chi N} P_{t,\bar{\xi}}(\mathbf{X}^N) \leq e^{\chi N_{\bar{\xi}}}.$$

Hence

$$J_{N_{\bar{\xi}}} \leq \exp\left\{ \chi \left(D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{|\bar{\xi}_j|}{|t-j|^{d+\varepsilon}} \right) \right\}. \tag{5.15}$$

By the convexity of the function e^x we obtain

$$J_{N_{\bar{\xi}}} \leq \exp\left\{ a D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{\chi |\bar{\xi}_j|}{a|t-j|^{d+\varepsilon}} \right\} \leq \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{1}{a|t-j|^{d+\varepsilon}} e^{a D_g \chi |\bar{\xi}_j|}.$$

And again, because of the convexity of e^x and property (g), we have

$$J_{N_{\bar{\xi}}} \leq \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{1}{a|t-j|^{d+\varepsilon}} e^{(1-aD_g)0 + aD_g \chi |\bar{\xi}_j|} \leq$$

$$\leq (1 - aD_g) + D_g \sum_{j \in \mathbb{Z}^d \setminus \{t\}} \frac{e^{\chi |\bar{\xi}_j|}}{|t - j|^{d+\varepsilon}}.$$

Therefore

$$J_{N_{\bar{\xi}}} \leq C^\star + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{t-j} e^{\chi |\bar{\xi}_j|},$$

where

$$C^\star = (1 - aD_g); \quad c_j = \frac{D_g}{|j|^{d+\varepsilon}}, \, j \in \mathbb{Z}^d \setminus \{0\}.$$

Finally, we have

$$\int_{\mathbf{X}} e^{\chi |\xi|} P_{t, \bar{\xi}}(d\xi) \leq C + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{t-j} e^{\chi |\bar{\xi}_j|},$$

where

$$C = C^\star + \exp\left\{ z e^{\chi + \beta B} \right\} - 1.$$

From property **(g)** it follows that

$$\sum_{j \in \mathbb{Z}^d \setminus \{0\}} c_j < 1. \quad \blacksquare$$

Lemma 5.1.4 *For any $\delta \in (0, 1)$, there exist bounded $\Lambda \subset \mathbb{Z}^d$ and constants δ_j, $j \in Z^d \setminus \{0\}$, such that*

$$\sum_{j \in Z^d \setminus \{0\}} \delta_j \leq \delta$$

and for any $\bar{\xi} \in T(\ddot{\Gamma}^t)$ and a measurable function $\varphi(\xi)$, $\xi \in \mathbf{X}$, $|\varphi(\xi)| \leq 1$, the following inequality holds

$$\left| \int_{\mathbf{X}} \varphi(\xi) P_{t, \bar{\xi}}(d\xi) - \int_{\mathbf{X}} \varphi(\xi) P_{t, \bar{\xi}_\Lambda}(d\xi) \right| \leq \sum_{j \in Z^d \setminus \{t\}} \delta_{j-t} e^{\chi |\bar{\xi}_j|},$$

where $\bar{\xi}_\Lambda$ is the projection of $\bar{\xi} \in T(\ddot{\Gamma}^t)$ on $\times_{i \in \Lambda} \mathbf{X}_i$.

Proof. The proof of this Lemma is completely analogous to the arguments occurred in Lemma 4 of [21] for constants d_t of the type

$$d_t = \frac{K}{|t|^{d+\varepsilon}}. \quad \blacksquare$$

Remark 5.1.4 *Lemmas 5.1.3 and 5.1.4 hold true if instead of the condition* **(V5)** *we assume that there exists a constant $C > 0$ such that for any x, $y \in \mathbb{R}^d$, $0 < |x - y| < g$*

$$V(x - y) \geq C.$$

The condition **(g)** *is replaced by the requirement $K < C/2a$ in this case.*

*   Indeed, the claim follows straightforward from the arguments used in Lemma 1 of [67].* ■

Proof of Theorem 5.1.1. Because of continuity of functions

$$f(\bar{\xi}, \Lambda) := \int_{\mathbf{X}} \varphi(\xi) P_{t,\bar{\xi}_\Lambda}(d\xi)$$

(see [21], [67]) and Lemmas 5.1.2, 5.1.3 and 5.1.4 (see [22] for details), there exists at least one Gibbs measure on $\mathcal{X}$ and, hence, measure μ on $\ddot{\Gamma}$. Moreover, as shown in [22], for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ there exists $C_\Lambda < \infty$ such that the following holds

$$\int_{\ddot{\Gamma}} e^{\chi|\sigma_\Lambda|} \mu(d\sigma) < C_\Lambda. \tag{5.16}$$

Therefore, for any $\chi > 0$, $z > 0$ and $\beta > 0$

$$\mathcal{G}_\chi(V, z, \beta) \neq \emptyset. \quad ■$$

Remark 5.1.5 *Theorem 5.2.1 holds if instead of* **(V5)** *we assume that there exists a constant $C > 0$ such that for any x, $y \in \mathbb{R}^d$, $0 < |x - y| < g$*

$$V(x - y) \geq C,$$

the potential V is stable, and the condition **(g)** *is replaced by $K < C/2a$.*

5.2   A modified approach to the existence problem and detailed properties of Gibbs states

5.2.1   The model

In the Section 5.2 we use the following conditions on the pair potential $\bar{V}$:

(V1) Symmetry: $\bar{V}(x, y) = \bar{V}(y, x)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

(V2) Translation invariance: for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and any $z \in \mathbb{R}^d$

$$\bar{V}(x + z, y + z) = \bar{V}(x, y)$$

We are able now introduce the function $V(x)$, $x \in \mathbb{R}^d$, by the equality

$$V(x - y) = \bar{V}(x, y)$$

(V3) Potential V is of $Dobrushin-Fisher-Ruelle$ (DFR) type, i.e., there exist $0 < d_1 < d_2 < \infty$ s.t.

$$V(x) \geq \frac{C_1}{|x|^{d+\delta_1}}, \quad |x| \leq d_1,$$

$$|V(x)| \leq \frac{C_2}{|x|^{d+\delta_2}}, \quad |x| \geq d_2$$

for some C_1, C_2, δ_1, $\delta_2 > 0$.

(V4) $V \in C(\mathbb{R}^d \setminus \{0\})$.

Let V satisfy **(V3)** with constants d_1, d_2, δ_1, δ_2, C_1, $C_2 > 0$. Without loss of generality, we will regard $d_1 \leq \min\{1, \varepsilon C_1\}$ for some $0 < \varepsilon < 1$ and $\delta_1 > \delta_2$. In the sequel, in this section we consider $g \leq d_1/2$.

Lemma 5.2.1 *Condition* **(V3)** *implies the following useful bounds.*
1. There exist $A > 0$ (independent of g) and $B \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$ holds

$$\sum_{\{x,y\} \subset \eta} V(x - y) \geq A \frac{|\eta|^2}{g^{\delta_1} |\Lambda|} - B|\eta|.$$

2. There exists constant $K > 0$ (independent of g) such that for all $i, j \in \mathbb{Z}^d$, $i \neq j$ and for any $x \in Q_i$, $y \in Q_j$

$$V(x - y) \geq -\frac{K}{(g|i - j|)^{d+\delta_2}}.$$

Proof. It is well-known from (DFR) criterion (see [76]) that condition $(\mathbf{V3})$ implies the second statement of Lemma 5.2.1 and superstability of V, i.e., there exist $A_0 > 0$, $B_0 \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$ holds

$$\sum_{\{x,y\}\subset\eta} V(x-y) \geq A_0 \sum_{i\in\Lambda} |\eta_i|^2 - B_0|\eta|.$$

Set

$$\bar{\alpha}(|x|) = \varepsilon C_1 \, \mathbb{1}_{[0,\,d_1]}(|x|) \left(\frac{1}{|x|^{d+\delta_1}} - \frac{1}{d_1^{d+\delta_1}} \right). \tag{5.17}$$

Obviously, the function $\bar{\alpha}$ is continuous decreasing and

$$\bar{\alpha}_0 := \lim_{t\to 0+0} \bar{\alpha}(t) = +\infty.$$

We can represent potential V as

$$V(x) = (V(x) - \bar{\alpha}(|x|)) + \bar{\alpha}(|x|).$$

For any $x : |x| \leq d_1$

$$V(x) - \bar{\alpha}(|x|) \geq \frac{(1-\varepsilon)C_1}{|x|^{d+\delta_1}} + \frac{\varepsilon C_1}{d_1^{d+\delta_1}} \geq \frac{(1-\varepsilon)C_1}{|x|^{d+\delta_1}}$$

and for any $x : |x| \geq d_2$

$$|V(x) - \bar{\alpha}(|x|)| = |V(x)| \leq \frac{C_2}{|x|^{d+\delta_2}}.$$

Hence, potential $V(x) - \bar{\alpha}(|x|)$ is of (DFR) type. Then from (DFR) criterion follows that there exists $B_1 \geq 0$ such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$ holds

$$\sum_{\{x,y\}\subset\eta} (V(x-y) - \bar{\alpha}(|x-y|)) \geq -B_1|\eta|. \tag{5.18}$$

Because $\bar{\alpha} \geq 0$

$$\sum_{\{x,y\}\subset\eta} \bar{\alpha}(|x-y|) \geq \sum_{i\in\Lambda} \sum_{\{x,y\}\subset\eta_i} \bar{\alpha}(|x-y|). \tag{5.19}$$

For any $t : 0 < t \leq g \leq d_1/2$:

$$\bar{\alpha}(t) \geq \varepsilon C_1 \left(\frac{1}{g^{d+\delta_1}} - \frac{1}{d_1^{d+\delta_1}} \right) \geq \left(1 - \frac{1}{2^{d+\delta_1}} \right) \frac{\varepsilon C_1}{g^{d+\delta_1}}. \tag{5.20}$$

Denote

$$A = \frac{\varepsilon C_1}{2} \left(1 - \frac{1}{2^{d+\delta_1}} \right).$$

The bound (5.20) implies

$$\sum_{i \in \Lambda} \sum_{\{x,y\} \subset \eta_i} \bar{\alpha}(|x - y|) \geq \sum_{i \in \Lambda} \frac{2A}{g^{d+\delta_1}} \cdot \frac{|\eta_i|(|\eta_i| - 1)}{2} =$$

$$= \frac{A}{g^{d+\delta_1}} \sum_{i \in \Lambda} |\eta_i|^2 - \frac{A}{g^{d+\delta_1}} |\eta|. \tag{5.21}$$

Set

$$B = B_1 + \frac{A}{g^{d+\delta_1}}.$$

Then, inequality (5.18), (5.19) and (5.21) give us

$$\sum_{\{x,y\} \subset \eta} V(x - y) \geq \frac{A}{g^{d+\delta_1}} \sum_{i \in \Lambda} |\eta_i|^2 - B|\eta|. \tag{5.22}$$

Because of Cauchy inequality

$$\sum_{i \in \Lambda} |\eta_i|^2 \geq \frac{1}{\sum_{i \in \Lambda} 1} \left(\sum_{i \in \Lambda} |\eta_i| \right)^2 = \frac{g^d |\eta|^2}{|\Lambda|}$$

the first statement of Lemma 5.2.1 is a direct consequence of (5.22).   ∎

The Hamiltonian $E_\Lambda^V : \Gamma_\Lambda \to \mathbb{R}$ for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ which corresponds to potential V is defined by

$$E_\Lambda^V(\eta) = \sum_{\{x,y\} \subset \eta} V(x - y), \ \eta \in \Gamma_\Lambda, \ |\eta| \geq 2.$$

For fixed V we will write for short $E_\Lambda = E_\Lambda^V$.

Consider a subset of the configuration space Γ:

$$\Gamma^t = \bigcup_{n=1}^{\infty} \Gamma_n^t,$$

where $\Gamma_n^t = \{ \gamma \in \Gamma \mid |\gamma_i| \leq n(\ln_+ |i|)^{\frac{1}{2}}, \ \forall i \in \mathbb{Z}^d \}$.

Definition 5.2.1 *Configuration $\gamma \in \Gamma$ is said to be tempered if $\gamma \in \Gamma^t$.*

For given $\bar{\gamma} \in \Gamma^t$ define the interaction energy between $\eta \in \Gamma_\Lambda$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \backslash \Lambda$ as

$$W_\Lambda(\eta|\bar{\gamma}) = \sum_{x \in \eta, \ y \in \bar{\gamma}_{\Lambda^c}} V(x - y).$$

The interaction energy is said to be well defined if for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma^t$ it is finite or $+\infty$.

Define

$$E_\Lambda(\eta|\bar{\gamma}) = E_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma})$$

and

$$Z_\Lambda(\bar{\gamma}) := \int_{\Gamma_\Lambda} \exp\{-E_\Lambda(\eta|\ \bar{\gamma})\}\lambda_z(d\eta)$$

the so-called partition function.

Lemma 5.2.2 *Let conditions* **(V1)**-**(V4)** *be fulfilled. Then for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma^t$ the interaction energy $W_\Lambda(\eta|\bar{\gamma})$ is well defined and partition function $Z_\Lambda(\bar{\gamma})$ is finite.*

Proof. Using representation $\Gamma_\Lambda := \bigsqcup_{N \in \mathbb{N}_0} \Gamma_\Lambda^{(N)}$ we have

$$Z_\Lambda(\bar{\gamma}) = \int_{\Gamma_\Lambda} e^{-E_\Lambda(\eta|\bar{\gamma})}\lambda_z(d\eta) = \sum_{N=0}^{\infty} \int_{\Gamma_\Lambda^{(N)}} e^{-E_\Lambda(\eta|\bar{\gamma})}\lambda_z(d\eta).$$

With **(V3)** and Lemma 5.2.1, it is not difficult to show that there exists $B \geq 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any $\eta \in \Gamma_\Lambda$, $\eta \geq 2$ holds

$$\sum_{\{x,y\} \subset \eta} V(x - y) \geq -B|\eta|.$$

In particular, potential V is bounded from below by $-2B$.

Define $\Lambda_{\mathcal{J}}$ as a union of all cubes Q_i which have nonempty intersection with Λ. Then, according to Lemma 5.2.1, the interaction energy can be estimated by

$$W_\Lambda(\eta|\bar{\gamma}) \geq -2B|\eta|\,|\bar{\gamma}_{\Lambda_{\mathcal{J}}\backslash\Lambda}| - K \sum_{i \in \Lambda} \sum_{j \in \mathbb{Z}^d \backslash \Lambda_{\mathcal{J}}} \frac{|\eta_i|\,|\bar{\gamma}_j|}{(g|i - j|)^{d+\delta_2}} \geq$$

$$\geq -|\eta| \left(2B|\bar{\gamma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}| + K \max_{i\in\Lambda} \sum_{j\in\mathbb{Z}^d\setminus\Lambda_{\mathcal{J}}} \frac{|\bar{\gamma}_j|}{(g|i-j|)^{d+\delta_2}} \right). \tag{5.23}$$

Let i_0 maximize the sum in (5.23). Then

$$W_\Lambda(\eta|\bar{\gamma}) \geq -|\eta| \left(2B|\bar{\gamma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}| + K \sum_{j\in\mathbb{Z}^d\setminus\Lambda_{\mathcal{J}}} \frac{|\bar{\gamma}_j|}{(g|i_0-j|)^{d+\delta_2}} \right).$$

Since $\bar{\gamma} \in \Gamma^t$, the series

$$S := \sum_{j\in\mathbb{Z}^d\setminus\Lambda_{\mathcal{J}}} \frac{|\bar{\gamma}_j|}{|i_0-j|^{d+\delta_2}}$$

is finite.

Therefore, the interaction energy is well defined. Moreover, the partition function can be estimated by

$$\exp\left\{ z|\Lambda|e^{B\left[1+2|\bar{\gamma}_{\Lambda_{\mathcal{J}}\setminus\Lambda}|\right]+KS} \right\} < \infty. \ \blacksquare$$

 In the following we will consider only tempered configurations.

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\gamma} \in \Gamma^t$. The finite volume Gibbs state on the space Γ_Λ with boundary configuration $\bar{\gamma}$ is defined by

$$P_{\Lambda,\bar{\gamma}}(d\eta) = \frac{\exp\left\{-E_\Lambda(\eta|\ \bar{\gamma})\right\}}{Z_\Lambda(\bar{\gamma})} \lambda_z(d\eta).$$

When $\bar{\gamma} = \emptyset$, let $P_{\Lambda,\emptyset}(d\eta) =: P_\Lambda(d\eta)$.

Let $\{\pi_\Lambda\}$ denote the specification associated with z and the Hamiltonian E (see [69]) which is defined by

$$\pi_\Lambda(A|\ \bar{\gamma}) = \int_{A'} P_{\Lambda,\bar{\gamma}}(d\eta)$$

where $A' = \{\eta \in \Gamma_\Lambda :\ \eta \cup (\bar{\gamma}_{\Lambda^c}) \in A\}$, $A \in \mathcal{B}(\Gamma)$ and $\bar{\gamma} \in \Gamma^t$.

A probability measure μ on Γ is called a Gibbs state for E and z if

$$\mu(\pi_\Lambda(A|\ \bar{\gamma})) = \mu(A)$$

for every $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is well known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [27] for more details.

Definition 5.2.2 *A measure μ on Γ is called tempered if μ is supported by the set Γ^t.*

The class of all tempered Gibbs states we denote by $\mathcal{G}_t(V, z)$.

5.2.2 Main results

As in Section 3.1 we consider a function $d_\alpha : \Gamma_\Lambda \times \Gamma_\Lambda \to \mathbb{R}_+$ which is defined by

$$d_\alpha(\eta_1, \eta_2) = \rho^\Lambda(\eta_1, \eta_2) + |E^\alpha(\eta_1) - E^\alpha(\eta_2)|, \quad \eta_1, \eta_2 \in \Gamma_\Lambda. \tag{5.24}$$

It is not difficult to show (see e.g. Chapter 3) that the function d_α is a metric on Γ_Λ, and, if Λ is a closed set, then the metric space $(\Gamma_\Lambda, d_\alpha)$ is a Polish space. Moreover, for any $C > 0$ and any closed set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the set

$$\{\eta \in \Gamma_\Lambda \mid E^\alpha(\eta) \leq C\}$$

is a relatively compact in $(\Gamma_\Lambda, d_\alpha)$. As a consequence, for any closed set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ function

$$h_\alpha(\eta) = e^{E^\alpha(\eta)} \tag{5.25}$$

is a compact function on Γ_Λ, i.e., for any $C > 0$

$$\{\eta \in \Gamma_\Lambda \mid h_\alpha(\eta) \leq C\}$$

is compact in $(\Gamma_\Lambda, d_\alpha)$.

We introduce an additional condition on the function α:

(V5) There exist $A > 0$ (independent of g) and $B \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$,

$$\sum_{\{x, y\} \subset \eta} V(x - y) - \sum_{\{x, y\} \subset \eta} \alpha(|x - y|) \geq A(\alpha, g, \Lambda)|\eta|^2 - B|\eta|,$$

where

$$A(\alpha, g, \Lambda) = A \frac{1}{g^{\delta_1}|\Lambda|} - \frac{\alpha_+}{2}.$$

Remark 5.2.1 *Condition* **(V5)** *holds, e.g., for*

$$\alpha(|x|) = \varepsilon C_1 \mathbb{1}_{[0, d_1]}(|x|) \left(\frac{1}{|x|^{d+\delta_1}} - \frac{1}{d_1^{d+\delta_1}} \right) + \frac{\varepsilon C_1}{d_1^{d+\delta_1}}. \tag{5.26}$$

Indeed, function α is continuous decreasing and $\alpha_0 = +\infty$. Because $d_1 \leq \varepsilon C_1$

$$\alpha_+ = \frac{\varepsilon C_1}{d_1^{d+\delta_1}} \geq 1.$$

Set as in (5.17)

$$\bar{\alpha}(|x|) = \varepsilon C_1 \mathbb{1}_{[0,\,d_1]}(|x|) \left(\frac{1}{|x|^{d+\delta_1}} - \frac{1}{d_1^{d+\delta_1}} \right).$$

As it was shown in Lemma 5.2.1, potential $V(x) - \bar{\alpha}(|x|)$ is of (DFR) type and hence there exist $A > 0$ (independent of g), and $B \geq 0$ such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and any $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$ holds

$$\sum_{\{x,y\} \subset \eta} (V(x-y) - \bar{\alpha}(|x-y|)) \geq A \frac{|\eta|^2}{g^{\delta_1}|\Lambda|} - B|\eta|. \tag{5.27}$$

Because $\alpha(|x|) = \bar{\alpha}(|x|) + \alpha_+$, bound (5.27) implies **(V5)**.   ∎

In the sequel we will consider function α of the form (5.26) which is constructed by the potential V.

We choose the size g of cubes Q_i, $i \in \mathbb{Z}^d$ small enough, such that the following properties hold

$$A(\alpha, g) = \frac{A}{g^{d+\delta_1}} - \frac{\alpha_+}{2} > 0, \tag{5.28}$$

$$K < \frac{1}{2a} \min \left\{ A g^{\delta_2 - \delta_1}, \left[\frac{\varepsilon C_1 A(\alpha, g) g^{d+\delta_2}}{2} \right]^{\frac{1}{2}} \right\}, \tag{5.29}$$

where

$$a := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|j|^{d+\delta_2}} < \infty$$

and A is the constant from **(V5)**. This can be done, because $\delta_1 > \delta_2$ and

$$A(\alpha, g) g^{d+\delta_2} = A g^{\delta_2 - \delta_1} - \frac{\alpha_+ g^{d+\delta_2}}{2} \to +\infty, \quad g \to 0. \tag{5.30}$$

The class of all measures $\mu \in \mathcal{G}_t(V, z)$ which satisfy

$$\int_\Gamma h_\alpha(\gamma_\Lambda) \mu(d\gamma) < \infty, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \tag{5.31}$$

we denote by $\mathcal{G}_t^\alpha(V, z)$.

Theorem 5.2.1 *Let conditions* **(V1)**-**(V4)** *be satisfied. Then for any* $z > 0$

$$\mathcal{G}_t^{\alpha}(V, z) \neq \emptyset.$$

The proof of this theorem is based on a general Dobrushin's existence criterion for lattice models (see [22]).

As in Section 5.1 we need to introduce a lattice structure associated with our continuous system, c.f. [67]. The construction of such lattice structure is very close to the construction occurred in the previous section. For the readers convenience, we repeat it with a necessary modification in the case of the space Γ.

For any $t \in \mathbb{Z}^d$ let us denote by $\mathbf{X}_t$ the configuration space $\Gamma_{\bar{Q}_t}$ in the closure $\bar{Q}_t$ of the cube Q_t, see (5.2). The space $\Gamma_{\bar{Q}_t}$ is endowed with the metric d_{α} (see (5.24)). Set

$$\mathcal{X} = \times_{t \in \mathbb{Z}^d} \mathbf{X}_t$$

the corresponding lattice configuration space. Having a configuration $\gamma \in \Gamma$ we construct a lattice configuration $\sigma = (\sigma(t))_{t \in \mathbb{Z}^d} \in \mathcal{X}$ in the following way. Set

$$\sigma(t) = \gamma \cap Q_t, \quad t \in \mathbb{Z}^d.$$

Denote this correspondence by $T : \Gamma \to \mathcal{X}$. For $\sigma = T\gamma$ we have $\sigma(t) \subset Q_t \subset \bar{Q}_t, \ t \in \mathbb{Z}^d$. Therefore,

$$T(\Gamma) \subset \mathcal{X}$$

and T is an injective map. The inverse map T^{-1} can be constructed as follows. If $\sigma \in T(\Gamma)$ then $T^{-1}\sigma := \gamma$ is defined by

$$\gamma = \bigcup_{t \in \mathbb{Z}^d} \sigma(t). \tag{5.32}$$

Because $\sigma \in T(\Gamma)$, configurations $\sigma(t)$ and $\sigma(s)$ do not intersect for $t \neq s$. The map T is a measurable embedding Γ into $\mathcal{X}$ and a bijection between Γ and $T(\Gamma) = \times_{k \in \mathbb{Z}^d} \Gamma_{Q_k}$.

Using constructed in such a way associated lattice structure and the analogous structure corresponding to the space $\ddot{\Gamma}$ in the previous section, one can maintain that every measure on Γ induces a measure on $\mathcal{X}$ and vice versa, every measure on $T(\Gamma)$ (correspondingly on $\mathcal{X}$) induces a measure on Γ (correspondingly on $\ddot{\Gamma}$).

For any $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, $\xi \in T(\Gamma_{\cup_{t \in \Lambda} Q_t})$ and $\bar{\sigma} \in T(\Gamma^t)$ the conditional energy $E_\Lambda(\xi \,|\, \bar{\sigma})$ is defined as

$$E_\Lambda(\xi \,|\, \bar{\sigma}) = E_\Lambda((T^{-1}\xi)_{\cup_{t \in \Lambda} Q_t} \,|\, (T^{-1}\bar{\sigma})_{\cup_{t \in \Lambda^c} Q_t}),$$

where $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$ and $(T^{-1}\sigma)_G$ is the restriction of the $T^{-1}\sigma \in \Gamma$ to the set $G \subset \mathbb{R}^d$.

Using Lemma 5.2.2 one can define finite volume Gibbs states for the lattice counterpart of the continuous model. Namely, for any $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ the finite volume Gibbs state $P_{\Lambda, \bar{\sigma}}$ under condition $\bar{\sigma} \in T(\Gamma^t)$ is given on $\times_{t \in \Lambda} \mathbf{X}_t$ by

$$P_{\Lambda, \bar{\sigma}}(d\xi_\Lambda) = \frac{\exp\left\{-E_\Lambda(\xi \,|\, \bar{\sigma})\right\}}{Z_\Lambda(\bar{\sigma})} d\lambda_z(\xi_\Lambda) = \frac{\exp\left\{-E_{\cup_{t \in \Lambda} Q_t}(\eta \,|\, \bar{\gamma})\right\}}{Z_\Lambda(\bar{\sigma})} d\lambda_z(\eta),$$

where

$$Z_\Lambda(\bar{\sigma}) = \int_{\Gamma_{\cup_{t \in \Lambda} Q_t}} e^{-E_{\cup_{t \in \Lambda} Q_t}(\eta \,|\, \bar{\gamma})} \lambda_z(d\eta) = Z_{\cup_{t \in \Lambda} Q_t}(\bar{\gamma}),$$

$(T^{-1}\xi)_{\cup_{t \in \Lambda} Q_t} = \eta$ and $T^{-1}\bar{\sigma} = \bar{\gamma}$.

The corresponding specifications are defined by

$$\pi_\Lambda(A| \,\bar{\sigma}) = \int_{A'} P_{\Lambda, \bar{\sigma}}(d\xi)$$

where $A' = \{\xi \in \times_{t \in \Lambda} \mathbf{X}_t : \ \xi \times \bar{\sigma}_{\mathbb{Z}^d \setminus \Lambda} \in A\}$, $A \in \mathcal{B}(\mathcal{X})$ and $\bar{\sigma}_{\mathbb{Z}^d \setminus \Lambda}$ is projection of $\bar{\sigma} \in T(\Gamma^t)$ on $\times_{t \in \mathbb{Z}^d \setminus \Lambda} \mathbf{X}_t$.

A probability measure μ on $\mathcal{X}$ is called a Gibbs state for E, z if

$$\mu(\pi_\Lambda(A| \,\bar{\sigma})) = \mu(A)$$

for every $A \in \mathcal{B}(\mathcal{X})$ and every $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$.

This relation is well known $Dobrushin - Lanford - Ruelle$ (DLR) equation. For more details, see [6], [67]

As in previous section, in this section we will need only single point Gibbs states, i.e. $\{P_{t, \bar{\sigma}} \,|\, t \in \mathbb{Z}^d, \ \bar{\sigma} \in T(\Gamma^t)\}$. Because, all spaces $\mathbf{X}_t$, $t \in \mathbb{Z}^d$ are isomorphic to the space $\mathbf{X}_0$, which we will denote for short by $\mathbf{X}$ we will consider for simplicity $\{P_{t, \bar{\sigma}} \,|\, t \in \mathbb{Z}^d, \ \bar{\sigma} \in T(\Gamma^t)\}$ on $\mathbf{X}$.

Suppose that Dobrushin's conditions for the lattice model with compact function

$$h_\alpha(\eta) = e^{E^\alpha(\eta)}$$

on $\mathbf{X}$ are fulfilled. Then, there exists Gibbs measure on $\mathcal{X}$ (see [22], [79]) and, hence, Gibbs measure μ on $\ddot{\Gamma}$. Using this measure later will be reconstructed tempered Gibbs measure on Γ^t. Thus, the existence of the Gibbs state for the lattice model implies the existence of the Gibbs state on the configuration space Γ, i.e., for the continuous one.

5.2.3 Proof

In this subsection we check Dobrushin's conditions for the lattice model with compact function

$$h_\alpha(\eta) = e^{E^\alpha(\eta)}$$

on $\mathbf{X}$ under assumptions **(V1)-(V4)**.

Lemma 5.2.3 *For any $t \in \mathbb{Z}^d$ and $\bar\gamma \in T(\Gamma^t)$, there exist $C > 0$ (independent of t) and $c_j \geq 0$, $j \in \mathbb{Z}^d \setminus \{0\}$, such that*

$$\sum_{j \in \mathbb{Z}^d \setminus \{0\}} c_j < 1,$$

$$\int_X h_\alpha(\eta) P_{t,\bar\gamma}(d\eta) \leq C + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{j-t} h_\alpha(\bar\gamma_j). \tag{5.33}$$

Proof. Using representation (2.2) we have

$$\int_{\mathbf{X}} h_\alpha(\eta) P_{t,\bar\gamma}(d\eta) = \tag{5.34}$$

$$= \sum_{N \leq N_{\bar\gamma}} \int_{\mathbf{X}^N} h_\alpha(\eta) P_{t,\bar\gamma}(d\eta) + \sum_{N > N_{\bar\gamma}} \int_{\mathbf{X}^N} h_\alpha(\eta) P_{t,\bar\gamma}(d\eta). \tag{5.35}$$

The number $N_{\bar\gamma}$ will be chosen later. First let us estimate the second term of (5.35)

$$I_{N_{\bar\gamma}} \equiv \sum_{N > N_{\bar\gamma}} \int_{\mathbf{X}^N} h_\alpha(\eta) P_{t,\bar\gamma}(d\eta) \leq \sum_{N > N_{\bar\gamma}} \int_{\mathbf{X}^N} h_\alpha(\eta) e^{-E(\eta|\bar\gamma)} \lambda_z(d\eta) =$$

$$= \sum_{N > N_{\bar\gamma}} \frac{z^N}{N!} \exp\left\{ \sum_{\{x,y\} \subset \eta} \alpha(|x - y|) - E(\eta) \right\} e^{-W(\eta|\bar\gamma)} \lambda_z(d\eta) \leq$$

$$\leq \sum_{N>N_{\bar{\gamma}}} \frac{z^N}{N!} e^{BN} \exp\left\{ N\left(-A(\alpha,g)N + K \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \right) \right\}.$$

In the first inequality we have used fact that the partition function $Z_{t,\bar{\gamma}}$ is greater than 1 and in the last one property **(V5)** and the second statement of Lemma 5.2.1.

Suppose that $\bar{\gamma} \in T(\Gamma_n^t)$, for some $n \in \mathbb{N}$. Then using fact that $A(\alpha,g) > 0$ (because of (5.28)) we have

$$\frac{K}{A(\alpha,g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \leq \frac{K}{A(\alpha,g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{n(\ln_+ |j|)^{\frac{1}{2}}}{(g|t-j|)^{d+\delta_2}} < \infty. \quad (5.36)$$

Choosing $N_{\bar{\gamma}}$ as the largest integer less or equal then

$$\frac{K}{A(\alpha,g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \quad (5.37)$$

we have for all $N > N_{\bar{\gamma}}$

$$A(\alpha,g)N \geq K \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}}.$$

This implies

$$I_{N_{\bar{\gamma}}} \leq \sum_{N>N_{\bar{\gamma}}} \frac{z^N}{N!} e^{BN} \leq \exp\left\{ze^B\right\} - 1.$$

Doing the same as for $I_{N_{\bar{\gamma}}}$ we are able to estimate the first term of (5.35) in the following way

$$J_{N_{\bar{\gamma}}} \equiv \sum_{N\leq N_{\bar{\gamma}}} \int_{\mathbf{X}^N} h_\alpha(\eta) P_{t,\bar{\gamma}}(d\eta) \leq$$

$$\leq \sum_{N\leq N_{\bar{\gamma}}} \frac{z^N}{N!} e^{BN} \exp\left\{ N\left(-A(\alpha,g)N + K \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \right) \right\}. \quad (5.38)$$

Because $A(\alpha,g) > 0$, the expression (5.38) can be estimated by

$$e^{ze^B} \exp\left\{ KN_{\bar{\gamma}} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \right\} \leq$$

$$\leq e^{ze^B} \exp\left\{ \frac{K^2}{A(\alpha,g)} \left(\sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{(g|t-j|)^{d+\delta_2}} \right)^2 \right\}. \tag{5.39}$$

Denote

$$C(g) := \frac{(Ka)^2}{g^{2(d+\delta_2)}A(\alpha,g)}.$$

Then using Cauchy inequality we bound (5.39) by

$$e^{ze^B} \exp\left\{ \frac{C(g)}{a} \left(\sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|^2}{|t-j|^{d+\delta_2}} \right) \right\} =$$

$$= e^{ze^B} \exp\left\{ \frac{C(g)}{a} \left(\sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|=1} \frac{|\bar{\gamma}_j|^2}{|t-j|^{d+\delta_2}} + \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{|\bar{\gamma}_j|^2}{|t-j|^{d+\delta_2}} \right) \right\} \leq$$

$$\leq e^{ze^B+C(g)} \exp\left\{ \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} C(g)|\bar{\gamma}_j|^2 \right\},$$

where

$$\tilde{a} := \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{1}{|t-j|^{d+\delta_2}} \leq a < \infty.$$

By the convexity of the function e^x we obtain

$$J_{N_{\bar{\gamma}}} \leq e^{ze^B+C(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} \exp\left\{ C(g)|\bar{\gamma}_j|^2 \right\}.$$

Let $T > 0$ be arbitrary positive number. Using inequality

$$\prod_{i=1}^{n} x_i \leq \frac{1}{n} \sum_{i=1}^{n} x_i^n \tag{5.40}$$

with $n = 2$ we obtain

$$J_{N_{\bar{\gamma}}} \leq e^{ze^B+C(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{e^T}{\tilde{a}|t-j|^{d+\delta_2}} \exp\left\{ C(g)|\bar{\gamma}_j|^2 - T \right\} \leq$$

$$\leq \frac{1}{2} e^{ze^B+C(g)+2T} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\geq 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} +$$

$$+ \frac{1}{2} \, e^{ze^B + C(g) - 2T} \sum_{j \in \mathbb{Z}^d \setminus \{t\} : |\bar{\gamma}_j| \geq 2} \frac{1}{\tilde{a}|t - j|^{d + \delta_2}} \exp\left\{ 2C(g)|\bar{\gamma}_j|^2 \right\} =$$

$$= \frac{1}{2} \, e^{ze^B + C(g) + 2T} + \frac{1}{2} e^{ze^B + C(g) - 2T} \sum_{j \in \mathbb{Z}^d \setminus \{t\} : |\bar{\gamma}_j| \geq 2} \frac{1}{\tilde{a}|t - j|^{d + \delta_2}} \exp\left\{ 2C(g)|\bar{\gamma}_j|^2 \right\}.$$

And again because of the convexity of the function e^x and the fact that

$$D(g) := \frac{8C(g)g^{d + \delta_2}}{\varepsilon C_1} < 1$$

which follows from (5.29) finally we have

$$J_{N_{\bar{\gamma}}} \leq \frac{1}{2} \, e^{ze^B + C(g) + 2T} + \frac{1}{2} \, e^{ze^B + C(g) - 2T} \times$$

$$\times \sum_{j \in \mathbb{Z}^d \setminus \{t\} : |\bar{\gamma}_j| \geq 2} \frac{1}{\tilde{a}|t - j|^{d + \delta_2}} \exp\left\{ (1 - D(g))0 + D(g)\frac{\varepsilon C_1 |\bar{\gamma}_j|^2}{4g^{d + \delta_2}} \right\} \leq$$

$$\leq \frac{1}{2} \, e^{ze^B + C(g) + 2T} + \frac{1}{2} \, (1 - D(g)) e^{ze^B + C(g) - 2T} +$$

$$+ \frac{1}{2} \, \exp\left\{ ze^B + C(g) - 2T \right\} D(g) \sum_{j \in \mathbb{Z}^d \setminus \{t\} : |\bar{\gamma}_j| \geq 2} \frac{1}{\tilde{a}|t - j|^{d + \delta_2}} \, e^{\frac{\varepsilon C_1 |\bar{\gamma}_j|^2}{4g^{d + \delta_2}}}.$$

Choosing $T = (ze^B + C(g))/2$ we have

$$J_{N_{\bar{\gamma}}} \leq C^\star + \sum_{j \in \mathbb{Z}^d \setminus \{t\} : |\bar{\gamma}_j| \geq 2} c^\star_{t-j} \, e^{\frac{\varepsilon C_1 |\bar{\gamma}_j|^2}{4g^{d + \delta_2}}},$$

where

$$C^\star = \frac{1}{2} \, (1 + e^{2(ze^B + C(g))} - D(g)); \quad c^\star_j = \frac{D(g)}{2\tilde{a}|j|^{d + \delta_2}}, \, j \in \mathbb{Z}^d \setminus \{0\}.$$

Using the fact that $\delta_1 > \delta_2$, for $|\bar{\gamma}_j| \geq 2$ we have

$$\frac{\varepsilon C_1 |\bar{\gamma}_j|^2}{4g^{d + \delta_2}} \leq \frac{\varepsilon C_1 |\bar{\gamma}_j|^2}{4g^{d + \delta_1}} \leq \frac{\alpha(g)|\bar{\gamma}_j|(|\bar{\gamma}_j| - 1)}{2} \leq \sum_{\{x, y\} \subset \bar{\gamma}_j} \alpha(|x - y|).$$

And now

$$\int_{\mathbf{X}} h_\alpha(\eta) P_{t,\bar\gamma}(d\eta) \leq C + \sum_{j\in\mathbb{Z}^d\setminus\{t\}} c_{t-j} h_\alpha(\bar\gamma_j),$$

where

$$C = \exp\{ze^B\} + C^\star - 1, \quad c_j = \begin{cases} c_j^\star, & \text{if } |\bar\gamma_j| \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

From (5.29) follows that $\sum_{j\in\mathbb{Z}^d\setminus\{0\}} c_j < 1$. ∎

The following two lemmas maintain the same results as lemma 5.1.3 and 5.1.4 but under conditions on potentials which are considered in the Section 5.2.

Lemma 5.2.4 *For any $\chi > 0$, $t \in \mathbb{Z}^d$ and $\bar\gamma \in T(\Gamma^t)$ there exist $C > 0$ (independent of g) and $c_j \geq 0$, $j \in \mathbb{Z}^d \setminus \{0\}$, such that*

$$\sum_{j\in\mathbb{Z}^d\setminus\{0\}:|\bar\gamma_j|\geq 2} c_j < 1$$

and

$$\int_X e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta) \leq C + \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar\gamma_j|\geq 2} c_{j-t} e^{\chi|\bar\gamma_j|}. \tag{5.41}$$

Proof. Similar to the proof of the Lemma 5.2.3 one can show that

$$\int_{\mathbf{X}} e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta) = \sum_{N\leq N_{\bar\gamma}} \int_{\mathbf{X}^N} e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta) + \sum_{N>N_{\bar\gamma}} \int_{\mathbf{X}^N} e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta), \tag{5.42}$$

and

$$I_{N_{\bar\gamma}} := \sum_{N>N_{\bar\gamma}} \int_{\mathbf{X}^N} e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta) \leq \exp\{ze^{\chi+B}\} - 1,$$

where $N_{\bar\gamma}$ is the largest integer less or equal then

$$\frac{K}{A(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar\gamma_j|}{|t-j|^{d+\delta_2}}$$

with $A(g) := Ag^{\delta_2-\delta_1}$.

To estimate the first term in (5.42) let us observe that

$$J_{N_{\bar\gamma}} := \sum_{N\leq N_{\bar\gamma}} \int_{\mathbf{X}^N} e^{\chi|\eta|} P_{t,\bar\gamma}(d\eta) = \sum_{N\leq N_{\bar\gamma}} e^{\chi N} P_{t,\bar\gamma}(\mathbf{X}^N) \leq e^{\chi N_{\bar\gamma}}.$$

Hence

$$J_{N_{\bar{\gamma}}} \le \exp\left\{ \chi\left(\frac{K}{A(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}} \frac{|\bar{\gamma}_j|}{|t-j|^{d+\delta_2}} \right) \right\} = \qquad (5.43)$$

$$= \exp\left\{ \chi\frac{K}{A(g)} \left(\sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|=1} \frac{|\bar{\gamma}_j|}{|t-j|^{d+\delta_2}} + \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{|\bar{\gamma}_j|}{|t-j|^{d+\delta_2}} \right) \right\} \le$$

$$\le e^{\chi a\frac{K}{A(g)}} \exp\left\{ \chi\frac{K}{A(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{|\bar{\gamma}_j|}{|t-j|^{d+\delta_2}} \right\}.$$

By the convexity of the function e^x we obtain

$$J_{N_{\bar{\gamma}}} \le e^{\chi a\frac{K}{A(g)}} \exp\left\{ \frac{\chi Ka}{A(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{|\bar{\gamma}_j|}{\tilde{a}|t-j|^{d+\delta_2}} \right\} \le$$

$$\le e^{\chi a\frac{K}{A(g)}} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} e^{\frac{\chi Ka|\bar{\gamma}_j|}{A(g)}}.$$

Let $T > 0$ be some positive number. Using inequality (5.40) with $n = 2$, analogously to the proof of the Lemma 5.2.3 we obtain

$$J_{N_{\bar{\gamma}}} \le \frac{1}{2}\, e^{\chi a\frac{K}{A(g)}+2T} + \frac{1}{2}e^{\chi a\frac{K}{A(g)}-2T} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} \exp\left\{ \frac{2\chi Ka|\bar{\gamma}_j|}{A(g)} \right\}.$$

And again because of the convexity of e^x and the fact that $\frac{2aK}{A(g)} < 1$ which follows from (5.29) we have

$$J_{N_{\bar{\gamma}}} \le \frac{1}{2} e^{\chi a\frac{K}{A(g)}+2T} + \frac{1}{2}e^{\chi a\frac{K}{A(g)}-2T} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{1}{\tilde{a}|t-j|^{d+\delta_2}} e^{(1-\frac{2aK}{A(g)})0+\frac{2aK}{A(g)}\chi|\bar{\gamma}_j|} \le$$

$$\le \frac{1}{2}\, e^{\chi a\frac{K}{A(g)}+2T} + \frac{1}{2}\left(1 - \frac{2Ka}{A(g)} \right) e^{\chi a\frac{K}{A(g)}-2T} +$$

$$+\frac{1}{2}\, \exp\left\{ \chi a\frac{K}{A(g)} - 2T \right\} \frac{2aK}{A(g)} \sum_{j\in\mathbb{Z}^d\setminus\{t\}:|\bar{\gamma}_j|\ge 2} \frac{e^{\chi|\bar{\gamma}_j|}}{\tilde{a}|t-j|^{d+\delta_2}}.$$

Choosing $T = (\chi a \frac{K}{A(g)})/2$ we obtain

$$J_{N_{\bar{\gamma}}} \leq C^{\star} + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{t-j} e^{\chi |\bar{\gamma}_j|},$$

where

$$C^{\star} = \frac{1}{2} e^{2\chi a \frac{K}{A(g)}} + \frac{1}{2}\left(1 - \frac{2Ka}{A(g)}\right); \quad c_j = \begin{cases} \frac{Ka}{A(g)\tilde{a}|j|^{d+\varepsilon}}, & \text{if } |\bar{\gamma}_j| \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Finally,

$$\int_{\mathbf{X}} e^{\chi |\eta|} P_{t,\bar{\gamma}}(d\eta) \leq C + \sum_{j \in \mathbb{Z}^d \setminus \{t\}} c_{t-j} e^{\chi |\bar{\gamma}_j|},$$

where $C = C^{\star} + \exp\{ze^{\chi+B}\} - 1$.

The inequality (5.29) implies $\sum_{j \in \mathbb{Z}^d \setminus \{0\}} c_j < 1$.   ∎

Lemma 5.2.5 *For any δ, $0 < \delta < 1$ there exist bounded $\Lambda \subset \mathbb{Z}^d$ and constants δ_j, $j \in Z^d \setminus \{0\}$, such that*

$$\sum_{j \in Z^d \setminus \{0\}} \delta_j \leq \delta$$

and for any $\bar{\gamma} \in T(\Gamma^t)$ and measurable function $\varphi(\eta)$, $\eta \in \mathbf{X}$, $|\varphi(\eta)| \leq 1$ the following holds

$$\left| \int_{\mathbf{X}} \varphi(\eta) P_{t,\bar{\gamma}}(d\eta) - \int_{\mathbf{X}} \varphi(\eta) P_{t,\bar{\gamma} \cap \Lambda^*}(d\eta) \right| \leq \sum_{j \in Z^d \setminus \{t\}} \delta_{j-t} h_\alpha(\bar{\gamma}_j),$$

where $\Lambda^* = \bigcup_{t \in \Lambda} Q_t$.

Proof. Due to the Lemma 5.2.4 the proof of this Lemma is completely analogous to arguments which was occurred in Lemma 4 of [21] for constants $\chi = \frac{1}{8}$ and

$$d_t = \frac{K}{|t|^{d+\delta_2}}. \quad \blacksquare$$

Proof of Theorem 5.2.1. Because of the continuity of functions

$$f(\bar{\gamma}, \Lambda^*) := \int_{\mathbf{X}} \varphi(\eta) P_{t,\bar{\gamma} \cap \Lambda^*}(d\eta)$$

(see [21], [67]), Lemmas 5.2.2, 5.2.3, 5.2.4 and 5.2.5 (see [22] for details), there exists at least one Gibbs measure on $\mathcal{X}$ and hence measure μ on $\ddot{\Gamma}$. Moreover, as shown in [22], for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ there exists $C_\Lambda < \infty$ such that the following holds

$$\int_{\ddot{\Gamma}} h_\alpha(\sigma_\Lambda)\mu(d\sigma) < C_\Lambda. \tag{5.44}$$

Therefore, the measure μ is supported by

$$\{\sigma \in \ddot{\Gamma} \mid h_\alpha(\sigma_{\Lambda_k}) < \infty, \ k \geq 1\}. \tag{5.45}$$

It is not difficult to see that the set (5.45) is a subset of Γ, see e.g. Chapter 3.

For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma \in \Gamma$ such that $|\gamma_\Lambda| \geq 2$ we have

$$\frac{\alpha(l_\Lambda)|\gamma_\Lambda|^2}{4} \leq \frac{\alpha(l_\Lambda)|\gamma_\Lambda|(|\gamma_\Lambda| - 1)}{2} \leq \sum_{\{x, y\} \subset \gamma_\Lambda} \alpha(|x - y|)$$

and, hence, (5.44) gives us immediately bound

$$\int_\Gamma e^{\varrho|\gamma_\Lambda|^2}\mu(d\gamma) < C_\Lambda, \ \ 0 \leq \varrho \leq \frac{\alpha(l_\Lambda)}{4}, \ \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d), \tag{5.46}$$

where

$$l_\Lambda = \sup_{x, y \in \Lambda} |x - y|.$$

Moreover, as shown in Chapter 4, in this case μ satisfies Ruelle's probability type bound, i.e., there exist constants $\alpha > 0$ such that for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $l_\Lambda \geq g$ and $N \in \mathbb{N}_0$

$$\mu(\{\gamma \mid |\gamma_\Lambda| \geq N\}) \leq C_\Lambda \exp\left\{-\alpha\frac{N^2}{l_\Lambda^d}\right\}. \tag{5.47}$$

and, hence, supported by Γ^t. ∎

The next theorem shows that the existence result can be extended on the class of more general potentials.

Theorem 5.2.2 *Let conditions* **(V1)**, **(V2)**, **(V4)** *be satisfied, and additionally the following conditions are fulfilled:*

1. There exist constant $K > 0$ (independent of g) and $\delta > 0$ such that for any i, $j \in \mathbb{Z}^d$, $i \neq j$ and for any $x \in Q_i$, $y \in Q_j$

$$V(x - y) \geq -\frac{K}{(g|i - j|)^{d+\delta}}.$$

2. The function α satisfies:

- *There exist $A_g(\alpha) > 0$ and $B \geq 0$ (may be g dependent) such that for any $\Lambda \in \mathcal{J}_g(\mathbb{R}^d)$ and $\eta \in \Gamma_\Lambda$, $|\eta| \geq 2$*

$$\sum_{\{x,y\} \subset \eta} V(x - y) - \sum_{\{x,y\} \subset \eta} \alpha(|x - y|) \geq A_g(\alpha) \sum_{i \in \Lambda} |\eta_i|^2 - B|\eta|. \quad (5.48)$$

-
$$\lim_{g \to 0} \alpha(g) g^{2(d+\delta)} A_g(\alpha) = +\infty. \quad (5.49)$$

Then for any $z > 0$

$$\mathcal{G}_t^\alpha(V, z) \neq \emptyset.$$

Proof. The proof is analogous to the proof of the Theorem 5.2.1 which is based on Lemmas 5.2.2, 5.2.3 and 5.2.4. The fulfillment of the latter Lemmas is ensured by the conditions (5.48) and (5.49). ∎

5.3 The case of multibody interaction

5.3.1 Interactions and Hamiltonians

We consider a general type of many-body interaction specified by a family of k-body potentials $V_k : \mathbb{R}^{dk} \to \mathbb{R}$, $k \geq 2$. About the potentials $\{V_k\}_{k \geq 2}$ we will assume:

A1. Finite range. There exists a constant $R > 0$, such that for any $k \geq 2$

$$V_k(x_1, ..., x_k) \equiv 0, \text{ if } \operatorname{diam}\{x_1, ..., x_k\} > R.$$

A2. Continuity.

$$V_k \in C(\widetilde{(\mathbb{R}^d)^k}), \ k \geq 2.$$

A3. Symmetry. For any $k \geq 2$, any $(x_1, ..., x_k) \in (\mathbb{R}^d)^k$, and any permutation π of numbers $\{1, \ldots, k\}$

$$V_k(x_1, ..., x_k) = V_k(x_{\pi(1)}, ..., x_{\pi(k)}).$$

A4. Translation invariance. For any $k \geq 2$, any $(x_1, ..., x_k) \in (\mathbb{R}^d)^k$, and any $a \in \mathbb{R}^d$

$$V_k(x_1, ..., x_k) = V_k(x_1 + a, ..., x_k + a).$$

We are able now to introduce the Hamiltonian $U^V : \Gamma_0 \to \mathbb{R} \cup \{\infty\}$, which corresponds to the family of potentials $V := \{V_k\}_{k \geq 2}$ and which is defined by

$$U^V(\eta) = \sum_{k \geq 2} \sum_{\{x_1, ..., x_k\} \subset \eta} V_k(x_1, ..., x_k), \ \eta \in \Gamma_0, \ |\eta| \geq 2.$$

For the fixed family of potentials V we will write for short $U = U^V$ and for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ we will sometimes write $U_\Lambda(\eta)$ instead of $U(\eta)$.

A5. Strong Superstability. For any $k \geq 2$ the potential V_k can be represented as

$$V_k = V_k^+ + V_k^{(st)},$$

where V_k^+ is a nonnegative function such that for any $(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k \backslash \widetilde{(\mathbb{R}^d)^k}$

$$V_k^+(x_1, \ldots, x_k) = +\infty,$$

and $V_k^{(st)}$ is stable, i.e. there exists a constant $B \geq 0$ such that for any configuration $\eta \in \Gamma_0$ holds

$$U^{V^{(st)}}(\eta) \geq -B|\eta|.$$

Let $\lambda \in \mathbb{R}_+$ be arbitrary. For each $r \in \mathbb{Z}^d$ we define an elementary cube

$$\Delta(r) = \{x \in \mathbb{R}^d \mid \lambda(r^i - 1/2) \leq x^i < \lambda(r^i + 1/2)\}.$$

These cubes form a partition of $\mathbb{R}^d$, which we denote by $\bar{\Delta}_\lambda$. We will sometimes write Δ instead of $\Delta(r)$, if a cube Δ is considered to be arbitrary and there is no reason to emphasize that it is centered at the concrete point $r \in \mathbb{Z}^d$. As before, by $\mathcal{J}_\lambda(\mathbb{R}^d)$ we denote all finite unions of cubes of the form $\Delta(r)$ (such sets are used in the construction of the Jordan measure).

Let $N \in \mathbb{N}$ and $k \geq N + 1$ be arbitrary. For any $X_N = \cup_{j=1}^N \Delta_j \in \mathcal{J}_\lambda(\mathbb{R}^d)$ we define

$$I_k^{k_1, ..., k_N | \bar{k}} (\Delta_1, ..., \Delta_N) :=$$

$$= \sup_{\substack{(x)_{k_i}^i \subset \Delta_i, \, 1 \leq i \leq N}} \sum_{\substack{\Delta_j' \subset X_N^c, \\ 1 \leq j \leq \bar{k}}}^* \sup_{y_1 \in \Delta_1', ..., y_{\bar{k}} \in \Delta_{\bar{k}}'} |V_k^{(st), -}(x_1^1, \ldots, x_{k_N}^N, y_1, ..., y_{\bar{k}})|, \quad (5.50)$$

where $\bar{k} \geq 1$, $k_i \geq 1$, $i = 1, \ldots, N$ such that $k_1 + \cdots + k_N + \bar{k} = k$, and

$$v_k^{k_1,\ldots,k_N}\left(\Delta_1, \ldots, \Delta_N\right) := \inf_{(x)_{k_i}^i \subset \Delta_i, \, 1 \leq i \leq N,} V_k^+(x_1^1, \ldots, x_{k_N}^N), \tag{5.51}$$

where $k_i \in \mathbb{N}_0$, $(x)_{k_i}^i = \{x_1^i, \ldots, x_{k_i}^i\}$, $1 \leq i \leq N$ such that $k_1 + \cdots + k_N = k$. $V_k^{(st),-}$ denotes the negative part of $V_k^{(st)}$, and the symbol $\sum^*$ means that the sum extends only over different cubes, i.e. $\Delta_i' \neq \Delta_j'$, $i \neq j$, $1 \leq i, j \leq \bar{k}$.

A6. Attraction-Repulsion relation. There exists $\lambda = \lambda_0 > 0$, such that for any $N \in \mathbb{N}$ and any $X_N = \cup_{j=1}^N \Delta_j \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ (we omit dependence on the cubes in the notations of (5.50) and (5.51)) the following holds

- for an arbitrary $\Delta \in \bar{\Delta}_{\lambda_0}$ and any $k \geq 2$

$$V_k(x_1, \ldots, x_k) \geq 0, \quad \{x_1, \ldots x_k\} \subset \Delta$$

- for an arbitrary $k \geq N + 1$

$$v_k^{k_1,\ldots,k_N} \geq 4\bar{I}_{k;\,k_1,\ldots,k_N}^{(N)}, \quad v_{N+1}^{k_1,\ldots,k_N} \geq 4(\bar{I}_{N+1}^{(N)} + B), \tag{5.52}$$

and

$$\bar{I}_{k;\,k_1,\ldots,k_N}^{(N)} = \sum_{l \geq 1} l I_{k+l}^{k_1,\ldots,k_N|\,l} < \infty, \quad \bar{I}_{N+1}^{(N)} = \sum_{l \geq 1} l I_{N+l}^{1,\ldots,1|\,l} < \infty, \tag{5.53}$$

$$k_1 + \cdots + k_N = k.$$

In the sequel we write $\bar{\Delta}$ instead of $\bar{\Delta}_{\lambda_0}$.

Remark 5.3.1 *By the definition, $V_k^{st,-}$ describes attractive part of k-body interaction. Therefore, $I_k^{k_1,\ldots,k_N|\bar{k}}(\Delta_1, \ldots, \Delta_N)$ describes only attractive part of k-body interaction of fixed particles in cubes $\Delta_1, \ldots, \Delta_N$ with "dilute configuration", i.e. no more then one particle is located in any cube Δ from $X_N^c = \mathbb{R}^d \setminus X_N$, $X_N = \cup_{j=1}^N \Delta_j$. Then, condition (5.53) means that the energy of k-body interaction decreases sufficiently fast with k. From the assumption **A6** and the definition of $\bar{I}_{k;\,k_1,\ldots,k_N}^{(N)}$ therein, it is clear that at least one cube from $\Delta_1, \ldots, \Delta_N$ contains more than one particle, and so $v_k^{k_1,\ldots,k_N}$ should be greater than contributions of all $k + l$-body attractive energies of interaction ($l \in N$) for sufficiently small λ.*

Remark 5.3.2 *From the definition of $I_k^{k_1,...,k_N|\bar{k}}(\Delta_1,...,\Delta_N)$ (see (5.50)) it is clear that*

$$I_k^{k_1,...,k_N|\bar{k}}(\Delta_1,...,\Delta_N) \leq C_k \lambda^{-d\bar{k}}, \quad \lambda \to 0,$$

where $C_k = C_k(\lambda) \geq 0$ are some constants. Moreover, if $V_k^{(st),-}$ is bounded from below on $(\mathbb{R}^d)^k \backslash \widetilde{(\mathbb{R}^d)^k}$, then $C_k(\lambda)$ has the following limit at $\lambda \to 0$:

$$C_k(0) = \int_{(\mathbb{R}^d)^{\bar{k}}} |V_k^{(st),-}(x_1^1,\ldots,x_{k_N}^N, y_1,...,y_{\bar{k}})| dy_1 \ldots dy_{\bar{k}},$$

where $x_1^1, \ldots, x_{k_N}^N$ some fixed points in $\mathbb{R}^d$. For example, if we would have only pair potential, to satisfy (5.52) the positive part of the potential $V_2^+(x_1, x_2)$ should behave like $|x_1 - x_2|^{-d-\varepsilon}$, $|x_1 - x_2| \to 0$, for some $\varepsilon > 0$.

In the case of all orders of interactions, the k-body potentials, for $k \geq 3$, can be chosen in such a way that constants $C_k, k \geq 3$ have behavior like $C^k/k!$, for some constant $C > 0$. Under such condition, $I_k^{k_1,...,k_N|\bar{k}}(\Delta_1,...,\Delta_N)$ will behave like $\lambda^{-d} C^{k+1} e^{C\lambda^{-d}}/k!$. Therefore, to satisfy (5.52), the positive part of the potentials $V_k^+(x_1,\ldots,x_k)$ should behave like

$$|x_i - x_j|^{-d-\varepsilon} \frac{C^{k+1}}{k!} e^{C|x_i-x_j|^{-d-\varepsilon}}, \quad |x_i - x_j| \to 0, \quad 1 \leq i,j \leq k$$

for some $\varepsilon > 0$.

For a given $\bar{\gamma} \in \Gamma$ define the interaction energy between $\eta \in \Gamma_\Lambda$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \setminus \Lambda$ as

$$W_\Lambda(\eta \mid \bar{\gamma}) = \sum_{k \geq 2} \sum_{\substack{m+n=k \\ m,n \geq 1}} \sum_{\substack{\{x_1,...,x_m\} \subset \eta \\ \{y_1,...,y_n\} \subset \bar{\gamma}_{\Lambda^c}}} V_k(x_1,\ldots,x_m,y_1,\ldots,y_n).$$

Define

$$U_\Lambda(\eta|\bar{\gamma}) = U_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma}).$$

A7. The order of interaction. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma$ the interaction energy $W_\Lambda(\eta \mid \bar{\gamma})$ does not become $-\infty$ and the partition function

$$Z_\Lambda(\bar{\gamma}) = \int_{\Gamma_\Lambda} \exp\{-U_\Lambda(\eta| \bar{\gamma})\} \lambda_\sigma(d\eta) < \infty.$$

Remark 5.3.3 *Assumption **A7** is important only for the next section, where the precise definition of the Gibbs state on the configuration space Γ will be given. In fact, for the results of the present chapter we do not need fulfillment of **A7** for all $\bar{\gamma} \in \Gamma$, but only for empty boundary configurations. In turn, this fact is automatically ensured by assumption **A5**.*

5.3.2 Gibbs specifications and correlation functions.

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\gamma} \in \Gamma$. The finite volume Gibbs state with boundary configuration $\bar{\gamma}$ for U, $z > 0$ and $\beta > 0$ is

$$\mu_\Lambda(d\eta| \bar{\gamma}) = \frac{\exp\left\{-\beta U_\Lambda(\eta| \bar{\gamma})\right\}}{Z_\Lambda(\bar{\gamma})} \lambda_{z\sigma}(d\eta).$$

Under assumption **A7**, the finite volume Gibbs state is well defined. When $\bar{\gamma} = \emptyset$, let $\mu_\Lambda(d\eta|\emptyset) \equiv \mu_\Lambda(d\eta)$.

The corresponding finite-volume correlation functions for boundary configuration $\bar{\gamma} \in \Gamma$ have the following form

$$\rho^\Lambda(\eta \mid \bar{\gamma}) = \frac{1}{Z_\Lambda(\bar{\gamma})} \int_{\Gamma_\Lambda} e^{-\beta U(\eta \cup \gamma|\bar{\gamma})} \lambda_\sigma(d\gamma), \quad \eta \in \Gamma_\Lambda. \tag{5.54}$$

Let $\{\pi_\Lambda\}$ denote the specification associated with z, β and the Hamiltonian U (see [69]), which is defined on Γ by

$$\pi_\Lambda(A| \bar{\gamma}) = \int_{A'} \mu_\Lambda(d\eta| \bar{\gamma}),$$

where $A' = \{\eta \in \Gamma_\Lambda : \eta \cup (\bar{\gamma}_{\Lambda^c}) \in A\}$, $A \in \mathcal{B}(\Gamma)$.

A probability measure μ on Γ is called a Gibbs state for U, β and z if

$$\mu(\pi_\Lambda(A| \bar{\gamma})) = \mu(A)$$

for every $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is the well known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [27] for more details. The class of all Gibbs states which correspond to the specifications $\{\pi_\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$ we denote by $\mathcal{G}(V, z, \beta)$.

5.3.3   Main results.

Theorem 5.3.1 *Suppose that the interaction family V satisfies the assumptions **A1-A6**. Then, for any $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ and any $\beta, z \geq 0$ there exists a constant $\xi = \xi(\beta, z)$ (independent of Λ) such that the finite volume correlation function $\rho^\Lambda(\eta) = \rho^\Lambda(\eta \mid \emptyset)$ satisfies the following inequality*

$$\rho^\Lambda(\eta) \leq \xi^{|\eta|} e^{-\frac{1}{2} U^+(\eta)}, \ \eta \in \Gamma_\Lambda. \tag{5.55}$$

Remark 5.3.4 *The estimate (5.55) without exponent factor at the right-hand side is the well-known Ruelle bound [76]. We call (5.55) a generalized Ruelle bound. For 2-body interaction it was obtained in [2], [73].*

As a consequence of Theorem 5.3.1 the following theorem is fulfilled.

Theorem 5.3.2 *Let the interaction family V satisfy **A1-A6**. Then for any $z \geq 0$ and $\beta \geq 0$*

$$\mathcal{G}(V, z, \beta) \neq \emptyset.$$

Proof. Existence of the corresponding Gibbs state follows from the arguments which are based on the results of the Chapter 3. Let $\psi \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be any positive function such that $\psi(x) \leq 1$, $x \in \mathbb{R}^d$, and let $\alpha(t)$, $t \in \mathbb{R}_+$ be any continuous decreasing function with the following properties:

(1) $\alpha_0 := \lim_{t \to 0+} \alpha(t) = +\infty$;

(2) $\alpha_+ := \lim_{t \to +\infty} \alpha(t) \geq 1$;

As shown in Chapter 3, for any $0 < D < \infty$ the set

$$\left\{ \gamma \in \Gamma \mid |E^{\alpha, \psi}(\gamma)| \leq D \right\}$$

is relatively compact in Γ, which is Polish space. Let us remind that

$$\Gamma^{\alpha, \psi} = \left\{ \gamma \in \Gamma \ \middle| \ \sum_{\{x,y\} \subset \gamma} \psi(x) \alpha(|x - y|) \psi(y) < \infty \right\}$$

and

$$E^{\alpha, \psi}(\gamma) = \sum_{\{x,y\} \subset \gamma} \psi(x) \alpha(|x - y|) \psi(y), \ \gamma \in \Gamma^{\alpha, \psi}.$$

In this section we consider α as any continuous decreasing function such that

$$\alpha(|x - y|) \leq e^{\frac{1}{2}V_2^+(x,y)}.$$

Obviously, chosen in such a way, this function satisfies the conditions above. Using the properties of the so-called K-transform (see [38]) and the Theorem 5.3.1, for any $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ we have

$$\int_\Gamma E^{\alpha,\psi}(\gamma)d\mu_\Lambda(\gamma) = \int_\Lambda \int_\Lambda \psi(x)\alpha(|x - y|)\psi(y)\rho_\Lambda^{(2)}(\{x,y\})dxdy < C,$$

where $C \in \mathbb{R}_+$ is some constant.

Therefore, by Prokhorov theorem the family of measures

$$\{\mu_\Lambda \mid \Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)\}$$

is relatively compact, which implies the existence of at least one limit measure μ when $\Lambda \nearrow \mathbb{R}^d$. We will prove that corresponding limit measure is Gibbsian. Let μ_{Λ_n}, $n \geq 1$, where $\Lambda_n \nearrow \mathbb{R}^d$, $n \to \infty$ be the sequence which converges (in the sense of the Prokhorov theorem) to the measure μ, and let ρ^{Λ_n}, ρ be the corresponding correlation functions. It is well-known (see [27]) that probability measure μ on Γ is Gibbs, iff μ fulfills the *Georgii-Nguyen-Zessin* equation (GNZ), i.e. for all positive, $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Gamma)$ measurable functions H the following holds

$$\int_\Gamma \sum_{x\in\gamma} H(x,\gamma)\mu(d\gamma) = \int_\Gamma \int_{\mathbb{R}^d} H(x,\gamma \cup \{x\})e^{-\beta W(\{x\}|\gamma)}\sigma(dx)\mu(d\gamma). \quad (5.56)$$

Moreover, using Mecke formula (see [27]), one can show that (5.56) holds for any measure μ_{Λ_n}, $n \geq 1$.

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. The σ-algebra $\mathcal{B}(\Gamma)$ is generated by sets of the form $A \cap \tilde{A}$ with $A \in \mathcal{B}_\Lambda(\Gamma)$, $\tilde{A} \in \mathcal{B}_{\mathbb{R}^d \setminus \Lambda}(\Gamma)$ and every measure on Γ is uniquely determined by its values on these sets.

Let us prove (5.56) for the function $H(x,\gamma) = \mathbb{1}_\Lambda(x)\mathbb{1}_A(\gamma)\mathbb{1}_{\tilde{A}}(\gamma)$. Let $n \in \mathbb{N}$ be arbitrary. Using the properties of the K-transform (see [38]) we have

$$\int_{\Gamma_{\Lambda_n}} \sum_{x\in\gamma} \mathbb{1}_\Lambda(x)\mathbb{1}_A(\gamma)\mathbb{1}_{\tilde{A}}(\gamma)\mu_{\Lambda_n}(d\gamma) \leq \int_{\Gamma_{\Lambda_n}} \sum_{x\in\gamma} \mathbb{1}_\Lambda(x)\mu_{\Lambda_n}(d\gamma) =$$

$$= \int_{\Lambda} \rho^{\Lambda_n}(x)\sigma(dx) \leq z\xi|\Lambda|. \tag{5.57}$$

The right hand side of (5.56) for the measure μ_{Λ_n} is bounded by

$$\int_{\mathbb{R}^d} \mathbb{1}_{\Lambda}(x) \int_{\Gamma_{\Lambda_n}} e^{-\beta W(\{x\}|\gamma)}\mu_{\Lambda_n}(d\gamma)\sigma(dx) =$$

$$= \int_{\mathbb{R}^d} \mathbb{1}_{\Lambda}(x)\rho^{\Lambda_n}(x)\sigma(dx) \leq z\xi|\Lambda|, \tag{5.58}$$

where we have used the definition of the correlation function and Fubini theorem. Hence, there exists some subsequence $\{\mu_{\Lambda_{n_k}}\}_{k\geq 1}$ which ensures the fulfillment of (5.56) for the limit measure μ. The proof for the general positive function H follows from the fact that any positive measurable function can be approximated by the simple functions.

5.3.4   The proof of Theorem 5.3.1

The proof is based on the expansion of the Lebesgue-Poisson integral for the correlation functions (5.54) into the series over some kind of dense configurations (see [73] and definition (3.4) therein).

5.3.5   Cluster expansion in densities of configurations.

The main idea of the construction consists in the use of the fact that if two or more particles are in one elementary cube $\Delta \in \bar{\Delta}$ then Gibbs factor $\exp[-\beta V_2(x_i, x_j)] \sim \exp[-\beta b]$, where

$$b = \inf_{\Delta \in \bar{\Delta}} \inf_{x_1, x_2 \in \Delta} V_2^+(x_1, x_2) \tag{5.59}$$

and $b \to \infty$, when $\lambda \to 0$. The configurations with this property will be called *dense* configurations, as opposed to *dilute* configurations, in which no more than one particle is situated in any cube. The main technical idea consists in separation of the dilute parts of configurations from the dense parts. In order to do this we define an indicator function for the configuration γ_{Λ}, $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ in the cube Δ:

$$\chi_n^{\Delta}(\gamma_{\Lambda}) = \chi_n^{\Delta}(\gamma_{\Delta}) = \begin{cases} 1, & \text{for } |\gamma_{\Delta}| = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then the indicator for *dilute* configurations is defined as

$$\chi_-^\Delta(\gamma_\Delta) \;=\; \chi_0^\Delta(\gamma_\Delta) + \chi_1^\Delta(\gamma_\Delta)$$

and for *dense* configurations as

$$\chi_+^\Delta(\gamma_\Delta) \;=\; \sum_{n\geq 2} \chi_n^\Delta(\gamma_\Delta).$$

To obtain decomposition we use the following partition of the unity:

$$1 \;=\; \prod_{\Delta\subset\Lambda} \left[\chi_-^\Delta(\gamma_\Delta) + \chi_+^\Delta(\gamma_\Delta)\right] \;=\; \sum_\omega \prod_{\Delta\subset\Lambda} \chi_{\omega(\Delta)}^\Delta(\gamma_\Delta), \qquad (5.60)$$

where ω is the map from $\bar\Delta \cap \Lambda := \{\Delta \in \bar\Delta \,:\, \Delta \subset \Lambda\}$ into the set $\{+, -\}$, such that $\omega(\Delta) = +$ or $-$ for any $\Delta \in \bar\Delta \cap \Lambda$. Inserting (5.60) into (5.54) for $\bar\gamma = \emptyset$, we get

$$\rho^\Lambda(\eta) \;=\; \frac{1}{Z_\Lambda} \sum_\omega \int_{\Gamma_\Lambda} \prod_{\Delta\subset\Lambda} \chi_{\omega(\Delta)}^\Delta(\gamma_\Delta) e^{-\beta U(\eta\cup\gamma)} \lambda_\sigma(d\gamma), \qquad (5.61)$$

where $Z_\Lambda = Z_\Lambda(\emptyset)$. Now we define the set

$$X = \bigcup_{\Delta\subset\Lambda\,:\,\omega(\Delta)=+} \Delta.$$

Then the sum over ω can be rewritten as the sum over all possible sets X in Λ. Namely,

$$\rho^\Lambda(\eta) \;=\; \frac{1}{Z_\Lambda} \sum_{\emptyset\subseteq X\subseteq\Lambda} \int_{\Gamma_\Lambda} \widetilde{\chi}_+^X(\gamma)\widetilde{\chi}_-^{X^c}(\gamma) e^{-\beta U(\eta\cup\gamma)} \lambda_\sigma(d\gamma),$$

where

$$\widetilde{\chi}_\pm^X(\gamma) \;=\; \prod_{\Delta\subset X} \chi_\pm^\Delta(\gamma_\Delta)$$

For any $X \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$, $X \subseteq \Lambda$ define graph $G_R(X)$ with vertices in the centers of all elementary cubes $\Delta \subset X$ and lines $l(\Delta, \Delta')$ iff $dist(\Delta, \Delta') \leq R$. The number of lines depends on graph $G_R(X)$.

Definition 5.3.1 *The set X is called R-connected if the corresponding graph $G_R(X)$ is connected in ordinary way.*

R-connected set X is denoted by X^R. Then, every set X can be represented as some fixed partition

$$\{X\}_n^R := \{X_1^R, \ldots, X_n^R \mid \mathrm{dist}(X_i^R, X_j^R) > R, \quad \text{for} \quad i \neq j\},$$

and so the sum over all possible X in Λ can be rewritten as the sum over all possible sets $\{X\}_n^R$ (for $n = 0$, $X = \emptyset$). Furthermore, we replace the sum over all such sets by the sum over $X_1^R, \ldots, X_n^R$ independently, and remove the conditions $dist(X_i^R, X_j^R) > R$ by introducing the *hard-core* potential

$$\chi_R^{cor}(X)_n = \begin{cases} 0, & \text{there exist } X_i^R,\ X_j^R,\ i \neq j,\ dist(X_i^R,\ X_j^R) \leq R, \\ 1, & \text{otherwise.} \end{cases}$$

Then we get

$$\rho^\Lambda(\eta) = \frac{1}{Z_\Lambda} \sum_{n \geq 0} \frac{1}{n!} \sum_{X_1^R \subseteq \Lambda} \cdots \sum_{X_n^R \subseteq \Lambda} \chi_R^{cor}(X)_n \times$$

$$\times \int_{\Gamma_\Lambda} \widetilde{\chi}_+^X(\gamma) \widetilde{\chi}_-^{X^c}(\gamma) e^{-\beta U(\eta \cup \gamma)} \lambda_\sigma(d\gamma). \tag{5.62}$$

In the sequel, having in mind only R-connected components of X, we drop index R in the notation X_i^R, and summation $\sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda}$, for simplicity, will be denoted by $\sum_{(X)_n}$. Now, the last step in arranging our decomposition is as follows. Define the set

$$X_0 = \bigcup_{\Delta \subset \Lambda \,:\, dist(\Delta, \eta) \leq R} \Delta.$$

This set is fixed for fixed variable of the correlation function $\rho^\Lambda(\eta)$. Now, for every $n \geq 0$ we split the sum over $(X)_n$ into two sums. The first one is over those X_j, which do not intersect the region X_0 and the second one over those which intersect X_0. To distinguish the sets X_j which do not intersect and do intersect X_0, the latter sets are denoted by Y_j. There are $n!/k!(n-k)!$ possibilities when any k sets X_j do not intersect X_0 and $(n-k)$ sets Y_j intersect X_0. So the final expansion is the following:

$$\rho^\Lambda(\eta) = \frac{1}{Z_\Lambda} \sum_{n \geq 0} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{(X)_k} \sum_{(Y)_{n-k}} \chi_R^{cor}((X)_k, (Y)_{n-k}) \times$$

$$\times \int_{\Gamma_\Lambda} \lambda_\sigma(d\gamma) \widetilde{\chi}_+^X(\gamma) \widetilde{\chi}_-^{X^c}(\gamma) e^{-\beta U(\eta \cup \gamma)}, \tag{5.63}$$

where

$$X = \widetilde{X}_k \cup \widetilde{Y}_{n-k} := \left[\bigcup_{i=1}^k X_i\right] \bigcup \left[\bigcup_{j=1}^{n-k} Y_j\right].$$

5.3.6 The main estimates.

As the first step, let us split the exponent in (5.63) into four parts: the part which corresponds to the positive part of the energy of the configuration η, the interactions of the particles inside the region $X_0 \cup \widetilde{Y}_{n-k}$, inside $\Lambda \setminus (X_0 \cup \widetilde{Y}_{n-k})$ and interactions between them. Note that interaction between $X_0 \cup \widetilde{Y}_{n-k}$ and $\widetilde{X}_k$ is zero due to the finite range of potential. Therefore, considering $\gamma \in \Gamma_\Lambda \; : \; \gamma \cap \eta = \emptyset$ we get

$$e^{-\beta U(\eta \cup \gamma)} = e^{-\beta U^+(\eta)} E_1 E_2 E_0,$$

where

$$E_1(X_0 \cup \widetilde{Y}_{n-k}) = e^{-\beta U^{st}(\eta)} \prod_{l=1}^{n-k} e^{\beta W(\eta \,|\, \gamma_{Y_l}) - \frac{1}{2}\beta U^+(\gamma_{Y_l}) - \beta U^{st}(\gamma_{Y_l})},$$

$$E_2(X_0 \cup \widetilde{Y}_{n-k} \,|\, (X_0 \cup X)^c) = e^{-\beta W(\eta \,|\, \gamma_{X_0 \setminus \widetilde{Y}_{n-k}})} \prod_{l=1}^{n-k} e^{-\beta[\frac{1}{2}U^+(\gamma_{Y_l}) + W(\gamma_{Y_l} \,|\, \gamma_{X^c})]},$$

and

$$E_0(\widetilde{Y}_{n-k}^c) = e^{-\beta U(\gamma_{\Lambda \setminus \widetilde{Y}_{n-k}})}.$$

Lemma 5.3.1

$$E_1 \le e^{\beta B |\eta|} \prod_{l=1}^{n-k} \prod_{\Delta \subset Y_l} e^{\beta B |\gamma_\Delta| - \frac{1}{2}\beta U^+(\gamma_\Delta)}. \tag{5.64}$$

Proof. Using **A5** we have

$$U^{(st)}(\eta \cup \gamma_{\widetilde{Y}_{n-k}}) \ge -B\left(|\eta| + \sum_{l=1}^{n-k} \sum_{\Delta \subset Y_l} |\gamma_\Delta|\right)$$

and

$$W^+(\eta \,|\, \gamma_{\widetilde{Y}_{n-k}}) \ge 0, \quad U^+(\gamma_{Y_l}) \ge \sum_{\Delta \subset Y_l} U^+(\gamma_\Delta). \quad \blacksquare$$

Lemma 5.3.2 *For any $\gamma \in \Gamma$ and $\bar{\gamma} \in \bar{\Gamma}_{X^c}$, $X \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$, $X \subseteq \Lambda$*

$$\frac{1}{4} U^+(\gamma_X) + W(\gamma_X \mid \bar{\gamma}) \geq -\bar{I}|\gamma_X|, \tag{5.65}$$

where $\bar{I} := \bar{I}_2^{(1)}$ (see (5.53)), and

$$\bar{\Gamma}_{X^c} = \{\gamma \in \Gamma_{X^c} \mid |\gamma \cap \Delta| \leq 1, \text{ for all } \Delta \subset X^c\}$$

Proof. See Appendix.   ∎

Let us define

$$\partial \eta = \bigcup_{\Delta \,:\, \eta \cap \Delta \neq \emptyset} \Delta.$$

Now using the property of infinite divisibility of measure λ_σ and estimate (5.65) we can calculate the part of integral in (5.63)

$$e^{-\frac{1}{2}\beta U^+(\eta)} \int_{\Gamma_{\widetilde{Y}_{n-k}}} \widetilde{\chi}_+^{\widetilde{Y}_{n-k}}(\gamma) E_1 E_2 \lambda_\sigma(d\gamma) \leq$$

$$\leq e^{-\frac{1}{4}\beta U^+(\eta) + \beta|\eta|\bar{I}} \int_{\Gamma_{\widetilde{Y}_{n-k}}} \widetilde{\chi}_+^{\widetilde{Y}_{n-k}}(\gamma) e^{-\beta W(\eta \,|\, \gamma_{(X_0 \cap \partial \eta) \setminus \widetilde{Y}_{n-k}})} E_1 \times$$

$$\times \prod_{l=1}^{n-k} e^{-\beta[\frac{1}{2}U^+(\gamma_{Y_l}) + W(\gamma_{Y_l} \,|\, \gamma_{X^c})]} \lambda_\sigma(d\gamma). \tag{5.66}$$

Assumption **A6**, estimate (5.64) and trivial inequality

$$U^+(\eta) \geq \sum_{\Delta \subset \partial \eta} U^+(\eta_\Delta)$$

gives us the bound for the integral (5.66)

$$e^{\beta|\eta|(\bar{I}+B) + \beta \sum_{\Delta \subset \partial \eta} \bar{I}|\eta_\Delta|} \prod_{l=1}^{n-k} \prod_{\Delta \subset Y_l} I_\Delta,$$

where

$$I_\Delta = \int_{\Gamma_\Delta} \chi_+^\Delta(\gamma_\Delta) e^{-\beta\frac{1}{2}U^+(\gamma_\Delta) + \beta(B+\bar{I})|\gamma_\Delta|} \lambda_\sigma(d\gamma). \tag{5.67}$$

Focusing only on the 2-body positive part of interaction and taking into account the definition (5.59) we can estimate the last integral by

$$I_\Delta \leq \varepsilon_1 = \frac{1}{2}z^2\lambda_0^{2d}e^{-\beta(\frac{1}{2}b-2\bar{I}-2B)}\exp\{z\lambda_0^d e^{-\beta(\frac{3}{2}b-\bar{I}-B)}\}, \qquad (5.68)$$

which is finite due to **A6**.

Now taking the maximum of E_0 in variable $\widetilde{Y}_{n-k}$ (we denote this maximum by $\bar{Y}_{n-k}$) and using elementary estimate

$$\chi_R^{cor}((X)_k, (Y)_{n-k}) \leq \chi_R^{cor}(X)_k \qquad (5.69)$$

we can estimate the sum over $(Y)_{n-k}$ by the following lemma:

Lemma 5.3.3 *(e.g. [62])*

$$\sum_{Y\cap X_0\neq\emptyset} \varepsilon_1^{\frac{|Y|}{\lambda^d}} \leq |\eta|c(d)\left(\frac{R}{\lambda}\right)^d\frac{\varepsilon}{1-\varepsilon} = |\eta|K, \qquad (5.70)$$

where $c(d)$ is a constant which depends only on d and $\varepsilon = 4c(d)\left(\frac{R}{\lambda}\right)^d\varepsilon_1$.

For the *proof* in our case see [73].

The last step is as follows. The expansion like (5.62) can be constructed for partition function Z_{Λ_1} with $\Lambda_1 \subset \Lambda$. Denote it by

$$Z_{\Lambda_1} = \sum_{k\geq 0}\frac{1}{k!}Z_{\Lambda_1}^{(k)}. \qquad (5.71)$$

Taking into account all previous estimates we get

$$\rho^\Lambda(\eta) \leq \frac{1}{Z_\Lambda}e^{-\frac{1}{2}\beta U^+(\eta_\Lambda)+\beta(2\bar{I}+B)|\eta|}\sum_{n\geq 0}\sum_{k=0}^n\frac{(|\eta|K)^{n-k}}{k!(n-k)!}Z_{\Lambda\backslash\bar{Y}_{n-k}}^k =$$

$$= \frac{1}{Z_\Lambda}e^{-\frac{1}{2}\beta U^+(\eta_\Lambda)+\beta(2\bar{I}+B)|\eta|}\sum_{k\geq 0}\frac{1}{k!}\sum_{l\geq 0}\frac{(|\eta|K)^l}{l!}Z_{\Lambda\backslash\bar{Y}_l}^k =$$

$$= e^{-\frac{1}{2}\beta U^+(\eta_\Lambda)+\beta(2\bar{I}+B)|\eta|}\sum_{l\geq 0}\frac{(|\eta|K)^l}{l!}\frac{Z_{\Lambda\backslash\bar{Y}_l}}{Z_\Lambda}. \qquad (5.72)$$

The fact that $Z_{\Lambda_1} \leq Z_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$ gives the inequality

$$\rho^\Lambda(\eta) \leq e^{-\frac{1}{2}\beta U^+(\eta)}e^{|\eta|(\beta(2\bar{I}+B)+K)}. \qquad \blacksquare$$

Appendix A

Proof of the lemma 5.3.2

Let $X = \cup_{j=1}^{N} \Delta_j$. Consider the configuration γ with $|\gamma_X| = m, |\gamma_{\Delta_1}| = m_1, ..., |\gamma_{\Delta_N}| = m_N$, $m_j \geq 1$ for $j = 1, ..., N$ and $m_1 + \cdots + m_N = m$. Let in the k-body interaction be involved $\bar{k} \geq 1$ particles from the dilute configuration $\bar{\gamma}_{X^c} \in \bar{\Gamma}_{X^c}$ and, correspondingly, q_1 particles of γ_X from Δ_1, which are situated in the points $x_1^{(1)}, ..., x_{q_1}^{(1)} \in \Delta_1$, ..., q_N particles $x_1^{(N)}, ..., x_{q_N}^{(N)}$ from Δ_N. It is clear that $q_1 + \cdots + q_N + \bar{k} = k$ and $0 \leq q_i \leq m_i$, $\bar{k} \geq 1$. Then the interaction energy between m particles of the configuration γ_X and $\bar{k}$ particles of dilute configuration $\bar{\gamma}_{X^c}$ can be written in the following form:

$$W_k(\gamma_X \mid \bar{\gamma}_{X^c}) = \sum_{\substack{0 \leq q_i \leq m_i, \bar{k} \geq 1 \\ q_1 + \cdots + q_N + \bar{k} = k}} \sum_{\{x_1^{(1)}, ..., x_{q_1}^{(1)}\} \in \gamma_{\Delta_1}} \cdots \sum_{\{x_1^{(N)}, ..., x_{q_N}^{(N)}\} \in \gamma_{\Delta_N}} \times$$

$$\times \sum_{\{y_1, ..., y_{\bar{k}}\} \in \bar{\gamma}_{X^c}} V_k(x_1^{(1)}, \ldots, x_{q_1}^{(1)}, \ldots, x_1^{(N)}, \ldots, x_{q_N}^{(N)}, y_1, \ldots, y_{\bar{k}}).$$

Then taking into account (5.50) we obtain

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \leq \sum_{\substack{0 \leq q_i \leq m_i, \bar{k} \geq 1 \\ q_1 + \cdots + q_N + \bar{k} = k}} \prod_{i=1}^{N} C_{m_i}^{q_i} I_k^{q_1, ..., q_N | \bar{k}}(\Delta_1, \ldots, \Delta_N), \qquad (A.1)$$

where $C_m^k = m!/k!(m-k)!$. Let in the sequence $q_1, \ldots, q_N$ be nonzero correspondingly $q_{l_i} = k_{l_i}$ particles from Δ_{l_i}, $i = 1, \ldots, M$ involved in k-body interaction. Changing in (A.1) to the summation over $k_{l_1}, \ldots, k_{l_M}$:

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \leq \qquad (A.2)$$

101

$$\leq \sum_{M=1}^{\min\{N,k-1\}} \sum_{1\leq l_1<l_2<\cdots<l_M\leq N} \sum_{\substack{1\leq k_{l_i}\leq m_{l_i},\bar{k}\geq 1 \\ k_{l_1}+\cdots+k_{l_M}+\bar{k}=k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} I_k^{k_{l_1},\ldots,k_{l_M}|\bar{k}}(\Delta_{l_1},\ldots,\Delta_{l_M})$$

Let among the cubes $\Delta_1,\ldots,\Delta_N$ be N_1 cubes with only one point of γ inside. Without loss of generality, we suppose that $m_j = 1$, $j = N - N_1 + 1,\ldots,N$. We suppose also that $1 \leq N_1 < N$. Split the summation over $1 \leq l_1 < l_2 < \cdots < l_M \leq N$ into the summation over $1 \leq l_1 < l_2 < \cdots < l_S \leq N - N_1$ over cubes $\Delta_1,\ldots,\Delta_{N-N_1}$ and the summation over $1 \leq l'_1 < l'_2 < \cdots < l'_{S'} \leq N_1$ over cubes $\Delta'_1,\ldots,\Delta'_{N_1}$. It is clear that $S + S' = M$ and S can take integer values from 0 to M. Therefore, we get additionally $M+1$ sums over S. Every value of $1 \leq l'_1 < \cdots < l'_{S'} \leq N_1$ corresponds to the dilute configuration. Hence, using the definition (5.50) we can apply the following formula:

$$\sum_{1\leq l'_1<l'_2<\cdots<l'_{S'}\leq N_1} I_k^{k_{l_1},\ldots,k_{l_S},1,\ldots,1|\bar{k}}(\Delta_{l_1},\ldots,\Delta_{l_S},\Delta_{l'_1},\ldots,\Delta_{l'_{S'}}) \leq$$

$$\leq I_k^{k_{l_1},\ldots,k_{l_S}|\bar{k}+S'}(\Delta_{l_1},\ldots,\Delta_{l_S}),$$

yielding
$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \leq$$

$$\leq \sum_{M=1}^{\min\{N-N_1,k-1\}} \sum_{1\leq l_1<l_2<\cdots<l_M\leq N-N_1} \sum_{l=0}^{\min\{N_1,k-M-1\}} \sum_{\substack{1\leq k_{l_i}\leq m_{l_i},\bar{k}\geq 1 \\ k_{l_1}+\cdots+k_{l_M}+\bar{k}+l=k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times I_k^{k_{l_1},\ldots,k_{l_M}|\bar{k}+l}(\Delta_{l_1},\ldots,\Delta_{l_M}) + N_* \sum_{l'_1=1}^{\min\{N_1,k-1\}} I_k^{1|k-1}(\Delta'_{l'_1}),$$

where $N_* = \min\{N, k-1\}$. Collecting the terms with $M = 1$, $k_{l_1} = 1$ in the first sum and the last sum, and selecting also the terms with $k_{l_1} = k_{l_2} = \cdots = k_{l_M} = 1$, summing up all inequalities in $k \geq 2$ and taking into account that $N_* \leq k - 1$, we get

$$-W(\gamma_X \mid \bar{\gamma}_{X^c}) \leq \bar{I}|\gamma_X| + W_1 + W_2,$$

where

$$W_1 = \sum_{M=2}^{N-N_1} \sum_{1\leq l_1<l_2<\cdots<l_M\leq N-N_1} \prod_{i=1}^{M} C_{m_{l_i}}^{1} \sum_{k\geq M+1} (k-M) I_k^{1,\ldots,1|k-M}(\Delta_{l_1},\ldots,\Delta_{l_M}),$$

$$W_2 = \sum_{M=1}^{N-N_1} \sum_{1 \le l_1 < l_2 < \cdots < l_M \le N-N_1} \sum_{k \ge M+1} \sum_{\substack{1 \le k_{l_i} \le m_{l_i} \\ k_{l_1} + \cdots + k_{l_M} = k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times \sum_{l \ge 1} l I_{k+l}^{k_{l_1}, \ldots, k_{l_M} \, |l} (\Delta_{l_1}, \ldots, \Delta_{l_M}).$$

Using the same arguments, one can get almost the same inequality for the positive part of energy:

$$U^+(\gamma_X) \ge U_0,$$

where

$$U_0 = \sum_{M=1}^{N-N_1} \sum_{1 \le l_1 < l_2 < \cdots < l_M \le N-N_1} \sum_{k \ge M+1} \sum_{\substack{1 \le k_{l_i} \le m_{l_i} \\ k_{l_1} + \cdots + k_{l_M} = k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times v_k^{k_{l_1}, \ldots, k_{l_M}} (\Delta_{l_1}, \ldots, \Delta_{l_M}).$$

Now it is clear from the assumptions **A6** that

$$\frac{1}{4} U_0 \ge W_1, \quad \text{and} \quad \frac{1}{4} U_0 \ge W_2,$$

which gives (5.65). It is not difficult to see (using direct computation) that condition $1 \le N_1 < N$ is not essential in the proof of Lemma 5.3.2. ■

Bibliography

[1] S. Albeverio, Yu. Daletzky, Yu. G. Kondratiev, and L. Streit. Non-Gaussian infinite dimensional analysis. *J. Funct. Anal.*, 138:311–350, 1996.

[2] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.

[3] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: The Gibbsian case. *J. Funct. Anal.*, 157:242–291, 1998.

[4] S. Albeverio, Yu. G. Kondratiev, and L. Streit. How to generalize white noise analysis to non-Gaussian spaces. In P.Blanchard, M.Sirugue-Collin, L. Streit, and D.Testard, editors, *Dynamics of Complex and Irregular Systems*, pages 120-130. World Scientific, Singapore, 1993.

[5] R. V. Ambartzumian and H. S. Sukiasian. Inclusion-Exclusion and Point Processes, *Acta Applicande Mathematicae* 22:15–31, 1991.

[6] V. Belitsky and E. A. Pechersky. Uniqueness of Gibbs state for non-ideal gas in $\mathbb{R}^d$: the case of multibody interaction. *J. Statist. Phys.*, 106, no.5-6:931–955, 2002.

[7] Yu. M. Berezansky. Commutative Jacobi field in Fock space. *Integral Equations Oper. Theory*, 4:1–31, 1998.

[8] Yu. M. Berezansky. On the theory of commutative Jacobi fields. *Methods Funct. Anal. Topol.*, 4:1–31, 1998.

[9] Yu. M. Berezansky. Poisson infinite-dimensional analysis as an example of analysis related to generalized translation operators. *Funct. Anal. Appl.*, 32:195–198, 1998.

[10] Yu. M. Berezansky and Yu. G. Kondratiev. Spectral Methods in Infinite-Dimensional Analysis. *Naukova Dumka, Kiev*, 1988. (In Russian). English translation: *Kluwer Academic Publishers, Dorbrecht*, 1995

[11] Yu. M. Berezansky and Yu. G. Kondratiev. Non-Gaussian analysis and hypergroups. *Funct. Anal. Appl.*, 29(3):188–191, 1995.

[12] Yu. M. Berezansky, Yu. G. Kondratiev, T. Kuna, and E.V. Lytvynov. On a spectral representation for correlation measures in configuration space analysis. *Methods Funct. Anal. Topology* 5(4):87-100, 1999.

[13] Yu. M. Berezansky and E.V. Lytvynov. Generalized white noise analysis assosiated with perturbed field theory. *Dopov. Akad. Nauk Ukr.*, (10):15-18, 1993.

[14] Yu. M. Berezansky, V. O. Livinsky, and E.V. Lytvynov. A generalization of Gaussian white noise analysis. *Methods of Functional Analysis and Topology*, 1(1):28–55, 1995.

[15] P. Billinsgley. Convergence of probability measures. New York-London-Sydney-Toronto: John Wiley & Sons, Inc. 1968.

[16] P. Billinsgley. Probability and measure. New York-Chichester-Brisbane-Toronto: John Wiley & Sons, Inc. 1979.

[17] N. Bourbaki. General topology. Application of real numbers in general topology. Functional spaces. Resume of results. Vocabulary. Translated from the third French edition by S. N. Krachkovskiy. Edited by D. A. Raykov. 408 pp. Izdat. Nauka, Moscow 1975.

[18] Yu. L. Daletsky. A biorthogonal analogy of the Hermite polynomials and the inversion of the Fourie transform with respect to a non Gaussian measure, *Funct. Anal. Appl.*, 25:68–70, 1991.

[19] R. L. Dobrushin. The description of a random field by means of conditional probabilities and conditions of its regularity. *Theory Probab. Appl.*, 13:197–224, 1968.

[20] R. L. Dobrushin. Gibbsian random fields;the general case. *Theory Probab. Appl.*, 3:22-28, 1969.

[21] R. L. Dobrushin. Gibbsian random fields for particles without hard core, *Theor. I Math. Fyz.* (in Russian) 4:101-118, 1970.

[22] R. L. Dobrushin. Prescribing a system of random variables by conditional distributions. *Theor. Veroat. Primen.*, 15:458–486, 1970 (transl. from the Russian).

[23] R. L. Dobrushin and E. A. Pechersky. A criterion of the uniqueness of Gibbs field in the non-compact case. *Lecture Notes in Mathematics*, 1021:97–110, 1982.

[24] R. L. Dobrushin, Ya. G. Sinai, and Yu. M. Suhov. Dynamical System of the Statistical Mechanics, *Itogi Nauki i Techniki, VINITI Akad. Nauk SSSR, ser. Sovremennye Problemy Matematiki, Fundamental'nye napravleniya* 2:235-284, 1985.

[25] H. O. Georgii. Canonical and grand canonical Gibbs states for continuum systems. *Comm. Math. Phys*, 48:31-51, 1976.

[26] H. O. Georgii. Canonical Gibbs Measures. *Lecture Notes in Math.*, volume 760, Springer Verlag, 1979.

[27] H. O. Georgii. Gibbs measures and Phase Transitions. *Walter de Gruyter* 1982.

[28] H. O. Georgii. Large deviations and the equivalence of ensembles for Gibbsian particle systems with superstable interaction. *Prob. Theory Related Fields*, 99:171-195, 1994.

[29] H. O. Georgii. The equivalence of ensembles for classical systems of particles. *Journal of Statistical Physics*, 80(5/6):1341-1378, 1995.

[30] H. O. Georgii and H. Zessin. Large deviations and the maximum entropy principle for marked point random fields. *Prob. Theory Related Fields*, 96:117–204, 1993.

[31] J. Ginibre. On the Asymptotic Exactness of the Bogoliubov Approximation for Many Boson Systems. *Commun. Math. Phys.* 8:26-51, 1968.

[32] W. Greenberg. Thermodynamic States of Classical Systems. *Commun. Math. Phys.*, 22:259-268, 1971.

[33] T. Hida. Brownian motion. volume 11 of *Appl. Math.*, Springer Verlag, New York, 1980.

[34] T. Hida, H. H. Kuo, J. Potthof, and L. Streit. White Noise. *An Infinite Dimensional Calculus.* Kluwer Academic Publishers, Dorbrecht, 1993.

[35] R. A. Holley and D. W. Stroock. Nearest neighbor birth and death processes on the real line. *Acta Math.* 140, no.1-2, 103-154, 1978.

[36] O. Kallenberg. Random Measures. Akademie-Verlag, Berlin, 3rd edition 1983.

[37] D. Klein and W. S. Yang. A characterization of first order phase transitions for superstable interactions in classical statistical mechanics. *Journal of Statistical Physics* 71:1043–1062, 1993.

[38] Yu. G. Kondratiev and T. Kuna. Harmonic analysis on configuration space I. General theory. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, 5, no.2: 201-233, 2002.

[39] Yu. G. Kondratiev and T. Kuna. Harmonic analysis on configuration space II. Gibbs states. In preparation, 2003.

[40] Yu. G. Kondratiev and T. Kuna, Correlation functionals for Gibbs measures and Ruelle bounds, *Methods Funct. Anal. Topology*, to appear.

[41] Yu. G. Kondratiev, T. Kuna, and J. L. Silva. Marked Gibbs measures via cluster expansion. *Methods of Functional Analysis and Topology*, 4(4):50–81, 1998.

[42] Yu. G. Kondratiev, T. Kuna, and M. J. Oliveira, Analytic aspects of Poissonian white noise analysis. SFB-256 Preprint No. 680, University Bonn, 2000. *Methods Funct. Anal. Topology*, to appear.

[43] Yu. G. Kondratiev, T. Kuna, and M. J. Oliveira, On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces. BiBoS Preprint n. 02-02-072, University of Bielefeld, 2002.

[44] Yu. G. Kondratiev and E. Lytvynov. Operators of gamma white noise calculus. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, 3:303-335, 2000.

[45] Yu. G. Kondratiev, E. Lytvynov, and M. Röckner. Non equilibrium spatial Birth-and-Death processes In preparation, 2003.

[46] Yu. G. Kondratiev, A. L. Rebenko, and M. Röckner. On diffusion dynamics for continuous systems with singular superstable interaction, Submitted to *Journal of Mathematical Physics* 2-0317.

[47] Yu. G. Kondratiev, J. L. Silva, and L. Streit. Generalized Appel Systems. *Methods Functional Analysis and Topology*, 3(2):28–61, 1997.

[48] Yu. G. Kondratiev, J. L. Silva, L. Streit, and G. F. Us. Analysis on Poisson and Gamma spaces. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, 1(1):91–117, 1998.

[49] Yu. G. Kondratiev, L. Streit, W. Westerkamp, and J. Yan. Generalized functions in infinite dimensional analysis, *Hiroshima Math. J.*, 28(2):213-260, 1998.

[50] T. Kuna. Studies on configuration space analysis and applications PhD thesis, University of Bonn, 1999.

[51] H. H. Kuo. Lectures on white noise analysis. *Soochow J. Math.*, 18:229–300, 1992.

[52] H. H. Kuo. White Noise Distribution Theory. CRC Press, Boca Raton, New York, London, and Tokyo, 1996.

[53] O. E. Lanford and D.Ruelle. Observables at infinity and states with short range correlations in statistical mechanics. *Comm. Math. Phys.*, 13:194-215, 1969.

[54] O. E. Lanford. Time Evolution of Large Classical Systems, Dynamical Systems: Theory and Applications., ed. J. Moser. *Lect. notes in Phys., Vol. 38*, 1975.

[55] J. Lebowitz and E. Pressuti. Statistical Mechanics of Systems of Unbounded Spins, *Commun. Math. Phys.* 50:195-218, 1976.

[56] A. Lenard. States of classical statistical mechanical systems of infinitely many particles. I, *Arch. Rational Mech. Anal.*, 59:219-239, 1975.

[57] A. Lenard. States of classical statistical mechanical systems of infinitely many particles. II. *Arch. Rational Mech. Anal.*, 59:241-256, 1975.

[58] J.T. Lewis, J.V. Pule, and P. de Smedt. The Superstability of Pair-Potentials of Positive Type. *Journal of Statistical Physics*, 35:381-385, 1968.

[59] E. Lytvynov. Multiple Wiener integrals and non-Gaussian white noises: a Jacobi field approach. *Methods of Functional analysis and Topology*, 1(1):61–85, 1995.

[60] E. Lytvynov. Fermion and boson random point processes as particle distributions of infinite free Fermi and Bose gases of finite density, *Reviews in Mathematical Physics* 14(10):1073-1098, 2002.

[61] R. A. Minlos. Limiting Gibbs distribution. *Funct. Anal. Appl.*, 1:141-150, 1967.

[62] V. Malyshev and R. Minlos. Gibbs random fields. The cluster expansion method. *Kluwer, Dordrecht* 1991.

[63] K. Matthes, J. Kerstan, and J. Mecke. Infinitely divisible point processes. John Wiley & Sons, Chichester-New York-Brisbane 1978.

[64] H. Moraal. The Kirkwood-Salsburg equation and the virial expansion for many-body potentials. *Phys. Lett.*, 59(A):9-10, 1976.

[65] X. X. Nguyen and H. Zessin. Integral and differential characterizations of the Gibbs process. *Math. Nachr.*, 88:105-115, 1979.

[66] M. J. Oliveira. Configuration Space Analysis and Poissonian White Noise Analysis PhD thesis, University of Lisboa, 2002.

[67] E. Pechersky and Yu. Zhukov. Uniqueness of Gibbs state for non-ideal gas in $\mathbb{R}^d$: The case of pair potentials. *Journal of Statistical Physics*, 97:145-172, 1999.

[68] C. Preston. Specifications and their Gibbs states. Manuscript. http://www.mathematik.uni-bielefeld.de/~preston

[69] C. Preston. Random Fields. In: *Lecture notes in Mathematics*, 534. Berlin Heidelberg, New York: Springer 1976.

[70] C. Preston. Canonical and microcanonical Gibbs states. *Z. Wahrsch. verw. Gebiete*, 46:125-158, 1979.

[71] A. Procacci and B. Scoppola. The Gas Phase of Continuous Systems of Hard Spheres Interacting via n-Body Potential. *Comm. Math. Phys.*, 211:487-496, 2000.

[72] S. Rachev. Probability metrics and the stability of stochastic models. John Wiley & Sons, Chichester-New York 1991.

[73] A. L. Rebenko. A New Proof of Ruelle's Superstability Bounds. *J. Stat. Phys.*, 91:815-826, 1998.

[74] A. L. Rebenko and G. V. Shchepan'uk. The Convergence of Cluster Expansions for Continuous Systems with Many-Body Interactions. *J. Stat. Phys.*, 88(3/4):665-689, 1997.

[75] D. Ruelle. Statistical Mechanics. New York, Benjamin 1969.

[76] D. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* 18:127-159, 1970.

[77] S. B. Shlosman. The method of reflective positivity in the mathematical theory of phase transitions of the first kind. *Uspekhi Mat. Nauk.*, 41, no.3(249):69–111, 240, 1986 (in Russian).

[78] J. L. Silva. Studies in Non-Gaussian Analysis. PhD thesis, University of Madeira, 1998.

[79] Ya. G. Sinai. Theory of phase Transitions: Rigorous Results. *Pergamon Press*, 1982.

[80] A. V. Skorokhod. On infinite system of stochastic differential equations. *J. Methods of Funct. Anal. and Topology* 4, 54–61, 1999.

[81] L. Streit. Introduction to white noise analysis. In A. L. Cardoso, M. de Faria, R. Potthoff, J. Seneor, and L. Streit, editors, *Stochastic Analysis and Applications in Physics*, pages 415-440. Kluwer Academic Publishers, Dorbrecht, 1994.

[82] W. Westerkamp. A primer in white noise analysis. In P.Blanchard, M.Sirugue-Collin, L. Streit, and D.Testard, editors, *Dynamics of Complex and Irregular Systems*, pages 188-202. World Scientific, Singapore, 1993.

[83] H. Zessin. The method of moments for random measures. *Z. Wahrsch. Verw. Gebiete* 62, no. 3:395-409, 1983.