Combinatorial and probabilistic aspects of lattice path models

(Kombinatorische und probabilistische Aspekte von Gitterwegmodellen)

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Uwe Schwerdtfeger
aus Dortmund

Berichterstatter: Prof. Dr. Michael Baake
PD Dr. Christoph Richard

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PD Dr. C. Richard, Prof. Dr. Dr. h.c. C. M. Ringel

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Chapter 1

Introduction

In this thesis we prove some asymptotic results on several classes of combinatorial objects. The results in Chapters 2, 3 and 4 are of probabilistic nature, i.e. a probability distribution is defined on these classes of objects and the behaviour of certain statistics in the limit of large size is computed. More precisely, in Chapters 2 and 3 these objects are symmetry classes of lozenge tilings of a hexagon and the intimately related plane partitions. For one symmetry class we are able to prove a macroscopic effect occurring in a randomly chosen tiling, namely "the arctic ellipse phenomenon" and fluctuations thereof, which are governed by laws known from random matrix theory (Chapter 2). In Chapter 3 a volume is assigned to these tilings and a Gaussian limit distribution for almost all relevant symmetry classes is proved. Chapters 4 and 5 are devoted to polygons on the square lattice. In Chapter 4 we study the limiting area distributions of all symmetry classes of staircase polygons, which in the non-trivial cases are expressed in terms of the Brownian excursion and meander distributions. The results of Chapter 5 concern subclasses of lattice polygons called prudent polygons and are combinatorial in nature. We derive some generating functions explicitly and give asymptotics of the enumeration sequences. In one case we prove the non-existence of a "nice" recursion formula for this sequence.

Apart from being natural objects arising in certain areas of mathematics, a strong connection to statistical physics or computer science is common to all these classes. The classes of Chapters 2 and 3 are related to dimer models [Els84], lattice path models ("vicious walkers") [GV85, KGV00] and models in crystallography. The polygons in Chapter 4 are strongly related to trees and algebraic languages on the one hand, and to models in polymer chemistry, and physics (Ising model for magnetism) on the other hand. The latter two relationships also hold for the polygons dealt with in Chapter 5.

Random Tilings

By a tiling we mean a covering of some domain in the plane (or of the entire plane) without gaps and overlaps with *translates* of polygons (tiles) taken from a finite set of polygons, the so-called prototiles. Notice that in our notation rotated copies of the same polygon lead to different prototiles, e.g. the lozenge tilings considered in Chapters 2 and 3 consist of $60°$-rhombi of side length one in three different orientations, referred to as *lozenges*. For a finite domain, which is also at hand in the examples here, one can consider collections of finitely many tilings which cover that domain and assign a probability to each of these tilings. Such a tiling will be referred to as a *random tiling*. The collection of examples focused upon in the present work are lozenge tilings of a hexagon H "with fixed boundary conditions" (i.e. no tile overlaps the boundary) equipped with a uniform distribution, see Figures 1.2 and 2.1.

In the 1970s Roger Penrose discovered a non-periodic, yet highly ordered tiling of the plane consisting of $72°$- and $36°$-rhombi [Pen79] assembled in an edge to edge manner, i.e. two tiles share a common vertex or edge or do not intersect at all. It features a fivefold rotational symmetry, which is forbidden for periodic patterns due to the crystallographic constraint. However, this tiling is "quasi-periodic" in the sense that, given $R > 0$, every patch of diameter R occurs with a positive frequency. Another tiling with this feature and with an eight-fold rotational symmetry is known as the Ammann-Beenker tiling [AGS92, Bee82]. It consists of $45°$-rhombi and squares which are also arranged edge to edge. Both tilings, respectively their vertex sets, can be obtained in several ways, e.g. by matching rules, substitution or as certain projections from higher dimensional lattices.

Apart from mathematicians also physicists got interested in those structures and their three-dimensional analogues as models for possible atomic configurations, in particular after Shechtman's [SBGC84] discovery of a quasicrystal, i.e. a metallic alloy whose atomic configuration exhibits a non-crystallographic symmetry. Since solids are often studied via their X-ray diffraction image, the question was raised what the diffraction image of such a quasi-periodic point set may look like. For crystals or periodic point sets the answer is given by a multidimensional version of Poisson's summation formula in terms of the reciprocal lattice[1] of the underlying lattice. The mathematically rigorous measure theoretic formalism which generalises to quasi-periodic point sets was developed in [Hof95]. At high temperatures quasicrystals were suggested to be modelled by randomised versions of the above quasiperiodic tilings, see [MB93] for a randomised version of the Amman Beenker tiling.

Further motivation to study random tilings from statistical physics and also graph theory comes from dimer models which may serve as an "important, even though physically oversimplified model of a system (e.g. solution or gas) containing diatomic molecules" [TF61]. The atoms of a dimer correspond to the vertices (sites) of a lattice and dimers are placed along edges (bonds) such that no site remains unoccupied. In terms of graph theory, one studies *perfect matchings* in certain graphs. For example, a perfect matching of (a patch of) the square lattice $(1/2, 1/2) + \mathbb{Z}^2$ corresponds bijectively to a configuration of 1×2- and 2×1-dominoes with corners in $\mathbb{Z}^2$ in the obvious way, just draw a rectangle around each edge. These domino tilings are not necessarily edge to edge. In a similar way we can look at the honeycomb lattice, which is the dual graph of the planar $60°$-triangular

[1] By the *reciprocal* lattice of a lattice L we mean the set of those vectors having integer scalar product with any vector in L.

lattice. Each vertex of the honeycomb lattice is the centre point of exactly one triangle and two vertices form an edge iff. the corresponding triangles share a side. A matching of the honeycomb lattice now maps to a configuration of 60°-rhombi, referred to as *lozenges*, simply by placing a lozenge upon each edge, such that the corresponding two triangles are covered. Thus a perfect matching of (a patch of) the honeycomb lattice corresponds bijectively to a lozenge tiling.

In a finite setting a natural, however in most cases intricate question to ask is: *What is the exact number of configurations?* Particularly nice answers have been given only for a few ensembles of tilings. One classical result is the exact number of tilings of a $2m \times 2n$-rectangle by dominoes, independently found in [Kas61] and [TF61]. It is given by

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \left(\frac{\pi j}{2m+1} \right) + \cos^2 \left(\frac{\pi k}{2n+1} \right) \right).$$

The result is obtained by expressing the generating function as a Pfaffian which can be evaluated in this form. Other regions of the square lattice for which the number of domino tilings were counted are the so-called Aztec Diamonds $\mathcal{A}_n$ consisting of all lattice unit squares $[i, i+1] \times [j, j+1]$, $i, j \in \mathbb{Z}$, inside the region $\{|x|+|y| \leq n+1\}$ [EKLP92a, EKLP92b], see Figure 1.1.

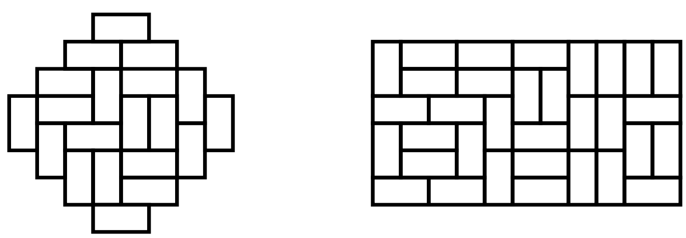

Figure 1.1: Domino tilings of an Aztec diamond and a rectangle

The authors derive the generating function for a refined enumeration of the tilings of $\mathcal{A}_n$ and find their total number to be $2^{n(n+1)/2}$. There is also a connection with Alternating-Sign Matrices and square ice pointed out in the two papers. Similarly, tilings of a hexagon of integer side lengths r, s, t, r, s, t with lozenges of side lengths one have been studied in [Els84]. The latter author used a bijection to *plane partitions*: If a drawing of such a tiling is viewed as a 3D object, it shows a pile of cubes inside an $r \times s \times t$-box with weakly descending columns and rows, which in turn represents a plane partition, see Chapter 3 below. Their number was shown by Mac Mahon [MM15] to be equal to

$$\prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2},$$

see also the beautiful monograph [Bre99] on this and other related topics in algebraic combinatorics. Mac Mahon also derived the generating function enumerating plane partitions fitting inside an $r \times s \times t$-box by their volume. Such generating functions were also found for several symmetry classes of plane partitions. The limiting distribution of the volume, assuming a uniform distribution on the tilings, is derived in Chapter 3 for several symmetry classes. This work is published in *Discrete Mathematics and Theoretical Computer Science: Conference Proceedings* [Sch08].

Common to the above three tiling models is the imposition of (rather "unphysical") *fixed boundary conditions*, as opposed to *periodic* and *free* boundary conditions. Boundary conditions may affect the *entropy per tile*

$$\lim_{\text{size of domain}\to\infty} \frac{\log \text{number of tilings}}{\text{number of tiles}}$$

with different boundary conditions. In many models there is evidence that free and periodic boundary conditions yield the same entropy in this so called *thermodynamic limit* "size of domain $\to \infty$" see e.g. [BJ96]. Whereas the choice of periodic or fixed boundary conditions does not have any influence on the entropy of the rectangular model, it has in the hexagonal tiling model, see [Els84].

Though being unphysical, tilings of domains with fixed boundaries are interesting from a mathematical point of view, since a "generic" tiling drawn uniformly at random of, say, a large hexagon (as well as a large Aztec Diamond) exhibits a remarkable macroscopic phenomenon: The tiling seems to be "frozen" in the corners of the bounding domain, i.e. only one species (orientation) of tiles occurs there. Towards the interior the tiling is unordered and all species appear with positive frequencies, see Figure 1.2. We call the frozen parts at the corners *arctic regions* and the unordered part the *temperate zone*. The remarkable effect is a sharp transition between the arctic regions and the temperate zone which is given by the inscribed circle in the Aztec Diamond [CEP96] and by the inscribed ellipse in the hexagon [CLP98]. Such boundary effects are also present for other shapes [KO07] or with probability distributions different from the uniform one [BGR09]. With the exact solutions (i.e. the aforementioned exact counts and refinements thereof) at hand these effects are quantifiable. A result of this flavour is proved in Chapter 2 for uniformly drawn lozenge tilings of a region called *half-hexagon*, which are equivalent to hexagon tilings symmetric in a horizontal axis, see also Figure 2.3 on page 35.

In [Joh02] it is pointed out that certain statistics on random tilings of the hexagon and the Aztec Diamond lead to probability measures resembling distributions from random matrix theory and non-intersecting Brownian motions. Particularly the latter matches up with bijections of tilings onto families of non-intersecting lattice paths [GV85] (see Figure 2.1 on page 18 for lozenge tilings). From these path models we obtain distributions of the form

$$\frac{1}{Z_N} \prod_{i=1}^{N} w_m(x_i) \prod_{1\le i<j\le N} (x_j - x_i)^2$$

by looking at the positions $x_1, \ldots, x_N$ of the paths (N their total number) after, say, m steps. Here w_m is a positive weight function and Z_N a normalisation constant. The

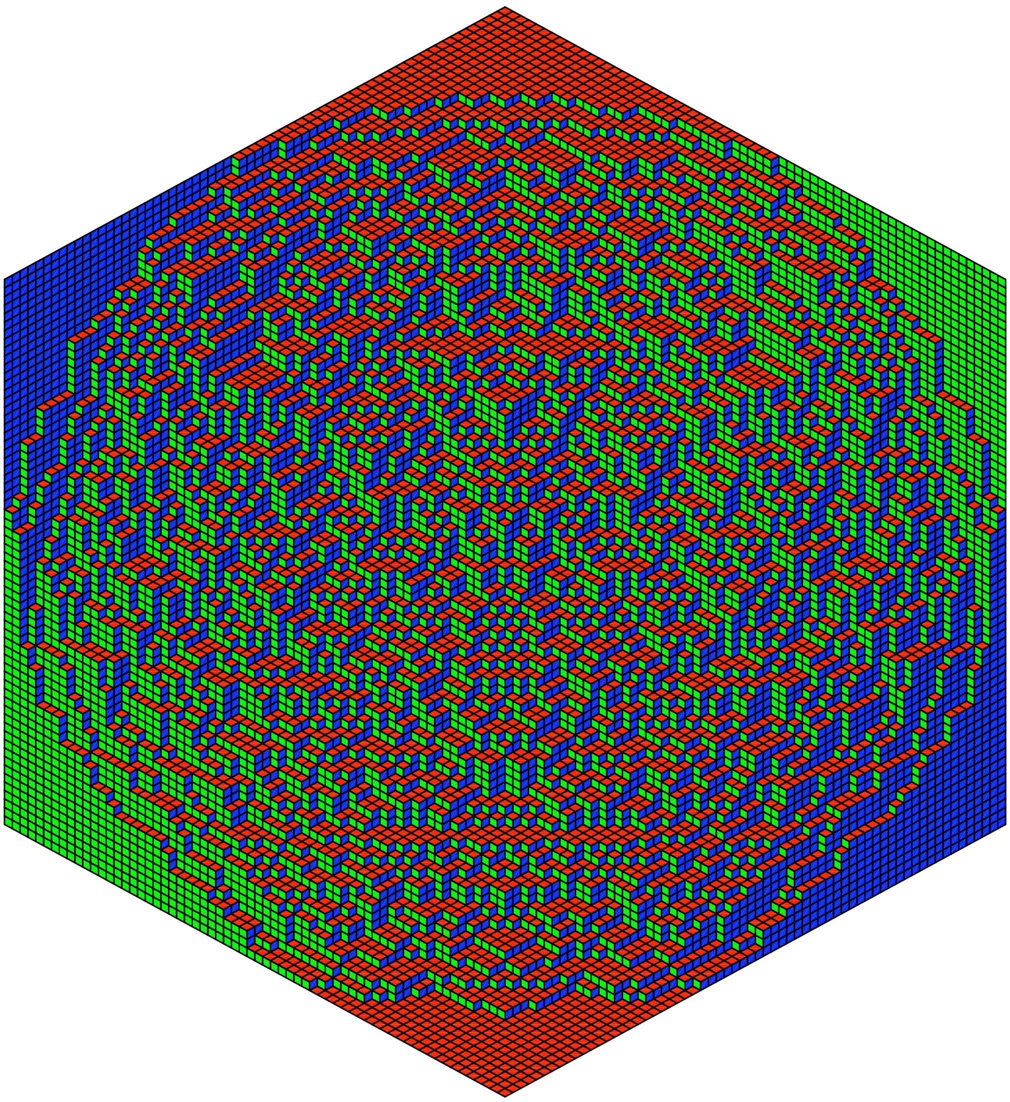

Figure 1.2: Arctic phenomenon in a random tiling of a regular hexagon of side length 64

continuous siblings of these distributions occur in the theory of random matrices as joint distributions of eigenvalues, cf. [For08a, Meh04]. Recently in [BKMM07] it was shown that in the discrete case certain distributions known from the *continuous* (random matrix) case also appear as universal limiting distributions for a class of *discrete* ensembles as above. The most prominent of those is the *Tracy-Widom distribution* [TW94] which governs the fluctuations of the largest eigenvalue of a GUE-matrix (a random Hermitian matrix with its entries normal distributed, cf. [For08a, Meh04]). In the tiling models it describes the fluctuations of the boundary of the arctic region.[2] It is natural to ask the same questions for the half-hexagon model, since similar distributions occur in the corresponding path model. This topic is related to research put forward in [FN08]. In this paper the authors show that the joint probability distribution of the "vertical tiles" in the half-hexagon model in a certain scaling limit coincides with the anti-symmetric GUE-minor process. We study this model with the number of paths and their common length growing simultaneously and proportionally. In addition to the aforementioned arctic phenomenon in this model we can also show the occurrence of some universal random matrix distributions.

The work of Chapter 2 was initiated during the author's stay at the University of Melbourne, where Professor Peter Forrester gave him a personal introductory course in the theory of orthogonal polynomials and random matrices and generously shared his insights how to obtain the orthogonal polynomials involved in the half-hexagon model.

Lattice polygons

The lattice path formulation of the tiling problems gives rise to the natural question for the distribution of the area, say, between two paths or, in a path model with a wall (half-hexagon model, see Figure 2.1), for the area a path encloses with the axis. This problem is well-studied for a single path, e.g. the *Bernoulli meander* and *Bernoulli excursions* (a.k.a. *Dyck paths*). The former are directed lattice paths consisting of n steps of the form $(i, j) \to (i + 1, j \pm 1)$ leading from $(0, 0)$ to (n, a), $a \in \mathbb{Z}_{\geq 0}$ never taking a step below the x-axis, the latter are such paths which are additionally conditioned to end in $(n, 0)$ (n even in this case). These paths occur in computer science, e.g. in connection with algebraic languages [BM92], in chemistry as models for polymers sticking to a wall [AW09] and in various fields of combinatorics and probability theory. The area functionals of those paths can be shown to converge weakly, respectively, to the area functional of the *Brownian meander* and the *Brownian excursion* (Brownian motions in the time interval $[0, 1]$ with similar constraints) [NT04, Tak91, Tak95]. For families of several non-intersecting paths little is known concerning area distributions. In [TW07] the authors computed the first moment of the area below the lowest and highest path in an ensemble of non-intersecting Brownian excursions.

We investigate the various symmetry classes of *staircase polygons* which can be viewed as a pair of directed lattice paths consisting of steps $(i, j) \to (i+1, j)$ and $(i, j) \to (i, j+1)$,

[2] For the Aztec Diamond a stronger result was shown in [Joh05].

sharing only their initial and terminal vertices, see Figure 4.1. We are interested in the so called *fixed perimeter ensembles*, where each staircase polygon with (half-)perimeter n is considered equally probable. Several functionals have been analysed in this ensemble, e.g. the area, radius of gyration and diagonal lengths [Lin07, Ric06]. The area limit law as $n \to \infty$ for the full ensemble of staircase polygons was first found in [Ric06], we re-derive it in Chapter 4, along with the limit laws for the subensembles fixed under the action of the subgroups of the dihedral group. This is done in a unified way, by analysing q-algebraic functional equations for the generating functions of the respective subensembles. Most of the limit laws are the area laws of Brownian excursions and meanders. This work is published in *Combinatorics, Probability and Computing*, [SRT10]. It is joint work with Christoph Richard and Balchandra Thatte. C. Richard suggested the techniques applied to the functional equations, B. Thatte supplied a functional equation for symmetric Dyck paths used at an earlier stage of the work. The unified approach to all symmetry classes was suggested and carried out by the present author.

Staircase polygons can also be viewed as a *solvable* subclass of the class of *self-avoiding polygons* (SAPs) on the two-dimensional square lattice. More general a *self-avoiding walk* (SAW) of *length* N on a d-dimensional hypercubic lattice $\mathbb{Z}^d$ is a nearest neighbour walk starting in the origin, which is not allowed to visit the same vertex twice.[3] A SAP is a SAW whose final vertex is adjacent to the starting vertex, see Figure 1.3. The most natural question to ask is for the *number* c_m of SAWs with m steps (p_m of SAPs of perimeter m, m even) or, equivalently, to ask for the respective generating functions. As easily as it is posed, as hard does it seem to provide rigorous results addressing these two questions in any dimension greater than one. For example, the seemingly obvious inequality $c_{m+1} \geq c_m$ took almost 40 years of study in the area until it was proved in [O'B90]. Notice that a SAW can get *trapped*, e.g. the rightmost walk in Figure 1.3 cannot be extended without violating self-avoidance. In two dimensions there is a result due to Rechnitzer which states that the anisotropic generating function[4] of SAPs on the square lattice cannot be D-finite. A (possibly multivariate) function $f(\mathbf{z})$ is *D-finite*, if the vector space over $\mathbb{C}(\mathbf{z})$ spanned by its derivatives is finite dimensional. In other words, the numbers of SAPs do not satisfy a recursion with a bounded number of terms and polynomial coefficients.

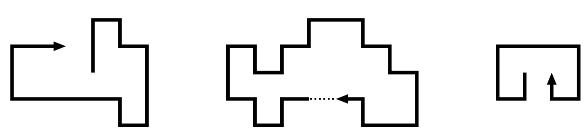

Figure 1.3: A SAW, a SAP and a trapped SAW

A more modest goal are rigorous asymptotics of those numbers. A first step in that

[3] In general one can define self-avoiding walks on an arbitrary undirected graph, e.g. the triangular or honeycomb lattice and also the graphs given by the vertices and edges of certain aperiodic tilings.

[4] The *anisotropic* generating function counts SAWs, resp. SAPs by their numbers of horizontal and vertical edges.

direction was to prove the existence of the *connective constants* [HM54, Ham61, MS93]

$$\lim_{m\to\infty} c_m^{1/m} \text{ and } \lim_{m\to\infty} p_m^{1/m},$$

i.e. the exponential growth rates. The existence of those limits follows from the inequalities

$$c_{m+n} \le c_m \cdot c_n \text{ and } p_{m+n} \ge \frac{p_m \cdot p_n}{d-1}. \tag{1.1}$$

The first is due to the fact that breaking an $m+n$-step SAW after m steps leaves one with two SAWs of respective lengths m and n. The second inequality is true since two arbitrary SAPs can be concatenated at "extremal vertices" (in 2D: highest vertex on the far right of the one polygon, lowest vertex at the far left of the other) [Ham61], yielding a SAP of perimeter $m+n$. In [Ham61] it is also shown that the two above limits coincide. That is, the numbers c_m and p_m are asymptotically of the form const. $\times\, \mu^m h(m)$, where $\lim_{m\to\infty} m^{-1}\log h(m) = 0$. The precise values for μ are unknown for any lattice in any dimension. Very crude bounds for μ for walks in $\mathbb{Z}^d$ are $d \le \mu \le 2d-1$ which follows from the inequalities

$$d^m \le c_m \le 2d(2d-1)^{m-1}.$$

The lower bound being the number of walks of m steps into the *positive* coordinate directions (which are necessarily SAWs), the upper bound being the number of walks *without immediate reversals*. For large d, μ is equal to $2d-1+O(d^{-1})$ [FS59, CS09], i.e. in very high dimensions, the principal effect of self-avoidance is the exclusion of immediate reversals. More sophisticated bounds can be found in [MS93]. For the two-dimensional square lattice extrapolation of series data from exact enumeration has led to high precision estimates of the exponential growth rate according to which μ is, at least numerically, indistinguishable from the positive root of $13x^4 - 7x^2 - 581$ [Gut84, GC01]. For the nearest neighbour SAWs on the two-dimensional honeycomb lattice (non-rigorous) arguments for $\mu = \sqrt{2+\sqrt{2}}$ were put forward in [Nie82].

Some light is shed on the leading asymptotic behaviour of the subexponential corrections $h(m)$. For the SAW model one wishes to prove the *scaling relations*

$$A\, c_m \sim \mu^m m^{\gamma-1} \quad (m\to\infty) \text{ and } \sum c_m z^m \sim \overline{A}\,(1-\mu z)^{-\overline{\gamma}} \quad (z \to 1/\mu)$$

with the so-called *critical exponents* γ and $\overline{\gamma}$.[5] In dimensions $d \ge 5$ the above power law behaviour is *proved* with $\gamma = \overline{\gamma} = 1$ and $A = \overline{A}$ [HS92b, HS92a, MS93]. For SAWs in dimensions $d \le 4$, there is not even a proof of the power law nature, though it is strongly expected. The critical exponents are believed to be $\gamma = 43/32$ [MS93] in $d = 2$ and $\gamma = 1.158\ldots$ in $d = 3$ dimensions [GHJ$^+$00]. In $d = 4$ dimensions $\gamma = 1$ is expected with a logarithmic correction. As opposed to the connective constant, the exponent γ is believed to depend *only* on the dimension and *not* on the particular lattice.

[5] Under certain regularity assumptions on the generating function, one has $\gamma = \overline{\gamma}$, e.g. Δ-regularity, see Appendix B.

For SAPs one also expects relations similar to those above, namely

$$A\,p_m \sim \mu^m m^{\alpha-3} \quad (m \to \infty) \text{ and } \sum p_m z^m \sim \overline{A}\,(1-\mu z)^{2-\overline{\alpha}} \quad (z \to 1/\mu)\,, \tag{1.2}$$

with critical exponents α and $\overline{\alpha}$ conjectured to be equal.[6] The existence of the asymptotic forms are unproved in any dimension $d > 1$, however, one can give rigorous bounds on the exponents in case of existence. By the inequality (1.1) $\alpha \leq 3$ and hence $\overline{\alpha} \leq 3$ is obvious. A more refined bound on $\overline{\alpha}$ is due to the estimate

$$\sum p_m z^m \leq \sum_{m=1}^{\infty} C m^{-(d-1)/2} (\mu z)^{m-1} \quad \text{for all } 0 \leq z \leq 1/\mu, \tag{1.3}$$

where C is a constant only depending on the dimension d [Mad91]. This implies that the SAP generating function can diverge at most like a square root for $d = 2$ and logarithmically for $d = 3$, when z approaches the critical point μ, whereas it remains bounded in all dimensions $d > 3$. To put it differently, $\overline{\alpha} \leq (7-d)/2$. For $d = 2$ the constant C can be chosen such that in inequality (1.3) also holds term-wise, i.e. $p_m \leq Cn^{-1/2}\mu^m$ [Mad95]. These rigorous bounds are far from optimal since non-rigorous methods and extrapolation of series data predict values $\alpha = \overline{\alpha} = 1/2$, $0.23\ldots$ and $2 - d/2$ for $d = 2$, $d = 3$ and $d \geq 4$, respectively.

An approach applied particularly in two dimensions is to study *solvable* subclasses of SAWs and SAPs, i.e. classes for which explicit expressions for the generating functions are available. One hopes to find sufficiently general models which exhibit similar properties as the conjectured ones for general SAWs and SAPs, such as the critical exponents, the mean squared end-to-end distance of SAWs or the area distribution of SAPs. Examples for the latter are the ensembles of staircase polygons of Chapter 4 and the bar graphs of Chapter 5 (see [Duc99] for the discussion of the area law). Though being much less rich than the full class of SAPs, these two classes led to deeper insight into the asymptotic area distribution of SAPs of a fixed large perimeter. Consider the half-perimeter and area generating function $P(t,q)$ of one such (solvable) ensemble, where t marks the half-perimeter and q the area. Common to both ensembles is that $P(t,q)$ satisfies a q-algebraic functional equation, i.e. an equation of the form

$$P(t,q) = G\left(P\left(t,q\right), P\left(tq,q\right), \ldots, P\left(tq^M,q\right), t, q\right) \tag{1.4}$$

with a suitable power series $G\left(y_0, y_1, \ldots, y_M, t, q\right)$, such that at $q = 1$ equation (1.4) becomes algebraic and hence $P(t,1)$ is algebraic. In Chapters 4 and 5 the half-perimeter generating functions of staircase polygons and bar graphs are both shown to be algebraic of degree two and to satisfy the relations (1.2) with $\alpha = \overline{\alpha} = 3/2$ at their respective critical points $1/\mu = 0.25$ and $0.2955977\ldots$. Now due to a general result first stated in [Duc99][7], a functional equation (1.4), a critical exponent $3/2$ in (1.2) and some properties of the series

[6] This definition of α is traditional in statistical mechanics.

[7] The result is stated there incorrectly and without a proof. A more general result is proved in [Ric09b].

G together imply, that the limiting distribution of the (properly scaled) sequence X_m of discrete random variables given by

$$\mathbb{P}(X_m = k) = \frac{\left[t^m q^k\right] P(t,q)}{[t^m]\, P(t,1)}$$

is the *Airy distribution.* From exact enumeration it was noticed that the generating function of *rooted* SAPs exhibits the same critical behaviour (with $\overline{\alpha} = \alpha = 3/2$), which gave rise to the question whether rooted SAPs also have a limiting area distribtion of Airy type. There are non-rigorous arguments assuming an equation (1.4) for SAPs and compelling numerical data supporting this conjecture [RGJ01].

In Chapter 5 we partially solve a subclass of rooted SAPs on the square lattice, which recently received some interest, the so-called *prudent polygons* (PP). These are characterised by the property that their boundary walks, w.l.o.g. starting in the origin, never take a step *towards* an already occupied vertex, i.e. a step is made "only if the road is perfectly clear". The full class of PPs remains unsolved, however we can give the generating functions of two natural subclasses, the one of which turns out to consist essentially of bar graphs. The second class is richer, it contains the bar graphs, but its half-perimeter generating function is proved not to be D-finite and hence non-algebraic. Hence the half-perimeter and area generating function does not satisfy an equation (1.4). We can show the critical exponents to be $\alpha = \overline{\alpha} = 3/2$. We have not studied the area of those objects yet. This work is accepted for publication in *European Journal of Combinatorics* [Sch10].

Part I

Hexagonal random tilings

Chapter 2

Random matrix distributions and arctic phenomenon in the half-hexagon model

Tilings of a hexagon with unit rhombi recently received interest of combinatorialists and probabilists alike. The former addressed enumerational questions, the first of which was answered by Mac Mahon in [MM15]. In the 1980s and 1990s combinatorialists also solved all enumerational problems for dihedral symmetry classes of such tilings, see [Bre99] and also Chapter 3. Probabilists, on the other hand, studied certain correlations of tiles in large tilings drawn uniformly at random and found distributions which also govern the fluctuations of eigenvalues of large random matrices [Joh02, BKMM07]. Further interest comes from a macroscopic effect a large random tiling exhibits, namely that it looks "periodic" at the corners, and "unordered" in the interior with a sharp transition along the inscribed ellipse of the hexagon [CLP98, Joh02, BKMM07], see Figure 1.2.

So far, the above mentioned symmetry classes have not received as much attention from the probabilistic community. In [FN08] the authors study tilings of the so-called half-hexagon and their close connection to a certain ensemble of random matrices. These tilings are equivalent to one symmetry class. We can extend the results of [Joh02, BKMM07] to the half-hexagon and, as a by-product, also prove an "arctic phenomenon" in that model, cf. Figure 2.3.

2.1 The models

The (Q, R, S)-*hexagon* is a hexagon with integer side lengths Q, R, S, Q, R, S and every angle equal to $120°$. We study tilings thereof with $60°$ unit rhombi referred to as *lozenges* or simply *tiles*. The hexagon is filled without gaps and overlap, and no tile juts out beyond the boundary ("fixed boundary conditions"). The tiles occur in three different species (orientations) referred to as *up-*, *vertical* and *down*-tiles, see figure 2.1. As we are interested in a certain symmetry class we restrict to hexagons with $Q = 2p$ and $R = S$.

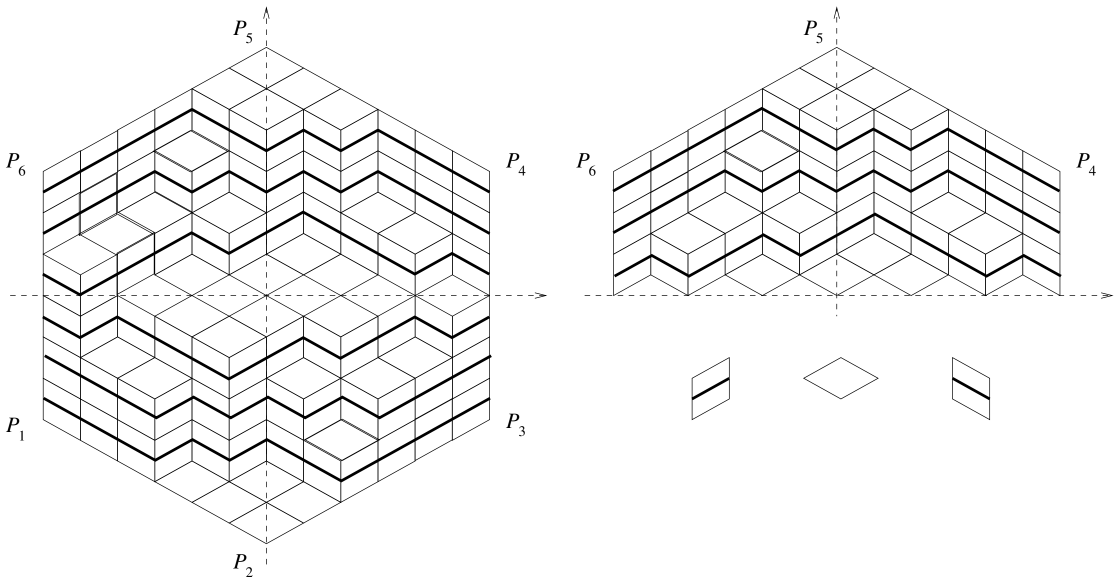

Figure 2.1: Tiling of a $(2p, R, R)$-hexagon (without symmetry), tiling of a (p, R, R)-half-hexagon, up-, vertical and down-tiles

To quantify things, we fix an ON-coordinate system and look at (symmetric) tilings of the $(2p, R, R)$-hexagon with corners $(\pm\sqrt{3}R/2, \pm p)$ and $(0, \pm(p + R/2)$. The corners are numbered in counterclockwise order starting with $P_1 = (-\sqrt{3}R/2, -p)$. If we focus upon the symmetry class of tilings w.r.t. the reflection in the x-axis, we can throw away the part below the x-axis and the chopped-in-half vertical tiles on the axis and obtain a tiling of the so-called (p, R, R)-*half-hexagon*, a model studied in [FN08]. Now the definition of the *arctic region at the corner* P_n, $n = 1, \ldots, 6$, is as follows [Joh02]. It is empty, if P_n is contained in two tiles of different species. Otherwise P_n is contained in a single tile T_0 at P_n, and a tile T' of the same species as T_0 belongs to this arctic region, if and only if there is a sequence $T_0, T_1, \ldots, T_k = T'$ all of the same species as T_0 such that T_i and T_{i+1} share an edge for $i = 0, \ldots, k-1$. The complement of the union of all arctic regions is called the *temperate region*.

Remark. If the side lengths of the hexagon tend to infinity in such a way that the ratios of each two side lengths tend to some constant, then the probability for an *empty* arctic region in a tiling decays exponentially. For tilings of the (p, R, R)-half-hexagon this is seen as follows. According to [KGV00] the number of those tilings is equal to

$$f(p) = \prod_{i=1}^{p} \frac{(l-1)!(p+l-1)!(2R+2l-2)!}{(2l-2)!(R+l+p-1)!(R+l-1)!}.$$

The arctic regions at P_4 and P_6 are always non-empty by the geometry of the problem. If the arctic region at P_5 is empty then P_5 is contained in an up- and a down-tile. But this completely determines the species of the tiles along the sides of the hexagon joining

P_5 and P_4 (all down-tiles) and P_5 and P_6 (all up-tiles). If we "slice off" these tiles we end up with a tiling of the $(p-1, R, R)$-hexagon. So in a uniform ensemble the probability for an empty arctic region is $f(p-1)/f(p)$, which leads to the claimed behaviour after some elementary calculations and an application of Stirling's formula to the occurring factorials. In the case of an (r, s, t)-hexagon one argues similarly employing the "$q = 1$"-version of formula (3.1).

If we fix a vertical line with abscissa $x = (-R+m)\sqrt{3}/2$, $0 \leq m \leq R$, that intersects the arctic region of P_6, then the lower boundary of that arctic zone is marked by the topmost vertical lozenge on that line, see figure 2.2.

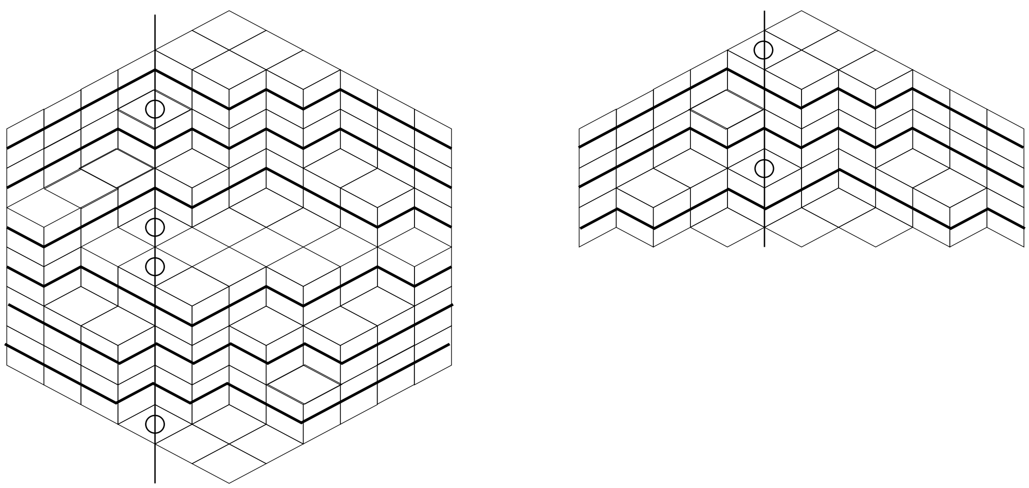

Figure 2.2: Configurations of intersection points. L_m consists of all intersection points of paths with the vertical line and the circles.

So, in order to asymptotically analyse the boundary of the arctic region at P_6 one has to study the asymptotic behaviour of the probabilities of finding a vertical lozenge at a given point of the line, i.e. the *one-point correlation function* (see Equation (2.5)) of the vertical lozenges on that line. In the arctic region at P_6 it should be asymptotically equal to zero (a *void*, cf. Section 2.3 below), and strictly between zero and one in the temperate region. The situation at P_5 is somewhat dual. If the line intersects the arctic region at P_5 (consisting of vertical tiles), the lower boundary is marked by the topmost non-vertical tile (up- or down-tile). The one-point correlation function of the vertical tiles should be asymptotically equal to one (a *saturated region*, cf. Section 2.3) in that arctic region and strictly between zero and one in the temperate region.

In order to compute the above mentioned correlation functions we need to know the probability of a given configuration of vertical tiles along a given line, assuming the uniform distribution of all tilings of the $(2p, R, R)$-hexagon, resp. (p, R, R)-half-hexagon. Recall that tilings of the full $(2p, R, R)$-hexagon map bijectively to families of $2p$ non-intersecting paths on the point lattice

$$L = \left\{ \left(\frac{-\sqrt{3}}{2}R, \frac{1}{2} \right) + q \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right) + r \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right),\ q, r \in \mathbb{Z} \right\}$$

with starting points in the set $S = S^+ \cup S^-$ and end points in $E = E^+ \cup E^-$, where

$$S^+ = \left\{ \left(-\frac{\sqrt{3}}{2}R, i + \frac{1}{2} \right),\ i = 0 \ldots p-1 \right\},\ E^+ = \left\{ \left(\frac{\sqrt{3}}{2}R, i + \frac{1}{2} \right),\ i = 0 \ldots p-1 \right\},$$

and S^- (resp. E^-) denotes the reflection of S^+ (resp. E^+) in the x-axis. Admissible steps are $(\sqrt{3}/2, \pm 1/2)$. To see this, connect in each up- and in each down-tile the midpoints of the vertical sides by a straight line segment (the decoration depicted in figure 2.1). In the same fashion tilings of the half-hexagon are mapped to families of p non-intersecting paths with starting points in S^+ and end points in E^+ which *do not touch the x-axis*. Denote by $L_m,\ m = 0, \ldots, 2R$ the intersection of the vertical line $x = (-R+m)\sqrt{3}/2$ with the lattice L and the $(2p, R, R)$-hexagon. L_m is the set of possible points where a family of lattice paths corresponding to a tiling can intersect the line after m steps. Denote by L_m^+ those points of L_m with positive ordinate.

Proposition 2.1.1 (Theorem 4.1 in [Joh02], Lemma 2.2 in [FN08])**.** *Consider the sets of families of $2p$ non-intersecting lattice paths with starting points in S and end points in E (tilings of the $(2p, R, R)$-hexagon) and of families of p such lattice paths with starting points in S^+ and end points in E^+ not touching the x-axis ((p, R, R)-half-hexagon) to be equipped with the respective uniform distributions.*

1. *Let $x_1 < x_2 < \ldots < x_{2p}$ be chosen such that $\left((-R+m)\sqrt{3}/2, x_j\right) \in L_m$ for $i = 1, \ldots, 2p$. Then the probability of a family of lattice paths to intersect the vertical line $x = (-R+m)\sqrt{3}/2$ at ordinates $x_1, \ldots, x_{2p}$ is equal to*

$$\tilde{P}_m(x_1, x_2, \ldots, x_{2p}) = \frac{1}{\tilde{Z}_m} \prod_{i=1}^{2p} \tilde{w}(x_i) \prod_{1 \le i < j \le 2p} (x_j - x_i)^2. \tag{2.1}$$

2. *Let $x_1 < x_2 < \ldots < x_p$ be chosen such that $\left((-R+m)\sqrt{3}/2, x_j\right) \in L_m^+$ for $i = 1, \ldots, p$. Then the probability of a family of p lattice paths not touching the x-axis to intersect the vertical line $x = (-R+m)\sqrt{3}/2$ at ordinates $x_1, \ldots, x_p$ is equal to*

$$\tilde{P}_m^{\mathrm{sym}}(x_1, x_2, \ldots, x_p) = \frac{1}{\tilde{Z}_m^{\mathrm{sym}}} \prod_{i=1}^{p} x_i^2 \tilde{w}(x_i) \prod_{1 \le i < j \le p} \left(x_j^2 - x_i^2\right)^2. \tag{2.2}$$

In both cases the weight function $\tilde{w}$ is even *and it is equal to*

$$\tilde{w}(z) = \frac{1}{(m/2 + p - 1/2 \pm z)!(R - m/2 + p - 1/2 \pm z)!}. \tag{2.3}$$

$\tilde{Z}_m$ and $\tilde{Z}_m^{\mathrm{sym}}$ are normalisation constants.

Probability distributions of the form (2.1) are known as *orthogonal polynomial ensembles*. Their continuous counterparts are well studied objects in the theory of random matrices, cf. [Meh04, For08a].

2.2 Discrete orthogonal polynomials

In the monograph [BKMM07] asymptotic results on general discrete orthogonal polynomial ensembles are established, which can be used to prove the arctic ellipse phenomenon in the full and half-hexagon. We briefly summarise the results which are relevant in our context.

2.2.1 The general ensembles

Given a positive weight function w_N defined on a set of nodes $X_N = \{x_{N,0}, \ldots, x_{N,N-1}\}$ contained in an interval $[a, b]$, the probability distribution on the set of k-tuples $(x_1, \ldots, x_k) \in X_N^k$ with $x_1 < x_2 < \ldots < x_k$ $(k \leq N)$, given by

$$p^{(N,k)}(x_1, x_2, \ldots, x_k) = \frac{1}{Z_k} \prod_{i=1}^{k} w_N(x_i) \prod_{1 \leq i < j \leq k} (x_j - x_i)^2 \tag{2.4}$$

is called *discrete orthogonal polynomial ensemble*. Here Z_k is a normalisation constant. We refer to this ensemble as $DOPE(N, k)$. $p^{(N,k)}(x_1, \ldots, x_k)$ can be viewed as the probability of finding a configuration of k particles located at the sites $x_1, \ldots, x_k \in X_N$. A scalar product on the set of complex valued functions on X_N is associated to the weight function w_N via

$$(f, g) \mapsto \sum_{i=0}^{N-1} w_N(x_{N,i}) \, f(x_{N,i}) \, \overline{g(x_{N,i})}.$$

By applying the Gram-Schmidt procedure to the sequence of monomials $1, x, \ldots x^{N-1}$ we obtain a family of *orthonormal polynomials* $p_{N,0}, \ldots, p_{N,N-1}$, i.e. the degree of $p_{N,j}$ is equal to j and the relation

$$\sum_{i=0}^{N-1} w_N(x_{N,i}) \, p_{N,k}(x_{N,i}) \, \overline{p_{N,l}(x_{N,i})} = \delta_{kl}$$

holds. Note that since the nodes, the weights and the coefficients of the $p_{N,k}$ are real, we can omit complex conjugation. Furthermore the leading coefficient $\gamma_{N,k}$ of $p_{N,k}$ is assumed to be positive. Denote by $\pi_{N,k} := \gamma_{N,k}^{-1} p_{N,k}$ the kth monic orthogonal polynomial. The m-point correlation functions $R_{N,k}^m(x_1, \ldots, x_m)$ describe the probability that a configuration of k particles contains particles at each of the m sites $x_1, \ldots, x_m$ $(m \leq k)$. In particular the one-point correlation function $R_{N,k}^1(x)$ equals the probability of finding a particle at x. We have

$$\begin{aligned} R_{N,k}^m(x_1, \ldots, x_m) &= \mathbb{P}(\text{particles at each of the sites } x_1, \ldots, x_m) \\ &= \det \left(K_{N,k}(x_i, x_j) \right)_{i,j=1,\ldots,m}, \end{aligned} \tag{2.5}$$

where for $x, y \in X_N$ the correlation kernel $K_{N,k}(x, y)$ is given by

$$\begin{aligned} K_{N,k}(x,y) &= \sqrt{w_N(x)w_N(y)}\sum_{n=0}^{k-1} p_{N,n}(x)p_{N,n}(y) \\ &= \sqrt{w_N(x)w_N(y)} \cdot \frac{\gamma_{N,k-1}}{\gamma_{N,k}} \cdot \frac{p_{N,k}(x)p_{N,k-1}(y) - p_{N,k}(y)p_{N,k-1}(x)}{x-y} \end{aligned} \tag{2.6}$$

if $x \neq y$, and otherwise

$$K_{N,k}(x,x) = w_N(x) \cdot \frac{\gamma_{N,k-1}}{\gamma_{N,k}} \cdot \left(p'_{N,k}(x)p_{N,k-1}(x) - p'_{N,k-1}(x)p_{N,k}(x)\right). \tag{2.7}$$

The second "=" in equation (2.6) is known as Christoffel-Darboux formula. The derivations of both the particular form of the correlation functions and of this latter summation formula are carried out in Appendix A, see also [For08a, Meh04].

2.2.2 Even weights

In the study of the half-hexagon problem we come across a probability distribution involving a weight function w_{2N} defined on a set of nodes $X_{2N} = Y_N \cup -Y_N$, where $Y_N = \{x_{N,0}, \ldots, x_{N,N-1}\}$ and $0 < x_{N,0} < \ldots < x_{N,N-1}$. Furthermore w_{2N} is assumed to be *even*, i.e. $w_{2N}(x) = w_{2N}(-x)$. In this section we are concerned with the probability distributions on the set of k-tuples in $(x_1, \ldots, x_k) \in Y_N^k$, $x_1 < \ldots < x_k$, given by

$$p_{\text{sym}}^{(N,k)}(x_1, x_2, \ldots, x_k) = \frac{1}{Z_k^{\text{sym}}} \prod_{i=1}^{k} x_i^2 w_{2N}(x_i) \prod_{1 \le i < j \le k} \left(x_j^2 - x_i^2\right)^2, \tag{2.8}$$

where again Z_k^{sym} is a normalisation constant, compare formula (2.2). This ensemble is referred to as $DOPE^{\text{sym}}(N, k)$. We can also obtain determinantal representations for the m-point correlation functions as in the orthogonal polynomial ensemble (2.4). To this end we need monic polynomials $q_j(z)$ of degree j, $j = 0, \ldots, N-1$ with the property

$$\sum_{x \in Y_N} x^2 w_{2N}(x) q_i\left(x^2\right) q_j\left(x^2\right) = \frac{\delta_{ij}}{\epsilon_i^2}.$$

Once these are at hand, we can repeat the computations of Section A.1 in Appendix A and find a determinantal representation of the m-point correlation function with kernel

$$K_{N,k}^{\text{sym}}(x,y) = \sqrt{x^2 w_{2N}(x)}\sqrt{y^2 w_{2N}(y)} \sum_{n=0}^{k-1} \epsilon_n^2 q_n(x) q_n(y).$$

Peter Forrester [For08b] pointed out how these polynomials q_j can be obtained. Consider the monic polynomials orthogonal w.r.t. w_{2N} on the set of nodes X_{2N},

$$\sum_{x \in X_{2N}} w_{2N}(x) \pi_{2N,i}(x) \pi_{2N,j}(x) = \frac{1}{\gamma_{2N,i}^2} \delta_{ij}.$$

Since w_{2N} is even, it follows easily from the Gram-Schmidt procedure that $\pi_{2N,2j}(x)$ is even and $\pi_{2N,2j+1}(x)$ is odd. In particular we have $\pi_{2N,2j+1}(x) = xq_j\left(x^2\right)$ for a monic polynomial q_j of degree j. The q_j satisfy

$$\begin{aligned}&\sum_{x\in Y_N} x^2 w_N(x) q_i\left(x^2\right) q_j\left(x^2\right)\\ =&\sum_{x\in Y_N} w_N(x)\pi_{2N,2i+1}\left(x\right)\pi_{2N,2j+1}\left(x\right)\\ =&\frac{1}{2}\sum_{x\in X_{2N}} w_N(x)\pi_{2N,2i+1}\left(x\right)\pi_{2N,2j+1}\left(x\right) = \frac{1}{2\gamma_{2N,2i+1}^2}\delta_{ij}\end{aligned}$$

and hence are the sought for polynomials. For $x, y \in Y_N$ the correlation kernel for the ensemble (2.8) can be written as

$$\begin{aligned}K_{N,k}^{\mathrm{sym}}(x,y) =&\sqrt{x^2 w_{2N}(x)}\sqrt{y^2 w_{2N}(y)}\sum_{n=0}^{k-1} 2\gamma_{2N,2n+1}^2 q_n\left(x\right) q_n\left(y\right)\\ =&\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\sum_{n=0}^{k-1} 2\gamma_{2N,2n+1}^2 \pi_{2N,2n+1}(x)\pi_{2N,2n+1}(y)\\ =&2\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\sum_{n=0}^{k-1} p_{2N,2n+1}(x)p_{2N,2n+1}(y)\\ =&\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\left[\sum_{n=0}^{2k-1} p_{2N,n}(x)p_{2N,n}(y) - \sum_{n=0}^{2k-1} p_{2N,n}(x)p_{2N,n}(-y)\right]\\ =&K_{2N,2k}(x,y) - K_{2N,2k}(x,-y).\end{aligned} \tag{2.9}$$

In the following we will show that for $x, y \in Y_N$ and $x, y > \varepsilon > 0$ the summand $K_{2N,2k}(x,-y)$ tends to zero in the considered limit, and hence the correlation kernels $K_{2N,2k}$ and $K_{N,k}^{\mathrm{sym}}$ are asymptotically the same.

2.2.3 Assumptions

In applications as the arctic circle phenomenon, the asymptotic behaviour of $p_{N,k}(z)$ and $K_{N,k}$ as N and k *simultaneously* tend to infinity plays a crucial role. The asymptotic results are obtained under some technical assumptions [BKMM07, Section 1.2] on the weight, nodes and degree. These are in particular fulfilled in the situation of the hexagonal tilings.

The nodes

We assume the nodes to be equidistributed in an interval $[a, b]$ of length 1, more precisely $b = a + 1$ and

$$x_{N,n} = a + \frac{2n+1}{2N},\ n = 0, \ldots, N-1. \tag{2.10}$$

Remark. In [BKMM07] the existence of a *node density function* ρ^0 is assumed which is real-analytic in a complex neighbourhood of $[a, b]$, strictly positive in $[a, b]$ and satisfies a normalisation condition and a certain quantisation rule. We simply assume the special case $\rho^0 \equiv 1$.

The weight function

We assume that we can write the weight function in the form

$$w_N\left(x_{N,n}\right) = e^{-NV_N\left(x_{N,n}\right)} \prod_{\substack{m=0 \\ n \neq m}}^{N-1} \left|x_{N,n} - x_{N,m}\right|^{-1}, \tag{2.11}$$

where $V_N(x)$ is a real-analytic function defined in a neighbourhood G of $[a, b]$. Furthermore

$$V_N(x) = V(x) + \frac{\eta(x)}{N}, \tag{2.12}$$

where $V(x)$ is a fixed real-analytic potential function independent of N and

$$\limsup_{N \to \infty} \sup_{z \in G} |\eta(z)| < \infty.$$

As opposed to $V(x)$, the correction $\eta(x)$ may depend on N.

The degree

The degree k and the number of nodes N are related by

$$k = cN + \kappa, \tag{2.13}$$

where $c \in (0, 1)$ and κ remains bounded as $N \longrightarrow \infty$.

Further assumptions are difficult to express explicitly in terms of the nodes and the weight and postponed to Section 2.3.1.

2.2.4 Particle-hole duality

We have interpreted the distribution (2.4) as the probability of a configuration of k particles (vertical tiles in the arctic circle problem). This also induces a distribution of *hole configurations*, i.e. the probability $\overline{p}^{(N,N-k)}(y_1, \ldots, y_{N-k})$ to find the sites $y_1, \ldots, y_{N-k}$ unoccupied. A computation [BKMM07, Section 3.2] shows that

$$\overline{p}^{(N,N-k)}(y_1, \ldots, y_{N-k}) = \frac{1}{\overline{Z}_{N,N-k}} \prod_{j=1}^{N-k} \overline{w}_N(y_j) \prod_{1 \leq i < j \leq N-k} (y_i - y_j)^2, \tag{2.14}$$

where

$$\overline{w}_N(y_j) = \frac{1}{w_N(y_j)} \prod_{\substack{n=0 \\ y_j \neq x_{N,n}}}^{N-1} (y_j - x_{N,n})^{-2}.$$

We call the ensemble (2.14) *dual* to (2.4). Note that $\overline{w}_N$ also has a representation (2.11) with V_N replaced by $-V_N$ and hence V by $-V$. Let $\overline{K}_{N,N-k}$ be the correlation kernel for the ensemble (2.14), then by [BKMM07, Propositions 7.2 and 7.3] we have for $x_{N,m}, x_{N,n}, x \in X_N$, $x_{N,m} \neq x_{N,n}$:

$$\begin{aligned} \overline{K}_{N,N-k}\left(x_{N,m}, x_{N,n}\right) &= (-1)^{m+n+1} K_{N,k}\left(x_{N,m}, x_{N,n}\right) \text{ and} \\ \overline{K}_{N,N-k}\left(x, x\right) &= 1 - K_{N,k}(x, x). \end{aligned} \tag{2.15}$$

Remark. The notion of duality here differs from the notion in [KS98].

2.3 The associated equilibrium energy problem

The asymptotic behaviour of $p_{N,k}$ and the asymptotic distribution of the zeroes in the interval $[a, b]$ can be expressed in terms of quantities arising in a related constrained variational problem [KR99]. Define a real-analytic function φ by

$$\varphi(x) := V(x) + \int_a^b \log|x-y| \mathrm{d}y. \tag{2.16}$$

Note that according to the representation (2.11) $w_N(x)$ is asymptotically equal to

$$w_N(x) \sim e^{-N\varphi(x) - \eta(x)}. \tag{2.17}$$

Further define a quadratic functional of Borel measures on $[a, b]$ by

$$E_c[\mu, V] := E_c[\mu] := c \int_a^b \int_a^b \log \frac{1}{|x-y|} \mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int_a^b \varphi(x)\mathrm{d}\mu(x). \tag{2.18}$$

Such *energy functionals* are encountered in electrostatics where the measure μ describes the distribution of charges in a conductor and $E_c[\mu]$ the energy of that distribution. The double integral models the interaction of the charges and the single integral the influence of the *external field* $\varphi(x)$. We are looking for a measure $\mu_{\min}^c$ which minimises $E_c[\mu]$ subject to the normalisation condition ("total charge")

$$\int_a^b \mathrm{d}\mu(x) = 1 \tag{2.19}$$

and the upper and lower constraints

$$0 \leq \int_{x \in B} \mathrm{d}\mu(x) \leq \frac{1}{c} \int_{x \in B} \mathrm{d}x \text{ for every a Borel set } B \text{ in } [a, b]. \tag{2.20}$$

We refer to $\mu^c_{\min}$ as the *equilibrium measure*. The latter constraints are due to the fact that all zeroes of $p_{N,k}$ are contained in the interval $[x_{N,0}, x_{N,N-1}]$ and a closed interval $[x_{N,n}, x_{N,n+1}]$ between two consecutive nodes contains at most one zero of $p_{N,k}$. Note that minimising $E_c[\mu]$ simply subjected to the normalisation condition (2.19) is formally like seeking a critical point of

$$F_c[\mu] = E_c[\mu] - l_c \int_a^b \mathrm{d}\mu(x)$$

with a Lagrange multiplier $l_c := l_c[V]$. When $\mu = \mu^c_{\min}$, l_c is a real constant. $\mu^c_{\min}$ is known to be unique and it has a piecewise analytic density $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x$. Points of non-analyticity are finite in number and do not occur in points where the upper and lower constraints $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x > 0$ and $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x < 1/c$ hold strictly and simultaneously.

2.3.1 Further assumptions: the equilibrium measure

The constraints give rise to the following

Definition 2.3.1. *A* band *is a maximal open subinterval of* $[a, b]$ *where* $\mu^c_{\min}$ *is a measure with a real-analytic density* $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x$ *which satisfies* $0 < \mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x < 1/c$. *A* void *is a maximal open subinterval of* $[a, b]$ *in which* $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x \equiv 0$, *i.e. meets the* lower *constraint. A* saturated region *is a maximal open subinterval of* $[a, b]$ *in which* $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x \equiv 1/c$ *and hence meets the* upper *constraint. If no stress is put on the active constraint, voids and saturated regions are referred to as* gaps.

As announced in Section 2.2.3 we make some further assumptions on the weight and the nodes which are expressed in terms of the equilibrium measure, cf. [BKMM07, Section 2.1.2]. For our applications we assume that there is exactly one non-empty band $I = (\alpha, \beta)$ and two non-empty gaps $[a, \alpha)$ and $(\beta, b]$ (a constraint being active at each end point a, b). Furthermore we make the following assumptions on the behaviour of $\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x$ at endpoints of bands. Let $z_0 \in \{\alpha, \beta\}$ be a band end point. If the gap at z_0 is a void, then

$$\lim_{x\to z_0,\, x\in I} \frac{1}{\sqrt{|x - z_0|}} \frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x) = K, \quad \text{with } 0 < K < \infty.$$

Similarly, if the gap at z_0 is a saturated region, we suppose that

$$\lim_{x\to z_0,\, x\in I} \frac{1}{\sqrt{|x - z_0|}} \left(\frac{1}{c} - \frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x) \right) = K, \quad \text{with } 0 < K < \infty.$$

So the constraints are met like a square root.

Remark. The one-point correlation function in the hexagonal tiling problem is shown to converge pointwise to the density of the corresponding equilibrium measure in [BKMM07, Theorem 3.12], see also Prop. 2.4.1. The band corresponds to the intersection of a vertical line with the temperate zone, the surrounding gaps to the intersection of the line with the arctic regions.

2.3.2 Quantities related to the equilibrium measure

We now define the quantities involved in the asymptotic expressions for $\pi_{N,k}$ in a gap Γ and the band I, cf. [BKMM07, Sect. 2.1.4]. The variational derivative of $E_c[\mu]$ evaluated at $\mu = \mu^c_{\min}$ is equal to

$$\frac{\delta E_c}{\delta \mu}(x) := \left.\frac{\delta E_c[\mu, V]}{\delta \mu}\right|_{\mu=\mu^c_{\min}}(x) = -2c\int_a^b \log|x-y|\mathrm{d}\mu^c_{\min}(x) + \varphi(x).$$

We have

$$\frac{\delta E_c}{\delta \mu}(x) - l_c \begin{cases} > 0 & \text{if } x \text{ is in a void,} \\ \equiv 0 & \text{if } x \text{ is in a band,} \\ < 0 & \text{if } x \text{ is in a saturated region.} \end{cases} \tag{2.21}$$

The function $\frac{\delta E_c}{\delta \mu} - l_c$ defined in a gap Γ extends analytically from the interior. Furthermore for a gap Γ we have the function

$$\overline{L}^{\Gamma}_c(z) := c\int_a^b \log|z-x|\mathrm{d}\mu^c_{\min}(x), \text{ for } z \in \Gamma,$$

which is analytic in z if $\Re z \in \Gamma$ and $\Im z$ sufficiently small.[1] Similarly define

$$\overline{L}^{I}_c(z) := c\int_a^b \log|z-x|\mathrm{d}\mu^c_{\min}(x), \text{ for } z \in I.$$

This function has an analytic continuation to a neighbourhood of $\overline{I} = [\alpha, \beta]$.

2.3.3 Equilibrium measure for the dual ensemble

There is also an equilibrium problem related to the dual ensemble (2.14), namely minimising the quadratic functional $E_{1-c}[\mu; -V]$ defined as in equation (2.18) under the constraints (2.19) and (2.20) (with c replaced by $1-c$). The unique solution is $\overline{\mu}^{1-c}_{\min}$ with density

$$\frac{\mathrm{d}\overline{\mu}^{1-c}_{\min}}{\mathrm{d}x}(x) = \frac{1}{1-c}\left(1 - c\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)\right).$$

In particular it follows that a void (a saturated region) for $\mu^c_{\min}$ is a a saturated region (void) for $\overline{\mu}^{1-c}_{\min}$. Furthermore we have the equality for the variational derivatives

$$\left.\frac{\delta E_{1-c}[\overline{\mu}, -V]}{\delta \overline{\mu}}\right|_{\overline{\mu}=\overline{\mu}^{1-c}_{\min}} = -\left.\frac{\delta E_c[\mu, V]}{\delta \mu}\right|_{\mu=\mu^c_{\min}},$$

and for the corresponding Lagrange multipliers we have

$$l_{1-c}[-V] = l_c[V].$$

[1] We denote by $\Re z$ the real part and by $\Im z$ the imaginary part of a complex number z.

2.4 Asymptotics of the polynomials and correlation functions

The main results in [BKMM07] are the asymptotic behaviours for $N \to \infty$ of the monic polynomials $\pi_{N,k}(z)$ for any $z \in \mathbb{C}$ and the kernels $K_{N,k}$. The statement of part of their results is easier under the following

Assumption VBV. The two gaps Γ_1 and Γ_2 enclosing the band I are assumed to be *voids.*

Remark. Since our goal is to extend asymptotic results for the $DOPE(2N, 2k)$ ensemble (full hexagon) to the $DOPE^{\mathrm{sym}}(N, k)$ ensemble (half hexagon), we study even weights which leads to symmetric equilibrium measures. Due to our additional "only-one-band-at-a-time"-assumption (cf. 2.3.1) we hence have a void-band-void or a saturated-band-saturated situation. By duality it suffices to study the former.

With the help of the functions defined in the last section we are now able to state the asymptotic results. The first is on leading coefficients.

Proposition 2.4.1 (Theorem 2.8 in [BKMM07])**.** *The leading coefficients $\gamma_{N,k}$ of $p_{N,k}(z)$ and $\gamma_{N,k-1}$ of $p_{N,k-1}(z)$ satisfy the asymptotic relations:*

$$\gamma_{N,k}^2 = \frac{4}{\beta - \alpha} e^{Nl_c + \gamma}(1 + O(1/N))$$

and

$$\gamma_{N,k-1}^2 = \frac{\beta - \alpha}{4} e^{Nl_c + \gamma}(1 + O(1/N))$$

where γ remains bounded as $N \longrightarrow \infty$.

Proposition 2.4.2 (Theorems 2.9, 2.13 and 2.15 in [BKMM07])**.** *In suitable neighbourhoods of the different subintervals of $[a, b]$ we have the following asymptotic representations of $\pi_{N,k}$.*

1. *Assume that J is a closed subinterval in a void Γ. Then there is a neighbourhood K_J of J, and function $A^\Gamma(z)$ analytic on K_J and uniformly bounded in N, such that*

   $$\pi_{N,k}(z) = e^{N\overline{L}_c^\Gamma(z)} \left(A^\Gamma(z) + O(1/N)\right)$$

   holds.

2. *Assume that F is a closed subinterval in a band I. Then there is a neighbourhood K_F of F, and a sequence of analytic functions $B_N^\Gamma(z)$ defined on K_J and uniformly bounded in N, such that*

   $$\pi_{N,k}(z) = e^{N\overline{L}_c^I(z)} \left(B_N^\Gamma(z) + O(1/N)\right).$$

3. *Let z_0 be a band endpoint. Then there is $r > 0$ and sequence of functions $C_N(z)$ analytic in $|z - z_0| < r$ and uniformly bounded for $N \to \infty$, such that for $|z - z_0| < r$*

$$\pi_{N,k}(z) = e^{N\overline{L}^I_c(z)} N^{1/6} \left(C_N(z) + O(1/N^{1/3}) \right).$$

Lemma 2.4.3 (Lemma 7.4 in [BKMM07])**.** *Under Assumption VBV we have the exact formula for any node $x \in X_N$*

$$\sqrt{w(x)} = \frac{1}{\sqrt{2\pi N}} e^{-\frac{1}{2}\eta(x)} e^{-\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(x) - l_c\right)} e^{-\frac{1}{2}(Nl_c+\gamma)} e^{-N\int_a^b \log|x-y| \mathrm{d}\mu^c_{\min}(y)} T_N(x)^{\frac{1}{2}}$$

where the function

$$T_N(z) = 2\cos\left(\frac{2\pi N(b-z)}{2}\right) \frac{1}{\prod_{x_{N,n}\in X_N}(z - x_{N,n})} e^{N\int_a^b \log|z-x|\mathrm{d}x}$$

is real analytic in (a, b) and bounded independently of N.

Remark. *i)* $T_N(z)$ has no poles at the nodes as the zeroes in the denominator are cancelled by the cosine. Furthermore the product in the denominator is asymptotically equal to the exponential for $N \to \infty$.
ii) The formula in Lemma 2.4.3 is only a special case of the more general formula in [BKMM07]. In general, a band can be enclosed by a void *and* a saturated region (a so-called *transition band*). The existence of transition bands necessitates different handling according to a node lying in one of two certain disjoint open subsets Σ_0^Δ and Σ_0^∇ of (a, b). In our case $\Sigma_0^\nabla = (a, b)$ and $\Sigma_0^\Delta = \varnothing$. Further quantities are involved, namely $\theta(z)$ (in our case equal to $-2\pi c \int_z^b d\mu^c_{\min}(x)$), $g^+(z)$ (here equal to $\int_a^b \log|x - y|\mathrm{d}\mu^c_{\min}(y) + i\theta(z)/2$) and Δ ($= \varnothing$).

Proposition 2.4.4 (Lemma 7.12 and 7.14 in [BKMM07])**.** *Let F be a fixed closed subset of the interval (a, b) such that $F \cap X_N \neq \varnothing$ and the end points of the band I are not in F (and hence bounded away from F). Then, for N sufficiently large, we have for all $x, y \in F \cap X_N$ the estimate*

$$\left| (x-y)K_{N,k}(x,y) e^{\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(x) - l_c\right)} e^{\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(y) - l_c\right)} \right| < \frac{C}{H(N)} \tag{2.22}$$

with $H(N) = N$ and a constant C only depending on F. Furthermore, if $x, y \in X_N$ are chosen from a sufficiently small neighbourhood G of the set $\{\alpha, \beta\}$ of the two band end points, the estimate holds with $H(N) = N^{2/3}$ and the constant C only depending on G.

Proof. Substitute the result of Lemma 2.4.3 and the assertions of Proposition 2.4.2 into the following formula (cf. (2.6))

$$\begin{aligned}(x-y)&K_{N,k}(x,y)\\ &= \sqrt{w_N(x)w_N(y)} \cdot \gamma^2_{2N,2k-1} \cdot (\pi_{N,k}(x)\pi_{N,k-1}(y) - \pi_{N,k}(y)\pi_{N,k-1}(x))\end{aligned}$$

and estimate the uniformly bounded parts by a constant. □

Remark. Recall that $\frac{\delta E_c}{\delta\mu}(z) - l_c$ is strictly positive for z in the interior of a void and identically zero in a band. So, if at least one of x, y lies in a void, $K_{N,k}(x, y)$ is exponentially small. Moreover, under the assumptions of Lemma 2.4.4, $K_{N,k}(x, y)$ can be asymptotically non-zero only if $x, y \in F \cap I$ and $|x - y| = O(1/N)$ or, for x, y close to a band end point, $|x - y| = O(1/N^{2/3})$.

More precisely we have the following results, the first of which treats the band case and involves the *sine kernel*

$$S(\xi, \eta) := \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)}.$$

Extend S to the diagonal by setting $S(\xi, \xi) := 1$.

Lemma 2.4.5 (Lemma 7.13 in [BKMM07]). *Fix x in the interior of the band $I = (\alpha, \beta)$ and let*

$$\delta(x) := \left[\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)\right]^{-1}.$$

Let ξ_N, η_N belong to a fixed bounded set $D \subset \mathbb{R}$ in such a way that the points defined by

$$w := x + \xi_N \frac{\delta(x)}{N},\ z := x + \eta_N \frac{\delta(x)}{N}$$

are both nodes in X_N. Consequently, $w, z \to x$ as $N \to \infty$. Then there is a constant C_D depending on D such that for sufficiently large N we have

$$\max_{\xi_N, \eta_N \in D} \left| K_{N,k}(w, z) - c\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x) S\left(\xi_N, \eta_N\right)\right| \le \frac{C_D}{N}.$$

The result for nodes close to a band end point is expressed in terms of the *Airy kernel*

$$A(\xi, \eta) := \frac{\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta) - \mathrm{Ai}'(\xi)\mathrm{Ai}(\eta)}{\xi - \eta}.$$

Lemma 2.4.6 (Lemma 7.16 in [BKMM07]). *Let β be the right band end point. For each fixed $M > 0$ there is a constant $C_\beta(M) > 0$, such that for N sufficiently large*

$$\max \left| K_{N,k}(x, y) - \left[\frac{(\pi c B_\beta)^{2/3}}{N^{1/3}}\right] A(\xi_N, \eta_N)\right| \le \frac{C_\beta(M)}{N^{2/3}}$$

holds, where the max *is taken over pairs of nodes $x, y \in X_N$ all satisfying*

$$\beta - MN^{-2/3} < x, y < \beta + MN^{-1/2},$$

the constant B_β is equal to (cf. Sect. 2.3.1)

$$B_\beta := \lim_{x \uparrow \beta} \frac{1}{\sqrt{\beta - x}} \frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x) > 0$$

and $\xi_N = (N\pi c B_\beta)^{2/3}(x - \beta)$ and $\eta_N = (N\pi c B_\beta)^{2/3}(y - \beta)$.

2.4.1 Main results: Asymptotics for $DOPE^{\rm sym}(N,k)$

We can now extend the results for $DOPE(2N,2k)$ to the ensemble $DOPE^{\rm sym}(N,k)$. By equation (2.9) the correlation kernel for $DOPE^{\rm sym}(N,k)$ is

$$K^{\rm sym}_{N,k}(x,y) = K_{2N,2k}(x,y) - K_{2N,2k}(x,-y).$$

We show that $K_{2N,2k}(x,-y)$ is asymptotically neglegible and the results of the previous section also hold for $DOPE^{\rm sym}(N,k)$.

Lemma 2.4.7. *Assume the situation from Section 2.2.2, i.e.* $[a,b] = [-1/2,1/2]$ *and* $X_{2N} = -Y_N \cup Y_N$ *for a set of nodes* $Y_N = \{(2n+1)/(4N),\ n = 0,\dots,N-1\}$ *and the weight function* w_{2N} *is supposed to be even. This implies the same symmetry for the equilibrium measure* $\mu^c_{\rm min}$. *Moreover, we have the following estimates.*

1. *Let* $x, y \in I \cap Y_N$ *be a pair of nodes with* $x, y > \delta_1$, *where* δ_1 *is a strictly positive constant independent of* N. *There is a constant* C *depending only on* δ_1 *such that*
$$|K_{2N,2k}(x,-y)| < \frac{C}{N}. \tag{2.23}$$
2. *For nodes* $x, y \in J \cap X_{2N}$, *where* J *is a closed interval in a gap* Γ, *there are positive constants* D_1, D_2 *depending on* J, *such that*
$$|K_{2N,2k}(x,-y)| < D_1 e^{-D_2 N}.$$
3. *Let* $M > 0$ *and* β *be the right band end point. Then there is a constant* C_M *such that for any pair of nodes* x, y *with*
$$\beta - r(2N)^{-2/3} < x, y < \beta + r(2N)^{-1/2}$$
we have
$$|K_{2N,2k}(x,-y)| < \frac{C_r}{N^{2/3}}. \tag{2.24}$$

Proof. The symmetry of the weight w_{2N} implies the symmetry of V_{2N} and V (cf. equations (2.11) and (2.12)). Hence the external field $\varphi(x)$ defined in (2.16) is even. It follows that the functional $E_c[\mu]$ in (2.18) is invariant under the transformation $\mu(x) \mapsto \mu(-x)$. By uniqueness the symmetry of $\mu^c_{\rm min}$ follows.

The other assertions follow directly from Lemma 2.4.4, since in the first assertion $|x - (-y)| > 2\delta_1$ is bounded away from zero. For the second assertion recall that $\frac{\delta E_c}{\delta \mu}(x) - l_c$ is negative and bounded away from zero if x is bounded away from the band. Hence we have exponential decay in this case. For the third assertion notice that $-\beta$ is the left band end point and $|x - (-y)|$ is asymptotically equal to 2β. $\square$

These estimates immediately imply the analogues of the above Lemmas 2.4.5 and 2.4.6.

Lemma 2.4.8. *Under the assumptions of Lemma 2.4.7 fix $x > 0$ in the interior of the band $I = (-\beta, \beta)$ and let*

$$\delta(x) := \left[\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)\right]^{-1}.$$

Let ξ_{2N}, η_{2N} belong to a fixed bounded set $\tilde{D} \subset \mathbb{R}$ such that

$$w := x + \xi_{2N}\frac{\delta(x)}{2N},\ z := x + \eta_{2N}\frac{\delta(x)}{2N}$$

are nodes in X_{2N} and hence $w, z \to x$ as $N \to \infty$. Then there is a constant $\tilde{C}_{\tilde{D}}$ depending on $\tilde{D}$ such that for sufficiently large N we have

$$\max_{\xi_{2N},\eta_{2N}\in\tilde{D}}\left|K^{\mathrm{sym}}_{N,k}(w,z) - c\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)S\left(\xi_{2N},\eta_{2N}\right)\right| \le \frac{\tilde{C}_{\tilde{D}}}{N}.$$

For $x = 0$ and $w = \xi_{2N}\delta(x)/2N,\ z = \eta_{2N}\delta(x)/2N$ we have

$$\max_{\xi_{2N},\eta_{2N}\in\tilde{D}}\left|K^{\mathrm{sym}}_{N,k}(w,z) - c\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(0)\left[S\left(\xi_{2N},\eta_{2N}\right) - S\left(\xi_{2N},-\eta_{2N}\right)\right]\right| \le \frac{\tilde{C}_{\tilde{D}}}{N}.$$

Proof. Combine the approximation in Lemma 2.4.5 with the estimate (2.23) in Lemma 2.4.7. □

Near the band end point, the kernel $K^{\mathrm{sym}}_{N,k}$ behaves like $K_{2N,2k}$, as the following Lemma shows.

Lemma 2.4.9. *Let β be the right band end point. For each fixed $M > 0$ there is a constant $\tilde{C}_\beta(M) > 0$ such that for N sufficiently large*

$$\max\left|K^{\mathrm{sym}}_{N,k}(x,y) - \left[\frac{(\pi c B_\beta)^{2/3}}{(2N)^{1/3}}\right]A(\xi_{2N},\eta_{2N})\right| \le \frac{\tilde{C}_\beta(M)}{N^{2/3}}$$

holds, where the max *is taken over pairs of nodes $x, y \in X_{2N}$ all satisfying*

$$\beta - M(2N)^{-2/3} < x, y < \beta + M(2N)^{-1/2},$$

with B_β as in Lemma 2.4.6 and $\xi_{2N} = (2N\pi c B_\beta)^{2/3}(x-\beta)$ and $\eta_{2N} = (2N\pi c B_\beta)^{2/3}(y-\beta)$.

Proof. This is obtained by combining estimate (2.24) with the approximation in Lemma 2.4.6. □

By plugging the above asymptotic results on the kernel $K^{\mathrm{sym}}_{N,k}$ into the determinantal expressions for the correlation functions, we obtain the main results of the chapter.

Theorem 2.4.1. *Denote by* $R_m^{(N,k)}(x_1,\ldots,x_m)$ *the* m*-point correlation function for the ensemble* $DOPE^{\mathrm{sym}}(N,k)$ *with an even weight* w_{2N}*. Fix* x *in the interior of* $(0,\beta)$ *and let*

$$\delta(x) := \left[\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)\right]^{-1}.$$

Let $\xi_N^{(1)},\ldots,\xi_N^{(m)}$ *belong to a fixed bounded set* $D\subset\mathbb{R}$ *in such a way that the points defined by*

$$x_j := x+\xi_N^{(j)}\frac{\delta(x)}{N},\ j=1,\ldots,m$$

are all nodes in X_N*. Consequently,* $x_j\to x$ *as* $N\to\infty$*. Then there is a constant* $C_{D,m}$ *depending on* D *and* m *such that for sufficiently large* N *we have*

$$\max_{\xi_N^{(1)},\ldots,\xi_N^{(m)}\in D}\left|R_m^{(N,k)}(x_1,\ldots,x_m)-\left(\frac{c}{\delta(x)}\right)^m\det\left(S\left(\xi_N^{(i)},\xi_N^{(j)}\right)\right)_{1\le i,j\le m}\right|\le\frac{C_{D,m}}{N}.$$

Furthermore, for a fixed closed interval F *in a gap* Γ *there are constants* K_F *and* $C_{F,m}$*, such that for sufficiently large* N

$$\max_{x_1,\ldots,x_m\in F}\left|R_m^{(N,k)}(x_1,\ldots,x_m)\right|\le C_{F,m}\frac{e^{-mK_FN}}{N^m}$$

holds in the void $\Gamma=(\beta,1/2)$*. In particular, for the one-point correlation functions* $R_1^{(N,k)}$ *we have for a sequence of nodes* $y_N\in X_N$*,* $y_N\to x>0$*,* $x\neq\beta$*, the pointwise limit*

$$\lim_{N\to\infty}R_1^{(N,k)}(y_N)=c\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x). \tag{2.25}$$

Remark. For the general ensemble $DOPE(N,k)$ these statements are Theorems 3.1, 3.3 and 3.5 in [BKMM07].

Close to the band end point we have

Theorem 2.4.2. *For each fixed* $M>0$*, each integer* m *and the right band end point* β *separating* I *from a void, there is a constant* $G_\beta^m(M)$ *such that for sufficiently large* N

$$\max\left|R_m^{(N,k)}(x_1,\ldots,x_m)-\left[\frac{(\pi cB_\beta)^{2/3}}{N^{1/3}}\right]^m\det\left(A(\xi_N^{(i)},\xi_N^{(j)})\right)_{1\le i,j\le m}\right|\le\frac{G_\beta^m(M)}{N^{(m+1)/3}},$$

where the max *is taken over nodes* $x_1,\ldots,x_m\in X_N$ *all satisfying*

$$\beta-MN^{-2/3}<x_j<\beta-MN^{-1/2},$$

the constant B_β *is equal to (cf. Sect. 2.3.1)*

$$B_\beta := \lim_{x\uparrow\beta}\frac{1}{\sqrt{\beta-x}}\frac{\mathrm{d}\mu^c_{\min}}{\mathrm{d}x}(x)>0$$

and $\xi_N^{(j)}=(N\pi cB_\beta)^{2/3}(x_j-\beta)$*.*

Remark. For $DOPE(N,k)$ this result is Theorem 3.7 in [BKMM07]. In this case an analogous result holds for the right band end point.

Another interesting statistic concerning particle systems is the fluctuation of extremal particles, e.g. the intersection point of a vertical line with the boundary of the arctic region in random tilings. Let $B \subset X_N$ be a set of nodes and m an integer such that $0 \leq m \leq \min(\#B, k)$. A well studied statistic is

$$\begin{aligned} A_m^{(N,k)}(B) &:= \mathbb{P}(\text{there are exactly } m \text{ particles in } B) \\ &= \frac{1}{m!}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \det\left(\mathbb{I} - t\, K_{N,k}|_B\right)\Big|_{t=1}, \end{aligned}$$

where $K_{N,k}$ is the operator on $\ell_2(X_N)$ with kernel $K_{N,k}(x,y)$ and $K_{N,k}|_B$ its restriction to $\ell_2(B)$. Denote by $x_{\max}$ the position of the rightmost particle. Since the one-point function converges pointwise to $c\,\mathrm{d}\mu^c_{\min}(x)/\mathrm{d}x$ one expects $x_{\max}$ near the band edge β. In this domain the correlation kernel approximates the Airy kernel. The latter kernel is the correlation kernel of the distribution of eigenvalues of a GUE matrix at the edge of the spectrum and the fluctuations of the largest eigenvalue are governed by the *Tracy-Widom distribution* ν [TW94] whose distribution function is equal to

$$\nu((\infty, s]) = \det\left(\mathbb{I} - \mathcal{A}|_{[s,\infty)}\right),$$

where $\mathcal{A}|_{[s,\infty)}$ is the trace class operator on $L_2[s,\infty)$ defined by the Airy kernel. The position rightmost particle (properly scaled) in $DOPE(N,k)$ is proved to be Tracy-Widom-distributed and the proof carries over verbatim to $DOPE^{\mathrm{sym}}(N,k)$.

Proposition 2.4.10 (Theorem 3.9 in [BKMM07])**.** *For the position of the rightmost particle $x_{\max}$ in the ensembles $DOPE(2N,2k)$ and $DOPE^{\mathrm{sym}}(N,k)$ we have the limiting distribution function*

$$\lim_{N\to\infty} \mathbb{P}\left(x_{\max} \leq \beta + \frac{s}{(2\pi N c B_\beta)^{2/3}}\right) = \det\left(\mathbb{I} - \mathcal{A}|_{[s,\infty)}\right).$$

2.5 Proof of the arctic phenomenon

We now apply the above results to prove the arctic ellipse phenomenon in the half-hexagon model, see Figure 2.3 for an illustration.

2.5.1 The Hahn and Associated Hahn ensemble

Let $[a,b] = [0,1]$ and $X_N = \{x_{N,n} = (2n+1)/2N,\ n = 0, \ldots, N-1\}$. The discrete orthogonal polynomial ensemble with weight function

$$w_N^{\mathrm{AHE}}(x_{N,n}; P, Q) = \frac{1}{n!(P-1+n)!(N-1+n)!(Q-1+N-1-n)!}$$

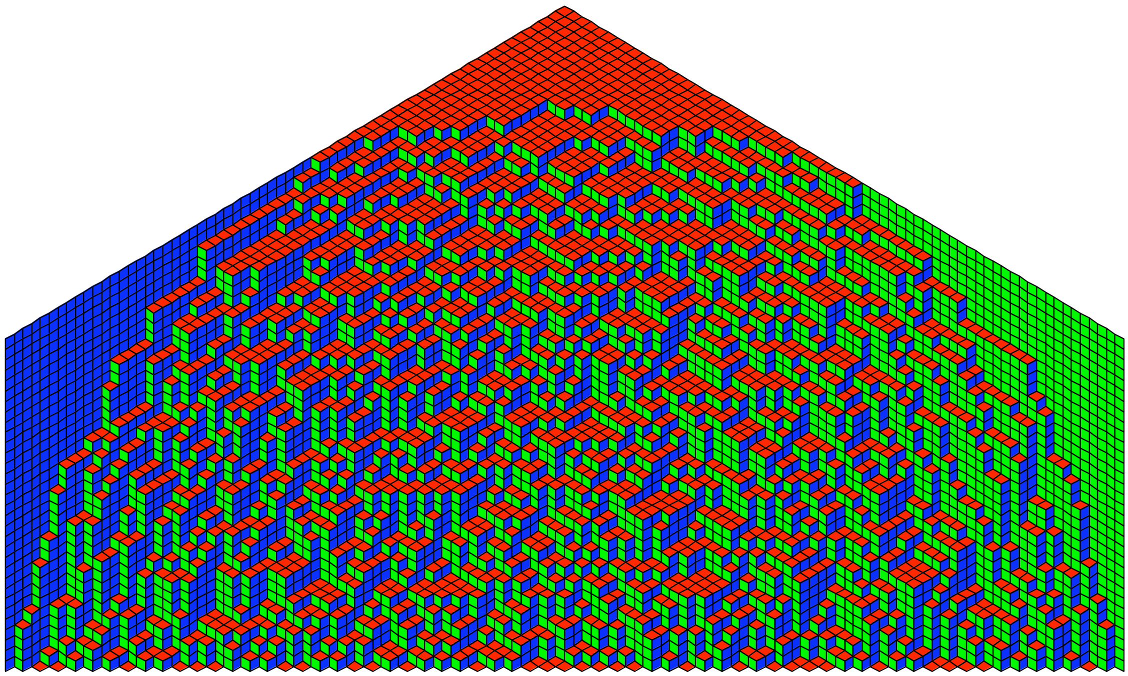

Figure 2.3: Arctic phenomenon in a $(32, 64, 64)$-half-hexagon

is called *Associated Hahn Ensemble* (AHE) [BKMM07, Joh02] with parameters P and Q. Its dual ensemble is known as *Hahn Ensemble* (HE) with weight $w^{\mathrm{HE}}(\,\cdot\,; P, Q)$. In [BKMM07, Sect. 2.4.2] the equilibrium measure $\mu^c_{\min}$ for the family $w^{\mathrm{HE}}(\,\cdot\,; AN{+}1, BN{+}1)$, $A, B > 0$, $c \in (0, 1)$ fixed is computed. It turns out that there is exactly one band interval. For $A = B$ it is an interval $(1 - \beta, \beta)$ enclosed by two gaps. If $c < c_A = \sqrt{A^2 + A} - A$ those two gaps are voids for HE (and hence saturated regions for AHE with c substituted by $1 - c$) and if $c > c_A$ they are saturated regions for HE (voids for AHE, c substituted by $1 - c$). For the right band end point one has

$$\beta = \frac{1}{2} + \frac{\sqrt{c(1-c)(2A+c)(2A+c+1)}}{2(A+c)}. \tag{2.26}$$

2.5.2 The parameters in the $(2p, R, R)$-hexagon

With the change of variables $z = n - (m/2 + p - 1/2)$ and the substitutions $2p + m = N$ and $R - m + 1 = P$ in the weight function $\tilde{w}(z)$ in (2.3), we see that $\tilde{w}$ is from the AHE family with the parameters P and Q both equal to $R - m + 1$. With view on the half-hexagon problem we consider the parameter m in the distributions in Prop. 2.1.1 to be even, $m = 2k$. This implies that the set L_{2k} (cf. Section 2.1) has an even number $2N$ of elements and the results of Section 2.2.2 are applicable. No asymptotic information is lost by this assumption, since looking at the model of non-intersecting paths, the ordinate of the intersection point of the jth path with the line $x = \sqrt{3}(2k+1)/2$ is equal to the respective ordinate on the two adjacent lines $x = \sqrt{3}(2k+1\pm 1)/2$ plus or minus $1/2$.

We choose R such that $R/p \to \lambda$ as $p \to \infty$. We scale the $(2p, R, R)$-hexagon by $1/p$, such that in the limit its intersection with he x-axis is the interval $[-\sqrt{3}\lambda/2, \sqrt{3}\lambda/2]$. The boundary of the inscribed ellipse of the rescaled hexagon is given by

$$\pm\sqrt{\lambda+1}\sqrt{1-\left(\frac{2}{\sqrt{3}\lambda}\tau\right)^2},\ \tau \in \left[-\frac{\sqrt{3}\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right]. \tag{2.27}$$

We want to show that the intersection of the temperate zone of the rescaled tiling with the vertical line $x = \tau$ is given by this formula. Let $\tau \in [-\sqrt{3}\lambda/2, 0]$, the case for positive τ follows by symmetry. We now express the parameters of the Hahn ensemble occurring in the tiling problem in terms of τ and p. As $p \to \infty$ we have the asymptotic ratios:

$$\begin{aligned}
&N/p \to (2 + 2\tau/\sqrt{3} + \lambda),\\
&2k/p = (2N - 2p)/p \to 2\tau/\sqrt{3} + \lambda,\\
&2k/2N \to c := (2\tau/\sqrt{3} + \lambda)/(2 + 2\tau/\sqrt{3} + \lambda),\\
&(P-1)/p = (R - 2k)/p \to -2\tau/\sqrt{3},\\
&A = (-2\tau/\sqrt{3})/(2 + 2\tau/\sqrt{3} + \lambda).
\end{aligned}$$

Plug this into the formula (2.26) shift by $-1/2$, such that the band interval is symmetric about 0 and scale by $2 + 2\tau/\sqrt{3} + \lambda$, (length of the intersection of the line $x = \tau$ with the rescaled hexagon). This yields formula (2.27) and thus proves the arctic ellipse phenomenon for the hexagon and half-hexagon. If we plug the expression for A into the expression for c_A in Sect. 2.5.1 and solve $c = c_A$ for τ, we obtain the abscissa of the tangent point of the ellipse, which reflects the transition from the void-band-void to the saturated-band-saturated situation.

Furthermore, for fixed τ the results 2.4.1 and 2.4.2 hold for the correlations of the vertical tiles in the temperate zone on the arctic boundary, and Prop. 2.4.10 for the fluctuations of the boundary.

2.6 Conclusion

We have proved some results for the correlations of tiles and an Arctic Ellipse Theorem for tilings of the half-hexagon, see Figure 2.3 for an illustration of the latter. It would be interesting to check whether similar results hold for the remaining dihedral symmetry classes, cf. Chapter 3. For example, (p, R, R)-hexagon tilings symmetric w.r.t. to the y-axis correspond to families of p lattice paths of R steps with "loose ends" in a similar fashion as described above. However, the above approach does not apply to the situation, since we do not have formulas for the measures of interest as in Prop. 2.1.1. One also would not expect orthogonal polynomial ensembles to show up on every vertical line $x = (-R + m)\sqrt{3}/2$, since this is not the case in the continuous model of non-intersecting Brownian bridges on the interval $[0, 1]$ [KT02]. In that model, close to 1 one observes a transition from the *Gaussian Unitary Ensemble* (which is an orthogonal polynomial ensemble) to the *Gaussian Orthogonal Ensemble*.

Chapter 3

Volume laws for plane partitions

It may strike a beholder with spacial perception that a lozenge tiling sooner or later protrudes from the plane and puts an image of a collection of stacked cubes into their mind. This may also have inspired Elser [Els84] who utilised a classic result from enumerative combinatorics due to Mac Mahon [MM15] to compute the entropy of hexagonal random tilings with fixed boundary conditions. Mac Mahon on the other hand was concerned with certain tableaux related to symmetric functions, so-called *plane partitions*. To the latter objects a volume is assigned in a natural way, whose limiting behaviour is focused upon in the present chapter. In [KO07] the authors also view the tilings as surfaces in three-space and prove that for a large boundary most of the suitably rescaled surfaces lie close to a limit surface. This implies that the sequence of properly rescaled volume random variables is concentrated to a fixed value, which we rederive here. Additionally we find the fluctuations of the centred and normalised volume random variable to be asymptotically Gaussian.

3.1 Plane Partitions

A *plane partition fitting inside an* (r, s, t)*-box* is an $r \times s$-array of non-negative integers $p_{i,j} \leq t$ with weakly decreasing rows and columns. It can be visualised as a pile Π of unit cubes in the box $\mathcal{B}(r, s, t) := [0, r] \times [0, s] \times [0, t]$ "flushed into the corner", see figure 3.1 and [Bre99]. To make that precise, let $c(i, j, k) = [i-1, i] \times [j-1, j] \times [k-1, k]$, $(i, j, k) \in \mathbb{N}^3$ be a unit cube with integer corners. Now

$$p_{i,j} = k \iff c(i, j, l) \in \Pi \text{ for } l = 1, \ldots, k.$$

By abuse of notation we denote both the array and the collection of cubes by Π. In the following we mean by "plane partition" one which fits inside an (r, s, t)-box. Figure 3.1 also illustrates the connection to lozenge tilings of a hexagon of side lengths r, s and t by orthogonal projection of the box and the "visible" faces of the cubes upon the hyperplane $x + y + z = 0$. Yet another interpretation is viewing such a pile of cubes as an order ideal in the product of three finite chains (total orders) with the respective lengths r, s and t, since $c(i, j, k) \in \Pi$ implies that also $c(i', j', k') \in \Pi$ for $i' \leq i$, $j' \leq j$ and $k' \leq k$. The volume vol(Π) of a plane partition is the sum of its parts or the number of unit cubes in the pile or the cardinality of the order ideal, respectively. If one side length, say t of the bounding box is one, such a pile can be regarded as an ordinary partition or *Ferrers diagram* of area equal to the volume fitting inside an $r \times s$-rectangle, see figure 3.2 in Section 3.4 for an illustration. Let $\mathcal{PP}(r,s,t)$ denote the set of all plane partitions fitting inside $\mathcal{B}(r,s,t)$. We asymptotically analyse the volume distribution on classes of plane partitions invariant under symmetry operations on the corresponding hexagonal tilings when the box is large.

The natural symmetry groups acting on hexagonal tilings are subgroups of the dihedral group $\mathcal{D}_6$. Recall that $\mathcal{D}_6$ is generated by a 60° rotation R and a reflection S. It consists of six rotations and six reflections, namely

$$\mathcal{D}_6 = \left\{ R^i,\ R^i S \,|\, i = 0, \ldots, 5 \right\}.$$

We first explain how $\mathcal{D}_6$ acts on plane partitions. To this end we define the *complement* of $\Pi \in \mathcal{PP}(r,s,t)$ as the plane partition Π^c given by the rule

$$c(i,j,k) \in \Pi^c \Longleftrightarrow c(r-i+1, s-j+1, t-k+1) \notin \Pi.$$

This is the complement of the pile of cubes Π in the box $\mathcal{B}(r,s,t)$, reflected in the plane $x = y$. Obviously, taking complements is an involution on $\mathcal{PP}(r,s,t)$. The corresponding symmetry operation of the hexagon is the rotation about 180°. Note that taking complements in general does *not* leave the volume invariant:

$$\mathrm{vol}\,(\Pi^c) = rst - \mathrm{vol}\,(\Pi)\,.$$

A clockwise rotation of a tiling by 120° is achieved by cyclically permuting the coordinate axes: $x \mapsto z,\ z \mapsto y,\ y \mapsto x$. This requires $r = s = t$ to make it a map from $\mathcal{B}(r,s,t)$ onto itself. So the counterclockwise rotation R of a tiling by 60° corresponds to taking the complement followed by that cyclic permutation of the coordinates. This composition yields one generator of $\mathcal{D}_6$. As the second one we may choose the reflection S which interchanges the x- and y-coordinates. This latter symmetry operation on its own only requires that $r = s$.

Now let G be a subgroup of $\mathcal{D}_6$. If G contains an element of the form R^{2j+1} or $R^{2j+1}S$ $(j = 0, 1, 2)$, i.e. one of precisely those elements involving an odd number of complements, then the volume is equal to $rst/2$ for every plane partition $\Pi \in \mathcal{PP}(r,s,t)$ invariant under G. This leaves essentially three symmetry subclasses with non-constant volume distribution, namely those which are fixed under subgroups of $\mathcal{S}_3$, the group of permutations of the coordinates generated by R^2 and S.

Call a plane partition $\Pi \in \mathcal{PP}(r,r,t)$ *symmetric* if the corresponding pile of cubes is symmetric about $x = y$, i.e. the cube $c(i,j,k)$ belongs to the pile, if and only if $c(j,i,k)$ does. A plane partition $\Pi \in \mathcal{PP}(r,r,r)$ is called *cyclically symmetric* if the corresponding tiling is invariant under a rotation about 120°, i.e. the cube $c(i,j,k)$ belongs to the pile if and only if $c(k,i,j)$ and $c(j,k,i)$ do. A plane partition in $\mathcal{PP}(r,r,r)$ invariant under both

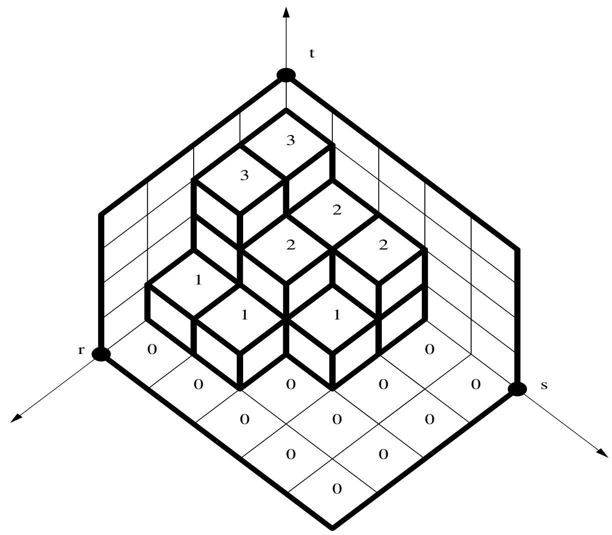

Figure 3.1: A plane partition in a $(4, 5, 4)$-box

actions and hence all permutations of the coordinate axes is called *totally symmetric*. Call these subsets $\mathcal{SPP}(r, t)$, $\mathcal{CSPP}(r)$ and $\mathcal{TSPP}(r)$ respectively.

It is possible to compute volume limit laws for some of those subclasses, as the generating functions counting plane partitions by volume are available for three of the above classes. For the following formulas we refer to [Bre99]. The volume generating function of $\mathcal{PP}(r, s, t)$ is

$$\prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}. \tag{3.1}$$

Symmetric plane partitions in $\mathcal{B}(r, r, t)$ have the volume generating function

$$\left(\prod_{i=1}^{r}\prod_{k=1}^{t}\frac{1-q^{2i+k-1}}{1-q^{2i+k-2}}\right)\left(\prod_{1\le i<j\le r}\prod_{k=1}^{t}\frac{1-q^{2+2(i+j+k-2)}}{1-q^{2(i+j+k-2)}}\right).$$

Here the first product corresponds to singleton orbits and the second product corresponds to doubleton orbits of $\mathcal{B}(r, r, t)$ under switching the first two coordinates. For the cyclically symmetric plane partitions we have the volume generating function

$$\left(\prod_{1\le i<j<k\le r}\frac{1-q^{3(i+j+k-1)}}{1-q^{3(i+j+k-2)}}\right)^2\left(\prod_{1\le i<k\le r}\frac{1-q^{3(2i+k-1)}}{1-q^{3(2i+k-2)}}\cdot\frac{1-q^{3(i+2k-1)}}{1-q^{3(i+2k-2)}}\right)$$
$$\times\left(\prod_{i=1}^{r}\frac{1-q^{3i-1}}{1-q^{3i-2}}\right),$$

where the first product runs over $\mathcal{C}_3$ orbits of $\mathcal{B}(r, r, r)$ with all coordinates distinct, the second runs over orbits with exactly two coordinates equal and the third corresponds to singleton orbits. The above generating functions are polynomials in q whose degree is the volume of the bounding box.

Unfortunately, there is no analogous result for the volume generating function on $\mathcal{TSPP}(r)$. However, there is a *conjecture* that the *orbit counting generating function* equals

$$\sum_{\Pi \in \mathcal{TSPP}(r)} q^{\mathrm{oc}(\Pi)} = \prod_{1 \le i \le j \le k \le r} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}, \tag{3.2}$$

where $\mathrm{oc}(\Pi)$ is the number of $\mathcal{S}_3$-orbits of cubes in Π. The $q = 1$ case of the conjecture was proved in [Ste95], and recently "a proof modulo a finite amount of routine calculations" of (3.2) was given in [KKZ08].

We equip $\mathcal{PP}(r, s, t)$, $\mathcal{SPP}(r, t)$, $\mathcal{CSPP}(r)$ and $\mathcal{TSPP}(r)$ with the uniform distribution ("uniform fixed bounding box ensemble"). Denote by $X = X_{rst}$ (resp. $= X_{rt},\ X_r$) the corresponding volume random variables. In the subsequent discussion we will drop subscripts as the side lengths are always r, s and t (resp. r and t, r). The probability generating function of this random variable is $P(q) := G(q)/G(1)$ where $G(q)$ is the respective generating function. In the $\mathcal{PP}(r, s, t)$ case this reads with the notion $\alpha_{ijk} := i + j + k - 1$

$$P(q) = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{(\alpha_{ijk} - 1)\,(1 - q^{\alpha_{ijk}})}{\alpha_{ijk}\,(1 - q^{\alpha_{ijk}-1})}. \tag{3.3}$$

In the following $g(x)^{(N)}$ denotes the N^{th} derivative of g w.r.t. x. Since X only takes finitely many integer values, moments of X of arbitrary order N exist and can be computed via

$$\mathbb{E}\left(X^N\right) = (-I)^N \left(P(e^{Ix})\right)^{(N)}\Big|_{x=0}, \tag{3.4}$$

where $P(e^{Ix})$ $(I^2 = -1)$ is the characteristic function of X.

3.2 Mean, variance and concentration properties

Denote by μ, μ_{spp}, μ_{cspp} and μ_{tspp} the mean value of the volume variables on $\mathcal{PP}(r, s, t)$, $\mathcal{SPP}(r, t)$, $\mathcal{CSPP}(r)$, and $\mathcal{TSPP}(r)$, respectively and by μ^{oc}_{tspp} the mean value of orbit counts on $\mathcal{TSPP}(r)$.

Lemma 3.2.1. *The volume distributions are symmetric about half the volume of the bounding box, and hence the expected volumes are $\mu = rst/2$, $\mu_{spp} = r^2 t/2$ and $\mu_{cspp} = \mu_{tspp} = r^3/2$. The expected number of orbits is $\mu^{\mathrm{oc}}_{tspp} = r(r + 1)(r + 2)/12$.*

Proof. Let v be the volume of the bounding box $\mathcal{B}$. The random variables X and $v - X$ are seen to be equally distributed as follows. Taking the complement is an involution which takes a plane partition of volume k to one of volume $v - k$. This operation respects symmetry, cyclic and total symmetry. Now the lemma follows as $\mathbb{E}(X) = \mathbb{E}(v - X)$. The same line of arguments works for orbit counts on $\mathcal{TSPP}(r)$, since an orbit is contained in Π if and only if it is not in Π^c and there is a total number of $r(r + 1)(r + 2)/6$ of $\mathcal{S}_3$-orbits in $\mathcal{B}$. $\square$

In order to prove the Gaussian limits we consider the characteristic functions $P(e^{Ix})$ of the random variables X. More precisely, we will study the logarithms of the characteristic functions, since sums are easier to handle than products. The following lemma enables us to compute explicit formulas for the variances of the random variables and to estimate the Taylor coefficients of the functions in question.

Lemma 3.2.2. *For positive real numbers α, c with $\alpha > c \geq 1$, we have the expansion*

$$\log\left(\frac{(\alpha-c)\left(1-e^{\alpha x}\right)}{\alpha\left(1-e^{(\alpha-c)x}\right)}\right) = \sum_{N\geq 1} H_{N,c}(\alpha)x^N. \tag{3.5}$$

Here $H_{N,c}$ is a polynomial of degree $N-1$ in α. In particular we have $H_{1,c}(\alpha) = \frac{c}{2}$ and $H_{2,c}(\alpha) = -\frac{c}{12}\alpha + \frac{c^2}{24}$. Furthermore there is a positive constant D, such that the inequality

$$|H_{N,c}(\alpha)| \leq D \cdot \alpha^{N-1} \cdot (2c)^N \tag{3.6}$$

holds for all $N \in \mathbb{N}$.

Proof. Define the function $g(t)$ by

$$g(t) = \log\left(\frac{e^t-1}{t}\right).$$

Then the lhs of (3.5) is easily seen to be equal to $g(\alpha x) - g((\alpha-c)x)$. We have the following series expansion for $g(t)$:

$$g(t) = \log\left(\frac{e^t-1}{t}\right) = \log\left(1 + \frac{1}{2}t + \frac{1}{3!}t^2 + \dots\right) = \sum_{N\geq 1} b_N t^N.$$

The numbers b_N can be expressed in terms of Bernoulli numbers. For our purposes it suffices to observe that the singularities of g of smallest modulus are $\pm 2\pi I$, so the numbers $|b_N|$ decay asymptotically like $(2\pi)^{-N}$ and hence are bounded by some constant D, cf. Section B.2. The Nth coefficient in the Taylor expansion of $g(\alpha x) - g((\alpha-c)x)$ about $x=0$ is in fact a polynomial of degree $N-1$, namely

$$H_{N,c}(\alpha) = b_N \cdot \left(\alpha^N - (\alpha-c)^N\right) = b_N \cdot \sum_{k=1}^{N} \binom{N}{k} (-1)^{k+1} c^k \alpha^{N-k}.$$

As $\alpha > c \geq 1$ it can be estimated as follows:

$$\left| b_N \cdot \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} c^k \alpha^{N-k} \right| \leq D \cdot \alpha^{N-1} \cdot c^N \cdot \sum_{k=1}^{N} \binom{N}{k} \leq D \cdot \alpha^{N-1} \cdot c^N \cdot 2^N.$$

This finishes the proof of Lemma 3.2.2. $\square$

Now we can easily compute the variances. Denote by σ, σ_{spp} and σ_{cspp} the standard deviation of the volume variables on $\mathcal{PP}(r,s,t)$, $\mathcal{SPP}(r,t)$ and $\mathcal{CSPP}(r)$, respectively. The formula for σ^2 is already shown in [Wil01].

Lemma 3.2.3. *We have*

1. $\sigma^2 = \frac{1}{12} rst(r+s+t)$,
2. $\sigma_{spp}^2 = \frac{1}{3}tr^3 + \frac{1}{6}t^2r^2 - \frac{1}{12}t^2r + \frac{1}{6}tr^2 - \frac{1}{3}tr$ *and*
3. $\sigma_{cspp}^2 = \frac{3}{4}r^4 - \frac{1}{2}r^2$.

Proof. Recall that the variance of a random variable Y can be obtained as $\mathbb{V}(Y) = -\log\left(P(e^{Ix})\right)''\Big|_{x=0}$, where $P(q)$ is the probability generating function of Y. In the $\mathcal{PP}(r,s,t)$ case we apply this to $P(q)$ as in (3.3) and obtain

$$-\log\left(P(e^{Ix})\right)''\Big|_{x=0} = -\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t} \log\left(\frac{(\alpha_{ijk}-1)\left(1-e^{\alpha_{ijk}Ix}\right)}{\alpha_{ijk}\left(1-e^{(\alpha_{ijk}-1)Ix}\right)}\right)''\Bigg|_{x=0},$$

where $\alpha_{ijk} = i+j+k-1$. According to Lemma 3.2.2, each summand on the rhs is equal to

$$2!H_{2,1}(\alpha_{ijk}) = -\frac{1}{6}\alpha_{ijk} + \frac{1}{12},$$

and a straightforward calculation now yields

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}\left(\frac{1}{6}(i+j+k-1) - \frac{1}{12}\right) = \frac{1}{12}\left(r^2st + rs^2t + rst^2\right) = \frac{1}{12}rst(r+s+t).$$

The other variances are calculated in an analogous way with suitable choices of α and c. $\square$

Remark. Assuming that formula (3.2) is correct, we can also compute the variance of the number of orbits in an analogous fashion as in the proof of Lemma 3.2.3. It turns out to be

$$\left(\sigma_{tspp}^{oc}\right)^2 = \frac{1}{24}r^2(r+1)(r+2).$$

Now we can investigate concentration properties of the families of volume random variables, i.e. we study the quotients of standard deviation and mean when the box gets large. According to Lemma 3.2.1 and Lemma 3.2.3 these quotients tend to zero if at least two side lengths of the bounding box tend to infinity. The latter is in particular satisfied when $r \to \infty$ in the $\mathcal{SPP}(r,t)$ and $\mathcal{CSPP}(r)$ case. In the general case of $\mathcal{PP}(r,s,t)$ we have

$$\left(\frac{\sigma}{\mu}\right)^2 = \frac{1}{3}\left(\frac{1}{st} + \frac{1}{rt} + \frac{1}{rs}\right). \tag{3.7}$$

If say r and s are unbounded, the right hand side of (3.7) tends to zero for $r, s \to \infty$ and the family is concentrated to the mean, i.e. for every $\delta > 0$ we have

$$\mathbb{P}\left(\left|\frac{X}{\mathbb{E}(X)} - 1\right| \leq \delta\right) \to 1 \qquad (r, s \to \infty).$$

On the other hand, if two coordinates are fixed, say r, s, then the quotient is bounded away from zero by $\frac{1}{3rs}$ for $t \to \infty$. These families are not concentrated. Similar considerations hold in the case of $\mathcal{SPP}(r,t)$ when r is fixed and $t \to \infty$. In the next section we investigate the limit laws in these two cases.

3.3 Limit laws

We first give a result for the concentrated families.

Proposition 3.3.1. *If at least two of r, s, t tend to infinity, the family*

$$\left(Y := \frac{X - \mu}{\sigma}\right)$$

of normalised volume random variables on $\mathcal{PP}(r,s,t)$ converges in distribution to a standard normal distributed random variable. The same statement holds for the volume variables on $\mathcal{SPP}(r,t)$ if at least $r \to \infty$ and on $\mathcal{CSPP}(r)$ if $r \to \infty$.

Proof. For $\mathcal{PP}(r,s,t)$. Let $r, s \to \infty$ and t vary arbitrarily. The characteristic function of Y is $\phi(x) := e^{-\mu I x/\sigma} P\left(e^{Ix/\sigma}\right)$, with P as in eq. (3.3). We prove that $\log(\phi(x)) \to -\frac{x^2}{2}$ for $x \in \mathbb{R}$ if $r, s \to \infty$. Then by Levy's continuity theorem (cf. [Fel71]) the assertion follows. By Lemma 3.2.1 the random variables Y and $-Y$ have the same distribution. So the characteristic function is real valued and the Taylor coefficients of ϕ and $\log(\phi)$ of odd order vanish. By Lemma 3.2.2 we have the Taylor expansion

$$\log(\phi(x)) = -\frac{x^2}{2} + \sum_{N \geq 2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} H_{2N,1}(i+j+k-1) \left(\frac{xI}{\sigma}\right)^{2N}. \tag{3.8}$$

According to the estimate (3.6) we can bound the modulus of the $2N$th summand, $N \geq 2$, by

$$D \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} 2^{2N} (i+j+k-1)^{2N-1} \left(\frac{|x|}{\sigma}\right)^{2N} \leq 4^N D |x|^{2N} \frac{rst(r+s+t)^{2N-1}}{\sigma^{2N}}. \tag{3.9}$$

Plugging the explicit expression for σ^2 of Lemma 3.2.3 into the rhs of (3.9) we obtain the estimate for the $2N$th summand, $N \geq 2$, in the expansion (3.8)

$$\left|\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} H_{2N,1}(i+j+k-1) \left(\frac{xI}{\sigma}\right)^{2N}\right| \leq D|x|^{2N} \cdot 48^N \left(\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}\right)^{N-1}. \tag{3.10}$$

Summing in (3.10) over $N \geq 2$ then yields the estimate

$$\left|\log(\phi(x)) + x^2/2\right| \leq 48D|x|^2 \frac{48|x|^2\left(\frac{1}{rs}+\frac{1}{rt}+\frac{1}{st}\right)}{1-48|x|^2\left(\frac{1}{rs}+\frac{1}{rt}+\frac{1}{st}\right)}.$$

The rhs tends to zero for any fixed real x if r and s tend to infinity. This proves the assertion for the $\mathcal{PP}(r,s,t)$ case. The other cases are shown with similar estimates. □

Remark. Assuming formula (3.2), we also obtain that the sequence of of orbit counting random variables is concentrated to the mean, and the sequence of centred and normalised random variables converges in distribution to a Gaussian random variable. Furthermore we find that the sequence of volume random variables X on $\mathcal{TSPP}(r)$ is concentrated to the mean. This is seen as follows. Denote by $\widetilde{X}$ the orbit count random variable. Each orbit of cubes in $\mathcal{B}(r,r,r)$ contains at most six cubes, exactly $r(r-1)$ orbits consist of three cubes and r orbits are singletons. Hence we have the estimate

$$6\widetilde{X} - 3r(r-1) - 5r \leq X \leq 6\widetilde{X}.$$

If we divide by $r^3/2$ and let $r \to \infty$, the rightmost and the leftmost term tend to one in probability.

Now we consider the non-concentrated case. If $r = s = 1$ we have a single column of unit cubes of height t. The volume clearly is uniformly distributed. So, if $rs > 1$ the volume random variable can be viewed as sum of dependent random variables with values in $\{0,\ldots,t\}$. Now the easiest guess is that for large t the dependence vanishes and the volume of a single column is uniformly distributed. This guess is the right one as the following proposition shows.

Proposition 3.3.2. *If r and s are fixed and t tends to infinity, the family*

$$\left(Z_t := \frac{X_{rst}}{t}\right)$$

of rescaled random variables converges in distribution to the (rs)-fold convolution of the uniform distribution on $[0,1]$. In the $\mathcal{SPP}(r,t)$ case the so rescaled sequence converges in distribution to the convolution of r factors of the uniform distribution on $[0,1]$ and $r(r-1)/2$ factors of the uniform distribution in $[0,2]$.

Proof. For $\mathcal{PP}(r,s,t)$. We show that the characteristic function of Z_t converges pointwise to $\left(\frac{e^{Ix}-1}{Ix}\right)^{rs}$, which is the characteristic function of the rs-fold convolution of the uniform distribution on $[0,1]$. The Fourier transform of Z_t is $P(e^{Ix/t})$, with P as in (3.3). We expand the product running from 1 to t in $P(e^{Ix/t})$. All but the last term in the numerator and the first term in the denominator cancel out:

$$P(e^{Ix/t}) = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{(i+j-1)(1-e^{Ix(i+j+t-2)/t})}{(i+j+t-1)(1-e^{Ix(i+j-2)/t})}.$$

The single factors are easily seen to converge to $\frac{e^{Ix}-1}{Ix}$. The $\mathcal{SPP}(r,t)$ case is worked out analogously. □

3.4 Ferrers Diagrams

A Ferrers diagram is a convex lattice polygon which contains both upper corners and the lower left corner of its smallest bounding rectangle. Here convex means, that the intersection of the polygon with any horizontal or vertical line is convex. When $t = 1$ a boxed plane partition can be viewed as a Ferrers diagram fitting inside an $r \times s$ rectangle. Then formula (3.1) reduces to the q-binomial coefficient $\left[{r+s \atop s}\right]_q$. The class of Ferrers diagrams with h rows and w columns has the area generating polynomial

$$q^{h+w-1} \left[{h+w-2 \atop h-1}\right]_q \tag{3.11}$$

since at least the $h + w - 1$ unit squares which constitute the top row and the leftmost column are contained in such a polygon, see figure 3.2 below.

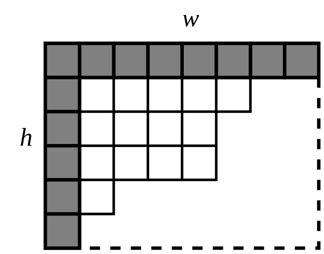

Figure 3.2: Each Ferrers diagram with h rows and w columns corresponds to a possibly empty one fitting inside a $h - 1 \times w - 1$ rectangle.

So, by Proposition 3.3.1 a Gaussian area limit law arises in this uniform fixed height and width ensemble, when both height and width tend to infinity.

In Chapters 4 and 5, we investigate so-called uniform fixed-perimeter ensembles, where all Ferrers diagrams of a fixed half-perimeter are considered equally likely. For fixed $m \geq 0$ the area generating function of Ferrers diagrams of half-perimeter $m + 2$ is obtained by summing formula (3.11) over all pairs $(h, w) \in \mathbb{N}^2$ with $h+w = m+2$. This can be written as

$$q^{m+1} \sum_{h=0}^{m} \left[\begin{array}{c} m \\ h \end{array} \right]_q . \tag{3.12}$$

Observe that the index of summation in (3.12) can be interpreted as the height minus one and that the total number of such Ferrers Diagrams is 2^m. So the following probability generating function (PGF)

$$2^{-m} q^{m+1} u \sum_{h=0}^{m} u^h \left[\begin{array}{c} m \\ h \end{array} \right]_q \tag{3.13}$$

describes the ensemble of Ferrers diagrams of half-perimeter $m + 2$ counted by height and area. The additional height parameter allows us to use the results for height- and width-ensembles discussed above. More precisely, we condition the area variable in the perimeter-ensemble on the height variable. In what follows we will prove that the joint distribution of the (properly rescaled) height and area variables in this ensemble converge in distribution to the two-dimensional standard normal distribution. The following proposition from [Fli81] (see also [Set61]) is tailor-made for this situation.

Proposition 3.4.1. *Let X_m, Y_m be real valued random variables and let Y_m be supported in a lattice $L_m := \{\alpha_m + k\delta_m | k \in \mathbb{Z}\}$, where $\delta_m > 0$ and $\alpha_m \in \mathbb{R}$, i.e. $\mathbb{P}(Y_m \in L_m) = 1$. Suppose Y_m satisfies a local limit law μ with a density $g(y)$ w.r.t. the Lebesgue measure on $\mathbb{R}$ (this implies $\delta_m \to 0$), i.e. for all $y \in \mathbb{R}$ and every sequence (y_m) with $y_m \in L_m$ and $y_m \to y$ we have $\mathbb{P}(Y_m = y_m)/\delta_m \to g(y)$. Suppose further that for μ-almost all $y \in \mathbb{R}$ the conditional distributions $\mathbb{P}(X_m \in \cdot\ | Y_m = y_m)$ converge weakly to a measure $\nu(\cdot, y)$ ($y_m \to y$, $y_m \in L_m$). Then the joint distribution of (X_m, Y_m) converges weakly to the measure ν determined by*

$$\nu(A \times B) := \int_B \nu(A, y)\mathrm{d}\mu(y)$$

for all Borel sets $A, B \subseteq \mathbb{R}$. □

For $q = 1$ formula (3.13) is simply $2^{-m}u(1+u)^m$, the PGF of the height random variable H_m. H_m is binomial distributed with the mean shifted by 1 and hence satisfies the assumptions of [FS09, Theorem IX.14] and hence a Gaussian local limit law arises. The mean and standard deviation of the height are easily computed to be asymptotically equal to $m/2$ and $\sqrt{m}/2$, respectively. So let $Y_m = 2(H_m - m/2)/\sqrt{m}$. The random variable Y_m is supported in $\mathbb{Z}/\sqrt{m}$. Let $y \in \mathbb{R}$ and $y_m \to y$, where each $y_m \in \mathbb{Z}/\sqrt{m}$. The area random variable A_m, conditioned on the event $\{Y_m = y_m\}$, has PGF

$$q^{m+1}\binom{m}{m/2 + y_m\sqrt{m}/2}^{-1}\begin{bmatrix} m \\ m/2 + y_m\sqrt{m}/2 \end{bmatrix}_q .$$

An application of Lemma 3.2.1 and Lemma 3.2.3 with $r = m/2 + y_m\sqrt{m}/2, s = m - r$ and $t = 1$ shows that the mean and the variance of the conditioned area variables are asymptotically equal to $m^2/8$ and $m^3/48$, respectively. So according to Proposition 3.3.1, the rescaled area variable $X_m := (A_m - m^2/8)/\sqrt{m^3/48}$ conditioned to $\{Y_m = y_m\}$ converges weakly to a Gaussian random variable with mean zero and variance one. Now Proposition 3.4.1 yields

Proposition 3.4.2. *Denote by A_m and H_m the random variables of area and height of a Ferrers diagram in the uniform fixed perimeter ensemble described by* (3.13)*. Let*

$$X_m = \frac{A_m - m^2/8}{\sqrt{m^3/48}}, \quad Y_m = \frac{H_m - m/2}{\sqrt{m}/2}.$$

Then, as $m \to \infty$, (X_m, Y_m) converges weakly to the two-dimensional standard normal-distribution. In particular, the normalised area random variables converge to the standard normal distribution. □

3.5 Conclusion

We have computed the volume limit laws for all symmetry classes of plane partitions with non-trivial volume distribution, except the class $\mathcal{TSPP}(r)$ of the totally symmetric ones. For the latter class we have proved, assuming (3.2), a limit law for the orbit counting random variable and the concentration property for the volume random variables in this ensemble. The limit law of the normalised volume random variables is likely to be a Gaussian, but we have not been able to prove that.

Special cases of the above results apply to uniform fixed height- and width-ensembles of Ferrers diagrams and a limit law for the uniform fixed-perimeter ensemble is found, supplementing studies of [PO95] on these ensembles.

Part II

Polygons

Chapter 4

Area laws for staircase polygons

We have seen in Chapter 2 how a random tiling of a hexagon maps to a collection of directed non-intersecting lattice paths on a hexagonal lattice. Since these paths only use steps in two lattice directions, we can map such a family to a family of paths on the square lattice only taking up-steps ($(i,j) \to (i,j+1)$) and right-steps ($(i,j) \to (i+1,j)$). In the case of an (r,s,t)-hexagon we have r paths, the ith path running from $(-i+1, i-1)$ to $(t-i+1, s+i-1)$. Each path can be viewed as a Ferrers diagram (cf. Section 3.4) fitting inside an $t \times s$-rectangle and the volume of a plane partition is the sum of their areas. Thus the volume can be viewed as an area functional on the set of families of paths. Other, more obvious area functionals are the area *between* two paths or the area a path encloses with a given line if the family is conditioned to stay on one side of that line (as those associated with symmetric tilings, cf. Section 2.1). In this chapter we investigate this problem in the special case of only two paths on the square lattice. We compute the area laws for all symmetry subclasses of such configurations in the limit of large path lengths. This model has been studied in combinatorics [BM96, Ric09b] as well as statistical physics [PB95, Ric06] under the name of *staircase polygons*, *parallelogram polygons* or *polyominoes*, where they serve as a simplified model of *self-avoiding polygons*, see also Chapter 5. Explicit expressions for the half-perimeter and area generating functions for all symmetry classes of staircase polygons are given in [LR01], however these are not amenable to a first principles approach as applied to formula 3.1 in Chapter 3. Instead we analyse *functional equations* satisfied by the half-perimeter and area generating functions of the respective classes and apply the moment method [Bil95, Section 30]. Some of the limit laws can also be obtained via bijections to related combinatorial objects (cf. Sections 4.2.3, 4.2.4 and 4.2.8), but we give a unified approach applicable to all symmetry subclasses.

4.1 The models and functional equations

We explain the models, introduce basic constructions, and fix the notation, following [Ric06]. We will then derive functional equations for the perimeter and area generating function of the symmetry subclasses of staircase polygons.

Consider two fully directed paths on the edges of the square lattice (i.e., paths stepping only up or right), which both start at the origin and end in the same vertex, but have no other vertex and no edge in common. The edge set of such a configuration is called a *staircase polygon*, if it is nonempty. For a given staircase polygon, consider the construction of moving the upper path one unit down and one unit to the right. For each path, remove its first and its last edge. The resulting object is a sequence of (horizontal and vertical) edges and staircase polygons, see Figure 4.1. The unit square yields the empty sequence.

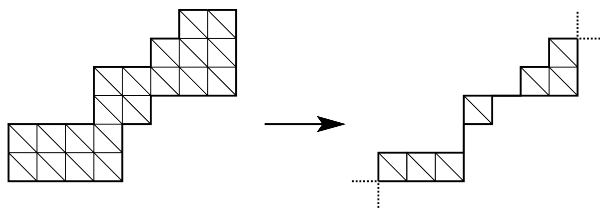

Figure 4.1: The set of staircase polygons is in one-to-one correspondence with the set of ordered sequences of edges and staircase polygons. A corresponding combinatorial bijection is characterised by shifting the upper path of a staircase polygon one unit down and one unit to the right, and by then removing the first and the last edge of each path.

It is easy to see that this construction describes a combinatorial bijection between the set $\mathcal{P}$ of staircase polygons and the set $\mathcal{Q}$ of ordered sequences of edges and staircase polygons. Let us denote the corresponding map by $f : \mathcal{P} \to \mathcal{Q}$. Thus, for a staircase polygon $P \in \mathcal{P}$, we have $f(P) = (Q_1, \ldots, Q_n) \in \mathcal{Q}$, where Q_i is, for $i = 1, \ldots, n$, either a single edge or a staircase polygon. We denote the single horizontal edge by e_h, and the single vertical edge by e_v. The image of the unit square is the empty sequence $n = 0$, which we occasionally identify with a single point, denoted by pt. Variants of this construction will be used below, in order to derive functional equations for the generating functions of symmetry subclasses.

The perimeter of a staircase polygon $P \in \mathcal{P}$ is defined to be the number of its edges. Since this number is always even, we consider in the sequel the *half-perimeter*. Notice that it equals the number of (negative) diagonals $n_0(P)$ plus one. The *area* $n_1(P)$ of a staircase polygon P is defined to be the number of its enclosed squares. It equals the sum of the lengths of its (negative) diagonals. See Figure 4.1 for an illustration. The *weight* of a staircase polygon P is the monomial $w_P(x, q) = x^{n_0(P)+1} q^{n_1(P)}$. The half-perimeter and area generating function of a subclass $\mathcal{C} \subseteq \mathcal{P}$ of staircase polygons is the (formal) power series

$$C(x, q) := \sum_{P \in \mathcal{C}} w_P(x, q).$$

Observe that for e_h, e_v, pt, and for $P \in \mathcal{P}$ we have [Ric06]

$$w_{f^{-1}(pt)}(x, q) = x^2 q, \qquad w_{f^{-1}(e_h)}(x, q) = x^3 q^2, \qquad w_{f^{-1}(e_v)}(x, q) = x^3 q^2,$$
$$w_{f^{-1}(P)}(x, q) = x^2 q \cdot w_P(xq, q).$$

For a polygon $P \in \mathcal{P}$, consider $f(P) = (Q_1, \ldots, Q_n)$. In order to retrieve P from $(Q_1, \ldots, Q_n)$, translate $P_i = f^{-1}(Q_i)$ in such a way that its lower left square coincides with the upper right square of P_{i-1}, for $i \in \{2, \ldots, n\}$. We say that P is the *concatenation* of $(P_1, \ldots, P_n)$, and write $P = c(P_1, \ldots, P_n)$. The weight $w_P(x, q)$ of P is retrieved from the weights of $P_1, \ldots, P_n$ via (see Section B.1 and [Ric06])

$$w_{c(P_1,\ldots,P_n)}(x, q) = \frac{1}{(x^2q)^{n-1}} w_{P_1}(x, q) \cdot \ldots \cdot w_{P_n}(x, q).$$

Denote by $\widetilde{\mathcal{P}} \subseteq \mathcal{P}$ the subset of polygons $\widetilde{P} = f^{-1}(P)$, where $P \in \mathcal{P} \cup \{e_h, e_v\}$. We have established a combinatorial bijection between the set $\mathcal{P}$ and the set of ordered sequences from $\widetilde{\mathcal{P}}$.

The group of point symmetries of the square lattice is the dihedral group $\mathcal{D}_4$. Its non-trivial subgroups are depicted in Figure 4.2. Note that the above decomposition respects any subgroup of the square lattice point symmetries. This observation is the key to deriving functional equations for the generating functions of the symmetry subclasses. In the proof of the following proposition, will treat two cases in some detail, the remaining ones being handled similarly.

$$\begin{array}{ccccccccc}
 & & & & \mathcal{D}_4 & & & & \\
 & & & / & | & \backslash & & & \\
 & & \langle d_1, d_2 \rangle & & \langle r \rangle & & \langle h, v \rangle & & \\
 & / & | & \backslash & | & / & | & \backslash & \\
\langle d_1 \rangle & & \langle d_2 \rangle & & \langle r^2 \rangle & & \langle h \rangle & & \langle v \rangle
\end{array}$$

Figure 4.2: The lattice of subgroups of $\mathcal{D}_4$. The rotation about $\pi/2$ is denoted by r, the reflections in the positive and the negative diagonal are denoted by d_1 and d_2, and the reflections in the horizontal and vertical axes are denoted by h and v. The trivial subgroup is omitted.

Proposition 4.1.1. *The half-perimeter and area generating functions of the staircase polygon symmetry subclasses satisfy the following functional equations.*

1. *Class $\mathcal{P}$ of all staircase polygons with generating function $P(x, q)$:*

$$P(x, q) = \frac{x^2q}{1 - 2xq - P(xq, q)}. \tag{4.1}$$

2. *Class $\mathcal{S}$ of $\langle r^2 \rangle$-symmetric staircase polygons with generating function $S(x, q)$:*

$$S(x, q) = \frac{1}{x^2q} \left(1 + 2xq + S(xq, q)\right) P(x^2, q^2). \tag{4.2}$$

3. *Class of $\langle d_1 \rangle$-symmetric staircase polygons with generating function $D_1(x, q)$:*

$$D_1(x, q) = \frac{x^2q}{1 - D_1(xq, q)}.$$

4. *Class of $\langle d_2 \rangle$-symmetric staircase polygons with generating function $D_2(x,q)$:*

$$D_2(x,q) = \frac{1}{x^2q}\left(1 + D_2(xq,q)\right) P(x^2,q^2).$$

5. *Class of $\langle d_1, d_2 \rangle$-symmetric staircase polygons with generating function $D_{1,2}(x,q)$:*

$$D_{1,2}(x,q) = \frac{1}{x^2q}\left(1 + D_{1,2}(xq,q)\right) D_1(x^2,q^2).$$

6. *Classes of $\langle h \rangle$-, $\langle v \rangle$-, and $\langle h, v \rangle$-symmetric staircase polygons with generating function $H(x,q)$:*

$$H(x,q) = x^2qH(xq,q) + x^2q\frac{1+xq}{1-xq}.$$

7. *Classes $\langle r \rangle$-symmetric staircase polygons with generating function $R(x,q)$:*

$$R(x,q) = x^2qR(xq,q) + x^2q.$$

Proof. Denote the induced group action $\alpha : \mathcal{D}_4 \times \mathcal{P} \to \mathcal{P}$ by $\alpha(g,P) = gP$.
1. The bijection described above implies the following chain of equalities, compare also [Ric06],

$$\begin{aligned}
P(x,q) &= \sum_{n=0}^{\infty} \sum_{(P_1,\dots,P_n)\in(\widetilde{\mathcal{P}})^n} w_{c(P_1,\dots,P_n)}(x,q) \\
&= \sum_{n=0}^{\infty} x^2q \sum_{(P_1,\dots,P_n)\in(\widetilde{\mathcal{P}})^n} \frac{w_{P_1}(x,q)}{x^2q} \cdot \ldots \cdot \frac{w_{P_n}(x,q)}{x^2q} \\
&= x^2q \sum_{n=0}^{\infty} \left(\frac{1}{x^2q} \sum_{P\in\widetilde{\mathcal{P}}} w_P(x,q) \right)^n \\
&= x^2q \frac{1}{1 - \frac{1}{x^2q}\left(w_{f^{-1}(e_h)}(x,q) + w_{f^{-1}(e_v)}(x,q) + \sum_{p\in\mathcal{P}} w_{f^{-1}(P)}(x,q) \right)} \\
&= \frac{x^2q}{1 - 2xq - P(xq,q)}.
\end{aligned} \tag{4.3}$$

2. For $P \in \mathcal{S}$, we have $f(P) = (P_1, \dots, P_n, C, r^2P_n, \dots, r^2P_1)$, where $C \in \mathcal{S} \cup \{e_v, e_h, pt\}$, and $P_i \in \mathcal{P} \cup \{e_v, e_h\}$ for $i = 1, \dots, n$, compare Figure 4.3. In analogy to the definition of $\widetilde{\mathcal{P}}$ above, define $\widetilde{\mathcal{S}} \subset \mathcal{P}$ as the pre-image of $\mathcal{S} \cup \{e_v, e_h, pt\}$ under f. Note that concatenation of $Q \in \mathcal{P}$ with the unit square results in Q again, and that we have $w_Q(x,q)^k = w_Q(x^k, q^k)$.

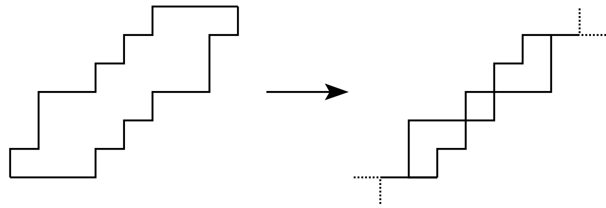

Figure 4.3: r^2-symmetric polygon and corresponding sequence of polygons and edges

With $P(x,q)$ as above, this yields

$$
\begin{aligned}
S(x,q) &= \sum_{n=0}^{\infty} \sum_{(P_1,\ldots,P_n,C)\in(\widetilde{\mathcal{P}})^n\times\widetilde{\mathcal{S}}} w_{c(P_1,\ldots,P_n,C,r^2P_n,\ldots,r^2P_1)}(x,q) \\
&= \sum_{C\in\widetilde{\mathcal{S}}} w_C(x,q) \sum_{n=0}^{\infty} \sum_{(P_1,\ldots,P_n)\in(\widetilde{\mathcal{P}})^n} \frac{w_{P_1}(x,q)^2}{(x^2q)^2}\cdot\ldots\cdot\frac{w_{P_n}(x,q)^2}{(x^2q)^2} \\
&= \left(w_{f^{-1}(pt)} + w_{f^{-1}(e_h)} + w_{f^{-1}(e_v)} + \sum_{C\in\mathcal{S}} w_{f^{-1}(C)}\right) \sum_{n=0}^{\infty} \left(\sum_{Q\in\widetilde{\mathcal{P}}} \frac{w_Q(x^2,q^2)}{(x^2q)^2}\right)^n \qquad (4.4) \\
&= \left(x^2q + 2x^3q^2 + x^2qS(xq,q)\right) \frac{1}{1-(2x^2q^2+P(x^2q^2,q^2))} \\
&= \frac{1+2xq+S(xq,q)}{x^2q}\cdot\frac{x^4q^2}{1-2x^2q^2-P(x^2q^2,q^2)} \\
&= \frac{1}{x^2q}\left(1+2xq+S(xq,q)\right)P(x^2,q^2)
\end{aligned}
$$

where the sum over $Q\in\widetilde{\mathcal{P}}$ in the third equation is treated as in equation (4.3). In the last step, we applied equation (4.1).

3. For a $\langle d_1\rangle$-symmetric polygon Q, we have $f(Q) = (P_1,\ldots P_n)$, with a $\langle d_1\rangle$-symmetric P_i for $i = 1,\ldots,n$. A calculation similar to that in equation (4.3) then yields the assertion.

4. For a $\langle d_2\rangle$-symmetric polygon Q, we have $f(Q) = (P_1,\ldots P_n, C, d_2P_n,\ldots,d_2P_1)$, where $P_i\in\mathcal{P}\cup\{e_h,e_v\}$ for $i=1,\ldots,n$, and where C is a *pt* or $\langle d_2\rangle$-symmetric. Now a computation similar to that in equation (4.4) yields the assertion.

5. For a $\langle d_1,d_2\rangle$-symmetric polygon Q, we have $f(Q) = (P_1,\ldots P_n, C, d_2P_n,\ldots,d_2P_1)$, with $\langle d_1\rangle$-symmetric P_i for $i=1,\ldots,n$, and where C is a *pt* or $\langle d_1,d_2\rangle$-symmetric. A computation similar to that of equation (4.4) yields the assertion.

6. Staircase polygons are also characterised by the property that they contain the lower left and the upper right corner of their smallest bounding rectangles. So the only staircase polygons with $\langle h\rangle$- or $\langle v\rangle$- symmetry are rectangles. f maps a rectangle Q either to a single rectangle, or to sequences of vertical (horizontal) edges, if the width (height) of Q is 1. This results in the above equation.

7. The only admissible polygons are squares. For a given half-perimeter, there is exactly one square. If $n > 1$, the function f maps a square of half-perimeter $2n$ to the square of half-perimeter $2n - 2$, and it maps the unit square to pt. We obtain the claimed equation. □

Remark. Equations of the above form appear in different contexts. Examples are classes of directed lattice paths, counted by length and area under the path [NT03], or classes of simply generated trees [MM78], counted by number of vertices and internal path length. This is due to combinatorial bijections between these classes.In the context of polygon models, equations appear for Class 1 in [BMV92] and for Class 6 in [Ric02], while Class 7 is trivial. Solutions of some equations may be given in explicit form, compare [BM96, PB95].

4.2 Area limit laws

In this section, we derive the limiting area laws for the various symmetry subclasses, in the uniform fixed perimeter ensemble. This will be achieved by an application of the moment method [Bil95, Sec. 30]. Such an approach has been used previously [Tak91, Tak95, NT03, Duc99] in similar contexts, using some involved computations. We will follow a streamlined version, based on the method of dominant balance [Ric09b], which finally allows to obtain the limit distribution by a mechanical calculation. In order to give a self-contained description of the method, we will treat the two cases $\mathcal{P}$ and $\mathcal{S}$ in detail, and then indicate the analogous arguments for the remaining subclasses.

4.2.1 Limit law for $\mathcal{P}$

A q-difference equation for the half-perimeter and area generating function

$$P(x,q) := \sum_{m,n} p_{m,n} x^m q^n$$

of all staircase polygons was derived in Proposition 4.1.1. Here $p_{m,n}$ is the number of staircase polygons of half-perimeter m and area n. We are interested in the distribution of area within a uniform ensemble where, for fixed half-perimeter m, each polygon has the same probability of occurrence. We introduce the discrete random variable X_m of area by

$$\mathbb{P}(X_m = n) = \frac{p_{m,n}}{\sum_n p_{m,n}}, \tag{4.5}$$

the sum in the denominator being finite since the square of its half-perimeter is an upper bound for the area of a polygon.

In the following, we will asymptotically analyse the moments of X_m which in turn, after proper rescaling, determine a unique limit distribution. The result can be expressed in terms of the *Airy distribution*, see [FL01, Jan07] for a discussion of its properties.

Definition 4.2.1. *A random variable Y is* Airy distributed *[FL01] if*

$$\frac{\mathbb{E}[Y^k]}{k!} = \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)} \frac{\phi_k}{\phi_0},$$

where $\gamma_k = 3k/2 - 1/2$, and where $\Gamma(z)$ is the Gamma function. The numbers ϕ_k satisfy for $k \in \mathbb{N}$ the quadratic recursion

$$\gamma_{k-1}\phi_{k-1} + \frac{1}{2}\sum_{l=0}^{k} \phi_l \phi_{k-l} = 0,$$

with initial condition $\phi_0 = -1$.

Remark. *i)* In the sequel, we shall make frequent use of *Carleman's condition*: A sequence of moments $\{M_m\}_{m\in\mathbb{N}}$ with the property $\sum_k (M_{2k})^{-1/(2k)} = \infty$ defines a unique random variable X with moments M_m, cf. [Fel71].
ii) Definition 4.2.1 is justified, since the above moment sequence satisfies Carleman's condition and hence uniquely determines Y. Explicit expressions can be given for its moments, its moment generating function, and its density. The name relates to the asymptotic expansion

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \mathrm{Ai}(s) \sim \sum_{k\geq 0} (-1)^k \frac{\phi_k}{2^k} s^{-\gamma_k} \qquad (s \to \infty),$$

where $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + tx)\mathrm{d}t$ is the Airy function. The Airy distribution appears in a variety of contexts. In particular, the random variable $Y/\sqrt{8}$ describes the Brownian excursion area.

We can now state the following result.

Theorem 4.2.1. *For staircase polygons of half-perimeter m, the moments of the appropriately normalised area random variables X_m converge to the moments of a rescaled Airy distributed random variable. More precisely, for $k = 1, 2, \ldots$ we have*

$$\mathbb{E}\left[\left(\frac{X_m}{m^{3/2}}\right)^k\right] \longrightarrow \mathbb{E}\left[\left(\frac{Y}{4}\right)^k\right] \qquad (m \to \infty),$$

where Y is Airy distributed. Furthermore we have convergence in distribution

$$\frac{X_m}{m^{3/2}} \xrightarrow{d} \frac{Y}{4} \qquad (m \to \infty).$$

Remark. *i)* The second assertion follows since Y is uniquely determined by its moments due to Carleman's condition. This implies that the normalised sequence $\{X_m/m^{3/2}\}_{m\in\mathbb{N}}$ converges to $Y/4$ in distribution, compare [Chu74, Thm. 4.5.5].
ii) The previous theorem is a special case of [Duc99, Thm. 3.1] and [Ric09b, Thm. 1.5].

In [Duc99], a limit distribution result is stated for certain algebraic q-difference equations, together with arguments of a proof using the moment method. In [Ric09b], a general multivariate limit distribution result is proved for certain q-functional equations, using the moment method and the method of dominant balance.

For the sake of completeness and to illustrate the method of dominant balance we rederive Theorem 4.2.1 following [Ric06]. Recall that $p_{m,n} = 0$ for $n > m^2$. Consequently $P(x, q)$ and all its mixed derivatives w.r.t. x and q are *formal power series in x whose coefficients are polynomials in q*, $P(x, q) \in \mathbb{C}[q][[x]]$. Hence the evaluation homomorphism $\mathbb{C}[q][[x]] \to \mathbb{C}[[x]]$, $G(x, q) \mapsto G(x, 1)$ is well defined. For a formal power series $G(x) \in R[[x]]$ with coefficients in an arbitrary ring R, we denote by $[x^m]\, G(x) \in R$ its mth coefficient. We are interested in the asymptotic behaviour of the moments of X_m which are given by

$$\mathbb{E}[X_m^k] = \frac{\sum_n n^k p_{m,n}}{\sum_n p_{m,n}} = \frac{[x^m]\left(q\frac{\partial}{\partial q}\right)^k P(x,q)\Big|_{q=1}}{[x^m]\, P(x,1)}.$$

For our purpose it is convenient and asymptotically equivalent to study the *factorial moments*. With the notation of the falling factorial $(a)_k = a(a-1)\cdots(a-k+1)$ those are given by

$$\mathbb{E}[(X_m)_k] = \frac{\sum_n (n)_k\, p_{m,n}}{\sum_n p_{m,n}} = \frac{[x^m]\, \frac{\partial^k}{\partial q^k} P(x,q)\Big|_{q=1}}{[x^m]\, P(x,1)} =: \frac{[x^m]\, P_{k,0}(x)}{[x^m]\, P_{0,0}(x)},$$

where we introduced the notation

$$P_{k,l}(x) := \frac{\partial^{k+l}}{\partial q^k \partial x^l} P(x,q)\Big|_{q=1}.$$

The formal power series $P_{k,l}(x)$ is called *factorial moment generating function of order* $(k, l) \in \mathbb{N}_0^2$ for the obvious reason. Explicit expressions may be obtained recursively from the functional equation equation (4.1), by implicit differentiation w.r.t. x and q, applying Faà di Bruno's formula [CS96]. In particular, simply setting $q = 1$ in equation (4.1) yields a quadratic equation for the half-perimeter generating function $P_{0,0}(x) = P(x, 1)$ whose relevant solution is

$$P_{0,0}(x) = P(x,1) = \frac{1}{4} - \frac{1}{2}\sqrt{1-4x} + \frac{1}{4}(1-4x). \tag{4.6}$$

In Lemma 4.2.2 below all factorial moment generating functions $P_{k,l}(x)$ turn out to be algebraic.

In summary, the asymptotic behaviour of the factorial moments is obtained by coefficient asymptotics of the generating functions $P_{k,l}(x)$. The latter are amenable to *singularity analysis* [FO90, FS09], a process converting the singular behaviour of an analytic function to precise asymptotic formulas for its Taylor coefficients, see also Section B.2.

To prove Theorem 4.2.1 we proceed in three steps.

1. We prove the existence of a certain local expansion for each factorial moment generating function about its singularity, by an application of the chain rule (or Faà di Bruno's formula [CS96]).

2. Then we will provide an explicit expression for the leading term in this expansion, by an application of the method of dominant balance.

3. The leading term is finally analysed with tools from singularity analysis, yielding the asymptotic behaviour of the corresponding moments.

The first step of our method is summarised by the following lemma. For its statement, recall that a function $f(u)$ is $\Delta(u_c)$-*regular*, for short Δ-regular, [FFK05] if it is analytic in the *indented disc* $\Delta = \Delta(u_c) = \{u : |u| \leq u_c + \eta, |\arg(u - u_c)| \geq \phi\}$ for some real numbers $u_c > 0$, $\eta > 0$ and ϕ, where $0 < \phi < \pi/2$. Note that $u_c \notin \Delta$, where we employ the convention $\arg(0) = 0$. The set of Δ-regular functions is closed under addition, multiplication, differentiation, and integration. Moreover, if $f(u) \neq 0$ in Δ, then $1/f(u)$ exists in Δ and is Δ-regular, see also Section B.2.

Lemma 4.2.2. *For $(k,l) \in \mathbb{N}_0^2$, the power series $P_{k,l}(x)$ has radius of convergence $1/4$ and is $\Delta(1/4)$-regular. It has a locally convergent expansion about $x = 1/4$, as in equation (4.6) for $(k,l) = (0,0)$, and for $(k,l) \neq (0,0)$ of the form*

$$P_{k,l}(x) = \sum_{r=0}^{\infty} \frac{d_{k,l,r}}{(1-4x)^{3k/2+l-r/2-1/2}}. \tag{4.7}$$

Remark. *i)* The exponent $3k/2 - 1/2 + l$ of the leading singular term in equation (4.7) might be guessed by computing the first few of the $P_{k,l}$ explicitly, a procedure nicknamed *moment pumping* in [FPV98]. In particular, from $P_{0,0}(x)$ and $P_{1,0}(x)$ the asymptotic behaviour of the mean area can be inferred, $\mathbb{E}[X_m] \sim A m^{3/2}$.
ii) The reasoning in the following proof may be used to show that all series $P_{k,l}(x)$ are algebraic.
iii) The coefficients $d_{k,l,0}$ might attain zero values at this stage. The recursion equation (4.13) given below however implies that all of them are non-zero. Moreover, it can be inferred from the proof below that the expansion (4.7) consists of finitely many terms only.
iv) For $\langle r^2 \rangle$-symmetric polygons, our proof below will use properties of the derivatives

$$\widetilde{P}_{k,l}(x) := \left. \frac{\partial^{k+l}}{\partial q^k \partial x^l} \left(P(x^2, q^2) \right) \right|_{q=1}.$$

These functions have all radius of convergence $1/2$, are $\Delta(1/2)$-regular, and have the same type of expansion as the functions $P_{k,l}(x)$ of the previous lemma, however with an infinite number of terms. This may be inferred from the previous lemma by an application of Faà di Bruno's formula [CS96].

Proof of Lemma 4.2.2. Due to the fact that $p_{m,n} = 0$ for $n > m^2$, it is seen that all functions $P_{k,l}(x)$ have the same radius of convergence. The statement of the theorem is true for $P_{0,0}(x) = P(x, 1)$, as follows from the explicit formula (4.6). For the general case, we argue by induction on (k, l), using the total order $\lhd$ defined by

$$(r, s) \lhd (k, l) \Leftrightarrow r + s < k + l \vee (r + s = k + l \wedge r < k),$$

chosen to be compatible with the combinatorics of derivatives. Define

$$H(x, q) := P(x, q)(1 - 2xq - P(xq, q)) - x^2 q,$$

compare equation (4.1). Fix $(k, l) \rhd (0, 0)$. An application of Leibniz' rule yields

$$\begin{aligned} \frac{\partial^{k+l}}{\partial q^k \partial x^l} \left(H(x, q) + x^2 q \right) = \sum_{(0,0) \unlhd (r,s) \unlhd (k,l)} \binom{k}{r} \binom{l}{s} \frac{\partial^{k+l-r-s}}{\partial q^{k-r} \partial x^{l-s}} \left(P(x, q) \right) \\ \cdot \frac{\partial^{r+s}}{\partial q^r \partial x^s} \left(1 - 2xq - P(xq, q) \right). \end{aligned} \tag{4.8}$$

In fact, terms corresponding to indices $(r, s) \unlhd (k, l)$ with $s > l$ or $r > k$ are zero. For the second derivative on the r.h.s. of equation (4.8), note that by the chain rule

$$\frac{\partial^r}{\partial q^r} \left(P(xq, q) \right) = \sum_{i=0}^{r} \binom{r}{i} q^{r-i} \left(\frac{\partial^r}{\partial x^{r-i} \partial q^i} P \right) (xq, q).$$

Taking further derivatives w.r.t. x, we may write

$$\begin{aligned} \frac{\partial^{r+s}}{\partial q^r \partial x^s} \left(P(xq, q) \right) = q^{r+s} \left(\frac{\partial^{r+s}}{\partial q^r \partial x^s} P \right) (qx, q) \\ + \sum_{(i,j) \lhd (r,s)} \left(\frac{\partial^{i+j}}{\partial q^i \partial x^j} P \right) (xq, q) \cdot w_{i,j}(x, q), \end{aligned} \tag{4.9}$$

for polynomials $w_{i,j}(x, q)$ in x and q, which satisfy $w_{i,j}(x, q) \equiv 0$ if $i < r$. By inserting equation (4.9) into equation (4.8) and setting $q = 1$, one observes that only the $(0, 0)$ and the (k, l) summand in equation (4.8) contribute terms with $P_{k,l}(x)$. The terms involving $P_{k,l}(x)$ sum up to

$$P_{k,l}(x) \left(1 - 2x - 2P_{0,0}(x) \right) = \sqrt{1 - 4x}\, P_{k,l}(x),$$

where we used equation (4.6). Now the claimed $\Delta(1/4)$-regularity of $P_{k,l}(x)$ follows from the induction hypothesis, by the closure properties of Δ-regular functions. For the particular singular expansion (4.7) note that, by induction hypothesis, each of the remaining terms in the summation in equation (4.8) has an expansion (4.7). Hence, the most singular exponent is bounded by

$$\left(\frac{3}{2}(k - r) + (l - s) - \frac{1}{2} \right) + \left(\frac{3}{2} r + s - \frac{1}{2} \right) = \frac{3}{2} k + l - 1.$$

We conclude that the leading singular exponent of $P_{k,l}(x)$ is at most $3k/2+l-1/2$, which yields the desired bound, and thus the remaining assertion of the Lemma.

□

The second and third step of our method yield a proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. For the second step we apply the method of dominant balance [Ric09b] in order to infer the coefficients $d_{k,0,0}$ in the expansion (4.7). Its idea consists in first replacing the factorial moment generating functions, which appear in the formal expansion of $P(x,q)$ about $q=1$,

$$P(x,q)=\sum_{k=0}^{\infty}(q-1)^k\frac{1}{k!}P_{k,0}(x),$$

by their singular expansion of Lemma 4.2.2, and then in studying the equation implied by the q-functional equation (4.1). Within the framework of locally convergent series we can thus write

$$\begin{aligned}
P(x,q)=&\frac{1}{4}+\sum_{k=0}^{\infty}(q-1)^k\frac{1}{k!}\sum_{r=0}^{\infty}\frac{d_{k,0,r}}{(1-4x)^{3k/2-r/2-1/2}}\\
=&\frac{1}{4}+(1-4x)^{1/2}\sum_{r=0}^{\infty}(1-4x)^{r/2}\sum_{k=0}^{\infty}\frac{d_{k,0,r}}{k!}\left(\frac{q-1}{(1-4x)^{3/2}}\right)^k \qquad (4.10)\\
=&\frac{1}{4}+(1-4x)^{1/2}\,F\left((1-4x)^{1/2},\frac{q-1}{(1-4x)^{3/2}}\right),
\end{aligned}$$

where $F(s,\epsilon)=\sum_r F_r(\epsilon)s^r\in\mathbb{C}[[s,\epsilon]]$ is a formal power series. In particular

$$F_0(\epsilon)=F(0,\epsilon)=\sum_k\frac{d_{k,0,0}}{k!}\epsilon^k=:\sum_k f_k\epsilon^k \qquad (4.11)$$

is a generating function of the sought for coefficients $d_{k,0,0}$, which determine the asymptotic form of the factorial moments (see step three below). We will use the q-functional equation (4.1) to derive a defining equation for $F_0(s)$. To this end use the above representation of $P(x,q)$ in terms of F in the q-difference equation, substitute $4x=1-s^2$, $q=1+s^3\epsilon$, and expand the functional equation to second order in s. Comparing terms involving s^2 yields a Riccati equation for the generating function $F_0(\epsilon)$,

$$\frac{3}{8}\epsilon^2\frac{\mathrm{d}}{\mathrm{d}\epsilon}F_0(\epsilon)-\frac{1}{8}\epsilon F_0(\epsilon)+F_0(\epsilon)^2-\frac{1}{4}=0. \qquad (4.12)$$

On the level of coefficients f_k of $F_0(\epsilon)$, we obtain the recursion

$$\frac{3k-4}{8}f_{k-1}+\sum_{l=0}^{k}f_lf_{k-l}=0, \qquad (4.13)$$

with initial condition $f_0 = d_{0,0,0} = -1/2$ inferred from equation (4.6). It is easily shown by induction that $f_k = 2^{-2k-1}\phi_k$, where the ϕ_k are the numbers occurring in the definition of the Airy distribution 4.2.1. In particular, all coefficients f_k are non-zero.

Remark. The f_k can also be obtained by proceeding more thoroughly in the proof of Lemma 4.7 [Tak91, Tak95]. Our approach based on generating functions can be easily implemented in computer algebra systems and it outputs the equations (4.12) and (4.13) mechanically.

As for the third and final step in deriving the limit distribution, recall that the functions $P_{k,0}(x)$ are $\Delta(1/4)$-regular. Thus the Transfer Theorem B.2.2 [FO90, Theorem 1] is applicable and we obtain the following asymptotic form for the factorial moments of X_m

$$\begin{aligned}\frac{\mathbb{E}[(X_m)_k]}{k!} &= \frac{1}{k!}\frac{[x^m]P_{k,0}(x)}{[x^m]P_{0,0}(x)} \sim \frac{1}{k!}\frac{[x^m]d_{k,0,0}(1-4x)^{-(3k/2-1/2)}}{[x^m]d_{0,0,0}(1-4x)^{1/2}}\\ &\sim \frac{f_k}{f_0}\frac{\Gamma(-1/2)}{\Gamma(3k/2-1/2)}m^{3k/2} = \frac{\phi_k}{\phi_0}\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_k)}\left(\frac{m^{3/2}}{4}\right)^k \qquad (m\to\infty).\end{aligned}$$

This shows that the factorial moment $\mathbb{E}[(X_m)_k]$ is asymptotically equal to the ordinary moment $\mathbb{E}[X_m^k]$ since the latter is a linear combination of the first k factorial moments. Furthermore it follows that the moments of the sequence of random variables $\left\{4m^{-3/2}X_m\right\}_{m\in\mathbb{N}}$ converge to those of Y, where Y is Airy distributed. It follows with [Chu74, Theorem 4.5.5] that the sequence of random variables $\left\{4m^{-3/2}X_m\right\}_{m\in\mathbb{N}}$ also converges in distribution to Y. This finishes the proof of Theorem 4.2.1. □

Remark. If we expand the functional equation to higher order in the above example, we obtain at order s^{r+2} a linear differential equation for the function $F_r(\epsilon)$, which is the generating function for the numbers $d_{k,0,r}$ in the expansion (4.7), compare [Ric02]. So we can mechanically obtain corrections to the asymptotic behaviour of the factorial moment generating functions, and hence to the moments of the limit distribution.

4.2.2 Limit law for $\mathcal{S}$

The above three step procedure can also be applied to derive a limit law for the area random variables X_m^{sym} in the uniform fixed perimeter ensemble of $\langle r^2\rangle$-symmetric staircase polygons. The result can be expressed in terms of the distribution of area of the Brownian meander, see [Tak95, Thms. 2,3] and the review [Jan07].

Definition 4.2.3. *The random variable Z of area of the Brownian meander is given by*

$$\frac{\mathbb{E}[Z^k]}{k!} = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_k)}\frac{\omega_k}{\omega_0}\frac{1}{2^{k/2}},$$

where $\alpha_k = 3k/2 + 1/2$. The numbers ω_k satisfy for $k\in\mathbb{N}$ the quadratic recursion

$$\alpha_{k-1}\omega_{k-1} + \sum_{l=0}^{k}\phi_l 2^{-l}\omega_{k-l} = 0,$$

with initial condition $\omega_0 = 1$, where the numbers ϕ_k appear in the Airy distribution.

Remark. The above moment sequence satisfies Carleman's condition and hence the above definition of the random variable Z is justified. Explicit expressions are known for the moment generating function and the distribution function.

The main result of this section is the following

Theorem 4.2.2. *The moments of the area random variables X_m^{sym} of $\langle r^2\rangle$-symmetric staircase polygons of half-perimeter m, appropriately normalised, converge to those of a rescaled meander distributed variable Z, more precisely*

$$\mathbb{E}\left[\left(\frac{X_m^{\mathrm{sym}}}{m^{3/2}}\right)^k\right] \to \mathbb{E}\left[\left(\frac{Z}{2}\right)^k\right] \qquad (m \to \infty).$$

Furthermore, we have convergence in distribution,

$$\frac{X_m^{\mathrm{sym}}}{m^{3/2}} \xrightarrow{d} \frac{Z}{2} \qquad (m \to \infty).$$

The first step in our proof is to prove the singular behaviour of the factorial moment generating functions

$$S_{k,l}(x) := \left.\frac{\partial^{k+l}}{\partial q^k \partial x^l} S(x,q)\right|_{q=1},$$

where $(k,l) \in \mathbb{N}_0^2$. As above, these series exist as formal power series and have the same radius of convergence. In analogy with Lemma 4.2.2 have the following lemma on local expansions about the dominant singularity.

Lemma 4.2.4. *For $(k,l) \in \mathbb{N}_0^2$, the power series $S_{k,l}(x)$ has radius of convergence $1/2$ and is $\Delta(1/2)$-regular. It has a locally convergent expansion about $x = 1/2$ of the form*

$$S_{k,l}(x) = \sum_{r \geq 0} \frac{s_{k,l,r}}{(1-2x)^{3k/2+l-r/2+1/2}}.$$

Remark. *i)* The following proof shows that all series $S_{k,l}(x)$ are algebraic.
ii) As opposed to the expansions $P_{k,l}(x)$, the singular expansions of the $S_{k,l}(x)$ consist of infinitely many non-zero terms.

Proof. The proof is analogous to that of Lemma 4.2.2. Elementary estimates show that all series $S_{k,l}(x)$ have the same radius of convergence. Setting $q = 1$ in functional equation (4.2), solving for $S_{0,0}(x)$ and expanding about $x = 1/2$ yields the assertion for $(k,l) = (0,0)$. We argue by induction on (k,l), using the total order $\lhd$. Fix $(k,l) \rhd (0,0)$. Differentiating equation (4.2) with Leibniz' Rule gives

$$\begin{aligned}\frac{\partial^{k+l}}{\partial q^k \partial x^l} S(x,q) = \sum_{(0,0) \unlhd (r,s) \unlhd (k,l)} \binom{k}{r}\binom{l}{s} \frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right) \\ \cdot \frac{\partial^{k+l-r-s}}{\partial q^{k-r} \partial x^{l-s}} \left(1 + 2xq + S(xq,q)\right).\end{aligned} \tag{4.14}$$

We argue as in the proof of Lemma 4.2.2 that only the $(0,0)$ summand on the right hand side of equation (4.14) contributes (k,l) derivatives of S, and that all other derivatives of S of order (r,s) satisfy $(r,s) \lhd (k,l)$. Setting $q=1$ in (4.14) and collecting all terms involving $S_{k,l}(x)$ on the left hand side gives

$$\left(1-\frac{\widetilde{P}_{0,0}(x)}{x^2}\right) S_{k,l}(x) = \frac{\widetilde{P}_{0,0}(x)}{x^2}(1+2x) + \sum_{(r,s)\unlhd(k,l)} \frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Bigg|_{q=1} \cdot \left(h_{r,s}(x) + \sum_{(i,j)\lhd(k,l)} a_{i,j} S_{i,j}(x)\right),$$

where the $h_{r,s}(x)$ are (at most linear) polynomials, and the $a_{i,j}$ are some real coefficients. Note also that the terms

$$\frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Bigg|_{q=1} = \sum_{i,j} \binom{r}{i}\binom{s}{j} \widetilde{P}_{i,j}(x) \frac{c_{i,j}}{x^{2+r-i} q^{1+s-j}}$$

are $\Delta(1/2)$-regular, with an expansion about $x=1/2$ having the same exponents as in the expansion (4.7), see the remark following Lemma 4.2.2. We thus get $\Delta(1/2)$-regularity of $S_{k,l}(x)$ by induction, and by the closure properties of Δ-regular functions. For the particular expansion, note that the right hand side has a locally convergent expansion about $1/2$ with most singular exponent $3k/2+l$, as the factor $\frac{\partial^{r+s}}{\partial q^r \partial x^s}\left(\frac{P(x^2,q^2)}{x^2 q}\right)\Big|_{q=1}$ has an expansion with most singular exponent $3r/2+s-1/2$, and the inner sum has by induction an expansion with most singular exponent at most $3(k-r)/2+(l-s)+1/2$. The first factor on the left hand side has a locally convergent expansion about $1/2$ starting with

$$\left(1-\frac{\widetilde{P}_{0,0}(x)}{x^2}\right) = -2\sqrt{2}\sqrt{1-2x} + O(1-2x) \qquad (x \to 1/2).$$

Solving for $S_{k,l}(x)$ yields the desired expansion. $\square$

With the first step of our programme completed we can now apply the method of dominant balance in the

Proof Theorem 4.2.2. Similarly to equation (4.10), we replace the factorial moment generating functions in the expansion of $S(x,q)$ about $q=1$ by the locally convergent Puiseux expansions (4.2.2) and obtain

$$S(x,q) = \frac{1}{(1-2x)^{1/2}}\, G\left((1-2x)^{1/2}, \frac{q-1}{(1-2x)^{3/2}}\right), \tag{4.15}$$

where $G(s,\epsilon) = \sum G_r(s)\epsilon^r$ is a formal power series in s and ϵ and

$$G(0,\epsilon) = G_0(\epsilon) = \sum_{k=0}^{\infty} \frac{s_{k,0,0}}{k!}\epsilon^k$$

is a generating function for the leading coefficients in the singular expansions of the functions $S_{k,0}(x)$. The functional equation (4.2) induces a recursion on the numbers $g_k := s_{k,0,0}/k!$, which determines the limit distribution, as we will see below. We insert equation (4.15) together with equation (4.10) into the functional equation, introduce $q = 1 + s^3\epsilon$ and $2x = 1 - s^2$, and expand the functional equation to order zero in s. This gives the linear inhomogeneous first order differential equation

$$\frac{3}{2}\epsilon^2 \frac{\mathrm{d}}{\mathrm{d}s} G_0(\epsilon) + \frac{1}{2}\epsilon\, G(\epsilon) + 2^{5/2} F_0\left(2^{-1/2}\epsilon\right) G_0(\epsilon) + 2 = 0, \tag{4.16}$$

where $F_0(s)$ is given by equation (4.11). On the level of coefficients, we have the recursion

$$\alpha_{k-1} g_{k-1} + 2^{5/2} \sum_{l=0}^{k} 2^{-l/2} f_l g_{k-l} = 0. \tag{4.17}$$

The initial condition $g_0 = s_{0,0,0} = 2^{-1/2}$ is obtained from the explicit solution $S_{0,0}(x)$ of the functional equation (4.2) with $q = 1$. If we set

$$g_k = \frac{\omega_k}{2^{3k/2+1/2}},$$

then the above recursion is identical to that occurring in the definition 4.2.3 of the meander distribution. In particular, all numbers g_k are non-zero. Since the functions $S_{k,0}(x)$ are $\Delta(1/2)$-regular, we may use the Transfer Theorem B.2.2 [FO90] to infer the asymptotic form of the factorial moments of X_m^{sym},

$$\begin{aligned}
\frac{\mathbb{E}[(X_m^{\mathrm{sym}})_k]}{k!} &= \frac{1}{k!} \frac{[x^m]S_{k,0}(x)}{[x^m]S_{0,0}(x)} \sim \frac{1}{k!} \frac{[x^m]s_{k,0,0}(1-2x)^{-(3k/2+1/2)}}{[x^m]s_{0,0,0}(1-2x)^{-1/2}} \\
&\sim \frac{g_k}{g_0} \frac{\Gamma(1/2)}{\Gamma(3k/2+1/2)} m^{3k/2} = \frac{1}{2^k} \frac{\omega_k}{\omega_0} \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_k)} \frac{1}{2^{k/2}} m^{3k/2} \qquad (m \to \infty).
\end{aligned}$$

The last term is, up to the factor $m^{3k/2}$, the k-th moment of $Z/2$, where Z is the meander area random variable. The previous estimate also implies that the factorial moments $\mathbb{E}[(X_m^{\mathrm{sym}})_k]$ are asymptotically equal to the ordinary moments $\mathbb{E}[(X_m^{\mathrm{sym}})^k]$. We hence have proved moment convergence and with [Chu74, Theorem 4.5.5] it follows that the sequence of random variables $\left\{2m^{-3/2} X_m^{\mathrm{sym}}\right\}_{m\in\mathbb{N}}$ converges in distribution to Z. This finishes the proof. □

Remark. As for the full class of staircase polygons, corrections to the asymptotic behaviour of the factorial moment generating functions can be mechanically obtained also for this example, by expanding the corresponding functional equation to higher orders in ϵ.

4.2.3 Limit law for $\langle d_1 \rangle$-symmetric polygons

These polygons always have even half-perimeter and hence we consider the area laws in the uniform fixed quarter-perimeter ensemble. Note that we have $\widetilde{D}_1(x, q) = D_1(x^{1/2}, q)$ for the generating function of the class $\langle d_1 \rangle$-symmetric polygons, counted by quarter-perimeter and area. The functional equation for $D_1(x, q)$ induces a similar one for $\widetilde{D}_1(x, q)$. Their factorial moment generating functions all have radius of convergence $1/4$, and a statement as in Lemma 4.2.2 can be formulated and proved almost verbatim for $\widetilde{D}_1(x, q)$. The method of dominant balance then yields a generating function for the leading coefficients in the singular expansions, a defining equation similar to equation (4.12), and a recursion similar to equation (4.13). We have the following result.

Theorem 4.2.3. *The area random variables X_m of $\langle d_1 \rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $m^{-3/2}$, converge in distribution to a random variable Y, which is Airy distributed. We also have moment convergence.*

The result can also be obtained via the obvious bijection to Dyck paths [LR01], which maps a polygon to, say its upper boundary paths, see figure 4.4. The area of a $\langle d_1 \rangle$-symmetric staircase polygons corresponds to twice the area the upper path encloses with the diagonal. The limit law of the area below a Dyck path has been studied in [Tak91] where a recursion for the moments is given. This latter result is applied in [NT03] to compute the area of a Brownian motion.

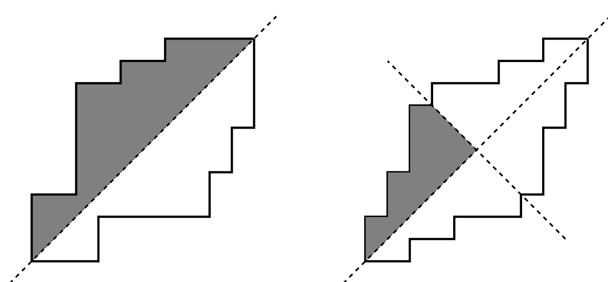

Figure 4.4: The area random variables of $\langle d_1 \rangle$-symmetric and of $\langle d_1, d_2 \rangle$-symmetric staircase polygons are essentially the Dyck path area and the discrete meander area, respectively.

4.2.4 Limit law for $\langle d_1, d_2 \rangle$-symmetric polygons

In this symmetry class, every polygon has even half-perimeter. So we define $\widetilde{D}_{12}(x, q) = D_{12}(x^{1/2}, q)$ as above, and obtain from the functional equation for $D_{12}(x, q)$ one for $\widetilde{D}_{12}(x, q)$, which involves $\widetilde{D}_1(x, q)$, resembling equation (4.2).

It can be argued, as in the proof Lemma 4.2.4, that all factorial moment generating functions $\frac{\partial^k}{\partial q^k} \widetilde{D}_{12}(x, q)\Big|_{q=1}$ have radius of convergence $1/2$, with singularities at $\pm 1/2$, where the leading singular behaviour of the coefficients is determined by the singularity at $1/2$.

We can apply the methods of Section 4.2.2, with the modifications of Section 4.2.3. This yields the following result.

Theorem 4.2.4. *The sequence of area random variables X_m of $\langle d_1, d_2\rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $m^{-3/2}$, converges in distribution to $2Z$, where Z is the meander area variable. We also have moment convergence.*

Alternatively, as in 4.2.3 there is a bijection from $\langle d_1, d_2\rangle$-symmetric staircase polygons to discrete meanders [LR01], see Figure 4.4. The polygon area is four times the meander area. The moments of the limiting distribution have been calculated in [Tak95].

4.2.5 Limit law for $\langle d_2\rangle$-symmetric polygons

In [LR01], a combinatorial bijection between $\langle d_2\rangle$-symmetric polygons and $\langle r^2\rangle$-symmetric polygons with even half-perimeter is described: cut a $\langle d_2\rangle$-symmetric polygon along the line of reflection, flip its upper right part, and glue the two parts together along the cut. So Theorem 4.2.2 translates to the $\langle d_2\rangle$-case.

Alternatively, one may apply the methods of Section 4.2.2, together with modifications similar to those of Section 4.2.3, to the quarter-perimeter and area generating function $\widetilde{D}_2(x,q) = D_2(x^{1/2}, q)$. Lemma 4.2.4 holds in this case, with $1/2$ replaced by $1/4$, and the method of dominant balance yields results similar to equations (4.16) and (4.17).

Theorem 4.2.5. *The area random variables X_m of $\langle d_2\rangle$-symmetric staircase polygons, indexed by quarter-perimeter m and scaled by $(2m)^{-3/2}$, converge in distribution to a random variable $Z/2$, where Z is the meander area random variable. We also have moment convergence.*

4.2.6 Limit law for $\langle r\rangle$-symmetric polygons

The class of staircase polygons with $\langle r\rangle$-symmetry is the class of squares. These may be counted by quarter-perimeter m. Since for given m there is exactly one square, they have, after scaling by m^{-2}, a concentrated limit distribution $\delta(x-1)$. This result can also be obtained from the q-difference equation in Proposition 4.1.1.

4.2.7 Limit law for $\langle h, v\rangle$- ($\langle h\rangle$-, $\langle v\rangle$-) symmetric polygons

The class of staircase polygons with $\langle h, v\rangle$-symmetry (or with $\langle h\rangle$- or $\langle v\rangle$-symmetry) is the class of rectangles. These have been discussed in [Ric09a]. The k-th moments of the area random variable X_m, with m half-perimeter, cf. equation (4.5), are given explicitly by

$$\mathbb{E}[X_m^k] = \sum_{l=1}^{m-1} (l(m-l))^k \frac{1}{m-1} \sim m^{2k} \int_0^1 (x(1-x))^k \mathrm{d}x = \frac{(k!)^2}{(2k+1)!} m^{2k} \qquad (m \to \infty),$$

where we used a Riemann sum approximation. Consider the normalised random variable

$$\widetilde{X}_m = 4X_m/m^2.$$

The moments of $\widetilde{X}_m$ converge as $m \to \infty$, and the limit sequence $M_k = \lim_{m\to\infty} \mathbb{E}_m[\widetilde{X}_m^k]$ satisfies Carleman's condition and hence defines a unique random variable with moments M_k. The corresponding distribution is the beta distribution $\beta_{1,1/2}$. We arrive at the following result.

Theorem 4.2.6. *The sequence $\widetilde{X}_m = 4X_m/m^2$ of area random variables of rectangles, with half-perimeter m scaled by $4/m^2$, converges in distribution to a $\beta_{1,1/2}$-distributed random variable. We also have moment convergence.*

One may also obtain this result by manipulating the associated q-functional equation in Proposition 4.1.1, see [Ric09a]. Expansions of the factorial moment generating functions about their singularity at $x = 1$ can be derived, and bounds for their most singular exponent can be given. The method of dominant balance can then be applied to obtain the leading singular coefficient of these expansions.

4.2.8 Staircase polygons, Dyck paths and discrete meanders

A combinatorial bijection between staircase polygons of perimeter $2m + 2$ and Dyck paths of length $2m$ has been described by Delest and Viennot [DV84]. The peaks of the path are in one-to-one correspondence with the columns of a staircase polygon (from left to right). Peak heights encode the respective column heights and the height of the valley between neighbouring peaks plus one encodes the number of edges along which neighbouring columns are glued together. The decomposition of a polygon sketched in Figure 4.1 corresponds to cutting the corresponding Dyck paths at its contacts with the axis. Within that bijection, the area of a staircase polygon corresponds to the sum of the peak heights of a Dyck path. Furthermore r^2-symmetric polygons are mapped to *symmetric* Dyck paths, which decompose in two identical discrete meanders [LR01], showing that the sum of peak heights parameter for meanders is in the limit distributed according to a meander *area* distribution.

4.2.9 Ferrers diagrams revisited

Ferrers diagrams (cf. Section 3.4) are also a subclass of staircase polygons, and the decomposition depicted in Figure 4.1 also leads to a q-functional equation and the factorial moment generating functions (which are all rational) can be extracted. However, the mean area and the standard deviation are asymptotically equal to $m^2/8$ and $\sqrt{m^3/48}$, respectively. Hence the sequence of area random variables is concentrated to the mean and scaling by the mean (as in the above cases) would thus lead to limit distribution with all its mass in one point. Centring and scaling by $m^{-3/2}$ on the other hand requires the study of more than just the leading singular terms of the expansions of the factorial generating functions about their dominant singularity.

4.3 Limit law for orbit counts

Let $\mathcal{H}$ be a subgroup of $\mathcal{D}_4$. By the Lemma of Burnside, the half-perimeter and area generating function $P_{\mathcal{H}}(x,q)$ of orbit counts w.r.t. $\mathcal{H}$ is given by [LR01]

$$P_{\mathcal{H}}(x,q) = \frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}} P_{\mathrm{Fix}(g)}(x,q) = \frac{1}{|\mathcal{H}|}\left(P(x,q)+R(x,q)\right),$$

where $|\mathcal{H}|$ denotes the cardinality of $\mathcal{H}$, and where $\mathrm{Fix}(g)\subseteq\mathcal{P}$ is the subclass of staircase polygons, which are fixed under $g\in\mathcal{D}_4$, with half-perimeter and area generating function $P_{\mathrm{Fix}(g)}(x,q)$. The series $P(x,q)$ is the full staircase polygon half-perimeter and area generating function, and $R(x,q)$ is the sum of generating functions of staircase polygons which are fixed under $g\in\mathcal{H}$, where $g\neq e$. Let $P(x,1)=\sum_m p_m x^m$ and $R(x,1)=\sum_m r_m x^m$. Due to the previous discussion, see also [LR01, Proposition 14], the number of polygons fixed by some non-trivial symmetry grows subexponentially w.r.t. the total number of polygons. This implies

$$\frac{m^\alpha r_m}{p_m}\to 0 \qquad (m\to\infty) \tag{4.18}$$

for any real number α. As a consequence, area limit distributions for orbit counts coincide with those for the full class of staircase polygons.

Theorem 4.3.1. *Let $\mathcal{H}$ be a subgroup of $\mathcal{D}_4$. Then the area limit law of the class $\mathcal{P}/\mathcal{H}$ coincides with that of $\mathcal{P}$.*

Proof. We show that both classes have asymptotically the same area moments. Since in all examples the limit distribution is uniquely determined by its moments, the claim follows.

Recall that, for polygons of half-perimeter m, their area n satisfies $1\leq n\leq m^2$. Let $P(x,q)=\sum p_{m,n}x^m q^n$ and $R(x,q)=\sum r_{m,n}x^m q^n$. By the relation (4.18), we have for $k\in\mathbb{N}_0$

$$\frac{\sum_n n^k r_{m,n}}{\sum_n n^k p_{m,n}}\leq \frac{m^{2k} r_m}{p_m}\to 0 \qquad (m\to\infty).$$

This implies for the coefficients of the moment generating functions the asymptotic estimate

$$\begin{aligned}[x^m]\left(q\frac{\partial}{\partial q}\right)^k P_{\mathcal{H}}(x,q)\Big|_{q=1} &= \frac{1}{|\mathcal{H}|}\sum_n n^k(p_{m,n}+r_{m,n})\\ &\sim \frac{1}{|\mathcal{H}|}\sum_n n^k p_{m,n} = \frac{1}{|\mathcal{H}|}[x^m]\left(q\frac{\partial}{\partial q}\right)^k P(x,q)\Big|_{q=1} \qquad (m\to\infty).\end{aligned}$$

We conclude that both classes have asymptotically the same area moments. $\square$

4.4 Conclusion

We analysed the symmetry subclasses of staircase polygons on the square lattice. Exploiting a simple decomposition for staircase polygons [Ric06], we obtained the area limit laws in the uniform fixed perimeter ensembles. This extends and completes previous results [LR01]. As expected, orbit counts with respect to different symmetry subgroups always lead to an Airy distribution. The enumeration of polygons fixed under a given symmetry group leads to a variety of area limit distributions, such as a concentrated distribution, the $\beta_{1,1/2}$-distribution, the Airy-distribution, or the Brownian meander area distribution.

One direction of further research is certainly on *self-avoiding polygons* (SAPs) on the square lattice. The general problem is intricate. So far a few numbers of SAPs of a given perimeter and area are found by exact enumeration yielding compelling numerical evidence for limiting area distribution to be Airy. On the other hand there are solved subensembles with convexity and directedness constraints, as e.g. convex polygons [BM96] and symmetry subclasses thereof [LRR98]. With some effort limit laws can be obtained from the decompositions given in the latter article applying the above methods. A further direction of research is on the analogous questions on a hexagonal lattice [GBL05].

One may also study the analogous problem of *perimeter* limit laws in a uniform ensemble where, for fixed area, every polygon occurs with the same probability. For the class of staircase polygons, the associated centred and normalised random variable is asymptotically Gaussian [FS09, Proposition IX.11], and the same result is expected to hold for the symmetry subclasses, apart from squares. Also, limit laws in non-uniform ensembles and for other counting parameters may be studied, compare [Ric09a].

Another direction of research is on functionals on families of (more than two) non-intersecting lattice paths. On the one hand one can consider families of Dyck paths which stay above the x-axis and study the area below the lowest or highest path. In a recent paper [TW07] on non-intersecting Brownian excursions the first moment of the area below the lowest path is computed. It would be interesting to find an analogous result in the discrete setting of "vicious walkers with a wall" [KGV00, KGV03]. So far this is only known for a single path [NT03, Ric09b]. Unfortunately, the decompositions of a single path do not carry over to families of several paths and e.g. the half-length generating functions for tuples of non-intersecting Dyck paths seem to be non-algebraic. Nor allow the known techniques for the enumeration of non-intersecting lattice paths [GV85, GOV98, KGV00, KGV03] to take the area below a single path into account, because edges are attached different weights depending upon which path they belong to.

Chapter 5

Enumeration of prudent polygons

The enumeration of self-avoiding walks (SAW) and polygons (SAP) on a lattice by the number of steps [MS93] is a long standing problem in combinatorics as well as probability theory. Extrapolation of series data from exact enumeration has led to high precision estimates of the exponential growth rate and subexponential corrections but an *exact* solution of either problem (i.e. finding the generating function) seems out of reach. Rechnitzer [Rec06] has shown that the anisotropic generating function of SAPs on the square lattice cannot be D-finite, i.e. if there is a recursion for the numbers, it is not a particularly nice one. At present one tries to find solvable subclasses with large exponential growth rates. This approach has been applied successfully in two dimensions, for example to subclasses with certain directedness and convexity properties (like staircase polygons). In the following we investigate some classes of highly non-convex polygons on the square lattice, so-called *prudent polygons*.

5.1 Prudent walks and polygons

One class of SAWs which recently received attention are so-called *prudent walks* (PW) [Duc05, Pre97], which never take a step towards an already occupied vertex. Note that a general prudent walk is not *reversible*, i.e. the walk traversed backwards from its terminal vertex to its initial vertex may not be prudent. Since SAWs are counted modulo translation, we may choose the initial vertex of a PW to be the origin $(0,0)$. The full problem of PW is unsolved, but recently Bousquet-Mélou [BM08] succeeded in enumerating a substantial subclass. We adopt the terminology of her paper and use the same methods to obtain the generating functions for the corresponding polygon models defined below. Every nearest neighbour walk on the square lattice has a minimal bounding rectangle containing it, referred to as the *box* of the walk. It is easy to see that each unit step of a prudent walk ends on the boundary of its current box. (This is *not* a characterisation of PWs, e.g. the walk $(0,0) \to (0,1) \to (1,1) \to (2,1) \to (2,0) \to (1,0)$ is not prudent.) This property allows the definition of the following subclasses. Call a PW *one-sided*, if every step ends on the *top* side and *two-sided*, if every step ends on *top* or on the *right* side of the current box. Similarly, a PW is referred to as *three-sided* if every step ends on the *left*, *top* or the *right* side of the current box and additionally each *left step* and each *right step* that ends on the *bottom side* (i.e. in the bottom left, resp. right corner) of the current box *inflates* the box to the left, resp. right.

Remark. *i)* As soon as the width of the box of a PW is greater than one, the latter additional condition becomes redundant. It rules out certain configurations which can occur only if the box width is equal to one, namely "downward zigzags" of width one, e.g. $\ldots \to (1,0) \to (1,-1) \to (0,-1) \to (0,-2) \to (1,-2) \to \ldots$. These needlessly complicate the computations below.
ii) Duchi [Duc05] introduced two-sided and three-sided PWs as *type-1* and *type-2* PWs, respectively, by forbidding certain subsequences of steps. In [DGJ07, GGJD09] the authors also employ this notation.

Explicit expressions for the generating functions of one-sided, two-sided and three-sided PWs have been found so far. The first class consists of partially directed walks and has a rational generating function. The second class was shown to have an algebraic generating function by Duchi [Duc05] and recently [BM08] the third class was solved and the generating function was found not to be D-finite. A (possibly multivariate) function $f(\mathbf{z})$ is *D-finite*, if the vector space over $\mathbb{C}(\mathbf{z})$ spanned by its derivatives is finite dimensional. In the univariate case this means that f is a solution of a homogenous linear ordinary differential equation with polynomial coefficients. A univariate D-finite function can at most have *finitely many singularities*, namely the zeroes of the coefficient of the highest order derivative. Guttmann [DGJ07, GGJD09, Gut06] proposed to study the polygon version of the problem, meaning walks, whose last vertex is adjacent to the starting vertex. As above, the property of being prudent demands a starting vertex and a terminal vertex. So prudent polygons are *rooted* polygons with a directed root edge. Note further that a prudent polygon (PP) which ends, say, to the right of the origin (i.e. in the vertex $(1,0)$) may never step right of the line $x = 1$, and furthermore if the walk hits that line it has to head directly to the vertex $(1,0)$. So prudent polygons are *directed* in the sense that they contain a corner of their box. Moreover, a k-sided PP can be interpreted as a $(k-1)$-sided PW confined in a half-plane. In this chapter we deal with the polygon versions of the two-sided and three-sided walks, referred to as two-sided and three-sided PPs. Enumeration of one-sided PPs is trivial, since these are simply rows of unit cells. We give explicit expressions for the half-perimeter generating functions of two-sided and three-sided PPs and show that the latter is not D-finite, which was also expected on numerical grounds. To our knowledge three-sided PPs are the first *exactly solved polygon model* with a non-D-finite half-perimeter generating function. Concerning the enumeration of the full class of PPs, we are able to give a system of functional equations satisfied by the generating function, however we have not been able to solve it so far. The situation is similar for the walk case.

Outline: In Section 5.2 we give functional equations for the generating functions which are based on decompositions of the classes in question, in Section 5.3 we solve those by the kernel method [BM96, BM08, BMJ06, MR09] and in Section 5.4 we study the analytic behaviour of the generating functions of two-sided and three-sided prudent polygons. Section 5.5 is dedicated to the random generation of PPs.

5.2 Functional equations

In the following we will count prudent polygons by half-perimeter and other, so-called *catalytic counting parameters*. The variables in the generating function (see Section B.1) marking the latter are called *catalytic variables*. Their introduction allows us to translate certain combinatorial decompositions into non-trivial functional equations for the associated generating functions [GJ83, FS09]. Furthermore, we identify a PP (a "closed" PW) with the collection of unit cells it encloses and build larger PPs from smaller ones by attaching unit cells in a prudent fashion, i.e. the new boundary walk with the same initial vertex remains prudent. A two-sided prudent polygon either ends at the vertex above the origin or at the vertex to the right of it. This partitions two-sided PPs into two subsets, which can be transferred into each other by the reflection in the diagonal $x = y$. So it suffices to enumerate prudent polygons ending on the top of their box. Here two cases occur, namely the "degenerate" case when the first steps of the walk are left. The resulting PP is simply a row of unit cells pointing to the left. These have a half-perimeter generating function $t^2 + t^3 + \ldots = t^2/(1-t)$. In the "generic case" such a PP is a bar graph turned upside down, i.e. a column convex polyomino containing the top side of its bounding box, cf. Figure 5.1. Denote by $B(t, u, w)$ the generating function of bar graphs counted by half-perimeter, width and height of the rightmost column (catalytic parameter), marked by t, u and w respectively. Here w is the catalytic variable. The width parameter is not a catalytic parameter. However, it will be important in the study of three-sided PPs. We follow the lines of [BM96].

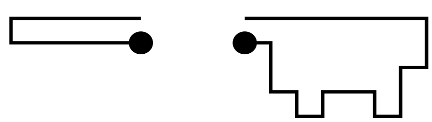

Figure 5.1: Degenerate (left) and generic 2-sided PPs ending on the top of the box

Lemma 5.2.1. *The generating function $B(t, u, w)$ of bar graphs satisfies the functional equation*

$$B\left(t,u,w\right)=u\left(\frac{t^{2}w}{1-wt}+\frac{wt\left(B\left(t,u,1\right)-B\left(t,u,w\right)\right)}{1-w}+\frac{B\left(t,u,w\right)t^{2}w}{1-wt}\right). \tag{5.1}$$

Proof. A bar graph is either a single column, or it is obtained by attaching a new column to the right side of a bar graph. The decomposition is sketched in Figure 5.2. Single columns of height ≥ 1 contribute $ut^2w/(1-wt)$ to the generating function. The polygons obtained by adding a column which is shorter than or equal to the old rightmost column contribute

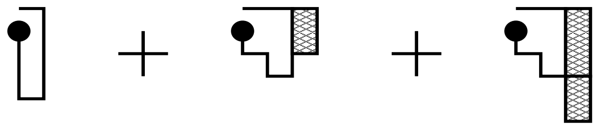

Figure 5.2: Illustration of the decomposition underlying functional equation (5.1)

the second summand. This is seen as follows. A polygon of half-perimeter n, width k and rightmost column height l contributing $t^n u^k w^l$ to $B(t,u,w)$ gives rise to l polygons whose rightmost column is shorter or equal. Their contribution sums up to

$$tu \sum_{j=1}^{l} t^n u^k w^j = tuw \, \frac{t^n u^k 1^l - t^n u^k w^l}{1-w}. \tag{5.2}$$

Summing this over all polygons gives the second summand. The third summand corresponds to adding a larger column. To this end duplicate the rightmost column and attach a non-empty column below the so obtained new rightmost column. A so obtained bar graph can be viewed as an ordered pair of a bar graph and a column. The generating function of those pairs is the third summand of the rhs. This finishes the proof. □

The walk constituting the boundary of a three-sided PP has $(0,0)$ as its initial vertex and $(1,0)$ or $(-1,0)$ or $(0,1)$ as its terminal vertex. Those walks with terminal vertex $(0,1)$ may not step above the line $y=1$ and they have to move directly to the vertex $(0,1)$ as soon as they step upon that line. This leads to two sorts of bar graphs either rooted on their left or on their right side, see Figure 5.3.

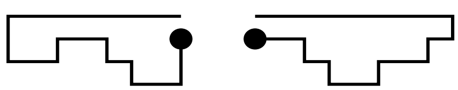

Figure 5.3: Three-sided PPs with terminal vertex $(0,1)$ are bar graphs.

So only those three-sided PPs are of further interest, which end in $(1,0)$ or $(-1,0)$. Both classes are transformed into each other by a reflection in the line $x=0$. We study those ending to the right of the origin in the vertex $(1,0)$. Again a degenerate and a generic case are distinguished, according to whether such a PP reaches its terminal vertex from below via the vertex $(1,-1)$ ("counterclockwise around the origin") or from above, via the vertex $(1,1)$ ("clockwise"). In the degenerate case we simply obtain a single column. In the generic case, a (possibly empty) sequence of initial down steps is followed by a left step. So denote by $R(t,u,w)$ the generating function of the generic three-sided PPs ending in the vertex $(1,0)$ counted by half-perimeter, the length of the top row and the distance of the top left corner of the top row and the top left corner of the box, marked by t, u, and w, respectively, cf. Figure 5.4. Here both u and w are catalytic variables.

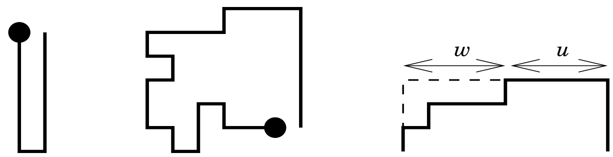

Figure 5.4: Degenerate and generic three-sided PPs, catalytic variables

Lemma 5.2.2. *The generating function $R(t,u,w)$ of generic three-sided PPs satisfies the functional equation*

$$\begin{aligned} R(t,u,w) = ut\left(B\left(t,u\right)+t\right) + \frac{ut\left(R\left(t,w,w\right)-R\left(t,u,w\right)\right)}{w-u} \\ + \frac{ut^2\left(R\left(t,u,w\right)-R\left(t,u,ut\right)\right)}{w-ut} + R\left(t,u,ut\right)ut\left(B\left(t,u\right)+t\right), \end{aligned} \tag{5.3}$$

where $B(t,u) := B(t,u,1)$ is the generating function of bar graphs counted by half-perimeter and width.

Proof. The decomposition we use is sketched in Figure 5.5.

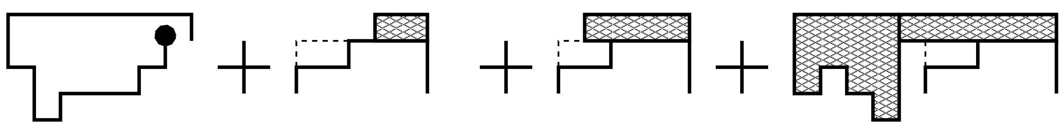

Figure 5.5: Illustration of the decomposition underlying functional equation (5.3)

The polygons in question contain the top right corner of their box. This corner is some point $(1,y)$. If $y=1$, then the PP is either the unit square containing $(0,0)$ and $(1,1)$ or a bar graph as above with that unit square glued to the right. This yields the first summand. A PP with $y>1$ is obtained in one of the following three ways from a PP with top right corner $(1,y-1)$. The first is to add a new row on top, which is shorter than or equal to the original top row. A similar computation as in (5.2) (with some additional book keeping on w) yields the second summand. The second way to obtain a larger PP from a smaller one is by adding a new row on top, which is longer than the original top row, but does not inflate the box to the left. Again a treatment similar to the computation in (5.2) yields the third summand. The third way to extend a PP is to add a row on top of length equal to the width of the box plus one and possibly an arbitrary bar graph. This finally yields the fourth summand and the functional equation is complete. □

Remark. As in the case of general SAPs [Ham61] we can define a concatenation of two three-sided PPs. Roughly speaking, the one PP can be enlarged by inserting the other one at the top left corner of the leftmost column, see Figure 5.6.

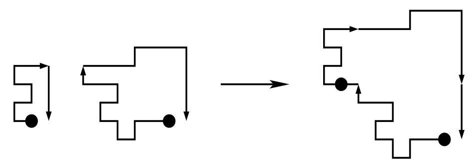

Figure 5.6: Concatenating two 3-sided PPs

The numbers $pp_3^{(m)}$ of three-sided PPs hence satisfy $pp_3^{(m+n)} \geq pp_3^{(m)} \cdot pp_3^{(n)}$. This implies the existence of a connective constant β, i.e. a representation $pp_3^{(m)} = \exp(\beta m + o(m))$. The precise value for β and the subexponential corrections are given in Section 5.4. The converse inequality holds for prudent walks, since breaking an $m+n$ step PW after m steps leaves one with a pair of prudent walks of respective lengths m and n.

We now turn to unrestricted PPs. They can be partitioned into eight subclasses according to their end point being $(1,0)$, $(0,1)$, $(-1,0)$ or $(0,-1)$ and their orientation (clockwise or counterclockwise around the origin). All eight classes can be transformed into each other by symmetry operations of the square. Hence it suffices to enumerate those PPs ending in $(1,0)$ which reach their endpoint via the vertex $(1,1)$ (clockwise). We denote this class by $\mathcal{F}$ and by $F(u,w,x) := F(t,u,w,x)$ its generating function. $G(u,w,x) := G(t,u,w,x)$ and $H(u,w,x) := H(t,u,w,x)$ are defined as the generating functions of the two auxiliary subclasses $\mathcal{G} \supseteq \mathcal{H}$ of $\mathcal{F}$ specified below. We have the following functional equation.

Lemma 5.2.3. *The power series F, G and H satisfy a system of functional equations. For $X = F, G, H$ the single equations are of the form*

$$X(u,w,x) = \frac{tux\left(X(w,w,x) - X(u,w,x)\right)}{w-u} + \frac{t^2ux\left(X(u,w,x) - X(u,ut,x)\right)}{w-ut} + I_X(u,w,x), \tag{5.4}$$

where the formal power series $I_X(u,w,x) := I_X(t,u,w,x)$ is equal to

$$I_X(u,w,x) := \begin{cases} G(x,x,u) & \text{if } X{=}F, \\ t^2ux + t^2uxF(x,xt,w) + xH(x,x,u) & \text{if } X{=}G, \\ t^2uxw^{-1}G(x,xt,w) & \text{if } X{=}H. \end{cases}$$

Proof. The proof relies on a decomposition similar to that of three-sided PPs. The PPs of the class $\mathcal{F}$ all contain the top right corner of their box. In the generating function $F(u,w,x)$ of $\mathcal{F}$ the variable u marks the length of the top row, w marks the distance of the top left corner of the top row to that of the box and x marks the height of the box.

We define the classes $\mathcal{G}$ and $\mathcal{H}$. $\mathcal{G}$ consists of the unit square together with those PPs in $\mathcal{F}$ which are obtained by attaching a piece (a collection of unit cells) on top of a given PP in $\mathcal{F}$, such that the top side of the box is shifted by one unit, the left side by at least one unit and the bottom side by an arbitrary number of units, see Figure 5.7.

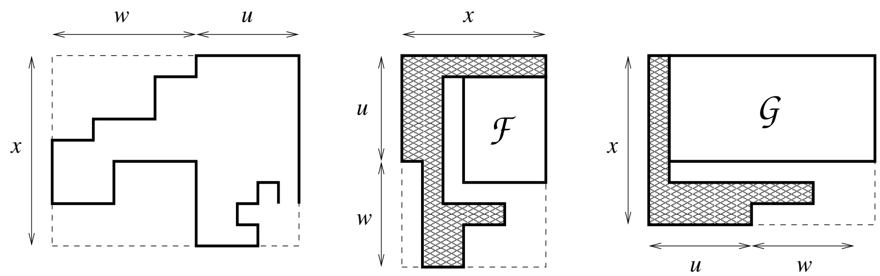

Figure 5.7: Illustrations of the classes $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, catalytic variables

Note, that the polygons in $\mathcal{G}$ contain the top left corner of their box. The catalytic variable u in the generating function $G(u, w, x)$ marks the length of the leftmost column, w marks the distance of the lower left corner of the leftmost column to the bottom left corner of the box and x marks the width of the box. The class $\mathcal{H}$ is the subclass of $\mathcal{G}$ obtained by glueing a piece to the left of the leftmost column and thereby shifting the left side of the box by exactly one unit and the bottom side by at least one unit, see Figure 5.7. The variable u in $H(u, w, x)$ marks the length of the bottom-most row, w marks the distance minus one of the lower right corner of that row to the bottom right corner of the box. The variable x marks the height of the box.

Now the functional equations are derived similarly to the three-sided case. Every PP in $\mathcal{F}$ can be extended by adding a new column on top which is shorter than or equal to the old top row or longer than the old top row, but does not inflate the box to the left. These two operations contribute the first and the second summand in the equation for $X = F$, as in the proof of equation (5.3). Inflating the box to the left yields a PP in $\mathcal{G}$, explaining the expression for I_F.

As for the functional equation $G(u, w, x)$ of the class $\mathcal{G}$ the first two summands on the rhs correspond to adding a new column to the left, the expression $xH(x, x, u)$ to adding a piece which shifts the bottom boundary of the box, in an analogous fashion as above. The unit square contributes t^2ux, the term $t^2uxF(x, xt, w)$ corresponds to the "minimal" polygons in $\mathcal{G}$ obtained by adding a top row on an arbitrary PP of length equal to the width of the box plus one.

The minimal PPs in $\mathcal{H}$ are those obtained by extending a PP in $\mathcal{G}$ adding a column to the left of length equal to the height of the box plus one. This explains the term for I_H The rest of the rhs corresponds to adding a new bottom row. $\square$

5.3 Solution by the kernel method

The following result has already been obtained in [PB95] as the solution of an algebraic equation which arises from a different decomposition of the class. We derive it here for completeness and to recall the "classic" kernel method as applied in [BM96].

Theorem 5.3.1. *The generating function $B(t,u) := B(t,u,1)$ of bar graphs counted by half-perimeter and width is equal to*

$$B(t,u) = \frac{1-t-u(1+t)t-\sqrt{t^2(1-t)^2u^2-2t\left(1-t^2\right)u+(1-t)^2}}{2tu}. \tag{5.5}$$

Proof. The functional equation 5.1 is equivalent to

$$\begin{aligned} 0 = \big(t^2uw(1-w) - uwt(1-wt) - (1-w)(1-wt)\big)\, B(t,u,w) \\ +tuw(1-wt)B(t,u,1) + t^2uw(1-w). \end{aligned} \tag{5.6}$$

The kernel equation

$$0 = \big(t^2uw(1-w) - uwt(1-wt) - (1-w)(1-wt)\big)$$

is a quadratic equation in the catalytic variable w and has the following unique *power series* solution $q(t,u)$

$$q(t,u) = \frac{1+(1-u)t+ut^2-\sqrt{t^2(1-t)^2u^2-2t\left(1-t^2\right)u+(1-t)^2}}{2t}. \tag{5.7}$$

Upon substituting $w = q(t,u)$ into Eq. (5.6), the terms with $B(t,u,w)$ are cancelled and we can solve for $B(t,u,1)$, which leads to (5.5). □

Remark. In principle, $B(t,u,w)$ can also be computed, by substituting the result for $B(t,u,1)$ into Eq. (5.6).

By setting $u = w = 1$ in the bar graph generating function, adding the contribution of the degenerate two-sided PPs and multiplication by 2, we obtain the following result so far conjectured by series extrapolation from exact enumeration data [DGJ07].

Corollary 5.3.1. *The generating function of two-sided prudent polygons is equal to*

$$\begin{aligned} PP_2(t) &= \frac{1}{t}\left(\frac{1-3t+t^2+3t^3}{1-t} - \sqrt{(1-t)\left(1-3t-t^2-t^3\right)}\right) \\ &= 4\,z^2 + 6\,z^3 + 12\,z^4 + 28\,z^5 + 72\,z^6 + 196\,z^7 + 552\,z^8 + 1590\,z^9 \\ &+ 4656\,z^{10} + 13812\,z^{11} + 41412\,z^{12} + 125286\,z^{13} + 381976\,z^{14} \\ &+ 1172440\,z^{15} + 3620024\,z^{16} + 11235830\,z^{17} + 35036928\,z^{18} \\ &+ 109715014\,z^{19} + 344863872\,z^{20} + \dots. \end{aligned} \tag{5.8}$$

Now we turn to the three-sided case. Note that the sum of the catalytic counting parameters, namely the length of the top row and the distance of its top left corner to the top left corner of the box, is equal to the width of the polygon. We have the following result for the generic three-sided PPs ending on the right. It is derived in a similar way as the corresponding result on PWs in [BM08].

Theorem 5.3.2. *The functional equation 5.3 has a unique power series solution. For the generating function $R(t,w,w)$ of generic three-sided prudent polygons ending on the right and counted by half-perimeter and width we have an explicit expression as a an infinite sum of formal power series*

$$R(t,w,w)=\sum_{k\geq 0} L\left(\left(tq^2\right)^k w\right)\prod_{j=0}^{k-1} K\left(\left(tq^2\right)^j w\right). \tag{5.9}$$

Here

$$q:=q(t,1)=\frac{t^2+1-\sqrt{1-4\,t+2\,t^2+t^4}}{2t}, \tag{5.10}$$

with $q(t,u)$ as in (5.7) *in the proof of Theorem 5.3.1. K and L are given by*

$$K(w)=\frac{(1-t)\,q-1-\left(\left(1-t+t^2\right)q-1\right)\left(B\left(t,qw\right)+t\right)w}{1-t\left(1+t\right)q-\left(t\left(1-t-t^3\right)q+t^2\right)\left(B\left(t,qw\right)+t\right)w} \tag{5.11}$$

and

$$L(w)=\frac{\left(1+t^2-\left(1-2\,t+2\,t^2+t^4\right)q\right)\left(B\left(t,qw\right)+t\right)w}{1-t\left(1+t\right)q-\left(t\left(1-t-t^3\right)q+t^2\right)\left(B\left(t,qw\right)+t\right)w}, \tag{5.12}$$

where $B(t,u)$ is the generating function of bar graphs as in (5.5).

Proof. The functional equation (5.3) is equivalent to

$$\begin{aligned}0=\big(ut^2(w-u)-ut(w-ut)-(w-u)(w-ut)\big)\,R(t,u,w)\\+\big(ut\left(B(t,u)+t\right)(w-u)(w-ut)-t^2u(w-u)\big)\,R(t,u,ut)\\+ut(w-ut)R(t,w,w)\\+ut(w-u)(w-ut)\left(B(t,u)+t\right).\end{aligned} \tag{5.13}$$

We first solve the kernel equation

$$\big(ut^2(w-u)-ut(w-ut)-(w-u)(w-ut)\big)=0$$

for u and w. The unique power series solutions are $U(t,w)=q(t)w$ resp. $W(t,u)=q(t)tu$, with $q(t)$ as in (5.10). We substitute $w=W(t,u)$ in Eq. (5.13) and obtain an expression for $R(t,u,ut)$ in terms of $R\left(t,qtu,qtu\right)$, namely

$$R\left(t,u,ut\right)=\frac{\left(q-1\right)R\left(t,qtu,qtu\right)+\left(1-q\right)\left(1-tq\right)\left(B\left(t,u\right)+t\right)u}{1-qt-\left(1-q\right)\left(1-tq\right)\left(B\left(t,u\right)+t\right)u}. \tag{5.14}$$

Substitute this into Eq. (5.13) and set $u=U(t,w)$. This relates $R(t,w,w)$ and $R\left(t,wtq^2,wtq^2\right)$ as follows:

$$R(t,w,w)=K(w)\cdot R\left(t,wtq^2,wtq^2\right)+L(w), \tag{5.15}$$

with $K(w)$ and $L(w)$ as in (5.11) and (5.12). $K(w)$ and $L(w)$ are a formal power series in t, which is seen as follows: $B\left(t,qw\right)$ is well-defined as a formal power series in t as

$\left[t^N\right] B(t,u)$ is a polynomial in u of degree at most $N-1$. Furthermore by the definition of B we see $B(t,u) = t^2u + O(t^3)$. The denominator is now easily seen to be $1 + O(t)$, so both $K(w)$ and $L(w)$ are well-defined as formal power series in t. Inspecting the first few coefficients we see $(1-t)q - 1 = O\left(t^3\right)$ and $1 - \left(1 - t + t^2\right) q = O\left(t^2\right)$, so the numerator of $K(w)$ is $O\left(t^3\right)$. In a similar way the numerator of $L(w)$ is seen to be $w \cdot O\left(t^2\right)$. Moreover we have $tq^2 = t + O\left(t^2\right)$. So we can iterate Eq. (5.15) and obtain formula (5.9). □

Remark. *i)* We have the following alternative expressions for $K(w)$ and $L(w)$:

$$K(w) = \frac{\left((1-q)(1-qt)qwt\left(B(t,qw)+t\right) + t^2q(q-1)\right)(q-1)}{q(1-qt)^2\left((1-q)qwt\left(B(t,qw)+t\right)+t\right)} \tag{5.16}$$

and

$$L(w) = \frac{(1-qt)(1-q^2t)(q-1)qtw\left(B(t,qw)+t\right)}{q(1-qt)^2\left((1-q)qwt\left(B(t,qw)+t\right)+t\right)}. \tag{5.17}$$

The expressions (5.11) and (5.12) were obtained by expressing powers of q in terms of q, e.g.

$$\begin{aligned} q^2 &= \left(t\left(t^2+1\right)q - t\right)/t^2, \\ q^3 &= \left(t\left(t^4+2t^2-t+1\right)q - t^3 - t\right)/t^3, \\ q^4 &= \left(tq\left(t^6+3t^4-2t^3+3t^2-2t+1\right) - t + t^2 - 2t^3 - t^5\right)/t^4. \end{aligned}$$

ii) In principle one could also compute $R(t,u,w)$. To obtain the generating function of all three-sided PPs we sum up the contributions of the degenerate PPs and those ending on top, multiply by two and obtain

$$PP_3(t) = 2\left(\frac{t^2}{1-t} + B(t,1) + R(t,1,1)\right).$$

The first few terms of the series $PP_3(t)$ are

$$\begin{aligned} PP_3(t) = {} & 6\,t^2 + 10\,t^3 + 24\,t^4 + 66\,t^5 + 198\,t^6 + 628\,t^7 + 2068\,t^8 + 7004\,t^9 + 24260\,t^{10} \\ & + 85596\,t^{11} + 306692\,t^{12} + 1113204\,t^{13} + 4085120\,t^{14} + 15131436\,t^{15} + 56495170\,t^{16} \\ & + 212377850\,t^{17} + 803094926\,t^{18} + 3052424080\,t^{19} + 11653580124\,t^{20} + \ldots. \end{aligned}$$

5.4 Analytic properties of the generating functions

So far we have considered the generating functions in question as *formal* power series. A crude estimate on the number of SAPs of half-perimeter n is 4^{2n} which is the total number of all nearest neighbour walks on the square lattice of length $2n$. So the series $PP_2(t)$ and $PP_3(t)$ converge at least in the open disc $\{|t| < 1/16\}$ and represent analytic functions there. This section deals with the analytic properties of these functions. We first discuss the analytic structure of the generating function of two-sided PPs.

Proposition 5.4.1. *The generating function $PP_2(t)$, cf.* (5.8), *is algebraic of degree 2, with its dominant singularity a square root singularity at $t = \rho$, where ρ is the unique real root of the equation*

$$\frac{1 - 4\,t + 2\,t^2 + t^4}{1 - t} = 1 - 3t - t^2 - t^3 = 0.$$

With $\theta = \sqrt[3]{26 + 6\sqrt{33}}$ the exact value for ρ can be written as

$$\rho = \frac{\theta^2 - \theta - 8}{3\theta} = 0.2955977\ldots.$$

The number $pp_2^{(m)}$ of two-sided PPs of half-perimeter m is asymptotically

$$pp_2^{(m)} \sim A \cdot \rho^{-m} \cdot m^{-3/2} \qquad (m \to \infty),$$

where

$$A = \frac{\sqrt{(-37 + 11\sqrt{33})\theta^2 + (-152 + 8\sqrt{33})\theta + 32}}{4\sqrt{6\pi\rho}} = 0.8548166\ldots.$$

Remark. *i)* The asymptotic form of the coefficients is inferred from the Transfer Theorem B.2.2, as PP_2 is easily seen to be $\Delta(\rho)$-regular.
ii) The generating function of two-sided prudent *walks* is algebraic with its dominant singularity a simple pole at $\overline{\sigma} = 0.403\ldots$. Its coefficients are hence asymptotically equal to $\kappa \cdot \overline{\sigma}^{-m}$, where $\kappa = 2.51\ldots$, cf. [BM08].
iii) The asymptotic numbers of bar graphs as well as the staircase polygons of Chapter 4 are of the form $\kappa \cdot \mu^n \cdot n^{-3/2}$. Furthermore, the area random variables in the fixed-perimeter ensembles of bar graphs are known to converge weakly to the Airy distribution [Duc99].

The analytic structure of $PP_3(t)$ is far more complicated due to the analytic structure of $R(t, 1, 1)$, which is stated in the main result Theorem 5.4.1. In what follows we make frequent use of the following facts about the series q :

Lemma 5.4.2. *The series q, $(1 - t)q - 1$, $q^2 t$, $t(1 + t)q$ and $t\,(1 - t - t^3)\,q + t^2$ have non-negative integer coefficients. For $|t| \le \rho$ we have the estimates*

$$|q| \le \frac{|t|^2 + 1}{2|t|},\ \left|q^2 t\right| \le 1,\ |(1 - t)q - 1| \le \rho,\ |1 - t(1 + t)q| \ge \rho.$$

Equality holds if and only if $t = \rho$. Furthermore

$$q\,(\rho) = \frac{\rho^2 + 1}{2\rho} = \frac{1}{\sqrt{\rho}}.$$

The singular behaviour of $B\left(t, q\,(q^2 t)^N\right)$ and $B(t, qw)$ plays an important role in the study of $R(t, 1, 1)$.

Lemma 5.4.3. *For $N \geq 0$ the dominant singularity of $B\left(t, q\left(q^2 t\right)^N\right)$ is σ_N, which is the unique solution in the interval $[0, \rho)$ of the equation*

$$u(t) - q\left(q^2 t\right)^N = \frac{1}{t} \cdot \frac{1-\sqrt{t}}{1+\sqrt{t}} - q\left(q^2 t\right)^N = 0.$$

In particular, $\sigma := \sigma_0 = \tau^2 = 0.2441312\ldots$, where τ is the unique real root of the polynomial $t^5 + 2t^2 + 3t - 2$. The sequence $\{\sigma_N,\ N \geq 0\}$ is monotonically increasing and converges to ρ. Furthermore $B(t, qw)$ is analytic in the polydisc $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$.

Proof. $B(t, u)$ is singular if and only if

$$t^2(1-t)^2 u^2 - 2t\left(1-t^2\right) u + (1-t)^2 = 0.$$

The relevant solution $u(t)$ with $u(\rho) = 1$ is

$$u(t) = \frac{1}{t} \cdot \frac{1-\sqrt{t}}{1+\sqrt{t}}.$$

$B\left(t, q\left(q^2 t\right)^N\right)$ is singular if $q\left(q^2 t\right)^N = u(t)$. This equation has a solution σ_N in the interval $(0, \rho)$, as $u(t) \to 1$ and $q\left(q^2 t\right)^N \to (\rho^2+1)/2\rho = 1/\sqrt{\rho} > 1$, for $t \to \rho$. Here u is strictly decreasing and $q\left(q^2 t\right)^N$ strictly increasing. We further see that σ_N converges to ρ, as for arbitrary fixed t with $0 < t < \rho$ we can chose N sufficiently large, such that $u(t) > q\left(q^2 t\right)^N$, see Lemma 5.4.2. So $\sigma_N \geq t$, which shows the convergence. Monotonicity follows, as $q\left(q^2 t\right)^{N+1} < q\left(q^2 t\right)^N$ for $t \in (0, \rho)$. All these singularities are square root singularities, as the expressions under the root are analytic in $|t| < \rho$. $B(t, qw)$ is singular, if $w = u(t)/q$ and hence

$$|w| = \frac{|u(t)|}{|q|} \geq \sqrt{\rho}\, u(\rho) = \sqrt{\rho},$$

with equality if and only if $t = \rho$. So there is no singularity inside the polydisc. □

Now we are ready to state the main result, which is proved in the subsequent lemmas.

Theorem 5.4.1. *The function $R(t, 1, 1)$ is analytic in the disc $\{|t| < \sigma\}$ with its unique dominant singularity a square root singularity at σ. Moreover it is meromorphic in the slit disc*

$$D_{\sigma,\rho} = \{|t| < \rho\} \setminus [\sigma, \rho),$$

and it has infinitely many square root singularities in the set $\{\sigma_N,\ N = 0, 1, 2, \ldots\}$. In particular, $R(t, 1, 1)$ is not D-finite.

Remark. *i)* From the proof of Lemma 5.4.5 below it can be inferred that $R(t, 1, 1)$ is $\Delta(\sigma)$-regular (Section B.2). By the Transfer Theorem B.2.2 the number $pp_3^{(m)}$ of three-sided PPs of half perimeter m is hence asymptotically equal to $\kappa \cdot \sigma^{-m} \cdot m^{-3/2}$ for some positive constant κ. In particular, two-sided PPs are exponentially rare among three-sided

PPs.
ii) The generating function of three-sided prudent *walks* has its dominant singularity a simple pole at $\overline{\sigma} = 0.403\ldots$, as in the two-sided case. It is meromorphic in some larger disc of radius $\overline{\rho} = \sqrt{2} - 1$ with infinitely many simple poles in the interval $[\overline{\sigma}, \overline{\rho})$. Its coefficients grow like $\kappa \cdot \overline{\sigma}^{-m}$, for some $\kappa > 0$ [BM08].

Possible singularities of $R(t,1,1)$ in $D_{\sigma,\rho}$ are zeroes of the denominators of $K(w)$ and $L(w)$, places, where the representation (5.9) diverges, and square root singularities of $B\left(t, q\left(q^2 t\right)^N\right)$. Now we investigate the analytic properties of the single summands in the representation (5.9).

Lemma 5.4.4. *1.* $K\left(\left(q^2 t\right)^N\right)$ *and* $L\left(\left(q^2 t\right)^N\right)$ *are analytic in* $\{|t| < \sigma_N\}$.

2. $K\left(\left(q^2 t\right)^N w\right)$ *and* $L\left(\left(q^2 t\right)^N w\right)$ *are analytic in* $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$.

Proof. With the above definition of $u(t)$ and a short computation we obtain the estimate

$$\left|B\left(t, q\left(q^2 t\right)^N\right)\right| < B(|t|, u(|t|)) = \sqrt{|t|}.$$

The denominator of $K(w)$ and $L(w)$ is

$$1 - T(t,w) = 1 - t\,(1+t)\,q - \left(t\left(1 - t - t^3\right)q + t^2\right)\left(B\left(t, qw\right) + t\right)w.$$

$T(t,w)$ is a power series in t and w with non-negative coefficients and $T(0,w) = 0$. Hence we have the estimate

$$T\left(t, \left(q^2 t\right)^N\right) \leq T\left(\sigma_N, \left(q\left(\sigma_N\right)^2 \sigma_N\right)^N\right) \leq T\left(\sigma_N, \frac{u\left(\sigma_N\right)}{q\left(\sigma_N\right)}\right).$$

A computation shows that the function $1 - T\left(t, u(t)/q(t)\right)$

$$1 - T\left(t, \frac{u(t)}{q(t)}\right) = 1 - t\,(1+t)\,q - \left(t\left(1 - t - t^3\right)q + t^2\right)\left(\sqrt{t} + t\right)\frac{u(t)}{q(t)}$$

has no zeroes in $[\sigma, \rho]$. This finishes the proof of the first assertion, as $K\left(\left(q^2 t\right)^N\right)$ and $L\left(\left(q^2 t\right)^N\right)$ do not have poles inside $\{|t| < \sigma_N\}$. Furthermore, the denominator $1 - T\left(t, \left(q^2 t\right)^N w\right)$ is analytic in the polydisc $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$, with the only singular point $(t,w) = (\rho, \sqrt{\rho})$ on its boundary. As above we see

$$\left|T\left(t, \left(q^2 t\right)^N w\right)\right| \leq T(|t|, |w|) \leq T\left(\rho, \frac{u(\rho)}{q(\rho)}\right) = T\left(\rho, \sqrt{\rho}\right),$$

and hence the denominator is non-zero in the domain in question and $K\left(\left(q^2 t\right)^N w\right)$ and $L\left(\left(q^2 t\right)^N w\right)$ are both analytic in the polydisc. $\square$

Lemma 5.4.5. *1. The series representation* (5.9) *of* $R(t,1,1)$ *is a series of algebraic functions, which converges compactly in the slit disc* $D_{\sigma,\rho} = \{|t| < \rho\} \setminus [\sigma,\rho)$ *to a meromorphic function.*

2. *Furthermore the corresponding representation of* $R(t,w,w)$ *converges compactly in the polydisc* $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$ *to an analytic function.*

3. *The Taylor expansion of* $R(t,w,w)$ *about* $(t,w) = (0,0)$ *converges absolutely in* $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$.

Proof. For the first assertion choose $0 < r < \rho$. We look at the disc $\{|t| \leq r\}$. The term independent of w in the numerator of $K(w)$ is strictly less than ρ for $|t| \leq r$ and the corresponding term in the denominator is strictly larger than ρ, see Lemma 5.4.2. So we can choose N large such that $\sigma_N > r$ and $\left|K\left((q^2t)^N\right)\right| < 1$ for $|t| \leq r$. Split the series at N. The summands for $k = 0, \ldots, N-1$ sum up to a function which is meromorphic in the slit disc $\{|t| \leq r\} \setminus [\sigma, r]$. In the rest of the series take out the common factors to obtain

$$\prod_{j=0}^{N-1} K\left((tq^2)^j\right) \sum_{k\geq 0} L\left((tq^2)^{N+k}\right) \prod_{j=0}^{k-1} K\left((tq^2)^{N+j}\right). \tag{5.18}$$

The first product is a meromorphic function in the slit disc. $L\left((tq^2)^{N+k}\right)$ is easily seen to converge uniformly to 0 in $|t| \leq r$ as $k \to \infty$. In $|t| \leq r$ all summands are holomorphic (see the above discussion) and the sum can be estimated by a geometric series and hence converges uniformly in the compact disc $\{|t| \leq r\}$. By Montel's theorem the limit of the sum is again analytic. This finishes the proof for the first assertion. The second assertion is proved along the similar lines. By the multivariate version of Montel's theorem [Sch05] the limit function is also analytic in the domain in question and thus the third assertion follows. ☐

Lemma 5.4.6. $R(t,1,1)$ *is singular at infinitely many of the* σ_N. *Furthermore,* $R(t,1,1)$ *is singular at* σ.

Proof. Terms singular at σ_N only show up in the summands for $k \geq N$. The sum of these (5.18) is equal to

$$\prod_{j=0}^{N-1} K\left((tq^2)^j\right) \left[L\left((tq^2)^N\right) + K\left((tq^2)^N\right) R\left(t, (tq^2)^{N+1}, (tq^2)^{N+1}\right)\right].$$

In order to show that the singularity σ_N does not cancel, only the term in square brackets is of interest. Singular terms show up in the numerators and the common denominator of $K\left((tq^2)^N\right)$ and $L\left((tq^2)^N\right)$. We now manipulate the expressions (5.16) and (5.17) for $K(w)$ and $L(w)$ in order to get rid of singular terms in the denominator, where the factor

$$(1-q)qwt\,(B(t,qw)+t)+t$$

leads to a singularity at σ_N for $w = (tq^2)^N$. Write

$$qwt\,(B(t,qw)+t) = A(w) - \phi(w),$$

where

$$A(w) = \frac{1}{2}\,(1+t-qw(1+t)t)$$

$$\phi(w) = \frac{1}{2}\sqrt{t^2(1-t)^2(qw)^2 - 2t\,(1-t^2)\,qw + (1-t)^2}.$$

Then $A\left((tq^2)^N\right)$ is analytic in $\{|t| < \rho\}$. After multiplication of the numerator and denominator with $(1-q)A(w)+t \;+\; (1-q)\phi(w)$ there is no more occurrence of ϕ in the denominator. We now have to collect the terms involving $\phi(w)$ in the numerators of $K(w)$ and $L(w)$. In the numerator of $K(w)$ the terms involving $\phi(w)$ sum up to

$$P_K(w)\phi(w) := t(q-1)^2(1-q^2t)\phi(w).$$

The terms involving $\phi(w)$ in the numerator of $L(w)$ sum up to

$$P_L(w)\phi(w) := (1-qt)(1-q^2t)(1-q)t\phi(w).$$

So the singularity at σ_N can only cancel if

$$-\frac{P_L\left((\sigma_N q(\sigma_N)^2)^N\right)}{P_K\left((\sigma_N q(\sigma_N)^2)^N\right)} = R\left(\sigma_N, \left(\sigma_N q(\sigma_N)^2\right)^{N+1}, \left(\sigma_N q(\sigma_N)^2\right)^{N+1}\right). \tag{5.19}$$

In order to prove that this equation can hold for at most finitely many of the σ_N, we show that for σ_N sufficiently close to ρ the lhs of eq. (5.19) is strictly decreasing while the rhs is strictly increasing. Since (σ_N) is monotonically increasing and converges to ρ this will finish the proof. We first prove the assertion on the rhs. The Taylor expansion of $R(t,w,w)$ about $(0,0)$ has non-negative coefficients and represents $R(t,w,w)$ in the polydisc $\{|t| < \rho\} \times \{|w| < \sqrt{\rho}\}$ by Lemma 5.4.5. By the definition of σ_N and $u(t)$ we have

$$\left(\sigma_N q\,(\sigma_N)^2\right)^{N+1} = u\,(\sigma_N)\,q\,(\sigma_N)\,\sigma_N.$$

The rhs of the last equation is strictly increasing for sufficiently large N and converges to $\sqrt{\rho}$ as $N \to \infty$. The sequence σ_N is also strictly increasing by Lemma 5.4.3. So for large enough N the sequence $R\left(\sigma_N, \left(\sigma_N q\,(\sigma_N)^2\right)^{N+1}, \left(\sigma_N q\,(\sigma_N)^2\right)^{N+1}\right)$ is strictly increasing. Now we turn to the lhs of eq. (5.19). A computation yields

$$-\frac{P_L\left((\sigma_N q(\sigma_N)^2)^N\right)}{P_K\left((\sigma_N q(\sigma_N)^2)^N\right)} = \frac{1-\sigma_N q(\sigma_N)}{q(\sigma_N)-1},$$

which easily seen to be ultimately strictly decreasing. This finishes the proof of Lemma 5.4.6. □

The Lemmas 5.4.4, 5.4.5 and 5.4.6 together constitute a proof of Theorem 5.4.1.

5.5 Random generation of prudent polygons

In [BM08] prudent walks of a given fixed length are generated uniformly at random with a refined version of a method proposed in [NW78]. We briefly describe a version of the method tailored to our particular needs. The main ingredient in the sampling procedure are *generating trees*. These are rooted trees with their nodes labelled in such a way that if two nodes bear the same label, then the multisets of the labels of their children are the same. In this section we present generating trees and sampling procedures for the various classes of prudent polygons.

The decompositions underlying the functional equations (5.1), (5.3) and (5.4) (see also Figures 5.2, 5.5 and 5.7) yield rules according to which a larger PP from the respective class can be constructed starting from a smaller one. We refine these building steps such that each step increases the half-perimeter by one. The result is a step-by-step procedure which allows to generate any PP of half-perimeter m in a *unique* way, starting from the unit square, such that after the kth step we have a PP of half-perimeter $k+2$, $k = 0, 1, \dots m-2$. To put it differently, we can organise the polygons in a rooted tree $\mathcal{T}$, with the unit square as the root and the polygons of half-perimeter m as the nodes on level $m-2$. So a random PP of half-perimeter m corresponds to a random path of length $m-2$ in that tree starting from the root. A polygon α is a *child* of a polygon π, if it is obtained by one of the following six construction steps.

1. Attaching a new top row which is shorter than or equal to the current top row,
2. attaching a unit square to the left side of the current top row,
3. attaching a new leftmost column which is shorter than or equal to the current leftmost column,
4. attaching a unit square to the bottom side of the leftmost column,
5. attaching a new bottom-most row which is shorter than or equal to the current bottom-most row, and
6. attaching a unit square to the right side of the bottom-most row.

Any of these steps, if applicable, increases the half-perimeter by one, see Figure 5.8 below.

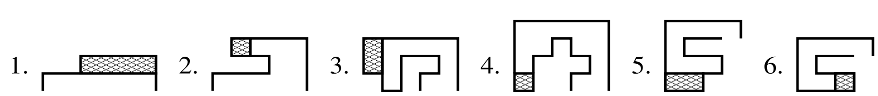

Figure 5.8: The types of steps used to obtain generating trees

Remark. *i)* Steps of types 2,4 and 6 are only admissible if the current top row (leftmost column, bottom row) is longer than or equal to the second row from top (second column

from the left, second row from the bottom). Additionally, a step of type 6 is forbidden, if the bottom row is only one unit shorter than the width of the box.
ii) In the proof of the functional equation (5.3) the steps of types 3 and 4 are encapsulated in the "attaching a bar graph to the left" operation. Hence any generic three-sided PP can be generated starting from the unit square by using only steps of the first four types.
iii) Building bar graphs only requires steps of type 1 and 2. Here we reflected the bar graphs discussed earlier in the line $x = y$.

In order to compute the appropriate probabilities according to which each step in the random path in the tree is chosen, we associate to each PP a label encoding the admissible steps which can be applied to enlarge it. We give a labelling for unrestricted PPs, since the labellings of the other classes are obtained as specialisations thereof. The labels are five-tuples $(a, e, k, l, p) \in \{B, L, T\} \times \{\mathrm{y,n}\} \times \mathbb{Z}^3_{\geq 0}$. The letter a encodes the last building step. It is equal to T (top), if the last step was of type 1 (which inflated the box to the top), or of type 2 but *without* inflating the box to the left. a is equal to L (left) if the last step was of type 2 *and* thereby inflating the box to the left, of type 3 or of type 4 but *without* inflating the box. Finally, a is equal to B (bottom) if the last building step was of type 4 *with* an inflation of the box, or of types 5 or 6.

If $a = T$, the parameter $e \in \{\mathrm{y,n}\}$ (yes/no) indicates if the current top row is longer than or equal to the second row from the top, and hence if a step of type 2 is applicable. Similarly, if $a = L$, e decides if a step of type 4 can be performed, i.e. if the leftmost column is shorter than or equal to the second but leftmost one. Finally, if $a = B$, e decides whether a step of type 6 can be performed.

The parameter k always denotes the length of the top row, and l is either the distance of the left end of the top row to the left side of the box or the length of the leftmost column or the length of the bottom row, depending on whether $a = T$ or $a = L$ or $a = B$ respectively. The unit square receives the label $(L, \mathrm{n}, 1, 1, 0)$.

Labels for the generating tree of PPs				
a	e	k	l	p
T	top row extendable?	top row	dist. of box to top row	height
L	left col. extendable?	top row	length of leftmost col.	dist. of box to left col.
B	bottom row extendable?	top row	length of bottom row	height

Remark. In the proof of equation (5.4) we introduced two subclasses $\mathcal{G}$ and $\mathcal{H}$. The polygons with $a = L$ (resp. $a = B$) are precisely those in $\mathcal{G}$ (resp. $\mathcal{H}$).

The construction steps yield the following rewriting rules for the labels associated with general PPs.

$$(T, \mathrm{n}, k, l, p) \to \begin{cases} (T, \mathrm{n}, i, l + k - i, p + 1), & i = 1, \ldots, k - 1 \\ (T, \mathrm{y}, k, l, p + 1) \end{cases} \tag{5.20}$$

$$(T,\mathrm{y},k,l,p) \rightarrow \begin{cases} (T,\mathrm{n},i,l+k-i,p+1), & i=1,\dots,k-1 \\ (T,\mathrm{y},k,l,p+1) & \\ (T,\mathrm{y},k+1,l-1,p), & \text{if } l \geq 1 \\ (L,\mathrm{n},k+1,1,p-1), & \text{if } l=0 \end{cases} \tag{5.21}$$

$$(L,\mathrm{n},k,l,p) \rightarrow \begin{cases} (T,\mathrm{n},i,k-i,l+p+1), & i=1,\dots,k-1 \\ (T,\mathrm{y},k,0,l+p+1) & \\ (L,\mathrm{n},k+1,i,l+p-i), & i=1,\dots,l-1 \\ (L,\mathrm{y},k+1,l,p) & \end{cases} \tag{5.22}$$

$$(L,\mathrm{y},k,l,p) \rightarrow \begin{cases} (T,\mathrm{n},i,k-i,l+p+1), & i=1,\dots,k-1 \\ (T,\mathrm{y},k,0,l+p+1) & \\ (L,\mathrm{n},k+1,i,l+p-i), & i=1,\dots,l-1 \\ (L,\mathrm{y},k+1,l,p) & \\ (L,\mathrm{y},k,l+1,p-1) & \text{if } p \geq 1 \\ (B,\mathrm{n},k,1,l+1) & \text{if } p=0 \end{cases} \tag{5.23}$$

$$(B,\mathrm{n},k,l,p) \rightarrow \begin{cases} (T,\mathrm{n},i,k-i,p+1), & i=1,\dots,k-1 \\ (T,\mathrm{y},k,0,p+1) & \\ (L,\mathrm{n},k+1,i,p-i), & i=1,\dots,p-1 \\ (L,\mathrm{y},k+1,l,0) & \\ (B,\mathrm{n},k,i,p+1), & i=1,\dots,l-1 \\ (B,\mathrm{y},k,l,p+1) & \text{if } k-1>l \\ (B,\mathrm{n},k,l,p+1) & \text{if } k-1=l \end{cases}$$

$$(B,\mathrm{y},k,l,p) \rightarrow \begin{cases} (T,\mathrm{n},i,k-i,p+1), & i=1,\dots,k-1 \\ (T,\mathrm{y},k,0,p+1) & \\ (L,\mathrm{n},k+1,i,p-i), & i=1,\dots,p-1 \\ (L,\mathrm{y},k+1,l,0) & \\ (B,\mathrm{n},k,i,p+1), & i=1,\dots,l-1 \\ (B,\mathrm{y},k,l,p+1) & \\ (B,\mathrm{n},k,l+1,p) & \text{if } k-1=l \\ (B,\mathrm{y},k,l+1,p) & \text{if } k-1>l \end{cases}$$

The labelled rooted tree generated according to these rewriting rules with its root labelled $(L,\mathrm{n},1,1,0)$ is a generating tree for unrestricted prudent polygons (more precisely, for the class $\mathcal{F}$, cf. Section 5.2). This tree is of course isomorphic to the tree $\mathcal{T}$ defined above having PPs as nodes, simply by replacing each PP by its label.

As mentioned above, choosing a PP of half-perimeter m uniformly at random is equivalent to choosing an $m-2$-step path starting from the root uniformly at random. This is achieved by picking each step in the path according to an appropriate probability which in turn can be expressed in terms of *extension numbers*. If π is a polygon of half-perimeter $m-s$ (a path of length $m-s-2$), then $EX(\pi, s)$ denotes the number of polygons of half-perimeter m which can be reached from π in s construction steps, or equivalently of extensions of length s of the path. Denote by $Ch(\pi)$ the set of polygons obtained from π in one step, i.e. the children of π in the generating tree. Now the right probability to choose $\alpha \in Ch(\pi)$ in the course of our random sampling procedure is equal to

$$\mathbb{P}(\alpha|\pi) = \frac{EX(\alpha, s-1)}{EX(\pi, s)}.$$

The numbers $EX(\pi, s)$ can be computed recursively, namely

$$EX(\pi, s) = \begin{cases} 1 & \text{if } s = 0, \\ \sum_{\alpha \in Ch(\pi)} EX(\alpha, s-1) & \text{otherwise.} \end{cases}$$

The crucial observation is that $EX(\pi, \cdot)$ *only depends on the label of* π, which allows an efficient computation.

For unrestricted PPs, in the first $m-2$ levels of the tree $O(m^3)$ different labels occur since none of the parameters exceeds m. It hence takes $O(m^4)$ operations to compute the all required extension numbers. We have implemented the procedure and computed these numbers up to $m = 80$. See Figure 5.12 at the end of the chapter for some samples.

Modifications for two-sided PPs. As remarked above, generating two-sided PPs only requires steps of types 1 and 2. The only required information for the building procedure is the length of the top row and if the top row is extendable. We hence only need labels (T, n, k) and (T, y, k) obtained from the labels $(T, \cdot, k, l, p)$ above by leaving the parameters l and p unconsidered. The rewriting rule (5.20) can be adapted unchanged (up to deleting the last two coordinates) and in (5.21) simply omit the last line (and the "if $l \geq 1$"-clause in the second but last line). The unit square receives the label $(T, \mathrm{y}, 1)$. There are $O(m)$ different labels in the first $m-2$ levels of the generating tree, and hence $O(m^2)$ extension numbers have to be computed. See Figure 5.10 for some samples of half-perimeter 250.

Modifications for three-sided PPs. For the generation of three-sided PPs steps 1, 2, 3 and 4 suffice. For an appropriate labelling we can hence dump down the $(B, \cdot, k, l, p)$ labels and use labels (T, n, k, l), (T, y, k, l), (L, n, k, l) and (L, y, k, l) obtained from the labels $(T, \cdot, k, l, p)$ and $(L, \cdot, k, l, p)$ by simply discarding the parameter p. The rewriting rules (5.20), (5.21) and (5.22) are adapted without change. In the rule (5.23) drop the last line. The unit square is labelled with $(L, \mathrm{n}, 1, 1)$. We have $O(m^2)$ different labels on the first $m-2$ levels of the tree and hence $O(m^3)$ extension numbers have to be computed. See Figure 5.11 for some samples of half-perimeter 250.

5.6 Conclusion

We have solved the class of two-sided and three-sided prudent polygons, the generating function being algebraic in the former and non-D-finite in the latter case. The analysis shows that two-sided PPs are exponentially rare among three-sided PPs which is different from the corresponding walk models where the growth rates are equal.

It would be nice to solve the class of general prudent polygons. The decompositions we found require three catalytic variables in the corresponding functional equations. This is also the case for the (unsolved) equation for the walk model.

Since the exponential growth rates of SAWs and SAPs are known to be equal [Ham61] it is also interesting to compare the exponential growth rates of k-sided PWs and PPs. To that end it suffices to study PPs ending in $(1,0)$. As already mentioned in the introduction, a k-sided PP ending in $(1,0)$ may never step right of the line $x=1$ and it heads towards the vertex $(1,0)$ as soon as it hits that line for the first time in a point $(1,y_0)$. Up to that step the boundary walk of that k-sided PP is genuinely $k-1$-sided. This yields an injective map sending a k-sided PP to a $k-1$-sided PW simply by reflecting the segment joining $(1,y_0)$ and $(1,0)$ in the line $y=y_0$, see Figure 5.9.

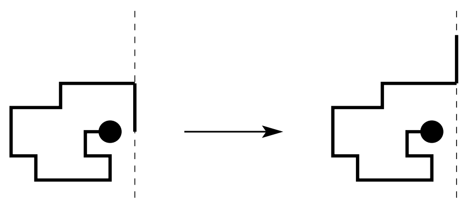

Figure 5.9: Embedding of k-sided PPs into $k-1$-sided PWs

We denote the so-obtained subclass of $k-1$-sided PWs by "embedded k-sided PPs". If we count PPs by full perimeter, their exponential growth rates become $1/\sqrt{\rho} = 1.83\ldots$ for two-sided PPs and $1/\sqrt{\sigma} = 2.02\ldots$ for three-sided PPs. It is known that the exponential growth rate of PWs is equal to $1+\sqrt{2} = 2.41\ldots$ in the one-sided case and equal to $2.48\ldots$ in the two- and three-sided cases [BM08]. The latter rate is also expected for unrestricted PWs [DGJ07, GGJD09]. Consequently, for $k=2,3$, our results imply that k-sided PPs are exponentially rare among k-sided PWs *and*, via embedding, among $k-1$-sided PWs. Furthermore, the rate of three-sided PPs is even smaller than that of one-sided PWs. This is not surprising looking at the pictures in Figure 5.11, as such a PP roughly consists of two "almost" one-sided PWs, one heading to the far left followed by an "almost directed" walk up and to the right (and the closing tail).

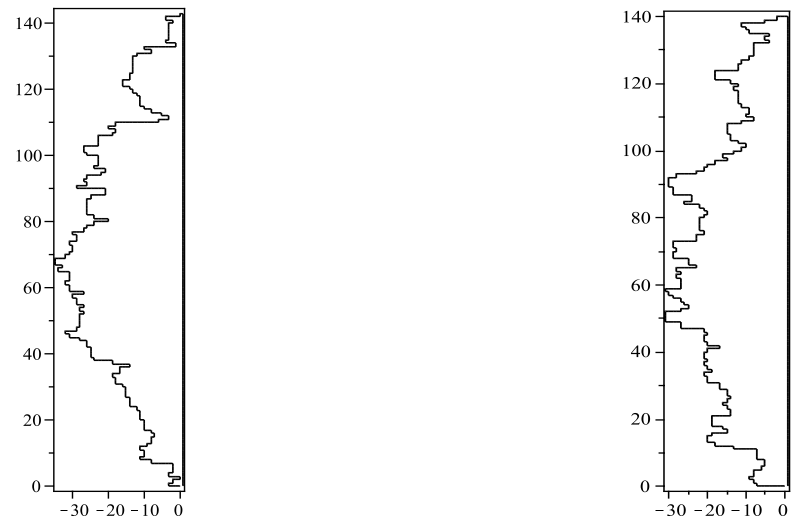

Figure 5.10: Random 2-sided PPs of half-perimeter 250

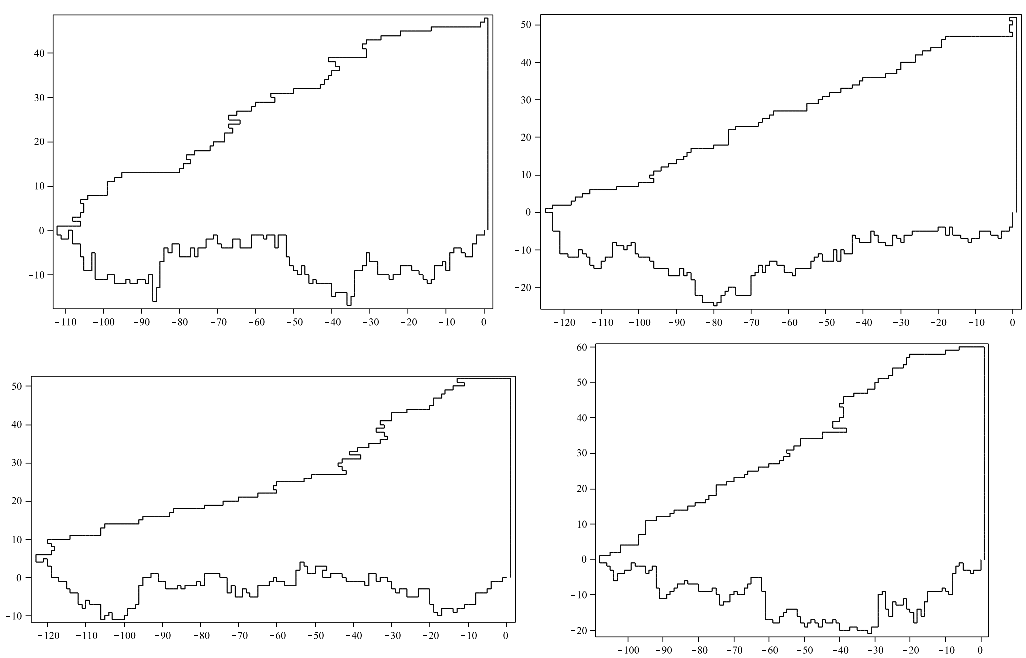

Figure 5.11: Random 3-sided PPs of half-perimeter 250

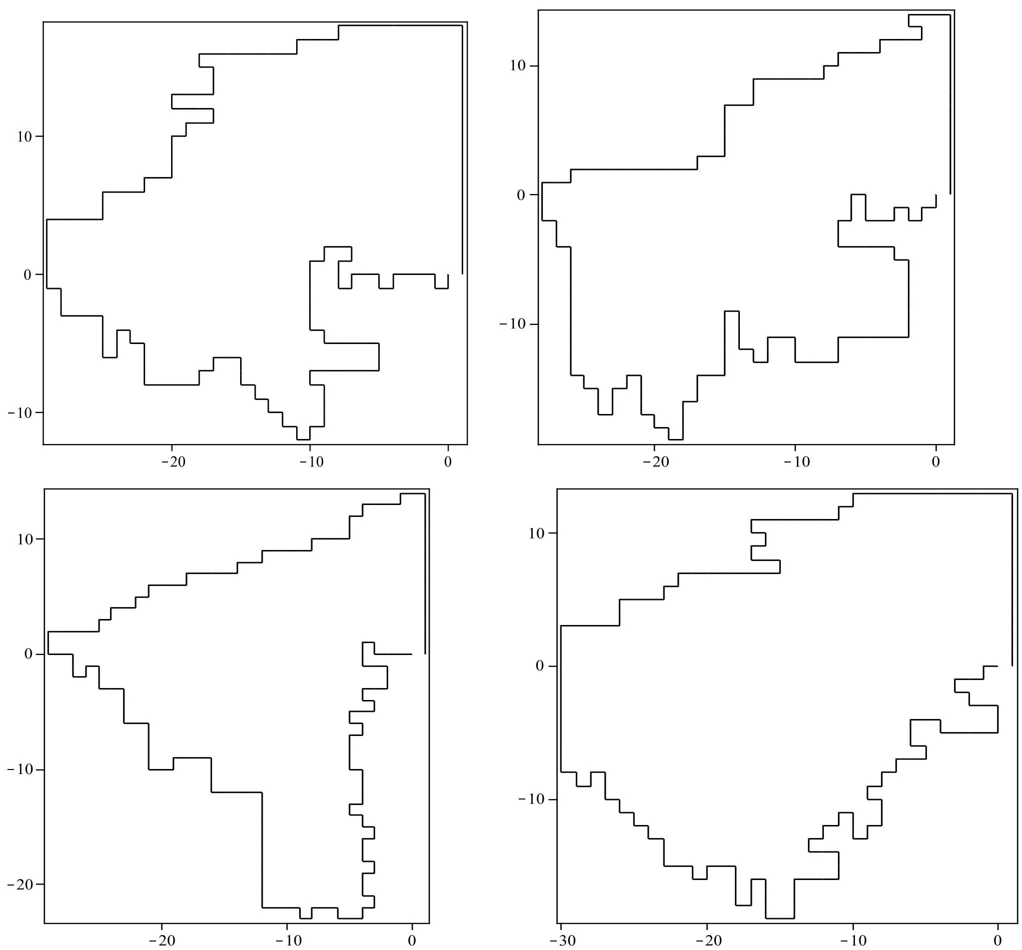

Figure 5.12: Random unrestricted PPs of half-perimeter 80

Appendix A

The basic calculations for orthogonal polynomial ensembles

We sketch a proof of the basic "random matrix calculations" which lead to *determinantal form* of the correlation functions and the Christoffel-Darboux formula (2.6) for the discrete orthogonal polynomial ensemble $DOPE(N,k)$. We follow the exposition in [For08a], see also [Meh04].

We recall some notation. $X_N = \{x_{N,0}, \ldots, x_{N,N-1}\} \subset \mathbb{R}$ is a set of nodes and w_N a positive weight function on X_N. We study the probability distribution

$$p^{(N,k)}(x_1, x_2, \ldots, x_k) = \frac{1}{Z_k} \prod_{i=1}^{k} w_N(x_i) \prod_{1 \le i < j \le k} (x_j - x_i)^2 \tag{A.1}$$

on the set of k-tuples $(x_1, \ldots, x_k) \in X_N^k$ with $x_1 < x_2 < \ldots < x_k$ $(k \le N)$. The number Z_k is a normalisation constant,

$$\begin{aligned} Z_k &= \sum_{x_1 < x_2 < \ldots < x_k} \prod_{i=1}^{k} w_N(x_i) \prod_{1 \le i < j \le k} (x_j - x_i)^2 \\ &= \frac{1}{k!} \sum_{x_1, \ldots, x_k} \prod_{i=1}^{k} w_N(x_i) \prod_{1 \le i < j \le k} (x_j - x_i)^2. \end{aligned}$$

A.1 Determinantal correlation functions

We interpret such a k-tuple as a configuration of k particles located at sites in X_N, where a site can be occupied by at most one particle. The condition $x_1 < x_2 < \ldots < x_k$ reflects the particles' being indistinguishable. In this context, it is natural to ask for the probability that $m \leq k$ given sites are covered by a configuration, i.e.

$$\begin{aligned}
R^m_{N,k}(x_1, \dots, x_m) &= \mathbb{P}(\text{particles at each of the sites } x_1, \dots, x_m) \\
&= \sum_{x_{m+1}<\dots<x_k} p^{(N,k)}(x_1, x_2, \dots, x_k) \\
&= \frac{1}{(k-m)!} \sum_{x_{m+1},\dots,x_k} p^{(N,k)}(x_1, x_2, \dots, x_k).
\end{aligned}$$

$R^m_{N,k}$ is called the m-point correlation function. In (not necessarily *discrete*) orthogonal polynomial ensembles the m-point correlation function can be expressed as an $m \times m$ determinant.

This representation involves the eponymous orthogonal polynomials. Denote by $(\pi_{N,i}(x))_{i=0,\dots,N-1}$ the *monic orthogonal polynomials* with respect to $w_N(x)$, i.e.

$$\deg \pi_{N,i} = i \text{ and } \sum_{x \in X_N} w_N(x) \pi_{N,i}(x) \pi_{N,j}(x) = \delta_{ij}/\gamma^2_{N,i}$$

and by $p_{N,i} = \gamma_{N,i}\pi_{N,i},\ \gamma_{N,i} > 0$, the respective *orthonormal* polynomials. Rewrite the measure (A.1) as follows:

$$\begin{aligned}
&\prod_{i=1}^k w_N(x_i) \prod_{1\leq i<j\leq k} (x_j - x_i)^2 \\
&= \det\left(\sqrt{w_N(x_j)}\pi_{N,i-1}(x_j)\right) \det\left(\sqrt{w_N(x_i)}\pi_{N,j-1}(x_i)\right) \qquad \text{(A.2)} \\
&= \prod_{i=1}^k 1/\gamma_i^2 \det\left(\sqrt{w_N(x_i)w_N(x_j)} \sum_{l=0}^{k-1} p_{N,l}(x_i)p_{N,l}(x_j)\right).
\end{aligned}$$

For the first "=" observe the square of the Vandermonde determinant in (A.1)

$$\prod_{1\leq i<j\leq k} (x_j - x_i) = \det\left(x_j^{i-1}\right)_{i,j=1,\dots,k}.$$

By elementary row manipulations the monomial x_j^{i-1} in the ith row can be replaced by the monic orthogonal polynomial $\pi_{i-1}(x_j)$. Then one of the two matrices is transposed and a factor $\sqrt{w_N(x_j)}$ is multiplied in the respective rows and columns. The second "=" is the product rule for determinants. Define the *correlation kernel* as

$$K_{N,k}(x,y) := \sqrt{w_N(x)w_N(y)} \sum_{l=0}^{k-1} p_{N,l}(x)p_{N,l}(y)$$

and note that due to orthonormality we have

$$\sum_{y\in X_N} K_{N,k}(x_i,y)K_{N,k}(y,x_j) = K_{N,k}(x_i,x_j) \text{ and } \sum_{y\in X_N} K_{N,k}(y,y) = k. \qquad \text{(A.3)}$$

We now prove a lemma which directly implies a determinantal expression for the correlation functions and which is also used in the proof of the Christoffel-Darboux formula below.

Lemma A.1.1. *For $M \leq k$ we have*

$$\sum_{x_M \in X_N} \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M} = (k - M + 1) \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M-1}.$$

Proof. Start with a Laplace expansion of the determinant along the bottom row and multiply the factor $K_{N,k}\left(x_m, x_l\right)$ $(l = 1, \ldots, M-1)$ into the rightmost column of the corresponding cofactor. This yields

$$\begin{aligned}
&\det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M} \\
=&\sum_{l=1}^{M-1} (-1)^{l+M} \det\left(K_{N,k}\left(x_i, x_j\right) \middle| K_{N,k}\left(x_i, x_M\right) K_{N,k}\left(x_M, x_l\right)\right) \\
&+ K_{N,k}\left(x_M, x_M\right) \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M-1},
\end{aligned}$$

where the indices i, j in the lth cofactor run from 1 to $M-1$ and j skips l. In the matrices in the sum only the rightmost column depends on x_M. Now sum over x_M and apply equations (A.3). This yields

$$\begin{aligned}
&\sum_{l=1}^{M-1} (-1)^{l+M} \det\left(K_{N,k}\left(x_i, x_j\right) \middle| K_{N,k}\left(x_i, x_l\right)\right) \\
&+ k \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M-1} \\
=&(-(M-1) + k) \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M-1},
\end{aligned}$$

as each determinant in the sum is equal to $-\det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,M-1}$ which is seen after some column operations. ☐

With the help of equations (A.2), (A.3) and the Lemma the correlation function can be evaluated as follows.

Theorem A.1.1. *The m-point correlation functions of a discrete orthogonal polynomial ensemble allow a determinantal representation,*

$$\begin{aligned}
R^m_{N,k}(x_1, \ldots, x_m) =&\mathbb{P}(\text{particles at each of the sites } x_1, \ldots, x_m) \\
=&\det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,m}.
\end{aligned} \tag{A.4}$$

Proof. We can rewrite the involved probabilities applying equation (A.2) as

$$\begin{aligned}
&R^m_{N,k}(x_1, \ldots, x_m) = \mathbb{P}(\text{particles at each of the sites } x_1, \ldots, x_m) \\
=&\frac{k!}{(k-m)!} \frac{\sum_{x_{m+1},\ldots,x_k} \prod_{i=1}^k w_N(x_i) \prod_{1 \leq i < j \leq k} (x_j - x_i)^2}{\sum_{x_1,\ldots,x_k} \prod_{i=1}^k w_N(x_i) \prod_{1 \leq i < j \leq k} (x_j - x_i)^2} \\
=&\frac{k!}{(k-m)!} \frac{\sum_{x_{m+1},\ldots,x_k} \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,k}}{\sum_{x_1,\ldots,x_k} \det\left(K_{N,k}\left(x_i, x_j\right)\right)_{i,j=1,\ldots,k}}.
\end{aligned}$$

To prove the theorem evaluate the numerator and the denominator in the last line by successively applying Lemma A.1.1 with $M = k, k-1, \dots, m+1$ and $M = k, k-1, \dots, 1$, respectively. □

A.2 Christoffel-Darboux formula

The following proof of the Christoffel-Darboux summation formula is also adapted from [For08a]. It uses the following lemma on alternating polynomials f, i.e. polynomials in several variables $x_1, \dots, x_M$ with the property

$$f\left(x_{\sigma(1)}, \dots, x_{\sigma(M)}\right) = \operatorname{sgn}(\sigma) f\left(x_1, \dots, x_M\right),$$

for every permutation σ on M letters, where $\operatorname{sgn}(\sigma)$ is the sign of σ.

Lemma A.2.1. *Let f be an alternating function in $x_1, \dots, x_M$ and let $g_0(z), \dots, g_{M-1}(z)$ be univariate polynomials. Then*

$$\begin{aligned} &\sum_{x_1 \in X_N} \cdots \sum_{x_M \in X_N} \det\left(g_{j-1}(x_i)\right)_{i,j=1,\dots,M} f\left(x_1, \dots, x_M\right) \\ &= M! \sum_{x_1 \in X_N} \cdots \sum_{x_M \in X_N} g_0(x_1) \cdots g_{M-1}(x_M) f\left(x_1, \dots, x_M\right). \end{aligned}$$

Proof. Leibniz' expansion of the determinant and f being alternating yield

$$\begin{aligned} &\sum_{\sigma} \sum_{x_1 \in X_N} \cdots \sum_{x_M \in X_N} \operatorname{sgn}(\sigma) g_0\left(x_{\sigma(1)}\right) \cdots g_{M-1}\left(x_{\sigma(M)}\right) \operatorname{sgn}(\sigma) f\left(x_{\sigma(1)}, \dots, x_{\sigma(M)}\right) \\ &= \sum_{\sigma} \sum_{x_1 \in X_N} \cdots \sum_{x_M \in X_N} g_0(x_1) \cdots g_{M-1}(x_M) f\left(x_1, \dots, x_M\right), \end{aligned}$$

after a change in the order of summation and renaming the variables. □

Now we can show

Theorem A.2.1. *The correlation kernel admits a representation*

$$\begin{aligned} K_{N,k}(x,y) &= \sqrt{w_N(x) w_N(y)} \sum_{n=0}^{k-1} p_{N,n}(x) p_{N,n}(y) \\ &= \sqrt{w_N(x) w_N(y)} \cdot \frac{\gamma_{N,k-1}}{\gamma_{N,k}} \cdot \frac{p_{N,k}(x) p_{N,k-1}(y) - p_{N,k}(y) p_{N,k-1}(x)}{x-y} \end{aligned}$$

if $x \neq y$, and otherwise

$$K_{N,k}(x,x) = w_N(x) \cdot \frac{\gamma_{N,k-1}}{\gamma_{N,k}} \cdot \left(p'_{N,k}(x) p_{N,k-1}(x) - p'_{N,k-1}(x) p_{N,k}(x)\right).$$

Proof. Evaluate the following function $f(x,y)$ in two ways.

$$f(x,y) = \sqrt{w_N(x)w_N(y)} \sum_{x_1,\ldots,x_{k-1}} \prod_{i=1}^{k-1} w_N(x_i) \prod_{1\le i<j\le k-1} (x_j - x_i)^2 \times \prod_{j'=1}^{k-1} (x_{j'} - x)(x_{j'} - y).$$

As above we can apply column operations to rewrite a Vandermonde factor in $f(x,y)$:

$$\prod_{1\le i<j\le k-1} (x_j - x_i) \prod_{j'=1}^{k-1} (x_{j'} - u) = \prod_{l=1}^{k} \frac{1}{\gamma_{l-1}} \det \begin{pmatrix} p_{N,j-1}(u) \\ p_{N,j-1}(x_i) \end{pmatrix}_{\substack{i=1,\ldots,k-1 \\ j=1,\ldots,k}}.$$

Similarly as in the equation (A.2) multiply the matrix in the determinant with $u = x$ with the transpose of the one with $u = y$ and multiply the factors $\sqrt{w_N(\cdot)}$ into the corresponding rows and columns. Then evaluate the sums by a successive applications of Lemma A.1.1. This yields

$$\begin{aligned} f(x,y) &= \sum_{x_1,\ldots,x_{k-1}} \prod_{l=1}^{k} \frac{1}{\gamma_{l-1}^2} \det \begin{pmatrix} K_{N,k}(x,y) \\ K_{N,k}(x_i,x_j) \end{pmatrix}_{\substack{i=1,\ldots,k-1 \\ j=1,\ldots,k}} \\ &= (k-1)! \prod_{l=1}^{k} \frac{1}{\gamma_{l-1}^2} K_{N,k}(x,y). \end{aligned} \tag{A.5}$$

As for the other evaluation of $f(x,y)$, observe that

$$\prod_{1\le i<j\le k-1} (x_j - x_i) \prod_{j'=1}^{k-1} (x_{j'} - x)(x_{j'} - y) = \frac{1}{x-y} \det \begin{pmatrix} \pi_{N,j-1}(x) \\ \pi_{N,j-1}(y) \\ \pi_{N,j-1}(x_i) \end{pmatrix}_{\substack{i=1,\ldots,k-1 \\ j=1,\ldots,k+1}}$$

holds, by another application of the Vandermonde formula and column operations. The remaining Vandermonde factor is rewritten as usual in terms of monic orthogonal polynomials. Plug this into $f(x,y)$ and observe that the factor in the second row is an alternating function in $x_1,\ldots,x_{k-1}$:

$$f(x,y) = \frac{\sqrt{w_N(x)w_N(y)}}{x-y} \sum_{x_1,\ldots,x_{k-1}} \det\left(\pi_{N,j-1}(x_i)\right)_{i,j=1,\ldots,k-1} \times \prod_{i=1}^{k} w_N(x_i) \det \begin{pmatrix} \pi_{N,j-1}(x) \\ \pi_{N,j-1}(y) \\ \pi_{N,j-1}(x_i) \end{pmatrix}_{\substack{i=1,\ldots,k-1 \\ j=1,\ldots,k+1}}.$$

Apply Lemma A.2.1 and multiply the factors $\pi_{N,i-1}(x_i)$ and $w_N(x_i)$ into the $(i+2)$th row:

$$f(x,y) = (k-1)! \frac{\sqrt{w_N(x) w_N(y)}}{x-y} \sum_{x_1,\ldots,x_{k-1}} \det \begin{pmatrix} \pi_{N,j-1}(x) \\ \pi_{N,j-1}(y) \\ w_N(x_i)\, \pi_{N,i-1}(x_i)\, \pi_{N,j-1}(x_i) \end{pmatrix}_{\substack{i=1,\ldots,k-1 \\ j=1,\ldots,k+1}}.$$

Now summation can be carried out row-wise, which leaves the $(i+2)$th row with an entry $1/\gamma_{N,i-1}^2$ on position $j=i$ and zeroes elsewhere. Evaluating the so obtained determinant gives the required expression for $f(x,y)$, namely

$$f(x,y) = (k-1)! \frac{\sqrt{w_N(x) w_N(y)}}{x-y} \left(\pi_{N,k-1}(x)\pi_{N,k-1}(y) - \pi_{N,k}(x)\pi_{N,k-1}(y)\right) \prod_{l=1}^{k-2} \frac{1}{\gamma_{N,l-1}^2}.$$

Equate this with equation (A.5) to obtain the first assertion. The second assertion follows by simply letting $y \to x$. $\square$

Appendix B

Generating functions and asymptotics

B.1 Combinatorial classes and generating functions

In the combinatorial enumeration tasks occurring in Chapters 4 and 5, we do not try to find the number of objects of a given size (polygons of a given half-perimeter) directly, but make frequent use of *generating functions*. This is due to the fact that a decomposition of a complicated object into simpler objects allows us to express the generating function of the former in terms of the generating functions of the latter. We give a short "dictionary" which mediates between such set-theoretic operations on the classes of objects and arithmetic operations on the involved generating functions.

A *combinatorial class* is a pair $(\mathcal{C}, |\cdot|_{\mathcal{C}})$ consisting of a countable set $\mathcal{C}$ of objects and a *size function*

$$|\cdot|_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbb{Z}_{\geq 0}$$

with the property that for all $n \in \mathbb{Z}_{\geq 0}$ the preimage of n under $|\cdot|_{\mathcal{C}}$ is *finite*. We refer to $(\mathcal{C}, |\cdot|_{\mathcal{C}})$ simply by $\mathcal{C}$ and sometimes drop the subscript in $|\cdot|_{\mathcal{C}}$. For $\alpha \in \mathcal{C}$ we call $w_\alpha(z) = z^{|\alpha|}$ the *weight* of α. The *generating function* of $\mathcal{C}$ is the formal power series

$$C(z) = \sum_{\alpha \in \mathcal{C}} w_\alpha(z) = \sum_{\alpha \in \mathcal{C}} z^{|\alpha|} = \sum_{n=0}^{\infty} c_n z^n,$$

where $c_n = \#\{\alpha \in \mathcal{C} \,|\, |\alpha| = n\}$. The set-theoretic operations we apply in the text are the *disjoint union* and *direct product* of two classes $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ and *finite sequences* of elements of $\mathcal{A}$. Denote the generating functions of $\mathcal{A}$ and $\mathcal{B}$ by $A(z)$ and $B(z)$, respectively. If $\mathcal{C} = \mathcal{A} \,\dot{\cup}\, \mathcal{B}$ and the size function is given by $|\alpha|_{\mathcal{C}} = |\alpha|_{\mathcal{A}}$, for $\alpha \in \mathcal{A}$ and by $|\alpha|_{\mathcal{C}} = |\alpha|_{\mathcal{B}}$, for $\alpha \in \mathcal{B}$, then the generating function is $C(z) = A(z) + B(z)$. Similarly, if we define $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ with the size function $|(\alpha, \beta)|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$ then the generating function is given by

$$C(z) = \sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} z^{|\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}} = A(z) \cdot B(z).$$

One further construction is the class of finite sequences of a class $\mathcal{A}$, namely

$$\mathcal{C} = \textsc{Seq}(\mathcal{A}) = \{\varepsilon\} \cup \mathcal{A} \cup \mathcal{A}^2 \cup \ldots \cup \mathcal{A}^n \cup \ldots,$$

where ε is the object of size 0 (empty sequence). If the size function is given by

$$|(\alpha_1, \ldots, \alpha_n)|_{\mathcal{C}} = |\alpha_1|_{\mathcal{A}} + \ldots + |\alpha_n|_{\mathcal{A}}$$

then the generating function is given by

$$C(z) = 1 + A(z) + A(z)^2 + \ldots + A(z)^n + \ldots = \frac{1}{1 - A(z)}.$$

Care has to be taken with the choice of the class $\mathcal{A}$, because $\mathcal{C} = \textsc{Seq}(\mathcal{A})$ to be a combinatorial class demands the preimage of 0 under $|\cdot|_{\mathcal{A}}$ to be *empty*. Otherwise, if $|\alpha|_{\mathcal{A}} = 0$ for some $\alpha \in \mathcal{A}$, then $(\alpha, \alpha, \ldots, \alpha) \in \mathcal{A}^n$ for every n and the preimage of 0 under $|\cdot|_{\mathcal{C}}$ is infinite. Actually every occurring value of $|\cdot|_{\mathcal{A}}$ is taken infinitely often by $|\cdot|_{\mathcal{C}}$.

In our applications the size parameter is the half-perimeter of some lattice polygons. It is desirable or even crucial to take more parameters into account, e.g. width or area. We can define a d-dimensional multiparameter

$$\chi : \mathcal{C} \longrightarrow \mathbb{Z}^d_{\geq 0}, \quad \alpha \mapsto (\chi_1(\alpha), \ldots, \chi_d(\alpha)).$$

The weight is refined to $w_\alpha(z, u_1, \ldots, u_d) = z^{|\alpha|} u_1^{\chi_1(\alpha)} \cdots u_d^{\chi_d(\alpha)}$ and the generating function becomes a multivariate power series

$$C(z, u_1, \ldots, u_d) = \sum_{\alpha \in \mathcal{C}} w_\alpha(z, u_1, \ldots, u_d) = \sum_{\alpha \in \mathcal{C}} z^{|\alpha|} u_1^{\chi_1(\alpha)} \cdots u_d^{\chi_d(\alpha)}.$$

If the classes $\mathcal{A}$ and $\mathcal{B}$ are endowed with d-dimensional multiparameters ξ and ζ, we can obtain a multiparameter χ on $\mathcal{A} \dot\cup \mathcal{B}$, $\mathcal{A} \times \mathcal{B}$ and $\textsc{Seq}(\mathcal{A})$ in a similar way as above, by case distinction in the first, and by taking sums in the latter cases. The above "dictionary" applies verbatim to this situation.

Remark. In the text we use slight modifications of the above schemes, e.g. for $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ we take $|\cdot|_{\mathcal{C}} = |\cdot|_{\mathcal{A}} + |\cdot|_{\mathcal{B}} + c$ and $\chi = \xi + \zeta + d$ where c, d are fixed constants. This leads only to minor changes in the dictionary.

B.2 Coefficient asymptotics

In Chapters 3, 4 and 5 asymptotic results on some (counting) sequences are derived. We recall some results which allow to infer asymptotic information on a given sequence with the help of generating functions following [FS09]. For a given sequence $(f_n)_{n\geq 0}$ of complex numbers, the formal power series $F(z) = \sum_{n\geq 0} f_n z^n$ is called the (ordinary) generating function of the sequence $(f_n)_{n\geq 0}$. In most cases, the f_n are non-negative integers as above in Section B.1. We also employ the notation $f_n = [z^n]\, F(z)$. Now the process of extracting asymptotics for f_n from F can be summarised in two principles.

First principle of coefficient asymptotics: The modulus of the dominant singularities of $F(z)$ dictates the exponential growth rate of the coefficients f_n.

Second principle of coefficient asymptotics: The nature of the dominant singularities dictates the subexponential corrections.

If the series $F(z)$ *converges* in some disc centred at 0 of positive radius, it represents some analytic function there (which we also denote by $F(z)$). This implies that the radius of convergence

$$R = \left(\limsup |f_n|^{1/n}\right)^{-1} \in [0, +\infty]$$

exists and is positive (possibly $= +\infty$). In the case $R < \infty$ this is equivalent to

$$f_n = \left(\frac{1}{R}\right)^n \theta(n), \text{ where } \limsup |\theta(n)|^{1/n} = 1. \tag{B.1}$$

It is well-known that there exists a *singularity of modulus* R of $F(z)$ [FS09, Theorem IV.5], every such singularity is called *dominant*. This yields the first principle. Furthermore, if $f_n \geq 0$ for all n, then R itself is a singular point of $F(z)$ by *Pringsheim's Theorem* [FS09, Theorem VI.5].

As for the second principle, the idea is to have a catalogue of "standard scale functions" whose coefficients' subexponential behaviour is known. If $F(z)$ behaves like such a function near a dominant singularity, it can be inferred that the coefficients of F and that standard scale function show the same asymptotic behaviour. To make that precise, we assume that F has a unique dominant singularity z_c on the real axis. Without loss of generality we can assume $z_c = 1$, as $f_n = [z^n]\, F(z) = z_c^{-n}\, [z^n]\, F(z_c z)$.

The family of standard functions which is of greatest interest in this thesis is discussed in

Theorem B.2.1 (Theorem VI.1 in [FS09])**.** *For the functions from the family*

$$\left\{(1-z)^{-\alpha},\ \alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}\right\}$$

we have the asymptotic expression for the coefficients

$$[z^n]\,(1-z)^{-\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Remark. A stronger result is stated in [FO90, FS09], namely a complete asymptotic expansion in descending powers of n.

On the one hand this can be proven by means of the binomial expansion and Stirling's formula. However, a method which generalises to more complicated functions starts at Cauchy's coefficient formula

$$[z^n]\,(1-z)^{-\alpha} = \frac{1}{2i\pi} \int (1-z)^{-\alpha} \frac{\mathrm{d}z}{z^{n+1}},$$

where the integral is over a suitable contour encircling 0. The rough idea is to substitute $z = (1 + t/n)$ in the integral which yields

$$\begin{aligned}
&\frac{1}{2i\pi}\int (1-z)^{-\alpha}\frac{\mathrm{d}z}{z^{n+1}} = \frac{n^{\alpha}}{2i\pi}\int \frac{(-t)^{-\alpha}}{\left(1+\frac{t}{n}\right)^{n+1}}\frac{\mathrm{d}t}{n}\\
&\sim \frac{n^{\alpha-1}}{2i\pi}\int (-t)^{-\alpha}e^{-t}\mathrm{d}t\left(1+O\left(\frac{1}{n}\right)\right) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right) \quad (n\to\infty).
\end{aligned}$$

The last "=" is achieved by deforming the path of integration into a *Hankel contour*, which results in the Hankel contour representation of $2i\pi/\Gamma(\alpha)$, see [FS09, Theorem B.1] or [Olv74]. This formal computation can be justified, see [FO90, FS09].

For a similar treatment of general functions $F(z)$ which behave like $(1-z)^{-\alpha}$ as $z \to 1$, we have to make some regularity assumptions. For real numbers $0 < u_c < \rho$ and $0 < \phi < \pi/2$ we call the *open indented disc*

$$\Delta(u_c, \rho, \phi) = \{z \in \mathbb{C} \,|\, |z| < \rho,\ z \neq u_c,\ |\arg(z-u_c)| < \phi\}$$

a $\Delta(u_c)$-*domain*. A function F is called $\Delta(u_c)$-*regular*, if it is analytic in some $\Delta(u_c)$-domain. If no stress is put on u_c we simply speak of Δ-regularity. Δ-regular functions are closed under addition, multiplication and if $F(z)$ is Δ-regular and $\neq 0$ in its Δ-domain, then $1/F(z)$ is also Δ-regular. For Δ-regular functions we have the following *Transfer Theorem*.

Theorem B.2.2 (Theorem VI.3 in [FS09])**.** *Let $F(z)$ be $\Delta(1)$-regular and $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.*

1. *If $F(z)$ satisfies in the intersection of a small neighbourhood of 1 with its Δ-domain the condition*
$$F(z) = O\left((1-z)^{-\alpha}\right) \quad (z \to 1),$$
then $[z^n]\, F(z) = O\left(n^{\alpha-1}\right)$.

2. *If $F(z)$ satisfies in the intersection of a small neighbourhood of 1 with its Δ-domain the condition*
$$F(z) = o\left((1-z)^{-\alpha}\right) \quad (z \to 1),$$
then $[z^n]\, F(z) = o\left(n^{\alpha-1}\right)$.

3. *If $F(z)$ satisfies in the intersection of a small neighbourhood of 1 with its Δ-domain the condition*
$$F(z) \sim (1-z)^{-\alpha} \quad (z \to 1),$$
then $[z^n]\, F(z) \sim \dfrac{n^{\alpha-1}}{\Gamma(\alpha)}$.

This theorem is proven starting from Cauchy's coefficient formula using a contour of integration which is very close to the boundary of the Δ-region. The third assertion follows from the other two since $F(z) \sim G(z) \Leftrightarrow F(z) = G(z) + o(G(z))$.

Remark. *i)* Results of that flavour hold for more general families of standard functions, e.g. involving logarithmic terms:

$$\left\{ (1-z)^{-\alpha} \, \frac{1}{z} \log\left(\frac{1}{1-z}\right)^{\beta}, \ \alpha \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}, \ \beta \in \mathbb{C} \right\}.$$

One has complete asymptotic expansions for the coefficients and the Transfer Theorem also holds in this situation.

ii) The Transfer Theorem also applies in a situation with a finite number of dominant singularities. The Δ-domain is replaced by a disc which is indented at each of the dominant singularities. Practically, one computes the contributions of the single singularities by means of Theorem B.2.2 and sums these up.

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