Feynman integrals, hypergeometric functions and nested sums

Dissertation

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Chapter 1

Prologue

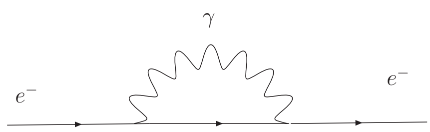
Belief that numbers could represent or describe the ”true” pattern or “reality” behind countless manifestations in nature is very old. One of the first and most prominent follower was certainly Pythagoras, who found that harmonic intervals in music are representable through simple natural numbers ratios. Plato went even further, claiming that the cosmos and everything within is just an imperfect image of an ideal mathematical world of numbers and ideas. Allthough nowdays nobody expects pure numbers or simple function to represent “real” laws of physics - instead we now talk of theories, notably the Standard Modell (SM) of particle physics where the physics is encoded in gauge invariant Lagrange densities - at the end of the day one still needs numbers, because physics is experiment oriented science and in order to compare the theoretical predictions with the experiment one needs to know how to extract numbers out of the theory. As of now there is no known solution of equations of motions for a realistic (i.e. four dimensional) QFT and one has to resort to some kind of approximation. One very successful scheme is perturbative QFT (pQFT) parts of which will be subject of this thesis.

1.1 Heuristics

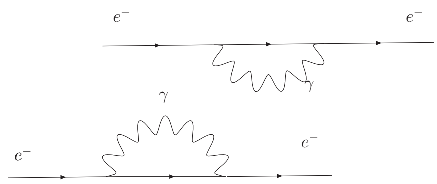
Can one describe pQFT in simple words? We’ll give it a try. One could say that QFT is a ”marriage” of quantum mechanics (QM) and special theory of relativity (SRT). We will just take the most prominent relations from both theories, from SRT the famous relation $E = mc^2$, which states that energy can be used to create particles and vice versa and from QM the Heisenberg uncertainty relation $[\vec{x}, \vec{p}] > ih/(2\pi)$, which states that one cannot simultaneously know to arbitrary precision the momentum and the spatial location of the particle. Using relativistic notion and combining E and $\vec{p}$ to four-momentum and t and $\vec{x}$ to space-time vector, we can say that in order to ”see” at short space-time distances involves high four-momenta and vice versa. The more we want to increase the resolution, the more momentum we need. This is the reason why we need bigger and bigger machines in order to detect smaller and smaller particles.

Let us now try to combine the two relation from SRT and QM and see what consequences this merge will have. For the sake of simplicity, we will take a freely propagation electron. Imagine now that we have the ability to look at the electron at very short distances. According to Heisenberg uncertainty relation, since we are ”looking” at very short distances, we have pretty certain information about the location of the electron, hence we are very uncertain about the electrons momentum, which can be very large. Here comes the second relation into the game. According to Einstein the electron with very high momentum could produce a photon and reabsorb it, all within the rules of Heisenberg uncertainty relation.

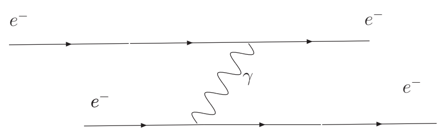
We can represent this schematically as:



We will call this photon a virtual photon, since it gets reabsorbed and cannot be directly measured. Let us now add another freely moving electron into the game. According to our previous considerations, the two electrons could look schematically something like this:



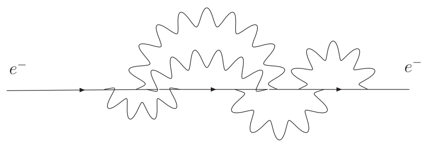
Imagine now that we bring the two electrons closer and closer together. At some point they will be so close together that the photon emmited from one electron, with certain probability, will be absorbed by the other electron, instead of being reabsorbed by the original. We could picture it like this:



In the last case however, we have momentum transfer between two electrons, in other word we have interaction.

Even in this simple picture, we have an idea of how particles could interact in QFT, namely through the exchange of a gauge particle, in the case of electrons - or speaking in terms of QFT, in the case of quantumelectrodynamics (QED) - through the exchange of a photon.

There is however one problem with the line of argumentation we have taken: Why did we include only one photon in our considerations? For all we know, going to smaller distance involving higher momenta could lead to a picture like this:



Actually it is far worse then that. We can imagine the photons in the above picture creating any particle-antiparticle pair, as long as the particles are electrically charged and these could again mediate gauge bosons themselves and so on ad infinitum. However, these diagrams contribute less to the process one is interested in and to see this we have to leave this simple picture we have obtained from only two relations and we have to bee more technical.

1.2 Amplitude and Feynman graphs

Particle experiments, like the upcoming experiments at Large Hadron Collider (LHC) at CERN, involves colliding beams of particles and measuring the cross section for the process. The simplest and most important collision is the one where two particles collide and a number of particles is created, of which some or all can be measured.

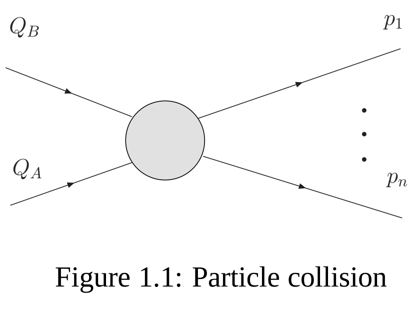


Figure 1.1: Particle collision

The cross section is given by the Golden Rule as

$$\sigma = \frac{1}{F} \int d\Phi_n |\mathcal{M}|^2, \tag{1.1}$$

where the Flux is given by

$$F = 4\sqrt{(Q_A Q_B)^2 - m_A^2 m_B^2}, \tag{1.2}$$

the phase space is

$$d\Phi_n = \Big\{ \prod_{i=1}^{n} \frac{d\vec{p}_i}{(2\pi)^3 2E_{\vec{p}_i}} \Big\} (2\pi)^4 \delta^{(4)} \Big(\sum_{j=1}^{n} p_j - Q_A - Q_B \Big), \tag{1.3}$$

and $\mathcal{M}$ is the amplitude. The amplitude contains the "physics", or the dynamics, whereas the phase space integrals contain the kinematics. Usually the phase space integrals are performed numerically using Monte Carlo methods.

How do we compute $\mathcal{M}$?

There is a graphical technique, which is by now folklore, to compute $\mathcal{M}$ and it consists of drawing all Feynman diagrams for a given process and translate the diagrams into mathematical expressions according to so called Feynman rules. These state that with every vertex, there comes a power of the coupling. For perturbative theory to work, the assumption is that the coupling is small, therefore every diagram with one more vertex contributes less. Additionally, every loop is equivalent to an integration over the inner momentum, which makes the diagrams hard to evaluate. So if every vertex brings a power of the coupling, which per definition should be small and every loop brings in an integration, then why do we care to evaluate multiloop diagrams, which contribute less and less and are hard to calculate?

Bigger and better experiments lead to the need of more accurate predictions from the theoretical side. This in turn means for perturbative calculations, that one has to evaluate multiloop diagrams if one wants to keep up with experiments. "This is the 'raison d'être' for loop calculations: A higher accuracy is reached by including more terms in the perturbative calculation" [80].

So far we have only been speaking of perturbative QFT in general terms. Now it is time to become more concrete and look at the physics which has been the initial phenomenological motivation for this thesis: finite temperature QCD.

1.3 Finite temperature QCD

Recently, large effort has been put in determination of the pressure of QCD. The motivation comes primarely from heavy-ion collisions at RHIC and the upcoming

LHC. Also, the pressure is of importance in cosmology for dark metter relic density computations.

Due to the fundamental property of asymptotic freedom [31, 58], we expect the coupling of QCD to approach zero, as we go to higher energies. This can be easily seen by taking the running of the coupling obtained from the leading order solution to the renormalization group equation:

$$g^2(\Lambda) = \frac{24\pi^2}{(11N - 2n_f)\ln(\Lambda/\Lambda_{QCD})} \tag{1.4}$$

where Λ is renormalization scale and $\Lambda_{QCD} \sim 150$ MeV the characteristic energy scale of the theory.

It is to expect that the behavior of QCD at high energies or small distances will be that of a free theory, hence justifying the use of perturbative methods. In terms of thermodynamical properties, this would mean that one can expect perturbative methods to produce reliable results in the limit of high temperatures.

However, computing the pressure perturbatively is not an easy task itself. The structure of the weak coupling expansion is not analytical in g^2. At high temperature and small coupling g, QCD develops a momentum scale hierarchy $2\pi T \gg gT \gg g^2T$. The first scale is the typical energy scale of a particle in a medium with temperature T. The other two scales are associated with the screening of color-electric and color-magnetic forces respectively. In order to account for this, effective field theory approach might be useful. It consists of separating different scales into effective theories, which reproduce static observables at successively longer distance scales, idea which is based on "dimensional reduction" [32, 2]. Let us first define the Lagrangian for QCD.

The Euclidean Lagrangian of QCD is given by:

$$\mathcal{L}_{QCD} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}\gamma_\mu D_\mu \psi \tag{1.5}$$

where $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ is the field strength tensor and $D_\mu \equiv \partial_\mu - igA_\mu \equiv \partial_\mu - igA^a_\mu T^a$ the covariant derivative, the $T^a, a = 1, \ldots, N^2 - 1$ are generators of the fundamental representation of SU(N) and f^{abc} are the structure coefficients of SU(N) given by $[T^a, T^b] = if^{abc}T^c$.
The partition function for QCD is:

$$\begin{aligned}
\mathcal{Z}_{QCD} &= \int_{periodic} \mathcal{D}A^a_\mu \int_{periodic} \mathcal{D}\bar{\eta}\mathcal{D}\eta \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi \\
& e^{\left\{-\int_0^\beta d\tau \int d^3x [\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}[\gamma_\mu D_\mu + m]\psi + \frac{1}{2\xi}(\partial_\mu A^a_\mu \partial_\nu A^a_\nu) + \bar{\eta}^a(\partial^2\delta^{ab} + gf^{abc}A^c_\mu \partial_\mu)\eta^b]\right\}} \\
&= \int_{periodic} \mathcal{D}A^a_\mu \int_{periodic} \mathcal{D}\bar{\eta}\mathcal{D}\eta \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi \quad e^{S_0 + S_{INT}}
\end{aligned}$$

(1.6)

where $\bar{\eta}$, η are the Faddeev-Popov ghosts, which have the same boundary conditions as the gauge fields.
Having the partition function one can derive the pressure using standard thermodynamic derivation. It is given as:

$$p_{QCD}(T) = \lim_{V\to\infty} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar{\psi}] \exp[-\int_0^{1/T} d\tau \int d^{3-2\epsilon}x \mathcal{L}_{QCD}]. \tag{1.7}$$

As already stated QCD is, even at high temperatures and small coupling g, a multiscale system. The reason is that of the gauge fields

$$\begin{aligned} A^a_\mu(x) &= T\sum_{-\infty}^{\infty} \exp[i\omega^b_n\tau] A^a_{\mu,n}(x) \\ \text{where} \quad & \omega^b_n = 2n\pi T \quad \text{are the Matsubara frequencies ,} \end{aligned} \tag{1.8}$$

the non-static modes gain effective masses that grow linearly with increasing temperature and then decouple, leaving the zero-modes of the gauge fields as true degrees of freedom contributing, since the fermionic fields even for $n = 0$ get effective masses. These zero modes can be described by an electrostatic scalar field $A^a_0(x)$ and magnetostatic gauge field $A^a_i(x)$ of a three dimensional effective theory, called electrostatic QCD (EQCD), with the Lagrangian:

$$\begin{aligned} \mathcal{L}_{EQCD} &= \frac{1}{2}TrF^2_{ij} + Tr[D_i, A_0]^2 + m^2_E TrA^2_0 + \frac{ig^3}{3\pi^2}\sum_f \mu_f TrA^3_0 + \\ &\quad +\lambda^{(1)}_E (TrA^2_0)^2 + \lambda^{(2)}_E TrA^4_0 + \text{higher order operators} \end{aligned} \tag{1.9}$$

with

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g_E f^{abc} A^b_i A^c_j \tag{1.10}$$

$$D_i = \partial_i - ig_E A_i \tag{1.11}$$

The pressure can then be expressed as:

$$p_{QCD}(T) = p_E(T) + \frac{T}{V}\ln \int \mathcal{D}A^a_i \mathcal{D}A^a_0 exp\{-S_E\} \tag{1.12}$$

where $p_E = p_{EQCD}$ is a parameter of the effective theory computable in perturbative full QCD [16]. With this theory one is able to compute the pressure of the full theory to the order g^5 [16]. This procedure of separating different scales in different effective theories can be applied further since there are still two dynamical scales gT and g^2T [14]. The non-perturbative scale g^2T which enters in the computation at order g^6, originates from the magnetostatic sector, that is from the fields A_i^a, so that we can write:

$$p_{QCD} = p_{EQCD} + p_{MQCD} + \frac{T}{V} \ln \int \mathcal{D}A_i^a exp\{-S_{MQCD}\} \tag{1.13}$$

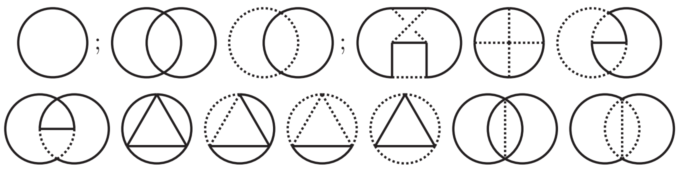
where

$$\begin{aligned} \mathcal{L}_{MQCD} &= \frac{1}{2} TrF_{ij}^a \\ F_{ij}^a &= \partial_i A_j^a - \partial_j A_i^a + g_M f^{abc} A_i^b A_j^c \end{aligned} \tag{1.14}$$

$g_M = g_{MQCD}$ is, analogous to p_E, computable through perturbative expansion of EQCD. The non-perturbative contribution has been determined numerically in [34, 35, 25].

1.4 EQCD

In this thesis we will be concerned only with EQCD, which is defined by the Lagrangian in eq. (1.9), which can be most easily obtained by first writing down the most general Lagrangian invariant under all the symmetries and then determining the parameters of the Lagrangian through matching computations in full QCD. The higher order, possibly non-renormalizable, operators would only contribute at g^7 order or higher [39]. Given the Lagrangian in eq. (1.9) one can write down all diagrams, carry out tensorial contractions and use integration by parts (IBP) identities to obtain up to four loops [61] the following set of master integrals.



(1.15)

In the picture above, the propagators have the form $\frac{1}{p^2+m^2}$, where the mass values are $m = m_E$ and $m = 0$ for A_0 and A_i fields respectively.

Allthough we need this set of master integrals in 3 dimensions for hot QCD it is also useful to compute the integrals in 4 dimensions, since some of them appear in different sets of master integrals, which contribute to different physical settings, for example in the calculation of the four loop QCD corrections to the electroweak ρ-parameter [63, 19, 13]. Therefore it would be useful to obtain a D-dimensional representation of the master integrals and have a method to expand in ϵ automatically. We will therefore try to find so called hypergeometric representations (see section 3.3) of master integrals and we will see that we can express some of them in terms of hypergeometric functions with half-integer coefficients, in 3 as well as in 4 dimensions. It is to expect that this feature is general for one-scale Feynman integrals, that is, integrals with one or more masses (for further example in other physical contexts see e.g. [23, 24]). It is therefore to expect that in various contexts hypergeometric functions with half-integer coefficients will arise and it would be of interest to have a general way of expanding these functions in ϵ. This is our main motivation for the implementation of a FORM package Hypsummer (see chapter 4) for the expansion of such functions to arbitrary order in terms of nested sums (see chapter 3).

At the end one can ask the question whether or not the methods used can be applied to full QCD at finite temperature as well. We speculate on this in chapter 6.

Chapter 2

Setting the stage

In this chapter we look at scalar Feynman integrals and introduce methods of rewriting them in terms of other integrals, especially in terms of the so called Mellin-Barnes type integrals. We also introduce the concept of master integrals, which will be important in this thesis. But first, let us look at some difficulties, which arise when dealing with Feynman integrals. The structure of the following sections follows roughly [80] and [68].

The main object of this thesis will be scalar integrals of the form:

$$\int \cdots \int \frac{d^4k_1 \ldots d^4k_l}{E_1^{\nu_1} \ldots E_N^{\nu_N}} \tag{2.1}$$

where k_i are loop momenta, ν_i are integer indices and the denominators are given by

$$E_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i p_j - m_r^2 \tag{2.2}$$

where the momenta p_i are either the loop momenta k_i or independent external momenta of the graph.
In this thesis we will only consider the class of integrals where the denominators determined by some matrix A contain momenta which are quadratic. The cases where denominators are linear with respect to loop and/or external momenta, will not be treated here although some of the methods used in this thesis are also applicable there, see [68].

Before going on to computation of these integrals, first let us see what are the difficulties in computing these integrals in the first place.

2.1 Regularization

Some of the loop integrals may be divergent. We call these integrals ill-defined quantities. A simple example is the two-point one-loop integral with zero external momentum:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} = \frac{1}{(4\pi)^2}\int_0^\infty dk^2\frac{1}{k^2} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{dx}{x}. \tag{2.3}$$

In fact, this integral is divergent as we take $k^2 \to 0$, as well as for $k^2 \to \infty$. We call the former infrared (IR) divergencies and the latter ultraviolet (UV) divergencies. These ill-defined integrals need to be regulated. The simplest method is by introducing a cut-off on the loop integral, rendering it finite. We will however use different method, which by now has become almost standard; it is called dimensional regularization (DR). The basic idea of DR is to extend the four dimensional momentum integration to a D-dimensional one, keeping D as an additional parameter, which can be rational or even complex. The result of the integration will then depend on D. Usually, one writes $D = 4 - 2\epsilon$ - allthough other dimensions are also of interest, e.g. $D = 3 - 2\epsilon$ in thermal field theory - and performs Laurent expansion in ϵ. In DR divergencies will manifest themselves as poles in $1/\epsilon$. In general, one finds that in l-loop integral UV divergencies can lead to poles $1/\epsilon^l$ and IR divergencies to poles $1/\epsilon^{2l}$ at worst. Renormalization absorbs UV divergencies and IR safe observables cancel in the final result, when summed over all degenerate states [43, 49].

2.2 Feynman parameters

We will now show how one can perform momentum loop integration at a cost of introducing integration over some additional parameters. The parameterization we choose, called Feynman parameterization, is defined by:

$$\prod_{i=1}^{n}\frac{1}{(-P_i)^{\nu_i}} = \frac{\Gamma(\nu)}{\prod_{i=1}^{n}\Gamma(\nu_i)}\int_0^1\left(\prod_{i=1}^{n}dx_i x_i^{\nu_i-1}\right)\frac{\delta(1-\sum_{i=1}^{n}x_i)}{(-\sum_{i=1}^{n}x_iP_i)^{\nu}} \tag{2.4}$$

with $\nu = \sum_{i=1}^{n}\nu_i$. Another widely used parameterization is so called Schwinger parameterization:

$$\frac{1}{(-P)^{\nu}} = \frac{1}{\Gamma(\nu)}\int_0^\infty dx x^{\nu-1}\exp(xP). \tag{2.5}$$

Any choice of Feynman parameters can be obtained using Schwinger parameterization and making appropriate changes in variables [68].

The standard procedure for computing loop integrals using Feynman parameters is to rewrite all propagators using eq. (2.4), then shift integration variables to complete the square in the momenta, perform Wick rotation going to Euclidian space where one can perform the integral over angles in terms of gamma functions, obtaining at the end result in terms of integrals over Feynman parameters. In general one can state, that a scalar l-loop integral, corresponding to a graph G, with n propagators and in D dimensions

$$I_G = \int \prod_{i=1}^{l} \frac{d^D k_i}{i\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}} \tag{2.6}$$

can be written in terms of Feynman parameters as [38]:

$$I_G = \frac{\Gamma(\nu - \frac{lD}{2})}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_0^1 \Big(\prod_{i=1}^{n} dx_i x_i^{\nu_i - 1} \Big) \delta\Big(1 - \sum_{i=1}^{n} x_i\Big) \frac{\mathcal{U}^{\nu - \frac{lD}{2} - \frac{D}{2}}}{\mathcal{F}^{\nu - \frac{lD}{2}}}. \tag{2.7}$$

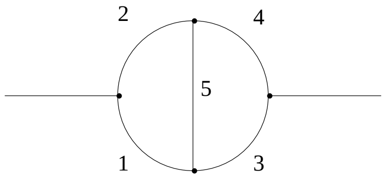
Polynomial functions $\mathcal{U}$ and $\mathcal{F}$ can be obtained from the topology of every given graph. Cutting l lines of a given connected l-loop graph, such that one gets connected 1-tree graph T gives a set of lines not belonging to this tree, called the chord $\mathcal{C}(T, G)$. The Feynman parameters x_i associated with each chord define a monomial of degree l. The set of all such trees T is called $\mathcal{T}_1$, a set of 1-trees. Elements of $\mathcal{T}_1$ define $\mathcal{U}$ as the sum over all monomials corresponding to the chord $\mathcal{C}(T, G)$. Cutting one more line on T$\in \mathcal{T}_1$ gives us two disconnected trees $(T_1, T_2) \in \mathcal{T}_2$, or a 2-tree. $\mathcal{T}_2$ is the set of all such pairs and the corresponding chord gives monomials of depth $l + 1$.
The square of the sum of momenta through the cut lines of one of the two disconnected trees T_1 or T_2 defines Lorentz invariant :

$$s_T = \Big(\sum_{j \in \mathcal{C}(T,G)} p_j^2 \Big)^2. \tag{2.8}$$

Defining $\mathcal{F}_0$ as sum over all monomials from $\mathcal{T}_2$ times minus the corresponding invariant s_T, one can define $\mathcal{F}$ as $\mathcal{F}_0$ plus additional piece involving internal masses m_j. In summary:

$$\begin{aligned}
\mathcal{U} &= \sum_{T \in \mathcal{T}_1} \Big(\prod_{j \in \mathcal{C}(T,G)} x_j \Big) \\
\mathcal{F}_0 &= \sum_{(T_1,T_2) \in \mathcal{T}_2} \Big(\prod_{j \in \mathcal{C}(T_1,G)} x_j \Big) (-s_T) \\
\mathcal{F} &= \mathcal{F}_0 + \mathcal{U} \sum_{j=1}^{n} x_j m_j^2.
\end{aligned} \tag{2.9}$$

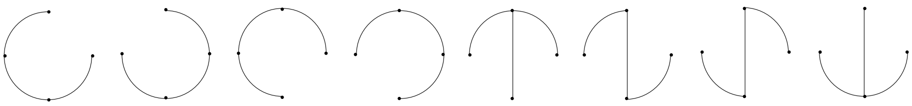
Let us demonstrate this on a scalar two-loop integral in the figure:



which reads

$$\int \frac{d^D k_1 d^D k_2}{(k_1^2)^{\lambda_1}((p-k_1)^2)^{\lambda_2}(k_2^2)^{\lambda_3}((p-k_2)^2)^{\lambda_4}((k_1-k_2)^2)^{\lambda_5}}. \tag{2.10}$$

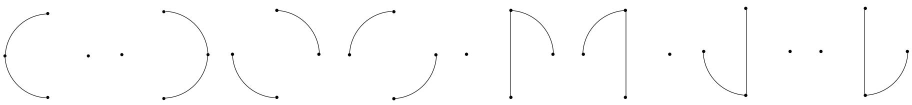
The set of all 1-trees is given in figure below:



and from that one can write

$$\mathcal{U} = (x_1 + x_2 + x_3 + x_4)x_5 + (x_1 + x_2)(x_3 + x_4). \tag{2.11}$$

Cutting one more line, one gets 2-trees:



which gives us the the $\mathcal{F}$ polynomials

$$\mathcal{F} = [(x_1 + x_2)x_3x_4 + (x_3 + x_4)x_1x_2 + (x_1 + x_3)(x_2 + x_4)x_5]p^2. \tag{2.12}$$

2.3 Tensor Integrals

So far we only considered scalar integrals. There are however also tensor integrals occurring on perturbative QFT, that is, integrals which have momenta in the numerator. These integrals can in general be reduced to scalar integrals. To see this let us assume we have written the tensor integral as integral over Feynman parameters and now we have loop momenta k in numerator as well. If we have an odd power of k, then the integral vanishes by symmetry. If the power is even, then the integral can be related to a scalar integral by Lorentz invariance [57]:

$$\begin{aligned}
\int \frac{d^D k}{i\pi^{D/2}} k^\mu k^\nu f(k^2) &= -\frac{1}{D} g^{\mu\nu} \int \frac{d^D k}{i\pi^{D/2}} (-k^2) f(k^2) \\
\int \frac{d^D k}{i\pi^{D/2}} k^\mu k^\nu k^\sigma k^\rho f(k^2) &= \frac{1}{D(D+2)} (g^{\mu\nu\rho\sigma} + g^{\mu\rho\nu\sigma} + g^{\mu\sigma\nu\rho}) \\
&\quad \int \frac{d^D k}{i\pi^{D/2}} (-k^2)^2 f(k^2)
\end{aligned} \tag{2.13}$$

Generalization to higher tensor structures can be achieved introducing shifting operators. Apart from a factor, the term $(-k^2)$ in numerator is equivalent to shifting dimension to $D \to D+2$. We can introduce an operator D^+ which does this shift and with this operator one can write:

$$\int \frac{d^D k}{i\pi^{D/2}} k^\mu k^\nu f(k^2) = -\frac{1}{2} g^{\mu\nu} D^+ \int \frac{d^D k}{i\pi^{D/2}} f(k^2) \tag{2.14}$$

In addition, shifting loop momenta $k' = k - xp$ introduces for tensor integrals Feynman parameters x_j in the numerator, which is equivalent to raising the power of the original propagator by one unit. Here we can also introduce an operator, which raises the power of the propagator. Using these one can write integrals with Feynman or Schwinger parameters in the numerator as a scalar integral, with the corresponding propagator raised to a higher power.

In summary: one can express all tensor integrals in terms of scalar integrals, which in turn may have higher powers of propagators and/or have shifted dimensions [70, 71].

2.4 Mellin-Barnes representations

Let us look at our general Feynman parameters representation of a scalar integral in eq. (2.7). In general, the integral depends on $\mathcal{U}$ and $\mathcal{F}$, which are homogenous functions of Feynman parameters. In the case that $\mathcal{U}$ and $\mathcal{F}$ are absent however, the parameter integrals can be performed easily using:

$$\int_0^1 \Big(\prod_{i=1}^n dx_i x_i^{\nu_i - 1} \Big) \delta\Big(1 - \sum_{i=1}^n x_i\Big) = \frac{\prod_{j=1}^n \Gamma(\nu_j)}{\Gamma(\nu_1 + \ldots + \nu_j)}. \tag{2.15}$$

We are going to try to reduce the general expression eq. (2.7) to the previous formula eq. (2.15).

To do this, Mellin-Barnes (MB) transformations comes in handy. It is defined as

$$\frac{1}{(A_1+A_2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{A_1^z}{A_2^{\lambda+z}}, \tag{2.16}$$

where the contour is chosen such that the poles of $\Gamma(-z)$ are to the right and the poles of $\Gamma(\lambda+z)$ are to the left. The MB transformation can be recursively applied to denominators with more then two terms, yielding:

$$\begin{aligned}&\frac{1}{(A_1+A_2+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1 \ldots \int_{-i\infty}^{i\infty} dz_{n-1}\\ &\times\Gamma(-z_1)\ldots\Gamma(-z_{n-1})\Gamma(\lambda+z_1+\ldots+z_{n-1})\\ &A_1^{z_1}\ldots A_{n-1}^{z_{n-1}}A_n^{-\lambda-z_1-\ldots-z_{n-1}}.\end{aligned} \tag{2.17}$$

We can use this representation to convert all the sums of monomials of $\mathcal{U}$ and $\mathcal{F}$ into a product, such that all x_j are of the form of LHS of eq. (2.15). Then we can integrate over x_i and obtain as a result gamma functions. In other words, we exchange the parameter integrals for multiple complex contour integrals. The contour integrals can in return be performed by closing the contour at infinity and summing up all the residues which lie inside. Since the integrand contains gamma functions, one has to use following residue formulas:

$$\begin{aligned} res(\Gamma(z+a), z=-a-n) &= \frac{(-1)^n}{n!}\\ res(\Gamma(-z+a), z=a+n) &= -\frac{(-1)^n}{n!}\end{aligned} \tag{2.18}$$

There are two strategies for obtaining ϵ-expansion using MB techniques. In the first, called strategy A [67, 9], one finds out the gamma functions in the integrand which contribute poles, shifts the contour and then take the residues. Let us demonstrate this strategy in a simple example:

$$I(a,b;m) \equiv \int \frac{d^Dk}{(k^2)^a((p-k)^2-m^2)^b}. \tag{2.19}$$

Using equation (2.16) we get:

$$\frac{1}{\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz(-m^2)^z\Gamma(-z)\Gamma(b+z)\int\frac{d^Dk}{(k^2)^a((p-k)^2)^{b+z}}. \tag{2.20}$$

Now, the last integral is a massless one-loop integral which is known analytically in terms of Gamma functions to be [68]:

$$\int\frac{d^Dk}{(k^2)^a((p-k)^2)^b} = \pi^{\frac{D}{2}}(p^2)^{\frac{D}{2}-a-b}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)\Gamma(a+b-\frac{D}{2})}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)}\ , \tag{2.21}$$

so using this result and linearly shifting the variable of integration $z = \frac{d}{2} - a - b - z$ (which does not change the separation of the contour, it only turns "left" into "right" and vice versa) we get:

$$I(a,b;m) = \frac{\Gamma(\frac{D}{2}-a)}{\Gamma(a)2\pi i} \int\limits_{-i\infty}^{i\infty} dz \frac{(-\frac{p^2}{m^2})^z \Gamma(-z)\Gamma(a+z)\Gamma(a+b-\frac{D}{2}+z)}{\Gamma(\frac{D}{2}+z)}. \tag{2.22}$$

Closing the contour on the right, we obtain:

$$\begin{aligned} I(a,b;m) &= \frac{\pi^{\frac{D}{2}}(-m^2)^{\frac{D}{2}-a-b}\Gamma(\frac{D}{2}-a)}{\Gamma(a)\Gamma(b)} \times \\ &\times \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{p^2}{m^2}\right)^j \frac{\Gamma(a+j)\Gamma(a+b-\frac{D}{2}+j)}{\Gamma(\frac{D}{2}+j)}, \end{aligned} \tag{2.23}$$

where we used the formula for the residues of Gamma functions eq. (2.18).

Please note the fact, which is of importance to us: In $D = 4-2\epsilon$ in the eq. (2.23), gamma functions have integer valued coefficients, given that a and b are integers. In $D = 3-2\epsilon$ however, which is the case in EQCD, we get half-integer coefficients. This was the initial reason that motivated us to write the package Hypsummer described in chapter 4. But let us now come back to the other strategy for evaluating MB integrals.

Strategy B [72] involves choosing an initial value for ϵ and a value of the real part of the integration variable z_i in such a way that the real parts of all the arguments in gamma functions in the numerator are positive. Then one can integrate over straight lines. Finally one lets $\epsilon \to 0$ and whenever the real part of the argument of some gamma function vanishes one crosses the pole and adds the corresponding residue, which has one integration less. On the remaining integral, one applies the same procedure. In fact, strategy B, being algorithmic, has already been implemented and published in the public Mathematica package MB.m by [20] and also implemented and used in [1] as well. Since we will not use this this strategy to expand integrals, we refer the reader to [72] for examples.
Recently, strategy A has also been implemented in Mathematica, the package is called MBresolve.m [69] and it needs the MB.m package. Once the singularities are resolved, all packages can perform numerical integration.

2.5 IBP and Master integrals

DR integrals have properties that one would expect from integrals, like linearity

$$\int d^D k (a_1 f_1(k) + a_2 f_2(k)) = a_1 \int d^D k f_1(k) + a_2 \int d^D k f_2(k), \qquad (2.24)$$

where a_i and b_i are constants, translation invariance

$$\int d^D k f(k+p) = \int d^D k f(k), \qquad (2.25)$$

where p is any vector and scaling law

$$\int d^D k f(\lambda k) = \lambda^{-D} \int d^D k f(k), \qquad (2.26)$$

where λ is a constant.
There is also a less trivial property which states that a derivative of an integral in DR with respect to mass or momentum equals the corresponding integral of the derivative. A corollary to this property leads to the possibility to integrate by parts and neglect the surface term:

$$\int d^D k \frac{\partial}{\partial k^\mu} v^\mu f(k, p_i) = 0 \ , \qquad (2.27)$$

where k is inner momentum, p_i are the external momenta and v can be either internal or external momentum. One write these integration by parts (IBP) identities and apply this set of equation to solve the so called reduction problem, i.e. to find out how a general Feynman integral of a given class can be expressed as a linear combination of some master integrals.
Let us look at a simple example [18]:

$$F(\lambda) = \int d^D k \frac{1}{(-k^2 + m^2)^\lambda} \qquad (2.28)$$

Writing down IBP identity

$$\int d^D k \frac{\partial}{\partial k^\mu} k^\mu \frac{1}{(-k^2 + m^2)^\lambda} = 0, \qquad (2.29)$$

gives us the following recurrence relation:

$$\begin{aligned}
& \delta^\mu_\mu F(\lambda) + 2\lambda k^2 F(\lambda+1) = 0 \\
\rightarrow \quad & DF(\lambda) - 2\lambda(-k^2 + m^2 - m^2) F(\lambda+1) = 0 \\
\rightarrow \quad & DF(\lambda) - 2\lambda F(\lambda) + 2\lambda m^2 F(\lambda+1) = 0 \\
\rightarrow \quad & F(\lambda+1) = \frac{2\lambda - D}{2\lambda m^2} F(\lambda). \qquad (2.30)
\end{aligned}$$

Since for $\lambda < 1$ $F(\lambda)$ is zero and for $\lambda > 1$ all $F(\lambda)$ can be expressed in terms of $F(1)$, we call $F(1)$ a master integral.

Let us summarize.

We have seen in this chapter that we only need to consider scalar integrals which may reduce to a smaller set of so called master integrals and that the momentum integrals can be traded for parameter integrals, which in turn can be written as complex contour integrals, which are of the form

$$\frac{1}{(2\pi i)^n} \int \prod_{l=1}^{n} dz_l \frac{\prod_i \Gamma(a_i + b_i\epsilon + \sum_j c_{ij} z_j)}{\prod_i \Gamma(a'_i + b'_i\epsilon + \sum_j c'_{ij} z_j)} \prod_k x_k^{d_k}, \tag{2.31}$$

where a_i and a'_i are integer, b_i, b'_i, c_{ij} and c'_{ij} are integers, x_k are ratios of kinematic invariants and/or masses and the exponents d_k are linear combination of ϵ and z-variables. Summing up all residues gives us the result of Feynman integrals in terms of multiple sums involving gamma functions. The expansion parameter ϵ will appear in the argument of some of these gamma functions. In order to get the Laurent expansion in ϵ, we need to know how to expand multiple sums with gamma functions around ϵ.

In the next chapter we will introduce objects, which will enable us to expand certain classes of multiple sums.

Chapter 3

Enter the actors

3.1 Introduction

In calculations of higher order radiative corrections, one encounters logarithms, classical polylogarithms and generalized polylogarithms [56]. At higher number of loops this set of functions may not suffice. As a consequence, people started to extend and generalize this class of functions to multiple polylogarithms [10, 30, 59, 28]. On the other hand, harmonic [7, 74] and Euler-Zagier [26, 81] sums have been used in calculation of higher order Mellin moments of deep inelastic structure functions [42, 76, 52]. Finally, in [53] generalization of harmonic and Euler-Zagier sums, called S- or Z-sums, were introduced, which at the same time encompassed all the multiple polylogarithms as certain special cases. The purely mathematical question, which numbers can appear as coefficients of Laurent expansion of Feynman integrals, has been addressed in [8].The answer is that integrals in Euclidian region, with all ratios of invariants and masses being rational have periods as coefficients of Laurent series. Periods can be defined as [8] complex numbers whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

We will introduce the Z-/S- sums, show some of their properties, which will be usefull in later chapters and summarize special cases and relate them to known functions. In the last section we introduce hypergeometric function and show the link to nested sums.

3.2 Nested sums

The Z-sums are defined recursively by[1]

$$Z(n) = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$$

$$Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i_i^m} Z(i-1; m_2, \ldots, m_k; x_2, \ldots, x_k) \quad (3.1)$$

where k is called the depth and $w = m_1 + m_2 + ... + m_k$ the weight of the Z-sum. Equivalent definition can be given by

$$Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 > i_2 > \ldots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} \quad (3.2)$$

Analogous definition can be given for the S-sums

$$S(n) = \begin{cases} 1 : n > 0 \\ 0 : n \leq 0 \end{cases}$$

$$S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_i}} S(i; m_2, \ldots, m_k; x_2, \ldots, x_k) \quad (3.3)$$

or

$$S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 \geq i_2 \geq \ldots \geq i_k > 1} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}} \quad (3.4)$$

Notice that the difference between the S- and Z-sums is the upper summation boundary, (i-1) for Z- and (i) for S-sums. With the help of the following formula, one can easily convert Z-sums into S-sums and vice versa

$$\begin{aligned} S(n; m_1, \ldots; x_1, \ldots) &= \sum_{i=1}^{n} \frac{x_1^{i_1}}{i^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_1^{i_1}}{i_1^{m_1}} S(i_2; m_3, \ldots; x_3, \ldots) \\ &+ S(n; m_1 + m_2, \ldots; x_1 x_2, x_3, \ldots) \end{aligned}$$

$$\begin{aligned} Z(n; m_1, \ldots; x_1, \ldots) &= \sum_{i=1}^{n} \frac{x_1^{i_1}}{i^{m_1}} \sum_{i_2=1}^{i_1} x_1^{i_1} i_1^{m_1} Z(i_2 - 1; m_3, \ldots; x_3, \ldots) \\ &- Z(n; m_1 + m_2, \ldots; x_1 x_2, x_3, \ldots) \end{aligned} \quad (3.5)$$

[1]This section follows closely the second chapter of [53]

Z-sums form an algebra, which means that the product of two Z-sums with the same upper summation, that is the same argument, can be written in terms of single Z-sums

$$
\begin{aligned}
& Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k) \times Z(n;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \\
= & \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} Z(i_1-1;m_2,\ldots,m_k;x_2,\ldots,x_k) \times \\
& \qquad \times Z(i_1-1,m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \\
+ & \sum_{i_2=1}^{n} \frac{x_1'^{i_2}}{i_2^{m'_1}} Z(i_2-1;m_1,\ldots,m_k;x_1,\ldots,x_k) \times \\
& \qquad \times Z(i_2-1,m'_2,\ldots,m'_l;x'_2,\ldots,x'_l) \\
+ & \sum_{i=1}^{n} \frac{(x_1x'_1)^i}{i^{m_1+m'_1}} Z(i-1;m_2,\ldots,m_k;x_2,\ldots,x_k) \times \\
& \qquad \times Z(i-1,m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)
\end{aligned} \tag{3.6}
$$

As one can see, one or both Z-sums on the RHS have reduced depth. Applying the formula recursively, since per definition it has an ending, leaves us with single Z-sums. For example:

$$
\begin{aligned}
& Z(n;m_1,m_2;x_1,x_2) \times Z(n;m_3;x_3) = \\
& Z(n;m_1,m_2,m_3;x_1,x_2,x_3) + Z(n;m_1,m_3,m_2;x_1,x_3,x_2) \\
& +Z(n;m_3,m_1,m_2;x_3,x_1,x_2) + Z(n;m_1,m_2+m_3;x_1,x_2x_3) \\
& +Z(n;m_1+m_3,m_2;x_1x_3,x_2)
\end{aligned} \tag{3.7}
$$

Similarly the product of two S-sums simplifies to sum of single S-sums:

$$
\begin{aligned}
& S(n;m_1,\ldots,m_k;x_1,\ldots,x_k) \times S(n;m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \\
= & \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} S(i_1;m_2,\ldots,m_k;x_2,\ldots,x_k) S(i_1,m'_1,\ldots,m'_l;x'_1,\ldots,x'_l) \\
+ & \sum_{i_2=1}^{n} \frac{x_1'^{i_2}}{i_2^{m'_1}} S(i_2;m_1,\ldots,m_k;x_1,\ldots,x_k) S(i_2,m'_2,\ldots,m'_l;x'_2,\ldots,x'_l) \\
- & \sum_{i=1}^{n} \frac{(x_1x'_1)^i}{i^{m_1+m'_1}} S(i;m_2,\ldots,m_k;x_2,\ldots,x_k) S(i,m'_2,\ldots,m'_l;x'_2,\ldots,x'_l)
\end{aligned} \tag{3.8}
$$

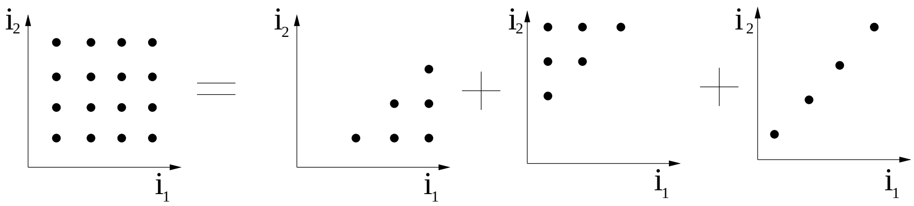


Figure 3.1: An intuitive, geometric picture for the multiplication of two Z-sums in eq. (3.6), taken from [53]

The proof for the equation (3.6) uses the triangle relation (see Fig. 3.1):

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{i-1} a_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{j-1} a_{ij} + \sum_{i=1}^{n} a_{ii} \tag{3.9}$$

The equation (3.6) actually states that the Z-sums form a so called Hopf Algebra (see appendix of [80]).
Since, in order to apply eq. (3.6), one needs to have nested sums with the same argument, it is usefull to know how to synchronize them. This can be done with the help of the following formulae:

$$\begin{aligned}
&Z(n+c-1; m_1,\ldots;x_1,\ldots) = \\
&Z(n-1; m_1,\ldots;x_1,\ldots) + \sum_{j=1}^{c-1} \frac{x_1^j x_1^n}{(n+j)^{m_1}} Z(n-1+j; m_2,\ldots;x_2,\ldots) \\
&S(n+c; m_1,\ldots;x_1,\ldots) = \\
&S(n; m_1,\ldots;x_1,\ldots) + \sum_{j=1}^{c} \frac{x_1^j x_1^n}{(n+j)^{m_1}} S(n+j; m_2,\ldots;x_2,\ldots).
\end{aligned} \tag{3.10}$$

The Z/S-sums are a fairly general object, in a lots of cases it wont be necessary to consider these general objects, but instead some simpler ones (see Fig.(3.2)). If one for example takes the index n in Z-sums to be infinity, one ends with the so called multiple polylogarithms of Goncharov [30]:

$$Z(\infty; m_1,\ldots,m_k; x_1,\ldots,x_k) = Li_{m_1,\ldots,m_k}(x_1,\ldots,x_k). \tag{3.11}$$

If, in addition to $n=\infty$ one also sets $x_1=\cdots=x_k=1$ then one gets multiple Zeta-values [12]:

$$Z(\infty; m_1,\ldots,m_k; 1,\ldots,1) = \zeta(m_1,\ldots,m_k). \tag{3.12}$$

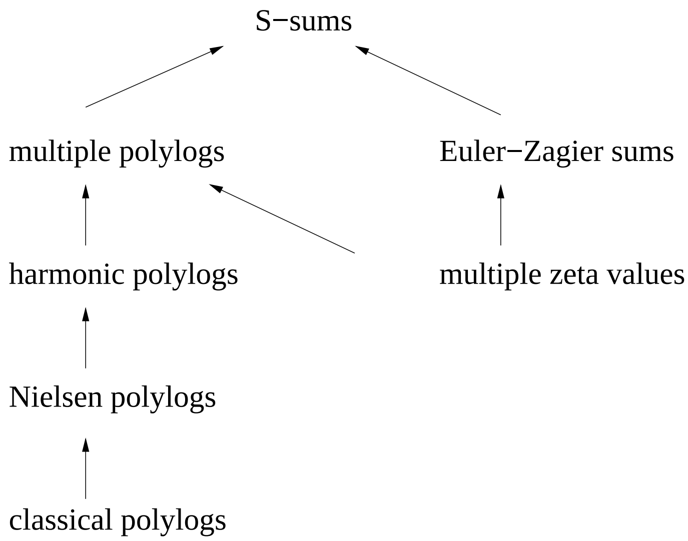


Figure 3.2: Inheritance diagram for S-sums from [53]

By taking only $x_1 = \cdots = x_k = 1$ and leaving n general, we get Euler-Zagier sums ([26] [81]):

$$Z(n; m_1, \ldots, m_k; 1, \ldots, 1) = Z_{m_1,\ldots,m_k}(n). \tag{3.13}$$

On the other hand, the S-sums for values $x_1 = \cdots = x_k = 1$ and $m_i > 0$ reduce to harmonic sums [74]:

$$S(n; m_1, \ldots, m_k; 1, \ldots, 1) = S_{m_1,\ldots,m_k}(n). \tag{3.14}$$

Multiple polylogs, in turn contain as a subset the classical polylogs $Li_n(x)$, Nielsen's generalized polylogs [56]:

$$S_{n,p}(x) = Li_{1,\ldots,1,n+1}(\underbrace{1, \ldots, 1}_{p-1}, x) \tag{3.15}$$

and harmonic polylogs introduced by Vermaseren and Remiddi [59]

$$H_{m_1,\ldots,m_k}(x) = Li_{m_k,\ldots,m_1}(\underbrace{1, \ldots, 1}_{k-1}, x). \tag{3.16}$$

In this work we will specially use multiple and harmonic polylogarithms, therefore we will take a closer look at these two subclasses of nested sums in the appendix. In the next section we will introduce hypergeometric functions, which are related to eq. (2.31) and later we will link those to nested sums.

3.3 Hypergeometric functions

3.3.1 Gauss function

The series

$$1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!}+\dots \tag{3.17}$$

is called the Gauss series or Gauss hypergeometric series or Gauss function [66]. The symbol ${}_2F_1(a,b;c;x)$ is usually reserved for it, where a,b,c are parameters of the function and x is called the argument. Introducing the following notation

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) \tag{3.18}$$

called the Pochhammer symbol, with $(a)_0 = 1$, then one can write

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3.19}$$

and the Gauss functions can be written as

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n x^n}{(c)_n n!}. \tag{3.20}$$

The series is convergent for all values of x, real or complex such that $|x| < 1$. In the case $|x| = 1$ it is convergent if Re$(c-a-b) > 0$.
The Gauss function has an integral representation, provided that $|x| < 1$ and Re$(c-b) > 0$ and Re$(b) > 0$, which is given by

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt. \tag{3.21}$$

The integral is also called Pochhammer integral.
Gauss function can also be represented as Barnes-type integral:

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-z)\Gamma(a+z)\Gamma(b+z)}{\Gamma(c+z)}(-x)^z dz. \tag{3.22}$$

provided that $|x| < 1$ and that $|\arg(-x)| < \pi$. Actually, the expression in eq. (2.22) is exactly of this form, so the result in eq. (2.23) can be also written as

$$I(a,b;m) = \frac{\pi^{\frac{D}{2}}(-m^2)^{\frac{D}{2}-a-b}\Gamma(\frac{D}{2}-a)\Gamma(a+b-\frac{D}{2})}{\Gamma(b)\Gamma(\frac{D}{2})} \times {}_2F_1\left(a, a+b-\frac{D}{2}; \frac{D}{2}; \frac{p^2}{m^2}\right). \quad (3.23)$$

There are number of relations which allow one to transform the parameters and argument of Gauss function. The most famous ones are Euler identity and Kummer identities [66]:

$$\begin{aligned} {}_2F_1(a,b;c;x) &= (1-x)^{-a}{}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right) \\ &= (1-x)^{-b}{}_2F_1\left(c-a, b; c; \frac{x}{x-1}\right) \\ &= (1-x)^{c-a-b}{}_2F_1(c-a, c-b; c; x). \end{aligned} \quad (3.24)$$

3.3.2 Generalized Gauss function

One can generalize the Gauss function, by adding equal number of further gamma functions in numerator and denominator in the series representation:

$$\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \dots (a_p)_n x^n}{(b_1)_n(b_2)_n \dots (b_{p-1})_n n!}. \quad (3.25)$$

The above series is called generalized Gauss function or generalized hypergeometric function, and for it we use the symbol

$${}_pF_{p-1}(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_{p-1}, x). \quad (3.26)$$

The above function is convergent when $|x| < 1$, for $x = 1$ if

$$Re\left(\sum_{i=1}^{p-1} b_i - \sum_{i=1}^{p} a_i\right) > 0 \quad (3.27)$$

and for $x = -1$ if

$$Re\left(\sum_{i=1}^{p-1} b_i - \sum_{i=1}^{p} a_i\right) > -1. \quad (3.28)$$

The integral representation is (we set now $q = p - 1$) [66]

$$
{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \frac{\Gamma(b_q)}{\Gamma(a_p)\Gamma(b_q-a_p)} \int\limits_0^1 t^{a_p-1}(1-t)^{b_q-a_p-1} \times {}_{p-1}F_{q-1}(a_1,\ldots,a_{p-1};b_1,\ldots,b_{q-1};xt)dt \tag{3.29}
$$

where Re(b_q) >Re$(a_p) > 0$ and $|\arg(1-x)| < \pi$.
The Barnes-type integral representation is given by:

$$
{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \frac{\prod\limits_{i=1}^{q}\Gamma(b_i)}{2\pi i\prod\limits_{i=1}^{p} a_i} \int\limits_{-i\infty}^{+i\infty} \frac{\Gamma(-z)\prod\limits_{i=1}^{p}\Gamma(a_i+z)}{\prod\limits_{i=1}^{q}\Gamma(b_i+z)}(-x)^z dz, \tag{3.30}
$$

provided that $|x| < 1$ and that $|\arg(-x)| < \pi$.

One interesting special case for $p = 3$ and $x = 1$ is Dixon's theorem [66], which states:

$$
{}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(d)\Gamma(s)\Gamma(e)}{\Gamma(a)\Gamma(d+e-a-c)\Gamma(d+e-a-b)} {}_3F_2(d-a,e-a,s;d+e-a-c,d+e-a-b;1), \tag{3.31}
$$

where $s = e + d - a - b - c$ and one must have Re$(s) > 0$ and Re$(a) > 0$ and which can be used to balance the hypergeometric function. That is: if say in eq. (3.31) $d = 1/2 + d'$ and all the other coefficients are integers, then, by applying Dixon's theorem one gets hypergeometric function with equal number of half-integer coefficients in numerator and denominator.

3.3.3 Appell functions

One can also generalize the Gauss function to two arguments, instead of just one. This leads to four possibilities:

$$
F_1(a,b_1,b_2;c;x_1,x_2) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{n+j}(b_1)_n(b_2)_j}{(c)_{n+j}} \frac{x_1^n i x_2^j}{n!j!} \tag{3.32}
$$

$$
F_2(a,b_1,b_2;c_1,c_2;x_1,x_2) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{n+j}(b_1)_n(b_2)_j}{(c_1)_n(c_2)_j} \frac{x_1^n x_2^j}{n!j!} \tag{3.33}
$$

$$F_3(a_1,a_2,b_1,b_2;c;x_1,x_2) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_1)_n(a_2)_j(b_1)_n(b_2)_j}{(c)_{n+j}}\frac{x_1^n x_2^j}{n!j!} \quad (3.34)$$

$$F_4(a,b;c_1,c_2;x_1,x_2) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{n+j}(b)_{n+j}}{(c_1)_n(c_2)_j}\frac{x_1^n x_2^j}{n!j!}. \quad (3.35)$$

(3.36)

In this thesis we will only be concerned with the first Appell function F_1 and the generalized form thereof

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_1)_{n+j}\cdots(a_p)_{n+j}}{(c_1)_{n+j}\cdots(c_p)_{n+j}}\frac{(e_1)_n\cdots(e_p)_n}{(g_1)_n\cdots(g_{p-1})_n}\frac{(i_1)_j\cdots(i_p)_j}{(l_1)_j\cdots(l_{p-1})_j}\frac{x_1^n}{n!}\frac{x_2^j}{j!} \quad (3.37)$$

which has the following contour integral representation:

$$\frac{\prod_{k=1}^{p_1-1}\Gamma(l_k)\prod_{k=1}^{p_2-1}\Gamma(g_k)\prod_{k=1}^{p_3-1}\Gamma(c_k)}{(2\pi i)^2\prod_{k=1}^{p_1}\Gamma(a_k)\prod_{k=1}^{p_2}\Gamma(e_k)\prod_{k=1}^{p_3}\Gamma(i_k)}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}dz_1dz_2(-x_1)^{z_1}(-x_2)^{z_2}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\prod_{k=1}^{p_1}\Gamma(a_k+z_1+z_2)\prod_{k=1}^{p_2}\Gamma(e_k+z_1)\prod_{k=1}^{p_3}\Gamma(i_k+z_2)}{\prod_{k=1}^{p_1-1}\Gamma(c_k+z_1+z_2)\prod_{k=1}^{p_2-1}\Gamma(g_k+z_1)\prod_{k=1}^{p_3-1}\Gamma(l_k+z_2)} \quad (3.38)$$

This representation, as well as representation eq. (3.30) of generalized hypergeometric function, is of the form eq. (2.31), therefore we expect some Feynman integrals to be expressible in terms of these hypergeometric functions. The generality of this statement is one of the main motivations for us to look closer at hypergeometric functions.
Let us now look at another general method for dealing with master integrals, which also has a link to hypergeometric functions.

3.4 Difference equations and hypergeometric functions

In a well known paper [46] a method has been introduced which is based on difference equations. One starts with an integral M and raises the power of one propagator to a number x, which one treats as a variable and one can fix other powers of propagators usually to one. Combining various IBP relations one obtains difference equation:

$$a_0(n)M(x)+a_1(x)M(x+1)+\ldots+a_r(x)M(x+r)=G(x), \quad (3.39)$$

where $a_i(x)$ are rational polynomials in x and ϵ and $G(x)$ contains Feynman integrals which have one or more propagators less then the original integral $M(x)$. For these integrals one obtains again similar difference equations and at the end one obtains a triangular system of difference equations. Starting with the simplest integral, i.e. the one with the least number of propagators, one can use various methods to solve the equations. The most widely used is by making Ansatz in form of factorial series [51, 46]

$$\mu^x \sum_{l=1}^{\infty} \frac{b_l x!}{\Gamma(x-K+l+1)}, \tag{3.40}$$

where the values of the parameters μ, b_l and K are to be determined. This method for obtaining high precision numerical values was successfully applied to various multiloop calculations, e.g. [47, 48, 62, 64]

The equation in eq. (3.39) is called r-order ordinary inhomogeneous difference equation. In case that the term $G(x)$ is zero, the equation is called homogenous. Similar to differential equations, difference equations of first order

$$M(x+1) = a(x)M(x) + G(x) \tag{3.41}$$

can be formally solved as

$$M(x) = \left[\prod_{i=x_0}^{x-1} a(i)\right] M(x_0) + \sum_{j=x_0}^{x-1} \left\{ \left[\prod_{i=j+1}^{x-1} a(i)\right] \right\} G(j), \tag{3.42}$$

where $M(x_0)$ is the initial value. In the case of a_i being fraction of polynomials with rational coefficients, the products give Pochhammer symbols. Therefore the solution is nothing else then a generalized hypergeometric function, assuming that $G(j)$ is given in terms of Pochhammer symbols and/or powers of argument j. That means that should we have a first order difference equation for a master integral, we can find automatically the hypergeometric representation and, in case the coefficients are balanced, we can expand it. Unfortunately, for difference equations of higher order, just like for differential equation, there is no formal solution. In this case one has to use more advanced and difficult methods, like Laplace transform [51, 46] or make an Ansatz for the solution in terms of functions one expects to appear [77].

The observation that (first order) difference equations, although a priori unrelated to Feynman integrals, can also be naturally expressed as hypergeometric functions strengthens the belief that hypergeometric functions are a natural representation of Feynman integrals. But let us now show the connection between hypergeometric functions and nested sums.

3.5 Relating nested sums and hypergeometric functions

As we have seen, hypergeometric functions can be represented as sums over Pochhammer symbols containing ϵ, a number and summation indices. How do these relate to nested sums introduces in first section of this chapter? Let us start with rewriting the Pochhammer symbols as products and manipulate the expression a bit:

$$\begin{aligned}
(1+\epsilon)_n &= \frac{(1)_n(1+\epsilon)_n}{(1)_n} \\
&= (1)_n \prod_{i=1}^{n} \frac{\epsilon+i}{i} \\
&= (1)_n \exp\Big[\ln\Big(\prod_{i=1}^{n} \frac{\epsilon+i}{i}\Big)\Big] \\
&= (1)_n \exp\Big[\sum_{i=1}^{n} \ln\Big(\frac{\epsilon+i}{i}\Big)\Big] \\
&= (1)_n \exp\Big[-\sum_{i=1}^{n}\sum_{k=1}^{\infty}\Big(\frac{(-1)^k}{k}\epsilon^k i^{-k}\Big)\Big] \\
&= (1)_n \exp\Big[-\sum_{k=1}^{\infty}\frac{(-\epsilon)^k}{k}\sum_{i=1}^{n}\frac{1}{i^k}\Big] \\
&= (1)_n \exp\Big[-\sum_{k=1}^{\infty}\frac{(-\epsilon)^k}{k} S(n;k;1)\Big].
\end{aligned} \tag{3.43}$$

This means that we can expand all Pochhammer symbols in hypergeometric functions in terms of nested sums. Since we have products of Pochhammer symbols, we will get products of nested sums, but using their algebra will allow us to systematically perform expansion in ϵ. The details of the implementation will be described in detail in next chapter.

Chapter 4

HypSummer

4.1 Introduction

In this chapter we will describe in detail the FORM package HypSummer, which expands in ϵ balanced higher transcendental functions of the form:

$$\sum_{n=0}^{\infty} \frac{(\frac{a_1}{2}+b_1\epsilon)_n \cdots (\frac{a_p}{2}+b_p\epsilon)_n}{(\frac{c_1}{2}+d_1\epsilon)_n \cdots (\frac{c_{p-1}}{2}+d_{p-1}\epsilon)_n} \frac{x^n}{n!}, \tag{4.1}$$

and

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a_1}{2}+b_1\epsilon)_{n+j} \cdots (\frac{a_p}{2}+b_p\epsilon)_{n+j}}{(\frac{c_1}{2}+d_1\epsilon)_{n+j} \cdots (\frac{c_p}{2}+d_p\epsilon)_{n+j}} \times \frac{(\frac{e_1}{2}+f_1\epsilon)_n \cdots (\frac{e_p}{2}+f_p\epsilon)_n}{(\frac{g_1}{2}+h_1\epsilon)_n \cdots (\frac{g_p}{2}+h_p\epsilon)_n}$$
$$\times \frac{(\frac{i_1}{2}+k_1\epsilon)_j \cdots (\frac{i_p}{2}+k_p\epsilon)_j}{(\frac{l_1}{2}+m_1\epsilon)_j \cdots (\frac{l_{p-1}}{2}+m_{p-1}\epsilon)_j} \frac{x^j}{j!}\frac{y^n}{n!} \tag{4.2}$$

where Pochhammer symbol is defined as $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$ and latin indices $a_i, b_i, \ldots,$ m_i are integer numbers. The first expression is the sum representation of the so called generalized hypergeometric function ${}_pF_{p-1}$ and the second one is the sum representation of the so called generalized first Appell function F_1. The term ”balanced” means in this context, that the number of integer and half-integer coefficients are equal for the corresponding summation index. In the case that all latin indices are even integer numbers, there already exists a FORM implementation called XSummer [54], C++ library called $nestedsums$ [78] as well as Mathematica implementation HypExp [50]. For the half-integer coefficients there has been published an upgrade of HypExp [37], which can deal with some number of classes of hypergeometric functions. There is however a general algorithm for expansion of generalized hypergeometric functions with balanced rational coefficients [79] based on algebraic manipulation of nested sums. By restricting ourselves to coefficients of the form $\frac{a}{2}$ which are the one needed in multiloop calculations[1] we considerably simplify the general algorithm A and B of [79]. In the following we describe the implementation and present some examples. Let us first look at the generalized hypergeometric functions.

4.2 Expansion of generalized hypergeometric functions

In order to expand the sum of eq. (4.1), we will first bring all Pochhammer symbols to the form $(\frac{1}{2}+\epsilon)_n$ and $(1+\epsilon)_n$ using the formula $\Gamma(x+1) = x\Gamma(x)$ [HypSummer $\rightarrow$ $GammaCracker.prc$][2]. The next step consists in expanding the Gamma functions in nested sums using [79]

$$\begin{aligned}\left(\frac{1}{2}+\epsilon\right)_n &= \left(\frac{1}{2}\right)_n \exp\left(-\sum_{k=1}^{\infty}\frac{(-2\epsilon)^k}{2k}[S_k(2n)-S_{-k}(2n)]\right)\\ \left(1+\epsilon\right)_n &= \left(1\right)_n \exp\left(-\sum_{k=1}^{\infty}\frac{(-2\epsilon)^k}{2k}[S_k(2n)+S_{-k}(2n)]\right)\end{aligned} \tag{4.3}$$

[HypSummer $\rightarrow$ $GammaExpandor.prc$] and also expanding the Gamma functions without summation index [HypSummer $\rightarrow$ $GammaepCracker.prc$]. Here one has to note that we expand integer as well as half-integer valued Gamma functions yielding nested sums with argument $2n$ in both cases, thus the package does not discriminate any more between the integers and half-integers and also purely integer valued generalized hypergeometric functions can expanded. Also one can see at this level already why the sums eq. (4.1) and eq. (4.2) have to be balanced. It is only when there are equal number of integer and half-integer valued gamma function with the same summation index, that the Pochhammer symbols in eq. (4.3) in front of the exponential function cancel. Now one can expand the exponential function to the desired order in ϵ and one gets products of nested sums, all with the same argument $2n$. Here the algebra of nested sums we mentioned in previous chapter, comes into play and reduces the products of nested sums into sums of single nested sums according to eq. (3.8) [HypSummer $\rightarrow$ $BasisS.prc$]. Also, the applied formula $\Gamma(x+1) = x\Gamma(x)$ brings possibly a great deal of polynomials in the denominator. To deal with those terms we use recursive general partial

[1] See remarks at the end of this chapter

[2] The text indicates the name of the FORM procedure in HypSummer

fractioning formula [HypSummer → $PartialCracker.prc$]:

$$\frac{1}{n+a}\frac{1}{n+b} = \delta_{a,b}\frac{1}{(n+a)^2} + (\Theta(a-b)+\Theta(b-a))\frac{1}{b-a}\left(\frac{1}{n+a}-\frac{1}{n+b}\right), \quad (4.4)$$

where $\Theta(x)$ is zero if $x \leq 0$ and one if $x > 0$. This leads us to the following cases:

$$\sum_{n=1}^{\infty}\frac{x^n}{(\frac{1}{2}+a+b\epsilon+n)^m}S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \quad (4.5)$$

$$\sum_{n=1}^{\infty}x^n n^k S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \quad (4.6)$$

$$\sum_{n=1}^{\infty}x^n S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \quad (4.7)$$

and the corresponding cases without the nested sums:

$$\sum_{n=1}^{\infty}\frac{x^n}{(\frac{1}{2}+a+b\epsilon+n)^m} \quad (4.8)$$

$$\sum_{n=1}^{\infty}x^n n^k \quad (4.9)$$

$$\sum_{n=1}^{\infty}x^n. \quad (4.10)$$

Let us consider the cases with nested sums first. We rewrite the eq. (4.5, 4.6, 4.7) as following:

$$\sum_{n=1}^{\infty}\frac{(\sqrt{x})^{2n}}{(\frac{1}{2}+a+b\epsilon+\frac{2n}{2})^m}S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \quad (4.11)$$

$$\sum_{n=1}^{\infty}(\sqrt{x})^{2n}\left(\frac{2n}{2}\right)^k S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \quad (4.12)$$

$$\sum_{n=1}^{\infty}(\sqrt{x})^{2n}S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n). \quad (4.13)$$

Now every summation index n is equipped with a 2, which means that we have a sum over a function with argument $2n$:

$$\sum_{n=1}^{\infty}f(2n) \quad (4.14)$$

and the next step will be to apply

$$\sum_{n=1}^{\infty} f(2n) = \frac{1}{2}\Big(\sum_{n=1}^{\infty} f(n) + \sum_{n=1}^{\infty}(-1)^n f(n)\Big). \tag{4.15}$$

This yields after some relabeling the following expressions [HypSummer → $SummConv.prc$]:

$$\sum_{n=1}^{\infty} \frac{(\pm\sqrt{x})^n}{(a+b\epsilon+n)^m} S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(n) \tag{4.16}$$

$$\sum_{n=1}^{\infty} (\pm\sqrt{x})^n n^k S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(n) \tag{4.17}$$

$$\sum_{n=1}^{\infty} (\pm\sqrt{x})^n S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(n) \tag{4.18}$$

and analogous terms without S-sums. Now we will convert the S-sums to Z-sums [HypSummer → $ConvStoZ.prc$] since Z-sums will be slightly more convenient to deal with. Taking eq. (4.16) we get

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a+b\epsilon+n)^m} Z_{m_1,\ldots,m_k;x_1,\ldots,x_k}(n-1) \tag{4.19}$$

and now we have to reduce the offset a to zero. We have to distinguish two cases $a < 0$ and $a > 0$. In the case of negative offset, we proceed as follows [HypSummer → $Summer2.prc$]:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(n-a+b\epsilon)^m} Z_{m_1\ldots}(n-1) &= \sum_{i=1}^{a-1} \frac{(\sqrt{x})^i}{(i-a+b\epsilon)^m} Z_{m_1,\ldots}(i-1) \\ +\frac{(\sqrt{x})^a}{b\epsilon^m} Z_{m_1,\ldots}(a-1) &+ \sum_{n=1}^{\infty} \frac{(\sqrt{x})^{n+a}}{(n+b\epsilon)^m} Z_{m_1,\ldots}(n+a-1). \end{aligned} \tag{4.20}$$

The last expression can be expanded in ϵ using

$$(n+\epsilon)^{-k} = \sum_{i=1}^{k} \frac{k!}{i!(k-i)!} n^i \epsilon^{k-i}, \tag{4.21}$$

which leaves us with

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{n^m} Z_{m_1,\ldots}(n+a-1). \tag{4.22}$$

The offset a in the argument of the Z-sum can be brought to zero using the eq. (3.10) [HypSummer $\to$ $Zsynch.prc$], which brings us to

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{n^m} Z_{m_1,\ldots}(n-1), \tag{4.23}$$

which is per definition $Z_{m,m_1,\ldots,m_k;\sqrt{x},x_k,\ldots,x_1}(\infty)$ or, using eq. (3.11) of previous chapter, $Li_{m_k,\ldots,m_1,m}(x_1,\ldots,x_k,\sqrt{x})$.
In the case that $a > 0$ we first expand the denominator in ϵ using eq. (4.21) from which we get

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a+n)^m} Z_{\ldots}(n-1). \tag{4.24}$$

Now we apply the following formula [HypSummer $\to$ $Summer21.prc$]:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a+n)^m} Z_{m_1,\ldots;x_1,\ldots}(n-1) = \\
&\frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a-1+n)^m} Z_{m_1,\ldots;x_1,\ldots}(n-1) \\
&- \sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a+n)^m} \frac{x_1^n}{n^{m_1}} Z_{m_2,\ldots;x_2,\ldots}(n-1).
\end{aligned} \tag{4.25}$$

In the first expression on the RHS of eq. (4.25) the offset a is now lowered by one and in the second expression of the above equation, the depth of the nested sum is reduced. Using eq. (4.25) recursively gives us terms

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{n^m} Z_{m_1,\ldots}(n-1), \tag{4.26}$$

which are the same as eq. (4.23) and/or terms like

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(a+n)^m}, \tag{4.27}$$

which we will treat further below (see eq. (4.32 ff.)).
Let us now treat the expression in eq. (4.17), the case without denominator but with a power of summation index. First we rewrite the Z-sum according to

$$Z_{m_1,\ldots;x_1,\ldots}(n-1) = \sum_{i_1=1}^{n-1} \frac{x_1^{i_1}}{i_1^{m_1}} Z_{m_2,\ldots;x_2,\ldots}(i_1-1). \tag{4.28}$$

Next thing we do is interchanging the two summations, leading to [HypSummer $\rightarrow NegSummer.prc$]:

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\sqrt{x})^n n^m Z_{m_1\ldots;x_1\ldots}(n-1) \\
&= \sum_{j=1}^{\infty}(\sqrt{x})^n n^m \sum_{i_1=1}^{n-1} \frac{x_1^{i_1}}{i_1^{m_1}} Z_{m_2\ldots;x_2\ldots}(i_1-1) \\
&= \sum_{i_1=1}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} Z_{m_2\ldots;x_2\ldots}(i_1-1) \sum_{n=i_1+1}^{\infty} (\sqrt{x})^n n^m. \qquad (4.29)
\end{aligned}$$

The last sum in the above equation can be done analytically using [HypSummer $\rightarrow NegLi.prc$]

$$\begin{aligned}
\sum_{n=i_1+1}^{\infty} (\sqrt{x})^n n^m &= \sum_{n=i_1+1}^{\infty} \left(\frac{\partial}{\partial\sqrt{x}}\right)^m (\sqrt{x})^n \\
&= \left(\frac{\partial}{\partial\sqrt{x}}\right)^m \sum_{n=i_1+1}^{\infty} (\sqrt{x})^n \\
&= \left(\frac{\partial}{\partial\sqrt{x}}\right)^m \frac{(\sqrt{x})^{i_1+1}}{1-\sqrt{x}}, \qquad (4.30)
\end{aligned}$$

which gives a finite number of polynomials in i_1 for any finite m. Using eq. (4.29) recursively we either reduce the depth of the Z-sum to zero, hence obtain terms like eq. (4.27), or we obtain terms with Z-sums of non-zero depth, but with denominators with positive powers of the summation index, that is terms like eq. (4.23). The last expression with nested sums, eq. (4.18), we can compute similarly using

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\sqrt{x})^n Z_{m_1\ldots;x_1\ldots}(n-1) \\
&= \sum_{j=1}^{\infty}(\sqrt{x})^n \sum_{i_1=1}^{n-1} \frac{x_1^{i_1}}{i_1^{m_1}} Z_{m_2\ldots;x_2\ldots}(i_1-1) \\
&= \sum_{i_1=1}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} Z_{m_2\ldots;x_2\ldots}(i_1-1) \sum_{n=i_1+1}^{\infty} (\sqrt{x})^n \\
&= \frac{\sqrt{x}}{1-\sqrt{x}} Z_{m_1,m_2\ldots;x_1\cdot\sqrt{x},x_2\ldots}(\infty). \qquad (4.31)
\end{aligned}$$

The three cases without nested sums eq. (4.8-4.10) can be done analogously using

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(n+a)^m} = \frac{1}{\sqrt{x}} \sum_{i=1}^{n} \frac{(\sqrt{x})^i}{(i+c-1)^m} - \frac{1}{c^m} + \frac{(\sqrt{x})^n}{(n+c)^m} \tag{4.32}$$

in case that the offset a is positive and in case it is negative we use

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{(n-a+b\epsilon)^m} = \sum_{i=1}^{a-1} \frac{(\sqrt{x})^i}{(i-a+b\epsilon)^m} + \frac{(\sqrt{x})^a}{b\epsilon^m} \quad + \quad \sum_{n=1}^{\infty} \frac{(\sqrt{x})^{n+a}}{(n+b\epsilon)^m}. \tag{4.33}$$

What is left to do is compute terms like

$$\sum_{n=1}^{\infty} \frac{(\sqrt{x})^n}{n^m} = Li_m(\sqrt{x}) \tag{4.34}$$

and

$$\sum_{n=1}^{\infty} (\sqrt{x})^n n^m = Li_{-m}(\sqrt{x}). \tag{4.35}$$

$Li_{-m}(x)$ is just a polynomial which can be done using the eq. (4.30) [HypSummer $\rightarrow NegLi.prc$]. At the end we have the result of our expansion of eq. (4.1) as a linear combination of multiple polylogarithms $Li_{m_k,\ldots,m_1,m}(1,\ldots,1,\sqrt{x})$. In the case that the argument $x = 1$ one needs to be a bit careful due to terms $\frac{1}{1-\sqrt{x}}$ comming from manipulations like eq. (4.31) and eq. (4.30). The case eq. (4.30) is not problematic, it is just

$$\sum_{n=i_1+1}^{\infty} n^m = \frac{n^{i_1+1}}{1-n}. \tag{4.36}$$

In the case of eq. (4.31) one cannot do anything similar. User of HypSummer has to make sure that the hypergeometric function one is expanding

$${}_PF_{P-1}\left[a_1,\ldots,a_P; b_1,\ldots,b_{P-1}; 1\right] \tag{4.37}$$

fullfills the convergence property

$$\sum_{i=1}^{P-1} b_i - \sum_{i=1}^{P} a_i > 0. \tag{4.38}$$

This implies that the expansion in ϵ commutes with the procedure of taking the limit $x \rightarrow 1$ in [HypSummer $\rightarrow arg1.prc$]. In the case, where argument is 1 and hypergoemetric function fullfills convergence property, multiple polylogs reduce to multiple zeta values and have a particularly compact representation.

4.3 Expansion of generalized first Appell functions

The generalized first Appell function can be written as:

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{a_1}{2}+b_1\epsilon)_{n+j}\cdots(\frac{a_p}{2}+b_p\epsilon)_{n+j}}{(\frac{c_1}{2}+d_1\epsilon)_{n+j}\cdots(\frac{c_p}{2}+d_p\epsilon)_{n+j}}\times\frac{(\frac{e_1}{2}+f_1\epsilon)_{n}\cdots(\frac{e_p}{2}+f_p\epsilon)_{n}}{(\frac{g_1}{2}+h_1\epsilon)_{n}\cdots(\frac{g_p}{2}+h_p\epsilon)_{n}}$$
$$\times\frac{(\frac{i_1}{2}+k_1\epsilon)_{j}\cdots(\frac{i_p}{2}+k_p\epsilon)_{j}}{(\frac{l_1}{2}+m_1\epsilon)_{j}\cdots(\frac{l_{p-1}}{2}+m_{p-1}\epsilon)_{j}}\frac{x^j}{j!}\frac{y^n}{n!} \quad = \quad 1 \quad +$$
$$+\sum_{j=1}^{\infty}\frac{(\frac{a_1}{2}+b_1\epsilon)_{j}\cdots(\frac{a_p}{2}+b_p\epsilon)_{j}}{(\frac{c_1}{2}+d_1\epsilon)_{j}\cdots(\frac{c_p}{2}+d_p\epsilon)_{j}}\frac{(\frac{i_1}{2}+k_1\epsilon)_{j}\cdots(\frac{i_p}{2}+k_p\epsilon)_{j}}{(\frac{l_1}{2}+m_1\epsilon)_{j}\cdots(\frac{l_{p-1}}{2}+m_{p-1}\epsilon)_{j}}\frac{x^j}{j!}$$
$$+\sum_{n=1}^{\infty}\frac{(\frac{a_1}{2}+b_1\epsilon)_{n}\cdots(\frac{a_p}{2}+b_p\epsilon)_{n}}{(\frac{c_1}{2}+d_1\epsilon)_{n}\cdots(\frac{c_p}{2}+d_p\epsilon)_{n}}\frac{(\frac{e_1}{2}+f_1\epsilon)_{n}\cdots(\frac{e_p}{2}+f_p\epsilon)_{n}}{(\frac{g_1}{2}+h_1\epsilon)_{n}\cdots(\frac{g_p}{2}+h_p\epsilon)_{n}}\frac{y^n}{n!}$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\frac{(\frac{a_1}{2}+b_1\epsilon)_{n}\cdots(\frac{a_p}{2}+b_p\epsilon)_{n}}{(\frac{c_1}{2}+d_1\epsilon)_{n}\cdots(\frac{c_p}{2}+d_p\epsilon)_{n}}\times\frac{(\frac{e_1}{2}+f_1\epsilon)_{n-j}\cdots(\frac{e_p}{2}+f_p\epsilon)_{n-j}}{(\frac{g_1}{2}+h_1\epsilon)_{n-j}\cdots(\frac{g_p}{2}+h_p\epsilon)_{n-j}}$$
$$\times\frac{(\frac{i_1}{2}+k_1\epsilon)_{j}\cdots(\frac{i_p}{2}+k_p\epsilon)_{j}}{(\frac{l_1}{2}+m_1\epsilon)_{j}\cdots(\frac{l_{p-1}}{2}+m_{p-1}\epsilon)_{j}}\frac{x^j}{j!}\frac{y^{n-j}}{(n-j)!} \tag{4.39}$$

The first two sums on the RHS of the above equation are just generalized hypergeometric functions, the last sum however,

$$\sum_{j=1}^{n-1}\frac{(\frac{e_1}{2}+f_1\epsilon)_{n-j}\cdots(\frac{e_p}{2}+f_p\epsilon)_{n-j}}{(\frac{g_1}{2}+h_1\epsilon)_{n-j}\cdots(\frac{g_p}{2}+h_p\epsilon)_{n-j}}\frac{(\frac{i_1}{2}+k_1\epsilon)_{j}\cdots(\frac{i_p}{2}+k_p\epsilon)_{j}}{(\frac{l_1}{2}+m_1\epsilon)_{j}\cdots(\frac{l_{p-1}}{2}+m_{p-1}\epsilon)_{j}}$$
$$\times\frac{x^j}{j!}\frac{x\prime^{n-j}}{(n-j)!} \tag{4.40}$$

we have to compute differently. Following the same same steps of expanding Pochhammer symbols in nested sums and using their algebra like we did for generalized hypergeometric functions, we get expressions like:

$$\sum_{j=1}^{n-1}\frac{x^j(x')^{n-j}}{(\frac{1}{2}+a+b\epsilon+n-j)^m}S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n-2j)\times$$
$$\times S_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2j) \tag{4.41}$$
$$\sum_{j=1}^{n-1}\frac{x^j(x')^{n-j}}{(\frac{1}{2}+a+b\epsilon+j)^m}S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n-2j)\times$$

$$\times S_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2j) \tag{4.42}$$

$$\sum_{j=1}^{n-1} x^j (x')^{n-j} j^k S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n-2j) \times \\ \times S_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2j) \tag{4.43}$$

$$\sum_{j=1}^{n-1} x^j (x')^{n-j} S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n) \times \\ \times S_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2j) \tag{4.44}$$

and the corresponding cases without the nested sums, which we will not describe here, since they can be computed using methods from previous section. Also, the eq. (4.41) can, via shifting the summation index $j \rightarrow n-j$ which is done automatically by HypSummer, be reduced to the eq. (4.42). Let us look closely at eq. (4.42-4.44). Rewriting the summand as

$$\sum_{j=1}^{n-1} \frac{(\frac{\sqrt{x}}{\sqrt{x'}})^{2j} (x')^n}{(\frac{1}{2}+a+b\epsilon+\frac{2j}{2})^m} S_{m_1,\ldots,m_k;x_1,\ldots,x_k}(2n-2j) \times \\ \times S_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2j) \tag{4.45}$$

and doing so analogously for eq. (4.43,4.44) and using again eq. (4.15) we obtain, after some relabeling and changing from S-sums to Z-sums:

$$\sum_{j=1}^{2n-1} \frac{(\sqrt{\frac{x}{x'}})^j}{(a+b\epsilon+j)^m} Z_{m_1,\ldots,m_k;x_1,\ldots,x_k}(j-1) \times \\ \times Z_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2n-j-1) \tag{4.46}$$

$$\sum_{j=1}^{2n-1} \left(\sqrt{\frac{x}{x'}}\right)^j j^k Z_{m_1,\ldots,m_k;x_1,\ldots,x_k}(j-1) \times \\ \times Z_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2n-j-1) \tag{4.47}$$

$$\sum_{j=1}^{2n-1} \left(\sqrt{\frac{x}{x'}}\right)^j Z_{m_1,\ldots,m_k;x_1,\ldots,x_k}(j-1) \times \\ \times Z_{m'_1,\ldots,m'_k;x'_1,\ldots,x'_k}(2n-j-1) \tag{4.48}$$

It suffices here to treat only eq. (4.46), the other expressions can be done analogously. We rewrite eq. (4.46) as

$$\sum_{j=1}^{2n-1} \sum_{j_2=1}^{j-1} \frac{(\sqrt{\frac{x}{x'}})^{j_2}}{(a+b\epsilon+j_2)^m} Z_{m_1,\ldots,m_k;x_1,\ldots,x_k}(j_2-1) \times$$

$$\frac{(x_1')^{j-j2}}{(j-j2)^{m_1'}} \times Z_{m_2',\ldots,m_k';x_2',\ldots,x_k'}(j-j2-1) \tag{4.49}$$

where the inner sum is again of the same type, but with the decreased depth of the nested sum. Recursive use of eq. (4.49) [HypSummer → $Ralga.prc$] gives us expressions like those in eq. (4.16-4.18):

$$\sum_{j_1=1}^{2n-1} \frac{(x_1)^{j_1}}{(a+b\epsilon+j_1)^{m_1}} \sum_{j_2=1}^{j_1-1} \frac{(x_2)^{j_2}}{(j_1-j_2)^{m_2}} \cdots \sum_{j_k=1}^{j_{k-1}-1} \frac{(x_k)^{j_k}}{(j_{k-1}-j_k)^{m_k}} \times \\ \times Z_{m_1',\ldots;x_1',\ldots}(j_k-1) \tag{4.50}$$

$$\sum_{j_1=1}^{2n-1} x_1^{j_1} j_1^{m_1} \sum_{j_2=1}^{j_1-1} \frac{(x_2)^{j_2}}{(j_1-j_2)^{m_2}} \cdots \sum_{j_k=1}^{j_{k-1}-1} \frac{(x_k)^{j_k}}{(j_{k-1}-j_k)^{m_k}} \times \\ \times Z_{m_1',\ldots;x_1',\ldots}(j_k-1) \tag{4.51}$$

$$\sum_{j_1=1}^{2n-1} x_1^{j_1} \sum_{j_2=1}^{j_1-1} \frac{(x_2)^{j_2}}{(j_1-j_2)^{m_2}} \cdots \sum_{j_k=1}^{j_{k-1}-1} \frac{(x_k)^{j_k}}{(j_{k-1}-j_k)^{m_k}} \times \\ \times Z_{m_1',\ldots;x_1',\ldots}(j_k-1) \tag{4.52}$$

and similar expressions without Z-sums. When summing back recursively, in order to account for the fact that the sums in eq. (4.49) have finite upper limit, we slightly modify some of the methods we used in previous section. In this case eq. (4.25) becomes

$$\sum_{j_k=1}^{j_{k-1}-1} \frac{x^{j_k}}{(a+j_k)^m} Z_{m_1,\ldots;x_1,\ldots}(j_k-1) = \\ \frac{1}{x} \sum_{j_k=1}^{j_{k-1}-1} \frac{x^{j_k}}{(a-1+j_k)^m} Z_{m_1,\ldots;x_1,\ldots}(j_k-1) \\ - \sum_{j_k=1}^{j_{k-1}-2} \frac{x^{j_k}}{(a+j_k)^m} \frac{x_1^{j_k}}{j_k^{m_1}} Z_{m_2,\ldots;x_2,\ldots}(j_k-1) \\ + \frac{x^{j_{k-1}}}{(a+j_{k-1})^m} Z_{m_1,\ldots;x_1,\ldots}(j_{k-1}-1), \tag{4.53}$$

and eq. (4.29) modifies to

$$\sum_{j_k=1}^{j_{k-1}-1} x^{j_k} j_k^m Z_{m_1,\ldots;x_1,\ldots}(j_k-1) =$$

$$\sum_{j_k=1}^{j_{k-1}-1} \frac{x_1^{j_k}}{j_k^{m_1}} Z_{m_2,\ldots;x_2,\ldots}(j_k-1) \times \left\{ \left(\frac{\partial}{\partial x}\right)^m \frac{x^{j_{k-1}}}{x-1} - \left(\frac{\partial}{\partial x}\right)^m \frac{x^{j_k}}{x-1} \right\} \tag{4.54}$$

and finally, eq. (4.31) modifies to

$$\sum_{j_k=1}^{j_{k-1}-1} x^{j_k} Z_{m_1,\ldots;x_1,\ldots}(j_k-1) = \frac{x}{1-x} Z_{m,m_1,\ldots;x\cdot x_1,\ldots}(j_{k-1}-1)$$
$$-\frac{x\cdot x^{j_{k-1}}}{1-x} Z_{m,m_1,\ldots;x\cdot x_1,\ldots}(j_{k-1}-1). \tag{4.55}$$

Applying eq. (4.53-4.55) and similar identities recursively to eq. (4.46-4.48) [HypSummer $\rightarrow$ $Ralga2.prc$] we can express eq. (4.40) as liner combination of $Z_{m_1,\ldots;x_1,\ldots}(2n-1)$. Transforming Z-sums to S-sums and synchronizing them, the double sum in eq. (4.39) results in

$$\sum_{n=1}^{\infty} \frac{(\frac{a_1}{2}+b_1\epsilon)_n \cdots (\frac{a_p}{2}+b_p\epsilon)_n}{(\frac{c_1}{2}+d_1\epsilon)_n \cdots (\frac{c_p}{2}+d_p\epsilon)_n} \frac{x^n}{n!} \times S_{m_1,\ldots;x_1,\ldots}(2n), \tag{4.56}$$

which can be computed with the algorithm from the previous section. The graphic fig. (4.1) shows the internal structure of HypSummer package.

4.4 Usage

In this section we use HypSummer to expand several hypergeometric and first Appell functions and compare the results with other packages or numerical results. In [53] examples of the expansion of several hypergeometric functions have been presented. We will use them to introduce the syntax of HypSummer and check the expansion result. The functions we want to expand are:

$$\begin{aligned}
(i) &\quad {}_2F_1(\epsilon, 2\epsilon; 1-3\epsilon; x), \\
(ii) &\quad {}_2F_1(1, -\epsilon; 1-\epsilon; x) \\
(iii) &\quad {}_3F_2(a-2\epsilon, -2\epsilon, 1-\epsilon; 1-2\epsilon, 1-2\epsilon; x) \\
(iv) &\quad {}_4F_3(\frac{1}{2}, 1, 2\epsilon, 2\epsilon; 2-\epsilon, \frac{1}{2}+\epsilon, 1+2\epsilon; 1) \\
(v) &\quad F_1(-2-\epsilon, \epsilon, \epsilon, 2, x, y)
\end{aligned} \tag{4.57}$$

(i) Let us look at the first hypergeometric function. The HypSummer input has to be as follows:

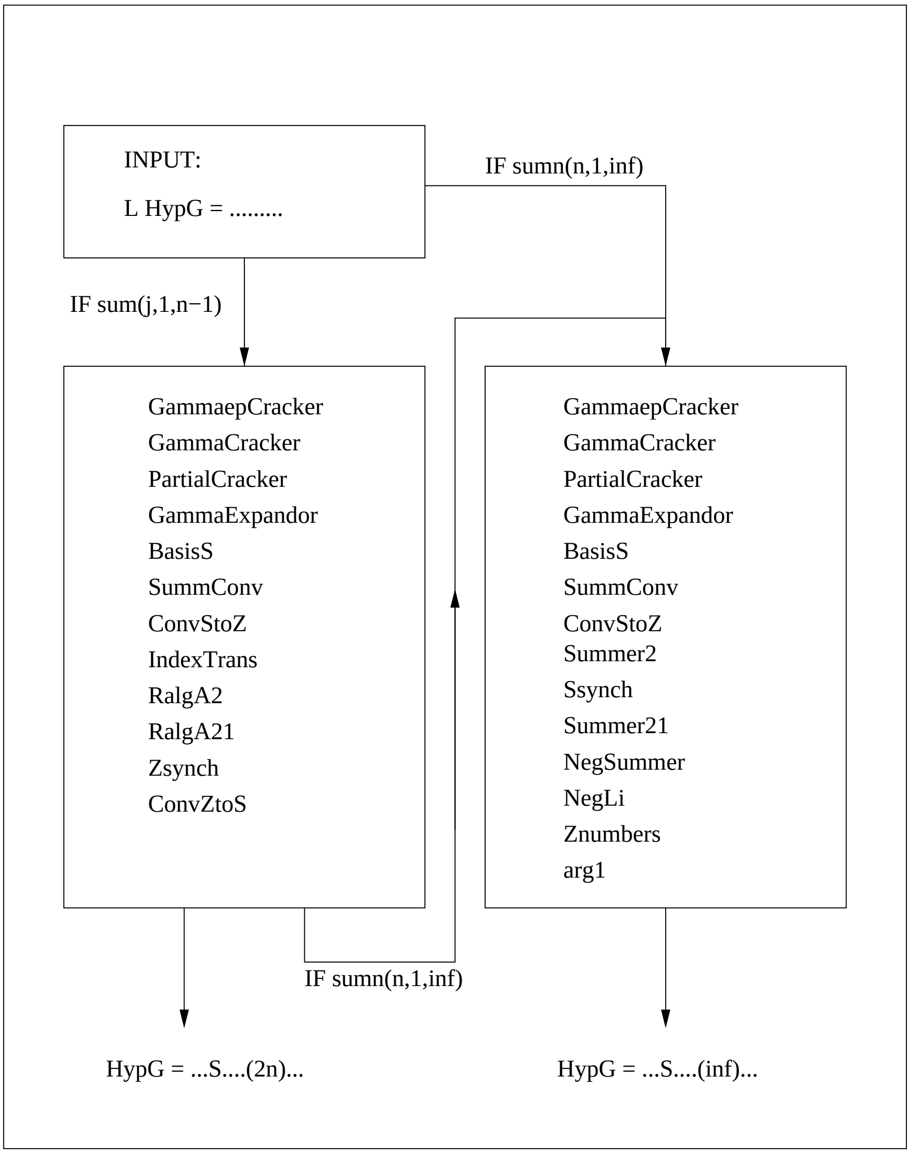


Figure 4.1: Internal structure of HypSummer

```
L f21 = 1+sumn(n,1,inf)*Po(0,ep,n)*Po(0,2*ep,n)*
        InvPo(1,-3*ep,n)*InvPo(1,0,n)*pow(x,n);
```

Here one can see that the objects one is putting in are $Po(a, ep, n)$ for Pochhammer symbols. For the half-integer cases we will write $Pohalf(1/2, a, ep, n)$. For inverse Pochhammer symbols one just need the self explanatory prefix Inv. The Pochhammer symbol $Po(a, ep, n)$ has three input slots, where the first one is reserved for the integer number or zero, but not a symbol, the second for the $a\epsilon$ and third slot is reserved for summation index. All of these slots have to be filled with something otherwise the program will not work. For example, if one has $\frac{\Gamma(-3+n)}{\Gamma(-3)}$ one would write $Po(-3, 0, n)$. For the Pochhammer symbols $Pohalf(1/2, a, ep, n)$ one has four slots where in the first one allways has to be $\frac{1}{2}$ and the other three are the same as for Po. For example $\frac{\Gamma(-3/2+\epsilon+n)}{\Gamma(-3/2+\epsilon)}$ would be $Pohalf(1/2, -2, ep, n)$. These definitions are chosen this way to simplify the internal pattern matching of the package. The result of (i) in eq. (4.57) up to order $\mathcal{O}(\epsilon^3)$ is

```
  f21 =

      + ep^2 * (
         + 4*Li(2, - (x)^(1/2))
         + 4*Li(2,(x)^(1/2))
         )

      + ep^3 * (
         + 24*Li(1,2,-1, - (x)^(1/2))
         + 24*Li(1,2,-1,(x)^(1/2))
         + 24*Li(1,2,1, - (x)^(1/2))
         + 24*Li(1,2,1,(x)^(1/2))
         + 24*Li(3, - (x)^(1/2))
         + 24*Li(3,(x)^(1/2))
         )

      + 1
        ;
```

Please notice that the result here is in different representation, then in [53], where the result is:

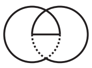
$$\begin{aligned} {}_2F_1(\epsilon, 2\epsilon; 1-3\epsilon; x) &= 1 + \epsilon^2(2Li(2,x)) \\ &+ \epsilon^3(12Li(1,2,1,x) + 6Li(3,x)) \end{aligned} \tag{4.58}$$

which is due to eq. (4.15). The result is still the same, as one can see using expressions like

$$Li(m, x^2) = 2^{m-1}\Big[Li(m,x) + Li(m,-x)\Big] \tag{4.59}$$

and generalizations thereof [79].
(ii),(iii) The result of the other two functions is:

$$\begin{aligned}
(ii) \qquad & {}_2F_1(1,-\epsilon;1-\epsilon;x) = 1 + (-Li(1,-\sqrt{x}) - Li(1,\sqrt{x}))\epsilon \\
& +2(-Li(2,-\sqrt{x}) - Li(2,\sqrt{x}))\epsilon^2 + 4(-Li(3,-\sqrt{x}) - Li(3,\sqrt{x}))\epsilon^3 \\
& +8(-Li(4,-\sqrt{x}) - Li(4,\sqrt{x}))\epsilon^4 + \mathcal{O}(\epsilon^5) \\
(iii) \qquad & {}_3F_2(-2\epsilon,-2\epsilon,1-\epsilon;1-2\epsilon,1-2\epsilon;x) = 1 + 8(Li(2,-\sqrt{x}) \\
& +Li(2,\sqrt{x}))\epsilon^2 + (-8Li(1,2,-1,\sqrt{x}) - 8Li(1,2,1,\sqrt{x}) \\
& -8Li(1,2,-1,-\sqrt{x}) - 8Li(1,2,1,-\sqrt{x}) + 48Li(3,-\sqrt{x}) \\
& +48Li(3,\sqrt{x}))\epsilon^3 + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{4.60}$$

which both agree with known values.
(iv) Another, rather nontrivial example is hypergeometric function which contribute to the graph :

$$ {}_4F_3(\frac{1}{2},1,2\epsilon,2\epsilon;2-\epsilon,\frac{1}{2}+\epsilon,1+2\epsilon;1) \tag{4.61}$$

Here we have half-integer valued coefficients which are balanced, therefore the function is expandable with HypSummer. The output from HypSummer is:

```
f43 =
    + ep^2 * (  - 4 + 4*z2 )

    + ep^3 * (  - 24 - 6*z3 + 8*z2 + 16*ln2 )

    + ep^4 * (  - 108 + 96*li4half - 12*z3 + 24*z2 - 96/
      5*z2^2 + 112*ln2 + 84*ln2*z3 - 48*ln2^2 - 24*ln2^2
      *z2 + 4*ln2^4 )

    + ep^5 * (  - 432 - 576*li5half + 451*z5 + 192*
      li4half - 28*z3 + 72*z2 + 32*z2*z3 - 192/5*z2^2 +
      560*ln2 - 576*ln2*li4half + 168*ln2*z3 - 336*ln2^2
       - 252*ln2^2*z3 - 48*ln2^2*z2 + 96*ln2^3 + 96*
      ln2^3*z2 + 8*ln2^4 - 96/5*ln2^5 )

    + 1;
```

In standard notation this gives:

$$
{}_4F_3\Big(\frac{1}{2},1,2\epsilon,2\epsilon;2-\epsilon,\frac{1}{2}+\epsilon,1+2\epsilon;1\Big) = 1+(-4+4\zeta_2)\epsilon^2
$$
$$
+\epsilon^3(-24-6\zeta_3+8\zeta_2+16\ln_2)+\epsilon^4(-108+96Li(4,\frac{1}{2})-12\zeta_3+24\zeta_2
$$
$$
-\frac{96}{5}\zeta_2^2+112\ln_2+84\ln 2\zeta_3-48\ln_2^2-24\ln_2^2\zeta_2+4\ln_2^4)+\mathcal{O}(\epsilon)^4 \quad (4.62)
$$

which coincides with the result given by HypExp2 from [37].
(v) Let us take an example of an Appell function:

$$
F_1(-2-\epsilon,\epsilon,\epsilon,2,x,y) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-2-\epsilon)_{n+j}(\epsilon)_j(\epsilon)_n}{(2)_{n+j}}\frac{x^j}{j!}\frac{y^n}{n!} \quad (4.63)
$$

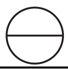
where we will take $x = y = 1$ in order to keep the output short. HypSummer gives:

```
f1 =
      + ep * (  - 5/3 )

      + ep^2 * ( 61/18 - 2*z2 )

      + ep^3 * (  - 239/108 - 2*z3 + 10/3*z2 )

      + ep^4 * ( 3853/648 + 10/3*z3 - 61/9*z2 + 2/5*z2^2 )

      + 1;
```

or

$$
F_1(-2-\epsilon,\epsilon,\epsilon,2,1,1) = 1-\frac{5}{3}\epsilon+(\frac{6}{18}-2\zeta_2)\epsilon^2
$$
$$
+(-\frac{239}{108}-2\zeta_3+\frac{10}{3}\zeta_2)\epsilon^3+(\frac{3853}{648}+\frac{10}{3}\zeta_3-\frac{61}{9}\zeta_2+\frac{2}{5}\zeta_2^2)\epsilon^4
$$
$$
+\mathcal{O}(\epsilon)^5 \quad (4.64)
$$

which coincides with known values.
One can also write in the input a number of basic functions which HypSummer can deal with (see table for all basic functions in HypSummer). For example, the two-loop integral:



$$
\frac{\quad}{J^2} \overset{d=3-2\epsilon}{=} -\frac{1-2\epsilon}{4\epsilon}\Big\{{}_2F_1(2\epsilon,1;1+\epsilon;\frac{1}{4})-2\epsilon\,{}_2F_1(\frac{1}{2},1;\frac{3}{2};\frac{1}{4})\Big\} \quad (4.65)
$$

where J is a massive one-loop tadpole, needs to be written in HypSummer as

```
L s3 = (1-2*ep)*den0(0,4*ep)*((1+sum(j,1,inf)*Po(0,2*ep,j)*
                        InvPo(1,ep,j)*pow(x1,j))-
               2*ep*(1+sum(j,1,inf)*Pohalf(1/2,0,ep,j)*
                       InvPohalf(1/2,1,0,j)*pow(x1,j)));
```

yielding

```
  s3 =

      + ep^-1 * (
         + 1/4
         )

      + ep * (
         - 2*Li(1, - 1/2)
         + Li(1,1,-1,1/2)
         + Li(1,1,1, - 1/2)
         - Li(2, - 1/2)
         - Li(2,1/2)
         )

      + ep^2 * (
         - 2*Li(1,1,-1,1/2)
         - 2*Li(1,1,1, - 1/2)
         + Li(1,1,1,-1,-1, - 1/2)
         + Li(1,1,1,-1,1,1/2)
         + Li(1,1,1,1,-1,1/2)
         + Li(1,1,1,1,1, - 1/2)
         - Li(1,2,-1, - 1/2)
         - Li(1,2,-1,1/2)
         - Li(1,2,1, - 1/2)
         - Li(1,2,1,1/2)
         + 2*Li(2, - 1/2)
         + 2*Li(2,1/2)
         - Li(2,1,-1, - 1/2)
         - Li(2,1,-1,1/2)
         - Li(2,1,1, - 1/2)
         - Li(2,1,1,1/2)
         + 2*Li(3, - 1/2)
         + 2*Li(3,1/2)
         )

      - 1/2
         + Li(1, - 1/2)
        ;
```

Name	Description	Standard notation/Example
sumn	Inf. summation symbol	$\sum_{n=1}^{\infty}$ = sumn(n,1,inf)
sum	Finite summation symbol	$\sum_{j=1}^{n-1}$ = sum(j,1,n-1)
den	Int. denominator	$\frac{1}{a+\epsilon+j}$ = den(a,ep,j)
den0	Int. denominator without sum. index	$\frac{1}{a+\epsilon}$ = den0(a,ep)
denhalf	Half-int. denominator	$\frac{1}{\frac{1}{2}+a+\epsilon+j}$ = denhalf(1/2,a,ep,j)
ep	Expansion parameter	ϵ
pow	Power function	x^j = pow(x,j)
powep	Power of exp. parameter	ϵ^a = powep(ep,a)
Gamma	Gamma function	$\Gamma(a+b\epsilon+n)$ = Gamma(a,b*ep,n)
Gammaep	Gamma function without summation index	$\Gamma(a+b\epsilon)$ = Gammaep(a,b*ep)
Po	Pochhammer symbol	$(a+\epsilon)_n$ = Po(a,ep,n)
Pohalf	Half-int. Pochhammer	$(\frac{1}{2}+a+\epsilon)_n$ = Pohalf(1/2,a,ep,n)
S(R(..),X(..),n)	S-sums	$S_{m_1,\ldots;x_1,\ldots}(n)$ = S(R(m1,..),X(x1,..),n)
Z(R(..),X(..),n-1)	Z-sums	$Z_{m_1,\ldots;x_1,\ldots}(n-1)$ = Z(R(m1,..),X(x1,..),n-1)

Table 1: Basic input objects for HypSummer

4.5 Remarks

Some remarks due to other package is in order. As already mentioned, in case that the sums we expand have integer valued coefficients, there are three packages already cited in section 4.1. In case of half-integer valued coefficients there is Mathematica package HypExp2 [37]. The package can expand certain hypergeometric functions, namely:

$$\begin{array}{lllll} 2_1^2, & 2_1^1, & 2_1^0, & 2_0^1, & \\ 3_2^3, & 3_2^2, & 3_1^1, & 3_1^0, & 3_0^1 \\ 4_1^1, & 4_3^3 & & & \end{array}$$

where P_a^b in this notation means that P is the depth of the hypergeometric function ${}_pF_{p-1}$ and a is the number of half-integer coefficients in denominator and b the number of half-integer coefficients in numerator. If $a = b$ we have balanced hypergeometric functions. The expansion algorithm of HypExp2 is different than the approach taken here. It reduces a given hypergeometric function to a basis function of the same type using differential operators and than it makes an Ansatz for the expansion of the basic functions of the corresponding type. Let us take a look at 2_a^b-type functions and compare with HypSummer. HypSummer can expand the balanced ones, but since 2_a^b is Gauss function, one can use the Euler relations in order to balance the case $a \neq b$. The case 4_a^b are both balanced, so HypSummer can expand them, which leaves the 3_a^b-type functions. Again, $a = b$ case can be done and 3_1^0-type can also be balanced using methods from previous chapter. The two remaining cases, 3_2^3 and 3_0^1 can unfortunately only be balanced in case that the argument is 1 using generalization of Dixon's theorem [66]. Also, one should say that the method HypExp2 uses is not in principle bounded to the mentioned hypergeometric functions above, it can be generalized to higher depths. One other remark also needs to be made here. Both HypExp2 and HypSummer expand hypergeometric functions whose coefficients are of the form $\frac{1}{2} + a + b\epsilon + j$. Recently, in [41] the two-loop massive sunset vacuum diagram like the one in eq. (4.65), but with two different masses, has been expressed in terms of two basic hypergeometric functions with coefficients having not half but quarter values and whose expansion in ϵ can be expressed in terms of elliptic integrals [44]. One of the two hypergeometric functions is balanced and could be in principle done using general Algorithm B from [79], the other however is 4_3^1-type. The class of multiple polylogarithms is not sufficient for the expansion of this functions.

Chapter 5

Applications

In this chapter we use the methods from chapters 2 and 3 and apply them to various Feynman integrals, first of all the set of EQCD master integrals. We first try to express all Feynman integrals in terms of hypergeometric functions. Then we use the package Hypsummer described in chapter 4 and expand the resulting hypergeometric functions in ϵ. We have seen that all scalar integrals can be expressed as Barnes-type integrals, but only one-fold integrals lead to generalized hypergeometric functions, which in turn can be expanded using Hypsummer. In order to achieve the minimal number of Mellin-Barnes integrations, we try to find Mellin-Barnes representations of subloop integrals and insert it in the given integral, which then might been computable in terms of gamma function.

5.1 EQCD master integrals

Let us start with the simplest example:

$$J = \int d^D k \frac{1}{k^2 + m^2} = \frac{\pi^{\frac{D}{2}} \Gamma(1 - \frac{D}{2})}{(m^2)^{1-\frac{D}{2}}} \tag{5.1}$$

This is also the first and only one-loop master integral of EQCD and we will use its integral measure for other integrals. However we will need later some other one-loop integrals, which will be used to compute more complicated integrals, therefore we will write them down here. Those are [68]:

$$\int \frac{d^D k}{p^{\lambda_1} (p-k)^{\lambda_2}} = \frac{\pi^{\frac{D}{2}} \Gamma(\lambda_1 + \lambda_2 - \frac{D}{2}) \Gamma(-\lambda_1 + \frac{D}{2}) \Gamma(-\lambda_2 + \frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(-\lambda_1 - \lambda_2 + D)(p^2)^{\lambda_1 + \lambda_2 - \frac{D}{2}}} \tag{5.2}$$

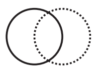
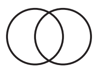
$$\int \frac{d^D k}{(p^2 + 1^2)^{\lambda_1} (p-k)^{\lambda_2}} = \frac{\pi^{\frac{D}{2}} \Gamma(-\lambda_2 + \frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \times$$

$$\times \int dz \frac{\Gamma(-\lambda_1 + \frac{D}{2} - z)\Gamma(-z)\Gamma(\lambda_1 + \lambda_2 - \frac{D}{2} + z)}{\Gamma(-\lambda_1 - \lambda_2 + D - z)(p^2)^{\lambda_1+\lambda_2-\frac{D}{2}+z}} \tag{5.3}$$

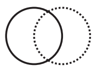
$$\int \frac{d^D k_1 d^D k_2}{(k_1^2)^{\lambda_1}((k_1 - k_2)^2)^{\lambda_2}(k_2 + 1^2)^{\lambda_3}} =$$
$$= \frac{\pi^D \Gamma(\lambda_1 + \lambda_2 - D + \lambda_3)\Gamma(\lambda_1 + \lambda_2 - \frac{D}{2})\Gamma(\lambda_1 - \frac{D}{2})\Gamma(-\lambda_2 + \frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\frac{D}{2})} \tag{5.4}$$

$$\int \frac{d^D k_1 d^D k_2}{(k_1^2 + 1^2)^{\lambda_1}((k_1 - k_2)^2)^{\lambda_2}(k_2 + 1^2)^{\lambda_3}} =$$
$$= \frac{\pi^D \Gamma(\lambda_1 + \lambda_2 - D + \lambda_3)\Gamma(\lambda_1 + \lambda_2 - \frac{D}{2})\Gamma(\lambda_3 + \lambda_2 - \frac{D}{2})\Gamma(-\lambda_2 + \frac{D}{2})}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_1 + 2\lambda_2 + \lambda_3 - D)\Gamma(\frac{D}{2})}. \tag{5.5}$$

Since there are no two loops master integrals for EQCD we go to three loops integrals of which there are two:

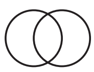
  (5.6)

The first one we will solve inserting the result of eq. (5.2), which will leave us with eq. (5.5). Since both results we are using are given in terms of gamma functions, the result of the master integrals is:



$$= \frac{\pi^{\frac{3D}{2}}\Gamma(4 - \frac{3D}{2})\Gamma(3 - D)^2\Gamma(2 - \frac{D}{2})\Gamma(-1 + \frac{D}{2})^2}{\Gamma(6 - 2D)\Gamma(\frac{D}{2})} \tag{5.7}$$

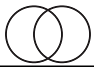
The second three loops integral is more difficult. We use the same method, however this time we insert eq. (5.3) instead, resulting again in eq. (5.5). Since eq. (5.3) is given in terms of a one-fold Barnes-type integral and eq. (5.5) in terms of gamma function, the master integral itself will be given in terms of generalized hypergeometric functions:



$$= \frac{\pi^{\frac{3(1+D)}{2}} Csc\left[\frac{D\pi}{2}\right]^2}{\Gamma\left[\frac{D}{2}\right]^2} \left(\frac{2^{2-D}\Gamma\left[4 - \frac{3D}{2}\right]\Gamma[3 - D]\Gamma\left[\frac{3}{2} - \frac{D}{2}\right]\Gamma\left[\frac{D}{2}\right]}{\Gamma[6 - 2D]} \times \right.$$
$$\times {}_2F_1\left[\left\{4 - \frac{3D}{2}, \frac{3}{2} - \frac{D}{2}\right\}, \left\{\frac{7}{2} - D\right\}, 1\right]$$
$$+ \frac{\sqrt{\pi} 2\Gamma\left[2 - \frac{D}{2}\right] {}_3F_2\left[\left\{-\frac{1}{2}, 1, 3 - D\right\}, \left\{\frac{5}{2} - \frac{D}{2}, \frac{D}{2}\right\}, 1\right]}{-3 + D}$$

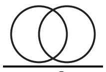
$$-\frac{\sqrt{\pi}8\Gamma\left[3-\frac{D}{2}\right]\Gamma\left[\frac{D}{2}\right]{}_3F_2\left[\left\{\frac{1}{2},2,4-D\right\},\left\{\frac{7}{2}-\frac{D}{2},1+\frac{D}{2}\right\},1\right]}{\left(20-9D+D^2\right)\Gamma\left[1+\frac{D}{2}\right]}$$
$$+\sqrt{\pi}\Gamma\left[2-\frac{D}{2}\right]{}_3F_2\left[\left\{1,2-\frac{D}{2},-\frac{1}{2}+\frac{D}{2}\right\},\left\{\frac{3}{2},-1+D\right\},1\right]\Bigg), \tag{5.8}$$

where $\pi Csc[\pi x] = \Gamma(x)\Gamma(1-x)$.
All hypergeometric functions are balanced in 3 as well as 4 dimension, therefore we can use Hypsummer to expand the result.



$$\frac{\quad}{J^3} \overset{D=4-2\epsilon}{=} -2-\frac{5}{3}\epsilon-\frac{1}{2}\epsilon^2-\epsilon^3(-\frac{103}{12})-\epsilon^4(-\frac{1141}{24}+\frac{112}{3}\zeta_3)$$
$$-\epsilon^5(-\frac{9055}{48}+256a_4+168\zeta_3-96\zeta_2^2-64\ln^2 2\zeta_2+\frac{32}{3}\ln^4 2-32\zeta_4)$$
$$-\epsilon^6(-\frac{63517}{96}+1536a_5-1240\zeta_5+1152a_4+\frac{1876}{3}\zeta_3-432\zeta_2^2$$
$$+576\ln 2\zeta_2^2-288\ln^2 2\zeta_2+128\ln^3 2\zeta_2+48\ln^4 2-\frac{64}{5}\ln^5 2-144\zeta_4$$
$$+192\zeta_4\ln 2)$$
$$-\epsilon^7(-\frac{418903}{192}+3840s6+9216a_6+6912a_5-5580\zeta_5+4288a_4$$
$$+\frac{6398}{3}\zeta_3-\frac{4880}{3}\zeta_3^2-1608\zeta_2^2-\frac{44288}{35}\zeta_2^3+2592\ln 2\zeta_2^2-1072\ln^2 2\zeta_2$$
$$-1728\ln^2 2\zeta_2^2+576\ln^3 2\zeta_2+\frac{536}{3}\ln^4 2-192\ln^4 2\zeta_2-\frac{288}{5}\ln^5 2$$
$$+\frac{64}{5}\ln^6 2-536\zeta_4-192\zeta_4\zeta_2+864\zeta_4\ln 2-576\zeta_4\ln^2 2-176\zeta_6)$$
$$-\epsilon^8(-\frac{2667781}{384}-\frac{87040}{7}s7b+\frac{74240}{7}s7a+55296a_7-\frac{772868}{7}\zeta_7$$
$$+17280s6+41472a_6+25728a_5-20770\zeta_5+14624a_4+\frac{20797}{3}\zeta_3$$
$$-7320\zeta_3^2+\frac{260720}{7}\zeta_2\zeta_5-5484\zeta_2^2+\frac{25024}{7}\zeta_2^2\zeta_3-\frac{199296}{35}\zeta_2^3$$
$$-\frac{74240}{7}\ln 2s6+\frac{92800}{7}\ln 2\zeta_3^2+9648\ln 2\zeta_2^2+\frac{189568}{35}\ln 2\zeta_2^3$$
$$+22320\ln^2 2\zeta_5-3656\ln^2 2\zeta_2-7776\ln^2 2\zeta_2^2+2144\ln^3 2\zeta_2$$
$$+3456\ln^3 2\zeta_2^2+\frac{1828}{3}\ln^4 2-864\ln^4 2\zeta_2-\frac{1072}{5}\ln^5 2+\frac{1152}{5}\ln^5 2\zeta_2$$
$$+\frac{288}{5}\ln^6 2-\frac{384}{35}\ln^7 2-1828\zeta_4+320\zeta_4\zeta_3-864\zeta_4\zeta_2+3216\zeta_4\ln 2$$

$$+1152\zeta_4 \ln 2\zeta_2 - 2592\zeta_4 \ln^2 2 + 1152\zeta_4 \ln^3 2 - 792\zeta_6 + 1056\zeta_6 \ln 2) \quad (5.9)$$



$$\frac{\cdot}{J^3} \overset{D=3-2\epsilon}{=} \frac{1}{\epsilon} - (-2 + 4\ln 2) - \Big(-16 - \frac{2\pi^2}{3} + 6\ln 2 - 4\ln 2^2 + \ln 4\Big)\epsilon - \frac{1}{3}\Big(-288 - 4\pi^2 + 192\ln 2 + 4\pi^2 \ln 2 - 18\ln^2 2 + 8\ln^3 2 - 2\ln 2\ln 8 + 114\zeta_3\Big)\epsilon^2 - \frac{1}{60}\Big(-34560 - 9600a_4 - 640\pi^2 - \frac{176\pi^4}{3} + 23040\ln 2 - 120\pi^2\ln 2 - 3840\ln^2 2 + 320\pi^2\ln^2 2 + 320\ln^3 2 - 480\ln^4 2 + 140\pi^2 \ln 4 + 4560\zeta_3 - 12960\ln 2\zeta_3\Big)\epsilon^3 - \Big(-864 - 160a_5 - 48\pi^2 - \frac{19\pi^4}{10} + \frac{61}{10}\pi^4(-1 + \ln 2) + 16\pi^2 \ln 2 - \frac{101}{90}\pi^4 \ln 2 - \frac{10}{3}\pi^2 \ln 2^2 - 2\pi^2 \ln^3 2 + \frac{2\ln^4 2}{3} + \frac{22\ln^5 2}{15} + \frac{2}{15}\ln 2(4320 + \ln 2(-720 + \ln 2(80 + (-5 + \ln 2)\ln 2))) - 280\zeta_3 + 9\pi^2\zeta_3 + 108\ln 2\zeta_3 + 54\ln^2 2\zeta_3 + 54\Big(8 + \ln^2 2 - \ln 4\Big)\zeta_3 + \frac{7}{9}\pi^2\Big(-144 + 2\ln 2\Big(24 + \ln^2 2 - \ln 8\Big) + 81\zeta_3\Big) + \frac{1445\zeta_5}{2}\Big)\epsilon^4 \quad (5.10)$$

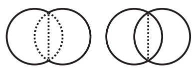
The numbers appearing in the results of all expansions in this thesis are defined in the appendix A3.

5.1.1 4-loop integrals

Now we come to four loop integrals, where we will try to use the same tactic we used on the two three loop integrals: insert simpler subloops in order to obtain integrals which can be written in terms of gamma functions.

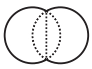
5 lines

Let us start with the integrals with 5 propagators. There are two of them:



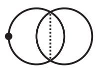
(5.11)

Just like in the case of three loop integrals, we insert eq. (5.2) twice for the first master integral obtaining eq. (5.5) which gives us:



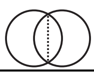
$$= \frac{\pi^{2d}\Gamma[5-2d]\Gamma\left(4-\frac{3d}{2}\right)^2\Gamma[3-d]\Gamma\left(-1+\frac{d}{2}\right)^3}{\Gamma[8-3d]\Gamma\left(\frac{d}{2}\right)}. \tag{5.12}$$

For the second master integral we insert first eq. (5.3), then eq. (5.2) obtaining again eq. (5.5), so that we can express the master integral as:



$$\begin{aligned}
&= \frac{-2(-1)^\lambda\pi^2 Csc[\epsilon\pi]^2\Gamma[1-\epsilon]\Gamma[-1+2\epsilon]\Gamma[-2+2\epsilon+\lambda]}{\Gamma[2-\epsilon]^2\Gamma[\lambda]\Gamma[-3+4\epsilon+\lambda]} \\
&\Gamma[-3+3\epsilon+\lambda]{}_3F_4\left(\begin{matrix}\frac{1}{2},-1+2\epsilon,-2+2\epsilon+\lambda,-3+3\epsilon+\lambda\\ 2-\epsilon,-\frac{3}{2}+2\epsilon+\frac{\lambda}{2},-1+2\epsilon+\frac{\lambda}{2}\end{matrix}\,\middle|\,1\right) \\
&+\frac{(-1)^\lambda 2^{-2+2\epsilon}\pi^{3/2}Csc[\epsilon\pi]^2\Gamma[1-\epsilon]\Gamma\left[-\frac{1}{2}+\epsilon\right]\Gamma[-2+3\epsilon]\Gamma[-3+3\epsilon+\lambda]}{\Gamma[2-\epsilon]\Gamma[\epsilon]\Gamma[\lambda]\Gamma[-5+6\epsilon+\lambda]} \\
&\Gamma[-4+4\epsilon+\lambda]{}_3F_4\left(\begin{matrix}-\frac{1}{2}+\epsilon,-2+3\epsilon,-3+3\epsilon+\lambda,-4+4\epsilon+\lambda\\ \epsilon,-\frac{5}{2}+3\epsilon+\frac{\lambda}{2},-2+3\epsilon+\frac{\lambda}{2}\end{matrix}\,\middle|\,1\right) \\
&+\frac{(-1)^\lambda\pi^2 Csc[\epsilon\pi]^2\Gamma[1-\epsilon]\Gamma[\epsilon]\Gamma[-1+\epsilon+\lambda]}{(-2+2\epsilon+\lambda)\Gamma[2-\epsilon]^3\Gamma[\lambda]} \\
&{}_3F_4\left(\begin{matrix}1,\frac{3}{2}-\epsilon,\epsilon,-1+\epsilon+\lambda,-2+2\epsilon+\lambda\\ 3-2\epsilon,2-\epsilon,-\frac{1}{2}+\epsilon+\frac{\lambda}{2},\epsilon+\frac{\lambda}{2}\end{matrix}\,\middle|\,1\right),
\end{aligned} \tag{5.13}$$

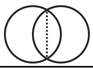
where the dot on the line means we have kept the power of the propagator a variable λ. In the result above D has already been set to $4-2\epsilon$, however, since no expansion has taken place the result is valid in all dimensions. Setting $\lambda = 1$ expanding the result using Hypsummer gives:



$$\begin{aligned}
\frac{}{J^4} &\overset{D=4-2\epsilon}{=} -1-\frac{\epsilon}{2}+\frac{17\epsilon^2}{36}+\frac{\epsilon^3}{216}-\frac{37207\epsilon^4}{1296}+\Bigg(-\frac{1976975}{7776} \\
&+\frac{1792\,\zeta_3}{9}\Bigg)\epsilon^5+\Bigg(-\frac{72443143}{46656}-\frac{256}{135}\left(17\pi^4+60\pi^2\ln_2^2-60\ln_2^4\right) \\
&+\frac{8192}{3}a_4+\frac{47488\,\zeta_3}{27}\Bigg)\epsilon^6+\Bigg(-\frac{2259199295}{279936}+\frac{128}{405}\Bigg(12(265-72\ln_2)\ln_2^4
\end{aligned}$$

$$+60\pi^2 \ln_2^2(-53 + 24\ln_2) + 17\pi^4(-53 + 72\ln_2)\Big) + \frac{217088}{9}a_4 + 32768a_5$$
$$+\frac{871360\,\zeta_3}{81} - \frac{87296\,\zeta_5]}{3}\Big)\epsilon^7 + \mathcal{O}(\epsilon^8)\ . \tag{5.14}$$

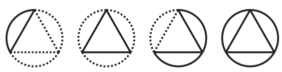
For 3-dim we have [5]:



$$\frac{\text{[diagram]}}{J^4} \overset{D=3-2\epsilon}{=} \frac{7}{4\epsilon} + 7 - 8\ln 2 + \epsilon(49 + 16\,\zeta_2 - 32\ln 2 + 16\ln^2 2)$$
$$+\epsilon^2(308 - 108\,\zeta_3 + 64\,\zeta_2 - 224\ln 2 - 64\,\zeta_2\ln 2 + 64\ln^2 2 - \frac{64}{3}\ln^3 2)$$
$$+\epsilon^3(1904 + 128\,a_4 - 432\,\zeta_3 + 448\,\zeta_2 + \frac{412}{5}\,\zeta_2^2 - 1408\ln 2 + 544\,\zeta_3\ln 2$$
$$-256\,\zeta_2\ln 2 + 448\ln^2 2 + 96\,\zeta_2\ln^2 2 - \frac{256}{3}\ln^3 2 + \frac{80}{3}\ln^4 2 + 426\,\zeta_4)$$
$$+\epsilon^4(11648 + 512\,a_5 - 3212\,\zeta_5 + 512\,a_4 - 3024\,\zeta_3 + 2816\,\zeta_2 - 1088\,\zeta_2\,\zeta_3$$
$$+\frac{1648}{5}\,\zeta_2^2 - 8704\ln 2 + 2176\ln 2\,\zeta_3 - 1792\ln 2\,\zeta_2 - \frac{1648}{5}\ln 2\,\zeta_2^2$$
$$+2816\ln^2 2 - 1088\ln^2 2\,\zeta_3 + 384\ln^2 2\,\zeta_2 - 1792/3\ln^3 2 - 128\ln^3 2\,\zeta_2$$
$$+\frac{320}{3}\ln^4 2 - 64/3\ln^5 2 + 1704\,\zeta_4 - 1704\,\zeta_4\ln 2)$$
$$+\epsilon^5(70784 - 256\,s_6 + 2048\,a_6 + 2048\,a_5 - 12848\,\zeta_5 + 3584\,a_4$$
$$-19008\,\zeta_3 + 3768\,\zeta_3^2 + 17408\,\zeta_2 - 4352\,\zeta_2\,\zeta_3 + \frac{11536}{5}\,\zeta_2^2 + \frac{7968}{35}\,\zeta_2^3$$
$$-53248\ln 2 + 13344\ln 2\,\zeta_5 + 15232\ln 2\,\zeta_3 - 11264\ln 2\,\zeta_2$$
$$+4352\ln 2\,\zeta_2\,\zeta_3 - \frac{6592}{5}\ln 2\,\zeta_2^2 + 17408\ln^2 2 - 4352\ln^2 2\,\zeta_3$$
$$+2688\ln^2 2\,\zeta_2 + \frac{3296}{5}\ln^2 2\,\zeta_2^2 - \frac{11264}{3}\ln^3 2 + \frac{4352}{3}\ln^3 2\,\zeta_3 - 512\ln^3 2\,\zeta_2$$
$$+\frac{2240}{3}\ln^4 2 + 128\ln^4 2\,\zeta_2 - \frac{256}{3}\ln^5 2 + \frac{128}{9}\ln^6 2 + 11928\,\zeta_4$$
$$+5592\,\zeta_4\,\zeta_2 - 6816\,\zeta_4\ln 2 + 3408\,\zeta_4\ln^2 2 + 11146\,\zeta_6) + \mathcal{O}(\epsilon^6)\ . \tag{5.15}$$

6 lines

There are four master integrals with 6 propagators:



(5.16)

Let us start with the one with two massive lines. Inserting eq. (5.2) squared leaves us with eq. (5.5) and we can write it in terms of gamma functions as:

$$= \frac{8^{D-3}\Gamma^3(\frac{1}{2})\Gamma(6-2D)\Gamma^3(\frac{D}{2})}{\sin(\frac{3D}{2})\Gamma(\frac{11-3D}{2})\Gamma^2(2-\frac{D}{2})\Gamma^2(D-2)}. \qquad (5.17)$$

The next master integrals has three massive lines and inserting one loop subintegral does not lead to a one-fold Barnes-type integral representation. However, the hypergeometric representation for this integral has been found solving the corresponding difference equation here [4]:

$$\frac{\cdot}{J^4} = \frac{2^{2-D}(D-2)\Gamma(6-2D)\Gamma(5-\frac{3D}{2})\Gamma(\frac{3-D}{2})\Gamma(\frac{D}{2})}{\sqrt{\pi}\Gamma^2(7-2D)\Gamma(1-\frac{D}{2})} \times$$
$$\times[{}_3F_2(1,9-3D,5-\frac{3D}{2};7-2D,7-2D;1) -$$
$$-{}_3F_2(1,9-3D,2-\frac{D}{2};7-2D,4-D;1)]. \qquad (5.18)$$

The expanion was given in [4] and we just present here the updated expansion in 3 dimensions up to ϵ^5

$$\frac{\cdot}{J^4} \stackrel{D=3-2\epsilon}{=} \frac{3}{16\epsilon}\,\zeta_2 + \Big(-\frac{9}{8}\,\zeta_2 + \frac{9}{4}\,\zeta_2\ln 2 - \frac{21}{8}\,\zeta_3\Big) + \epsilon\Big(\frac{3}{2}\ln^4 2$$
$$+\frac{9}{4}\,\zeta_2 - \frac{27}{2}\,\zeta_2\ln 2 + \frac{9}{2}\,\zeta_2\ln^2 2 - \frac{207}{40}\,\zeta_2^2 + \frac{63}{4}\,\zeta_3 + 36\,a_4\Big)$$
$$+\epsilon^2\Big(-9\ln^4 2 + \frac{18}{5}\ln^5 2 - \frac{3}{2}\,\zeta_2 + 27\,\zeta_2\ln 2 - 27\,\zeta_2\ln^2 2 + 18\,\zeta_2\ln^3 2$$
$$+\frac{621}{20}\,\zeta_2^2 - \frac{621}{10}\,\zeta_2^2\ln 2 - \frac{63}{2}\,\zeta_3 - \frac{87}{8}\,\zeta_3\,\zeta_2 + \frac{4743}{16}\,\zeta_5 - 216\,a_4$$
$$-432\,a_5\Big) + \epsilon^3\Big(1836\,s6 + 5184\,a_6 + 2592\,a_5$$
$$-\frac{14229}{8}\,\zeta_5 + 432\,a_4 + 21\,\zeta_3 - \frac{5655}{8}\,\zeta_3^2 + 288\,\zeta_2\,a_4$$
$$+\frac{261}{4}\,\zeta_2\,\zeta_3 - \frac{621}{10}\,\zeta_2^2 - \frac{145029}{280}\,\zeta_2^3 - 18\ln 2\,\zeta_2 + \frac{243}{2}\ln 2\,\zeta_2\,\zeta_3$$
$$+\frac{1863}{5}\ln 2\,\zeta_2^2 + 54\ln^2 2\,\zeta_2 - \frac{2223}{5}\ln^2 2\,\zeta_2^2 - 108\ln^3 2\,\zeta_2 + 18\ln^4 2$$
$$+66\ln^4 2\,\zeta_2 - \frac{108}{5}\ln^5 2 + \frac{36}{5}\ln^6 2 - \frac{1527}{4}\,\zeta_4\,\zeta_2\Big)$$
$$-\epsilon^4(\frac{81360}{7}s7b - \frac{72864}{7}s7a - 62208a_7 + \frac{12311091}{112}\zeta_7 - 11016s6$$

$$
\begin{aligned}
&-31104a_6 - 5184a_5 + \frac{14229}{4}\zeta_5 - 288a_4 - 1440\zeta_3 a_4 + \frac{16965}{4}\zeta_3^2 \\
&-3456\zeta_2 a_5 - \frac{1803411}{56}\zeta_2\zeta_5 - 1728\zeta_2 a_4 - \frac{261}{2}\zeta_2\zeta_3 \\
&+207/5\zeta_2^2 - \frac{676203}{140}\zeta_2^2\zeta_3 + \frac{435087}{140}\zeta_2^3 + \frac{72864}{7}\ln 2 s6 \\
&-\frac{91080}{7}\ln 2\zeta_3^2 - 729\ln 2\zeta_2\zeta_3 - \frac{3726}{5}\ln 2\zeta_2^2 - \frac{292707}{70}\ln 2\zeta_2^3 \\
&-\frac{42687}{2}\ln^2 2\zeta_5 - 36\ln^2 2\zeta_2 + 1089\ln^2 2\zeta_2\zeta_3 \\
&+\frac{13338}{5}\ln^2 2\zeta_2^2 + 216\ln^3 2\zeta_2 - \frac{8892}{5}\ln^3 2\zeta_2^2 - 12\ln^4 2 - 60\ln^4 2\zeta_3 \\
&-396\ln^4 2\zeta_2 + \frac{216}{5}\ln^5 2 + \frac{792}{5}\ln^5 2\zeta_2 - \frac{216}{5}\ln^6 2 + \frac{432}{35}\ln^7 2 \\
&+\frac{10689}{2}\zeta_4\zeta_3 + \frac{4581}{2}\zeta_4\zeta_2 - 4581\zeta_4\ln 2\zeta_2) + \epsilon^5(\frac{874368}{7}s8d \\
&+\frac{976320}{7}s8c + \frac{9563184}{7}s8b + \frac{11691495}{56}s8a + 746496a_8 \\
&-\frac{488160}{7}s7b + \frac{437184}{7}s7a + 373248a_7 - \frac{36933273}{56}\zeta_7 + 22032s6 \\
&+62208a_6 + 3456a_5 - \frac{4743}{2}\zeta_5 - \frac{1627776}{7}\zeta_3 a_5 + \frac{7655985}{56}\zeta_3\zeta_5 + 8640\zeta_3 a_4 \\
&-\frac{16965}{2}\zeta_3^2 - \frac{2571264}{7}\zeta_2 s6 + 41472\zeta_2 a_6 + 20736\zeta_2 a_5 + \frac{5410233}{28}\zeta_2\zeta_5 \\
&+3456\zeta_2 a_4 + 87\zeta_2\zeta_3 - \frac{9006387}{56}\zeta_2\zeta_3^2 + \frac{2063808}{35}\zeta_2^2 a_4 + \frac{2028609}{70}\zeta_2^2\zeta_3 \\
&-\frac{435087}{70}\zeta_2^3 - \frac{525208177}{9800}\zeta_2^4 - \frac{437184}{7}\ln 2 s6 + \frac{546480}{7}\ln 2\zeta_3^2 \\
&-\frac{5815017}{14}\ln 2\zeta_2\zeta_5 + 1458\ln 2\zeta_2\zeta_3 + \frac{2484}{5}\ln 2\zeta_2^2 - \frac{5517639}{35}\ln 2\zeta_2^2\zeta_3 \\
&+\frac{878121}{35}\ln 2\zeta_2^3 - \frac{488160}{7}\ln^2 2 s6 + 128061\ln^2 2\zeta_5 - \frac{137295}{7}\ln^2 2\zeta_3^2 \\
&-6534\ln^2 2\zeta_2\zeta_3 - \frac{26676}{5}\ln^2 2\zeta_2^2 - \frac{18513}{7}\ln^2 2\zeta_2^3 - 144\ln^3 2\zeta_2 \\
&-\frac{115236}{7}\ln^3 2\zeta_2\zeta_3 + \frac{53352}{5}\ln^3 2\zeta_2^2 + 360\ln^4 2\zeta_3 + 792\ln^4 2\zeta_2 \\
&-\frac{20148}{7}\ln^4 2\zeta_2^2 - \frac{144}{5}\ln^5 2 + \frac{67824}{35}\ln^5 2\zeta_3 - \frac{4752}{5}\ln^5 2\zeta_2 + \frac{432}{5}\ln^6 2 \\
&+\frac{1584}{5}\ln^6 2\zeta_2 - \frac{2592}{5}\ln 2^7 + \frac{648}{35}\ln 2^8 - 73296\zeta_4 a_4 - 32067\zeta_4\zeta_3 \\
&-4581\zeta_4\zeta_2 + \frac{105363}{10}\zeta_4\zeta_2^2 + 27486\zeta_4\ln 2\zeta_2 - 9162\zeta_4\ln^2 2\zeta_2 - 3054\zeta_4\ln^4 2
\end{aligned}
$$

$$
-\frac{58649}{2}\zeta_6\zeta_2) + \mathcal{O}(\epsilon^6) \tag{5.19}
$$

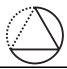
The next master integral is with four massive lines. Inserting eq. (5.3) times eq. (5.2) gives us again eq. (5.5) and we can write it down in terms of hypergeometric functions:

$$
\begin{aligned}
&= \frac{-Cot(\frac{D\pi}{2})Csc(\frac{D\pi}{2})\Gamma(6-2D)\Gamma(\frac{3}{2}-\frac{D}{2})\Gamma(2-\frac{D}{2})\Gamma(-1+\frac{D}{2})^3}{\pi^{-(\frac{3}{2}+2D)}8^{(3-D)}(1+2Cos(D\pi))\Gamma(4-D)\Gamma(-2+D)\Gamma(\frac{D}{2})} \\
&-\frac{((4^{-3+D}\pi^{\frac{3}{2}+2D}Cos(D\pi)Csc(\frac{D\pi}{2})^2\Gamma(2-\frac{D}{2})^2\Gamma(-1+\frac{D}{2})^2\Gamma(-\frac{5}{2}+D))}{(2Cos(\frac{D\pi}{2})+Cos(\frac{3D\pi}{2}))\Gamma(-2+D)\Gamma(\frac{D}{2})\Gamma(-3+\frac{3D}{2})} \\
&{}_3F_2\Big(1, 2-\frac{D}{2}, -\frac{5}{2}+D; \frac{3}{2}, -3+\frac{3D}{2}, 1\Big) \\
&+\frac{2^{-7+2D}\pi^{\frac{5}{2}+2D}Csc(\frac{D\pi}{2})^2\Gamma(5-\frac{3D}{2})\Gamma(2-\frac{D}{2})\Gamma(-1+\frac{D}{2})^2 Sec(\frac{D\pi}{2})}{\Gamma(\frac{9}{2}-D)\Gamma(-2+D)\Gamma(\frac{D}{2})^2} \\
&{}_3F_2\Big(\frac{1}{2}, 1, 5-\frac{3D}{2}; \frac{9}{2}-D, \frac{D}{2}, 1\Big).
\end{aligned} \tag{5.20}
$$

In 4 dimensions Hypsummer computes

$$
\begin{aligned}
&\frac{\quad}{J^4} \overset{D=4-2\epsilon}{=} \frac{2}{3} + \frac{4}{3}\epsilon + \frac{2}{3}\epsilon^2 - \frac{4(11-4\zeta_3)}{3}\epsilon^3 - \frac{4(435+\pi^4-250\zeta_3)}{15}\epsilon^4 \\
&-\Big(\frac{(326\pi^4)}{45} + \frac{64\pi^2\ln^2 2}{3} - \frac{8(-241+8\ln^4 2+192a_4+149\zeta_3+36\zeta_5)}{3}\Big)\epsilon^5 \\
&-\Big(\frac{9328}{3} + \frac{2126\pi^4}{45} + \frac{8\pi^6}{21} - \frac{2416}{45}\pi^4\ln 2 + \frac{448}{3}\pi^2\ln^2 2 \\
&-\frac{512}{9}\pi^2\ln^3 2 - \frac{448\ln^4 2}{3} + \frac{512\ln^5 2}{15} - 3584a_4 - 4096a_5 - \frac{5864\zeta_3}{3} \\
&-\frac{64\zeta_3^2}{3} + 2784\zeta_5\Big)\epsilon^6 \\
&-\Big(14032 - 12288s6 + \frac{2182\pi^4}{9} + \frac{25408\pi^6}{945} - \frac{16912}{45}\pi^4\ln 2 + \frac{2368}{3}\pi^2\ln^2 2 \\
&+\frac{9664}{45}\pi^4\ln^2 2 - \frac{3584}{9}\pi^2\ln^3 2 - \frac{2368\ln^4 2}{3} + \frac{1024}{9}\pi^2\ln^4 2 \\
&+\frac{3584\ln^5 2}{15} - \frac{2048\ln^6 2}{45} - 18944a_4 - 28672a_5 - 32768a_6 - 8760\zeta_3 + \frac{32}{15}\pi^4\zeta_3 \\
&+\frac{14872\zeta_3^2}{3} + 20736\zeta_5 - 1328\zeta_7\Big)\epsilon^7 + \mathcal{O}(\epsilon^8),
\end{aligned} \tag{5.21}
$$

and in 3 dimensions

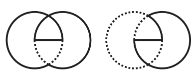


$$
\begin{aligned}
&\frac{\quad}{J^4} \overset{D=3-2\epsilon}{=} \frac{\pi^2}{32\epsilon} - \Big(-\frac{1}{16}\pi^2(-3+\ln 4) + \frac{7\zeta_3}{4}\Big) \\
&-\frac{\Big(-89\pi^4 + 60\pi^2(-9+2\ln 2(9+\ln 2)) - 240\Big(48a_4 + 2\ln 2^4 + 63\zeta_3\Big)\Big)\epsilon}{1440} \\
&-\Big(48a_4 + 32a_5 + 21\zeta_3 + \frac{1}{720}\Big(-96\ln 2^4(-15+\ln 4) - 89\pi^4(-3+\ln 4) \\
&+5\pi^2\Big(36 + 16\ln 2^3 - 8\ln 2(27+\ln 512) + 381\zeta_3\Big)\Big) + \frac{403\zeta_5}{16}\Big)\epsilon^2 \\
&-\Bigg(-192a_5 - 128a_6 - \frac{13159\pi^6}{60480} - \frac{32}{3}a_4\Big(9+\pi^2\Big) + 52s_6 - \frac{4}{45}\ln 2^4 \\
&(45 + 2(-9+\ln 2)\ln 2) - \frac{1}{3}\pi^2\ln 2\Big(-3 + \ln 2^3 + \ln 2(-3+\ln 4)\Big) \\
&-\frac{1}{120}\pi^4(89 - 178\ln 2 + \ln 4\ln 8) \\
&+(-14 + \frac{1}{8}\pi^2(-127+5\ln 4))\zeta_3 - \frac{253\zeta_3^2}{4} - \frac{1209\zeta_5}{8}\Bigg)\epsilon^3 \\
&-\Big(64(a_4 + 8(a_5+a_6)) - 208s_6 + \frac{\pi^6(99321 - 64862\ln 2)}{30240} \\
&+\frac{8}{45}\ln 2^4(15 + 4(-6+\ln 2)\ln 2) + \frac{40}{3}\Big(24a_4 + \ln 2^4\Big)\zeta_3 \\
&+393\zeta_3^2 + \frac{1}{540}\pi^4(267 + 4\ln 2(-444 + (87 - 160\ln 2)\ln 2 + 360\ln 4) \\
&+2037\zeta_3) + 403\zeta_5 + \frac{1}{720}\pi^2(60(768a_4 + 512a_5 + 291\zeta_3) - 8\ln 2(4\ln 2 \\
&(15 + \ln 2(-10 + \ln 2(-125 + 14\ln 2))) + 75(-27+\ln 8192)\zeta_3) \\
&+28095\zeta_5)\Big)\epsilon^4 + \mathcal{O}(\epsilon^5).
\end{aligned} \tag{5.22}
$$

The last master integral is the one with all 6 massive lines. We have only been able to find two-fold Barnes-type representation, which leads to unbalanced hypergeometric double sums.

7 lines

There are two master integrals:



(5.23)

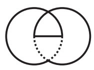
Let us start with the first one, the master integral with 5 massive lines. The one loop integrals eq. (5.2,5.3) will not be of any use here. However there exist hypergeometric representation of the two loop two-point function which we can use [17]:

$$\int\int \frac{d^D k_1 d^D k_2}{(k_1^2+m^2)(k_2^2)((k_1-q)^2+m^2)((k_2-q)^2)((k_1-k_2)^2+m^2)} =$$
$$\frac{\pi^d \Gamma(1+\epsilon)^2}{m^{4\epsilon+2}(1+\epsilon)(1-2\epsilon)}$$
$$\left(\frac{1}{(1+\epsilon)(1-\epsilon)}\, {}_4F_3(1,1+\epsilon,1+\epsilon,1+2\epsilon;\frac{3}{2}+\epsilon,2+\epsilon,2-\epsilon;-\frac{q^2}{4m^2})+\right.$$
$$\frac{(1+\epsilon)\Gamma(1-\epsilon)^2 m^{2\epsilon}}{2\epsilon\Gamma(1-2\epsilon)q^{2\epsilon}}\, {}_3F_2(1,1,1+\epsilon;\frac{3}{2},2;-\frac{q^2}{4m^2})$$
$$\left.-\frac{1}{2\epsilon}\, {}_3F_2(1,1+\epsilon,1+\epsilon;\frac{3}{2},2+\epsilon;-\frac{q^2}{4m^2})\right), \tag{5.24}$$

where $\epsilon = 2 - D/2$ but until we expand the result will be valid in all dimensions. In order to insert this into the expression for our master integral, we need to write the hypergeometric functions in terms of Barnes-type integrals, as:

$$
\begin{aligned}
{}_4F_3\left(\begin{array}{c} 1,1+\epsilon,1+\epsilon,1+2\epsilon \\ 2-\epsilon,\frac{3}{2}+\epsilon,2+\epsilon \end{array}\middle| -\frac{q^2}{4m^2}\right) &= \frac{\Gamma(\frac{3}{2}+\epsilon)\Gamma(2+\epsilon)\Gamma(2-\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)\Gamma(1+\epsilon)} \\
&\int_{+i\infty}^{-i\infty} dz \frac{\Gamma(z)\Gamma(1-z)\Gamma^2(1+\epsilon-z)\Gamma(1+2\epsilon-z)}{\Gamma(\frac{3}{2}+\epsilon-z)\Gamma(2-\epsilon-z)\Gamma(2+\epsilon-z)}\left(\frac{4m^2}{q^2}\right)^z \\
{}_3F_2\left(\begin{array}{c} 1,1,1+\epsilon \\ \frac{3}{2},2 \end{array}\middle| -\frac{q^2}{4m^2}\right) &= \frac{\Gamma(\frac{3}{2})\Gamma(2)}{\Gamma(1+\epsilon)} \\
&\int_{+i\infty}^{-i\infty} dz \frac{\Gamma(z)\Gamma^2(1-z)\Gamma(1+\epsilon-z)}{\Gamma(\frac{3}{2}-z)\Gamma(2--z)}\left(\frac{4m^2}{q^2}\right)^z \\
{}_3F_2\left(\begin{array}{c} 1,1+\epsilon,1+\epsilon \\ \frac{3}{2},2+\epsilon \end{array}\middle| -\frac{q^2}{4m^2}\right) &= \frac{\Gamma(\frac{3}{2})\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+\epsilon)} \\
&\int_{+i\infty}^{-i\infty} dz \frac{\Gamma(z)\Gamma(1-z)\Gamma^2(1+\epsilon-z)}{\Gamma(\frac{3}{2}-z)\Gamma(2+\epsilon-z)}\left(\frac{4m^2}{q^2}\right)^z .
\end{aligned} \tag{5.25}
$$

Note that we decided to take the countour to the left. We can insert the result in the our master integral and obtain again eq. (5.5). Closing the contour again on the left we get:



$$= \frac{\pi^{9-4\epsilon}\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(2+\epsilon)\Gamma(-1+2\epsilon)\Gamma(-1+3\epsilon)}{2^{-1+2\epsilon}\epsilon(-1+3\epsilon)(-1+\epsilon+2\epsilon^2)\Gamma(\frac{3}{2}-\epsilon)\Gamma(2-\epsilon)\Gamma(-\frac{1}{2}+2\epsilon)}$$

$$pFq\left(\left\{-\frac{1}{2}+\epsilon,-1+3\epsilon,-1+3\epsilon\right\},\left\{3\epsilon,-\frac{1}{2}+2\epsilon\right\},1\right)$$

$$+\frac{2^{1-4\epsilon}\pi^{9-4\epsilon}\Gamma(2-3\epsilon)\Gamma(1-\epsilon)^3\Gamma(\epsilon)\Gamma(1+\epsilon)\Gamma(-1+3\epsilon)\Gamma(-1+4\epsilon)}{\epsilon(-1+3\epsilon)\Gamma(2-2\epsilon)\Gamma(\frac{3}{2}-\epsilon)\Gamma(2-\epsilon)\Gamma(-\frac{1}{2}+3\epsilon)}$$

$$pFq\left(\left\{-\frac{1}{2}+\epsilon,-1+3\epsilon,-1+4\epsilon\right\},\left\{3\epsilon,-\frac{1}{2}+3\epsilon\right\},1\right)$$

$$+\frac{\pi^{8-4\epsilon}\Gamma(1-\epsilon)\Gamma(-1+\epsilon)\Gamma(\epsilon)^2\Gamma(2+\epsilon)}{2\epsilon^2(-1+\epsilon+2\epsilon^2)\Gamma(2-\epsilon)}$$

$$pFq\left(\left\{\frac{1}{2},1,2\epsilon,2\epsilon\right\},\left\{2-\epsilon,\frac{1}{2}+\epsilon,1+2\epsilon\right\},1\right)$$

$$+\frac{2^{-4\epsilon}\pi^{\frac{19}{2}-4\epsilon}Csc(2\epsilon\pi)\Gamma(1-\epsilon)^2\Gamma(-1+\epsilon)\Gamma(3\epsilon)\Gamma(1+\epsilon)}{\epsilon^2\Gamma(2-2\epsilon)\Gamma(2-\epsilon)\Gamma\left(\frac{1}{2}+2\epsilon\right)}$$

$$pFq\left(\left\{\frac{1}{2},1,2\epsilon,3\epsilon\right\},\left\{2-\epsilon,\frac{1}{2}+2\epsilon,1+2\epsilon\right\},1\right)$$

$$+\frac{2^{2-2\epsilon}\pi^{8-4\epsilon}\Gamma(1-\epsilon)\Gamma(-1+\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(3\epsilon)\Gamma(\frac{3}{2}+\epsilon)\Gamma(2+\epsilon)}{\epsilon(1-5\epsilon^2+4\epsilon^4)\Gamma(\frac{1}{2}+2\epsilon)\Gamma(1+2\epsilon)}$$

$$pFq\left(\left\{\frac{1}{2},2\epsilon,2\epsilon,3\epsilon\right\},\left\{2-\epsilon,\frac{1}{2}+2\epsilon,1+2\epsilon\right\},1\right)$$

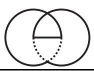
$$+\frac{2^{2-4\epsilon}\pi^{\frac{17}{2}-4\epsilon}\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)^2\Gamma(-1+2\epsilon)\Gamma(-2+3\epsilon)}{\epsilon\Gamma(2-\epsilon)\Gamma\left(-\frac{1}{2}+2\epsilon\right)}$$

$$pFq\left(\left\{1,1,\frac{3}{2}-2\epsilon,1+\epsilon\right\},\left\{\frac{3}{2},2,3-3\epsilon\right\},1\right)$$

$$+\frac{\pi^{9-4\epsilon}\Gamma(3-2\epsilon)\Gamma(-1+\epsilon)\Gamma(1+\epsilon)\Gamma(2+\epsilon)\Gamma(-2+2\epsilon)}{\epsilon(1+\epsilon)(-1+\epsilon+2\epsilon^2)\Gamma(\frac{3}{2}-\epsilon)\Gamma(2-\epsilon)\Gamma(-\frac{1}{2}+\epsilon)}$$

$$pFq\left(\left\{1,\frac{3}{2}-\epsilon,1+\epsilon,1+\epsilon\right\},\left\{\frac{3}{2},3-2\epsilon,2+\epsilon\right\},1\right)$$
$$+\frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(\frac{3}{2}+\epsilon)\Gamma(2+\epsilon)\Gamma(-1+2\epsilon)\Gamma(-1+3\epsilon)\Gamma(-1+4\epsilon)}{2^{-3+2\epsilon}\pi^{-\frac{17}{2}+4\epsilon}(-1+3\epsilon)(1-5\epsilon^2+4\epsilon^4)\Gamma(\frac{3}{2}-\epsilon)\Gamma(1+2\epsilon)\Gamma(-\frac{1}{2}+3\epsilon)}$$
$$pFq\left(\left\{-\frac{1}{2}+\epsilon,-1+3\epsilon,-1+3\epsilon,-1+4\epsilon\right\},\left\{\epsilon,3\epsilon,-\frac{1}{2}+3\epsilon\right\},1\right)$$
$$+\frac{2\pi^{9-4\epsilon}\Gamma(3-2\epsilon)\Gamma(-1+\epsilon)\Gamma(1+\epsilon)\Gamma(2+\epsilon)\Gamma(-2+2\epsilon)}{(1+\epsilon)(1-5\epsilon^2+4\epsilon^4)\Gamma(\frac{3}{2}-\epsilon)\Gamma(2-\epsilon)\Gamma(-\frac{1}{2}+\epsilon)}$$
$$pFq\left(\left\{1,\frac{3}{2}-\epsilon,1+\epsilon,1+\epsilon,1+2\epsilon\right\},\left\{3-2\epsilon,2-\epsilon,\frac{3}{2}+\epsilon,2+\epsilon\right\},1\right) \tag{5.26}$$

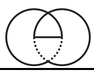
All hypergeometric functions have balanced half-integer valued coefficients and using Hypsummer we get:



$$\frac{\quad}{J^4} \overset{D=4-2\epsilon}{=} -\frac{1}{6}+\epsilon\Big(-\frac{5}{6}\Big)+\epsilon^2\Big(-\frac{11}{3}-\zeta_3\Big)$$
$$+\epsilon^3\Big(-\frac{44}{3}+\frac{2}{3}\zeta_3-\frac{3}{2}\zeta_4\Big)+\epsilon^4\Big(-\frac{166}{3}+53\,\zeta_5+\frac{31}{3}\,\zeta_3-\frac{24}{5}\,\zeta_2^2-3\,\zeta_4\Big)$$
$$+\epsilon^5\Big(-\frac{602}{3}+154\,\zeta_5+128a_4+\frac{38}{3}\,\zeta_3-128\,\zeta_3^2-\frac{584}{5}\,\zeta_2^2$$
$$+\frac{2732}{35}\,\zeta_2^3-32\ln_2^2\,\zeta_2+\frac{16}{3}\ln_2^4+\frac{159}{2}\,\zeta_4-78\,\zeta_4\,\zeta_2-425\,\zeta_6\Big)$$
$$+\epsilon^6\Big(-\frac{2122}{3}+1920s7b+1920s7a+\frac{27591}{2}\,\zeta_7+1280a_5$$
$$-353\,\zeta_5+128a_4-\frac{784}{3}\,\zeta_3+3360\,\zeta_3a_4-\frac{736}{3}\,\zeta_3^2-5640\,\zeta_2\,\zeta_5$$
$$-\frac{2048}{5}\,\zeta_2^2-\frac{7824}{5}\,\zeta_2^2\,\zeta_3-\frac{4808}{35}\,\zeta_2^3-1920\ln_2 s6+2400\ln_2\,\zeta_3^2$$
$$+\frac{4736}{5}\ln_2\,\zeta_2^2+336\ln_2\,\zeta_2^3-32\ln_2^2\,\zeta_2-840\ln_2^2\,\zeta_2\,\zeta_3$$
$$+\frac{320}{3}\ln_2^3\,\zeta_2+\frac{16}{3}\ln_2^4+140\ln_2^4\,\zeta_3-\frac{32}{3}\ln_2^5+543\,\zeta_4$$
$$-375\,\zeta_4\,\zeta_3+132\,\zeta_4\,\zeta_2-768\,\zeta_4\ln_2-250\,\zeta_6\Big)$$
$$+\epsilon^7\Big(-\frac{7322}{3}+\frac{2884}{5}s8a+3840s7b+3840s7a+28255\,\zeta_7+4224s6$$
$$+11776a_6-1280a_5+4802\,\zeta_5-3840a_4+22784a_4^2-\frac{8528}{3}\,\zeta_3$$

$$-39872\,\zeta_3 a_5 + 36903\,\zeta_3\,\zeta_5 + 6720\,\zeta_3 a_4 - \frac{7652}{3}\,\zeta_3^2 - 11280\,\zeta_2\,\zeta_5$$
$$+\frac{192}{5}\,\zeta_2^2 - \frac{107776}{5}\,\zeta_2^2 a_4 - 3168\,\zeta_2^2\,\zeta_3 - \frac{133628}{35}\,\zeta_2^3 + \frac{6193972}{875}\,\zeta_2^4$$
$$-3840\ln_2 s6 + 4800\ln_2\,\zeta_3^2 + \frac{9088}{5}\ln_2\,\zeta_2^2 - \frac{94304}{5}\ln_2\,\zeta_2^2\,\zeta_3$$
$$+672\ln_2\,\zeta_2^3 + 960\ln_2^2\,\zeta_2 - 11392\ln_2^2\,\zeta_2 a_4 - 1680\ln_2^2\,\zeta_2\,\zeta_3$$
$$-\frac{20864}{5}\ln_2^2\,\zeta_2^2 + \frac{26944}{5}\ln_2^2\,\zeta_2^3 - \frac{320}{3}\ln_2^3\,\zeta_2 - \frac{9968}{3}\ln_2^3\,\zeta_2\,\zeta_3$$
$$-160\ln_2^4 + \frac{5696}{3}\ln_2^4 a_4 + 280\ln_2^4\,\zeta_3 - \frac{736}{3}\ln_2^4\,\zeta_2 + \frac{7888}{15}\ln_2^4\,\zeta_2^2$$
$$+\frac{32}{3}\ln_2^5 + \frac{4984}{15}\ln_2^5\,\zeta_3 + \frac{736}{45}\ln_2^6 - \frac{1424}{3}\ln_2^6\,\zeta_2 + \frac{356}{9}\ln_2^8 + 2544\,\zeta_4$$
$$+6432\,\zeta_4 a_4 - 750\,\zeta_4\,\zeta_3 + 1542\,\zeta_4\,\zeta_2 - \frac{17388}{5}\,\zeta_4\,\zeta_2^2 - 3984\,\zeta_4\ln_2$$
$$+5628\,\zeta_4\ln_2\,\zeta_3 + 3216\,\zeta_4\ln_2^2 - 1608\,\zeta_4\ln_2^2\,\zeta_2 + 268\,\zeta_4\ln_2^4$$
$$+\frac{8361}{4}\,\zeta_4^2 + 2605\,\zeta_6 - 3600\,\zeta_6\,\zeta_2 - \frac{70903}{4}\,\zeta_8\Big) + \mathcal{O}(\epsilon^8) \tag{5.27}$$

For expansion in three dimesion we have [5]:



$$\frac{}{J^4} = \frac{1}{4}\,\zeta_2 - \frac{1}{2}\ln^2 2 + \epsilon(-4\,\zeta_3 - \frac{5}{2}\,\zeta_2 + \frac{9}{2}\ln 2\,\zeta_2 + 5\ln^2 2 + \ln^3 2)$$
$$+\epsilon^2(30\,a_4 + 40\,\zeta_3 + 13\,\zeta_2 - \frac{1}{4}\,\zeta_2^2 - \frac{21}{4}\ln 2\,\zeta_3 - 45\ln 2\,\zeta_2 - 26\ln^2 2$$
$$-\frac{23}{2}\ln^2 2\,\zeta_2 - 10\ln^3 2 + \frac{1}{12}\ln^4 2) + \epsilon^3(-28\,a_5 - \frac{2103}{16}\,\zeta_5 - 300\,a_4$$
$$-208\,\zeta_3 - 54\,\zeta_2 - 13\,\zeta_2\,\zeta_3 + \frac{5}{2}\,\zeta_2^2 + 28\ln 2\,a_4 + \frac{105}{2}\ln 2\,\zeta_3 + 234\ln 2\,\zeta_2$$
$$+\frac{361}{5}\ln 2\,\zeta_2^2 + 108\ln^2 2 + \frac{213}{4}\ln^2 2\,\zeta_3 + 115\ln^2 2\,\zeta_2 + 52\ln^3 2$$
$$-\frac{14}{3}\ln^3 2\,\zeta_2 - \frac{5}{6}\ln^4 2 + \frac{12}{5}\ln^5 2) + \epsilon^4(-\frac{9}{8} - 278\,s6 + 24a_6 + 280a_5$$
$$+\frac{12531}{8}\,\zeta_5 + 1560a_4 + 1632\,\zeta_3 + \frac{4325}{16}\,\zeta_3^2 - 552\,\zeta_2 - 12\,\zeta_2 a_4 + 604\,\zeta_2\,\zeta_3$$
$$-1063\,\zeta_2^2 - \frac{34901}{210}\,\zeta_2^3 - \frac{3319}{8}\ln 2 - 184\ln 2a_5 + \frac{1755}{8}\ln 2\,\zeta_5$$
$$-280\ln 2a_4 - 393\ln 2\,\zeta_3 + 316\ln 2\,\zeta_2 - \frac{317}{2}\ln 2\,\zeta_2\,\zeta_3 + \frac{2324}{5}\ln 2\,\zeta_2^2$$
$$-432\ln^2 2 - 92\ln^2 2a_4 - \frac{1065}{2}\ln^2 2\,\zeta_3 - 1150\ln^2 2\,\zeta_2 - 175\ln^2 2\,\zeta_2^2$$

$$-216\ln^3 2 - \frac{199}{2}\ln^3 2\,\zeta_3 + \frac{172}{3}\ln^3 2\,\zeta_2 - 33\ln^3 2\,\zeta_2\,\zeta_3 + 16\ln^3 2\,\zeta_2^2$$
$$+\frac{13}{3}\ln^4 2 + \frac{73}{3}\ln^4 2\,\zeta_2 - 24\ln^4 2\,\zeta_2^2 - 24\ln^5 2 + 8\ln^5 2\,\zeta_2 - \frac{133}{45}\ln^6 2$$
$$-\frac{16}{3}\ln^6 2\,\zeta_2) + \mathcal{O}(\epsilon^5) \tag{5.28}$$

For the second master integral we were not been able to find the appropriate hypergeometric representation that would enable us to use Hypsummer. The second master is governed by a difference equation of first order [65] for which the master integral in eq. (5.13) is needed for all λ. Unfortunately, the hypergeometric representation we have found in eq. (5.13) is not balanced for a general λ, it is balanced if one sets λ to any fixed integer value. Therefore the formal solution of the difference equation for the second master integral in eq. (5.23) gives hypergeometric unbalanced double sum, which we cannot expand.

8 lines

There is only one master integral for EQCD with 8 lines:

$$Wa[1] = \frac{\text{(diagram)}}{J^4} \tag{5.29}$$

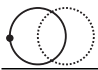
It obeys the difference equation of first order given by [65]:

$$Wa[1+x] = \frac{(3-D)(80+9(-6+D)D)(5-2D+X)^2}{4(-3+D-X)(-6+2D-X)(-5+2D-X)(-11+3D-X)}$$
$$\times\frac{(6-2D+X)^2(3D-2(4+X))Ga3(X)}{(-10+3D-X)(-9+3D-X)(-8+3D-X)}$$
$$+\frac{(-3+D)(-10+3D)(6-2D+X)^2 VBc8(X)}{2(-6+2D-X)(-11+3D-X)(-10+3D-X)}$$
$$+\frac{(-8+3D)(-4+2D-X)(5-2D+X)^2(6-2D+X)^2}{8(-3+D-X)(-6+2D-X)(-5+2D-X)(-11+3D-X)}$$
$$\times\frac{(3D-2(3+X))(5D-2(8+X))(3D-2(4+X))BBa(X)}{(-10+3D-X)(-9+3D-X)(-8+3D-X)(-7+3D-X)}$$
$$-\frac{(-8+3D)(D-2X)(-13+4D-X)(D-2(1+X))Ba(1)J(X)}{16(-3+D-X)(-11+3D-X)(-10+3D-X)}$$
$$+\frac{(-3+D)(-2+D)(3D-2(4+X))Vc3(X)J(1)}{4(-6+2D-X)(-11+3D-X)}$$
$$-\Big[(-2+D)(-2+D-X)\Big[(-3+D)^2(664+D(-448+75D))$$
$$-5(-3+D)(212+D(-143+24D))X + (626+D(-422+71D))X^2$$

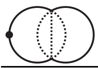
$$-(-56+19D)X^3+2X^4\Big](3D-2(3+X))(3D-2(4+X))Ba(X)J(1)\Big]$$
$$/\Big[8(-3+D-X)(-6+2D-X)(-5+2D-X)(-11+3D-X)\times$$
$$\times(-10+3D-X)(-9+3D-X)(-8+3D-X)\Big]$$
$$-\frac{(-2+D)^2(D-2X)(9-3D+X)^2(D-2(1+X))J(X)J(1)^3}{16(-3+D)(-3+D-X)(-6+2D-X)(-11+3D-X)(-9+3D-X)}$$
$$-\frac{(-3+D)(5-2D+X)(6-2D+X)\big(4-\frac{3D}{2}+X\big)Ta(X)}{(9-3D+X)(10-3D+X)(11-3D+X)}+$$
$$\frac{\big(5-\frac{3D}{2}+X\big)(3-D+X)Wa(X)}{X(11-3D+X)} \tag{5.30}$$

where some of the integrals on the RHS can be represented using gamma functions, namely:

$$Ta(x)=\frac{\text{[diagram]}}{J^4}=\frac{\Gamma(\frac{3d}{2})\Gamma(1-\frac{3d}{2})\Gamma(\frac{d}{2})^3\Gamma(5-2d+x)\Gamma(4-\frac{3d}{2}+x)}{\Gamma(2-\frac{d}{2})^2\Gamma(d-2)^2\Gamma(x)\Gamma(9-3d+x)} \tag{5.31}$$

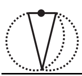


$$Ba(x)=\frac{\text{[diagram]}}{J^3}=\frac{\Gamma(3-d)\Gamma(\frac{d}{2})\Gamma(3-\frac{3d}{2}+x)\Gamma(2-d+x)}{\Gamma(1-\frac{d}{2})\Gamma(2-\frac{d}{2})\Gamma(x)\Gamma(5-2d+x)} \tag{5.32}$$

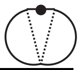


$$BBa(x)=\frac{\text{[diagram]}}{J^4}=-\frac{3(d-2)\Gamma(8-3d)\Gamma(\frac{5}{2}-\frac{d}{2})\Gamma(\frac{d}{2})^2\Gamma(4-2d+x)}{4^{3-d}\Gamma(\frac{11}{2}-\frac{3d}{2})\Gamma(2-\frac{d}{2})^4\Gamma(x)\Gamma(7-3d+x)}$$
$$\times\Gamma(3-\frac{3d}{2}+x). \tag{5.33}$$

The other have first order inhomogeneous difference equations which can be solved in terms of hypergeometric sums:



$$Ga(x)=\frac{\text{[diagram]}}{J^4}==\frac{(6-2D)_{-1+x}(5-\frac{3D}{2})_{-1+x}}{(9-3D)_{-1+x}(4-d)_{-1+x}}Ga(1)+$$
$$+\sum_{j=1}^{x-1}\frac{(6-2D+j)_{x-1-j}(5-\frac{3D}{2}+j)_{x-1-j}}{(9-3D+j)_{x-1-j}(4-D+j)_{x-1-j}}\times$$
$$\frac{2(-2+D)J(1)\Gamma(2-D)\Gamma(\frac{D}{2})\Gamma(5-\frac{3D}{2}+j)\Gamma(4-D+j)}{(3-D+j)^2(8-3D-j)\Gamma(1-\frac{D}{2})^2\Gamma(1+j\Gamma(6-2D+j)} \tag{5.34}$$



$$Vc3(x) = \frac{\bullet}{J^3} = -\frac{(D-2)\Gamma(4-\frac{3D}{2}+x)\Gamma(2-\frac{D}{2}+x)}{(D-3-x)\Gamma(1-\frac{D}{2})\Gamma(1+x)\Gamma(5-\frac{D}{2}+x)}$$
$$\times {}_4F_3\left(\begin{array}{c} 1, 6-2D+x, 3-D+x, 2-\frac{D}{2}+x \\ 1+x, 5-\frac{3D}{2}+x, 4-D+x \end{array} \middle| 1\right) \tag{5.35}$$

$$VBc(x) = \frac{\bullet}{J^4} = \frac{(7-2D)_{-1+x}}{(11-3D)_{-1+x}} VBc(1) +$$
$$+ \sum_{j=1}^{x-1} \frac{\Gamma(6-2D+x)\Gamma(11-3D+j)}{\Gamma(7-2D+j)\Gamma(10-3D+x)} \left[\frac{(-8+3D)(-6+2D-j)}{4(-3+D-j)(-10+3D-j)} \times \right.$$
$$\frac{(-5+2D-j)(-4+2D-j)(3D-2(3+j))(3D-2(4+j))BBa(j)}{(-9+3D-j)(-8+3D-j)(-7+3D-j)j} +$$
$$\left. + \frac{(-8+3D)(D-2j)(D-2(1+j))Ba(1)J(j)}{8(-3+D-j)(-10+3D-j)j} \right] \tag{5.36}$$

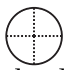
where

$$Ga(1) = \frac{2D-5}{2(D-3)} BBa(1) - \frac{D-2}{2(D-3)} Ba(1) \tag{5.37}$$

and

$$\begin{aligned} VBc(1) &= \frac{2}{3D-10} \Big[(D-3)Ta(1) \\ &\quad - \frac{3D-8}{2(D-3)} \Big(\frac{2D-5}{3} BBa(1) - \frac{D-2}{4} Ba(1) \Big) \Big] \end{aligned} \tag{5.38}$$

are the initial values taken from [64].

The eq. (5.30) is a first order difference equation like the one in eq. (3.41), and given the initial value, it is formally solvable, where the formal solution is given in eq. (3.42). The problem is that our master integral  is in the difference equation language $Wa(1)$, in other words it is the initial value. The way we are going to solve the difference equation is by using the boundary condition at infinity, following the procedure in [4]. We will take formal solution eq. (3.42) and write it formally as equation for the initial value:

$$Wa[x_0] = Wa[x] \left[\prod_{i=x_0}^{x-1} \frac{1}{a(i)} \right] - \sum_{j=x_0}^{x-1} G(j) \left[\prod_{i=x_0}^{j} \frac{1}{a(i)} \right] \tag{5.39}$$

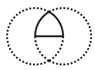
where

$$\prod_{i=x_0}^{x-1} \frac{1}{a(i)} = \frac{\Gamma(x)\Gamma(11-3D+x)\Gamma(3-D+x_0)\Gamma(5-\frac{3D}{2}+x_0)}{\Gamma(5-\frac{3D}{2}+x)\Gamma(3-D+x)\Gamma(11-3D+x_0)\Gamma(x_0)} \tag{5.40}$$

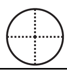
and similarly

$$\prod_{i=x_0}^{j} \frac{1}{a(i)} = \frac{\Gamma(1+j)\Gamma(12-3D+j)\Gamma(3-D+x_0)\Gamma(5-\frac{3D}{2}+x_0)}{\Gamma(6-\frac{3D}{2}+j)\Gamma(4-D+j)\Gamma(11-3D+x_0)\Gamma(x_0)}. \tag{5.41}$$

The LHS of eq. (5.39) does not depend on x, therefore the the RHS cannot depend on x as well. We are free to take the formal limit $x \to \infty$ and use the fact that $Wa(x)$ in this limit reduces to:

$$Wa(x) \overset{x\to\infty}{=} J(x\to\infty) \times$$  . (5.42)

Using Stirling's formula $\Gamma(x+a)/\Gamma(x+b) = x^{a-b}(1+\mathcal{O}(x^{-1}))$ the first term on RHS of eq. (5.39) goes like x^{3-D} for $x >> 1$. This is one in 3 dimensions and it vanishes in 4 dimensions.
Since some of the terms of $G(j)$ are given in terms of generalized hypergeometric functions eq. (5.34, 5.35, 5.36), the result of $Wa(1)$ will be given in terms of at most generalized first Appell functions. In 3 dimensions the sums are unbalanced and hence cannot be expanded in ϵ.
The first coefficient is [64]:

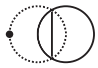
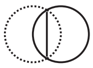


$$\frac{\quad}{J^4} = 5\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4). \tag{5.43}$$

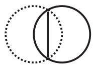
The reason why we could not compute any coefficients is that the expansion starts giving contributions at $\mathcal{O}(\epsilon^{-2})$ for most of the summands, which of course at the end cancel. However we need also to expand the solution for $Wa(1)$ to $\mathcal{O}(\epsilon^7)$ in order to get the $\mathcal{O}(\epsilon^3)$ right. That means computing very large expression from $\mathcal{O}(\epsilon^{-2} - \epsilon^7)$ and here we come into problems with time. Further improvements on the algorithmic implementations should be made in order to speed the computation up. This is however only a technical problem, not a conceptual one.

5.1.2 Additional master integrals

In this section we will try to apply the same methods as in previous chapter on some of the four loops master integrals needed for the QCD corrections to electroweak ρ-parameter [63, 19, 13]. The master integrals have been computed to high precision using Laporta algorithm and/or so called Pade approximation in [19, 13] For the simplest case of master integrals with 5 lines we have two cases:

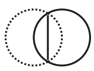
  (5.44)

where the dot in this case means that the propagator is risen to power two. Using similar technique like in th case of eq. (5.11) we obtain following hypergeometric representations:



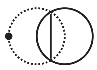
$$= -\frac{\pi^{2D}\Gamma\left[4-\frac{3D}{2}\right]\Gamma[3-D]\Gamma\left[\frac{1}{2}(-2+D)\right]^2}{\Gamma[8-3D]\Gamma[6-2D]\Gamma\left[\frac{D}{2}\right]}$$
$$\Bigg(\Gamma[5-2D]\Gamma[6-2D]\Gamma\left[4-\frac{3D}{2}\right]\Gamma\left[\frac{D}{2}-1\right]$$
$$\times {}_3F_2\left(\begin{matrix}5-2D, 4-\frac{3D}{2}, 3-D\\ \frac{9}{2}-\frac{3D}{2}, 2-\frac{D}{2}\end{matrix}\middle|\frac{1}{4}\right) + \Gamma(8-3D)\Gamma(3-D)\Gamma\left(1-\frac{D}{2}\right)$$
$$\times\Gamma\left(2-\frac{D}{2}\right){}_3F_2\left(\begin{matrix}4-\frac{3D}{2}, 3-D, 2-\frac{D}{2}\\ \frac{7}{2}-D, \frac{D}{2}\end{matrix}\middle|\frac{1}{4}\right)\Bigg) \quad (5.45)$$

Both have unbalanced half-integer coefficients in 4 dimensions and cannot be expanded in ϵ using Hypsummer. However, unlike the other unbalanced function, these functions are "only" ${}_3F_2$ functions and can be expanded using HypExp2 [37]. The first few terms of the expansion are [1]:



$$= \frac{1}{4\epsilon^4} + \frac{1}{\epsilon^3} + \frac{97+4\pi^2}{48\epsilon^2} + \frac{833+96\pi^2-96\zeta_3}{288\epsilon} + \mathcal{O}(\epsilon^0) \quad (5.46)$$

The same master integral with one massles line risen to power two is given as:

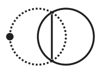


$$= \frac{\Gamma(6-2D)\Gamma^2(5-\frac{3D}{2})\Gamma(4-D)\Gamma(-2+\frac{D}{2})\Gamma^2(-1+\frac{D}{2})}{\pi^{-2D}\Gamma(10-3D)\Gamma(\frac{D}{2})}$$
$$\times {}_3F_2\left(\begin{matrix}6-2D, 5-\frac{3D}{2}, 4-D\\ \frac{11}{2}-\frac{3D}{2}, 2-\frac{D}{2}\end{matrix}\middle|\frac{1}{4}\right)$$

[1] after we multiply the result with $\left((e^{\epsilon\gamma})/(i\pi^{D/2})\right)^4$ in order to compare the results with numerics in [13]

$$+\frac{\Gamma(5-\frac{3D}{2})\Gamma(4-D)^2\Gamma(1-\frac{D}{2})\Gamma(3-\frac{D}{2})\Gamma(\frac{D}{2}-2)\Gamma(\frac{3D}{2}-1)}{\pi^{-2D}\Gamma(8-2D)\Gamma(\frac{D}{2})}$$
$$\times{}_3F_2\left(\begin{array}{c} 5-\frac{3D}{2},4-D,3-\frac{D}{2} \\ \frac{9}{2}-D,\frac{D}{2} \end{array}\middle|\,\frac{1}{4}\right) \tag{5.47}$$

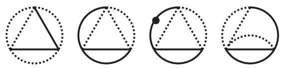
and the expansion goes as:



$$= \quad -\frac{1}{4\epsilon^4}-\frac{9}{8\epsilon^3}+\frac{-30-\pi^2}{12\epsilon^2}+\mathcal{O}(\epsilon^{-1}), \tag{5.48}$$

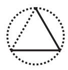
where we multiplied the expansion result with $\Big((e^{\epsilon\gamma})/(i\pi^{D/2})\Big)^4$ to match values in [13].

There are four more master integrals we were able to compute:

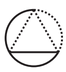


(5.49)

There are with six line and can be represented in terms of hypergeometric functions along the line similar to eq. (5.16). They can be written as:



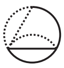
$$= \frac{\pi Csc[\frac{D\pi}{2}]\Gamma[6-2D]\Gamma[-1+\frac{D}{2}]^4\Gamma[-3+D]}{\Gamma[-2+D]\Gamma[\frac{D}{2}]\Gamma[-3+\frac{3D}{2}]}$$
$$pFq\left[\{6-2D,3-D,-2+D\},\left\{4-D,2-\frac{D}{2}\right\},1\right]$$
$$-\frac{(-8+3D)\pi^2 Csc[\frac{D\pi}{2}]Csc[\frac{3D\pi}{2}]\Gamma[2-\frac{D}{2}]\Gamma[-1+\frac{D}{2}]^4}{(-4+D)\Gamma[-2+D]\Gamma[\frac{D}{2}]^2\Gamma[-3+\frac{3D}{2}]}$$
$$pFq\left[\left\{5-\frac{3D}{2},2-\frac{D}{2},-3+\frac{3D}{2}\right\},\left\{3-\frac{D}{2},\frac{D}{2}\right\},1\right] \tag{5.50}$$

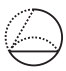


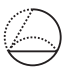
$$= \frac{2^{3-D}\pi^{\frac{1}{2}+D}\Gamma[6-2D]\Gamma[5-\frac{3D}{2}]\Gamma[2-\frac{D}{2}]^2\Gamma[-1+\frac{D}{2}]^4\Gamma[-4+\frac{3D}{2}]}{\Gamma[4-D]\Gamma[-\frac{1}{2}+\frac{D}{2}]\Gamma[-2+D]\Gamma[\frac{D}{2}]}$$
$$pFq\left[\left\{6-2D,\frac{3}{2}-\frac{D}{2}\right\},\{4-D\},4\right]$$
$$+\frac{2\pi^D\Gamma[5-\frac{3D}{2}]\Gamma[4-D]\Gamma[1-\frac{D}{2}]\Gamma[2-\frac{D}{2}]\Gamma[-1+\frac{D}{2}]^3\Gamma[-3+D]}{\Gamma[3-\frac{D}{2}]\Gamma[-2+D]\Gamma[\frac{D}{2}]}$$
$$pFq\left[\left\{\frac{1}{2},1,5-\frac{3D}{2}\right\},\left\{3-\frac{D}{2},\frac{D}{2}\right\},4\right]$$

$$+\frac{2^{-5+2D}\pi^{\frac{1}{2}+D}\Gamma[4-\frac{3D}{2}]\Gamma[3-D]\Gamma[2-\frac{D}{2}]^2\Gamma[-1+\frac{D}{2}]^3}{\Gamma[\frac{7}{2}-D]\Gamma[\frac{D}{2}]^2}$$
$$pFq\left[\left\{1,2-\frac{D}{2},-\frac{5}{2}+D\right\},\left\{\frac{D}{2},-3+\frac{3D}{2}\right\},4\right] \tag{5.51}$$

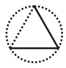

$$= \frac{\pi^{\frac{1}{2}+D}\Gamma[7-2D]\Gamma[6-\frac{3D}{2}]\Gamma[2-\frac{D}{2}]\Gamma[3-\frac{D}{2}]\Gamma[-2+\frac{D}{2}]}{2^{10-3D+2(-7+2D)}\Gamma[5-D]\Gamma[-\frac{3}{2}+\frac{D}{2}]\Gamma[-2+D]\Gamma[\frac{D}{2}]}$$
$$\Gamma[-1+\frac{D}{2}]^3\Gamma[-5+\frac{3D}{2}]pFq\left[\left\{7-2D,\frac{5}{2}-\frac{D}{2}\right\},\{5-D\},4\right]$$
$$+\frac{2^{-10+3D-2(-5+\frac{3D}{2})}\pi^{D}\Gamma[5-\frac{3D}{2}]\Gamma[4-D]\Gamma[2-\frac{D}{2}]^2\Gamma[-1+\frac{D}{2}]^3\Gamma[-3+D]}{\Gamma[3-\frac{D}{2}]\Gamma[-2+D]\Gamma[\frac{D}{2}]}$$
$$pFq\left[\left\{\frac{1}{2},1,5-\frac{3D}{2}\right\},\left\{3-\frac{D}{2},-1+\frac{D}{2}\right\},4\right]$$
$$+\frac{2^{-10-2(-2+\frac{D}{2})+3D}\pi^{\frac{1}{2}+D}\Gamma[5-\frac{3D}{2}]\Gamma[3-D]\Gamma[2-\frac{D}{2}]^2\Gamma[-1+\frac{D}{2}]^3}{\Gamma[\frac{7}{2}-D]\Gamma[\frac{D}{2}]^2}$$
$$pFq\left[\left\{1,2-\frac{D}{2},-\frac{5}{2}+D\right\},\left\{\frac{D}{2},-4+\frac{3D}{2}\right\},4\right] \tag{5.52}$$

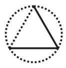

$$= -\frac{2\Gamma[6-2D]\Gamma\left[5-\frac{3D}{2}\right]\Gamma[3-D]\Gamma\left[2-\frac{D}{2}\right]\Gamma\left[-1+\frac{D}{2}\right]^4}{(-4+D)\Gamma\left[8-\frac{5D}{2}\right]\Gamma[-2+D]\Gamma\left[\frac{D}{2}\right]}$$
$$pFq\left[\left\{6-2D,5-\frac{3D}{2},2-\frac{D}{2}\right\},\left\{8-\frac{5D}{2},3-\frac{D}{2}\right\},1\right] \tag{5.53}$$

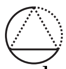
The expansion of  can be done with Hypsummer

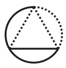

$$= \frac{5}{24\epsilon^4}+\frac{55}{48\epsilon^3}+\frac{993+44\pi^2}{288\epsilon^2}+\frac{2931+484\pi^2+1088\,\zeta_3}{576\epsilon}$$
$$+\frac{-82395+13220\pi^2-528\pi^4+59840\,\zeta_3}{5760}$$

$$+\Big(-\frac{42485}{256}-\frac{121\pi^4}{240}+\frac{1327\,\zeta_3}{36}+\pi^2\Big(\frac{335}{576}+\frac{77\,\zeta_3}{27}\Big)+\frac{172\,\zeta_5}{3}\Big)\epsilon+ \\ +\mathcal{O}(\epsilon^2) \quad (5.54)$$

as well as the expansion of 


$$= \frac{1}{12\epsilon^4}+\frac{5}{12\epsilon^3}+\frac{21+\pi^2}{36\epsilon^2}+\frac{-201-\pi^2+74\,\zeta_3}{36\epsilon} \\ +\frac{1}{360}\Big(-21150-650\pi^2-63\pi^4+5860\,\zeta_3\Big)+\frac{1}{1080} \\ (65268\,\zeta_5-404190-1533\pi^4+93300\,\zeta_3+10\pi^2(-1803+362\,\zeta_3))\epsilon \\ +\mathcal{O}(\epsilon^2) \quad (5.55)$$

The  and  however contain unbalanced half-integer coefficient, but they can be done using HypExp2 [37], which gives:


$$= \frac{1}{3\epsilon^4}+\frac{23}{12\epsilon^3}+\left(\frac{65}{12}+\frac{\pi^2}{9}\right)\frac{1}{\epsilon^2}+\mathcal{O}(\epsilon^{-1}) \quad (5.56)$$

and


$$= \frac{1}{4\epsilon^4}+\frac{11}{12\epsilon^3}+\frac{13+\pi^2}{12\epsilon^2}+\mathcal{O}(\epsilon^{-1}), \quad (5.57)$$

where the normalization in all four expansions has been taken such that the numerics can be directly compared with [13]. Further coefficients can be computed without problems, it is just a mather of CPU time.

This concludes this chapter dealing with scalar integrals at temperature $T=0$. In the next chapter we will speculate on possible application of similar methods in finite temperature field theory.

Chapter 6

Heating things up

In this chapter we want to see what happens if we naively apply the methods we used so far to $T \neq 0$ QFT and whether it may potentially be of any use. The main difference to the zero temperature case, is - at the level of Feynman integrals - that we have instead of four dimensional scalar integrals the so called scalar sum-integrals, which we define as:

$$\sum\!\!\!\!\!\!\int_k \equiv T \sum_{k_0} \int_k \equiv T \sum_{k_0} \int \frac{d^{3-3\epsilon}k}{(2\pi)^{3-2\epsilon}} \tag{6.1}$$

where $k_0 = 2\pi n T$, $n \in \mathcal{Z}$ are the bosonic Matsubara momenta. Let us look at the simplest example of a scalar massive tadpole integral, with general power of propagator:

$$\sum\!\!\!\!\!\!\int_k \frac{1}{(P^2+m^2)^{\lambda_1}} = \mathcal{I}(n=0,m^2) + 2T \sum_{n=1}^{\infty} \int_p \frac{1}{(p^2+m^2+(2\pi n T)^2)^{\lambda}} \tag{6.2}$$

where the $\mathcal{I}(n=0,m^2)$ contains zero-mode and we will not look at it, it can be given in terms of gamma functions. Since we have only one propagator in the rest of the expression, we start with MB transformations obtaining:

$$\begin{aligned} & 2T \sum_{n=1}^{\infty} \int_p \int dz \frac{\Gamma(-z)\Gamma(z+\lambda)}{\Gamma(\lambda) 2\pi i} \frac{(2\pi n T)^{2z}}{(p^2+m^2)^{\lambda+z}} \\ &= 2T \int dz \frac{\Gamma(-z)(4\pi^2 T^2)^z \zeta(-2z) \Gamma(\lambda+z-3/2+\epsilon)}{(4\pi)^{3/2-\epsilon}(m^2)^{\lambda+z-3/2+\epsilon}\Gamma(\lambda)2\pi i}. \end{aligned} \tag{6.3}$$

Closing the contour to the right and picking the poles of $\Gamma(\lambda+z-3/2+\epsilon)$ gives us:

$$\frac{T(2^2\pi^2T^2)^{\frac{3}{2}-\epsilon-\lambda}}{2^{2-2\epsilon}\pi^{\frac{3}{2}-\epsilon}\Gamma(\lambda)}\sum_{j=0}^{\infty}\frac{\Gamma(\lambda+\epsilon-\frac{3}{2}+j)\zeta(-\frac{3}{2}\epsilon+2\lambda+2j)}{\Gamma(1+j)}\left(\frac{-m^2}{4\pi^2T^2}\right)^j \tag{6.4}$$

or

$$\frac{2^{1-2\lambda}\pi^{\frac{3}{2}-\epsilon-2\lambda}T^{4-2\epsilon-2\lambda}}{\Gamma(\lambda)}\Bigg\{\Gamma(\lambda+\epsilon-\frac{3}{2})\zeta(-3+2\epsilon+2\lambda)+$$
$$\sum_{j=1}^{\infty}\frac{\Gamma(\lambda+\epsilon-\frac{3}{2}+j)\zeta(-\frac{3}{2}\epsilon+2\lambda+2j)}{\Gamma(1+j)}\left(\frac{-m^2}{4\pi^2T^2}\right)^j\Bigg\}. \tag{6.5}$$

The first part is $m=0$ limit and the rest is expansion in $\frac{m}{T}$.
Let us now look at the next more complicated example, a massless self-energy one-loop integral:

$$\begin{aligned}\Pi(P) &= \sum\!\!\!\!\!\!\int_k \frac{1}{(Q^2)^{\lambda_1}((Q-P)^2)^{\lambda_2}}\\ &= T\sum_{q_0}\int_q \frac{1}{(q^2+q_0^2)^{\lambda_1}((q-p)^2+q_0^2-2p_0q_0+p_0^2)^{\lambda_2}}.\end{aligned} \tag{6.6}$$

For the sake of simplicity, we do not consider Matsubara zero-mode, since it basically coresponds to eq. (2.19), which can be written in terms of Gauss function. We now can use Feynman parameterisation and Mellin-Barnes transformation to write the self-energy integral as contour integral. Since we know the result of the integral without p_0, q_0 in eq. (2.21), we will skip Feynman parameterisation and use MB transform right away:

$$\begin{aligned}\Pi'(P) &= T\sum_{q_0}{}'\int_q\int dz_1\ldots dz_4\frac{\Gamma(-z_1)\ldots\Gamma(-z_4)\Gamma(\lambda_1+z_1)\Gamma(\lambda_2+z_{234})}{\Gamma(\lambda_1)\Gamma(\lambda_2)(2\pi i)^4}\\ &\quad\times\frac{(q_0^2)^{z_1}}{(q^2)^{\lambda_1+z_1}}\frac{(q_0^2)^{z_2}(-2p_0q_0)^{z_3}(p_0^2)^{z_4}}{((q-p)^2)^{\lambda_1+z_{234}}}\end{aligned} \tag{6.7}$$

where we use the notation $z_{123\ldots}=z_1+z_2+\ldots$. Now we can perform momentum integration using eq. (2.21) and also the sum over Matsubara momenta by splitting the summation to two parts, $-\infty$ to -1 and 1 to ∞, leading to:

$$\Pi(P) = T\int dz_1\ldots dz_4\frac{\Gamma(-z_1)\ldots\Gamma(-z_4)}{\Gamma(\lambda_1)\Gamma(\lambda_2)(2\pi i)^4\Gamma(D-\lambda_{12}-z_{1234})}$$

$$
\begin{aligned}
&\times\ (p_0)^{2z_4+z_3}(-2)^{z_3}\pi^{D/2}(p^2)^{D/2-\lambda_{12}-z_{1234}} \\
&\times\ \Gamma(-D/2+\lambda_{12}+z_{1234})\Gamma(D/2-\lambda_1-z_1)\Gamma(D/2-\lambda_2-z_{234}) \\
&\times\ \left(1+(-1)^{2z_{12}+z_3}\right)\zeta(-2z_{12}-z_3) \qquad (6.8)
\end{aligned}
$$

This integral is to complicated to evaluate analytically. However, the general structure of sum-integrals seems to be like this:

$$
\frac{1}{(2\pi i)^n}\int\prod_{l=1}^{n}dz_l\frac{\prod_i\Gamma(a_i+\sum_j c_{ij}z_j)\zeta(a_i''+\sum_j c_{ij}''z_j)}{\prod_i\Gamma(a_i'+\sum_j c_{ij}'z_j)}\prod_k x_k^{d_k} \qquad (6.9)
$$

This is, apart from ζ-functions, the same structure as in eq. (2.31). Since the integration is over the complex plane, and arguments of the ζ-function are complex, the $\zeta(s)$-function is meromorphic function for Re$(s) > 1$ and it has unique analytical continuation to entire complex plane, excluding the point $s = 1$, where it has a simple pole. Therefore it can be assumed that the strategies for resolving the singularities of MB representation and performing numerical integration can be applied here too. Since Matsubara frequencies lead to great proliferation of MB integrals, the question is how fast the numerical convergence will be.

Introducing finite chemical potential generates, since it is equivalent to a shift of Matsubara frequencies by a constant imaginary term, generalized zeta functions, also known as Hurwitz zeta functions. Since it has the same analytical properties as zeta function, adding finite chemical potential should pose no additional problems.

Chapter 7

Epilogue

We have seen that the most important class of Feynman integrals are scalar integrals, since all other cases are reducible to these. Furthermore, one can reduce the set of integrals for one particular problem to the set of so called master integrals, which in turn have to be computed. In general, one can write these integrals as Barnes-type integrals. This has the advantage that the poles can be resolved in an algorithmic manner, which enables the numerical calculation. In the case of one-fold MB integrals, one can perform the integration by picking up the poles and summing over them, resulting in generalized hypergeometric series. These can in turn be expanded in some cases using nested sums. For integer coefficients there are existing packages on the "market", however many hypergeometric series have half-integer coefficients, for which, in case of balanced coefficients, algorithms exist, but, at time we started the work, no package. Motivated by the fact that in various theories master integrals can be represented through hypergoemetric functions and that some of these, in case of single scale integrals, have half-integer coefficients, we have implemented in FORM a package for expansion in ϵ of balanced generalized hypergeometric and first Appell functions. Using MB representation of subloop integrals we were able to find suitable, that is one-fold MB representations, of number of master integrals of EQCD. For those complicated integrals, which have many massive lines and/or are non-planar, we were not able to find suitable hypergeometric representation, which would enable us to use Hypsummer. In those case where we obtained suitable representation, Hypsummer was successfully applied, producing analytical solution, which coincides with known numerical values. We tried to apply the same strategy on other set of master integrals, the one contributing to QCD correction of the so called ρ-parameter. Here, the results were disappointing. Only for a few integrals of simple topology could we find the appropriate representation, and even there the hypergeometric sums were mostly unbalanced. This limits very much the applicability of the methods we used to obtain analytical results and the question must be asked, whether or not the method of finding hypergeometric representations and expanding them using nested sums is going to be of any use in calculations in cases of single- or many-scaled integrals. Here, clearly, new approaches and/or further developments in hypergeometric functions are needed.

The situation in finite temperature is even worse. Technically speaking, finite temperature field theory is one case where we encounter many-scale integrals. The so called Matsubara frequencies act as different mass terms, therefore, when applying the methods to finite temperature, we were not even been able to compute simple diagrams analytically. However, it seems possible, at least in principle, to apply MB approach and to obtain numerical values. We have shown it on the simplest possible case. Like any other integrals, the integrals in finite temperature can be represented as Barnes-type integrals and maybe similar methods for resolution of singularities can be applied here as well. Once the poles are resolved one can interchange integration and summation and numerically compute the integrals. Since, as of now, there are no better alternatives and according to [33] there is “the need to develop novel computational techniques, in order to be able to complete...[the] task in systematic fashion“. Altough it is far from it this methods might be worth looking at.

Another research area that is interesting concerning this thesis is the application of nested sums to difference equations. As we have seen, for first order difference equations one immediately obtains hypergeometric sums. There are more advanced methods used to find solutions of higher order difference equations in terms of nested sums [77]. Since derivation of difference equations is algorithmic [46], having an algorithms for solution in terms of nested sums, regardless of the order of the difference equation, would be a major step forward. The existing package Sigma [60] was applied on difference equations we had for master integrals of EQCD, however without result. Maybe the more general methods along the lines of work of [60, 6, 55] will be able to solve them.

Appendix A

Special cases of nested sums

A.1 Multiple polylogarithms

The result of the expansion given by Hypsummer is in terms of multiple polylogarithms [30, 10], therefore we will give additional information about this class of functions. Multiple polylogarithms have a nested sums representation, since there are the special case of Z-sums in the case that the argument is infinity, as well as an iterated integral representation. Let us first introduce functions:

$$G(z_1, \dots, z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \cdots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k} \tag{A.1}$$

where $z_k \neq 0$.

Introducing $g(z; y) = \frac{1}{y-z}$ we have

$$\frac{d}{dy} G(z_1, \dots, z_k; y) = g(z_1; y) G(z_2, \dots, z_k; y) \tag{A.2}$$

and

$$G(z_1, \dots, z_k; y) = \int_0^y dt g(z_1; t) G(z_2, \dots, z_k; t) \tag{A.3}$$

and by defining additionally $G(0, \dots, 0; y) = \frac{1}{k!}(\ln y)^k$ for all k-values being zeros on can introduce the notation

$$G_{m_1, \dots, m_k}(z_1, \dots, z_k; y) = G(0, \dots, 0, z_1, \dots, z_{k-1}, 0, \dots, 0, z_k; y). \tag{A.4}$$

One can then find the relation to multiple polylogarithms

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = (-1)^k Li_{m_1,\ldots,m_k}(\frac{y}{z_1},\frac{z_1}{z_2},\ldots,\frac{z_{k-1}}{z_k}) \qquad \text{(A.5)}$$

and the inverse relation

$$Li_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}(\frac{y}{x_1},\frac{1}{x_1 x_2},\ldots,\frac{x_1}{\ldots}x_k;1). \qquad \text{(A.6)}$$

Using eq. (A.1) one can define integral representation for multiple polylogarithms as

$$Li_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k \int_0^1 \left(\frac{dt}{t}\circ\right)^{m_1-1}\frac{dt}{t-b_1} \left(\frac{dt}{t}\circ\right)^{m_2-1}\frac{dt}{t-b_2}\circ\ldots\circ\left(\frac{dt}{t}\circ\right)^{m_k-1}\frac{dt}{t-b_k}, \qquad \text{(A.7)}$$

where $b_j = \frac{1}{x_1 x_2 \ldots x_j}$ and

$$\int_0^y \frac{dt}{t-a_n}\circ\ldots\circ\frac{dt}{t-a_1} = \int_0^y \frac{dt_n}{t_n-a_n}\int_0^{t_n}\frac{dt_{n-1}}{t_{n-1}-a_{n-1}}\times\ldots\times\int_0^{t_1}\frac{dt_1}{t_1-a_1} \qquad \text{(A.8)}$$

and the short notation

$$\int_0^y \left(\frac{dt}{t-a_n}\circ\right)^m \frac{dt}{t-a} = \int_0^y \frac{dt}{t-a_n}\circ\ldots\circ\frac{dt}{t-a}. \qquad \text{(A.9)}$$

It is also of great importance to have a possibility to compute numerical values of multiple polylogarithms at fixed values. To do that one uses

$$\begin{aligned} &G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = \\ &\int_0^y \left(\frac{dt}{t}\circ\right)^{m_1-1}\frac{dt}{t-z_1}\left(\frac{dt}{t}\circ\right)^{m_2-1}\frac{dt}{t-z_2}\cdots\left(\frac{dt}{t}\circ\right)^{m_k-1}\frac{dt}{t-z_k} = \\ &\sum_{j_1=1}^{\infty}\cdots\sum_{j_k=1}^{\infty}\frac{1}{(j_1+\ldots+j_k)^{m_1}}\left(\frac{y}{z_1}\right)^{j_1}\times \\ &\times\frac{1}{(j_2+\ldots+j_k)^{m_2}}\left(\frac{y}{z_2}\right)^{j_2}\cdots\frac{1}{j_k^{m_k}}\left(\frac{y}{z_k}\right)^{j_k} \end{aligned} \qquad \text{(A.10)}$$

and one transfers all arguments into a region where one has a converging power series expansion.

A.2 Harmonic polylogarithms

When expanding generalized hypergeometric functions of type eq. (4.1) one ends up with multiple polylogarithms of the form $Li_{m_1,\ldots,m_k}(1,\ldots,1,x)$ which are harmonic polylogs (HPL's). On their own, one can define HPL's recursively as following [59]:

$$\begin{aligned} H(\underbrace{0,\ldots,0}_{n};x) &= \frac{1}{n!}\log^n x \\ H(a,a_1,\ldots,a_k;x) &= \int_0^x f_a(t)H(a_1,\ldots,a_k;x)dt \end{aligned} \tag{A.11}$$

for general vector of length, or weight n, where $a_i = 1, 0, -1$ and functions $f_a(x)$ are

$$f_1(x) = \frac{1}{1+x} \quad , \quad f_0(x) = \frac{1}{x} \quad , \quad f_{-1}(x) = \frac{1}{1-x} \tag{A.12}$$

The beginning of the recursion also has to be given, in this case that would be the lowest weight:

$$\begin{aligned} H(1;x) &= \int_0^x \frac{1}{1-t}dt &= -\ln(1-x) \\ H(0;x) &= \int_0^x \frac{1}{t}dt &= \ln(x) \\ H(-1;x) &= \int_0^x \frac{1}{1+t}dt &= \ln(1+x) \end{aligned} \tag{A.13}$$

An alternative definition would be:

$$\frac{d}{dx}H(a,a_1,\ldots,a_k;x) = f_a(x)H(a_1,\ldots,a_k;x) \tag{A.14}$$

From the equations above, it is easy to see that HPL's are a generalization of Nielsen polylogarithms [56]. Historically, that was the reason for their introduction [59].

HPL's also form an algebra, so one can write, just like in case of S/Z-sums, the product of two HPL's (with the same argument) as a sum of single HPL's of higher weight. For example:

$$\begin{aligned} H(a_1,a_2;x)H(b_1,b_2;x) &= H(a_1,a_2,b_1,b_2;x) + H(a_1,b_1,a_2,b_2;x) \\ &+ H(a_1,b_1,b_2,a_2;x) + H(b_1,a_1,a_2,b_2;x) \\ &+ H(b_1,a_1,b_2,a_2;x) + H(b_1,b_2,a_1,a_2;x) \end{aligned} \tag{A.15}$$

Notice, that in the above formula the relative order of the elements of a vector $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ respectively, is preserved. This is due to shuffle algebra [59]. The general formula is then:

$$H(a_1, \dots, a_{k1}; x) H(b_1, \dots, b_{k2}; x) = \sum_{c_i \in a_i \bigcup^{>} b_i} H(c_1, \dots, c_{k_1+k_2}; x) \quad \text{(A.16)}$$

where the symbol $\bigcup^{>}$ stands for the fact mentioned earlier, namely that the internal order of the elements a_i and b_i respectively is preserved.
The HPL's can be Mellin transformed and Taylor expanded. Since we do not need Mellin transforms and the Taylor expansion of HPL's, we refer the interested reader to original literature [59].

What we are interested in are the HPL's with argument x=1. These are actually nothing else then Euler-Zagier sums at infinity, which are nothing else then multiple zeta values (MZV) for positive a's, or colored MZV for arbitrary a's.

$$\begin{aligned} H(a;1) &= \zeta(a) \quad , \quad a > 0 \\ H(-a;1) &= (1-2^{1-a})\zeta(a) \quad , \quad a > 0 \\ H(a_1, \dots, a_k; 1) &= (-1)^{\#(a_i<0)} \zeta(\bar{a}_1, \dots, \bar{a}_k) \quad , \quad k > 1 \end{aligned} \quad \text{(A.17)}$$

where ζ's are:

$$\zeta(a_1, \dots, a_k) = \sum_{i_1}^{\infty} \sum_{1}^{i_1-1} \cdots \sum_{1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{sgn(a_j)^{i_j}}{i_j^{|a_j|}} \quad \text{(A.18)}$$

and vector $\bar{a} = (a_1, sgn(a_1)a_2, \dots, sgn(a_{i-1})a_i, \dots, sgn(a_{k-1})a_k)$.
The MZV's themselves possess an algebra, which means that they can be expressed in terms of a few mathematical constants, like powers of π, ζ-functions and certain polylogarithms. For the relations see for example [10], [11] and next section of the appendix.

A.3 Special values of harmonic sums

Here are the definitions of numbers appearing in the results in the expansions in terms of harmonic sums as well as numerical values.

$$
\begin{aligned}
\ln 2 &= -S_{-1}(\infty) \\
\zeta_{n\geq 2} &= S_n(\infty) \\
a_{n\geq 3} &= Li_n(\frac{1}{2}) \\
s6 &= S_{-5,-1}(\infty) \approx 0.98744142640329971377 \\
s7a &= S_{-5,1,1}(\infty) \approx -0.95296007575629860341 \\
s7b &= S_{5,-1,-1}(\infty) \approx 1.02912126296432453422 \\
s8a &= S_{5,3}(\infty) \approx 1.04178502918279188338 \\
s8b &= S_{-7,-1}(\infty) \approx 0.99644774839783766598 \\
s8c &= S_{-5,-1,-1,-1}(\infty) \approx 0.98396667382173367092 \\
s8d &= S_{-5,-1,1,1}(\infty) \approx 0.99996261346268344769 \\
s9a &= S_{7,-1,-1}(\infty) \approx 1.00640196269235635900 \\
s9b &= S_{-7,-1,1}(\infty) \approx 0.99842952512288855439 \\
s9c &= S_{-6,-2,-1}(\infty) \approx -0.98747515763691525588 \\
s9d &= S_{-5,-1,1,1,1}(\infty) \approx 1.00219817413397743629 \\
s9e &= S_{-5,-1,-1,-1,1}(\infty) \approx 0.98591171955244547261 \\
s9f &= S_{-5,-1,-1,1,-1}(\infty) \approx 0.97848117128116624247
\end{aligned}
\tag{A.19}
$$

Bibliography

[1] C. Anastasiou and A. Daleo, “Numerical evaluation of loop integrals,” JHEP **0610** (2006) 031 [arXiv:hep-ph/0511176].

[2] T. Appelquist and R. D. Pisarski, “High-Temperature Yang-Mills Theories And Three-Dimensional Quantum Chromodynamics,” Phys. Rev. D **23** (1981) 2305.

[3] C. Bauer,A. Frink and R. Kreckel, arXiv:cs/0004015.

[4] E. Bejdakic and Y. Schroder, “Hypergeometric representation of a four-loop vacuum bubble,” Nucl. Phys. Proc. Suppl. **160** (2006) 155 [arXiv:hep-ph/0607006].

[5] E. Bejdakic, “Multiloop Bubbles for hot QCD,” Nucl. Phys. A **820** (2009) 263C [arXiv:0810.3097 [hep-ph]].

[6] I. Bierenbaum, J. Blumlein, S. Klein and C. Schneider, PoS A **CAT2007** (2007) 082 [arXiv:0707.4659 [math-ph]].

[7] J. Blumlein and S. Kurth, “Harmonic sums and Mellin transforms up to two-loop order,” Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241].

[8] C. Bogner and S. Weinzierl, “Periods and Feynman integrals,” J. Math. Phys. **50** (2009) 042302 [arXiv:0711.4863 [hep-th]].

[9] E. E. Boos, A. I. Davydychev, “A Method of evaluating massive Feynman integrals,” Theor. Math. Phys. **89** (1991) 1052 [Teor. Mat. Fiz. **89** (1991) 56].

[10] J. M. Borwein, D. M. Bradley and D. J. Broadhurst, “Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k,” arXiv:hep-th/9611004.

[11] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, "Special Values of Multiple Polylogarithms," Trans. Am. Math. Soc. **353** (2001) 907 [arXiv:math.ca/9910045].

[12] J.M.Borwein, D.M.Bradley, D.J.Broadhurst, P.Lisonek, math.CA/9910045

[13] R. Boughezal and M. Czakon, "Single scale tadpoles and O(G(F) m(t)**2 alpha(s)**3) corrections to the rho parameter," Nucl. Phys. B **755**, 221 (2006) [arXiv:hep-ph/0606232].

[14] E. Braaten, "Solution to the perturbative infrared catastrophe of hot gauge theories," Phys. Rev. Lett. **74** (1995) 2164 [arXiv:hep-ph/9409434].

[15] E. Braaten and A. Nieto, "Effective field theory approach to high temperature thermodynamics," Phys. Rev. D **51** (1995) 6990 [arXiv:hep-ph/9501375].

[16] E. Braaten and A. Nieto, "Free Energy of QCD at High Temperature," Phys. Rev. D **53** (1996) 3421 [arXiv:hep-ph/9510408].

[17] D. J. Broadhurst, J. Fleischer and O. V. Tarasov, "Two loop two point functions with masses: Asymptotic expansions and Taylor series, in any dimension," Z. Phys. C **60** (1993) 287 [arXiv:hep-ph/9304303].

[18] K. G. Chetyrkin and F. V. Tkachov, "Integration By Parts: The Algorithm To Calculate Beta Functions In 4 Loops," Nucl. Phys. B **192** (1981) 159.

[19] K. G. Chetyrkin, M. Faisst, J. H. Kuhn, P. Maierhofer and C. Sturm, Phys. Rev. Lett. **97** (2006) 102003 [arXiv:hep-ph/0605201].

[20] M. Czakon, "Automatized analytic continuation of Mellin-Barnes integrals," Comput. Phys. Commun. **175** (2006) 559 [arXiv:hep-ph/0511200].

[21] A. I. Davydychev and J. B. Tausk, "Two loop selfenergy diagrams with different masses and the momentum expansion," Nucl. Phys. B **397** (1993) 123.

[22] A. I. Davydychev, "Explicit results for all orders of the epsilon-expansion of certain massive and massless diagrams," Phys. Rev. D **61** (2000) 087701 [arXiv:hep-ph/9910224].

[23] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. B **605** (2001) 266 [arXiv:hep-th/0012189].

[24] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. B **699** (2004) 3 [arXiv:hep-th/0303162].

[25] F. Di Renzo, M. Laine, V. Miccio, Y. Schroder and C. Torrero, JHEP **0607** (2006) 026 [arXiv:hep-ph/0605042].

[26] L.Euler,
Novi Comm.Acad.Sci.Petropol. 20, 140 (1775)

[27] J. Fleischer and M. Y. Kalmykov, “Single mass scale diagrams: Construction of a basis for the epsilon-expansion,” Phys. Lett. B **470** (1999) 168 [arXiv:hep-ph/9910223].

[28] T. Gehrmann and E. Remiddi, “Two-Loop Master Integrals for $\gamma^* \to 3$ Jets: The planar topologies,” Nucl. Phys. B **601** (2001) 248 [arXiv:hep-ph/0008287].

[29] P. H. Ginsparg, “First Order And Second Order Phase Transitions In Gauge Theories At Finite Temperature,” Nucl. Phys. B **170** (1980) 388.

[30] A.B.Goncharov,
Math.Res.Lett. 5 ,497 (1998)
http://www.math.uiuc.edu/K-theory/0297

[31] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343.

[32] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53** (1981) 43.

[33] A. Gynther, M. Laine, Y. Schroder, C. Torrero and A. Vuorinen, JHEP **0704** (2007) 094 [arXiv:hep-ph/0703307].

[34] A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, “Plaquette expectation value and gluon condensate in three dimensions,” JHEP **0501** (2005) 013 [arXiv:hep-lat/0412008].

[35] A. Hietanen and A. Kurkela, JHEP **0611** (2006) 060 [arXiv:hep-lat/0609015].

[36] Michael E Hoffman, “Quasi-shuffle products“ Journal of Algebraic Combinatorics,11,49 (2000) [math.QA/9907173]

[37] T. Huber and D. Maitre, “HypExp 2, Expanding Hypergeometric Functions about Half-Integer Parameters,” arXiv:0708.2443 [hep-ph].

[38] C. Itzykson and J. B. Zuber, “Quantum Field Theory,” *New York, Usa: Mcgraw-hill (1980) 705 P.(International Series In Pure and Applied Physics)*

[39] K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, “Four-loop vacuum energy density of the SU(N(c)) + adjoint Higgs theory,” JHEP **0304**, 036 (2003) [arXiv:hep-ph/0304048].

[40] M. Y. Kalmykov, B. F. L. Ward and S. A. Yost, “Multiple (inverse) binomial sums of arbitrary weight and depth and the all-order epsilon-expansion of generalized hypergeometric functions with one half-integer value of parameter,” JHEP **0710** (2007) 048 [arXiv:0707.3654 [hep-th]].

[41] M. Y. Kalmykov and B. A. Kniehl, “Towards all-order Laurent expansion of generalized hypergeometric functions around rational values of parameters,” Nucl. Phys. B **809** (2009) 365 [arXiv:0807.0567 [hep-th]].

[42] D. I. Kazakov and A. V. Kotikov, “TOTAL ALPHA-s CORRECTION TO DEEP INELASTIC SCATTERING CROSS-SECTION RATIO, R = sigma-L / sigma-t IN QCD. CALCULATION OF LONGITUDINAL STRUCTURE FUNCTION,” Nucl. Phys. B **307** (1988) 721 [Erratum-ibid. B **345** (1990) 299].

[43] T. Kinoshita, “Mass Singularities Of Feynman Amplitudes,” J. Math. Phys. **3** (1962) 650.

[44] B. A. Kniehl, A. V. Kotikov, A. Onishchenko and O. Veretin, “Two-loop sunset diagrams with three massive lines,” Nucl. Phys. B **738** (2006) 306 [arXiv:hep-ph/0510235].

[45] Mikko Laine. *Thermal Field Theory*, Lecture given at Bielefeld University in 2003.

[46] S. Laporta, “High-precision calculation of multi-loop Feynman integrals by difference equations,” Int. J. Mod. Phys. A **15** (2000) 5087 [arXiv:hep-ph/0102033].

[47] S. Laporta, “High-precision epsilon-expansions of three-loop master integrals contributing to the electron g-2 in QED,” Phys. Lett. B **523** (2001) 95 [arXiv:hep-ph/0111123].

[48] S. Laporta, “High-precision epsilon expansions of massive four-loop vacuum bubbles,” Phys. Lett. B **549** (2002) 115 [arXiv:hep-ph/0210336].

[49] T. D. Lee and M. Nauenberg, “Degenerate Systems and Mass Singularities,” Phys. Rev. **133** (1964) B1549.

[50] D. Maitre, “HPL, a Mathematica implementation of the harmonic polylogarithms,” arXiv:hep-ph/0507152.

[51] Milne-Thomson L.M. *The Calculus of Finite Differences*,
London MACMILLAN & CO LTD. 1965.

[52] S. Moch and J. A. M. Vermaseren, "Deep inelastic structure functions at two loops," Nucl. Phys. B **573** (2000) 853 [arXiv:hep-ph/9912355].

[53] S. Moch, P. Uwer and S. Weinzierl, "Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals," J. Math. Phys. **43** (2002) 3363 [arXiv:hep-ph/0110083].

[54] S. Moch and P. Uwer, "xSummer: Transcendental functions and symbolic summation in Form," arXiv:math-ph/0508008.

[55] S. Moch and C. Schneider, PoS A **CAT2007** (2007) 083 [arXiv:0709.1769 [math-ph]].

[56] N.Nielsen,
Nova Acta Leopoldina(Halle 90),123 (1909)

[57] G. Passarino and M. J. G. Veltman, "One Loop Corrections For E+ E- Annihilation Into Mu+ Mu- In The Weinberg Model," Nucl. Phys. B **160** (1979) 151.

[58] H. D. Politzer, Phys. Rev. Lett. **30** (1973) 1346.

[59] E. Remiddi and J. A. M. Vermaseren, "Harmonic polylogarithms," Int. J. Mod. Phys. A **15**, 725 (2000) [arXiv:hep-ph/9905237].

[60] C. Schneider "Symbolic Summation Assists Combinatorics" Sem. Lothar. Combin. 56, pp. 1-36. 2007. ISSN 1286-4889. Article B56b

[61] Y. Schröder, "Automatic reduction of four-loop bubbles," Nucl. Phys. Proc. Suppl. **116** (2003) 402

[62] Y. Schroder and A. Vuorinen, "High-precision evaluation of four-loop vacuum bubbles in three dimensions," arXiv:hep-ph/0311323.

[63] Y. Schroder and M. Steinhauser, Phys. Lett. B **622**, 124 (2005) [arXiv:hep-ph/0504055].

[64] Y. Schroder and A. Vuorinen, "High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles," JHEP **0506**, 051 (2005) [arXiv:hep-ph/0503209].

[65] Y.Schroder, "unpublished"

[66] L.J. Slater, "Generalized Hypergeometric Functions"
Cambridge University Press, (1966)

[67] V. A. Smirnov, "Analytical result for dimensionally regularized massless on-shell double box," Phys. Lett. B **460**, 397 (1999) [arXiv:hep-ph/9905323].

[68] V. A. Smirnov, "Feynman integral calculus," *Berlin, Germany: Springer (2006) 283 p*

[69] A. V. Smirnov and V. A. Smirnov, "On the Resolution of Singularities of Multiple Mellin-Barnes Integrals," Eur. Phys. J. C **62** (2009) 445 [arXiv:0901.0386 [hep-ph]].

[70] O. V. Tarasov, "Connection between Feynman integrals having different values of the space-time dimension," Phys. Rev. D **54**, 6479 (1996) [arXiv:hep-th/9606018].

[71] O. V. Tarasov, "Generalized recurrence relations for two-loop propagator integrals with arbitrary masses," Nucl. Phys. B **502**, 455 (1997) [arXiv:hep-ph/9703319].

[72] J. B. Tausk, "Non-planar massless two-loop Feynman diagrams with four on-shell legs," Phys. Lett. B **469**, 225 (1999) [arXiv:hep-ph/9909506].

[73] F. V. Tkachov, "A Theorem On Analytical Calculability Of Four Loop Renormalization Group Functions," Phys. Lett. B **100** (1981) 65.

[74] J. A. M. Vermaseren, "Harmonic sums, Mellin transforms and integrals," Int. J. Mod. Phys. A **14**, 2037 (1999) [arXiv:hep-ph/9806280].

[75] J. A. M. Vermaseren, "New features of FORM," arXiv:math-ph/0010025.

[76] J. A. M. Vermaseren and S. Moch, "Mathematics for structure functions," Nucl. Phys. Proc. Suppl. **89** (2000) 131 [arXiv:hep-ph/0004235].

[77] J. A. M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005) 3 [arXiv:hep-ph/0504242].

[78] S. Weinzierl, "Symbolic Expansion of Transcendental Functions," Comput. Phys. Commun. **145** (2002) 357 [arXiv:math-ph/0201011].

[79] S. Weinzierl, "Expansion around half-integer values, binomial sums and inverse binomial sums," J. Math. Phys. **45** (2004) 2656 [arXiv:hep-ph/0402131].

[80] S. Weinzierl, "The art of computing loop integrals," arXiv:hep-ph/0604068.

[81] D.Zagier, First European Congress of Mathematics, Voll 2 Birkhauser, Boston, 497 (1994)

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