Stochastic Dynamics with Singular Lower Order Terms in Finite and Infinite Dimensions

Dissertation
zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

vorgelegt von
Peng Jin
im Oktober 2009

Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706.

Contents

Introduction

Diffusion processes are stochastic processes describing the physical phenomenon of diffusion. Their mathematical theories are firmly based on modern probability theory, or more precisely, It$\hat{o}$ calculus. Using It$\hat{o}$'s stochastic calculus, it is possible to characterize the infinitesimal motion of a diffusion particle. The dynamics of a diffusion particle in $\mathbb{R}^d$ is usually governed by a stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{1}$$

where $\sigma(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$, $b(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable and W_t is an r-dimensional Brownian motion. Because of intuitive physical meanings, the matrix $a(t, x) := \sigma(t, x)\sigma^T(t, x)$ is called the diffusion matrix and $b(t, x)$ called the drift vector.

The basic existence and uniqueness theories for (1) were already established by the end of last 60's. It was K. It$\hat{o}$ who gave the first existence and uniqueness theorem to (1). He proved that if $\sigma(t, x), b(t, x)$ are uniformly Lipschitz with respect to x and are at most of linear growth, then there exists a unique strong solution to (1). Then in 1969, using the martingale problem methods, Stroock and Varadhan proved that there exists a unique weak solution to (1) if $a(t, x)$ is bounded continuous, everywhere positive definite and $b(t, x)$ is bounded measurable.

However, the above mentioned existence and uniqueness results require the coefficients of (1) to be locally bounded, namely $a(t, x), b(t, x)$ could not be singular. Motivated from applications, of course also mathematically important, many people studied the existence and uniqueness problem for (1) when the coefficients are not locally bounded (cf. [Stu93, ES84, Por90, BC03, KR05], and the list is far from complete). Now it is an accepted fact that, if the diffusion matrix is "nice", then very mild assumptions on the drift vector still ensure that (1) has a unique weak solution.

Now let's look at the simplest case in which $a(t, x)$ is everywhere the identity matrix. The stochastic differential equation (1) becomes

$$dX_t = b(t, X_t)dt + dW_t. \tag{2}$$

The solution to (2), if it exists, is usually called Brownian motion with drift b. To solve (2), both probabilistic and analytic methods can be used. We know that, under Novikov's condition holding for $b(t, x)$, the equation (2) can be solved through Girsanov transformation. Several authors adopted this approach to solve (2) under various assumptions on $b(t, x)$ (cf. [Stu93, Por90]).

Apart from probabilistic methods, one can also use modern PDE theories. We could look at the corresponding Kolmogorov's backward equation

$$\frac{\partial u}{\partial s} + \frac{1}{2}\triangle u + b(s, x) \cdot \nabla u = 0. \tag{3}$$

If $b(s, x)$ is smooth and has compact support, it is well-known that for (3) there exists a classical fundamental solution which is exactly the transition density function for the diffusion process described by (2). When $b(s, x)$ is merely bounded and measurable, classical fundamental solutions for (3) do not exist in general. However, D.G. Aronson's work (cf. [Aro68]) tell us there still exists a fundamental solution $p(s, x; t, y)$ for (3) in a weak sense when $b(s, x)$ only satisfies some integrability condition. Using this weak fundamental solution $p(s, x; t, y)$ as the transition probability density of the desired process, N.I. Portenko constructed a weak solution to (2) for a broad class of drift vectors b.

Recently, Bass and Chen used another method to solve (2) (cf. [BC03]). They proved that if the drift $b(t, x)$ is independent of time (i.e. $b(t, x) = b(x)$) and each component $b^i(x)$ belongs to the Kato class $\mathcal{K}_{d-1}$ (cf. Example 1.1.2 for the definition), then (2) has a unique weak solution. In fact they could even allow the drift to be a Radon measure, but then the notion of a solution to (2) would be a little bit different from the usual sense. Their method is based on constructing the resolvent S^λ of the desired process described by (2).

The above mentioned results concerning the stochastic differential equation (2) dealt only with weak solutions. In the paper [KR05], Krylov and Röckner considered existence and uniqueness of strong solutions to (2). They proved that if $b(t, x)$ is locally in $L^{p,q}$ (cf. Example 1.1.2 for the definition) with $p \geq 2$ and $\frac{d}{2p} + \frac{1}{q} < \frac{1}{2}$, then (2) has a unique strong solution up to an explosion time.

In this work, we aimed to construct diffusion processes with singular coefficients using analytic methods. In chapter 1 and chapter 2, we consider the stochastic differential equation (2) for a new class of singular drift vector $b(t,x)$. In chapter 3 we turn to the infinite dimensional case and construct the Glauber dynamics of an unbounded spin system on a graph. To be more precise, we now explain the contents and main results of this thesis chapter by chapter.

Weak fundamental solution for a parabolic equation with singular lower order terms

From the work of N.I. Portenko, we have seen that fundamental solutions of second order parabolic equations are very helpful for the construction of diffusion processes. Therefore in chapter 1 we study a class of second order parabolic equations of the following form

$$\nabla(a(t,x)\cdot\nabla u(t,x)) + b(t,x)\cdot\nabla u(t,x) + V(t,x)u(t,x) - \partial_t u(t,x) = 0 \quad (4)$$

in the domain $[0,T]\times\mathbb{R}^d$, where $T<\infty$. Here we use the notation

$$\nabla(a(t,x)\cdot\nabla u) = \sum_{i,j=1}^{d}\partial_{x_i}a_{ij}\partial_{x_j}u, \quad b\cdot\nabla u = \sum_{j=1}^{d} b_j\partial_{x_j}u.$$

There has been a lot of work on weak fundamental solutions of (4) under various assumptions on the coefficients. In particular, it was Qi S. Zhang who first introduced time-dependent Kato classes to study (4). In [Zha96a, Zha97a], Qi S. Zhang studied the special case of (4) in which $V\equiv 0$. There he assumed that $a(t,x)$ is uniformly elliptic and Hölder continuous, $|b(t,x)|$ has compact support and belongs to $\mathcal{TK}^c_{d-1}$ (cf. Definition 1.1.1) for any $c>0$, then he proved Gaussian bounds of the corresponding weak fundamental solution. He also treated the case in which $b\equiv 0$ and $V(t,x)$ has compact support and belongs to the class $\mathcal{TK}^c_{d-2}$ (cf. Definition 1.1.9) for any $c>0$, see [Zha96b, Zha97b] for more details.

Then in [LS00], Liskevich and Semenov studied the full form of (4). For the principle part they only assumed the matrix $a(t,x)$ to be measurable and uniformly elliptic, without any additional continuity conditions. For the zero order term $V(t,x)$ they assumed similar conditions like [Zha96b, Zha97b], but they dropped the restriction that $V(t,x)$ is compactly supported. For compensation they imposed more restrictive assumptions on the first order term $b(t,x)$ than in [Zha96a, Zha97a]. Under these conditions they proved that (4) has a unique weak fundamental solution. However, in order to drop the restriction that $b(t,x)$ and $V(t,x)$ are compactly supported, they used a very sophisticated argument (cf. [LS00, page 538]).

We now state our assumptions on the coefficients of (4). We assume that the matrix $a(t,x) = (a_{ij}(t,x))$ is symmetric and uniformly elliptic, Hölder continuous in t, x and $\frac{\partial}{\partial x_i} a_{ij}(t,x)$ are bounded and Hölder continuous in x. Under these assumptions on $a(t,x)$, it is well-known (cf. [Fri64, LSU67, Aro68]) that the equation

$$\nabla(a(t,x)\cdot\nabla u(t,x)) - \partial_t u(t,x) = 0$$

has a classical fundamental solution $p(t,x;s,y)$ satisfying the following estimates: there exist positive constants $c_0, \alpha_0, C_0, \alpha$ such that for any $x, y \in \mathbb{R}^d$ and $0 \le s < t \le T$,

$$\text{(i)}\ \frac{c_0}{(t-s)^{\frac{d}{2}}}\exp(-\alpha_0\frac{|x-y|^2}{t-s}) \le p(t,x;s,y) \le \frac{C_0}{(t-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s}),$$

$$\text{(ii)}\ |\nabla_x p(t,x;s,y)| \le \frac{C_0}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s}).$$

For the first order term, we assume $|b|$ to be in the time-dependent Kato class $\mathcal{TK}^{\alpha_1}_{d-1}$ for some $\alpha_1 < \frac{\alpha}{2}$, namely

$$\lim_{h\to 0} N_h^{\alpha_1}(|b|) = 0,$$

where

$$N_h^{\alpha_1}(|b|) := \sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha_1\frac{|x-y|^2}{t-s})|b(t,x)|dxdt+$$
$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha_1\frac{|x-y|^2}{t-s})|b(s,y)|dyds.$$

For the zero order term, we assume $V \in \mathcal{TK}^{\alpha_2}_{d-2}$ for some $\alpha_2 < \frac{\alpha}{4}$, namely

$$\lim_{h\to 0} M_h^{\alpha_2}(V) = 0,$$

where

$$M_h^{\alpha_2}(V) := \sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d}{2}}}\exp(-\alpha_2\frac{|x-y|^2}{t-s})|V(t,x)|dxdt+$$
$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d}{2}}}\exp(-\alpha_2\frac{|x-y|^2}{t-s})|V(s,y)|dyds.$$

Here our assumptions on the lower order terms $b(t,x), V(t,x)$ are weaker than in [Zha96a, Zha96b]. In particular we don't assume that $b(t,x), V(t,x)$ are compactly supported.

Under the above assumptions, our main result of this chapter, Theorem 1.4.10, states that there exists a unique weak fundamental solution

$$G(t,x;s,y),\ 0 \le s < t \le T,\ x,y \in \mathbb{R}^d$$

for the parabolic equation (4). To prove this theorem, the main difficulty lies in the existence part, namely how to construct such a $G(t,x;s,y)$. To this end we use a general scheme as in [Zha96a, Zha96b, LS00]. We first consider the equation

$$\nabla(a(t,x) \cdot \nabla u(t,x)) + b_n(t,x) \cdot \nabla u(t,x) + V_k(t,x) \cdot u(t,x) - \partial_t u(t,x) = 0 \quad (5)$$

where $b_n(t,x)$ and $V_k(t,x)$ are bounded smooth and approximate $b(t,x)$ and $V(t,x)$ respectively in a reasonable way. For the parabolic equation (5), there exists the fundamental solution $G_{nk}(t,x;s,y)$. Then we prove that $G_{nk}(t,x;s,y)$ converges locally uniformly to a function $G(t,x;s,y)$. This kind of convergence was first proved in the special case when $b(t,x) = b(x)$ is time-indepedent, $b(x) \in \mathcal{K}_{d-1}$ and $V \equiv 0$ in the paper [KS06], where Kim and Song studied the transition probability densities of the Markov process constructed in [BC03]. Here we do it in the more general time-depedent case and we have to overcome many technical difficulties. Then we verify that $G(t,x;s,y)$ is indeed a weak fundamental solution to the parabolic equation (4). The uniqueness of such a weak fundamental solution $G(t,x;s,y)$ can be proved similarly to [LS00].

It should be pointed out that the method of chapter 1 can also be applied to the backward parabolic equation (3). Therefore some results of chapter 1 will be used in chapter 2 to study the corresponding diffusion processes.

Diffusions with time-dependent singular drift

In chapter 2 we study the stochastic differential equation

$$\begin{cases} dX_t = dW_t + B(t,X_t)dt, & t \ge s \\ X_t = x, & 0 \le t \le s \end{cases} \quad (6)$$

with a new class of time-dependent singular drift terms. Here we only consider weak solutions to (6). It is well-known that existence and uniqueness of weak solutions to (6) is equivalent to the martingale problem for the operator L being well-posed, where

$$L_t = \frac{1}{2}\triangle + B(t,x)\cdot\nabla$$

Now we state our assumption on the drift term. We assume $|B(t,x)|$ to be in the forward-Kato class $\mathcal{FK}_{d-1}^{\alpha}$ for some $\alpha < \frac{1}{2}$, namely

$$\lim_{h\to 0} N_h^{\alpha,+}(|B|) = 0,$$

where

$$N_h^{\alpha,+}(|B|) := \sup_{(s,x)\in[0,\infty)\times\mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha\frac{|x-y|^2}{t-s})|B(t,y)|dydt.$$

We should note that the forward-Kato class $\mathcal{FK}_{d-1}^{\alpha}$ is strictly larger than the time-dependent Kato class $\mathcal{TK}_{d-1}^{\alpha}$. In section 2.1 we shall give an example which belongs to $\mathcal{TK}_{d-1}^{\alpha}$ but does not belong to $\mathcal{FK}_{d-1}^{\alpha}$.

Under the above assumption, we prove Theorem 2.2.22. It states that the martingale problem for

$$L_t = \frac{1}{2}\triangle + B(t,x)\cdot\nabla$$

is well-posed, or equivalently, the stochastic differential equation (6) has a unique weak solution for every starting point (s,x). This is the main result of this chapter. We should note that $\mathcal{FK}_{d-1}^{\alpha}$ includes the (time-independent) Kato class $\mathcal{K}_{d-1}$, therefore our work extends the results of [BC03].

In section 2.3 we further assume $|B(t,x)| \in \mathcal{TK}_{d-1}^{\alpha'}$ for some $\alpha' < \frac{1}{4}$, then from the results of Chapter 1, we can easily prove Theorem 2.3.4, which tells us that the solution X_t of (6), as a Markov process, has a transition density function $q(s,x;t,y)$ satisfying two sided Gaussian estimates.

Construction of Glauber dynamics for an unbounded spin system on a graph

To construct diffusion processes, we can also use Dirichlet form methods. After Fukushima discovered the connection between symmetric Markov processes and symmetric Dirichlet forms, this methodology has been implemented in great generality. One advantage of this method is that it still works in infinite dimensional cases (cf. [AR91, MR92]). In chapter 3 we use Dirichlet form methods to construct the Glauber dynamics for an unbounded spin system on a graph.

In this chapter, we consider an unbounded spin system which was first studied in [Pas07a, Pas07b]. More precisely, let $\mathbb{G}(\mathbb{V},\mathbb{E})$ be a connected simple graph consisting of a countable set of vertices $v \in \mathbb{V}$ and a set of unordered edges $e \in \mathbb{E}$. For each vertex v, let m_v be the degree of v. We assume that $\mathbb{G}(\mathbb{V},\mathbb{E})$ is of uniformly bounded degree, i.e.

$$m_{\mathbb{G}} := \sup_{v\in\mathbb{V}} m_v < \infty.$$

Then we can define

$$\delta_{\mathbb{G}} := \inf\big\{\delta > 0 : \sum_{v\in\mathbb{V}} e^{-\delta\rho(v,o)} < \infty\big\},$$

where $o \in \mathbb{V}$ is a fixed vertex and $\rho(v,o)$ is the combinatorial distance between vertices v and o. Suppose that to each vertex $v \in \mathbb{V}$, there corresponds a particle performing one-dimensional oscillation. The configuration space $\Omega := \mathbb{R}^{\mathbb{V}}$ of this unbounded spin system consists of all real sequence $x = (x_v)_{v\in\mathbb{V}}$. We assume that the potential energy of each configuration $x \in \Omega$ is given by the formal Hamiltonian

$$H(x) = \sum_v V_v(x_v) + \frac{1}{2}\sum_{v\sim v'} W_{vv'}(x_v, x_{v'}),$$

where the sums are running over all $v \in \mathbb{V}$ and ordered pairs $(v,v') \in \mathbb{V}^2$ with $v \sim v'$ ($\sim$ means v and v' are adjacent, namely $\rho(v,v') = 1$). Here we assume $W_{vv'} \equiv 0$ if v and v' are not adjacent.

For the interaction potential $W_{vv'}$, we assume that $W_{vv'}(\cdot,\cdot)$ is measurable and there exist constants $C_W, J \geq 0$ such that for all $v \sim v'$ and $x_v, x_{v'} \in \mathbb{R}$

$$|W_{vv'}(x_v, x_{v'})| \leq \frac{1}{2}J(C_W + |x_v|^2 + |x_{v'}|^2).$$

For the self-potential V_v, we assume that $V_v(\cdot)$ is measurable and there exist constants $p \geq 2, A_V > m_{\mathbb{G}}J(e^{\delta_{\mathbb{G}}} + \frac{1}{2}), B_V \in \mathbb{R}, C_V > 0$, such that for all $v \in \mathbb{V}$ and $x_v \in \mathbb{R}$

$$A_V|x_v|^2 + B_V \leq V_v(x_v) \leq C_V(1 + |x_v|^p).$$

It should be emphasized that here we merely assume the potential functions to be measurable. This is much weaker than the conditions in [Pas07a, Pas07b], where $W_{vv'}$ and V_v were assumed to be twice continuously differentiable.

For each $\delta > 0$ we set

$$\|x\|_\delta := \left[\sum_{v\in\mathbb{V}} |x_v|^2 \exp\{-\delta\rho(v,o)\}\right]^{1/2}$$

and $\Omega_\delta := \Big\{x \in \Omega : \|x\|_\delta < \infty\Big\}$. The tempered configuration space is defined as

$$\Omega^t := \bigcap_{\delta > \delta_{\mathbb{G}}} \Omega_\delta.$$

Under the above assumptions, we aim to construct the stochastic evolution of this spin system, which is usually called Glauber dynamics, on the tempered configuration space Ω^t. Since the potential functions are only measurable, we can not construct the Glauber dynamics by solving the corresponding infinite system of stochastic differential equations (cf. [Pas07a, Pas07b]). We have to use Dirichlet form methods. To do that we first need to find a good reference measure on the tempered configuration space. Therefore in section 3.2 we adapt the methods of [Pas07a] to prove the existence of tempered Gibbs measures, which are mathematical descriptions of equilibrium states of the spin system.

In section 3.3 we fix some tempered Gibbs measure μ on the tempered configuration Ω^t. Using the general framework in [AR90], we can define a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\Omega^t;\mu)$. Then we use the standard arguments to show that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Using the correspondence between Markov processes and quasi-regular Dirichlet forms, thus we can construct the Glauber dynamics on the tempered configuration space.

Acknowledgements

First of all, I would like to thank my supervisors Prof. Michael Röckner and Prof. Zhi-Ming Ma. They have always been patient to me and have encouraged me all the time. They also gave me the precious opportunity to study mathematics in Beiijng and Bielefeld.

I am also very indebted to Dr. Tetyana Pasurek, for many instructive discussions and valuable suggestions. Moreover, I also would like to thank my colleagues and friends in Bielefeld and Beijing, in particular Prof. Jia-An Yan, Prof. Shun-Long Luo, Prof. Fu-Zhou Gong, Dr. Shunxiang Ouyang, Dr. Wei Liu, Dr. Sven Struckmeier, Dr. Felipe Torres Tapia, Dr. Olga Byegunova, Sven Wiesinger, Jonas Tölle, Lukasz Derdziuk and Nazim Hikmet Tekmen.

Finally the financial support of Deutsche Forschungsgemeinscaft (DFG) and Chinese Academy of Sciences is gratefully acknowledged.

Chapter 1

Weak fundamental solution for a parabolic equation with singular lower order terms

In this chapter, we consider a parabolic equation in the following form

$$\nabla(a(t,x)\cdot\nabla u(t,x)) + b(t,x)\cdot\nabla u(t,x) + V(t,x)\cdot u(t,x) - \partial_t u(t,x) = 0 \quad (*)$$

in the domain $[0,T]\times\mathbb{R}^d$ where $T<\infty$. We assume that the matrix $a(t,x)=(a_{ij}(t,x))$ is uniformly elliptic, Hölder continuous in t,x and $\frac{\partial}{\partial x_i}a_{ij}$ are bounded and Hölder continuous in x. The lower order coefficients $b(t,x)$ and $V(t,x)$ are assumed to be in some proper time-dependent Kato classes (cf. Assumption (1.2.2) below).

Under these conditions we prove that there exists a unique weak fundamental solution to the above equation $(*)$. In section 1.1 we introduce several time-dependent Kato classes and study some of their properties. In section 1.2 we make precise assumptions on the coefficients of $(*)$ and introduce the notion of weak fundamental solutions. In section 1.3 we first consider the equation

$$\nabla(a(t,x)\cdot\nabla u(t,x)) + b_n(t,x)\cdot\nabla u(t,x) + V_k(t,x)\cdot u(t,x) - \partial_t u(t,x) = 0$$

where $b_n(t,x)$ and $V_k(t,x)$ are bounded smooth and approximate $b(t,x)$ and $V(t,x)$ respectively in a reasonable way. Since $b_n(t,x)$ and $V_k(t,x)$ are bounded smooth, the above equation has a unique fundamental solution $G_{nk}(t,x;s,y)$. Then we prove that $G_{nk}(t,x;s,y)$ converges locally uniformly to a function $G(t,x;s,y)$. In Section 1.4 we verify that $G(t,x;s,y)$ is indeed a weak fundamental solution to $(*)$. The uniqueness of weak fundamental solutions to $(*)$ can be proved with the same methods used in [LS00].

1.1 Time-dependent Kato classes

In the study of Schrödinger equations, the (time-independent) Kato class of functions plays a very important role. It was first introduced by T.Kato to show the essential self-adjointness of the Schrödinger operator $-\triangle + V$ on $C_0^\infty(\mathbb{R}^d)$. In [Zha96a] and [Zha96b], Qi S. Zhang generalized the notion of Kato class to the time-dependent case. In this section we will explore some properties of the classes $\mathcal{TK}_{d-1}^c$ and $\mathcal{TK}_{d-2}^c$ introduced by Qi S. Zhang. These classes will be used later as assumptions on the lower order terms of the equation $(*)$.

1.1.1 Time-dependent Kato class $\mathcal{TK}_{d-1}^c$ and its properties

Definition 1.1.1. A measurable function f on $[0,\infty) \times \mathbb{R}^d$ is said to be in the time-dependent Kato class $\mathcal{TK}_{d-1}^c$ if

$$\lim_{h\to 0} N_h^c(f) = 0,$$

where

$$N_h^c(f) := \sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt + \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d} \int_{t-h}^t \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(s,y)|dyds.$$

Here $c > 0$ is a given constant and $f(\cdot,\cdot)$ is extended to $\mathbb{R}\times\mathbb{R}^d$ by 0.

We use the notation $\mathcal{TK}_{d-1}^c$ here because this class is the natural extension of the (time-independent) Kato class $\mathcal{K}_{d-1}$.

Example 1.1.2. (a) (Time-indepedent Kato class $\mathcal{K}_{d-1}$) Suppose $d \geq 3$. Let a measurable function $f : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ be time-indepedent, i.e. $f(t,x) = f(x)$, and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x;r)} \frac{|f(y)|}{|x-y|^{d-1}} dy = 0. \tag{1.1}$$

Then $f \in \mathcal{TK}_{d-1}^c$ for any $c > 0$. The reader is referred to [KS06, Proposition 2.3] for a proof of this fact. The class of functions which satisfy (1.1) is called (time-independent) Kato class $\mathcal{K}_{d-1}$.

(b) If a measurable function f on $[0,\infty)\times\mathbb{R}^d$ is bounded, then $f\in\mathcal{TK}^c_{d-1}$ for any $c>0$.
(c) For $p,q\in[1,\infty]$ we denote $L^p=L^p(\mathbb{R}^d)$, $L^{p,q}=L^q(\mathbb{R},L^p)$. If a measurable function f on $[0,\infty)\times\mathbb{R}^d$ has compact support and $f\in L^{p,q}$ (here f is regarded as 0 outside $[0,\infty)\times\mathbb{R}^d$) with $\frac{d}{2p}+\frac{1}{q}<\frac{1}{2}$, then $f\in\mathcal{TK}^c_{d-1}$ for any $c>0$, see [Zha97a, Proposition 2.1] for a proof.

Remark 1.1.3. If $f\in\mathcal{TK}^c_{d-1}$, then f is locally integrable. Since $\lim_{h\to0}N^c_h(f)=0$, we can find a small enough $h>0$ such that $N^c_h(f)<\infty$. For any $(t',x')\in[0,\infty)\times\mathbb{R}^d$, let $s=t'-\frac{h}{2}$, then

$$\begin{aligned}N^c_h(f)\geq&\int_s^{t'+\frac{h}{2}}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-x'|^2}{t-s})|f(t,x)|dxdt\\ \geq&\int_{s+\frac{h}{4}}^{t'+\frac{h}{4}}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-x'|^2}{t-s})|f(t,x)|dxdt\\ \geq&C\int_{s+\frac{h}{4}}^{t'+\frac{h}{4}}\int_{|x-x'|\leq h}|f(t,x)|dxdt,\end{aligned}$$

where

$$C=\inf_{\substack{t\in[t'-\frac{h}{4},t'+\frac{h}{4}]\\ |x-x'|\leq h}}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-x'|^2}{t-s})>0.$$

Therefore $\int_{s+\frac{h}{4}}^{t'+\frac{h}{4}}\int_{|x-x'|\leq h}|f(t,x)|dxdt<\infty$ and f is locally integrable.

Next we prove some properties of the class $\mathcal{TK}^c_{d-1}$ and these properties will be used in the subsequent sections.

Proposition 1.1.4. *Suppose $f(t,x)\in\mathcal{TK}^c_{d-1}$, then $N^c_l(f)<\infty$ for any $l>0$.*

Proof. For any $s<s_1<t$, we have the following inequality

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-c\frac{|z-y|^2}{s_1-s})\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s_1)^{\frac{d+1}{2}}}\exp(-c\frac{|x-z|^2}{t-s_1})dz\\ \geq&\frac{1}{(t-s)^{\frac{1}{2}}}\int_{\mathbb{R}^d}\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-c\frac{|z-y|^2}{s_1-s})\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s_1)^{\frac{d}{2}}}\exp(-c\frac{|x-z|^2}{t-s_1})dz\\ =&\frac{1}{(t-s)^{\frac{1}{2}}}\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s)^{\frac{d}{2}}}\exp(-c\frac{|y-x|^2}{t-s})\\ =&\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|y-x|^2}{t-s})\end{aligned}\tag{1.2}$$

Suppose $h > 0$ is such that $N_h^c(f) < \infty$, then

$$\begin{aligned}&\int_s^{s+2h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt\\ \le& N_h^c(f)+\int_{s+h}^{s+2h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt\end{aligned}$$

Let $s + h = s_1$, then by (1.2)

$$\begin{aligned}&\int_{s+h}^{s+2h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt\\ \le&\int_{\mathbb{R}^d}\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(h)^{\frac{d}{2}}}\exp(-c\frac{|z-y|^2}{h})dz\int_{s+h}^{s+2h}\int_{\mathbb{R}^d}\frac{1}{(t-s_1)^{\frac{d+1}{2}}}\exp(-c\frac{|x-z|^2}{t-s_1})|f(t,x)|dxdt\\ \le&\int_{\mathbb{R}^d}\frac{(2c)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-c\frac{|z-y|^2}{s_1-s})dz\cdot N_h^c(f)\\ \le& N_h^c(f)\end{aligned}$$

Therefore we get

$$\int_s^{s+2h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt\le 2N_h^c(f).$$

Similarly we can prove for all $n\in\mathbb{N}$

$$\int_s^{s+nh}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt\le nN_h^c(f).$$

Then it is easy to see that the propostion is true. □

The following lemma is taken from [LS00, Proposition 2.4], for the readers' convenience we give a proof here.

Lemma 1.1.5. *Suppose $f \in \mathcal{TK}_{d-1}^c$ and f is considered to be 0 outside $[0,\infty)\times\mathbb{R}^d$, then for any nonnegative $\phi\in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}}\phi(\xi)d\xi = 1$, we have $N_h^c(f*\phi)\le N_h^c(f)$. Here $f*\phi$ denotes the convolution of ϕ and f, namely $f*\phi(\xi)=\int f(\xi-\eta)\phi(\eta)d\eta$.*

Proof. For each fixed $(s,y)\in\mathbb{R}\times\mathbb{R}^d$, let

$$g_{s,y}(t,x)=\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s}).$$

Then

$$\begin{aligned}
&\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f*\phi|(t,x)dxdt\\
&=\int_s^{s+h}\int_{\mathbb{R}^d} g_{s,y}(t,x)\Big|\int_{\mathbb{R}^{d+1}} f(t-\tau,x-z)\phi(\tau,z)d\tau dz\Big|dxdt\\
&\le\int_s^{s+h}\int_{\mathbb{R}^d} g_{s,y}(t,x)\int_{\mathbb{R}^{d+1}}|f(t-\tau,x-z)|\phi(\tau,z)d\tau dzdxdt\\
&=\int_{\mathbb{R}^{d+1}}\phi(\tau,z)d\tau dz(\int_s^{s+h}\int_{\mathbb{R}^d} g_{s,y}(t,x)|f(t-\tau,x-z)|dxdt).
\end{aligned}$$

Let $x-z=x', t-\tau=t'$, then

$$\begin{aligned}
&\int_s^{s+h}\int_{\mathbb{R}^d} g_{s,y}(t,x)|f(t-\tau,x-z)|dxdt\\
&=\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t-\tau,x-z)|dxdt\\
&=\int_{s-\tau}^{s-\tau+h}\int_{\mathbb{R}^d}\frac{1}{(t'-(s-\tau))^{\frac{d+1}{2}}}\exp(-c\frac{|x-z-y'|^2}{(t'-(s-\tau))})|f(t',x')|dx'dt'\\
&\le\sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt.
\end{aligned}$$

Note that $\int_{\mathbb{R}^{d+1}}\phi(\tau,z)d\tau dz=1$, therefore we have

$$\begin{aligned}
&\sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f*\phi(t,x)|dxdt\\
&\le\sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt.
\end{aligned}$$

Similarly we can show

$$\begin{aligned}
&\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f*\phi(s,y)|dyds\\
&\le\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(s,y)|dyds.
\end{aligned}$$

Therefore $N_h^c(f*\phi)\le N_h^c(f)$. $\square$

Lemma 1.1.6. *Suppose $f \in \mathcal{TK}^c_{d-1}$ and f is considered to be 0 outside $[0,\infty) \times \mathbb{R}^d$, then there exist nonnegative functions $\varphi_n \in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}} \varphi_n(\xi)d\xi = 1$ such that $f * \varphi_n \in C_b^\infty(\mathbb{R}^{d+1})$. Moreover, φ_n can be chosen such that*

$$supp(\varphi_n) \in \big\{x \in \mathbb{R}^{d+1} : |x| \leq \frac{1}{n}\big\}.$$

Proof. First we can find a nonnegative $\phi \in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}} \phi(\xi)d\xi = 1$ and

$$supp(\phi) \subset \big\{\xi \in \mathbb{R}^{d+1} : |\xi| \leq \frac{1}{2}\big\}.$$

Define $\phi_n(\xi) = n^{(d+1)}\phi(n\xi)$, then $\int_{\mathbb{R}^{d+1}} \phi_n(\xi)d\xi = 1$ and

$$supp(\phi_n) \subset \big\{\xi \in \mathbb{R}^{d+1} : |\xi| \leq \frac{1}{2n}\big\}.$$

Let

$$g_n(t,x) = \begin{cases} \frac{C_n}{t^{(d+1)/2}} \cdot \exp(-c\frac{|x|^2}{t}), & \text{if } 0 < t \leqslant \frac{1}{2n},\ |x| \leqslant \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

where C_n is chosen such that $\int_{\mathbb{R}^{d+1}} g_n(t,x)dxdt = 1$.

Then for any $(t,x) \in \mathbb{R}^{d+1}$,

$$\begin{aligned} |f * g_n|(t,x) &= |\int_{\mathbb{R}^{d+1}} g_n(t-s, x-y)f(s,y)dyds| \\ &\leq \int_{t-\frac{1}{2n}}^t \int_{\mathbb{R}^d} \frac{C_n}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(s,y)|dyds \leq C_n N^c_{\frac{1}{2n}}(f), \end{aligned}$$

namely $f * g_n$ is bounded.

Let $\varphi_n = g_n * \phi_n$, then

$$\begin{aligned} f * \varphi_n &= f * (g_n * \phi_n) = (f * g_n) * \phi_n \\ &= \int_{\mathbb{R}^{d+1}} \phi_n(\xi - \eta) \cdot \big(f * g_n(\eta)\big)d\eta \end{aligned}$$

Since $f * g_n$ is bounded and $\phi_n \in C_0^\infty(\mathbb{R}^{d+1})$, we have $f * \varphi_n \in C_b^\infty(\mathbb{R}^{d+1})$. $\square$

Now suppose that $f \in \mathcal{TK}^c_{d-1}$, by Remark 1.1.3 we know that f is locally integrable. For any compact $K \subset [0,\infty) \times \mathbb{R}^d$, we can define a finite measure $\mu(d\xi) := \mathbf{1}_K(\xi) \cdot |f|(\xi)m(d\xi)$ on $([0,\infty) \times \mathbb{R}^d, \mathcal{B})$, where m is the Lebesgue measure on $\mathbb{R}^{d+1}$. The following lemma is just a straightforward computation.

Lemma 1.1.7. *For each $(s, y) \in [0, \infty) \times \mathbb{R}^d$, define*

$$g_{s,y}(t, x) := \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s}), \ \text{if } (t, x) \in (s, \infty) \times \mathbb{R}^d$$

and $g_{s,y}(t, x) := 0$ if $(t, x) \in [0, s] \times \mathbb{R}^d$. Then the family $\{g_{s,y}(t, x)\}_{(s,y)\in[0,\infty)\times\mathbb{R}^d}$ is uniformly integrable with respect to the measure μ.

Proof. For any $a > 0$, let $h(a) := a^{-\frac{2}{d+1}}$. Then

$$\begin{aligned}
&\int_{\{(t,x):g_{s,y}(t,x)>a\}} g_{s,y}(t, x) d\mu \\
=&\int_{\{(t,x):g_{s,y}(t,x)>a\}\cap K} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(t, x)|dxdt \\
\leq&\int_s^{s+h(a)} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(t, x)|dxdt \\
\leq&N^c_{h(a)}(f).
\end{aligned}$$

Since $h(a)$ tends to 0 as $a \to \infty$ and $f \in \mathcal{TK}^c_{d-1}$, so we have

$$\lim_{a\to\infty} \int_{\{(t,x):g_{s,y}(t,x)>a\}} g_{s,y}(t, x) d\mu = 0.$$

Therefore $\{g_{s,y}(t, x)\}_{(s,y)\in[0,\infty)\times\mathbb{R}^d}$ is uniformly integrable with respect to the measure μ. □

The following proposition is an improved version of [LS00, Proposiiton 2.4(ii)]. It plays a crucial role in the subsequent sections of this chapter.

Proposition 1.1.8. *Let φ_n be as in Lemma 1.1.6 and $f \in \mathcal{TK}^c_{d-1}$, then for any compact $K \subset [0, \infty) \times \mathbb{R}^d$,*

$$\lim_{n\to\infty} N^c_h(\mathbf{1}_K|f * \varphi_n - f|) = 0.$$

Proof. For fixed $(s, y) \in [0, \infty) \times \mathbb{R}^d$, let

$$A = [s, s+h] \times \mathbb{R}^d, \ \xi = (t, x), \ g_{s,y}(t, x) = \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})$$

Then

$$\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f*\varphi_n-f|\cdot\mathbf{1}_K(t,x)dxdt$$
$$=\int_A|f*\varphi_n-f|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi=\int_A|\int_{\mathbb{R}^{d+1}}f(\xi-\eta)\varphi_n(\eta)d\eta-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$=\int_A|\int_{\mathbb{R}^{d+1}}(f(\xi-\eta)-f(\xi))\varphi_n(\eta)d\eta|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$\le\int_{\mathbb{R}^{d+1}}\varphi_n(\eta)d\eta\int_A|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi.$$

Set $K^1:=\{\xi\in\mathbb{R}^{d+1}:d(\xi,K)\leqslant 1\}$, then by Lusin's theorem, for a given $\delta>0$, there exists a closed set $F^\delta\subset K^1$ and a continuous function f_δ on $\mathbb{R}^{d+1}$ with compact support such that

$$m(K^1\setminus F^\delta)<\delta\quad\text{and}\quad f_\delta=f\text{ on }F^\delta,$$

here m is the Lebesgue measure. If $|\eta|\leqslant 1$, then

$$\int_A|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$=\int_{A\cap K^1}|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$\le\int_{A\cap F^\delta\cap(F^\delta+\eta)}|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi+$$
$$+\int_{\left(K^1\setminus F^\delta\right)\bigcup\left((K^1\setminus F^\delta)+\eta\right)}|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$\le\int_{A\cap F^\delta\cap(F^\delta+\eta)}|f_\delta(\xi-\eta)-f_\delta(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi+\int_C|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi$$
$$=\mathrm{I}+\mathrm{II}$$

where $C=\left(K^1\setminus F^\delta\right)\bigcup\left((K^1\setminus F^\delta)+\eta\right)$ with $m(C)<2\delta$.

Suppose $\epsilon>0$ is any given constant. By Lemma 1.1.7, the family $\{g_{s,y}(\xi)\}_{(s,y)}$ is uniformly integrable with respect to the finite measure $\mathbf{1}_{K^1}(\xi)\cdot|f|(\xi)m(d\xi)$ and note that $m(C)<2\delta$, we can choose δ small enough such that

$$\begin{aligned}\mathrm{II}&=\int_C|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi\\&\le\int_C|f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi+\int_{C+\eta}|f(\xi)|g_{s',y'}(\xi)\cdot\mathbf{1}_{K^1}(\xi)d\xi<\epsilon.\end{aligned}$$

Since the above f_δ is continuous with compact support, then we can choose n_0 large enough such that $|f_\delta(\xi-\eta)-f_\delta(\xi)|<\epsilon$ whenever $|\eta|\leq \frac{1}{n_0}$. Since

$$supp(\varphi_n)\in \big\{x\in\mathbb{R}^{d+1}:|x|\leq \frac{1}{n}\big\},$$

we have for $n\geq n_0$,

$$\begin{aligned}\mathrm{I}=&\int_{A\cap F^\delta\cap(F^\delta+\eta)}|f_\delta(\xi-\eta)-f_\delta(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi\\ \leq&\epsilon\int_{A\cap F^\delta\cap(F^\delta+\eta)}g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi\leq\epsilon\int_A g_{s,y}(\xi)d\xi.\end{aligned}$$

But $\int_A g_{s,y}(\xi)d\xi$ is a constant, so we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^{d+1}}\varphi_n(\eta)d\eta\int_A|f(\xi-\eta)-f(\xi)|g_{s,y}(\xi)\cdot\mathbf{1}_K(\xi)d\xi=0.$$

Therefore we have

$$\lim_{n\to\infty}\sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})\mathbf{1}_K|f*\varphi_n-f|(t,x)dxdt=0.$$

In the same way we can also prove

$$\lim_{n\to\infty}\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})\mathbf{1}_K|f*\varphi_n-f|(s,y)dyds=0.$$

□

1.1.2 Time-dependent Kato class $\mathcal{TK}^c_{d-2}$

Similarly to the above section we just collect the same facts to $\mathcal{TK}^\alpha_{d-2}$.

Definition 1.1.9. A measurable function f on $[0,\infty)\times\mathbb{R}^d$ is said to be in the time-dependent Kato class $\mathcal{TK}^c_{d-2}$ if

$$\lim_{h\to 0}M^c_h(f)=0,$$

where

$$\begin{aligned}M^c_h(f):=&\sup_{(s,y)\in\mathbb{R}\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,x)|dxdt+\\ &\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\int_{t-h}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(s,y)|dyds.\end{aligned}$$

Here $c>0$ is a given constant and $f(\cdot,\cdot)$ is regarded as 0 outside $[0,\infty)\times\mathbb{R}^d$.

We use the notation $\mathcal{TK}^c_{d-2}$ here because this class is the natural extension of the (time-independent) Kato class $\mathcal{K}_{d-2}$.

Example 1.1.10. (a) (Time-indepedent Kato class $\mathcal{K}_{d-2}$) Suppose $d \geq 3$. Let a measurable function $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be time-indepedent, i.e. $f(t, x) = f(x)$, and

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x;r)} \frac{|f(y)|}{|x-y|^{d-2}} dy = 0. \tag{1.3}$$

Then $f \in \mathcal{TK}^c_{d-2}$ for any $c > 0$, see [KS06, Proposition 2.3] for a proof. The class of functions which satisfy (1.3) is called (time-independent) Kato class $\mathcal{K}_{d-2}$.
(b) If a measurable function f on $[0, \infty) \times \mathbb{R}^d$ is bounded, then $f \in \mathcal{TK}^c_{d-2}$ for any $c > 0$.
(c) If a measurable function f on $[0, \infty) \times \mathbb{R}^d$ has compact support and $f \in L^{p,q}$ with $\frac{d}{2p} + \frac{1}{q} < 1$, then $f \in \mathcal{TK}^c_{d-2}$ for any $c > 0$. The reader is referred to [Zha96b, Proposition 2.1] for a proof.

Corresponding lemmas also hold in this section.

Proposition 1.1.11. *(i) If $f \in \mathcal{TK}^c_{d-2}$, then f is locally integrable.*
(ii) If $f(t, x) \in \mathcal{TK}^c_{d-2}$, then $M^c_l(f) < \infty$ for any $l > 0$.

Lemma 1.1.12. *Suppose $f \in \mathcal{TK}^c_{d-2}$, then for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}} \phi(\xi) d\xi = 1$, we have $M^c_h(f * \phi) \leq M^c_h(f)$.*

Lemma 1.1.13. *Suppose $f \in \mathcal{TK}^c_{d-2}$ and f is considered to be 0 outside $[0, \infty) \times \mathbb{R}^d$, then there exist nonnegative functions $\Phi_n \in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}} \Phi_n(\xi) d\xi = 1$ such that $f * \Phi_n \in C_b^\infty(\mathbb{R}^{d+1})$. Moreover, Φ_n can be chosen such that*

$$supp(\Phi_n) \in \{\xi \in \mathbb{R}^{d+1} : |\xi| \leq \frac{1}{n}\}.$$

Proposition 1.1.14. *Let Φ_n be as in Lemma 1.1.13 and $f \in \mathcal{TK}^c_{d-2}$, then for any compact $K \subset [0, \infty) \times \mathbb{R}^d$,*

$$\lim_{n \to \infty} M^c_h(\mathbf{1}_K |f * \Phi_n - f|) = 0.$$

1.2 Assumptions on the coefficients and the notion of a weak fundamental solution

In this section we give our assumptions on the coefficients of the equation

$$\nabla(a(t, x) \cdot \nabla u(t, x)) + b(t, x) \cdot \nabla u(t, x) + V(t, x) \cdot u(t, x) - \partial_t u(t, x) = 0 \quad (*)$$

in $[0,T]\times\mathbb{R}^d$, where $T<\infty$ is fixed throughout this chapter. Here we use the notation

$$\nabla(a(t,x)\cdot\nabla u)=\sum_{i,j=1}^{d}\partial_{x_i}a_{ij}\partial_{x_j}u,\quad b\cdot\nabla u=\sum_{j=1}^{d}b_j\partial_{x_j}u.$$

Assumption 1.2.1. We assume $a(t,x)=(a_{ij}(t,x))$ satisfy:
(i) The matrix $a(t,x)$ is uniformly elliptic, i.e. there exist constants $\lambda_0,\lambda_1>0$ such that

$$\lambda_0|\xi|^2\leqslant\sum_{i,j=1}^{d}a_{ij}(t,x)\xi_i\xi_j\leqslant\lambda_1|\xi|^2.$$

(ii) Each $a_{ij}(t,x)$ are Hölder continuous in t and x, i.e. there exists constant $0<\beta\le 1$ such that for all $x,x'\in\mathbb{R}^d, t,t'\in[0,T]$,

$$|a_{ij}(t,x)-a_{ij}(t',x')|\leqslant A(|x-x'|^{\beta}+|t-t'|^{\frac{\beta}{2}}).$$

(iii) $\frac{\partial}{\partial x_i}a_{ij}(t,x)$ are bounded and Hölder continuous in x.

Under Assumption 1.2.1, we know that for the equation

$$\nabla(a(t,x)\cdot\nabla u(t,x))-\partial_t u(t,x)=0,$$

there exists a classical fundamental solution $p(t,x;s,y)$ satisfying the following estimates: there exist positive constants c_0,α_0,C_0,α such that for any $x,y\in\mathbb{R}^d, 0\le s<t\le T$

$$\text{(i)}\ \frac{c_0}{(t-s)^{\frac{d}{2}}}\exp(-\alpha_0\frac{|x-y|^2}{t-s})\le p(t,x;s,y)\le\frac{C_0}{(t-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s}),\tag{1.4}$$

$$\text{(ii)}\ |\nabla_x p(t,x;s,y)|\le\frac{C_0}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s}).\tag{1.5}$$

The above estimates can be found in [Aro68, LSU67, Fri64].

For the lower order terms of $(*)$, we need the following assumptions.

Assumption 1.2.2. $|b|\in\mathcal{TK}_{d-1}^{\alpha_1}$ for some $\alpha_1<\frac{\alpha}{2}$.

Assumption 1.2.3. $V\in\mathcal{TK}_{d-2}^{\alpha_2}$ for some $\alpha_2<\frac{\alpha}{4}$.

Here α is the constant appearing in the Gaussian esitmates (1.4) and (1.5) for $p(t,x;s,y)$.

Now we introduce the notions of weak solutions and weak fundamental solutions to $(*)$.

Definition 1.2.4. Suppose that $0 \leq s < T$, a weak solution of

$$\nabla(a \cdot \nabla u) + b \cdot \nabla u + V \cdot u - \partial_t u = 0 \tag{$*$}$$

in $[s,T] \times \mathbb{R}^d$ is a function u such that

$$\begin{aligned} u \in C([s,T]; L^2(\mathbb{R}^d)) &\cap L^2((s,T); H^1(\mathbb{R}^d)), \\ b \cdot \nabla u \in &L^1((s,T) \times \mathbb{R}^d), \\ Vu \in &L^1((s,T) \times \mathbb{R}^d), \\ \int_s^T \int_{\mathbb{R}^d} (\nabla u \cdot a \cdot \nabla \phi - &\phi b \cdot \nabla u - Vu\phi - u\partial_t \phi) dxdt = 0, \\ &\forall \phi \in C_0^\infty([s,T] \times \mathbb{R}^d), \end{aligned}$$

where H^1 denotes the Sobolev space of square integrable functions with the distributional derivatives in L^2.

Definition 1.2.5. A function $G(t,x;s,y)$ is called a weak fundamental solution to the parabolic equation $(*)$ if

$$u_s(t,x) = \int_{\mathbb{R}^d} G(t,x;s,y) f(y) dy, \ \ s \leq t \leq T$$

is a weak solution of $(*)$ in $[s,T] \times \mathbb{R}^d$ for all $0 \leq s \leq T$ and all $f \in L^1 \cap L^\infty$.

1.3 Construction of $G(t,x;s,y)$

In order to solve the original equation $(*)$, as an intermediate step, we first consider equations with smooth coefficients.

Since $|b| \in \mathcal{TK}_{d-1}^{\alpha_1}$, by Lemma 1.1.6 and Proposition 1.1.8, we can find a sequence of functions $\varphi_n \in C_0^\infty(\mathbb{R}^{d+1})$ such that

$$b_n := b * \varphi_n = (b^1 * \varphi_n, \cdots, b^d * \varphi_n) \in C_b^\infty(\mathbb{R}^{d+1})$$

and for any compact set $K \subset [0,\infty) \times \mathbb{R}^d, h > 0$,

$$\lim_{n \to \infty} N_h^{\alpha_1}(\mathbf{1}_K |b_n - b|) = 0. \tag{1.6}$$

Remark 1.3.1. From Lemma 1.1.5, it is easily seen that there exists a constant $\kappa > 1$ such that for any $h > 0$ and $n \in \mathbb{N}$

$$N_h^{\alpha_1}(|b_n|) \leq \kappa N_h^{\alpha_1}(|b|). \tag{1.7}$$

Similarly, since $V \in \mathcal{TK}^{\alpha_2}_{d-2}$, by Lemma 1.1.13 and Proposition 1.1.14, we can find $\Phi_k \in C_0^\infty(\mathbb{R}^{d+1})$ such that

$$V_k = V * \Phi_k \in C_b^\infty(\mathbb{R}^{d+1})$$

and for any compact set $K \subset [0, \infty) \times \mathbb{R}^d, h > 0$,

$$\lim_{k\to\infty} M_h^{\alpha_2}(\mathbf{1}_K |V_k - V|) = 0. \tag{1.8}$$

Let $G_{nk}(t, x; s, y)$ be the weak fundamental solution for the parabolic equation with smooth coefficients b_n, V_k:

$$\nabla(a \cdot \nabla u) + b_n \cdot \nabla u + V_k u - \partial_t u = 0. \tag{1.9}$$

In this section we construct a function $G(t, x; s, y)$ as a limit of $G_{nk}(t, x; s, y)$.

1.3.1 A priori estimates

In this section we explain Qi S. Zhang's method to obtain two-sided Gaussian estimates for the fundamental solution $G_{nk}(t, x; s, y)$ of the parabolic equation (1.9), for more details see [Zha97a] and [Zha97b].

In this section we will use the following three inequalities (1.11)-(1.13) very often; their proofs can be found in [Zha97a] (see also [Ria07]). First we introduce some notations, let

$$\begin{aligned}\Gamma_c(t, x; s, y) &:= \frac{1}{(t-s)^{\frac{d}{2}}} \exp(-c\frac{|x-y|^2}{t-s}),\\ \Psi_c(t, x; s, y) &:= \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s}).\end{aligned} \tag{1.10}$$

Lemma 1.3.2. *Let $0 < c_1 < c_2$, then for any c with $0 < c < (c_2 - c_1) \wedge c_1$, there exists a constant $C > 0$ depending on c_1, c_2, c such that for any $s < \tau < t$ and $x, y, z \in \mathbb{R}^d$,*

$$(i)\ \frac{\Gamma_{c_1}(t, x; \tau, z)\Gamma_{c_2}(\tau, z; , s, y)}{\Gamma_{c_1}(t, x; s, y)} \le C\big(\Gamma_c(t, x; \tau, z) + \Gamma_c(\tau, z; s, y)\big) \tag{1.11}$$

$$(ii)\ \frac{\Gamma_{c_1}(t, x; \tau, z)\Psi_{c_2}(\tau, z; , s, y)}{\Psi_{c_1}(t, x; s, y)} \le C\big(\Psi_c(t, x; \tau, z) + \Psi_c(\tau, z; s, y)\big) \tag{1.12}$$

$$(iii)\ \frac{\Psi_{c_1}(t, x; \tau, z)\Psi_{c_2}(\tau, z; , s, y)}{\Psi_{c_1}(t, x; s, y)} \le C\big(\Psi_c(t, x; \tau, z) + \Psi_c(\tau, z; s, y)\big) \tag{1.13}$$

Let us first look at the equation

$$\nabla(a \cdot \nabla u) + b_n \cdot \nabla u - \partial_t u = 0.$$

Since $b_n \in C_b^\infty(\mathbb{R}^{d+1})$, there exists a weak fundamental solution $q_n(t,x;s,y)$ for this parabolic equation. In [Zha97a] it was showed that $q_n(t,x;s,y)$ satisfies Gaussian bounds, namely there exist $\alpha', c_q, C_q > 0$ such that for any $0 \le s < t \le T$ and $x, y \in \mathbb{R}^d$,

$$\text{(i)}\ \frac{c_q}{(t-s)^{\frac{d}{2}}} \exp(-\alpha' \cdot \frac{|x-y|^2}{t-s}) \le q_n(t,x;s,y) \le \frac{C_q}{(t-s)^{\frac{d}{2}}} \exp(-\frac{\alpha}{2} \cdot \frac{|x-y|^2}{t-s}) \tag{1.14}$$

$$\text{(ii)}\ |\nabla_x q_n(t,x;s,y)| \le \frac{C_q}{(t-s)^{\frac{d+1}{2}}} \exp(-\frac{\alpha}{2} \cdot \frac{|x-y|^2}{t-s}). \tag{1.15}$$

where the constant C_q does not depend on n and only depends on the rate at which $N_h^{\alpha_1}(|b|)$ goes to 0 as $h \to 0$. For convenience here we sketch the ideas of the proof.

First we prove (1.14). By Duhamel's formula,

$$q_n(t,x;s,y) = p(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z) b_n(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau. \tag{1.16}$$

This is an integral equation, so we can formally write

$$q_n(t,x;s,y) = \sum_{i=0}^\infty J_n^i(t,x;s,y), \tag{1.17}$$

where the convergence of the series on the right-hand is shown below and $J_n^i(t,x;s,y)$ are defined inductively in the following way:

$$\begin{aligned}
J_n^0(t,x;s,y) &= p(t,x;s,y),\\
J_n^1(t,x;s,y) &= \int_s^t \int_{\mathbb{R}^d} p(t,x;\tau,z) b_n(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau,\\
&\vdots\\
J_n^{i+1}(t,x;s,y) &= \int_s^t \int_{\mathbb{R}^d} J_n^i(t,x;\tau,z) b_n(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau.
\end{aligned}$$

Recall that $\alpha_1 < \frac{\alpha}{2}$. Then by Lemma 1.3.2(ii), there exists a constant $C_1 > 0$ such that for all $x, y \in \mathbb{R}^d$, $s < t$,

$$\frac{\Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)\Psi_\alpha(\tau,z;,s,y)}{\Gamma_{\frac{\alpha}{2}}(t,x;s,y)} \le C_1\big(\Psi_{\alpha_1}(t,x;\tau,z)+\Psi_{\alpha_1}(\tau,z;s,y)\big) \tag{1.18}$$

and

$$\frac{\Psi_{\frac{\alpha}{2}}(t,x;\tau,z)\Psi_\alpha(\tau,z;,s,y)}{\Psi_{\frac{\alpha}{2}}(t,x;s,y)} \le C_1\big(\Psi_{\alpha_1}(t,x;\tau,z)+\Psi_{\alpha_1}(\tau,z;s,y)\big). \tag{1.19}$$

If $t > s$, then

$$\begin{aligned}
&|J_n^1(t,x;s,y)| \\
=&|\int_s^t\int_{\mathbb{R}^d} p(t,x;\tau,z)b_n(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau| \\
\le& C_0^2\int_s^t\int_{\mathbb{R}^d}\Gamma_\alpha(t,x;\tau,z)|b_n(\tau,z)|\Psi_\alpha(\tau,z;s,y)dzd\tau \\
\le& C_0^2\int_s^t\int_{\mathbb{R}^d}\Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|b_n(\tau,z)|\Psi_\alpha(\tau,z;s,y)dzd\tau \\
\le& C_0^2C_1\int_s^t\int_{\mathbb{R}^d}\Gamma_{\frac{\alpha}{2}}(t,x;s,y)(\Psi_{\alpha_1}(t,x;\tau,z)+\Psi_{\alpha_1}(\tau,z;s,y))|b_n(\tau,z)|dzd\tau \\
\le& C_0^2C_1N_{t-s}^{\alpha_1}(|b_n|)\cdot\Gamma_{\frac{\alpha}{2}}(t,x;s,y).
\end{aligned} \tag{1.20}$$

By Remark 1.3.1, $N_h^{\alpha_1}(|b_n|) \le \kappa N_h^{\alpha_1}(|b|)$. By induction it is easy to get that for all $i \ge 1$

$$|J_n^i(t,x;s,y)| \le C_0\big(\kappa C_0C_1N_{t-s}^{\alpha_1}(|b|)\big)^i\cdot\Gamma_{\frac{\alpha}{2}}(t,x;s,y). \tag{1.21}$$

If we choose $h_1 > 0$ sufficiently small such that $\kappa C_0C_1N_{h_1}^{\alpha_1}(|b|) < 1$, then

$$\begin{aligned}
q_n(t,x;s,y) &= \sum_{i=0}^\infty J_n^i(t,x;s,y) \\
&\le \frac{C_0}{1-\kappa C_0C_1N_{h_1}^{\alpha_1}(|b|)}\cdot\Gamma_{\frac{\alpha}{2}}(t,x;s,y), \quad 0 < t-s \le h_1.
\end{aligned} \tag{1.22}$$

To prove the Gaussian lower bound, we can choose $h_0 > 0$ sufficiently small such that

$$\begin{aligned}
q_n(t,x;s,y) &= \sum_{i=0}^{\infty} J_n^i(t,x;s,y) \\
&\geq p(t,x;s,y) - \sum_{i=1}^{\infty} |J_n^i(t,x;s,y)| \\
&\geq c_0 e^{-\alpha_0} \frac{1}{(t-s)^{\frac{d}{2}}} - \frac{C_0}{1-\kappa C_0 C_1 N_{t-s}^{\alpha_1}(|b|)} \frac{1}{(t-s)^{\frac{d}{2}}} \\
&\geq \frac{1}{2} c_0 e^{-\alpha_0} \frac{1}{(t-s)^{\frac{d}{2}}},
\end{aligned}$$

when $\frac{|x-y|^2}{t-s} \leq 1$ and $0 < t-s \leq h_0$. By a rescaling argument, it is then proved in [Zha96a] that there exist α', c_0' such that for $0 < t-s \leq h_0$

$$q_n(t,x;s,y) \geq c_0' \Gamma_{\alpha'}(t,x;s,y). \tag{1.23}$$

Therefore for $0 \leq s < t \leq T$ with $|t-s| \leq h_1 \wedge h_0$, we have

$$c_0' \Gamma_{\alpha'}(t,x;s,y) \leq q_n(t,x;s,y) \leq \frac{C_0}{1-\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|)} \cdot \Gamma_{\frac{\alpha}{2}}(t,x;s,y).$$

Then using the reproducing property of $q_n(t,x;s,y)$, we easily get the Gaussian bounds (1.14). Namely, there exist $c_q, C_q > 0$ such that for any $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}^d$,

$$c_q \Gamma_{\alpha'}(t,x;s,y) \leq q_n(t,x;s,y) \leq \frac{C_q}{(t-s)^{\frac{d}{2}}} \exp(-\frac{\alpha}{2} \cdot \frac{|x-y|^2}{t-s}) \tag{1.24}$$

Remark 1.3.3. (i) From the above argument it is easily seen that we can also define $J^i(t,x;s,y)$ inductively by

$$\begin{aligned}
J^0 =& p(t,x;s,y), \\
J^1 =& \int_s^t \int_{\mathbb{R}^d} p(t,x;\tau,z) b(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau, \\
&\vdots \\
J^{i+1} =& \int_s^t \int_{\mathbb{R}^d} J^k(t,x;\tau,z) b(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau, \ i \geq 1.
\end{aligned}$$

Similarly to (1.21) we can prove

$$|J^i(t,x;s,y)| \le C_0\big(\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)\big)^i \cdot \Gamma_{\frac{\alpha}{2}}(t,x;s,y). \tag{1.25}$$

(ii) Since h_1 is such that $\kappa C_0 C_1 N^{\alpha_1}_{h_{t-s}}(|b|) < 1$, now we can define

$$q(t,x;s,y) := \sum_{i=0}^{\infty} J^i(t,x;s,y),\ 0 < t-s \le h_1.$$

It is easily seen that for any $x, y \in \mathbb{R}^d$ and $0 < t-s \le h_1$,

$$q(t,x;s,y) = p(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} q(t,x;\tau,z) b(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau.$$

Now we try to explain how to get the gradient estimate (1.15) for $q_n(t,x;s,y)$. Formally taking ∇_x in the both sides of the equation (1.16), we have

$$\begin{aligned}\nabla_x q_n(t,x;s,y) =& \nabla_x p(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} \nabla_x q_n(t,x;\tau,z) \cdot b_n(\tau,z) \nabla_z p(\tau,z;s,y) dz d\tau \\ =& \nabla_x p(t,x;s,y) + \sum_{i=1}^{\infty} I^i_n(t,x;s,y),\end{aligned}$$

where $I^i_n(t,x;s,y)$ are defined inductively by

$$\begin{aligned} I^1_n(t,x;s,y) &= \int_s^t \int_{\mathbb{R}^d} \nabla_x p(t,x;\tau,z) b_n(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau, \\ &\vdots \\ I^{i+1}_n(t,x;s,y) &= \int_s^t \int_{\mathbb{R}^d} I^i_n(t,x;\tau,z) b_n(\tau,z) \cdot \nabla_z p(\tau,z;s,y) dz d\tau,\ i \ge 1. \end{aligned}$$

It is easily seen that

$$I^i_n(t,x;s,y) = \nabla_x J^i_n(t,x;s,y). \tag{1.26}$$

Now we use the estimate (1.5) of $p(t,x;s,y)$ and Lemma 1.3.2(iii). Similarly to the above method used to get (1.21), we can show that

$$|I^i_n(t,x;s,y)| \le C_0\big(\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)\big)^i \cdot \Psi_{\frac{\alpha}{2}}(t,x;s,y). \tag{1.27}$$

Since h_1 is such that $\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|) < 1$, then

$$
\begin{aligned}
|\nabla_x q_n(t,x;s,y)| &= \Big|\sum_{i=0}^{\infty} I_i^n(t,x;s,y)\Big| \\
&\le \frac{C_0}{1-\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|)} \cdot \Psi_{\frac{\alpha}{2}}(t,x;s,y), \quad |t-s| \le h_1.
\end{aligned}
$$

Then using the reproducing property of $q_n(t,x;s,y)$, we easily get the estimates (1.15), namely for any $0 \le s < t \le T, x, y \in \mathbb{R}^d$

$$
|\nabla_x q_n(t,x;s,y)| \le \frac{C_q}{(t-s)^{\frac{d+1}{2}}} \exp(-\frac{\alpha}{2}\cdot\frac{|x-y|^2}{t-s}). \tag{1.28}
$$

Remark 1.3.4. From the above arguments it is clear that we can also define $I^i(t,x;,s,y)$ inductively by

$$
\begin{aligned}
I^0 =& \nabla_x p(t,x;s,y), \\
I^1 =& \int_s^t \int_{\mathbb{R}^d} \nabla_x p(t,x;\tau,z) b(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau, \\
&\vdots \\
I^{i+1} =& \int_s^t \int_{\mathbb{R}^d} I^k(t,x;\tau,z) b(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau.
\end{aligned}
$$

We can also show

$$
|I^i(t,x;s,y)| \le C_0\big(\kappa C_0 C_1 N_{t-s}^{\alpha_1}(|b|)\big)^i \cdot \Psi_{\frac{\alpha}{2}}(t,x;s,y). \tag{1.29}
$$

From the definition of $J^i(t,x;s,y)$ and $q(t,x;s,y)$ it follows that

$$
I^i(t,x;s,y) = \nabla_x J^i(t,x;s,y),\ 0 < t-s \le h_1, \tag{1.30}
$$

and therefore

$$
\nabla_x q(t,x;s,y) = \nabla_x p(t,x;s,y) + \sum_{i=1}^{\infty} I^i(t,x;s,y),\ 0 < t-s \le h_1.
$$

Now we come to the equation

$$
\nabla(a\cdot\nabla u) + b_n\cdot\nabla u + V_k u - \partial_t u = 0. \tag{1.31}
$$

Suppose $G_{nk}(t,x;s,y)$ is the fundamental solution for the above equation, then again by Duhamel's formula we have

$$G_{nk}(t,x;s,y) = q_n(t,x;s,y) + \int_s^t G_{nk}(t,x;\tau,z)V_k(\tau,z)q_n(\tau,z;s,y)dzd\tau. \tag{1.32}$$

Therefore we can write

$$G_{nk}(t,x;s,y) = \sum_{i=0}^{\infty} J^i_{nk}(t,x;s,y),$$

where

$$\begin{aligned} J^0_{nk} &= q_n(t,x;s,y), \\ J^1_{nk} &= \int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z)V_k(\tau,z)q_n(\tau,z;s,y)dzd\tau, \\ &\vdots \\ J^{i+1}_{nk} &= \int_s^t \int_{\mathbb{R}^d} J^i_n(t,x;\tau,z)V_k(\tau,z)q_n(\tau,z;,s,y)dzd\tau, \ i \geq 1. \end{aligned}$$

Recall that now $\alpha_2 < \frac{\alpha}{4}$. By Lemma 1.3.2(i), there exists a constant C_2 such that for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$\frac{\Gamma_{\frac{\alpha}{4}}(t,x;\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;,s,y)}{\Gamma_{\frac{\alpha}{4}}(t,x;s,y)} \leq C_2\big(\Gamma_{\alpha_2}(t,x;\tau,z) + \Gamma_{\alpha_2}(\tau,z;s,y)\big) \tag{1.33}$$

Using the a priori estimates (1.14) for $q_n(t,x;s,y)$, we can easily get the following estimates:

$$|J^i_{nk}(t,x;s,y)| \leq C_q\big(C_qC_2M^{\alpha_2}_{t-s}(V)\big)^i \cdot \Gamma_{\frac{\alpha}{4}}(t,x;s,y), \quad 0 < t-s < h_1. \tag{1.34}$$

Then we can find an sufficiently small h_2 such that $h_2 < h_1$, $C_qC_2M^{\alpha_2}_{h_2}(V) < 1$, and

$$\begin{aligned} G_{nk}(t,x;s,y) &= \sum_{i=0}^{\infty} J^i_{nk}(t,x;s,y) \\ &\leq \frac{C_q}{1 - C_qC_2M^{\alpha_2}_{t-s}(V)} \cdot \Gamma_{\frac{\alpha}{4}}(t,x;s,y) \\ &\leq \frac{C_q}{1 - C_qC_2M^{\alpha_2}_{h_2}(V)} \cdot \Gamma_{\frac{\alpha}{4}}(t,x;s,y), \quad 0 < t-s \leq h_2. \end{aligned} \tag{1.35}$$

The Gaussian lower bound for G_{nk} can be proved similarly to (1.23). There exist $h_0', c^{''}, \alpha^{''} > 0$ such that for $0 < t - s < h_0'$,

$$G_{nk}(t, x; s, y) \geq c^{''} \Gamma_{\alpha^{''}}(t, x; s, y).$$

Then by the reproducing property of $G_{nk}(t, x; s, y)$, there exist positive constants c_G, C_G such that for $0 \leq s < t \leq T$,

$$c_G \cdot \Gamma_{\alpha''}(t, x; s, y) \leq G_{nk}(t, x; s, y) \leq C_G \Gamma_{\frac{\alpha}{4}}(t, x; s, y). \tag{1.36}$$

Remark 1.3.5. (i) From the above argument it is easily seen that we can also define $J_G^i(t, x; , s, y)$ inductively as

$$\begin{aligned}
J_G^0 &= q(t, x; s, y), \\
J_G^1 &= \int_s^t \int_{\mathbb{R}^d} q(t, x; \tau, z) V_k(\tau, z) q(\tau, z; s, y) dz d\tau, \\
&\vdots \\
J_G^{i+1} &= \int_s^t \int_{\mathbb{R}^d} J^i(t, x; \tau, z) V_k(\tau, z) q(\tau, z; s, y) dz d\tau, \ i \geq 1.
\end{aligned}$$

Similarly we can prove

$$|J_G^i(t, x; s, y)| \leq C_q \big(C_q C_2 M_{t-s}^{\alpha_2}(V)\big)^i \cdot \Gamma_{\frac{\alpha}{4}}(t, x; s, y), \quad 0 < t - s \leq h_2. \tag{1.37}$$

(ii) Since h_2 is such that $C_q C_2 M_{h_2}^{\alpha_2}(V) < 1$, we can define

$$G(t, x; s, y) = \sum_{i=0}^{\infty} J_G^i(t, x; s, y), \ 0 < t - s \leq h_2.$$

It is easily seen that for $0 \leq s < t \leq T, \ 0 < t - s \leq h_2$

$$G(t, x; s, y) = q(t, x; s, y) + \int_s^t \int_{\mathbb{R}^d} G(t, x; \tau, z) V(\tau, z) q(\tau, z; s, y) dz d\tau.$$

Remark 1.3.6. To get the formula (1.32), we used the fact that, for each fixed $(s, y) \in [0, T) \times \mathbb{R}^d$, the function $(t, x) \mapsto G_{nk}(t, x; s, y)$ for $t > s$ is a solution to the equation (1.31). We should note that for fixed $(t, x) \in (0, T] \times \mathbb{R}^d$, the function of $(s, y) \mapsto G_{nk}(t, x; s, y)$ for $s < t$ satisfies the adjoint equation

$$\nabla(a \cdot \nabla u) - \nabla(b_n u) + V_k u + \partial_s u = 0.$$

Similarly, the function of $(s,y) \mapsto q_n(t,x;s,y)$ solves

$$\nabla(a\cdot\nabla u) - \nabla(b_n u) + \partial_s u = 0.$$

So we can use Duhamel's principle to get

$$G_{nk}(t,x;s,y) = q_n(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z)V_k(\tau,z)G_{nk}(\tau,z;s,y)dzd\tau. \tag{1.38}$$

Sometimes it is more convenient to use this expression of $G_{nk}(t,x;s,y)$ instead of (1.32).

1.3.2 Convergence of $G_{nk}(t,x;s,y)$ to $G(t,x;s,y)$

In this section we will prove that $G_{nk}(t,x;s,y)$ converges locally uniformly to $G(t,x;s,y)$ as n and k goes to ∞. This kind of convergence is inspired by [KS06], where they only considered the case in which $V\equiv 0$ and $b(t,x)=b(x)$ belongs to the time-independent Kato class $\mathcal{K}_{d-1}$.

Recall that h_1 is the constant which appears in (1.22). By (1.22) we know

$$q_n(t,x;s,y) = \sum_{i=0}^{\infty} J_n^i(t,x;s,y), \quad 0 < t-s \le h_1.$$

From Remark 1.3.3, we also have

$$q(t,x;s,y) = \sum_{i=0}^{\infty} J^i(t,x;s,y), \quad 0 < t-s \le h_1.$$

Lemma 1.3.7. *Let δ be any constant such that $0<\delta<h_1$. Let $K_1, K_2 \subset \mathbb{R}^d$ be compact sets and $\theta := \{(s,t) : 0 \le s < t \le h_1\}$. Then*

$$\lim_{n\to\infty} \sup_{\substack{(s,t)\in\theta, |t-s|\ge\delta \\ x\in K_1, y\in K_2}} |J_n^1(t,x;s,y) - J^1(t,x;s,y)| = 0.$$

Proof. By (1.4) and (1.5)

$$
\begin{aligned}
&|J_n^1(t,x;s,y) - J^1(t,x;s,y)| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} p(t,x;\tau,z) b_n(\tau,z)\cdot \nabla_z p(\tau,z;s,y)dzd\tau - \\
&\qquad - \int_s^t \int_{\mathbb{R}^d} p(t,x;\tau,z) b(\tau,z)\cdot \nabla_z p(\tau,z;s,y)dzd\tau\Big| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} p(t,x;\tau,z)(b_n-b)(\tau,z)\cdot \nabla_z p(\tau,z;s,y)dzd\tau\Big| \\
\le& C_0^2 \int_s^t \int_{\mathbb{R}^d} \Gamma_\alpha(t,x;\tau,z)|b_n-b|(\tau,z)\Psi_\alpha(\tau,z;s,y)dzd\tau \qquad (1.39)\\
\le& C_0^2\Big(\int_s^t\int_{|z|>k} + \int_s^t\int_{|z|\le k}\Big)\Gamma_\alpha(t,x;\tau,z)|b_n-b|(\tau,z)\Psi_\alpha(\tau,z;s,y)dzd\tau \\
=&C_0^2(\mathrm{I}+\mathrm{II})
\end{aligned}
$$

From Remark 1.3.1, we know for any $h>0$ and $n\in\mathbb{N}$

$$N_h^{\alpha_1}(|b_n|) \le \kappa N_h^{\alpha_1}(|b|).$$

Then for $0<h_\epsilon<\frac{\delta}{2}$ and k large enough such that

$$|x-z| \ge \frac{1}{2}|z|,\ \forall x\in K_1, |z|\ge k,$$

we have

$$
\begin{aligned}
\mathrm{I} =& \int_s^t\int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}}\exp(-\alpha\cdot\frac{|x-z|^2}{t-\tau})|b_n-b|\frac{1}{(\tau-s)^{\frac{d+1}{2}}}\exp(-\alpha\cdot\frac{|z-y|^2}{\tau-s})dzd\tau \\
\le& \int_s^{t-h_\epsilon}\int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}}\exp(-\alpha\cdot\frac{|x-z|^2}{t-\tau})|b_n-b|\frac{1}{(\tau-s)^{\frac{d+1}{2}}}\exp(-\alpha\cdot\frac{|z-y|^2}{\tau-s})dzd\tau + \\
&+\int_{t-h_\epsilon}^t\int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}}\exp(-\alpha\cdot\frac{|x-z|^2}{t-\tau})|b_n-b|\frac{1}{(\tau-s)^{\frac{d+1}{2}}}\exp(-\alpha\cdot\frac{|z-y|^2}{\tau-s})dzd\tau \\
\le& h_\epsilon^{-\frac{d}{2}}\exp(-\alpha\cdot\frac{k^2}{4h_1})\int_s^{t-h_\epsilon}\int_{|z|>k}|b_n-b|\frac{1}{(\tau-s)^{\frac{d+1}{2}}}\exp(-\alpha\cdot\frac{|z-y|^2}{\tau-s})dzd\tau + \\
&+\frac{2^{\frac{d}{2}}}{\delta^{\frac{d}{2}}}\int_{t-h_\epsilon}^t\int_{|z|>k}\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\exp(-\alpha\cdot\frac{|x-z|^2}{t-\tau})|b_n-b|dzd\tau \\
\le& (\kappa+1)h_\epsilon^{-\frac{d}{2}}\exp(-\alpha\cdot\frac{k^2}{4h_1})N_{h_1}^{\alpha_1}(|b|) + (\kappa+1)(\frac{2}{\delta})^{\frac{d}{2}}N_{h_\epsilon}^{\alpha_1}(|b|)
\end{aligned}
$$

Given any $\epsilon > 0$, we can first choose h_ϵ sufficiently small such that

$$(\kappa+1)(\frac{2}{\delta})^{\frac{d}{2}} N^{\alpha_1}_{h_\epsilon}(|b|) < \frac{\epsilon}{3},$$

and then find large enough k such that

$$(\kappa+1)h_\epsilon^{-\frac{d}{2}} \exp(-\alpha \cdot \frac{k^2}{4h_1}) N^{\alpha_1}_{h_1}(|b|) < \frac{\epsilon}{3}.$$

For II we have

$$\begin{aligned}
\mathrm{II} &= \int_s^t \int_{|z|\le k} \Gamma_\alpha(t,x;\tau,z)|b_n - b|(\tau,z)\Psi_\alpha(\tau,z;s,y)dzd\tau \\
&\le \int_s^t \int_{|z|\le k} \Gamma_\alpha(t,x;\tau,z)(\mathbf{1}_{[0,h_1]\times\{|z|\le k\}}|b_n - b|)(\tau,z)\Psi_\alpha(\tau,z;s,y)dzd\tau \\
&\le C_1 N^{\alpha_1}_{h_1}((\mathbf{1}_{[0,h_1]\times\{|z|\le k\}}|b_n - b|)) \cdot \Gamma_{\frac{\alpha}{2}}(t,x;s,y) \\
&\le (\delta)^{-\frac{d}{2}} C_1 N^{\alpha_1}_{h_1}(\mathbf{1}_{[0,h_1]\times\{|z|\le k\}}|b_n - b|).
\end{aligned}$$

Since $[0,h_1]\times\{|z|\le k\}$ is compact, from (1.6), we have

$$\lim_{n\to\infty} N^{\alpha_1}_{h_1}(\mathbf{1}_{([0,h_1]\times\{|z|\le k\})} \cdot |b_n - b|) = 0.$$

So we can find n_0 such that II $< \frac{\epsilon}{3}$ when $n \ge n_0$. Therefore if $n \ge n_0$, we have

$$\sup_{\substack{(s,t)\in\theta, |t-s|\ge\delta \\ x\in K_1, y\in K_2}} |J^i_n(t,x;s,y) - J^1(t,x;s,y)| < C_0^2\epsilon.$$

Thus the lemma is proved. □

Remark 1.3.8. Recall that $\theta := \{(s,t) : 0 \le s < t \le h_1\}$. Then for any compact $K \subset \theta$, we can always find some $\delta > 0$ such that $K \subset \{(s,t) \in \theta : |t-s| \ge \delta\}$, therefore from the above lemma it follows that for any $K_1, K_2 \subset \mathbb{R}^d$ compact,

$$\lim_{n\to\infty} \sup_{\substack{(s,t)\in K \\ x\in K_1, y\in K_2}} |J^1_n(t,x;s,y) - J^1(t,x;s,y)| = 0.$$

Lemma 1.3.9. *For any compact sets $K \subset \theta$ and $K_1, K_2 \subset \mathbb{R}^d$, we have*

$$\lim_{n\to\infty} \sup_{\substack{(s,t)\in K \\ x\in K_1, y\in K_2}} |J^i_n(t,x;s,y) - J^i(t,x;s,y)| = 0, \ \forall i \ge 1.$$

Proof. The proof goes by induction argument. For $i = 1$, this has been proved. Now suppose that the lemma is true for i, then for $(s,t) \in \theta, |t-s| \geq \delta$, we have

$$\begin{aligned}
&|J_n^{i+1}(t,x;s,y) - J^{i+1}(t,x;s,y)| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} J_n^i(t,x;\tau,z)b_n(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau- \\
&\quad -\int_s^t \int_{\mathbb{R}^d} J^i(t,x;\tau,z)b(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau\Big| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} J_n^i(t,x;\tau,z)(b_n-b)(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau- \\
&\quad -\int_s^t \int_{\mathbb{R}^d} (J^i-J_n^i)(t,x;\tau,z)b(\tau,z)\cdot\nabla_z p(\tau,z;s,y)dzd\tau\Big| \\
\leq& C_0(\int_s^t \int_{\mathbb{R}^d} |J_n^i|(t,x;\tau,z)|(b_n-b)|\Psi_\alpha(\tau,z;s,y)dzd\tau \\
&\quad +\int_s^t \int_{\mathbb{R}^d} |J^i-J_n^i|(t,x;\tau,z)|b|\Psi_\alpha(\tau,z;s,y)dzd\tau) \\
=&C_0(\mathrm{I}+\mathrm{II}).
\end{aligned}$$

For I, we have

$$\mathrm{I} \leq C_0\big(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|)\big)^i \int_s^t \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|b_n-b|(\tau,z)\Psi_\alpha(\tau,z;s,y)dzd\tau.$$

So we can do the similar procedure as as we did with (1.39) to get

$$\lim_{n\to\infty} \mathrm{I}(t,x;s,y) = 0$$

uniformly for $(s,t) \in \theta, |t-s| \geq \delta$ and $x \in K_1, y \in K_2$.

For II, we have

$$\begin{aligned}
&\int_s^t \int_{\mathbb{R}^d} |(J^i-J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)|\cdot|\nabla_z p(\tau,z;s,y)|dzd\tau \qquad (1.40)\\
=&\int_s^{t-h_\epsilon} \int_{\mathbb{R}^d} |(J^i-J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)|\cdot|\nabla_z p(\tau,z;s,y)|dzd\tau+ \\
&\quad +\int_{t-h_\epsilon}^t \int_{\mathbb{R}^d} |(J^i-J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)|\cdot|\nabla_z p(\tau,z;s,y)|dzd\tau \\
\leq&\int_s^{t-h_\epsilon} \int_{\mathbb{R}^d} |(J^i-J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)|\cdot|\nabla_z p(\tau,z;s,y)|dzd\tau+
\end{aligned}$$

$$
\begin{aligned}
&+ 2C_0^2\big(\kappa C_0 C_1 N_{h_\epsilon}^{\alpha_1}(|b|)\big)^i \int_{t-h_\epsilon}^{t}\int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|b(\tau,z)|\Psi_\alpha(\tau,z;s,y)dzd\tau \\
\leq &\int_s^{t-h_\epsilon}\int_{\mathbb{R}^d} |(J^i - J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)|\cdot|\nabla_z p(\tau,z;s,y)|dzd\tau + 2C_0^2(\frac{2}{\delta})^{\frac{d}{2}}\big(\kappa C_0 C_1 N_{h_\epsilon}^{\alpha_1}(|b|)\big)^{i+1} \\
\leq &\mathrm{II}_1 + \mathrm{II}_2
\end{aligned}
$$

For II_1, we can choose k large enough such that

$$
|x-z| \geq \frac{1}{2}|z|,\ \forall x \in K_1, |z| \geq k,
$$

then

$$
\begin{aligned}
\mathrm{II}_1 = &\int_s^{t-h_\epsilon}\Big(\int_{|z|>k} + \int_{|z|\leq k}\Big)|(J^i - J_n^i)(t,x;\tau,z)|\cdot|b(\tau,z)||\nabla_z p(\tau,z;s,y)|dzd\tau \\
\leq &\ 2C_0^2(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|))^i \int_s^{t-h_\epsilon}\int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|b(\tau,z)|\Psi_\alpha(\tau,z;s,y)dzd\tau + \\
&\sup_{\substack{(\tau,t)\in\theta, |t-\tau|\geq h_\epsilon \\ x\in K_1, |z|\leq k}} |(J^i - J_n^i)(t,x;\tau,z)| \int_s^{t-h_\epsilon}\int_{|z|\leq k} |b(\tau,z)|\Psi_\alpha(\tau,z;s,y)dzd\tau \\
\leq &\ 2C_0^2(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|))^i h_\epsilon^{-\frac{d}{2}} \exp(-\alpha\cdot\frac{k^2}{8h_1})N_{h_1}^{\alpha_1}(|b|) + C' \sup_{\substack{(\tau,t)\in\theta, |t-\tau|\geq h_\epsilon \\ x\in K_1, |z|\leq k}} |(J^i - J_n^i)(t,x;\tau,z)|,
\end{aligned}
$$

where κ is the constant from Remark 1.3.1.

Given $\forall\epsilon > 0$, we can first choose h_ϵ sufficiently small such that $\mathrm{II}_2 < \frac{\epsilon}{3}$, and then find large enough k such that

$$
2C_0^2(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|))^i h_\epsilon^{-\frac{d}{2}} \exp(-\alpha\cdot\frac{k^2}{8h_1})N_{h_1}^{\alpha_1}(|b|) < \frac{\epsilon}{3}.
$$

Since

$$
\lim_{n\to\infty} \sup_{\substack{(\tau,t)\in\theta, |t-\tau|\geq h_\epsilon \\ x\in K_1, |z|\leq k}} |(J^i - J_n^i)(t,x;\tau,z)| = 0,
$$

We get

$$
\lim_{n\to\infty} \mathrm{II}(t,x;s,y) = 0
$$

uniformly for $(s,t)\in\theta, |t-s|\geq\delta$ and $x\in K_1, y\in K_2$. For a general compact set $K\subset\theta$, the statement is still true. □

Remark 1.3.10. From (1.26) and (1.30), recall that

$$
I_n^i(t,x;s,y) = \nabla_x J_n^i(t,x;s,y)
$$

and

$$I^i(t,x;s,y) = \nabla_x I^i(t,x;s,y).$$

If we replace $J_n^i(t,x;s,y), J_n(t,x;s,y)$ with $I_n^i(t,x;s,y)$ and $I^i(t,x;s,y)$ in the above lemma, it is still true. The idea of the proof is the same, so we omit it.

Theorem 1.3.11. *For any compact sets $K \subset \theta$ and $K_1, K_2 \subset \mathbb{R}^d$, $q_n(t,x;s,y)$ converges uniformly to $q(t,x;s,y)$ on $\{(t,x;s,y) : (s,t) \in K, x \in K_1, y \in K_2\}$.*

Proof. For $(s,t) \in \theta, |t-s| \geq \delta$, $q_n(t,x;s,y) = \sum_{i=0}^{\infty} J_i^n(t,x;s,y)$, and

$$|J_n^i(t,x;s,y)| \leq C_0\big(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|)\big)^i \cdot \Gamma_{\frac{\alpha}{2}}(t,x;s,y) \leq \frac{C_0\big(\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|)\big)^i}{(\delta)^{\frac{d}{2}}},$$

since $\kappa C_0 C_1 N_{h_1}^{\alpha_1}(|b|) < 1$, the above series converges absolutely. Using lemma 1.3.9 and a standard argument we can easily get

$$q_n(t,x;s,y) \to q(t,x;s,y)$$

uniformly for $(s,t) \in \theta, |t-s| \geq \delta$ and $x \in K_1, y \in K_2$. For a general compact subset K of θ, the lemma still holds. □

From Remark 1.3.10, we also have the following theorem.

Theorem 1.3.12. *For any compact sets $K \subset \theta$ and $K_1, K_2 \subset \mathbb{R}^d$, $\nabla_x q_n(t,x;s,y)$ converges uniformly to $\nabla_x q(t,x;s,y)$ on $\{(t,x;s,y) : (s,t) \in K, x \in K_1, y \in K_2\}$.*

Now we proceed to prove the convergence of $G_{nk}(t,x;s,y)$ to $G(t,x;s,y)$. Recall that $h_2 > 0$ is the constant which makes the inequality (1.35) hold. By (1.35), we know for $0 < t-s \leq h_2$,

$$G_{nk}(t,x;s,y) = \sum_{i=0}^{\infty} J_{nk}^i(t,x;s,y).$$

From Remark 1.3.5, we also have

$$G(t,x;s,y) = \sum_{i=0}^{\infty} J^i(t,x;s,y), \quad 0 < t-s \leq h_2.$$

Lemma 1.3.13. *Let δ be a constant such that $0 < \delta < h_2$, $K_1, K_2 \subset \mathbb{R}^d$ be compact sets, then*

$$\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in\theta,|t-s|\geq\delta \\ x\in K_1, y\in K_2}} |J^i_{nk}(t,x;s,y) - J^i_G(t,x;s,y)| = 0, \ \forall i \geq 1.$$

Proof. Step 1: We consider $i = 1$. Then

$$\begin{aligned}
&|J^1_{nk}(t,x;s,y) - J^1_G(t,x;s,y)| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z)V_k(\tau,z)q_n(\tau,z;s,y)dzd\tau - \\
&-\int_s^t \int_{\mathbb{R}^d} q(t,x;\tau,z)V(\tau,z)q(\tau,z;s,y)dzd\tau\Big| \\
=&\Big|\int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z)V_k(\tau,z)(q_n-q)(\tau,z;s,y)dzd\tau \\
&+\int_s^t \int_{\mathbb{R}^d} q_n(t,x;\tau,z)(V_k-V)(\tau,z)q(\tau,z;s,y)dzd\tau \\
&+\int_s^t \int_{\mathbb{R}^d} (q_n-q)(t,x;\tau,z)V(\tau,z)q(\tau,z;s,y)dzd\tau\Big| \\
=&|\mathrm{I}+\mathrm{II}+\mathrm{III}|
\end{aligned}$$

For I, by (1.24),

$$\begin{aligned}
|\mathrm{I}| \leq& C_q\int_s^t \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k|(\tau,z)|q_n-q)|(\tau,z;s,y)dzd\tau \\
\leq& 2C_q\int_s^{s+h_\epsilon} \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k|(\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;s,y)dzd\tau dzd\tau \\
&+C_q\int_{s+h_\epsilon}^t (\int_{|z|>k} + \int_{|z|\leq k})\Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k|(\tau,z)|q_n-q|(\tau,z;s,y)dzd\tau
\end{aligned}$$

Then use the same method as we estimated (1.40), we get

$$\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in\theta,|t-s|\geq\delta \\ x\in K_1, y\in K_2}} |\mathrm{I}(t,x;s,y)| = 0.$$

The second summand is estimated as follows:

$$
\begin{aligned}
|\mathrm{II}| \leq & C_q \int_s^t \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k - V|(\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;s,y)dzd\tau \\
\leq & C_q\Big(\int_s^t \int_{|z|>k} + \int_s^t \int_{|z|\leq k}\Big)\Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k - V|(\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;s,y)dzd\tau \\
= & C(\mathrm{II}_1 + \mathrm{II}_2)
\end{aligned}
$$

For II_1, we have

$$
\begin{aligned}
\mathrm{II}_1 = & \int_s^t \int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|x-z|^2}{t-\tau}\Big)|V_k - V| \frac{1}{(\tau-s)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|z-y|^2}{\tau-s}\Big)dzd\tau \\
\leq & \int_s^{t-h_\epsilon} \int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|x-z|^2}{t-\tau}\Big)|V_k - V| \frac{1}{(\tau-s)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|z-y|^2}{\tau-s}\Big)dzd\tau \\
& + \int_{t-h_\epsilon}^t \int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|x-z|^2}{t-\tau}\Big)|V_k - V| \frac{1}{(\tau-s)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|z-y|^2}{\tau-s}\Big)dzd\tau \\
\leq & h_\epsilon^{-\frac{d}{2}} \exp\Big(-\alpha\cdot\frac{k^2}{8h_2}\Big)\int_s^{t-h_\epsilon}\int_{|z|>k}|V_k - V| \frac{1}{(\tau-s)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|z-y|^2}{\tau-s}\Big)dzd\tau + \\
& + \frac{2^{\frac{d}{2}}}{(t-s)^{\frac{d}{2}}}\int_{t-h_\epsilon}^t \int_{|z|>k} \frac{1}{(t-\tau)^{\frac{d}{2}}} \exp\Big(-\frac{\alpha}{2}\cdot\frac{|x-z|^2}{t-\tau}\Big)|V_k - V|dzd\tau \\
\leq & 2h_\epsilon^{-\frac{d}{2}} \exp\Big(-\alpha\cdot\frac{k^2}{8h_2}\Big)M_{h_2}^{\alpha_2}(V) + 2\Big(\frac{2}{\delta}\Big)^{\frac{d}{2}}M_{h_\epsilon}^{\alpha_2}(V)
\end{aligned}
$$

Therefore we can first choose h_ϵ sufficiently small such that $2(\frac{2}{\delta})^{\frac{d}{2}}M_{h_\epsilon}^{\alpha_2}(V) < \frac{\epsilon}{3}$, and then find large enough k such that $2h_\epsilon^{-\frac{d}{2}}\exp(-\alpha\cdot\frac{k^2}{8h_2})M_{h_2}^{\alpha_2}(V) < \frac{\epsilon}{3}$.

For II_2 we have

$$
\begin{aligned}
\mathrm{II}_2 = & \int_s^t \int_{|z|\leq k} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)|V_k - V|(\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;s,y)dzd\tau \\
\leq & \int_s^t \int_{|z|\leq k} \Gamma_{\frac{\alpha}{2}}(t,x;\tau,z)(\mathbf{1}_{[0,h_2]\times\{|z|\leq k\}}|V_k - V|)(\tau,z)\Gamma_{\frac{\alpha}{2}}(\tau,z;s,y)dzd\tau \\
\leq & C_2 M_{h_2}^{\alpha_2}(\mathbf{1}_{[0,h_2]\times\{|z|\leq k\}}|V_k - V|)\cdot\Gamma_{\frac{\alpha}{2}}(s,x;t,y) \\
\leq & (\delta)^{-\frac{d}{2}} C_2 M_{h_2}^{\alpha_2}(\mathbf{1}_{[0,h_2]\times\{|z|\leq k\}}|V_k - V|).
\end{aligned}
$$

Since $[0,h]\times\{|z|\leq k\}$ is compact, we have

$$
\lim_{n\to\infty} M_{h_2}^{\alpha_2}(\mathbf{1}_{([0,h_2]\times\{|z|\leq k\})}(V_k - V) = 0.
$$

So we can find $n_0 > 0$ such that when $n, k \geq n_0, \mathrm{II} < \frac{\epsilon}{3}$.

The third term III can be done as the first term I. So we have

$$\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in\theta,|t-s|\geq\delta \\ x\in K_1, y\in K_2}} |J^1_{nk}(t,x;s,y) - J^1_G(t,x;s,y)| = 0.$$

Step 2: Suppose now the lemma holds for i. Then for $i+1$ we have

$$\begin{aligned}
&|J^{i+1}_{nk}(t,x;s,y) - J^{i+1}_G(t,x;s,y)| \\
=&|\int_s^t \int_{\mathbb{R}^d} J^i_{nk}(t,x;\tau,z)V_k(\tau,z)q_n(\tau,z;s,y)dzd\tau - \\
&- \int_s^t \int_{\mathbb{R}^d} J^i(t,x;\tau,z)V(\tau,z)q(\tau,z;s,y)dzd\tau| \\
=&|\int_s^t \int_{\mathbb{R}^d} J^i_{nk}(t,x;\tau,z)V_k(\tau,z)(q_n - q)(\tau,z;s,y)dzd\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} J^i_{nk}(t,x;\tau,z)(V_k - V)(\tau,z)q(\tau,z;s,y)dzd\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} (J^i_{nk} - J^i)(t,x;\tau,z)V(\tau,z)q(\tau,z;s,y)dzd\tau
\end{aligned}$$

Then the rest of the proof is very similar to *step* 1, so we omit it. □

The following theorem is an easy consequence of the above lemma.

Theorem 1.3.14. *For any compact sets $K \subset \theta$, $K_1, K_2 \subset \mathbb{R}^d$, $G_{nk}(t,x;s,y)$ converges uniformly to $G(t,x;s,y)$ on $\{(t,x;s,y) : (s,t) \in K, x \in K_1, y \in K_2\}$.*

1.3.3 How to define $G(t,x;s,y)$ for $0 \leq s < t \leq T$

So far we have only defined $G(t,x;s,y)$ locally for $0 < t - s \leq h_2$ and $x, y \in \mathbb{R}^d$. Now we use the reproducing property of $G_{nk}(t,x;s,y)$ to define $G(t,x;s,y)$ for all $0 \leq s < t \leq T$.

Theorem 1.3.15. *Suppose that $K_1, K_2 \subset \mathbb{R}^d$ are compact sets and $\delta \in (0, h_2)$. Let $\theta_T := \{(s,t) : 0 \leq s < t \leq T\}$, then $G_{nk}(t,x;s,y)$ converges uniformly on $\{(s,t) : (s,t) \in \theta_T, t-s \geq \delta\} \times K_1 \times K_2$.*

Proof. We only look at the case $T = \frac{3}{2}h_2$, for general case the lemma can be proved similarly.

We define

$$A := \{(s,t) : 0 \le s < t \le h_2,\ t-s \ge \delta\},\ B := \{(s,t) : \frac{1}{2}h_2 \le s < t \le \frac{3}{2}h_2,\ t-s \ge \delta\},$$

$$C := \{(s,t) : 0 \le s \le \frac{1}{2}h_2,\ \ h_2 \le t \le \frac{3}{2}h_2\}.$$

We easily see that $\{(s,t) : (s,t) \in \theta_T, t-s \ge \delta\} \subset A \cup B \cup C$.

For $(s,t) \in B$, $G(t,x;s,y)$ is already defined and with almost the same proof of theorem 1.3.14, we have

$$\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in B \\ x\in K_1, y\in K_2}} |G_{nk}(t,x;s,y) - G(t,x;s,y)| = 0.$$

From theorem 1.3.14 we also have

$$\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in A \\ x\in K_1, y\in K_2}} |G_{nk}(t,x;s,y) - G(t,x;s,y)| = 0.$$

So if we can show $G_{nk}(t,x;s,y)$ converges uniformly on $C \times K_1 \times K_2$, then we are done.

For $0 \le s \le \frac{1}{2}h_2, h_2 \le t \le \frac{3}{2}h_2$ and $x \in K_1, y \in K_2$, by the reproducing property of $G_{nk}(t,x;s,y)$,

$$\begin{aligned}
&|G_{nk}(t,x;s,y) - \int_{\mathbb{R}^d} G(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz| \\
=&\Big| \int_{\mathbb{R}^d} G_{nk}(t,x;\frac{3}{4}h_2,z)G_{nk}(\frac{3}{4}h_2,z;s,y)dz - \int_{\mathbb{R}^d} G(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz\Big| \\
=&\Big| \int_{\mathbb{R}^d} G_{nk}(t,x;\frac{3}{4}h_2,z)(G_{nk}-G)(\frac{3}{4}h_2,z;s,y)dz \\
&\qquad - \int_{\mathbb{R}^d} (G-G_{nk})(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz\Big|
\end{aligned}$$

Since K_1, K_2 are compact sets, we can find $R > 0$ large enough such that

$$4C_G^2 \int_{|z|>R} \Gamma_{\frac{\alpha}{4}}(t,x;\frac{3}{4}h_2,z)\Gamma_{\frac{\alpha}{4}}(\frac{3}{4}h_2,z;s,y)dz < \frac{\epsilon}{2}$$

for any $0 \le s \le \frac{1}{2}h_2, h_2 \le t \le \frac{3}{2}h_2$ and $x \in K_1, y \in K_2$.

Then

$$
\begin{aligned}
&|G_{nk}(t,x;s,y) - \int_{\mathbb{R}^d} G(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz| \\
\leq & 4C_G^2 \int_{|z|>R} \Gamma_{\frac{\alpha}{4}}(t,x;\frac{3}{4}h_2,z)\Gamma_{\frac{\alpha}{4}}(\frac{3}{4}h_2,z;s,y)dz \\
&+ \int_{|z|\leq R} \Gamma_{\frac{\alpha}{4}}(t,x;\frac{3}{4}h_2,z)|G_{nk}-G|(\frac{3}{4}h_2,z;s,y)dz \\
&+ \int_{\mathbb{R}^d} |G-G_{nk}|(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz \\
\leq & \frac{\epsilon}{2} + C' \sup_{\substack{0\leq s\leq \frac{1}{2}h_2 \\ y\in K_2, |z|\leq R}} |G_{nk}-G|(\frac{3}{4}h_2,z;s,y) + C' \sup_{\substack{h_2\leq t\leq \frac{3}{2}h_2 \\ x\in K_1, |z|\leq R}} |G-G_{nk}|(t,x;\frac{3}{4}h_2,z).
\end{aligned}
$$

Therefore

$$
\lim_{n,k\to\infty} |G_{nk}(t,x;s,y) - \int_{\mathbb{R}^d} G(t,x;\frac{3}{4}h_2,z)G(\frac{3}{4}h_2,z;s,y)dz| = 0, \text{ uniformly on } C\times K_1\times K_2.
$$

$\square$

Remark 1.3.16. (i) From the above theorem, we can therefore define

$$
G(t,x;s,y) := \lim_{n,k\to\infty} G_{nk}(t,x;s,y), \ 0\leq s<t\leq T.
$$

(ii) By Theorem 1.3.15, for any compact $K\subset\{(s,t): 0\leq s<t\leq T\}, K_1, K_2\subset\mathbb{R}^d$, we have

$$
\lim_{n,k\to\infty} \sup_{\substack{(s,t)\in K \\ x\in K_1, y\in K_2}} |G_{nk}(t,x;s,y) - G(t,x;s,y)| = 0. \tag{1.41}
$$

(iii) Recall that $G_{nk}(t,x;s,y)$ satisfies Gaussian lower and upper bounds, namely there exist constants $c_G, C_G>0$ such that

$$
c_G\cdot\Gamma_{\alpha''}(t,x;s,y) \leq G_{nk}(t,x;s,y) \leq C_G\cdot\Gamma_{\frac{\alpha}{4}}(t,x;s,y) \tag{1.42}
$$

for all $0\leq s<t\leq T$ and $x,y\in\mathbb{R}^d$. Therefore we also have

$$
c_G\cdot\Gamma_{\alpha''}(t,x;s,y) \leq G(t,x;s,y) \leq C_G\cdot\Gamma_{\frac{\alpha}{4}}(t,x;s,y) \tag{1.43}
$$

for all $0\leq s<t\leq T$ and $x,y\in\mathbb{R}^d$.

1.4 Existence and uniqueness of weak fundamental solution

In last section we constructed $G(t,x;s,y)$ as limit of $G_{nk}(t,x;s,y)$. Since each $G_{nk}(t,x;s,y)$ is a weak fundamental solution for

$$\nabla(a\cdot\nabla u)+b_n\cdot\nabla u+V_k u-\partial_t u=0,$$

in the limit case, we would expect $G(t,x;s,y)$ to be a weak fundamental solution to the parabolic equation

$$\nabla(a\cdot\nabla u)+b\cdot\nabla u+Vu-\partial_t u=0. \tag{$*$}$$

In this section we prove that this is indeed the case. Thereafter we will also show that weak fundamental solution for $(*)$ is unique.

Let $0\le s<T$, for any $f\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, we define $u(s,x)=f(x)$ and

$$u(t,x)=\int_{\mathbb{R}^d}G(t,x;s,y)f(y)dy,\quad s<t\le T.$$

Theorem 1.4.1. *$u(t,x)$ is a weak solution to $(*)$ in $[s,T]\times\mathbb{R}^d$, namely*

$$\begin{aligned}
u\in C([s,T];L^2(\mathbb{R}^d))\cap L^2((s,T);H^1(\mathbb{R}^d)),&\\
b\cdot\nabla u\in& L^1((s,T)\times\mathbb{R}^d),\\
Vu\in& L^1((s,T)\times\mathbb{R}^d),\\
\int_s^T\int_{\mathbb{R}^d}(\nabla u\cdot a\cdot\nabla\phi-&\phi b\cdot\nabla u-Vu\phi-u\partial_t\phi)dxdt=0,\\
&\forall\phi\in C_0^\infty([s,T]\times\mathbb{R}^d).
\end{aligned}$$

We prove this theorem through the following several lemmas.

We define $u_{nk}(s,x)=f(x)$ and

$$u_{nk}(t,x)=\int_{\mathbb{R}^d}G_{nk}(t,x;s,y)f(y)dy,\quad s<t\le T.$$

Since $G_{nk}(t,x;s,y)$ is a weak fundamental solution for

$$\nabla(a\cdot\nabla u)+b_n\cdot\nabla u+V_l u-\partial_t u=0,$$

we have

$$
\begin{aligned}
u_{nk} \in C([s,T];L^2(\mathbb{R}^d)) &\cap L^2((s,T);H^1(\mathbb{R}^d)),\\
b_n \cdot \nabla u_{nk} &\in L^1((s,T)\times \mathbb{R}^d),\\
V_k u_{nk} &\in L^1((s,T)\times \mathbb{R}^d),
\end{aligned}
$$

$$
\int_s^T \int_{\mathbb{R}^d} (\nabla u_{nk}\cdot a \cdot \nabla\phi - \phi b_n \cdot \nabla u_{nk} - u_{nk}V_k\phi - u_{nk}\partial_t\phi)dxdt = 0,
$$

$$
\forall \phi \in C_0^\infty([s,T]\times\mathbb{R}^d).
$$

Lemma 1.4.2. *Let $0<\delta<T-s$, then*

$$
\lim_{n,k\to\infty} \sup_{s+\delta\le t\le T} \|u_{nk}(t,\cdot)-u(t,\cdot)\|_2 = 0,
$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R}^d)$.

Proof. For $t\in[s+\delta,T]$,

$$
\begin{aligned}
&\|u_{nk}(t,\cdot)-u(t,\cdot)\|_2^2 = \int_{\mathbb{R}^d}(u_{nk}(t,x)-u(t,x))^2dx\\
=&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} G_{nk}(t,x;s,y)f(y)dy - \int_{\mathbb{R}^d} G(t,x;s,y)f(y)dy\right)^2 dx\\
\le&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |G_{nk}(t,x;s,y)-G(t,x;s,y)||f(y)|dy\right)^2 dx.
\end{aligned}
$$

Since $G_{nk}(t,x;s,y), G(t,x,s,y)$ satisfy Gaussian bounds (1.42) and (1.43), then we can use Jensen's inequality to get

$$
\begin{aligned}
&\|u_{nk}(t,\cdot)-u(t,\cdot)\|_2^2\\
\le& C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |G_{nk}(t,x;s,y)-G(t,x;s,y)||f(y)|^2dydx\\
=& C\int_{\mathbb{R}^d}|f(y)|^2dy(\int_{\mathbb{R}^d}|G_{nk}(t,x;s,y)-G(t,x;s,y)|dx)\\
\le& C'\int_{|y|>K_1}|f(y)|^2dy + C\int_{|y|\le K_1}|f(y)|^2dy(\int_{\mathbb{R}^d}|G_{nk}(t,x;s,y)-G(t,x;s,y)|dx).
\end{aligned}
$$

Since $f(x)\in L^2(\mathbb{R}^d)$, we can choose K_1 large enough s.t.

$$
C'\int_{|y|>K_1}|f(y)|^2dy < \frac{\epsilon}{3}. \tag{1.44}
$$

For the second term we have

$$
\begin{aligned}
&\int_{|y|\le K_1} |f(y)|^2 dy \Big(\int_{\mathbb{R}^d} |G_{nk}(t,x;s,y) - G(t,x;s,y)| dx \Big) \\
\le &\int_{|y|\le K_1} |f(y)|^2 dy \Big(C'' \int_{|x|>K_2} \Gamma_{\frac{\alpha}{4}}(t,x;s,y) dx + \int_{|x|\le K_2} |G_{nk} - G|(t,x;s,y) dx \Big) \\
\le &C'' \int_{|y|\le K_1} |f(y)|^2 dy \Big(\int_{|x|>K_2} \frac{1}{\delta^{\frac{d}{2}}} e^{-\frac{\alpha}{8}\cdot\frac{|K_2-K_1|^2}{T}} dx \Big) \\
&\quad + \int_{|y|\le K_1} |f(y)|^2 dy \int_{|x|\le K_2} |G_{nk} - G|(t,x;s,y) dx
\end{aligned}
$$

We can choose K_2 large enough s.t.

$$
C'' \int_{|y|\le K_1} |f(y)|^2 dy \Big(\int_{|x|>K_2} \frac{1}{\delta^{\frac{d}{2}}} e^{-\frac{\alpha}{8}\cdot\frac{|K_2-K_1|^2}{T}} dx \Big) < \frac{\epsilon}{3C}. \tag{1.45}
$$

By Remark 1.3.16(ii), we have

$$
\lim_{n,k\to\infty} \sup_{\substack{|x|\le K_2, |y|\le K_1 \\ \delta\le t-s\le T}} |G_{nk} - G|(t,x;s,y) = 0,
$$

and therefore

$$
\lim_{n,k\to\infty} \int_{|y|\le K_1} |f(y)|^2 dy \int_{|x|\le K_2} |G_{nkl} - G|(t,x;s,y) dx) = 0. \tag{1.46}
$$

So with (1.44), (1.45) and (1.46) we get

$$
\lim_{n,k\to\infty} \sup_{s+\delta\le t\le T} \|u_{nk}(t,\cdot) - u(t,\cdot)\|_2 = 0.
$$

□

Lemma 1.4.3. $\lim_{t\downarrow s} \|u(t,\cdot) - f(\cdot)\|_2 = 0$.

Proof. Recall that $p(t,x;s,y)$ is the weak fundamental solution to

$$
\nabla(a\cdot\nabla u) - \partial_t u = 0,
$$

and hence

$$
\lim_{t\downarrow s} \Big\| \int_{\mathbb{R}^d} p(t,x;s,y) f(y) dy - f(x) \Big\|_2 = 0. \tag{1.47}
$$

By Remark 1.3.3, for $0 < t - s \le h_1$, we have

$$
q(t,x;s,y) = \sum_{i=0}^{\infty} J^i(t,x;s,y) = p(t,x;s,y) + \sum_{i=1}^{\infty} J^i(t,x;s,y),
$$

and

$$|J^i(t,x;s,y)| \le C_0\big(\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)\big)^i \cdot \Gamma_{\frac{\alpha}{2}}(t,x;s,y).$$

Therefore

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} q(t,x;s,y)f(y)dy - f(x)\Big\|_2 \\
=&\Big\| \int_{\mathbb{R}^d} p(t,x;s,y)f(y)dy - f(x) + \int_{\mathbb{R}^d}\sum_{i=1}^{\infty} J^i(t,x;s,y)f(y)dy\Big\|_2 \\
\le&\Big\| \int_{\mathbb{R}^d} p(t,x;s,y)f(y)dy - f(x)\Big\|_2 + \Big\|\int_{\mathbb{R}^d}\sum_{i=1}^{\infty} J^i(t,x;s,y)f(y)dy\Big\|_2
\end{aligned}$$

But

$$\begin{aligned}
&\Big\|\int_{\mathbb{R}^d}\sum_{i=1}^{\infty} J^i(t,x;s,y)f(y)dy\Big\|_2^2 = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sum_{i=1}^{\infty} J^i(t,x;s,y)f(y)dy\Big)^2 dx \\
\le&\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\sum_{i=1}^{\infty} J^i(t,x;s,y)|\cdot|f(y)|dy\Big)^2 dx \\
\le&\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\frac{\kappa C_0^2 C_1 N^{\alpha_1}_{t-s}(|b|)}{1-\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)}\Gamma_{\frac{\alpha}{2}}(t,x;s,y)|f(y)|dy\Big)^2 dx \\
\le&\Big(\frac{\kappa C_0^2 C_1 N^{\alpha_1}_{t-s}(|b|)}{1-\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)}\Big)^2\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\Gamma_{\frac{\alpha}{2}}(t,x;s,y)|f(y)|dy\Big)^2 dx \\
\le&C'\Big(\frac{\kappa C_0^2 C_1 N^{\alpha_1}_{t-s}(|b|)}{1-\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)}\Big)^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_{\frac{\alpha}{2}}(t,x;s,y)|f(y)|^2 dydx \\
\le&C'\Big(\frac{\kappa C_0^2 C_1 N^{\alpha_1}_{t-s}(|b|)}{1-\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)}\Big)^2\int_{\mathbb{R}^d}|f(y)|^2 dy\Big(\int_{\mathbb{R}^d}\Gamma_{\frac{\alpha}{2}}(t,x;s,y)dx\Big) \\
\le&C''\Big(\frac{\kappa C_0^2 C_1 N^{\alpha_1}_{t-s}(|b|)}{1-\kappa C_0 C_1 N^{\alpha_1}_{t-s}(|b|)}\Big)^2\int_{\mathbb{R}^d}|f(y)|^2 dy.
\end{aligned}$$

Since $\lim_{t\downarrow s} N^{\alpha_1}_{t-s}(|b|) = 0$, together with (1.47), we have

$$\lim_{t\downarrow s}\|\int_{\mathbb{R}^d} q(t,x;s,y)f(y)dy - f(x)\|_2 = 0. \tag{1.48}$$

From Remark 1.3.5, we know

$$G(t,x;s,y) = \sum_{i=0}^{\infty} J^i_G(t,x;s,y) = q(t,x;s,y) + \sum_{i=1}^{\infty} J^i_G(t,x;s,y)$$

and

$$|J_G^i(t,x;s,y)| \le C_q\big(C_q C_2 M_{t-s}^{\alpha_2}(V)\big)^i \cdot \Gamma_{\frac{\alpha}{4}}(t,x;s,y), \quad 0 < t-s \le h_2 \quad (1.49)$$

Similarly to the proof of (1.48), now we can use (1.48) and (1.49) to get

$$\lim_{t\downarrow s} \| \int_{\mathbb{R}^d} G(t,x;s,y)f(y)dy - f(x)\|_2 = 0. \tag{1.50}$$

□

Remark 1.4.4. From the proof of the above lemma, we see that the rate at which $\|u(t,\cdot)-f(\cdot)\|_2$ goes to 0 as $t \downarrow s$ depends only on $M_{t-s}^{\alpha_2}(V), N_{t-s}^{\alpha_1}(|b|)$ and the rate of

$$\lim_{t\downarrow s} \| \int_{\mathbb{R}^d} p(t,x;s,y)f(y)dy - f(x)\|_2 = 0. \tag{1.51}$$

If we change $u(t,\cdot)$ to $u_{nk}(t,\cdot)$ in the above lemma, by the same estimates we see that the rate at which $\|u_{nk}(t,\cdot)-f(\cdot)\|_2$ goes to 0 depends only on $M_{t-s}^{\alpha_2}(V), N_{t-s}^{\alpha_1}(|b|)$ and the rate of (1.51). In particular, it does not depend on n, k.

From Lemma 1.4.2, Lemma 1.4.3 and Remark 1.4.4, a simple "$\epsilon-\delta$" argument leads us to the following corollary.

Corollary 1.4.5. *$u \in C([s,T];L^2(\mathbb{R}^d))$ and*

$$\lim_{n,k\to\infty} \sup_{s\le t\le T} \|u_{nk}(t,\cdot)-u(t,\cdot)\|_2 = 0.$$

Next we show that $u \in L^2((s,T);H^1(\mathbb{R}^d))$.

The following lemma is a time-depedent version of Lemma 2.11 in [KLSU04].

Lemma 1.4.6. *There is a constant $C>0$ independent of n,k such that*

$$\int_s^T \int_{\mathbb{R}^d} \nabla u_{nk} \cdot a \cdot \nabla u_{nk} dx dt \le C.$$

Proof. This is a modification of the proof of Lemma 2.11 in [KLSU04].

Let $s < t_1 < t_2 < T,\ 0 < \epsilon < \frac{t_2-t_1}{2}$ and define

$$\eta(t) = \begin{cases} 0 & t \in [0,t_1]\cup[t_2,T] \\ \frac{1}{\epsilon}(t-t_1) & t \in (t_1, t_1+\epsilon) \\ 1 & t \in [t_1+\epsilon, t_1-\epsilon] \\ \frac{1}{\epsilon}(t_2-t) & t \in (t_2-\epsilon, t_2). \end{cases}$$

Using the estimates for $G_{nk}(t,x;s,y)$, it is easy to verify that

$$\eta u_{nk} \in H_0^1((s,T)\times\mathbb{R}^d).$$

Therefore

$$\int_s^T \int_{\mathbb{R}^d} (\nabla u_{nk}\cdot a\cdot\nabla(\eta u_{nk}) - (\eta u_{nk}) b_n\cdot\nabla u_{nk} - u_{nk} V_k(\eta u_{nk}) - u_{nk}\partial_t(\eta u_{nk}))dxdt = 0,$$

or

$$\begin{aligned}
&\int_s^T \eta\nabla u_{nk}\cdot a\cdot\nabla u_{nk} dxdt \\
=&\int_s^T \int_{\mathbb{R}^d} (\eta u_{nk}\partial_t u_{nk} + u_{nk}^2\partial_t\eta)dxdt + \int_s^T \int_{\mathbb{R}^d} (\eta u_{nk}) b_n\cdot\nabla u_{nk} dxdt + \int_s^T \int_{\mathbb{R}^d} \eta V_k |u_{nk}|^2 dxdt \\
=&I_1 + I_2 + I_3.
\end{aligned}$$

$$I_1 = \frac{1}{\epsilon}\int_{t_1}^{t_1+\epsilon} \|u_{nk}\|_2^2(t)dt - \frac{1}{\epsilon}\int_{t_2-\epsilon}^{t_2} \|u_{nk}\|_2^2(t)dt + \frac{1}{2}\int_{t_1}^{t_2} \eta\partial_t\big(\|u_{nk}\|_2^2\big)(t)dt.$$

A direct computation yields

$$\lim_{\epsilon\to 0} I_1 = \frac{1}{2}\big(\|u_{nk}\|_2^2(t_2) + \|u_{nk}\|_2^2(t_1)\big).$$

From the proof of Lemma 1.4.2, we know that for any $s\le t\le T$,

$$\|u_{nk}(t,\cdot)\|_2^2 \le C\int_{\mathbb{R}^d} |f(y)|^2 dy,$$

where C only depends on the constant C_G which appears in the Gaussian bounds for $G_{nk}(t,x;s,y)$ and $G(t,x;s,y)$. Therefore we have

$$\lim_{\epsilon\to 0} |I_1| \le C\int_{\mathbb{R}^d} |f(y)|^2 dy. \tag{1.52}$$

For I_2, we have

$$\begin{aligned}
&|I_2| \le \int_s^T \int_{\mathbb{R}^d} |b_n(t,x)|\cdot|\nabla u_{nk}(t,x)|\cdot|u_{nk}|(t,x)dxdt \\
\le&\int_s^T \int_{\mathbb{R}^d} |b_n(t,x)|\cdot\Big|\int_{\mathbb{R}^d} \nabla_x G_{nk}(t,x;s,y)f(y)dy\Big|\cdot\Big|\int_{\mathbb{R}^d} G_{nk}(t,x;s,y)f(y)dy\Big|dxdt \\
\le&C_1\|f\|_\infty\cdot\int_s^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |b_n(t,x)|\cdot|\nabla_x G_{nk}(t,x;s,y)|\cdot|f(y)|dydxdt \\
\le&C_1\|f\|_\infty\cdot\int_{\mathbb{R}^d} |f(y)|dy\bigg(\int_s^T \int_{\mathbb{R}^d} |b_n(t,x)|\cdot|\nabla_x G_{nk}(t,x;s,y)|dxdt\bigg).
\end{aligned}$$

From (1.38), we know

$$\nabla_x G_{nk}(t,x;s,y) = \nabla_x q_n(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} \nabla_x q_n(t,x;\tau,z) V_k(\tau,z) G_{nk}(\tau,z;s,y) dz d\tau. \tag{1.53}$$

Therefore by Fubuni's theorem and Proposition 1.1.4 and 1.1.11,

$$\begin{aligned}
&\int_s^T \int_{\mathbb{R}^d} |b_n(t,x)| \cdot |\nabla_x G_{nk}(t,x;s,y)| dx dt \\
=&\int_s^T \int_{\mathbb{R}^d} |b_n(t,x)| \cdot \Big| \nabla_x q_n(t,x;s,y) + \int_s^t \nabla_x q_n(t,x;\tau,z) V_k(\tau,z) G_{nk}(\tau,z;s,y) dz d\tau \Big| dx dt \\
\le & C_2 \int_s^T \int_{\mathbb{R}^d} |b_n(t,x)| \Psi_{\frac{\alpha}{2}}(t,x;s,y) dx dt \\
&\qquad + \int_s^T \int_{\mathbb{R}^d} |b_n(t,x)| dx dt \big(| \int_s^t \nabla_x q_n(t,x;\tau,z) V_k(\tau,z) G_{nk}(\tau,z;s,y) dz d\tau | \big) \\
\le & C_3 + \int_s^T \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{4}}(\tau,z;s,y) V_k(\tau,z) dz d\tau \big(\int_\tau^T |b_n(t,x)| \Psi_{\frac{\alpha}{2}}(t,x;\tau,z) dx dt \big) \\
\le & C_4
\end{aligned}$$

The constants C_1, C_2, C_3, C_4 depend only on the quantity $N_h^{\alpha_1}(|b|)$ and $M_h^{\alpha_2}(|V|)$ and are the same for all n, k. So we have

$$|I_2| \le C_1 C_4 \|f\|_\infty \int_{\mathbb{R}^d} |f(y)| dy. \tag{1.54}$$

For I_3, we have

$$\begin{aligned}
|I_3| \le & \int_s^T \int_{\mathbb{R}^d} |V_k(t,x)| \cdot |u_{nk}(t,x)|^2 dx dt \\
\le & \int_s^T \int_{\mathbb{R}^d} |V_k(t,x)| \cdot \Big| \int_{\mathbb{R}^d} G_{nk}(t,x;s,y) f(y) dy \Big|^2 dx dt \\
\le & C_6 \int_s^T \int_{\mathbb{R}^d} |V_k(t,x)| \int_{\mathbb{R}^d} G_{nk}(t,x;s,y) |f(y)|^2 dy dx dt \\
\le & C_7 \int_{\mathbb{R}^d} |f(y)|^2 dy \Big(\int_s^T \int_{\mathbb{R}^d} |V_k(t,x)| \Gamma_{\frac{\alpha}{4}}(t,x;s,y) dx dt \Big) \\
\le & C_8 \int_{\mathbb{R}^d} |f(y)|^2 dy
\end{aligned} \tag{1.55}$$

Combining (1.52), (1.54) and (1.55) we show that

$$\int_{t_1}^{t_2} \nabla u_{nk} \cdot a \cdot \nabla u_{nk} dx dt \le C,$$

where C does not depend on n, k. Since t_1, t_2 are arbitrary, the lemma is proved. $\square$

Corollary 1.4.7. *$u \in L^2((s,T); H^1(\mathbb{R}^d))$ and there exists a subsequence of ∇u_{nk} converges to ∇u weakly in $L^2((s,T), L^2(\mathbb{R}^d))$.*

Proof. It is easy to check that there exists a constant $C > 0$ independent of n, k such that

$$\int_s^T \int_{\mathbb{R}^d} |u_{nk}|^2(t,x)dxdt < C \int_{\mathbb{R}^d} |f(y)|^2 dy.$$

Then the lemma follows from Corollary 1.4.5 and Lemma 1.4.6. $\square$

Lemma 1.4.8. $b \cdot \nabla u, Vu \in L^1((s,T) \times \mathbb{R}^d)$.

Proof. We only prove $b \cdot \nabla u \in L^1((s,T) \times \mathbb{R}^d)$, the claim for Vu is proved similarly. From (1.38) and (1.41), we can get

$$G(t,x;s,y) = q(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} q(t,x;\tau,z)V(\tau,z)G(\tau,z;s,y)dzd\tau,$$

and therefore

$$\nabla_x G(t,x;s,y) = \nabla_x q(t,x;s,y) + \int_s^t \int_{\mathbb{R}^d} \nabla_x q(t,x;\tau,z)V(\tau,z)G(\tau,z;s,y)dzd\tau. \tag{1.56}$$

By Fubuni's theorem and Proposition 1.1.4 and 1.1.11,

$$\begin{aligned}
&\int_s^T \int_{\mathbb{R}^d} |b(t,x) \cdot \nabla_x G(t,x;s,y)| dxdt \\
\leq &\int_s^T \int_{\mathbb{R}^d} |b(t,x)| \cdot \Big| \nabla_x q(t,x;s,y) + \int_s^t \nabla_x q(t,x;\tau,z)V(\tau,z)G(\tau,z;s,y)dzd\tau \Big| dxdt \\
\leq &C_1 \int_s^T \int_{\mathbb{R}^d} |b(t,x)| \Psi_{\frac{\alpha}{2}}(t,x;s,y)dxdt \\
&\quad + \int_s^T \int_{\mathbb{R}^d} |b(t,x)| dxdt \big(\int_s^t \big| \nabla_x q(t,x;\tau,z) \big| V(\tau,z)G(\tau,z;s,y)dzd\tau \big) \\
\leq &C_2 + \int_s^T \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{4}}(\tau,z;s,y)V(\tau,z)dzd\tau \big(\int_\tau^T |b(t,x)| \Psi_{\frac{\alpha}{2}}(t,x;\tau,z)dxdt \big) \\
< &\infty.
\end{aligned}$$

$\square$

Lemma 1.4.9. *For $\forall \phi \in C_0^\infty([s,T]\times\mathbb{R}^d)$, we have*

$$\int_s^T \int_{\mathbb{R}^d} (\nabla u \cdot a \cdot \nabla\phi - \phi b \cdot \nabla u - Vu\phi - u\partial_t\phi) dxdt = 0.$$

Proof. For each u_{nk}, we know that

$$\int_s^T \int_{\mathbb{R}^d} (\nabla u_{nk} \cdot a \cdot \nabla\phi - \phi b_n \cdot \nabla u_{nk} - u_{nk}V_k\phi - u_{nk}\partial_t\phi) dxdt = 0.$$

By Corollary 1.4.5, we know that u_{nk} converges to u strongly in $L^2((s,T), L^2(\mathbb{R}^d))$, therefore

$$\lim_{n,k\to\infty} \int_s^T \int_{\mathbb{R}^d} u_{nk}\partial_t\phi dxdt = \int_s^T \int_{\mathbb{R}^d} u\partial_t\phi dxdt.$$

From Corollary 1.4.7, there exists a subsequence of ∇u_{nk} converges to ∇u weakly in $L^2((s,T), L^2(\mathbb{R}^d))$, for simplicity, we denote it still by ∇u_{nk}. So

$$\lim_{n,k\to\infty} \int_s^T \int_{\mathbb{R}^d} \nabla u_{nk} \cdot a \cdot \nabla\phi dxdt = \int_s^T \int_{\mathbb{R}^d} \nabla u \cdot a \cdot \nabla\phi dxdt.$$

If we can prove

$$\lim_{n,k\to\infty} \int_s^T \int_{\mathbb{R}^d} \phi b_n \cdot \nabla u_{nk} dxdt = \int_s^T \int_{\mathbb{R}^d} \phi b \cdot \nabla u dxdt \tag{1.57}$$

and

$$\lim_{n,k\to\infty} \int_s^T \int_{\mathbb{R}^d} u_{nk}V_k\phi dxdt = \int_s^T \int_{\mathbb{R}^d} Vu\phi dxdt, \tag{1.58}$$

then we are done. Here we only prove (1.57), because (1.58) can be done similarly.

To prove (1.57), we have

$$\begin{aligned}
&\Big|\int_s^T \int_{\mathbb{R}^d} \phi b_n \cdot \nabla u_{nk} dxdt - \int_s^T \int_{\mathbb{R}^d} \phi b \cdot \nabla u dxdt\Big| \\
=&\Big|\int_s^T \int_{\mathbb{R}^d} \phi(b_n - b) \cdot \nabla u_{nk} dxdt - \int_s^T \int_{\mathbb{R}^d} \phi b \cdot (\nabla u - \nabla u_{nk}) dxdt\Big| \\
\le&\Big|\int_s^T \int_{\mathbb{R}^d} \phi(b_n - b) \cdot \nabla u_{nk} dxdt\Big| + \Big|\int_s^T \int_{\mathbb{R}^d} \phi b \cdot (\nabla u - \nabla u_{nk}) dxdt\Big| \\
=&I_1 + I_2
\end{aligned}$$

By (1.53) and (1.56)

$$
\begin{aligned}
I_1 =&\Big|\int_s^T\int_{\mathbb{R}^d}\phi(b_n-b)\cdot\nabla u_{nk}dxdt\Big|\\
=&\Big|\int_s^T\int_{\mathbb{R}^d}\phi(b_n-b)\big(\int_{\mathbb{R}^d}\nabla_x G_{nk}(t,x;s,y)f(y)dy\big)dxdt\Big|\\
=&\Big|\int_s^T\int_{\mathbb{R}^d}\phi(b_n-b)\int_{\mathbb{R}^d}\nabla_x q_n(t,x;s,y)f(y)dydxdt\\
&\quad+\int_s^T\int_{\mathbb{R}^d}\phi(b_n-b)\big(\int_{\mathbb{R}^d}\int_s^t\int_{\mathbb{R}^d}\nabla_x q_n(t,x;\tau,z)V_k(\tau,z)G_{nk}(\tau,z;s,y)f(y)dzd\tau dy\big)dxdt\Big|\\
=&|I_{11}+I_{12}|.
\end{aligned}
$$

By Fubini's theorem,

$$
\begin{aligned}
|I_{11}|\le&\int_{\mathbb{R}^d}|f(y)|dy\big(\int_s^T\int_{\mathbb{R}^d}|\phi(b_n-b)|\Psi_{\frac{\alpha}{2}}(t,x;s,y)dxdt\big)\\
\le& C_1\int_{\mathbb{R}^d}|f(y)|dy\cdot N^{\alpha_1}_{T-s}(|\phi(b_n-b)|).
\end{aligned}
$$

Since ϕ is of compact support, we have

$$\lim_{n,k\to\infty} I_{11}=0.$$

For I_{12}, we again use Fubini's theorem, which yields

$$
\begin{aligned}
&|I_{12}|=\Big|\int_{\mathbb{R}^d}f(y)dy\big(\int_s^T\int_{\mathbb{R}^d}\int_s^t\int_{\mathbb{R}^d}\phi(b_n-b)\nabla_x q_n(t,x;\tau,z)V_k(\tau,z)G_{nk}(\tau,z;s,y)f(y)dzd\tau dxdt\big)\Big|\\
&\le\int_{\mathbb{R}^d}|f(y)|dy\big(\int_s^T\int_{\mathbb{R}^d}|V_k(\tau,z)|G_{nk}(\tau,z;s,y)dzd\tau\int_\tau^T\int_{\mathbb{R}^d}|\phi(b_n-b)|\Psi_{\frac{\alpha}{2}}(t,x;\tau,z)dxdt\big)\\
&\le C_2\int_{\mathbb{R}^d}|f(y)|dy\cdot M^{\alpha_2}_{T-s}(V)\cdot N^{\alpha_1}_T(|\phi(b_n-b)|).
\end{aligned}
$$

Since $\lim_{n\to\infty}N^{\alpha_1}_T(|\phi(b_n-b)|)=0$, we also get

$$\lim_{n,k\to\infty} I_{12}=0.$$

Therefore we proved

$$\lim_{n,k\to\infty} I_1=0.$$

The term I_2 can be estimated by similar methods, so we get

$$\lim_{n,k\to\infty}\int_s^T\int_{\mathbb{R}^d}\phi b_n\cdot\nabla u_{nk}dxdt=\int_s^T\int_{\mathbb{R}^d}\phi b\cdot\nabla u dxdt.$$

□

Thus we have proved Theorem 1.4.1. So now we can state the main theorem of this section.

Theorem 1.4.10. $G(t, x; s, y), 0 \leq s < t \leq T$, *is the unique weak fundamental solution for the equation*

$$\nabla(a \cdot \nabla u) + b \cdot \nabla u + Vu - \partial_t u = 0. \tag{$*$}$$

Proof. By Theorem 1.4.1, we know $G(t, x; s,)$ is a weak fundamental solution to $(*)$. Uniqueness can be proved in the same way as [LS00], Lemma 4.6 and 4.7. $\square$

Chapter 2

Diffusions with time-dependent singular drift

In the paper [BC03], Chen and Bass proved existence and uniqueness of weak solutions to the following stochastic differential equation

$$dX_t = dW_t + B(X_t)dt, \; X_0 = x \tag{2.1}$$

where the drift term belongs to the (time-independent) Kato class $\mathcal{K}_{d-1}$ and can be very singular. In fact they could even allow B to be a Radon measure, but then the notion of a solution to (2.1) would be a little bit different from the usual sense. Their method is based on constructing the resolvent S^λ for the process X_t. Later P. Kim and R. Song studied the process X_t with singular drift B thoroughly (see [KS08, KS07a, KS07b, KS06] et al); among many other things, they obtained two sided estimates for the heat kernel and Green function of X_t.

In this chapter we study the time-dependent version of (2.1), namely

$$\begin{cases} dX_t = dW_t + B(t, X_t)dt, & t \geq s. \\ X_t = x, & 0 \leq t \leq s. \end{cases} \tag{2.2}$$

We assume $|B(t,x)|$ to be in the forward-Kato class $\mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$, then we prove existence and uniqueness of weak solutions to (2.2). Basic ideas are taken from [BC03], but we have to extend them to the time-dependent case. We should note that $\mathcal{FK}^{\alpha}_{d-1}$ includes the (time-independent) Kato class $\mathcal{K}_{d-1}$, therefore our work extends the results of [BC03].

If we further assume $|B(t,x)| \in \mathcal{TK}^{\alpha'}_{d-1}$ for some $\alpha' < \frac{1}{4}$, then from the results of Chapter 1, we can also get two-sided Gaussian estimates for the transition density function of X_t. These are time-dependent versions of the results of P. Kim and R. Song.

2.1 Forward Kato class $\mathcal{FK}_{d-1}^c$

In the time-dependent case, the definition of Kato class is very subtle. In this section we introduce the forward Kato class $\mathcal{FK}_{d-1}^c$, which is strictly larger than $\mathcal{TK}_{d-1}^c$.

Definition 2.1.1. A measurable function f on $[0,\infty)\times\mathbb{R}^d$ is said to be in the forward Kato class $\mathcal{FK}_{d-1}^c$ if

$$\lim_{h\to 0} N_h^{c,+}(f)=0,$$

where

$$N_h^{c,+}(f):=\sup_{(s,x)\in[0,\infty)\times\mathbb{R}^d}\int_s^{s+h}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{t-s})|f(t,y)|dydt.$$

Here $c>0$ is a given constant.

By definition, $\mathcal{FK}_{d-1}^c$ includes the time-dependent Kato class $\mathcal{TK}_{d-1}^c$. However, they are not the same.

Example 2.1.2. For any given $c>0$, we have $\mathcal{FK}_{d-1}^c\neq\mathcal{TK}_{d-1}^c$.

Let

$$f(t,x)=\begin{cases}-\frac{1}{(1-t)^{1/2}\ln(1-t)} & \text{if } \frac{1}{2}\leqslant t<1,\ |x|\leqslant 3d(1-t)^{\frac{1}{2}}\\ 0 & \text{otherwise}\end{cases}$$

If we fix $(t,y)=(1,0)$, then for all $0<h<1$,

$$\begin{aligned}&\int_{1-h}^1\int_{\mathbb{R}^d}\frac{1}{(1-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x|^2}{1-s})|f(s,x)|dxds\\
=&\int_{1-h}^1\int_{|x|\leqslant 3d(1-s)^{\frac{1}{2}}}\frac{1}{(1-s)^{\frac{d+1}{2}}}\exp(-c\frac{|x|^2}{1-s})|f(s,x)|dxds\\
=&\int_{1-h}^1\frac{1}{(1-s)\ln\frac{1}{(1-s)}}ds\int_{|x|\leqslant 3d(1-s)^{\frac{1}{2}}}\frac{1}{(1-s)^{\frac{d}{2}}}\exp(-c\frac{|x|^2}{1-s})dx,\ \text{let }\frac{x}{(1-s)^{\frac{1}{2}}}=x'\\
=&\int_{1-h}^1\frac{1}{(1-s)\ln\frac{1}{(1-s)}}ds\int_{|x'|\leqslant 3d}\exp(-c|x'|^2)dx,\\
\geq& C\int_{1-h}^1\frac{1}{(1-s)\ln\frac{1}{(1-s)}}ds,\ \text{where } C>0 \text{ is some constant.}\end{aligned}$$

Since $\int_{1-h}^1 \frac{1}{(1-s)\ln\frac{1}{(1-s)}}ds$ diverges, we have

$$N_h^\alpha(f) \geqslant \int_{1-h}^1 \int_{\mathbb{R}^d} \frac{1}{(1-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x|^2}{1-s})|f(s,x)|dxds = \infty,$$

so $f \notin \mathcal{TK}^c_{d-1}$. Next we show that $f \in \mathcal{FK}^c_{d-1}$.

For $\frac{1}{2} \leq s < 1$, we have

$$\lim_{s\to 1}\int_s^1 \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt = 0$$

In fact,

$$\begin{aligned}
&\lim_{s\to 1}\int_s^1 \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt \\
\leq &\lim_{s\to 1}\int_s^1 \frac{1}{(t-s)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\ln\frac{1}{(1-t)}}dt \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d}{2}}} \exp(-c\frac{|y|^2}{t-s})dy \\
\leq &\lim_{s\to 1} C\int_s^1 \frac{1}{(t-s)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\ln\frac{1}{(1-t)}}dt, \quad \text{let} \quad r = \frac{t-s}{1-s}, \\
\leq &\lim_{s\to 1} C\int_0^1 \frac{1}{r^{\frac{1}{2}}(1-r)^{\frac{1}{2}}\ln\frac{1}{(1-s)(1-r)}}dr \\
\leq &\lim_{s\to 1} \frac{1}{\ln\frac{1}{1-s}} C\int_0^1 \frac{1}{r^{\frac{1}{2}}(1-r)^{\frac{1}{2}}}dr = 0
\end{aligned}$$

For any given $\epsilon > 0$, we can find a constant $s_0 > \frac{1}{2}$ such that

$$\int_s^1 \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt < \frac{\epsilon}{4}, \quad \text{if} \quad s_0 \leq s < 1.$$

We set

$$C_1 = \sup_{\frac{1}{2}\leqslant t\leqslant \frac{s_0+1}{2}} \frac{1}{(1-t)^{\frac{1}{2}}\ln\frac{1}{(1-t)}}.$$

Let $h_0 > 0$ be sufficiently small such that

$$\int_0^{h_0} \int_{\mathbb{R}^d} \frac{1}{t^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t})dydt < \frac{\epsilon}{4C_1}.$$

Let $h < \frac{h_0}{2} \wedge \frac{(1-s_0)}{2}$. Then for $s < s_0$, we have

$$\begin{aligned}
&\int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt \\
&\leq \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s}) \cdot C_1 dydt \\
&\leq C_1 \cdot \frac{\epsilon}{4C_1} \leq \frac{\epsilon}{4}.
\end{aligned}$$

If $s \geq s_0$, then

$$\begin{aligned}
&\int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt \\
&\leq \int_s^{1} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt < \frac{\epsilon}{4}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{(s,x)\in\mathbb{R}\times\mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{t-s})|f(t,y)|dydt \\
&\leq \sup_{s\in\mathbb{R}} \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-c\frac{|y|^2}{t-s})|f(t,y)|dydt < \epsilon.
\end{aligned}$$

We have proved that $\lim_{h\to\infty} N_h^{c,+}(f) = 0$ and $f \in \mathcal{FK}_{d-1}^c$.

Now we state some properties of $\mathcal{FK}_{d-1}^c$, the proofs are similar to the case of $\mathcal{TK}_{d-1}^c$.

Proposition 2.1.3. *(i) If $f \in \mathcal{FK}_{d-1}^c$, then f is locally integrable.*
(ii) If $f(t,x) \in \mathcal{FK}_{d-1}^c$, then $N_l^{c,+}(f) < \infty$ for any $l > 0$.

Lemma 2.1.4. *Suppose $f \in \mathcal{FK}_{d-1}^c$, then for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^{d+1})$ with*

$$\int_{\mathbb{R}^{d+1}} \phi(\xi)d\xi = 1,$$

we have $N_h^{c,+}(f * \phi) \leq N_h^{c,+}(f)$.

Proposition 2.1.5. *Given a non-negative function $\phi \in C_0^\infty(\mathbb{R}^{d+1})$ with*

$$\int_{\mathbb{R}^{d+1}} \phi(\xi)d\xi = 1.$$

Let

$$\phi_n(\xi) = n^{(d+1)}\phi(n\xi).$$

Suppose $f \in \mathcal{FK}_{d-1}^c$, then for any compact set $K \subset [0,\infty) \times \mathbb{R}^d$,

$$\lim_{n\to\infty} N_h^{c,+}(\mathbf{1}_K|f * \phi_n - f|) = 0.$$

2.2 Brownian motion with time-dependent singular drift

In this section we study the following stochastic differential equation

$$\begin{cases} dX_t = dW_t + B(t, X_t)dt, & t \geq s. \\ X_t = x, & 0 \leq t \leq s. \end{cases} \tag{2.3}$$

Throughout the rest of this chaper we impose

Assumption 2.2.1. $|B(\cdot,\cdot)| \in \mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$.

Under the above assumption, we prove there exits a unique weak solution to (2.3) for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$. Instead of dealing with equation (2.3) directly, we use the equivalent formulation of the martingale problem due to Stroock and Varadhan. Namely we will prove that the martingale problem for the generator

$$L_t = \frac{1}{2}\triangle + B(t, x) \cdot \nabla$$

is well-posed. We will use the same method as in [BC03] and adapt it to the time-dependent case. The idea is to construct the space-time resolvent S^λ of the process X_t. If the drift $B(t, x)$ has support in a very small comact set, then we will see that the space-time resolvent S^λ of X_t can be expressed in terms of the space-time resolvent of Brownian motion. So heuristically, we can first solve (2.3) locally. Then after a standard gluing argument we will also get a global solution. To prove uniqueness we need to use the techniques from Srtoock and Varadhan's martingale problem approach.

2.2.1 The local martingale problem and martingale problem

As well-known, (2.3) is equivalent to the local martingale problem of Stroock and Varadhan. As compared with (2.3), using the local martingale problem approach has several advantages. Now we let

$$L_t = \frac{1}{2}\triangle + B(t, x) \cdot \nabla,$$

and $\Omega = C([0, \infty); \mathbb{R}^d)$ be the space of continuous trajectories from $[0, \infty)$ into $\mathbb{R}^d$. Given $t \geq 0$ and $\omega \in \Omega$, let $X_t(\omega) := \omega(t)$. Let

$$\mathcal{M}_t = \sigma(X_s : 0 \leq s \leq t),$$

and

$$\mathcal{M} = \sigma\big(\cup_{t\geq 0} \mathcal{M}_t \big).$$

Definition 2.2.2. Given $(s, x) \in [0, \infty) \times \mathbb{R}^d$, a solution to the local martingale problem for L_t starting from (s, x) is a probability measure $\mathbf{P}^{s,x}$ on $(\Omega = C([0, \infty); \mathbb{R}^d), \mathcal{M})$ with the following properties:

$$\mathbf{P}^{s,x}(X_t = x, 0 \leqslant t \leqslant s) = 1 \quad \text{and}$$

$$f(X_t) - \int_s^t L_u f(X_u))du$$

is a $(\mathbf{P}^{s,x}, \mathcal{F}_t)$ local martingale after time s for all $f \in C^2(\mathbb{R}^d)$. Here the filtration $\mathcal{F}_t$ is the augmentation of $\mathcal{M}_t$ w.r.t. $\mathbf{P}^{s,x}$, i.e.

$$\mathcal{F}_t = \mathcal{G}_{t+}, \mathcal{G}_t = \sigma(\mathcal{M}_t, \mathcal{N}).$$

According to [KS91, Proposition 4.11], in our case the martingale problem and the local martingale problem for L_t are equivalent. Note that here the second order term in L_t is nothing but $\frac{1}{2}\triangle$.

Definition 2.2.3. Given $(s, x) \in [0, \infty) \times \mathbb{R}^d$, a solution to the martingale problem for L_t starting from (s, x) is a probability measure $\mathbf{P}_{s,x}$ on $(\Omega = C([0, \infty); \mathbb{R}^d), \mathcal{M})$ with the properties that

$$\mathbf{P}^{s,x}(X_t = x, 0 \leqslant t \leqslant s) = 1 \quad \text{and}$$

$$f(X_t) - \int_s^t L_u f(X_u))du$$

is a $(\mathbf{P}^{s,x}, \mathcal{F}_t)$ martingale after time s for all $f \in C_0^\infty(\mathbb{R}^d)$.

We say that the martingale problem for L_t is *well-posed* if, for each (s, x) there is exactly one solution to that martingale problem staring from (s, x).

Remark 2.2.4. In Definition 2.2.3, we have used the filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions. But it is more convenient to deal with $\{\mathcal{M}_t\}$ itself because it does not depend on the probability measure and is countably generated. Suppose $f \in C_0^\infty(\mathbb{R}^d)$ and let

$$M_t^f := f(X_t) - \int_s^t L_u f(X_u))du.$$

If $\{M_t^f, \mathcal{M}_t\}$ is a martingale after time s, then so is $\{M_t^f, \mathcal{F}_t\}$. The reason is simple. For any $s \leq t_1 < t_2$, if

$$E[M_{t_2}^f | \mathcal{M}_{t_1+\frac{1}{n}}] = M_{t_1+\frac{1}{n}}^f,$$

then let n goes to infinity, we get

$$E[M^f_{t_2}|\mathcal{M}_{t_1+}] = M^f_{t_1}, a.e.$$

Therefore $\{M^f_t, \mathcal{F}_t\}$ is also a martingale after time s.

2.2.2 Some gradient estimates for R^λ

In this subsection we derive some gradient estimates for the space-time resolvent R^λ of Brownian motion. Recall that the transition density function $p(s,x;t,y)$ of Brownian motion is given by

$$p(s,x;t,y) = \frac{1}{(2\pi)^{\frac{d}{2}}(t-s)^{\frac{d}{2}}} \exp(-\frac{1}{2}\cdot\frac{|x-y|^2}{t-s}).$$

For $\alpha < \frac{1}{2}$, it is easy to verify that there exists a constant $C_1 > 1$ such that for any $0 \leq s < t$ and $x, y \in \mathbb{R}^d$,

$$|\nabla_x p(s,x;t,y)| \leqslant \frac{C_1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha\frac{|x-y|^2}{t-s}). \tag{2.4}$$

For any $\lambda > 0$ and any bounded measurable function f on $[0,\infty) \times \mathbb{R}^d$, let R^λ be the space-time resolvent of Brownian motion, namely

$$R^\lambda f(s,x) := \int_s^\infty e^{-\lambda(t-s)}dt \int_{\mathbb{R}^d} p(s,x;t,y)f(t,y)dy$$

Lemma 2.2.5. *If $f \in \mathcal{FK}^\alpha_{d-1}$ and $supp(f) \subset [s_1, s_1+h] \times \mathbb{R}^d$ for some $s_1 \geqslant 0, h > 0$, then*

$$|\nabla R^\lambda f| \leqslant C_1 \cdot N_h^{\alpha,+}(f),$$

where C_1 is the constant appearing in (2.4).

Proof.

$$\begin{aligned}
&|\nabla_x R^\lambda f(s,x)| \\
=&|\nabla_x \int_s^\infty e^{-\lambda(t-s)}dt \int_{\mathbb{R}^d} p(s,x;t,y)f(t,y)dy| \\
=&|\int_s^\infty e^{-\lambda(t-s)}dt \int_{\mathbb{R}^d} \nabla_x p(s,x;t,y)f(t,y)dy| \\
\leqslant& \int_s^\infty \int_{\mathbb{R}^d} C_1 \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha\frac{|x-y|^2}{t-s})|f(t,y)|dydt
\end{aligned}$$

If $s \geqslant s_1$, then

$$\begin{aligned}
&|\nabla_x R^\lambda f(s,x)| \\
\leqslant &\int_s^{s+h}\int_{\mathbb{R}^d} C_1 \frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s})|f(t,y)|dydt \\
\leqslant &C_1\cdot N_h^{\alpha,+}(f)
\end{aligned}$$

If $s < s_1$, then

$$\begin{aligned}
&|\nabla_x R^\lambda f(s,x)| \\
\leqslant &\int_{s_1}^{s_1+h}\int_{\mathbb{R}^d} C_1 \frac{1}{(t-s)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|x-y|^2}{t-s})|f(t,y)|dydt \\
\leqslant &\int_{\mathbb{R}^d}\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|z-x|^2}{s_1-s})dz\int_{s_1}^{s_1+h}\int_{\mathbb{R}^d}\frac{C_1}{(t-s_1)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|y-z|^2}{t-s_1})|f(t,y)|dydt \qquad (2.5)\\
\leqslant &C_1\int_{\mathbb{R}^d}\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|z-x|^2}{s_1-s})dz\cdot N_{\alpha,h}^+(f) \\
\leqslant &C_1\cdot N_{\alpha,h}^+(f)
\end{aligned}$$

In fact, to get (2.5) we need the following inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|z-x|^2}{s_1-s})\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s_1)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|y-z|^2}{t-s_1})dz \\
\geqslant &\frac{1}{(t-s)^{\frac{1}{2}}}\int_{\mathbb{R}^d}\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(s_1-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|z-x|^2}{s_1-s})\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s_1)^{\frac{d}{2}}}\exp(-\alpha\frac{|y-z|^2}{t-s_1})dz \\
= &\frac{1}{(t-s)^{\frac{1}{2}}}\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s)^{\frac{d}{2}}}\exp(-\alpha\frac{|y-x|^2}{t-s}) \\
= &\frac{(2\alpha)^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}(t-s)^{\frac{d+1}{2}}}\exp(-\alpha\frac{|y-x|^2}{t-s})
\end{aligned}$$

□

Similar to the above lemma, we have the following estimate for R^λ.

Lemma 2.2.6. *Suppose $f \in \mathcal{FK}_{d-1}^\alpha$ and supp(f) $\subset [s_1, s_1+h]\times\mathbb{R}^d$ for some $s_1 \geqslant 0$, $0 < h < 1$, then*

$$|R^\lambda f| \leqslant N_h^{\alpha,+}(f)$$

Proof. The proof is similar to Lemma 2.2.5, we only need to note that if $0 < t - s < 1$, then

$$p(s,x;t,y) \leq \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha \frac{|x-y|^2}{t-s}).$$

□

The following lemma is well-known, but for the reader's convenience we give a proof here.

Lemma 2.2.7. *For each $\lambda > 0$, there exists a constant $C_\lambda > 0$ such that for any g bounded measurable on $[0,\infty) \times \mathbb{R}^d$, we have*

$$|\nabla R^\lambda g| \leqslant C(\lambda) \cdot \|g\|_\infty$$

Proof.

$$\begin{aligned}
&|\nabla_x R^\lambda g(s,x)| \\
=&|\nabla_x \int_s^\infty e^{-\lambda(t-s)} dt \int_{\mathbb{R}^d} p(s,x;t,y) g(t,y) dy| \\
=&| \int_s^\infty e^{-\lambda(t-s)} dt \int_{\mathbb{R}^d} \nabla_x p(s,x;t,y) g(t,y) dy| \\
\leqslant& \int_s^\infty e^{-\lambda(t-s)} dt \int_{\mathbb{R}^d} C_1 \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha \cdot \frac{|x-y|^2}{t-s}) |g(t,y)| dy dt \\
\leqslant& C_1 \|g\|_\infty \int_s^\infty e^{-\lambda(t-s)} dt \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp(-\alpha \cdot \frac{|x-y|^2}{t-s}) dy \\
\leqslant& C(\lambda) \cdot \|g\|_\infty
\end{aligned}$$

Thus we have proved our lemma. □

2.2.3 Well-posedness of the martingale problem: local case

In this subsection we will get the well-posedness of martingale problem for L_t if the drift B has support in a very small compact set.

Recall Assumption 2.2.1 that $|B(\cdot,\cdot)| \in \mathcal{FK}^\alpha_{d-1}$ for some $\alpha < \frac{1}{2}$. We first consider smooth approximations of the singular drift B. We can find a nonnegative function $\phi \in C_0^\infty(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}} \phi(\xi) d\xi = 1$.

Let

$$\phi_n(\xi) = n^{(d+1)}\phi(n\xi),$$

and define

$$B_n := B * \phi_n = (B^1 * \phi_n, \cdots, B^d * \phi_n).$$

Remark 2.2.8. From Lemma 2.1.4, it is easily seen that there exists a constant $\kappa > 1$ such that for any $h > 0$

$$N_h^{\alpha,+}(|B_n|) \leq \kappa N_h^{\alpha,+}(|B|). \tag{2.6}$$

For this subsection we impose an additional assumption on B.

Assumption 2.2.9. There exist $(s_1, x_1) \in [0, \infty) \times \mathbb{R}^d$ such that

$$\mathrm{supp}(B) \subset [s_1, s_1 + \epsilon_1] \times \left\{x \in \mathbb{R}^d : |x - x_1| \leq 1\right\}$$

and

$$N_{2\epsilon_1}^{\alpha,+}(|B|) < \frac{1}{2\kappa C_1}.$$

Remark 2.2.10. Since now B has compact support, by Lemma 2.1.5 we have for any $h > 0$,

$$\lim_{n\to\infty} N_h^{\alpha,+}(|B_n - B|) = 0. \tag{2.7}$$

Since B_n is smooth and has compact support, for each starting point $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exists a unique probability measure $\mathbf{P}_n^{s,x}$ on $\big(\Omega = C([0,\infty); \mathbb{R}^d), \mathcal{M}\big)$ which solves the martingale problem for the generator

$$\frac{1}{2}\triangle + B_n(t, x) \cdot \nabla \tag{2.8}$$

For any $\lambda > 0$ and any bounded measurable function f on $[0, \infty) \times \mathbb{R}^d$, define

$$S_n^\lambda f(s, x) := E_n^{s,x}[\int_s^\infty e^{-\lambda(t-s)} f(t, X_t) dt],$$

where $E_n^{s,x}[\cdot]$ means taking expectation under the measure $\mathbf{P}_n^{s,x}$ on $\big(\Omega = C([0,\infty); \mathbb{R}^d), \mathcal{M}\big)$.

Now we want to get an exact expression of the space-time resolvent S_n^λ. Recall that R^λ is the space-time resolvent for Brownian motion. For any $f \in \mathcal{B}_b([0,\infty) \times \mathbb{R}^d)$, since $\frac{\partial R^\lambda f}{\partial x_i}$ exists and is continuous, we can define the operator

$$BR^\lambda(f) := (B, \nabla R^\lambda f) = \sum_{i=1}^d B^i \cdot \frac{\partial R^\lambda f}{\partial x_i}.$$

Similarly we can define

$$B_n R^\lambda(f) := (B_n, \nabla R^\lambda f) = \sum_{i=1}^d B_n^i \cdot \frac{\partial R^\lambda f}{\partial x_i}.$$

Lemma 2.2.11. *If $g \in \mathcal{B}_b([0,\infty) \times \mathbb{R}^d)$, then*

$$S_n^\lambda g(s,x) = \sum_{k=0}^\infty R^\lambda (B_n R^\lambda)^k g(s,x),$$

where the convergence on the right-hand side is uniform with respect to $(s,x) \in [0,\infty) \times \mathbb{R}^d$.

Proof. Since B_n is smooth and has compact support, Brownian motion with such a drift B_n has a transition density function $q_n(s,x;t,y)$. Recall that $p(s,x;t,y)$ is the transition density function of Brownian motion. Then by Duhamel's formular, we get that

$$q_n(s,x;t,y) = p(s,x;t,y) + \int_s^t \int_{\mathbb{R}^d} q_n(s,x;\tau,z) B_n(\tau,z) \cdot \nabla_z p(\tau,z;t,y) dz d\tau. \tag{2.9}$$

By (2.4) and noting that $|B_n| \in \mathcal{TK}_{d-1}^{\frac{\alpha}{4}}$, we can apply the same arguments of section 1.3.1 in chapter 1. In particular we get the same Gaussian bounds for $q_n(t,x;s,y)$ as (1.14).

By (2.9), we can consider the difference between S_n^λ and R^λ. If f is bounded measurable, then

$$\begin{aligned}
&S_n^\lambda f(s,x) - R^\lambda f(s,x)\\
=&\int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} q_n(s,x;t,y) f(t,y) dy dt - \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} p(s,x;t,y) f(t,y) dy dt\\
=&\int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} (q_n(s,x;t,y) - p(s,x;t,y)) f(t,y) dy dt\\
=&\int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} f(t,y) (\int_s^t \int_{\mathbb{R}^d} q_n(s,x;\tau,z) B_n(\tau,z) \cdot \nabla_z p(\tau,z;t,y) dz d\tau) dy dt.
\end{aligned}$$

Since f is bounded and B_n has compact support, by (1.14) and (1.20) we have

$$\begin{aligned}
&\int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} f(t,y) \Big(\int_s^t \int_{\mathbb{R}^d} q_n(s,x;\tau,z) |B_n(\tau,z)| \cdot |\nabla_z p(\tau,z;t,y)| dz d\tau \Big) dy dt\\
\leqslant& C \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} f(t,y) \Big(\int_s^t \int_{\mathbb{R}^d} \Gamma_{\frac{\alpha}{2}}(s,x;\tau,z) |B_n(\tau,z)| \Psi_\alpha(\tau,z;t,y) dz d\tau \Big) dy dt\\
\leqslant& C \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} f(t,y) \Gamma_{\frac{\alpha}{2}}(s,x;t,y) dy dt < \infty,
\end{aligned}$$

where $\Gamma_{\frac{\alpha}{2}}$ and Ψ_α are introduced in (1.10). Therefore we can apply Fubini's theorem to get

$$
\begin{aligned}
&S_n^\lambda f(s,x) - R^\lambda f(s,x) \\
&= \int_s^\infty \int_{\mathbb{R}^d} q_n(s,x;\tau,z)dzd\tau\Big(\int_\tau^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} f(t,y) B_n(\tau,z)\cdot\nabla_z p(\tau,z;t,y)dydt\Big) \\
&= \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(\tau-s)} q_n(s,x;\tau,z)dzd\tau\Big(\int_\tau^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-\tau)} f(t,y) B_n(\tau,z)\cdot\nabla_z p(\tau,z;t,y)dydt\Big) \\
&= \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(\tau-s)} q_n(s,x;\tau,z)\big(B_n(\tau,z)\cdot\nabla_z\big(\int_\tau^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-\tau)} p(\tau,z;t,y) f(t,y)dydt\big)dzd\tau
\end{aligned}
$$

Namely we have shown that

$$S_n^\lambda f - R^\lambda f = S_n^\lambda B_n R^\lambda f.$$

For any bounded measurable function g on $[0,\infty)\times\mathbb{R}^d$, taking $f = B_n R^\lambda g$ in the above formula, we get

$$S_n^\lambda B_n R^\lambda g - R^\lambda B_n R^\lambda g = S_n^\lambda B_n R^\lambda B_n R^\lambda g,$$

therefore we get

$$S_n^\lambda g = R^\lambda g + R^\lambda B_n R^\lambda g + S_n^\lambda (B_n R^\lambda)^2 g.$$

Similarly, after i steps, we get to

$$S_n^\lambda g = \sum_{k=0}^{i} R^\lambda (B_n R^\lambda)^k g + S_n^\lambda (B_n R^\lambda)^{i+1} g.$$

If n is large enough, we can ensure that $supp(B_n) \subset [s_1 - \frac{\epsilon_1}{2}, s_1 + \frac{3\epsilon_1}{2}] \times K$ for some compact set $K \subset \mathbb{R}^d$. we also have

$$N_{2\epsilon_1}^{\alpha,+}(|B_n|) \leq \kappa N_{2\epsilon_1}^{\alpha,+}(|B|) < \frac{1}{2C_1}.$$

Claim 1. *$supp((B_n R^\lambda)^k g) \subset [s_1 - \frac{\epsilon_1}{2}, s_1 + \frac{3\epsilon_1}{2}] \times K$ and $(B_n R^\lambda)^k g \in \mathcal{FK}_{d-1}^\alpha$. Moreover*

$$N_{2\epsilon_1}^{\alpha,+}((B_n R^\lambda)^k g) < C_\lambda \|g\|_\infty (\frac{1}{2})^k. \tag{2.10}$$

When $k = 1$, by Lemma 2.2.7 we have

$$|B_n R^\lambda g| \leqslant |B_n| \cdot |\nabla R^\lambda g| \leqslant |B_n| C_\lambda \|g\|_\infty$$

So

$$N^{\alpha,+}_{2\epsilon_1}(B_nR^\lambda g) < C_\lambda\|g\|_\infty(\frac{1}{2}).$$

Suppose that the claim is true for k, then by Lemma 2.2.5

$$|(B_nR^\lambda)^{k+1}g| \leqslant |B_n|\cdot|\nabla R^\lambda(B_nR^\lambda)^k g| \leqslant |B_n|\cdot C_1\cdot N^{+}_{\alpha,2\epsilon_1}((B_nR^\lambda)^k g)$$

$$\leqslant C_\lambda\|g\|_\infty|B_n|\cdot C_1(\frac{1}{2})^k.$$

Therefore

$$N^{\alpha,+}_{2\epsilon_1}((B_nR^\lambda)^{k+1}g) \leqslant C_\lambda\|g\|_\infty C_1(\frac{1}{2})^k\cdot\frac{1}{2C_1} \leqslant C_\lambda\|g\|_\infty(\frac{1}{2})^{k+1}$$

So the claim is proved.

Now we have that

$$\begin{aligned}
&|S^\lambda_n(B_nR^\lambda)^{k+1}g|\\
=&|S^\lambda_n(B_n\nabla R^\lambda(B_nR^\lambda)^k g)|\\
\leqslant&S^\lambda_n(|B_n||\nabla R^\lambda(B_nR^\lambda)^k g|)\\
\leqslant&C_1N^{\alpha,+}_{2\epsilon_1}((B_nR^\lambda)^k g)S^\lambda_n(|B_n|)\\
\leqslant&C'(\frac{1}{2})^{k+1},
\end{aligned}$$

where $C'>0$ is a constant. With the same argument, we can also get

$$|R^\lambda(B_nR^\lambda)^k g| \leqslant C_1R^\lambda(|B_n|)(\frac{1}{2})^{k-1}.$$

From Lemma 2.2.6, we have

$$|R^\lambda(|B_n|)| \leq N^{\alpha,+}_{2\epsilon_1}(|B_n|) \leq \frac{1}{2C_1}.$$

Therefore

$$|R^\lambda(B_nR^\lambda)^k g| \leq (\frac{1}{2})^k. \tag{2.11}$$

Now it is clear that

$$S^\lambda_n g = \sum_{k=0}^{\infty} R^\lambda(B_nR^\lambda)^k g. \tag{2.12}$$

So the lemma is proved. $\square$

Remark 2.2.12. If we check the proof of the above lemma, the only thing we need to ensure the convergence in (2.12) is that

$$N_{2\epsilon_1}^{\alpha,+}(|B_n|) < \frac{1}{2C_1},$$

and we can exactly do the same thing for B, it means that for any bounded measurable function g, we can define $S^\lambda g$ as follows,

$$S^\lambda g(s,x) = \sum_{k=0}^{\infty} R^\lambda (BR^\lambda)^k g(s,x).$$

Moreover, for each term, we know

$$|R^\lambda (BR^\lambda)^k g| \leqslant (\frac{1}{2})^k. \tag{2.13}$$

Now we are ready to the prove that the above definition of S^λ is exactly what we need. we verify that S^λ is the limit of S_n^λ.

Lemma 2.2.13. *For each bounded measurable function g on $[0,\infty) \times \mathbb{R}^d$, $S_n^\lambda g(s,x)$ converges to $S^\lambda g(s,x)$ uniformly with respect to $(s,x) \in [0,\infty) \times \mathbb{R}^d$ as $n \to \infty$.*

Proof. We first show that

$$\lim_{n\to\infty} R^\lambda B_n R^\lambda g(s,x) = R^\lambda B R^\lambda g(s,x),$$

and the convergence is uniform with respect to $(s,x) \in [0,\infty) \times \mathbb{R}^d$. In fact, by Lemma 2.2.6 and Lemma 2.2.7,

$$\begin{aligned}
&|R^\lambda B_n R^\lambda g(s,x) - R^\lambda B R^\lambda g(s,x)| \\
=&|R^\lambda (B_n - B) R^\lambda g(s,x)| \\
\leqslant& C_\lambda \|g\|_\infty R^\lambda |B_n - B|(s,x) \\
\leqslant& C_\lambda \|g\|_\infty N_{2\epsilon_1}^{\alpha,+}(|B - B_n|),
\end{aligned}$$

By (2.7), we have

$$\lim_{n\to\infty} R^\lambda B_n R^\lambda g(s,x) = R^\lambda B R^\lambda g(s,x), \quad \text{uniformly for } (s,x) \in [0,\infty) \times \mathbb{R}^d.$$

Suppose now we have

$$\lim_{n\to\infty} R^\lambda (B_n R^\lambda)^k g(s,x) = R^\lambda (BR^\lambda)^k g(s,x), \quad \text{uniformly for } (s,x) \in [0,\infty) \times \mathbb{R}^d,$$

and

$$\lim_{n\to\infty} N^{\alpha,+}_{2\epsilon_1}[(B_nR^\lambda)^k g - (BR^\lambda)^k g] = 0.$$

Then

$$\begin{aligned}
&|(B_nR^\lambda)^{k+1}g - (BR^\lambda)^{k+1}g| \\
=&|(B_nR^\lambda)^{k+1}g - B_n\nabla R^\lambda(BR^\lambda)^k g + B_n\nabla R^\lambda(BR^\lambda)^k g - (BR^\lambda)^{k+1}g| \\
\leqslant&|B_n||\nabla R^\lambda(B_nR^\lambda)^k g - \nabla R^\lambda(BR^\lambda)^k g)| + |B_n - B||\nabla R^\lambda(BR^\lambda)^k g| \\
\leqslant&C_1 \cdot N^+_{\alpha,2\epsilon_1}((B_nR^\lambda)^k g - (BR^\lambda)^k g)|B_n| + C_1 \cdot |B_n - B|N^+_{\alpha,2\epsilon_1}((BR^\lambda)^k g).
\end{aligned}$$

Similarly to (2.10), we can show

$$N^{\alpha,+}_{2\epsilon_1}((BR^\lambda)^k g) < C_\lambda\|g\|_\infty(\frac{1}{2})^k.$$

So we get

$$\lim_{n\to\infty} N^{\alpha,+}_{2\epsilon_1}[(B_nR^\lambda)^{k+1} g - (BR^\lambda)^{k+1} g] = 0.$$

Then

$$\begin{aligned}
&|R^\lambda(B_nR^\lambda)^{k+1}g - R^\lambda(BR^\lambda)^{k+1}g| \\
\leqslant&C_0N^{\alpha,+}_{2\epsilon_1}[(B_nR^\lambda)^{k+1}g - (BR^\lambda)^{k+1}g] \to 0,
\end{aligned}$$

as $n \to \infty$, and the convergence is uniform with respect to $(s,x) \in [0,\infty) \times \mathbb{R}^d$.

Then by (2.11) and (2.13), the lemma can be proved easily. □

Lemma 2.2.14. *$S^\lambda_n|B_n|(s,x)$ tends to 0 uniformly for $(s,x) \in [0,\infty) \times \mathbb{R}^d$ as $\lambda \to \infty$, and the rate is independent of n.*

Proof. Since $|B_n|$ is bounded, by Lemma 2.2.11, we have

$$S^\lambda_n|B_n| = \sum_{k=0}^{\infty} R^\lambda(B_nR^\lambda)^k|B_n|.$$

Similarly to (2.10), we can get

$$N^{\alpha,+}_{2\epsilon_1}((B_nR^\lambda)^k|B_n|) < (\frac{1}{2})^{k+1}.$$

Therefore

$$
\begin{aligned}
S_n^\lambda|B_n| \leqslant& \sum_{k=0}^{\infty} R^\lambda\big(|B_n| \cdot \big|\nabla R^\lambda (B_n R^\lambda)^{k-1}(|B_n|)\big|\big) \\
\leqslant& \sum_{k=0}^{\infty} R^\lambda\big(|B_n|\big) \cdot C_1 \cdot N_{2\epsilon_1}^{\alpha,+}((B_n R^\lambda)^{k-1}|B_n|) \\
\leqslant& C_1 \cdot R^\lambda|B_n|.
\end{aligned}
$$

For $\forall\epsilon > 0$, we can find a $\delta > 0$ such that

$$
N_\delta^{\alpha,+}(|B|) \leqslant \frac{\epsilon}{4\kappa C_1}.
$$

By (2.6) and noting that $supp(B_n) \subset [s_1 - \frac{\epsilon_1}{2}, s_1 + \frac{3\epsilon_1}{2}] \times K$, we have

$$
\begin{aligned}
& S_n^\lambda|B_n|(s,x) \\
\leqslant& C_1 \cdot R^\lambda|B_n|(s,x) \\
\leqslant& C_1 \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda(t-s)} p(s,x;t,y)|B_n|(t,y)dydt \\
\leqslant& C_1\Big(\int_s^{s+\delta}\int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d}{2}}} \exp(-\alpha\frac{|x-y|^2}{t-s})|B_n|(t,y)dydt \\
& \qquad + \int_{s+\delta}^\infty \int_{\mathbb{R}^d} e^{-\lambda\delta} p(s,x;t,y)|B_n|(t,y)dydt\Big) \\
\leqslant& C_1 \cdot N_\delta^{\alpha,+}(|B_n|) + C_1 e^{-\lambda\delta} \int_{s+\delta}^{s_1+\frac{3\epsilon_1}{2}} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d}{2}}} \exp(-\alpha\frac{|x-y|^2}{t-s})|B_n|(t,y)dydt \\
\leqslant& C_1\kappa \cdot N_\delta^{\alpha,+}(|B|) + C_1 e^{-\lambda\delta} N_{2\epsilon_1}^{\alpha,+}(|B_n|) \\
\leqslant& C_1\kappa \cdot N_\delta^{\alpha,+}(|B|) + C_1\kappa e^{-\lambda\delta} N_{2\epsilon_1}^{\alpha,+}(|B|) \\
\leqslant& \frac{\epsilon}{2} + \frac{1}{2} e^{-\lambda\delta}
\end{aligned}
$$

If $\lambda \to \infty$, then $S_n^\lambda|B_n| < \epsilon$. □

Using the above lemma and doing exactly the same calculation as in the time-independent case (cf. [BC03, Theorem 4.3]), we prove the following proposition, which implies the tightness of the family $\mathbf{P}_n^{s,x}$. To be complete, we include the proof here.

Proposition 2.2.15. *Let $\beta, \epsilon, T > 0$, then $\exists\delta > 0$ not depending on (s,x) nor n, s.t.*

$$
\mathbf{P}_n^{s,x}(\sup_{t,t' \leqslant T, |t-t'| \leqslant \delta} |X_t - X_{t'}| > \beta) < \epsilon.
$$

Proof. By Markov property, it is enough to show that

$$\mathbf{P}_n^{s,x}(\sup_{s\le t\le s+\delta}|X_t-x|>\beta)<\epsilon.$$

We know that under the measure $\mathbf{P}_n^{s,x}$,

$$dX_t^n=dW_t+B_n(t,x)dt,$$

where W_t is the Brownian motion. For Brownian motion W_t, it is known that there exits small enough $\delta>0$ such that

$$\mathbf{P}_n^{s,x}(\sup_{s\le t\le s+\delta}|W_t-W_s|>\frac{\beta}{2})<\frac{\epsilon}{2}.$$

If we can find $\delta>0$ such that

$$\mathbf{P}_n^{s,x}(\int_s^{s+\delta}|B_n(u,X_u)|du>\frac{\beta}{2})<\frac{\epsilon}{2},$$

then we are done. Let $\theta=\delta^{-1}$, then

$$\begin{aligned}
&\mathbf{P}_n^{s,x}(\int_s^{s+\delta}|B_n(u,X_u)|du>\frac{\beta}{2})\le\frac{2}{\beta}E_n^{s,x}\int_s^{s+\delta}|B_n(u,X_u)|du\\
\le&\frac{2e}{\beta}E_n^{s,x}\int_s^{s+\delta}e^{-\theta(t-s)}|B_n(u,X_u)|du\le\frac{2e}{\beta}E_n^{s,x}\int_s^{\infty}e^{-\theta(t-s)}|B_n(u,X_u)|du\\
\le&\frac{2e}{\beta}S_n^\theta|B_n|(s,x)
\end{aligned}$$

By Lemma 2.2.14, we can find large enough $\theta>0$, independent of (s,x) and n such that

$$S_n^\theta|B_n|(s,x)<\frac{\epsilon}{2}.$$

The lemma is proved. □

Corollary 2.2.16. *Let $\beta\in(0,1]$, then there exists $\delta<1$, which does not depends on (s,x), such that if*

$$\tau_s:=\epsilon_1\wedge\inf\{r\ge 0:|X_{s+r}-X_s|>\beta\},$$

then

$$\sup_n E_n^{s,x}e^{-\tau_s}\leqslant\delta.$$

Proof. See the proof of [BC03, Corollary 4.4]. □

By Proposition 2.2.15 and [SV06, Theorem 1.3.2], we know that the family $\{\mathbf{P}_n^{s,x}\}_{n\geq 1}$ of probability measures on $\big(\Omega = C([0,\infty);\mathbb{R}^d), \mathcal{M}\big)$ is tight, therefore we can find a subsequence which converges under the weak topology. Suppose $\mathbf{P}^{s,x} = \lim_{k\to\infty} \mathbf{P}_{n_k}^{s,x}$ is the limit point, then for any $\lambda > 0$ and $g \in C_0(\mathbb{R}^d)$,

$$\begin{aligned}
&E_{\mathbf{P}^{s,x}} \int_0^\infty e^{-\lambda(t-s)} g(X_t)dt \\
=& \lim_{k\to\infty} E_{\mathbf{P}_{n_k}^{s,x}} \int_0^\infty e^{-\lambda(t-s)} g(X_t)dt \\
=& \lim_{k\to\infty} S_{n_k}^\lambda g(s,x) \\
=& S^\lambda g(s,x).
\end{aligned}$$

Since λ is arbitrary, by uniqueness of the Laplace transform, we get that one dimensional distribution of $(X_t, \mathbf{P}^{s,x})$ is determined. Similarly, using Markov property, multidimensional distributions of the process $(X_t, \mathbf{P}^{s,x})$ are also determined. That means that $\{\mathbf{P}_n^{s,x}\}_{n\geq 1}$ has only one limit point.

Theorem 2.2.17. $\mathbf{P}_n^{s,x}$ *converges weakly to* $\mathbf{P}^{s,x}$*, for each* (s,x)*. Moreover,*

$$E^{s,x}\Big(\int_s^\infty e^{-\lambda(t-s)} f(t,X_t)dt\Big) = S^\lambda f(s,x).$$

Now we need to show that the measure $\mathbf{P}^{s,x}$ is indeed a solution to the maringale problem for

$$L_t = \frac{1}{2}\triangle + B(t,x)\cdot\nabla$$

Theorem 2.2.18. $\mathbf{P}^{s,x}$ *is a solution to the martingale problem for* L_t*, starting from* (s,x)*.*

Proof. This is a modification of the proof of [BC03] Proposition 4.9. We need to show that

$$f(X_t) - \int_s^t L_u f(X_u))du$$

is a $\mathbf{P}^{s,x}$-martingale after time s for $\forall f \in C_0^\infty(\mathbb{R}^d)$. Since $\mathbf{P}_n^{s,x}$ solves the martingale problem for

$$\frac{1}{2}\triangle + B_n(t,x)\cdot\nabla,$$

therefore

$$f(X_t) - \int_s^t \Big(\frac{1}{2}\triangle f + \sum_{i=1}^d B_n^i(u,x)\frac{\partial f}{\partial x_i}\Big)(X_u)du$$

is a $\mathbf{P}_n^{s,x}$-martingale after time s.

For any $s \leq t_1 \leq t_2$, $0 \leq r_1 \leq \cdots \leq r_l \leq t_1$ and $g_1, \cdots, g_l \in C_0(\mathbb{R}^d)$, let

$$Y = \prod_{j=1}^{l} g_j(X_{rj}),$$

where $l \in \mathbb{N}$ is arbitrary chosen. Then

$$\begin{aligned}
&E_n^{s,x}\Big[Y\big(f(X_{t_1}) - \int_s^{t_1} (\frac{1}{2}\triangle f + \sum_{i=1}^{d} B_n^i(u,x)\frac{\partial f}{\partial x_i})f(X_u)du\big)\Big] \\
=&E_n^{s,x}\Big[Y(f(X_{t_2}) - \int_s^{t_2} (\frac{1}{2}\triangle f + \sum_{i=1}^{d} B_n^i(u,x)\frac{\partial f}{\partial x_i})f(X_u)du\big)\Big] \qquad (2.14)
\end{aligned}$$

By the weak convergence of $\mathbf{P}_n^{s,x}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} E_n^{s,x}\Big[Y\big(f(X_{t_1}) - \int_s^{t_1} \frac{1}{2}\triangle f(X_u)du\big)\Big] \\
&= E^{s,x}\Big[Y\big(f(X_{t_1}) - \int_s^{t_1} \frac{1}{2}\triangle f(X_u)du\big)\Big].
\end{aligned}$$

Therefore we only need to show that

$$\lim_{n\to\infty} E_n^{s,x}\Big[Y\int_s^{t_1} B_n^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big] = E^{s,x}\Big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big]$$

For each fixed $k \geq 1$, we have

$$\begin{aligned}
&\Big|E_k^{s,x}\big[Y\int_s^{t_1} B_n^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E_k^{s,x}\big[Y\int_s^{t_1} B_m^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
=&\Big|E_k^{s,x}\big[Y\int_s^{t_1} (B_n^i - B^i)(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E_k^{s,x}\big[Y\int_s^{t_1} (B_m^i - B^i)(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
\leq&C \cdot E_k^{s,x}\int_s^{t_1} (|B_n^i - B^i| + |B_m^i - B^i|)\Big|\frac{\partial f}{\partial x_i}\Big|(u,X_u)du \\
\leq&C_1 \cdot S_k^{\lambda}(|B_n^i - B^i| + |B_m^i - B^i|)(s,x) \\
\leq&C_2 \cdot R^{\lambda}(|B_n^i - B^i| + |B_m^i - B^i|)(s,x) \\
\leq&C_3 \cdot N_{2\epsilon_1}^{\alpha,+}(|B_n^i - B^i| + |B_m^i - B^i|) \to 0, \text{ as } n,m \to \infty. \qquad (2.15)
\end{aligned}$$

Similarly to (2.15), we have

$$\begin{aligned}
&\Big|E^{s,x}\big[Y\int_s^{t_1} B_n^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
\leq&C_3 \cdot N_{2\epsilon_1}^{\alpha,+}(|B_n^i - B^i|) \to 0, \quad \text{as } n \to \infty. \qquad (2.16)
\end{aligned}$$

By (2.15) and (2.16), for any given $\epsilon > 0$, we can find n_1, which is independent of k, such that when $n, m \geq n_1$,

$$\Big|E^{s,x}\big[Y\int_s^{t_1} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| < \epsilon$$

and

$$\Big|E^{s,x}_k\big[Y\int_s^{t_1} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}_k\big[Y\int_s^{t_1} B^i_m(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| < \epsilon.$$

Note that there exists n_2 such that for $n \geq n_2$,

$$\Big|E^{s,x}_n\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| < \epsilon$$

Now if $n \geq \sup\{n_1, n_2\}$, we have

$$\begin{aligned}
&\Big|E^{s,x}_n\big[Y\int_s^{t_1} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
\leq&\Big|E^{s,x}_n\big[Y\int_s^{t_1} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}_n\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
&\quad + \Big|E^{s,x}_n\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
&\qquad + \Big|E^{s,x}\big[Y\int_s^{t_1} B^i_{n_1}(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big] - E^{s,x}\big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\big]\Big| \\
\leq&3\epsilon
\end{aligned}$$

So we get that

$$\lim_{n\to\infty} E^{s,x}_n\Big[Y\int_s^{t_1} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big] = E^{s,x}\Big[Y\int_s^{t_1} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big].$$

Similarly

$$\lim_{n\to\infty} E^{s,x}_n\Big[Y\int_s^{t_2} B^i_n(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big] = E^{s,x}\Big[Y\int_s^{t_2} B^i(u,x)\frac{\partial f}{\partial x_i}(X_u)du\Big].$$

Let $n \to \infty$ in (2.14), then we get that

$$f(X_t) - \int_s^t L_u f(X_u)du$$

is a $\mathbf{P}^{s,x}$-martingale after time s. □

Now we prove the uniqueness of solutions to the martingale problem for L_t. This can be done with the same method as in the time-indepedent case.

Theorem 2.2.19. *If there exists another probability measure $\mathbf{Q}^{s,x}$ which solves the martingale problem for L_t starting from (s,x), then for all $f \in C_b(\mathbb{R}^d)$ we have $\mathbf{Q}^{s,x}[f(X_t)] = \mathbf{P}^{s,x}[f(X_t)]$, $\forall t \geq 0$.*

Proof. This is a modification of [BC03] Proposition 5.1. We define a sequence of stopping times

$$\sigma_n := \inf\{t \geq s : \int_s^t |B(u, X_u)| du > n\}, \quad \tau_n := \sigma_n \wedge n,$$

and construct $\mathbf{Q}_n^{s,x}$ in the following way:

$$\mathbf{Q}_n^{s,x}(A \cap (C \circ \theta_{\tau_n})) = E_{\mathbf{Q}^{s,x}}[\mathbf{P}^{\tau_n, X_{\tau_n}}(C); A], \ \forall A \in \mathcal{M}_{\tau_n}, C \in \mathcal{M},$$

where θ_t is the usual shift operators on Ω so that $\theta_t(\omega)(t) = \omega(t+s)$. Then $\mathbf{Q}_n^{s,x}$ again solves the martingale problem for L_t, starting from (s,x), and

$$\begin{aligned}
&E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)}|B(t,X_t)|dt] \\
=&E_{\mathbf{Q}^{s,x}}[\int_s^{\tau_n} e^{-\lambda(t-s)}|B(t,X_t)|dt] + E_{\mathbf{Q}^{s,x}}[e^{-\lambda(\tau_n - s)}E_{\mathbf{P}^{\tau_n, X_{\tau_n}}}\int_{\tau_n}^\infty e^{-\lambda(t-\tau_n)}|B(t,X_t)|dt] \\
\leq&E_{\mathbf{Q}^{s,x}}[\int_s^{\tau_n} |B(t,X_t)|dt] + E_{\mathbf{Q}^{s,x}}[S^\lambda |B|(\tau_n, X_{\tau_n})] \\
<&\infty.
\end{aligned}$$

If now $f \in C^2([0,\infty) \times \mathbb{R}^d)$ with $|f|, |\nabla f|$ bounded, then by Ito's formula,

$$\begin{aligned}
&f(t,X_t) - f(s,X_s) \\
=&\int_s^t \nabla f(u,X_u) \cdot dX_u + \int_s^t \frac{\partial f}{\partial u}(u,X_u)du + \frac{1}{2}\int_s^t \triangle f(u,X_u)du \\
=&\text{"}Martingale\text{"} + \int_s^t (\frac{\partial f}{\partial u} + L_u f)(u,X_u)du.
\end{aligned}$$

Taking expectations, we have

$$E_{\mathbf{Q}_n^{s,x}}[f(t,X_t)] - f(s,x) = E_{\mathbf{Q}_n^{s,x}}[\int_s^t (\frac{\partial f}{\partial u} + L_u f)(u,X_u)du]$$

Multiplying both sides with $e^{-\lambda(t-s)}$ and integrating w.r.t. t from s to ∞, we get

$$\begin{aligned}
&E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)} f(t,X_t)dt] \\
=&\frac{1}{\lambda}f(s,x) + E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)} \int_s^t (\frac{\partial f}{\partial u} + L_u f)(u,X_u)dudt] \\
=&\frac{1}{\lambda}f(s,x) + \frac{1}{\lambda}E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)} (\frac{\partial f}{\partial t} + L_t f)(t,X_t)dt]
\end{aligned}$$

Define the linear functional $V_n^\lambda f$ by

$$V_n^\lambda f(s,x) := E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)} f(t,X_t)dt],$$

then we have

$$\lambda V_n^\lambda f(s,x) = f(s,x) + V_n^\lambda[(\frac{\partial f}{\partial u} + L_u f)](s,x)$$

Note that for given $g \in C_b^1([0,\infty)\times\mathbb{R}^d)$, we have $f = R^\lambda g \in C^2([0,\infty)\times\mathbb{R}^d)$ and $|\nabla f|$ is bounded. Substituting this f in the above equation and noting that $(\lambda - L_u^0 - \frac{\partial}{\partial u})R^\lambda g = g$, we get

$$V_n^\lambda g = R^\lambda g + V_n^\lambda B R^\lambda g. \tag{2.17}$$

After an standard approximation procedure, the above equation holds for any bounded continuous function g. Then it is easy to prove that the above equation still holds for any bounded measurable function g.

Since $V_n^\lambda|B| = E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)}|B(t,X_t)|dt] < \infty$, using (2.17) we get

$$V_n^\lambda B R^\lambda g = R^\lambda B R^\lambda g + V_n^\lambda (BR^\lambda)^2 g.$$

After iteration in k steps we arrive at

$$V_n^\lambda g = \sum_{i=0}^k R^\lambda (BR^\lambda)^i g + V_n^\lambda (BR^\lambda)^{k+1} g.$$

But

$$|V_n^\lambda (BR^\lambda)^{k+1} g| \le \|\nabla R^\lambda (BR^\lambda)^k g\|_\infty E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)}|B(t,X_t)|dt] \to 0,$$

as $k \to \infty$.

Therefore we have

$$E_{\mathbf{Q}_n^{s,x}}[\int_s^\infty e^{-\lambda(t-s)}g(t,X_t)dt] = \sum_{i=0}^\infty R^\lambda(BR^\lambda)^i g(s,x) = E_{\mathbf{P}^{s,x}}[\int_s^\infty e^{-\lambda(t-s)}g(t,X_t)dt].$$

By the uniqueness of the Laplace transform, for $\mathbf{Q}_n^{s,x}$ we have

$$\mathbf{Q}_n^{s,x}[f(X_t)] = \mathbf{P}^{s,x}[f(X_t)], \text{ for all } f \in C_b(\mathbb{R}^d) \text{ and } t \geq s.$$

Then we can finally get

$$\begin{aligned}
&\mathbf{Q}^{s,x}[f(X_t)] \\
=&\mathbf{Q}^{s,x}[f(X_t), t < \tau_n] + \mathbf{Q}^{s,x}[f(X_t), t \geq \tau_n] \\
=&\lim_{n\to\infty} \mathbf{Q}_n^{s,x}[f(X_t), t < \tau_n] \\
=&\lim_{n\to\infty} \mathbf{Q}_n^{s,x}[f(X_t), t < \tau_n] + \lim_{n\to\infty} \mathbf{Q}_n^{s,x}[f(X_t), t \geq \tau_n] \\
=&\mathbf{P}^{s,x}[f(X_t)].
\end{aligned}$$

□

Lemma 2.2.20. *Suppose that $\mathbf{Q}^{s,x}$ is a solution to the martingale problem for L_t starting from (s,x). For a given $t \geqslant s$, define $Q_\omega := \mathbf{Q}^{s,x}|\mathcal{M}_t$ which is the regular conditional distribution of $\mathbf{Q}^{s,x}$ under $\mathcal{M}_t$. Then there exists a set $N \in \mathcal{M}_t$ such that $\mathbf{Q}^{s,x}(N) = 0$ and Q_ω solves the martingale problem for L_t starting from $(t, \omega(t))$ with $\omega \notin N$.*

Proof. Let $\{f_n : f_n \in C_0^\infty(\mathbb{R}^d),\ n \geq 1\}$ be dense in $C_0^\infty(\mathbb{R}^d)$. By [SV06, Theorem 1.2.10], for each f_n there exists $N_n \in \mathcal{M}_t$ such that $\mathbf{Q}^{s,x}(N_n) = 0$ and

$$M_{f_n}(u) := f_n(X_u) - f_n(X_t) - \int_t^u Lf_n(r, X_r)dr$$

is a martingale after time t with respect to $(\Omega, \mathcal{M}_u, Q_\omega)$ for all $\omega \notin N_n$.

We define a sequence of stopping times

$$\tau_n := \inf\{u \geq s : \int_s^u |B(r, X_r)|dr > n\}.$$

It is easy to see that

$$\mathbf{Q}^{s,x}\big(\{\omega : \tau_n(\omega) \to \infty\}\big) = 1.$$

Therefore there exists $N_\tau \in \mathcal{M}_t$ such that

$$Q_\omega(\{\omega : \tau_n(\omega) \to \infty\}) = 1 \quad \text{for all } \omega \notin N_\tau.$$

Let

$$N := N_\sigma \cup (\cup_{n\geq 1} N_n).$$

Then for any $\omega \notin N$, $M_{f_n}(u \wedge \sigma_n)$ is again a martingale with respect to $(\Omega, \mathcal{M}_u, Q_\omega)$.

For any $f \in C_0^\infty(\mathbb{R}^d)$, we can find f_{n_k} such that $f_{n_k} \to f$ in $C_0^\infty(\mathbb{R}^d)$ as $k \to \infty$. Then

$$M_{f_{n_k}}(u \wedge \tau_n) \to M_f(u \wedge \tau_n)$$

bounded and pointwise as $k \to \infty$.

Hence $(M_f(u \wedge \tau_n), \mathcal{M}_u, Q_\omega)$ is a martingale after t for any $\omega \notin N$. Then for any $\omega \notin N$, $(M_f(u), \mathcal{M}_u, Q_\omega)$ is a local martingale after time t. So we know that Q_ω solves the local martingale problem for L_t starting from (t, ω_t) for any $\omega \notin N$. Since the second order term in L_t is $\frac{1}{2}\triangle$, by [KS91, Proposition 4.11], the local martingale problem for L_t is equivalent to the martingale problem for L_t. Thus Q_ω solves the martingale problem for L_t starting from (t, ω_t) for all $\omega \notin N$. □

Now from the Lemma 2.2.20 and Theorem 2.2.19, by standard arguments one can get the uniqueness of solutions to the martingale problem for L_t.

Theorem 2.2.21. $\mathbf{P}^{s,x}$ *is the unique solution to the martingale problem for* L_t *starting from* (s, x)*. Therefore the martingale problem for*

$$L_t = \frac{1}{2}\triangle + B(t,x) \cdot \nabla$$

is well-posed.

Proof. The proof is the same as the proof of [SV06, Theorem 6.2.3]. □

2.2.4 Well-posedness of the martingale problem: general case

In last section, under Assumption 2.2.9, we proved that the martingale problem for L_t is well-posed. This means that although B is very singular, we can still solve the martingale problem for L_t locally. However, martingale problem can always be reduced to local considerations. Therefore in this section we remove the additional technical Assumption 2.2.9. Now we come back to the general case, namely we only impose Assumption 2.2.1:

$$|B(\cdot,\cdot)| \in \mathcal{FK}_{d-1}^{\alpha} \text{ for some } \alpha < \frac{1}{2}$$

The procedure to extend to general B is standard and is essentially the same as in the time-independent case.

Theorem 2.2.22. *If $|B(\cdot,\cdot)| \in \mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$, then the martingale problem for*

$$L_t = \frac{1}{2}\triangle + B(t,x)\cdot\nabla$$

is well-posed.

Proof. This is a modification of the proof of [BC03, Theorem 2.6]. Since $|B(\cdot,\cdot)| \in \mathcal{FK}^{\alpha}_{d-1}$, we can find sufficiently small $\epsilon_1 > 0$ such that

$$N^{\alpha,+}_{2\epsilon_1}(|B|) < \frac{1}{2\kappa C_1}.$$

For each $(s,x) \in [0,\infty)\times\mathbb{R}^d$, let

$$R_{(s,x)} := [s, s+\epsilon_1] \times \left\{y \in \mathbb{R}^d : |y-x| \le 1\right\}$$

and $\mathbf{P}^{s,x}$ be the solution we constructed in last section with drift vector $\tilde{B}_{(s,x)}$, where $\tilde{B}_{(s,x)}(t,y) := \mathbf{1}_{R_{(s,x)}}(t,y)B(t,y)$.

Now fix $(s,x) \in [0,\infty)\times\mathbb{R}^d$ and let $T_0 = s$. Define

$$T_{i+1} = \inf\{t \ge T_i : (t, X_t) \notin R_{(T_i, X_{T_i})}\}.$$

Let $Q_1 = \mathbf{P}^{s,x}$ and define inductively

$$Q_{i+1}\big(A \cap (C\circ\theta_{T_i})\big) := E_{Q_i}\left[\mathbf{P}^{T_i,X_{T_i}}(C); A\right],\ A \in \mathcal{M}_{T_i},\ C \in \mathcal{M}.$$

It is clear that $Q_m|\mathcal{M}_{T_k} = Q_k|\mathcal{M}_{T_k}$ if $m \ge k$, therefore we can define

$$Q(A) = Q_k(A) \text{ if } A \in \mathcal{M}_{T_k}.$$

Since

$$E_{Q_{i+1}}\left[e^{-T_{i+1}}\right] = E_{Q_{i+1}}\left[e^{-(T_{i+1}-T_i)}e^{-T_i}\right] = E_{Q_i}\left[e^{-T_i}E_{\mathbf{P}^{T_i,X_{T_i}}}\left[e^{-(T_{i+1}-T_i)}\right]\right],$$

from Corollary 2.2.16, we know $E_{\mathbf{P}^{T_i,X_{T_i}}}[e^{-(T_{i+1}-T_i)}] < \delta < 1$, therefore by induction

$$E_{Q_{i+1}}\left[e^{-T_{i+1}}\right] \le \delta^{i+1}.$$

So we have $E_Q[e^{-T_i}] \le \delta^i$ and therefore

$$\lim_{i\to\infty} T_i = \infty, \text{ a.e. under } Q.$$

It is routine to check that Q is a solution. Furthermore, the uniqueness also holds by standard arguments (cf. [Bas98, Section 6.3] and [SV06, Section 6.6]). □

2.3 Transition density function estimates under further conditions

In Section 2.2 we proved existence and uniqueness of weak solutions to the stochastic differential equation

$$\begin{cases} dX_t = dW_t + B(t, X_t)dt, & t \geq s. \\ X_t = x, & 0 \leq t \leq s. \end{cases}$$

with drift terms $B(t, x)$ such that $|B(\cdot, \cdot)| \in \mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$. This process need not to have a transition density function. In this section we will impose further conditions on the drift term B and prove that then the process will have a continuous transition density admitting two-sided Gaussian estimates. To this end we assume B to be in the time-dependent Kato class $\mathcal{TK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{4}$.

Assumption 2.3.1.

$$|B(\cdot, \cdot)| \in \mathcal{TK}^{\alpha}_{d-1} \text{ for some } \alpha < \frac{1}{4}.$$

The reason why we use the class $\mathcal{TK}^{\alpha}_{d-1}, \alpha < \frac{1}{4}$, comes from the gradient estimate of Brownian heat kernel $p(s, x; t, y)$. Namely if $\alpha < \frac{1}{4}$, then we can find $C > 0$ and α_0 with $2\alpha < \alpha_0 < \frac{1}{2}$ such that for any $0 \leq s < t$ and $x, y \in \mathbb{R}^d$,

$$|\nabla_x p(s, x; t, y)| \leq \frac{C}{(t-s)^{\frac{d+1}{2}}} \exp\big(-\alpha_0 \frac{|x-y|^2}{t-s}\big). \tag{2.18}$$

The gradient estimate (2.18) reminds us of the perturbation techniques we used in Chapter 1.

Under Assumption 2.3.1, we prove that $(X_t, \mathbf{P}^{s,x})$ has a continuous transition function $q(s, x; t, y)$ satisfying two-sided Gaussian estimates. Since the method is similar to what we have already used in Chapter 1, here we just state the main theorem and omit the details of the proof.

We should remark that these are analogs of the results of [KS06] in the time-dependent case.

In Chapter **1**, we considered a parabolic equation with singular lower order terms. In view of the estimates (2.18), the methods in chapter **1** can also be applied here.

Since $|B(\cdot,\cdot)| \in \mathcal{TK}^{\alpha}_{d-1}, \alpha < \frac{1}{4}$, by Lemma 1.1.6 and Lemma 1.1.8, we can find a sequence of functions $\varphi_n \in C_0^\infty(\mathbb{R}^{d+1})$ such that

$$B_n := B * \varphi_n = (B^1 * \varphi_n, \cdots, B^d * \varphi_n) \in C_b^\infty(\mathbb{R}^{d+1})$$

and for any $h > 0$ and compact set $K \subset [0,\infty) \times \mathbb{R}^d$,

$$\lim_{n\to\infty} N_h^\alpha(\mathbf{1}_K |B_n - B|) = 0. \tag{2.19}$$

Since $B_n \in C_b^\infty(\mathbb{R}^{d+1})$, there exists a unique solution $\mathbf{P}_n^{s,x}$ to the martingale problem for

$$\frac{1}{2}\triangle + B_n(t,x)\cdot\nabla$$

starting from (s,x). Moreover, the Markov process $(X_t^n, \mathbf{P}_n^{s,x})$ has a continuous transition density $q_n(s,x;t,y)$ which satisfies Kolmogorov's backward equation:

$$\frac{\partial q_n(s,x;t,y)}{\partial s} + \frac{1}{2}\triangle q_n(s,x;t,y) + B_n(s,x)\cdot\nabla_x q_n(s,x;t,y) = 0. \tag{2.20}$$

From the gradient estimate (2.18) and the fact that $2\alpha < \alpha_0 < \frac{1}{2}$, we can use the same method as that of Section 1.3.1 to get the following two Lemmas.

Lemma 2.3.2. *For each $T > 0$, there exist constants c_T, C_T, α' such that for any $0 < t - s \le T$ and $x, y \in \mathbb{R}^d$,*

(i) $\dfrac{c_T}{(t-s)^{\frac{d}{2}}}\exp(-\alpha'\cdot\dfrac{|x-y|^2}{t-s}) \le q_n(s,x;t,y) \le \dfrac{C_T}{(t-s)^{\frac{d}{2}}}\exp(-\dfrac{\alpha_0}{2}\cdot\dfrac{|x-y|^2}{t-s})$;

(ii) $|\nabla_x q_n(s,x;t,y)| \le \dfrac{C_T}{(t-s)^{\frac{d+1}{2}}}\exp(-\dfrac{\alpha_0}{2}\cdot\dfrac{|x-y|^2}{t-s})$.

where the constants c_T, C_T depend only on T and the rate at which $N_h^\alpha(|B|)$ tends to 0 as $h \to 0$.

Lemma 2.3.3. *Let $\theta := \{(s,t) : 0 \le s < t\}$. Suppose that $K_1, K_2 \subset \mathbb{R}^d$ and $K \subset \theta$ are compact sets, then both $q_n(s,x;t,y)$ and $\nabla_x q_n(s,x;t,y)$ converge uniformly on $\{(s,t) : (s,t) \in K\} \times K_1 \times K_2$.*

By Lemma 2.3.2 and 2.3.3 we can prove the main theorem of this section.

Theorem 2.3.4. *Under Assumption 2.3.1, the Markov process* $(X_t, \mathbf{P}^{s,x})$ *has a continuous transition density function* $q(s, x; t, y)$*. Moreover, for each* $T > 0$*, there exist constants* c_T, C_T, α' *such that for any* $0 < t - s \leq T$ *and* $x, y \in \mathbb{R}^d$*,*

$$\text{(i) } \frac{c_T}{(t-s)^{\frac{d}{2}}} \exp(-\alpha' \cdot \frac{|x-y|^2}{t-s}) \leq q(s,x;t,y) \leq \frac{C_T}{(t-s)^{\frac{d}{2}}} \exp(-\frac{\alpha_0}{2} \cdot \frac{|x-y|^2}{t-s}),$$

$$\text{(ii) } |\nabla_x q(s,x;t,y)| \leq \frac{C_T}{(t-s)^{\frac{d+1}{2}}} \exp(-\frac{\alpha_0}{2} \cdot \frac{|x-y|^2}{t-s}),$$

where the constants c_T, C_T *depend only on* T *and the rate at which* $N_h^\alpha(|B|)$ *tends to* 0 *as* $h \to 0$*.*

Proof. This is a modification of the proof of [KS06] Theorem 3.14. From Lemma 2.3.3, we know that the transition density $q_n(s, x; t, y)$ for $(X_t^n, \mathbf{P}_n^{s,x})$ converges uniformly on compact sets. We denote the limit as $q(s, x; t, y)$. Like Theorem 1.3.12, it is easy to show that $\nabla_x q_n(s, x; t, y)$ converges to $\nabla_x q(s, x; t, y)$ uniformly on compact sets. If we can show that $q(s, x; t, y)$ is the transition density of $(X_t, \mathbf{P}^{s,x})$, then we are done.

Using the same arguments in the proof of [KS06] Theorem 3.14, we claim that, as probability measures on $\big(\Omega = C([0,\infty); \mathbb{R}^d), \mathcal{M}\big)$, $\mathbf{P}_n^{s,x}$ converges weakly to $\mathbf{P}^{s,x}$. Now we fix $(s, x) \in [0, \infty) \times \mathbb{R}^d$ and $t > s$. For any $f \in C_0(\mathbb{R}^d)$, we have

$$\begin{aligned}
\mathbf{P}^{s,x}(f(X_t)) &= \lim_{n\to\infty} \mathbf{P}_n^{s,x}(f(X_t)) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} q_n(s,x;t,y) f(y) dy \\
&= \int_{\mathbb{R}^d} q(s,x;t,y) f(y) dy.
\end{aligned}$$

Therefore $(X_t, \mathbf{P}^{s,x})$ has transition density $q(s, x; t, y)$. The theorem is proved. □

Chapter 3

Construction of Glauber dynamics for an unbounded spin system on a graph

Recently unbounded spin systems on a graph were investigated in [Pas07a] and [Pas07b]. Under certain assumptions on the graph and the potential functions, the author obtained existence of tempered Gibbs measures and ergodicity of the corresponding Glauber dynamics.

In this chapter we consider the same model as in [Pas07a], but impose weaker conditions on the potential functions. Apart from some growth conditions, we merely assume the potential functions to be measurable. The main aim is to construct the corresponding Glauber dynamics. Not like the previous chapters, here we come to an infinite dimensional space. To overcome this difficulty we will use the Dirichlet form methods. We adapt the methods of [Pas07a] to get existence of tempered Gibbs measures; then we show that the corresponding Dirichlet form is quasi-regular. Using the correspondence between Markov processes and Dirichlet forms, we can construct the Glauber dynamics.

3.1 Descriptions of the model

Unbounded spin systems on a lattice are very important in statistical mechanics. The properties of the spin system necessarily depend on the specific geometrical structure of the lattice. In [Pas07a], T. Pasurek studied unbounded spin systems on a graph. In this section we consider the same model as in [Pas07a], but under more general assumptions.

3.1.1 Spin system on an infinite graph

Here, instead of the lattice we consider a general infinite graph $\mathbb{G}(\mathbb{V},\mathbb{E})$ consisting of a countable set of vertices $v \in \mathbb{V}$ and a set of unordered edges $e = [v, v'] \in \mathbb{E}$. The graph $\mathbb{G}(\mathbb{V},\mathbb{E})$ is assumed to be connected and simple, i.e. without isolated vertices, loops, and multiple edges. Naturally we have the combinatorial distance $\rho(v, v')$ on $\mathbb{V}$ which is the length of the shortest path connecting $v, v' \in \mathbb{V}$. If $\rho(v, v') = 1$, we say that v and v' are adjacent and denote this by $v \sim v'$. For each vertex v, we define its vicinity $\partial v := \{v' \in \mathbb{V} | v \sim v'\}$ and the degree $m_v := |\partial v|$. We assume throughout this chapter that $\mathbb{G}(\mathbb{V},\mathbb{E})$ is of uniformly bounded degree, i.e.

$$m_{\mathbb{G}} := \sup_{v\in\mathbb{V}} m_v < \infty.$$

Remark 3.1.1. For any $\mathbb{G}(\mathbb{V},\mathbb{E})$ of uniformly bounded degree, it is easy to see that there exists $\delta > 0$ such that

$$\sum_{v\in\mathbb{V}} e^{-\delta\rho(v,o)} < \infty,$$

for some (and hence, for each) fixed vertex $o \in \mathbb{V}$. We define

$$\delta_{\mathbb{G}} := \inf \Big\{ \delta > 0 : \sum_{v\in\mathbb{V}} e^{-\delta\rho(v,o)} < \infty \Big\}.$$

Now we introduce spin systems on $\mathbb{G}(\mathbb{V},\mathbb{E})$. For a spin system on $\mathbb{G}(\mathbb{V},\mathbb{E})$, we mean that to each vertex $v \in \mathbb{V}$, there corresponds a particle performing one-dimensional oscillation and we use $x_v \in \mathbb{R}$ to denote the state of the particle at vertex v. Here for simplicity we consider the one-dimensional spins, but our method also works for the multidimensional case. The configuration space $\Omega := \mathbb{R}^{\mathbb{V}}$ of this system consists of all real sequence $x = (x_v)_{v\in\mathbb{V}}$. We assume that the potential energy of each configuration $x \in \Omega$ is given by a formal Hamiltonian

$$H(x) = \sum_{v} V_v(x_v) + \frac{1}{2} \sum_{v\sim v'} W_{vv'}(x_v, x_{v'}),$$

where the sums are running over all $v \in \mathbb{V}$ and ordered pairs $(v, v') \in \mathbb{V}^2$ with $\rho(v, v') = 1$. The self-potentials V_v and interactions $W_{vv'}$ are assumed to satisfy the following conditions:

Assumption 3.1.2. $W_{vv'}(\cdot,\cdot)$ is a measurable function on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and there exist constants $C_W, J \geq 0$ such that for all $v \sim v'$ and $x_v, x_{v'} \in \mathbb{R}$

$$|W_{vv'}(x_v, x_{v'})| \leq \frac{1}{2}J(C_W + |x_v|^2 + |x_{v'}|^2).$$

Assumption 3.1.3. $V_v(\cdot)$ is a measurable function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and there exist constants $p \geq 2, A_V > m_{\mathbb{G}} J(e^{\delta_{\mathbb{G}}} + \frac{1}{2}), B_V \in \mathbb{R}$ and $C_V > 0$ such that for all $v \in \mathbb{V}$ and $x_v \in \mathbb{R}$

$$A_V|x_v|^2 + B_V \leq V_v(x_v) \leq C_V(1 + |x_v|^p).$$

Here apart from some growth conditions, we merely assume $V_v, W_{vv'}$ to be measurable.

Remark 3.1.4. We suppose that the interaction potentials $W_{vv'}$ are symmetric in $v, v' \in \mathbb{V}$.

3.1.2 Local specification and Gibbs measures

In this section we study Gibbs measures which describe the equilibrium states of our spin system.

Recall that the configuration space $\Omega = \{\omega = (\omega_v)_{v\in\mathbb{V}} : \omega_v \in \mathbb{R}\}$ of our spin system is the infinite product space of $\mathbb{V}$ copies of one dimensional Euclidean space, let $\mathcal{F}$ be the product σ-algebra on Ω.

For each $v \in \mathbb{V}$, let

$$\begin{aligned} \sigma_v : \Omega \to & \mathbb{R} \\ \omega \mapsto & \omega_v \end{aligned}$$

be the projection onto the v'th coordinate. Similarly for each $\Lambda \Subset \mathbb{V}$, by

$$\sigma_\Lambda : \Omega \to \mathbb{R}^\Lambda$$

we denote the projection onto the coordinates in Λ. Let $\mathcal{F}_\Lambda$ be the σ-algebra generated by σ_Λ.

Definition 3.1.5. (i) A real function f on Ω is called a *local function* if f is $\mathcal{F}_\Lambda$-measurable for some $\Lambda \Subset \mathbb{V}$. A subset $A \subset \Omega$ is called a *cylinder set* if $\mathbf{1}_A$ is a local function.

(ii) A function $f : \Omega \to \mathbb{R}$ is called quasilocal if there is a sequence $(f_n)_{n\geq 1}$ of local functions f_n such that $\lim_{n\to\infty} \sup_{\omega\in\Omega} |f(\omega) - f_n(\omega| = 0$.

We fix an inverse temperature $\beta > 0$ and define the local specification

$$\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}.$$

This is a family of probability kernels

$$\mathcal{B}(\Omega) \times \Omega \ni (B, y) \mapsto \pi_\Lambda(B|y) \in [0, 1],$$

where

$$\pi_\Lambda(B|y) := Z_\Lambda^{-1} \int_{\Omega_\Lambda} \exp\big\{ -\beta H_\Lambda(x_\Lambda|y)\big\} \mathbf{1}_B(x_\Lambda \times y_{\Lambda^c}) dx_\Lambda$$

and $\mathbf{1}_B$ is the indicator function of $B \in \mathcal{B}(\Omega)$. Here

$$Z_\Lambda(y) := \int_{\Omega_\Lambda} \exp\big\{ -\beta H_\Lambda(x_\Lambda|y)\big\} dx_\Lambda$$

is the normalization factor and

$$H_\Lambda(x_\Lambda|y) := \sum_{v \in \Lambda} V_v(x_v) + \frac{1}{2} \sum_{\substack{v \in \Lambda \\ v' \in \Lambda \cap \partial v}} W_{vv'}(x_v, x_{v'}) + \sum_{\substack{v \in \Lambda \\ v' \in \Lambda^c \cap \partial v}} W_{vv'}(x_v, y_{v'})$$

is the potential energy within the finite volume Λ with boundary condition $y \in \Omega$.

Remark 3.1.6. By construction the local specification $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ satisfies:

$$\int_\Omega \pi_\Lambda(B|x)\pi_{\Lambda'}(dx|y) = \pi_{\Lambda'}(B|y), \forall \Lambda \subseteq \Lambda', B \in \mathcal{B}(\Omega), y \in \Omega.$$

This is usually referred as the consistency property.

Because we are considering finite range potentials here, by Proposition 2.24(a) and Example 2.25(i) in [Geo88], we can get the following proposition.

Proposition 3.1.7. *The specification $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ is quasilocal, namely for any $\Lambda \Subset \mathbb{V}$ and bounded quasilocal function f, $\pi_\Lambda f$ is again quasilocal.*

Let $\mathcal{P}(\Omega, \mathcal{F})$ denote the set of all probability measures on $(\Omega, \mathcal{F})$. Now we give the definition of Gibbs measures.

Definition 3.1.8. $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ is called a Gibbs measure for $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ if it satisfies the DLR equilibrium equation

$$\int_\Omega \pi_\Lambda(B|x)\mu(dx) = \mu(B), \tag{3.1}$$

for all $\Lambda \Subset \mathbb{V}$ and $B \in \mathcal{B}(\Omega)$.

3.1.3 Tempered configuration space Ω^t

Throughout the rest part of this chapter, we fix a vertex $o \in \mathbb{V}$. For each $\delta > 0$ we define

$$\|x\|_\delta := \left[\sum_{v\in\mathbb{V}} |x_v|^2 \exp\big\{-\delta\rho(v,o)\big\}\right]^{1/2}$$

and $\Omega_\delta := \Big\{x \in \Omega : \|x\|_\delta < \infty\Big\}$. The tempered configuration space is

$$\Omega^t := \bigcap_{\delta > \delta_{\mathbb{G}}} \Omega_\delta.$$

We endow Ω^t with the topology of the projective limit generated by the norms $\|\cdot\|_\delta$, $\delta > \delta_{\mathbb{G}}$.

Suppose $\delta_n > \delta_{\mathbb{G}}, \delta_n \downarrow \delta_{\mathbb{G}}$, on each Ω_{δ_n}, there is a natural distance

$$d_n(x,y) := \Big(\sum_{v\in\mathbb{V}} |x_v - y_v|^2 \exp\big\{-\delta_n\rho(v,o)\big\}\Big)^{1/2}.$$

For any $x, y \in \Omega^t$, define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x,y)}{1 + d_n(x,y)}.$$

It is easy to see that this metric induces the same topology as the original one and with this metric Ω^t becomes a Polish space.

Definition 3.1.9. A Gibbs measure μ is called tempered if $\mu(\Omega^t) = 1$.

Later we restrict ourselves to the tempered configuration space Ω^t and construct the Glauber dynamics on Ω^t.

3.2 Existence of tempered Gibbs measures

In order to prove the existence of Gibbs measure, we first need to adopt a topology on $\mathcal{P}(\Omega, \mathcal{F})$. Here we take the topology of local convergence (cf. [Geo88, Chapter 4]) because our potential functions are only assumed to be meausurable.

Definition 3.2.1. A net $(\mu_\alpha)_{\alpha\in D}$ in $\mathcal{P}(\Omega,\mathcal{F})$ converges to μ under the topology of local convergence if $\lim_D \mu_\alpha(A) = \mu(A)$ for all $A \in \mathcal{F}^0$ where $\mathcal{F}^0$ is the algebra generated by all cylinder sets in Ω.

Definition 3.2.2. A net $(\mu_\alpha)_{\alpha\in D}$ in $\mathcal{P}(\Omega,\mathcal{F})$ is said to be locally equicontinuous if for each $\Lambda \Subset \mathbb{V}$ and each sequence $(A_m)_{m\geq 1}$ in $\mathcal{F}_\Lambda$ with $A_m \downarrow \emptyset$

$$\lim_{m\to\infty} \limsup_{\alpha\in D} \mu_\alpha(A_m) = 0.$$

Let $\triangle := A_V - \frac{1}{2} m_{\mathbb{G}} J$, then by the assumptions on V_v, we have

$$\triangle > m_{\mathbb{G}} J e^{\delta_{\mathbb{G}}}.$$

Now suppose that $v \in \mathbb{V}$, $\Lambda \Subset \mathbb{V}$, $y \in \Omega$, and κ is an arbitrary constant such that

$$m_{\mathbb{G}} J e^{\delta_{\mathbb{G}}} < \kappa < \triangle,$$

we define

$$n_v(\Lambda|y) := \ln\left\{ \int_\Omega \exp\{\beta\kappa|x_v|^2\}\pi_\Lambda(dx|y) \right\}.$$

The following lemma is taken from [Pas07a], for the reader's convenience we put the proof in the appendix to this chapter.

Lemma 3.2.3. *For any $\delta \in \left(\delta_{\mathbb{G}}, \ln \frac{\kappa}{m_{\mathbb{G}} J}\right)$, there exists $\Upsilon_\delta := \Upsilon_\delta(\beta,\kappa) > 0$ such that uniformly for all $y \in \Omega_\delta$*

$$\limsup_{\Lambda\uparrow\mathbb{V}} \left[\sum_{v\in\Lambda} n_v(\Lambda|y) \exp\{-\delta\rho(v,o)\} \right] \leq \beta\Upsilon_\delta.$$

Now we fix a sequence $\Lambda_n \uparrow \mathbb{V}$.

Lemma 3.2.4. *For a given $y \in \Omega^t$, we define $\mu_n := \pi_{\Lambda_n}(\cdot|y)$. Then the sequence μ_n is locally equicontinuous in $\mathcal{P}(\Omega,\mathcal{F})$ and therefore has a cluster point.*

Proof. Let $K_l = [-l, l], l \geq 1$, according to [Geo88, Corollary 4.13], we only need to prove

$$\lim_{l\to\infty} \limsup_n \mu_n(x_v \notin K_l) = 0, \ \forall v \in \mathbb{V}.$$

From Lemma 3.2.3, for each fixed $v \in \mathbb{V}$, we have

$$\limsup_{\Lambda\uparrow\mathbb{V}} \ln\left\{ \int_\Omega \exp\{\beta\kappa|x_v|^2\}\pi_\Lambda(dx|y) \right\} \exp\{-\delta\rho(v,o)\} \leq \beta\Upsilon_\delta.$$

By Jensen's inequality,

$$\limsup_{\Lambda\uparrow\mathbb{V}} \int_\Omega \beta\kappa|x_v|^2\pi_\Lambda(dx|y) \le \beta\Upsilon_\delta \exp\{\delta\rho(v,o)\}.$$

Therefore

$$\begin{aligned}&\limsup_n \mu_n(x_v \notin K_l) = \limsup_n \pi_{\Lambda_n}((x_v \notin K_l)|y)\\ \le&\limsup_n \int_{\{x:|x_v|>l\}} \frac{|x_v|^2}{l^2}\pi_{\Lambda_n}(dx|y) \le \frac{1}{l^2}\limsup_n \int_\Omega |x_v|^2\pi_{\Lambda_n}(dx|y)\\ \le&\frac{1}{l^2\kappa}\beta\Upsilon_\delta\exp\{\delta\rho(v,o)\}.\end{aligned}$$

So

$$\lim_{l\to\infty}\limsup_n \mu_n(x_v \notin K_l) = 0.$$

□

Fix some $y \in \Omega^t$ and suppose that μ is a cluster point of $\mu_n = \pi_{\Lambda_n}(\cdot|y)$. Since our specification is quasilocal, μ is in fact a Gibbs measure. Now we show that μ is tempered.

Theorem 3.2.5. *μ is a tempered Gibbs measure.*

Proof. According to [Geo88, Proposition 4.15], there is a subsequence μ_{n_k} which converges to μ. From Lemma 3.2.3, by Jensen's inequality,

$$\limsup_{\Lambda\uparrow\mathbb{V}} \left[\int_\Omega \sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\pi_\Lambda(dx|y)\right] \le \Upsilon_\delta/\kappa.$$

For each fixed $\Lambda \Subset \mathbb{V}$, there exists k large enough such that $\Lambda \subset \Lambda_{n_k}$. So

$$\limsup_{k\to\infty} \left[\int_\Omega \sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\mu_{n_k}(dx)\right] \le \Upsilon_\delta/\kappa.$$

Recall that for any $A \in \mathcal{F}_\Lambda$, we have $\lim_k \mu_{nk}(A) = \mu(A)$. Then for any $M > 0$,

$$\begin{aligned}&\int_\Omega M \wedge \left(\sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\right)\mu(dx)\\ =&\lim_{k\to\infty}\left[\int_\Omega M \wedge \left(\sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\right)\mu_{n_k}(dx)\right]\\ \le&\limsup_{k\to\infty}\left[\int_\Omega \sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\mu_{n_k}(dx)\right]\\ \le&\Upsilon_\delta/\kappa.\end{aligned}$$

Since M is arbitrary, by monotone convergence theorem, we get

$$\int_\Omega \sum_{v\in\Lambda} |x_v|^2 \exp\{-\delta\rho(v,o)\}\mu(dx) \le \Upsilon_\delta/\kappa,$$

then clearly

$$\int_\Omega \sum_{v\in\mathbb{V}} |x_v|^2 \exp\{-\delta\rho(v,o)\}\mu(dx) \le \Upsilon_\delta/\kappa,$$

which implies $\mu(\Omega_\delta) = 1$. But it is true for any δ such that $\delta_{\mathbb{G}} < \delta < \kappa$, now it follows $\mu(\Omega^t) = 1$. $\square$

3.3 The Dirichlet form and construction of the Glauber dynamics

In this section, we will construct the Glauber dynamics on Ω^t. From now on, we fix some tempered Gibbs measure μ on Ω^t.

For each vertex $v \in \mathbb{V}$, we denote the unit vector at v by $e_v := (\delta_{vv'})_{v'\in\mathbb{V}}$. Now we fix a $v \in \mathbb{V}$ and for $x = (x_v)_{v\in\mathbb{V}} \in \Omega^t$ define

$$\tau_v(x) := x - x_v \cdot e_v.$$

Let $E_v := \{\tau_v(x) : x \in \Omega^t\}$, then E_v is a closed subspace of Ω^t. For each $y \in E_v, s \in \mathbb{R}$, define

$$\rho_v(y,s) = Z_v(y)^{-1} \exp\big\{-\beta H_v(y,s)\big\},$$

here

$$Z_v(y) := \int_{\mathbb{R}} \exp\big\{-\beta H_v(y,s)\big\} ds$$

is the normalization factor and

$$H_v(y,s) := V_v(s) + \sum_{v'\in\partial v} W_{vv'}(s, y_{v'}).$$

Since μ is a tempered Gibbs measure, by definition, for any bounded measurable function $u(x)$ on Ω^t, we have

$$\int_{\Omega^t} u(x)\mu(dx) = \int_{E_v}\int_{\mathbb{R}} u(y + s\cdot e_v)\rho_v(y,s)\mu_v(dx),$$

where $\mu_v := \tau_v(\mu)$.

From Assumption 3.1.2 and Assumption 3.1.3, for $\forall\ y \in E_v, s \in \mathbb{R}$

$$\begin{aligned}H_v(y,s) \le& C_V(1+|s|^p) + \sum_{v'\in\partial v} \frac{1}{2} J(C_W + |s|^2 + |y_{v'}|^2)\\ \le& C_V + \frac{1}{2} m_{\mathbb{G}} C_W J + \sum_{v'\in\partial v} \frac{1}{2}|y_{v'}|^2 J + C_V|s|^p + \frac{1}{2} m_{\mathbb{G}} J|s|^2\end{aligned}$$

Therefore

$$Z_v(y)^{-1} \exp\big\{-\beta C_V - \frac{1}{2}\beta J(m_{\mathbb{G}} C_W + \sum_{v'\in\partial v} |y_{v'}|^2) - \beta(C_V|s|^p + \frac{1}{2} m_{\mathbb{G}} J|s|^2)\big\} \le \rho_v(y,s)$$

For each fixed $y \in E_v$, the function $s \mapsto \rho_v(y,s)$ satisfies the condition (H) in [AR90, Section 2].

Now we denote the dual space of Ω^t by $(\Omega^t)'$ and let

$$l^2(\mathbb{V}) := \{x \in \Omega : \sum_{v\in V} |x_v|^2 < \infty\}.$$

Then we easily see that

$$(\Omega^t)' \subset l^2(\mathbb{V}) \subset \Omega^t$$

densely and continuously. It follows that for any $l \in (\Omega^t)'$

$$\sum_{v\in\mathbb{V}} |\langle l, e_v\rangle|^2 < \infty. \tag{3.2}$$

Now let

$$\begin{aligned}\mathcal{F}C_b^\infty := \{u : \Omega^t \to \mathbb{R} : &\text{ there exist } l_1, \cdots, l_m \in E' \text{ and } f \in C_b^\infty(\mathbb{R}^m)\\ &\text{such that } u(x) = f(l_1(x), \cdots, l_m(x)), x \in \Omega^t\}.\end{aligned}$$

By [AR90, Theorem 3.2], the bilinear form

$$\mathcal{E}_v(u,v) := \int \frac{\partial u}{\partial e_v}\frac{\partial v}{\partial e_v} d\mu,\ u, v \in \mathcal{F}C_b^\infty$$

is closable and we denote the closure as $\big(\mathcal{E}_v, D(\mathcal{E}_v)\big)$.

We define a bilinear form

$$\begin{aligned}D(\bar{\mathcal{E}}) :=& \Big\{u \in \mathcal{F}C_b^\infty : \sum_{v\in\mathbb{V}} \mathcal{E}_v(u,u) < \infty\Big\}\\ \bar{\mathcal{E}}(u,v) :=& \sum_{v\in\mathbb{V}} \mathcal{E}_v(u,v),\ u, v \in D(\bar{\mathcal{E}}).\end{aligned}$$

By (3.2) and [AR90, Theorem 3.8], we know that

$$D(\bar{\mathcal{E}}) = \mathcal{F}C_b^\infty,$$

$(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ is closable and the closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form. Next we want to show that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular (cf. [MR92, Chap.IV, Definition 3.1], namely the followings hold for $(\mathcal{E}, D(\mathcal{E}))$:

(i)There exists an $\mathcal{E}$-nest consisting of compact sets.

(ii) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions.

(iii) There exist $u_n \in D(\mathcal{E}), n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous μ-versions, $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $\Omega^t \setminus N$.

Then we can use the correspondence of quasi-regular Dirichlet forms and Markov processes to construct the Glauber dynamics.

First we need to prove the following lemma.

Lemma 3.3.1. *For any $y \in \Omega^t$, define $v(x) : \Omega^t \to \mathbb{R}$ by*

$$v(x) = d(x, y),$$

then $v(x) \in D(\mathcal{E})$.

Proof. Since

$$v(x) = d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x, y)}{1 + d_n(x, y)},$$

we first show that

$$\frac{d_n(x, y)}{1 + d_n(x, y)} \in D(\mathcal{E}).$$

We arrange the countable vertexes of $\mathbb{G}(\mathbb{V}, \mathbb{E})$ in a sequence:

$$v_1, v_2, \cdots, v_m, \cdots\cdots$$

Then

$$\frac{d_n(x, y)}{1 + d_n(x, y)} = \lim_{m \to \infty} \frac{\left(\sum_{1 \le i \le m} (x_{v_i} - y_{v_i})^2 a_i\right)^{\frac{1}{2}}}{1 + \left(\sum_{1 \le i \le m} (x_{v_i} - y_{v_i})^2 a_i\right)^{\frac{1}{2}}} \tag{3.3}$$

where $a_i := e^{-\delta_n \rho(v_i, o)}$. Define $f(z) : \mathbb{R}^m \to \mathbb{R}$ as

$$f(z_1, \cdots, z_m) := \frac{\left(\sum_{1 \le i \le m} (z_i - y_{v_i})^2 a_i\right)^{\frac{1}{2}}}{1 + \left(\sum_{1 \le i \le m} (z_i - y_{v_i})^2 a_i\right)^{\frac{1}{2}}}.$$

Then for $f(z)$, let $f_n(z) = f * \varphi_n(z)$, here $\varphi_n(z) = n^m\varphi(nz)$ and $\varphi \in C_0^\infty(\mathbb{R}^m)$ nonnegative, $\int_{\mathbb{R}^m} \varphi(z)dz = 1$. Then by direct computation and noting that $0 < a_i < 1$ we have

$$\begin{aligned}\sum_{i=1}^m |\frac{\partial f_n(z)}{\partial z_i}|^2 &= \sum_{i=1}^m |\int_{\mathbb{R}^m} \frac{\partial f}{\partial z_i}(z-w)\varphi_n(w)dw|^2 \\ \leq \int_{\mathbb{R}^m} \sum_{i=1}^m |\frac{\partial f}{\partial z_i}&(z-w)|^2 \varphi_n(w)dw \leq 1, \text{ for } \forall z \in \mathbb{R}^m.\end{aligned}$$

Then $g_n(x) := f_n(x_{v_1}, \cdots, x_{v_m}) \in \mathcal{F}C_b^\infty$ and

$$\begin{aligned}\mathcal{E}(g_n, g_n) &= \int_{\Omega^t} \sum_{i=1}^m |\frac{\partial g_n(x)}{\partial e_{v_i}}|^2 \mu(dx) \\ &= \int_{\Omega^t} \sum_{i=1}^m |\frac{\partial f_n}{\partial z_i}(x_{v_1}, \cdots, x_{v_m})|^2 \mu(dx) \leq 1.\end{aligned}$$

But $\lim_{n\to\infty} f_n(x_{v_1}, \cdots, x_{v_m}) = f(x_{v_1}, \cdots, x_{v_m})$, namely

$$\lim_{n\to\infty} g_n(x) = \frac{\big(\sum_{1\leq i\leq m}(x_{v_i} - y_{v_i})^2 a_i\big)^{\frac{1}{2}}}{1 + \big(\sum_{1\leq i\leq m}(x_{v_i} - y_{v_i})^2 a_i\big)^{\frac{1}{2}}}.$$

By [MR92, Chap.I, Lemma 2.12], we have

$$f(x_{v_1}, \cdots, x_{v_m}) = \frac{\big(\sum_{1\leq i\leq m}(x_{v_i} - y_{v_i})^2 a_i\big)^{\frac{1}{2}}}{1 + \big(\sum_{1\leq i\leq m}(x_{v_i} - y_{v_i})^2 a_i\big)^{\frac{1}{2}}} \in D(\mathcal{E})$$

and $\mathcal{E}\big(f(x_{v_1}, \cdots, x_{v_m}), f(x_{v_1}, \cdots, x_{v_m})\big) \leq 1$.

Similarly, by (3.3), we get

$$\frac{d_n(x,y)}{1 + d_n(x,y)} \in D(\mathcal{E}),\ \mathcal{E}\left(\frac{d_n(x,y)}{1 + d_n(x,y)}, \frac{d_n(x,y)}{1 + d_n(x,y)}\right) \leq 1.$$

Now it is easily seen that $v(x) \in D(\mathcal{E})$. $\square$

Theorem 3.3.2. $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular and therefore there exists a diffusion process* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ *on* Ω^t *which is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$. *This diffusion process is usually called the Glauber dynamics.*

Proof. Note that for any $\delta > \delta_{\mathbb{G}}, \Omega^t \subset \Omega_\delta$ densely and continuously. But Ω_δ is a separable Hilbert space and therefore we can find $l_n \in (\Omega_\delta)', n \in \mathbb{N}$, such that

$$\{\sin l_n | n \in \mathbb{N}\}$$

separates the points of Ω_δ. Here $(\Omega_\delta)'$ denotes the dual space of Ω_δ. Since l_n, restricted on Ω^t, is again a continuous linear functional on Ω^t and therefore $\sin l_n \in \mathcal{F}C_b^\infty$. Clearly $\{\sin l_n | n \in \mathbb{N}\}$ also separates the points of Ω^t.

Since Ω^t is separable, we can choose a countable dense set $\{y_m | m \in \mathbb{N}\}$ in Ω^t. For each $m \in \mathbb{N}$ define $v_m : \Omega^t \to \mathbb{R}$ by

$$v_m(x) = d(x, y_m).$$

By the preceding lemma, we know $v_m(x) \in D(\mathcal{E})$.

For each $v \in \mathbb{V}$ and $x \neq y_m$,

$$\begin{aligned}
\frac{\partial v_m(x)}{\partial e_v} =& \frac{\partial v_m(x + t \cdot e_v)}{\partial t}\Big|_{t=0} \\
=& \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \left(\frac{d_n(x + t \cdot e_v, y_m)}{1 + d_n(x + t \cdot e_v, y_m)} \right)' \Big|_{t=0} \\
=& \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{1}{\big(1 + d_n(x, y_m)\big)^2} \cdot \frac{(x_v - y_v) \cdot e^{-\delta_n \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}}}.
\end{aligned}$$

Therefore for $\forall\ x \neq y_m$

$$|\frac{\partial v_m(x)}{\partial e_v}| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_v - y_v| \cdot e^{-\delta_n \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}}},$$

so

$$\begin{aligned}
&\sum_{v \in \mathbb{V}} |\frac{\partial v_m(x)}{\partial e_v}|^2 \leq \sum_{v \in \mathbb{V}} \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_v - y_v| \cdot e^{-\delta_n \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}}} \right)^2 \\
=& \sum_{v \in \mathbb{V}} \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_v - y_v| \cdot e^{-\delta_n \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}}} \right) \left(\sum_{l=1}^{\infty} \frac{1}{2^l} \frac{|x_v - y_v| \cdot e^{-\delta_l \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_l \rho(o,v)}\big)^{\frac{1}{2}}} \right) \\
=& \sum_{\substack{v \in \mathbb{V} \\ n \geq 1, l \geq 1}} \frac{1}{2^{n+l}} \frac{|x_v - y_v|^2 \cdot e^{-(\delta_n + \delta_l) \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}} \big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_l \rho(o,v)}\big)^{\frac{1}{2}}} \\
=& \sum_{n,l \leq 1} \frac{1}{2^{n+l}} \frac{\sum_{v \in \mathbb{V}} |x_v - y_v|^2 \cdot e^{-(\delta_n + \delta_l) \rho(o,v)}}{\big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n \rho(o,v)}\big)^{\frac{1}{2}} \big(\sum_{v \in \mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_l \rho(o,v)}\big)^{\frac{1}{2}}}
\end{aligned}$$

$$= \sum_{n,l\le 1} \frac{1}{2^{n+l}} \cdot \frac{\left(\sum_{v\in\mathbb{V}} |x_v - y_v|^2 \cdot e^{-(\delta_n+\delta_l)\rho(o,v)}\right)^{\frac{1}{2}}}{\left(\sum_{v\in\mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_n\rho(o,v)}\right)^{\frac{1}{2}}} \frac{\left(\sum_{v\in\mathbb{V}} |x_v - y_v|^2 \cdot e^{-(\delta_n+\delta_l)\rho(o,v)}\right)^{\frac{1}{2}}}{\left(\sum_{v\in\mathbb{V}} (x_v - y_v)^2 \cdot e^{-\delta_l\rho(o,v)}\right)^{\frac{1}{2}}}$$
$$\le \sum_{n,l\le 1} \frac{1}{2^{n+l}} = 1.$$

It is easily seen that $\mu(y_m) = 0$, therefore

$$\sum_{v\in\mathbb{V}} |\frac{\partial v_m(x)}{\partial e_v}|^2 \le 1 \quad \text{for } \mu - a.e.\ x \in \Omega^t.$$

Then we can use the same method as [MR92, Chap.IV, Proposition 4.2] to prove that there exists an $\mathcal{E}$-nest consisting of compact sets. Thus we have proved that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

□

3.4 Appendix

In this section we give a proof of Lemma 3.2.3. The following proof is taken from [Pas07a].

Proof of Lemma 3.2.3

Proof. From Assumption 3.1.2, for any $v \in \mathbb{V}$ and $x, y \in \Omega$

$$\sum_{v'\in\partial v} |W_{vv'}(x_v, y_{v'})| \le \frac{m_{\mathbb{G}}J}{2}|x_v|^2 + \frac{J}{2}\sum_{v'\in\partial v}(C_W + |y_{v'}|^2). \tag{3.4}$$

By (3.4) and the definition of $\pi_v(dx|y)$,

$$\int_\Omega \exp\left\{\beta\kappa|x_v|^2\right\}\pi_v(dx|y)$$
$$\le (X_v/Y_v) \cdot \exp\left\{\beta J(m_{\mathbb{G}}C_W + \sum_{v'\in\partial v}|y_{v'}|^2)\right\},$$

where

$$X_v := \int_{\mathbb{R}} \exp\left\{ -\beta\left[V_v(x_v) - (\kappa + \frac{m_{\mathbb{G}}J}{2})|x_v|^2\right]\right\}dx_v,$$
$$Y_v := \int_{\mathbb{R}} \exp\left\{ -\beta\left[V_v(x_v) + \frac{m_{\mathbb{G}}J}{2}|x_v|^2\right]\right\}dx_v.$$

Using the upper and lower bounds in Assumption 3.1.3, one observes that

$$X := \sup_v X_v \leq \exp\{-\beta B_V\} \int_{\mathbb{R}} \exp\Big\{ - \beta(\triangle - \kappa)x_v|^2\Big\} dx_v < \infty,$$
$$Y := \inf_v Y_v \geq \int_{\mathbb{R}} \exp\Big\{ - \beta\big[C_V(1+|x_v|^p) + \frac{m_{\mathbb{G}}J}{2}|x_v|^2\big]\Big\} dx_v > 0.$$

Therefore we get

$$\int_\Omega \exp\{\beta\kappa|x_v|^2\}\pi_v(dx|y) \leq \exp\Big\{\beta\Big(\Upsilon + \sum_{v'\in\partial v} J|y_{v'}|^2\Big)\Big\}, \tag{3.5}$$

where $\Upsilon := \beta^{-1}\ln\{X/Y\} + C_W m_{\mathbb{G}} J$.

Recall that

$$n_v(\Lambda|y) := \ln\Big\{\int_\Omega \exp\{\beta\kappa|x_v|^2\}\pi_\Lambda(dx|y)\Big\}.$$

Integrating in (3.5) with respeact to $\pi_\Lambda(dx|y)$ with $y \in \Omega_\delta$ and using the consistency property we arrive at

$$\begin{aligned} n_v(\Lambda|y) \leq& \beta\Big(\Upsilon + \sum_{v'\in\partial^+\Lambda} J_{vv'}|y_{v'}|^2\Big) \\ &+ \ln\Big\{\int_\Omega \exp\Big(\sum_{v'\in\Lambda} \kappa^{-1}J\beta\kappa|x_{v'}|^2\Big)\pi_\Lambda(dx|y)\Big\} \\ \leq& \beta\Big(\Upsilon + \sum_{v'\in\partial^+\Lambda} J_{vv'}|y_{v'}|^2\Big) + \kappa^{-1}\sum_{v'\in\Lambda} J_{vv'}\cdot n_{v'}(\Lambda|y), \end{aligned} \tag{3.6}$$

where $\partial^+\Lambda := \{v' \in \Lambda^c|\rho(v',\Lambda) = 1\}$ and $J_{vv'} = J$ if $v \sim v'$, otherwise $J_{vv'} = 0$. Here we used the multiple Hölder inequality $\mu(\prod_{i=1}^n f_i^{\alpha_i}) \leq \prod_{i=1}^n(\mu(f_i))^{\alpha_i}$, valid for $\mu \in \mathcal{P}(\Omega)$, functions $f_i \geq 0$, and $\alpha_i \in \mathbb{R}_+, \sum_{i=1}^n \alpha_i \leq 1$.

Multiplying both sides of (3.6) by $\exp\{-\delta\rho(v,0)\}$ and then summing over $v \in \Lambda$ one gets

$$\begin{aligned} &\sum_{v\in\Lambda} n_v(\Lambda|y)\exp\{-\delta\rho(v,o)\} \\ &\leq \frac{\beta}{1 - k^{-1}m_{\mathbb{G}}J\cdot\exp\delta}\cdot[\Upsilon\|\mathbf{1}_\Lambda\|_\delta^2 + \|y_{\Lambda^c}\|_\delta^2 m_{\mathbb{G}}J\cdot\exp\delta]. \end{aligned}$$

Since $\|y\|_\delta < \infty$ for $y \in \Omega_\delta$, we finally conclude that

$$\limsup_{\Lambda \uparrow \mathbb{V}} \left[\sum_{v \in \Lambda} n_v(\Lambda|y) \exp\{-\delta\rho(v,o)\} \right] \leq \beta \Upsilon_\delta,$$

where $\Upsilon_\delta := \frac{\Upsilon \cdot \|\mathbf{1}\|_\delta^2}{1 - k^{-1} m_{\mathbb{G}} J \cdot \exp \delta}$. $\square$

Bibliography

[AR90] Sergio Albeverio and Michael Röckner. Classical Dirichlet forms on topological vector spaces—closability and a Cameron-Martin formula. *J. Funct. Anal.*, 88(2):395–436, 1990.

[AR91] S. Albeverio and M. Röckner. Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. *Probab. Theory Related Fields*, 89(3):347–386, 1991.

[Aro68] D. G. Aronson. Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:607–694, 1968.

[Bas98] Richard F. Bass. *Diffusions and elliptic operators*. Probability and its Applications (New York). Springer-Verlag, New York, 1998.

[BC03] Richard F. Bass and Zhen-Qing Chen. Brownian motion with singular drift. *Ann. Probab.*, 31(2):791–817, 2003.

[ES84] H. J. Engelbert and W. Schmidt. On exponential local martingales connected with diffusion processes. *Math. Nachr.*, 119:97–115, 1984.

[Fri64] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[Geo88] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1988.

[KLSU04] Vladimir Kondratiev, Vitali Liskevich, Zeev Sobol, and Oleksiy Us. Estimates of heat kernels for a class of second-order elliptic operators with applications to semi-linear inequalities in exterior domains. *J. London Math. Soc. (2)*, 69(1):107–127, 2004.

[KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[KS06] Panki Kim and Renming Song. Two-sided estimates on the density of Brownian motion with singular drift. *Illinois J. Math.*, 50(1-4):635–688 (electronic), 2006.

[KS07a] Panki Kim and Renming Song. Boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains. *Math. Ann.*, 339(1):135–174, 2007.

[KS07b] Panki Kim and Renming Song. Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts. *J. Math. Anal. Appl.*, 332(1):57–80, 2007.

[KS08] Panki Kim and Renming Song. On dual processes of non-symmetric diffusions with measure-valued drifts. *Stochastic Process. Appl.*, 118(5):790–817, 2008.

[LS00] Vitali Liskevich and Yuli Semenov. Estimates for fundamental solutions of second-order parabolic equations. *J. London Math. Soc. (2)*, 62(2):521–543, 2000.

[LSU67] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[MR92] Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.

[Pas07a] T. Pasurek. Ergodicity of the Glauber dynamics on infinite graphs. unpublished, 2007.

[Pas07b] T. Pasurek. Theory of Gibbs measures with unbounded spins: probabilistic and analytical aspects. Habilitation Thesis, Universität Bielefeld, 2007.

[Por90] N. I. Portenko. *Generalized diffusion processes*, volume 83 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1990. Translated from the Russian by H. H. McFaden.

[Ria07] Lotfi Riahi. Dirichlet Green functions for parabolic operators with singular lower-order terms. *JIPAM. J. Inequal. Pure Appl. Math.*, 8(2):Article 36, 24 pp. (electronic), 2007.

[Stu93] Wolfgang Stummer. The Novikov and entropy conditions of multidimensional diffusion processes with singular drift. *Probab. Theory Related Fields*, 97(4):515–542, 1993.

[SV06] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[Zha96a] Qi Zhang. A Harnack inequality for the equation $\nabla(a\nabla u)+b\nabla u = 0$, when $|b| \in K_{n+1}$. *Manuscripta Math.*, 89(1):61–77, 1996.

[Zha96b] Qi Zhang. On a parabolic equation with a singular lower order term. *Trans. Amer. Math. Soc.*, 348(7):2811–2844, 1996.

[Zha97a] Qi S. Zhang. Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$. *Manuscripta Math.*, 93(3):381–390, 1997.

[Zha97b] Qi S. Zhang. On a parabolic equation with a singular lower order term. II. The Gaussian bounds. *Indiana Univ. Math. J.*, 46(3):989–1020, 1997.