Interacting particle systems on graphs

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To my wife Simone,
my daughter Ayla
and to all our children who will be born in the future.

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Contents

Chapter 1

Introduction

This thesis deals with the theory of *Gibbs measures*, which is a traditional branch of classical statistical physics and also a part of modern probability theory. The mathematical notion of a Gibbs measure dates back to the pioneering papers of R. L. Dobrushin [Do 1968, Do 1970] and O. E. Lanford and D. Ruelle [LaRu 1969, Ru 1969]. Gibbs measures are used to describe equilibrium states of a physical system which consists of a very large (i.e., tending to infinity) number of interacting particles. Physically, we attempt to explain the macroscopic behavior of matter on the basis of its microscopic structure. The underlying structure in the whole manuscript is a general infinite graph $G(\mathbb{V}, \mathbb{E})$. For this case we obtain a series of results concerning existence, uniqueness and non-uniqueness of the corresponding Gibbs measures. A well-studied subclass of the graph $G(\mathbb{V}, \mathbb{E})$ is the lattice $\mathbb{Z}^d$, $d \geq 2$. For this case we refer to the Habilitation [Pa 2008] by T. Pasurek. Our aim is, however, to develop new methods of studying Gibbs measures, which do not involve any information about translation invariance or symmetries of the underlying graphs.

Along with physics, there are numerous research fields where we face *interacting particle systems on graphs*, especially in biology and economy. In these areas the number of applications on graphs including, e.g., complex systems and irregular structures, is increasing enormously, see [AlBa 2002] by R. Albert and A.-L. Barabasi, [Bo 1998] by B. Bollobás, [BuCa 2005] by R. Burioni and D. Cassi and [Ly 2000] by R. Lyons. The World Wide Web, ecological networks, disease networks, neural networks and, of course, society are just some examples. Originally, applications on such systems appeared at the beginning of the last century in statistical physics, e.g. for mathematical descriptions of gases or the magnetization of ferromagnets in the case of the famous Ising model. In all these systems a *huge number of members* and the *interaction between them* is fundamental. So, they are *large interaction networks*. But, in common, one only has the opportunity to gain partial information inside a small part of the entire system. Comparing the whole system with these small parts we can regard the whole system as an *infinite system*. A basic mathematical task is then to give well-defined (infinite dimensional) probability measures depending on the microscopic characteristics of all constituents and allowing to compute equilibrium expectations of the system and hence to explain its macroscopical behavior. Here Gibbs measures are a mathematically exact framework to describe such infinite systems.

The construction of the graph model is the following: We look at an interacting system with a large number of particles. We start with a set of vertices $\mathbb{V}$ which labels the particles of the system (e.g. their equilibrium positions). The possible states of each particle are described by the set $\mathbb{R}^\nu$, which will give rise to some difficulties since we are in the *non-compact* spin space situation. Having specified these two sets, we can describe a particular state (e.g. configuration) of the total system by a suitable element $\sigma := (\sigma(x))_{x\in\mathbb{V}}$ of the product space

$$\Omega := [\mathbb{R}^\nu]^{\mathbb{V}} := \{\sigma = (\sigma(x))_{x\in\mathbb{V}} \mid \sigma : \mathbb{V} \to \mathbb{R}^\nu\}.$$

Respectively, Ω is called the *configuration space*. In mathematical language, we consider an interacting system of spins (or particles) living on a graph $G(\mathbb{V}, \mathbb{E})$. To each $x \in \mathbb{V}$ there corresponds the variable $\sigma(x)$ called *spin* which takes values in $\mathbb{R}^\nu$. The presence of interaction between the two particles marked by $x, y \in \mathbb{V}$ means that the corresponding vertices are joint by the edge $(x, y) \in \mathbb{E}$. To develop a theory of such systems, we have to restrict ourselves to a physically reasonable class of the graphs $G(\mathbb{V}, \mathbb{E})$ which satisfy a basic geometrical condition. This is the so-called *uniformly bounded degree* condition. Let us explain this crucial restriction. Defining the *degree* $m(x)$ as the number of nearest neighbors of the vertex x, we impose the condition $sup_{x\in\mathbb{V}} m(x) < \infty$. Note that until now there is no adequate theory of Gibbs measures with *unbounded degree*. However, for the existence of tempered Gibbs measures in some explicit ferromagnetic models we will be able to skip this restriction, see Section 3.6. Next, we specify a formal Hamiltonian $E(\sigma)$ which assigns to each configuration $\sigma \in \Omega$ its potential energy

$$E(\sigma) := \sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} W_{xy}(\sigma(x), \sigma(y)) + \sum_{x\in\mathbb{V}} U_x(\sigma(x)),$$

where $W_{xy}(\sigma(x), \sigma(y))$ is the *pair interaction potential*, $U_x(\sigma(x))$ the *self-interaction potential* and the infinite sum $\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}}$ is taken over all unordered pairs $x, y \in \mathbb{V}$ of nearest neighbors. The equilibrium state of such systems with Hamiltonian $E(\sigma)$ is described by the probability measure

$$\mu(d\sigma) = \frac{1}{Z} e^{-\beta E(\sigma)} \prod_{x \in \mathbb{V}} d\sigma(x)$$

on the configuration space Ω. The notation $d\sigma(x)$ refers to the Lebesgue measure on $\mathbb{R}^\nu$, $\beta := \frac{1}{k_B T} > 0$ is the *inverse* (absolute) *temperature* with k_B denoting the Boltzmann constant, and $Z > 0$ is a *normalizing constant* (or *partition function*). The measures μ are called *Gibbs measures* (or *Gibbs states*). One of the main question we deal with in this manuscript is: how we can ensure that such a measure really exists on an infinite graph $G(\mathbb{V}, \mathbb{E})$? The main tool is the DLR (Dobrushin-Lanford-Ruelle) equation $\mu \pi_\Lambda = \mu$, $\Lambda \Subset \mathbb{V}$, whereby the Gibbs measures $\mu \in \mathcal{G}$ can be rigorously defined in terms of its *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$. So, these are probability measures on the space Ω, which have prescribed conditional probabilities $\mu_\Lambda(d\sigma|\xi)$ with respect to the boundary conditions ξ fixed outside finite regions Λ. In turn, each $\mu_\Lambda(d\sigma|\xi)$ is constructed by means of the corresponding local Hamiltonian $E_\Lambda(d\sigma|\xi)$.

After establishing the existence result (see Chapter 3) we will discuss uniqueness and non-uniqueness of Gibbs measures (see respectively Chapters 4 and 5). Then there arises the important physical notion of phase transition (or magnetization in ferromagnetic systems). A basic ambition is to classify a given specification as admitting either a unique Gibbs measure or multiple ones. Generally, a description of a spin system includes several parameters (such as temperature) and the aim is to classify the interval of parameter values into two regimes, one where the Gibbs measure is unique, and the other where there are multiple Gibbs measures. A common example is the *Ising model of ferromagnetism.* Considering the lattice $\mathbb{Z}^d$ for $d \geq 2$, we attach on each vertex a particle with spin either up (+1) or down (-1). Each particle interacts with its nearest neighbors favoring alignment of the spins. This effect decreases with the temperature T of the system, so that at high temperature the spins behave almost independently, while at low temperature there are large connected regions. So, the Ising model on the lattice with no external magnetic field admits a unique Gibbs measure when the temperature is above a known critical value $T_c = \frac{1}{\beta_c}$, and multiple Gibbs measures when the temperature is below T_c. One of our aims will be to obtain similar results for *general ferromagnetic models* with *unbounded spins living on irregular graphs.*

1.1 Overview of the thesis

We give a brief summary of the Chapters 2-6. The main results of each chapter are pointed out.

Main definitions and constructions

This chapter is devoted to general aspects of the theory of Gibbs measures. In contrast to the lattice case the theory of Gibbs measures on graphs requires new *concepts* and *techniques*. The main issues are that on graphs the *translation invariance* and the *definition of dimension* is absent. Therefore, even to state well-known classical results is quite difficult. Hence, we introduce here *notions*, *concepts* and some *graph properties* which will be crucial in the whole work. As mentioned above, the main property assumed for a graph is the *uniformly bounded degree* condition, that is

$$\sup_{x\in\mathbb{V}} m(x) < \infty, \tag{1.1}$$

where $m(x)$ is the *number of nearest neighbors*. Until now there is no adequate theory of Gibbs measures on *unbounded degree* graphs besides some special results for ferromagnetic harmonic interactions by comparison methods, see Section 5.2 below, or for underlying graphs with certain repulsive properties for heavy vertices, see [KoKoPa 2009] by Y. G. Kondratiev, Y. Kozitsky and T. Pasurek. Furthermore, we introduce the family of *local Hamiltonians* $E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ corresponding to potentials satisfying the basic Assumptions (**W**) and (**U**). The Assumption (**W**) is the so-called *polynomial growth condition* and Assumption (**U**) the so-called *stability condition.* Since we consider *unbounded spin systems*, e.g. with $\sigma(x) \in \mathbb{R}^\nu$, we restrict ourselves to certain subsets Ω^t of *exponentially tempered configurations* and to corresponding *tempered Gibbs measures* $\mu \in \mathcal{P}(\Omega)$ supported by Ω^t. In Section 2.6 we construct Gibbs measures μ by using the standard *Dobrushin-Lanford-Ruelle* (DLR) approach, see [Do 1968]. This gives us a rigorous definition of μ, cf. (2.8), as Markov random fields on $\mathbb{V}$ determined by means of their *local specification* $\Pi := \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$. The common literature on the general theory of Gibbs measures are the monographs [Pr 1976] and [Ge 1988]. In the end of this chapter we present the so-called *Wasserstein distance* for probability distributions, with some important measurability properties for the so-called *optimal couplings.* The justification of the notion Wasserstein distance is a very delicate issue, since it was introduced several times by different authors, first of all by L. Kantorovich in [Ka 1940, Ka 1942, Ka 1948], L. Kantorovich and G. Rubinstein in [KaRu 1958] and by Vasershtein in [Va 1969]. A detailed exposition of these developments is given in the monographs [RaRü 1998a, RaRü 1998b] by S. Rachev and L. Rüschendorf. We also refer to [Ra 1991] by S. Rachev (see Chapter 5) and to [Vi 2005] by C. Villani.

Existence problem

The existence problem goes back to R. L. Dobrushin's papers [Do 1968, Do 1970], where the general existence criterion for Gibbs measures was first given.

In Chapter 3 we, firstly, give an overview about the fundamental methods concerning the existence of Gibbs measures and, in particular, present the fundamental *Dobrushin's existence criterion*, cf. Theorem 3.1. As it is typical for classical lattice or graph systems with non-compact spin spaces, e.g. $\mathbb{R}^\nu$, even the initial question of whether the set $\mathcal{G}^t$ of tempered Gibbs measures is non-void is far from evident. The most known results are related to the simplest case of ferromagnetic translation invariant systems on a lattice, see [CaOlPePr 1978, BH-K 1982, Si 1982, PrFo 1991, Ze 1996, Yo 2001]. Therefore, we introduce a new approach to the existence problem which is based on particular *exponential estimates* for the one-point stochastic kernels $\pi_x(d\sigma|\xi)$ of the local specification, see Lemma 3.3. Initial steps in this direction were done for the lattice case in [KoPa 2007] by Y. Kozitsky and T. Pasurek, [Pa 2008] by T. Pasurek and in the author's Diploma thesis [Tek 2006]. In Section 3.3 we prove the main technical Lemmas 3.3 and 3.5 from which the *existence* of at least one measure μ in the set $\mathcal{G}^t$ of tempered Gibbs measures on an infinite graph $G(\mathbb{V}, \mathbb{E})$ then follows, see Theorem 3.7 in Section 3.4. The idea is the following: As soon as a certain *exponentially bound* in Lemma 3.3 for the one-point probability kernels $\pi_x(d\sigma|\xi)$ has been established, using the so-called *consistency property*, we get for all $\pi_\Lambda(d\sigma|\xi)$, $\Lambda \subset \mathbb{V}$, *uniform bounds* as in Lemma 3.5. This yields immediately, relying on the so-called *relatively compact* property of $\pi_\Lambda(d\sigma|\xi)$ in the topology $\mathcal{W}_t$, that $\mathcal{G}^t$ is not empty. On this way, we also get *a priori bounds* on all points of the set $\mathcal{G}^t$, see Theorem 3.8, which can be proven without knowing anything about the existence of such measures. It should be mentioned that the exponential bound we prove for $\pi_x(d\sigma|\xi)$ is stronger than the one in Dobrushin's criteria, but our bound gives additional information about the Gibbs measures. In Section 4.7 we also study a non-trivial example for the existence of tempered Gibbs measures which was not previously covered in the literature. The crucial new issue is that we consider Hamiltonians with possibly *unbounded* interaction strength. In the last section of this chapter we extend this method to *infinite* *range potentials* and *multi-particle interactions.*

Uniqueness problem

In Chapter 4 we develop an analytical approach to the uniqueness of tempered Gibbs measures and give the main result for this problem in Theorem 4.2. In general, to show that the set $\mathcal{G}^t$ is a singleton one needs more detailed information about the structure of the interactions as compared with the assumptions which guarantee the existence of such measures. We apply the *Dobrushin uniqueness criterion* which was first given in the famous paper [Do 1970] by R. L. Dobrushin. The necessary estimates for the *Wasserstein distance* between the corresponding one-point conditional distributions differing only at one site, are reduced to estimates of variances for Lipschitz continuous functions. The case of a lattice $\mathbb{Z}^d$ was studied in [Kü 1982] by H. Künsch, [Gr 1979] by L. Gross, [Roy 1977] by G. Royer, [AlKoRö 2003] by S. Albeverio, Y. G. Kondratiev and M. Röckner and [Pa 2008] by T. Pasurek. Concerning the quantum lattice systems we also refer to [AlKoRöTs 1997a], [AlKoRöTs 1997b] and [AlKoRöTs 2000] by Albeverio, Y. G. Kondratiev, M. Röckner and T. V. Tsikalenko (Pasurek). In Section 4.3 we apply the result of Theorem 4.2 to ferromagnetic pair interaction potentials of the form $W_{xy}(\sigma(x), \sigma(y)) := w_{xy}(\sigma(x) - \sigma(y))$ with convex functions $w_{xy} : \mathbb{R} \to \mathbb{R}_+$. After discussing possible extensions in Section 4.4 we present some concrete examples in Section 4.5, which are *basic* for the whole manuscript.

In Section 4.6 we present a *generalized version* of the Dobrushin uniqueness criterion which involves the original criterion as a part. Originally, this criterion only considers *one-point* probability kernels $\pi_x(d\sigma|\xi)$ of the local specification Π, which describe the influence of a site on another site, see [Do 1970] and Section 3.2 and 4.1. The general criterion considers the influence of larger volumes on other volumes. Such extensions of Dobrushin's uniqueness criterion first appeared in [W 2005] by D. Weitz, in [WiTa 2006] by S. Winkler and S. Tatikonda and in [ZhZh 2008] by H. Zhou and Z. Zheng. The same principal conditions in all these papers are that they only regard *finite* graphs with *compact spin spaces.* Our main aim in this section is to extend this construction to any *Polish spaces* X_i, $i \in \mathbb{I}$, and any underlying index set $\mathbb{I}$. The main application of this technique will be given in Section 4.7, where we consider interactions of possibly *unbounded strength* and only for convenience a lattice case. So, a new issue is that this lattice model provides points which may have *very strong interaction strength* with their neighbors. In Theorem 4.18 we give an uniqueness result with a sufficient condition for the lattice case. The main and unexpected point here is that the sufficient condition is independent of the unbounded interaction strength as soon as heavy points are not too close to each other. An extension of this theorem to the graph systems is given in Theorem 4.66.

Ferromagnetic models

Chapter 5 is entirely devoted to *ferromagnetic models* with scalar spins. The first section is a preliminary section which is organized as following. In Subsection 5.1.1 we present the ferromagnetic model under consideration and necessary assumptions for the existence of tempered Gibbs measures in the set $\mathcal{G}^t$. In Subsection 5.1.2 we present the so-called *correlation inequalities*, e.g. *Brascamp-Lieb*, *Fortuin-Kasteleyn-Ginibre* (FKG), *Griffiths-Kelly-Sherman* (GKS), Griffiths *first and second inequality*, which are a powerful tool for showing existence and uniqueness results and also for phase transitions. They describe moments of Gibbs measures in ferromagnetic system, for a detailed introduction we refer to [GlJa 1981, Si 1982, BrLi 1976]. In Subsection 5.1.3 we will define a *partial order* on $\mathcal{G}^t$.

In Section 5.2 we give an existence result for ferromagnetic systems on general graphs. A new and very important issue of this section is that we consider any graph $G(\mathbb{V},\mathbb{E})$ with possibly *unbounded degree*, i.e. $\sup_{x\in\mathbb{V}} m(x) \leq +\infty$. For this result we use the notion of *right-* and *left-dominators* of the self interaction potential U_x, which was introduced in a quite different context by H. Osada and H. Spohn in [OsSp 1999]. In Section 5.3 we give a general uniqueness criterion for *ferromagnetic scalar models*, see Lemma 5.24 and Proposition 5.25. In contrast to the standard moment problems, such criterion uses information about the first moments of the Gibbs measures only. Originally this criterion came from the paper by J. Lebowitz and E. Presutti [LP 1976], but we suggest an alternative short proof applying the Wasserstein distance. In Subsection 5.3.2 we establish a *new comparison criterion* for two ferromagnetic systems with different one-particle potentials, cf. Proposition 5.25. The main issue is to compare the initial model (5.1) with the so-called *(lower- and upper-)reference models.* In such reference (typically, polynomial) models one can establish uniqueness or phase transitions in a much simpler way. By the comparison criterion, we then immediately conclude the same qualitative behavior in the initial model.

In Section 5.4 we will present a new method showing phase transition in *unbounded* spin systems by using the so-called *Wells inequality.* Since this result is based on the classical Ising model we open this section with a short introduction on the well-known Ising model. This model was first introduced by W. Lenz in [Lz 1920] and his student E. Ising in [Is 1925] in order to describe spontaneous magnetization (i.e. phase transitions) of a ferromagnetic attractive substance on $\mathbb{Z}^d$ with spin space $\{+1, -1\}$. In this context we have the classical Theorem 5.26, which claims the existence of a threshold for phase transition of Gibbs measures on a lattice $\mathbb{Z}^d$, $d \geq 2$. This result was then extended to general trees by R. Lyons in [Ly 1989], see Theorem 5.28. Since these results are only for the spin space $\{+1, -1\}$, our aim is to develop a method of proving phase transitions for systems of *unbounded continuous spins on infinite graphs.* In this area we have the important new Theorem 5.29. It gives the critical inverse temperature of the model (5.1) with the *spin space* $\mathbb{R}$ and *double-well potentials* on *general trees.* It should be stressed that the classical techniques of proving phase transitions, like the reflection positivity method and the Peierls argument, do not apply on general graphs.

In Subsection 5.4.3 we will introduce concepts leading step by step to the justification of Theorem 5.29. There we introduce the so-called *Wells' moment inequality.* It is a relation between expectation values of an even probability measure and a certain Dirac measure. The Wells' inequality claims that the polynomial moments of the Dirac measure is always smaller then those of the even probability measure. The first *published* proof of this inequality can be found in the paper [BrLePf 1981] by J. Bricmont, J. L. Lebowitz and C. E. Pfister. However, we draw attention to a misprint in the calculation of the published proof. We will formulate and prove this inequality in Theorem 5.32 for the case of *unbounded spins with multi-particle interactions*, which is the most possible generalization. Although papers on this inequality are very rare, it seems to be a *fundamental tool* in the theory of phase transitions. Wells' inequality gives an *elementary new method* to prove the existence of phase transitions. In Subsection 5.4.4 we establish Wells' inequality for some polynomial models with the even double-well potentials like $V(s) := s^4 - \kappa s^2$ and $V(s) := s^{2n} - \kappa s^2$ with $n > 2$. Now we are on the stage to prove the main Theorem 5.29 for phase transitions via Wells' inequality.

Appendix

In Chapter 6, which is the appendix of this thesis, we introduce the Romberg Integration. It is a numerical method in calculating (via Computer software) the exact threshold for Wells' inequality corresponding to general anharmonic potentials.

Chapter 2

Main definitions and constructions

The construction of spin systems on *graphs* are highly natural since they emerge in numerous research fields. As we described in the introduction of this thesis graphs can be very useful since we are frequently confronted with ecological networks, disease networks, neural networks and society. Mathematically, graphs describe a set of *interacting particles* in a best possible generalization. So, such systems can be constructed in a very complex way, which will be studied in detail in the following subsections. As a matter of fact, the knowledge about the *geometry* of the system is crucial. Since the interplay between probabilistic models, on the one hand, and geometry of the graph on the other hand, is very strong, small changes in the geometry lead to new physical properties of the system. Therefore, the study of statistical models on graphs requires the introduction of new techniques and concepts with respect to the well-known case of lattices. For detailed explanations on interacting particle systems we also refer to [Lig 1985], [Ge 1988], [Bo 1998], [Wo 2000], [AlBa 2002], [BuCa 2005].

In this preliminary chapter for given *interaction potentials*, we define the *local specifications* and the corresponding *Gibbs measures*. We recall a standard setting generally known by the monographs [Ge 1988] and [Pr 1976]. In Section 2.1 we give a background of the main notions of the graph theory. In Section 2.2 we introduce a set of infinite graphs which can be reasonably used in statistical mechanics. A crucial assumption on the graph is the so-called *uniformly bounded degree* restriction. In Section 2.3 we construct a *configuration space*, which describes all possible realizations of the system. We give the heuristic *infinite-volume energy functional* which is called the *Hamiltonian*. In Section 2.4 the assumptions on the interaction potentials are given. In Section 2.5 we define a reasonable class of tempered configurations with *exponentially growth*. In the last Section 2.6 we construct a family of *local specifications* and define the corresponding *tempered Gibbs measures*.

2.1 Background on graphs

We start this section with the introduction of the main elements of the graph theory which is used in the whole manuscript.

Definition 2.1. *A* graph $G(\mathbb{V},\mathbb{E})$ *consists of a countable set of* vertices $\mathbb{V}$ *and of a set* $\mathbb{E}$ *of unordered pairs of vertices. Such vertices are said to be* nearest neighbors or adjacent*, we write* $x \sim y$ *if* $(x,y) \in \mathbb{E}$*. The elements of* $\mathbb{E}$ *are called* edges*. If* $\mathbb{V}$ *is finite,* $G(\mathbb{V},\mathbb{E})$ *is called a* finite graph*. The graph is called* simple *if there do not exist elementary* loops *and* multiple edges*, this means that each existing edge connects two different vertices and there is at most one edge between two vertices.*

Definition 2.2. *A* path *from* x_0 *to* x_k *in* $G(\mathbb{V},\mathbb{E})$ *is a sequence of edges connecting the vertices* x_0 *and* x_k *in the form* $(x_0,x_1),(x_1,x_2),...,(x_{k-1},x_k)$*, where* $x_i \neq x_j$ *for* $i \neq j$ *and* $x_i, x_j \in \mathbb{V}$*. The* length *of a path from* x_0 *to* x_k *is defined as the number of edges between them, here it is* k*.*

Definition 2.3. *A graph* $G(\mathbb{V},\mathbb{E})$ *is said to be* connected *if for any two vertices* $x, y \in \mathbb{V}$ *there is a path between them.*

The graph $G(\mathbb{V},\mathbb{E})$ is naturally provided with an intrinsic metric.

Definition 2.4. *For each vertex* $x, y \in \mathbb{V}$ *we introduce the* combinatorial (or chemical) distance $d(x,y)$*, defined as the length of the shortest path connecting the vertices* x *and* y*. The vertices* $x, y \in \mathbb{V}$ *are then nearest neighbors if the combinatorial distance* $d(x,y)$ *is equal to one.*

Definition 2.5. *For each vertex* $x \in \mathbb{V}$ *we define its* degree $m(x)$ *as the number of edges coming to the vertex* x *(the number of connected neighbors, or we call it also the number of nearest neighbors.), i.e., for* $x, y \in \mathbb{V}$ *and* $e = (x,y) \in \mathbb{E}$ *we define*

$$m(x) = \#\{e \in \mathbb{E} \mid x \in e\}.$$

Corresponding to this we define $\varphi(x)$ *the set of the nearest neighbors of* $x \in \mathbb{V}$ *as*

$$\varphi(x) := \{y \in \mathbb{V} \mid e = (x,y) \in \mathbb{E}\}. \tag{2.1}$$

With the combinatorial distance $d(x, y)$ we introduce the balls:

Definition 2.6. *The ball $B_r(x)$ with radius $r \in \mathbb{N}$ and the center $x \neq y$ is the subset of $\mathbb{V}$ defined as*

$$B_r(x) := \{y \in \mathbb{V} \mid 0 < d(x, y) \leq r\}.$$

We define $m_r(x)$ as the number of vertices in $B_r(x)$. In particular, $B_1(x) = \varphi(x)$ and $m_1(x) = m(x)$.

2.2 Geometrical conditions on graphs

The graph $G(\mathbb{V}, \mathbb{E})$ under consideration has a countable set of vertices $x \in \mathbb{V}$ and a set of unordered edges $e = (x, y) \in \mathbb{E}$. It is always simple, connected and endowed with the combinatorial distance $d(x, y)$. Almost in the whole manuscript, except in Section 5.2, the main restriction on the graph is the *uniformly bounded degree* condition, that is

$$m := \sup_{x \in \mathbb{V}} m(x) < \infty. \tag{2.2}$$

This condition is very essential in the next chapters in order to apply the so-called Dobrushin's existence and uniqueness criteria, see Remark 3.2.

According to Remark 2.8 below for every graph with the uniformly bounded degree condition (2.2) there exists a nonnegative number $\gamma_0 \leq \log m$ so that, for all $\gamma > \gamma_0$ and each initial point $x_0 \in \mathbb{V}$, it holds

$$\sum_{x \in \mathbb{V}} e^{-\gamma d(x, x_0)} < \infty. \tag{2.3}$$

In addition, we require a special subclass of the graph $G(\mathbb{V}, \mathbb{E})$ which fulfills a uniform extension of the assumption (2.3). It is in a certain sense close to a regular lattice. We call the graph $G(\mathbb{V}, \mathbb{E})$ *regular*, if it satisfies the following assumption:

Assumption (G). *There exists a $\gamma_0 \geq 0$ such that for every $\gamma > \gamma_0$*

$$\sup_{x_0 \in \mathbb{V}} \sum_{x \in \mathbb{V}} e^{-\gamma d(x, x_0)} < \infty.$$

This condition is needed to get *a priori estimates* on Gibbs measures *uniformly* with respect to the points of the graph $G(\mathbb{V}, \mathbb{E})$. In particular, it is true if $\mathbb{V} = \mathbb{Z}^d$. In the latter case Assumption $(\mathbf{G})$ holds just with $\gamma_0 = 0$.

Remark 2.7. *By the uniformly bounded degree condition (2.2) it holds*

$$m_r := \sup_{x \in \mathbb{V}} m_r(x) < \infty, \tag{2.4}$$

cf. Definition 2.6. In particular, one has

$$m_r \leq m^r. \tag{2.5}$$

Remark 2.8. *The constant γ_0 appearing in the condition (2.3) depends on the geometry of the graph. If no more information on the graph is available, one can roughly estimate γ_0 with the help of the number m. We will manifest this in the following lines. Indeed, for every graph with the uniformly bounded degree condition* (2.2) *there exists a positive number $\gamma_0 \leq \log m$ so that, for all $\gamma > \gamma_0$ and each initial point $x_0 \in \mathbb{V}$, it holds* (2.3)*. So, for $x_0 \in \mathbb{V}$ one can calculate using* (2.5)

$$\begin{aligned}
\sum_{x \in \mathbb{V}} e^{-\gamma d(x,x_0)} &< m_1(x_0)e^{-\gamma} + m_2(x_0)e^{-2\gamma} + m_3(x_0)e^{-3\gamma} + \cdots \\
&\leq m^1 e^{-\gamma} + m^2 e^{-2\gamma} + m^3 e^{-3\gamma} + \cdots \\
&= \sum_{N=0}^{\infty} e^{-N\gamma} m^N \\
&= \sum_{N=0}^{\infty} (e^{-\gamma} m)^N < \infty,
\end{aligned}$$

if $\log m < \gamma$. The number γ_0 is surely finite since we only deal with connected graphs, which is, of course, very natural and given by definition in interacting particle systems. In the case of $\mathbb{V} = \mathbb{Z}^d$ we can take $\gamma_0 = 0$.

2.3 Configuration spaces and Hamiltonians

We consider an interacting system of spins living on a graph $G(\mathbb{V}, \mathbb{E})$ always equipped with the stated properties of the last sections. To each $x \in \mathbb{V}$ correspond variables $\sigma(x)$ called *spins* which take values in $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$. In statistical physics we have the following general interpretation: The system of particles we consider performs a ν-dimensional oscillation, typically in time, around their non-stable point of equilibrium with vector displacements $\sigma(x)$. As the *configuration space* we define the space of all sequences over $\mathbb{V}$

$$\Omega := (\mathbb{R}^\nu)^{\mathbb{V}} := \{\sigma = (\sigma(x))_{x \in \mathbb{V}} \mid \sigma : \mathbb{V} \to \mathbb{R}^\nu\}. \tag{2.6}$$

Its elements $\sigma \in \Omega$ will be called *configurations*, and the *values* of a configuration $\sigma \in \Omega$ at each *vertex* $x \in \mathbb{V}$ are the *single spins* $\sigma(x) \in \mathbb{R}^\nu$. We equip Ω with the *product topology* and with the corresponding Borel σ-algebra $\mathcal{B}(\Omega)$. Recall that the product topology is the weakest topology such that all finite volume projections

$$\Omega \ni \sigma \to \mathbb{P}_\Lambda \sigma := \sigma_\Lambda := (\sigma(x))_{x\in\Lambda} \in (\mathbb{R}^\nu)^{|\Lambda|} =: \Omega_\Lambda, \quad \Lambda \Subset \mathbb{V},$$

are continuous, and the Borel σ-algebra $\mathcal{B}(\Omega)$ coincides with the σ-algebra generated by all cylinder sets

$$\{\sigma \in \Omega \mid \sigma_\Lambda \in B_\Lambda\}, \quad B_\Lambda \in \mathcal{B}(\Omega_\Lambda), \quad \Lambda \Subset \mathbb{V}.$$

Let $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega_\Lambda)$ denote the set of all *probability measures* respectively on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$.

In this chapter we restrict ourselves to systems with pair interactions. However, possible generalizations with *multi-particle interactions* will be discussed in Subsection 3.6.3. We will use multi-particle interactions for instance in Section 5.4.3.

Systems with pair interactions are described by means of the following heuristic *infinite-volume energy functional* (also called *Hamiltonian*)

$$E(\sigma) := \sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} W_{xy}(\sigma(x),\sigma(y)) + \sum_{x\in\mathbb{V}} U_x(\sigma(x)), \tag{2.7}$$

where W_{xy} are the *pair interaction potentials* and U_x the *self interaction potentials*. In the whole chapter, the infinite sum $\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}}$ is taken over all unordered pairs $e = (x, y) \in \mathbb{E}$ of nearest neighbors. For possible generalizations, e.g. infinite range interactions, we refer to the Section 3.6.

As a matter of fact, the infinite-volume energy functional (2.7) cannot be defined directly as a mathematical object. But it can be represented by the family of *local Hamiltonians* $E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ indexed by finite volumes $\Lambda \Subset \mathbb{V}$ and for given *boundary conditions* $\xi \in \Omega$, see sections below.

Heuristically, the Gibbs measures μ we are interested in have the following representation. They are *probability measures* on $(\Omega, \mathcal{B}(\Omega))$ in the form of

$$\mu(d\sigma) = \frac{1}{Z} e^{-\beta E(\sigma)} \prod_{x\in\mathbb{V}} d\sigma(x), \tag{2.8}$$

where

$$Z := \int_\Omega e^{-\beta E(\sigma)} \prod_{x\in\mathbb{V}} d\sigma(x),$$

is a *normalizing constant* and $d\sigma(x)$ is the *Lebesgue measure* on the spin space $\mathbb{R}^\nu$ and $\beta := \frac{1}{k_B T} > 0$ the fixed *inverse temperature* with k_B denoting the *Boltzmann constant*. We will define μ rigorously in Section 2.6 using the so-called *Dobrushin-Lanford-Ruelle* (DLR) framework.

In order to describe our constructions properly we introduce the following *notations*. We denote by $|\cdot|$ and $(\cdot,\cdot)$ respectively the *norm* and the *inner product* in the Euclidean space $\mathbb{R}^\nu$. For a finite set $\Lambda \Subset \mathbb{V}$ we denote its *cardinality*, which is the *number of elements* in Λ, by $|\Lambda|$ and by $\Lambda^c := \mathbb{V}\backslash\Lambda$ its *complement*. We write $\Lambda \Subset \mathbb{V}$ whenever Λ is not empty and $|\Lambda| < \infty$. A sequence of finite volumes $\mathcal{L} := (\Lambda_N)_{N\in\mathbb{N}}$ is called *cofinal* if it is ordered by inclusion and exhausts the entire graph $G(\mathbb{V},\mathbb{E})$. Furthermore, $\Lambda \nearrow \mathbb{V}$ means the limit along any cofinal sequence $\mathcal{L} := (\Lambda_N)_{N\in\mathbb{N}}$.

2.4 Conditions on interaction potentials

Throughout the manuscript, the interaction potentials are given by *continuous functions*

$$U_x : \mathbb{R}^\nu \to \mathbb{R}, \qquad U_x(0) = 0, \qquad x \in \mathbb{V}, \tag{2.9}$$

$$W_{xy} = W_{yx} : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}, \qquad x \sim y \in \mathbb{V}. \tag{2.10}$$

Without loss of generality, we suppose that the pair interaction potentials W_{xy} are invariant with respect to all interchanges of the coordinates $x, y \in \mathbb{V}$ and variables $\sigma(x), \sigma(y) \in \mathbb{R}^\nu$.

We impose the following assumptions on the interaction potentials W_{xy} and U_x:

Assumption (W). *There exist constants $R \geq 2$ and $C, J \geq 0$, such that for all $x, y \in \mathbb{V}$ with $x \sim y$ and $s, t \in \mathbb{R}^\nu$, we have*

$$|W_{xy}(s,t)| \leq J(C + |s|^R + |t|^R).$$

The number J is the strength *of the pair interaction.*

Assumption (U). *There exists a continuous function $U : \mathbb{R}^\nu \to \mathbb{R}$ and constants $P > R$ and $A, B > 0$, such that for all $x \in \mathbb{V}$ and $s \in \mathbb{R}^\nu$, we have*

$$A|s|^P + B \leq U_x(s) \leq U(s).$$

Remark 2.9. *The Assumptions (**W**), the so-called* polynomial growth condition, *and (**U**), the so-called* stability condition, *stated above are fundamental. Only for simplicity in Assumption (**W**) it is supposed that J is restricted* *to be the same number for each pair $(x, y) \in \mathbb{E}$. A generalization on all $x, y \in \mathbb{V}$ (not only nearest neighbors) can be given with further conditions, see Section 3.6. The main new issue in Section 3.5 is that we give an existence result for pair interactions with* unbounded interaction strength J_{xy}, *such that* $\sup_{x,y} |J_{xy}| = +\infty$. *On some stages we will modify the Assumptions (**W**) and (**U**), particularly in the next section we will weaken the Assumption (**U**) in order to give an existence result for the case where $P = R$, cf. Remark 2.11.*

Remark 2.10. *The above assumptions on the interaction potentials are fulfilled for a considerably large family of interactions with relations to physical models. Here, the* polynomials *given by*

$$U_x(s) := \sum_{q=1}^{p} a_x^{(q)} |s|^{2q}, \tag{2.11}$$

$$W_{xy}(s,t) := \sum_{q=1}^{r} b_{xy}^{(q)} |s-t|^{2q}, \tag{2.12}$$

with $r, p \in \mathbb{N}$, $r < p$, and with coefficients $a_x^{(q)}$ and $b_{xy}^{(q)}$ in $\mathbb{R}$, so that $a_x^p > 0$.

Remark 2.11. *The existence result can also be demonstrated for the case where $P = R$ if we change Assumption (**U**). For more accurate analysis see Section 3.3.*

2.5 Exponentially tempered configurations

It is typical that *unbounded spin systems* give rise to a restriction to certain subsets $\Omega^t \subset \Omega$ of admissible configurations $\sigma \in \Omega^t$, or rather to probability measures $\mu \in \mathcal{P}(\Omega)$ supported by such Ω^t. The most suitable choice of Ω^t depends on the conditions imposed on the interaction. Remind that we only consider pair interactions, extensions will be given in Section 3.6. In order to introduce reasonable configurations we establish a family of *weighted Banach spaces.*

Definition 2.12. *Let us fix any initial point $x_0 \in \mathbb{V}$ and define the* family of Banach spaces

$$\Omega_\gamma := \left\{ \sigma \in \Omega \,\middle|\, \|\sigma\|_\gamma := \left(\sum_{x \in \mathbb{V}} |\sigma(x)|^R e^{-\gamma d(x,x_0)} \right)^{1/R} < \infty \right\}, \tag{2.13}$$

*indexed by all $\gamma > \gamma_0$ with $\gamma_0 \leq \log m$, from Assumption (**G**).*

Recall that $R \geq 2$ is the largest order of the polynomial growth of W_{xy} allowed by Assumption (**W**). *Then the set of* (exponentially) tempered configurations *is defined as*

$$\Omega^t := \bigcap_{\gamma > \gamma_0} \Omega_\gamma.$$

Note that for $\gamma_1 > \gamma_2$ one has $\Omega_{\gamma_1} \supset \Omega_{\gamma_2}$. If $\sigma \in \Omega^t$ then $\sigma \in \Omega_\gamma$ for all $\gamma > \gamma_0$. For later use we define for finite $\Lambda \Subset \mathbb{V}$, $x_0 \in \mathbb{V}$ and $\gamma > \gamma_0$

$$\|\sigma_\Lambda\|_\gamma := \left(\sum_{x \in \Lambda} |\sigma(x)|^R e^{-\gamma d(x,x_0)} \right)^{1/R}. \tag{2.14}$$

Remark 2.13. Ω^t *will always be considered as a* Polish space *(i.e., a separable complete metrizable space) equipped, for example, with the (Fréchet-) metric*

$$\varrho(\sigma_1, \sigma_2) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\|\sigma_1 - \sigma_2\|_{\gamma_N}}{1 + \|\sigma_1 - \sigma_2\|_{\gamma_N}}$$

with any $(\gamma_N)_{N \in \mathbb{N}}$, such that $\gamma_N \to \gamma_0$ for $N \to \infty$.

Remark 2.14. *Actually the sets Ω_γ and Ω^t do not depend on the choice of the initial point x_0. Choosing any two initial points x_0 and y_0, the corresponding norms $\|\sigma\|_{\gamma,x_0} := \|\sigma\|_\gamma$ and $\|\sigma\|_{\gamma,y_0} := (\sum_{x \in V} e^{-\gamma d(x,y_0)} |\sigma(x)|^R)^{1/R}$ on Ω_γ are* equivalent, *i.e., there exists a constant $C > 0$ such that for all $\sigma \in \Omega^t$*

$$\frac{1}{C} \|\sigma\|_{\gamma,y_0} \leq \|\sigma\|_{\gamma,x_0} \leq C \|\sigma\|_{\gamma,y_0}. \tag{2.15}$$

Proof. The proof is just an application of the triangle inequality. Fixing $x_0, y_0 \in \mathbb{V}$, it is enough to show for all $x \in \mathbb{V}$ that

$$\frac{1}{C} e^{-\gamma d(x,y_0)} \leq e^{-\gamma d(x,x_0)} \leq C e^{-\gamma d(x,y_0)}.$$

However, we have the following inequality immediately

$$d(x, y_0) - d(x_0, y_0) \leq d(x, x_0) \leq d(x, y_0) + d(y_0, x_0),$$

from which follows

$$e^{\gamma d(x,y_0)} e^{-\gamma d(x_0,y_0)} \leq e^{\gamma d(x,x_0)} \leq e^{\gamma d(x,y_0)} e^{\gamma d(y_0,x_0)}.$$

This implies the assertion with $C := e^{\gamma d(x_0,y_0)}$.

□

Remark 2.15. *In the subsequent we shall crucially use the fact that the embedding $\Omega_\gamma \subset \Omega_{\gamma'}$ is compact if $\gamma < \gamma'$. This means that for all $r \in (0, \infty)$ the balls*

$$B_\gamma(r) := \{\sigma \in \Omega_\gamma \mid \|\sigma\|_\gamma \le r\} \tag{2.16}$$

are (closed) compact sets in $\Omega_{\gamma'}$.

2.6 Tempered Gibbs measures

Constructing the Gibbs measure μ we use the standard *Dobrushin-Lanford-Ruelle* (DLR) approach, see [Do 1968]. The common literature on the general theory of Gibbs measures are the monographs [Pr 1976] and [Ge 1988]. This gives us a rigorous definition of μ, cf. (2.8), as a Gibbs field on $\mathbb{V}$ determined by means of their *local specification* $\Pi := \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda \Subset \mathbb{V}}$. Namely, for any finite subset $\Lambda \Subset \mathbb{V}$ we define the family of *stochastic (probability) kernels* $\pi_\Lambda : \mathcal{B}(\Omega) \times \Omega \to [0, 1]$ by the following formula: For all $\xi \in \Omega$ and for $B \in \mathcal{B}(\Omega)$ we define

$$\pi_\Lambda(B|\xi) := \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\} \quad \mathbf{1}_B(\sigma_\Lambda, \xi_{\Lambda^c}) \prod_{x\in\Lambda} d\sigma(x), \tag{2.17}$$

where $\mathbf{1}_B$ is the *indicator function* on B. Defining

$$E_\Lambda(\sigma_\Lambda) := \sum_{\substack{x,y\in\Lambda \\ x\sim y}} W_{xy}(\sigma(x), \sigma(y)) + \sum_{x\in\Lambda} U_x(\sigma(x)), \tag{2.18}$$

we have

$$E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}) := E_\Lambda(\sigma_\Lambda) + \sum_{\substack{x\in\Lambda, y\in\Lambda^c \\ x\sim y}} W_{xy}(\sigma(x), \xi(y)) \tag{2.19}$$

which is the *local energy or local Hamiltonian* in the finite volume $\Lambda \Subset \mathbb{V}$ corresponding to the boundary condition $\xi \in \Omega$ in the complement Λ^c, i.e., $\xi_{\Lambda^c} := (\xi(y))_{y\in\Lambda^c}$. The *normalizing constant*, which is also called *partition function*, is defined by

$$Z_\Lambda(\xi) := \int_{\Omega_\Lambda} \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\} \quad \prod_{x\in\Lambda} d\sigma(x). \tag{2.20}$$

The second term on the right hand side of (2.19) makes sense for all $\xi \in \Omega$ because we only consider the interaction of the nearest neighbors. Otherwise, we should restrict ξ to the subset Ω^t of Ω, see Section 3.6. Recall that the sums $\sum_{\substack{x,y\in\Lambda\\ x\sim y}}$ and $\sum_{\substack{x\in\Lambda, y\in\Lambda^c\\ x\sim y}}$ are taken over all *unordered* pairs of nearest neighbors. It is also convenient to consider the *finite volume projections* of the kernels π_Λ on Ω_Λ given by

$$\mu_{\Lambda,\xi}(d\sigma_\Lambda) := \pi_\Lambda(d\sigma|\xi) \circ \mathbb{P}_\Lambda^{-1} \in \mathcal{P}(\Omega_\Lambda), \tag{2.21}$$

which is called the *local Gibbs distribution* under the boundary condition ξ.

By the above construction, the stochastic kernels (2.17) satisfy the *consistency property*, see [Pr 1976] and [Ge 1988], i.e., for all $\Lambda \subset \Lambda' \Subset V$

$$\pi_{\Lambda'}\pi_\Lambda = \pi_{\Lambda'}.$$

More precisely, this means that for all $B \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$

$$\int_\Omega \pi_\Lambda(B|\sigma)\pi_{\Lambda'}(d\sigma|\xi) = \pi_{\Lambda'}(B|\xi). \tag{2.22}$$

Next we define $\mathcal{G}$ and $\mathcal{G}^t$ the sets of *Gibbs measures* respectively *tempered Gibbs measures.*

Definition 2.16. *A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called a* Gibbs measure*, or* Gibbs state*, for the* local specification $\Pi := \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$*, if and only if it satisfies the* DLR equilibrium equations*, i.e., for all $\Lambda \Subset \mathbb{V}$, $B \in \mathcal{B}(\Omega)$, we have that $\mu\pi_\Lambda(B) = \mu(B)$, namely,*

$$\int_\Omega \pi_\Lambda(B|\xi)\mu(d\xi) = \mu(B). \tag{2.23}$$

For fixed $\beta > 0$, we denote by $\mathcal{G}$ the set of all Gibbs measures corresponding to the Hamiltonian (2.7).

Definition 2.17. *By $\mathcal{P}(\Omega^t)$ we denote the subset of* tempered measures *supported by Ω^t, i.e.,*

$$\mathcal{P}(\Omega^t) := \{\mu \in \mathcal{P}(\Omega) | \mu(\Omega^t) = 1\}. \tag{2.24}$$

Respectively, the subset of tempered Gibbs measures $\mathcal{G}^t$ *consists of all $\mu \in \mathcal{G}$ which are supported by Ω^t, i.e.,*

$$\mathcal{G}^t := \mathcal{G} \cap \mathcal{P}(\Omega^t). \tag{2.25}$$

In the subsequent discussion we shall crucially use that for all $\lambda > 0$,

$$\int_{\Omega} \exp\,(\lambda \sum_{x\in\Lambda} \mid \sigma(x) \mid^{R}) \pi_{\Lambda}(d\sigma|\xi) < \infty, \tag{2.26}$$

which immediately follows from Assumptions (**W**) and (**U**).

Remark 2.18. *As we get from Theorem 1.33 in [Ge 1988] and Theorem 8.1 in [Pr 2005], $\mu \in \mathcal{P}(\Omega)$ is a Gibbs measure for the corresponding local specification $\{\pi_{\Lambda}(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ if it satisfies the DLR equation* (2.23) *just for all* one-point sets $\Lambda := \{x\}$. *Therefore, all needed information about the measures $\mu \in \mathcal{G}^t$ could be gained from the family of their* one-point probability kernels

$$\pi_{\{x\}}(d\sigma|\xi) := \frac{1}{Z_{\{x\}}(\xi)} \int_{\mathbb{R}^{\nu}} \exp\{-\beta E_{\{x\}}(\sigma(x)|\xi)\} \mathbf{1}_{B}(\sigma(x), \xi_{\Lambda\setminus\{x\}}) d\sigma(x), \tag{2.27}$$

where,

$$E_{\{x\}}(\sigma(x)|\xi) := \sum_{y\in\varphi(x)} W_{xy}(\sigma(x), \xi(y)) + U_{x}(\sigma(x)), \tag{2.28}$$

and

$$Z_{\{x\}}(\xi) := \int_{\mathbb{R}^{\nu}} \exp\{-\beta E_{\{x\}}(\sigma(x)|\xi)\} \mathbf{1}_{B}(\sigma(x), \xi_{\Lambda\setminus\{x\}}) d\sigma(x).$$

The finite volume projection of the one-point probability kernels $\pi_{\{x\}}$ is then defined by

$$\mu_{\{x\},\xi}(d\sigma(x)) := \pi_{\{x\}}(d\sigma|\xi) \circ \mathbb{P}^{-1}_{\{x\}} \in \mathcal{P}(\Omega_{\{x\}}). \tag{2.29}$$

For the sake of simplicity we write $\pi_{x}(d\sigma|\xi)$ and $E_{x}(\sigma(x)|\xi)$ instead of $\pi_{\{x\}}(d\sigma|\xi)$ respectively $E_{\{x\}}(\sigma(x)|\xi)$. We do the same for the corresponding objects indexed by $\{x\}$.

Let $(C_b(\Omega), \|\cdot\|_{\sup})$ be the Banach space of all *bounded continuous functions* $f : \Omega \to \mathbb{R}$ equipped with the supremum norm $\|f\|_{\sup} := \sup_{\xi\in\Omega}|f(\xi)|$. By $\mathcal{W}$ we denote the *weak topology* on the set $\mathcal{P}(\Omega)$. It is defined as the roughest topology on $\mathcal{P}(\Omega)$ such that the mappings

$$\mu \mapsto \int_{\Omega} f d\mu$$

are continuous for all $f \in C_b(\Omega)$. For each $\mu \in \mathcal{P}(\Omega)$ its local base is given by

$$V_{f_1,\dots,f_n;\varepsilon}(\mu) := \left\{ \nu \in \mathcal{P}(\Omega) : \left| \int_\Omega f_i d\mu - \int_\Omega f_i d\nu \right| < \varepsilon, 1 \le i \le n \right\}, \quad (2.30)$$

where $f_1, ..., f_n$ are functions from $\mathcal{C}_b(\Omega)$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Similarly, replacing $\mathcal{C}_b(\Omega)$ by $\mathcal{C}_b(\Omega_\gamma)$ in (2.30), we obtain the weak topology $\mathcal{W}_\gamma$ on $\mathcal{P}(\Omega_\gamma)$. With these topologies the sets $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega_\gamma)$ become Polish spaces, cf. [P 1967] Theorem 6.5. Note that the weak topology $\mathcal{W}_\gamma$ is stronger than the weak topology $\mathcal{W}$ restricted to Ω_γ.

In our case we immediately have the Feller property:

Lemma 2.19 (Feller property)**.** *For every* $\Lambda \Subset \mathbb{V}$ *and any* $f \in C_b(\Omega)$ *the mapping*

$$\begin{aligned} \Omega \quad &\ni \quad \xi \mapsto \pi_\Lambda(f|\xi) \\ &:= \quad \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\sigma_\Lambda|\xi_{\Lambda^c}) \exp\left\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\right\} \prod_{x\in\Lambda} d\sigma(x) \end{aligned} \quad (2.31)$$

belongs to $C_b(\Omega)$*, thus continuous and bounded in* Ω*. Moreover,* $f \mapsto \pi_\Lambda f$ *is a contraction on* $C_b(\Omega)$*.*

A direct consequence of the Feller property is the following lemma which suggests an obvious way of constructing $\mu \in \mathcal{G}$. Later on, we will show the relatively compactness of the family $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$, which will then give us the existence of at least one element in $\mathcal{G}$.

Lemma 2.20. *For each fixed* $\xi \in \Omega$*, any accumulation point* $\mu \in \mathcal{P}(\Omega)$ *in* $\mathcal{W}$ *of the family* $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$*, as* $\Lambda \nearrow V$*, is the desired Gibbs measure.*

Proof. A measure $\mu \in \mathcal{P}(\Omega)$ solves (2.23) if and only if for any $f \in C_b(\Omega)$ and all $\Lambda \Subset V$,

$$\int_\Omega f(\sigma)\mu(d\sigma) = \int_\Omega \pi_\Lambda(f|\sigma)\mu(d\sigma). \quad (2.32)$$

Let $\{\pi_{\Lambda_k}(d\sigma|\xi_k)\}_{k\in\mathbb{N}}$ converge in $\mathcal{W}$ to some $\mu \in \mathcal{P}(\Omega)$. For every $\Lambda \Subset V$, one finds a $k_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_k$ for all $k > k_\Lambda$. Then by the consistency property (2.22), one has

$$\int_\Omega f(\sigma)\pi_{\Lambda_k}(d\sigma|\xi_k) = \int_\Omega \pi_\Lambda(f|\sigma)\pi_{\Lambda_k}(d\sigma|\xi_k).$$

Now by Remark 3.17, one can pass to the limit $k \to \infty$ and obtain (2.32).

$\square$

2.7 Wasserstein distance

The Wasserstein distance is used to measure the distance between probability distributions. Historically, it has its origin in the so-called *optimal transport* problem, which was introduced several times by different authors. It was first formulated by the French geometer G. Monge in his famous work [Mon 1781]. The problem considered by Monge describes the smallest cost at which the total production can be transported to the consumers.

Many years later Monge's problem had a revival through the Russian mathematician L. Kantorovich, without the knowledge that Monge's problem already existed. He considered this problem in a more extended way and introduced the *duality theorem* in [Ka 1940] and the problem of *optimal transport* in [Ka 1942]. He finally connected his problem to Monge's problem in [Ka 1948]. Later in [KaRu 1958] L. Kantorovich and G. Rubinstein achieved a more explicit duality theory. In 1975 he was then awarded the Nobel Prize for economics.

Since that time these techniques are broadly used in other mathematical disciplines, for example in *statistics*, *probability theory* and especially in *mathematical physics*. In the mathematical physics this distance is traditionally named after Vasershtein (or Wasserstein), who rediscovered it in his paper [Va 1969]. So, in the seminal paper [Do 1970], Dobrushin used the Wasserstein distance to study Gibbs random fields. Only a particular case, see Remark (2.22), is called *Kantorovich-Rubinstein distance*, even though Kantorovich was the first who introduced this distance. A detailed exposition with applications of these developments is given in the monographs [RaRü 1998a, RaRü 1998b] by S. Rachev and L. Rüschendorf. We also refer to [Ra 1991] by S. Rachev (see Chapter 5) and [Vi 2005] by C. Villani.

In [Do 1970] Dobrushin used the Wasserstein distance and the so-called Kantorovich-Rubinstein duality relation (2.36) to show the famous Dobrushin uniqueness criterion (4.1). However, a problem concerning the measurability of the so-called *optimal couplings* occurs on *unbounded spin spaces*. In some works, such as [La 1971], [Fö 1982], [Kü 1982], [Ge 1988], [BaKuMePr 2007] and [FoGuMé 2007], this problem is partially overcome by using the Kantorovich-Rubinstein duality relation (2.36) for Wasserstein distances.

Let us now collect facts about the Wasserstein distance, which can be found in detail in [Ra 1991] and [Vi 2005]. We also refer to [Du 1999]. In particular, Items (iv) and (v) about the *measurability* of optimal couplings were recently proved in [Pa 2008]. Let us define the *Wasserstein distance* on a general Polish space, which, of course, is applicable to the former sections.

Definition 2.21 (Wasserstein distance). *Let (X, ρ) be a Polish space. Let $\mathcal{P}_1(X)$ denote the subset of all probability measures μ on $(X, \mathcal{B}(X))$ having finite moments*

$$\int_X \rho(x, x_0)\mu(dx) < \infty, \tag{2.33}$$

for some $x, x_0 \in X$ and therefore for all. For a pair $\mu, \tilde{\mu} \in \mathcal{P}_1(X)$, we define the Wasserstein distance

$$\mathbf{W}_\rho(\mu, \tilde{\mu}) := \inf_{P \in \mathcal{C}(\mu, \tilde{\mu})} \int_{X^2} \rho(x, \tilde{x}) P(dx, d\tilde{x}), \tag{2.34}$$

where the infimum is taken over all couplings $P \in \mathcal{C}(\mu, \tilde{\mu})$*, that is, probability measures $P \in \mathcal{P}(X \times X)$ with the marginal distributions μ and $\tilde{\mu}$. Note that this is called the L^1-Wasserstein distance. In the literature one also finds the L^p-Wasserstein distance, which is then defined, for $p \geq 1$, by*

$$\mathbf{W}_\rho(\mu, \tilde{\mu}) := \inf_{P \in \mathcal{C}(\mu, \tilde{\mu})} \left(\int_{X^2} \rho(x, \tilde{x})^p P(dx, d\tilde{x}) \right)^{1/p}. \tag{2.35}$$

Remark 2.22. *The L^1-Wasserstein distance is also commonly called the* Kantorovich-Rubinstein *distance, since Kantorovich is one of the founding fathers, see the introduction of this section.*

Let us now discuss some topics concerning the Wassertein distance.

(i) Convergence: From Theorem 6.1 in [Vi 2005] we know that $(\mathcal{P}_1(X), \mathbf{W}_\rho)$ becomes itself a *Polish space*, whereby the convergence $\mathbf{W}_\rho(\mu, \mu_n) \to 0$, as $n \to 0$, is equivalent to the weak convergence of the measures $\mu_n \to \mu$ combined with the convergence of their moments (2.33), see Theorem 6.8 in [Vi 2005]. Since $\mathcal{P}_1(X)$ is closed as a subset in $(\mathcal{P}(X), \mathcal{W})$, it can also be considered as the Polish space equipped by the weak topology $\mathcal{W}$.

(ii) Optimal couplings: The infimum in the Definition 2.21 can always be attained at some $P \in \mathcal{C}(\mu, \tilde{\mu})$, which is either *unique* or *infinitely many*, see Theorem 4.1 in [Vi 2005]. Such minimizing coupling will be called *optimal* and the set of optimal couplings will be denoted by $\mathcal{C}^*(\mu, \tilde{\mu})$. This set is a convex compact set in $\mathcal{P}(X \times X)$ equipped with the corresponding topology of weak convergence, see Corollary 5.20 in [Vi 2005].

(iii) Duality relation for the Kantorovich-Rubinstein distance (L^1-Wasserstein distance): This relation says that the following two definitions of the Wasserstein distance are equivalent for any pair of $\mu, \tilde{\mu} \in$

$\mathcal{P}_1(X)$. We have the equivalence

$$\begin{aligned} \mathbf{W}_\rho(\mu,\tilde{\mu}) &:= \inf_{P\in\mathcal{C}(\mu,\tilde{\mu})} \int_{X^2} \rho(x,\tilde{x}) P(dx,d\tilde{x}) \\ &= \sup_{f\in Lip_1(X,\rho)} \left| \int_X f(x)\mu(dx) - \int_X f(x)\tilde{\mu}(d\tilde{x}) \right|, \end{aligned} \tag{2.36}$$

where

$$Lip_1(X,\rho) := \left\{ f : X \to \mathbb{R} \middle| [f] := \sup_{x\neq\tilde{x}} \frac{|f(x)-f(\tilde{x})|}{\rho(x,\tilde{x})} \leq 1 \right\}$$

is the unit ball in the space of Lipschitz continuous functions on X. For more detail we refer to Theorem 2.5.6 in [Ra 1991] and Theorem 5.9 in [Vi 2005].

(iv) Measurable selection: Consider the product space $X\times X$ with the metric

$$\tilde{\rho}[(x_1,x_2),(\tilde{x}_1,\tilde{x}_2)] := \rho(x_1,\tilde{x}_1) + \rho(x_2,\tilde{x}_2)$$

for $(x_1,x_2),(\tilde{x}_1,\tilde{x}_2) \in X\times X$. In a similar way one may equip $\mathcal{P}_1(X\times X)$ with the Wasserstein metric $\mathbf{W}_{\tilde{\rho}}$. Then

$$\mathcal{P}_1(X)\times\mathcal{P}_1(X) \ni (\mu,\tilde{\mu}) \to \mathcal{C}^*(\mu,\tilde{\mu}) \subset \mathcal{P}_1(X\times X)$$

can be regarded as a *multifunction* taking values in nonempty closed subsets $\mathcal{C}^*(\mu,\tilde{\mu})$ of the Polish space $(\mathcal{P}_1(X\times X),\mathbf{W}_{\tilde{\rho}})$ so that there exists a measurable selection P of the random set $\mathcal{C}^*(\mu,\tilde{\mu})$ which is a function

$$\mathcal{P}_1(X)\times\mathcal{P}_1(X) \ni (\mu,\tilde{\mu}) \to P(\mu,\tilde{\mu}) \in \mathcal{C}^*(\mu,\tilde{\mu}).$$

See Subsection 4.4.1 (v) in [Pa 2008] for the simple proof, which mainly uses the *fundamental selection theorem* for multifunctions, see also Theorems III.6 and III.8 in [CaVa 1977] or Theorem 2.13 in [Mo 2005].

(v) Measurable dependence on a parameter: Let the marginals $\mu_\lambda, \tilde{\mu_\lambda} \in \mathcal{P}(X)$ vary in a measurable way with respect to some abstract parameter λ. Then by (iv) there exists a *measurable realization* of the optimal coupling

$$\lambda \to P_\lambda \in \mathcal{C}^*(\mu_\lambda,\tilde{\mu_\lambda}) \subset \mathcal{P}_1(X\times X).$$

Therefore, the Wasserstein distance $\mathbf{W}(\mu_\lambda,\tilde{\mu_\lambda})$ is also measurable in λ.

In the DLR setting all this applies to the mappings $\xi \to \pi_\Lambda(d\sigma|\xi) \in (\mathcal{P}(\Omega),\mathcal{W})$ which, by construction, are known to be measurable.

Corollary 2.23. *For any local specification* $\Pi = \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda \Subset \mathbb{I}}$ *obeying for all* $\xi \in \Omega$ *and* $x \in \Lambda \Subset \mathbb{V}$

$$\int_\Omega |\sigma(x)| \pi_\Lambda(d\sigma|\xi) < \infty,$$

the Wasserstein distance $\mathbf{W}(\mu_\Lambda(d\sigma|\xi), \mu_\Lambda(d\sigma|\eta))$ *is a measurable function of* $(\xi, \eta) \in \Omega \times \Omega$.

Remark 2.24. *Note that the Items (iv) and (v) will be crucial for proving uniqueness criteria for Gibbs measures via the so-called* reconstruction procedure *first developed by Dobrushin, see Section 4.6.*

Chapter 3

Existence problem

Dealing with the set of tempered Gibbs measures $\mathcal{G}^t$ on systems with unbounded spins gives rise to the question whether $\mathcal{G}^t$ is not empty. Therefore, the *existence problem* is the first step in any study of Gibbs measures. In fact, in contrast to the case of compact spin spaces, it is not obvious to confirm that $\mathcal{G}^t$ is not empty. We will present a *new approach* for the existence problem. The main issue is to establish certain exponential estimates for the one-point stochastic kernels $\pi_x(d\sigma|\xi)$ with weak dependence on the boundary conditions $\xi \in \Omega$, see Lemma 3.3. Note that these estimates are *stronger* than those required in the fundamental *Dobrushin's existence criterion*, see Section 3.2. First steps in this field was done for the lattice case in [Pa 2008] by T. Pasurek and in the author's Diploma thesis [Tek 2006]. Other methods, especially applicable to the ferromagnetic systems, will be presented in Section 3.1. In this chapter we adopt this essentially new approach to the situation of a graph $G(\mathbb{V}, \mathbb{E})$.

In Section 3.3 we prove the main technical Lemmas 3.3 and 3.5 from which the *existence* of at least one $\mu \in \mathcal{G}^t$ follows, see Theorem 3.7 in Section 3.4. We even get *a priori bounds* on all points of the set $\mathcal{G}^t$, see Theorem 3.8. It is self-evident that we can extend this method to *finite range potentials*, to *infinite range potentials* and to general *multi-particle interactions*, see Section 3.6. In Subsection 3.5 the main new issue is that we give an existence result for pair interactions with *unbounded interaction strength.*

3.1 Overview of fundamental methods

In this section we give a review on literature. As it is typical for systems with non-compact spin spaces, e.g., $\mathbb{R}^\nu$ in the case of the classical lattice or graph systems, even the initial question of whether the set $\mathcal{G}^t$ of tempered Gibbs measures is non-void is far from evident. In order to give the reader an insight into the subject, we would like to present a systematic account of the fundamental methods within the DLR-approach and their applications to unbounded spin systems.

(i) General Dobrushin's criterion for existence of Gibbs distributions [Do 1970]. The validity of the sufficient conditions of the Dobruhsin existence theorem for some ferromagnetic classical lattice systems with scalar spins in $\mathbb{R}$ has been verified with different methods, e.g., in [BH-K 1982] by J. Bellissard and R. Høegh-Krohn, [CaOlPePr 1978] by M. Cassandro, E. Olivieri, A. Pellegrinotti and E. Presutti, [Si 1982] by Ya. G. Sinai and [PrFo 1991] by B. Prum and J.C. Fort. Thereafter, for lattice systems of vector spins it has been verified by S. Albeverio, Y. G. Kondratiev, T. Pasurek, and M. Röckner in [AlKoPaRö 2005] using the so-called *Intergration by Parts*(IbP)-formulas for $\mu \in \mathcal{G}^t$ and even under less restrictive assumptions on the interaction potentials than in the previous literature. With some new techniques it has been verified for quantum systems by T. Pasurek in [Pa] and Y. Kozitsky and T. Pasurek in [KoPa 2007].

(ii) Ruelle's technique of superstability estimates. This well-known technique has been introduced by D. Ruelle in [Ru 1969] for continuous systems and by J. L. Lebowitz and E. Presutti in [LP 1976] for lattice systems. This technique in particular requires that the interaction is *translation invariant* and the many-particle potentials have *at most quadratic growth*. So, the method cannot be directly extended to general graphs.

(iii) Cluster expansions. This method is one of the most powerful for the study of Gibbs measures, but it works only in a *perturbative regime*, i.e., when an effective parameter of the interaction is small. For this we refer to the monographs [GlJa 1981] by J. Glimm and A. Jaffe and [MaMi 1991] by V. Malyshev and R. Minlos.

(iv) Method of correlation inequalities. For some specific classes of interactions for classical lattice systems one can use the so-called *correlation inequalities* (such as FKG, GKS, Lebowitz, Brascamp-Lieb etc.) to study existence and uniqueness problems, see e.g. Subsections 5.1.2 and 5.4.3. This method involves more detailed information on the interactions, e.g., whether they are ferromagnetic or convex. For review on different correlation inequalities see the monographs [Ge 1988] by H.-O. Georgii, [Pr 1976] by C. Preston and [GlJa 1981] by J. Glimm and A. Jaffe.

(v) Method of reflection positivity. As a part of (iv), this technique can be applied to *translation invariant* systems with *pair interactions* and gives the existence of so-called *periodic* Gibbs states. Moreover, this method can also be used to study phase transitions in ferromagnetic spin models. We refer to the monographs [Ge 1988] by H.-O. Georgii, [GlJa 1981] by J. Glimm and A. Jaffe, and to the works, e.g., [BaKo 1992] by V. S. Barbulyak and Y. G. Kondratiev, [PaKh 1987] by L. A. Pastur and B. A. Khoruzhenko and [DrLaPe 1979] by W. Driesler, L. Landau and J. F. Perez. Because of absence of proper symmetries, the reflection positivity method is not applicable to graph systems.

3.2 Dobrushin's existence criterion

R. L. Dobrushin plays a pioneering role in establishing the theory of *Gibbs measures*. In his papers [Do 1968] and [Do 1970] he gave a general existence criterion for the first time, see Theorem 1 in [Do 1970]. We also refer to Theorem 1.3 in [Si 1982]. In this section we present an elementary new approach to check the sufficient conditions of Dobrushin's existence criterion (3.2). In our context, the main condition of this criterion supposes that for a given specification $\Pi = \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$, for all $x \in \mathbb{V}$ and $\xi \in \Omega$, the one-point probability kernels $\pi_x(d\sigma|\xi)$ should satisfy the following condition:

Theorem 3.1 (Existence Criterion). *There exists a certain compact function $h : \mathbb{R}^\nu \to \mathbb{R}_+ \cup \{+\infty\}$ and nonnegative constants A and I_{xy}, $x \neq y$, so that $\mathbf{I} = (I_{xy})_{\mathbb{V}^2}$ is a* strictly contractive matrix *with the entries $I_{xy} \geq 0$, that is,*

$$\| \mathbf{I} \| := \sup_{x\in\mathbb{V}} \sum_{y\in\mathbb{V}} I_{xy} < 1, \tag{3.1}$$

and for all $x \in \mathbb{V}$ and $\xi \in \Omega$ we have

$$\int_\Omega h(\sigma(x))\pi_x(d\sigma|\xi) \leq A + \sum_{y\in\mathbb{V}} I_{xy} h(\xi(y)). \tag{3.2}$$

Note that the continuous function h is called compact *if for any $t \in \mathbb{R}$, the level set $\{s \in \mathbb{R}^\nu | h(s) \leq t\}$ is compact in $\mathbb{R}^\nu$.*

Then there exists at least one Gibbs measure such that

$$\sup_{x\in\mathbb{V}} \int_\Omega h(\sigma(x))\pi_x(d\sigma|\xi) < \infty$$

.

The existence criterion yields that, for any boundary condition $\xi \in \Omega$ so that $\sup_{x\in\mathbb{V}} h(\xi(x)) < \infty$, the family $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ is relatively compact in the weak topology $\mathcal{W}$ on $\mathcal{P}(\Omega)$. Recall that the family of probability measures $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ is called *relatively compact* if from any sequence $\{\pi_{\Lambda_n}(d\sigma|\xi)\}_{\Lambda_n\Subset\mathbb{V}} \subset \{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ it is possible to select a weakly convergent subsequence. In the case of infinite range interactions we need a stronger topology, see Section 3.6. Furthermore, any measure $\mu \in \mathcal{G}^t$ constructed in such a way obeys the *a priori bound* $\sup_{x\in\mathbb{V}} \int_\Omega h(\sigma(x))\mu(d\sigma|\xi) < \infty$. If, additionally, $h(s) \geq |s|^R$, $s \in \mathbb{R}^\nu$, this yields that, for all $\gamma > \gamma_0$, $\mu \in \mathcal{P}(\Omega_\gamma)$ and hence $\mu \in \mathcal{G}^t$ is not empty. It is difficult to check (3.2) directly, which was mainly done for translation invariant, ferromagnetic systems by asymptotic methods, cf. [Si 1982, LP 1976, PrFo 1991]. Therefore, we adopt elementary new technics in proving Dobrushin's existence criterion for the graph system. An advantage of our approach is that it can be easily extended to multi (or infinite) dimensional spin spaces and to multi-particle interactions. In this new approach, instead of proving (3.2) directly, we prove the stronger *exponential bound*,

$$\int_\Omega \exp\{h(\sigma(x))\}\pi_x(d\sigma|\xi) \leq \exp\{\Gamma + \sum_{y\in\mathbb{V}} I_{xy} h(\xi(y))\}. \tag{3.3}$$

Then by Jensen's inequality (3.3) immediately implies the Dobruhsin bound (3.2). Let us recall Jensen's inequality: For any $\mu \in \mathcal{P}(\Omega)$ and $0 \leq f \in L^1(\mu)$, it holds

$$\exp\left(\int_\Omega f d\mu\right) \leq \int_\Omega \exp(f) d\mu. \tag{3.4}$$

One more principal difference from the previous papers is that the choice of the function $h(\sigma(x))$ will now crucially depend on the growth of the Hamiltonian $E(\sigma)$, see Corollary 3.4.

Remark 3.2. *Note that for the case of* unbounded degree*, i.e., $sup_{x\in\mathbb{V}} m(x) = +\infty$, the properties of the graph $G(\mathbb{V},\mathbb{E})$ changes drastically. In particular, both Dobrushin's existence, see Theorem 3.1, and uniqueness, see Theorem 4.1, conditions do not apply directly. That is because the Dobrushin interdependence matrices $\boldsymbol{I}$ and $\mathcal{D}$ are no longer strictly contractive in $l^\infty(\mathbb{V})$. Until now there is no adequate theory of Gibbs measures with unbounded degree besides some special results for ferromagnetic harmonic interactions by comparison methods, see Section 5.2 below, or for underlying graphs with certain repulsive properties for heavy vertices, see [KoKoPa 2009] by Y. G. Kondratiev, Y. Kozitsky and T. Pasurek.*

3.3 Moment estimates for the local specification

First we prove certain *moment estimates*, more precisely *exponentially bounds*, on the one-point kernels $\pi_x(d\sigma|\xi)$ subject to the fixed boundary condition $\xi \in \Omega$. Then we extend this bound to arbitrary volumes Λ by the consistency property. The latter allows to prove the relatively compactness of the family $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda \Subset \mathbb{V}}$ in the weak topology $\mathcal{W}$, which by Lemma 2.20 guaranties that $\mathcal{G}^t$ is not empty. We consider here a slightly more general setup as in Section 2.4. The assumption which we will introduce is weaker than Assumption (**U**). It enables us to regard the potentials W_{xy} and U_x with the *same* order of polynomial growth, which means that we take $P = R$:

Assumption ($\bar{\mathbf{U}}$)**.** *There exists a continuous function $U : \mathbb{R}^\nu \to \mathbb{R}$ and constants $A_1 > 0$, $B_1 \in \mathbb{R}$, such that for all $x \in \mathbb{V}$ and $s \in \mathbb{R}^\nu$*

$$A_1|s|^R + B_1 \leq U_x(s) \leq U(s). \tag{3.5}$$

Furthermore, the constant A_1 is chosen large enough, so that the following relation holds:

$$A_1 > 2mJ. \tag{3.6}$$

Let us comment on the differences between Assumptions (**U**) and ($\bar{\mathbf{U}}$). If $P > R$, (**U**) is stronger and immediately implies the validity of ($\bar{\mathbf{U}}$) with arbitrary large A_1. For the case $P = R$, i.e., Assumption ($\bar{\mathbf{U}}$) holds, we introduce a strictly positive *stability parameter*

$$\delta := A_1 - mJ, \tag{3.7}$$

which by (3.6) fulfills $\delta > mJ$. In fact, the strictly positiveness of A_1 guarantees that the specification $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda \Subset \mathbb{V}}$ is well-defined and the integrability condition (2.26) holds with any positive $\kappa < \delta$.

In our case, we can choose constant entries I in the matrix $\mathbf{I}$, that is

$$I_{xy} = \begin{cases} I, & x \sim y; \\ 0, & \text{otherwise}. \end{cases} \tag{3.8}$$

Then the one-point probability kernels $\pi_x(d\sigma|\xi)$ should satisfy, for all $x \in \mathbb{V}$, the estimate

$$\int_\Omega h(\sigma(x))\pi_x(d\sigma|\xi) \leq A + I \sum_{y \in \varphi(x)} h(\xi(y)), \tag{3.9}$$

such that

$$\parallel \mathbf{I} \parallel = mI < 1 \tag{3.10}$$

holds. Recall that $\varphi(x) = B_1(x)$ is the set of all nearest neighbors of $x \in \mathbb{V}$, see (2.1).

The key technical result is the following:

Lemma 3.3. *Suppose that Assumption* ($\mathbf{W}$) *and* ($\bar{\mathbf{U}}$) *hold. Then for every* $\kappa < \delta$ *there exists a corresponding* $\Gamma = \Gamma(\beta, \kappa) > 0$ *such that, for all* $x \in \mathbb{V}$ *and* $\xi \in \Omega$, *we have that*

$$\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma|\xi) \le \exp\{\Gamma + 2J\beta \sum_{y\in\varphi(x)} |\xi(y)|^R\}. \tag{3.11}$$

Proof. By Assumption ($\mathbf{W}$), one has for all $\sigma(x)$, $\xi(y) \in \mathbb{R}^\nu$

$$\begin{aligned} &\sum_{y\in\varphi(x)} |W_{xy}(\sigma(x), \xi(y))| \le \sum_{y\in\varphi(x)} J(C + |\sigma(x)|^R + |\xi(y)|^R) \\ \le\ & m(x)J|\sigma(x)|^R + \sum_{y\in\varphi(x)} J(C + |\xi(y)|^R) \\ \le\ & mJ|\sigma(x)|^R + \sum_{y\in\varphi(x)} J(C + |\xi(y)|^R). \end{aligned}$$

By this estimate and the definition (2.27) of $\pi_x(d\sigma|\xi)$, one has

$$\begin{aligned} &\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma(x)|\xi) \\ =\ & \int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\frac{1}{Z_x(\xi)}\exp\{-\beta E_x(\sigma(x)|\xi)\}d\sigma(x) \\ =\ & \frac{1}{Z_x(\xi)}\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R - \beta U_x(\sigma(x)) - \beta\sum_{y\in\varphi(x)} W_{xy}(\sigma(x), \xi(y))\}d\sigma(x) \\ =\ & \frac{\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R - \beta U_x(\sigma(x)) - \beta\sum_{y\in\varphi(x)} W_{xy}(\sigma(x), \xi(y))\}d\sigma(x)}{\int_\Omega \exp\{-\beta U_x(\sigma(x)) - \beta\sum_{y\in\varphi(x)} W_{xy}(\sigma(x), \xi(y))\}d\sigma(x)} \\ \le\ & \frac{\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R - \beta U_x(\sigma(x)) + \beta\sum_{y\in\varphi(x)} |W_{xy}(\sigma(x), \xi(y))|\}d\sigma(x)}{\int_\Omega \exp\{-\beta U_x(\sigma(x)) - \beta\sum_{y\in\varphi(x)} |W_{xy}(\sigma(x), \xi(y))|\}d\sigma(x)} \\ \le\ & \frac{X_x(\kappa)}{Y_x}\exp\{2\beta\sum_{y\in\varphi(x)} J(C + |\xi(y)|^R)\} \\ \le\ & \frac{X_x(\kappa)}{Y_x}\exp\{2mJ\beta C + 2J\beta\sum_{y\in\varphi(x)} |\xi(y)|^R\}, \end{aligned}$$

where

$$X_x(\kappa) := \int_\Omega \exp\{-\beta(U_x(\sigma(x)) - (\kappa + mJ)|\sigma(x)|^R)\}d\sigma(x)$$

and

$$Y_x := \int_\Omega \exp\{-\beta(U_x(\sigma(x)) + mJ|\sigma(x)|^R)\}d\sigma(x).$$

In short, we get the inequality

$$\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma|\xi) \leq \frac{X_x(\kappa)}{Y_x}\exp\{2mJ\beta C + 2J\beta \sum_{y\in\varphi(x)} |\xi(y)|^R\}.$$

Using the upper bound in ($\bar{\mathbf{U}}$) to estimate $\inf_x Y_x$, and the lower bound in ($\bar{\mathbf{U}}$) to estimate $\sup_x X_x(\kappa)$, one easily observes that

$$\begin{aligned} X(\kappa) &:= \sup_x X_x(\kappa) \\ &\leq \exp(-\beta B_1)\int_{\mathbb{R}^\nu} \exp(-\beta(A_1 - mJ - \kappa)|\sigma(x)|^R)d\sigma(x) < \infty, \end{aligned}$$

and

$$Y := \inf_x Y_x \geq \int_{\mathbb{R}^\nu} \exp(-\beta(U(\sigma(x)) + mJ|\sigma(x)|^R))d\sigma(x) > 0.$$

Then it follows

$$0 < Y := \inf_x Y_x < \sup_x X_x(\kappa) =: X(\kappa) < \infty.$$

This proves the required estimate (3.11) with

$$\Gamma := \Gamma(\beta,\kappa) := 2mJ\beta C + \log\left(\frac{X(\kappa)}{Y}\right). \tag{3.12}$$

□

The application of Jensen's inequality gives us the following important corollary. Note that we can choose any $\kappa > 2mJ$ if we assume (**U**) instead of ($\bar{\mathbf{U}}$).

Corollary 3.4. *Let us choose in* (3.11) *any* $\kappa \in (2mJ, \delta)$. *Then the kernels* $\pi_x(d\sigma|\xi)$ *obey the Dobrushin bound* (3.9), *with*

$$A := \frac{\Gamma}{\beta\kappa}, \quad I := \frac{2J}{\kappa}, \quad mI < 1,$$

and with the compact function

$$\mathbb{R}^\nu \ni \sigma(x) \mapsto h(\sigma(x)) := |\sigma(x)|^R.$$

Proof. We have by Jensen's inequality

$$\begin{aligned} &\exp\left\{\beta\kappa \int_\Omega |\sigma(x)|^R \pi_x(d\sigma|\xi)\right\} \\ \leq\ & \int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma|\xi) \\ \leq\ & \exp\left\{\Gamma + 2J\beta \sum_{y\in\varphi(x)} |\xi(y)|^R\right\}. \end{aligned}$$

This is equivalent to

$$\int_\Omega |\sigma(x)|^R \pi_x(d\sigma|\xi) \leq \frac{\Gamma}{\beta\kappa} + \sum_{y\in\varphi(x)} \frac{2J}{\kappa}|\xi(y)|^R,$$

which gives us the Dobrushin bound (3.2).

□

Taking a step forward we gain similar moment estimates for $\pi_\Lambda(d\sigma|\xi)$ *uniformly in volumes* $\Lambda \Subset V$. We define for all $\kappa < \delta$

$$n_x(\Lambda|\xi) := \log\left(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_\Lambda(d\sigma|\xi)\right). \tag{3.13}$$

Note that by Lemma (3.3) $n_x(\Lambda|\xi)$ is nonnegative and finite.

Lemma 3.5. *Given any* $\kappa < \delta$, $\gamma > \gamma_0$ *and* $x_0 \in \mathbb{V}$, *there exists a finite* $\Gamma_{\gamma,x_0} := \Gamma_{\gamma,x_0}(\beta,\kappa) > 0$ *such that uniformly for all* $\xi \in \Omega^t$

$$\limsup_{\Lambda\nearrow\mathbb{V}} \left(\sum_{x\in\Lambda} n_x(\Lambda|\xi)\cdot e^{-\gamma d(x,x_0)}\right) \leq \Gamma_{\gamma,x_0}, \tag{3.14}$$

where the constant Γ_{γ,x_0} *is given below by* (3.40). *In particular, under Assumption* (**G**), *uniformly for all* $x_0 \in \mathbb{V}$ *and all* $\xi \in \Omega^t$

$$\limsup_{\Lambda\nearrow\mathbb{V}} \left(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_\Lambda(d\sigma|\xi)\right) \leq \exp(\Gamma_\gamma) \tag{3.15}$$

with finite $\Gamma_\gamma := \sup_{x_0\in V}\Gamma_{\gamma,x_0}$, *for each* $\gamma > \gamma_0$.

Proof. Without loss of generality, one may choose $\kappa \in (2J, \delta)$, so that

$$\sum_{y\in\varphi(x)} 2J\kappa^{-1} \le \frac{2Jm}{\kappa} < 1. \tag{3.16}$$

Integrating both sides of the exponential bound (3.11) with respect to the measure $\pi_\Lambda(d\sigma|\xi)$ and taking into account the consistency property (2.22), we come to the following estimate with Γ given by (3.36)

$$\begin{aligned}
& n_x(\Lambda|\xi) \\
:= \ & \log\Big(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_\Lambda(d\sigma|\xi)\Big) \\
= \ & \log\Big(\int_\Omega\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma|\eta)\pi_\Lambda(d\eta|\xi)\Big) \\
\le \ & \log\Big(\int_\Omega \exp\Big\{\Gamma + \sum_{y\in\varphi(x)} 2J\beta|\eta(y)|^R\Big\}\pi_\Lambda(d\eta|\xi)\Big) \\
= \ & \log\Big(\int_\Omega \exp\Big\{\Gamma + \sum_{y\in\varphi(x)\cap\Lambda^c} 2J\beta|\eta(y)|^R\Big\} \\
\cdot \ & \exp\Big\{\sum_{y\in\varphi(x)\cap\Lambda} 2J\beta|\eta(y)|^R\Big\}\pi_\Lambda(d\eta|\xi)\Big) \\
= \ & \Gamma + \sum_{y\in\varphi(x)\cap\Lambda^c} 2J\beta|\xi(y)|^R \\
+ \ & \log\Big(\int_\Omega \exp\Big\{\sum_{y\in\varphi(x)\cap\Lambda} 2J\beta\kappa^{-1}\kappa|\eta(y)|^R\Big\}\pi_\Lambda(d\eta|\xi)\Big) \\
\le \ & \Gamma + \sum_{y\in\varphi(x)\cap\Lambda^c} 2J\beta|\xi(y)|^R + \sum_{y\in\varphi(x)\cap\Lambda} 2J\kappa^{-1}n_y(\Lambda|\xi).
\end{aligned} \tag{3.17}$$

We have used here the assumption (3.16) and the multiple Hölder inequality

$$\int\Big(\prod_{i=1}^n \phi_i^{\alpha_i}\Big)d\mu \le \prod_{i=1}^n\Big(\int \phi_i d\mu\Big)^{\alpha_i},$$

where μ is a probability measure, $\phi_i \ge 0$ are functions, and $\alpha_i \ge 0$ are numbers such that $\sum_{i=1}^n \alpha_i \le 1$. In our case, this condition is fulfilled by the assumption (3.16). We convince ourself of the last inequality in (3.17) by calculating the following:

$$
\begin{aligned}
& \log\left(\int_\Omega \exp\Big\{\sum_{y\in\varphi(x)\cap\Lambda} 2J\beta\kappa^{-1}\kappa|\eta(y)|^R\Big\}\pi_\Lambda(d\eta|\xi)\right) \\
= \;& \log\left(\int_\Omega \prod_{y\in\varphi(x)\cap\Lambda} \left(\exp\{\beta\kappa|\eta(y)|^R\}\right)^{2J\kappa^{-1}}\pi_\Lambda(d\eta|\xi)\right) \\
\leq \;& \log\left(\prod_{y\in\varphi(x)\cap\Lambda}\left(\int_\Omega \exp\{\beta\kappa|\eta(y)|^R\}\pi_\Lambda(d\eta|\xi)\right)^{2J\kappa^{-1}}\right) \\
= \;& \sum_{y\in\varphi(x)\cap\Lambda} 2J\kappa^{-1}\log\left(\int_\Omega \exp\{\beta\kappa|\eta(y)|^R\}\pi_\Lambda(d\eta|\xi)\right) \\
= \;& \sum_{y\in\varphi(x)\cap\Lambda} 2J\kappa^{-1} n_y(\Lambda|\xi).
\end{aligned}
$$

Now, fixing arbitrary $x_0\in\mathbb{V}$, we shall take the sum in (3.17) over $x\in\Lambda$ with the weights $e^{-\gamma d(x,x_0)}$. We set

$$\|\,\mathbf{1}_\Lambda\,\|_{\gamma,x_0} := \sum_{x\in\Lambda} e^{-\gamma d(x,x_0)},$$

which is finite by Assumption (**G**), see Section 2.2. Denoting

$$n(\Lambda|\xi) := \sum_{x\in\Lambda} e^{-\gamma d(x,x_0)} n_x(\Lambda|\xi),$$

we obtain that

$$
\begin{aligned}
& n_{x_0}(\Lambda|\xi) \\
& \quad \leq \; n(\Lambda|\xi) \\
& \quad \leq \; \Gamma\sum_{x\in\Lambda} e^{-\gamma d(x,x_0)} \\
& \qquad +2J\beta\sum_{x\in\Lambda} e^{-\gamma d(x,x_0)}\sum_{y\in\varphi(x)\cap\Lambda^c}|\xi(y)|^R e^{-\gamma d(y,x_0)}e^{\gamma d(y,x_0)} \\
& \qquad +2J\kappa^{-1}\sum_{x\in\Lambda} e^{-\gamma d(x,x_0)}\sum_{y\in\varphi(x)\cap\Lambda} n_y(\Lambda|\xi) e^{-\gamma d(y,x_0)}e^{\gamma d(y,x_0)} \\
& \quad \leq \; \Gamma\sum_{x\in\Lambda} e^{-\gamma d(x,x_0)} + 2J\beta\sum_{x\in\Lambda}\sum_{y\in\varphi(x)\cap\Lambda^c} e^{-\gamma d(y,x_0)}|\xi(y)|^R e^{\gamma d(x,y)} \\
& \qquad +2J\kappa^{-1}\sum_{x\in\Lambda}\sum_{y\in\varphi(x)\cap\Lambda} e^{-\gamma d(y,x_0)} n_y(\Lambda|\xi) e^{\gamma d(x,y)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Gamma \sum_{x \in \Lambda} e^{-\gamma d(x,x_0)} \\
&+ 2J\beta \sum_{y \in \Lambda^c} e^{-\gamma d(y,x_0)} |\xi(y)|^R \sum_{x \in \Lambda \cap \varphi(y)} e^{\gamma d(x,y)} \\
&+ 2J\kappa^{-1} \sum_{y \in \Lambda} e^{-\gamma d(y,x_0)} n_y(\Lambda|\xi) \sum_{x \in \Lambda \cap \varphi(y)} e^{\gamma d(x,y)}.
\end{aligned}
$$

Since

$$\sup_{x,y} \sum_{x \in \varphi(y)} e^{\gamma d(x,y)} \leq C(\gamma) < \infty$$

this yields

$$
\begin{aligned}
n(\Lambda|\xi) \leq \Gamma \|\mathbf{1}_\Lambda\|_{\gamma,x_0} &+ 2J\beta C(\gamma) \|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \\
&+ 2J\kappa^{-1} C(\gamma) n(\Lambda|\xi),
\end{aligned}
$$

or equivalently,

$$n(\Lambda|\xi) \leq \frac{1}{1 - 2J\kappa^{-1}C(\gamma)} \left(\|\mathbf{1}_\Lambda\|_{\gamma,x_0} \Gamma + 2J\beta C(\gamma) \|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \right).$$

Since $n_{x_0}(\Lambda|\xi) \leq n(\Lambda|\xi)$ we have

$$n_{x_0}(\Lambda|\xi) \leq \frac{1}{1 - 2J\kappa^{-1}C(\gamma)} \left(\|\mathbf{1}_\Lambda\|_{\gamma,x_0} \Gamma + 2J\beta C(\gamma) \|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \right)$$

Since $\|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \to 0$ with $\Lambda \nearrow \mathbb{V}$ the latter implies

$$
\begin{aligned}
\limsup_{\Lambda \nearrow \mathbb{V}} n_{x_0}(\Lambda|\xi) &\leq \limsup_{\Lambda \nearrow \mathbb{V}} n(\Lambda|\xi) \\
&\leq \frac{\Gamma}{1 - 2J\kappa^{-1}C(\gamma)} \|\mathbf{1}\|_{\gamma,x_0} \\
&=: \Gamma_{\gamma,x_0} < \infty. \qquad (3.18)
\end{aligned}
$$

Finally, by Assumption (**G**) $\|\mathbf{1}_\Lambda\|_{\gamma,x_0}$ is uniformly bounded by

$$\|\mathbf{1}\|_\gamma := \sup_{x_0 \in \mathbb{V}} \sum_{x \in \mathbb{V}} e^{-\gamma d(x,x_0)} < \infty,$$

and we have

$$\sup_{x_0 \in \mathbb{V}} \Gamma_{\gamma,x_0} =: \Gamma_\gamma < \infty,$$

which completes the proof of the Lemma.

□

Remark 3.6. *In place of the Assumptions* ($\bar{\mathbf{U}}$)*, where we take the same polynomially growth order for the potentials W_{xy} and U_x ($P = R$), we can use the stronger initial Assumption* ($\mathbf{U}$) *where we have $P > R$. In this situation, all the former statements, especially Lemmas 3.3 and 3.5, hold for all $\beta > 0$ and $\kappa > 2mJ$.*

3.4 Existence and a priori bounds for Gibbs measures

Now we are on the stage to prove *existence* and *a priori bounds* for Gibbs measures. Below the main Theorems 3.7 and 3.8 characterize the set of tempered Gibbs measures $\mathcal{G}^t$. The methods of proving the following statements are strongly encouraged by the work [Pa 2008]. However, the original concept relies on the paper [BH-K 1982] by J. Bellissard and R. Høegh-Krohn.

The idea is the following: As soon as the exponentially bound in Lemma 3.3 for the one-point probability kernels $\pi_x(d\sigma|\xi)$ has been established, using the consistency property, we get for $\pi_\Lambda(d\sigma|\xi)$ *uniform bounds* as in Lemma 3.5. This yields immediately, connected with the *relatively compact* property of $\pi_\Lambda(d\sigma|\xi)$ in the topology $\mathcal{W}_t$, that $\mathcal{G}^t$ is not empty. As the last principle result in this section in Theorem 3.8 we obtain *a priori moment bounds* for all measures $\mu \in \mathcal{G}^t$. Since in Lemma 3.5 the bounds on $\pi_\Lambda(d\sigma|\xi)$ are asymptotically independent, as $\Lambda \nearrow \mathbb{V}$ on the initial data ξ, using the DLR equation we get the same uniform bounds on all $\mu \in \mathcal{G}^t$. Note that we show the relatively compactness of $\pi_\Lambda(d\sigma|\xi)$ in the stronger topology $\mathcal{W}_\gamma$ in Theorem 3.7 below. This gives us immediately the relatively compactness of $\pi_\Lambda(d\sigma|\xi)$ in the weaker topology $\mathcal{W}$, which is only needed for the pair interaction case. In turn, the relatively compactness in the topology $\mathcal{W}_\gamma$ enables us to treat the interactions of infinite range, see Section 3.6. Now we formulate and prove the main theorem of the whole chapter:

Theorem 3.7. *Let Assumptions* ($\mathbf{G}$), ($\mathbf{W}$) *and* ($\bar{\mathbf{U}}$) *are fulfilled. Then for any boundary condition $\xi \in \Omega^t$, the family $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda \Subset \mathbb{V}}$ is relatively compact in the weak topology $\mathcal{W}_t$. All its limit points are tempered Gibbs measures supported by Ω^t, which means that $\mathcal{G}^t$ is not empty.*

Proof. For any fixed $\xi \in \Omega^t$ and $\kappa < \delta$, by the estimate (3.14), Assumption (**G**) and Jensen's inequality one calculates

$$\begin{aligned}
&\limsup_{\Lambda\nearrow\mathbb{V}} \beta\kappa \sum_{x\in\Lambda} \int_\Omega |\sigma(x)|^R e^{-\gamma d(x,x_0)} \pi_\Lambda(d\sigma|\xi) \\
\leq\ & \limsup_{\Lambda\nearrow\mathbb{V}} \sum_{x\in\Lambda} \log\left(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\} \pi_\Lambda(d\sigma(x)|\xi)\right) e^{-\gamma d(x,x_0)} \\
\leq\ & \Gamma_{\gamma,x_0}.
\end{aligned}$$

Taking into account that $\|\sigma_\Lambda\|_\gamma^R := \sum_{x\in\Lambda} |\sigma(x)|^R e^{-\gamma d(x,x_0)}$ and by Jensen's inequality we conclude that

$$\limsup_{\Lambda\nearrow\mathbb{V}} \int_\Omega \|\sigma_\Lambda\|_\gamma^R \pi_\Lambda(d\sigma|\xi) \leq \frac{\Gamma_{\gamma,x_0}}{\beta\kappa}. \tag{3.19}$$

Therefore, for each $\xi \in \Omega^t$ one finds a corresponding finite $\Gamma_\gamma(\xi) > 0$ such that

$$\sup_{\Lambda\Subset\mathbb{V}} \int_\Omega \|\sigma\|_\gamma^R \pi_\Lambda(d\sigma|\xi) \leq \Gamma_\gamma(\xi). \tag{3.20}$$

Since the embedding $\Omega_\gamma \subset \Omega_{\gamma'}$ is compact if $\gamma < \gamma'$ (cf. Remark 2.15), by Prohorov's theorem, this implies the relatively compactness of $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ in the weak topology $\mathcal{W}_{\gamma'}$. So, there exists at least one limit point, say $\mu \in \mathcal{P}(\Omega_{\gamma'})$, for this family. By Fatou's lemma, this μ satisfies

$$\begin{aligned}
\int_\Omega \|\sigma\|_\gamma^R \mu(d\sigma) &\leq \limsup_{\Lambda\Subset V} \int_\Omega \|\sigma\|_\gamma^R \pi_\Lambda(d\sigma|\xi) \\
&\leq \Gamma_\gamma(\xi),
\end{aligned} \tag{3.21}$$

since $\Omega \ni \sigma \mapsto \|\sigma\|_\gamma \in \mathbb{R}\cup\{+\infty\}$ is a lower semi-continuous function. Hence, μ is supported by Ω^t. Finally, by Lemma 2.20 every such μ is Gibbsian.

□

An important consequence of Lemma 3.5 is the following theorem giving an *uniform integral bound* for all tempered Gibbs measures. Basically, we can prove such an uniform integral bound without knowing anything about the existence of such measures, which is the reason why we are speaking about *a priori estimates*. We will see that the bound (3.22) below is uniform for all $\mu \in \mathcal{G}^t$ and explicitly depends on the inverse temperature β and parameters of the interaction. Note that this result cannot be obtained from Dobrushins's criterion alone.

Theorem 3.8. *Let the same assumptions as in Theorem* (3.7) *hold. For every* $\kappa < \delta$*, there exists a positive constant* $C = C(\beta, \kappa)$*, such that uniformly for all* $\mu \in \mathcal{G}^t$

$$\sup_{x\in\mathbb{V}} \int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\mu(d\sigma) \leq C. \tag{3.22}$$

Proof. Let us choose any Banach space Ω_γ with $\gamma > \gamma_0$. By the DLR equation (2.23), B. Levi's monotone convergence theorem and Lemma 3.5, one has for all $N > 0$

$$\begin{aligned}
& \int_\Omega \exp\{(\beta\kappa|\sigma(x)|^R \wedge N)\}\mu(d\sigma(x)) \\
= \; & \lim_{\Lambda\nearrow\mathbb{V}} \int_{\Omega_\gamma}\int_\Omega \exp\{(\beta\kappa|\sigma(x)|^R \wedge N)\}\pi_\Lambda(d\sigma|\xi)\mu(d\xi) \\
= \; & \int_{\Omega_\gamma}\left(\limsup_{\Lambda\nearrow\mathbb{V}}\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R \wedge N\}\pi_\Lambda(d\sigma|\xi)\right)\mu(d\xi) \\
= \; & \limsup_{\Lambda\nearrow\mathbb{V}}\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R \wedge N\}\pi_\Lambda(d\sigma|\xi) \\
\leq \; & \exp(\Gamma_\gamma),
\end{aligned} \tag{3.23}$$

where we define $(\kappa|\sigma(x)|^R \wedge N) := \min(\kappa|\sigma(x)|^R, N)$. Hence, again with B. Levi's theorem, we obtain for all $\mu \in \mathcal{G}^t$

$$\begin{aligned}
& \int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\mu(d\sigma) \\
= \; & \limsup_{N\to\infty}\int_\Omega \exp\{(\beta\kappa|\sigma(x)|^R \wedge N)\}\mu(d\sigma(x)) \\
\leq \; & \exp(\Gamma_\gamma).
\end{aligned} \tag{3.24}$$

Then, by Jensen's inequality, we conclude that

$$\sup_{x\in\mathbb{V}} \int_\Omega |\sigma(x)|^R\mu(d\sigma) < \frac{\Gamma_\gamma}{\beta\kappa}. \tag{3.25}$$

The latter implies, by Chebyshev's inequality and Assumption (**G**), that any $\mu \in \mathcal{G}^t$ is actually supported by every Ω_γ whenever $\gamma > \gamma_0$. Hence (3.24) yields the desired estimate (3.22) with constant $C := \exp(\Gamma_\gamma)$, which is the same for all $\mu \in \mathcal{G}^t$.

$\square$

The above a priori bound (3.22) allows to describe the support properties of $\mu \in \mathcal{G}^t$ more precisely.

Corollary 3.9. *Suppose that the graph $G(\mathbb{V}, \mathbb{E})$ satisfies for some $x_0 \in \mathbb{V}$ and $\lambda > 0$ that*

$$(\mathbf{G}_\lambda) \qquad \sum_{x \in \mathbb{V}} (1 + d(x_o, x))^{-\lambda} < \infty. \tag{3.26}$$

Then, the Gibbs measures $\mu \in \mathcal{G}^t$ are indeed supported by the smaller set

$$\begin{aligned} \Omega(b) \;:=\; & \Big\{ \sigma \in \Omega | \exists \Lambda_\sigma \Subset \mathbb{V}, \quad \forall x \in (\Lambda_\sigma)^c \Big| \\ & |\sigma(x)|^2 \leq b \log\left(1 + d(x_o, x)\right) \Big\}, \end{aligned} \tag{3.27}$$

where $b > \frac{\lambda}{\kappa\beta}$.

Proof. We follow the scheme of the proof of Lemma 3.1 in [LP 1976], see also [Pa 2008]. We write the complement of the set (3.27) as follows:

$$[\Omega(b)]^c = \bigcap_{\Lambda \Subset \mathbb{V}} \bigcup_{x \in \Lambda^c} \Omega_x(b), \tag{3.28}$$

where

$$\Omega_x(b) := \left\{ \sigma \in \Omega \middle| |\sigma(x)|^2 \leq b \log\left(1 + d(x_o, x)\right) \right\}.$$

By Chebyshev's inequality and the estimate (3.22),

$$\mu([\Omega_x(b)]^c) \leq C(\beta, \kappa) \cdot (1 + d(x_o, x))^{-b\beta\kappa}. \tag{3.29}$$

Therefore, by (3.28) and (3.29), for any cofinal sequence $\mathcal{L} \nearrow \mathbb{V}$

$$\mu([\Omega(b)]^c) \leq C(\beta, \kappa) \lim_{\Lambda \in \mathcal{L}} \sum_{x \in \Lambda^c} (1 + d(x_o, x))^{-b\beta\kappa}. \tag{3.30}$$

By Assumption $(\mathbf{G}_\lambda)$ the series in (3.30) is convergent for $b > \frac{\lambda}{\kappa\beta}$, which yields the desired result $\mu([\Omega(b)]^c) = 0$.

$\square$

Remark 3.10. *For $\mathbb{V} = \mathbb{Z}^d$, the property* (3.26) *holds with any $d > \kappa\beta$.*

3.5 Interactions with unbounded intensity

In this subsection we study a non-trivial example for the existence of tempered Gibbs measures for the following model. The crucial difference is that we consider Hamiltonians with possibly *unbounded* interaction strength. Namely, *harmonic pair interactions* with intensity $J_{xy} \geq 0$ such that

$$\sup_{x,y \in \mathbb{V}} J_{xy} = +\infty.$$

The system is described by the following heuristic *infinite-volume energy functional*

$$E(\sigma) := \sum_{\substack{x,y \in \mathbb{V} \\ x \sim y}} J_{xy}(\sigma(x), \sigma(y))_{\mathbb{R}^d} + \sum_{x \in \mathbb{V}} U_x(\sigma(x)), \tag{3.31}$$

where $(\cdot, \cdot)_{\mathbb{R}^d}$ is the scalar product in $\mathbb{R}^d$, U_x the *self interaction potentials* and the infinite sum $\sum_{\substack{x,y \in \mathbb{V} \\ x \sim y}}$ is taken over all unordered pairs $e = (x, y) \in \mathbb{E}$ of nearest neighbors. We suppose that the Assumption (**U**) holds for the self interaction potentials $U_x(\sigma(x))$ with some $P > 2$. We define the spaces of tempered configurations as usual by

$$\Omega^t := \bigcap_{\gamma > \gamma_0} \Omega_\gamma,$$

where we define for $R > 2$

$$\Omega_\gamma := \left\{ \sigma \in \Omega \,\middle|\, \|\sigma\|_\gamma := \left(\sum_{x \in \mathbb{V}} |\sigma(x)|^R e^{-\gamma d(x, x_0)} \right)^{1/R} < \infty \right\}. \tag{3.32}$$

Respectively, the subset of tempered Gibbs measures $\mathcal{G}^t$ consists of all $\mu \in \mathcal{G}$ which are supported by Ω^t.

Immediately, by *Young's inequality*

$$ab \leq \frac{a^R}{R} + b^{\frac{R}{R-1}} \left(1 - \frac{1}{R}\right) \quad a, b > 0,$$

we have for any $\epsilon > 0$ the following bound

$$\left| J_{xy} \cdot (\sigma(x), \sigma(y))_{\mathbb{R}^d} \right| \leq \epsilon^{\frac{R-2}{R}} |J_{xy}|^{\frac{R}{R-2}} + \epsilon \left[|\sigma(x)|^R + |\sigma(y)|^R \right], \tag{3.33}$$

which is essential for the existence result. However, for the existence of tempered Gibbs measures μ there emerges a principle condition. Let us define $\tilde{J}_x := \sup_{y \in \partial x} |J_{xy}|$, then we have the following assumption:

Assumption (WJ). *There exists a nonnegative number γ_0 such that, for all $\gamma > \gamma_0$ and each initial point $x_0 \in \mathbb{V}$, it holds*

$$C(\gamma, \tilde{J}) := \sum_{x\in\mathbb{V}} |\tilde{J}_x|^{\frac{R}{R-2}} e^{\gamma d(x,x_0)} < \infty. \tag{3.34}$$

With Assumption (**WJ**) we can follow the same scheme of showing existence of tempered Gibbs measures as in Sections 3.3 and 3.4. The corresponding changes in the proofs are the dependence of $\Gamma(x)$ on $x \in \mathbb{V}$. However, we cannot obtain an *uniform in x integral bound* for all tempered Gibbs measures as was obtained by Theorem 3.8 for bounded interaction strength. Let us state the main new statements pointing out the difference to the previous scheme.

Lemma 3.11. *Suppose that the assumption of this subsection hold. Then for every $P > R > 2$ and $\kappa > 0$ there exists a corresponding $\Gamma(x) > 0$ such that, for all $x \in \mathbb{V}$ and $\xi \in \Omega$, we have that*

$$\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma|\xi) \le \exp\{\Gamma(x) + 2\beta \sum_{y\in\varphi(x)} |J_{xy}|^{\frac{R}{R-2}}|\xi(y)|^R\}, \tag{3.35}$$

Proof. By (3.33), one has for all $\sigma(x)$, $\xi(y) \in \mathbb{R}^\nu$

$$\sum_{y\in\varphi(x)} \left| J_{xy} \cdot (\sigma(x), \xi(y))_{\mathbb{R}^d} \right| \le \sum_{y\in\varphi(x)} \left(\epsilon^{\frac{R-2}{R}} |J_{xy}|^{\frac{R}{R-2}} + \epsilon \Big[|\sigma(x)|^R + |\xi(y)|^R \Big] \right).$$

This leads to

$$\begin{aligned} & \int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\}\pi_x(d\sigma(x)|\xi) \\ \le\ & \frac{X_x(\kappa)}{Y_x} \exp\{2\beta \sum_{y\in\varphi(x)} \epsilon^{\frac{R-2}{R}} |J_{xy}|^{\frac{R}{R-2}} + 2\beta\epsilon \sum_{y\in\varphi(x)} |\xi(y)|^R\}, \end{aligned}$$

where

$$X_x(\kappa) := \int_\Omega \exp\{-\beta(U_x(\sigma(x)) - \kappa\epsilon|\sigma(x)|^R)\} d\sigma(x)$$

and

$$Y_x := \int_\Omega \exp\{-\beta U_x(\sigma(x))\} d\sigma(x).$$

Using the upper bound in (**U**) to estimate $\inf_x Y_x$, and the lower bound in (**U**) to estimate $\sup_x X_x(\kappa)$, one easily observes that

$$\begin{aligned} X(\kappa) &:= \sup_x X_x(\kappa) \\ &\leq \exp(-\beta B) \int_{\mathbb{R}^\nu} \exp(-\beta(A - \kappa\epsilon)|\sigma(x)|^R) d\sigma(x) < \infty, \end{aligned}$$

and

$$Y := \inf_x Y_x \geq \int_{\mathbb{R}^\nu} \exp(-\beta U(\sigma(x))) d\sigma(x) > 0.$$

This proves the required estimate (3.35) with

$$\Gamma(x) := 2\beta \sum_{y \in \varphi(x)} \epsilon^{\frac{R-2}{R}} |J_{xy}|^{\frac{R}{R-2}} + \log\left(\frac{X(\kappa)}{Y}\right). \tag{3.36}$$

□

Now, defining for all $\kappa > 0$

$$n_x(\Lambda|\xi) := \log\left(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\} \pi_\Lambda(d\sigma|\xi)\right), \tag{3.37}$$

we have the following.

Lemma 3.12. *Given any $\kappa > 0$, $\gamma > \gamma_0$ and $x_0 \in \mathbb{V}$, there exists a finite $\Gamma_{\gamma,x_0}(x) :=> 0$ such that uniformly for all $\xi \in \Omega^t$*

$$\limsup_{\Lambda \nearrow \mathbb{V}} \left(\sum_{x \in \Lambda} n_x(\Lambda|\xi) \cdot e^{-\gamma d(x,x_0)}\right) \leq \Gamma_{\gamma,x_0}(x), \tag{3.38}$$

where the constant $\Gamma_{\gamma,x_0}(x)$ is given below by (3.40). In particular

$$\limsup_{\Lambda \nearrow \mathbb{V}} \left(\int_\Omega \exp\{\beta\kappa|\sigma(x)|^R\} \pi_\Lambda(d\sigma|\xi)\right) \leq \exp(\Gamma_{\gamma,x_0}(x)). \tag{3.39}$$

Proof. Fixing an arbitrary $x_0 \in \mathbb{V}$, we shall take the sum in (3.17) over $x \in \Lambda$ with the weights $e^{-\gamma d(x,x_0)}$. We set

$$\| \Gamma(x) \|_{\gamma,x_0} := \sum_{x \in \Lambda} \Gamma(x) e^{-\gamma d(x,x_0)},$$

which is finite by Assumption (**WJ**). Denoting

$$n(\Lambda|\xi) := \sum_{x\in\Lambda} e^{-\gamma d(x,x_0)} n_x(\Lambda|\xi),$$

we obtain that

$$\begin{aligned} & n_{x_0}(\Lambda|\xi) \\ & \qquad \leq \; n(\Lambda|\xi) \\ & \qquad \leq \; \sum_{x\in\Lambda} \Gamma(x) e^{-\gamma d(x,x_0)} \\ & \qquad + \; 2\beta\epsilon^{\frac{R-2}{R}} \sum_{y\in\Lambda^c} e^{-\gamma d(y,x_0)} |\xi(y)|^R \sum_{x\in\Lambda\cap\varphi(y)} |\tilde{J}_x|^{\frac{R}{R-2}} e^{\gamma d(x,y)} \\ & \qquad + \; 2\kappa^{-1}\epsilon^{\frac{R-2}{R}} \sum_{y\in\Lambda} e^{-\gamma d(y,x_0)} n_y(\Lambda|\xi) \sum_{x\in\Lambda\cap\varphi(y)} |\tilde{J}_x|^{\frac{R}{R-2}} e^{\gamma d(x,y)}. \end{aligned}$$

This yields

$$\begin{aligned} n(\Lambda|\xi) \leq \|\Gamma(x)\|_{\gamma,x_0} \; & + \; 2\beta\epsilon^{\frac{R-2}{R}} C(\gamma,\tilde{J}) \|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \\ & + \; 2\kappa^{-1}\epsilon^{\frac{R-2}{R}} C(\gamma,\tilde{J}) n(\Lambda|\xi) < +\infty. \end{aligned}$$

Since $n_{x_0}(\Lambda|\xi) \leq n(\Lambda|\xi)$ we have

$$n_{x_0}(\Lambda|\xi) \leq \frac{1}{1 - 2\kappa^{-1}\epsilon^{\frac{R-2}{R}} C(\gamma,\tilde{J})} \left(\|\Gamma(x)\|_{\gamma,x_0} + 2\beta\epsilon^{\frac{R-2}{R}} C(\gamma,\tilde{J}) \|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \right)$$

Since $\|\xi_{\Lambda^c}\|^R_{\gamma,x_0} \to 0$ with $\Lambda \nearrow \mathbb{V}$ the latter implies

$$\begin{aligned} \limsup_{\Lambda\nearrow\mathbb{V}} n_{x_0}(\Lambda|\xi) \;\; & \leq \;\; \frac{\|\Gamma(x)\|_{\gamma,x_0}}{1 - 2\kappa^{-1}\epsilon^{\frac{R-2}{R}} C(\gamma,\tilde{J})} \\ & =: \;\; \Gamma_{\gamma,x_0}(x) < \infty, \end{aligned} \tag{3.40}$$

which completes the proof of the Lemma.

□

Then we have the following theorem.

Theorem 3.13. *Let the assumptions of this subsection hold. Then for any boundary condition $\xi \in \Omega^t$, the family $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\Subset\mathbb{V}}$ is relatively compact in the weak topology $\mathcal{W}_t$. All its limit points are tempered Gibbs measures supported by Ω^t, which means that $\mathcal{G}^t$ is not empty.*

Remark 3.14. *Up to our knowledge, the above model is the first example of unbounded spin systems with pair interactions of unbounded strength ever studied in the literature. Furthermore, the same method can be used to treat the pair interactions with unbounded intensities $J_{xy}(\omega)$ with a random parameter ω.*

3.6 Further Extensions

3.6.1 Finite range potentials

Considering the graph $G(\mathbb{V},\mathbb{E})$ as a particular case of discrete metric spaces, we can generalize the situation by introducing variable pair interaction potentials $W_{xy}(\sigma(x),\sigma(y))$ for pairs $x,y\in\mathbb{V}$ (not only for nearest neighbors). Let us first suppose that the pair interaction potentials W_{xy} are of *finite range*. This means that there exists a constant $r>0$ such that for every $x,y\in\mathbb{V}$ with $d(x,y)>r$, we have

$$W_{xy}\equiv 0.$$

Hereby we change the sum in the Hamiltonian (2.7) taken over W_{xy}. Respectively, in the definition of the Hamiltonian (2.7) we take the sum over all $x,y\in\mathbb{V}$ with $d(x,y)\le r$. Now we can replace the Assumption (**W**) by the following condition.

Assumption ($\mathbf{W_1}$). *For all $x,y\in\mathbb{V}$ with $d(x,y)\le r$, $\sigma(x),\sigma(y)\in\mathbb{R}^\nu$ and for some $J>0$, we have*

$$|W_{xy}(\sigma(x),\sigma(y))|\le J(C+|\sigma(x)|^R+|\sigma(y)|^R).$$

In this generalized situation the proofs we have presented go through with only small technical changes. We list here the main issues we have to change.

We define the *local Hamiltonian* as follows: Defining

$$E_\Lambda(\sigma_\Lambda):=\sum_{\substack{x,y\in\Lambda\\ d(x,y)\le r}}W_{xy}(\sigma(x),\sigma(y))+\sum_{x\in\Lambda}U_x(\sigma(x)),\tag{3.41}$$

we have

$$E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}):=E_\Lambda(\sigma_\Lambda)+\sum_{\substack{x\in\Lambda,y\in\Lambda^c\\ d(x,y)\le r}}W_{xy}(\sigma(x),\xi(y)).\tag{3.42}$$

Then, in all the theorems we make the following changes: The summations for $y \in \varphi(x)$ will be changed into $y \in B_r(x)$, where we define $B_r(x) := \{y \in \mathbb{V} \mid 0 < d(x,y) \leq r\}$ as in Definition 2.6. And m into m_r, where we define m as the maximum of $m_r(x)$, which is the number of vertices in $B_r(x)$, see Section 2.6. For the case of P=R we assume

$$A_1 > 2Jm_r.$$

3.6.2 Infinite range potentials

Here we remove the finite range condition, which means that we have no restrictions onto $x, y \in \mathbb{V}$. Hereby the sum in the Hamiltonian (2.7) is taken over all $x, y \in \mathbb{V}$. However, the local Hamiltonians (2.19) make only sense with the following supplementary condition. We assume that the variable interaction strength $J_{xy} \geq 0$ decreases while growing distance between vertices.

Assumption (J). *The matrix $\boldsymbol{J}$ is exponentially decreasing, that is, for every $\gamma \geq 0$,*

$$\|\boldsymbol{J}\|_\gamma := \sup_{x \in \mathbb{V}} \sum_{y \in \mathbb{V}} J_{xy} e^{\gamma d(x,y)} < \infty.$$

Now we change Assumption $(\mathbf{W_1})$ by skipping any restriction on $x, y \in \mathbb{V}$:

Assumption $(\mathbf{W_J})$. *For all $x, y \in \mathbb{V}$, $\sigma(x), \sigma(y) \in \mathbb{R}^\nu$, we have*

$$|W_{xy}(\sigma(x), \sigma(y))| \leq J_{xy}(C + |\sigma(x)|^R + |\sigma(y)|^R).$$

If we suppose that Assumption (J) holds and we are in the case of $P = R$ then we also have to change Assumption $(\bar{\mathbf{U}})$ taking into account that we have variable interaction strength J_{xy}. We then choose the A_1 large enough so that

$$A_1 > 2\|J\|_0. \tag{3.43}$$

And we impose a strictly positive stability parameter $\delta := A_1 - \|J\|_0$, which by (3.43) fulfills $\delta > \|J\|_0$. In this generalized situation we introduce the following change. We modify the definition of a local Gibbs specification, cf. Section 2.6, including the possibility for $\pi_\Lambda(d\sigma|\xi)$ to vanish if ξ do not belong to the subset Ω^t. For any finite subset $\Lambda \Subset \mathbb{V}$ we define the family of *stochastic kernels* $\pi_\Lambda : \mathcal{B}(\Omega) \times \Omega \to [0,1]$ by the following formula: For all $\xi \in \Omega$ and for $B \in \mathcal{B}(\Omega)$ we define

$$\pi_\Lambda(B|\xi) \\ := \begin{cases} \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\} \quad \mathbf{1}_B(\sigma_\Lambda, \xi_{\Lambda^c}) \prod_{x\in\Lambda} d\sigma(x), & \xi \in \Omega^t; \\ 0, & \xi \notin \Omega^t. \end{cases}$$

where $\mathbf{1}_B$ is the *indicator function* on B. Otherwise, the local Hamiltonians $E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ may be divergent for some $\xi \in \Omega$. This modification holds the DLR theory consistent. Especially, we have the following claims, describing the regularity properties of the kernels $\pi_\Lambda(d\sigma|\xi)$.

Lemma 3.15. *Let the Assumptions* $(\mathbf{W_J})$ *and* $(\bar{\mathbf{U}})$ *hold. Then for* $\Lambda \Subset \mathbb{V}$ *and* $\gamma > \gamma_0$*, the map* $\Omega_\gamma \times \Omega_\gamma \ni (\sigma, \xi) \longmapsto E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ *is continuous. Furthermore, for every finite radius ball* $B_\gamma(r) := \{\sigma \in \Omega_\gamma \mid \|\sigma\|_\gamma \le r\}$, $r > 0$*, it holds*

$$-\infty < \inf_{\substack{\sigma\in\Omega \\ \xi\in B_\gamma(r)}} E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}), \tag{3.44}$$

and

$$\sup_{\sigma,\xi\in B_\gamma(r)} |E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})| < +\infty. \tag{3.45}$$

Proof. Since the functions $W_{xy} : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}$ and $U_x : \mathbb{R}^\nu \to \mathbb{R}$ are continuous, the map $(\sigma, \xi) \longmapsto E_\Lambda(\sigma_\Lambda)$ is continuous and locally bounded. Moreover,

$$\begin{aligned} &|\sum_{y\in\Lambda^c} W_{xy}(\sigma(x), \xi(y))| \le \sum_{y\in\Lambda^c} |W_{xy}(\sigma(x), \xi(y))| \\ \le\ & \sum_{y\in\Lambda^c} J_{xy}(C + |\sigma(x)|^R) + \sum_{y\in\Lambda^c} J_{xy}|\xi(y)|^R \\ \le\ & \|J\|_0(C + |\sigma(x)|^R) + \|J\|_\gamma \|\xi_{\Lambda^c}\|_\gamma^R e^{\gamma d(x,y)}, \end{aligned} \tag{3.46}$$

where we used Assumption $(\mathbf{W_J})$. This yields the continuity of the map $(\sigma, \xi) \longmapsto E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ and the upper bound (3.45). To prove the lower bound we apply the Assumption ($\bar{\mathbf{U}}$). Then we have

$$\begin{aligned}
& E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}) \\
\geq \; & A_1 \sum_{x\in\Lambda} |\sigma(x)|^R + B_1|\Lambda| + \sum_{x,y\in\Lambda} W_{xy}(\sigma(x),\xi(y)) \\
+ \; & \sum_{x\in\Lambda, y\in\Lambda^c} W_{xy}(\sigma(x),\xi(y)) \\
\geq \; & (B_1 - 2C\|J\|_0)|\Lambda| + (A_1 - 2\|J\|_0) \sum_{x\in\Lambda} |\sigma(x)|^R \\
- \; & \|J\|_\gamma \|\xi_{\Lambda^c}\|_\gamma^R \sum_{x\in\Lambda} e^{\gamma d(x,y)},
\end{aligned} \tag{3.47}$$

which gives us the lower bound (3.44).

□

Using Lebesgue's dominated convergence theorem, a direct corollary of Lemma 3.15 is the following.

Corollary 3.16. *For all $\Lambda \Subset \mathbb{V}$, the partition function $\Omega_\gamma \ni \xi \mapsto Z_\Lambda(\xi)$ with values in the interval $(0,\infty)$ is continuous. Moreover, for any $r > 0$ and $\gamma > \gamma_0$,*

$$0 < \inf_{\xi\in B_\gamma(r)} Z_\Lambda(\xi) \leq \sup_{\xi\in B_\gamma(r)} Z_\Lambda(\xi) < \infty. \tag{3.48}$$

Lemma 3.17 (Feller property)**.** *For every $\Lambda \Subset \mathbb{V}$ and any $f \in C_b(\Omega_\gamma)$ the mapping*

$$\begin{aligned}
\Omega_\gamma \;\ni\; & \xi \mapsto \pi_\Lambda(f|\xi) \\
:= \; & \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\sigma_\Lambda|\xi_{\Lambda^c}) \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\} \prod_{x\in\Lambda} d\sigma(x)
\end{aligned} \tag{3.49}$$

belongs to $C_b(\Omega_\gamma)$, thus continuous and bounded in Ω_γ. Moreover, $f \mapsto \pi_\Lambda f$ is a contraction on $C_b(\Omega_\gamma)$.

Proof. For $f \in C_b(\Omega_\gamma)$ and $\xi \in \Omega_\gamma$ we define

$$\pi_\Lambda(f|\xi) := \int_{\Omega_\Lambda} F_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}) d\sigma_\Lambda, \tag{3.50}$$

where we set

$$F_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}) := \frac{1}{Z_\Lambda(\xi)} f(\sigma_\Lambda|\xi_{\Lambda^c}) \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\}$$

. By Lemma 3.15 and Corollary 3.16 the integrand $F_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ is continuous in both variables and the map

$$\Omega_\gamma \ni \xi \mapsto \sup_{\sigma_\Lambda \in \Omega_\Lambda} |F_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})| \tag{3.51}$$

is locally bounded. This allows us to use Lebesgue's dominated convergence theorem, which yields the continuity of $\xi \mapsto \pi_\Lambda(f \mid \xi)$. Apparently, we have

$$\|\pi_\Lambda f\|_{\sup} \leq \|f\|_{\sup}, \tag{3.52}$$

which completes the proof.

□

Now, in order to give an existence result for tempered Gibbs measures we can go through the same scheme as in Sections 3.3 and 3.4 with only small changes in the formulation of the Lemmas 3.3 and 3.5.

3.6.3 Multi-particle interactions

In this subsection we discuss in a few words a *multi-particle interaction* model with *infinite range* which is a further generalization of the model (2.7). The method described in the previous sections can be applied without any principal changes to the multi-particle interaction system. On a large scale, we define the interaction as a *family of potentials* $(W_\Delta)_{\Delta \Subset \mathbb{V}}$ indexed by all finite sets $\Delta \Subset \mathbb{V}$, where each $W_\Delta : (\mathbb{R}^\nu)^{|\Delta|} \to \mathbb{R}$ is a *continuous function.* They are invariant under permutations of its coordinates. For $\sigma \in \Omega^t$ the local Hamiltonian in the finite volume $\Lambda \Subset \mathbb{V}$ corresponding to the boundary condition $\xi \in \Omega^t$ on the complement Λ^c is then given by

$$E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c}) := \sum_{\substack{\Delta \Subset \mathbb{V} \\ \Delta \subset \Lambda}} W_\Delta(\sigma_\Delta) + \sum_{\substack{\Delta \Subset \mathbb{V} \\ \Delta\cap\Lambda \neq \emptyset \\ \Delta\cap\Lambda^c \neq \emptyset}} W_\Delta(\sigma_{\Delta\cap\Lambda}|\xi_{\Delta\cap\Lambda^c}). \tag{3.53}$$

Similarly to (2.17) – (2.20), we define the local Hamiltonians $E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})$ and the stochastic kernels $\pi_\Lambda(d\sigma|\xi)$ corresponding to the boundary conditions $\xi \in \Omega^t$. For this more general model all previous statements for the Gibbs measures $\mu \in \mathcal{G}^t$ apply with the following assumptions. We suppose the Assumption (**U**) holds for the self interaction potential $U_x := W_{\{x\}}$ and introduce a new assumption on the multi-particle interaction potential W_Δ with $|\Delta| \geq 2$. In order to describe a more realistic model we directly use variable interaction strength J_Δ of the multi-particle interaction which forces us to give an additional assumption.

Assumption ($\mathbf{W}_\Delta$). *There exist constants $R \geq 2$ and C, $J_\Delta \geq 0$, such that for each $\Delta \Subset \mathbb{V}$ with $|\Delta| \geq 2$ and for all $\sigma_\Delta = (\sigma(x_1), \cdots, \sigma(x_{|\Delta|})) \in (\mathbb{R}^\nu)^{|\Delta|}$, we have*

$$\left|W_\Delta(\sigma(x_1), \cdots, \sigma(x_{|\Delta|}))\right| \leq J_\Delta \Bigg(C + \sum_{n=1}^{|\Delta|} |\sigma(x_n)|^R \Bigg). \tag{3.54}$$

Assumption ($\mathbf{J}_\Delta$). *The interaction strength J_Δ is exponentially decreasing as the diameter of the sets Δ grow, that is, for every $\gamma \geq 0$*

$$|||\mathbf{J}|||_\gamma := \sup_{x_1 \in \mathbb{V}} \sum_{\substack{\Delta : x_1 \in \Delta \\ |\Delta| \geq 2}} J_\Delta e^{\gamma \max_{2 \leq n \leq |\Delta|} d(x_1, x_n)} < \infty. \tag{3.55}$$

As a main difference to the model (2.7) in the associated results there emerges a new matrix $\tilde{J} = (\tilde{J}_{x_1 x_2})_{\mathbb{V}^2}$ with the entries

$$\tilde{J}_{x_1 x_2} := \sum_{\substack{\Delta : \{x_1, x_2\} \subset \Delta \\ |\Delta| \geq 2}} J_\Delta \tag{3.56}$$

and norms $\|\tilde{\mathbf{J}}\|_\gamma \leq |||\mathbf{J}|||_\gamma$. This new matrix emerges in the following inequality which is necessary in the proof of Lemma 3.3: For all $\sigma, \xi \in \Omega^t$ we have

$$\begin{aligned} & \sum_{\substack{\Delta : x \in \Delta \\ |\Delta| \geq 2}} \left|W_\Delta(\sigma(x) | \xi_{\Delta \setminus \{x\}})\right| \\ \leq \; & \|\mathbf{J}\|_0 \left(C + |\sigma(x)|^R\right) + \sum_{y \neq x} \tilde{J}_{xy} \, |\xi(y)|^R \, . \end{aligned} \tag{3.57}$$

The existence results for tempered Gibbs measures differ only in the formulation of the exponential bound (3.11).

3.6.4 Interactions with heavy tails

In the recent paper of C. Roberto [Ro 2008] the so-called potentials of *sub-exponential growth* are considered. They introduce the self interaction potential $U : \mathbb{R} \to \mathbb{R}$ defined by $U(\sigma(x)) := |\sigma(x)|^P$, for $P \in (0, 1]$, which they call *sub-exponential like laws.* For the pair interaction potential W_{xy} they assume $\| W \|_\infty < \infty$, $\| W' \|_\infty < \infty$ and $\| W'' \|_\infty < \infty$, which is obviously too strong to ensure the existence result. As could be seen from the proof of Lemmas 3.3 and 3.5, it is enough to have for $0 < R < P \in (0,1]$ that

$$|W_{xy}(\sigma(x),\sigma(y))| \leq J(1+|\sigma(x)|^R+|\sigma(x)|^R).$$

The corresponding one-point reference measure $\mu(d\sigma(x))$ on $\mathbb{R}$ is given for $P \in (0,1]$ by the so-called sub-exponential laws

$$\mu(d\sigma(x)) = \frac{1}{Z}e^{-|\sigma(x)|^P}d\sigma(x),$$

where

$$Z := 2\Gamma(1+\frac{1}{P}).$$

Considering the model

$$\mu(d\sigma) = \frac{1}{Z}e^{-\sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} W_{xy}(\sigma(x),\sigma(y))}\prod_{x\in\mathbb{Z}^d}\mu(d\sigma(x))$$

with the corresponding one-point reference measure $\mu(d\sigma(x))$ we can ensure the existence of Gibbs measures without principle difficulties.

Chapter 4

Uniqueness problem

To describe the characteristics of a spin system, when it is in thermodynamical equilibrium, is one of the most important problems in statistical mechanics. One of the main issues is determining whether a spin system admits one or more states in thermodynamical equilibrium, that is, whether the system admits a *unique* or *multiple* Gibbs measures. In [Do 1970] R. L. Dobrushin was the first who gave a general sufficient condition for this issue, which has become widely known as the *Dobrushin uniqueness criterion*. Since we are dealing with unbounded spin systems this gives rise to regard the set of tempered Gibbs measures $\mathcal{G}^t$. In general, to show that the set $\mathcal{G}^t$ is a singleton one needs more detailed information about the structure of the interactions as compared with the assumptions which guarantee the existence of such measures. The case of classical systems on lattice $\mathbb{Z}^d$ was studied in [Kü 1982] by H. Künsch, [Gr 1979] by L. Gross and [Roy 1977] by G. Royer. Concerning the quantum lattice systems we also refer to [AlKoRö 2003] by S. Albeverio, Y. G. Kondratiev and M. Röckner, and [AlKoRöTs 1997a], [AlKoRöTs 1997b] and [AlKoRöTs 2000] by Albeverio, Y. G. Kondratiev, M. Röckner and T. V. Tsikalenko (Pasurek) and [Pa 2008] by T. Pasurek.

First of all we give the *Dobrushin uniqueness criterion* in Section 4.1 from which the desired uniqueness result follows. In Section 4.2 we give the main statement, see Theorem 4.2, for the uniqueness of tempered Gibbs measures. In Section 4.3 we apply this result to a ferromagnetic interaction potential given by the pair interaction potential $W_{xy}(\sigma(x), \sigma(y)) := w_{xy}(\sigma(x) - \sigma(y))$. After discussing possible extensions in Section 4.4 we present some concrete examples, which are *basic* for the whole manuscript. In the last section of this chapter we discuss a *generalized version* of the Dobrushin uniqueness criterion for unbounded spin spaces which involves the original criterion, see Section 4.6.

4.1 Dobrushin uniqueness criterion

In order to apply the *Dobrushin uniqueness criterion* we impose the following assumption on the self interaction potentials U_x.

Assumption $(\mathbf{U_2})$. *Additionally, to the Assumption* $(\mathbf{U})$ *we suppose that the self interaction potentials can be splitted into two terms of the form* $U_x := U_{x,1} + U_{x,2}$. *Here* $U_{x,1} \in C^2(\mathbb{R}^\nu)$ *is a* strictly convex function *which is* twice continuously differentiable *and* $U_{x,2} \in C_b(\mathbb{R}^\nu)$ *is continuous and globally bounded. Furthermore, this decomposition satisfies the following: We define the Hessian of* $U_{x,1}$, *which is a* $\nu \times \nu$*-symmetric matrix of second derivatives, as follows*

$$U''_{x,1}(s) := \partial_s^2 U_{x,1}(s) = \begin{pmatrix} \frac{\partial^2 U_{x,1}(s)}{\partial s_1 \partial s_1} & \cdots & \frac{\partial^2 U_{x,1}(s)}{\partial s_1 \partial s_\nu} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U_{x,1}(s)}{\partial s_\nu \partial s_1} & \cdots & \frac{\partial^2 U_{x,1}(s)}{\partial s_\nu \partial s_\nu} \end{pmatrix},$$

where we denote $s := (s_i)_{i=1}^\nu \in \mathbb{R}^\nu$. *We assume that* $U''_{x,1}(s)$ *is a* uniformly positive definite *matrix, i.e., there exists a constant* $a > 0$ *so that for all* $x \in \mathbb{V}$ *and* $\varphi, s \in \mathbb{R}^\nu$

$$(U''_{x,1}(s)\varphi, \varphi) \geq a^2 |\varphi|^2 \tag{4.1}$$

and that

$$\delta := \sup_{x \in \mathbb{V}} \delta(U_{x,2}) < \infty,$$

where we define the total oscillation *by*

$$\delta(U_{x,2}) := \sup_{s \in \mathbb{R}^\nu} U_{x,2}(s) - \inf_{s \in \mathbb{R}^\nu} U_{x,2}(s).$$

In the subsequent we will give the original Dobrushin uniqueness criterion (cf. Theorem 4 in [Do 1970]) adjusted to our setting. The Dobrushin criterion is based on a comparison of the measures $\pi_x(d\sigma|\xi)$, for different boundary conditions ξ, in the *Wasserstein distance*, which we already introduced in Section 2.7. We recall this distance related to the *Euclidean metric* $|\cdot|$ on $\mathbb{R}^\nu$:

$$W(\pi_x(d\sigma|\xi), \pi_x(d\sigma|\eta)) := \sup_{f \in \mathrm{Lip}_1(\mathbb{R}^\nu)} \left| \int_{\mathbb{R}^\nu} f \pi_x(d\sigma|\xi) - \int_{\mathbb{R}^\nu} f \pi_x(d\sigma|\eta) \right|,$$

where

$$\mathrm{Lip}_1(\mathbb{R}^\nu) := \left\{ f : \mathbb{R}^\nu \to \mathbb{R} \,\middle|\, [f] := \sup_{\sigma(x) \neq \sigma'(x)} \frac{|f(\sigma(x)) - f(\sigma'(x))|}{|\sigma(x) - \sigma'(x)|} \leq 1 \right\}$$

is the *unit ball* in the space of Lipschitz continuous functions on $\mathbb{R}^\nu$.
The *Dobrushin interdependence matrix* $\mathcal{D} = (\mathcal{D}_{xy})_{x,y\in\mathbb{V}}$ is then defined by

$$\mathcal{D}_{xy} := \sup_{\substack{\xi,\eta\in\Omega, z\in\mathbb{V} \\ \xi(z)=\eta(z)\forall z\neq y}} \left\{ \frac{W(\pi_x(d\sigma|\xi), \pi_x(d\sigma|\eta))}{|\xi(y)-\eta(y)|} \right\},$$

for all $x \neq y$ and by $\mathcal{D}_{xy} = 0$ for $x = y$. Then the *Dobrushin uniqueness criterion* states the following:

Theorem 4.1 (**Dobrushin uniqueness criterion**). *Suppose that the Dobrushin matrix $\mathcal{D} = (\mathcal{D}_{xy})_{x,y\in\mathbb{V}}$ is $l^\infty(\mathbb{V})$-contractive, i.e.,*

$$\|\mathcal{D}\| := \sup_{x\in\mathbb{V}} \sum_{y\in\mathbb{V}} \mathcal{D}_{xy} < 1. \tag{4.2}$$

Then the set of all Gibbs measures $\mathcal{G}$ such that

$$\sup_{x\in\mathbb{V}} \int_{\mathbb{R}^\nu} |\sigma(x)|\mu(d\sigma(x)) < \infty$$

is a singleton.

4.2 Uniqueness for tempered Gibbs measures

In this section we consider the same interaction potential as in Section 2.4. We only remind the reader that we are in the situation of the general pair interaction potential

$$W_{xy}(\sigma(x), \sigma(y)), \tag{4.3}$$

for *nearest neighbors*. We assume the following conditions on this potential, which will give us the *positive definiteness* of the matrix of its second partial derivatives.

Assumption ($\mathbf{W_2}$). *Additionally to the Assumption* ($\mathbf{W}$), *we suppose that all $W_{xy} \in C^2(\mathbb{R}^\nu \times \mathbb{R}^\nu)$ and assume the following: First of all, we fix $t \in \mathbb{R}^\nu$ and look at the function $\mathbb{R}^\nu \ni s \mapsto W_{xy}(s,t)$, and define the $\nu\times\nu$ - symmetric matrix (Hessian)*

$$\partial_s^2 W_{xy}(s,t), \quad x,y \in \mathbb{V}.$$

For our considerations this matrix should be uniformly positive definite, *that is, there exists a finite $b_-^2 > 0$ such that for all s, $t \in \mathbb{R}^\nu$ and for all $\varphi \in \mathbb{R}^\nu$, we have*

$$(\partial_s^2 W_{xy}(s,t)\varphi, \varphi) \geq J b_-^2 |\varphi|^2, \quad x,y \in \mathbb{V}.$$

Now for arbitrary $s, t \in \mathbb{R}^\nu$, we also introduce the corresponding second derivative of the function $W_{xy} : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}$,

$$\partial^2_{st} W_{xy}(s,t), \quad x, y \in \mathbb{V}.$$

We impose the following assumption on $\partial^2_{st} W_{xy}(s,t)$: There exists a finite $b_+ > 0$ such that for all s, $t \in \mathbb{R}^\nu$ and for all $\varphi \in \mathbb{R}^\nu$,

$$\sup_{s,t \in \mathbb{R}^\nu} \left| (\partial^2_{st} W_{xy}(s,t), \varphi) \right| \le J b_+^2 |\varphi|^2, \quad x, y \in \mathbb{V}.$$

Together with these assumptions the next theorem gives us a condition on the parameters of the system from which the validity of Dobrushin uniqueness criterion follows. The main theorem of this chapter is:

Theorem 4.2. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{b_+^2 e^{2\beta\delta}}{b_-^2 + \frac{a^2}{Jm}} < 1. \tag{4.4}$$

Then

$$|\mathcal{G}^t| = 1.$$

Proof. By Theorems 3.7 and 3.8, the set $\mathcal{G}^t$ is nonempty and all its elements obey the moment estimates (3.22). We use the Dobrushin uniqueness criterion and consider the one-point probability kernels $\pi_x(d\sigma|\xi)$. Using (2.27) we have

$$\begin{aligned} & \pi_x(d\sigma|\xi) \\ = \ & \frac{1}{Z_x(\xi)} \exp\{-\beta E_x(\sigma(x)|\xi)\} d\sigma(x) \\ = \ & \frac{1}{Z_x(\xi)} \exp\Big\{ -\beta U_x(\sigma(x)) - \beta \sum_{y \in \varphi(x)} W_{xy}(\sigma(x), \xi(y)) \Big\} d\sigma(x), \end{aligned} \tag{4.5}$$

where $Z_x(\xi)$ is the normalizing constant. Our aim is to check that the Dobrushin matrix $\mathcal{D} := (\mathcal{D}_{xy})_{x,y\in\mathbb{V}}$ fulfills the following condition

$$\|\mathcal{D}\| = \sup_{x \in \mathbb{V}} \sum_{y \in \mathbb{V}} \mathcal{D}_{xy} < 1. \tag{4.6}$$

To this end, we show that under conditions imposed on the potentials we have the following estimate for all $x, y \in \mathbb{V}$

$$\mathcal{D}_{xy} < \frac{Jb_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x)b_-^2}, \tag{4.7}$$

which together with (4.4) gives us (4.2). The idea of proving (4.7) is strongly motivated by [AlKoRöTs 2000].

Given some $f \in \mathrm{Lip}_1(\mathbb{R}^\nu)$, we define a mapping

$$\mathbb{R}^\nu \ni \xi(y) \mapsto F(\xi(y)) := \langle f \rangle_{\pi_x(d\sigma|\xi)} := \int_{\mathbb{R}^\nu} f(\sigma(x))\pi_x(d\sigma|\xi)$$

for $x, y \in \mathbb{V}$, $x \neq y$. As usual $\mathrm{Cov}_{\pi_x(d\sigma|\xi)}(f,g)$ and $\mathrm{Var}_{\pi_x(d\sigma|\xi)}(f)$ denote the *covariance* and *variance* with respect to the measure $\pi_x(d\sigma|\xi)$, i.e.,

$$\mathrm{Cov}_{\pi_x(d\sigma|\xi)}(f,g) := \langle fg \rangle_{\pi_x(d\sigma|\xi)} - \langle f \rangle_{\pi_x(d\sigma|\xi)} \langle g \rangle_{\pi_x(d\sigma|\xi)}$$

$$\mathrm{Var}_{\pi_x(d\sigma|\xi)}(f) := \int_{\mathbb{R}^\nu} (f - \langle f \rangle_{\pi_x(d\sigma|\xi)})^2 d\pi_x(d\sigma|\xi). \tag{4.8}$$

Under our assumptions, the function $F : \mathbb{R}^\nu \to \mathbb{R}$ is Fréchet differentiable. We calculate its partial derivative in direction $\varphi \in \mathbb{R}^\nu$

$$\begin{aligned}
(\nabla F(\xi(y)), \varphi) &= \frac{1}{Z_x^2(\xi)} \Bigg[-\beta \int_{\mathbb{R}^\nu} f(\sigma(x)) \left(\frac{\partial W_{xy}(\sigma(x), \xi(y))}{\partial \xi(y)}, \varphi \right) \\
&\quad \times e^{-\beta E_x(\sigma(x)|\xi)} d\sigma(x) Z_x(\xi) \\
&\quad + \beta \int_{\mathbb{R}^\nu} \left(\frac{\partial W_{xy}(\sigma(x), \xi(y))}{\partial \xi(y)}, \varphi \right) e^{-\beta E_x(\sigma(x)|\xi)} d\sigma(x) \\
&\quad \times \int_{\mathbb{R}^\nu} f(\sigma(x)) e^{-\beta E_x(\sigma(x)|\xi)} d\sigma(x) \Bigg] \\
&= -\beta \cdot \mathrm{Cov}_{\pi_x(d\sigma(x)|\xi)} \left(f(\sigma(x)), \left(\frac{\partial W_{xy}(\sigma(x), \xi(y))}{\partial \xi(y)}, \varphi \right) \right),
\end{aligned}$$

which can be estimated by the Cauchy-Schwarz inequality by

$$\begin{aligned}
|(\nabla F(\xi(y)), \varphi)| &\leq \beta \left(\mathrm{Var}_{\pi_x(d\sigma|\xi)} f(\sigma(x)) \right)^{1/2} \\
&\quad \times \left(\mathrm{Var}_{\pi_x(d\sigma|\xi)} \left(\frac{\partial W_{xy}(\sigma(x), \xi(y))}{\partial \xi(y)}, \varphi \right) \right)^{1/2}.
\end{aligned} \tag{4.9}$$

Let us first assume that $U_{x,2} = 0$, which implies by Assumption ($\mathbf{U_2}$) and ($\mathbf{W_2}$) that

$$\begin{aligned}(\partial^2_{\sigma(x)} E_x(\sigma(x) \mid \xi)\varphi, \varphi) &= (U''_x(\sigma(x))\varphi, \varphi) + \sum_{y \in \varphi(x)} (\partial^2_{\sigma(x)} W_{xy}(\sigma(x), \xi(y))\varphi, \varphi) \\ &\geq a^2|\varphi|^2 + \sum_{y \in \varphi(x)} J b_-^2 |\varphi|^2 \\ &= [a^2 + Jm(x)b_-^2] \cdot |\varphi|^2 > 0. \end{aligned} \tag{4.10}$$

In other words, the measure $\pi_x(d\sigma|\xi)$ is log-concave. This enables us to estimate the variances in (4.9) by the *Poincaré inequality* and the Corollary 1.4 in [Le 2001]. The Poincaré inequality is given for all $f \in C_b^1(\mathbb{R}^\nu)$ by

$$\mathrm{Var}_{\pi_x(d\sigma|\xi)}(f) \leq \frac{1}{C_{SG}} \int_{\mathbb{R}^\nu} |f'(\sigma(x))|^2 \pi_x(d\sigma|\xi). \tag{4.11}$$

This inequality is valid uniformly for all $\pi_x(d\sigma|\xi)$ with the *spectral gap* constant C_{SG}, which by (4.10) and Corollary 1.4 in [Le 2001] fulfills the estimate

$$C_{SG} \geq \beta[a^2 + Jm(x)b_-^2]. \tag{4.12}$$

Then the Poincaré inequality standardly leads, for all $f \in \mathrm{Lip}_1(\mathbb{R}^\nu)$, to the estimate

$$\begin{aligned}\mathrm{Var}_{\pi_x(d\sigma|\xi)}(f) &\leq \frac{[f]^2}{C_{SG}} \\ &\leq \frac{1}{\beta(a^2 + Jm(x)b_-^2)}[f]^2 \\ &\leq \frac{1}{\beta(a^2 + Jm(x)b_-^2)}. \end{aligned} \tag{4.13}$$

Especially, by Assumption ($\mathbf{W_2}$) we have

$$\begin{aligned} &\mathrm{Var}_{\pi_x(d\sigma|\xi)}\left(\frac{\partial W_{xy}(\sigma(x), \xi(y))}{\partial \xi(y)}, \varphi\right) \\ \leq\ & \frac{1}{C_{SG}} \int_{\mathbb{R}^\nu} |(\partial^2_{\sigma(x)\xi(y)} W_{xy}(\sigma(x), \xi(y)), \varphi)|^2 \pi_x(d\sigma|\xi) \\ \leq\ & \frac{1}{\beta(a^2 + Jm(x)b_-^2)} \sup_{\sigma(x), \xi(y) \in \mathbb{R}^\nu} (\partial^2_{\sigma(x)\xi(y)} W_{xy}(\sigma(x), \xi(y)), \varphi)^2 \\ \leq\ & \frac{J^2 b_+^4}{\beta(a^2 + Jm(x)b_-^2)}. \end{aligned} \tag{4.14}$$

Adding a bounded potential $U_{x,2}$ with the total oscillation $\delta(U_{x,2}) < \infty$ we obtain by the well-known *perturbation argument* of Lemma 1.2 in [Le 2001] the extra factor $e^{2\beta\delta(U_{x,2})}$ in the constant C_{SG}. Hence, (4.9) implies that

$$\begin{aligned} |(\nabla F(\xi(y)), \varphi)| &\leq \frac{\beta J b_+^2 e^{2\beta\delta(U_{x,2})}}{C_{SG}} \\ &\leq \frac{J b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2} \end{aligned}$$

and then the *mean-value theorem* gives for any $f \in \mathrm{Lip}_1(\mathbb{R}^\nu)$ and for all $\xi, \eta \in \Omega$ such that $\xi = \eta$ off $z \in \mathbb{V}$,

$$\begin{aligned} \left| \int f\ \pi_x(d\sigma|\xi) - \int f\ \pi_x(d\sigma|\eta) \right| &\leq \frac{\beta J b_+^2 e^{2\beta\delta(U_{x,2})}}{C_{SG}} |\xi(y) - \eta(y)| \\ &\leq \frac{J b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2} |\xi(y) - \eta(y)|, \end{aligned}$$

or in terms of the Wasserstein distance

$$\begin{aligned} W(\pi_x(d\sigma|\xi), \pi_x(d\sigma|\eta)) &\leq \frac{\beta J b_+^2 e^{2\beta\delta(U_{x,2})}}{C_{SG}} |\xi(y) - \eta(y)|, \\ &\leq \frac{J b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2} |\xi(y) - \eta(y)|. \end{aligned}$$

After having checked (4.7), we are in the position to check the sufficient condition (4.2) of the Dobrushin uniqueness criterion. Using the estimate (4.7) we obtain that, for $y \sim x$,

$$\begin{aligned} \mathcal{D}_{xy} &\leq \frac{\beta J b_+^2 e^{2\beta\delta(U_{x,2})}}{C_{SG}} \\ &\leq \frac{J b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2}. \end{aligned}$$

Therefore we get

$$\begin{aligned} \sum_{y \in \mathbb{V}} \mathcal{D}_{xy} &\leq \sum_{y \in \mathbb{V}} \frac{J b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2} \\ &= \frac{Jm(x) b_+^2 e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x) b_-^2} \\ &= \frac{b_+^2 e^{2\beta\delta(U_{x,2})}}{\frac{a^2}{Jm(x)} + b_-^2}. \end{aligned}$$

Since $m := \sup_{x\in\mathbb{V}} m(x) < \infty$, the latter implies

$$\sup_{x\in\mathbb{V}} \sum_{y\in\mathbb{V}} \mathcal{D}_{xy} < \frac{b_+^2 e^{2\beta\delta}}{\frac{a^2}{Jm} + b_-^2}.$$

Now, using the assumption (4.4) of the theorem, we see that

$$\sup_{x\in\mathbb{V}} \sum_{y\in\mathbb{V}} \mathcal{D}_{xy} < 1$$

and conclude then that $|\mathcal{G}^t| = 1$.

$\square$

We would like to give a comment on the condition (4.4) of the latter theorem. This assumption holds for the next four situations: If the oscillation δ is equal to zero; if β is small enough, which means that we have a high temperature; if J is small enough, which means that we have a small intensity of the interaction; or if a^2 is large enough, which means that we have a convex self interaction potential $U_{x,1}$. In order to fix these ideas we give a corollary.

Corollary 4.3. *Consider the spin system* (2.7) *for nearest neighbor pair interaction potentials satisfying the Assumptions* ($\mathbf{U_2}$) *and* ($\mathbf{W_2}$)*. Then, for every* $\beta_0 > 0$ *there exists a number* $\mathcal{J} := \mathcal{J}(\beta_0)$ *such that* $|\mathcal{G}^t| = 1$ *for all* $\beta \le \beta_0$ *and* $J < \mathcal{J}$*.*

4.3 General ferromagnetic interaction potentials

Let us consider the following pair interaction potentials

$$W_{xy}(\sigma(x), \sigma(y)) := w_{xy}(\sigma(x) - \sigma(y)),$$

so that $w_{xy} : \mathbb{R}^\nu \to \mathbb{R}$ are smooth convex functions with a condition on their second derivatives as follows: There exist finite b_-^2, $b_+^2 > 0$ and $J > 0$ such that for every $q \in \mathbb{R}^\nu$

$$Jb_-^2 \cdot \mathrm{Id}_\nu \le w''_{xy}(q) \le Jb_+^2 \cdot \mathrm{Id}_\nu,$$

where Id_ν is the identity matrix of the order $\nu \times \nu$. The following statement is a corollary of Theorem 4.2.

Corollary 4.4. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{b_+^2 e^{2\beta\delta}}{\frac{a^2}{Jm} + b_-^2} < 1.$$

Then

$$|\mathcal{G}^t| = 1.$$

Proof. The statement is just a particular case of Theorem 4.2.

Example 4.5. *The simplest example of a pair interaction potential where the uniqueness criterion holds is the following harmonic one: We define, for $x \sim y$, that*

$$W_{xy}(\sigma(x), \sigma(y)) := \frac{J}{2} |\sigma(x) - \sigma(y)|^2 \tag{4.15}$$

with intensity $J > 0$. In this case we have $b_-^2 = b_+^2 = 1$ so that

$$\mathcal{D}_{xy} < \frac{J e^{2\beta\delta(U_{x,2})}}{a^2 + Jm(x)}$$

with the sufficient condition

$$\|\mathcal{D}\| \leq \frac{Jme^{2\beta\delta}}{a^2 + Jm} < 1. \tag{4.16}$$

The uniqueness of $\mu \in \mathcal{G}^t$ comes true by choosing sufficiently small one of the following parameters, of course, with dependence on the other fixed parameters in condition (4.16)*: the inverse temperature β, the intensity of the pair interaction J or the total oscillation δ.*

Corollary 4.6. *Suppose that the parameters of the considered system* (4.15) *satisfy the inequality*

$$\frac{e^{2\beta\delta}}{\frac{a^2}{mJ} + 1} < 1.$$

Then

$$|\mathcal{G}^t| = 1.$$

4.4 Extensions

A possible extension can be achieved if we consider the *finite range* potentials, see Section 3.6, $W_{xy}(\sigma(x), \sigma(y))$ such that for $d(x, y) > r$ it follows that $W_{xy} \equiv 0$. Then we have the following theorem with a slightly different sufficient condition for the Dobrushin uniqueness criterion.

Theorem 4.7. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{b_+^2 e^{2\beta\delta}}{b_-^2 + \frac{a^2}{\|J\|_0}} < 1,$$

where we have

$$\|\boldsymbol{J}\|_0 := \sup_{x\in\mathbb{V}} \sum_{y\in\mathbb{V}} J_{xy} < \infty.$$

Then

$$|\mathcal{G}^t| = 1.$$

The proof is only a similar repetition of the proof of Theorem 4.2. Here we have the very useful corollary:

Corollary 4.8. *Consider the spin system* (2.7) *with the finite range pair interaction potentials satisfying the Assumptions* $(\mathbf{U_2})$ *and* $(\mathbf{W_2})$. *Then, for every* $\beta_0 > 0$ *there exists a number* $\mathcal{J} := \mathcal{J}(\beta_0)$ *such that* $|\mathcal{G}^t| = 1$ *for all* $\beta \le \beta_0$ *and* $\|J\|_0 < \mathcal{J}$.

Further extensions can be achieved for *infinite range potentials* and *multiparticle interaction potentials.*

4.5 Basic Examples

Now we present our *basic* examples for the self interaction potentials satisfying the Assumption $(\mathbf{U_2})$. These examples will be also the main objects for the application of the so-called Wells' inequality to prove possible phase transitions, see Section 5.4.3.

4.5.1 φ^4-potential

Let us consider the scalar case, i.e. $\nu = 1$. We fix parameters J, $\beta > 0$ and suppose that the self interaction potential $U_x : \mathbb{R} \to \mathbb{R}$ is given by the following double-well potential

$$U_x(\sigma) = \sigma^4 - t^2\sigma^2, \quad \sigma \in \mathbb{R}.$$

We show that for all $|t| < t_0$ the self interaction potential satisfies the conditions of the Dobrushin's uniqueness theorem. With this aim we now decompose U_x as follows:

$$U_x(\sigma) = U_{x,1}(\sigma) + U_{x,2}(\sigma)$$

with

$$U_{x,1}(\sigma) = \begin{cases} U_x(\sigma), & |\sigma| \geq |t|; \\ \frac{2}{3t^2}(\sigma^6 - t^6) - \sigma^2(\sigma^2 - t^2), & |\sigma| < |t|, \end{cases}$$

and

$$U_{x,2}(\sigma) = \begin{cases} 0, & |\sigma| \geq |t|; \\ -\frac{2}{3t^2}(\sigma^6 - t^6) + 2\sigma^2(\sigma^2 - t^2), & |\sigma| < |t|. \end{cases}$$

Now we would like to check that

$$\inf_{\sigma\in\mathbb{R}} U''_{x,1}(\sigma) = \frac{t^2}{5}.$$

For $|\sigma| \leq |t|$ we have

$$\begin{aligned} U'_{x,1}(\sigma) &= \frac{4}{t^2}\sigma^5 - 4\sigma^3 + 2t^2\sigma. \\ U''_{x,1}(\sigma) &= \frac{20}{t^2}\sigma^4 - 12\sigma^2 + 2t^2. \\ U'''_{x,1}(\sigma) &= \frac{80}{t^2}\sigma^3 - 24\sigma. \end{aligned}$$

We now solve $U'''_{x,1}(\sigma) = 0$ and obtain $\sigma = \sqrt{\frac{3}{10}}t$, which we put into $U''_{x,1}$ and calculate

$$\begin{aligned} \inf_{\sigma\in\mathbb{R}} U''_{x,1}(\sigma) &= U''_{x,1}\left(\sqrt{\frac{3}{10}}t\right) \\ &= \frac{20}{t^2}\frac{9}{100}t^4 - 12\frac{3}{10}t^2 + 2t^2 \\ &= \frac{20}{100}t^2 \\ &= \frac{t^2}{5}. \end{aligned}$$

It is also easy to observe that

$$\delta(U_{x,2}) \leq 2\sup_{\sigma\in\mathbb{R}} |U_{x,2}(\sigma)| \leq \frac{7}{3}t^4.$$

One can write the uniqueness condition (4.17) as

$$\begin{aligned} e^{\beta\delta(U_{x,2})} &< 1 + \frac{a^2}{m_r J} \\ &\leq 1 + \frac{\inf_{\sigma\in\mathbb{R}} U''_{x,1}(\sigma)}{m_r J} \\ &= 1 + \frac{t^2}{5m_r J}. \end{aligned} \tag{4.17}$$

On the other hand, for $|t| > 0$ so small such that $\beta\delta(U_{x,2}) \leq \frac{7}{3}\beta t^4 < 1$, we have the estimate

$$e^{\beta\delta(U_{x,2})} \leq 1 + \frac{7}{3}\beta t^4 + \left(\frac{7}{3}\beta t^4\right)^2.$$

This means that (4.17) holds as soon as

$$\frac{7}{3}\beta t^4 + \left(\frac{7}{3}\beta t^4\right)^2 < \frac{t^2}{5m_r J},$$

which is true for all $|t| < t_0(J, \beta)$.

4.5.2 General φ^4-potential

We again fix parameters J, $\beta > 0$ and suppose that $U_x : \mathbb{R} \to \mathbb{R}$ is a φ^4-potential of the form

$$U_x(\sigma) = s\sigma^4 - t\sigma^2, \quad \sigma \in \mathbb{R}, \quad s, t > 0. \tag{4.18}$$

This potential has a double-well shape with the minima $\sigma = \pm\sqrt{\frac{t}{2s}}$. The depth of the wells is equal to

$$|\inf_{\sigma\in\mathbb{R}} U_x(\sigma)| = \frac{t^2}{4s}.$$

With a simple transformation of the decomposition of Example 1 for the potential (4.18) we obtain a sufficient condition for the uniqueness of the corresponding Gibbs measures of the form

$$\frac{7t^2}{3s}\beta + \left(\frac{7t^2}{3s}\beta\right)^2 < \frac{t}{5m_r J}. \tag{4.19}$$

Here we observe that the uniqueness can be obtained in case of a sufficiently small depth of the wells, which is proportional to $\frac{t^2}{s}$.

Space-scaling of the general φ^4-potential

At this point we can also consider the space-scaling of the above potential (4.18), that is

$$U_x^\epsilon(\sigma) := U_x(\epsilon^{-1}\sigma) = \frac{s}{\epsilon^4}\sigma^4 - \frac{t}{\epsilon^2}\sigma^2, \quad \sigma \in \mathbb{R}, \quad s, t, \epsilon > 0.$$

The uniqueness condition (4.19) refers now to the form

$$\frac{7t^2}{3s}\beta + \left(\frac{7t^2}{3s}\beta\right)^2 < \frac{t}{5m_r J\epsilon^2}.$$

This condition is always valid for small enough $\epsilon > 0$. Fixing the depth of the wells, by making ϵ small, we get the wells of the potential (4.18) coming close together. As a result we have a rigorous proof of the well-known physical phenomenon that pressure can remove the critical behavior of the system.

4.5.3 General φ^{2n}-potential

We fix parameters $n \geq 2$, $t > 0$ and suppose that the self interaction potential $U_x : \mathbb{R} \to \mathbb{R}$ is given by the formula

$$U_x(\sigma) = \sigma^{2n} - t^{2n-2}\sigma^2, \quad \sigma \in \mathbb{R}.$$

We can decompose the latter potential as follows:

$$U_{x,1}(\sigma) = \begin{cases} U_x(\sigma), & |\sigma| \geq t; \\ \frac{2(n-1)}{t^2(n+1)}(\sigma^{2n+2} - t^{2n+2}) - \sigma^{2n} + \sigma^2 t^{2n-2}, & |\sigma| < t, \end{cases}$$

and

$$U_{x,2}(\sigma) = \begin{cases} 0, & |\sigma| \geq t; \\ -\frac{2(n-1)}{t^2(n+1)}(\sigma^{2n+2} - t^{2n+2}) + 2\sigma^{2n} - 2\sigma^2 t^{2n-2}, & |\sigma| < t. \end{cases}$$

From these formulas it is easy to see that we have for all $n \geq 2$ and $\sigma \in \mathbb{R}$ estimates for the convexity of $U_{x,1}$ and the oscillation of $U_{x,2}$ of the form

$$\inf_{\sigma\in\mathbb{R}} U''_{1,x}(\sigma) \geq C_1 t^{2n-2},$$

$$\delta(U_{x,2}) \leq C_2 t^{2n}$$

with some constants $C_1, C_2 > 0$. Then the uniqueness condition (4.17) can be written as

$$e^{C_2\beta t^{2n}} < 1 + \frac{C_1 t^{2n-2}}{m_r J}.$$

For $t < 1$ we satisfy the uniqueness condition by choosing n big enough. In this case, we make the depth of the wells (which is proportional to t^{2n}) small for big n.

4.6 Generalized Dobrushin uniqueness criterion

In this section we discuss some results in the literature concerning the famous *Dobrushin uniqueness criterion.* Along with Dobrushin's formulation, we will state a more generalized version. Originally, this criterion only concerns *one-point* probability kernels $\pi_x(d\sigma|\xi)$ of the local specification Π, which describes the influence of a site on another site, see [Do 1970] by R. L. Dobrushin and Sections 3.2 and 4.1. In a later work, by R. L. Dobrushin and S. B. Shlosman [DoSh 1985] they gave a more general condition which describe the influence of larger volumes, also known as *blocks*, on other volumes. However, other than the original Dobrushin condition, their condition is not applicable to the case when the underlying structure is a general graph $G(\mathbb{V}, \mathbb{E})$. It is only applicable to the lattice $\mathbb{Z}^d$, where they use crucially the *translation invariance* property. Other versions of the *Dobrushin-Shlosman condition* were given by D. W. Stroock and B. Zegarlinski in [StZe 1992], but still in the context of $\mathbb{Z}^d$. Recently, further extensions appear in [W 2005] by D. Weitz, in [WiTa 2006] by S. Winkler and S. Tatikonda and in [ZhZh 2008] by H. Zhou and Z. Zheng. In [W 2005] the author introduced the influence of sites on each other via blocks which was extended in [WiTa 2006], where they introduce the influence on blocks affected by the change of sites. A further extension was done in [ZhZh 2008] where they show the influence of blocks to blocks. The same principal conditions in these papers are that they only regard *compact spin spaces.* Our main aim in this section will be to extend this to any *Polish spaces* X_i, $i \in \mathbb{I}$, and any underlying index set $\mathbb{I}$. In this subsection we extend Dobrushin's uniqueness criterion and give new insights into the underlying theory. In order to do that we strongly use the *Wasserstein probability distance* corresponding to the metrics $\rho_i(\sigma_i, \tilde{\sigma}_i)$.

4.6.1 The abstract model

The abstract formulation on the problem is as follows: Let $\mathbb{I}$ be a *countably infinite, metrizable discrete index set* with a metric d. To each $i \in \mathbb{I}$, let us attach a *Polish space* (X_i, ρ_i) with the metric ρ_i. Then the *configuration space* is defined as an infinite product

$$X := \prod_{i \in \mathbb{I}} X_i.$$

Let us denote the finite subsets of $\mathbb{I}$, as usual, by Λ. We write $\Lambda \Subset \mathbb{I}$ whenever Λ is not empty and the cardinality $|\Lambda| < \infty$. Then we define for $\Lambda \Subset \mathbb{I}$ the corresponding *local configuration space*

$$X_\Lambda := \prod_{i \in \Lambda} X_i.$$

The elements $\sigma_i \in X_i$ are called *(single) spin variables* and the sequences $\sigma := (\sigma_i)_{i \in \mathbb{I}} \in X$ respectively $\sigma_\Lambda := (\sigma_i)_{i \in \Lambda} \in X_\Lambda$ are called *configurations*. We consider for all $\gamma > \gamma_0$ (with $\gamma_0 > 0$ to be specified below) the following weighted Polish spaces

$$X_\gamma := \left\{ \sigma \in X \,\middle|\, \sum_{i \in \mathbb{I}} \rho_i(\sigma_i, o_i) e^{-\gamma d(i_0, i)} < \infty \right\}$$

with the metrics

$$\rho_\gamma(\sigma, \tilde{\sigma}) := \sum_{i \in \mathbb{I}} \rho_i(\sigma_i, \tilde{\sigma}_i) e^{-\gamma d(i_0, i)},$$

constructed for some fixed initial points $o_i \in X_i$ and $i_0 \in \mathbb{I}$. Additionally, for all $\gamma > \gamma_0$ and for this (and hence for each) initial point $i_0 \in \mathbb{I}$, we assume the *summability condition*

$$\mathbb{I}_\gamma := \sum_{i \in \mathbb{I}} e^{-\gamma d(i_0, i)} < \infty. \tag{4.20}$$

Then the set of (exponentially) *tempered configurations* is defined as

$$X^t := \bigcap_{\gamma > \gamma_0} X_\gamma.$$

We endow X with the *product topology* and with the corresponding Borel σ-algebra $\mathcal{B}(X)$. Recall that the product topology is the weakest topology such that all finite volume projections

$$X \ni \sigma \to \mathbb{P}_\Lambda \sigma := \sigma_\Lambda := (\sigma_i)_{i \in \Lambda} \in X_\Lambda, \quad \Lambda \Subset \mathbb{I},$$

are continuous, and the Borel σ-algebra $\mathcal{B}(X)$ coincides with the σ-algebra generated by all cylinder sets

$$\{\sigma \in X \mid \sigma_\Lambda \in B_\Lambda\}, \quad B_\Lambda \in \mathcal{B}(X_\Lambda), \quad \Lambda \Subset \mathbb{I}.$$

Let $\mathcal{P}(X)$, $\mathcal{P}(X_\Lambda)$ and $\mathcal{P}(X^t)$ denote the set of all *probability measures* respectively on $(X, \mathcal{B}(X))$, $(X_\Lambda, \mathcal{B}(X_\Lambda))$ and $(X^t, \mathcal{B}(X^t))$. By $\mathcal{P}(X^t)$ we denote the subset of *tempered probability measures* supported by X^t, which means

$$\mathcal{P}(X^t) := \{\mu \in \mathcal{P}(X) | \mu(X^t) = 1\}. \tag{4.21}$$

Remark 4.9. *Each space X_γ strongly depends on the choice of the initial sequence $(o_i)_{i\in\mathbb{I}}$, but the choice of the initial point $i_0 \in \mathbb{I}$ is not relevant.*

Definition 4.10. *Let $\Pi := \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{I}}$ be a family of* measure kernels *with the following properties:*

- *For all fixed $\xi \in X^t$, $\pi_\Lambda(d\sigma|\xi)$ is a* probability measure *supported by X^t and for all fixed $\xi \notin X^t$, $\pi_\Lambda(d\sigma|\xi) \equiv 0$.*
- $X \ni \xi \mapsto \pi_\Lambda(B|\xi)$ measurable function *for each fixed $B \in \mathcal{B}(X)$ and $\Lambda \Subset \mathbb{I}$.*
- *It satisfies the* consistency property *(see [Gi 1969, Pr 1976]): For all $\Lambda \subset \Lambda' \Subset \mathbb{I}$ we have*

$$\pi_{\Lambda'}\pi_\Lambda = \pi_{\Lambda'}, \tag{4.22}$$

which means that for all $B \in \mathcal{B}(X)$ and $\xi \in X$

$$\int_X \pi_\Lambda(B|\sigma(x))\pi_{\Lambda'}(d\sigma|\xi) = \pi_{\Lambda'}(B|\xi). \tag{4.23}$$

Then we call $\Pi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{I}}$ a local specification.

Remark 4.11. *In the previous sections we dealt with concrete specifications constructed via local Hamiltonians, see Section 2.6. Our definition of specification generalizes the corresponding definitions in [Gi 1969] and [Pr 1976]. In contrast to the previous literature the main new situation here is that the interaction may have infinite range. In particular, the local Hamiltonians $E_\Lambda(\sigma|\xi)$ may not be defined for some $\xi \in X$, see Subsection 3.6.2. To ensure furthermore the DLR framework, in Definition 4.10 we defined $\pi_\Lambda(d\sigma|\xi) \equiv 0$ for $\xi \notin X^t$. Hence we only consider tempered Gibbs measures $\mu \in \mathcal{G}^t$, which are supported by the tempered configurations $\sigma \in X^t$.*

Definition 4.12. *A probability measure μ on $(X, \mathcal{B}(X))$ is called a* Gibbs measure *for the* local specification $\Pi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{I}}$, *if it satisfies the* DLR equilibrium equations*: For all $\Lambda \Subset \mathbb{I}$, we have that*

$$\mu\pi_\Lambda = \mu.$$

More precisely, this means the following for all $B \in \mathcal{B}(X)$

$$\int_X \pi_\Lambda(B|\xi)\mu(d\xi) = \mu(B). \tag{4.24}$$

For fixed $\beta > 0$, we denote by $\mathcal{G}$ the set of all Gibbs measures corresponding to the specification Π. Then the subset of tempered Gibbs measures *$\mathcal{G}^t$ consists of all $\mu \in \mathcal{G}$ which are supported by X^t, that means*

$$\mathcal{G}^t := \mathcal{G} \cap \mathcal{P}(X^t). \tag{4.25}$$

Remark 4.13. *As already discussed, in many reasonable cases all information about the measures $\mu \in \mathcal{G}^t$ could be gained from the family of their* one-point probability kernels *$\pi_{\{i\}}(d\sigma|\xi)$, which is enough to prove the uniqueness of such Gibbs measures. For the sake of simplicity we write $\pi_i(d\sigma|\xi)$. Let us define, for $i \in \mathbb{I}$ and $\xi \in X^t$, the one-point projections as*

$$\mu_i(d\sigma_i|\xi) := \pi_i(d\sigma|\xi) \circ \mathbb{P}_i^{-1} \in \mathcal{P}(X_i),$$

where $\mathcal{P}(X_i)$ is the set of probability measures on X_i.

In the following we generalize the Dobrushin uniqueness criterion from Section 4.1 to this new setting. The Dobrushin criterion is based on a comparison of the measures $\mu_i(d\sigma_i|\xi)$, for $i \in \mathbb{I}$ and different boundary conditions ξ, in the *Wasserstein probability distance*, see Section 2.7. We introduce this distance related to the metrics $\rho_i(\sigma_i, \tilde{\sigma}_i)$ on X_i by

$$\begin{aligned} &\mathbf{W}_{\rho_i}(\mu_i(d\sigma_i|\xi), \mu_i(d\sigma_i|\eta) := \\ &\inf_{P \in \mathcal{C}(\mu_i(d\sigma_i|\xi), \mu_i(d\sigma_i|\eta))} \int_{X_i^2} \rho_i(\sigma, \tilde{\sigma}) P(d\sigma, d\tilde{\sigma}). \end{aligned} \tag{4.26}$$

So, the *Dobrushin interdependence matrix* $\mathcal{D} := (\mathcal{D}_{ij})_{i,j \in \mathbb{I}}$ is then defined, for all $i \neq j$, by

$$\mathcal{D}_{ij} := \sup_{\substack{\xi, \eta \in X^t, \\ \xi_k = \eta_k \forall k \neq j}} \left\{ \frac{W_{\rho_i}(\mu_i(d\sigma_i|\xi), \mu_i(d\sigma_i|\eta))}{\rho_j(\xi_j, \eta_j)} \right\}, \tag{4.27}$$

where the supremum is taken only over the tempered $\xi, \eta \in X^t$ and for all $i = j$ by

$$\mathcal{D}_{ij} := 0.$$

Naturally, we should assume for $\xi \in X$ and $i \in \mathbb{I}$ that

$$\int_{X_i^2} \rho_i(\sigma_i, o_i) \mu_i(d\sigma_i|\xi) < \infty.$$

4.6.2 Main theorem

Theorem 4.14 (Generalized Dobrushin uniqueness criterion)**.** *Let us assume the contraction condition*

$$\|\mathcal{D}\|_\gamma := \sup_{i\in\mathbb{I}} \sum_{\substack{j\in\mathbb{I}\\ j\neq i}} \mathcal{D}_{ij} e^{\gamma d(i,j)} < 1. \tag{4.28}$$

Then there is at most one tempered Gibbs measure $\mu \in \mathcal{G}^t$, such that

$$\sum_{i\in\mathbb{I}} \Big(\int_X \rho_i(\sigma_i, o_i)\mu(d\sigma) \Big) e^{-\gamma d(i,i_o)} < \infty. \tag{4.29}$$

Remark 4.15. *The condition* (4.28) *guarantees, for $\gamma > \gamma_0$, that the matrix $\mathcal{D} := (\mathcal{D}_{ij})_{i,j\in\mathbb{I}}$ generates a linear bounded operator in each of the Banach spaces*

$$l^1_\gamma(\mathbb{I}) := \left\{ \sigma = (\sigma_i)_{i\in\mathbb{I}} \in X \,\middle|\, \|\sigma\|_{l^1_\gamma} := \sum_{i\in\mathbb{I}} |\sigma_i| e^{-\gamma d(i_0,i)} < \infty \right\},$$

$$l^\infty_\gamma(\mathbb{I}) := \left\{ \sigma = (\sigma_i)_{i\in\mathbb{I}} \in X \,\middle|\, \|\sigma\|_{l^\infty_\gamma} := \sup_{i\in\mathbb{I}} (|\sigma_i| e^{-\gamma d(i_0,i)}) < \infty \right\}.$$

Note that $\|\mathcal{D}\|_\gamma$ is just the operator norm of the matrix $\mathcal{D}$ in $l^\infty_\gamma(\mathbb{I})$.

Proof of Theorem 4.14. Let us assume that there exist two different Gibbs measures $\mu, \tilde{\mu} \in \mathcal{G}^t$ obeying (4.29). We will estimate the Wasserstein distance in (X_γ, ρ_γ) between the measures μ and $\tilde{\mu}$ (see Section 2.7)

$$\mathbf{W}_{\rho_\gamma}(\mu, \tilde{\mu}) := \inf_{P\in\mathcal{C}(\mu,\tilde{\mu})} \int_{X^2_\gamma} \rho_\gamma(\sigma, \tilde{\sigma}) P(d\sigma, d\tilde{\sigma}), \tag{4.30}$$

where the infimum is taken over all couplings $P \in \mathcal{C}(\mu, \tilde{\mu})$. Our aim is to show that $\mathbf{W}_{\rho_\gamma}(\mu, \tilde{\mu}) = 0$ which then implies $\mu = \tilde{\mu}$. Let $P \in \mathcal{C}^*(\mu, \tilde{\mu})$ be an optimal coupling, which exists by Section 2.7 (ii). This implies

$$\mathbf{W}_{\rho_\gamma}(\mathbb{P}\mu, \mathbb{P}\tilde{\mu}) = \sum_{i\in\mathbb{I}} M_i e^{-\gamma d(i_0,i)}, \tag{4.31}$$

where we define for all $i \in \mathbb{I}$

$$M_i := \int_{X^2_\gamma} \rho_i(\sigma_i, \tilde{\sigma}_i) P(d\sigma, d\tilde{\sigma}).$$

The central issue is to verify that the Dobrushin condition (4.27) implies that for all $i \in \mathbb{I}$

$$M_i \leq \sum_{\substack{j\in\mathbb{I}\\ i\neq j}} D_{ij} M_j. \tag{4.32}$$

Let us suppose for a moment that (4.32) holds. Note that by our construction the vector $M := (M_i)_{i\in\mathbb{I}}$ belongs to the Banach space $l^\infty_\gamma(\mathbb{I})$ defined in Remark 4.15. The matrix $\mathcal{D} = (D_{ij})_{i,j\in\mathbb{I}}$ generates a bounded operator in $l^\infty_\gamma(\mathbb{I})$ whose norm does not exceed $\|\mathcal{D}\|_\gamma$. Then we get by iteration of (4.32) that for all $N \in \mathbb{N}$ and $(\mathcal{D}^N M)_i := \sum_{j\in\mathbb{I}} D^{(N)}_{ij} M_j$ it holds

$$M_i \leq (\mathcal{D}M)_i \leq (\mathcal{D}^N M)_i, \tag{4.33}$$

where $D^{(N)}_{ij}$ are the entries of the N-th power of the matrix $\mathcal{D}$. Using this result we get

$$\begin{aligned}
\|M\|_{l^\infty_\gamma} &= \sup_{i\in\mathbb{I}}(M_i e^{-\gamma d(i_o,i)}) \\
&\leq \sup_{i\in\mathbb{I}}(\sum_{j\in\mathbb{I}} D^{(N)}_{ij} M_j e^{-\gamma d(i_o,i)}) \\
&= \sup_{i\in\mathbb{I}}((\mathcal{D}^N M)_i e^{-\gamma d(i_o,i)}) \\
&= \|\mathcal{D}^N M\|_{l^\infty_\gamma}.
\end{aligned} \tag{4.34}$$

This immediately implies for all $N \in \mathbb{N}$ that

$$\begin{aligned}
\|M\|_{l^\infty_\gamma} &\leq \|\mathcal{D}^N\|_\gamma \|M\|_{l^\infty_\gamma} \\
&\leq \|\mathcal{D}\|^N_\gamma \|M\|_{l^\infty_\gamma}
\end{aligned} \tag{4.35}$$

Since, by assumption $\|\mathcal{D}\|_\gamma < 1$, we have that $\lim_{N\to\infty}\|\mathcal{D}\|^N_\gamma = 0$, which proves that $M_i = 0$ for all $i \in \mathbb{I}$. So, $\mu = \tilde{\mu}$.

It remains to verify (4.32). Fixing some $i \in \mathbb{I}$, we apply to $P(d\sigma, d\tilde{\sigma})$ the Dobrushin's *reconstruction procedure*, which first appeared in [Do 1968] and [Do 1970] for discrete spin spaces. The possibility of such reconstruction in our case comes from Section 2.7, see also Remark 4.17. As a result of this reconstruction we get a new measure $\tilde{P} \in \mathcal{P}(\mu, \tilde{\mu})$ with the same marginals $\mathbb{P}\mu$ and $\mathbb{P}\tilde{\mu}$, which we define as follows. Let

$$X^2_\gamma \ni (\xi, \eta) \to \pi^*_i(d\sigma_i d\tilde{\sigma}_i|\xi, \eta) \in \mathcal{C}^*(\mu_i(d\sigma_i|\xi), \mu_i(d\sigma_i|\eta)) \tag{4.36}$$

be a *measurable* mapping in (ξ, η) so that

$$\int_{X^2_i} \rho_i(\sigma_i, \tilde{\sigma}_i)\pi^*_i(d\sigma_i d\tilde{\sigma}_i|\xi, \eta) = \mathbf{W}_{\rho_i}(\mu_i(d\sigma_i|\xi), \mu_i(d\sigma_i|\eta). \tag{4.37}$$

Since $X_\gamma \ni \xi \to \mu_i(d\sigma_i|\xi) \in \mathcal{P}(X_i)$ is a measurable mapping by Definition 4.10, (4.36) of the optimal coupling between $\mu_i(d\sigma_i|\xi)$ and $\mu_i(d\sigma_i|\eta)$ does exist by Section 2.7 (iv),(v). Then $\tilde{P}$ is uniquely determined by the duality

$$\int_{X_\gamma^2} f(\sigma,\tilde{\sigma})\tilde{P}(d\sigma,d\tilde{\sigma}) :=$$
$$\int_{X_\gamma^2}\left(\int_{X_i^2} f(\sigma_i \times \xi_{\{i\}^c}, \tilde{\sigma}_i \times \eta_{\{i\}^c})\pi_i^*(d\sigma_i d\tilde{\sigma}_i|\xi,\eta)\right)P(d\xi,d\eta), \quad (4.38)$$

which holds on all bounded continuous cylinder functions $f \in C_b(X_\Lambda \times X_\Lambda)$ with $\Lambda \Subset \mathbb{I}$. Combining (4.37) and (4.38) we obtain for all $j \neq i$ that

$$M_j = \tilde{M}_j. \quad (4.39)$$

Using (4.27) we obtain

$$\begin{aligned} \tilde{M}_i &\leq \sum_{\substack{j\in\mathbb{I}\\ j\neq i}} D_{ij}\int_{X^2}\rho_j(\xi_j,\eta_j)\tilde{P}(d\sigma,d\tilde{\sigma}) \\ &= \sum_{\substack{j\in\mathbb{I}\\ j\neq i}} D_{ij}\tilde{M}_j. \end{aligned} \quad (4.40)$$

On the other hand,

$$\sum_{j\in\mathbb{I}} \tilde{M}_j e^{-\gamma d(i_0,i)} \geq \mathbf{W}_{\rho_\gamma}(\mathbb{P}\mu,\mathbb{P}\tilde{\mu}) = \sum_{j\in\mathbb{I}} M_j e^{-\gamma d(i_0,i)}, \quad (4.41)$$

together with (4.39) and (4.40) this yields the required estimate

$$M_i \leq \tilde{M}_i \leq \sum_{\substack{j\in\mathbb{I}\\ j\neq i}} D_{ij}M_j. \quad (4.42)$$

□

Remark 4.16. *Indeed, with* (4.35), *the following weaker assumption is sufficient for the uniqueness*

$$r_{sp}(\mathcal{D}) = \lim_{N\to\infty}\|D^N\|_\gamma^{\frac{1}{N}} \leq \|\mathcal{D}\|_\gamma < 1, \quad (4.43)$$

where $r_{sp}(\mathcal{D})$ is the spectral radius *of the operator $\mathcal{D}$ in $l_\gamma^\infty(\mathbb{I})$. Nevertheless,* (4.28) *is more convenient for applications, since it can be easily verified in terms of the Dobrushin coefficients D_{ij}.*

Remark 4.17. *Let us comment on Dobrushin's reconstruction procedure. On the easier setting of discrete spin spaces, which was the case in [Do 1970], we do not need any measurability properties for the optimal couplings. However, in a further work of Dobrushin and Shlosman in [DoSh 1985], they assumed in a more general situation the existence of the optimal measurable coupling but without any evidence. In general, particularly for unbounded spins, the existence of the optimal coupling, which should be measurable in ξ, η, is not clear. For clarification we refer to Section 2.7.*

4.7 Interactions with unbounded intensity

In this section we give a non-trivial example of applying the generalized version of Dobrushin's criteria. We consider an interacting system of scalar spins on a lattice $\mathbb{Z}^d \subset \mathbb{R}^d$, $d \geq 2$, equipped with the Euclidean distance $|\cdot|$. As the *configuration space* we define the space of all sequences over $\mathbb{Z}^d$

$$\mathbb{R}^{\mathbb{Z}^d} := \{\sigma = (\sigma(x))_{x\in\mathbb{Z}^d} | \sigma : \mathbb{Z}^d \to \mathbb{R}\}. \tag{4.44}$$

A new issue is that this lattice model provides points which have very strong interaction strength with their neighbors. We will call such points *heavy points*. Besides the usual attractive pair interaction with strength $J > 0$ between all neighbors in $\mathbb{Z}^d$ there appear an additional interaction between the heavy points and their neighbors. Let us collect the heavy points in a sequence of the following form $\{o_i\}_{i\in\mathbb{N}} \subset \mathbb{Z}^d$. The main technical assumption on this sequence is that for all $i, j \in \mathbb{N}$ we have

$$d(o_i, o_j) > 3. \tag{4.45}$$

This assumption means that the heavy points are not directly connected with each other. To each $o_i \in \mathbb{Z}^d$ there corresponds an additional interaction with the strength $I_i \in (0, \infty)$, where in general $\sup_{i\in\mathbb{N}} I_i = +\infty$. Let us enumerate the neighbors of each o_i by the sequence $\{\gamma_{i,n}\}_{n=1}^{2d} := (\gamma_{i,1}, \cdots, \gamma_{i,2d})$. For convenience we set $\gamma_{i,0} := o_i$. The model we are dealing with is a *ferromagnetic system* with quadratic pair interaction. But a principle difference from the previous considerations is that I_i are not uniformly bounded. The system is described by the following heuristic *infinite-volume energy functional*

(Hamiltonian)

$$\begin{aligned} E(\sigma) &:= \sum_{x\in\mathbb{Z}^d} U_x(\sigma(x)) + \frac{J}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} (\sigma(x)-\sigma(y))^2 \\ &+ \frac{1}{2}\sum_{i\in\mathbb{N}} I_i \sum_{n=1}^{2d} (\sigma(o_i)-\sigma(\gamma_{i,n}))^2. \end{aligned} \tag{4.46}$$

We define the spaces of tempered configurations as usual by

$$\Omega^t := \bigcap_{\gamma>0} \Omega_\gamma,$$

where

$$\Omega_\gamma := \left\{ \sigma \in \mathbb{R}^{\mathbb{Z}^d} \,\middle|\, \sum_{x\in\mathbb{Z}^d} |\sigma(x)| e^{-\gamma|x|} < \infty \right\}.$$

By $\mathcal{P}(\Omega^t)$ we denote the subset of tempered measures supported by Ω^t, i.e.,

$$\mathcal{P}(\Omega^t) := \{\mu \in \mathcal{P}(\Omega) | \mu(\Omega^t) = 1\}. \tag{4.47}$$

Respectively, the subset of tempered Gibbs measures $\mathcal{G}^t$ consists of all $\mu \in \mathcal{G}$ which are supported by Ω^t, i.e.,

$$\mathcal{G}^t := \mathcal{G} \cap \mathcal{P}(\Omega^t). \tag{4.48}$$

Since, we cannot define directly the infinite-volume Gibbs measure we define as usually the local Gibbs measures in finite volumes corresponding to the boundary condition ξ. The Gibbs measure is defined by the DLR equilibrium equations, see Definition 4.12. The existence problem for this system will be discussed below in Section 5.2. We note that the standard Dobrushin technique we used in Section 3 is not applicable because of unboundedness of the sequence I_i, $i \in \mathbb{N}$.

In this section we discuss the uniqueness problem of such Gibbs measures. In this case we impose as usually some additional assumptions for the self interaction potentials U which are similar to those in Section 4.1. By analogy with the previous Assumption ($\mathbf{U_2}$), we suppose that the self interaction potentials can be splitted into two terms of the form

$$U := U_1 + U_2,$$

where $U_1 \in C^2(\mathbb{R})$ is a *twice continuously differentiable*, *strictly convex function* and $U_2 \in C_b(\mathbb{R})$ is *continuous* and *globally bounded*. More precisely, for the second derivative of U_1, we assume that there exists a constant $a > 0$ such that

$$U_1''(s) \geq a^2,$$

for all $s \in \mathbb{R}$. Furthermore we define the *total oscillation* of U_2 by

$$\delta := \delta(U_2) := \sup_{s\in\mathbb{R}} U_2(s) - \inf_{s\in\mathbb{R}} U_2(s) < \infty.$$

The standard Dobrushin uniqueness criterion from Section 4.1 is however not applicable to this new model. The reason can be briefly explained as follows.

Considering the local Gibbs measure for *isolated points* x, i.e. those which are not direct neighbors of any heavy point o_i, provides no difficulties. In this case we have for some $i \in \mathbb{N}$ such that $d(x, o_i) = 2$ the following one point probability measure

$$\begin{aligned} \mu(d\sigma(x)|\xi) &= \frac{1}{Z} \exp\Bigg\{ - \beta U(\sigma(x)) - \beta \sum_{\substack{y\in\partial x \\ y_{isolated}}} \frac{J}{2}(\sigma(x) - \xi(y))^2 \\ &- \beta \sum_{y\in\partial x\cap\partial o_i} \frac{J}{2}(\sigma(x) - \xi(y))^2 \Bigg\} d\sigma(x), \end{aligned} \tag{4.49}$$

which gives us the standard Dobrushin matrix coefficient (see the proof of Theorem 4.2) obeying the following bound

$$\mathcal{D}_{xy} \leq \frac{Je^{2\beta\delta}}{a^2 + 2dJ}. \tag{4.50}$$

Then the Dobrushin uniqueness criterion can be achieved by the sufficient condition

$$\sup_{x_{isolated}} \sum_{y\in\partial x} \mathcal{D}_{xy} \leq \frac{e^{2\beta\delta}}{\frac{a^2}{2dJ} + 1} < 1. \tag{4.51}$$

This condition surely holds for appropriate parameters of the system (e.g. for small β or J as well as large a^2). But if we consider all points $x \in \mathbb{Z}^d$, including the heavy points o_i, this sufficient condition cannot be fulfilled by any choice of the parameters β, J or a^2. This is because in (4.51) instead of J we have $J + I_i$ with the additional interaction strength I_i, which is unbounded. Precisely, if x is a heavy point, i.e., $x = o_i$ such that $d(x, y) = 1$, then

$$\mathcal{D}_{xy} \leq \frac{(J + I_i)e^{2\beta\delta}}{a^2 + 2d(J + I_i)}. \tag{4.52}$$

And we get

$$\sup_{x_{heavy}} \sum_{y \in \partial x} \mathcal{D}_{xy} \leq \sup_{i \in \mathbb{N}} \frac{e^{2\beta\delta}}{\frac{a^2}{2d(J+I_i)} + 1} = e^{2\beta\delta} > 1, \tag{4.53}$$

where the supremum is taken over all heavy points $x = o_i$. Since $I_i \nearrow \infty$ for any positive $\delta > 0$ this term is even bigger than one. The sufficient condition of Dobrushin is not fulfilled. To overcome this problem we will try to apply the modification of Dobrushin's criteria stated in Section 4.6, see Theorem 4.14.

Now we will give the main theorem of this chapter. The proof uses the same concepts as in the proof of Theorem 4.2 to get the Dobrushin coefficient matrix, but at the first step we should construct a special partition of the lattice $\mathbb{Z}^d$ and the configuration space $\mathbb{R}^{\mathbb{Z}^d}$. The main and unexpected point here is that the sufficient condition is independent of I_i. Let us now give the theorem:

Theorem 4.18. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{2d^2 J e^{2(2d+1)\beta\delta}}{a^2} < 1. \tag{4.54}$$

Then independently of the additional interaction intensities I_i the set $\mathcal{G}^t$ is a singleton, i.e.,

$$|\mathcal{G}^t| = 1.$$

Proof. The main idea is to arrange a decomposition of $\mathbb{Z}^d$ on isolated points and proper blocks including heavy points and their neighbors. First we construct the corresponding *blocks* V_i. Each block V_i is composed of a center $o_i \in V_i$ and their neighbors, that means V_i has $2d + 1$-points. Let us collect these points as $\gamma_i := \{\gamma_{i,n}\}_{n=0}^{2d}$, where we define $\gamma_{i,0} := o_i$. All other points from $\mathbb{Z}^d \setminus \bigcup_{i \in \mathbb{N}} V_i$ are called *isolated.* The corresponding block spin variables we denote by $\sigma(\gamma_i) := (\sigma(\gamma_{i,0}), \cdots, \sigma(\gamma_{i,2d})) \in \mathbb{R}^{2d+1}$. We can represent the configurations in the following way:

$$(\sigma(x))_{x \in \mathbb{Z}^d} \cong (\sigma(x))_{\substack{x \in \mathbb{Z}^d \\ x_{isolated}}} \cup \left\{ \bigcup_{i \in \mathbb{N}} \sigma(\gamma_i) \right\}.$$

Then the set of *(exponentially) tempered configurations* is defined as

$$X^t := \bigcap_{\gamma > 0} X_\gamma, \tag{4.55}$$

where we define

$$
\begin{aligned}
X_\gamma \quad := \quad & \Bigg\{\sigma \in \mathbb{R}^{\mathbb{Z}^d} \Bigg| \sum_{\substack{x\in\mathbb{Z}^d \\ x_{isolated}}} |\sigma(x)| e^{-\gamma|x|} \\
& + \sum_{i\in\mathbb{N}} \left[\sum_{n=0}^{2d} |\sigma(\gamma_{i,n})|\right] e^{-\gamma|o_i|} < \infty \Bigg\}. \qquad (4.56)
\end{aligned}
$$

By this definition we see that the sets X_γ and Ω_γ coincide. Then obviously we can generalize the notion of the Gibbs specification $\tilde{\pi}_\Lambda(d\sigma|\xi)$, where the finite volumes Λ will be constructed of isolated points and blocks. It is clear that the uniqueness of Gibbs measures for such specification will imply the uniqueness of the Gibbs measures for the initial lattice system.

Our aim is to check the Dobrushin criteria from Theorem 4.14 and to give the Dobrushin matrix coefficient for the blocks V_i and one-points. Given a boundary condition $\xi \in \Omega$, the corresponding local Gibbs measure for the block V_i is

$$
\mu(d\sigma(\gamma_i)|\xi) := \frac{1}{Z_i(\xi)} \exp\{-\beta E(\sigma(\gamma_i))\} \prod_{n=0}^{2d} d\sigma(\gamma_{i,n}), \qquad (4.57)
$$

where

$$
\begin{aligned}
E(\sigma(\gamma_i)) \quad := \quad & \sum_{n=0}^{2d} U(\sigma(\gamma_{i,n})) \\
& + \frac{\beta}{2}(J + I_i) \sum_{n=0}^{2d} (\sigma(\gamma_{i,0}) - \sigma(\gamma_{i,n}))^2 \\
& + \beta\frac{J}{2} \sum_{n=1}^{2d} \sum_{\substack{l_m\in\partial\gamma_{i,n} \\ l_m \neq o_i}} (\sigma(\gamma_{i,n}) - \xi(l_m))^2, \qquad (4.58)
\end{aligned}
$$

is the corresponding local Hamiltonian which describes interactions between each single block and the boundary configuration ξ. Here l_m are the neighbors of $\gamma_{i,n}$ outside of the block V_i. If x is an isolated point, the corresponding one-point Gibbs measures are given by (4.49). The main issue is to estimate the Dobrushin's coefficients $\mathcal{D} := (\mathcal{D}_{V_i,l_m})$, for each i and m which are defined by

$$
\mathcal{D}_{V_i,l_m} := \sup_{\substack{\xi,\eta\in\Omega^t, l_n\in\mathbb{Z}^d \\ \xi(l_n)=\eta(l_n)\forall l_n\neq l_m}} \left\{ \frac{W(\mu(d\sigma(\gamma_i)|\xi), \mu(d\sigma(\gamma_i)|\eta))}{|\xi(l_m) - \eta(l_m)|} \right\}. \qquad (4.59)
$$

To this end, we claim that under conditions imposed on the potentials we have the following estimate for all $i, m \in \mathbb{N}$

$$\mathcal{D}_{V_i, l_m} < \frac{2dJe^{2(2d+1)\beta\delta}}{a^2}, \tag{4.60}$$

which gives us (for appropriate parameters J, a^2 and β of the system) the sufficient condition

$$\sup_{i \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathcal{D}_{V_i, l_m} < 1. \tag{4.61}$$

Let us first assume that $U_2 = 0$, which implies by the convexity assumptions imposed on U and the Hessian matrix for $E(\sigma(\gamma_i))$ that

$$\partial^2_{\sigma(\gamma_i)} E(\sigma(\gamma_i)|\xi) \quad \geq \quad a^2 > 0. \tag{4.62}$$

This is the case because the Hessian matrix for the second and third sums of $E(\sigma(\gamma_i))$ in (4.58) is positive semi definite and therefore negligible in getting the estimate (4.60), see the proof of Theorem 4.2. This is the main issue of our proof since here we drop any dependence on the interaction strength I_i. Then by [Le 2001], the spectral gap constant for the measure (4.57) is given by

$$C_{SG} \geq \beta a^2, \tag{4.63}$$

uniformly for all boundary condition ξ. Adding bounded potentials $U_2 = U_2(\sigma(\gamma_i))$, $0 \leq i \leq 2d$, with the total oscillation $\delta < \infty$ leads by the well-known *perturbation argument* (see Lemma 1.2 in [Le 2001]) to the extra factor $e^{2(2d+1)\beta\delta}$. Then we get for the spectral gap constant

$$C_{SG} \geq e^{-2(2d+1)\beta\delta} \beta a^2.$$

Now we follow the scheme of the proof of Theorem 4.2. With the Poincaré inequality and the *mean-value theorem* we have then the following bound for the Wasserstein distance whenever $\xi = \eta$ off l_m

$$\begin{aligned} W(\mu(d\sigma(\gamma_i)|\xi), \mu(d\sigma(\gamma_i)|\eta)) \quad &\leq \quad \frac{\beta 2dJ}{C_{SG}} |\xi(l_m) - \eta(l_m)| \\ &\leq \quad \frac{2dJe^{2(2d+1)\beta\delta}}{a^2} |\xi(l_m) - \eta(l_m)|, \end{aligned}$$

which readily implies (4.60). After having checked (4.60), we are in the position to check the sufficient condition of the Dobrushin uniqueness criterion

(4.61). Using the estimate (4.60) we obtain, for $l_m \in \partial\gamma_{i,n}$, that

$$\begin{aligned}\sum_m \mathcal{D}_{V_i,l_m} &< \sum_m \frac{2dJe^{2(2d+1)\beta\delta}}{a^2} \\ &= \frac{2d^2Je^{2(2d+1)\beta\delta}}{a^2}.\end{aligned}$$

In this estimate we take into account that every point $\gamma_{i,n}$, $0 \leq n \leq 2d$, could have at most d neighbors l_m outside the block. The last estimation implies

$$\sup_{i\in\mathbb{N}} \sum_m \mathcal{D}_{V_i,l_m} < \frac{2d^2Je^{2(2d+1)\beta\delta}}{a^2} < 1. \tag{4.64}$$

Now, using the assumption (4.54) of the theorem, we see that the right hand side of the last inequality is smaller than one.

As we already mentioned before, for isolated points x we have the standard Dobrushin matrix coefficient (4.50). For the isolated points x the Dobrushin uniqueness criterion can be achieved by the sufficient condition

$$\sup_{x_{isolated}} \sum_{y\in\partial x} \mathcal{D}_{xy} \leq \frac{e^{2\beta\delta}}{\frac{a^2}{2dJ}+1} < 1, \tag{4.65}$$

which is obviously weaker than (4.64). Hence, having (4.64) we immediately conclude that $|\mathcal{G}^t| = 1$.

□

The last Theorem can surely be extended to the case of graphs $G(\mathbb{V},\mathbb{E})$ with $m := \sup_{x\in\mathbb{V}} m(x)$, where $m(x)$ is the number of edges coming to the vertex x (number of its nearest neighbors). Then the Theorem generalizes in the following way:

Theorem 4.19. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{m^2Je^{2(m+1)\beta\delta}}{a^2} < 1. \tag{4.66}$$

Then independently of the additional interaction intensities I_i the set $\mathcal{G}^t$ is a singleton, i.e.,

$$|\mathcal{G}^t| = 1.$$

Remark 4.20. *(i) The conditions* (4.54) *and* (4.66) *can be surely obtained by choosing* β *or* J *small enough, or* a^2 *big enough. One should also mention that these conditions are depending on the geometry of the system, that means* (4.54) *and* (4.66) *depends on the number of nearest neighbors* $2d$ *respectively* m.
(ii) In principle, we can consider the sequence of heavy points o_i*, obeying* $d(o_i, o_j) > 2$. *Sufficient condition for the uniqueness are still given by Theorems 4.18 and 4.19.*

Chapter 5

Ferromagnetic models

In Section 5.1 we introduce the *model* under consideration and some important *tools*, which are specific for attractive interactions. Then in Section 5.2 we give an existence result for ferromagnetic systems on general graphs. A new important issue of this section is that we consider any graph $G(\mathbb{V}, \mathbb{E})$ with possibly *unbounded degree*. The proof uses the so-called *right- and left-dominators*, which can be also found in [OsSp 1999] by H. Osada and H. Spohn, in a rather different context. Section 5.3 is devoted to the *uniqueness problem*, where we consider *ferromagnetic models* for the case of *one dimensional continuous spins*. With some important tools of Section 5.1 we give a so-called *comparison* criterion, in Subsection 5.3.2, for uniqueness and phase transitions of Gibbs measures. There we compare the initial model with certain *reference* models. In Section 5.4 we present a new method showing phase transition in *unbounded* spin systems. Since this result is based on the classical Ising model we open this Section with an introduction on the famous Ising model, see Subsection 5.4.1. In Subsection 5.4.3 we introduce a special correlation inequality which seems to be very useful for studying phase transitions. It was first discovered in D. Wells PhD thesis [We 1977] and is called *Wells' inequality*. This inequality provides us an elementary new way showing phase transitions. In Subsection 5.4.4 we consider the basic examples of Section 4.5 and give concrete thresholds so that the Wells' inequality holds. Our main Theorem 5.29 gives sufficient conditions for phase transitions in φ^4 models on a *general tree*.

5.1 Preliminary constructions

In this preliminary section we introduce all the necessary tools for the subsequent sections. In Section 5.1.1 we present the ferromagnetic model we are considering and necessary assumptions for the existence of tempered Gibbs measures in the set $\mathcal{G}^t$. Then, in Section 5.1.2, we present the so-called *correlation inequalities*, e.g. FKG, GKS, Griffiths, Brascamp-Lieb, which are needed for both existence and uniqueness results. In Section 5.1.3 we will define a *partial order* on $\mathcal{G}^t$.

5.1.1 The model and assumptions

Let us introduce a standard ferromagnetic system with scalar spins $\sigma(x) \in \mathbb{R}$ described by the following formal Hamiltonian

$$E(\sigma) = -\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} J_{xy}\sigma(x)\sigma(y) + \sum_{x\in\mathbb{V}} U_x(\sigma(x)), \tag{5.1}$$

where U_x are the *self interaction potentials* and, for $x \neq y$, $J_{xy} \geq 0$ is the *interaction strength* between each x and y. In our context we set for the *harmonic pair interaction* $W_{xy}(\sigma(x), \sigma(y)) := -J_{xy}\sigma(x)\sigma(y)$. The pair interaction $W_{xy}(\sigma(x), \sigma(y))$ satisfies Assumptions (**W**) and (**J**), i.e.,

$$|W_{xy}(\sigma(x), \sigma(y))| \leq \frac{J_{xy}}{2}(\sigma(x)^2 + \sigma(y)^2)$$

. To guarantee the existence of tempered Gibbs measures $\mu \in \mathcal{G}^t$ we assume:

Assumption (U). *There exist a continuous function $U : \mathbb{R}^\nu \to \mathbb{R}$ and constants $P > 2$ and $A, B > 0$, such that for all $x \in \mathbb{V}$ and $\sigma(x) \in \mathbb{R}^\nu$*

$$A|\sigma(x)|^P + B \leq U_x(\sigma(x)) \leq U(\sigma(x)).$$

We define the set of tempered Gibbs measures $\mathcal{G}^t$ in a usual way, see Chapter 2. Under the above assumptions the existence of $\mu \in \mathcal{G}^t$ follows from the results of that chapter.

5.1.2 Correlation inequalities

Correlation inequalities for moments of Gibbs measures are a powerful tool to study uniqueness and phase transitions in ferromagnetic system. In this section we will briefly introduce them, for detailed presentation we refer e.g. to [GlJa 1981] by J. Glimm and A. Jaffe.

Let us begin this subsection with definitions, which are crucial for this and the subsequent sections.

Definition 5.1. *For two different configurations $\Omega \ni \sigma = (\sigma(x))_{x\in\mathbb{V}}$ and $\Omega \ni \tilde{\sigma} = (\tilde{\sigma}(x))_{x\in\mathbb{V}}$ defined on a graph $\mathbb{V}$ we write $\sigma \le \tilde{\sigma}$ if for all $x \in \mathbb{V}$ it holds $\sigma(x) \le \tilde{\sigma}(x)$.*

Definition 5.2. *A function $f : \Omega \to \mathbb{R}$ is in the set $\mathcal{F}C_b(\Omega)$, defined as the set of all* bounded continuous cylinder function*, if it can be represented for some $\phi \in C_b(\mathbb{R}^n)$, $x_i \in \mathbb{V}$, $1 \le i \le n$, and $n \in \mathbb{N}$ as*

$$f(\sigma) = \phi(\sigma(x_1), \cdots, \sigma(x_n)). \tag{5.2}$$

A function $f \in \mathcal{F}C_b(\Omega)$ is called increasing*, if $\sigma \le \tilde{\sigma}$ implies $f(\sigma) \le f(\tilde{\sigma})$. We then denote the* set of all increasing bounded cylinder functions *by $\mathcal{F}C_b^+(\Omega)$.*

We consider the ferromagnetic model (5.1). For this model we give the *Fortuin-Kasteleyn-Ginibre* (FKG) inequality, cf. [GlJa 1981].

Proposition 5.3 (FKG inequality)**.** *For all $\Lambda \Subset \mathbb{V}$, $\xi \in \Omega^t$ and any $f, g \in \mathcal{F}C_b^+(\Omega)$, it follows that*

$$\pi_\Lambda(f \cdot g|\xi) \ge \pi_\Lambda(f|\xi) \cdot \pi_\Lambda(g|\xi). \tag{5.3}$$

This yields, in particular, that for all such functions f it holds

$$\xi \le \tilde{\xi} \Rightarrow \pi_\Lambda(f|\xi) \prec \pi_\Lambda(f|\tilde{\xi}). \tag{5.4}$$

Indeed, the above inequalities hold also for any continuous increasing functions f, g, for which the corresponding integrals exist.

The polynomial moments and covariances of general ferromagnets are nonnegative due to the Griffiths inequalities, cf. [GlJa 1981], p. 74-76. We recall it here.

Proposition 5.4 (Griffiths inequalities)**.** *For all nonnegativ boundary conditions $\xi \in \Omega^t$, all $x_1, \cdots, x_{n+m} \in \Lambda \Subset \mathbb{V}$, and all $n, m \in \mathbb{N}$, it holds*

$$\int_{\Omega^t} \left(\prod_{i=1}^{n} \sigma(x_i) \right) \pi_\Lambda(d\sigma|\xi) \ge 0, \tag{5.5}$$

and

$$\begin{aligned} & \int_{\Omega^t} \left(\prod_{i=1}^{n} \sigma(x_i) \right) \left(\prod_{i=n+1}^{n+m} \sigma(x_i) \right) \pi_\Lambda(d\sigma|\xi) \\ \geq & \int_{\Omega^t} \left(\prod_{i=1}^{n} \sigma(x_i) \right) \pi_\Lambda(d\sigma|\xi) \cdot \int_{\Omega^t} \left(\prod_{i=n+1}^{n+m} \sigma(x_i) \right) \pi_\Lambda(d\sigma|\xi). \end{aligned} \tag{5.6}$$

For even ferromagnets, the above proposition is extended by the *Griffiths-Kelly-Sherman* (GKS) inequalities, cf. [Si 1979] by B. Simon, p. 119-124.

Proposition 5.5 (GKS inequalities)**.** *Let for all $x \in \mathbb{V}$ the self interaction potential have the form*

$$U_x(s) = u_x(s), \tag{5.7}$$

where u_x is continuous. Let also the continuous functions $f_1, \cdots, f_{n+m} : \mathbb{R} \to \mathbb{R}$ be polynomially bounded and such that every f_i is increasing nonnegative on $\mathbb{R}_+$ and either even or odd on the whole $\mathbb{R}$. Then the following inequalities hold for all $x_1, \cdots, x_{n+m} \in \Lambda \Subset \mathbb{V}$, and $N, M \in \mathbb{N}$,

$$\int_{\Omega^t} \left(\prod_{i=1}^{n} f_i(\sigma(x_i)) \right) \pi_\Lambda(d\sigma|0) \geq 0, \tag{5.8}$$

and

$$\begin{aligned} & \int_{\Omega^t} \left(\prod_{i=1}^{n} f_i(\sigma(x_i)) \right) \left(\prod_{i=n+1}^{n+m} f_i(\sigma(x_i)) \right) \pi_\Lambda(d\sigma|0) \\ \geq & \int_{\Omega^t} \left(\prod_{i=1}^{n} f_i(\sigma(x_i)) \right) \pi_\Lambda(d\sigma|0) \\ \cdot & \int_{\Omega^t} \left(\prod_{i=n+1}^{n+m} f_i(\sigma(x_i)) \right) \pi_\Lambda(d\sigma|0). \end{aligned} \tag{5.9}$$

We also introduce a general inequality, which is the so-called *Brascamp-Lieb* inequality proved in [BrLi 1976] by H. J. Brascamp and E. H. Lieb, see also [Gia 2003] by G. Giacomin.

Proposition 5.6 (Brascamp-Lieb inequality)**.** *Let $U(s)$ be a convex function on $\mathbb{R}^n$, and let A be a real, positive definite, $n \times n$ matrix. Assume* $\exp\{-(As, s) - U(s)\} \in L^1(ds)$ *and define the probability measure*

$$\mu_U(ds) = \frac{1}{Z} \exp\{-(As, s) - U(s)\} ds.$$

If $U \equiv 0$ we write $\mu_A(ds)$, which is the Gaussian measure. Let $\varphi \in \mathbb{R}^n$ and $\alpha \geq 1$. Then

$$\int_{\mathbb{R}^n} \left| (\varphi, s)_{\mathbb{R}^n} - \int_{\mathbb{R}^n} (\varphi, s)_{\mathbb{R}^n} \mu_U(ds) \right|^\alpha \mu_U(ds) \leq \int_{\mathbb{R}^n} \left| (\varphi, s)_{\mathbb{R}^n} \right|^\alpha \mu_A(ds), \tag{5.10}$$

where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the scalar product on $\mathbb{R}^n$.

5.1.3 Partial order on the set $\mathcal{G}^t$

In this section we establish a *partial order* on the set of tempered Gibbs measures $\mathcal{G}^t$. The Definition (5.2) of the set $\mathcal{F}C_b^+(\Omega)$ brings us to the notion of a *partial order* or, to be precisely, *stochastic domination* on the set of probability measures $\mathcal{P}(\Omega)$.

Definition 5.7. *For μ, $\tilde{\mu} \in \mathcal{P}(\Omega)$ we say μ is* stochastically dominated *by $\tilde{\mu}$, i.e. $\mu \prec \tilde{\mu}$, if for all $f \in \mathcal{F}C_b^+(\Omega)$ it holds*

$$\mu(f) \leq \tilde{\mu}(f). \tag{5.11}$$

For the lattice $\mathbb{Z}^d$ it is well known by [LP 1976, BH-K 1982] that there exist the unique *maximal* and *minimal* elements in the set of tempered Gibbs measures $\mathcal{G}^t$, which we will denote by μ_+ respectively μ_-. Furthermore, μ_+ and μ_- are extreme elements in $\mathcal{G}^t$, also well known from [LP 1976, BH-K 1982, KoPa 2007].

Definition 5.8. *Let us define the set $\mathcal{P}_1(\Omega)$ of all $\mu \in \mathcal{P}(\Omega)$ for which all their first moments are finite, i.e., for all $x \in \mathbb{V}$*

$$\int_\Omega |\sigma(x)| \mu(d\sigma) < \infty.$$

Lemma 5.9. *Consider a pair of probability measures μ, $\tilde{\mu} \in \mathcal{P}_1(\Omega)$. If for all $f \in \mathcal{F}C_b^+(\Omega)$, we have that*

$$\mu(f) = \tilde{\mu}(f), \tag{5.12}$$

then $\mu = \tilde{\mu}$. In other words this means that the set $\mathcal{F}C_b^+(\Omega)$ is a measure defining class for $\mathcal{P}_1(\Omega)$.

Proof. For all $x \in \mathbb{V}$ the assumption (5.12) and (3.22) implies the identity

$$\int_\Omega \sigma(x) \mu(d\sigma) = \int_\Omega \sigma(x) \tilde{\mu}(d\sigma). \tag{5.13}$$

Then, we fix some $x_1, \cdots, x_n \in \mathbb{V}$ for $n \in \mathbb{N}$ and construct for the measures μ and $\tilde{\mu}$ the corresponding projections P_n and $\tilde{P}_n$ on $\mathbb{R}^n$. This means that for a cylinder function $f \in \mathcal{F}C_b^+(\Omega)$ with a corresponding ϕ (see Definition 5.2), each P_n obeys

$$\int_\Omega f(\sigma)\mu(d\sigma) = \int_{\mathbb{R}^n} \phi(\sigma(x_1), \cdots, \sigma(x_n))P_n(d\sigma). \tag{5.14}$$

For $\tilde{P}_n$ a similar equality holds. Then by assumption (5.12), it especially follows, for all increasing $\phi \in C_b(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} \phi(\sigma(x_1), \cdots, \sigma(x_n))P_n(d\sigma) \leq \int_{\mathbb{R}^n} \phi(\sigma(x_1), \cdots, \sigma(x_n))\tilde{P}_n(d\sigma). \tag{5.15}$$

An elegant idea is to use here *Strassen's theorem*, see e.g. [Li 1992] by T. Lindvall. Consider

$$M := \left\{(\sigma, \tilde{\sigma}) \in \mathbb{R}^{2n} \mid \sigma(x_i) \leq \tilde{\sigma}(x_i), \quad 1 \leq i \leq n\right\}, \tag{5.16}$$

which is the closed set in $\mathbb{R}^{2n}$. Taking into account the stochastical order $P_n \prec \tilde{P}_n$, by *Strassen's theorem* a coupling $\hat{P} \in \mathcal{C}(P_n, \tilde{P}_n)$ exists such that $\hat{P}(M) = 1$. Therefore, the Wasserstein distance, see Section 2.7, between P_n and $\tilde{P}_n$ can be estimated as

$$\begin{aligned} \mathbf{W}(P_n, \tilde{P}_n) &\leq \int_M |\sigma - \tilde{\sigma}| \, \hat{P}(d\sigma, d\tilde{\sigma}) \leq \sum_{i=1}^n \int_{\mathbb{R}^{2n}} (\tilde{\sigma}(x_i) - \sigma(x_i))\hat{P}(d\sigma, d\tilde{\sigma}) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} \sigma(x_i) \left[\tilde{P}_n(d\sigma) - P_n(d\sigma)\right] = 0, \end{aligned} \tag{5.17}$$

where the latter equality is implied by (5.13). So, we prove that $P_n = \tilde{P}_n$ for all $n \in \mathbb{N}$. Since by Kolmogorov's theorem each measure is uniquely determined by its finite volume projections, we have that $\mu = \tilde{\mu}$.

□

We have the following basic corollary coming from the above proof, see the estimate (5.17).

Corollary 5.10. *Consider a pair of probability measures μ, $\tilde{\mu}$ on $\mathcal{P}_1(\Omega)$, such that μ is stochastically dominated by $\tilde{\mu}$, i.e., $\mu \prec \tilde{\mu}$. If all their first moments coincide, i.e.* (5.13) *holds, then $\mu = \tilde{\mu}$.*

Let us discuss how to construct μ_+ and $\mu_- \in \mathcal{G}^t$ directly.

Remark 5.11. *(i) Let us assume that the graph $G(\mathbb{V}, \mathbb{E})$ obeys Assumption $(\mathbf{G}_\lambda)$. Choose $b \geq \frac{\lambda}{\kappa\beta}$ the same as in Corollary 3.9. Let us define the constant boundary condition $\hat{\xi} = (\hat{\xi}(x))_{x\in\mathbb{V}}$ as*

$$\hat{\xi}(x) := (b \log(1 + d(x_o, x))^{1/2}, \quad \forall x \in \mathbb{V}. \tag{5.18}$$

Then $\hat{\xi}$ belongs to Ω^t and also to the subset $\Omega(b)$ of Ω^t defined in (3.27)*. By Corollary 3.9 and for every $\xi \in \Omega(b)$ one can find a set $\bar{\Lambda} \Subset \mathbb{V}$ so that $\xi(x) \leq \hat{\xi}(x)$ for all $x \in \bar{\Lambda}^c$. Therefore, for any cofinal sequence $\mathcal{L}$ one can find $\bar{\Lambda} \in \mathcal{L}$ such that for large enough $\Lambda \in \mathcal{L}$ containing $\bar{\Lambda}$*

$$\pi_\Lambda(\cdot \mid \xi) \prec \pi_\Lambda(\cdot \mid \hat{\xi}). \tag{5.19}$$

As we know from Theorem 3.7, the local specification $\{\pi_\Lambda(d\sigma|\xi)\}_{\Lambda\in\mathcal{L}}$ is relatively compact in the weak topology $\mathcal{W}_t$. Let $\hat{\mu}$ be any of the accumulation points of $\{\pi_\Lambda(d\sigma|\hat{\xi})\}_{\Lambda\in\mathcal{L}}$. Then, by Lemma 2.20 this $\hat{\mu}$ belongs to $\mathcal{G}^t$ and it dominates every element of the set of extreme Gibbs measures $ex(\mathcal{G}^t)$, which actually means $\hat{\mu} = \mu_+$, see [BH-K 1982, KoPa 2007]. Therefore, the maximal element is unique. Similar arguments are true for the accumulation points of $\{\pi_\Lambda(d\sigma| - \hat{\xi})\}_{\Lambda\in\mathcal{L}}$. Thus, for every cofinal sequence $\mathcal{L}$ we have for plus/minus boundary conditions

$$\lim_{\mathcal{L}} \pi_\Lambda(\cdot \mid \pm\hat{\xi}) =: \mu_\pm. \tag{5.20}$$

(ii) If the Assumption $(\mathbf{G}_\lambda)$ is not valid to construct the Gibbs measures $\mu_\pm$ we are forced to define the boundary condition $\hat{\xi}$ in a different way. For some $\gamma_1 > \gamma_0$ we define

$$\hat{\xi}(x) := b \exp\{\gamma_1 d(x_o, x)\}, \quad \forall x \in \mathbb{V}.$$

Under the Assumption ($\bar{\boldsymbol{U}}$) in (3.3) *we can repeat all the scheme of constructing the limiting points $\mu_\pm := \lim_{\mathcal{L}} \pi_\Lambda(\cdot \mid \pm\hat{\xi})$. They are Gibbs measures supported by the larger set Ω_{γ_2}, where $\gamma_2 > \gamma_0 + 2\gamma_1$. Nevertheless, $\mu_\pm$ satisfy the a priori bounds*

$$\int \exp\{\delta|\sigma(x)|^2\}\mu_\pm(d\sigma) < \infty,$$

for all $\delta > 0$, and hence they are supported by any Ω_γ, $\gamma > \gamma_0$. So, we have that $\mu_\pm \in \mathcal{G}^t$.

5.2 Existence results for ferromagnetic systems on general graphs

A new important issue of this section is that we consider any graph $G(\mathbb{V},\mathbb{E})$ with possibly *unbounded degree*, that means we study graphs with

$$\sup_{x\in\mathbb{V}} m(x) \leq +\infty.$$

The *configuration space* is defined by the space of all sequences over $\mathbb{V}$

$$\Omega := (\mathbb{R}^\nu)^{\mathbb{V}} := \{\sigma = (\sigma(x))_{x\in\mathbb{V}} \mid \sigma : \mathbb{V} \to \mathbb{R}^\nu\}. \tag{5.21}$$

Let us consider the following formal Hamiltonian, corresponding to the ferromagnetic pair interaction of strength $J > 0$:

$$E(\sigma) := \frac{J}{2} \sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} (\sigma(x) - \sigma(y))^2 + \sum_{x\in\mathbb{V}} U_x(\sigma(x)), \tag{5.22}$$

where the sum $\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}}$ runs over all unordered pairs of the nearest neighbors. The corresponding Gibbs measure is heuristically given by

$$\mu(d\sigma) = \frac{1}{Z} e^{-\beta E(\sigma)} \prod_{x\in\mathbb{V}} d\sigma(x). \tag{5.23}$$

As usual we define the Gibbs measure μ rigorously by using the DLR framework, see Definition (2.16). In order to do that we consider the corresponding local specification: For any finite volume $\Lambda \Subset \mathbb{V}$ and boundary condition $\xi \in \Omega$, we set

$$\begin{aligned}
\pi_\Lambda(d\sigma_\Lambda|\xi) \quad &:= \quad \frac{1}{Z} \exp\Bigg\{ -\frac{J}{2} \sum_{\substack{x,y\in\Lambda\\ x\sim y}} (\sigma(x) - \sigma(y))^2 \\
&- \quad \frac{J}{2} \sum_{\substack{x\in\Lambda, y\in\Lambda^c\\ x\sim y}} (\sigma(x) - \xi(y))^2 \\
&- \quad \sum_{x\in\Lambda} U_x(\sigma(x)) \Bigg\} \prod_{x\in\Lambda} d\sigma(x) \\
&= \quad \frac{1}{Z} \exp\Bigg\{ -\frac{J}{2} \sum_{\substack{x,y\in\Lambda\\ x\sim y}} (\sigma(x) - \sigma(y))^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{J}{2}\sum_{x\in\Lambda} m_{\Lambda^c}(x)(\sigma(x)-\bar{\xi}_{\Lambda^c}(x))^2 \\
&- \sum_{x\in\Lambda} U_x(\sigma(x))\Big\}\prod_{x\in\Lambda} d\sigma(x),
\end{aligned}
\tag{5.24}
$$

where

$$\bar{\xi}_{\Lambda^c}(x) := \frac{\sum_{y\in\partial x\cap\Lambda^c}\xi(y)}{m_{\Lambda^c}(x)}$$

with

$$m_{\Lambda^c}(x) := \#\{y\in\Lambda^c | y\sim x\}.$$

In particular, we have

$$
\begin{aligned}
\pi_x(d\sigma(x)|\xi) &:= \frac{1}{Z}\exp\Big\{-\frac{J}{2}\sum_{y\in\partial x}(\sigma(x)-\sigma(y))^2 \\
&- U_x(\sigma(x))\Big\}d\sigma(x) \\
&= \frac{1}{Z}\exp\Big\{-\frac{J}{2}m(x)(\sigma(x)-\bar{\xi}(x))^2 \\
&- U_x(\sigma(x))\Big\}d\sigma(x),
\end{aligned}
$$

where $\bar{\xi}(x)=\frac{\sum_{y\in\partial x}\xi(y)}{m(x)}$ is the average boundary condition around the point x.

Remark 5.12. *As is seen from* (5.24) *the model* (5.22) *is equivalent to the ferromagnetic model with the following heuristic Hamiltonian*

$$
\begin{aligned}
E(\sigma) &:= -J\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}}\sigma(x)\sigma(y) \\
&+ \frac{J}{2}\sum_{x\in\mathbb{V}} m(x)\sigma(x)^2+\sum_{x\in\mathbb{V}}U_x(\sigma(x)),
\end{aligned}
$$

since they have the same local Gibbs specification $\pi_\Lambda(d\sigma_\Lambda|\xi)$. As compared with the ferromagnetic model (5.69)*, see Section 5.4.1, we have an additional stabilizing term $m(x)\sigma(x)^2$, which enables us to study the existence and uniqueness of the corresponding Gibbs measures for the model* (5.22)*. Whereas the model* (5.69) *is more convenient to study phase transitions, see Section 5.4.1 below.*

In order to give an existence result for arbitrary graph $G(\mathbb{V},\mathbb{E})$ and intensity $J>0$ we need some preliminary constructions. Firstly we introduce the notion of *right-* and *left-dominators*, see [OsSp 1999] by H. Osada and H. Spohn.

Definition 5.13. *Let $f:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ and $g:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ be functions. We call g a* right-dominator *(respectively* left-dominator*) of f if it satisfies the following conditions:*
(i) g is convex and finite on at least two different points.
(ii) $f-g$ is nondecreasing (respectively nonincreasing) in $s\in\mathbb{R}$. For the case that $f(s)=\infty$ and $g(s)=\infty$ we set $f(s)-g(s)=0$.
(iii) There exists a constant $a>0$ such that the second derivative fulfills $g''(s)\geq a^2$ for all s such that $g(s)<\infty$.

Remark 5.14. *If f and g are differentiable, then a right-dominator satisfies $f'(s)\geq g'(s)$ and a left-dominator $f'(s)\leq g'(s)$ with g' strictly increasing. We say a dominator g is symmetric around $m\in\mathbb{R}$ if $g(s-m)=g(|s-m|)$ for all s. Note that if g is a right-dominator (respectively left-dominator) of f which is symmetric around m (respectively $-m$), then there exist right-dominators (respectively left-dominator) of f symmetric around n (respectively $-n$) for all $n\geq m$. Indeed, we easily see that*

$$R_n^+(g(s))=\begin{cases} g(s), & \text{for } s\geq n;\\ g(2n-s), & \text{for } s\leq n.\end{cases} \tag{5.25}$$

$$R_n^-(g(s))=\begin{cases} g(-2n-s), & \text{for } s\geq -n;\\ g(s), & \text{for } s\leq -n,\end{cases} \tag{5.26}$$

are such dominators.

For simplicity we suppose that the self interaction potentials can be splitted into two functions of the form

$$U_x := U_{x,1}+U_{x,2},$$

where $U_{x,1}\in C^2(\mathbb{R})$ is *twice continuously differentiable, strictly convex function* and $U_{x,2}\in C_b^1(\mathbb{R})$ is a *continuously differentiable* function with *bounded derivative.* Furthermore, $U_{x,1}(s)$ is *symmetric*, i.e., $U_{x,1}(s)=U_{x,1}(-s)$.

Additionally, we assume that there exist constants $a,b>0$ such that for all $x\in\mathbb{V}$ and $s\in\mathbb{R}$

$$U''_{x,1}(s)\geq a^2, \quad |U'_{x,2}(s)|<b$$

Theorem 5.15. *For the system* (5.22) *and for the given conditions on the self interaction potential U_x the set $\mathcal{G}$ of Gibbs measures is not empty, i.e.,*

$$\mathcal{G} \neq 0.$$

Moreover, for any boundary condition $\xi \in \Omega$ with $\sup_y |\xi(y)| < \infty$, there exist a limit point $\mu = \lim_{\Lambda \nearrow \mathbb{V}} \pi_\Lambda(d\sigma_\Lambda|\xi) \in \mathcal{G}$, which satisfies

$$\sup_{x \in \mathbb{V}} \int_\Omega |\sigma(x)| \mu(d\sigma) < \infty.$$

Remark 5.16. *Let U_x be symmetric, i.e., $U_x(s) = U_x(-s)$, $s \in \mathbb{R}$, and consider the zero boundary condition $\xi \equiv 0$. Then we have*

$$\int_{\mathbb{R}^\Lambda} \sigma(x) \pi_\Lambda(d\sigma_\Lambda|0) = 0, \tag{5.27}$$

which comes from the symmetry of the interaction.

Remark 5.17. *Below we shall crucially use the* generalized FKG inequality, *see [Pr2 1974] by C. J. Preston. We write $\sigma \leq \tilde{\sigma}$ if for all $x \in \mathbb{V}$ it holds $\sigma(x) \leq \tilde{\sigma}(x)$ and $\sigma_\Lambda \leq \tilde{\sigma}_\Lambda$ if for all $x \in \Lambda$ it holds $\sigma(x) \leq \tilde{\sigma}(x)$. Let $F : \mathbb{R}^\Lambda \to \mathbb{R}$ be an* increasing nonnegative function *on $\mathbb{R}^\Lambda$, which means it satisfies $F(\sigma_\Lambda) \leq F(\tilde{\sigma}_\Lambda)$ for all $\sigma_\Lambda \leq \tilde{\sigma}_\Lambda$, see Section 5.1.3. Let a potential V be such that $U - V$ is nondecreasing in s. Then, for all $\xi \leq \tilde{\xi}$,*

$$\int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^U(d\sigma_\Lambda|\xi) \leq \int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^V(d\sigma_\Lambda|\tilde{\xi}), \tag{5.28}$$

where we write π_Λ^U and π_Λ^V in order to indicate the dependence of the corresponding Gibbs specification on the one particle potential U respectively V. Of course, one or both integrals in (5.28) *could be plus infinity. The inequality* (5.28) *allows us to compare the moment of the initial specification $\pi_\Lambda^U(d\sigma_\Lambda|\xi)$ with the corresponding moments of the specification $\pi_\Lambda^V(d\sigma_\Lambda|\tilde{\xi})$.*

In particular, we have for any right-dominator *U_{right} and for any* left-dominator *U_{left} of U the following inequalities*

$$\int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^U(d\sigma_\Lambda|\xi) \leq \int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^{U_{right}}(d\sigma_\Lambda|\tilde{\xi}). \tag{5.29}$$

$$\int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^U(d\sigma_\Lambda|\xi) \geq \int_{\mathbb{R}^\Lambda} F(\sigma_\Lambda) \pi_\Lambda^{U_{left}}(d\sigma_\Lambda|\tilde{\xi}). \tag{5.30}$$

Proof of Theorem 5.15. The main step is to construct a right-dominator of U under the imposed conditions. We introduce the shifted potential $U_{x,A} := U_{x,1}(\cdot - A)$, for some $A > 0$. Then obviously for all $x \in \mathbb{R}$

$$
\begin{aligned}
U'_x(s) - U'_{x,A}(s) &= U'_{x,1}(s) - U'_{x,1}(s-A) + U'_{x,2}(s) \\
&\geq a^2 A - b \geq 0,
\end{aligned}
\tag{5.31}
$$

if

$$
A \geq \frac{b}{a^2}.
$$

Therefore under this condition we can choose $U_{x,A}$ as the right-dominator of U, which is symmetric with respect to $s = A$. Analogously, $U_{x,-A} := U_{x,1}(\cdot + A)$ is the left-dominator, which is symmetric with respect to $s = -A$.

Let us denote $[s]_+ := s \vee 0$ and $[s]_- := -s \vee 0$, which are monotone functions of s. Fix the boundary condition $\xi \leq \tilde{\xi}$ such that $\xi(x) = 0$ and $\tilde{\xi}(x) = A$ for all $x \in \mathbb{V}$. And consider the corresponding local Gibbs distributions $\pi_\Lambda^{U_A}(d\sigma_\Lambda|A)$ and $\pi_\Lambda^{U_{-A}}(d\sigma_\Lambda|-A)$. By the FKG inequality in Remark 5.17 we have for any $x \in \Lambda \Subset \mathbb{V}$

$$
\begin{aligned}
\int_{\mathbb{R}^\Lambda} |\sigma(x)| \pi_\Lambda(d\sigma_\Lambda|0) &= \int_{\mathbb{R}^\Lambda} [\sigma(x)]_+ + [\sigma(x)]_- \pi_\Lambda(d\sigma_\Lambda|0) \\
&\leq \int_{\mathbb{R}^\Lambda} [\sigma(x)]_+ \pi_\Lambda^{U_A}(d\sigma_\Lambda|A) + \int_{\mathbb{R}^\Lambda} [\sigma(x)]_- \pi_\Lambda^{U_{-A}}(d\sigma_\Lambda|-A) \\
&\leq 2A + \int_{\mathbb{R}^\Lambda} [\sigma(x) - A]_+ \pi_\Lambda^{U_A}(d\sigma_\Lambda|A) \\
&+ \int_{\mathbb{R}^\Lambda} [\sigma(x) + A]_- \pi_\Lambda^{U_{-A}}(d\sigma_\Lambda|-A),
\end{aligned}
\tag{5.32}
$$

where $\pi_\Lambda^{U_A}(d\sigma_\Lambda|A)$ and $\pi_\Lambda^{U_{-A}}(d\sigma_\Lambda|-A)$ are defined as the following

$$
\begin{aligned}
\pi_\Lambda^{U_{\pm A}}(d\sigma_\Lambda|\pm A) &:= \frac{1}{Z} \exp\Big\{ -\frac{J}{2} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} (\sigma(x) - \sigma(y))^2 \\
&- \frac{J}{2} \sum_{x\in\Lambda} m_{\Lambda^c}(x)(\sigma(x) \mp A)^2 \\
&- \sum_{x\in\Lambda} U_{x,\pm A}(\sigma(x)) \Big\} \prod_{x\in\Lambda} d\sigma(x).
\end{aligned}
\tag{5.33}
$$

In the second line of the inequality (5.32) we use that $[s]_+$ is growing and $[s]_-$ is decaying, which leads to the FKG inequality with the boundary conditions

A and $-A$ respectively. Since

$$\int_{\mathbb{R}^\Lambda} [\sigma(x) - A]_+ \pi_\Lambda^{U_A}(d\sigma_\Lambda|A) = \frac{1}{2}\int_{\mathbb{R}^\Lambda} |\sigma(x) - A| \pi_\Lambda^{U_A}(d\sigma_\Lambda|A)$$

and

$$\int_{\mathbb{R}^\Lambda} [\sigma(x) + A]_- \pi_\Lambda^{U_{-A}}(d\sigma_\Lambda| - A) = \frac{1}{2}\int_{\mathbb{R}^\Lambda} |\sigma(x) + A| \pi_\Lambda^{U_{-A}}(d\sigma_\Lambda| - A)$$

we finally get

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} |\sigma(x)|\pi_\Lambda(d\sigma_\Lambda|0) \leq 2A \quad &+ \quad \frac{1}{2}\int_{\mathbb{R}^\Lambda} |\sigma(x) - A|\pi_\Lambda^{U_A}(d\sigma_\Lambda|A) \\ &+ \quad \frac{1}{2}\int_{\mathbb{R}^\Lambda} |\sigma(x) + A|\pi_\Lambda^{U_{-A}}(d\sigma_\Lambda| - A). \qquad (5.34)\end{aligned}$$

Note that due to symmetry of $U_{x,A}$ with respect to $s = A$ and $U_{x,-A}$ with respect to $s = -A$ we have

$$\int_{\mathbb{R}^\Lambda} \sigma(x)\pi_\Lambda^{U_A}(d\sigma_\Lambda|A) = A \qquad (5.35)$$

and

$$\int_{\mathbb{R}^\Lambda} \sigma(x)\pi_\Lambda^{U_{-A}}(d\sigma_\Lambda| - A) = -A. \qquad (5.36)$$

Now we apply the classical *Brascamp-Lieb* inequality (5.6) (see also [BrLi 1976]), using the presentation $U_{x,A}(s) = \frac{a^2}{2}(s-A)^2 + \tilde{U}_{x,A}(s)$ with some convex and symmetric function $\tilde{U}_{x,A}(s)$ with respect to $s = A$ such that $\tilde{U}''_{x,A}(s) \geq 0$. Then dropping the convex terms $(\sigma(x) - \sigma(y))^2$ and $\tilde{U}_{x,A}(\sigma(x))$ we have for all $n \geq 1$ that

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} |\sigma(x) - A|^{2n}\pi_\Lambda^{U_A}(d\sigma_\Lambda|A) \quad &\leq \quad \int_{\mathbb{R}^\Lambda} |\sigma(x) - A|^{2n} g_\Lambda^A(d\sigma_\Lambda) \\ &= \quad \int_{\mathbb{R}^\Lambda} |\sigma(x) - A|^{2n} \prod_{x\in\Lambda} g^A(d\sigma(x)) \\ &= \quad C(n, a) < \infty, \qquad (5.37)\end{aligned}$$

where g_Λ^A is the Gaussian measure on $\mathbb{R}^\Lambda$ of the following form

$$g_\Lambda^A(d\sigma_\Lambda) := \frac{1}{Z}\exp\{-\frac{a^2}{2}\sum_{x\in\Lambda}(\sigma(x) - A)^2\}\prod_{x\in\Lambda} d\sigma(x), \qquad (5.38)$$

with the normalizing constant

$$Z = \int_{\mathbb{R}^\Lambda} \exp\{-\frac{a^2}{2}\sum_{x\in\Lambda}(\sigma(x)-A)^2\}\prod_{x\in\Lambda} d\sigma(x).$$

In particular, by $g^A(d\sigma(x))$ we denote the corresponding one-dimensional measure on $\mathbb{R}$. Take note that the last line in (5.37) is independent of A and Λ. In a similar way, one considers the integral

$$\int_{\mathbb{R}^\Lambda} |\sigma(x)+A|\pi_\Lambda^{U_{-A}}(d\sigma_\Lambda|-A).$$

Combining the inequalities (5.34) and (5.37) for $n=2$ we get that for all $\Lambda \Subset \mathbb{V}$

$$\int_{\mathbb{R}^\Lambda} |\sigma(x)|\pi_\Lambda(d\sigma_\Lambda|0) \leq 2A + C, \tag{5.39}$$

where the constant C comes from (5.37) and

$$A \geq \frac{b}{a^2}.$$

So,

$$\sup_{x\in\Lambda}\int_{\mathbb{R}^\Lambda} |\sigma(x)|\pi_\Lambda(d\sigma_\Lambda|0) < \infty,$$

which implies tightness for $\{\pi_\Lambda(d\sigma_\Lambda|0)\}_{\Lambda\Subset\mathbb{V}}$ in the locally weak topology and, hence, existence of a limit point $\mu \in \mathcal{G}$.

Analogously one can deal with boundary conditions $\xi \neq 0$, $\sup_x |\xi(y)| < \infty$, by taking $\tilde{A} = A \vee \max_{y\in\mathbb{V}} |\xi(y)|$, where A is the minimal shift for which $U_{x,A}$ is the right-dominator, cf. (5.31).

□

Remark 5.18. *In fact, instead of the Brascamp-Lieb inequality, see [BrLi 1976], in* (5.37) *we can also use the result of Giacomin in [Gia 2003], which says that for any* monotone $F : \mathbb{R}^+ \to \mathbb{R}^+$ *we have*

$$\int_{\mathbb{R}^\Lambda} F(|\sigma(x)-M_A|)\pi_\Lambda^{U_A}(d\sigma_\Lambda|A) \leq \int_{\mathbb{R}^\Lambda} F(|\sigma(x)|)g_\Lambda^{A=0}(d\sigma(x)), \tag{5.40}$$

where M_A is the meridian of the random variable $\sigma(x)$ under $\pi_\Lambda^{U_A}(d\sigma_\Lambda|A)$, that means $\pi_\Lambda^{U_A}(\cdot|A)\{\sigma(x)\leq M_A\} = \frac{1}{2}$. In our case obviously we have $M_A = A$.

Example 5.19. *Additionally to the class of potentials U_x as described above, let us list some important classes of one-particle potentials for which one can construct dominators. This examples are due to H. Osada and H. Spohn, see [OsSp 1999].*

(i) $U(s) = |s|^q + P(s) + L(s)$, $s \in \mathbb{R}$ where $q \geq 2$, P is a polynomial of degree $deg(P) \leq q-1$, and L is a Lipschitz continuous function.
(ii) $U(s) = e^{|s|} + Q(s)$, where $Q'(s)$ has at most polynomial growth.

Remark 5.20. *By Theorem 5.15 we construct Gibbs measure $\mu \in \mathcal{G}$ satisfying $\sup_{x\in\mathbb{V}} \int_{\mathbb{R}^\Lambda} |\sigma(x)|\mu(d\sigma) < \infty$. Such measures are supported by the tempered configurations*

$$\Omega^\alpha := \{\sigma \in \Omega | \sum_{x\in\mathbb{V}} \alpha(x)|\sigma(x)| < \infty\},$$

where $\{\alpha(x)\}_{x\in\mathbb{V}}$ is any nonnegative sequence over $\mathbb{V}$ such that

$$\sum_{x\in\mathbb{V}} \alpha(x) < \infty.$$

For a general graph of unbounded degree, we cannot guarantee that $\alpha(x)$ can be chosen as $\exp\{-\gamma d(o,x)\}$ with some $\gamma > 0$, which was the case in the previous chapters.

Remark 5.21. *In the full analogy to the proof of Theorem 5.15 the method applies to the case of ferromagnetic systems on the lattice $\mathbb{Z}^d$ with unbounded interacting intensities $(J + I_i)$, see our example for the uniqueness problem in Section 4.7. In this case we construct a tempered Gibbs measure $\mu \in \mathcal{G}$ such that $\sup_{x\in\mathbb{V}} \int_{\mathbb{R}^\Lambda} |\sigma(x)|\mu(d\sigma) < \infty$ and hence $\mu(\Omega^t) = 1$, where Ω^t is the set of exponentially tempered configurations defined for all $\gamma > 0$ as*

$$\Omega^t := \{\sigma \in \Omega | \sum_{x\in\mathbb{Z}^d} |\sigma(x)| e^{-\gamma|x|} < \infty\}.$$

5.3 Uniqueness and a comparison criterion for ferromagnetic systems

In Section 5.3.1 we give a general *uniqueness criterion* for ferromagnetic scalar models which will be fundamental in the subsequent sections. The remarkable correlation inequalities will crucially help us to give a so-called *comparison* criterion, in Section 5.3.2, for uniqueness and phase transitions of Gibbs measures. There we compare the initial model with certain *reference* (and as rule *simpler*) models. In particular, this opens a possibility to study non-polynomial one particle potentials $U_x(s)$ on the basis of the knowledge about their asymptotic behavior at the infinity.

5.3.1 Uniqueness criterion

Now, we would like to present a simple criterion for the uniqueness of $\mu \in \mathcal{G}^t$ in ferromagnetic models. Originally, this criterion came from the paper by J. Lebowitz and E. Presutti [LP 1976], but we give a short alternative proof which employs the Wasserstein distance via Corollary 5.10. For the quantum case similar criterion can be found in [KoPa 2007].

Theorem 5.22 (Uniqueness criterion). *For the scalar ferromagnetic model* (5.1) *the following properties are equivalent*
(i) $\mathcal{G}^t$ *is a singleton.*
(ii) For all $x \in \mathbb{V}$ *it holds*

$$\int_\Omega \sigma(x)\mu_+(d\sigma) = \int_\Omega \sigma(x_0)\mu_-(d\sigma), \tag{5.41}$$

where μ_+ *and* μ_- *are the* maximal *respectively* minimal *Gibbs states in* $\mathcal{G}^t$.
(iii) For all $x \in \mathbb{V}$ *and for any pair of boundary conditions* ξ, $\tilde{\xi} \in \Omega^t$ *it holds for every cofinal sequence* $\mathcal{L}$ *that*

$$\lim_{\mathcal{L}} \left(\int_\Omega \sigma(x)\pi_\Lambda(d\sigma|\xi) - \int_\Omega \sigma(x)\pi_\Lambda(d\sigma|\tilde{\xi}) \right) = 0. \tag{5.42}$$

Proof. i)$\Leftrightarrow$ ii):

For the extreme unique maximal and minimal elements of $\mathcal{G}^t$ we have the following inequality. For all $\mu \in \mathcal{G}^t$ and for all $f \in \mathcal{F}C_b^+(\Omega)$ we have

$$\mu_-(f) \leq \mu(f) \leq \mu_+(f). \tag{5.43}$$

Therefore, $|\mathcal{G}^t| = 1$ is equivalent to $\mu_+ = \mu_-$. By Corollary 5.10 this is equivalent to the identity valid for all $x \in \mathbb{V}$.

$$\int_\Omega \sigma(x)\mu_+(d\sigma) = \int_\Omega \sigma(x)\mu_-(d\sigma). \tag{5.44}$$

ii)$\Leftrightarrow$ iii): Let $\xi, \tilde{\xi}$ be given as well as a cofinal sequence $\mathcal{L}$. By Remark 5.11, one can find some $\bar{\Lambda} \Subset \mathbb{V}$ such that for all $\Lambda \in \mathcal{L}$ with $\bar{\Lambda} \subset \Lambda$ it holds

$$\pi_\Lambda(d\sigma \mid -\hat{\xi}) \prec \pi_\Lambda(d\sigma \mid \xi) \tag{5.45}$$

and

$$\pi_\Lambda(d\sigma \mid \tilde{\xi}) \prec \pi_\Lambda(d\sigma \mid \hat{\xi}), \tag{5.46}$$

where $\hat{\xi}$ is defined in the Remark 5.11(ii). Then, by the triangular inequality, the definition of stochastic domination and using (5.45) and (5.46), we receive

$$\begin{aligned}
& \left| \int_{\Omega^t} \sigma(x) \left[\pi_\Lambda(d\sigma \mid \xi) - \pi_\Lambda(d\sigma \mid \tilde{\xi}) \right] \right| \\
\leq \;& \left| \int_{\Omega^t} \sigma(x) \pi_\Lambda(d\sigma \mid \xi) - \int_{\Omega^t} \sigma(x) \pi_\Lambda(d\sigma \mid \hat{\xi}) \right| \\
+ \;& \left| \int_{\Omega^t} \sigma(x) \pi_\Lambda(d\sigma \mid \hat{\xi}) - \int_{\Omega^t} \sigma(x) \pi_\Lambda(d\sigma \mid \tilde{\xi}) \right| \\
\leq \;& \int_{\Omega^t} \sigma(x) \left[\pi_\Lambda(d\sigma \mid \hat{\xi}) - \pi_\Lambda(d\sigma \mid \hat{\xi}) \right],
\end{aligned} \tag{5.47}$$

which by (5.20) and (ii) implies (iii). Conversely, choosing the boundary conditions $\hat{\xi}$ respectively $-\hat{\xi}$ instead of ξ respectively $\tilde{\xi}$, (iii) immediately implies (ii).

$\square$

Corollary 5.23. *Suppose that U is symmetric, i.e., $U(s) = U(-s)$. Then the zero spontaneous magnetization $\mu_\pm(\sigma(x)) = 0$ for the minimal and maximal Gibbs measures implies uniqueness of the tempered Gibbs measures $\mathcal{G}^t$.*

5.3.2 Comparison criterion

Let us impose further conditions on J_{xy} and U_x. We assume that the interaction strength between the nearest neighbors is *uniformly nonzero*, i.e.,

$$J := \inf_{\substack{x,y \in \mathbb{V} \\ x \sim y}} J_{xy} > 0. \tag{5.48}$$

For the self interaction potentials $U_x \in \mathcal{C}(\mathbb{R}_+)$ we suppose that they are *even continuous functions* and can therefore be written, for all $s \in \mathbb{R}$, in the form

$$U_x(s) = U_x(-s) := u_x(s^2). \tag{5.49}$$

Our main assumption is that the *lower bound* of U_x can be chosen, uniformly for all $s \in \mathbb{R}$, as

$$U(s) = u(s^2), \tag{5.50}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a *convex function*. Furthermore, we suppose that the function $U_x(s) - U(s)$ is increasing on $\mathbb{R}_+$, that means, for all $s \leq \tilde{s}$

$$u_x(s) - u(s) \leq u_x(\tilde{s}) - u(\tilde{s}). \tag{5.51}$$

In particular, the *upper bound* in Assumption **U** can be chosen for all $s \geq 2$ as the polynomials given by

$$U(s) = \sum_{q=1}^{p} b^{(q)} s^{2q}, \tag{5.52}$$

with the nonnegative coefficient $b^{(q)} \geq 0$, for $q \geq 2$. In particular, $b^{(1)}$ could be negative which corresponds to the double-well potential $U(s)$.

Additionally, to our initial model (5.1) we introduce the so-called *reference models*. The first model, which we will introduce, is the *lower reference model*. For $\Lambda \Subset \mathbb{V}$ and $\sigma \in \Omega^t$ we set the local Hamiltonians

$$E_\Lambda^{low}(\sigma) := -\sum_{x,y \in \mathbb{V}} \tilde{J}_{xy} \sigma(x)\sigma(y) + \sum_{x \in \mathbb{V}} U(\sigma(x)), \tag{5.53}$$

where $\tilde{J}_{xy} = J$ for $x \sim y$ and zero otherwise. The second model is the *upper reference model*. For $\Lambda \Subset \mathbb{V}$, we set

$$E_\Lambda^{up}(\sigma) := -\sum_{x,y \in \mathbb{V}} J_{xy} \sigma(x)\sigma(y) + \sum_{x \in \mathbb{V}} u(\sigma(x)^2), \tag{5.54}$$

where J_{xy} satisfies Assumption (**J**). Together with the assumptions of Section 5.1.1 we surely have achieved all the results in Chapter 3 concerning the existence for the low/up reference model. The *extreme elements* are correspondingly denoted by $\mu_\pm^{low}$ and $\mu_\pm^{up}$. Since the potentials of both reference models have the same form as in (5.7), all the statements of Section 5.1.2 hold true.

Lemma 5.24. *For every $x \in \mathbb{V}$, it holds that*

$$\mu_+^{low}(\sigma(x)) \leq \mu_+(\sigma(x)) \leq \mu_+^{up}(\sigma(x)). \tag{5.55}$$

Proof. By (5.20) we have for any cofinal sequence $\mathcal{L}$ that

$$\int_{\Omega^t} \sigma(x)\mu_{\pm} = \lim_{\mathcal{L}} \int_{\Omega^t} \sigma(x)\pi_\Lambda(d\sigma \mid \pm\hat{\xi}), \tag{5.56}$$

with $\hat{\xi}$ defined by Remark 5.11. Therefore, it is sufficient to prove for all $x \in \Lambda \Subset \mathbb{V}$ that

$$\pi_\Lambda^{low}(\sigma(x) \mid \hat{\xi}) \leq \pi_\Lambda(\sigma(x) \mid \hat{\xi}) \leq \pi_\Lambda^{up}(\sigma(x) \mid \hat{\xi}). \tag{5.57}$$

First we prove

$$\pi_\Lambda^{low}(\sigma(x) \mid \hat{\xi}) \leq \pi_\Lambda(\sigma(x) \mid \hat{\xi}). \tag{5.58}$$

Let us introduce the following family of measures parameterized by $s, t \in [0, 1]$

$$\begin{aligned} \mu_\Lambda^{(s,t)}(d\sigma_\Lambda) : &= \frac{1}{Y(s,t)} \exp\Bigg(\sum_{x,y\in\Lambda} \tilde{J}_{xy}\sigma(x)\sigma(y) + \sum_{x\in\Lambda} \sigma(x)\eta_x^s \\ &- \sum_{x\in\Lambda} U(\sigma(x)) + s \sum_{x,y\in\Lambda} [J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y) \\ &- t\sum_{x\in\Lambda} [U_x(\sigma(x)) - U(\sigma(x))] \Bigg) \chi_\Lambda(d\sigma_\Lambda), \end{aligned} \tag{5.59}$$

where we define

$$\begin{aligned} \eta^s(x) : &= \sum_{y\in\Lambda^c} \tilde{J}_{xy}\tilde{\xi}(y) + s \sum_{y\in\Lambda^c} [J_{xy} - \tilde{J}_{xy}]\tilde{\xi}(y) \\ &\geq \sum_{y\in\Lambda^c} \tilde{J}_{xy}\tilde{\xi}(y) > 0. \end{aligned} \tag{5.60}$$

The partition function is given by

$$\begin{aligned} Y(s,t) &= \int_{\Omega^t} \exp\Bigg(\sum_{x,y\in\Lambda} \tilde{J}_{xy}\sigma(x)\sigma(y) + \sum_{x\in\Lambda} \sigma(x)\eta^s(x) \\ &- \sum_{x\in\Lambda} U(\sigma(x)) + s \sum_{x,y\in\Lambda} [J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y) \\ &- t\sum_{x\in\Lambda} [U_x(\sigma(x)) - U(\sigma(x))] \Bigg) \chi_\Lambda(d\sigma_\Lambda). \end{aligned} \tag{5.61}$$

Since the η_x^s is positive, the first moments of the measure (5.59), i.e.,

$$\mu_\Lambda^{(s,t)}(\sigma(x)) := \int_{\Omega^t} \sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda), \tag{5.62}$$

satisfies the GKS inequalities in Proposition 5.5. Therefore, for any $x \in \Lambda$ and $s, t \in [0,1]$, the function

$$\phi(s,t) = \mu_\Lambda^{(s,t)}(\sigma(x)), \tag{5.63}$$

is continuous and increasing in both variables. Indeed, taking into account (5.48), we have

$$\begin{aligned}
\frac{\partial}{\partial s}\phi(s,t) &= \frac{\partial}{\partial s}\int_{\Omega^t} \sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda) \\
&= \int_{\Omega^t} \sigma(x)\left[\frac{1}{Y(s,t)^2}\left(Y(s,t)\frac{\partial}{\partial s}\exp(\dots)\right.\right. \\
&- \left.\left.\exp(\dots)\frac{\partial}{\partial s}Y(s,t)\right)\right]\chi_\Lambda(d\sigma_\Lambda) \\
&= \int_{\Omega^t} \sigma(x)\left[\frac{1}{Y(s,t)}\left(\sum_{x\in\Lambda}\sum_{y\in\Lambda^c}[J_{xy} - \tilde{J}_{xy}]\tilde{\xi}(y)\sigma(x)\right.\right. \\
&+ \left.\left.\sum_{x,y\in\Lambda}[J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y)\right)\exp(\dots)\right]\chi_\Lambda(d\sigma_\Lambda) \\
&- \int_{\Omega^t} \sigma(x)\left[\frac{1}{Y(s,t)^2}\exp(\dots)\int_{\Omega^t}\left(\sum_{x\in\Lambda}\sum_{y\in\Lambda^c}[J_{xy} - \tilde{J}_{x,y}]\tilde{\xi}(y)\sigma(x)\right.\right. \\
&+ \left.\left.\sum_{x,y\in\Lambda}[J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y)\right)\exp(\dots)\chi_\Lambda(d\sigma_\Lambda)\right]\chi_\Lambda(d\sigma_\Lambda) \\
&= \int_{\Omega^t} \sigma(x)\left(\sum_{x\in\Lambda}\sum_{y\in\Lambda^c}[J_{xy} - \tilde{J}_{xy}]\tilde{\xi}(y)\sigma(x)\right. \\
&+ \left.\sum_{x,y\in\Lambda}[J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y)\right)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda) \\
&- \int_{\Omega^t} \sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\cdot\int_{\Omega^t}\left(\sum_{x\in\Lambda}\sum_{y\in\Lambda^c}[J_{xy} - \tilde{J}_{xy}]\tilde{\xi}(y)\sigma(x)\right. \\
&+ \left.\sum_{x,y\in\Lambda}[J_{xy} - \tilde{J}_{xy}]\sigma(x)\sigma(y)\right)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda) \\
&= \left(\sum_{x\in\Lambda}\sum_{y\in\Lambda^c}[J_{xy} - \tilde{J}_{xy}]\tilde{\xi}(y)\right)\cdot\left(\int_{\Omega^t}\sigma(x)\sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\right. \\
&- \left.\int_{\Omega^t}\sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\cdot\int_{\Omega^t}\sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\sum_{x,y\in\Lambda}[J_{xy}-\tilde{J}_{xy}]\right)\cdot\Bigg(\int_{\Omega^t}\sigma(x)\sigma(x)\sigma(y)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda) \\
&- \int_{\Omega^t}\sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\cdot\int_{\Omega^t}\sigma(x)\sigma(y)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\Bigg)\geq 0, \qquad (5.64)
\end{aligned}
$$

where $\exp(\dots)$ is a short hand for the exponential function under the integral 5.61. Following the same scheme of the last calculation we also get

$$
\begin{aligned}
\frac{\partial}{\partial t}\phi(s,t) &= \sum_{x\in\Lambda}\Bigg(\int_{\Omega^t}\sigma(x)(U(\sigma(x))-U_x(\sigma(x)))\mu_\Lambda^{(s,t)}(d\sigma_\Lambda) \\
&- \int_{\Omega^t}\sigma(x)\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\cdot\int_{\Omega^t}U(\sigma(x))-U_x(\sigma(x))\mu_\Lambda^{(s,t)}(d\sigma_\Lambda)\Bigg) \\
&= \sum_{x\in\Lambda}\Bigg(\mu_\Lambda^{(s,t)}\left[\sigma(x)(U(\sigma(x))-U_x(\sigma(x)))\right] \\
&- \mu_\Lambda^{(s,t)}[\sigma(x)]\cdot\mu_\Lambda^{(s,t)}[(U(\sigma(x))-U_x(\sigma(x)))]\Bigg)\geq 0. \qquad (5.65)
\end{aligned}
$$

At the same time by (5.59) and (5.63) we also have

$$
\begin{aligned}
\phi(0,0) &= \int_{\Omega^t}\sigma(x)\frac{1}{Y(0,0)}\exp\Bigg(\sum_{x,y\in\Lambda}\tilde{J}_{xy}\sigma(x)\sigma(y) \\
&+ \sum_{x\in\Lambda}\sum_{y\in\Lambda^c}\tilde{J}_{xy}\sigma(x)\hat{\xi}(y)-\sum_{x\in\Lambda}U(\sigma(x))\Bigg)\chi_\Lambda(d\sigma_\Lambda) \\
&= \pi_\Lambda^{low}(\sigma(x)|\hat{\xi}), \qquad (5.66)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(1,1) &= \int_{\Omega^t}\sigma(x)\frac{1}{Y(1,1)}\exp\Bigg(\sum_{x,y\in\Lambda}J_{xy}\sigma(x)\sigma(y) \\
&+ \sum_{x\in\Lambda}\sum_{y\in\Lambda^c}J_{xy}\sigma(x)\hat{\xi}(y)-\sum_{x\in\Lambda}U_x(\sigma(x))\Bigg)\chi_\Lambda(d\sigma_\Lambda) \\
&= \pi_\Lambda(\sigma(x)|\hat{\xi}). \qquad (5.67)
\end{aligned}
$$

Therefore we get (5.58). To prove

$$\pi_\Lambda(\sigma(x)|\hat{\xi})\leq\pi_\Lambda^{up}(\sigma(x)|\hat{\xi}), \qquad (5.68)$$

we have to change the variables in (5.59) as follows. Set $s=1$, $t\in[0,1]$ and $u(\sigma(x)^2)$ instead of $U(\sigma(x))$. Afterwards we recapitulate the above steps.

□

The following statements, whose confirmation follows from Theorem 5.22 and Lemma 5.24, gives us a proper *uniqueness criterion* for the reference models.

Proposition 5.25 (Comparison Criterion for the reference models)**.** *The initial model* (5.1) *undergoes a phase transition if the low-reference model does so. The uniqueness of tempered Gibbs measures of the up-reference model implies that* $|\mathcal{G}^t| = 1$.

We would like to point out that the above criterion first occurred both for the classical and for the quantum spin systems on the lattice $\mathbb{Z}^d$ in [KoPa 2007]. Here, we give a generalization onto graphs for classical spin systems.

5.4 Phase transitions

In this section we will present a new method showing phase transition in *unbounded* spin systems. Since this result is based on the classical Ising model we open this Section with an introduction on the famous Ising model. After describing the spontaneous magnetization phenomenon (i.e. phase transition) we give the crucial new Theorem 5.29 of this Section. In the Section 5.4.3 we will introduce concepts leading step by step to the justification of Theorem 5.29.

5.4.1 The Ising model

The most popular model in the statistical equilibrium mechanics is the *classical ferromagnetic Ising model.* It has played a crucial role in the history of statistical physics. This model was first introduced by W. Lenz in [Lz 1920] and his student E. Ising in [Is 1925] in order to describe *spontaneous magnetization* (i.e. *phase transitions*) of a ferromagnetic (attractive) substance on $\mathbb{Z}^d$. This setup means the following: The lattice $\mathbb{Z}^d$ represents the positions of atoms in a regular substance and each atom is endowed with a magnetic moment (with only the two values $+1$ and -1). As we already mentioned in the introduction the classical Ising model fails to produce the spontaneous magnetization phenomenon in one dimension, which was Ising's main claim in [Is 1925]; for an updated proof we also refer to the monograph [Bov 2006] by A. Bovier. However, Ising's heuristic, but incorrect, argumentation for extending this result to higher dimensions is the reason why the model has been forgotten for several years. Later the model had its revival through Peierls in [Pe 1936]. For a detailed discussion and historical comments we refer to the manuscripts [Bov 2001, Bov 2006], [GeHäMa 2000] and [EvKePeSc 2000].

Today, it is a standard model for magnetism on a lattice $\mathbb{Z}^d$, $d \geq 2$, and the most studied model in statistical mechanics. The main issue was to discover critical temperatures in order to verify the exact threshold between uniqueness and non-uniqueness of Gibbs measures. Especially in [On 1944] by L. Onsager a critical value on $\mathbb{Z}^2$ was obtained. A substantial progress in the higher dimensions ($d \geq 3$) was achieved e.g. in [Pr 1974] by C. J. Preston and [Ge 1988] by H.-O. Georgii. The next natural and important step is to consider the Ising model on general graphs, in particular on trees, see [Ly 1989] by R. Lyons. In the next subsections we will reflect these results.

Gibbs states in Ising model

The *ferromagnetic Ising model* describes a system of atoms in a piece of iron which is naturally regarded as a regular lattice. Each atom has some inherent random magnetization and they interact with each other. The mathematical setup can be given as follows. As the spin space we chose $\{+1, -1\}$ with the spins denoted by $\sigma(x) = \pm 1$ for all $x \in \mathbb{V}$. The formal Hamiltonian has the form

$$E(\sigma) = -\sum_{\substack{x,y \in \mathbb{V} \\ x \sim y}} \sigma(x)\sigma(y).$$

To proceed rigorously, we consider, for each $\Lambda \Subset \mathbb{V}$, the following local Hamiltonian with boundary conditions $\xi(y) = \pm 1$, $y \in \Lambda^c$, by

$$E_\Lambda(\sigma_\Lambda|\xi) = -\sum_{\substack{x,y \in \Lambda \\ x \sim y}} \sigma(x)\sigma(y) - \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c \\ x \sim y}} \sigma(x)(\xi(y)). \tag{5.69}$$

The *local Gibbs measure* on $\Omega_\Lambda := \{-1, +1\}^\Lambda$ corresponding to the local Hamiltonian (5.69) at inverse temperature $\beta > 0$ is given by

$$\mu_\Lambda^\beta(d\sigma_\Lambda|\xi) := \frac{1}{Z_\Lambda^\beta(\xi)} \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\nu_\Lambda(d\sigma_\Lambda), \tag{5.70}$$

where

$$Z_\Lambda^\beta(\xi) := \int \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\nu_\Lambda(d\sigma_\Lambda), \tag{5.71}$$

is the *normalizing constant* and $\nu_\Lambda(d\sigma_\Lambda) := \prod_{x\in\Lambda} \nu_x(d\sigma(x))$ is the *product measure* on the *product space* $\mathbb{R}^\Lambda := \prod_{x\in\Lambda} \mathbb{R}$. The *reference measure* $\nu_x(d\sigma(x))$ is the Dirac measure concentrated on $\{+1, -1\}$ such that $\nu_x(\{\pm 1\}) = \frac{1}{2}$. For a detailed construction we refer to Section 5.4.3.

The Gibbs measures in the infinite volume are defined through their local specification $\mu_\Lambda^\beta(d\sigma_\Lambda|\xi)$ via the DLR equilibrium equations (2.23). By a general compactness argument, the set $\mathcal{G}$ of Gibbs measures (at a given temperature β) is not empty, see e.g. [Si 1982]. In the case of $\xi(y) = 1$ for all $y \in \mathbb{V}$ we will denote the corresponding local Gibbs measures by $\mu_\Lambda^\beta(d\sigma_\Lambda|+)$. Analogously, for the boundary condition $\xi(y) = -1$ we will define $\mu_\Lambda^\beta(d\sigma_\Lambda|-)$. Then we define the *infinite volume limit* of the local Gibbs measures by $\mu_\pm = \lim_{\Lambda\nearrow\mathbb{V}} \mu_\Lambda^\beta(d\sigma_\Lambda|\pm)$. Note that μ_+ and μ_- are respectively the unique maximum and minimum elements in the set $\mathcal{G}$, see Subsection 5.1.3.

5.4.2 Spontaneous magnetization

As we described above the phenomenon of *spontaneous magnetization* corresponds to a *phase transition*, and therefore to the *non-uniqueness* of Gibbs measures. In fact, the occurrence of a phase transition on the lattice $\mathbb{Z}^d$, $d \geq 2$, depends on the inverse temperature $\beta > 0$ of the system. To be precisely, there will be a sharp critical inverse temperature $\beta_c > 0$ for which a phase transition occurs if $\beta > \beta_c$, which is the so-called *low temperature regime*. In this case the interaction of the particles becomes so strong that a *long range order* appears, i.e., the Gibbs measures prefer configurations with either spins with $+1$ or spins with -1, and this preference even survives in the infinite volume limit. Thus, there exist two different Gibbs measures μ_+ and μ_- corresponding to the boundary condition $\xi(y) = +1$ respectively $\xi(y) = -1$, for all $y \in \mathbb{Z}^d$, $d \geq 2$. In contrast, when $\beta < \beta_c$, which is the so-called *high temperature regime*, the interaction is not strong enough to produce any long range order, so that the boundary condition is negligible in the infinite volume limit and the Gibbs measure is hence unique.

The critical behavior of the system is monotonically depending on β, which means if $\beta_1 < \beta_2$ and a phase transition occurs for $\beta = \beta_1$, then it also occurs for $\beta = \beta_2$. This monotonicity was originally proved by using the Griffiths inequalities in [Gr 1967]. Together with Dobrushin's uniqueness condition in [Do 1968] the following classical theorem arises.

Theorem 5.26. *For the ferromagnetic Ising model on $\mathbb{Z}^d$, $d \geq 2$, there exists a critical inverse temperature $\beta_c = \beta_c(d) \in (0, \infty)$ such that for $\beta < \beta_c$ the* *model has a unique Gibbs measure while for $\beta > \beta_c$ there are multiple Gibbs measures.*

The classical proof of this theorem can be found in [Pe 1936], see also [Do 1965] and [Do2 1968]. Allowing for β_c the value ∞ we get for $\mathbb{Z}^1$ the critical value $\beta_c = \infty$. This means that there is a unique Gibbs measure for all $\beta > 0$. For $\mathbb{Z}^2$ the critical value has been found by L. Onsager in [On 1944] to be $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$, which was a very notable result. In the paper [AbMa-Lö 1973] by D. B. Abraham and A. Martin-Löf it was also shown that the Ising model has the unique Gibbs measure exactly at this critical value $\beta = \beta_c$. For higher dimensions a rigorous calculation of the critical value is beyond our knowledge. It is supposed that uniqueness holds at a certain critical value in all dimensions $d \geq 2$, but until now this is only known for $d = 2$ and $d \geq 4$, see [AiFe 1986] by M. Aizenman and R. Fernandez.

The next step in studying phase transitions is to consider trees instead of the lattice $\mathbb{Z}^d$. With the notion of a *tree* we mean a countable connected graph which has no loops or cycles and which is locally finite, i.e., each vertex belongs only to a finite number of edges. The study of the Ising model on trees was initiated in [KuKiWa 1953] by M. Kurata, R. Kikuchi and T. Watari. They discussed the case for *regular* trees of degree $b+1$ and found that the critical value is $\beta_c = coth^{-1}b$. We define $coth^{-1}b$ as the inverse of the cotangens hyperbolicus, which can be also represented as $\frac{1}{2}\log(\frac{b+1}{b-1})$. This means that there is a unique Gibbs measure for $\beta < coth^{-1}b$ and phase transition for $\beta > coth^{-1}b$. We also refer to the basic books [Pr 1974] by C. J. Preston and [Ge 1988] by H.-O. Georgii. An extension to *general* trees was given by R. Lyons in [Ly 1989]. In order to recall that result, we have to introduce the notion of *branching number*, see [Ly 1989] and [Ly 1990].

Definition 5.27. *Let T be tree with a root vertex o. If $x \in T$ is a vertex, we write $d(x,o)$, which is the combinatorial distance of x to o. A* cutset*, Δ, is a finite set of vertices such that every infinite path from o intersects Δ. The* branching number of T*, denoted by $br(T)$, is defined by*

$$br(T) := \inf\left\{\lambda > 0 \middle| \inf_{\Delta} \sum_{x\in\Delta} \frac{1}{\lambda^{|x|}} = 0\right\}. \tag{5.72}$$

Obviously, for a regular tree of degree $b+1$ we have just $br(T) = b$.

Theorem 5.28. *Let T be a general tree, then its critical β_c equals $coth^{-1}br(T)$, where $br(T)$ is the* branching number*. This means for $\beta < \beta_c$ we have uniqueness and $\beta > \beta_c$ we have phase transition.*

For big $br(T)$ we would have $coth^{-1}br(T) \searrow 0$. This means that adding edges and vertices to a tree can only increase its critical temperature, see [Lig 1985], Theorem IV. 1.21. Take note, that the situation for $\beta = \beta_c$ is not clear.

Standard methods for proving phase transitions on a lattice $\mathbb{Z}^d$ are the Peierls argument [Pe 1936] and the reflection positivity method [GlJa 1981] and [FrSiSp 1976]. The Peierls argument crucially uses the translation-invariance of the interaction, see the classical references [Si 1982] and [Za 2000]. The reflection positivity method exploits special symmetries of the underlying lattice and involves, as a part, the so-called infrared (Gaussian) bounds on two-point correlation function. Some (mathematically non-rigorous) attempts to generalize the infrared bounds to general graphs were done by the physicists R. Burioni and D. Cassi in [BuCa 1998] and [BuCa 2005]. To summarize, both methods do not apply to our graph systems because of the absence of translation invariance and proper symmetries.

So, our aim is to develop an alternative method of proving phase transitions for systems of *unbounded continuous* spins on infinite graphs. In this area we have the following important new theorem. In the next section we will introduce concepts that justify this theorem.

Theorem 5.29 (Main Theorem)**.** *Let T be a general tree with the branching number $br(T) < \infty$. Let us consider the scalar ferromagnetic model* (5.1) *with the even self interaction potential (double-well potential)*

$$V(s) := s^4 - \kappa s^2, \quad s \in \mathbb{R},$$

for a given $\kappa > 0$. Then the critical inverse temperature of this system equals

$$\beta_c = \frac{8coth^{-1}br(T)}{\kappa J}.$$

This means that we have phase transition for $\beta > \beta_c$, which is the case for big κ or big J.

Remark 5.30. *(i) We also can consider double-well potentials $V(s) := s^{2n} - \kappa s^2$, with integer $n > 2$ and $k > 0$, with a different β_c for each $n \in \mathbb{N}$. (ii) Combining this with the comparison criterion we can treat non-polynomial $U(s)$.*

5.4.3 Wells' inequality and applications

In this section we study a correlation inequality which seems to be very useful for Gibbs measures. It was first discovered in D. Wells PhD thesis

[We 1977]. This so-called *Wells' inequality* is a relation between certain moments, i.e., expectation values of an even probability measure and a proper Dirac measure. The Wells' inequality claims that the polynomial moments of the Dirac measure is always smaller then those of the even probability measure. The first published proof of this inequality can be found in the paper [BrLePf 1981] by J. Bricmont, J. L. Lebowitz and C. E. Pfister. However, we would like to draw attention to a misprint in the calculation of the published proof which is important since subsequent papers are based on this proof. For example in the paper [OsSp 1999] by H. Osada and H. Spohn we find a wrong condition for Wells' inequality which is weaker then the correct condition. It is mentionable that the calculations in the original paper [We 1977] are correct. However, precisely because [We 1977] is inaccessible for the wide audience, we are obliged to give an accurate proof of Wells' inequality. Although papers on this inequality are very rare, it seems to be a *fundamental tool* in the theory of phase transitions. As will be shown below, Wells' inequality gives an elementary new method to prove the existence of phase transitions. In Subsection 5.4.4 we consider the basic examples of Section 4.5 and give concrete thresholds so that the Wells' inequality holds.

The model

Let us introduce the model. On the configuration space $\Omega = \mathbb{R}^{\mathbb{V}}$, which is defined in (5.21), we consider an infinite spin system with multi-particle ferromagnetic interaction. For each $x \in \mathbb{V}$ we define an even probability measure $\nu_x(ds) = \nu(ds)$ on $\mathbb{R}$, which will be called the *reference measure*. We associate with each site $x \in \mathbb{V}$ a *spin variable* $\sigma(x) \in \mathbb{R}$. For all $\Lambda \Subset \mathbb{V}$ we have the *configuration* $\sigma_\Lambda := (\sigma(x))_{x\in\Lambda} \in \mathbb{R}^\Lambda$. By $\nu_\Lambda(d\sigma_\Lambda) = \prod_{x\in\Lambda} \nu_x(d\sigma(x))$ we define the *product measure* on $\mathbb{R}^\Lambda$. For each $x \in \mathbb{V}$, the *reference measure* is defined as

$$\nu_x(ds) = \frac{1}{Z} \exp\{-U(s)\} ds, \tag{5.73}$$

where $U(s)$ is the self interaction potential satisfying

$$U(s) = U(-s)$$

and

$$U(s) \geq C|s|^P - B,$$

for $s \in \mathbb{R}$ and $P > 2$.

Let A be a *multi-index with compact support*, i.e., $A := (\alpha_x)_{x\in\mathbb{V}}$, $\alpha_x \in \mathbb{Z}_+$, so that $\alpha_x \neq 0$ for finitely many $x \in \mathbb{V}$. We define the *set of all such multi-indices* by

$$\mathcal{A} := (\mathbb{Z}_+)_0^{\mathbb{V}} := \{A := (\alpha_x)_{x\in\mathbb{V}} \mid \alpha_x \in \mathbb{Z}_+, \alpha_x = 0 if \forall x \in \Lambda^c, \Lambda \Subset \mathbb{V}\}.$$

Let us denote the *support of* $A \in \mathcal{A}$ by $\Lambda_A := suppA \Subset \mathbb{V}$. We define the *cardinality* of $A \in \mathcal{A}$ as

$$|A| := \sum_{x\in\mathbb{V}} \alpha_x. \tag{5.74}$$

By $\mathcal{A}_R$ we define the subset of all $A \in \mathcal{A}$ with $|A| \le R < \infty$. For $A \in \mathcal{A}$ we define the *product* of the corresponding spin variables as

$$\sigma^A := \prod_{x\in\mathbb{V}} \sigma(x)^{\alpha_x} = \prod_{x\in\Lambda_A} \sigma(x)^{\alpha_x}. \tag{5.75}$$

For any $A \in \mathcal{A}$ and $\Lambda \Subset \mathbb{V}$ we have the decomposition $A = A_\Lambda \cup A_{\Lambda^c}$, so that $suppA_\Lambda \Subset \Lambda$ and $suppA_{\Lambda^c} \Subset \Lambda^c$. Then, for each $\Lambda \Subset \mathbb{V}$, we define the *local Hamiltonian* as a function on $\mathbb{R}^\Lambda$ by

$$E_\Lambda(\sigma_\Lambda|\xi) := -\sum_{A:\Lambda_A\subset\Lambda} J_A\sigma^A - \sum_{\substack{A:\Lambda_A\cap\Lambda\neq\emptyset \\ \Lambda_A\cap\Lambda^c\neq\emptyset}} J_A(\sigma^{A_\Lambda}\xi^{A_{\Lambda^c}}), \tag{5.76}$$

where $\xi_{\Lambda^c} = (\xi(y))_{x\in\Lambda^c}$ is a fixed *boundary condition.* The Hamiltonian (5.76) is called to be *general spin Ising ferromagnet with multi-particle interactions.* We assume that the *interaction strength* J_A is equal zero if $|A| > R$ or $diam(suppA) > R_0$ for some $0 < R < P$ and $R_0 > 0$. Take note that $\sigma^{A_\Lambda} := \prod_{x\in suppA_\Lambda} \sigma(x)^{\alpha_x}$ and $\xi^{A_{\Lambda^c}} := \prod_{x\in suppA_{\Lambda^c}} \xi(x)^{\alpha_x}$, so that $\sigma^A = \sigma^{A_\Lambda}\xi^{A_{\Lambda^c}} = \prod_{x\in\Lambda_A} \sigma(x)^{\alpha_x}\xi(x)^{\alpha_x}$. So, the Hamiltonian (5.76) is well-defined. We assume the above Hamiltonian to be *ferromagnetic*, i.e., $J_A \ge 0$ for all $A \in \mathcal{A}$. The corresponding *local Gibbs measures* (at the inverse temperature β) are given by

$$\mu_\Lambda(d\sigma_\Lambda|\xi) := Z_\Lambda^{-1}(\xi)\exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\nu_\Lambda(d\sigma_\Lambda), \tag{5.77}$$

where

$$Z_\Lambda(\xi) := \int_{\mathbb{R}^\Lambda} \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\nu_\Lambda(d\sigma_\Lambda).$$

We denote the *expectation value* of σ^A with respect to $\mu_\Lambda(\sigma_\Lambda|\xi)$ as

$$\begin{aligned}\langle\sigma^A\rangle^\xi_{\mu_\Lambda} &:= Z_\Lambda^{-1}(\xi)\int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\nu_\Lambda(d\sigma_\Lambda) \\ &= \int_{\mathbb{R}^\Lambda} \sigma^A \mu_\Lambda(d\sigma_\Lambda|\xi).\end{aligned}$$

For the *local Gibbs measures* (5.77) we assume, for all $A \in \mathcal{A}$,

$$\int_{\mathbb{R}^\Lambda} \sigma^{|A|} \mu_\Lambda(d\sigma_\Lambda|\xi) < \infty. \tag{5.78}$$

The finiteness of all polynomial moments can be achieved through growth conditions on the Hamiltonian $E_\Lambda(\sigma_\Lambda|\xi)$. For more details we refer to the first chapter of this manuscript. With the Assumptions ($\mathbf{W_\Delta}$) and ($\mathbf{J_\Delta}$) from Subsection 3.6.3 we guarantee the existence of Gibbs measures $\mu \in \mathcal{G}$ as well as the finiteness of all moments in (5.78).

In order to state Wells' inequality we need some more definitions. For $a > 0$ and $s \in \mathbb{R}$ the *Dirac measures* $\delta_a(ds)$ and $\delta_{-a}(ds)$ on $\mathbb{R}$ we define

$$\delta_{\pm a}(ds) := \frac{1}{2}[\delta_a(ds) + \delta_{-a}(ds)].$$

For all $\Lambda \Subset \mathbb{V}$ we define $\delta_{\pm a}(d\sigma_\Lambda) := \prod_{x \in \Lambda} \delta_{\pm a}(d\sigma(x))$ as the *product measure* on $\mathbb{R}^\Lambda$. For $A \in \mathcal{A}$ we introduce the corresponding *local Gibbs measure* by

$$\mu^{IS}_{\Lambda,\pm a}(d\sigma_\Lambda|\xi) := Z^{-1}_{\Lambda,\delta_{\pm a}}(\xi) \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\delta_{\pm a}(d\sigma_\Lambda), \tag{5.79}$$

where

$$Z_{\Lambda,\delta_{\pm a}}(\xi) := \int_{\mathbb{R}^\Lambda} \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\delta_{\pm a}(d\sigma_\Lambda), \tag{5.80}$$

is the corresponding partition function and $E_\Lambda(\sigma_\Lambda|\xi)$ is the same Hamiltonian as in (5.76). We denote the *expectation value* of σ^A with respect to $\mu^{IS}_{\Lambda,\pm a}(d\sigma_\Lambda|\xi)$ as

$$\begin{aligned} \langle \sigma^A \rangle^\xi_{\mu^{IS}_{\Lambda,\pm a}} &:= Z^{-1}_{\Lambda,\delta_{\pm a}}(\xi) \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right)\delta_{\pm a}(d\sigma_\Lambda) \\ &= \int_{\mathbb{R}^\Lambda} \sigma^A \mu^{IS}_\Lambda(d\sigma_\Lambda|\xi). \end{aligned} \tag{5.81}$$

Remark 5.31. *In the case of $a = 1$ and pair interactions with nearest neighbors we have the standard Ising model* (5.1).

Well's moment inequality

We now present and prove *Wells' inequality.* Originally, in the PhD thesis of D. Wells [We 1977], this inequality was proved for *unbounded spins with pair interactions.* Whereas the published proof in [BrLePf 1981] was only in the case of *compact spins* but with *multi-particle interactions*. We prove it for the case of *unbounded spins with multi-particle interactions*, which is the most possible generalization. We now present the main theorem of this section.

Theorem 5.32 (Wells' inequality). *Let, for all $x \in \mathbb{V}$, $\nu_x = \nu$ be an even probability measure on $\mathbb{R}$ with $\nu(\{0\}) < 1$ and let the Hamiltonian (5.76) be ferromagnetic as described above. Then there exists a constant $a > 0$ such that for all $A \in \mathcal{A}$ and $\Lambda \Subset \mathbb{V}$ and for every nonnegative boundary condition $\xi = (\xi(y))_{y\in\Lambda^c} \in \Omega_{\Lambda^c}$, it holds*

$$\langle \sigma^A \rangle^{\xi}_{\mu_\Lambda} \geq \langle \sigma^A \rangle^{\xi}_{\mu^{IS}_{\Lambda,\pm a}}. \tag{5.82}$$

Proof. To prove this inequality we introduce a duplicate graph system with the same Hamiltonian (5.76) labeled by the duplicate variables $\sigma_\Lambda = (\sigma(x))_{x\in\Lambda}$ and $\tilde{\sigma}_\Lambda = (\tilde{\sigma}(x))_{x\in\Lambda}$. The claim (5.82) is equivalent to the following. For a proper $a > 0$ we have to show

$$\frac{1}{Z_{\Lambda,\mu_\Lambda}(\xi) Z_{\Lambda,\delta_{\pm a}}(\xi)} \Bigg(Z_{\Lambda,\delta_{\pm a}}(\xi) \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right) \prod_{x\in\Lambda} \nu_x(d\sigma(x))$$
$$- Z_{\Lambda,\mu_\Lambda}(\xi) \int_{\mathbb{R}^\Lambda} \tilde{\sigma}^A \exp\left(-\beta E_\Lambda(\tilde{\sigma}_\Lambda \mid y)\right) \prod_{x\in\Lambda} \delta_{\pm a}(d\tilde{\sigma}(x)^2) \Bigg) \geq 0. \tag{5.83}$$

Since $Z_{\Lambda,\mu_\Lambda}(\xi) Z_{\Lambda,\delta_{\pm a}}(\xi)$ is positive it is enough to show that the term in the bracket is positive. So,

$$\begin{aligned}
& Z_{\Lambda,\delta_{\pm a}}(\xi) \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right) \prod_{x\in\Lambda} \nu_x(d\sigma(x)) \\
& - Z_{\Lambda,\mu_\Lambda}(\xi) \int_{\mathbb{R}^\Lambda} \tilde{\sigma}^A \exp\left(-\beta E_\Lambda(\tilde{\sigma}_\Lambda \mid y)\right) \prod_{x\in\Lambda} \delta_{\pm a}(d\tilde{\sigma}(x)) \\
= & \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right) \exp\left(-\beta E_\Lambda(\tilde{\sigma}_\Lambda \mid y)\right) \prod_{x\in\Lambda} \nu_x(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)) \\
& - \int_{\mathbb{R}^\Lambda} \tilde{\sigma}^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda|\xi)\right) \exp\left(-\beta E_\Lambda(\tilde{\sigma}_\Lambda \mid y)\right) \prod_{x\in\Lambda} \nu_x(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)) \\
= & \int_{\mathbb{R}^\Lambda} (\sigma^A - \tilde{\sigma}^A) \exp\left[\left(-\beta E_\Lambda(\sigma_\Lambda|\xi) - \beta E_\Lambda(\tilde{\sigma}_\Lambda \mid y)\right)\right] \prod_{x\in\Lambda} \nu_x(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)) \\
= & \int_{\mathbb{R}^\Lambda} (\sigma^A - \tilde{\sigma}^A) \exp\Bigg[\beta \sum_{A:\Lambda_A \subset \Lambda} J_A(\sigma^A + \tilde{\sigma}^A)
\end{aligned}$$

$$+\beta \sum_{\substack{A:\Lambda_A\cap\Lambda\neq\emptyset \\ \Lambda_A\cap\Lambda^c\neq\emptyset}} J_A \xi^{A_{\Lambda^c}}(\sigma^{A_\Lambda}+\tilde{\sigma}^{A_\Lambda})\Big] \prod_{x\in\Lambda} \nu_x(d\sigma(x))\delta_{\pm a}(d\tilde{\sigma}(x)). \qquad (5.84)$$

So, (5.83) will surely hold if (5.84) is nonnegative. The right-hand side of (5.84) adopts the following form, where we stress that all constants are included in the interaction constant $J_A \geq 0$.

$$\int_{\mathbb{R}^\Lambda} (\sigma^A - \tilde{\sigma}^A) \exp\left(\beta \sum_{A:\Lambda_A\subset\Lambda} J_A(\sigma^A+\tilde{\sigma}^A)\right) \prod_{x\in\Lambda} \nu_x(d\sigma(x))\delta_{\pm a}(d\tilde{\sigma}(x)).$$

Now, by expansion of the exponential function in terms of power series and interchanging the summation and integral signs, we obtain

$$\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^\Lambda} (\sigma^A - \tilde{\sigma}^A)\left(\beta \sum_{A:\Lambda_A\subset\Lambda} J_A(\sigma^A+\tilde{\sigma}^A)\right)^m \prod_{x\in\Lambda} \nu_x(d\sigma(x))\delta_{\pm a}(d\tilde{\sigma}(x)). \qquad (5.85)$$

Hence, it is sufficient to show that each term in (5.85) is positive. Since we are in the ferromagnetic case $J_A \geq 0$ and $\beta > 0$, through expanding the sum over $A : \Lambda_A \subset \Lambda$, it is enough to show for all $m \in \mathbb{N}$ that

$$\int_{\mathbb{R}^\Lambda} (\sigma^A - \tilde{\sigma}^A)(\sigma^A+\tilde{\sigma}^A)^m \prod_{x\in\Lambda} \nu_x(d\sigma(x))\delta_{\pm a}(d\tilde{\sigma}(x)) \geq 0. \qquad (5.86)$$

Let $|\Lambda| = k$ be the cardinality of Λ and, for $A \in \mathcal{A}$ so that $\Lambda_A \subset \Lambda$, $\sigma^A = \sigma_{x_1}^{\alpha_{x_1}} \cdot \ldots \cdot \sigma_{x_k}^{\alpha_{x_k}}$. Then we have

$$\begin{aligned}
\sigma^A - \tilde{\sigma}^A &= \prod_{q=1}^{k} \sigma_{x_q}^{\alpha_{x_q}} - \prod_{q=1}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}} \\
&= \frac{1}{2}(\sigma_{x_1}^{\alpha_{x_1}} + \tilde{\sigma}_{x_1}^{\alpha_{x_1}})(\prod_{q=2}^{k} \sigma_{x_q}^{\alpha_{x_q}} - \prod_{q=2}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}}) + \frac{1}{2}(\sigma_{x_1}^{\alpha_{x_1}} - \tilde{\sigma}_{x_1}^{\alpha_{x_1}})(\prod_{q=2}^{k} \sigma_{x_q}^{\alpha_{x_q}} + \prod_{q=2}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}})
\end{aligned}$$

and

$$\begin{aligned}
\sigma^A + \tilde{\sigma}^A &= \prod_{q=1}^{k} \sigma_{x_q}^{\alpha_{x_q}} + \prod_{q=1}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}} \\
&= \frac{1}{2}(\sigma_{x_1}^{\alpha_{x_1}} + \tilde{\sigma}_{x_1}^{\alpha_{x_1}})(\prod_{q=2}^{k} \sigma_{x_q}^{\alpha_{x_q}} + \prod_{q=2}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}}) + \frac{1}{2}(\sigma_{x_1}^{\alpha_{x_1}} - \tilde{\sigma}_{x_1}^{\alpha_{x_1}})(\prod_{q=2}^{k} \sigma_{x_q}^{\alpha_{x_q}} - \prod_{q=2}^{k} \tilde{\sigma}_{x_q}^{\alpha_{x_q}}).
\end{aligned}$$

By iteration, we obtain a factorized form of the polynomials $\sigma^A - \tilde{\sigma}^A$ and $\sigma^A + \tilde{\sigma}^A$ with nonnegative coefficients, see Lemma 4.1.2. in [GlJa 1981]. Then the integral (5.86) becomes by its linearity a sum of integrals of the form

$$c \int_{\mathbb{R}^\Lambda} \prod_{x\in\Lambda} (\sigma(x) - \tilde{\sigma}(x))^{n_x} (\sigma(x) + \tilde{\sigma}(x))^{m_x} \prod_{x\in\Lambda} \nu_x(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)), \quad (5.87)$$

where $c > 0$ coming from the positive coefficients of the polynomial and $m_x, n_x \in \mathbb{N}$. Hence, the integral (5.87) decomposes into a product of one-dimensional parts

$$c \prod_{x\in\Lambda} \int_{\mathbb{R}^2} (\sigma(x) - \tilde{\sigma}(x))^{n_x} (\sigma(x) + \tilde{\sigma}(x))^{m_x} \nu_x(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)). \quad (5.88)$$

Now we are reduced to show for all $x \in \mathbb{V}$, all $m_x, n_x \in \mathbb{N}$, $m_x \geq n_x$ and any even probability measure $\nu_x = \nu$ that

$$\int_{\mathbb{R}^2} (\sigma(x) - \tilde{\sigma}(x))^{n_x} (\sigma(x) + \tilde{\sigma}(x))^{m_x} \nu(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)) \geq 0. \quad (5.89)$$

Obviously, for m_x, n_x both even, the integral (5.89) is nonnegative. For the case where exactly one of m_x or n_x is odd, by using the symmetry of the measures $\nu(d\sigma(x))$ and $\delta_{\pm a}(d\tilde{\sigma}(x))$, i.e., they are symmetric under $\sigma(x) \to -\sigma(x)$, the integral (5.89) is again nonnegative. Indeed, problems occur when m_x, n_x are both odd. In this case we integrate over the Dirac measure which gives us, for $m_x \geq n_x$,

$$\begin{aligned}
& \int_{\mathbb{R}^2} (\sigma(x) - \tilde{\sigma}(x))^{n_x} (\sigma(x) + \tilde{\sigma}(x))^{m_x} \nu(d\sigma(x)) \delta_{\pm a}(d\tilde{\sigma}(x)) \\
= \; & \frac{1}{2} \int_{\mathbb{R}} (\sigma(x) - a)^{m_x} (\sigma(x) + a)^{n_x} + (\sigma(x) + a)^{m_x} (\sigma(x) - a)^{n_x} \nu(d\sigma(x)) \\
= \; & \frac{1}{2} \int_{\mathbb{R}} (\sigma(x) - a)^{m_x} \left(\frac{\sigma(x)^2 - a^2}{\sigma(x) - a} \right)^{n_x} \\
& + (\sigma(x) + a)^{m_x} \left(\frac{\sigma(x)^2 - a^2}{\sigma(x) + a} \right)^{n_x} \nu(d\sigma(x)) \\
= \; & \frac{1}{2} \int_{\mathbb{R}} (\sigma(x)^2 - a^2)^{n_x} \Big[(\sigma(x) - a)^{m_x - n_x} \\
& + (\sigma(x) + a)^{m_x - n_x} \Big] \nu(d\sigma(x)). \qquad (5.90)
\end{aligned}$$

Since $m_x - n_x$ is even, the term $(\sigma(x) - a)^{m_x - n_x} + (\sigma(x) + a)^{m_x - n_x}$ is an increasing function of $|\sigma(x)|$. Hence, on the interval $|\sigma(x)| \leq a$

$$\begin{aligned} & (\sigma(x)^2 - a^2)^{n_x} \Big[(\sigma(x) - a)^{m_x - n_x} + (\sigma(x) + a)^{m_x - n_x} \Big] \\ \geq \;& (\sigma(x)^2 - a^2)^{n_x} (2a)^{m_x - n_x}, \end{aligned} \tag{5.91}$$

since $(\sigma(x)^2 - a^2)^{n_x} \leq 0$ and $(\sigma(x) - a)^{m_x - n_x} + (\sigma(x) + a)^{m_x - n_x} \leq (2a)^{m_x - n_x}$. On the interval $|\sigma(x)| \geq a$ we have the same as (5.91), since $(\sigma(x)^2 - a^2)^{n_x} \geq 0$ and $(\sigma(x) - a)^{m_x - n_x} + (\sigma(x) + a)^{m_x - n_x} \geq (2a)^{m_x - n_x}$. Therefore, we can estimate for all $a > 0$ the integral (5.90) from below by

$$\frac{(2a)^{m_x - n_x}}{2} \int_{\mathbb{R}} \Big(\sigma(x)^2 - a^2 \Big)^{n_x} \nu(d\sigma(x)). \tag{5.92}$$

Now we search for the constant a satisfying the claim of the theorem. We finally calculate the integral in (5.92).

$$\begin{aligned} & \int_{\mathbb{R}} \Big(\sigma(x)^2 - a^2 \Big)^{n_x} \nu(d\sigma(x)) \\ = \;& \int_{|\sigma(x)| < a} \Big(\sigma(x)^2 - a^2 \Big)^{n_x} \nu(d\sigma(x)) \\ & + \int_{a \leq |\sigma(x)| \leq \sqrt{2}a} \Big(\sigma(x)^2 - a^2 \Big)^{n_x} \nu(d\sigma(x)) \\ & + \int_{|\sigma(x)| > \sqrt{2}a} \Big(\sigma(x)^2 - a^2 \Big)^{n_x} \nu(d\sigma(x)) \\ \geq \;& -2a^{2n_x} \nu([0, a]) + 2 \Big(2a^2 - a^2 \Big)^{n_x} \nu([\sqrt{2}a, +\infty)), \end{aligned} \tag{5.93}$$

where we drop the nonnegative integral over $[a, \sqrt{2}a]$. Note that the additional 2 appears because of the symmetry in the domain of the integration. The right-hand side of the inequality (5.93) should be nonnegative which is equivalent to

$$2a^{2n_x} \Big(- \nu([0, a]) + \nu([\sqrt{2}a, +\infty)) \Big) \geq 0.$$

Clearly, we can choose $a > 0$ such that

$$\nu([\sqrt{2}a, +\infty)) \geq \nu([0, a]) \tag{5.94}$$

holds. Indeed, we have that

$$\nu([\sqrt{2}a, +\infty)) \to \nu((0, +\infty)),$$

and

$$\nu([0,a]) \to \nu(\{0\}),$$

if $a \to 0$. So, the inequality (5.94) always holds for small enough a, which proves the theorem.

The more delicate and interesting situation appears for big a. Precisely, we are searching for the largest possible $a^\star$ so that the inequality (5.94) holds.

□

The above arguments lead to the following corollary.

Corollary 5.33. *Let everything be as in Theorem 5.32. Choose the largest possible constant $a^\star > 0$ so that we have*

$$\nu([\sqrt{2}a^\star, +\infty)) \geq \nu([0,a^\star]). \tag{5.95}$$

Then for a fixed boundary condition ξ such that $\xi(y) \geq 0$ for all $y \in \Lambda^c$, the Wells' inequality (5.82) holds.

Remark 5.34. *We can also consider different reference measures ν_x provided the condition $\nu_x([\sqrt{2}a, +\infty)) \geq \nu_x([0,a])$ holds for some $a > 0$ uniformly for all $x \in \mathbb{V}$.*

Remark 5.35. *The method of duplication of variables is standard in proving correlation inequalities (see the books [Si 1974] and [Si 1979] of B. Simon and the papers [Gi 1969] and [Gi 1970] of J. Ginibre and [Le 1974] of J. L. Lebowitz). Indeed, the proof of Well's inequality is quite similar to the proof of Griffiths' second inequality in the book [GlJa 1981] of J. Glimm and A. Jaffe, see also* (5.6) *in Section 5.1.2.*

Remark 5.36. *As mentioned in the introduction, the published proof of Wells' inequality in [BrLePf 1981] includes an essential misprint, which leads to false conclusions. It provided a weaker condition than actually needed for the validity of the inequality (5.94). The misprint is on page 276 in the paper [BrLePf 1981], where the downwards estimation of the integral (5.93) is incorrect. Being accurate, there is a constant 2 missing. Hence, in that paper we have the weaker condition*

$$2\nu([\sqrt{2}a, +\infty)) \geq \nu([0,a]), \tag{5.96}$$

which causes a bigger constant $a > 0$ than we should have for the validity of Wells' inequality (5.82). We would like to mention that this mistake continues in the paper [OsSp 1999] of H. Osada and H. Spohn, see also [RoZa 1998] by S. Romano and V. A. Zagrebnov.

Remark 5.37. *By scaling we can compare the given anharmonic model with the classical ferromagnetic Ising model with spins ± 1 and many-body interactions. For this purpose we essentially use the identity*

$$\langle \sigma^A \rangle^{a\xi}_{\mu^{IS}_{\Lambda,\pm a}} = a^{|A|} \langle \sigma^A \rangle^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}, \quad A \in \mathcal{A}, \tag{5.97}$$

where we should mention that the corresponding Hamiltonians differ in the temperature constant or equivalently in the interaction strength. For the concrete definition of $\tilde{\mu}^{IS}_{\Lambda,\pm 1}$ see the calculation below. The equality (5.97) follows immediately by the following calculation

$$\begin{aligned}
& \langle \sigma^A \rangle^{a\xi}_{\mu^{IS}_{\Lambda,\pm a}} \\
= \; & Z^{-1}_{\Lambda,\delta_{\pm a}}(a\xi) \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta E_\Lambda(\sigma_\Lambda | a\xi)\right) \delta_{\pm a}(d\sigma_\Lambda) \\
= \; & Z^{-1}_{\Lambda,\delta_{\pm a}}(a\xi) \int_{\mathbb{R}^\Lambda} \sigma^A \exp\Bigg(\beta \sum_{B:\Lambda_B \subset \Lambda} J_B \sigma^B \\
+ \; & \beta \sum_{\substack{B:\Lambda_B \cap \Lambda \neq \emptyset \\ \Lambda_B \cap \Lambda^c \neq \emptyset}} J_B \sigma^{B_\Lambda} (a\xi)^{B_{\Lambda^c}} \Bigg) \delta_{\pm a}(d\sigma_\Lambda) \\
= \; & \tilde{Z}^{-1}_{\Lambda,\delta_{\pm 1}}(\xi) a^{|A|} \int_{\mathbb{R}^\Lambda} \sigma^A \exp\Bigg(\beta \sum_{B:\Lambda_B \subset \Lambda} J_B a^{|B|} \sigma^B \\
+ \; & \beta \sum_{\substack{B:\Lambda_B \cap \Lambda \neq \emptyset \\ \Lambda_B \cap \Lambda^c \neq \emptyset}} J_B a^{|B_\Lambda|} \sigma^{B_\Lambda} a^{|B_{\Lambda^c}|} \xi^{B_{\Lambda^c}} \Bigg) \delta_{\pm 1}(d\sigma_\Lambda) \\
= \; & \tilde{Z}^{-1}_{\Lambda,\delta_{\pm 1}}(\xi) a^{|A|} \int_{\mathbb{R}^\Lambda} \sigma^A \exp\left(-\beta \tilde{E}_\Lambda(\sigma_\Lambda | \xi)\right) \delta_{\pm 1}(d\sigma_\Lambda) \\
= \; & a^{|A|} \langle \sigma^A \rangle^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}},
\end{aligned} \tag{5.98}$$

where $\tilde{E}_\Lambda$ corresponds to the interactions $\tilde{J}_B := J_B a^{|B|}$, $B \in \mathcal{A}$. By analogous estimations we get for the normalizing constant the equality

$$Z^{-1}_{\Lambda,\delta_{\pm a}}(a\xi) = \tilde{Z}^{-1}_{\Lambda,\delta_{\pm 1}}(\xi).$$

This means, for growing a we gather stronger interaction. If we fix the interaction then we would get lower temperature $T = \frac{1}{\beta}$ with growing a. So, $\tilde{\mu}^{IS}_{\Lambda,\pm 1}$ describes the classical Ising model with spins ± 1. Together with Well's inequality, this relation yields to a new method which shows phase transitions. We will discuss this issue in detail in the next subsection.

5.4.4 Phase transition via Wells' inequality

A beautiful application of Wells' inequality (5.82) is a new method showing phase transition in general ferromagnetic systems by only knowing the phase transition in the *Ising model*. This method was first realized by D. Wells for some spin systems on a lattice. Below we extend this method to the more general situation of unbounded spin systems on graphs. From Remark (5.97) it follows for pair interaction potentials with $|A| = 2$ that

$$\langle \sigma^A \rangle^{a\xi}_{\mu^{IS}_{\Lambda,\pm a}} = a^2 \langle \sigma^A \rangle^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}, \tag{5.99}$$

where $\tilde{J}_{xy} = J_{xy} a^2$. Now, for this model, we assume that Wells' inequality

$$\langle \sigma^A \rangle^{a\xi}_{\mu_\Lambda} \geq \langle \sigma^A \rangle^{a\xi}_{\mu^{IS}_{\Lambda,\pm a}}, \tag{5.100}$$

holds for all $A \in \mathcal{A}$, $\xi \in \Omega$ and for a fixed $a, \beta > 0$. With (5.99) the right side of (5.100) is equal to $a^2 \langle \sigma^A \rangle^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}$. We define the average magnetization of $\langle \sigma^A \rangle^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}$ by $m^{\xi}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}$. Let us consider the plus-boundary condition, which is $\xi(y) = +a$ (resp. $\xi(y) = +1$) for all $y \in \mathbb{V}$, abbreviated by $\xi = +a$ (resp. $\xi(y) = +$). Since we have $\langle \sigma^A \rangle^{+a}_{\mu_\Lambda} \geq a^2 \langle \sigma^A \rangle^{+}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}$ it follows for the plus-boundary condition that

$$m^{+a}_{\mu_\Lambda} \geq m^{+}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}. \tag{5.101}$$

This shows us the following important fact. If there is a magnetization in the classical Ising model for a finite subset $\Lambda \Subset \mathbb{V}$, i.e., $m^{+}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}} > 0$, then we have magnetization in the general ferromagnetic model, i.e., $m^{+a}_{\mu_\Lambda} > 0$. Since most of the results on phase transition in literature are for the case of spin space $\{+1, -1\}$ it would be of particular interest to have inequalities in the form (5.101). Moreover, if the *Ising model* undergoes a phase transition, i.e., $\lim_{\mathcal{L}} m^{+}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}} > 0$ for some cofinal sequence $\mathcal{L}$, then we would have the same in the general model. Since this is an essential observation we summarize it in the next lemma.

Lemma 5.38. *Assume the existence of the Gibbs measures for the model* (5.76)*. Assume that Wells' inequality holds. Then for $|A| = 2$ and pair interaction potentials $J_{xy} = J$ we have for $\tilde{\beta}(a, J) = \beta a^2 J$ that*

$$\lim_{\mathcal{L}} m^{+a}_{\mu_\Lambda} \geq \lim_{\mathcal{L}} m^{+}_{\tilde{\mu}^{IS}_{\Lambda,\pm 1}}. \tag{5.102}$$

So, if for a given β, the infinite volume Gibbs measure is non-unique in the classical Ising model, then it is non-unique in the general ferromagnetic model, with the proper relation between the intensities.

Application

In this subsection we would like to present some concrete examples where the Wells' condition (5.95) holds. We will take a look at the very popular *double-well potentials* in physical science, which we already know from Section 4.5 as our basic examples.

φ^4 potential $V(s) := s^4 - \kappa s^2$

For $\kappa \geq 0$ we define the *double-well potential*

$$V(s) := s^4 - \kappa s^2, \quad s \in \mathbb{R}. \tag{5.103}$$

The reference measure is an even probability measure on $\mathbb{R}$

$$\nu(ds) = \frac{1}{Z} \exp\left(-s^4 + \kappa s^2\right) ds, \tag{5.104}$$

with the normalizing constant

$$Z = \int_{\mathbb{R}} \exp\left(-s^4 + \kappa s^2\right) ds. \tag{5.105}$$

The following proposition gives a dependence between κ and the number a forthcoming from the Wells' condition (5.95). It says that for growing $\kappa \geq 0$ the constant $a \geq 0$ also grows. The interesting point is that we can give a concrete number, even though it is not the largest one, for all $\kappa \geq 0$ and all corresponding φ^4 models (5.104) such that the condition (5.95) holds. In the last part of this subsection we give for $\kappa = 1, \cdots, 10$ some numerical values of a such that the condition (5.95) holds.

Proposition 5.39. *Let the probability measure ν be defined as in* (5.104) *and let $a := \frac{\sqrt{\kappa}}{2\sqrt{2}}$. Then*

$$\nu([\sqrt{2}a, +\infty)) \geq \nu([0, a]). \tag{5.106}$$

Now we are on the level to prove the main Theorem 5.29, which gives a sufficient condition for phase transitions for systems of *unbounded continuous* spins on infinite graphs. We would like to recall this result.

Theorem 5.40 (Main Theorem)**.** *Let T be a general tree with the branching number $br(T) < \infty$. Let us consider the model* (5.1) *corresponding to the spin space $\mathbb{R}$ with the even self interaction potential (double-well potential)*

$$V(s) := s^4 - \kappa s^2, \quad s \in \mathbb{R}$$

for some $\kappa \geq 0$. Then the critical inverse temperature of this system equals

$$\beta_c = \frac{8coth^{-1}br(T)}{\kappa J}. \tag{5.107}$$

Proof Let us consider the scaling (5.99) correspondingly to the classical Ising model at inverse temperature $\tilde{\beta} := J\beta a^2$. By Proposition 5.39 we have for $a := \frac{\sqrt{\kappa}}{2\sqrt{2}}$ that the Wells' condition (5.95) holds for the double-well potential V. And by Theorem 5.28, see [Ly 1989], the critical inverse temperature $\tilde{\beta}_c$ of the classical Ising model (5.69) equals to $coth^{-1}br(T)$. Then for the critical inverse temperature β_c of the general ferromagnetic Ising model (5.1) it yields (5.107).

□

Proof of Proposition 5.39. We calculate the lower bound for $\nu([\frac{\sqrt{\kappa}}{2},\infty])$ and the upper bound for $\nu([0,\frac{\sqrt{\kappa}}{2\sqrt{2}}])$. If the upper bound of $\nu([0,\frac{\sqrt{\kappa}}{2\sqrt{2}}])$ is still smaller or equal to the lower bound of $\nu([\frac{\sqrt{\kappa}}{2},\infty])$ then (5.106) holds. We now calculate the lower bound of the left-hand side of (5.106)

$$\begin{aligned}
& \nu\left(\left[\frac{\sqrt{\kappa}}{2},\infty\right]\right) \\
\geq\ & \nu\left(\left[\frac{\sqrt{\kappa}}{2},\frac{\sqrt{\kappa}}{\sqrt{2}}\right]\right) + \nu\left(\left[\frac{\sqrt{\kappa}}{\sqrt{2}},\sqrt{\kappa}\right]\right) \\
=\ & \frac{1}{Z}\int_{\frac{\sqrt{\kappa}}{2}}^{\frac{\sqrt{\kappa}}{\sqrt{2}}} \exp\left(-s^4+\kappa s^2\right)ds + \frac{1}{Z}\int_{\frac{\sqrt{\kappa}}{\sqrt{2}}}^{\sqrt{\kappa}} \exp\left(-s^4+\kappa s^2\right)ds \\
\geq\ & \frac{1}{Z}\left(\frac{\sqrt{\kappa}}{\sqrt{2}}-\frac{\sqrt{\kappa}}{2}\right)\exp\left(-\left(\frac{\sqrt{\kappa}}{2}\right)^4+\kappa\left(\frac{\sqrt{\kappa}}{2}\right)^2\right) \\
+\ & \frac{1}{Z}\left(\sqrt{\kappa}-\frac{\sqrt{\kappa}}{\sqrt{2}}\right)\exp\left(-(\sqrt{\kappa})^4+\kappa(\sqrt{\kappa})^2\right) \\
=\ & \frac{1}{Z}\left[\frac{(2-\sqrt{2})\sqrt{\kappa}}{2\sqrt{2}}\exp\left(\frac{3\kappa^2}{16}\right)+\frac{(\sqrt{2}-1)\sqrt{\kappa}}{\sqrt{2}}\right] =: (1),
\end{aligned} \tag{5.108}$$

where we get the first estimate by cutting the integration limits and the second estimate by using the fact that $\frac{\sqrt{\kappa}}{\sqrt{2}}$ is the maximum of the function $\exp\left(-s^4+\kappa s^2\right)$. In the next step we compute the upper bound of the right-hand side of (5.106)

$$\begin{aligned}
&\nu\left(\left[0,\frac{\sqrt{\kappa}}{2\sqrt{2}}\right]\right)\\
=\ & \frac{1}{Z}\int_0^{\frac{\sqrt{\kappa}}{2\sqrt{2}}}\exp\left(-s^4+\kappa s^2\right)ds\\
\le\ & \frac{1}{Z}\frac{\sqrt{\kappa}}{2\sqrt{2}}\exp\left(-\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}\right)^4+\kappa\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}\right)^2\right)\\
=\ & \frac{1}{Z}\frac{\sqrt{\kappa}}{2\sqrt{2}}\exp\left(\frac{31\kappa^2}{256}\right) =: (2). \qquad (5.109)
\end{aligned}$$

It remains to show $(1) \ge (2)$, which is equivalent to

$$\frac{(2-\sqrt{2})\sqrt{\kappa}}{2\sqrt{2}}\exp\left(\frac{3\kappa^2}{16}\right)+\frac{(\sqrt{2}-1)\sqrt{\kappa}}{\sqrt{2}} \ge \frac{\sqrt{\kappa}}{2\sqrt{2}}\exp\left(\frac{31\kappa^2}{256}\right) \qquad (5.110)$$

or

$$2\sqrt{2}-2 \ge \exp\left(\frac{31\kappa^2}{256}\right)-(2-\sqrt{2})\exp\left(\frac{3\kappa^2}{16}\right). \qquad (5.111)$$

Now, defining the function $f(\kappa) := \exp\left(\frac{31\kappa^2}{256}\right)-(2-\sqrt{2})\exp\left(\frac{3\kappa^2}{16}\right)$ we have to show, for all $\kappa \ge 0$, that

$$2\sqrt{2}-2 \ge f(\kappa). \qquad (5.112)$$

To convince us that this inequality is accurate we have to calculate the maximum of $f(\kappa)$. The derivative of $f(\kappa)$ is given by

$$f'(\kappa) = \kappa\frac{31}{128}\exp\left(\frac{31\kappa^2}{256}\right)-\kappa\frac{3}{8}(2-\sqrt{2})\exp\left(\frac{3\kappa^2}{16}\right). \qquad (5.113)$$

So, we have an extremum in $\kappa = 0$ and $\kappa = \pm\kappa_0 \approx \pm 1.2132$. Calculating the second derivative of $f(\kappa)$ we get

$$\begin{aligned}
f''(\kappa) &= \left[\left(\frac{31}{128}\right)^2\kappa^2+\frac{31}{128}\right]\exp\left(\frac{31\kappa^2}{256}\right)\\
&- \left[\left(\frac{3}{8}\right)^2(2-\sqrt{2})\kappa^2+\frac{3}{8}(2-\sqrt{2})\right]\exp\left(\frac{3\kappa^2}{16}\right). \qquad (5.114)
\end{aligned}$$

For $\kappa = 0$ we have $\frac{d^2}{dk^2}f(0) > 0$ and for $\kappa = \pm\kappa_0$ we have $\frac{d^2}{dk^2}f(\pm\kappa_0) < 0$. Therefore, we have a minimum in $\kappa = 0$ and a maximum in $\kappa = \pm\kappa_0$. However, for $\kappa = \pm\kappa_0$ we compute

$$f(\pm\kappa_0) \approx 0.4233. \qquad (5.115)$$

For the maximum of the function $f(\kappa)$ we get the inequality (5.112), since it holds

$$0.8284 \approx 2\sqrt{2} - 2 \geq f(\pm\kappa_0). \tag{5.116}$$

Hence, the inequality (5.112) holds for all $\kappa \geq 0$.

$\square$

Remark 5.41. *For Wells' inequality (5.82) we can also consider different reference measures $\nu_x = \frac{1}{Z}\exp(-s^4 + \kappa s^2)ds$, differing in κ for each $x \in \mathbb{V}$. Indeed, the condition $\nu([\sqrt{2}a, \infty]) \geq \nu([0, a])$ holds for $a = \frac{1}{2\sqrt{2}}$ uniformly for all $x \in \mathbb{V}$ if we choose $\kappa_x \geq 1$, see Remark* (5.34).

We would like to describe the behavior of the number a for fixed $\kappa \geq 0$. In order to do that we introduce the following function

$$\phi(a) := \int_{\sqrt{2}a}^{\infty} \exp\left(-s^4 + \kappa s^2\right)ds - \int_0^a \exp\left(-s^4 + \kappa s^2\right)ds. \tag{5.117}$$

Since by the definition, $\phi(a)$ is a monotonically decreasing function, exactly one number a exists so that $\phi(a) = 0$, which then is the largest possible constant $a^\star$ fulfilling the Wells' condition (5.95).

Next, we would like to describe the change in κ for fixed number a. So, we introduce similarly as in (5.117) the following function

$$\Phi(\kappa) := \int_{\sqrt{2}a}^{\infty} \exp\left(-s^4 + \kappa s^2\right)ds - \int_0^a \exp\left(-s^4 + \kappa s^2\right)ds \tag{5.118}$$

If $\Phi(\kappa)$ is a monotonically increasing function we then know for increasing κ that the integral within the limits $[\sqrt{2}a, +\infty)$ increases while the integral within the limits $[0, a]$ decreases. Without loss of generality we optimize this growing by correcting the integration limits for each κ by defining $a = \frac{\sqrt{\kappa}}{2\sqrt{2}}$. This statement gives us the following important fact: For growing κ we can choose a bigger a.

Proposition 5.42. *Let $a = \frac{\sqrt{\kappa}}{2\sqrt{2}}$ so that the inequality* (5.106) *holds, then the function $\Phi(\kappa)$ is strictly monotonically increasing.*

Proof. We have the following derivative

$$\Phi'(\kappa) := \int_{\frac{\sqrt{\kappa}}{2}}^{\infty} s^2 \exp\left(-s^4 + \kappa s^2\right)ds - \int_0^{\frac{\sqrt{\kappa}}{2\sqrt{2}}} s^2 \exp\left(-s^4 + \kappa s^2\right)ds, \tag{5.119}$$

which should be strictly positive. Using the same technique as in the proof of Proposition (5.39) we calculate the lower sum of the integral in the limits $[\frac{\sqrt{\kappa}}{2}, +\infty)$ and the upper sum of the integral in the limits $[0, \frac{\sqrt{\kappa}}{2\sqrt{2}}]$. If the upper sum of $\int_0^{\frac{\sqrt{\kappa}}{2\sqrt{2}}} s^2 \exp(-s^4 + \kappa s^2) ds$ is still smaller than the lower sum of $\int_{\frac{\sqrt{\kappa}}{2}}^{\infty} s^2 \exp(-s^4 + \kappa s^2) ds$ then $\Phi'(\kappa) > 0$. Similarly as in the proof of Proposition (5.39) we estimate

$$
\begin{aligned}
& \int_{\frac{\sqrt{\kappa}}{2}}^{\infty} s^2 \exp\left(-s^4 + \kappa s^2\right) ds \\
\geq\; & \int_{\frac{\sqrt{\kappa}}{2}}^{\frac{\sqrt{\kappa}}{\sqrt{2}}} s^2 \exp\left(-s^4 + \kappa s^2\right) ds + \int_{\frac{\sqrt{\kappa}}{\sqrt{2}}}^{\sqrt{\kappa}} s^2 \exp\left(-s^4 + \kappa s^2\right) ds \\
\geq\; & \left(\frac{\sqrt{\kappa}}{\sqrt{2}} - \frac{\sqrt{\kappa}}{2}\right)\left(\frac{\sqrt{\kappa}}{2}\right)^2 \exp\left(-\left(\frac{\sqrt{\kappa}}{2}\right)^4 + \kappa\left(\frac{\sqrt{\kappa}}{2}\right)^2\right) \\
+\; & \left(\sqrt{\kappa} - \frac{\sqrt{\kappa}}{\sqrt{2}}\right)(\sqrt{\kappa})^2 \exp\left(-(\sqrt{\kappa})^4 + \kappa(\sqrt{\kappa})^2\right) \\
=\; & \frac{(2-\sqrt{2})\kappa\sqrt{\kappa}}{8\sqrt{2}} \exp\left(\frac{3\kappa^2}{16}\right) + \frac{(\sqrt{2}-1)\kappa\sqrt{\kappa}}{\sqrt{2}} =: (1),
\end{aligned}
\tag{5.120}
$$

and

$$
\begin{aligned}
& \int_0^{\frac{\sqrt{\kappa}}{2\sqrt{2}}} s^2 \exp\left(-s^4 + \kappa s^2\right) ds \\
\leq\; & \frac{\sqrt{\kappa}}{2\sqrt{2}}\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}\right)^2 \exp\left(-\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}\right)^4 + \kappa\left(\frac{\sqrt{\kappa}}{2\sqrt{2}}\right)^2\right) \\
=\; & \frac{\kappa\sqrt{\kappa}}{16\sqrt{2}} \exp\left(\frac{31\kappa^2}{256}\right) =: (2).
\end{aligned}
\tag{5.121}
$$

It remains to show $(1) \geq (2)$, which is equivalent to

$$
\frac{(2-\sqrt{2})\kappa\sqrt{\kappa}}{8\sqrt{2}} \exp\left(\frac{3\kappa^2}{16}\right) + \frac{(\sqrt{2}-1)\kappa\sqrt{\kappa}}{\sqrt{2}} \geq \frac{\kappa\sqrt{\kappa}}{16\sqrt{2}} \exp\left(\frac{31\kappa^2}{256}\right) \tag{5.122}
$$

or

$$
16(\sqrt{2}-1) \geq \exp\left(\frac{31\kappa^2}{256}\right) - (4-2\sqrt{2}) \exp\left(\frac{3\kappa^2}{16}\right). \tag{5.123}
$$

Now, defining the function $f(\kappa) := \exp\left(\frac{31\kappa^2}{256}\right) - (4-2\sqrt{2}) \exp\left(\frac{3\kappa^2}{16}\right)$ we have to show, for all $\kappa \geq 0$, that

$$
16(\sqrt{2}-1) \geq f(\kappa). \tag{5.124}
$$

To make sure that this inequality is true we have to work out the maximum of $f(\kappa)$. So, the derivative of $f(\kappa)$ is given by

$$f'(\kappa) = \kappa\frac{31}{128}\exp\left(\frac{31\kappa^2}{256}\right) - \kappa\frac{3}{8}(4-2\sqrt{2})\exp\left(\frac{3\kappa^2}{16}\right), \tag{5.125}$$

which gives us an extremum in $\kappa = 0$. Calculating the second derivative of $f(\kappa)$ as

$$\begin{aligned} f''(\kappa) &= \left[\left(\frac{31}{128}\right)^2\kappa^2 + \frac{31}{128}\right]\exp\left(\frac{31\kappa^2}{256}\right) \\ &- \left[\left(\frac{3}{8}\right)^2(4-2\sqrt{2})\kappa^2 + \frac{3}{8}(4-2\sqrt{2})\right]\exp\left(\frac{3\kappa^2}{16}\right), \end{aligned} \tag{5.126}$$

we get for $\kappa = 0$ that $f''(0) < 0$. Hence, in $\kappa = 0$ there is a maximum. However, we compute

$$\begin{aligned} f(0) &= \exp(0) - (4-2\sqrt{2})\exp(0) \\ &= 2\sqrt{2} - 3. \end{aligned} \tag{5.127}$$

This, of course, yields to the following true inequality

$$16(\sqrt{2}-1) \geq f(0) = 2\sqrt{2} - 3. \tag{5.128}$$

Thus the inequality (5.124) holds for all $\kappa \geq 0$.

$\square$

We now use a computer algorithm for finding the critical values of a. With this algorithm we can calculate the area under graphs of certain functions without primitives. This numerical method we use is the so-called *Romberg integration*, which we will introduce in the appendix. In Figure 5.1 we calculate for $\kappa = 1, \cdots, 10$ the following function

$$\phi(a) := \int_{\sqrt{2}a}^{\infty} \exp\left(-s^4 + \kappa s^2\right)ds - \int_0^a \exp\left(-s^4 + \kappa s^2\right)ds. \tag{5.129}$$

From the picture we can conclude how big the number a can be chosen so that $\phi(a)$ is still positive. The largest possible a so that $\phi(a) \geq 0$, is the optimal $a^\star$ solving the Wells' condition (5.95). From Figure 5.2, which is an enlargement of Figure 5.1, we can read the following $a^\star$: For $k = 1$, $a^\star \approx 0,53$, for $k = 2$, $a^\star \approx 0,72$, for $k = 3$, $a^\star \approx 0,93$, for $k = 4$, $a^\star \approx 1,11$, for $k = 5$, $a^\star \approx 1,26$, etc. .

Remark 5.43. *Compared with the next example, defining* $a = \frac{\sqrt{\kappa}}{2\sqrt{2}}$ *works in this case while defining* $a = \frac{\kappa}{2\sqrt{2}}$ *does not work anymore. Indeed, for already* $\kappa = 3$ *we would get* $a = \frac{\kappa}{2\sqrt{2}} = \frac{3}{2\sqrt{2}} = 1,0605$ *which is of course bigger than* $a^\star \approx 0,93$.

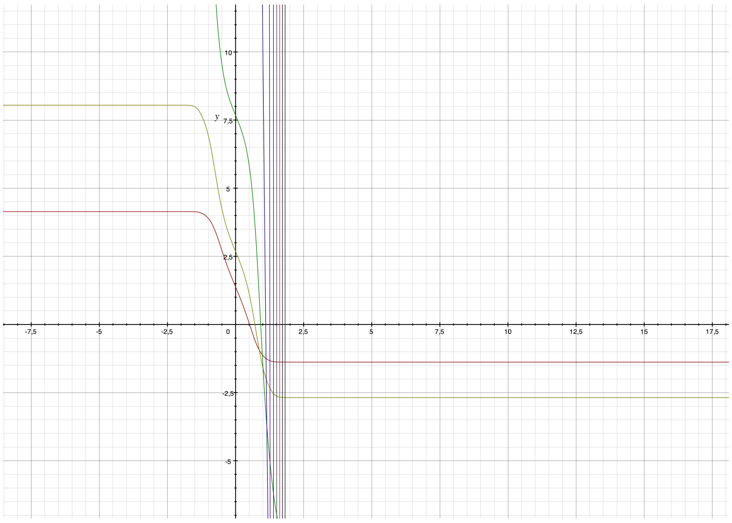

Figure 5.1: Function $\phi(a)$ for k=1,...,10

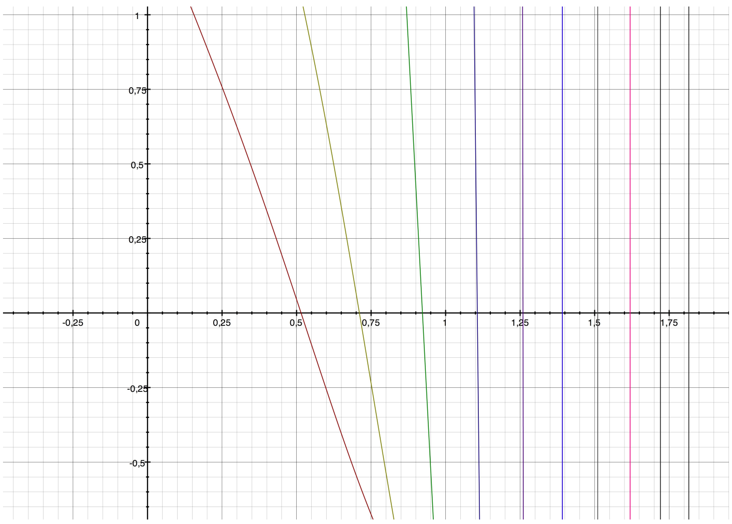

Figure 5.2: Enlargement of Figure 5.1

φ^{2n} **potential** $V(s) := s^{2n} - \kappa s^2$

For $n > 2$ and $\kappa \in \mathbb{N}$ let ν be an even probability measure on $\mathbb{R}$ given by the *double-well potential*

$$V(s) := s^{2n} - \kappa s^2. \tag{5.130}$$

Let us define ν similarly as in the first example. First of all we would like to mention that this potential behaves strongly different than the potentials in the first example. We easily see that $f(s) := \exp\left(-s^{2n} + s^2\right)$ tends for growing n to the function $\exp\left(s^2\right)$ in the interval (-1,1) and zero outside of this interval. The Figure (5.3) gives an intuition for this. And for higher order of a fixed κ we have bigger values of the corresponding functions while growing n.

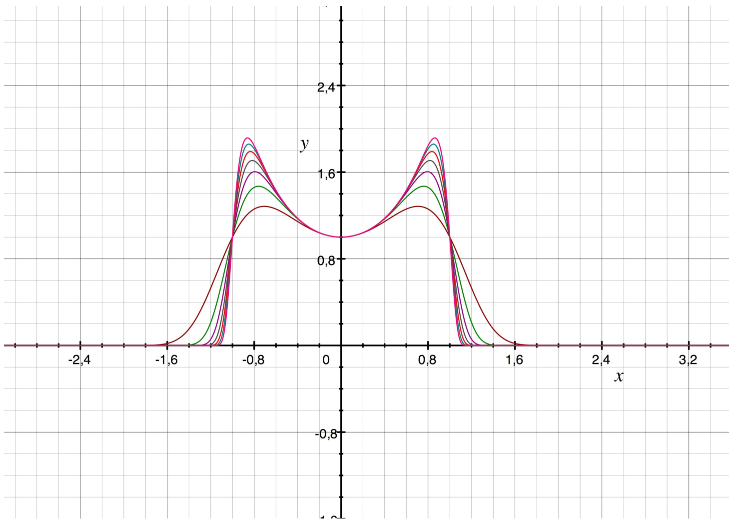

Figure 5.3: Function $f(s) := \exp\left(-s^{2n} + s^2\right)$ for n=2,...,8

In Figure 5.4 we calculate for $n = 2, \cdots, 10$ the following function with $\kappa = 1$

$$\phi(a) := \int_{\sqrt{2}a}^{\infty} \exp\left(-s^{2n} + s^2\right) ds - \int_0^a \exp\left(-s^{2n} + s^2\right) ds \tag{5.131}$$

From the picture we can infer how big the number a can be chosen so that $\phi(a)$ is still positive. The interval, where we can choose such an a is relatively short. We are searching for the optimal $a \leq a^\star$ solving the Wells' condition (5.95). From the enlarged Figure 5.5 we can read the following $a^\star$: For $n = 2$, $a^\star \approx 0,513$, for $n = 3$, $a^\star \approx 0,493$, for $n = 4$, $a^\star \approx 0,492$, for $n = 5$, $a^\star \approx 0,494$, etc. .

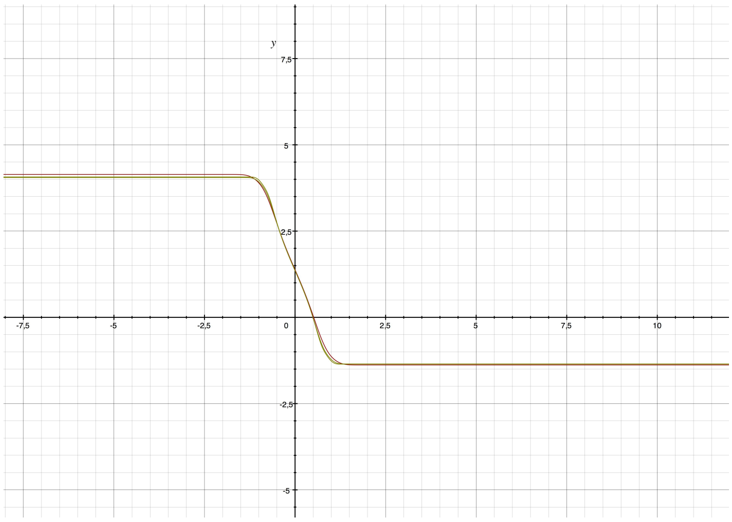

Figure 5.4: Function $\phi(a)$ for n=2,...,10

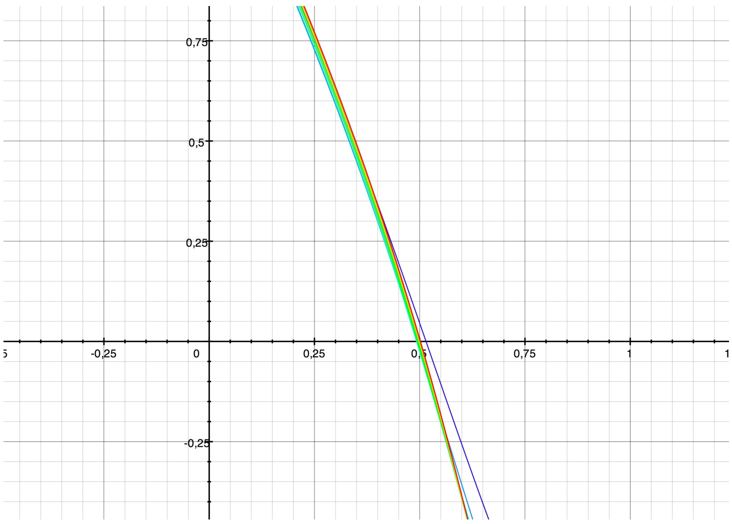

Figure 5.5: Enlargement of Figure 5.4

After some calculations with the computer we know that for big n the number a changes very slowly and it seems to converge to a fixed number, which is supposed to be 0,519. For instance, for $n = 33$ we have $a^\star \approx 0,513$ and for $n = 400$ $a^\star \approx 0,518$.

Chapter 6

Appendix

6.1 Romberg Integration: A method of numerical extrapolation

The Norwegian mathematician Werner Romberg first described this systematic extrapolation procedure in connection with a related numerical integration formula called the *trapezoidal rule* in his paper [Ro 1955]. This procedure of extrapolation to provide successively more accurate approximations to an integral is known as *Romberg integration*. Romberg integration has a very important advantage compared to trapezoidal rule integration that a much smaller number of subdivisions are needed for a required accuracy. For a summary about the Romberg integration we also refer to the paper [BaRuSt 1963] by F.L. Bauer, H. Rutishauser and E. Stiefel. There is expanded literature on the classical Romberg method. In any book of numerical analysis one can find this method. For more details and variants the interested reader should consult to the review paper [Lyn 1986] by J. N. Lyness and [Joy 1971] by D. C. Joyce.

In the last subsections we essentially used the famous Romberg method in order to calculate some crucial integrals since the functions we used there have no primitives. This forces us to use the numerical integration method of Romberg. By using this method we apply a computer to derive approximate solutions. Instead of spending a few hours with an exhausting pencil and paper analysis (see Example (6.1)) we are more efficient using the computer which only needs a few seconds for the calculation in order to present us the result. In some parts of applied physics and mathematics it is widely believed that for some tasks one cannot prevent applying computers to do some complicated work.

Now let us describe the Romberg method. Since it is based on the trapezoidal rule, we introduce at first this numerical integration formula. Let f be a function on $[a,b]$. Let $a = s_0 < s_1 < \cdots < s_n = b$ be a partition of $[a,b]$ with the same distance h_n between s_i and s_{i+1}. We define $t(i) := a + ih_n$. The number n can be regarded as the total number of interpolation points. Consider the *composite trapezoidal rule* for the function f over the above partition which is based on the function values at the end points of each subinterval, that is

$$\begin{aligned}\int_a^b f(s)ds &\approx \frac{h_n}{2}(f(a) + 2\sum_{i=1}^{n-1} f(t(i)) + f(b)) \\ &= \frac{h_n}{2}(f(a) + f(b)) + h_n \sum_{i=1}^{n-1} f(t(i)). \qquad (6.1)\end{aligned}$$

Suppose that there are 2^n subintervals and define $h_n := \frac{b-a}{2^n}$, then the above formula can be rewritten as

$$R(n,0) = \frac{h_n}{2}(f(a) + f(b)) + h_n \sum_{i=1}^{2^n-1} f(t(i)). \qquad (6.2)$$

For $n = 0$ we have $h_0 = b - a$ and

$$R(0,0) = \frac{h_0}{2}(f(a) + f(b)), \qquad (6.3)$$

which is the *simple trapezoidal rule*. Proceeding for $n = 1$ we have $h_1 = \frac{b-a}{2}$ and

$$\begin{aligned}R(1,0) &= \frac{h_1}{2}(f(a) + f(b)) + h_1 f(a + h_1) \\ &= \frac{1}{2}R(0,0) + h_1 f(a + h_1). \qquad (6.4)\end{aligned}$$

For this equality we only need to calculate the function in the additional point $a + h_1$. For $n = 2$ we have $h_2 = \frac{b-a}{4}$ and

$$\begin{aligned}R(2,0) &= \frac{h_2}{2}(f(a) + f(b)) + h_2 \sum_{i=1} 3f(t(i)) \\ &= \frac{h_2}{2}(f(a) + f(b)) + h_2 f(a + 2h_2) + h_2(f(a + h_2) + f(a + 3h_2)) \\ &= \frac{1}{2}R(1,0) + h_2(f(a + h_2) + f(a + 3h_2)). \qquad (6.5)\end{aligned}$$

Again, we see that for this equality we only have to know the function values in two additional points. So, $R(2,0)$ is given in terms of $R(1,0)$ and $R(1,0)$ is given in terms of $R(0,0)$. Generally, if $R(n,0)$ is available, then $R(n+1,0)$ can be computed by the so-called *recursive trapezoidal formula*

$$R(n+1,0) = \frac{1}{2}R(n,0) + h_n \sum_{k=1} 2^{n-1} f(a + (2k-1)h), \tag{6.6}$$

where we define $h_n := \frac{b-a}{2^n}$.

The convergence of the sequence $R(n,0)$ to the integral $\int_a^b f(s)ds$ when n tends to infinity has been studied in [La 1963] by P. J. Laurent. Especially, P. J. Laurent proved that the sequence h_n cannot be chosen arbitrarily for ensuring the convergence. So, the choice of h_n we made here is also obligatory for the Romberg method which we will describe next. It is an algorithm which produces a triangular alignment of numbers, which are numerical estimates of the definite integral of a function $f(x)$ with integration limits $[a,b]$. This algorithm, called *Romberg's extrapolation formula*, is given by

$$R(n+1,m+1) = R(n+1,m) + \frac{R(n+1,m) - R(n,m)}{4^{m+1}-1}, \tag{6.7}$$

for $n, m \geq 0$. These numbers can be arranged in a triangular alignment which is presented in the Figure 6.1. The first column of the Figure 6.1 contains the

$$\begin{array}{cccccc}
\mathrm{R(0,0)} & & & & & \\
\mathrm{R(1,0)} & \mathrm{R(1,1)} & & & & \\
\mathrm{R(2,0)} & \mathrm{R(2,1)} & \mathrm{R(2,2)} & & & \\
\mathrm{R(3,0)} & \mathrm{R(3,1)} & \mathrm{R(3,2)} & \mathrm{R(3,3)} & & \\
\vdots & \vdots & \vdots & \vdots & & \\
\mathrm{R(N,0)} & \mathrm{R(N,1)} & \mathrm{R(N,2)} & \mathrm{R(N,3)} & \cdots & \mathrm{R(N,N)}
\end{array}$$

Figure 6.1: Triangular alignment of the Romberg method.

estimate of the definite integral using the recursive trapezoidal formula (6.6). The other entries are generated using Romberg's extrapolation formula (6.7). Note that each element in $(m+1)$-th column depends on two elements from the m-th column.

Example 6.1. *Let us apply the Romberg algorithm for the following integral on* $[0,1]$*:*

$$f(s) = \exp(s^2). \tag{6.8}$$

First we calculate with the recursive trapezoidal formula for the subintervals 1, 2, 4, 8.

$$\begin{aligned}
R(0,0) &= \frac{1}{2}(f(0)+f(1)) \\
&= \frac{1}{2}(1+e^1) = 1,859140614 \\
R(1,0) &= \frac{1}{2}R(0,0) + \frac{1}{2}f(0+\frac{1}{2}) \\
&= 0,929570457 + 0,642012708 \\
&= 1,571583165 \\
R(2,0) &= \frac{1}{2}R(1,0) + \frac{1}{4}(f(0+\frac{1}{4}) + f(0+3\frac{1}{4})) \\
&= 0,785791582 + 0,704887279 \\
&= 1,490678862 \\
R(3,0) &= \frac{1}{2}R(2,0) + \frac{1}{8}(f(0+\frac{1}{8}) + f(0+\frac{3}{8}) + f(0+\frac{5}{8}) + f(0+\frac{7}{8})) \\
&= 0,74533943 + 0,724372845 \\
&= 1,469712276. \qquad (6.9)
\end{aligned}$$

These estimates are entries of the first column. We complete the triangular alignment by using the formula (6.7)*. For instance* $R(2,2)$ *is calculated as*

$$\begin{aligned}
R(1,1) &= R(1,0) + \frac{1}{3}(R(1,0) - R(0,0)) \\
&= 1,571583165 + \frac{1}{3}(1,571583165 - 1,859140614) \\
&= 1,475730682 \qquad (6.10)
\end{aligned}$$

We list the triangular alignment

1,859140614			
1,571583165	*1,475730682*		
1,490678862	*1,463710761*	*1,462909433*	
1,469712276	*1,462723414*	*1,461736067*	*1,461717452.*

Note that by computing $R(5,1)$, $R(6,1)$, *etc. we can improve the accuracy of the estimate 1,461717452. In order to have a comparison we estimate this integral with a computer program which gives us the number 1,462652.*

Bibliography

[AbMa-Lö 1973] D. B. Abrham and A. Martin-Löf, *The transfer matrix for a pure phase in the two-dimensional Ising model.*, Commun. Math. Phys. 32, 245-268, 1973.

[AiFe 1986] M. Aizenman and R. Fernandez, *On the critical behavior of the magnetization in high-dimensional Ising models*, J. Stat. Phys. 44, 393-454, 1986.

[AlBa 2002] R. Albert and A.-L. Barabasi, *Statistical mechanics of complex networks*, Rev. Modern Physics 74, 47-97, 2002.

[AlKoPaRö 2005] S. Albeverio, Y. G. Kondratiev, T. Pasurek and M. Röckner, *Euclidean Gibbs measures of quantum crystals: existence, uniqueness and a priori estimates*, 29-54 in J.-D. Deuschel, A. Greven, Eds., "Interacting Stochastic Systems", Springer, 2005.

[AlKoPaRö 2001] S. Albeverio, Y. G. Kondratiev, T. Pasurek and M. Röckner, *Glauber dynamics for quantum lattice systems*, Rev. Math. Phys. Vol. 13, No. 1, 51-124, 2001.

[AlKoRö 2003] S. Albeverio, Y. G. Kondratiev and M. Röckner, *Symmetrizing measures for infinite dimensional diffusions: an analytic approach*, 475-486 in S. Hildebrandt, H. Karcher, Eds., "Geometric Analysis and Nonlinear Differential Equations", Springer, Berlin, 2003.

[AlKoRöTs 1997a] S. Albeverio, Y. G. Kondratiev, M. Röckner and T. V. Tsikalenko, *Uniqueness of Gibbs states on loop lattices*, C. R. Acad. Sci. Paris, Ser. I, 324, 1401-1406, 1997.

[AlKoRöTs 1997b] S. Albeverio, Y. G. Kondratiev, M. Röckner and T. V. Tsikalenko, *Uniqueness of Gibbs states for quantum lattice systems*, Probability Theory and Related Fields, 108, 193-218, 1997.

[AlKoRöTs 2000] S. Albeverio, Y. G. Kondratiev, M. Röckner and T. V. Tsikalenko, *A priori estimates for symmetrizing measures and their applications to Gibbs states*, J. Funct. Anal. 171, 366-400, 2000.

[BaKuMePr 2007] F. Baffioni, T. Kuna, I. Merola and E. Presutti, *A relativized Dobrushin uniqueness condition and applications to Pirogov-Sinai models*, SFB-701 Preprint 07-041, Universität Bielefeld, 2007.

[BaRuSt 1963] F.L. Bauer, H. Rutishauser and E. Stiefel, *New Aspects in numerical Quadrature*, Proc. of Symposia in Applied Mathematics 15, Am. Math. Soc., 1963.

[BH-K 1982] J. Bellissard and R. Høegh-Krohn, *Compactness and the maximal Gibbs states for random Gibbs fields on a lattice*, Commun. Math. Phys. 84, 297-327, 1982.

[BaKo 1992] V. S. Barbulyak and Y. G. Kondratiev, *The quasiclassical limit for the Schrödinger operator and phase transitions in quantum statistical physics*, Func. Anal. Appl., 26(2), 61-64, 1992.

[Bo 1998] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics 184, Springer-Verlag, New York, Berlin, Heidelberg, 1998.

[Bov 2001] A. Bovier, *Statistical Mechanics of Disordered Systems*, Lecture Notes, Weierstrass-Institut fr Angewandte Analysis und Stochastik, 2001.

[Bov 2006] A. Bovier, *Statistical Mechanics of Disordered Systems: A Mathematical Perspective*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2006.

[BuCa 1998] R. Burioni and D. Cassi, *Magnetization bound for classical spin models on graphs*, http://arxiv.org/pdf/cond-mat/9808046v1, 1998.

[BuCa 2005] R. Burioni and D. Cassi, *Random walks on graphs: ideas, techniques and results*, J. Phys. A: Math. Gen. 38 (2005) R45-R78 .

[BrLi 1976] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindner theorems, including inequalities for log-concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22, 366-389, 1976.

[BrLePf 1981] J. Bricmont, J. L. Lebowitz and C. E. Pfister, *Periodic Gibbs States of Ferromagnetic Spin Systems*, J. Stat. Phys. 24, 269-278, 1981.

[CaOlPePr 1978] M. Cassandro, E. Olivieri, A. Pellegrinotti and E. Presutti, *Existence and uniqueness of DLR measures for unbounded spin systems*, Z. Wahrsch. verw. Gebiete 41, 313334, 1978.

[CaVa 1977] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lect. Notes Math. 580, Springer-Verlag, Berlin Heidelberg New York, 1977.

[Do 1965] R. L. Dobrushin, *Existence of a phase transition in two and three dimensional Ising model.*, Theory Prob. Appl. 10, 193-213, 1965.

[Do 1968] R. L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory Prob. Appl. 13, 197-224, 1968.

[Do2 1968] R. L. Dobrushin, *The problem of uniqueness of a Gibbsian random field and the problem of phase transition*, Funct. Anal. Appl. 2, 302312, 1968.

[Do 1970] R. L. Dobrushin, *Prescribing a system of random variables by conditional distributions*, Theory Prob. Appl. 15, 458-486, 1970.

[DoSh 1985] R. L. Dobrushin and S. B. Shlosman, *Constructive criterion for the uniqueness of a Gibbs field*, Statistical mechanics and dynamical systems, J. Fritz, A. Jaffe, and D. Szasz, Birkhäuser, Boston, 347-370, 1985.

[DoPe 1982] R. L. Dobrushin and E. A. Pecherski, *A criterion of the uniqueness of Gibbsian fields in the non-compact case*, Lectures Notes in Mathematics 1021, 97-110, 1982.

[DrLaPe 1979] W. Driesler, L. Landau and J. F. Perez, *Estimates of critical length and critical temperatures for classical and quantum lattice systems*, J. Stat. Phy., 20, 123-162, 1979.

[Du 1999] R. M. Dudley, *Real Analysis and Probability*, CRC Press, 1999.

[EvKePeSc 2000] W. Evans, C. Kenyon, Y. Peres and L. J. Schulman, *Broadcasting on Trees and the Ising Model*, Ann. Appl. Probab. 10, 410-433, 2000.

[Fö 1982] H. Föllmer, *A covariance estimate for Gibbs measures*, J. Funct. Anal. 46, 387-395, 1982.

[FoGuMé 2007] J. Fontbona, H. Guérin and S. Méléard, *Measurability of optimal transportation and convergence rate of Landau type interacting particle systems*, Preprint, arXiv:math-pr/0703432, 2007.

[FoKaGi 1971] C. Fortuin, P. Kasteleyn and J. Ginibre, *Correlation inequalities on some partially oriented sets*, Commun. Math. Phys. 22, 89-103, 1971.

[FrSiSp 1976] J. Fröhlich, B. Simon and T. Spencer, *Infrared bounds, phase transitions and continuous symmetry breaking*, Commun. Math. Phys., 50, 79-85, 1976.

[Ge 1988] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, Studies in Mathematics 9, Walter de Gruyter, Berlin, New York, 1988.

[GeHäMa 2000] H.-O. Georgii, O. Häggström and C. Maes, *The random geometry of equilibrium phases*, Academic Press 18, 1-142, London, 2000.

[Gia 2003] G. Giacomin, *On stochastic domination in the Brascamp-Lieb framework*, Math. Proc. Cambridge Philos. Soc., 134, pp. 507-514, 2003.

[Gi 1969] J. Ginibre, *Simple Proof and Generalization of Griffiths' Second Inequality*, Phys. Rev. Lett. 23, 828-830, 1969.

[Gi 1970] J. Ginibre, *General Formulation of Griffiths' Inequalities*, Comm. Math. Phys. 16, 310-328, 1970.

[Gr 1979] L. Gross, *Decay of correlations in classical lattice models at high temperature*, Comm. Math. Phys. 68, 9-27, 1979.

[GlJa 1981] J. Glimm and A. Jaffe, *Quantum Physics, A Funtional Integral Point of View, Second Edition*, Springer, Berlin, 1981.

[Gr 1967] R. Griffiths, *Correlations in Ising Ferromagnets I*, J. Math. Phys. 8, 478-483, 1967.

[Gr2 1967] R. Griffiths, *Correlations in Ising Ferromagnets II*, J. Math. Phys. 8, 484-489, 1967.

[Is 1925] E. Ising, *Beitrag zur Theorie der Ferromagnetismus*, Z. Phys. 31, 253-258, 1925.

[JoSt 1999] J. Jonasson and J. F. Steif, *Amenability and phase transitions in the Ising model*, J. Theoret. Probab. 12, 549-559, 1999.

[Joy 1971] D. C. Joyce, *Survey of extrapolation processes in numerical analysis.*, SIAM Rev. 13, 435-490, 1971.

[Ka 1940] L. V. Kantorovich, *On an effective method of solving certain classes of extremal problems*, Dokl. Akad. Nauk. USSR 28, 212-215, 1940.

[Ka 1942] L. V. Kantorovich, *On the translocation of masses.*, Dokl. Akad. Nauk. USSR 37, 199-201, 1942.

[Ka 1948] L. V. Kantorovich, *On a problem of Monge* Uspekhi Mat. Nauk. 3, 225-226, 1948. English translation in J. Math. Sci. 133, 4, 1383, 2006.

[KaRu 1958] L. V. Kantorovich and G. S. Rubinshtein, *On a space of totally additive functions* Vestn. Leningrad. Univ. 13, 7, 52-59, 1958.

[KoPa 2005] Y. Kozitsky and T. Pasurek, *Euclidean Gibbs measures of quantum anharmonic crystals*, BiBoS Preprint 05-05-180, 47 pages, 2005.

[KoPa 2007] Y. Kozitsky and T. Pasurek, *Euclidean Gibbs measures of interacting quantum anharmonic oscillators*, J. of Stat. Phys., Vol. 127, 2007.

[KoKoPa 2009] Y. G. Kondratiev, Y. Kozitsky and T. Pasurek, *Gibbs random fields with unbounded spins on unbounded degree graphs*, arXiv:0904.3207, 2009.

[Kü 1982] H. Künsch, *Decay of correlations under Dobrushin's uniqueness condition and its applications*, Commun. Math. Phys. 84, 207-222, 1982.

[KuKiWa 1953] M. Kurata, R. Kikuchi and T. Watari, *A theory of cooperative phenomena. III. Detailed discussions of the cluster variation method.*, J. Chem. Phys. 21, 434-448, 1953.

[La 1971] O. E. Lanford, *Selected Topics in Functional Analysis*, In: Statistical Mechanics and Quantum Field Theory, 109-214, Gordon and Breach, New York, 1971.

[LaRu 1969] O. E. Lanford and D. Ruelle, *Observables at infinity and states with short range correlation in statistical mechanics*, Commun. Math. Phys., 194-215, 1969.

[La 1963] P. J. LAURENT, *Un thorme de convergence pour le procd d'extrapolation de Richardson*, C. R. Acad. Sci. Paris 256, 1435-1437, 1963.

[Le 1974] J. L. LEBOWITZ, *GHS and other Inequalities*, Comm. Math. Phys. 35, 87-92, 1974.

[LP 1976] J. L. LEBOWITZ AND E. PRESUTTI, *Statistical mechanics of systems of unbounded spins*, Comm. Math. Phys. 50 , 195-218, 1976.

[Le 2001] M. LEDOUX, *Logarithmic sobolev inequalities for unbounded spin systems revisited*, Lecture Notes in Mathematics 1755, 1167-194. Springer, 2001.

[Lz 1920] W. LENZ, *Beitrag zum verständnis der magnetischen Erscheinungen in festen Körpern*, Phys. Zeitschr. 21, 613-615, 1920.

[Lig 1985] T. M. LIGGETT, *Interacting Particle Systems.*, Berlin, Heidelberg, New York, Springer, 1985.

[Li 1992] T. LINDVALL, *Lectures on the Coupling Method*, John Wiley and Sons, Inc., New York Chichester Brisbane Toronto Singapore, 1992.

[Lyn 1986] J. N. LYNESS, *Numerical integration. Part A: extrapolation methods for multi-dimensional quadrature.*, In J. L. Mohamed and J. E. Walsh, eds., Numerical Algorithms, Oxford, Clarendon Press, 105-124, 1986.

[Ly 1989] R. LYONS, *The Ising model and percolation on trees and tree-like graphs.*, Comm. Math. Phys. 125, 337-353, 1989.

[Ly 1990] R. LYONS, *Random walks and percolation on trees*, The Annals of Probability, Vol 18, No. 3, 931-958, 1990.

[LyPe] R. LYONS AND Y. PERES, *Probability on Trees and Networks*, http://research.microsoft.com/en-us/um/people/peres/, in progress.

[Ly 2000] R. LYONS, *Phase transitions on nonamenable graphs*, J. Math. Phys. 41, 1099-1126, 2000.

[Mo 2005] I. MOLCHANOV, *Theory of Random Sets*, Springer-Verlag, Berlin Heidelberg New York, 2005.

[MaMi 1991] V. MALYSHEV AND R. MINLOS, *Gibbsian Random Fields*, Kluwer Academic Publishes, Dordrecht, Boston, London, 1991.

[Mon 1781] G. Monge, *Mémoire sur la théorie des dèblais et des remblais.*, Histoire de l'Académie des sciences de Paris, avec les Mémoires de mathématique et de physique pour la même année, 666-704, 1781.

[On 1944] L. Onsager, *Crystal statistics, I. A two-dimensional model with an order-disorder transformation.*, Phys. Rev., 65, 117-149, 1944.

[OsSp 1999] H. Osada and H. Spohn, *Gibbs Measures Relative to Brownian Motion*, Ann. Prob. 27 No. 3, 1183-1207, 1999.

[P 1967] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

[Pa] T. Pasurek, *Dobrushin's compactness criterion for Gibbs measures on loop spaces* (in preparation).

[Pa 2008] T. Pasurek, *Theory of Gibbs measures with unbounded spins: probabilistic and analytical aspects*, Habilitation Thesis, University of Bielefeld, 2008.

[Pe 1936] R. Peierls, *On Ising's model of ferromagnetism.*, Proc. Cambridge Phil. Soc., 32, 477-481, 1936.

[Pr2 1974] C. J. Preston, *A generalization of the FKG inequality*, Comm. Math. Phys., 36, 223-241, 1974.

[Pr 1974] C. J. Preston, *Gibbs States on Countable Sets*, Cambridge University Press, London, New York, Cambridge Tracts in Mathematics, No. 68, 1974.

[Pr 1976] C. J. Preston, *Random Fields*, Lecture Notes in Mathematics, Springer-Verlag, New York, 1976.

[Pr 2005] C. J. Preston, *Specifications and their Gibbs States*, Preprint, Available at http://www.math.uni-bielefeld.de, 2005.

[PaKh 1987] L. A. Pastur and B. A. Khoruzhenko, *Phase transitions in quantum models of rotators and ferroelectrics*, Theor. Math. Phys., 73, 1094-1104, 1987.

[PrFo 1991] B. Prum and J.C. Fort, *Stochastic Processes on a Lattice and Gibbs Measures*, Mathematical Physics Studies, Kluwer Academic Publishes, Dordrecht Boston London, 1991.

[Ra 1991] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, John Wiley & Sons Ltd., New York, 1991.

[Ro 2008] C. Roberto, *Slow decay of Gibbs measures with heavy tails*, arXiv:0811.2733v1, 2008.

[RaRü 1998a] S. T. Rachev and L. Rüschendorf, *Mass Transportation Problems. Vol. I: Theorey. Probability and its Applications.*, Springer, 1998a.

[RaRü 1998b] S. T. Rachev and L. Rüschendorf, *Mass Transportation Problems. Vol. II: Applications. Probability and its Applications.*, Springer, 1998b.

[ReSi 1975] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, London 1975.

[ReSi 1980] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis, Revised and Enlarged Edition*, Academic Press, New York, London 1980.

[RoZa 1998] S. Romano and V. A. Zagrebnov, *Orientational ordering transition in a continuous-spin ferrofluid*, Physica A, 253, 483-497, 1998.

[Ro 1955] W. Romberg, *Vereinfachte numerische Integration*, Det. Kong. Norske Videnskabers Selskab Forhandlinger 28, Nr. 7, Trondheim 1955.

[Roy 1977] G. Royer, *Étude des champs Euclidiens sur un resau $\mathbb{Z}^\nu$*, J. Math. Pures Appl., 56, 455-478, 1977.

[Ru 1969] D. Ruelle, *Statistical Mechanics. Rigorous Results*, Benjamin, New York Amsterdam, 1969.

[Si 1974] B. Simon, *The $P(\Phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton, New Jersey, 1974.

[Si 1979] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, Princeton, New Jersey, 1979.

[Si 1982] Ya. Sinai, *Theory of Phase Transitions. Rigorous Results*, Pergamon Press, Oxford, 1982.

[StZe 1992] D. W. Stroock and B. Zegarlinski, *The logarithmic Sobolev inequality for discrete spin systems on a lattice*, Comm. Math. Phys. 149, 175-194, 1992.

[Tek 2006] N. H. Tekmen, *Gibbs measures on infinite graphs with finite range Hamiltonians*, University of Bielefeld, 2006.

[Yo 2001] N. Yoshida, *The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on the lattice.* Ann. Inst. H. Poincar Prob. Statist. 37, 223-243, 2001.

[Va 1969] L. N. Vasershtein, *Markov processes over denumerable products of spaces describing large system of automata.* Problemy Peredači Informacii 5 (3), 64-72, 1969.

[Vi 2005] C. Villani, *Optimal Transport, Old and New*, Lecture Notes Ecole d'Eté de Probabilités de Saint-Flour, 2005; Manuscript (2006) available online at http://www.umpa.ens-lyon.fr/ cvillani.

[W 2005] D. Weitz, *Combinatorial Criteria for Uniqueness of Gibbs Measures*, Random Structures Algorithms 27 (4), 445-475, 2005.

[We 1977] D. Wells, *Some Moment Inequalities and a Result on Multivariable Unimodality*, Thesis, Indiana Univerity, 1977.

[Wo 2000] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics 138, Cambridge University Press, 2000.

[WiTa 2006] S. Winkler and S. Tatikonda, *Criteria for Rapid Mixing of Gibbs Samplers and Uniqueness of Gibbs Measures*, 44th Allerton Conference on Communication, Control, and Computing, Monticello, 2006.

[Za 2000] M. Zahradnik, *Contour methods and Pirogov-Sinai theory for continuous spin models*, Amer. Math. Soc. Transl. Ser. 2, 197-220, 2000.

[Ze 1996] B. Zegarlinski, *The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice*, Commun. Math. Phys. 175, 401-432, 1996.

[ZhZh 2008] H. Zhou and Z. Zheng, *Generalized Criteria for Uniqueness of Gibbs Measures*, Accepted by International Conference on Pattern Recognition, Florida, 2008.