J-invariant of semisimple algebraic groups

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Contents

Introduction

Let G be a semisimple linear algebraic group of inner type over a field F and X be a projective homogeneous G-variety such that G splits over the function field of X. We call such a variety generically split. In the present paper we address the problem of computing the Chow motive $\mathcal{M}(X)$ of X or, in other words, providing a direct sum decomposition of $\mathcal{M}(X)$ into indecomposable summands.

When the group G is isotropic this problem was solved by B. Köck [Ko91] (in the split case), V. Chernousov, S. Gille and A. Merkurjev [CGM05] (in the case of an isotropic X) and P. Brosnan [Br05] (in the general case). In all these proofs one constructs a (relative) cellular filtration on X, which allows to express the motive of the total space X in terms of motives of the base. Since the latter consists of homogeneous varieties of anisotropic groups, it reduces the problem to the case of anisotropic G.

When G is an orthogonal group and X is an anisotropic quadric, $\mathcal{M}(X)$ can be computed following the works of M. Rost [Ro98] (Pfister quadrics), N. Karpenko, A. Merkurjev and A. Vishik (general case). For Severi-Brauer varieties we refer to the paper by N. Karpenko [Ka96]. For some exceptional varieties the motivic decompositions was found by J.-P. Bonnet [Bo03] (G_2-case) and by S. Nikolenko, N. Semenov, K. Zainoulline [NSZ] (F_4-case). To obtain all these results one essentially uses Rost Nilpotence Theorem which says that in order to provide a desired decomposition it is enough to provide it over the algebraic closure with the property that all respective idempotents are defined over the base field. This reduces the problem to finding rational idempotents in the Chow ring $\mathrm{CH}^*(\bar{X} \times \bar{X})$. Observe that in all cases above the respective idempotents were detected using specific geometrical properties of X.

We uniformize all these proofs. The key idea comes from the paper [Kc85] by V. Kac, where he invented the notion of p-exceptional degrees – numbers which encode the information about the Chow ring of a split group $\bar{G}$ modulo a torsion prime p. The results of N. Karpenko, A. Merkurjev [KM05] and K. Zainoulline [Za06] concerning canonical p-dimensions of algebraic groups tell us that there is a strong interrelation between those numbers and the 'size' of the subgroup of rational cycles in $\mathrm{CH}^*(\bar{X} \times \bar{X})$. All this together lead to the notion of J-invariant $J_p(G)$ of a group G modulo p (see Definition 4.5); in the case of orthogonal groups this invariant was introduced by A. Vishik in slightly different terms. Our main observation is that $J_p(G)$ characterizes the motivic decomposition of X with $\mathbb{Z}/p$-coefficients.

The paper is organized as follows. In Chapter 1 we recall the definition of Chow motives and show how to find certain rational cycles using the 'generic point' diagram. In Chapter 2 we provide several 'idempotent lifting' tools which will be used in the sequel. In particular, we show that decompositions of motives with $\mathbb{Z}/m$-coefficients, where $m = 2, 3, 4, 6$ can be always lifted to integers. In Chapter 3 using the motivic version of the result of D. Eddidin and W. Graham on cellular fibrations we prove Theorem 3.9 generalizing and simplifying the results of paper [CPSZ]. In Chapter 4 we introduce the notion of J-invariant, consider the case of the variety of complete flags (Theorem 4.8) and extend the obtained result to arbitrary generically split projective homogeneous varieties (Theorem 4.21). In Chapter 5 we describe properties of J-invariant and its relations to the canonical dimension and splitting behavior of a group. Chapter 6 is devoted to examples of motivic decompositions.

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Chapter 1

Chow motives

1.1. In the present paper we work with the category of Grothendieck-Chow motives $Chow(F;\Lambda)$ over a field F with coefficients in Λ, where Λ is a commutative ring with identity. Recall briefly the definition of this category (we refer to [Ka01, §2] and [Ma68] for details).

A *correspondence* between X and Y is an element of $\mathrm{CH}^*(X\times Y;\Lambda) = \mathrm{CH}^*(X\times Y)\otimes\Lambda$. A correspondence ϕ between X and Y determines a homomorphism $\phi_*\colon \mathrm{CH}^*(X;\Lambda)\to\mathrm{CH}^*(Y;\Lambda)$ called the *realization* of ϕ. There is a bilinear composition rule

$$\circ\colon\ \mathrm{CH}^*(Y\times Z;\Lambda)\times\mathrm{CH}^*(X\times Y;\Lambda)\to\mathrm{CH}^*(X\times Z;\Lambda)$$

compatible with the realization. The identity element is given by the *diagonal* $\Delta_X\in\mathrm{CH}^{\dim X}(X\times X)$. A correspondence ϕ between X and Y may be viewed as a correspondence between Y and X as well; we call this correspondence a *transpose* of ϕ and denote it by ϕ^t.

An object of $Chow(F;\Lambda)$ is a pair (X,φ) consisting of a smooth projective variety X and an idempotent (or *projector*) $\phi\in\mathrm{CH}^{\dim X}(X\times X;\Lambda)$. The group of morphisms $\mathrm{Hom}((X,\phi),(Y,\phi'))$ equals $\phi'\circ\mathrm{CH}^{\dim Y}(X\times Y;\Lambda)\circ\phi$; the composition of morphisms is the usual composition of correspondences. We denote an object (X,Δ_X) by $\mathcal{M}(X;\Lambda)$ and call it the *motive* of X with coefficients in Λ. Observe that $Chow(F;\Lambda)$ is a tensor category, where the tensor product is induced by the usual product of varieties over F.

Note that the motive of a projective line splits as a direct sum of two motives $\mathcal{M}(\mathbb{P}^1;\Lambda)=\Lambda\oplus\Lambda(1)$, where Λ is the motive of a point and $\Lambda(1)$ is called *Lefschetz* motive. For a given motive M and $i\in\mathbb{Z}$ we denote by $M(i)$ the tensor product $M\otimes\Lambda(1)^{\otimes i}$ and call it the *twist* (or *shift*) of M.

In the case $\Lambda = \mathbb{Z}$ we will often omit the coefficients in the notation.

1.2 Definition. We say L is a *splitting field* of a variety X or, equivalently, a variety X *splits* over L if the motive $\mathcal{M}(X;\Lambda)$ splits over L as a direct sum of twisted Lefschetz motives. To simplify the notation we will write $\mathrm{CH}^*(\bar{X};\Lambda)$ for $\mathrm{CH}^*(X_L;\Lambda)$ and $\overline{\mathrm{CH}}^*(X;\Lambda)$ for the image of the restriction map $\mathrm{res}\colon \mathrm{CH}^*(X;\Lambda) \to \mathrm{CH}^*(\bar{X};\Lambda)$. Elements of $\overline{\mathrm{CH}}^*(X;\Lambda)$ will be called *rational* cycles. Observe that $\mathrm{CH}^*(\bar{X};\Lambda)$ and $\overline{\mathrm{CH}}^*(X;\Lambda)$ don't depend on the choice of a splitting field.

1.3 Example. Let G be a semisimple linear algebraic group over F, X be a projective G-homogeneous variety. If G becomes quasi-split over K as a linear group (that is G_K contains a Borel subgroup defined over K) then K is a splitting field of X for $\Lambda = \mathbb{Z}$ (see [CGM05]) and therefore for any ring Λ.

1.4. Assume X has a splitting field. Observe that the Chow ring $\mathrm{CH}^*(\bar{X};\Lambda)$ is a free Λ-module. Denote by $P(\mathrm{CH}^*(\bar{X};\Lambda),t) = \sum_{i\geq 0} \mathrm{rk}_\Lambda \mathrm{CH}^i(\bar{X};\Lambda)\cdot t^i$ the respective Poincaré polynomial.

According to [KM05, Rem. 5.6] there is the Künneth decomposition $\mathrm{CH}^*(\bar{X}\times\bar{X};\Lambda) = \mathrm{CH}^*(\bar{X};\Lambda)\otimes \mathrm{CH}^*(\bar{X};\Lambda)$ and Poincare duality. The latter means that for a given Λ-basis of $\mathrm{CH}^*(\bar{X};\Lambda)$ there is a dual one with respect to the pairing $(\alpha,\beta)\mapsto \deg(\alpha\cdot\beta)$.

Note that for correspondences in $\mathrm{CH}^*(\bar{X}\times\bar{X};\Lambda)$ the composition rule is given by the formula $(\alpha_1\times\beta_1)\circ(\alpha_2\times\beta_2) = \deg(\alpha_1\beta_2)(\alpha_2\times\beta_1)$, the realization is given by $(\alpha\times\beta)_*(\gamma) = \deg(\alpha\gamma)\beta$ and the transpose by $(\alpha\times\beta)^t = \beta\times\alpha$.

1.5 Lemma. *Let X and Y be two smooth projective varieties such that $F(Y)$ is a splitting field of X and Y has a splitting field. Consider the projection on the first summand in the Künneth decomposition*

$$\mathrm{pr}_0\colon\ \mathrm{CH}^r(\bar{X}\times\bar{Y};\Lambda) = \bigoplus_{i=0}^{r} \mathrm{CH}^{r-i}(\bar{X};\Lambda)\otimes \mathrm{CH}^i(\bar{Y};\Lambda) \to \mathrm{CH}^r(\bar{X};\Lambda).$$

Then for any $\rho\in \mathrm{CH}^r(\bar{X};\Lambda)$ we have $\mathrm{pr}_0^{-1}(\rho)\cap \overline{\mathrm{CH}}^r(X\times Y;\Lambda)\neq\emptyset$.

Proof. Lemma follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^r(X\times Y;\Lambda) & \xrightarrow{\mathrm{res}_{L/F}} & \mathrm{CH}^r(\bar{X}\times\bar{Y};\Lambda) & & \\
\downarrow & & \downarrow & \searrow^{\mathrm{pr}_0} & \\
\mathrm{CH}^r(X_{F(Y)};\Lambda) & \xrightarrow{\simeq} & \mathrm{CH}^r(\bar{X}_{\bar{F}(\bar{Y})};\Lambda) & \xrightarrow{\simeq} & \mathrm{CH}^r(\bar{X};\Lambda)
\end{array}$$

where the vertical arrows are taken from the localization sequence for Chow groups and, hence, are surjective and the bottom horizontal maps are isomorphisms. $\square$

We will extensively use the following version of Rost Nilpotence Theorem.

1.6 Lemma. *Let X be a smooth projective variety such that it splits over any field K over which it has a rational point. Then for any α in the kernel of the natural map* $\mathrm{End}(\mathcal{M}(X;\Lambda)) \to \mathrm{End}(\mathcal{M}(\bar{X};\Lambda))$ *we have* $\alpha^{\circ(\dim X+1)} = 0$.

Proof. Follows from [EKM, Theorem 68.1]. $\square$

Chapter 2

Lifting of idempotents

2.1. Given a $\mathbb{Z}$-graded ring A^* and two idempotents ϕ_1, $\phi_2 \in A^0$ we say ϕ_1 and ϕ_2 are orthogonal if $\phi_1\phi_2 = \phi_2\phi_1 = 0$. We say an element θ_{12} provides an isomorphism of degree d between idempotents ϕ_1 and ϕ_2 if $\theta_{12} \in \phi_2 A^{-d}\phi_1$ and there exist $\theta_{21} \in \phi_1 A^d \phi_2$ such that $\theta_{12}\theta_{21} = \phi_2$ and $\theta_{21}\theta_{12} = \phi_1$.

2.2. Consider the graded ring $\mathrm{End}^*(\mathcal{M}(X;\Lambda))$, where

$$\mathrm{End}^i(\mathcal{M}(X;\Lambda)) = \mathrm{CH}^{\dim X+i}(X \times X;\Lambda),$$

with respect to the usual composition of correspondences. Note that an isomorphism θ_{12} of degree d between ϕ_1 and ϕ_2 provides an isomorphism between the motives (X,ϕ_1) and $(X,\phi_2)(d)$. By $\overline{\mathrm{End}}^*(\mathcal{M}(X;\Lambda))$ we denote the subring of $\mathrm{End}^*(\mathcal{M}(\bar{X};\Lambda))$ consisting of rational cycles.

2.3. Given a $\mathbb{Z}$-graded Λ-module V^* we denote by $\mathrm{End}^*(V^*)$ the graded ring whose d-th component consists of all endomorphisms of V^* of degree d. Note that using Poincaré duality one can identify $\mathrm{End}^*(\mathcal{M}(\bar{X};\Lambda))$ with $\mathrm{End}^*(\mathrm{CH}^*(\bar{X};\Lambda))$.

2.4 Definition. Let $f\colon A^* \to B^*$ be a homomorphism of $\mathbb{Z}$-graded rings. We say that f is *decomposition preserving* if given a family $\phi_i \in B^0$ of pair-wise orthogonal idempotents such that $\sum_i \phi_i = 1_B$, there exists a family of pair-wise orthogonal idempotents $\varphi_i \in A^0$ such that $\sum_i \varphi_i = 1_A$ and each $f(\varphi_i)$ is isomorphic to ϕ_i by means of an isomorphism of degree 0. We say f is *strictly decomposition preserving* if, moreover, one can choose φ_i such that $f(\varphi_i) = \phi_i$. We say f is *isomorphism preserving* if for any idempotents φ_1 and φ_2 in A^0 and any isomorphism θ_{12} of degree d between idempotents

$f(\varphi_1)$ and $f(\varphi_2)$ in B^0 there exists an isomorphism ϑ_{12} of degree d between φ_1 and φ_2. We say f is *strictly isomorphism preserving* if, moreover, one can choose ϑ_{12} such that $f(\vartheta_{12}) = \theta_{12}$.

2.5 Lemma. *Let $f\colon A^* \to B^*$ and $g\colon B^* \to C^*$ be homomorphisms such that $g \circ f$ is decomposition (resp. isomorphism) preserving and g is isomorphism preserving. Then f is decomposition (resp. isomorphism) preserving.*

Proof. Obvious. □

2.6 Lemma. *Assume we are given a cartesian square*

$$\begin{array}{ccc} A^* & \xrightarrow{f} & B^* \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ C^* & \xrightarrow{g} & D^*. \end{array}$$

(it means that $\operatorname{Ker} g \subset \operatorname{Im} i$) such that g is strictly decomposition (resp. strictly isomorphism) preserving. Then f is strictly decomposition (resp. strictly isomorphism) preserving.

Proof. An easy diagram chase. □

2.7 Lemma. *Let $f\colon A^* \to B^*$ be a surjective homomorphism such that the kernel of the restriction of f to A^0 consists of nilpotent elements. Then f is strictly decomposition and strictly isomorphism preserving.*

Proof. We show that f is strictly decomposition preserving. Suppose we are given pair-wise orthogonal idempotents $\phi_1, \dots, \phi_m$ in B^0 whose sum is the identity. The proof goes by induction on m. Let $m = 2$. Choose ψ such that $f(\psi) = \phi_1$. Then $f(\psi(1-\psi)) = 0$ and therefore $\psi^n(1-\psi)^n = 0$ for some n. Split the expression $(\psi + (1-\psi))^{2n-1}$ into two summands $\varphi_1 = \sum_{k=n}^{2n-1} \binom{2n-1}{k} \psi^k (1-\psi)^{2n-1-k}$ and $\varphi_2 = \sum_{k=n}^{2n-1} \binom{2n-1}{k} \psi^{2n-1-k}(1-\psi)^k$. Now $\varphi_1 + \varphi_2 = 1$ and $\varphi_1\varphi_2 = 0$; it means that φ_1 and φ_2 are orthogonal idempotents. It is easy to see that $f(\varphi_1) = \phi_1$ and $f(\varphi_2) = \phi_2$.

Now consider the general case. Choose an idempotent φ_m such that $f(\varphi_m) = \phi_m$ and consider the ring $(1-\varphi_m)A^0(1-\varphi_m)$. Its image under f is $(1-\phi_m)B^0(1-\phi_m)$ and therefore contains idempotents $\phi_1, \dots, \phi_{m-1}$ whose sum is $1 - \phi_m$ which is the identity in that ring. Applying the induction hypothesis we can find pair-wise orthogonal idempotents φ_i, $i \le m-1$, whose sum is $1 - \varphi_m$, such that $f(\varphi_i) = \phi_i$.

Now the fact that f is strictly isomorphism preserving follows from the following more general lemma. □

2.8 Lemma. *Let A, B be two rings, A^0, B^0 be their subrings, $f^0\colon A^0 \to B^0$ be a ring homomorphism, $f\colon A \to B$ be a map of sets satisfying the following conditions:*

- *$f(\alpha)f(\beta)$ equals either $f(\alpha\beta)$ or 0 for all $\alpha, \beta \in A$;*
- *$f^0(\alpha)$ equals $f(\alpha)$ if $f(\alpha) \in B^0$ or 0 otherwise;*
- *$\operatorname{Ker} f^0$ consists of nilpotent elements.*

Let φ_1 and φ_2 be two idempotents in A^0, ψ_{12} and ψ_{21} be elements in A such that $\psi_{12}A^0\psi_{21} \subset A^0$, $\psi_{21}A^0\psi_{12} \subset A^0$, $f(\psi_{21})f(\psi_{12}) = f(\varphi_1)$, $f(\psi_{12})f(\psi_{21}) = f(\varphi_2)$. Then there exist elements $\vartheta_{12} \in \varphi_2A^0\psi_{12}A^0\varphi_1$ and $\vartheta_{21} \in \varphi_1A^0\psi_{21}A^0\varphi_2$ such that $\vartheta_{21}\vartheta_{12} = \varphi_1$, $\vartheta_{12}\vartheta_{21} = \varphi_2$, $f(\vartheta_{12}) = f(\varphi_2)f(\psi_{12}) = f(\psi_{12})f(\varphi_1)$, $f(\vartheta_{21}) = f(\varphi_1)f(\psi_{21}) = f(\psi_{21})f(\varphi_2)$.

Proof. Since $\operatorname{Ker} f^0$ consists of nilpotents, f^0 sends non-zero idempotents in A^0 to non-zero idempotents in B^0; in particular, $f(\varphi_1) = f^0(\varphi_1) \neq 0$, $f(\varphi_2) = f^0(\varphi_2) \neq 0$. Observe that

$$f(\psi_{12})f(\varphi_1) = f(\psi_{12})f(\psi_{21})f(\psi_{12}) = f(\varphi_2)f(\psi_{12})$$

and, similarly, $f(\psi_{21})f(\varphi_2) = f(\varphi_1)f(\psi_{21})$. Changing ψ_{12} to $\varphi_2\psi_{12}\varphi_1$ and ψ_{21} to $\varphi_1\psi_{21}\varphi_2$ we may assume that $\psi_{12} \in \varphi_2A\varphi_1$ and $\psi_{21} \in \varphi_1A\varphi_2$. We have

$$f^0(\varphi_2) = f(\varphi_2) = f(\psi_{12})f(\psi_{21}) = f(\psi_{12}\psi_{21}) = f^0(\psi_{12}\psi_{21});$$

therefore $\alpha = \psi_{12}\psi_{21} - \varphi_2 \in A^0$ is nilpotent, say $\alpha^n = 0$. Note that $\varphi_2\alpha = \alpha = \alpha\varphi_2$. Set $\alpha^\vee = \varphi_2 - \alpha + \ldots + (-1)^{n-1}\alpha^{n-1} \in A^0$; then $\alpha\alpha^\vee = \varphi_2 - \alpha^\vee$, $\varphi_2\alpha^\vee = \alpha^\vee = \alpha^\vee\varphi_2$ and $f(\varphi_2) = f^0(\varphi_2) = f^0(\alpha^\vee) = f(\alpha^\vee)$. Therefore setting $\vartheta_{21} = \psi_{21}\alpha^\vee$ we have $\vartheta_{21} \in \varphi_1A\varphi_2$, $\psi_{12}\vartheta_{21} = \varphi_2$ and $f(\vartheta_{21}) = f(\psi_{21})$.

Now $\vartheta_{21}\psi_{12}$ is an idempotent. We have

$$f^0(\varphi_1) = f(\varphi_1) = f(\vartheta_{21})f(\psi_{12}) = f(\vartheta_{21}\psi_{12}) = f^0(\vartheta_{21}\psi_{12});$$

therefore $\beta = \vartheta_{21}\psi_{12} - \varphi_1 \in A^0$ is nilpotent. Note that $\beta\varphi_1 = \beta = \varphi_1\beta$. Now $\varphi_1 + \beta = (\varphi_1 + \beta)^2 = \varphi_1 + 2\beta + \beta^2$ and therefore $\beta(1 + \beta) = 0$. But $1 + \beta$ is invertible and hence we have $\beta = 0$. It means that $\vartheta_{21}\psi_{12} = \varphi_1$ and we can set $\vartheta_{12} = \psi_{12}$. □

2.9 Corollary. *The map* $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p^n)) \to \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/p))$ *is strictly decomposition and strictly isomorphism preserving.*

2.10 Lemma. *Let* $m = m_1m_2$ *be a product of two coprime integers. Then the map* $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m)) \to \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m_1)) \times \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m_2))$ *is an isomorphism.*

Proof. Follows from Chinese Remainder Theorem. □

2.11 Definition. We say that a field extension E/F is *rank preserving* with respect to X if the restriction map $\mathrm{res}_{E/F}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

2.12 Lemma. *Assume* X *has a splitting field. Then for any finite rank preserving field extension* E *over* F *we have* $[E:F]\cdot\overline{\mathrm{CH}}^*(X_E) \subset \overline{\mathrm{CH}}^*(X)$.

Proof. Let L be a splitting field containing E. Let γ be any element in $\overline{\mathrm{CH}}^*(X_E)$. By definition there exists $\alpha \in \mathrm{CH}^*(X_E)$ such that $\gamma = \mathrm{res}_{L/E}(\alpha)$. Since $\mathrm{res}_{E/F} \otimes \mathbb{Q}$ is an isomorphism, there exists an element $\beta \in \mathrm{CH}^*(X)$ and a non-zero integer n such that $\mathrm{res}_{E/F}(\beta) = n\alpha$. By projection formula

$$n \cdot \mathrm{cores}_{E/F}(\alpha) = \mathrm{cores}_{E/F}(\mathrm{res}_{E/F}(\beta)) = [E:F]\cdot\beta.$$

Applying $\mathrm{res}_{L/E}$ to both sides we obtain $n(\mathrm{res}_{L/E}(\mathrm{cores}_{E/F}(\alpha))) = n[E:F]\cdot\gamma$. Therefore, $\mathrm{res}_{L/E}(\mathrm{cores}_{E/F}(\alpha)) = [E:F]\cdot\alpha$. □

From now on we assume that X is a smooth projective variety which has a splitting field, with a property that the kernel of the map

$$\mathrm{End}^*(\mathcal{M}(X_E;\Lambda)) \to \mathrm{End}^*(M(\bar{X};\Lambda))$$

consists of nilpotent elements for all extensions E/F and all rings Λ. Say, that is the case when X satisfies the condition of Lemma 1.6.

2.13 Lemma. *The map* $\mathrm{End}^*(\mathcal{M}(X_E;\Lambda)) \to \overline{\mathrm{End}}^*(\mathcal{M}(X_E;\Lambda))$ *is strictly decomposition and strictly isomorphism preserving for any extension* E/F.

Proof. Follows from Lemma 2.7. □

2.14 Lemma. *Assume* X *has a splitting field,* E/F *is a field extension of degree coprime with* m*, which is rank preserving with respect to* $X \times X$*. Then the map* $\mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m)) \to \mathrm{End}^*(\mathcal{M}(X_E;\mathbb{Z}/m))$ *is decomposition and isomorphism preserving.*

Proof. By Lemma 2.12 we have $\overline{\mathrm{End}}^*(\mathcal{M}(X_E;\mathbb{Z}/m)) = \overline{\mathrm{End}}^*(\mathcal{M}(X;\mathbb{Z}/m))$. Now apply Lemma 2.13 and Lemma 2.5 with $A^* = \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m))$, $B^* = \mathrm{End}^*(\mathcal{M}(X_E;\mathbb{Z}/m))$, $C^* = \overline{\mathrm{End}}^*(\mathcal{M}(X_E;\mathbb{Z}/m))$. □

2.15 Lemma. *Let V^* be a graded $\mathbb{Z}$-module whose components are free and have finite ranks. Then the reduction map $\mathrm{End}^*(V^*) \to \mathrm{End}^*(V^* \otimes_{\mathbb{Z}} \mathbb{Z}/m)$ strictly preserves decompositions with the property that the graded components of $\mathrm{Im}\,\phi_i$ are free $\mathbb{Z}/m$-modules.*

Proof. We are given a decomposition $V^k \otimes_{\mathbb{Z}} \mathbb{Z}/m = \oplus_i W_i^k$, where W_i^k is the k-graded component of $\mathrm{Im}\,\phi_i$. Present V^k as a direct sum $V^k = \bigoplus_i V_i^k$ of free $\mathbb{Z}$-modules such that $\mathrm{rk}_{\mathbb{Z}} V_i^k = \mathrm{rk}_{\mathbb{Z}/m} W_i^k$. Fix a $\mathbb{Z}$-basis $\{v_{ij}^k\}_j$ of V_i^k. For each W_i^k choose a basis $\{w_{ij}^k\}_j$ such that the linear transformation D^k of $V^k \otimes_{\mathbb{Z}} \mathbb{Z}/m$ sending each $v_{ij}^k \otimes 1$ to w_{ij}^k has determinant 1. By Lemma 2.16 there is a lifting $\tilde{D}^k$ of D^k over $\mathbb{Z}$. So we obtain $V^k = \bigoplus_i \tilde{W}_i^k$, where $\tilde{W}_i^k = \tilde{D}^k(V_i^k)$ satisfies $\tilde{W}_i^k \otimes_{\mathbb{Z}} \mathbb{Z}/m = W_i^k$. Define φ_i on each V^k to be the projection onto $\tilde{W}_i^k$. □

2.16 Lemma. *The map $\mathrm{SL}_l(\mathbb{Z}) \to \mathrm{SL}_l(\mathbb{Z}/m)$ induced by the reduction modulo m is surjective.*

Proof. Since $\mathbb{Z}/m$ is a semi-local ring, the group $\mathrm{SL}_l(\mathbb{Z}/m)$ is generated by elementary matrices (see [HOM89, Theorem 4.3.9]). □

2.17 Lemma. *In the statement of Lemma 2.15 assume additionally that $(\mathbb{Z}/m)^\times = \{\pm 1\}$. Then the reduction map is strictly isomorphism preserving.*

Proof. Let φ_1, φ_2 be two idempotents in $\mathrm{End}^*(V^*)$; denote by V_i^k the k-graded component of $\mathrm{Im}\,\varphi_i$. An isomorphism θ_{12} between $\varphi_1 \otimes 1$ and $\varphi_2 \otimes 1$ of degree d can be identified with a family of isomorphisms $\theta_{12}^k \colon V_1^k \otimes \mathbb{Z}/m \to V_2^{k-d} \otimes \mathbb{Z}/m$. Now by Lemma 2.16 we can lift them to isomorphisms $\vartheta_{12}^k \colon V_1^k \to V_2^{k-d}$, and we are done. □

2.18 Lemma. *Assume X has a splitting field of degree m which is rank preserving with respect to $X \times X$. Then the map*

$$\mathrm{End}^*(\mathcal{M}(X)) \to \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m))$$

preserves decompositions with the property that $\mathrm{Im}\,\mathrm{res}(\phi_i)$ are free $\mathbb{Z}/m$-modules, where

$$\mathrm{res}\colon\ \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m)) \to \mathrm{End}^*(\mathcal{M}(\bar{X};\mathbb{Z}/m))$$

is the restriction. If additionally $(\mathbb{Z}/m)^\times = \{\pm 1\}$ *then this map is isomorphism preserving.*

Proof. Consider the diagram

$$\begin{array}{ccc}
\mathrm{End}^*(\mathcal{M}(X)) & \twoheadrightarrow & \mathrm{End}^*(\mathcal{M}(X;\mathbb{Z}/m)) \\
\downarrow & & \downarrow \\
\overline{\mathrm{End}}^*(\mathcal{M}(X)) & \twoheadrightarrow & \overline{\mathrm{End}}^*(\mathcal{M}(X;\mathbb{Z}/m)) \\
\downarrow & & \downarrow \\
\mathrm{End}^*(\mathcal{M}(\bar{X})) & \twoheadrightarrow & \mathrm{End}^*(\mathcal{M}(\bar{X};\mathbb{Z}/m)).
\end{array}$$

The bottom arrow strictly preserves decompositions with the property stated by Lemma 2.15 with $V^* = \mathrm{CH}^*(\bar{X})$; in the case this map is isomorphism preserving by Lemma 2.17. By Lemma 2.12 the bottom square is cartesian and therefore we may apply Lemma 2.6 and obtain that the middle arrow preserves decompositions and isomorphisms as well. Now in the top square vertical arrows are decomposition and isomorphism preserving by Lemma 2.13. It remains to apply Lemma 2.5. □

Chapter 3

Motives of fibered spaces

3.1 Definition. Let X be a smooth projective variety over a field F. We say a smooth projective morphism $f\colon Y \to X$ is a *cellular fibration* if it is a locally trivial fibration whose fiber $\mathcal{F}$ is cellular, i.e., has a decomposition into affine cells.

3.2 Lemma. *Let $f\colon Y \to X$ be a cellular fibration. Then $\mathcal{M}(Y)$ is (non-canonically but compatible with base change) isomorphic to $\mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.*

Proof. We follow the proof of [EG97, Prop. 1]. Define the morphism

$$\varphi\colon \bigoplus_{i\in\mathcal{I}} \mathcal{M}(X)(\operatorname{codim} B_i) \to \mathcal{M}(Y)$$

to be the direct sum $\varphi = \bigoplus_{i\in\mathcal{I}} \varphi_i$, where each φ_i is given by the cycle $[\mathrm{pr}_Y^*(B_i)\cdot \Gamma_f] \in \mathrm{CH}^*(X\times Y)$ produced from the graph cycle Γ_f and the chosen (non-canonical) basis $\{B_i\}_{i\in\mathcal{I}}$ of $\mathrm{CH}^*(Y)$ over $\mathrm{CH}^*(X)$. The realization of φ coincides exactly with an isomorphism of abelian groups $\mathrm{CH}^*(X)\otimes \mathrm{CH}^*(\mathcal{F}) \to \mathrm{CH}^*(Y)$ constructed in [EG97, Prop. 1]. By Manin's identity principle [Ma68] φ is an isomorphism and we are done. □

3.3 Lemma. *Let G be a linear algebraic group over a field F, X be a projective homogeneous G-variety and Y be a G-variety. Let $f\colon Y\to X$ be a G-equivariant projective morphism. Assume that the fiber of f over $F(X)$ is isomorphic to $\mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$ over F. Then f is a locally trivial fibration with the fiber $\mathcal{F}$.*

Proof. By the assumptions, we have $Y\times_X \operatorname{Spec} F(X) \simeq (\mathcal{F}\times X)\times_X \operatorname{Spec} F(X)$ as schemes over $F(X)$. Since $F(X)$ is a direct limit of $\mathcal{O}(U)$ taken over all non-empty affine open subsets U of X, by [EGA IV, Corollaire 8.8.2.5] there exists U such that $f^{-1}(U) = Y \times_X U$ is isomorphic to $(\mathcal{F} \times X) \times_X U \simeq \mathcal{F} \times U$ as a scheme over U. Since G acts transitively on X and f is G-equivariant, the map f is a locally trivial fibration. □

3.4 Corollary. *Let X be a projective G-homogeneous variety, Y be a projective variety such that $Y_{F(X)} \simeq \mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$. Then the projection map $X \times Y \to X$ is a locally trivial fibration with the fiber $\mathcal{F}$.*

Proof. Apply Lemma 3.3 to the projection map $X \times Y \to X$. □

3.5 Corollary. *In the statement of Corollary 3.4 assume that $\mathcal{F}$ is cellular. Then $\mathcal{M}(X \times Y) \simeq \mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.*

Proof. Follows from Lemma 3.2. □

3.6. Let G be a semisimple (connected) linear algebraic group over a field F, X be a projective G-homogeneous variety. Denote by $\mathcal{D}$ the Dynkin diagram of G. Galois descent shows that one can choose a quasi-split group G_0 over F with the same Dynkin diagram, a parabolic subgroup P of G_0 and a cocyle $\xi \in H^1(F, G_0)$ such that G is isogenic to ${}_\xi G_0$ and X is isomorphic to ${}_\xi(G_0/P)$. G is called of *inner type* if one can take split G_0 and of *strongly inner type* if one can take simply-connected split G_0.

3.7 Lemma. *Let G be a semisimple linear algebraic group over F, X and Y be projective G-homogeneous varieties corresponding to parabolic subgroups P and Q of G_0, $Q \leq P$. Denote by $f\colon Y \to X$ the natural map corresponding to the quotient map $G_0/Q \to G_0/P$. If G becomes quasi-split over $F(X)$ then f is a cellular fibration with the fiber $\mathcal{F} = P/Q$.*

Proof. Since G becomes quasi-split over $F(X)$, the fiber of f over $F(X)$ is isomorphic to $(P/Q)_{F(X)} = \mathcal{F}_{F(X)}$. Now apply Lemma 3.3 and note that $\mathcal{F}$ is cellular. □

3.8 Example. Let $P = P_\Theta$ be the standard parabolic subgroup of a quasi-split group G_0, corresponding to a $*$-invariant subset Θ of the respective Dynkin diagram $\mathcal{D}$ (enumeration of roots follows Bourbaki). In this notation the Borel subgroup corresponds to the empty set. Let ξ be a cocycle in $H^1(F, G_0)$; set $G = {}_\xi G_0$ and $X = {}_\xi(G_0/P)$. We denote by q the degree of a splitting field of G_0. In the cases of A_n, D_n, E_6 and E_7 we denote by d the index of the associated central simple algebra over F or over a quadratic extension of F (note that $d = 1$ if G_0 is simply-connected). Analyzing Tits indices ([Ti66, Table II]; cf. [KR94, §7]) we see that G becomes quasi-split over $F(X)$ (or, in other words, X is generically split) in the following cases. If G_0 is split it suffices to require that the subset $\mathcal{D} \setminus \Theta$ contains one of the following vertices k:

G_0	${}^1\mathrm{A}_n$	B_n	C_n	${}^1\mathrm{D}_n$
k	$\gcd(k,d) = 1$	$k = n$; any k in the Pfister case	k is odd;	$k = n - 1$; $k = n$ if $2 \nmid n$ or $d = 1$; any k in the Pfister case

G_0	G_2	F_4	${}^1\mathrm{E}_6$	E_7	E_8
k	any	$k = 1,2,3$; any k if $q = 3$	$k = 3,5$; $k = 2,4$ if $d = 1$; any k if $d = 1$ and $q = 3$	$k = 2,5$; $k = 3,4$ if $d = 1$; $k \neq 7$ if $q = 3$	$k = 2,3,4,5$; any k if $q = 5$

By the Pfister case we mean the case when the cocycle ξ corresponds to a Pfister form or its maximal neighbor.

If G_0 is quasi-split but not split it suffices to require that the subset $\mathcal{D}\setminus\Theta$ contains one of the following $*$-invariant subset K:

G_0	${}^2\mathrm{A}_n$, $m = [\frac{n+1}{2}]$	${}^2\mathrm{E}_6$
K	$K = \{m, n+1-m, m-1, n+2-m\}$; $K = \{m, n+1-m\}$ if $\gcd(m,d) = 1$	$K = \{3,5\}$

G_0	${}^2\mathrm{D}_n$	${}^3\mathrm{D}_4$, ${}^6\mathrm{D}_4$
K	$K = \{n-2, n-1, n\}$; $K = \{n-1, n\}$ if $2 \mid n$ or $d = 1$	$K = \{1,3,4\}$

Case-by-case arguments of paper [CPSZ] show that under certain conditions the Chow motive of a twisted flag variety X can be expressed in terms of the motive of a minimal flag. These conditions cover almost all twisted flag varieties corresponding to groups of types A_n and B_n together with some examples of types C_n, G_2 and F_4. Using the following theorem we provide a uniform proof of these results as well as extend it for groups of types D_n and exceptional types.

3.9 Theorem. *Let Y and X be taken as in Lemma 3.7. Then the Chow motive $\mathcal{M}(Y)$ of Y is isomorphic to a direct sum of twisted copies of the* *motive* $\mathcal{M}(X)$*, i.e.,*

$$\mathcal{M}(Y) \simeq \bigoplus_{i\geq 0} \mathcal{M}(X)(i)^{\oplus c_i},$$

where $\sum c_i t^i = P(\mathrm{CH}^*(\bar{Y}), t)/P(\mathrm{CH}^*(\bar{X}), t)$.

Proof. Follows from Lemma 3.7 and Lemma 3.2. □

3.10 Remark. The explicit formula for $P(\mathrm{CH}^*(\bar{X}), t)$ involves the degrees of basic polynomial invariants of G_0 and is provided in [Hi82, Ch. IV, Cor. 4.5].

Chapter 4

Complete flag varieties

4.1. Let G_0 be a split semisimple linear algebraic group with a maximal split torus T and a Borel subgroup B containing T. Let $G = {}_\xi G_0$ be a twisted form of G_0 given by a cocyle $\xi \in H^1(F, G_0)$ and $X = {}_\xi(G_0/B)$ be the corresponding *variety of complete flags*. Observe that the group G splits over any field K over which X has a rational point, in particular, over the function field $F(X)$. According to [De74] $\mathrm{CH}^*(\bar{X})$ can be expressed in purely combinatorial terms and therefore depends only on type of G and not on the base field F.

4.2. Let p be a torsion prime of G_0. Let $\hat{T}$ denote the group of characters of T and $S^*(\hat{T})$ be the symmetric algebra. By R^* we denote the image of the characteristic map $c\colon S^*(\hat{T}) \to \mathrm{CH}^*(\bar{X}; \mathbb{Z}/p)$ (see [Gr58, (4.1)]). According to [KM05, Thm.6.4.(i)] we have $R^* \subset \overline{\mathrm{CH}}^*(X; \mathbb{Z}/p)$.

Consider the Chow ring $\mathrm{CH}^*(\bar{G}; \mathbb{Z}/p)$ of the split group $\bar{G}$ and the induced by the quotient map $\pi\colon \mathrm{CH}^*(\bar{X}; \mathbb{Z}/p) \to \mathrm{CH}^*(\bar{G}; \mathbb{Z}/p)$. According to [Gr58, Rem. 2°] π is surjective with the kernel generated by R^+, where R^+ stands for the ideal of constant-free elements in R^*. In particular, $\mathrm{CH}^*(\bar{G}; \mathbb{Z}/p)$ depends only on type of G and p and does not depend on the base field F.

4.3. The explicit presentation of $\mathrm{CH}^*(\bar{G}; \mathbb{Z}/p)$ in terms of generators and relations is known for all types of G and all p. The most uniform description can be found in the paper by V. Kac [Kc85]. Namely, by [Kc85, Thm. 3]

$$\mathrm{CH}^*(\bar{G}; \mathbb{Z}/p) \simeq (\mathbb{Z}/p)[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}),$$

where x_i is a generator of codimension d_i, $p \nmid d_i$, and the numbers $d_i p^{k_i}$,

$i = 1, \dots, r$, are known as *p-exceptional degrees* of $\bar{G}$. We assume that the order of x_i is compatible with their codimension, that is $d_i \le d_j$ when $i \le j$.

4.4. We will use the standard notation concerning multi-indices (or tuples). Given an r-tuple $M = (m_1, \dots, m_r)$ denote $x^M = \prod_{i=1}^r x_i^{m_i}$ and $|M| = \operatorname{codim} x^M = \sum_{i=1}^r d_i m_i$. Operations between r-tuples are assumed to be componentwise. Denote also $\binom{M}{L} = \prod_{i=1}^r \binom{m_i}{l_i}$.

We will write $M \preccurlyeq L$ if $m_i \le l_i$ for all i. Note that $\preccurlyeq$ is just a partial order. We also introduce a well-order on the set of all r-tuples, usually called *DegLex*. Namely, we will write $M \le N$ if either $|M| < |N|$, or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \ne n_i$. Obviously the order is compatible with addition.

4.5 Definition. For each $i = 1, \dots, r$ let j_i be the smallest non-negative integer such that the image of $\overline{\mathrm{CH}}^*(X; \mathbb{Z}/p)$ under π contains an element with the leading term $x_i^{p^{j_i}}$ (with respect to the DegLex order). Clearly $j_i \le k_i$. Define the J-invariant of G modulo p to be the r-tuple $J_p(G) = (j_1, \dots, j_r)$.

4.6 Example. In the case when G corresponds to the *generic* G_0-torsor we have $\overline{\mathrm{CH}}^*(X; \mathbb{Z}/p) = R^*$ (see [KM05, Theorem 6.4 (2)]) and, therefore, $J_p(G) = (k_1, \dots, k_r)$.

4.7 Example. Let ϕ be a quadratric form with trivial discriminant. A. Vishik defined $J(\phi)$ in terms of rationality of cycles on the maximal orthogonal Grassmannian (see [Vi05, Definition 5.11] or [EKM, § 88]). Using Theorem 3.9 one can show that $J(\phi)$ can be expressed in terms of $J_2(\mathrm{O}^+(\phi)) = (j_1, \dots, j_r)$ as follows:

$$J(\phi) = \{2^l d_i \mid i = 1, \dots, r,\ 0 \le l \le j_i - 1\}.$$

Since all d_i are odd, $J_2(\mathrm{O}^+(\phi))$ is determined by $J(\phi)$ as well.

4.8 Theorem. *Given G and p with $J_p(G) = (j_1, \dots, j_r)$ the motive of X is isomorphic to the direct sum*

$$\mathcal{M}(X; \mathbb{Z}/p) \simeq \bigoplus_{i \ge 0} \mathcal{R}(i)^{\oplus c_i},$$

where the motive $\mathcal{R}$ is indecomposable,

$$P(\bar{\mathcal{R}}, t) = \prod_{i=1}^r \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}},$$

and c_i are the coefficients of the polynomial

$$\sum_{i\geq 0} c_i t^i = P(\mathrm{CH}^*(\bar{X}), t)/P(\bar{\mathcal{R}}, t).$$

Fix preimages e_i of x_i in $\mathrm{CH}^*(\bar{X})$. Set $K = (k_1, \dots, k_r)$ and $N = p^K - 1$.

4.9 Claim. *The Chow ring $\mathrm{CH}^*(\bar{X}; \mathbb{Z}/p)$ is a free R^*-module with a basis $\{e^M\}$, $M \preccurlyeq N$.*

Proof. Note that R^+ is a nilpotent ideal in R^*. Applying Nakayama Lemma we obtain that $\{e^M\}$ generate $\mathrm{CH}^*(\bar{X}; \mathbb{Z}/p)$. By [Kc85, (2)] $\mathrm{CH}^*(\bar{X}; \mathbb{Z}/p)$ is a free R^*-module, hence, for the Poincaré polynomials we have

$$P(\mathrm{CH}^*(\bar{X}; \mathbb{Z}/p), t) = P(\mathrm{CH}^*(\bar{G}; \mathbb{Z}/p), t) \cdot P(R^*, t).$$

Substituting $t = 1$ we obtain that

$$\mathrm{rk}_{\mathbb{Z}/p} \mathrm{CH}^*(\bar{X}; \mathbb{Z}/p) = \mathrm{rk}_{\mathbb{Z}/p} \mathrm{CH}(\bar{G}; \mathbb{Z}/p) \cdot \mathrm{rk}_{\mathbb{Z}/p} R^*.$$

To finish the proof observe that $\mathrm{rk}_{\mathbb{Z}/p} \mathrm{CH}^*(\bar{G}; \mathbb{Z}/p)$ coincides with the number of generators $\{e^M\}$. □

Set $d = \dim X - |N| = \deg(P(R^*, t))$.

4.10 Claim. *The pairing $R^* \times R^{d-*} \to \mathbb{Z}/p$ given by $(\alpha, \beta) \mapsto \deg(e^N \alpha\beta)$ is non-degenerated.*

Proof. We have to show that for any $\alpha \in R^*$ there exists β such that $\deg(e^N \alpha\beta) \neq 0$. Let $\alpha^\vee$ be a Poincare dual of α. Expanding $\alpha^\vee$ we obtain

$$\alpha^\vee = \sum_{M \preccurlyeq N} e^M \beta_M, \text{ where } \beta_M \in R^*.$$

Note that if $M \neq N$ then $\operatorname{codim} \alpha\beta_M > d$, therefore, $\alpha\beta_M = 0$. So we can set $\beta = \beta_N$. □

Fix a homogeneous basis $\{\alpha_i\}$ of R^* and its dual $\{\beta_j\}$ with respect to the pairing introduced in 4.10.

4.11 Claim. *For $|M| \leq |N|$ we have*

$$\deg(e^M \alpha_i \beta_j) = \begin{cases} 1, & M = N \textit{ and } i = j \\ 0, & \textit{otherwise} \end{cases}$$

Proof. If $M = N$, then it follows from the definition of the dual basis. Assume $|M| < |N|$. If $\deg(e^M \alpha_i \beta_j) \neq 0$, then $\operatorname{codim}(\alpha_i\beta_j) > d$, a contradiction with the fact that $\alpha_i\beta_j \in R^*$. Hence, we reduced to the case $M \neq N$ and $|M| = |N|$. Since $|M| = |N|$, $\operatorname{codim}(\alpha_i\beta_j) = d$ and, hence, $R^+\alpha_i\beta_j = 0$. From the other hand side there exists i such that $m_i \geq p^{k_i}$ and $e^{p^{k_i}} \in \mathrm{CH}^*(\bar{X}; \mathbb{Z}/p)R^+$. Hence, $e^M\alpha_i\beta_j = 0$. □

4.12. Given two pairs (M, t) and (L, s), where M, L are r-tuples and t, s are integers, we will write $(M, t) \leq (L, s)$ iff $M \leq L$ and in the case $M = L$ $t \leq s$ (the *lexicographical* order). Consider the following filtration on the ring $\mathrm{CH}^*(\bar{X})$: the (M, t)-th term $\mathrm{CH}^*(\bar{X})_{M,t}$ is the subring generated by elements $e^I\alpha$ with $I \leq M$, $\alpha \in R^*$, $\operatorname{codim}\alpha \leq t$. The *associated graded ring* is defined as follows:

$$A^* = \bigoplus_{M,t} A^{M,t}, \text{ where } A^{M,t} = \mathrm{CH}^*(\bar{X})_{M,t} / \bigcup_{(L,s)<(M,t)} \mathrm{CH}^*(\bar{X})_{L,s}.$$

Actually the unions stabilize at finite steps. As usual, A^* can be equipped with a structure of a graded ring. Clearly $A^{M,t}$ consists of the images of elements $e^M\alpha$ with $\alpha \in R^*$, $\operatorname{codim}\alpha \leq t$ when $M \preccurlyeq N$; such an image will be denoted by $e^M\alpha$ too. Since $\mathrm{rk}_{\mathbb{Z}/p} A^* = \mathrm{rk}_{\mathbb{Z}/p} \mathrm{CH}^*(\bar{X})$, $A^{M,t}$ is trivial when $M \not\preccurlyeq N$. We also consider the subring $\overline{\mathrm{CH}}^*(X)$ of rational cycles with the induced filtration. The associated graded ring will be denoted by A^*_{rat}; it may be naturally identified with a subring of A^*.

Similarly, consider the filtration on the ring $\mathrm{CH}^*(\bar{X} \times \bar{X})$ whose (M, t)-th term is generated by elements $e^I\alpha \times e^L\beta$, $I + L \leq M$, $\alpha, \beta \in R^*$, $\operatorname{codim}\alpha + \operatorname{codim}\beta \leq t$. The associated graded ring will be denoted by B^*. It is easy to see that B^* is isomorphic to $A^* \otimes_{\mathbb{Z}/p} A^*$ as a graded ring. The graded ring associated to $\overline{\mathrm{CH}}^*(X \times X)$ will be denoted by B^*_{rat}.

4.13. The key observation is that due to Claim 4.11 we have

$$\mathrm{CH}^*(\bar{X} \times \bar{X})_{M,t} \circ \mathrm{CH}^*(\bar{X} \times \bar{X})_{L,s} \subset \mathrm{CH}^*(\bar{X} \times \bar{X})_{M+L-N,s+t-d},$$

and therefore we have the correctly defined composition law

$$\circ \colon B^{M,t} \times B^{L,s} \to B^{M+L-N,s+t-d}.$$

In particular, $B^{N+*,d+*}$ can be viewed as a graded ring with respect to the composition.

Similarly, $(\mathrm{CH}^*(\bar{X} \times \bar{X})_{M,t})_*(\mathrm{CH}^*(\bar{X})_{L,s}) \subset \mathrm{CH}^*(\bar{X})_{M+L-N}$, and therefore we have the realization map $* \colon B^{M,t} \times A^{L,s} \to A^{M+L-N,s+t-d}$.

4.14 Claim. *The elements* $e_l \otimes 1 - 1 \otimes e_l$, $l = 1, \ldots, r$, *belong to* B^*_{rat}.

Proof. Since X splits over $F(X)$, by Lemma 1.5 there exists a cycle in $\overline{\mathrm{CH}}^{d_l}(X \times X; \mathbb{Z}/p)$ of the form $\xi = e_l \times 1 + \sum_i \mu_i \times \nu_i + 1 \times \mu$, where $\operatorname{codim} \mu_i, \operatorname{codim} \nu_i < d_l$. Then the cycle

$$\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi) = (e_l \times 1 - 1 \times e_l) \times 1 + \sum_i (\mu_i \times 1 - 1 \times \mu_i) \times \nu_i$$

is rational in $\mathrm{CH}^*(\bar{X} \times \bar{X} \times \bar{X}; \mathbb{Z}/p)$. Applying Corollary 3.5 to the variety $X \times X \times X \to X$ we see that the pull-back map $\mathrm{pr}_3^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times X \times X)$ has a left inverse, say, δ_3. Passing to a splitting field we obtain a map $\delta_3\colon \mathrm{CH}^*(\bar{X} \times \bar{X} \times \bar{X}) \to \mathrm{CH}^*(X)$ which is left inverse to pr_3^*, preserves codimension and respects rationality of cycles. Hence we obtain a desired rational cycle

$$\delta_3(\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi)) = e_l \times 1 - 1 \times e_l + \sum_i (\mu_i \times 1 - 1 \times \mu_i)\delta_3(\nu_i)$$

whose image in B^*_{rat} is $e_l \otimes 1 - 1 \otimes e_l$. □

4.15 Claim. *The elements* $e_l^{p^{j_l}}$, $l = 1, \ldots r$, *belong to* A^*_{rat}.

Proof. Follows immediately from the definition of the J-invariant. □

We will write $(e \otimes 1 - 1 \otimes e)^M$ for $\prod_{i=1}^r (e_i \otimes 1 - 1 \otimes e_i)^{m_i}$.

4.16 Claim. *Let* α *be an element of* R^*, $\alpha^\vee$ *be a dual, that is* $\deg(e^N \alpha \alpha^\vee) = 1$. *Then we have*

$$((e \otimes 1 - 1 \otimes e)^M (\alpha^\vee \otimes 1))_*(e^L \alpha) = \binom{M}{M + L - N} (-1)^{M+L-N} e^{M+L-N}.$$

Proof. Direct computations using Claim 4.11. □

Set for brevity $J = J_p(G)$.

4.17 Claim. *The elements* $e^{p^J L} \alpha_i$, $L \preccurlyeq p^{K-J} - 1$, *form a basis of* A^*_{rat} *over* $\mathbb{Z}/p$.

Proof. Clearly, these elements are linearly independent. Assume there exists a homogeneous element which can not be presented as a linear combination of these elements; choose such an element $e^L\alpha$ with the smallest L. Obviously L can not be presented as $p^J M$; it means that there exists an index i such that in the presentation $l_i = p^s l_i'$ with $p \nmid l_i'$ we have $s < j_i$. We show that $L = (0, \dots, l_i, \dots, 0)$; indeed, otherwise we can set $M = N - L + (0, \dots, l_i, \dots, 0)$ and obtain the element $((e \otimes 1 - 1 \otimes e)^M(\alpha^\vee \otimes 1))_*(e^L\alpha)$ which by Claim 4.16 has the degree $(0, \dots, l_i, \dots, 0)$. Assume that $l_i' > 1$. By Lucas' theorem on binomial coefficients we have $p \nmid \binom{p^{k_i}-p^s(l_i'-1)-1}{p^s}$. Set $M = N + (0, \dots, p^{k_i} - p^s(l_i' - 1) - 1, \dots, 0)$; then applying Claim 4.16 again we obtain an element of degree $(0, \dots, p^s, \dots, 0)$, a contadiction. It means that $l_i' = 1$. Let γ be a representative of $e_i^{p^s}$ in $\overline{\mathrm{CH}}^*(X)$; then the element $\pi(\gamma)$ has the leading term x_i^s with $s < j_i$, a contradiction to the definition of the J-invariant. □

4.18 Claim. *The elements* $(e\otimes 1-1\otimes e)^S(e^{p^J L}\alpha_i \otimes e^{p^J M}\beta_j)$, $L, M \preccurlyeq p^{K-J}-1$, $S \preccurlyeq p^J - 1$, *form a basis of* B^*_{rat} *over* $\mathbb{Z}/p$.

Proof. Clearly, these elements are linearly independent and their number is $p^{|2K-J|}(\mathrm{rk}_{\mathbb{Z}/p} R^*)^2$. On the other hand, by Corollary 3.5, Lemma 4.9 and Lemma 4.17 we have

$$\begin{aligned}\mathrm{rk}_{\mathbb{Z}/p} B^*_{rat} &= \mathrm{rk}_{\mathbb{Z}/p} \overline{\mathrm{CH}}^*(X \times X; \mathbb{Z}/p) = \mathrm{rk}_{\mathbb{Z}/p} \overline{\mathrm{CH}}^*(X; \mathbb{Z}/p) \cdot \mathrm{rk}_{\mathbb{Z}/p} \mathrm{CH}^*(\bar{X}; \mathbb{Z}/p) \\ &= \mathrm{rk}_{\mathbb{Z}/p} A^*_{rat} \cdot p^{|K|} \mathrm{rk}_{\mathbb{Z}/p} R^* = p^{|2K-J|}(\mathrm{rk}_{\mathbb{Z}/p} R^*)^2.\end{aligned}$$

□

4.19 Claim. *The elements*

$$\theta_{L,M,i,j} = (e \otimes 1 - 1 \otimes e)^{p^J-1}(e^{p^J L}\alpha_i \otimes e^{p^J(p^{K-J}-1-M)}\beta_j), \ L, M \preccurlyeq p^{K-J} - 1,$$

satisfy the relation $\theta_{L,M,i,j} \circ \theta_{L',M',i',j'} = \delta_{LM'}\delta_{ij'}\theta_{L'Mi'j}$.

Proof. Follows from Claim 4.11. □

Proof of Theorem 4.8. Consider the projection map

$$f^0 \colon \overline{\mathrm{CH}}^*(X \times X)_{N,d} \to B^{N,d}_{rat}.$$

Its kernel is nilpotent, and therefore by Lemma 2.7 there exist pair-wise orthogonal idempotents $\varphi_{L,i}$ in $\overline{\mathrm{CH}}^*(X \times X)$ which map to $\theta_{L,L,i,i}$ and whose sum is the identity. Their $(N+d)$-graded components also have these properties and therefore we may assume that $\varphi_{L,i}$ belong to $\overline{\mathrm{CH}}^{\dim X}(X \times X)$.

We show that $\varphi_{L,i}$ are indecomposable. Claim 4.18 and Claim 4.19 show that the ring $B^{N,d}_{rat}$ is isomorphic to a direct product of matrix rings over $\mathbb{Z}/p$:

$$B^{N,d}_{rat} \simeq \prod_s \mathrm{End}((\mathbb{Z}/p)^{p^{|K-J|}\,\mathrm{rk}_{\mathbb{Z}/p} R^s}).$$

Under this identification elements $\theta_{L,L,i,i}$ correspond to idempotents of rank 1 and therefore are indecomposable. Since f^0 preserves isomorphisms, $\varphi_{L,i}$ are indecomposable as well.

We show that $\varphi_{L,i}$ is isomorphic to $\varphi_{M,j}$. In the ring B^*_{rat} mutually inverse isomorphisms between them are given by $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$. Let

$$f\colon \overline{\mathrm{CH}}^*(X \times X) \to B^*_{rat}$$

be the *leading term* map; it means that for any $\xi \in \overline{\mathrm{CH}}^*(X \times X)$ we find the smallest degree (I, s) such that ξ belongs to $\overline{\mathrm{CH}}^*(X \times X)_{I,s}$ and set $f(\xi)$ to be the image of γ in $B^{I,s}_{rat}$. Note that f is not a homomorphism but satisfies the condition that $f(\xi)\circ f(\eta)$ equals either $f(\xi\circ\eta)$ or 0. Choose preimages $\psi_{L,M,i,j}$ and $\psi_{M,L,j,i}$ of $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$ by means of f. Applying Lemma 2.8 we obtain mutually inverse isomorphisms $\vartheta_{L,M,i,j}$ and $\vartheta_{M,L,j,i}$ between $\varphi_{L,i}$ and $\varphi_{M,j}$. It remains to take their homogeneous components of the appropriate degrees.

Now applying Lemma 1.6 and Lemma 2.13 we obtain the desired motivic decomposition. □

4.20 Remark. The proof actually shows that every direct summand of $\mathcal{M}(X; \mathbb{Z}/p)$ is isomorphic to a direct sum of twisted copies of $\mathcal{R}$. Indeed, in the ring $B^{N,d}_{rat}$ any idempotent is isomorphic to a sum of idempotents $\theta_{L,L,i,i}$, and the map f^0 preserves isomorphisms. It is no wonder: results of [CM06] show that for motives of G-homogeneous varieties with $\mathbb{Z}/p$-coefficients the Krull-Schmidt Theorem holds.

In the sequel we will denote the motive $\mathcal{R}$ introduced in Theorem 4.8 by $\mathcal{R}_p(G)$.

4.21 Theorem. *Let X be a projective G-homogeneous variety, where G is a semisimple group of inner type which splits over $F(X)$. Then the motive of*

X is isomorphic to the direct sum

$$\mathcal{M}(X;\mathbb{Z}/p) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i},$$

where a_i are the coefficients of the polynomial

$$\sum_{i\geq 0} a_i t^i = P(\mathrm{CH}^*(\bar{X}),t)/P(\bar{\mathcal{R}}_p(G),t).$$

Proof. Let Y be the variety of complete G-flags. Apply Theorem 3.9 and Remark 4.20. □

We describe some properties of $\mathcal{R}_p(G)$ in the following theorem.

4.22 Theorem. *Let G and G' be two semisimple groups of inner type, X and X' be corresponding varieties of complete flags.*

- **(base change)** *For any field extension E/F we have*

  $$\mathcal{R}_p(G)_E \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G_E)(i)^{\oplus a_i},$$

  where $\sum a_i t^i = P(\bar{\mathcal{R}}_p(G),t)/P(\bar{\mathcal{R}}_p(G_E),t)$.

- **(transfer argument)** *If E/F is a field extension of degree coprime to p then $J_p(G_E) = J_p(G)$ and $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$. Moreover, if $\mathcal{R}_p(G_E) \simeq \mathcal{R}_p(G'_E)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

- **(comparison theorem)** *If G splits over $F(X')$ and G' splits over $F(X)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

Proof. The first claim follows from Theorem 4.8 and Remark 4.20. To prove the second claim note that E is rank preserving with respect to X and $X \times X$ by Lemma 4.24 below. Now $J_p(G_E) = J_p(G)$ by Lemma 2.12, and hence $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$ by the first claim. The remaining part of the claim follows from Lemma 2.14 applied to the variety $X \coprod X'$.

Now we prove the last claim. The variety $X\times X'$ is the variety of complete $G\times G'$-flags. Applying Corollary 3.5 we can express $\mathcal{M}(X\times X';\mathbb{Z}/p)$ in terms of $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$. Now apply Remark 4.20. □

4.23 Corollary. *We have $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G_{an})$, where G_{an} is the anisotropic kernel of G.*

4.24 Lemma. *Let G be a group of inner type, X be a projective G-homogeneous variety. Then any field extension E/F is rank preserving with respect to X and $X \times X$.*

Proof. By [Pa94, Theorem 2.2 and 4.2] the restriction map $K_0(X) \to K_0(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Now the Chern character $ch\colon K_0(X) \otimes \mathbb{Q} \to \mathrm{CH}^*(X) \otimes \mathbb{Q}$ is an isomorphism and respects pull-backs, hence E is rank preserving with respect to X. It remains to note that $X \times X$ is $G \times G$-homogeneous variety. $\square$

Chapter 5

Properties of J-invariant

5.1. Recall (see [Br03]) that if the characteristic of the base field F is different from p then one can construct *Steenrod p-th power operations*

$$S^l\colon \mathrm{CH}^*(X;\mathbb{Z}/p) \to \mathrm{CH}^{*+l(p-1)}(X;\mathbb{Z}/p)$$

such that $S^0 = \mathrm{id}$, S^l restricted to $\mathrm{CH}^l(X;\mathbb{Z}/p)$ coincides with the taking to the p-th power, and the total Steenrod operation $S^\bullet = \sum_{l\geq 0} S^l$ is a homomorphism of $\mathbb{Z}/p$-algebras compatible with pull-backs. In the case of varieties over the field of complex numbers S^l compatible with their topological counterparts: reduced power operations $\mathcal{P}^l$ if $p \neq 2$ and Steenrod squares Sq^{2l} if $p = 2$ (recall that CH^* in this case may be viewed as a subring in H^{2*}).

When X is the variety of complete G-flags the action of Steenrod operations on $\mathrm{CH}^*(\bar{X})$ can be described in purely combinatorial terms (see [Du05]) and therefore does not depend on the base field. Since Steenrod operations respect pull-back they respect rationality as well.

Over the field of complex numbers $\mathrm{CH}^*(\bar{G})$ may be identified with the image of the pull-back map $H^{2*}(\bar{X}) \to H^{2*}(\bar{G})$. An explicit description of this image and formulae describing the action of $\mathcal{P}^l$ and Sq^l on $H^*(\bar{G})$ are given in [IKT76, KoMi77, BB65].

5.2. Assume that in $\mathrm{CH}^*(\bar{G})$ we have $S^l(x_i) = x_m$ and $S^l(x_{i'}) < x_m$ with respect to the order DegLex when $i' < m$. Then $j_m \leq j_i$. Indeed, by definition there exists a cycle $\alpha \in \overline{\mathrm{CH}}^*(X)$ such that the leading term of $\pi(\alpha)$ is $x_i^{p^{j_i}}$. Applying $S^{lp^{j_i}}$ we obtain a rational cycle whose image under π has the leading term $x_m^{p^{j_i}}$.

5.3. We summarize information about restrictions on J-invariant which can be obtained using the method described in 5.2 into the following table (numbers d_i and k_i are taken from [Kc85, Table II]).

G_0	p	r	d_i	k_i	j_i
SL_n/μ_m, $m \mid n$	$p \mid m$	1	1	$p^{k_1} \parallel n$	
PGSp_n, $2 \mid n$	2	1	1	$2^{k_1} \parallel n$	
SO_n	2	$[\frac{n+1}{4}]$	$2i-1$	$[\log_2 \frac{n-1}{2i-1}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$
Spin_n	2	$[\frac{n-3}{4}]$	$2i+1$	$[\log_2 \frac{n-1}{2i+1}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i}{l}$
PGO_{2n}	2	$[\frac{n+2}{2}]$	1, $i=1$	$2^{k_1} \parallel n$	
			$2i-3$, $i \geq 2$	$[\log_2 \frac{2n-1}{2i-3}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-2}{l}$
Ss_{2n}, $2 \mid n$	2	$\frac{n}{2}$	1, $i=1$	$2^{k_1} \parallel n$	
			$2i-1$, $i \geq 2$	$[\log_2 \frac{2n-1}{2i-1}]$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$
G_2, F_4, E_6	2	1	3	1	
F_4, E_6^{sc}, E_7	3	1	4	1	
E_6^{ad}	3	2	1, 4	2, 1	
E_7^{sc}	2	3	3, 5, 9	1, 1, 1	$j_1 \geq j_2 \geq j_3$
E_7^{ad}	2	4	1, 3, 5, 9	1, 1, 1, 1	$j_2 \geq j_3 \geq j_4$
E_8	2	4	3, 5, 9, 15	3, 2, 1, 1	$j_1 \geq j_2 \geq j_3$
E_8	3	2	4, 10	1, 1	$j_1 \geq j_2$
E_8	5	1	6	1	

We give some applications of the notion of J-invariant. First, as a by-product of the proof of Theorem 4.8 we obtain the following expression for the canonical p-dimension of the variety of complete flags (cf. [EKM, Theorem 90.3] for the case of quadrics).

5.4 Theorem. *In the notation of Theorem 4.8 we have*

$$cd_p(X) = \sum_{i=1}^{r} d_i(p^{j_i} - 1).$$

Proof. Follows from Claim 4.17 and [KM05, Theorem 5.8]. □

Let for a moment X be any smooth projective variety which has a splitting field.

5.5 Lemma. *For any* $\phi, \psi \in \mathrm{CH}^*(\bar{X} \times \bar{X})$ *one has*

$$\deg((\mathrm{pr}_2)_*(\phi \cdot \psi^t)) = \mathrm{tr}((\phi \circ \psi)_*).$$

Proof. Choose a homogeneous basis $\{e_i\}$ of $\mathrm{CH}^*(\bar{X})$; let $\{e_i^\vee\}$ be its Poincaré dual basis. Since both sides are bilinear, it suffices to check the assertion for $\phi = e_i \times e_j^\vee$, $\psi = e_k \times e_l^\vee$. Now the both sides equal $\delta_{il}\delta_{jk}$. □

Denote by $d(X)$ the greatest common divizor of the degrees of all zero cycles on X and by $d_p(X)$ its p-primary component.

5.6 Corollary. *For any* $\phi \in \overline{\mathrm{CH}}^*(X \times X; \mathbb{Z}/m)$ *we have*

$$\gcd(d(X), m) \mid \mathrm{tr}(\phi_*).$$

Proof. Set $\psi = \Delta_{\bar{X}}$ and apply Lemma 5.5. □

5.7 Lemma. *Let X be a variety which has a splitting field. Assume that $\mathcal{M}(X; \mathbb{Z}/p)$ has a direct summand M. Then*

1. $d_p(X) \mid P(\bar{M}, 1)$;
2. *if* $d_p(X) = P(\bar{M}, 1)$ *and the kernel of the map*

$$\mathrm{End}(\mathcal{M}(X; \mathbb{Z}/p)) \to \mathrm{End}(\mathcal{M}(\bar{X}; \mathbb{Z}/p))$$

consists of nilpotents then M is indecomposable.

Proof. Set $q = d_p(X)$ for brevity. Let an idempotent $\phi \in \mathrm{End}(\mathcal{M}(X); \mathbb{Z}/p)$ present M. By Lemma 2.9 there exists an idempotent $\varphi \in \mathrm{End}(\mathcal{M}(X); \mathbb{Z}/q)$ such that $\varphi \mod p = \phi$. Then $\mathrm{res}(\varphi) \in \mathrm{End}(\mathcal{M}(\bar{X}); \mathbb{Z}/q)$ is a rational idempotent. Since every projective module over $\mathbb{Z}/q$ is free, we have

$$\mathrm{tr}(\mathrm{res}(\varphi)_*) = \mathrm{rk}_{\mathbb{Z}/q}(\mathrm{res}(\varphi)_*) = \mathrm{rk}_{\mathbb{Z}/p}(\mathrm{res}(\phi)_*) = P(\bar{M}, 1) \mod q,$$

and the first claim follows by Corollary 5.6. The second claim follows from the first, since the assumption implies that for any nontrivial direct summand M' of M we have $P(\bar{M}', 1) < P(\bar{M}, 1)$. □

5.8. Let G be a group of inner type. Denote by $n(G)$ the greatest common divisor of degrees of all finite splitting field of G and by $n_p(G)$ its p-primary component. Note that $n(G) = d(X)$ and $n_p(G) = d_p(X)$, where X is the variety of complete G-flags.

We have the following estimation on $n_p(G)$ in terms of J-invariant (cf. [EKM, Proposition 88.11] for the case of quadrics).

5.9 Theorem. *For any group G of inner type with $J_p(G) = (j_1, \dots, j_r)$ we have*

$$n_p(G) \le p^{\sum_i j_i}.$$

Proof. Follows from Theorem 4.8 and Lemma 5.7. □

5.10 Corollary. *The following statements are equivalent:*

- $J_p(G) = (0, \dots, 0)$;
- $n_p(G) = 1$;
- $\mathcal{R}_p(G) = \mathbb{Z}/p$.

Proof. If $J_p(G) = (0, \dots, 0)$ then $n_p(G) = 1$ by Theorem 5.9. If $n_p(G) = 1$ then there exists a splitting field L of degree m prime to p, and therefore $\mathcal{R}_p(G) = \mathbb{Z}/p$ by the transfer argument (see Theorem 4.22). The remaining implication is obvious. □

Finally, we give some kind of a 'reduction formula' (cf. [EKM, Corollary 88.7] for the case of quadrics).

5.11 Theorem. *Let G be a group of inner type, X be the variety of complete G-flags, Y be a projective variety such that the map $\mathrm{CH}^l(Y) \to \mathrm{CH}^l(Y_{F(x)})$ is surjective for all $x \in X$ and $l \le n$. Then $j_i(G) = j_i(G_{F(Y)})$ for all i such that $p^{j_i(G_{F(Y)})} d_i \le n$.*

Proof. Indeed, by [EKM, Lemma 88.5] the map $\mathrm{CH}^l(X) \to \mathrm{CH}^l(X_{F(Y)})$ is surjective for all $l \le n$, and therefore $j_i(G) \le j_i(G_{F(Y)})$. The converse inequality is obvious. □

5.12 Corollary. $J_p(G) = J_p(G_{F(t)})$.

Proof. Take $Y = \mathbb{P}^1$ and apply Theorem 5.11. □

Chapter 6

Examples

Types A_n **and** C_n Let G be a group of inner type A_n or C_n corresponding to a central simple algebra $A = M_m(D)$, where D is a division algebra of index d over a field F. Let p be a prime divisor of d ($p = 2$ in the case of C_n). Let X_Θ be the projective homogeneous G-variety given by a subset Θ of vertices of the respective Dynkin diagram such that $p \nmid j$ for some $j \notin \Theta$. Then the Chow motive of X_Θ modulo p decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{p,2}$ such that $P(\bar{\mathcal{R}}_{p,2}, t) = \frac{1-t^{p^{j_1}}}{1-t}$, $p^{j_1} \mid \deg(A)$. Using the comparison theorem we see that $\mathcal{R}_{p,2}$ depends only on D, so we may assume $m = 1$. Now $p^{j_1} \mid \mathrm{ind}(D)$, but on the other hand side $n_p(G) \le p^{j_1}$ by Theorem 5.9. Therefore we have $p^{j_1} \parallel \mathrm{ind}(D)$.

Now we identify $\mathcal{R}_{p,2}$. Present D in the form $D_p \otimes_F D'$, where $p \nmid \mathrm{ind}\, D'$. By the transfer argument passing to a splitting field of D' will not affect the motive $\mathcal{R}_{p,2}$ up to an isomorphism; so we may assume $D = D_p$. But in this case $\mathcal{M}(\mathrm{SB}(D_p); \mathbb{Z}/p)$ is isomorphic to $\mathcal{R}_{p,2}$ by dimensional reasons. Finally, we have $J = (j_1)$, where $\mathrm{ind}(D_p) = p^{j_1}$, and $\mathcal{R}_{p,2} \simeq \mathcal{M}(\mathrm{SB}(D_p); \mathbb{Z}/p)$.

Types B_n **and** D_n Let $G = \mathrm{O}^+(\phi)$, where ϕ is a k-fold Pfister form or its maximal Pfister neighbor. Assume $J_2(G) \neq (0, \dots, 0)$; by Springer's theorem this holds iff ϕ doesn't split. The Chow motive of any projective homogeneous G-variety X modulo 2 decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{2,k}$ known as the *Rost motive* (see [Ro98]). According to Lemma 2.18 this decomposition can be lifted to $\mathbb{Z}$. Observe that the notation $\mathcal{R}_{2,k}$ and $\mathcal{R}_{p,2}$ agree when $k = 2$, $p = 2$, since 2-fold Pfister quadrics correspond to quaternion algebras.

Now we compute $J_2(G)$. Let Y be a projective quadric corresponding to ϕ; then G splits over $F(Y)$ and Y splits over $F(x)$ for any $x \in X$. It is known that $\mathrm{CH}^l(\bar{Y})$ for $l < 2^{k-1} - 1$ is generated by $\mathrm{CH}^1(\bar{Y})$ and therefore is rational. Applying Theorem 5.11 we see that $j_i(G) = 0$ for $i < r = 2^{k-2}$. Therefore, we have $J_2(G) = (0, \dots, 0, 1)$ and $P(\bar{\mathcal{R}}_{2,k}, t) = 1 + t^{2^{k-1}-1}$.

Type G_2 Let $G = \mathrm{Aut}(\mathbb{O})$, where $\mathbb{O}$ is an octonion algebra. Assume $J_2(G) \neq (0)$; this holds iff G doesn't split and in this case $J_2(G) = (1)$. The Chow motive of any projective homogeneous G-variety X modulo 2 decomposes into a direct sum of shifted copies of $\mathcal{R}_{2,3}$, where, as the comparison theorem shows, $\mathcal{R}_{2,3}$ is the Rost motive of a quadric given by the Pfister form $N_{\mathbb{O}}$. By Lemma 2.18 this decomposition can be lifted to $\mathbb{Z}$. This result was proved first in [Bo03].

Type F_4 Let G be a group of type F_4. Let X_Θ be the projective homogeneous G-variety corresponding to a subset Θ of vertices of the respective Dynkin diagram.

p=2 Assume $J_2(G) \neq (0)$; this holds iff G does not split over a cubic field extension and in this case $J_2(G) = (1)$. For any $\Theta \neq \{1, 2, 3\}$ the Chow motive of X_Θ modulo 2 decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{2,3}$. The comparison theorem and the transfer argument show that $\mathcal{R}_{2,3}$ is the Rost motive of the Pfister quadric given by the norm of the coordinate algebra of G. In the case when G is reduced (that is splits over a quadratic extension) this decomposition can be lifted to $\mathbb{Z}$ by Lemma 2.18.

p=3 Assume $J_3(G) \neq (0)$; this holds iff G is not reduced and in this case $J_3(G) = (1)$. For any Θ the Chow motive of X_Θ modulo 3 decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{3,3}$, $P(\bar{\mathcal{R}}_{3,3}, t) = 1 + t^4 + t^8$. If G splits over a cubic field extension then this decomposition can be lifted to $\mathbb{Z}$ by Lemma 2.18. This result was proved first in [NSZ].

$\mathbb{Z}$-coefficients If $\Theta \neq \{1, 2, 3\}$ using Lemma 2.10 and Lemma 2.18 we obtain that the motive of X_Θ over $\mathbb{Z}$ decomposes into a direct sum of shifted copies of some motive $\mathcal{R}$ such that

$$\mathcal{R} \otimes \mathbb{Z}/2 = \bigoplus_{i\in\{0,1,2,6,7,8\}} \mathcal{R}_{2,3}(i),$$
$$\mathcal{R} \otimes \mathbb{Z}/3 = \bigoplus_{i\in\{0,1,2,3\}} \mathcal{R}_{3,3}(i).$$

Type E_6 Let G be a group of type E_6. Let X_Θ be the projective homogeneous G-variety corresponding to a subset Θ of vertices of the respective Dynkin diagram.

p=2 Assume $J_2(G) \neq (0)$; this holds iff the cohomological invariant $f_3(G) \neq 0$ and in this case $J_2(G) = (1)$. For any $\Theta \not\supset \{2,3,4,5\}$ the Chow motive of X_Θ modulo 2 decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{2,3}$. The comparison theorem and the transfer argument show that $\mathcal{R}_{2,3}$ is the Rost motive of the Pfister quadric corresponding to $f_3(G)$. In the case when G is strongly inner and isotropic the same decomposition holds with integer coefficients.

p=3 We consider only the case when G is strongly inner. Assume $J_3(G) \neq (0)$; this holds iff G is anisotropic and in this case $J_3(G) = (1)$. For any Θ the Chow motive of X_Θ modulo 3 decomposes into a direct sum of shifted copies of $\mathcal{R}_{3,3}$, where, as comparison theorem shows, $\mathcal{R}_{3,3}$ is the same as in F_4-case (to be precise, one should take a group G' of type F_4 with $g_3(G') = g_3(G)$). If G splits over a cubic field extension then this decomposition can be lifted to $\mathbb{Z}$.

$\mathbb{Z}$-coefficients If G is strongly inner and $\Theta \neq \{1,2,3\}$ using Lemma 2.10 and Lemma 2.18 we obtain that the motive of X_Θ over $\mathbb{Z}$ decomposes into a direct sum of shifted copies of $\mathcal{R}$, where $\mathcal{R}$ is the same as in F_4-case.

Type E_7**, p=3** Let G be a group of type E_7. Assume that $J_3(G) \neq (0)$; this holds iff the cohomological invariant $g_3(G) \neq 0$ and in this case $J_3(G) = (1)$. Let X_Θ be the projective homogeneous G-variety given by a subset Θ of vertices of the respective Dynkin diagram, $\Theta \neq \{1,2,3,4,5,6\}$. Then the Chow motive of X_Θ modulo 3 decomposes into a direct sum of shifted copies of $\mathcal{R}_{3,3}$, where, as comparison theorem shows, $\mathcal{R}_{3,3}$ is the same as in F_4-case (to be precise, one should take a group G' of type F_4 with $g_3(G') = g_3(G)$). If G splits over a cubic field extension then the decomposition can be lifted to $\mathbb{Z}$.

Type E_8**, p=5** Let G be a group of type E_8. Assume that $J_5(G) \neq (0)$; this holds iff the Rost-Serre invariant modulo 5 $h_3(G) \neq 0$ (see [Ch94]) and in this case $J_5(G) = (1)$. The motive of any projective G-homogeneous variety X modulo 5 decomposes into a direct sum of shifted copies of some indecomposable motive $\mathcal{R}_{5,3}$, where $P(\bar{\mathcal{R}}_{5,3}, t) = 1 + t^6 + t^{12} + t^{18} + t^{24}$.

Bibliography

[BB65] P.F. Baum, W. Browder. The cohomology of quotients of classical groups. Topology **3** (1965), 305–336.

[Bo03] J.-P. Bonnet. Un isomorphisme motivique entre deux variétés homogènes projectives sous l'action d'un groupe de type G_2. Doc. Math. **8** (2003), 247–277.

[Br03] P. Brosnan. Steenrod Operations in Chow rings. Trans. Amer. Math. Soc. **355** (2003), no. 5, 1869–1903.

[Br05] P. Brosnan. On motivic decompositions arising from the method of Białynicki-Birula. Invent. Math. **161** (2005), no. 1, 91–111.

[BT65] A. Borel, J. Tits. Groupes réductifs. Publications Math. IHES, **27** (1965), 55–151.

[CPSZ] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline. Chow motives of twisted flag varieties. Compositio Math. **142** (2006), 1063–1080.

[Ch94] V. Chernousov. A remark on the $(\bmod 5)$-invariant of Serre for groups of type E_8. Math. Notes **56** (1994), no. 1-2, 730–733.

[CM06] V. Chernousov, A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem, to appear in Transf. of Groups **11** (2006).

[CGM05] V. Chernousov, S. Gille, A. Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. Duke Math. J. **126** (2005), no. 1, 137–159.

[De74] M. Demazure. Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

[Du05] H. Duan, X. Zhao. A unified formula for Steenrod operations in flag manifolds. Preprint, 2005. Available from `http://arxiv.org/math.AT/0306250`

[EG97] D. Edidin, W. Graham. Characteristic classes in the Chow ring. J. Algebraic Geom. 6 (1997), no. 3, 431–443.

[EKM] R. Elman, N. Karpenko, A. Merkurjev. Quadratic forms and algebraic cycles. Book in progress (2006), available from `http://www.math.uni-bielefeld.de/~karpenko/`

[Ful] W. Fulton. Intersection Theory. Second edition, Springer-Verlag, Berlin-Heidelberg, 1998.

[Ga06] S. Garibaldi. Cohomological Invariants. Lens lectures 2006. `http://www.mathcs.emory.edu/~skip/lens-ci/lens-ci.html`

[Gr58] A. Grothendieck. La torsion homologique et les sections rationnelles. Exposé 5 in Anneaux de Chow et applications.(French) Séminaire C. Chevalley; 2e année: 1958.

[HOM89] A.J. Hahn, O.T. O'Meara. The classical groups and K-theory. Springer-Verlag, Berlin, 1989.

[Hi82] H. Hiller, Geometry of Coxeter groups. Research Notes in Math. 54, Pitman Adv. Publ. Prog. 1982.

[IKT76] K. Ishitoya, A. Kono, H. Toda. Hopf algebra structure of mod 2 cohomology of sumple Lie groups. Publ. RIMS 12 (1976), 141–167.

[Ka96] N. Karpenko. The Grothendieck Chow-motifs of Severi-Brauer varieties. St. Petersburg Math. J. **7** (1996), no. 4, 649–661.

[Ka00] N. Karpenko. Criteria of motivic equivalence for quadratic forms and central simple algebras. Math. Ann. **317** (2000), no. 3, 585–611.

[Ka01] N. Karpenko. Characterization of minimal Pfister neighbours via Rost projectors. J. Pure Appl. Algebra **160** (2001), 195–227.

[Kc85] V. Kac. Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. Invent. Math. **80** (1985), 69–79.

[KM05] N. Karpenko, A. Merkurjev. Canonical p-dimension of algebraic groups. Advances in Mathematics, article in press.

[Ko91] B. Köck. Chow Motif and Higher Chow Theory of G/P. Manuscripta Math. **70** (1991), 363–372.

[KoMi77] A. Kono, M. Mimura. Cohomology operations and the Hopf algebra structure of the compact exceptional Lie groups E_7 and E_8. Proc. London Math. Soc. **35** (1977), no. 3, 345–358.

[KR94] I. Kersten, U. Rehmann. General Splitting of Reductive Groups. Tohoku Math. J. **46** (1994), 35–70.

[Ma68] Y. Manin. Correspondences, motives and monoidal transformations. Math. USSR Sbornik **6** (1968), 439–470.

[NSZ] S. Nikolenko, N. Semenov, K. Zainoulline. Motivic decomposition of anisotropic varieties of type F_4 into generalized Rost motives. Max-Planck-Institut für Mathematik, Preprint Series 2005 (90).

[Pa94] I. Panin. On the algebraic K-theory of twisted flag varieties. K-theory **8** (1994), 541–585.

[Ro98] M. Rost. The motive of a Pfister form. Preprint, 1998. `http://www.mathematik.uni-bielefeld.de/~rost/`

[Ti66] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Symp. Pure Math.)*, Amer. Math. Soc., Providence, R.I., 1966.

[Ti90] J. Tits. Strongly inner anisotropic forms of simple algebraic groups. J. of Algebra **131** (1990), 648–677.

[Ti92] J. Tits. Sur les degrés des extensions de corps déployant les groupes algébriques simples. C.R. Acad. Sci. Paris Sér. I **315** (1992), 1131–1138.

[To04] B. Totaro. Splitting fields for E_8-torsors. Duke Math. J. **121** (2004), no. 3, 425–455.

[Vi05] A. Vishik. On the Chow groups of quadratic Grassmannians. Doc. Math. **10** (2005), 111–130.

[Za06] K. Zainoulline. Canonical p-dimensions of algebraic groups and degrees of basic polynomial invariants. London Math. Bull. (2007), 5 p., to appear.