# Two-Sided Matchings and Cliques

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# Contents

# Introduction

Matching problems occur in various situations. A classical example in our social life is the *marriage market*. Here, we have a group of women on the one side and a group of men on the other side. In general, women want to marry men and vice versa. Each individual has his own ideas about his first choice, second choice and so on. The main question now is: How should the women (men) be matched with the men (women)? One possibility to solve this problem may be to determine marriages randomly. But then we will have a lot of unhappy married couples and consequently divorces after a short time for sure. Ergo, the random choice proves to be not very reasonable. Obviously, the suggested matching does not respect the individuals' expectations. For example, a marriage should satisfy a certain degree of happiness, however this may be quantified from an economic point of view. The marriage market is a typical one-to-one sided matching problem.

Another less social but rather economic example for a matching problem is the *job market*. We have employers, offering jobs, on the one side and employees, searching for jobs, on the other side. There are various reasons, why an employer prefers one employee compared to another (skills, team abilities, flexibility a.s.o.). The same holds for the employees. Comparing two firms, an employee always can state for which one he prefers to work. The reasons for their choices may be interest in the job, money or the work atmosphere. Again, the structure of the problem - and hence the obvious question - is the same. How to match each other such that both parties are satisfied? In this context we can face all kinds of matching problems. In a one-to-one sided matching context, only one employee is working for one firm. In

the many-to-one sided matching context several employees are working for one firm and in the many-to-many sided matching problem an employee is hired by several firms simultaneously.
In both scenarios durable solutions are required. This leads to a central desirable property in matching markets: *stability*. If there are no two individuals, one from each side of the market, who prefer to be matched instead of staying with their current partners in the matching then we call the matching stable. This is the common definition of a stable matching. Obviously one could imagine, there exist a lot of further properties that point out one matching compared to another. If a society agreed on the properties that a solution of a matching problem should satisfy, the next fundamental question occurs. How can we get a matching that satisfies our desired properties? One possibility is the use of *procedures*. Roughly speaking, given the individuals' preferences, a procedure selects matchings. Rules are defined such that at some point the procedure stops. It consists of different rounds. Its design depends on the purpose it serves. In general, it is desirable to focus on procedures that are easy to understand, since this leads to a higher acceptance of its application. In addition, its design is influenced by the outcome, we are interested in most, e.g. such that its solution satisfies stability (For this, see Gale & Shapley's (1962) deferred acceptance procedure which is the basic reference in matching theory.). Hence, a procedure is a matter of stepwise determination of a solution, whereas specific rules define how to proceed at each step.

In connection with the allocation of internships for medical students which started at the beginning of the twentieth century in the US, another problem arose. Some pairs of students became couples during their education or even were couples before. Arriving on the internship market, they search for neighboring jobs. Using the existing procedures causes instabilities.
An even more general similar problem is given on the many-to-one sided job market. What about the fact, if employees do not only care about their employers but as well about their colleagues? This is one of the main questions, we want to work on

in this thesis. We think of this kind of question being of special interest, because people spend a lot of time of their lives in offices, hospitals or other institutions, interacting with their colleagues. Consequently, if they enjoy to work for the company but they do not like their colleagues, this will influence their work as well as their personal life. Matching agents in a firm with colleagues, with whom they get along well, is beneficial for both sides. Again in this context, we have to determine the properties, judged mostly important to be fulfilled. Thereby, different aspects may be taken into account. We can focus on the individuals' interests, we can put the main emphasis on welfaristic aspects or the practicability of a solution is judged to be important.

Therefore, we have chosen several procedural approaches. We think of our procedures being especially appropriate to select matchings, since the course of action is always transparent, a solution will not "appear from nowhere". The acteurs' interests are sometimes more, sometimes less taken into account, but they are never completely ignored. On the one hand we will have several different procedures, given a fixed set of acteurs. On the other hand we will vary the set of acteurs and consequently the procedures have to be modified or new procedures are added.

It is also interesting to observe the influence, the selection of a procedure has on the solution as well as the individuals' behavior itself. The individuals, participating in a procedure, should agree on the properties that should be satisfied by the solution. Another aspect by choosing a procedure is the fact that an individual often can improve his situation by not telling the truth about his preferences. Thus, the individuals or a central planner have to decide, how important it is that everyone has no incentive to misrepresent his interests. These are the basic questions we will answer in this thesis.

The structure of the thesis is as follows:

The first chapter gives an overview of the existing literature which we divide into different fields. One field illustrates the literature in economic theory, the other focuses on literature in a more practical context of matching medical students to internship places in hospitals. Finally, we range this thesis into the literature. The second chapter of the thesis describes the model of a job market (for agents), in general, explains the agents' preference relations and their representation and gives a precise definition of a matching. We introduce eligible properties of a matching and a possible restriction on the preferences. Three different procedures are introduced, two of them will be illustrated and formally described by graphs. The last section of Chapter 2 introduces some aspects about strategic behavior and discusses incentive compatibility. Chapter 3 provides and discusses results. We prove the existence of the solutions of the different procedures. We discuss the quality of the procedural solutions and the quality of the procedures itself. The fourth chapter focuses on the firms' preferences. In the first part of the chapter, we integrate the firms into the basic model of Chapter 2, they are treated as if they were agents as well. We get a job market for participants and then we restart our analysis. The second part discusses another approach. In this job market, we assume that the agents have no influence on the choice of their jobs, we only deal with the firms' preferences. Again we establish a specific model, similar to Chapter 2. Two procedures are introduced and the properties of their solutions as well as the quality of the procedures are discussed. In the last chapter we summarize our results and give various ideas to extend the models, we have introduced in this thesis.

# Chapter 1

# Survey of the Literature

The reader may be interested in the main contributions already made in the last decades. Therefore, we provide a rough overview. This chapter falls into two main sections. The first one deals with a general historical background. The second delimits the thesis from other research domains and embeds it in the literature relevant for our issue.

The overview of the literature, given in this chapter, does not call for being a complete description of the existing literature in two-sided matching theory. It should be rather seen as a selection of subjectively most important results. In addition, some papers are reviewed in corresponding sections of subsequent chapters.

## 1.1 Historical Background

We have divided this section into two main subsections. The *theoretical evolution* illuminates publications in mathematical and economic journals. In contrast to that, the second subsection, *practical evolution*, discusses articles published in medical journals. Health professionals analyze the annual internship allocation for postgraduate medical students.

This structure of the section seemed to be the most natural one. Independently from each other, the theoretical as well as the "real" world developed procedures to solve matching problems, which is the reason why we should give credit to both approaches.

### 1.1.1 Theoretical Evolution

In the world of economic theory it is generally accepted that the "fathers" of two-sided matching theory are Gale & Shapley (1962). The college admission problem, they introduce in their work, is as follows. There are colleges on the one side and applicants on the other side. Colleges have preferences over applicants and vice versa. Hence, many applicants are matched with one college. A special case of this model is the marriage problem where one woman is matched with at most one man. Gale & Shapley (1962) show for the marriage problem as well as for the college admission problem the existence of a *stable* set of matchings. Stable means that no woman and no man can obtain a more preferred matching by single or pairwise deviations. The proof of the theorem is constructive and results in the *deferred acceptance procedure*[1]. Furthermore, this procedure yields a men (or women) optimal matching.

Although Gale & Shapley (1962) is the basic reference in economics, several mathematicians worked on the marriage problem in the first half of the 20th century.[2] In his book Jacobs (1983, Chapter 2 §1) summarizes the exact genesis of the so called *marriage theorem*, as it was termed by Weyl in 1949 and proven by Hall in 1935. An equivalent version of this theorem already appeared as König's theorem in 1916. Several alternative proofs are given by various mathematicians.

Roughly, the content of the theorem is the following: We have two finite sets, called women and men. A mapping from the set of women into the power set of men is called a *system of friendships*. Each woman has a set of male friends. An injective

[1] For a short overview see Section B.0.1.

[2] Never mentioned in the economic literature of two-sided matching markets.

mapping from the set of women into the set of men is called a *marriage*. Now, a marriage is *compatible* with a system of friendships if each woman marries one of her friends. Furthermore, the system of friendships satisfies the *party condition* if for any subset of women, say $D$ many, the number of corresponding male friends exceeds $D$. Now the marriage theorem claims: There exists at least one compatible marriage if and only if the party condition is satisfied.
What does this purely combinatorial theorem have to do with Gale & Shapley's (1962) marriage market? Men's preferences are totally ignored and women's preferences only discriminate between friends and non-friends. Neither woman would want to reject the partner assigned to her by a compatible marriage, as she can marry one of her best choices (in terms of her preferences). Therefore, in the Gale & Shapley (1962) context, there is no pair of man and woman who strictly prefers to be matched to each other. In this spirit, one may find the origins of matching theory in combinatorics.

Coming back to our literature overview, the question rose, whether incentive compatibility can be guaranteed. Dubins & Freedman (1981) show that on a college admission market students cannot get a better college by misrepresenting their preferences. A direct consequence of this result is given by Gale & Sotomayor (1985). In the deferred acceptance procedure, it will almost always be better for a woman not to state her true preferences if men are proposing. They analyze the strategic possibilities for the women and describe their best competitive behavior. Roth (1982) discusses the connection between stability and incentives. It turns out that no matching procedure exists, which always yields a stable matching and which does not give agents (e.g. women and men in the marriage market problem) incentives to misrepresent their true preferences. However, procedures do exist (e.g. the deferred acceptance procedure) where we get stable matchings and the agents on one side of the market have no incentives to misrepresent their preferences (see Roth (1984b)). Furthermore, Roth (1984c) shows that a women (men)-optimal stable matching is the best stable matching for each woman (man) and the worst for each man (woman). A good overview of foundations and further results on strategic behavior, incomplete information and other interesting research directions can be found in Roth & Sotomayor (1990).

We want to mention some further research directions that became more and more important. Based on Roth's (1985) result that the college admission problem is not equivalent to the marriage market problem, as asserted before, more recent works may be categorized by the type of a matching problem that is analyzed. For the college admission problem Roth & Sotomayor (1989) discuss the specific properties of the set of stable outcomes. Sönmez (1996), Sotomayor (2000), Alcade & Romero-Medina (2000) and Abdulkadiroglu & Sönmez (2003) discuss in different ways strategy-proofness and mechanism design issues.

Roth & Sotomayor (1990) also dedicate several chapters to a cooperative game theoretic approach using *assignment games*[3], first discussed in Shapley & Shubik (1972). The paper fills the gap that salaries were not considered for jobs, yet. This model is not directly related to our work, but it plays an important role in the literature, and therefore is worth being briefly mentioned. Agents on both sides of the market want to exchange one unit of an indivisible commodity[4], one side supplies the good, the other demands. It is exchanged for money, consequently we are looking for suitable assignments of buyers and sellers. Shapley & Shubik (1972) show that the outcomes in the core of an assignment game are the solution of a linear programming problem, which is dual to the optimal assignment problem. Furthermore, every core outcome is competitive. About ten years later, Crawford & Knoer (1981) study competitive adjustment processes in labor market models. As usual, there are heterogeneous but perfectly informed firms and workers. It is also shown that the selected equilibrium is the most preferred by the agents who made the offers. Kelso & Crawford (1982) provide existence results for equilibria, obtained by competitive adjustment problems and they show that an equilibrium in such markets is stable, provided that

[3]The marriage problem is a special case of the assignment game without side-payments.

[4]e.g. a car or a house

workers are gross substitutes from the firms' point of view. The agents who make the offers are always favored. This is also generalized in their paper. Demange & Gale (1985) discuss the problem of non-uniqueness of equilibria and the problem of manipulability for two-sided matching markets (with monetary transfers) using the Walrasian mechanism. A lot more publications follow. To get a nice overview, we refer the interested reader to Roth & Sotomayor (1990, Section 8.6) and for the current literature to the bibliography of two-sided matching on Roth's web page[5].

Another type of matching model is the so-called *roommate problem*, which was also introduced by Gale & Shapley (1962). Just one set of agents is involved, let us say students. Each student is looking for a roommate in a dormitory. They have preferences over each other. Therefore, the marriage problem where women (men) only have preferences over members of the opposite sex can be viewed as a specific case of the roommate problem. A stable matching may not always exist in this context (shown by Gale & Shapley (1962)). Irving (1984) describes an algorithm that determines whether a stable matching exists and if so, the algorithm will find such a matching. Then, Tan (1991) shows the existence of a stable matching for strict preferences and Chung (2000) guarantees the existence even in the case of weak preferences, by introducing a particular condition.

### 1.1.2 Practical Evolution

At the beginning of the twentieth century internships for postgraduate medical students were compulsory for the first time. This opportunity was beneficial for both sides of the market. Students gained experience in medical education and hospitals got cheap labor. From the beginning on, the number of positions available exceeded the number of graduates.[6] A direct consequence was a considerable competition among the hospitals. Each hospital made appointments with the students earlier

[5] http://kuznets.fas.harvard.edu/ aroth/alroth.html

[6] For partial evidence see also Graettinger (1976).

and earlier in the year. This led to an unsatisfactory situation for the students as well as for the hospitals. The students' school year was interrupted by interviews and the hospitals now invited students without knowing their grades. In 1926, Darrach (1927, published) informed the Association of American Medical Colleges (AAMC) that "It has been decided to defer the appointments of interns at the Presbyterian Hospital in the City of New York until some time in April." But this commitment was not maintained in the long run. About ten years later, Fitz (1939) suggested schools, which were members of the AAMC, should agree on not to release information before a fixed date. Furthermore, hospitals should wait for a date given by the AAMC, before they start to make appointments with candidates. These ideas were never applied for unknown reasons. Next, Turner (1945) formulated several conditions to resolve the problem of selecting interns. In particular, he again claimed fixed dates for medical schools to release information about the students and for the hospitals to require a firm acceptance of an offered internship. Furthermore, for successful implementation, it was necessary that Turner's instructions were followed nationwide. Indeed, these rules caused appointments to take place later in the senior year. But now another problem arose. The time period between a hospital's offer of an internship and the student's acceptance of the position was extended, because students waited for possible offers of more preferred hospitals in their ranking list. From 1945-1951 this problem resulted in a permanent absurd time reduction from the time when the offer was made to the requirement of acceptance. Therefore, Mullin (1950) and Mullin & Stalnaker (1951) introduced a procedure with a central clearing agency to eliminate the failures of the previous one[7] and to insure a fair principle of distribution. Students and hospitals announced their rankings of each other to a central institution, which in return arranges a matching according to some specific rules. A trial run was decided and executed in 1950-1951. But before the procedure was implemented to the real market, again another failure emerged. If

[7] e.g. unfair pressure on students to make early commitments, incoming telegraph offers at different points of time and so on.

students misrepresented their true rankings, they probably would be matched with a more preferred hospital. A slight modification, discussed by Mullin & Stalnaker (1952), called the NIMP[8] algorithm was finally implemented. The algorithm was very successful and remains in use until these days.
Only ten years after Gale & Shapley's (1962) publication, the authors heard about the NIMP procedure and detected that in order to prove the existence of a stable matching they had developed a much simpler procedure than proposed by the NIMP. It is an astonishing observation, both the practical and the theoretical world independently achieved the same result.

Over the years, the number of couples among the students increased. Getting internships in neighboring hospitals, given the NIMP algorithm, is difficult. Therefore, in the 1970s more and more couples negotiated directly with the hospitals. The question of a "couple mechanism" arose. In a mechanism used until 1983, each couple submits a ranking-order list for each spouse and assigned one priority in the match. Elias & Elias (1980) discussed the drawbacks of this description of couples' preferences and proposed that each couple should submit a ranking-order list expressing their preferences over pairs of positions. In spite of these suggestions of improvement, the problem of unstable matchings persisted. In the theoretical world this is shown by Roth (1984a). The set of stable matchings may be empty in a market, in which the set of agents exhibits couples. Furthermore in this paper, the author related the evolution of the labor market for medical interns and residents with the given results in game theory.

Recent publications on practical issues are nicely summarized in a paper of Roth (2003) to which we refer for further reading. For instance, he reviews the article of Roth & Peranson (1997) who propose a newly designed applicant-proposing algorithm and compare it with the existing NRMP algorithm. It turns out that the

[8]National Intern Matching Program. In 1968 it was renamed the National Intern and Resident Matching Program (NIRMP), and in 1978 renamed the National Resident Matching Program (NRMP), to reflect changes in the structure of postgraduate medical training.

differences are very small. The authors conclude both algorithms perform similarly. Medical internship allocation is still a good example for matching market theory. Theory and practice are closely related, although they pursue different paths.

## 1.2 The Thesis and the Literature

In this section we contrast our work with the existing literature that is closely related to the topics in this work and we search for similarities of recently published (or even not yet published) papers.

As already mentioned in Subsection 1.1.2, Roth (1984a) first showed that in the hospital-intern problem with couples, the set of stable matchings may be empty. Aldershof & Carducci (1996) examine several results for the stable marriage problem and show that they do not hold, if some interns are couples who express their preferences over pairs of hospitals. In the following year an article by Dutta & Masso (1997) is published. They study the consequences for stability in standard two-sided matchings, when the composition of a worker's colleagues can affect the preferences over firms. They introduce different preference restrictions for agents and for firms. First the set of agents may consist of single agents and couples. Workers who are singles have the classical[9] strict preferences. Couples' preferences fulfill the property of *togetherness* and firms satisfy *group substitutability*, which guarantees that there are no complementarities among agents.[10] If such a preference structure is given, the authors describe a modified deferred acceptance procedure, the so called *multi-stage deferred acceptance algorithm*[11]. Its outcome is a matching and their first theorem claims that this matching is in the core of any market with couples. Next they show, if individuals have $\mathcal{F}$*-lexicographic*[12] preferences over firm-colleague

[9] like in the marriage market described by Gale & Shapley (1962)

[10] For detailed definitions see Dutta & Masso (1997).

[11] For a brief overview see Section B.0.3.

[12] One firm together with a subset of agents is strictly preferred by one agent to another firm

pairs and firms have *substitutable*[13] preferences, then the set of matchings in the core is nonempty, because the set of not weakly blocked matchings is a subset of the core. The remaining two results of Dutta & Masso (1997) are less relevant in our context, because we do not deal with their further introduced preference restrictions. The work of Dutta & Masso (1997) is closest to the context of our thesis. Nevertheless, our model exhibits a number of differences. They will be discussed more detailed in the corresponding sections of Chapter 2, once we have introduced the model.

Klaus & Klijn (2005) introduce *weakly responsive preferences*[14] and then show that this notion of preferences guarantees stability, if couples are looking for jobs in some labor market. In subsequent works Klaus & Klijn (2004b) focus on paths to stability for matching markets with couples given weakly responsive preferences for couples, as well as Klaus & Klijn (2004a) discuss a fair and efficient stable matching, when couples' preferences are responsive. In Klaus, Klijn & Masso (2003) the authors show that the NRMP algorithm may not find an existing stable matching, even if couples' preferences are responsive.
Finally, all papers deal with two-sided matchings where couples are present, i.e. a student or an intern does not only care about where to study or in which hospital he will work but he also cares about to which university or hospital his partner is matched. This extension of the agents' preferences is similar to the structure in our thesis but we will take a step further. Agents will care about the colleagues they will have to work with in a firm.

Before we start to introduce the basic model, we want to delimit it from another research field. The concept of *hedonic coalitions* was first introduced by Drèze & Greenberg (1980). In their model an agent's utility depends not only on his con-

together with another subset of agents if and only if he strictly prefers the firm to the other firm. (We introduce a formal definition in Chapter 2 later or see also Dutta & Masso (1997).)

[13]A firm regards agents as substitutes rather than complements since it continues to want to employ an agent even if some of the others become unavailable.

[14]For a detailed definition see Klaus & Klijn (2005).

sumption bundle but also on the members of the coalition which he belongs to. More precisely, the authors refer to private and public goods, whereas the public good depends upon the coalition. In particular, they study stability of coalition structures in their model. They show, under which conditions a coalition structure is *individually stable*[15] and prove that such an "individual stable equilibrium" may fail to exist. Next they introduce *contractual individual stability*[16]. Now, if transfers among coalitions are allowed the existence of "individually stable contractual equilibria" can be guaranteed.

A more recent work of Bogomolnaia & Jackson (2002) gives new insights in hedonic settings. They consider only purely hedonic settings in the sense that an agent's utility only depends on the members of his coalition. If the agents' preferences are restricted to (additively) separable and symmetric preferences, then the set of individually stable coalition partitions is nonempty.

We observe the following similarities and differences to our work. Both have in common that the agents' preferences depend on the members of their coalition or in other words on the colleagues with whom they will work. In our context an agent's utility depends on a firm and colleagues. If we interpret the firm as a "discrete consumption good", then our scenario becomes similar to Drèze & Greenberg (1980). Nevertheless, we have problems to motivate Drèze & Greenberg's (1980) assumption that the utility function is strictly increasing in the private good in our model. There is no obvious explanation for an assumption that an agent's utility arises if he would work for more than one firm. Rather the opposite assumption may be more intuitive. Next, the size of coalitions is not fixed in Drèze & Greenberg (1980) and Bogomolnaia & Jackson (2002). By moving from one coalition to another, agents reduce and enlarge coalitions. In this context their notions of stability make sense.

[15] A coalition partition is *individually stable* if no single individual has incentives and opportunities to change the coalition.

[16] An individual can only change a coalition, if his move is beneficial to himself, to all members of the coalition which he joins and also to all members of the coalition he leaves.

In our context, the size of a coalition, respectively the number of agents who can work for a firm, is fixed in advance. Thus, neither the notion of individual stability nor contractual individual stability is applicable in our context. Hedonic settings are not the context we deal with in our model. The idea to embed the thesis in this field seems to be rather artificial than natural.[17]

Many things are different in this thesis compared to the existing literature. First, we establish two models only focusing on one side of the market; agents' preferences on bundles consist of a firm and colleagues. Furthermore, concerning the preferences in general only few restrictions are made. Only one model comprises both sides' preferences simultaneously. Furthermore, except of Pareto efficiency, none of the properties, we will introduce, are made in this context, yet. Of course, the main ideas may not be completely unfamiliar. For instance, another notion of stability is introduced. We give three different procedural approaches to select a set of matchings. A new (formal) graph theoretic representation is given for some of the procedures, which leads to new questions and already well known questions, e.g. incentive compatibility. Roughly, these are some of the main differences compared to the existing literature.

[17] I would like to thank Professor Matthew O. Jackson for helpful comments and hints during my stay at Caltech in 2004.

# Chapter 2

# The Basic Model

The basic model is a many-to-one matching model which means that many agents on the one side are matched with one agent on the other side. In particular, we interpret the many agents as employees and the one agent on the other side as a firm. Hence, we identify our matching model with a *job market*. Traditionally, in job market models agents have preferences over firms and firms have preferences over agents. In our model agents also have preferences over their colleagues, a fact rarely discussed in the literature. Agents spend a lot of time per week at their place of employment which stands to reason that they do not only care about the firm they work for, but they also care about with whom they have to work. We neglect the firms' preferences in this chapter, because first we entirely want to concentrate on the agents' concerns. In Chapter 4 we will reintegrate the firms' interests in different manners.

Once the agents are allocated to a job and colleagues, natural questions arise concerning the quality of this matching and as well of the quality of the way how we got this matching. One may ask whether some agents prefer another job and colleagues. Another important question is whether we can distinguish between some allocations. Are their allocations that are "more fair or just" than others? Or are agents able to influence the results by cheating? A lot of different questions that we are not able to answer straight away. Therefore, we introduce definitions and techniques, in short the tools to have a basis for our analysis.

In the first section we introduce the model of a job market where agents are matched with colleagues and firms. Next we define properties of matchings such that we are able to distinguish between matchings. Before we start to develop procedures to select matchings, we restrict the agents' preferences in a natural way, motivated by the real market. A graph theoretic construction is given to facilitate illustrations and to enable a formal description for some procedures. We define three different procedures. The last section of the chapter focuses on the agents' strategic behavior, mechanisms and the introduction of incentive compatibility.

## 2.1 The Framework

On the one side, we have agents looking for a job in a firm. On the other side, each firm seeks for employees. The number of jobs may be different for all firms. Once an agent got a job and colleagues as well, he can precisely state whether he got his first best choice, his second best choice and so on. This section provides a basis to permit further analysis. The model is described by the following quantities. Let

- $H$ be the *set of firms*, $|H| = m$
- $S$ be the *set of agents*, $|S| = n$.

Define two bijections $h, s$ by:

$$h : \{1, \ldots, m\} \to H \text{ whereas } k \mapsto h_k \ ,$$
$$s : \{1, \ldots, n\} \to H \text{ whereas } i \mapsto s_i \ .$$

We order $H$ and $S$ by

$$h_k > h_{k'} \Leftrightarrow k > k' ,$$
$$s_i > s_{i'} \Leftrightarrow i > i' ,$$

and restrict ourselves to use the indices $k, l$ or $k', l'$ exclusively for firms and $i, j$ or $i', j'$ for agents. We use the simpler notation, $k$ for a firm and $i$ for an agent, whenever misunderstandings are impossible. In examples we use the notation $h_k, h_{k'}$ and $s_i, s_{i'}$ to facilitate the identification of firms and agents, if confusions are possible.[1] Let

- $\kappa_k \in \mathbb{N}$ be *the number of jobs in firm* $k \in H$.[2] Each firm may have a different number of jobs to fill. Adding $\kappa_1, \ldots, \kappa_m$, we get the total number of available jobs, $\sum_{k=1}^{m} \kappa_k = \kappa$. The total number of jobs equals the total number of agents, $\kappa = n$.

We suppose that the total number of jobs and the total number of agents is equal to concentrate on the main objectives and to facilitate the analysis. All jobs will be filled, and all participating firms have at least one open job. Let

- $\mathbf{C}_q := \{C \subseteq S \mid |C| = q\}$ with $0 \leq q \leq |S|$ be the set of subsets of $S$ with $q$ agents. Each $C \in \mathbf{C}_q$ is called a *q-clique*. We simply use the term *clique*, if we do not specify the size of the subsets.

- $\mathbf{C}_{q-1}^{-i} := \{C \subseteq S \mid |C| = q - 1,\ i \notin C\}$ be the set of subsets with $q - 1$ agents and without agent $i \in S$. We call $C \in \mathbf{C}_{q-1}^{-i}$ *colleagues* of agent $i$. For $q = 1$, we have $\mathbf{C}_{q-1}^{-i} = \{\emptyset\}$.

- $b_i := (k, C)$ with $k \in H$ and $C \in \mathbf{C}_{\kappa_k-1}^{-i}$ be a *bundle of agent* $i$ and $B_i^S := \cup_{k=1}^{m} \{(k, C) \mid C \in \mathbf{C}_{\kappa_k-1}^{-i}\}$ be the set of all *bundles* of agent $i$ with $b_i \in B_i^S$.

- every agent $i$ have a *strict preference relation* $\succ_i^S$ on $B_i^S$ induced by $\succeq_i^S$, that is transitive and total[3] (if $b_i \neq b_i'$ then either $b_i \succ_i^S b_i'$ or $b_i' \succ_i^S b_i$, but not

[1] This mainly occurs in Chapter 4.
[2] We use $\mathbb{N} = \{1, 2, 3, \ldots\}$.
[3] which implies completeness

both). $\succ_i^S$ is sometimes called linear ordering. We write $\succ^S = (\succ_1^S, \ldots, \succ_n^S)$ as *preference profile*, $\mathcal{P}_i^S$ denotes the *set of all possible preference profiles of agent* $i$, for all $i \in S$. And we denote the *set of all possible preference profiles* with $\mathcal{P}^S = \mathcal{P}_1^S \times \ldots \times \mathcal{P}_n^S$.

- the firms be indifferent about their future employees.

Any $\kappa_k$-clique can work for firm $k$. But if $\kappa_l = \kappa_k$, for $l, k \in H$, a $\kappa_k$-clique can work for firm $l$ as well. Only the size of the cliques is important. There are $\binom{n}{q}$ different $q$-cliques. A firm can only employ as many agents as it has disposable jobs. Consequently, all cliques of size $q$ can work for firms that offer $q$ jobs.
The agents' preference ordering is restricted to the bundles consisting of a firm and of colleagues. We cannot derive the agents' preferences over bundles from preferences over firms on the one hand or from preferences over colleagues on the other hand. The reason for that is dependency between the components of the bundles, a firm and the colleagues. What do we mean by dependency? If an agent preferred a bundle with firm $k$ to a bundle with firm $l$ irrespective of the colleagues within the bundles, then we could partly derive the strict preferences over the bundles from the preferences over the firms.[4] Obviously, these considerations tend to a lexicographic structure, but first we have chosen a more general case. Agents have preferences over bundles. No restrictions are made. Nevertheless, we will later discuss lexicographic preferences, as well for which a detailed justification is given in the corresponding Subsection 2.2.2.

Agents face a set of different firms and the diversification even increases, if the firms are combined with colleagues. On the other hand, we assume firms to be indifferent. One possible interpretation of this scenario is the following. A firm simply "takes who it gets". We have exactly one job for each agent. So, if finally an agent is the

[4]The same would be true, if an agent preferred the bundle with clique $C$ to the bundle with clique $C'$ irrespective of the firms within the bundles.

only candidate for a job, he gets it. Implicitly, we assume that it is always better for a firm to have a job filled no matter with whom.
Specific situations lead to other interpretations of this model given in the following.[5] Agents are identified with citizens looking forward to live in a city together with other citizens. The capacity of housing in each city is limited, $\kappa_k$ for city $k$. Now, agents have preferences over the cities, they would like to live in, and over the agents (neighbors), they would like to live with in the city. However, the cities themselves are indifferent between all possible groups of citizens.
Thirdly, we have a number of different projects in a firm and a set of employees, having preferences over the projects and the colleagues they will work with in the project.[6] Again projects are indifferent in respect of the agents' composition.
These are only three possible scenarios, we have in mind with our model. To facilitate formulations and with regard to the extensions of the model, we will only use the firm-agent interpretation which is the most common one in economic literature.

**Remark 2.1.1**
If $\kappa_k = 1$ for each $k \in H$, agents do not have preferences over colleagues because each firm only has one job to fill. We are in the classical one-to-one matching context and we can apply the classical results as discussed, for example in Roth & Sotomayor (1990).

To collect the components, we have established until now, we describe our *job market (for agents)* by the tuple

$$\Upsilon_S = (H, S, (\kappa_k)_{k\in H}, \succ^S).$$

Next, we give some simple examples for how the agents' preferences look like. In the first example, all firms have the same amount of jobs to fill. Then we construct

[5]The author would like to thank Mark J. Machina and other participants of the First Illinois Workshop on Economic Theory 2003 for helpful comments.

[6]The author thanks Matthias G. Raith for pointing to this interpretation.

an example which brings us back to a classical one-to-one matching model. The third example describes the most general case. Each firm offers a different number of jobs.

**Example 2.1.2**

Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$. The agents' preferences can be described in the following way.

$(h_1, \{s_2\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_2, \{s_3\}) \succ_1^S (h_2, \{s_4\})$

$(h_2, \{s_4\}) \succ_2^S (h_2, \{s_1\}) \succ_2^S (h_1, \{s_4\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_1, \{s_1\}) \succ_2^S (h_1, \{s_3\})$

$(h_2, \{s_4\}) \succ_3^S (h_2, \{s_2\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_1, \{s_2\}) \succ_3^S (h_1, \{s_1\}) \succ_3^S (h_1, \{s_4\})$

$(h_2, \{s_3\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (h_2, \{s_1\}) \succ_4^S (h_1, \{s_3\}) \succ_4^S (h_1, \{s_2\}) \succ_4^S (h_1, \{s_1\})$. □

**Example 2.1.3**

$H = \{h_1, h_2, h_3, h_4\}$, $\kappa_k = 1$ for all $k \in H$, $S = \{s_1, s_2, s_3, s_4\}$. Agents have strict preferences over firms. Each firm only has one job to fill. This leads to the former context of the classical marriage market problem and can be easily resolved by the deferred acceptance procedure given in Gale & Shapley (1962). □

**Example 2.1.4**

Let $H = \{h_1, h_2\}$ with $\kappa_1 = 1, \kappa_2 = 3$ and $S = \{s_1, s_2, s_3, s_4\}$. The agents' preferences are given by:

$(h_2, \{s_2, s_3\}) \succ_1^S (h_2, \{s_3, s_4\}) \succ_1^S (h_1, \emptyset) \succ_1^S (h_2, \{s_2, s_4\})$

$(h_2, \{s_1, s_4\}) \succ_2^S (h_1, \emptyset) \succ_2^S (h_2, \{s_3, s_4\}) \succ_2^S (h_2, \{s_1, s_3\})$

$(h_2, \{s_1, s_4\}) \succ_3^S (h_2, \{s_2, s_4\}) \succ_3^S (h_1, \emptyset) \succ_3^S (h_2, \{s_1, s_2\})$

$(h_1, \emptyset) \succ_4^S (h_2, \{s_2, s_3\}) \succ_4^S (h_2, \{s_1, s_2\}) \succ_4^S (h_2, \{s_1, s_3\})$. □

**Definition 2.1.5 (Matching)**

A mapping $\mu : S \to H$ is called a *matching*, if for all $k \in H$ we have $\mu^{-1}(k) \in \mathbf{C}_{\kappa_k}$.[7] Therefore, we define $C_{\mu,k} := \mu^{-1}(k)$ as the *$\kappa_k$-clique under $\mu$* for firm $k \in H$. An alternative description of the matching $\mu$ is $M_\mu = (C_{\mu,1}, \ldots, C_{\mu,m})$. All matchings are given by $\mathcal{M} := \{M_\mu \mid \mu \text{ is a matching}\}$. Respectively, we define $\mathbf{C}^{-i}_{\kappa_{\mu(i)}-1} \ni C^{-i}_{\mu,\mu(i)} := C_{\mu,\mu(i)} \setminus \{i\}$.

Each agent $i$ is assigned to exactly one firm $k$. There exists a job for each agent, because $\kappa = n$ and each agent can only work for one firm. The mapping $\mu$ is surjective. Only if each firm has one job to fill, hence $\kappa_k = 1$ for all $k \in H$, we get a bijective mapping. If so, we have $m = n$. A matching generates a partition of $S$. The pre-image of each firm $k$ contains exactly as many agents as the firm has available jobs, namely $\kappa_k$.

If we have cliques $(C_1, \ldots, C_m)$ with the properties

$$C_k \in \mathbf{C}_{\kappa_k} \qquad \forall\, k \in H\,, \tag{2.1}$$

$$C_k \cap C_l = \emptyset \qquad \forall\, k, l \in H,\, k \neq l\,, \tag{2.2}$$

then this also describes a matching $\mu$ (which is implicitly given by $C_k = \mu^{-1}(k)$) Clearly for any job market a matching exists. One possible matching is given by $C_1 = \{s_1, \ldots, s_{\kappa_1}\}, \ldots, C_m = \{s_{(\sum_{k=1}^{m-1} \kappa_k)+1}, \ldots, s_{(\sum_{k=1}^{m-1} \kappa_k)+\kappa_m}\}$.

One may wonder why we have not chosen the more common definition of many-to-one matchings given in Roth & Sotomayor (1990). The simple answer is, it does not precisely fit into our model. Their definition of a matching describes a function from the set $H \cup S$ into the set of unordered families of elements of $H \cup S$. We mainly focus on the fact that agents are matched to firms. Of course, firms are also matched to agents, but we neglect this aspect in the basic model, because as already mentioned we are more interested on the agents' side, first. Furthermore,

[7] With a slight abuse of notation, we write $\mu^{-1}(k)$ instead of $\mu^{-1}(\{k\})$ as $\mu^{-1}$ is defined on subsets of $H$.

the traditional definition of matchings admits that an agent may not get a job and a firms' position may remain unfilled. Up to now, we have excluded these alternatives. We do not admit unfilled positions, if there are still agents without a job on the market.

**Remark 2.1.6**

Given our strict preference relation $\succ_i^S$ on $B_i^S$, we can deduce (weak) preferences over matchings $M, M' \in \mathcal{M}$. A matching $M$ is strictly preferred to $M'$ by agent $i$, if and only if agent $i$ strictly prefers the firm and the colleagues, he is matched with in $M$. But agent $i$ is indifferent between two matchings, if he always works in the same firm together with the same colleagues, regardless of which matching, $M$ or $M'$, is given. The formal description is: Let $M(\succ_i^S)$ be the preference relation derived from the agents' preferences $\succ_i^S$ on $B_i^S$ by $M\, M_P(\succ_i^S)\, M'$ if and only if $b_i \succ_i^S b_i'$ and $M\, M_I(\succ_i^S)\, M'$ if and only if $b_i = b_i'$. Denote by $M_P(\succ_i^S)$ and $M_I(\succ_i^S)$ its strict and indifferent component. Agent $i$ gets the bundle $b_i$ if we face the matching $M$.[8] We write $b_i = (\mu(i), C_{\mu,\mu(i)}^{-i})$ where $\mu$ is another description of $M$.

Next, we discuss how the agents' preferences can be represented. We do not follow the usual description of utility functions, even if the concept of ordinality holds in our description as well. Our interest mainly focuses on the positions of the bundles. We want to know whether a bundle is the first, the second or the tenth most preferred one. Therefore, we introduce a *ranking function for agents*. The motivation for this becomes completely evident when we introduce desirable properties of matchings in Section 2.2.

**Definition 2.1.7 (Agents' Ranking Function)**

A mapping $v_i^S : B_i^S \to \{1, \ldots, |B_i^S|\}$ with $v_i^S(b_i) > v_i^S(b_i')$, if and only if $b_i \succ_i^S b_i'$, is an *agent's ranking function* for each $i \in S$. $v_i^S(b_i)$ is called the *rank of* $b_i$ with $b_i \in B_i^S$.

[8] $b_i'$ if we face the matching $M'$.

Given the agents' preferences, we assume a decreasing order of bundles. The number of different bundles is the same for all agents. Each agent faces $\sum_{k=1}^{m} \binom{n-1}{\kappa_k-1}$ bundles. Now, the agents' ranking function assigns downward counting. The most preferred bundle of each agent, denoted by $b_i^{(1)}$, is assigned to $\sum_{k=1}^{m} \binom{n-1}{\kappa_k-1} = |B_i^S|$, the second most, $b_i^{(2)}$, to $\sum_{k=1}^{m} \binom{n-1}{\kappa_k-1} - 1 = |B_i^S| - 1$ and so on. The least preferred bundle, $b_i^{(|B_i^S|)}$, is assigned to 1. Therefore, all agents have exactly the same codomain. We deal with a joint normalization. Typically, the agents' ranking functions themselves are not identical.

Thus, the basic model is described. Firms have a given number of jobs to fill. Agents have preferences over bundles, each consisting of a firm and colleagues. Groups of agents, working for firms, are defined, and matchings deliver job distributions.

The reader may also suspect that some matchings are "better" than others. In this context, better may be measured in ranks. If an agent's ranking function maps all agents' bundles onto their highest rank, hence their most preferred bundle, this is for sure a "better" matching for all agents than a matching where all agents get their lowest rank. But this is only a first vague trial to distinguish matchings. In the next section, we introduce a couple of desirable properties for matchings in more detail.

## 2.2 Properties

Given the set of all matchings, some may appear to be "better" or "more desirable" than others. However, this is a vague description, hard to quantify or analyze. Therefore in this section, we introduce several properties which seem to be reasonable requirements to qualify a matching "more desirable" than one which does not satisfy one of these properties at all. After having defined a property, we check its existence as well. The "most plausible" property becomes questionable, if there is no matching satisfying it.

In the second subsection, we focus on agents' preferences. Having a short look on a real job market, agents first take their choice, concerning a firm for which they would like to work. Then they care about their potential colleagues. Therefore, we introduce *lexicographic* preferences.[9] A consequence of this natural restriction is that we get an appealing existence result.

### 2.2.1 Properties of Matchings

**Definition 2.2.1 (Pareto Efficiency)**
A matching $M \in \mathcal{M}$ is *Pareto efficient*, if there is no other matching $M' \in \mathcal{M}$, in which no agent is worse off and at least one agent is strictly better off in the sense of Remark 2.1.6.

This is the standard definition of Pareto efficiency. Consider a Pareto efficient matching. If an agent $i \in S$ resigns from his job in firm $k \in H$ and he accepts a job in another firm $l \in H$, the only reason to do this is with prospect of a higher rank. The action has two consequences. First, another agent $j \in S$ in firm $l \in H$ will lose his job. And second, the cliques in firm $k \in H$ and $l \in H$ are changing. We assume the jobless agent may take the unfilled job in firm $k \in H$. Anyway, it is the only job still available on the market. Since we deal with a matching, satisfying Pareto efficiency, either one of the agents in the clique of firm $l$ or in the clique of firm $l$, except agent $i$, will get a lower rank than he had before. Thus, agent $i \in S$ deteriorated at least one other agent's rank, by improving its own one. This holds for all agents who deviate from a Pareto efficient matching.
Note that the definition of Pareto efficiency is a notion of "agent Pareto efficiency", since the firms' preferences do not enter the model for the moment.

**Theorem 2.2.2**
There always exists a Pareto efficient matching $M \in \mathcal{M}$.

[9]Dutta & Masso (1997) already describe this restriction on preferences in their paper.

*Proof:* The number of matchings on the job market for agents is finite. So, the number of possible Pareto improvements is also finite, consequently, a Pareto efficient matching always exists. □

Pareto efficiency is often an essential requirement for economic outcomes. It basically provides an indication of the desirability of an outcome. Given a Pareto efficient outcome (in our context a matching), we do not waste resources (ranks).

**Definition 2.2.3 (Agent-max-min)**

A matching $\bar{M} \in \mathcal{M}$ fulfills the *agent-max-min property*, if

$$\bar{M} \in \underset{M_\mu \in \mathcal{M}}{\operatorname{argmax}} \ \min_{i \in S} \ v_i^S(\mu(i), C_{\mu,\mu(i)}^{-i}).$$

We take an arbitrary matching. Within this matching, we take the minimal rank of all agents' ranks $v_i^S(\mu(i), C_{\mu,\mu(i)}^{-i})$. An agent $i \in S$ with the lowest rank gets a less preferred bundle, compared to what the others get. Now, we determine the minimal rank for all matchings. Hence we get a set, consisting of all minimal ranks of all matchings. From this, we take the maximal rank[10]. This maximal rank corresponds to the set of matchings, which possess the agent-max-min property. In other words, the set of matchings, satisfying the agent-max-min property, contains those matchings, in which the agents with the worst position get the best position, compared to all other agents who get the worst position in the remaining matchings. Therefore, the assumption that all agents face the same range of the ranking function is crucial. The agent-max-min property requires that interpersonal comparisons of preferences are possible. The most preferred bundle is assigned to the same rank by all agents, the second most preferred and so on. The agent-max-min property becomes meaningless, if agents have the same rank e.g. for one's most preferred and one's least preferred bundle. Basically, what we intend to compare is not the rank itself, it is the fact whether an agent gets his first second and so on most preferred

[10]The solution does not have to be unique. Of course, the same holds for the minimal rank.

bundle.

The motivation for this property is to treat agents as equal as possible, regarding the ranks they get. For this concept see also Rawls's (1999) theory of justice. Assume in a matching $M \in \mathcal{M}$ agents in some cliques get their highest rank whereas others get their lowest rank. At the same time in another matching $M' \in \mathcal{M}$ all agents get their second most preferred bundle. Then the matching $M'$ is "better" than the matching $M$, according to the agent-max-min property, because the agents with their worst rank in $M'$ get a higher rank than the agents with the worst rank in $M$. One may think within a matching, the agent-max-min property searches to minimize the gap between the "best and the worst", but this is not the case. We take two matchings, $\tilde{M}$ and $\bar{M}$, whereas in $\tilde{M}$ all agents are matched with their most preferred firm and agents and in $\bar{M}$ all agents are matched with their less preferred firm and colleagues. In both matchings the gap, regarding the agents' rank, is zero. Thus, we can not state a difference, regarding this. But taking the agent-max-min property into account, we can clearly determine that $\tilde{M}$ is "better" than $\bar{M}$.

We can restrict the agent-max-min property to cliques as well, therefore it becomes a local property. Here, we only focus on cliques in any firm. This is formally described by the next definition.

**Definition 2.2.4 (Clique-max-min)**

A $\kappa_k$-clique $\bar{C} \in \mathbf{C}_{\kappa_k}$ fulfills the *clique-max-min property* for firm $k \in H$, if

$$\bar{C} \in \underset{C \in \mathbf{C}_{\kappa_k}}{\operatorname{argmax}} \; \min_{i \in C} \; v_i^S(k, C \setminus \{i\}).$$

We take an arbitrary clique $C \in \mathbf{C}_{\kappa_k}$. Within this $\kappa_k$-clique, we take the minimal rank over all agents' ranks who are elements/members of the clique. This has to be done for all cliques of size $\kappa_k$. Hence, we get a set of ranks consisting of all minimal ranks of all $\kappa_k$-cliques. Now, we take the maximal rank over the minimal ranks. That maximal rank corresponds to a set of $\kappa_k$-cliques[11]. This set of $\kappa_k$-cliques for

[11] Obviously, this set of $\kappa_k$-cliques may consist of one element only.

firm $k \in H$ fulfills the clique-max-min property. Thus, the agents with the worst position in a $\kappa_k$-clique for firm $k \in H$ get the best position, compared to all other agents who get the worst position in the remaining $\kappa_k$-cliques.

Another observation is that the same clique may correspond to different matchings, hence $C_{\mu,k} = C_{\hat{\mu},k}$. This does not matter for the determination of the set of cliques satisfying the clique-max-min property for a firm $k$. If two firms offer the same number of jobs, then they face the same cliques as well. Thus, the set of cliques, satisfying the clique-max-min property, will also be identical.

Comparing the agent-max-min property with the clique-max-min property, we note that the first one is a condition for matchings, it focuses on all firms. The second one is a condition for cliques, it focuses on one firm only. Take a matching where all agents get their highest rank. It satisfies the agent-max-min property as well as the clique-max-min property for each clique in the matching. But the next example shows that neither the agent-max-min property nor the clique-max-min property has to imply the satisfaction of the other.

**Example 2.2.5**

Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$. The agents' preferences can be described in the following way.

$(h_1, \{s_2\}) \succ_1^S (h_2, \{s_3\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_2, \{s_4\})$

$(h_1, \{s_1\}) \succ_2^S (h_1, \{s_4\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_2, \{s_1\}) \succ_2^S (h_1, \{s_3\}) \succ_2^S (h_2, \{s_4\})$

$(h_2, \{s_4\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_2, \{s_2\}) \succ_3^S (h_1, \{s_4\}) \succ_3^S (h_1, \{s_1\}) \succ_3^S (h_1, \{s_2\})$

$(h_1, \{s_3\}) \succ_4^S (h_2, \{s_1\}) \succ_4^S (h_1, \{s_1\}) \succ_4^S (h_2, \{s_3\}) \succ_4^S (h_1, \{s_2\}) \succ_4^S (h_2, \{s_2\})$.

The matching $M = (\{s_1, s_4\}, \{s_2, s_3\})$ satisfies the agent-max-min property. Furthermore, let $\tilde{\mu}$ and $\hat{\mu}$ be two other matchings, whereas the cliques $C_{\tilde{\mu},1} = \{s_1, s_2\}$ and $C_{\hat{\mu},2} = \{s_1, s_3\}$ satisfy the clique-max-min property for firm 1 and firm 2, respectively. The cliques do not have to be disjoint, because they belong to different matchings.

Comparing both, we observe that the agents in matching $M$, satisfying the agent-max-min property, have lower ranks than the agents in the cliques, $C_{\tilde{\mu},1}$ and $C_{\hat{\mu},2}$, satisfying the clique-max-min property. Furthermore, the agent-max-min property and the clique-max-min property are not satisfied simultaneously. The cliques, satisfying the clique-max-min property, correspond to the matchings, $M_{\tilde{\mu}}$ and $M_{\hat{\mu}}$ which are both different compared to $M$. □

Our next property, *stability*, is often discussed in different contexts of matching models. We will analyze the notion of stability more precisely. Gale & Shapley (1962) proved the existence of a stable matching for every marriage market. In their context, a matching is stable if it is not blocked[12] by any individual or any pair of agents (a man and a woman). However, this notion of stability is no longer appropriate in our context for several reasons. First of all, we have only taken the agents side into account so far. Firms do not play an active role at the moment, in the sense that our notion of blocking will only deal with pairs of agents. Then, in the marriage market model, women only have preferences over men. In our model agents have preferences over firms and colleagues. Therefore, we consider another, more appropriate notion of stability which also seems to be more intuitive in our framework.

Consider a matching $M \in \mathcal{M}$, otherwise described by $\mu$. A matching is $t_a$-*blocked* - blocked by *t*rade among *a*gents - by two agents $(i, j)$ with $i \in C_{\mu,k}$ and $j \in C_{\mu,l}$, if both get a higher rank by exchanging their jobs. So, agent $i$ prefers to work in firm $l$ together with the remaining agents[13] as well as agent $j$ prefers to work with the remaining agents in firm $k$. In exchanging their jobs, they both improve their situations, i.e. they get a higher rank. This should be excluded in $t_a$-stable matchings.

[12]Consider two disjoint sets (same size), e.g. set of women and a set of men and a matching. If a woman and a man are not matched to one another in this matching, but they prefer each other to their assignments, they will block the matching (see also Roth & Sotomayor (1990), p. 21).

[13]Agent $j$ does not have a job in firm $l$ any longer.

**Definition 2.2.6** **($t_a$-Stability)**

A matching is *$t_a$-stable*, if it is not $t_a$-blocked by any pair of two agents working in different firms[14].

$t_a$-stability seems to be an intuitive requirement. A matching does not persist very long, if two agents can improve their ranks by deviating, i.e. by forming a $t_a$-blocking coalition.

The concept of $t_a$-stability only focuses on the $t_a$-blocking coalition. If two agents exchange their jobs on the one hand another matching is generated and on the other hand, they influence the other agents' outcomes working in the two firms. Consequently, there may be again two agents who will $t_a$-block the matching. This may continue infinitely often, of course the same matchings will reappear again and again, because of the finiteness of the set of matchings. This observation leads us to the next theorem.

**Theorem 2.2.7**

For any preference profile $\succ^S \in \mathcal{P}^S$ there does not always exist a $t_a$-stable matching.

The proof of Theorem 2.2.7 is given by Example 2.2.8. Furthermore, we use the opportunity to illustrate the matchings satisfying Pareto efficiency.[15]

**Example 2.2.8**

Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$ with their preferences.

$(\mathbf{h_1}, \{\mathbf{s_3}\}) \succ_1^S (h_2, \{s_4\}) \succ_1^S (\mathbf{h_2}, \{\mathbf{s_3}\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_1, \{s_2\}) \succ_1^S (h_2, \{s_2\})$

$(\mathbf{h_2}, \{\mathbf{s_4}\}) \succ_2^S (h_1, \{s_3\}) \succ_2^S (\mathbf{h_1}, \{\mathbf{s_4}\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_1, \{s_1\}) \succ_2^S (h_2, \{s_1\})$

$(\mathbf{h_2}, \{\mathbf{s_2}\}) \succ_3^S (h_1, \{s_1\}) \succ_3^S (\mathbf{h_1}, \{\mathbf{s_2}\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_2, \{s_4\}) \succ_3^S (h_1, \{s_4\})$

$(\mathbf{h_1}, \{\mathbf{s_1}\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (\mathbf{h_2}, \{\mathbf{s_1}\}) \succ_4^S (h_1, \{s_2\}) \succ_4^S (h_2, \{s_3\}) \succ_4^S (h_1, \{s_3\})$.

We start with an arbitrary matching, let us say $M_1 = (\{s_1, s_4\}, \{s_2, s_3\})$. This

[14] Of course, exchanging jobs in the same firm does not change their ranks at all.

[15] We already illustrated the max-min properties in the former example.

will be $t_a$-blocked by the agents $(\mathbf{s_1}, \mathbf{s_2})$, agent $\mathbf{s_1}$ prefers the bundle $(\mathbf{h_2}, \{\mathbf{s_3}\})$ and agent $\mathbf{s_2}$ prefers the bundle $(\mathbf{h_1}, \{\mathbf{s_4}\})$ they will exchange their positions. We obtain the matching $M_2 = (\{s_2, s_4\}, \{s_1, s_3\})$ which again will be $t_a$-blocked (this time, by $(\mathbf{s_3}, \mathbf{s_4})$) and we get $M_3 = (\{s_2, s_3\}, \{s_1, s_4\})$. Now, this matching is not $t_a$-stable because the pair of agents $(\mathbf{s_1}, \mathbf{s_2})$ will deviate, and we get $M_4 = (\{s_1, s_3\}, \{s_2, s_4\})$. Finally, we come back to $M_1$, since $(\mathbf{s_3}, \mathbf{s_4})$ will $t_a$-block. We created a cycle that certainly does not contain a $t_a$-stable matching. We still have to verify the two remaining matchings. $M_5 = (\{s_1, s_2\}, \{s_3, s_4\})$ and $M_6 = (\{s_3, s_4\}, \{s_1, s_2\})$. Both will be $t_a$-blocked. For the first one the agents $(\mathbf{s_1}, \mathbf{s_4})$ can profitably deviate and for the second one $(\mathbf{s_1}, \mathbf{s_3})$ will exchange their positions. This is an example for non-existence of a $t_a$-stable matching.

Nevertheless, we have two Pareto efficient matchings, viz $M_1, M_4$. □

Let us summarize our results. We have introduced several useful properties which allow us to differentiate between matchings such that we can e.g. say: This matching satisfies this or that property. We still have some problems with $t_a$-stability, because we cannot guarantee existence, yet. In this context, the next question arises. Do we deal with a realistic assumption, if two agents can exchange their positions as often as they like, because of the general structure of the preferences? The answer is clearly, no, because no firm will hire and dismiss employees ten times, on the one hand. On the other hand, there exist possible restrictions on preferences that can be justified. These considerations are discussed in the next subsection.

### 2.2.2 Lexicographic Preferences

It may be more significant to deal with results based on the most general form of preferences. But on the other hand one may ask whether this general form is an appropriate representation in our model. In the real world first of all, each agent cares about getting a job and then about the colleagues, he will be matched with.

Nobody will reject a job in his most preferred firm, because of his future colleagues. We take Dutta & Masso's (1997) restriction on the agents' preferences. They call the agents' preferences *lexicographic*, if it is possible to extract preferences over the set of firms.

**Definition 2.2.9 (Lexicographic Preferences)**
Agent $i$'s preference relation $\succ_i^S \in \mathcal{P}_i^S$ is called *lexicographic*, if there is a strict ordering $H(\succ_i^S)$ on $H$ such that for all $(k, C) \in H \times \mathbf{C}^{-i}_{\kappa_k - 1}$, $(l, C') \in H \times \mathbf{C}^{-i}_{\kappa_l - 1}$,

$$(k, C) \succ_i^S (l, C') \Leftrightarrow k\, H(\succ_i^S)\, l \qquad (k \neq l).$$

We now have strict lexicographic preferences. The agents' priority lies on the choice of the firms, only then they focus on their future colleagues. There is his most preferred firm, his second most preferred firm and so on. Within a firm, agents can state as well, which of two bundles they prefer, because of the underlying preference structure on $B_i^S$. Note, that in different firms an agent may rank the bundles with the same cliques differently.
Let $\mathcal{P}_i^{lex}$ denote the *set of all possible lexicographic preference profiles of agent* $i$, for all $i \in S$. And we denote the *set of all possible lexicographic preference profiles* with $\mathcal{P}^{lex} = \mathcal{P}_1^{lex} \times \ldots \times \mathcal{P}_n^{lex}$.

One consequence of lexicographic preferences now is that, if two agents $t_a$-block a matching, each one leaves the firm and starts to work in another one, no one will ever reapply in the former one, because all combinations of bundles containing the former firm, he originally worked for, will be a deterioration for the agent. Therefore, we restart our analysis of existence of $t_a$-stability, given the assumption of lexicographic preferences.

**Theorem 2.2.10**
If all agents have lexicographic preferences, then a $t_a$-stable matching exists.

*Proof:* Assume each matching $M \in \mathcal{M}$ is $t_a$-blocked. Since the set of matchings is finite, there exists a cycle such that $t_a$-blocking never ends. We denote the set of matchings generating a cycle by $Z$. Let $M, M', M'' \in Z$ be the matchings in the cycle. Now, $i$ is an agent who $t_a$-blocks $M$ together with another one. We get a matching $M'$, whereas $M' \, M_P(\succ_i^S) \, M$. In analyzing $M'$, we get either $i$ is again an element of the blocking pair, then $M'' \, M_P(\succ_i^S) M' \, M_P(\succ_i^S) \, M$ or $i$ is no element of the blocking pair, nevertheless, $M'' \, M_P(\succ_i^S) \, M$. Because of the lexicographic preferences, each matching in the sequence will improve agent $i$ (at least not deteriorate). Since $M \in Z$, we come back to $M$ after finitely many steps. So $M \, M_P(\succ_i^S) \, M$. This is a contradiction. Consequently, there always exists a $t_a$-stable matching. □

In this section, we have introduced an intuitive restriction to the agents' preferences. And given this restriction, $t_a$-stability can always be guaranteed. In the next section we give a precise description of the construction of a set of matchings.

## 2.3 Selection of Matchings

In this section, we develop a new graph theoretic representation, how to determine a set of matchings in our specific framework after finitely many steps. The basic motivation clearly is the fact that we are looking for a distinct illustration of the procedures, which we introduce in the second subsection. A graph theoretic representation seems to be a natural one.

Knuth (1976) introduced the analysis of algorithms on the basis of the formation of stable matchings. Balinski & Ratier (1997) and Balinski & Ratier (1998) first had the idea to illustrate the marriage problem in form of directed graphs. The main purpose of their papers is to propose another approach via directed graphs. For a short informal introduction in their *reduction algorithm* see Section B.0.2. The authors also compare the reduction algorithm with the *deferred acceptance procedure* (for a brief introduction see this time Section B.0.1). Therefore, we refer the

interested reader to Balinski & Ratier (1997).
Our construction of graphs does not possess many similarities unless the fact that preferences are indicated by directed edges. They only face one directed graph, whereas we will construct several directed graphs in each firm. They deal with men's (women's) preferences over women (men). Our agents do not only have preferences over firms, but also over other agents. This will be also taken into account in the graph theoretic representation. Furthermore, we only take the men's preferences into account. There are many other differences. Therefore, we think it is more appropriate to construct our own formal representation.

Before we enter into the subsections, we want to explain our comprehension of *procedures*. The notion is not well defined in the literature, yet. In the model we search for matchings. In a procedure this will be done iteratively, *round by round*. Depending on the given rules of a procedure, agents act in each round. Between two rounds a central planner checks, whether the procedure stops or continues. We face a progression of rounds that stops, if a matching is reached. But this is only a more or less technical description, it still does not explain why an agent should prefer to participate on the determination of matchings via our procedures instead of any other way to be matched with a firm and colleagues. The main motivation is the fact that in each step the agents' interests are kept. We can state even more, a step is determined by the agents' belongings. Next at any time it is easy for each agent to understand, why the procedure continues or stops. For two of our three procedures in this chapter, we even have a graph theoretic representation which further illuminates the course of action.

In the first subsection, we introduce the definitions and the general constructions of our directed graphs. The second subsection introduces three different procedures to determine a set of matchings. According to the former graph theoretic description, we now define when and in which order the edges emerge in the graphs for the first and the second procedure. Therefore, the agents' preferences come into play. One may think of a central planer who collects all the information and, according to the rules of the procedure, a set of matchings is selected. First, we focus more on the technical description of the procedures, but an economic motivation follows.

### 2.3.1 Graph Theoretic Representation

Before we enter into the formal description, we want to give an informal introduction to the main idea of our graph theoretic representation.
For each firm we construct directed graphs. The vertices are subsets of the set of agents. Roughly, the edges in a firm correspond to the agents' bundles. What does this mean? One could imagine all agents assembled in one room, they talk to each other. According to different bundles, within these conversations each agent asks other agents whether they would like to work with him in this or that firm. These informal invitations for joint work will be reflected in directed graphs by directed edges. A bundle contains the information in which firm which edges are given.

**Definition 2.3.1 (Directed Graph)**
A *directed graph* is a pair $G = (V, E)$ of (finite) sets. $V$ represents the set of vertices and $E \subseteq V \times V$ is the set of directed edges.

Each edge $(v, v') = e \in E$ has a starting vertex in $V$ and an ending vertex in $V$. A directed edge with the same starting and ending vertex is called a *loop*. Formally, the set of loops is given by $D_V := \{(v, v) \mid v \in V\}$.

**Definition 2.3.2 (Complete)**
A directed graph $G = (V, E)$ is called *complete* with or without loops, if $E = V \times V$ or $E = V \times V \setminus D_V$, respectively.

This is the most general description of directed graphs, loops and completeness. Coming back to our basic model, we identify a set of vertices in a directed graph $G$

with the agents in a $\kappa_k$-clique, hence, $V = C \in \mathbf{C}_{\kappa_k}$. The set of edges is given by $E_{k,C} \subseteq C \times C$. Since each edge reveals an agent's informal invitation for joint work, $E_{k,C} = C \times C$ implies all possible informal invitations in a $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$. Thus, we summarize a *directed graph in firm $k$ by $G_{k,C} = (C, E_{k,C})$*. The vertices in a directed graph in firm $k$ are always fix. Only the set of edges $E_{k,C}$ is flexible in each directed graph. One combinatorial remark, we have $\binom{n}{\kappa_k}$ different directed graphs in a firm $k$. Note that a directed edge corresponds to different directed graphs. An agent may have the same colleague in different $\kappa_k$-cliques. We continue with some more notation. $G_k = (G_{k,C})_{C \in \mathbf{C}_{\kappa_k}}$ with $k \in H$ is a *collection of directed graphs in firm $k$* and $\mathcal{G}_k$ is the *set of all such collections of directed graphs in firm $k$*. And finally, $G = (G_1, \ldots, G_m) \in \mathcal{G}$ is *one family of directed graphs*. We denote by $\mathcal{G}$ the *set of all such families of directed graphs.*

Now, we have directed graphs on the one hand and bundles on the other hand. Formally, we have to introduce how bundles are transformed into edges in the corresponding firms. This will be done in the next two definitions and illustrated in an example. Furthermore, we will explain in more detail the idea of an informal invitation.

**Definition 2.3.3 (Transformation)**
Let $E^i_{k,C} = \{(v, v') \in E_{k,C} \mid v = i\}$ be the set of edges in firm $k$ and in graph $G_{k,C} = (C, E_{k,C})$, $C \in \mathbf{C}_{\kappa_k}$ with the initial vertex $i$. A mapping[16]

$$\begin{aligned} tr_i : H \times \mathbf{C}^{-i}_{\kappa_k - 1} &\rightarrow H \times \mathcal{P}E^i_{k,C} \\ (k, C) &\mapsto tr_i(k, C) = (k, \{(i, i_1), \ldots, (i, i_{\kappa_k - 1})\}) \qquad i \in S \end{aligned}$$

transforms a bundle $b_i$ (with $C = \{i_1, \ldots, i_{\kappa_k - 1}\}$) into edges in the corresponding directed graph.

[16] $\mathcal{P}E^i_{k,C}$ denotes the power set of $E^i_{k,C}$.

There exists only one directed graph in firm $k$ with the edges $(i, i_1), \ldots, (i, i_{\kappa_k - 1})$. The order of the edges in the image of the mapping does not matter.

**Definition 2.3.4 (Projection)**

Let $proj : H \times \mathcal{P}E^i_{k,C} \rightarrow \mathcal{P}E^i_{k,C}$ with $tr_i(k, C) \mapsto proj\,(tr_i(k, C)) = \{(i, i_1), \ldots, (i, i_{\kappa_k - 1})\}$ (with $C = \{i_1, \ldots, i_{\kappa_k - 1}\}$) be the projection on $\mathcal{P}E^i_{k,C}$ which "cuts" $H$.

So, we are now able to represent each bundle of any agent in a corresponding directed graph in form of directed edges. The next example illustrates this representation of bundles.

**Example 2.3.5**

Firm 1 has four open jobs to fill. We take the following bundles of four agents[17], $b_1 = (1, \{2, 3, 4\})$, $b_2 = (1, \{1, 3, 4\})$, $b_3 = (1, \{1, 2, 4\})$ $b_4 = (1, \{1, 2, 3\})$ and we apply the transformation function and the projection function:

$tr_1(1, \{2, 3, 4\}) = (1, \{(1, 2), (1, 3), (1, 4)\})$

$tr_2(1, \{1, 3, 4\}) = (1, \{(2, 1), (2, 3), (2, 4)\})$

$tr_3(1, \{1, 2, 4\}) = (1, \{(3, 1), (3, 2), (3, 4)\})$

$tr_4(1, \{1, 2, 3\}) = (1, \{(4, 1), (4, 2), (4, 3)\})$

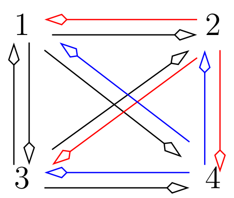
$proj(tr_1(1, \{2, 3, 4\})) = \{(1, 2), (1, 3), (1, 4)\}$

$proj(tr_2(1, \{1, 3, 4\})) = \{(2, 1), (2, 3), (2, 4)\}$

$proj(tr_3(1, \{1, 2, 4\})) = \{(3, 1), (3, 2), (3, 4)\}$

$proj(tr_4(1, \{1, 2, 3\})) = \{(4, 1), (4, 2), (4, 3)\}$.

The corresponding directed graph in firm 1 is: $G_{1,\{1,2,3,4\}}$




Each agent's bundle is illustrated in the graph. We face a complete directed graph without loops in firm 1. □

[17]We neglect the complete representation of the agents' preferences, the total number of firms and so on, because they are not relevant for the purpose of the example.

As we already indicated at the beginning of this subsection, agents approach each other and distribute informal invitations for joint work. They signal with whom they would like to constitute a $\kappa_k$-clique. No affirmative nor negative answer is expected. Agents simply reveal their interest for joint work with others. Therefore, we introduce *questions* and *offers* before we continue with the next subsection.

**A Question**

We identify each directed edge $(i, i_j)$ with a *question* of agent $i$ for joint work. Agent $i$ asks agent $i_j$ whether he would like to work with him in the corresponding $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$. The same question may belong to different cliques. It is ambiguous. If $\kappa_k - 1 = 0$, then the firm $k$ only has one open job. An agent can only ask trivially himself for joint work. Hence, a directed edge $(i, i)$ is a loop. In Example 2.3.5 we see that e.g. agent 2 asks agent 1, agent 3 and agent 4 for joint work in firm 1.

**An Offer**

Given one agent's bundle $b_i$ which we transform into edges, we identify all these directed edges with an *offer* of agent $i$. An offer is the union of agent $i$'s $\kappa_k - 1$ single questions for joint work in firm $k$. If $\kappa_k - 1 = 0$ or $\kappa_k - 1 = 1$ questioning himself or questioning only one other agent coincides with an offer. One and the same question belongs to different offers, if $\kappa_k - 1 \geq 2$. Thus, the same colleague can be asked more than once for joint work.

We have introduced a number of new definitions and terms, but until now, the reader may miss the connection between one another. What does complete graphs have to do with questions or offers? Let us take a firm and let $G_{k,C} = (C, E_{k,C})$ be a complete graph. Focusing on any agent within this graph, each edge $(i, \cdot) \in E_{k,C}$ can be interpreted as a question. Agent $i$ can only ask $\kappa_k - 1$ agents for joint work. These $\kappa_k - 1$ questions form an offer, and we have a $\kappa_k$-clique in firm $k$, if all agents within this clique made their offers in this $\kappa_k$-clique in firm $k$. Thus, each agent asked his $\kappa_k - 1$ questions and he got exactly $\kappa_k - 1$ counter questions, one for each question he asked. Hence, a bundle is interpreted as an offer and all offers in a complete directed graph in a firm $k$ constitute a $\kappa_k$-clique.

We have introduced some basic graph theoretic terminology. Furthermore, we have established a graph theoretic representation of the bundles on the one hand and the meaning of directed edges on the other hand. Thus, we now have the basic tools to deliver a formal description of two of our three procedures in the next subsection.

### 2.3.2 The Procedures

This subsection focuses on a description of three procedures. We assume each agent knows the procedure that will be applied and agrees on participating.
We identify a procedure with an *outcome correspondence (for agents)*. More precisely, we have $g : \mathcal{P}^S \Rightarrow \mathcal{M}$ with $(\succ_1^S, \ldots, \succ_n^S) \longmapsto g(\succ_1^S, \ldots, \succ_n^S)$. The correspondence $g$ selects for each vector of $n$ agents' preferences a set of matchings. This set will be called the *solution of a procedure*. We describe the course of action from a given preference profile to obtaining a matching as a *sequence* of families of directed graphs in the first procedure and in the second procedure. As we have seen in the former subsection, each element $G_k \in G$ of a family of directed graphs, $G \in \mathcal{G}$, contains a collection of directed graphs in a firm $k$. Roughly speaking, each family of directed graphs is identified with *a round* $r$ in the procedure and the family of directed graphs in round $r$ is related to the family of directed graphs in round $r+1$. However, the graph theoretic representation is not appropriate for the third one, but we also face different rounds, before the solution is established.
For each procedure, except the third one, we first deliver an intuitive and then a formal description. The third procedure goes without much formalism equally well. Each procedure selects cliques for the $m$ firms in a different manner and, consequently, the solutions of the procedures may be different, too. Furthermore, the

point of time[18], at which the procedures stop may also be different. Each procedure stops, if a set of matchings is selected. We also show that each procedure leads to a solution.

**Procedure 1a (P1a)**

1. All agents, $i \in S$, state their preferences $\succ_i^S \in \mathcal{P}_i^S$ on $B_i^S$.
2. Each agent $i$ takes his most preferred bundle $b_i^{(r)},\ k \in H,\ C \in \mathbf{C}_{\kappa_k-1}^{-i}$ among those, he has not selected, yet, and makes an offer in $C$, i.e. asks the agents $j \in C$ for joint work in firm $k$. The offers are transformed in the corresponding firm graphs to edges.
3. The following situations may appear after a round:
   - Given the edges of this round and the former ones, no complete directed graph is created in any firm $k$, i.e. no additional $\kappa_k$-clique emerged in any firm $k$.
     $\Rightarrow$ The offers of this round and the former rounds remain valid. We go back to *step 2*.
   - One or several additional $\kappa_k$-clique(s) emerge(s) in the same or in different firms. It may be the first emerging $\kappa_k$-clique(s). Two cases have to be distinguished.
     - Together with the former $\kappa_k$-cliques, it does not constitute a matching, i.e. equations (2.1) and (2.2) are not satisfied.
       $\Rightarrow$ The offers of the former rounds remain valid. Already constructed $\kappa_k$-cliques are still potential candidates for a matching. We go back to *step 2*.

[18] If we take time as a discrete parameter, one could identify round one with point of time one, round two with point of time two and so on.

– It constitutes a matching, i.e. equations (2.1) and (2.2) are satisfied.
$\Rightarrow$ The procedure stops.

The solution of **P1a** may not be unique. We denote the corresponding outcome correspondence by $g^{\mathbf{P1a}}$. It delivers for each given preference profile a set of matchings. Only if the procedure stops after the first round, the solution has to be unique.
This is our intuitive description how **P1a** works. But as already announced, we also have a formal representation in mind. This will be given next.

**Formal Representation of P1a**

We construct a sequence of families of directed graphs $G^0, G^1, G^2, \dots \in \mathcal{G}$, each element $G^r = (G_1^r, \dots, G_m^r)$ is identified with a round.[19] The vertices in all directed graphs are fixed, but the edges vary. In $G^0$, the set of edges $E_{k,C}^{(0)} = \emptyset$ for all $k \in H$ and for all $C \in \mathbf{C}_{\kappa_k}$. The procedure starts with $G_{k,C}^0 = (C, E_{k,C}^{(0)})$ for all $k \in H$ and for all $C \in \mathbf{C}_{\kappa_k}$.

1. All agents, $i \in S$, state their preferences $\succ_i^S \in \mathcal{P}_i^S$ on $B_i^S$.

2. Now in each round $r$, all agents make their offers, one offer per agent.[20] The function $tr_i$ transforms each bundle $b_i^{(r)}$ of round $r$ into the edges in the corresponding firm. Applying the projection function, we get the corresponding edges $E_{k,C}^{(r)}$ for all $k \in H$ and for all $C \in \mathbf{C}_{\kappa_k}$. Facing any directed graph in e.g. $G_k^r$, we now have $G_{k,C}^r = (C, E_{k,C}^r)$ with $E_{k,C}^r := \cup_{t=1}^r E_{k,C}^{(t)}$. Each $G_k^r$, with $k \in H$ collects all edges up to round $r$.

3. After each round, we have to check whether complete directed graphs were constructed, i.e. for each $k \in H$ directed graphs $G_{k,C}^r$ such that $E_{k,C}^r = C \times C \setminus D_C$, if $C$ consists of more than one agent and $E_{k,C}^r = C \times C$, if $|C| = 1$. Each complete directed graph (with or without loops) is a $\kappa_k$-clique in the

[19] $\mathcal{G}$ is a family of directed graphs. For a more detailed description see Subsection 2.3.1.

[20] The agents' first offers correspond to their most preferred bundles, the second to the second and so on (Therefore, see Definition 2.1.7 and below.).

corresponding firm $k$. In case that there are complete directed graphs, i.e. $\kappa_k$-cliques[21], we have to check validity of the equations (2.1) and (2.2). If a matching occurs, the procedure stops, if not, it continues.

**Remark 2.3.6**

To detect a matching, we first check equation (2.1). If there exists at least one clique in each firm, the validity of equation (2.2) still has to be shown. We take two cliques in different firms, occurring in *step 3* of the formal representation of the procedure and check whether their intersection is empty. If so, we take another clique in a third firm into account and intersect the union of the former two with the third one. If this intersection is again empty, we continue in the same way. This goes on until we included a clique in the last firm. If the intersection between the first two cliques is not empty and only one clique occurred in each firm, so far, a matching is not given, yet. If there are several cliques in one of the two first firms or in both, we have to check the pairwise emptiness of the union of all combinations of cliques in the different firms. If one is empty, take another clique in a third firm into account. We restart to check the emptiness of unions. If we have an empty intersection over all firms, equation (2.2) is satisfied as well. We got a matching, the procedure stops.

In **P1a** all agents make offers, round per round, as long as a set of matchings occurs. Then the procedure stops. The sequence of graphs is finite[22], because there exist only finitely many bundles for each agent. Non-emptiness of the solution of **P1a** will be proved in the next theorem.

**Theorem 2.3.7**

The procedure **P1a** always selects a set of matchings.

*Proof:* The proof is divided into two parts. i) **P1a** generated a set of matchings before all agents made their last offer. ii) **P1a** did not generate a set of matchings before

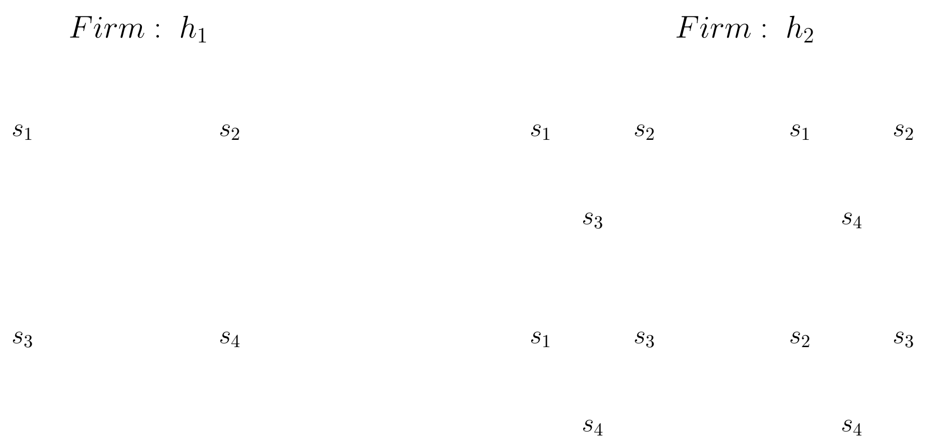
[21] There may occur more than one in the same round in the same firm or in different firms.

[22] If $r = \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$, we face the last round.

all agents made their last offer. Then all agents make their last offer. Consequently, all directed graphs in all firms, corresponding to all cliques in all firms, are now complete directed graphs. One possible combination of $\kappa_k$-cliques, generated by **P1a**, is $C_{\mu,1}, \ldots, C_{\mu,m}$, with $C_{\mu,1} = \{s_1, \ldots, s_{\kappa_1}\}$, $C_{\mu,2} = \{s_{\kappa_1+1}, \ldots, s_{\kappa_1+\kappa_2}\}, \ldots, C_{\mu,m} = \{s_{\sum_{l=1}^{m-1}\kappa_l}, \ldots, s_{\sum_{l=1}^{m}\kappa_l}\}$. All these cliques together satisfy the conditions (2.1) and (2.2). Therefore, the procedure **P1a** stops and generated a set of matchings at least containing the matching $M = \{C_{\mu,1}, \ldots, C_{\mu,m}\}$. □

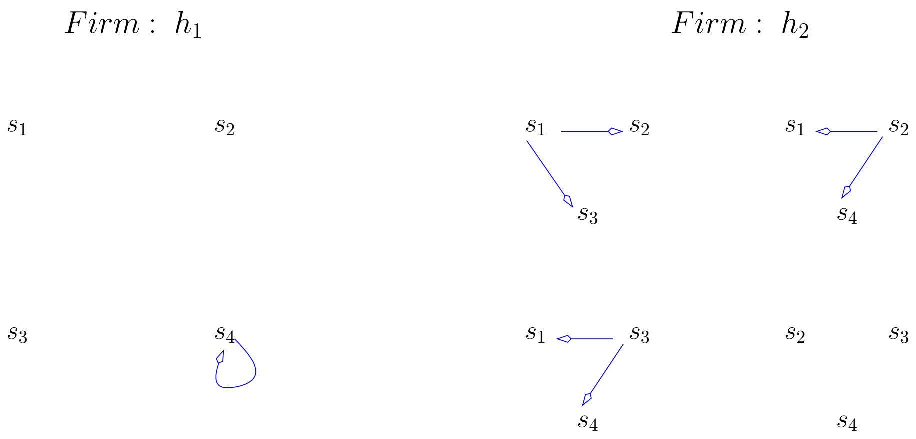
Before we pass into the next procedures, we briefly want to give an example to illustrate the course of action of our first procedure.

**Example 2.3.8 (Example 2.1.4 cont.)**
Each agent reveals his preferences. Therefore see Example 2.1.4. In the first family of directed graphs $G^0$, the set of edges $E^{(0)}_{k,C} = \emptyset$ is empty for all $k \in H$ and for all $C \in \mathbf{C}_{\kappa_k}$. This is illustrated in the first figure. We face four possible cliques in each firm. In firm $h_1$ a $\kappa_1$-clique consists of one agent, in firm $h_2$ a $\kappa_2$-clique consists of three agents.

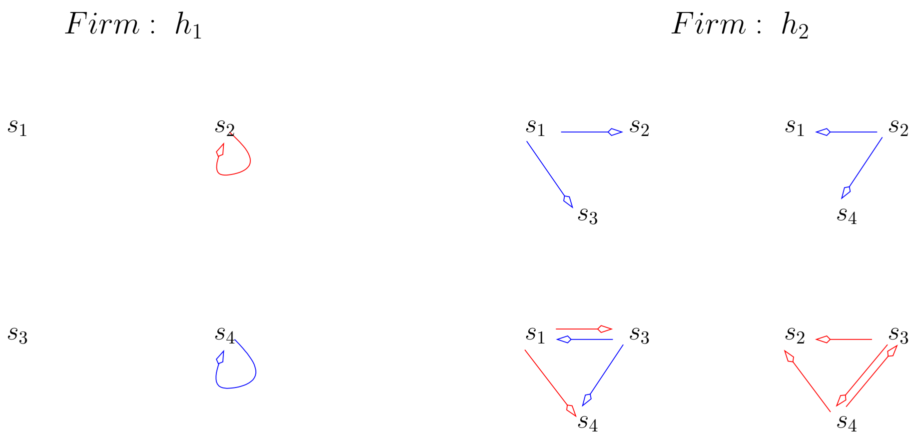



Now, in the first round $r = 1$, all agents make their offers, consisting of their most preferred bundles $b_i^{(1)}$, i.e. $b_1^{(1)} = (h_2, \{s_2, s_3\})$, $b_2^{(1)} = (h_2, \{s_1, s_4\})$, $b_3^{(1)} = (h_2, \{s_1, s_4\})$, $b_4^{(1)} = (h_1, \{\emptyset\})$. These bundles are transformed into edges, and we get: $E^{(1)}_{2,\{1,2,3\}} = \{(1,2), (1,3)\}$, $E^{(1)}_{2,\{1,2,4\}} = \{(2,1), (2,4)\}$, $E^{(1)}_{2,\{1,3,4\}} = \{(3,1), (3,4)\}$,

$E^{(1)}_{1,\{4\}} = \{(4,4)\}$. The next figure shows the directed graphs in the firms after the first round.



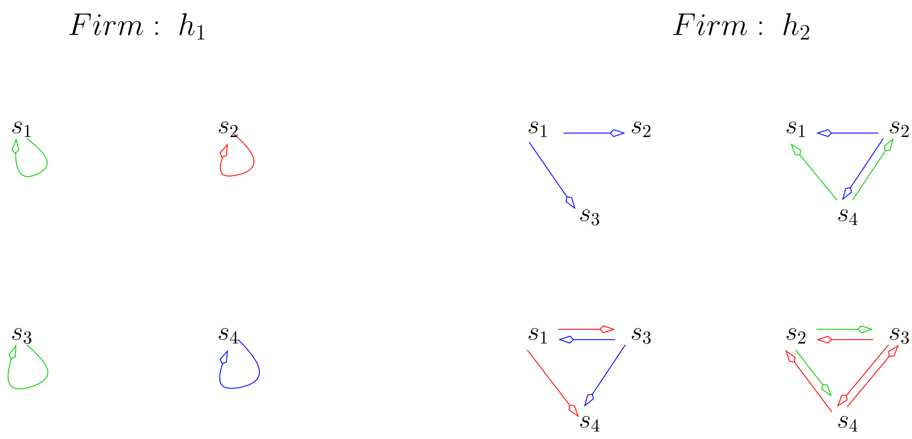

We observe one $\kappa_1$-clique in firm $h_1$, namely $C = \{4\}$. Thus, no matching occurred after the first round. The offers remain valid, and we enter the second round. Here, the edges $E^{(2)}_{2,\{1,3,4\}} = \{(1,3),(1,4)\}$, $E^{(2)}_{1,\{2\}} = \{(2,2)\}$, $E^{(2)}_{2,\{2,3,4\}} = \{(3,2),(3,4),(4,2),(4,3)\}$ accrue.[23] We get the next figure.




We now have two complete directed graphs in firm $h_1$, $C = \{4\}$ and $C' = \{2\}$, but still no one in firm $h_2$. Therefore, we enter the third round. The offers, transformed in edges are: $E^{(3)}_{1,\{1\}} = \{(1,1)\}$, $E^{(3)}_{2,\{2,3,4\}} = \{(2,3),(2,4)\}$, $E^{(3)}_{1,\{3\}} = \{(3,3)\}$,

[23] The transformations of $b^{(2)}_3$ and $b^{(2)}_4$ are summarized in $E_{2,\{2,3,4\}}$, because both agents make an offer in the same firm $h_2$ and the same $\kappa_2$-clique $\tilde{C} = \{2,3,4\}$.

$E^{(3)}_{2,\{1,2,4\}} = \{(4,1),(4,2)\}$. Again we supplement the directed graphs by the edges given in round 3.




Now we face four complete directed graphs, $C = \{4\}$, $C' = \{2\}$, $C'' = \{1\}$, $C''' = \{3\}$ in firm $h_1$, and one in firm $h_2$, $\tilde{C} = \{2,3,4\}$. Equation (2.1) is satisfied, we face at least one clique in each firm. Equation (2.2) also holds. The only empty intersection is $C'' \cap \tilde{C} = \emptyset$ and $|C''| + |\tilde{C}| = 4$. Hence, after the third round, we get a matching, the procedure stops. □

Before we continue to describe the second and the third procedure, we have to make some additional comments. In the following, **we restrict the set of preference profiles to lexicographic preferences**. We already motivated this restriction in Subsection 2.2.2, it is a natural assumption that an agent first cares about the firm, for which he will work, and then about his colleagues. In this framework we apply the procedures **P2** and **P3a**. Furthermore, we need the next lemma to guarantee the operability of the procedures.

**Lemma 2.3.9**

Let $\{i \in S \mid k\,H(\succ_i^S)\,l,\ \forall\ l \in H \setminus \{k\}\} =: \mathcal{S}_k$ be the set of agents who mostly prefer firm $k$. For any lexicographic preference profile $\succ^S \in \mathcal{P}^{lex}$ there always exists at least one firm $k$ for which $|\mathcal{S}_k| \geq \kappa_k$.

*Proof:* Assume for all firms $k \in H$, we only have $(\kappa_k - 1)$ agents who mostly prefer firm $k$, i.e. $|\mathcal{S}_k| < \kappa_k$ for all $k \in H$. Consequently, the total number of agents who can make $(\kappa_k - 1)$ offers to each firm is $\sum_{k=1}^{m}(\kappa_k - 1)$. But this is smaller than $\sum_{k=1}^{m} \kappa_k = n$. So, the assumption was wrong. It always exists at least one firm $k$ with $|\mathcal{S}_k| \geq \kappa_k$. □

**Remark 2.3.10**

The Lemma 2.3.9 also holds for preference profiles $\succ^S \in \mathcal{R}^S$, but we are only interested in lexicographic preference profiles.

Given these preliminary remarks, we now can pass into the detailed description of the procedures, restricted to lexicographic preference profiles. We start with an intuitive description of procedure **P2** and it follows a graph theoretic representation. In the following we talk about *selecting firms and agents*. The idea is to consider *selected* firms and agents separately.

Again we face different rounds, but now in any round $r$, some agents are "active", whereas others are not. With active we mean, they will make their offers, i.e. they question other agents in the room for joint work. The remaining agents are "inactive" in the sense that they are only standing in the room without participating, because they were not selected before.

**Procedure 2 (P2)**

1. All agents, $i \in S$, state their lexicographic preferences $\succ_i^S \in \mathcal{P}_i^{lex}$ on $B_i^S$.

2. First among all firms still on the market, we select those, for which Lemma 2.3.9 holds. We also take all agents $i \in |\mathcal{S}_k| \geq \kappa_k$ of these selected firms.

3. We only focus on the selected firms and the corresponding agents. Each selected agent $i$ takes his most preferred bundle $b_i^{(r)} = (k, C),\ k \in H,\ C \in \mathbf{C}_{\kappa_k - 1}^{-i}$ among those, he has not taken, yet and which only consists of a selected firm

and selected agents. He makes an offer in $C$, i.e. questions the agents $j \in C$ for joint work in firm $k$. The offers are transformed into edges. Agents make offers until (a) $\kappa_k$-clique(s) emerges.

4. The agents in such a $\kappa_k$-clique are matched with the firm and all together leave the market. We get a *reduced job market for agents*. We restart with *step 2* given the reduced job market, if there are still firms and agents on the market, otherwise we got a matching, and the procedure stops.

From an economic point of view what happens is the following. First the agents apply for a job in a firm. Then the procedure only focuses on the firms, which have more applicants than positions available. A clique is chosen, for which all its agents make their offers first. In other words, the agents of this clique revealed their willingness for joint work first. Why do we not match the agents first to the firms, in which the number of applicants is smaller than the number of available positions? Because we are interested in treating agents as equal as possible. This will definitely become more obvious, if we discuss the "quality" of our solution. We ask the reader to be patient.
Before we continue with the formal description of the procedure, we introduce some notational simplifications to facilitate the reading. The new terms are basically used in the formal representation of **P2**. They describe the status quo at the beginning of the procedure as well as it provides us with precise definitions of the transitions from one round into the next one.

**Notation 2.3.11**
For each $\succ_i^S \in \mathcal{P}_i^{lex}$ and $t \geq 0$ define

- $H^0 := H$, $S^0 := S$
- $\mathcal{S}_k^t := \{i \in S^t \,|\, k\, H(\succ_i^S)\, l,\ \forall\, l \in H^t \setminus \{k\}\} \qquad (k \in H^t)$
- $\mathcal{H}^t := \{k \in H^t \mid |\mathcal{S}_k^t| \geq \kappa_k\}$

- $H^{t+1} := H^t \setminus \mathcal{H}^t$.

Generally, we are not able to define the set of agents $S^t$, $(t \geq 1)$, participating in a specific round, in advance since this set depends on the course of the procedure itself. Hence, we describe the set of agents $S^{t+1}$ in the formal representation of **P2**.

**Formal Representation of P2**

1. All agents, $i \in S$, state their lexicographic preferences $\succ_i^S \in \mathcal{P}_i^{lex}$ on $B_i^S$.

2. Taking the (reduced) job market for agents $\Upsilon_S^t = (H^t, S^t, (\kappa_k)_{k \in H^t}, \succ^{S^t}|_{H^t})$, we select the set of firms $\mathcal{H}^t$. At the beginning of the procedure, $t = 0$, we start with the job market for agents $\Upsilon_S = (H, S, (\kappa_k)_{k \in H}, \succ^S)$ with $H = H^0$, $S = S^0$ and $\Upsilon_S = \Upsilon_S^0$.

3. We treat the firms $k \in \mathcal{H}^t$ separately. In each firm $k \in \mathcal{H}^t$ we construct a sequence in $\mathcal{G}_k^t \ni G_k^{(t)_0}, G_k^{(t)_1}, \ldots$ of collections of directed graphs as follows.[24] In $G_k^{(t)_0}$, the set of edges is given by $E_{k,C_k}^{((t)_0)} = \emptyset$ for all $k \in \mathcal{H}^t$ and for all $C_k \in \mathbf{C}_{\kappa_k}$. So, we have $G_{k,C_k}^{(t)_0} = (C_k, E_{k,C_k}^{((t)_0)})$ for all $k \in \mathcal{H}^t$ and for all $C_k \in \mathbf{C}_{\kappa_k}$. Now in each round $(t)_r$, all agents in $\mathcal{S}_k^t$ make their offers, one offer per agent.[25] Applying the transformation function $tr_i$ and the projection function $proj$, we get for each bundle $b_i^{((t)_r)}$ the corresponding set of edges $E_{k,C_k}^{((t)_r)}$ in firm $k \in \mathcal{H}^t$ for $C_k \in \mathbf{C}_{\kappa_k}$. Facing any directed graph in e.g. $G_k^{(t)_r}$, we now have $G_{k,C_k}^{(t)_r} = (C_k, E_{k,C_k}^{(t)_r})$, where $E_{k,C_k}^{(t)_r} = \cup_{q=0}^{r} E_{k,C_k}^{((t)_q)}$. It collects all edges up to round $(t)_r$. After each round $(t)_r$, we have to check, whether complete directed graphs are constructed in all $k \in \mathcal{H}^t$. After finitely many rounds there occurs at least one such complete directed graph, corresponding to, say $C_k^t$ $(k \in \mathcal{H}^t)$.[26] All

[24]The upper index $(t)_\cdot$ of a collection of directed graphs in firm $k$, $G_k^{(t)_\cdot}$, illustrates that we only face selected firms and agents of the reduced job market $\Upsilon_S^t$. Within these firms we may face several rounds, hence a sequence of collections of directed graphs.

[25]The agents' first offers correspond to their most preferred bundles in firm $k$, the second to the second given $\Upsilon_S^0$ and so on.

[26]In case there are more than one, we may choose arbitrarily.

firms $k \in \mathcal{H}^t$ and agents $i \in \cup_{k \in \mathcal{H}^t} C_k^t$ leave the job market. We are now able to define the set of agents. We get $S^{t+1} := S^t \setminus \cup_{k \in \mathcal{H}^t} C_k^t$. We enter the next *step*.

4. If there are still firms and agents on the job market, we define the next reduced job market for agents $\Upsilon_S^{t+1}$ and restart with *step 2*. If there do not remain firms and agents on the market, the procedure stops. Equation (2.1) and (2.2) are satisfied.

We denote the outcome correspondence by **P2**. The solution does not have to be unique, several matchings can be selected. We denote the corresponding outcome correspondence by $g^{\mathbf{P2}}$. During the course of action of **P2**, it may happen that two complete directed graphs ($\kappa_k$-cliques) are simultaneously determined in a firm $k$. Then the procedure follows different "paths", because both cliques belong to different matchings. We continue two separated courses of action, each time starting at the point where one of the two simultaneously determined $\kappa_k$-cliques (together with the firm $k$) leaves the market.
A collection of directed graphs $G_k$ contains all possible directed graphs for firm $k$. We could have restricted a collection of directed graphs to those directed graphs that only consist of agents in $\mathcal{S}_k$, but in favor of the simplicity of the formal representation, we have taken all directed graphs in a firm. Some of them will always consist of an empty set of edges. On the other hand in favor of a better overview, we will neglect the directed graphs in a graph theoretic representation, whenever the agents in these graphs are not selected.
Selected agents, not matched to a firm $k \in \mathcal{H}^t$, do not leave the market. They restart to make offers in other firms which are still on the market. They will be selected again and after finitely many selections they are also matched and then they leave the market.
A solution of the procedure is always reached. Taking a selected firm and the corresponding selected agents, a clique is always generated after finitely many offers,

because there are more applicants than jobs available. This holds for all selected sets. Since the number of agents and firms is finite, the procedure stops after finitely many steps. This will be confirmed by the next theorem.

**Theorem 2.3.12**

The procedure **P2** always selects a set of matchings.

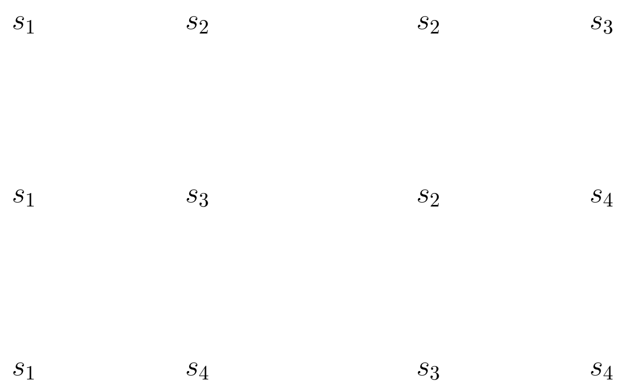
*Proof:* The proof is divided into two parts. i) **P2** generated a $\kappa_k$-clique in a firm $k \in \mathcal{H}^t$, before all agents made their last offer in firm $k$. ii) **P2** did not generate a $\kappa_k$-clique before all agents in $k \in \mathcal{H}^t$ made their last offer. Then all agents in firm $k \in \mathcal{H}^t$ make their last offer. All directed graphs in firm $k \in \mathcal{H}^t$ are now complete directed graphs, since $|\mathcal{S}_k^t| \geq \kappa_k$. A $\kappa_k$-clique is generated.
This argumentation holds for all selected firms and agents, and because the set of firms and agents is finite, we always get a matching. □

Again we want to give an example, after having introduced different versions to describe **P2**. The graph theoretic representation shows how easy it is to apply **P2** and how obviously a complete directed graph can be located.

**Example 2.3.13**

Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$. We assume that the agents have lexicographic preferences.
$(h_1, \{s_2\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_2, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_2, \{s_4\})$
$(h_2, \{s_1\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_2, \{s_4\}) \succ_2^S (h_1, \{s_1\}) \succ_2^S (h_1, \{s_4\}) \succ_2^S (h_1, \{s_3\})$
$(h_1, \{s_1\}) \succ_3^S (h_1, \{s_2\}) \succ_3^S (h_1, \{s_4\}) \succ_3^S (h_2, \{s_4\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_2, \{s_2\})$
$(h_1, \{s_2\}) \succ_4^S (h_1, \{s_1\}) \succ_4^S (h_1, \{s_3\}) \succ_4^S (h_2, \{s_3\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (h_2, \{s_1\})$.
Each agent reveals his preferences. Next we select the set of of firms with $|\mathcal{S}_k^0| \geq \kappa_k$, namely $\mathcal{H}^0 = \{1\}$. A graph theoretic representation for the selected firm $h_1$, before the agents start to make their offers is given by the next figure.

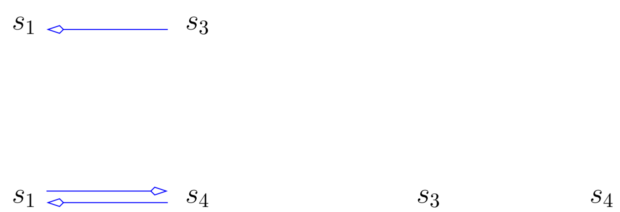
$Firm: \; h_1$




We have six directed graphs, whereas the set of edges is still empty. $\mathcal{S}_1^0 = \{1,3,4\}$ is the set of agents who mostly prefer firm 1. Since only agents in $\mathcal{S}_1^0$ will make offers, we neglect - as already mentioned - all directed graphs containing agent 2. Formally, we have to take them into account each round as well.

Now, in the first round $(t)_r = 1_0$, all selected agents make their offers, consisting of their most preferred bundles in firm 1, i.e. $b_1^{(0_0)} = (h_1, \{s_4\})$, $b_3^{(0_0)} = (h_1, \{s_1\})$, $b_4^{(0_0)} = (h_1, \{s_1\})$. These bundles are transformed into edges and we get $E_{1,\{1,4\}}^{(0_0)} = \{(1,4),(4,1)\}$ and $E_{1,\{1,3\}}^{(0_0)} = \{(3,1)\}$. The next figure illustrates the selected firm and agents after the first round.

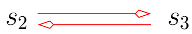
$Firm: \; h_1$




We observe a complete directed graph, a $\kappa_1$-clique, namely $C = \{1,4\}$. Firm $h_1$ and the agents $s_1, s_4$ leave the market. We get a reduced job market for agents, $\Upsilon_S^1 = \{H \setminus \{1\}, S \setminus \{1,4\}, \kappa_2, \succ^{S\setminus\{1,4\}}|_2\}$. We restart with *step 2.* $\mathcal{H}^1 = \{2\}$,

$\mathcal{S}_2^1 = \{2, 3\}$. In the second round $(t)_r = 2_0$ the bundles, $b_2^{(1_0)} = (h_2, \{s_3\})$ and $b_3^{(1_0)} = (h_2, \{s_2\})$ are transformed into $E_{2,\{2,3\}}^{(1_0)} = \{(2, 3), (2, 3)\}$. Hence we get the complete directed graph $C' = \{2, 3\}$ in firm $h_2$.

$Firm: \ h_2$




No firm and agents remain on the job market. Equation (2.1) and (2.2) are satisfied as well. Hence, after the second round, we get a matching, the procedure stops. □

Now, we introduce the last procedure. Roth & Sotomayor (1990) propose in their Example 4.3 an alternative procedure to the deferred acceptance algorithm to match men with women. Our next procedure is similar to this one. We adjusted it to our framework.

**Procedure 3a (P3a)**

1. All agents, $i \in S$, state their lexicographic preferences $\succ_i^S \in \mathcal{P}_i^{lex}$ on $B_i^S$.

2. The order of the $n$ given agents is fixed with an equal distribution. The chosen order enters the procedure.

3. First we check for each firm, which is still on the market, whether Lemma 2.3.9 is satisfied. This has to be the case for at least one firm. In the following, the procedure only focuses on these selected firms and agents.

4. In a selected firm we choose the agent with the lowest index, according to the order determined in *step 2*. The agent then chooses his most preferred $\kappa_k - 1$ future colleagues among those who were also selected for the same firm. These $\kappa_k$ agents together with the firm leave the market. The set of bundles is reduced by these agents and firms. **P3a** goes back to *step 3*.

5. The procedure stops if one firm remains, hence, the number of agents equals the number of offered jobs. The remaining clique together with the already selected ones fulfill the conditions (2.1) and (2.2).

**Remark 2.3.14**

At the beginning of **P3a**, each agent is assigned to a number with the same probability. Therefore, we can talk about treating the agents equally. Each agent will be with the same probability the one who chooses his colleagues. Each solution occurs with the same probability. Strictly speaking, the range of the outcome function $g^{\mathbf{P3a}}$ is not the set of matchings, but a probability distribution on the set of matchings. Thus, each matching occurs with a specific probability. It is easy to see that the procedure **P3a** always generates a matching. We do the analysis without further probabilistic formalism, simply to avoid notation. But the reader should have in mind that particularity of the third procedure.

We pass on a formal representation of **P3a**. First of all it does not fit into our graph theoretic framework. Complete directed graphs are not constructed, because only one agent determines a clique and second it is also easy to understand the procedure with the intuitive description.

These are the procedures, we will analyze in the following in respect of its solutions and themselves. Their criteria for selecting a set of matchings is always different. **P1a** waits until all offers, belonging to a matching, are made. **P2** does the same, but only for a subset of agents. Instead of that, **P3a** determines with equal probability only one agent who selects the clique, in which he will work for a firm. Nevertheless, all procedures have in common that the set of matchings is constructed stepwise and the agents are always treated equally.

Before we enter the next section, we briefly want to come back to a literature reference, because now we are able to make comparisons in more detail. As already mentioned in Chapter 1 the publication of Dutta & Masso (1997) is closest to the context of our thesis. Nevertheless, there exist a lot of differences. Firms do not play an active role in our model, yet. In Dutta & Masso's (1997) work firms always can reject or accept workers. In addition, the size of groups of colleagues is not exogenously given as it is in Dutta & Masso (1997). Properties like Pareto efficiency and the agent-(clique-)max-min property neither occur in their paper. And we define a new notion of stability. Furthermore, their multi-stage deferred acceptance algorithm can be interpreted as an extension of the classical deferred-acceptance algorithm. Our procedures do not bear a lot of resemblance to both of them. Offers are only made on one side of the market, an offer cannot be rejected. In **P2** for each firm a clique is only established, if all agents within this clique asked the others for joint work. In **P3a** not even all agents can actively participate on the determination of the cliques.

## 2.4 Strategic Behavior

Until now, we took the agents' preferences as given, no strategic behavior was assumed. As from now, agents behave strategically. This is a realistic assumption, because it provides the opportunity that agents may take advantage of situations. The structure of procedures, the fact that agents interact and act strategically has a profound influence on the solution of the procedures. Mechanism design studies the institutions, through which agents are interacting and their strategic behavior. The design of an institution, e.g. an election procedure, may have a crucial influence on the outcome of the procedure and on the agents' strategic behavior. Our main interest centers on *incentive compatibility.* We want to analyze whether our procedures prompt the agents to report their preferences truthfully or whether they can take advantage of lying.
In this section we briefly want to introduce some basics of this broad research field. It is far from giving a complete introduction of the topic. It will rather provide us

with the basic tools to answer the question we are interested in most. For more detailed surveys of this research field see e.g. Jackson (2000), Jackson (2001) or Osborne & Rubinstein (1994).

### 2.4.1 The Mechanism

Initially, we have to discuss the *information structure* within our setting. We again take the image, already introduced in Subsection 2.3.1. All agents are assembled in one room. They talk to each other and during these conversations each agent finds out the other agents' preferences. We assume that each agent gets to know the others' true preferences. Next, the agents will have to report their preferences to the central planner. This is, where strategic behavior comes into play. Not stating the true preferences may be advantageous for an agent.

But first we want to introduce the notion of a mechanism. In Subsection 2.3.2 we described our outcome correspondence (for agents), $g : \mathcal{P}^S \Rightarrow \mathcal{M}$.[27] Each vector of $n$ agents' preferences is mapped to a set of matchings. Now, strategically acting agents have to know the consequences of their behavior. Hence, a correspondence is not precise enough in specifying an outcome, given a preference profile. Therefore, we analyze strategic behavior along *selections of* $g$, i.e along an *outcome function* $\tilde{g} : \mathcal{P}^S \rightarrow \mathcal{M}$ with $\tilde{g}(\succ^S) \in g(\succ^S)$. Each preference profile is assigned to exactly one matching. This crucial detail allows agents to anticipate the consequences of their strategic behavior.

**Definition 2.4.1 (Direct Mechanism)**

A *direct mechanism* is a pair $(S, \tilde{g})$, where $S$ is a set of agents and $\tilde{g} : \mathcal{P}^S \rightarrow \mathcal{M}$ is an outcome function. *Each strategy profile*, here each preference profile, is associated with a matching.

[27] This is the general description of our outcome correspondences. Certainly, each procedure has its own outcome correspondence, $g^{\mathbf{P1a}}, g^{\mathbf{P2}}, g^{\mathbf{P3a}}$.

The mechanism is called *direct*, because an agent has to report his preference relation. We only treat direct mechanisms, therefore, we sometimes neglect the word direct to shorten descriptions.

Now, any pair $(M(\succ^S), \tilde{g})$ with $\succ^S \in \mathcal{P}^S$ induces a *noncooperative game*, given by $\Gamma(M(\succ^S), \tilde{g}) = (\mathcal{P}_1^S, \ldots, \mathcal{P}_n^S, M(\succ_1^S) \circ \tilde{g}, \ldots, M(\succ_n^S) \circ \tilde{g})$. $\succ^S$, and consequently $M(\succ^S)$ describes the agents' true preferences over matchings, with which the image of the outcome function $\tilde{g}$ is assessed. This leads to the above used expression $M(\succ_1^S) \circ \tilde{g}, \ldots, M(\succ_n^S) \circ \tilde{g}$.

The next definition determines how to represent an agent's preference relation.

**Definition 2.4.2 (Utility Function)**

An agent's true preference relation $M(\succ_i^S)$ over matchings with $\succ_i^S \in \mathcal{P}_i^S$ is represented by a *utility function* $u_i^S : \mathcal{M} \to \mathbb{R}$.

$M(\succ_i^S) \circ \tilde{g}$ reflects agent i's true preference relation over strategy profiles. If $M(\succ_i^S)$ is represented by $u_i^S : \mathcal{M} \to \mathbb{R}$, then $u^S \circ \tilde{g}$ represents $M(\succ_i^S) \circ \tilde{g}$ and, therefore, can be seen as *payoff functions* in $\Gamma(M(\succ^S), \tilde{g})$.

To facilitate distinction and to simplify further representation in this section, we denote in the following an element of the set of all strategy profiles with $(R_1^S, \ldots, R_n^S) \in \mathcal{P}^S$ and if we write $(\succ_1^S, \ldots, \succ_n^S) \in \mathcal{P}^S$, we want to refer to the agents' underlying true preferences.

## 2.4.2 Incentive Compatibility

Before we define incentive compatibility, we have to describe the agents' strategic behavior more detailed. Each agent $i$ announces his strategy $R_i^S \in \mathcal{P}_i^S$. We get a vector $(R_1^S, \ldots, R_n^S) \in \mathcal{P}^S$. In the following, we sometimes want to focus only on one agent's strategy of the whole strategy profile. Therefore, we write $(R_1^S, \ldots, R_n^S) = (R_{-i}^S, R_i^S)$. $R_i^S$ is agent $i$'s strategy and $R_{-i}^S$ are the strategies of the remaining $n-1$

agents.

The problem with multi-person decisions is that no agent knows which strategies the others will play. Once one strategy of agent $i$ will be a better response to the other agents' strategies and once another one. Agent i's outcome not only depends on his own strategy but also on the others' strategies. These interdependencies are a fundamental property of noncooperative games. Nevertheless, it may happen that there is a strategy profile $\bar{R}_i^S$ such that for each agent $i$, $\bar{R}_i^S$ is the best (most preferred) strategy choice, given all other agents report the strategies $\bar{R}_{-i}^S$. This leads to the next definition.

**Definition 2.4.3 (Nash Equilibrium)**

A strategy $\bar{R}^S \in \mathcal{P}^S$ is called a *Nash equilibrium* in $\Gamma(M(\succ^S), \tilde{g})$, if

$$\tilde{g}(\bar{R}_1^S, \ldots, \bar{R}_n^S) \;\; M(\succ_i^S) \;\; \tilde{g}(\bar{R}_{-i}^S, R_i^S) \qquad \forall\, R_i^S \in \mathcal{P}_i^S.$$

Agents act through the mechanism, they choose a strategy and the outcome function determines their colleagues and the firm. Each chosen strategy is based on the agents' true preferences. Whether an agent's reported preference coincides with his true preferences depends on the design of the mechanism. There may be situations, in which lying is a better strategy than telling the truth, simply because some agents are matched with a more preferred bundle of colleagues and a firm than by stating their true preferences. But there may also exist mechanisms where agents have no incentive to lie. All agents report their true preferences as if they are their best strategy, given all others report truthfully, too. Therefore, we get the next definition.

**Definition 2.4.4 (Incentive Compatibility)**

A mechanism $(S, \tilde{g})$ is *incentive compatible*, if $\succ^S \in \mathcal{P}^S$ is a Nash equilibrium in $\Gamma(M(\succ^S), \tilde{g})$ for each $\succ^S \in \mathcal{P}^S$.

This section was only a short introduction in some aspects of mechanism design. Of course, the research field includes many more interesting directions. We took this choice, because incentive compatibility is the property, we are interested in most. In the next chapter we will analyze which of our mechanisms are incentive compatible.

# Chapter 3

# Results

In this chapter, we will state various results for our basic model. The first section is divided into two subsections. In Subsection 3.1.1 we focus on the solutions (outcome correspondences) which we obtain in applying our procedures. We investigate whether solutions of the procedures satisfy the properties or at least some of them, introduced in a former section. The second subsection contrasts the solutions of the three different procedures. The second section is again divided into two subsections. In the first, we discuss incentive compatibility and in the second we compare our results given the three mechanisms. We conclude the chapter with a discussion about the procedures versus the solutions.

## 3.1 Properties of Procedural Solutions

The first procedure is defined for a general domain of preference profiles. A direct comparison with the other two procedures is only possible if we restrict the first procedure on lexicographic preferences, because **P2** and **P3a** are inapplicable to the general case of preference relations. We can foreclose that the solutions of the procedures will fulfill different properties. This is an appealing result, because according to the properties a society judges as most important, it only has to choose the appropriate procedure.

### 3.1.1 Outcome Correspondences

In this subsection, we discuss the different outcome correspondences of the procedures by means of our properties. We will analyze the "quality" of each solution. Of course, we are not able to classify the solutions of the different procedures, such that we can say "one is better than another one". At best, we can state the more properties are satisfied the better. In case this classification is impossible, it depends on where we put the main emphasis on. Mainly, we distinguish between *welfare aspects and practicability aspects.* Welfare aspects consider solutions that work out advantageously for the society. Obviously, this is a broad description, but this is on purpose. Different societies may judge different properties as the most important one. Practicability aspects deal with the simple fact how easy it is to establish a solution.

**Theorem 3.1.1**

For any preference profile $\succ^S \in \mathcal{P}^S$ the solution $g^{\mathbf{P1a}}(\succ^S)$

1. coincides with the set of matchings that fulfill the agent-max-min property and

2. contains at least one Pareto efficient matching.

*Proof:* 1. This proof is divided into two parts.

"$\Rightarrow$" If $M \in g^{\mathbf{P1a}}(\succ^S)$, then it fulfills the agent-max-min property.

**P1a** stops after round $r$. The worst rank an agent gets in a matching $M \in g^{\mathbf{P1a}}(\succ^S)$ is the one corresponding to round $r$. If these matchings do not fulfill the agent-max-min property, **P1a** would have stopped in an earlier round. Consequently,

$M \in g^{\mathbf{P1a}}(\succ^S)$ satisfies the agent-max-min property.

"$\Leftarrow$" If $M \notin g^{\mathbf{P1a}}(\succ^S)$, then it does not fulfill the agent-max-min property. The agent with the lowest rank in $M$ gets a lower rank than those who get the lowest rank in the set of matchings selected by **P1a**. Thus, $M$ does not fulfill the agent-max-min property.

2. Assume there does not exist a Pareto efficient matching in the solution of **P1a**. It follows that all $M \in g^{\mathbf{P1a}}(\succ^S)$ are Pareto dominated. Let $M' \in \mathcal{M}$ be a Pareto efficient matching that Pareto dominates some matching $M'' \in g^{\mathbf{P1a}}(\succ^S)$. Together with the first statement of this theorem this means that the worst rank in $M'$ is the same as the worst rank in $M''$, which implies $M' \in g^{\mathbf{P1a}}(\succ^S)$. □

In the last chapter we have shown that a solution of **P1a** always exists. In addition, we have proved that the solution always includes a Pareto efficient matching and it satisfies the agent-max-min property. Now, before we start the interpretation of the solution $g^{\mathbf{P1a}}(\succ^S)$ we want to discuss the property of $t_a$-stability.

We have already shown that a $t_a$-stable matching does not always exist in the general case. If non-existence is given, it is, of course, needless to check whether the solution of **P1a** fulfills the $t_a$-stability property. However, even if there are $t_a$-stable matchings, **P1a** may not find such a matching as the following theorem shows.

**Theorem 3.1.2**

Let $\succ^S \in \mathcal{P}^S$ be a preference profile such that a $t_a$-stable matching $M \in \mathcal{M}$ exists. Then, $M$ need not be an element of $g^{\mathbf{P1a}}(\succ^S)$.

*Proof:* The proof is given by an example.[1] Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$. Preferences are given by:

$(h_2, \{s_3\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_2, \{s_4\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_1, \{s_2\})$

[1] In colors highlighted matchings are the solution of the procedure or matchings by means of which we want to illustrate specific issues. Bold faced and colored bundles indicate the rank a pair of agents gets, if they $t_a$-block the corresponding colored matching.

$(\mathbf{h_1}, \{\mathbf{s_1}\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_1, \{s_3\}) \succ_2^S (h_1, \{s_4\}) \succ_2^S (h_2, \{s_4\}) \succ_2^S (h_2, \{s_1\})$
$(h_1, \{s_1\}) \succ_3^S (h_2, \{s_2\}) \succ_3^S (h_1, \{s_2\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_1, \{s_4\}) \succ_3^S (h_2, \{s_4\})$
$(\mathbf{h_2}, \{\mathbf{s_3}\}) \succ_4^S (h_1, \{s_1\}) \succ_4^S (h_2, \{s_1\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (h_1, \{s_3\}) \succ_4^S (h_1, \{s_2\})$.
The **P1a** only yields the matching $M = (\{s_1, s_4\}, \{s_2, s_3\})$, which is $t_a$-blocked by the pair $(\mathbf{s_2}, \mathbf{s_4})$. The agents exchange their jobs in the different firms $h_1$ and $h_2$ and, consequently, each one is matched with a strictly more preferred firm and colleague. Another matching, not selected by the procedure, is the following: $M' = (\{s_2, s_3\}, \{s_1, s_4\})$. This matching is $t_a$-stable, because there does not exist a pair of agents, such that by exchanging their jobs (and therefore colleagues as well) both get a higher rank. □

In this given example, the procedure **P1a** selects a matching that is not $t_a$-stable, although at least one $t_a$-stable matching exists. Our last attempt to get $t_a$-stability in applying **P1a** is to restrict the agents' preferences on lexicographic preferences. But again a simple example will disprove the conjecture. This result is stated in the next theorem.

**Theorem 3.1.3**

For any lexicographic preference profile $\succ^S \in \mathcal{P}^{lex}$ the solution $g^{\mathbf{P1a}}(\succ^S)$ does not have to contain an element, which is $t_a$-stable.

*Proof:* With lexicographic preferences, we can guarantee existence of a $t_a$-stable matching, (see Theorem 2.2.10). Again, an example (already introduced in Example 2.3.13) proves the theorem.
We repeat the setting to facilitate the illustration of matchings. Let $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4\}$. We assume that the agents have lexicographic preferences.
$(h_1, \{s_2\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_2, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_2, \{s_4\})$
$(h_2, \{s_1\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (\mathbf{h_2}, \{\mathbf{s_4}\}) \succ_2^S (h_1, \{s_1\}) \succ_2^S (h_1, \{s_4\}) \succ_2^S (h_1, \{s_3\})$
$(\mathbf{h_1}, \{\mathbf{s_1}\}) \succ_3^S (h_1, \{s_2\}) \succ_3^S (h_1, \{s_4\}) \succ_3^S (h_2, \{s_4\}) \succ_3^S (h_2, \{s_1\}) \succ_3^S (h_2, \{s_2\})$

$(h_1, \{s_2\}) \succ_4^S (h_1, \{s_1\}) \succ_4^S (h_1, \{s_3\}) \succ_4^S (h_2, \{s_3\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (h_2, \{s_1\})$.
The matching $M = (\{s_1, s_2\}), \{s_3, s_4\})$ is the unique solution of **P1a**. But this matching is not $t_a$-stable, because both agents in the pair $(\mathbf{s_2}, \mathbf{s_3})$ can ameliorate their ranks by exchanging their jobs. Hence, they will $t_a$-block the solution of **P1a**. The matching $M' = (\{s_1, s_4\}, \{s_2, s_3\})$ is $t_a$-stable. □

Now, let us come to the classification of our solution. We start to discuss the welfare aspects. Pareto efficiency clearly is a welfaristic property. If a matching is Pareto efficient no one can get a higher rank without making someone worse off. Thus, it is optimal for the society as a whole, we maximize some of the ranks over all matchings. Unfortunately, the solution of **P1a** contains Pareto efficient as well as non Pareto efficient matchings, hence, this welfaristic property cannot be guaranteed.
The agent-max-min property focuses on another aspect of welfare. According to Rawls principle of distributive justice (see Rawls (1999)) it favors the agent with the worst rank in a matching who gets the highest rank over all lowest ranks of all matchings. This fact can be interpreted as a fairness aspect. It favors the "weaker" or it supports the "less favored". If fairness is one of the societies' goals, then it is appropriate to choose procedure **P1a**, since it always generates a solution satisfying the agent-max-min property.
Let us come to the aspects of practicability. One basic assumption is the existence of a matching. This is guaranteed with Theorem 2.3.7. But the solution does not have to be unique. Therefore, if there are several matchings in the solution, the problem arises, which one to select. Another lack is the fact that, in general, the solution is not $t_a$-stable. We view $t_a$-stability as a practicability aspect as well. There are often pairs of agents who will $t_a$-block the solution. These are two reasons that make the solution of **P1a** less practicable.
To summarize, using **P1a** we get a solution that always contains Pareto efficient matchings and that fulfills the agent-max-min property, given the most general form of preference profiles. It does not necessarily fulfill $t_a$-stability. There always

exists a solution, but it does not have to be unique. To facilitate practicability, one could always select the set of Pareto efficient matchings within the solution of the procedure **P1a**. This would at least reduce the number of matchings be considered for a selection.

From now on, we restrict the preference profiles to lexicographic preferences (see Definition 2.2.9). We start to analyze the outcome correspondence of procedure **P2**. Recall that **P2** first selects the firms and agents for which the number of applicants is greater or equal to the number of available jobs. Then the procedure selects cliques among the selected agents in the selected firms. Agents and firms abandon the market, the procedure restarts until one firm and a set of agents remain. Again we investigate what kind of properties are satisfied given the solution of **P2**.

**Theorem 3.1.4**
For any lexicographic preference profile $\succ^S \in \mathcal{P}^{lex}$ each set of cliques of any matching in the solution $g^{\mathbf{P2}}(\succ^S)$, determined on $\Upsilon^t_S$ with $t = 0, 1, 2, \ldots$, satisfies the clique-max-min property on $\Upsilon^t_S$.

*Proof:* This proof is divided into two parts.
"$\Rightarrow$" If $C$ is a clique appearing in $M \in g^{\mathbf{P2}}(\succ^S)$, then it fulfills the clique-max-min property on $\Upsilon^t_S$, where it was determined.
After round $\binom{|\mathcal{S}^t_k|-1}{\kappa_k-1}$ no further offers are made in firm $k \in \mathcal{H}^t$.[2] The worst rank an agent gets in a clique $C$ is the one corresponding to round $\binom{|\mathcal{S}^t_k|-1}{\kappa_k-1}$. If this set of cliques does not fulfill the clique-max-min property, a complete directed graph would have been constructed for firm $k$ in an earlier round. Consequently, all established complete graphs $C$ in firm $k$ satisfy the clique-max-min property. This holds for all firms $k \in \mathcal{H}^t$.
"$\Leftarrow$" If $C$ is a clique not appearing in $M \in g^{\mathbf{P2}}(\succ^S)$, then it does not fulfill the

[2] $\binom{|\mathcal{S}^t_k|}{\kappa_k}$ is the number $\kappa_k$-cliques which can be established. Consequently, $\binom{|\mathcal{S}^t_k|-1}{\kappa_k-1}$ is the number of offers each selected agent can make.

clique-max-min property on $\Upsilon^t_S$, where it was determined.

The agent with the lowest rank in $C$ gets a lower rank than those who get the lowest rank in the set of cliques selected by **P2** given $\Upsilon^t_S$. Thus, $C$ does not fulfill the clique-max-min property. □

**Theorem 3.1.5**

For any lexicographic preference profile $\succ^S \in \mathcal{P}^{lex}$ any matching in the solution $g^{\mathbf{P2}}(\succ^S)$ is

1. Pareto efficient and
2. $t_a$-stable.

*Proof:* 1. Assume there is a matching $M \in g^{\mathbf{P2}}(\succ^S)$ that is not Pareto efficient. Then, there exists another matching $M' \in \mathcal{M}$ such that no agent in $M'$ is worse off and at least one agent is strictly better off. Thus, at least one agent gets a better rank either in the same firm or in another one. If he gets a better rank in the same firm in another clique $C'$, the other agents in this clique $C'$ will have to get a better rank as well, since we assume strict preferences. But then this clique would already have been chosen by the procedure **P2** in an earlier round, consequently $M \notin g^{\mathbf{P2}}(\succ^S)$. The same argumentation holds, if the agent gets a better rank in another firm, and again we get $M \notin g^{\mathbf{P2}}(\succ^S)$. Thus, all $M \in g^{\mathbf{P2}}(\succ^S)$ are Pareto efficient.

2. According to Lemma 2.3.9, there always exists at least one firm $k$ with $|\mathcal{S}_k| \geq \kappa_k$. The procedure **P2** starts and selects a $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ in this firm. No agent $i \in C$ will be a member of a $t_a$-blocking coalition, because there is no firm $l \in H \setminus \{k\}$ such that agent $i$ gets a higher rank in a firm $l$ in another clique $\bar{C} \in \mathbf{C}_{\kappa_l}$. We restart with a reduced job market for agents. Again Lemma 2.3.9 holds for at least one firm $k'$. The procedure selects a $\kappa_{k'}$-clique $C' \in \mathbf{C}_{\kappa_{k'}}$ in this firm. No agent in this $\kappa_{k'}$-clique will find a $t_a$-blocking partner (The only $t_a$-blocking partner, they are interested in,

are working in firm $l$, if firm $k'$ is their second most preferred firm.). But agents in the former constructed $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ can only deteriorate their rank. This argumentation goes on until the last clique is formed. Thus, any $M \in g^{\mathbf{P2}}(\succ^S)$ is $t_a$-stable. □

Again we want to classify the solution of **P2**. We start the discussion in the same order, first the welfare aspects. The solution of **P2** always satisfies Pareto efficiency. It selects the matchings where no agent can get a higher rank without deteriorating someone else's rank. The agent-max-min property is not satisfied. For this, see Example A.0.5 (setting 1.) in Appendix A and compare the outcome correspondences $g^{\mathbf{P1a}}(\succ^S)$ and $g^{\mathbf{P2}}(\succ^S)$, given the preferences in setting 1. But each selected clique, determined on $\Upsilon^t_S$, $t = 0, 1, 2, \ldots$ in the job market for agents satisfy the clique-max-min property on $\Upsilon^t_S$. This local aspect is very interesting. In general, agents are less concerned about global[3] issues than about what is going on in the firm they would like to work for. It is likely that agents are content if they know that their firm tries to favor the best among the worst workers. This property is advantageous for the daily situations at work. Often the global aspect is of less interest for the agents. Only from the society's point of view we could favor the agent-max-min property, because it selects the agent with the highest rank among all agents who ever get the worst rank in a matching.

Coming to the practicability aspect, existence of the solution is guaranteed by Theorem 2.3.12. But again it does not have to be unique. The selection problem among the matchings in the solution exists again. On the other hand the solution is always $t_a$-stable, which is an important result. Once we have determined the solution, we know for sure, that no pair of agents will ever have an incentive to $t_a$-block the solution. It is very practicable in the sense that it persists.

[3]The term global in this context is related to the agent-max-min property, because it takes all agents' preferences into account.

To summarize, the solution **P2** is Pareto efficient, satisfies the clique-max-min property for all cliques on the corresponding reduced job market, it always exists, but does not have to be unique, and it is $t_a$-stable.
Before we want to compare the solutions, we analyze the last procedure **P3a**.

**Theorem 3.1.6**
Let $\succ^S \in \mathcal{P}^{lex}$ be a lexicographic preference profile and fix an order of the agents. Then the solution $g^{\mathbf{P3a}}(\succ^S)$ w.r.t. to this order[4] is

1. a singleton,
2. Pareto efficient and
3. $t_a$-stable.

*Proof:* 1. This statement is obvious. In each firm there is only one agent who selects his colleagues. Hence, $m$ agents are selecting their $\sum_{k=1}^{m} \kappa_k - 1$ colleagues. Altogether exactly the $n$ agents are matched.
2. Assume $(C_1, \ldots, C_m) = M \in g^{\mathbf{P3a}}(\succ^S)$ is not Pareto efficient. Then there exists another matching $(C'_1, \ldots, C'_m) = M' \in \mathcal{M}$ such that no agent is worse off and at least one is strictly better off. W.l.o.g let $C_1$ be a clique chosen first by the agent with the lowest index $i$ in the set of agents who applied for a job in this firm (see Lemma 2.3.9, there always exists a $k$ such that $|\mathcal{S}_k| \geq \kappa_k$). Then $C_1 = C'_1$, because if $C_1 \neq C'_1$ agent $i$ will be worse off. The job market for agents is reduced by $C_1$. W.l.o.g let $C_2$ be a clique chosen first by the agents with the lowest index $j$ in the set of agents who applied for a job in this firm. Then $C_2 = C'_2$, because if $C_2 \neq C'_2$ agent $j$ would be worse off in a clique in the same firm 2. He only would be strictly better off in firm 1, if $1\, H(\succ^S_j)\, 2$, but then agent $i$ would be strictly worse off. This argumentation goes on until all cliques are checked. It follows $M = M'$, hence, $M \in g^{\mathbf{P3a}}(\succ^S)$ is Pareto efficient.

[4]See discussion in Subsection 2.3.2.

3. According to Lemma 2.3.9, there always exists at least one firm $k$ with $|\mathcal{S}_k^t| \geq \kappa_k$. In **P3a** the agent with the lowest index selects a $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ in this firm. No agent $i \in C$ will be a member of a $t_a$-blocking coalition, because there is no firm $l \in H \setminus \{k\}$ such that agent $i$ gets a higher rank in firm $l$ and in another firm in a clique $\bar{C} \in \mathbf{C}_{\kappa_l}$ than in the former one. We restart with a reduced job market for agents. Again Lemma 2.3.9 holds for at least one firm $k'$ ($|\mathcal{S}_{k'}^{t+1}| \geq \kappa'_k$). The agent with the lowest index in the firm $k'$ selects a $\kappa_{k'}$-clique $C' \in \mathbf{C}_{\kappa_{k'}}$ in this firm. No agent in this $\kappa_{k'}$-clique will find a $t_a$-blocking partner either. Agents in the former constructed $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ can only deteriorate their rank. This argumentation goes on until the last clique is formed. Thus, $M \in g^{\mathbf{P3a}}(\succ^S)$ is $t_a$-stable. □

The welfare aspects of the solution of **P3a** are characterized by Pareto efficiency. It is needless to analyze max-min properties, because in each firm only one agent determines the clique. For all other members of a clique, selected by this agent, it may be the worst rank within the firm.
The practicability aspect is supported by $t_a$-stability and uniqueness - a property we have not found in one of the previous solutions, yet. Uniqueness makes a lot of things easier. We do not have to think of, which matching to select if we face several matchings. Needless to say that this is an appealing result.

### 3.1.2 Comparisons

All procedures are distinguished by the fact that their solutions satisfy a different set of properties. To make comparisons easier, we give in Appendix A an Example A.0.5 with lexicographic preferences (setting 1.). We apply the procedures **P1a** and **P2** to this example and get different solutions. For the third procedure, we do not need an example, because once a fixed order of agents is given, the agent with the lowest index in a selected firm will choose his most preferred bundle and the procedure continues until a matching is reached. There does not exist a graph theoretic illustration. The next table gives an overview of our results.

| | $g^{\mathbf{P1a}}(H(\succ^S))$ | $g^{\mathbf{P2}}(H(\succ^S))$ | $g^{\mathbf{P3a}}(H(\succ^S))$ |
|---|---|---|---|
| welfare aspects | $\supset$ Pareto efficiency<br>agent-max-min | = Pareto efficiency<br>clique-max-min on $\Upsilon_S^t$ | = Pareto efficiency |
| practicability aspects | existence | existence<br><br>$t_a$-stability | existence<br>uniqueness<br>$t_a$-stability |

In the table we can see, a lot of properties are satisfied simultaneously. They are not in opposition to one another, e.g. Pareto efficiency to agent-max-min property or Pareto efficiency to $t_a$-stability. But it is outstanding that the agent-max-min property and $t_a$-stability do not appear in the same solution. We want to analyze this observation in more detail.

A simple thought experiment shows that this does not imperatively have to be the case. If we face a preference profile such that, independently whether we apply **P1a**, **P2** or **P3a**, all agents are always matched with their most preferred firm and their most preferred colleagues, then the solution satisfies the agent-max-min property as well as $t_a$-stability. Hence, both properties can be fulfilled by the same solution. Nevertheless, the "later" a matching with the agent-max-min property is detected by **P1a** the more likely it is not $t_a$-stable, because more and more possibilities to $t_a$-block appear for the agents.

Another interesting aspect is the selection of a matching in $g^{\mathbf{P1a}}(\succ^S)$ and $g^{\mathbf{P2}}(\succ^S)$. Both solutions do not have to be unique. In $g^{\mathbf{P1a}}(\succ^S)$ a central planner will have to select a matching to get rid of the dilemma. In the second case, one could also think of a central planner who chooses for a selected firm a clique if their appears more than one during the procedure is running. Each time he may decide by what is best for one firm, not for all firms at the same time. On the one hand this clearly solves the problem, which matching to choose, if the solution is not unique. But on the other hand, it creates the problem that the central planner gets the "power", and it rises the question, how he should choose among matchings or cliques. Another approach could be, once the solution of **P1a** is determined, all firms come together and choose a matching. Or in **P2**, if there occur two cliques in one firm, the firm can determine its future group of employees and the not chosen agents go back on the market. But by giving decision power to the firms, occurs another problem. Applying **P1a** how should e.g. two firms agree on a matching, if one prefers $M \in g^{\mathbf{P1a}}(\succ^S)$ and the other $M' \in g^{\mathbf{P1a}}(\succ^S)$? The same problem appears, given **P2**. If two firms together with agents are selected and if after round $r$ in both two cliques are determined, e.g $C^1, C^2$ in firm $h_k$ and $C^3, C^4$ in firm $h_l$, with $C^1 \cap C^3 = \emptyset$ and $C^2 \cap C^4 = \emptyset$ and firm $h_k$ prefers $C^1$ whereas firm $h_l$ prefers $C^4$. Then it is unclear how the parties should agree on two cliques as well. If there is only one firm in round $r$ which has to select between two or more cliques, this problem is not given at all. But all this is a thought experiment, since we assumed in Chapter 2 that firms are indifferent about the agents (or cliques) that will fill their vacant jobs.

If we focus on the practicability aspect, the third procedure **P3a** is outstanding. But beside of this positive feature, there are only $m$ agents involved in the process of determining $m$ cliques. Of course, the underlying probability process makes it equally likely for all agents to be one of these $m$ agents, but this justification is probably difficult to understand for agents on the real market, if they were not chosen and now have to work with their less preferred colleagues in a firm.

We can summarize, depending on whether a society judges welfare aspects or practicability aspects as more important, it chooses the appropriate procedure to get a set of matchings. And in more detail, if specific properties have to be satisfied, the procedure should be chosen according to these.

## 3.2 The Mechanisms and Strategic Behavior

In this section we focus on incentive compatibility introduced in Subsection 2.4.2. For this, we check whether telling the truth is always the best strategy. In advance, we can state that we only get one positive result. In the second subsection we try to relate properties of the solution with incentive compatibility to get a better understanding whether these properties are mutual exclusive or complementary.

### 3.2.1 Incentive Compatibility

To analyze incentive compatibility, we simply take Example A.0.1 (setting 1.) for **P1a** and Example A.0.2 (setting 1.) in Appendix A for **P2**. Again, we have chosen lexicographic preferences to facilitate further comparisons.

We discuss the mechanisms in the same order they were introduced. Checking incentive compatibility of **P1a** in Example A.0.1 (setting 1.) we see that agent $s_5$ can take advantage of the mechanism **M1a** in misrepresenting his preferences. If he announces a different ranking order in firm $h_2$, he is matched with firm $h_1$ instead of with firm $h_2$. Hence taking **M1a**, agents may have an incentive not to state their true preferences. Lying may finally lead to a higher rank than telling the truth. This is a typical example where incentive compatibility is not satisfied.

Taking **P2** (Example A.0.2 (setting 1.)) the analysis of the second mechanism **M2** is similar. Again we find an agent, this time agent $s_5$ who will not represent his true preferences and, therefore improves his rank. Hence, we have shown that neither **M1a** nor **M2** is incentive compatible. Therefore, we now focus on our last mechanism **M3a**. We can foreclose that this time we get a positive result.

**Theorem 3.2.1**

The mechanism **M3a** is incentive compatible, given a fixed order of agents.

*Proof:* Each agent whenever he is determined by the procedure **P3a** to be the one who can choose his colleagues, can select his most preferred bundle in his most preferred firm among those which are still on the market. So, there does not exist an incentive to misrepresent preferences. If he is not the one who chooses his colleagues, a modification in his preference ranking will have no influence on the cliques which will be selected by the procedure. So, again the agent cannot benefit from not telling the truth. Therefore, the procedure is incentive compatible. □

We may interpret incentive compatibility as a practicability property. A mechanism that can be manipulated by not stating the true preferences is, obviously less attractive, because agents can take advantage of lying. Consequently, from this point of view mechanism **M3a** is more attractive than **M1a** and **M2**.

### 3.2.2 Comparisons

Comparing the mechanisms, the question arises whether there exists a correlation between the lack of incentive compatibility and the properties of the solutions. We neglect Pareto efficiency, since it holds more or less for all solutions. The question of a correlation is answered easily, if we focus on $t_a$-stability, because the solutions of the second and the third procedures are both $t_a$-stable, but **M2** is not incentive compatible whereas **M3a** is. Hence, there cannot be a correlation between $t_a$-stability and incentive compatibility. It remains to take a closer look on the max-min properties.

Procedure **P1a** does not stop before the last complete graph, e.g. w.l.o.g. $C_m$ in firm $m$, belonging to a matching, is determined. Consequently, in other firms several complete graphs are already given. Let us assume that only one constellation $C_1, \ldots, C_{m-1}$ together with $C_m$ constitutes a matching and the constellation $C_1, \ldots, C_{m-1}$ is less preferred by all its members than another constellation of, during the procedure already constructed, cliques $C'_1, \ldots, C'_{m-1}$. This is exactly the

point, where lying becomes interesting. Agents can extend the number of rounds such that, by misrepresenting their preferences, they anticipate the matching, **P1a** would have selected, given the true preferences and at the same time they take advantage of, because they now will become members of more preferred cliques. Hence a matching, satisfying agents-max-min property and incentive compatibility is rather an exception. The same argumentation holds for the procedure **P2**, it only has to be reduced on cliques and the clique-max-min property.
In procedure **P3a** we leave the choice of cliques within a firm to only one agent. Definitely, the agent will choose his most preferred colleagues who are still on the market no matter how he is ranked by the colleagues. The selection of one agent who will determine his colleagues in a firm is the reason for incentive compatibility[5] and at the same time the reason for the lack of max-min properties. Finally, we can state, in general incentive compatibility and max-min properties are not satisfied simultaneously.

## 3.3 The Procedures versus the Solutions

All procedures are easy to apply on the job market for agents. In **P1a** all agents make their offers until a set of offers constitutes a set of matchings. In **P2** cliques are selected stepwise, again only if in a clique all agents made the corresponding offers. The procedure stops if all cliques together constitute a set of matchings. In **P3a** only one agent per firm determines a clique. This is done in all firms, therefore we also get a matching.
We observe in the first two procedures all agents are integrated in the selection process of a set of matchings whereas in **P3a** only $m$ agents are engaged in this process. Different perspectives lead to different interpretations of this phenomenon. One could state, we should favor **P3a** because it is the easiest way - given the three

[5]Therefore, see also the proof of Theorem 3.2.1.

procedures - to select a set of matchings. The agents are still treated equally, since the order of the agents and hence the ones who will select a clique are determined by an equal distribution. Nevertheless, some criticism arises immediately. Once the agent is chosen he acts like a dictator. No matter of the interests of the other agents who applied for this firm, he chooses his most preferred clique. After being matched, the other agents may be dissatisfied because now they probably have to work with less preferred colleagues in this firm.
This aspect is taken into account by the first and the second procedure and is reflected in welfare aspects of the solution, namely the max-min properties. These procedures take all agents' preferences into account. Thereby, **P2** is more competitive in the sense that first cliques are determined in those firms, in which there are more applicants are given than jobs available. Both procedures only stop if in **P1a** all agents made their offers in a matching or in **P2** all agents in all firms made their offers in different cliques. Hence, both procedures put more weight on all agents' interests than **P3** does.

If we now face the solutions versus the procedures, it turns out that the solution of **P1a** is not $t_a$-stable, but satisfies the agent-max-min property. A bit "more competition" leads in **P2** to a $t_a$-stable matching and still the clique-max-min property on reduced job markets for agents can be guaranteed. **P3a** guarantees $t_a$-stability and uniqueness. One could argue, we should favor **P2** and **P3a** according to the solutions, since $t_a$-stability is a crucial property as well as Pareto efficiency (only a subset of the solution of **P1a**). But choosing between these two procedures depends on whether the society attaches importance to either some sort of clique-max-min property or uniqueness. Instead of that if we take the fact into account that the mechanism **M3a** is incentive compatible, it becomes easier to justify that we favor **P3a**.

# Chapter 4

# The Basic Model and the Firms

Until now, we only focused on the agents' preferences to get a better understanding of their preferences over firms and colleagues. But nevertheless, the assumption that firms do not care about whom to hire, is fairly unrealistic. Therefore, we want to take the firms' preferences into account in this chapter.
We discuss two different approaches to model the firms' preferences. Both of them do not appear in the literature, yet. The first one is basically an extension of the model in Chapter 2. In the past, agents had preferences over bundles consisting of one firm and a subset of colleagues. Each agent was assigned to a firm together with colleagues. Now we extend the model. Roughly speaking, we embed the firms in the basic model of Chapter 2, as if they were agents. We call the union of agents and firms *participants*. Each participant is assigned to a firm together with other participants. Two obvious motivations, a technical and an economic one, cause this extension. First, Chapter 2 and Chapter 3 provide everything to incorporate the firms in such a manner, it simply seems to be a natural next step. And second, it is common fact that firms are working for other firms. A well known example is the car industry. Several firms are employed by a firm to deliver different parts of a car. And the employing firm employs agents and itself as well to finalize the

car. Since we want to stay as close as possible to our first model where a clique of agents worked for one firm we restrict the extension to a model where the only firm employed by a firm is itself. The remaining participants working for the firm are agents. After having adjusted the terminology, we are again able to apply suitable modifications of the procedures **P1a, P2, P3a**. In fact, we will modify **P3a** slightly. This new version will be called **P3b**. We then check which properties are satisfied and we also discuss incentive compatibility. The second last subsection compares the results given in Chapter 3 and Chapter 4 by means of examples.
The second approach is a contraposition to the model in Chapter 2. It may be seen as a more classical approach due to other many-to-one sided matchings. We have agents on the one side and firms on the other. Here, we only focus on the firms' preferences over groups of agents they want to hire. No further restrictions are made on the preferences. We introduce another notion of stability, which we will call $t_f$*-stability* and two new procedures. Again we discuss the outcomes of the procedures. Within this section we often face the results with those of the former chapters to get a better understanding of the different structures of the models.

## 4.1 Participants

We want to integrate the firms in the agents' settings. Therefore, we enlarge the set of agents by the firms, such that we get a set of *participants*. So, on the one side we now have the participants and on the other side there are the firms they have to be matched with, no matter the fact that firms are present on both sides now. We adjust the terminology of cliques, number of job offerings and preferences. In this extension of the model given in Chapter 2, we want to stay close to our original question, how to model a job market, if agents do not only care about their future colleagues. A lack of description in Chapter 2 clearly is the fact that firms' preferences are not taken into account at all. Therefore, in this section we want to

drop this assumption. For not changing to many issues at the same time, we restrict the participants' set of preferences to *admissible preferences*, we want to assume, a firm can only be matched with itself and the remaining open positions have to be filled exclusively with agents.[1] So, for the first time we are able to analyze the agents' as well as the firms' interests.
The example of the car industry is a further extension where more than one firm can be employed by another firm. We will not further discuss this kind of job market here.

### 4.1.1 The Framework

First, we introduce the slightly different notation of the model. The order of the presentation is the same as in Chapter 2. The notation, introduced there, remains valid and will also be used in this section. Let

- $P = S \cup H$ be the set of participants with $p, p' \in P$ representing participants who could either be an agent or a firm.
- $\tilde{\kappa}_k = \kappa_k + 1$ be the number of offered jobs per firm $k \in H$. Adding $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_m$, we get the total number of jobs, $\sum_{k=1}^{m} \tilde{\kappa}_k = \tilde{\kappa} = m + n$.
- $\mathbf{D}_{q+1} = \{D \subseteq P \mid |D| = q + 1\}$ with $0 \leq q + 1 \leq |P|$ be the set of subsets with $q + 1$ participants. Each $D \in \mathbf{D}_{q+1}$ is called a $q + 1$-*clique*.
- $\mathbf{D}_q^{-p} = \{D \subseteq P \mid |D| = q, p \notin D\}$ be the set of subsets with $q$ *colleagues* (for participant $p$).[2]

[1]For sure this restriction may be easily released, since it is quite common in a century of outsourcing that firms are employed by other firms. What kind of impact this would have on the model, is indeed a more challenging question which we will not investigate in this thesis.

[2]In this section, colleagues include agents as well as firms.

- $\cup_{k=1}^{m}\{(k,D) \mid D \in \mathbf{D}_{\kappa_k}^{-p}\} =: B_p^P$ be the set of all *bundles* of participant $p$ with $b_p \in B_p^P$.

- Every participant $p$ has a *strict preference relation* $\succ_p^P$ on $B_p^P$. $\succ_p^P$ is a linear ordering. We write $\succ^P = (\succ_1^P, \ldots, \succ_{m+n}^P)$ as *preference profile*, $\mathcal{P}_p^P$ denotes the *set of all possible preferences of participant* $p$, for all $p \in P$. And we denote the *set of all possible preference profiles* with $\mathcal{P}^P = \mathcal{P}_1^P \times \ldots \times \mathcal{P}_{m+n}^P$.

**Remark 4.1.1**
$\mathbf{D}_{q+1}$ is the analogon to $\mathbf{C}_q$ of Chapter 2. $\mathbf{D}_q^{-p}$ is the analogon to $\mathbf{C}_{q-1}^{-i}$. But the subsets now consist of agents and firms and in both sets the subsets consist of one additional element. An element of $D \in \mathbf{D}_q^{-p}$ has the same number of elements as an element $C \in \mathbf{C}_q$, and $C \in \mathbf{D}_q^{-p}$ but not necessarily $D \in \mathbf{C}_q$.

Analogous to Chapter 2, if $q+1 = \tilde{\kappa}_k = \tilde{\kappa}_l$ holds, then the $q+1$-clique can work for firm $k$ as well as for firm $l$. Nevertheless, we observe several differences compared to the former model. First, the possible $q+1$-cliques consist of participants, hence of agents and firms. Second, the participants now have preferences over bundles $(k, D)$, whereas $D \in \mathbf{D}_q^{-p}$. Every participant has preferences over a bundle consisting of one firm, and a subset of agents and firms, namely $\kappa_k$ colleagues. Furthermore, the firms now have preferences, too. Another difference is the number of jobs on the market. Each subset $D \in \mathbf{D}_q^{-p}$ consists of $q$ colleagues (= participants). In the first model the agents have preferences over $q-1$ colleagues (= agents). Consequently, each firm has one more vacant job. Therefore, we now have $m+n$ jobs on the market.

To summarize the components we have established until now, we describe our *job market (for participants)* by the tuple

$$\Upsilon_P = (H, P, (\tilde{\kappa}_k)_{k\in H}, \succ^P).$$

This is the most general form of the model. The structure of the job market for participants $\Upsilon_P$ is the same as for agents, $\Upsilon_P = \Upsilon_{S\cup H}$. No restrictions are given for the ratio between firms and agents in a $q+1$-clique, yet. But, as already mentioned in the introduction of this section, we are more interested in $q+1$-cliques with one firm $k$ and $\kappa_k$ agents. Especially, we want to focus on the case, where $\kappa_k$ agents are working for firm $k$ and firm $k$ is employed by itself. Thus, we face the same structure as in Chapter 2, except that we integrate each firm as an employee in its own firm.

**Definition 4.1.2 (Admissible)**

1. A bundle $(l, D) \in B_k^P$ consisting of a firm $l$ and of $\kappa_l$ colleagues is called *admissible for firm* $k$, if $l = k$ and $D \in \mathbf{C}_{\kappa_k}$ holds.
2. A bundle $(l, D) \in B_i^P$ consisting of a firm $l$ and of $\kappa_l$ colleagues is called *admissible for agent* $i$, if $D \in \mathbf{D}_{\kappa_l}^{-i}$ and $D \cap H = \{l\}$ holds.

A bundle $(k, D)$ is admissible for a firm $k \in P$, if $D$ only consists of agents. Each firm $k \in P$ will have preferences over $\kappa_k$ colleagues only consisting of agents. But in fact, each firm offers $\tilde{\kappa}_k = \kappa_k + 1$ jobs. It counts itself as well as an employee.[3]
A bundle $(k, D)$ is admissible for an agent $i \in P$, if $D$ contains exactly one firm, more precisely the one for which the $\kappa_k$ colleagues including agent $i$ will work, and as a direct consequence it consists of $\kappa_k - 1$ agents. Hence, an agent will only work for one firm together with the firm.
This clearly technical definition is necessary to remain in the context of the former model (Chapter 2). We still want to match agents with firms, including the firms preferences, now. If one is looking for an interpretation of Definition 4.1.2, we can think of firms who do not want to hire other firms except themselves and agents who only want to work for one firm.

[3]This is analogous to Chapter 2. There, each agent only has preferences over his $\kappa_k - 1$ colleagues in a firm, since he applies for the $\kappa_k$th job.

**Remark 4.1.3**

1. For each $k \in H$ the number of admissible bundles is $\binom{n}{\kappa_k}$.

2. For each $i \in S$ the number of admissible bundles is $\sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$.

3. For each $p \in P$ the number of all bundles is $|B_p^P| = \sum_{k=1}^{m} \binom{m+n-1}{\kappa_k}$.

The number of admissible bundles for firms differ, if they have different $\kappa_k$'s, i.e. if they offer a different number of jobs. The number of admissible bundles is for all agents the same, because the number of possible combinations of colleagues per firm is for all agents the same as well as the number of firms. The different number of possible admissible bundles for firms and for agents results from the different structure of the set of admissible bundles. A firm $k$ focuses on all possible combinations of $\kappa_k$ agents out of the set of all agents $n$. However, an agent focuses on all possible combinations of $\kappa_k - 1$ agents out of the set of $n-1$ agents, given each firm $k$. Either the number of admissible bundles for agents is greater or equal to the number of admissible bundles for firms or vice versa, both scenarios may appear.[4] Now, we come to a subclass of preferences introduced in this section.

**Definition 4.1.4 (Admissible Preferences)**
A participants' $p \in P$ strict preference relation $\succ_p^P$ over bundles is called *admissible*, if for all admissible bundles $(k, D)$ and non-admissible bundles $(k', D')$ the property $(k, D) \succ_p^P (k', D')$ holds. We denote the set of all admissible preference profiles with $\mathcal{P}^{adm} = \mathcal{P}_1^{adm} \times \ldots \times \mathcal{P}_m^{adm}$.

Now, before we define an admissible matching, we have to describe the corresponding set $\mathbf{D}^k$ of *admissible* $\tilde{\kappa}_k$*-cliques*. Each element of this set consists of $\kappa_k$ agents and of firm $k$. Hence, we have

$$\mathbf{D}^k = \{D \subseteq P \mid |D| = \tilde{\kappa}_k,\ D \cap H = \{k\}\}.$$

[4] This will be proved in Lemma 4.1.9.

**Definition 4.1.5 ((Admissible) Matching)**

A mapping $\tilde{\mu} : P \to H$ is called an *(admissible) matching* (for the job market $\Upsilon_P$), if for all $k \in H$ we have $\tilde{\mu}^{-1}(k) \in \mathbf{D}^k$. Therefore, we define $D_{\tilde{\mu},k} := \tilde{\mu}^{-1}(k)$ as the $\tilde{\kappa}_k$-*clique under* $\tilde{\mu}$ for each firm $k \in H$. An alternative description of the (admissible) matching $\tilde{\mu}$ is $M_{\tilde{\mu}} := (D_{\tilde{\mu},1}, \ldots, D_{\tilde{\mu},m})$. All (admissible) matchings are given by $\tilde{\mathcal{M}} := \{M_{\tilde{\mu}} \mid \tilde{\mu} \text{ is an (admissible) matching}\}$.

This is the definition of a matching introduced in Chapter 2 applied to the extended model. If we have cliques $(D_1, \ldots, D_m)$ with the properties

$$D_k \in \mathbf{D}^k \qquad \forall\ k \in H, \tag{4.1}$$

$$D_k \cap D_l = \emptyset \qquad \forall\ k, l \in H, \tag{4.2}$$

then this also describes an admissible matching $\tilde{\mu}$ (which is implicitly given by $D_k = \tilde{\mu}^{-1}(k)$).

**Definition 4.1.6 (Participants' Ranking Function)**

A mapping $v_p^P : B_p^P \to \{1, \ldots, |B_p^P|\}$ with $v_p^P(b_p) > v_p^P(b'_p)$, if and only if $b_p \succ_p^P b'_p$ is a *ranking function* for each $p \in P$. $v_p^P(b_p)$ is called the *rank of* $b_p$.

Analogous to Chapter 2, we get an ordering such that $b_p^{(1)}$ is the most preferred bundle of participant $p \in P$, $b_p^{(2)}$ the second most and finally $b_p^{(|B_p^P|)}$ his least preferred bundle. Again our ranking functions are assigning downward counting. $|B_p^P|$ is the rank of $p$'s most preferred bundle $b_p^{(1)}$, $|B_p^P| - 1$ the rank of his second most preferred bundle $b_p^{(2)}$ and finally 1 is the rank of his least preferred bundle $b_p^{(|B_p^P|)}$.
This is the general description. Even if we still can distinguish between agents and firms in the set of participants, in the new model they are treated in the same manner. The basic structure stays the same, if we focus on admissible matchings.

In the next subsection we want to modify the procedures introduced in Chapter 2 and we again want to discuss their solutions. Therefore, we first briefly revise the properties introduced in Subsection 2.2.1.

Because the structure of a job market for agents and for participants is the same, the definitions for Pareto efficiency and the max-min properties in $\Upsilon_P$ (we call the agent-max-min property now *participant-max-min property*) also remain the same as in $\Upsilon_S$, see Subsection 2.2.1. The same holds for $t_a$-stability whereas in particular the firm never $t_a$-block, because in admissible $\tilde{\kappa}_k$-cliques $D \in \mathbf{D}^k$ a firm always employs itself and at the same time is his most preferred "employer". Furthermore, if a firm is part of a $t_a$-blocking pair, this leads to a non-admissible matching.

### 4.1.2 P1a, P2, P3b and their Outcomes

After having established the new framework, we will now investigate the results we get if the participants announce their preferences and we apply our procedures **P1a, P2, P3b**. The difference between **P3a** and **P3b** will also be introduced in this subsection. Comparisons and interpretations are given in the next subsection. At the moment, we focus on the procedures and their outcomes. We always assume admissible preferences. We keep the order and start with **P1a**.

**P1a** is applied in the same way as before. We construct the same structure of graphs in each firm. The only difference now is that firms also make offers to the agents and the agents do not only ask other agents but also firms for joint work. Thus, we roughly have to exchange the word "agents" by "participants" in the description of **P1a**.[5] Furthermore, each directed graph in each firm consists of one more vertex. The procedure stops, if the equations (4.1) and (4.2) are satisfied. Clearly, we get a theorem analogous to Theorem 3.1.1 for the solution correspondence $g^{\mathbf{P1a}}$.

[5]The same has to be done for the other procedures, discussed in this subsection.

**Theorem 4.1.7**

For any admissible preference profile, $\succ^P \in \mathcal{P}^{adm}$, the solution of $g^{\mathbf{P1a}}(\succ^P)$

1. coincides with the set of matchings that satisfy the participant-max-min property and

2. contains at least one Pareto efficient matching.

The basic structure of the participants' preferences is the same with respect to the fact that the number of participants in each clique increased by one. Therefore, the proof of Theorem 4.1.7 is the same as the one of Theorem 3.1.1. The solution always includes a Pareto efficient matching. It also satisfies the participant-max-min property, because in all other matchings the agent with the fewest rank will get again less compared to the one given by the procedure. However it is not clear whether **P1a** produces an admissible matching. Consequently, before we start to discuss the next procedure, we want to show when a solution of **P1a** only consisting of admissible matchings can be guaranteed.

**Theorem 4.1.8**

Given an admissible preference profile $\succ^P \in \mathcal{P}^{adm}$ and let $\max_{k \in H} \binom{n}{\kappa_k} \leq \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$, then the solution $g^{\mathbf{P1a}}(\succ^P)$ is a set of admissible matchings.

*Proof:* After $\bar{r} = |\sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}|$ rounds all offers consisting of admissible bundles are made by all participants. The proof is divided into two parts. 1.) **P1a** generated a set of matchings before all participants made their last offer consisting of an admissible bundle. 2.) **P1a** did not generate a set of matchings before all participants made their last offer consisting of an admissible bundle. Then all participants make the last offer. Consequently, all cliques consisting of one firm and $\kappa_k$ agents (admissible $\tilde{\kappa}_k$-cliques) in all firms are now complete directed graphs. Among others **P1a** generated the admissible matchings $(D_{\tilde{\mu},1}, \ldots, D_{\tilde{\mu},m})$ with $D_{\tilde{\mu},1} = \{s_1, \ldots, s_{\kappa_1}, h_1\}$, $D_{\tilde{\mu},2} =$

$\{s_{\kappa_1+1},\ \ldots,\ s_{\kappa_1+\kappa_2}, h_2\}, \ldots,\ D_{\tilde{\mu},m} = \{s_{\sum_{l=1}^{m-1}\kappa_l+1}, \ldots, s_{\sum_{l=1}^{m-1}\kappa_l+\kappa_m}, h_m\}$. Therefore, the procedure **P1a** stops and generated a set of matchings at least consisting of $M = (\{s_1, \ldots, s_{\kappa_1}, h_1\}, \ldots,\ \{s_{\sum_{l=1}^{m-1}\kappa_l+1}, \ldots, s_{\sum_{l=1}^{m-1}\kappa_1+\kappa_m}, h_m\})$. □

Of course it immediately arises the natural question, under which circumstances the unequation $\max_{k\in H}\binom{n}{\kappa_k} \leq \sum_{k=1}^{m}\binom{n-1}{\kappa_k-1}$ is satisfied. The answer is given by the next Lemma.

**Lemma 4.1.9**

1. If $m = 2$, then $\max_{k\in H}\binom{n}{\kappa_k} = \sum_{k=1}^{m}\binom{n-1}{\kappa_k-1}$.
2. If $m > 2$, $\kappa_k = \kappa_l =: \bar{\kappa}$ for all $k, l \in H$, then $\max_{k\in H}\binom{n}{\kappa_k} = \sum_{k=1}^{m}\binom{n-1}{\kappa_k-1}$.
3. If $m > 2$, $\kappa_k = \kappa_l =: \bar{\kappa}$ for all $k, l \in H \setminus \{k'\}$ with $\kappa_{k'} =: \hat{\kappa}$, $\bar{\kappa} \neq \hat{\kappa}$, then $\max_{k\in H}\binom{n}{\kappa_k} > \sum_{k=1}^{m}\binom{n-1}{\kappa_k-1}$.

*Proof:* 1. W.l.o.g. we let $\max_{k=1,2}(\binom{n}{\kappa_1}\binom{n}{\kappa_2}) = \binom{n}{\kappa_1}$ for the left side of the equation. then the right side is given by

$$\binom{n-1}{\kappa_1-1} + \binom{n-1}{\kappa_2-1} = \binom{n-1}{\kappa_1-1} + \binom{n-1}{n-\kappa_1-1}. \tag{4.3}$$

It is well known (here applied to the left side) that

$$\binom{n}{\kappa_1} = \binom{n-1}{\kappa_1} + \binom{n-1}{\kappa_1-1}. \tag{4.4}$$

Equating (4.3) with (4.4), we have to show

$$\begin{aligned}
\binom{n-1}{\kappa_1} + \binom{n-1}{\kappa_1-1} &= \binom{n-1}{\kappa_1-1} + \binom{n-1}{n-\kappa_1-1} \\
\Leftrightarrow \quad \binom{n-1}{\kappa_1} &= \binom{n-1}{n-\kappa_1-1} \\
\Leftrightarrow \quad \frac{(n-1)!}{(n-1-\kappa_1)!\kappa_1!} &= \frac{(n-1)!}{((n-\kappa_1-1)-(n-1))!(n-\kappa_1-1)!} \\
\Leftrightarrow \quad \frac{(n-1)!}{(n-1-\kappa_1)!\kappa_1!} &= \frac{(n-1)!}{\kappa_1!(n-\kappa_1-1)!}.
\end{aligned}$$

2. We have $\max_{k\in H}\binom{n}{\kappa_k} = \binom{n}{\bar{\kappa}}$ for the left side of the equation. Since $\kappa_k = \kappa_l =: \bar{\kappa}$ for all $k, l \in H$, it follows for the right side of the equation that,

$$\sum_{k=1}^{m}\binom{n-1}{\kappa_k-1} = m\binom{n-1}{\bar{\kappa}-1}.$$

Hence, we have to show

$$\binom{n}{\bar{\kappa}} = m\binom{n-1}{\bar{\kappa}-1}. \tag{4.5}$$

Since the number of vacant jobs in all firms equals the number of participants on the market, we have $m(\bar{\kappa}+1) = m+n \Leftrightarrow m\bar{\kappa} = n \Leftrightarrow m = \frac{n}{\bar{\kappa}}$. It follows by inserting in equation (4.5)

$$\begin{aligned}\binom{n}{\bar{\kappa}} &= \frac{n}{\bar{\kappa}}\binom{n-1}{\bar{\kappa}-1}\\ &= \frac{n}{\bar{\kappa}}\frac{(n-1)!}{((n-1)-(\bar{\kappa}-1))!(\bar{\kappa}-1)!}\\ &= \frac{n!}{(n-\bar{\kappa})!\bar{\kappa}!}.\end{aligned}$$

3. Since $\kappa_k = \kappa_l =: \bar{\kappa}$ for all $k, l \in H \setminus \{k'\}$ with $\kappa_{k'} =: \hat{\kappa}$, $\bar{\kappa} \neq \hat{\kappa}$, and the number of vacant jobs equals the number of participants, we have

$$\begin{aligned}(m-1)(\bar{\kappa}+1)+(\hat{\kappa}+1) &= m+n\\ \Leftrightarrow\ m\bar{\kappa}+m-\bar{\kappa}-1+\hat{\kappa}+1 &= m+n\\ \Leftrightarrow\ (m-1)\bar{\kappa}+\hat{\kappa} &= n\\ \Leftrightarrow\ m &= \frac{n-\hat{\kappa}}{\bar{\kappa}}+1.\end{aligned}$$

For the right side of the unequation, which we want to prove, we can write

$$\sum_{k=1}^{m}\binom{n-1}{\kappa_k-1} = (m-1)\binom{n-1}{\bar{\kappa}-1} + \binom{n-1}{\hat{\kappa}-1}.$$

Now we have to distinguish between two cases.

*Case 1.* $\binom{n}{\bar\kappa} < \binom{n}{\hat\kappa}$, then $\max_{k\in H}\binom{n}{\kappa_k} = \binom{n}{\hat\kappa}$ and we have to show

$$
\begin{aligned}
& & \binom{n}{\hat\kappa} &> (m-1)\binom{n-1}{\bar\kappa-1} + \binom{n-1}{\hat\kappa-1} \\
& & &> \frac{n-\hat\kappa}{\bar\kappa}\frac{(n-1)!}{(n-\bar\kappa)!(\bar\kappa-1)!}\frac{n}{n} + \binom{n-1}{\hat\kappa-1} \\
& & &> \frac{n-\hat\kappa}{n}\binom{n}{\bar\kappa} + \binom{n-1}{\hat\kappa-1} \\
\Leftrightarrow & & \frac{n}{n-\hat\kappa}\binom{n}{\hat\kappa} &> \binom{n}{\bar\kappa} + \frac{(n-1)!}{(n-\hat\kappa)!(\hat\kappa-1)!}\frac{n}{n-\hat\kappa}\frac{\hat\kappa}{\hat\kappa} \\
& & &> \binom{n}{\bar\kappa} + \binom{n}{\hat\kappa}\frac{\hat\kappa}{n-\hat\kappa} \\
\Leftrightarrow & & \frac{n-\hat\kappa}{n-\hat\kappa}\binom{n}{\hat\kappa} &> \binom{n}{\bar\kappa} \\
\Leftrightarrow & & \binom{n}{\hat\kappa} &> \binom{n}{\bar\kappa}.
\end{aligned}
$$

*Case 2.* $\binom{n}{\bar\kappa} > \binom{n}{\hat\kappa}$, then $\max_{k\in H}\binom{n}{\kappa_k} = \binom{n}{\bar\kappa}$ and we have to show

$$
\begin{aligned}
& & \binom{n}{\bar\kappa} &> (m-1)\binom{n-1}{\bar\kappa-1} + \binom{n-1}{\hat\kappa-1} \\
& & \binom{n}{\bar\kappa} &> \frac{n-\hat\kappa}{n}\binom{n}{\bar\kappa} + \binom{n-1}{\hat\kappa-1} \\
\Leftrightarrow & & \frac{n}{n-\hat\kappa}\binom{n}{\bar\kappa} &> \binom{n}{\bar\kappa} + \binom{n}{\hat\kappa}\frac{\hat\kappa}{n-\hat\kappa} \\
\Leftrightarrow & & \frac{n-(n-\hat\kappa)}{n-\hat\kappa}\binom{n}{\bar\kappa} &> \binom{n}{\hat\kappa}\frac{\hat\kappa}{n-\hat\kappa} \\
\Leftrightarrow & & \binom{n}{\bar\kappa} &> \binom{n}{\hat\kappa}.
\end{aligned}
$$

□

**Corollary 4.1.10**

From the proof of 3. in Lemma 4.1.9 it follows that given $m > 2$ and $\kappa_k \neq \kappa_l$ for all $k, l \in H$, the inequality $\max_{k \in H} \binom{n}{\kappa_k} > \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$ always holds.

*Proof:* Given the assumptions that $m > 2$ and $\kappa_k \neq \kappa_l$ for all $k, l \in H$, it follows immediately that that $\binom{n}{\kappa_k} \neq \binom{n}{\kappa_l}$ for all $k, l \in H$, because $\binom{n}{\kappa_k} = \binom{n}{\kappa_l}$ holds, if and only if $\kappa_k = \kappa_l$ or $\kappa_l = n - \kappa_k$ is satisfied.
This proof refers to *Case 1.* of part 3. in Lemma 4.1.9.
We assume that $\max_{k \in H} \binom{n}{\kappa_k} = \binom{n}{\kappa_l} =: \binom{n}{\hat{\kappa}}$ and $\binom{n}{\hat{\kappa}} > \binom{n}{\kappa_k}$ for all $k \in H \setminus \{l\}$. Because of the proof of part 3. (*Case 1.*) in Lemma 4.1.9, it follows that $\max_{k \in H} \binom{n}{\kappa_k} > \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$. □

Reflecting the last results, we observe for procedure **P1a** that the results from Chapter 2 cannot be taken over right away. Only if $\max_{k \in H} \binom{n}{\kappa_k} \leq \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$ is satisfied, we have a sufficient condition such that **P1a** always yields a set of admissible matchings. At the same time, it is not a necessary condition. Hence, if $\max_{k \in H} \binom{n}{\kappa_k} > \sum_{k=1}^{m} \binom{n-1}{\kappa_k - 1}$, the solution $g^{\mathbf{P1a}}(\succ^P)$ may still only consist of admissible matchings. Nevertheless, the requirement reduces the quality of the solution. And one may ask whether it is appropriate to apply **P1a** at all for the given framework. It is also unclear what happens if **P1a** selects a set of matchings consisting of non-admissible matchings or a set of matchings consisting of non-admissible as well as of admissible matchings. In the thesis we simply exclude these cases. We already mentioned that we only want to focus on a particular model. If this model is extended in such a way that firms are employed by other firms and more than one firm can work for another one, the restriction that only admissible matchings are allowed, vanishes. But this research direction is not part of our work. Finally, coming back to the framework, given in this chapter, we can foreclose that the next procedures **P2** and **P3a** will always yield a set of admissible matchings.

In the following, we again assume some sort of lexicographic preferences[6]. We restrict the class of preferences on *lexicographic admissible preferences.*

**Definition 4.1.11 (Lexicographic Admissible Preferences)**
Participant $p$'s admissible preferences are called *lexicographic admissible*, if there is a strict ordering $H(\succ_p^P)$ on $H$ such that for all $(k, D) \in H \times \mathbf{D}_{\kappa_k}^{-p}$, $(l, D') \in H \times \mathbf{D}_{\kappa_l}^{-p}$ the following condition is satisfied:

$$(k, D) \succ_p^P (l, D') \Leftrightarrow k \, H(\succ_p^P) \, l \qquad (k \neq l).$$

$\mathcal{P}_p^{lexad}$ is the *set of all lexicographic admissible preference profiles of participant* $p$ for all $p \in P$. We denote the *set of all lexicographic admissible preference profiles* with $\mathcal{P}^{lexad} = \mathcal{P}_1^{lexad} \times \ldots \times \mathcal{P}_{m+n}^{lexad}$. If $p$ is an agent the condition is identical to Definition 2.2.9 of lexicographic preferences in Chapter 2.
If $p \in P$ is a firm one could interprete the condition such that the firm provided that a firm or a manager/owner of the firm always mostly prefers to work in his own firm.

Before we start to study the solution of the procedures we have to adjust Lemma 2.3.9. The content remains the same, but the subclass of preferences is different and we now have to consider participants instead of agents.

**Lemma 4.1.12**
Let $\{p \in P \mid \; k \, H(\succ_p^P) \, l, \; \forall \, l \in H \setminus \{k\}\} =: \tilde{\mathcal{S}}_k$ be the set of participants who mostly prefer firm $k$. For any lexicographic admissible preference profile $\succ_p^P \in \mathcal{P}^{lexad}$ there always exists at least one firm $k$, for which $|\tilde{\mathcal{S}}_k| \geq \tilde{\kappa}_k$.

Each firm always mostly prefers to work for itself. Focusing on the agents only, the proof is exactly the same as the one for Lemma 2.3.9. Now, we are ready to

[6] Definition 2.2.9 introduces lexicographic preferences.

concentrate on the results in the new context. It turns out that most of the results from Chapter 3 can easily be carried over.

**Theorem 4.1.13**
For any lexicographic admissible preference profile $\succ^P \in \mathcal{P}^{lexad}$ each set of cliques of any matching in the solution $g^{\mathbf{P2}}(\succ^P)$, determined on $\Upsilon_P^t$ with $t = 0, 1, 2, \ldots$, satisfies the clique-max-min property on $\Upsilon_P^t$.

**Theorem 4.1.14**
For any lexicographic preference profile $\succ^P \in \mathcal{P}^{lexad}$ any matching in the solution $g^{\mathbf{P2}}(\succ^P)$ is

1. Pareto efficient and

2. $t_a$-stable.

The proof is the same as in Theorem 4.1.14. Again squeezing the firm in the agents' model framework does not have any impact on the results.
Again we have to take care of the fact that the procedure should yield admissible matchings. To remind the reader of procedure **P2**, selected firms and agents make their offers until a set of cliques is reached. They leave the market, and again on the reduced job market (for participants) selected participants make their offers. This goes on, until no participants remain on the market.

**Theorem 4.1.15**
For any $\succ^P \in \mathcal{P}^{lexad}$ the solution $g^{\mathbf{P2}}(\succ^P)$ only consists of admissible matchings.

*Proof:* Each selected set of firms and agents make their offers until a set of admissible cliques[7] is determined. Admissible cliques leave the job market. The remaining

[7]An admissible matching consists of admissible cliques.

firms' and agents' most preferred bundles are admissible. Again a set of firms and agents is selected. Offers are made until a set of admissible cliques is reached. The corresponding participants leave the job market. This goes on until an admissible matching is determined. □

Next, we discuss the third procedure. As already mentioned, it will be slightly modified. We change some steps of the former procedure **P3a**. Basically, in **P3a** we first choose an order of agents with equal distribution. The agent with the lowest index then chooses his colleagues among the other selected agents in a firm. One motivation to choose a random order was to treat the agents equally, another was to get a rule how to determine the agent who will choose his colleagues. Now, we are in a much better situation. We always have a participant in the selected set of participants who seems to be a natural candidate to choose his colleagues. Obviously, it is the firm itself. Therefore, we drop the random selection of an order of the agents, hence *step 2* of the former **P3a**. *Step 3* does not change. In the modified procedure the validity of Lemma 2.3.9 is checked for each firm and the corresponding firms and agents are selected. In *step 4* not the agent with the lowest index, but the selected firm now chooses its most preferred $\kappa_k$ colleagues (employees at the same time) among those who where also selected for the same firm. The $\kappa_k$ agents together with the firm leave the market. We get a reduced job market for participants. The procedure goes back to *step 2* (*step 3* in the former **P3a**). And finally, it stops when no more participants are on the job market. We will call the modified version of **P3a** in the following **P3b**.

**Theorem 4.1.16**

For any $\succ^P \in \mathcal{P}^{lexad}$, the solution $g^{\mathbf{P3b}}(\succ^P)$ is

1. Pareto efficient and

2. $t_a$-stable.

The proof is the same as the one for Theorem 3.1.6. Theorem 4.1.16 is an appealing result. The firms now choose their future employees among those who applied for the jobs. Moreover, all properties of **P3a** remain valid for the solution of **P3b**. We get a Pareto efficient matching and no two participants want to exchange their colleagues and their firms.
It remains to verify whether the solution of **P3b** only consists of admissible matchings. Without the firms' preferences we always get an admissible matching[8]. Integrating the firms' preferences in **P3a** does not change this result, because of Lemma 4.1.9. At latest, if all agents make their last offer, consisting of an admissible bundle, an admissible clique is determined.

### 4.1.3 Comparisons

In this subsection we pause for some comparisons between the results of this chapter and of the former Chapter 3. Therefore, we consider two simple examples. Both are described at length in Appendix A. Example A.0.4 is exclusively given for procedure **P1a**. No lexicographic preferences are assumed. Example A.0.5 will be used to apply the second procedure.

Example A.0.4 discusses the framework given in Chapter 2 and in Section 4.1. In the first setting, the firms do not have any preferences. In the second, we include the firms' preferences. Here, the agents' preferences are extended by the firms and the firms are added. Now, applying **P1a** to the framework of Chapter 2 (Example A.0.4 (setting 1.)) leads to the solution $M_1$. If we include the firms' preferences (Example A.0.4 (setting 2.)), the procedure **P1a** delivers $M_2$. A graph theoretical illustration of both scenarios is given in Example A.0.4.
Comparing the solutions, in $M_2$ agent $s_3$ is matched with a more preferred firm and colleague than in $M_1$. At the same time the opposite is true for agent $s_4$. Intuitively,

[8]We do not call it admissible in Chapter 2.

one may have expected, by ignoring the firms' preferences, that the matching, selected by **P1a**, were strictly preferred by all agents. But this does not - as we have shown in the Example A.0.4 - have to be the case. The reason for that lies in the procedure itself. Each complete graph of the matching has to be constructed before **P1a** stops. In setting 1. of Example A.0.4 the procedure **P1a** stops after all agents having made four offers. If we include the firms' preferences each complete graph of a firm increases by one vertex, namely the firm itself. Now after four offers of all participants, only the admissible cliques $\{s_3, h_1\}$ and $\{s_1, s_2, h_2\}$ form complete graphs. In clique $\{s_4, s_5, h_3\}$ the firm $h_3$ did not make its offer, yet. Hence, we do not have a matching. After six offers the admissible matching $M_2$ occurs such that the procedure stops. The matching $M_1$ is still not completed, because $\{s_4, s_5\}$ is only firm $h_3$'s seventh most preferred bundle. The matching $M_2$ occurs before. Thus we have seen, integrating the firms' preferences on the agents' side may lead to a completely different solution. Nevertheless, both solutions contain at least one Pareto efficient (admissible) matching and satisfy the agent-max-min property. Thus, the welfare aspects remain identical.

We introduce a second example (A Example A.0.5 ) and proceed in the same manner. In the first setting, the firms' preferences are excluded, it refers to Chapter 2 and Chapter 3. In the second, they are added. All agents and then participants have lexicographic (admissible) preferences. With the aid of this example, we can compare the results of procedures **P2**, applied to the different preference settings. Again graph theoretical illustrations are given at length in the example, too.

Applying procedure **P2** to Example A.0.5 (setting 1.), we get a unique solution $M^1 = (\{s_1, s_4\}, \{s_2, s_5\}, \{s_3\})$. First the clique $\{s_1, s_4\}$ is selected in firm $h_1$. Next, the other two cliques are selected simultaneously.

Next, we have to apply the same procedure to participants. For this, we take the extended version of the preferences given in Example A.0.5 setting 2. Firms are no longer indifferent. We get a solution consisting of three admissible matchings, $M^1 = (\{s_1, s_4, h_1\}, \{s_2, s_5, h_2\}, \{s_3, h_3\})$ (which is the same as above), $M^2 =$

$(\{s_1, s_3, h_1\}, \{s_4, s_5, h_2\}, \{s_2, h_3\})$ and $M^3 = (\{s_1, s_3, h_1\}, \{s_2, s_5, h_2\}, \{s_4, h_3\})$. Both solutions (with and without the firms' preferences) satisfy Pareto efficiency and each clique in the reduced job market for agents $\Upsilon_S^t$ (for participants $\Upsilon_P^t$), $t = 1, 2, \ldots$ in the matchings fulfills the clique-max-min property. But we also observe that the solution $M^1$ of **P2** is some sort of a subset[9] of the solution of **P2** given the participants' preferences. This is coincidence. A clique that would have been matched given only the agents' preferences, does not have to be selected at all, given the participants' preferences. For example assume the agents are matched after the first round but the firms only rank these colleagues on the third rank. In the second round two other agents and the same firm are matched because they all have each other on their second rank of the preference list. Then, these agents and the firm are matched, they leave the market. Therefore, the solution of **P2**, given a job market for agents $\Upsilon_S$, cannot be a subset of the solution of **P2**, given a job market for participants, $\Upsilon_P$. If an example yields $g^{\mathbf{P2}}(\succ^P) \subseteq g^{\mathbf{P2}}(\succ^S)$, it is again only a coincidence. In each selected set of participants we also have the firm now. This additional participant may restrict the number of selected cliques. For example a firm $k$ has $\tilde{\kappa}_k = 3$ jobs. If we only take the agents' preferences into account (with $\kappa_k = 3$ we have two vacant jobs in firm $k$) and we have three selected agents, we may have three different complete graphs after two rounds. Now, if we add the firms' preferences, we only can have two complete graphs after two rounds. Thus, the number of different paths for potential matchings, we will follow diminished by one. Hence, in the job market for agents, cliques may emerge in **P2** that are not at all selected in the job market for participants. This is the explanation for $g^{\mathbf{P2}}(\succ^P) \nsubseteq g^{\mathbf{P2}}(\succ^S)$. Finally, no inclusion of the two solutions holds in general. In one example, it is better for some agents if the firms' preferences are included in the other it is not. This is a counterintuitive observation. It seems to be plausible that extending the set of agents by the firms and, consequently, taking the interests

[9] With "some sort of a subset" we mean, the same agents working in the same firm (and with the same firm).

of $m + n$ participants into account, yields a deterioration for the agents. But as discussed above this does not have to be the case.

A comparison of the solutions of the last procedures (**P3a** and **P3b**) is more difficult since we modified the procedure in this chapter. Instead of letting chance choose, the agent who selects his colleagues, now the firm is a natural candidate to choose. The solutions only coincide if the agent who will choose the clique has the same preferences as the firm does.

### 4.1.4 Incentive Compatibility

Again we want to discuss incentive compatibility, because it is an important criterion for the acceptance of a procedure. To give a better illustration, we base the analysis on Example A.0.1 (setting 2.) and on Example A.0.2 (setting 2.) given in Appendix A.

We start with procedure **P1a** given the lexicographic admissible[10] preferences of participants in Example A.0.1 (setting 2). It turns out that both, agents and firms, have an incentive to misrepresent their preferences, in order to be matched to a strictly preferred clique. Hence, the procedure **P1a** does not necessarily have to be incentive compatible, even if we face a job market for participants now. Neither the agents nor the firms may have an incentive to state their true preferences.

Next we discuss procedure **P2** on the basis of Example A.0.2 (setting 2). All participants have lexicographic admissible preferences. Analyzing the example, it turned out that again neither an agent nor a firm may have an incentive to state his true preferences. Both an agent $s_5$ and a firm $h_1$ get a better matching in Example A.0.2 (setting 2.), if they are lying. Hence, also this procedure is not incentive compatible.

[10] Of course, a more general framework of preferences is possible, but again to facilitate the comparisons between the solution of different procedures, we already restricted the preferences for **P1a**.

Nothing changed with the integration of firms' preferences.

Last, we investigate incentive compatibility for **P3b**, which arose as a modification of the incentive compatible procedure **P3a** in $\Upsilon_S$. We again start to analyze the agents and then the firms. We take Example A.0.3 to show that there exist one agent who will be able to ameliorate his outcome by misrepresenting his preferences. As we can easily see, only the fact that now for sure the firms select their future employees (and no longer a randomly selected agent) is sufficient to turn an incentive compatible procedure into a non-incentive compatible one for the agents. So, agents have incentives to misrepresent their preferences. But this statement is not true for the firms. Each selected firm chooses its future employees among the set of given applicants. Misrepresenting their preferences such that a less preferred bundle gets a higher rank as it has, only results in selecting a less preferred bundle. The firm only cheats itself. This cannot be beneficial for the firm.
Summarizing, procedure **P1a** is not necessarily incentive compatible for any participant. The same holds for procedure **P2**. In procedure **P3b** lying is only worth for the agents.

## 4.2 The Firms

This section seems to be a natural consequence of the former chapters. First we only discussed the agents' preferences, then we included the firms' preferences and now, we only want to focus on the firms' preferences.
Each firm has several vacant jobs. A common way to introduce preferences over groups of agents is to deal with *responsive preferences* or to restrict the analysis on *substitutable preferences*, already mentioned in Chapter 1 (or see also Roth & Sotomayor (1990)). We deviate from the classical way of preference restrictions in this setting, because we do not want to assume agents to be substitutes in firms. We rather want to take into account that a worker is influenced by his colleagues.

He may have colleagues, he really get along with such that their team work is much more productive as if one of them would be replaced by another worker. Firms are aware of these influences. Consequently, they have preferences over groups of agents, no particular structure is assumed. On the other hand, we totally ignore the agents' interests in this section.
Similar to the first part of this thesis, we set up a general framework and then introduce the properties a matching should satisfy - but this time from the firms' perspective. In particular, we define another notion of stability for the firms. Furthermore, we discuss two procedures, similar to **P1a** and **P3a** in which matchings are selected based on the firms' preferences.

### 4.2.1 The Framework

The framework is closely related to the classical "many-to-one" matching contexts. In this section, we focus on the "one" side. We neglect the agents' preferences and discuss the firms' preferences over groups of agents at length. The main setting is exactly the same as in Chapter 2. Each firm $k$ has a fixed number of vacant jobs $\kappa_k$ and faces all possible $\kappa_k$-cliques (in $\mathbf{C}_{\kappa_k}$) it could hire, given the total number $n$ of agents on the job market. We have the same sets of firms $H$ and of agents $S$. The firms' preferences are described as follows.

Each firm $k$ has a strict preference relation $\succ_k^H$ on $\mathbf{C}_{\kappa_k}$. $\succ_k^H$ is a linear ordering. We write $\succ^H = (\succ_1^H, \ldots, \succ_m^H)$ as *preference profile*. $\mathcal{P}_k^H$ denotes the *set of all possible preference profiles of firm k*, for all $k \in H$. And we denote the *set of all preference profiles* with $\mathcal{P}^H = \mathcal{P}_1^H \times \ldots \times \mathcal{P}_m^H$.

Let us pause for some discussion. Roth (1985) first introduced preferences over groups of agents. He defined *responsive preferences*[11]. In principal, a firm's preference relation is called responsive to its preferences over individual agents if, for any

[11] The forerunner of weakly responsive preferences.

two cliques that differ in only one individual agent, it prefers the clique with the more preferred agent. Preferences over groups can be still led back to preferences over singles. Next, there is also the definition of *substitutability*[12], which generalizes the definition of responsiveness. The economic interpretation of this property is the following. Each firm regards agents as substitutes rather than as complements. Even if subsets of agents become unavailable, because e.g. they decided to work for another firm, a firm still wants to employ an agent who is an element of its most preferred subset.

Here we do not restrict the firms' preferences over $\kappa_k$-cliques in such a manner. Taking an arbitrary $\kappa_k$-clique, firm $k$ always can state, according to its preferences, whether it prefers this bundle given another one or not. But we are not able to derive preferences over single agents or arbitrary subsets of agents. In our context the idea of substitutes, which lies behind the definition of responsiveness and substitutability is no longer applicable. This creates room for the interpretation of complements, which we will explain next. We have two cliques only differing in one agent, assume $s_i$ in the first clique, $s_j$ in the second. The firm $k$ prefers the clique with $s_i$ to the clique with $s_j$. Next, we take two other cliques again only differing in the same agents, $s_i$ and $s_j$, but the composition of the remaining $\kappa_k - 1$ agents is different. The firm now prefers the clique with $s_j$ instead of the clique with $s_i$. Consequently, the different subsets of agents within the cliques cause a reversal of the preferences on $s_i$ and $s_j$ if we try to derive preference relations over single agents. We do not regard agents as simple substitutes. Preferences over single agents depend on the other employees in the clique. We cannot derive preferences over single agents from preferences over cliques. Assuming responsive preferences, this was possible. The same argumentation holds comparing our preference relation with substitutable preferences. We have a set of the most preferred agents for a firm. If a firm's preferred set of agents from $S$ includes $s_i$, so will its preferred set of agents from any subset of $S$ that still includes $s_i$. This is no longer the case.

[12]See also Roth & Sotomayor (1990), p. 174.

An intuitive application for this preference structure is a coach's preferences over his soccer team. Assume, we have two $\kappa_k$-cliques ($\kappa_k = 11$, $k = 1, 2$) only differing in one agent, one agent is a forward the other agent is a defender. We cannot automatically conclude that the coach always likes the $\kappa_k$-clique with the forward better than the one with the defender. His preferences between the two different individuals may vary according to the composition of the remaining $\kappa_k - 1 = 10$ agents. His preferences over the single agent clearly depend on the whole team.

Comparing the firms' preferences with those of the agents and the participants in the former models reveals a different domain. Firms only have preferences over cliques, subsets of agents of a fixed size. Agents (in the model of Chapter 2) or participants (in Section 4.1) have to think about whether they like the firm they work for and whether they will get along with their colleagues. Firms only care about their future employees/cliques. In later sections we will see what kind of impact this difference has on the properties and the procedures.

After having introduced the basic framework of the *job market* that only deals with the firms' preferences, we summarize:

$$\Upsilon_H = (H, S, (\kappa_k)_{k \in H}, \succ^H).$$

Considering the matching functions, $\mu$ or $\tilde{\mu}$, we note that a mapping from the set of agents into the set of firms is not the right description in this section. But if we consider the inverse function of the mapping $\mu$, we get a correspondence, illustrating firms that are matched with cliques. Formalizing this observation, we get the next definition.

**Definition 4.2.1 (Inverse Matching)**

A correspondence $\hat{\mu} : H \Rightarrow S$ is called *inverse matching*, if for all $k \in H$ we have $\hat{\mu}(k) \in \mathbf{C}_{\kappa_k}$. We define $C_{\hat{\mu},k} := \hat{\mu}(k)$ as the $\kappa_k$-clique under $\hat{\mu}$ for each firm $k$.[13] An alternative description of the inverse matching $\hat{\mu}$ is $M = (C_{\hat{\mu},1}, \ldots, C_{\hat{\mu},m})$. All inverse matchings are given by $\hat{\mathcal{M}} := \{M_{\hat{\mu}} \mid \hat{\mu} \text{ is an inverse matching}\}$.

This definition is not really new. We deal with the inverse function[14] of the agents' matching function $\mu$. Again an alternative description of an inverse matching is given by the equations (2.1) and (2.2).

**Remark 4.2.2**

Given our strict preference relation $\succ_k^H$ on $\mathbf{C}_{\kappa_k}$, we can deduce (weak) preferences over matchings $M, M' \in \hat{\mathcal{M}}$. A matching $M$ is strictly preferred to $M'$ by firm $k$, if and only if firm $k$ strictly prefers the clique, it is matched with in $M$. But firm $k$ is indifferent between two matchings, if he always gets the same employees, regardless of which matching, $M$ or $M'$, is given. The formal description is: Let $M(\succ_k^H)$ be the preference relation on $\hat{\mathcal{M}}$ derived from the firms' preferences $\succ_k^H$ on $\mathbf{C}_{\kappa_k}$ given by $M\, M_P(\succ_k^H)\, M'$ if and only if $C \succ_k^H C'$ and $M\, M_I(\succ_k^H)\, M'$ if and only if $C = C'$. Denote by $M_P(\succ_k^H)$ and $M_I(\succ_k^H)$ its strict and indifferent component. Firm $k$ gets clique $C$, if we face the matching $M$.[15]

**Definition 4.2.3 (Firms' Ranking Function)**

We define a mapping $v_k^H : \mathbf{C}_{\kappa_k} \to \{1, \ldots, |\mathbf{C}_{\kappa_k}|\}$ with $v_k^H(C) > v_k^H(C')$, if and only if $C \succ_k^H C'$ is a *firm's ranking function* for each $k \in H$. $v_k^H(C)$ is called the *rank* of $C$ with $C \in \mathbf{C}_{\kappa_k}$.

[13]Of course, $C_{\mu,k} = C_{\hat{\mu},k}$, we decided to use $C_{\hat{\mu},k}$, because it fits better in the terminology of this section.

[14]to be precise "inverse correspondence"

[15]$C'$ if it faces the matching $M'$.

Each firm $k \in H$ assigns a rank to each $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ that may work for it. We assume that the ranking function is discrete and increasing, the more preferred a clique is, the higher the rank it gets. Furthermore, we again assume that all firms assign their most preferred clique the same rank, their second most preferred and so on. Depending on $\kappa_k$, the firms will not rank the same number of cliques. Consequently, the highest rank must be equal to the $\max_{k \in H} |\mathbf{C}_{\kappa_k}|$.

### 4.2.2 Properties

There are several reasons why some matchings seem to be "better" for the firms than others. To give the word "better" a more precise meaning, we introduce some well defined properties. They already turned out to be meaningful when we only discussed the agents preferences. We again discuss Pareto efficiency and a max-min property, but this time from the firms' perspective. Next, we introduce a notion of stability. Again only two firms will be able to block a matching.

**Definition 4.2.4 (Firms' Pareto Efficiency)**
A matching $M \in \hat{\mathcal{M}}$ is *firm Pareto efficient*, if there is no other matching $M' \in \hat{\mathcal{M}}$, in which no firm is worse off and at least one firm is strictly better off in the sense of Remark 4.2.2.

If a firm $k$ gets a strictly preferred $\kappa_k$-clique in another matching, then at least one other firm $l$ gets a strictly less preferred $\kappa_l$-clique. A firm Pareto efficient matching always exists. For a proof of this claim, we refer to Theorem 2.2.2.[16]

The basic idea of the *max-min property* remains the same as in Section 2.2. Focusing on the firms' side, changes the definition such that we have to take the firms' ranking functions and the inverse matching $\hat{\mu}$.

[16] We use the fact that the set of matchings in the job market and the set of firms are finite.

**Definition 4.2.5 (Firm-max-min)**

A matching $\bar{M} \in \hat{\mathcal{M}}$ fulfills the *firm-max-min property* if

$$\bar{M} \in \underset{M_{\hat{\mu}} \in \hat{\mathcal{M}}}{\operatorname{argmax}} \ \min_{k \in H} \ v_k^H(C_{\hat{\mu},k}).$$

Facing any matching, we take the minimal rank over all firms' ranks. Then facing all matchings, we determine the maximal rank over all such minimal ranks. This set of matchings fulfills the firm-max-min property. The economic interpretation is the same as in Chapter 2. Because of finiteness of $\hat{\mathcal{M}}$, at least one matching satisfying the firm-max-min property always exists.

Now we come to the last property, the notion of stability. First, we give some basic motivation. For sure, each firm is interested in avoiding high fluctuations in manpower. To become acquainted with a new job not only takes some time but is also costly for the firm. Another aspect, the evidence of which becomes most enlightened, if we focus on a soccer team (or other sports teams), is the efficiency[17] of a team or a group of agents. The longer a group of agents plays soccer together, the more they can coordinate and the better - or the more successful - they become. Even if we do not explicitly model these aspects, they still are very good motivations why stability is a desirable property.

Before giving a precise definition of stability, we first have to introduce some further notation. Considering two firms, we define $\mathbf{E}_{k,l} := \{(c_k, c_l) \mid c_k \subseteq C \in \mathbf{C}_{\kappa_k}, c_l \subseteq C \in \mathbf{C}_{\kappa_l}, |c_k| = |c_l| \leq \min\{\kappa_k, \kappa_l\}\}$. $\mathbf{E}_{k,l}$ is the set of all pairs of subgroups of agents in two firms with an equal amount of agents. The maximal number of agents in $|c_k|$[18] is always equal to the minimal number of jobs, one of the two firms is offering. We call $\mathbf{E}_{k,l}$ the *exchange set*, because it consists of all possible subgroups of agents, which two firms can exchange. Now, we consider a matching $M \in \hat{\mathcal{M}}$. We say, a matching $M$ is $t_f$*-blocked* "blocked by *t*rade among *f*irms" by two firms

[17] demotic use of the term

[18] therefore also in $c_l$

$(k, l)$ with $k, l \in H$, if both get a higher rank by exchanging a pair of subgroups of agents $(c_k, c_l) \in \mathbf{E}_{k,l}$. It immediately becomes clear, why we established the set $\mathbf{E}_{k,l}$. Firms cannot exchange agents, if one firm would like to exchange more agents than the other firm has vacant jobs.

Several (economic) interpretations are at hand. Beyond the firm setting where different departments in one firm may want to exchange their employees to increase the team skills, one may think of a soccer team playing for a specific club. If soccer defenders do not get along very well with the mid fielders, the coach may prefer to exchange some players with another soccer club to "increase the soccer clubs' rank". Two clubs or firms always give away and get back the same number of agents.

As long as firms can get a higher rank by exchanging subsets of their employees, a permanent matching is not given. Therefore, we are interested in establishing stability.

**Definition 4.2.6 ($t_f$-Stability)**
A matching $M \in \hat{\mathcal{M}}$ is *$t_f$-stable*, if it is not $t_f$-blocked by any two firms.

**Remark 4.2.7**
The set of firm Pareto optimal matchings is a subset of the set of $t_f$-stable matchings.

We want to discuss the relation between firm Pareto efficiency and $t_f$-stability. The notion of $t_f$-stability is restricted to two firms that exchange subsets of agents if they both reach a higher rank. The notion of firm Pareto efficiency is similar. If a matching is not firm Pareto efficient, we can make someone, i.e. a firm, better off and no one worse off while only two firms exchange a subset of agents. But in contrast to $t_f$-stability, this phenomenon is not restricted to two firms. Several firms can be involved into firm Pareto improvement. Therefore, a firm Pareto efficient matching is in particular $t_f$-stable whereas a $t_f$-stable matching does not necessarily have to be firm Pareto efficient.

We come to another comparison. What are the differences or similarities between $t_a$-stability and $t_f$-stability? The first notion of stability defines two agents who can $t_a$-block a matching by exchanging their jobs - and therefore their colleagues as well - in different firms. The second notion focuses on two firms that can $t_f$-block. The firms exchange subsets of agents who work for them. Thus, in former models agents (or participants) only can exchange themselves whereas in this section firms are able to exchange all kind of subsets[19] of agents working for them. Both definitions of stability focus on an exchange of agents, once the blocking behavior is initiated by the agents themselves, once by the firms.
Another interesting aspect is the investigation how the exchange of agents influences other agents not involved into blocking. If a pair of agents $t_a$-block a matching, this changes the ranks of all agents they worked with before, too. It has no influence on agents working in firms the $t_a$-blocking pair of agents does not work for. But if a pair of firms exchanges a subset of agents, this has no influence on other firms at all and the agents' belongings are not taken into account in this model.

**Theorem 4.2.8 (Existence of $t_f$-stable matchings)**
For any preference profile $\succ^H \in \mathcal{P}^H$ a $t_f$-stable matching always exists.

*Proof:* Assume each matching $M \in \hat{\mathcal{M}}$ is $t_f$-blocked. Since the set of matchings is finite, there exists a cycle such that $t_f$-blocking never ends. Denote the set of matchings contained in this cycle by $Z$. Let $M, M', M'' \in Z$ be matchings in the cycle. Now, firm $k$ $t_f$-blocks $M$ together with another firm. We get a matching $M'$ whereas $M' \, M_P(\succ_k^H) \, M$. In analyzing $M'$, we get either $k$ is again an element of the blocking pair, then $M'' \, M_P(\succ_k^H) \, M' \, M_P(\succ_k^H) \, M$ or $k$ is no element of the $t_f$-blocking pair, nevertheless $M'' \, M_P(\succ_k^H) \, M$. Since $M \in Z$, we come back to $M$ after finitely many steps. So $M \, M_P(\succ_k^H) \, M$. This is a contradiction. Consequently, there always exists a $t_f$-stable matching. □

[19]Of course, given the restriction that $(c_k, c_l) \in \mathbf{E}_{k,l}$.

We emphasize again that no assumption was made to restrict the class of preferences. The key of this existence result is the fact that exchanging subsets of agents between two firms does not affect the other firms' ranks.

### 4.2.3 Procedures and Results

Again we want to discuss procedures, which select matchings. All procedures are related to those we already introduced in former sections. But the fact that we deal with the firms' preferences requires adjustments. We again[20] identify a procedure with an *outcome correspondence (for firms)*. We have $\hat{g} : \mathcal{P}^H \Rightarrow \hat{\mathcal{M}}$ with $(\succ_1^H, \ldots, \succ_m^H) \mapsto \hat{g}(\succ_1^H, \ldots, \succ_m^H)$. The correspondence selects for each vector of $m$ firms' preferences a set of matchings. We focus on **P1a** and **P3a**. We skip **P2**, the reason for that becomes evident, once we have analyzed the results of **P1a**. We start with a variation of **P1a**.

**Procedure 1b (P1b)**

1. Each firm $k$ announces its most preferred $\kappa_k$-clique $C \in \mathbf{C}_{\kappa_k}$ it would like to hire among those it has not announced, yet.

2. We check whether (2.1) and (2.2) are satisfied (given all announced cliques in this round and in the former rounds).

   - If so, the procedure stops, we get a set of matchings.
   - If not, we reenter *step 1.* All announcements of former rounds remain valid.

The solution of the procedure **P1b** is denoted by $\hat{g}^{\mathbf{P1b}}(\succ^H)$. The structure of this procedure is slightly different compared to **P1a**. We cannot construct complete

[20]For comparisons see Subsection 2.3.2.

graphs any longer. In procedure **P1a** agents make offers among agents. Each offer consists of questions. To get a matching in **P1a**, it is necessary to ask all agents in the clique and being asked by all agents in the clique (part of the matching). Whereas in this model firms only announce their most preferred clique, second most preferred and so on, they want to hire. Once each pairwise intersection between all cliques is empty ($C_k \cap C_l = \emptyset$ for all $k, l \in H$), such that all agents have a job, the procedure stops. Hence, the main difference is the fact that a firm does not have to wait for some sort of counter announcements. Procedure **P1a** did not stop before all offers in a clique and in all firms were made such that a matching occurred. In procedure **P1b** it suffices if all announcements in all firms constructing a matching are made. Obviously, this may include more than one matching. A direct consequence of this difference is that the graph theoretic illustration of complete graphs, introduced in Subsection 2.3.1, is not applicable in this context.

**Theorem 4.2.9**

For any preference profile $\succ^H \in \mathcal{P}^H$, the solution $\hat{g}^{\mathbf{P1b}}(\succ^H)$

1. contains at least one Pareto efficient matching,

2. contains at least one $t_f$-stable matching and

3. coincides with the matchings that fulfill the firm-max-min property.

*Proof:* 1. and 3. are analogous to the proofs of Theorem 3.1.1.
2. Assume there does not exist a $t_f$-stable matching in the solution of **P1b**. Then all $M \in \hat{g}^{\mathbf{P1b}}$ are $t_f$-blocked by a pair of firms. Let $(h_k, h_l)$ be such a pair of firms that $t_f$-blocks any matching $M$. By exchanging subsets of agents $(c_k, c_l) \in \mathbf{E}_{k,l}$, both firms get a higher rank in a new matching $M'$. But this matching is already selected by **P1b**. Thus, the assumption was wrong. □

The most interesting result is for sure the fact that the solution of **P1b** contains a $t_f$-stable matching whereas the solution of **P1a** is not $t_a$-stable. And even more;

Given the most general form of preferences for **P1a**, a $t_a$-stable matchings does not have to exist at all. Why is this result so different? If a matching is $t_a$-blocked and, consequently, two agents exchange their jobs, they improve their ranks. But at the same time, they also change the ranks of their former colleagues. Therefore, a $t_a$-blocking pair of agents may induce another pair of agents, which will $t_a$-block afterwards. According to Theorem 2.2.7, there does not have to exist a $t_a$-stable matching at all. Now, if two firms $t_f$-block a matching, they both will get a better rank. But the more important observation is the fact that they do not influence the outcome of the other firms. This makes it a much easier concept of stability.

The second procedure or its modification becomes superfluous. First **P1b** already generates a $t_f$-stable matching and second the structure of the firms' preferences is different. To introduce lexicographic preferences in this context seems to be rather artificial than a natural consequence.

The next obvious question is, whether the procedure **P1b** is incentive compatible. A simple example shows that is not.

**Example 4.2.10**

We have $H = \{h_1, h_2\}$. Each firm has two vacant jobs, $\kappa_1 = \kappa_2 = 2$, hence we have $S = \{s_1, s_2, s_3, s_4\}$ agents on the job market. The firms' preferences[21] are:

$(s_1, s_2) \succ_1^H (s_1, s_3) \succ_1^H (s_3, s_4) \succ_1^H (s_1, s_4) \succ_1^H \dots$

$(s_1, s_2) \succ_2^H (s_3, s_4) \succ_2^H (s_1, s_3) \succ_2^H (s_2, s_3) \succ_2^H \dots$ □

Applying **P1b**, we get $M = (\{s_1, s_2\}, \{s_3, s_4\})$. Now, if firm $h_2$ announces the preferences $(s_1, s_2) \succ_2^H (s_2, s_3) \succ_2^H (s_1, s_3) \succ_2^H \dots$ instead of its true preferences, then the procedure selects the matching $M' = (\{s_3, s_4\}, \{s_1, s_2\})$ which is strictly better for firm $h_2$. It gets its most preferred clique. The second firm simply has to forestall the matching $M$. Therefore, incentive compatibility is not satisfied.

[21] We forgo to give a complete description of the preferences.

The second procedure we want to discuss in this section is a procedure proposed by Roth & Sotomayor (1990). We call it in the following procedure **P3c** to stay in our terminology and because the course of action is basically the same as in **P3a** and **P3b**.

**Procedure 3c (P3c)**

We fix the order of the firms with an equal distribution. The first firm in this order chooses the agents, which he mostly wants to hire. The firm and the chosen agents leave the job market. The firm with the lowest index in the set of remaining firms now chooses its future employees. Again the firm and the chosen agents leave the job market. The procedure stops when the last firm took its choice.[22]

**Theorem 4.2.11**

Let $\succ^H \in \mathcal{P}^H$ be a preference profile and fix an order of firms. Then the solution $\hat{g}^{\mathbf{P3c}}(\succ^H)$ w.r.t. to this order is

1. singleton,

2. firm Pareto efficient and

3. $t_f$-stable.

*Proof:* 1. There are $m$ firms. According to the fixed order, each firm $k$ selects its $\kappa_k$-clique. Hence, $m$ firms are selecting $\sum_{k=1}^{m} \kappa_k$ agents. This equals the number of agents in one matching.

2. Assume $(C_1, \ldots, C_m) = M \in \hat{g}^{\mathbf{P3a}}(\succ^H)$ is not Pareto efficient. Then there exists another matching $(C_1', \ldots, C_m') = M' \in \hat{\mathcal{M}}$ such that no agent is worse off and at least one is strictly better off. W.l.o.g let $C_1$ be a clique chosen first by the firm with the lowest index $k$ in the set of firms. Then $C_1 = C_1'$, because if $C_1 \neq C_1'$ firm $k$ will

[22]This, of course, is not really taking its choice, since only one clique left on the market.

be worse off. The job market is reduced by $C_1$. W.l.o.g let $C_2$ be a clique chosen first by the firms with the lowest index $l$ in the remaining set of firms. Then $C_2 = C_2'$, because if $C_2 \neq C_2'$ firm $l$ would be worse off. He only could be strictly better off, if $C_1 \succ_l^H C_2$, but then firm $k$ would be strictly worse off. This argumentation goes on until all cliques are checked. It follows $M = M'$, hence, $M \in \hat{g}^{\mathbf{P3c}}(\succ^H)$ is firm Pareto efficient.

3. Assume the solution of the procedure is not $t_f$-stable, then there are two firms $k, k'$, which want to exchange a subgroup of agents $(c_k, c_l) \in \mathbf{E}_{k,l}$. We know one firm chose its clique in an earlier round than the other. W.l.o.g., it is $k$. If $k$ exchanges a subgroup of agents it gets a better outcome. It got this subgroup from the firm $k'$ which chose in a later round, because a firm that chooses before firm $k$, would already have taken the clique with the more preferred subset. But then, the subgroup of agents was already available for it, when it had chosen its actual clique. So, $k$ cannot be a partner in a $t_f$-blocking coalition. This holds for all possible $t_f$-blocking coalition. Therefore, the solution of **P3c** is always $t_f$-stable. □

The procedure does not fulfill the firm-max-min property. Given the fixed order of firms, each firm selects among the remaining agents the clique it prefers most. The fact that its choice is always restricted on the remaining agents, is the main reason for the lack of the firm-max-min property. For example, if we have two firms $H = \{h_1, h_2\}$ with $\kappa_1 = \kappa_2 = 2$. $h_1$ chooses first its most preferred clique. We assume for firm $h_2$ it only remains its less preferred clique, but there also exists a matching where both firms would get their second most preferred cliques. Clearly, the matching, determined by the procedure, cannot satisfy the firm-max-min property.

**Theorem 4.2.12**

The mechanism **M3c** is incentive compatible given a fixed order of firms.

*Proof:* Assume a firm misrepresents its preferences. It gets its most preferred clique given the untruthfully stated preferences. If this is not conform with his most preferred bundle given his true preferences, he gets a lower rank. Hence, stating the true preferences is incentive compatible. □

The fact that a firm only has preferences over cliques, slightly changes the results. We had to introduce another notion of stability and we also had to modify the procedures **P1a** and **P3a**. **P2** becomes superfluous because of the different underlying structure of the preferences. Nevertheless, a lot of results remain valid. The solution of **P1b** contains at least one firm Pareto efficient matching and fulfills the firm-max-min property. Furthermore, it produces a $t_f$-stable matching. However, **P1b** is not incentive compatible. The second procedure **P3c** delivers a singleton solution that is firm Pareto efficient and $t_f$-stable. Furthermore, the procedure is incentive compatible.

## 4.3 Conclusion

In this chapter we have addressed the firms' requests. For this, we have chosen two different approaches. The first one integrates the firms into the agents' side, the second one leaves the firms on the other side. Thus on the one hand, we take all parties into account whereas on the other hand we only focus on the firms' belongings.

If we compare the results of Chapter 3 and Chapter 4, we observe that considering agents or participants does not influence the properties of the solution $g^{\mathbf{P1a}}$. Given a preference profile in $\mathcal{P}^S$ or $\mathcal{P}^P$, the solution contains at least one Pareto efficient matching and fulfills the agent-max-min property. The solutions of **P2** in the presence of lexicographic preferences or lexicographic admissible preferences are also identical. The same holds for the solutions of **P3a** and **P3b**, even if the procedures are slightly different. Furthermore, given the preference profiles in $\mathcal{P}^S$ the mechanism **M3a** is the only one which is incentive compatible. Given the participants' preference profiles $\mathcal{P}^P$, none of the procedures is entirely[23] incentive compatible. In

[23]for all sides of the market

this context the results are changing, if we add the firms on the agents' side.

The second section of this chapter dealt with the firms' preferences only. We introduce another notion of stability. The procedures are also modified. The main difference in comparison to the former models is the fact that the solution of the procedure **P1b** is already $t_f$-stable. This is caused by the different structure of the bundles, over which the firms have preferences and, consequently, the slightly different notion of stability. The solution $\hat{g}^{P3a}$ is singleton, firm Pareto efficient and $t_f$-stable. Furthermore, the procedure itself is incentive compatible. Compared to $g^{\mathbf{P3a}}$, nothing has changed.

Summarizing, different models and procedures lead to different solutions and depending on what kind of properties a society judges as most important it can choose the appropriate model.

# Chapter 5

# Concluding Comments

First, we want to give a short overview of the models, we have introduced in the thesis. We defined new properties for the solution of a particular matching problem, new procedures that lead to a solution were described and we investigated incentive compatibility of the procedures. We briefly want to abstract and evaluate the insights, we gained during our analysis. And finally, we will illustrate some rough ideas about possible extensions that arose during the formation of the thesis.

In the first chapter, the literature overview illustrates the parallel evolution of algorithms in the practical as well as in the theoretical world, given typical two-sided matching problems. It is quite interesting to see that, independently from each other, both approaches reached the same solution even if the inventors' skills were totally different. In addition, it is also amazing that it took ten years until practioners and theorists learned about each others' solutions of the problem. The last section embeds the thesis in the literature which is not obvious. There are few references in the literature in which agents are having preferences over firms as well as over other agents (colleagues).

The second and the third chapter belong together. A new model is introduced for many-to-one sided matching problems, whereas we put the main emphasis on the agents' side. They have preferences over firms and colleagues. In this context max-min properties and a new notion of stability, namely $t_a$-stability, are defined. We judge "favoring the least" and a "durable matching on the job market" as highly desirable properties. The new framework leads to new procedures that yield a set of matchings. Depending on the chosen procedure, different properties are fulfilled by the solutions (matchings). Hence according to the properties a society judges to be most important, it (or a central planner) can choose the appropriate procedure. Furthermore, two of the three procedures are supported by a graph theoretical representation. This makes it especially easy to follow the course of action. Besides the quality of the solution and an easy illustration, we also focused on the quality of the procedures themselves. It turned out not to be trivial at all to develop procedures inducing agents to state their true preferences, hence to be incentive compatible. Only the third procedure satisfies this property.

Chapter 4 discusses the firms' preferences. Firms are no longer indifferent, they do care about their future employees which is indeed a plausible assumption. In the first section we simply integrated the firms in the model of Chapter 2 as if they were agents. The results only change in such a way that we have $m$ more agents ($m$ firms) on the agents' side. On the firms' side nothing changed. The procedures may determine different solutions, but the welfare and practical aspects are not affected by this sort of extension. In the second section we excluded the agents' preferences and only focused on the firms. Therefore, two modifications of the first and the third procedure (introduced in Chapter 2) are established. The basic structure of this model remains similar, compared to the former ones. Nevertheless, we had to adjust a couple of things. Firms only have preferences over cliques, hence over bundles, only consisting of elements of the other side. This has various consequences for the analysis. First the notion of stability becomes different. In Chapter 2 agents exchanged themselves and influenced other agents' ranks simultaneously. In Section 4.2 firms are exchanging subsets of agents which has no influence on other firms. Furthermore, the existence of $t_f$-stable matchings is given without any additional restrictions on the preferences. A graph theoretical representation is no longer applicable, and a modification of the procedures becomes necessary. Finally, the separate analysis of each side showed that they are not symmetric and revealed a lot of differences and new insights. Regarding firms as if they were employed by themselves is a simple way to include the agents' and the firms' preferences which leads to reasonable results.

All models do not exist in the literature, yet. During their description we made some simplifying assumptions. Dropping these restrictions, would be a first natural extension of the models. Secondly, further ideas about possible modifications came to our mind during the genesis of the thesis.
We always assumed a job market with equal numbers of vacant jobs and agents, searching for a job. This clearly is an unrealistic assumption. On the one hand nowadays, we typically have an excess supply of labor. But on the other hand, we could as well think of an excess supply of jobs. In this case an obvious application would be the number of vacant jobs for high qualified agents. Hence, to model an excess of supply or demand of jobs should be one of the first next steps. Another aspect is the fact that not every job is acceptable for an agent, i.e. the agent likes the job at least as well as remaining unemployed, and probably not every clique is acceptable for a firm. An agent may prefer to stay unemployed as well as the firms could decide to leave positions unfilled instead of hiring non-qualified persons (or groups of agents). This is a well known requirement in the marriage market problem. A man or a woman may prefer to stay alone instead of being matched with an unacceptable mate. To include these requirements could be another extension that further approaches the models to a representation of the real job market. Next, uniqueness of the solution cannot be guaranteed by all procedures, yet. We did not discuss in detail how to select a matching being an element of the solution or who should select. This is a shortcoming, preventing the application of the models to the real job market.

Further research raises the question whether a more separate treatment of the two sides of the market is possible without identifying the firms as agents. One could try to merge the framework of Chapter 2 and Section 4.2. In this context several problems are immediately at hand. What could be an appropriate procedure that comprises both sides' interests? Does there exist a set of matchings satisfying $t_a$- and $t_f$-stability? Or, is it necessary to think of other properties a matching should satisfy? Plenty of further questions arise. In this regard, we would like to close the short summary of possible extensions and we leave the reader alone in order that he can fantasize on his own about other interesting extensions.

# Appendix A

# Examples

**Example A.0.1 (P1a and Incentive Compatibility)**

1. Let $H = \{h_1, h_2\}$ with $\kappa_1 = 3$, $\kappa_2 = 2$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$. The agents have lexicographic preferences.

   $(h_1, \{s_2, s_3\}) \succ^S_{s_1} (h_1, \{s_2, s_4\}) \succ^S_{s_1} (h_1, \{s_3, s_4\}) \succ^S_{s_1} (h_1, \{s_2, s_5\})$
   $\succ^S_{s_1} (h_1, \{s_4, s_5\}) \succ^S_{s_1} (h_1, \{s_3, s_5\}) \succ^S_{s_1} (h_2, \{s_3\}) \succ^S_{s_1} (h_2, \{s_2\})$
   $\succ^S_{s_1} (h_2, \{s_4\}) \succ^S_{s_1} (h_2, \{s_5\})$

   $(h_1, \{s_1, s_3\}) \succ^S_{s_2} (h_1, \{s_1, s_4\}) \succ^S_{s_2} (h_1, \{s_3, s_4\}) \succ^S_{s_2} (h_1, \{s_1, s_5\})$
   $\succ^S_{s_2} (h_1, \{s_4, s_5\}) \succ^S_{s_2} (h_1, \{s_3, s_5\}) \succ^S_{s_2} (h_2, \{s_3\}) \succ^S_{s_2} (h_2, \{s_1\})$
   $\succ^S_{s_2} (h_2, \{s_5\}) \succ^S_{s_2} (h_2, \{s_4\})$

   $(h_1, \{s_2, s_4\}) \succ^S_{s_3} (h_1, \{s_4, s_5\}) \succ^S_{s_3} (h_1, \{s_1, s_2\}) \succ^S_{s_3} (h_1, \{s_2, s_5\})$
   $\succ^S_{s_3} (h_1, \{s_1, s_4\}) \succ^S_{s_3} (h_1, \{s_1, s_5\}) \succ^S_{s_3} (h_2, \{s_5\}) \succ^S_{s_3} (h_2, \{s_4\})$
   $\succ^S_{s_3} (h_2, \{s_2\}) \succ^S_{s_3} (h_2, \{s_1\})$

   $(h_1, \{s_1, s_2\}) \succ^S_{s_4} (h_1, \{s_2, s_3\}) \succ^S_{s_4} (h_1, \{s_3, s_5\}) \succ^S_{s_4} (h_1, \{s_1, s_5\})$
   $\succ^S_{s_4} (h_1, \{s_2, s_5\}) \succ^S_{s_4} (h_1, \{s_1, s_3\}) \succ^S_{s_4} (h_2, \{s_3\}) \succ^S_{s_4} (h_2, \{s_5\})$
   $\succ^S_{s_4} (h_2, \{s_2\}) \succ^S_{s_4} (h_2, \{s_1\})$

   $(h_1, \{s_3, s_4\}) \succ^S_{s_5} (h_1, \{s_2, s_4\}) \succ^S_{s_5} (h_1, \{s_2, s_3\}) \succ^S_{s_5} (h_1, \{s_1, s_4\})$

$\succ^S_{s_5} (h_1, \{s_1, s_3\}) \succ^S_{s_5} (h_1, \{s_1, s_2\}) \succ^S_{s_5} (h_2, \{s_3\}) \succ^S_{s_5} (h_2, \{s_4\})$
$\succ^S_{s_5} (h_2, \{s_1\}) \succ^S_{s_5} (h_2, \{s_2\})$

2. Let $P = \{s_1, s_2, s_3, s_4, s_5, h_1, h_2\}$ with $\tilde{\kappa}_1 = 4$, $\tilde{\kappa}_2 = 3$. The participants have lexicographic admissible preferences.

$(h_1, \{s_2, s_3, h_1\}) \succ^P_{s_1} (h_1, \{s_2, s_4, h_1\}) \succ^P_{s_1} (h_1, \{s_3, s_4, h_1\})$
$\succ^P_{s_1} (h_1, \{s_2, s_5, h_1\}) \succ^P_{s_1} (h_1, \{s_4, s_5, h_1\}) \succ^P_{s_1} (h_1, \{s_3, s_5, h_1\})$
$\succ^P_{s_1} (h_2, \{s_3, h_2\}) \succ^P_{s_1} (h_2, \{s_2, h_2\}) \succ^P_{s_1} (h_2, \{s_4, h_2\}) \succ^P_{s_1} (h_2, \{s_5, h_2\}) \ldots$

$(h_1, \{s_1, s_3, h_1\}) \succ^P_{s_2} (h_1, \{s_1, s_4, h_1\}) \succ^P_{s_2} (h_1, \{s_3, s_4, h_1\})$
$\succ^P_{s_2} (h_1, \{s_1, s_5, h_1\}) \succ^P_{s_2} (h_1, \{s_4, s_5, h_1\}) \succ^P_{s_2} (h_1, \{s_3, s_5, h_1\})$
$\succ^P_{s_2} (h_2, \{s_3, h_2\}) \succ^P_{s_2} (h_2, \{s_1, h_2\}) \succ^P_{s_2} (h_2, \{s_5, h_2\}) \succ^P_{s_2} (h_2, \{s_4, h_2\}) \ldots$

$(h_1, \{s_2, s_4, h_1\}) \succ^P_{s_3} (h_1, \{s_4, s_5, h_1\}) \succ^P_{s_3} (h_1, \{s_1, s_2, h_1\})$
$\succ^P_3 (h_1, \{s_2, s_5, h_1\}) \succ^P_3 (h_1, \{s_1, s_4, h_1\}) \succ^P_3 (h_1, \{s_1, s_5, h_1\})$
$\succ^P_{s_3} (h_2, \{s_5, h_2\}) \succ^P_{s_3} (h_2, \{s_4, h_2\}) \succ^P_{s_3} (h_2, \{s_2, h_2\}) \succ^P_{s_3} (h_2, \{s_1, h_2\}) \ldots$

$(h_1, \{s_1, s_2, h_1\}) \succ^P_{s_4} (h_1, \{s_2, s_3, h_1\}) \succ^P_{s_4} (h_1, \{s_3, s_5, h_1\})$
$\succ^P_{s_4} (h_1, \{s_1, s_5, h_1\}) \succ^P_{s_4} (h_1, \{s_2, s_5, h_1\}) \succ^P_{s_4} (h_1, \{s_1, s_3, h_1\})$
$\succ^P_{s_4} (h_2, \{s_3, h_2\}) \succ^P_{s_4} (h_2, \{s_5, h_2\}) \succ^P_{s_4} (h_2, \{s_2, h_2\}) \succ^P_{s_4} (h_2, \{s_1, h_2\}) \ldots$

$(h_1, \{s_3, s_4, h_1\}) \succ^P_{s_5} (h_1, \{s_2, s_4, h_1\}) \succ^P_{s_5} (h_1, \{s_2, s_3, h_1\})$
$\succ^P_{s_5} (h_1, \{s_1, s_4, h_1\}) \succ^P_{s_5} (h_1, \{s_1, s_3, h_1\}) \succ^P_{s_5} (h_1, \{s_1, s_2, h_1\})$
$\succ^P_{s_5} (h_2, \{s_3, h_2\}) \succ^P_{s_5} (h_2, \{s_4, h_2\}) \succ^P_{s_5} (h_2, \{s_1, h_2\}) \succ^P_{s_5} (h_2, \{s_2, h_2\}) \ldots$

$(h_1, \{s_2, s_3, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_2, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_2, s_5\})$
$\succ^P_{h_1} (h_1, \{s_2, s_3, s_5\}) \succ^P_{h_1} (h_1, \{s_2, s_4, s_5\}) \succ^P_{h_1} (h_1, \{s_1, s_2, s_3\})$
$\succ^P_{h_1} (h_2, \{s_1, s_3, s_4\}) \succ^P_{h_1} (h_2, \{s_1, s_4, s_5\}) \succ^P_{h_1} (h_2, \{s_3, s_4, s_5\})$
$\succ^P_{h_1} (h_2, \{s_1, s_4, s_5\}) \succ^P_{h_1} (h_2, \{s_1, s_3, s_5\}) \ldots$

$(h_2, \{s_2, s_3\}) \succ^P_{h_2} (h_2, \{s_1, s_2\}) \succ^P_{h_2} (h_2, \{s_2, s_5\}) \succ^P_{h_2} (h_2, \{s_4, s_5\})$
$\succ^P_{h_2} (h_2, \{s_2, s_4\}) \succ^P_{h_2} (h_2, \{s_3, s_4\}) \succ^P_{h_2} (h_2, \{s_3, s_5\}) \succ^P_{h_2} (h_2, \{s_1, s_3\})$
$\succ^P_{h_2} (h_2, \{s_1, s_4\}) \succ^P_{h_2} (h_2, \{s_1, s_5\}) \succ^P_{h_2} \ldots$

Incentive Compatibility of **P1a** given setting 1.:

If all agents made seven offers, **P1a** stops. It yields the matching $M_1 = (\{s_1, s_2, s_4\}, \{s_3, s_5\})$. But if agent $s_5$ misrepresents his preferences, such that

$$(h_1, \{s_3, s_4\}) \succ_{s_5}^{S} (h_1, \{s_2, s_4\}) \succ_{s_5}^{S} (h_1, \{s_2, s_3\}) \succ_{s_5}^{S} (h_1, \{s_1, s_4\}) \succ_{s_5}^{S} (h_1, \{s_1, s_3\}) \succ_{s_5}^{S} (h_1, \{s_1, s_2\}) \succ_{s_5}^{S} (h_2, \{s_1\}) \succ_{s_5}^{S} (h_2, \{s_2\}) \succ_{s_5}^{S} (h_2, \{s_3\}) \succ_{s_5}^{S} (h_2, \{s_4\}),$$

then **P1a** stops after eight offers per agent. It delivers the matchings $M_2 = (\{s_3, s_4, s_5\}, \{s_1, s_2\})$ and $M_3 = (\{s_1, s_2, s_5\}, \{s_3, s_4\})$. Both matchings have a higher rank for agent $s_5$ than he gets if he is telling the truth.

Incentive Compatibility of **P1a** given setting 2.:

Given a job market for participants and assuming all participants state their true preferences, **P1a** determines the matching $M_1' = (\{s_1, s_2, s_4, h_1\}, \{s_3, s_5, h_2\})$.
If agent $s_5$ misrepresents its preferences again such that

$$(h_1, \{s_3, s_4, h_1\}) \succ_{s_5}^{P} (h_1, \{s_2, s_4, h_1\}) \succ_{s_5}^{P} (h_1, \{s_2, s_3, h_1\}) \succ_{s_5}^{P} (h_1, \{s_1, s_4, h_1\}) \succ_{s_5}^{P} (h_1, \{s_1, s_3, h_1\}) \succ_{s_5}^{P} (h_1, \{s_1, s_2, h_1\}) \succ_{s_5}^{P} (h_2, \{s_1, h_2\}) \succ_{s_5}^{P} (h_2, \{s_2, h_2\}) \succ_{s_5}^{P} (h_2, \{s_3, h_2\}) \succ_{s_5}^{P} (h_2, \{s_4, h_2\}) \ldots,$$

**P1a** determines the matchings $M_3' = (\{s_1, s_2, s_5, h_1\}, \{s_3, s_4, h_2\})$ and $M_2' = (\{s_3, s_4, s_5, h_1\}, \{s_1, s_2, h_2\})$.

If firm $h_2$ misrepresents its preferences such that

$$(h_2, \{s_1, s_2\}) \succ_{h_2}^{P} (h_2, \{s_2, s_5\}) \succ_{h_2}^{P} (h_2, \{s_4, s_5\}) \succ_{h_2}^{P} (h_2, \{s_2, s_4\}) \succ_{h_2}^{P} (h_2, \{s_3, s_4\}) \succ_{h_2}^{P} (h_2, \{s_1, s_4\}) \succ_{h_2}^{P} (h_2, \{s_1, s_5\}) \succ_{h_2}^{P} (h_2, \{s_1, s_3\}) \succ_{h_2}^{P} (h_2, \{s_2, s_3\}) \succ_{h_2}^{P} (h_2, \{s_3, s_5\}) \succ_{h_2}^{P} \ldots,$$

**P1a** determines the matchings $M_3' = (\{s_1, s_2, s_5, h_1\}, \{s_3, s_4, h_2\})$. □

**Example A.0.2 (P2 and Incentive Compatibility)**

1. Let $H = \{h_1, h_2\}$ with $\kappa_1 = 2$, $\kappa_2 = 3$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$. The agents have lexicographic preferences.

   $(h_1, \{s_5\}) \succ^S_{s_1} (h_1, \{s_2\}) \succ^S_{s_1} (h_1, \{s_4\}) \succ^S_{s_1} (h_1, \{s_3\}) \succ^S_{s_1} (h_2, \{s_4, s_5\}) \succ^S_{s_1} (h_2, \{s_3, s_5\}) \succ^S_{s_1} (h_2, \{s_2, s_3\}) \succ^S_{s_1} (h_2, \{s_2, s_5\}) \succ^S_{s_1} (h_2, \{s_2, s_4\}) \succ^S_{s_1} (h_2, \{s_3, s_4\})$

   $(h_1, \{s_3\}) \succ^S_{s_2} (h_1, \{s_1\}) \succ^S_{s_2} (h_1, \{s_5\}) \succ^S_{s_2} (h_1, \{s_4\}) \succ^S_{s_2} (h_2, \{s_4, s_5\}) \succ^S_{s_2} (h_2, \{s_3, s_5\}) \succ^S_{s_2} (h_2, \{s_1, s_3\}) \succ^S_{s_2} (h_2, \{s_1, s_5\}) \succ^S_{s_2} (h_2, \{s_1, s_4\}) \succ^S_{s_1} (h_2, \{s_3, s_4\})$

   $(h_1, \{s_1\}) \succ^S_{s_3} (h_1, \{s_4\}) \succ^S_{s_3} (h_1, \{s_5\}) \succ^S_{s_3} (h_1, \{s_2\}) \succ^S_{s_3} (h_2, \{s_4, s_5\}) \succ^S_{s_3} (h_2, \{s_2, s_5\}) \succ^S_{s_3} (h_2, \{s_1, s_2\}) \succ^S_{s_3} (h_2, \{s_1, s_5\}) \succ^S_{s_3} (h_2, \{s_1, s_4\}) \succ^S_{s_3} (h_2, \{s_2, s_4\})$

   $(h_1, \{s_3\}) \succ^S_{s_4} (h_1, \{s_2\}) \succ^S_{s_4} (h_1, \{s_1\}) \succ^S_{s_4} (h_1, \{s_4\}) \succ^S_{s_4} (h_2, \{s_3, s_5\}) \succ^S_{s_4} (h_2, \{s_2, s_5\}) \succ^S_{s_4} (h_2, \{s_1, s_2\}) \succ^S_{s_4} (h_2, \{s_1, s_5\}) \succ^S_{s_4} (h_2, \{s_1, s_3\}) \succ^S_{s_4} (h_2, \{s_2, s_3\})$

   $(h_1, \{s_3\}) \succ^S_{s_5} (h_1, \{s_4\}) \succ^S_{s_5} (h_1, \{s_2\}) \succ^S_{s_5} (h_1, \{s_1\}) \succ^S_{s_5} (h_2, \{s_3, s_4\}) \succ^S_{s_5} (h_2, \{s_2, s_4\}) \succ^S_{s_5} (h_2, \{s_1, s_2\}) \succ^S_{s_5} (h_2, \{s_1, s_4\}) \succ^S_{s_5} (h_2, \{s_1, s_3\}) \succ^S_{s_5} (h_2, \{s_2, s_3\})$

2. Let $P = \{s_1, s_2, s_3, s_4, s_5, h_1, h_2\}$ with $\tilde{\kappa}_1 = 4$, $\tilde{\kappa}_2 = 3$. The participants have lexicographic admissible preferences.

   $(h_1, \{s_5, h_1\}) \succ^S_{s_1} (h_1, \{s_2, h_1\}) \succ^S_{s_1} (h_1, \{s_4, h_1\}) \succ^S_{s_1} (h_1, \{s_3, h_1\}) \succ^S_{s_1} (h_2, \{s_4, s_5, h_2\}) \succ^S_{s_1} (h_2, \{s_3, s_5, h_2\}) \succ^S_{s_1} (h_2, \{s_2, s_3, h_2\}) \succ^S_{s_1} (h_2, \{s_2, s_5, h_2\}) \succ^S_{s_1} (h_2, \{s_2, s_4, h_2\}) \succ^S_{s_1} (h_2, \{s_3, s_4, h_2\}) \ldots$

   $(h_1, \{s_3, h_1\}) \succ^S_{s_2} (h_1, \{s_1, h_1\}) \succ^S_{s_2} (h_1, \{s_5, h_1\}) \succ^S_{s_2} (h_1, \{s_4\}) \succ^S_{s_2} (h_2, \{s_4, s_5, h_2\}) \succ^S_{s_2} (h_2, \{s_3, s_5, h_2\}) \succ^S_{s_2} (h_2, \{s_1, s_3, h_2\}) \succ^S_{s_2} (h_2, \{s_1, s_5, h_2\}) \succ^S_{s_2} (h_2, \{s_1, s_4, h_2\}) \succ^S_{s_1} (h_2, \{s_3, s_4, h_2\}) \ldots$

$$(h_1,\{s_1,h_1\}) \succ^S_{s_3} (h_1,\{s_4,h_1\}) \succ^S_{s_3} (h_1,\{s_5,h_1\}) \succ^S_{s_3} (h_1,\{s_2,h_1\})$$
$$\succ^S_{s_3} (h_2,\{s_4,s_5,h_2\}) \succ^S_{s_3} (h_2,\{s_2,s_5,h_2\}) \succ^S_{s_3} (h_2,\{s_1,s_2,h_2\})$$
$$\succ^S_{s_3} (h_2,\{s_1,s_5,h_2\}) \succ^S_{s_3} (h_2,\{s_1,s_4,h_2\}) \succ^S_{s_3} (h_2,\{s_2,s_4,h_2\}) \ldots$$

$$(h_1,\{s_3,h_1\}) \succ^S_{s_4} (h_1,\{s_2,h_1\}) \succ^S_{s_4} (h_1,\{s_1,h_1\}) \succ^S_{s_4} (h_1,\{s_4,h_1\})$$
$$\succ^S_{s_4} (h_2,\{s_3,s_5,h_2\}) \succ^S_{s_4} (h_2,\{s_2,s_5,h_2\}) \succ^S_{s_4} (h_2,\{s_1,s_2,h_2\})$$
$$\succ^S_{s_4} (h_2,\{s_1,s_5,h_2\}) \succ^S_{s_4} (h_2,\{s_1,s_3,h_2\}) \succ^S_{s_4} (h_2,\{s_2,s_3,h_2\}) \ldots$$

$$(h_1,\{s_3,h_1\}) \succ^S_{s_5} (h_1,\{s_4,h_1\}) \succ^S_{s_5} (h_1,\{s_2,h_1\}) \succ^S_{s_5} (h_1,\{s_1,h_1\})$$
$$\succ^S_{s_5} (h_2,\{s_3,s_4,h_2\}) \succ^S_{s_5} (h_2,\{s_2,s_4,h_2\}) \succ^S_{s_5} (h_2,\{s_1,s_2,h_2\})$$
$$\succ^S_{s_5} (h_2,\{s_1,s_4,h_2\}) \succ^S_{s_5} (h_2,\{s_1,s_3,h_2\}) \succ^S_{s_5} (h_2,\{s_2,s_3,h_2\}) \ldots$$

$$(h_1,\{s_3,s_5\}) \succ^P_{h_1} (h_1,\{s_1,s_2\}) \succ^P_{h_1} (h_1,\{s_4,s_5\}) \succ^P_{h_1} (h_1,\{s_2,s_5\})$$
$$\succ^P_{h_1} (h_1,\{s_2,s_4\}) \succ^P_{h_1} (h_1,\{s_3,s_4\}) \succ^P_{h_1} (h_1,\{s_2,s_3\}) \succ^P_{h_1} (h_1,\{s_1,s_3\})$$
$$\succ^P_{h_1} (h_1,\{s_1,s_4\}) \succ^P_{h_1} (h_1,\{s_1,s_5\}) \succ^P_{h_1} \ldots$$

$$(h_2,\{s_1,s_2,s_3\}) \succ^P_{h_2} (h_2,\{s_1,s_2,s_4\}) \succ^P_{h_2} (h_2,\{s_1,s_2,s_5\})$$
$$\succ^P_{h_2} (h_2,\{s_2,s_3,s_5\}) \succ^P_{h_2} (h_2,\{s_1,s_4,s_5\}) \succ^P_{h_2} (h_2,\{s_1,s_3,s_5\})$$
$$\succ^P_{h_2} (h_2,\{s_1,s_3,s_4\}) \succ^P_{h_2} (h_2,\{s_1,s_2,s_5\}) \succ^P_{h_2} (h_2,\{s_3,s_4,s_5\})$$
$$\succ^P_{h_2} (h_2,\{s_2,s_3,s_4\}) \succ^P_{h_2} (h_2,\{s_1,s_3,s_5\}) \ldots$$

Incentive Compatibility of **P2** given setting 1.:

The second procedure selects all agents, since they all prefer firm $h_1$ to $h_2$. After all agents having made two offers, the first emerging complete graphs in firm $h_1$ consist of the agents $s_3, s_4$ and $s_1, s_2$. Hence **P2** selects the matchings $M_1 = (\{s_3,s_4\},\{s_1,s_2,s_5\})$ or $M_2 = (\{s_1,s_2\},\{s_3,s_4,s_5\})$. W.l.o.g. the outcome function $\tilde{g} : \mathcal{P}^S \longrightarrow \mathcal{M}$ maps the preference relation $\succ^S$ on $M_1$.

Now, if agent $s_5$ misrepresents his preferences such that $(h_1,\{s_1\}) \succ^S_{s_5} (h_1,\{s_3\})$
$\succ^S_{s_5} (h_1,\{s_4\}) \succ^S_{s_5} (h_1,\{s_2\}) \succ^S_{s_5} (h_2,\{s_3,s_4\}) \succ^S_{s_5} (h_2,\{s_2,s_4\})$
$\succ^S_{s_5} (h_2,\{s_1,s_2\}) \succ^S_{s_5} (h_2,\{s_1,s_4\}) \succ^S_{s_5} (h_2,\{s_1,s_3\}) \succ^S_{s_5} (h_2,\{s_2,s_3\})$

he achieves a better rank. **P2** determines the matching $M_3 = (\{s_1,s_5\},\{s_2,s_3,s_4\})$.

Incentive Compatibility of **P2** given setting 2.:

If all participants state their preferences truthfully, the procedure determines the matching $M_2' = (\{s_1, s_2, h_1\}, \{s_3, s_4, s_5, h_2\})$. Analyzing the example, it turns out that we find an agent as well as a firm that gets a higher rank by not revealing their true preferences. This time agent $s_3$ gets a higher rank if he states the preferences:

$(h_1, \{s_5, h_1\}) \succ^P_{s_3} (h_1, \{s_4, h_1\}) \succ^S_{s_3} (h_1, \{s_2, h_1\}) \succ^P_{s_3} (h_1, \{s_1, h_1\})$
$\succ^P_{s_3} (h_1, \{s_2, s_4, h_1\}) \succ^P_{s_3} (h_1, \{s_1, s_2, h_1\}) \succ^P_{s_3} (h_2, \{s_1, s_4, h_2\})$
$\succ^P_{s_3} (h_2, \{s_2, s_5, h_2\}) \succ^P_{s_3} (h_2, \{s_4, s_5, h_2\}) \succ^P_{s_3} (h_2, \{s_1, s_5, h_2\}) \ldots.$
We get the matching $M_4 = (\{s_3, s_5, h_1\}, \{s_1, s_2, s_4, h_2\})$.

Whereas firm $h_1$ gets its most preferred rank, if it misrepresents its preferences, such that

$(h_1, \{s_3, s_5\}) \succ^P_{h_1} (h_1, \{s_4, s_5\}) \succ^P_{h_1} (h_1, \{s_2, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_4\}) \succ^P_{h_1} (h_1, \{s_2, s_5\})$
$\succ^P_{h_1} (h_1, \{s_1, s_3\}) \succ^P_{h_1} (h_2, \{s_3, s_4\}) \succ^P_{h_1} (h_2, \{s_1, s_5\}) \succ^P_{h_1} (h_2, \{s_3, s_4, s_5\})$
$\succ^P_{h_1} (h_2, \{s_1, s_2\}) \succ^P_{h_1} (h_2, \{s_2, s_3\}) \ldots$

the solution is $M_3' = (\{s_1, s_5, h_1\}, \{s_2, s_3, s_4, h_2\})$. □

**Example A.0.3 (P3b and Incentive Compatibility)**

Let $P = \{h_1, h_2, h_3, s_1, s_2, s_3, s_4, s_5\}$ and $\tilde{\kappa}_1 = \tilde{\kappa}_2 = 3 \quad \tilde{\kappa}_3 = 2$. The lexicographic admissible preferences of the participants are given by:

$(h_3, \{h_3\}) \succ^P_{s_1} (h_1\{s_2, h_1\}) \succ^P_{s_1} (h_1, \{s_3, h_1\}) \succ^P_{s_1} (h_1, \{s_5, h_1\}) \succ^P_{s_1} (h_1, \{s_4, h_1\})$
$\succ^P_{s_1} (h_2, \{s_2, h_2\}) \succ^P_{s_1} (h_2, \{s_5, h_2\}) \succ^P_{s_1} (h_2, \{s_3, h_2\}) \succ_{s_1} (h_2, \{s_4, h_2\}) \ldots$

$(h_1, \{s_1, h_1\}) \succ^P_{s_2} (h_1, \{s_4, h_1\}) \succ^P_{s_2} (h_1, \{s_5, h_1\}) \succ^P_{s_2} (h_1, \{s_3, h_1\}) \succ^P_{s_2} (h_3, \{h_3\})$
$\succ^P_{s_2} (h_2, \{s_1, h_2\}) \succ_{s_2} P(h_2, \{s_5, h_2\}) \succ_{s_2} P(h_2, \{s_4, h_2\}) \succ_{s_2} P(h_2, \{s_3, h_2\}) \ldots$

$(h_1, \{s_4, h_1\}) \succ_{s_3} P(h_1, \{s_5, h_1\}) \succ_{s_3} P(h_1, \{s_2, h_1\}) \succ_{s_3} P(h_1, \{s_1, h_1\})$
$\succ^P_{s_3} (h_3, \{h_3\}) \succ^P_{s_3} (h_2, \{s_4, h_2\}) \succ^P_{s_3} (h_2, \{s_1, h_2\}) \succ^P_{s_3} (h_2, \{s_5, h_2\})$
$\succ^P_{s_3} (h_2, \{s_2, h_2\}) \ldots$

$(h_1, \{s_3, h_1\}) \succ^P_{s_4} (h_1, \{s_1, h_1\}) \succ^P_{s_4} (h_1, \{s_5, h_1\}) \succ^P_{s_4} (h_1, \{s_2, h_1\})$
$\succ^P_{s_4} (h_2, \{s_2, h_2\}) \succ^P_{s_4} (h_2, \{s_5, h_2\}) \succ^P_{s_4} (h_2, \{s_1, h_2\}) \succ^P_{s_4} (h_2, \{s_3, h_2\})$

$\succ^P_{s_4} (h_3, \{h_3\}) \ldots$

$(h_3, \{h_3\}) \succ^P_{s_5} (h_2, \{s_4, h_2\}) \succ^P_{s_5} (h_2, \{s_3, h_2\}) \succ^P_{s_5} (h_2, \{s_2, h_2\})$
$\succ^P_{s_5} (h_2, \{s_1, h_2\}) \succ^P_{s_5} (h_1, \{s_4, h_1\}) \succ^P_{s_5} (h_1, \{s_1, h_1\}) \succ^P_{s_5} (h_1, \{s_3, h_1\})$
$\succ^P_{s_5} (h_1, \{s_2, h_1\}) \ldots$

$(h_1, \{s_1, s_2\}) \succ^P_{h_1} (h_1, \{s_3, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_5\})$
$\succ^P_{h_1} (h_1, \{s_2, s_3\}) \succ^P_{h_1} (h_1, \{s_2, s_4\}) \succ^P_{h_1} (h_1, \{s_2, s_5\}) \succ^P_{h_1} (h_1, \{s_1, s_3\})$
$\succ^P_{h_1} (h_1, \{s_3, s_5\}) \succ^P_{h_1} (h_1, \{s_4, s_5\}) \ldots$

$(h_2, \{s_1, s_3\}) \succ^P_{h_2} (h_2, \{s_3, s_5\}) \succ^P_{h_2} (h_3, \{s_2, s_3\}) \succ^P_{h_2} (h_2, \{s_2, s_5\})$
$\succ^P_{h_2} (h_2, \{s_3, s_4\}) \succ^P_{h_2} (h_2, \{s_1, s_2\}) \succ^P_{h_2} (h_2, \{s_4, s_5\}) \succ^P_{h_2} (h_2, \{s_2, s_4\})$
$\succ^P_{h_2} (h_2, \{s_1, s_5\}) \succ^P_{h_2} (h_2, \{s_1, s_4\}) \ldots$

$(h_3, \{s_5\}) \succ^P_{h_3} (h_3, \{s_1\}) \succ^P_{h_3} (h_3, \{s_3\}) \succ^P_{h_3} (h_3, \{s_4\}) \succ^P_{h_3} (h_3, \{s_2\}) \ldots$

Incentive Compatibility of **P3b** given admissible lexicographic preferences:

First, firm $h_1$ chooses among the set of selected agents $s_2, s_3, s_4$ those he mostly prefers, namely the agents $s_3$ and $s_4$. The firm $h_3$ selects agents $s_5$, given the set of agents $\{s_1, s_5\}$. Hence, we get the matching $M_1 = (\{s_3, s_4, h_1\}, \{s_1, s_2, h_2\}, \{s_5, h_3\})$. But now agent $s_1$ can improve his situation if he misrepresents his preferences in the following way:

$(h_1, \{s_2, h_1\}) \succ^P_{s_1} (h_1, \{s_3, h_1\}) \succ^P_{s_1} (h_1, \{s_5, h_1\}) \succ^P_{s_1} (h_1, \{s_4, h_1\}) \succ^P_{s_1} (h_3, \{h_3\})$
$\succ^P_{s_1} (h_2, \{s_2, h_2\}) \succ^P_{s_1} (h_2, \{s_5, h_2\}) \succ^P_{s_1} (h_2, \{s_3, h_2\}) \succ^P_{s_1} (h_2, \{s_4, h_2\}) \ldots$

Then agent $s_1$ is matched with firm $h_1$ instead of his less preferred firm $h_2$, because firm $h_1$ prefers the $\tilde{\kappa}_1$-clique $C = \{s_1, s_2\}$ to $C' = \{s_3, s_4\}$. **P3b** selects the matching $M_2 = (\{s_1, s_2, h_1\}, \{s_3, s_4, h_2\}, \{s_5, h_3\})$. The clique working in firm $h_3$ remains the same (red colored). □

**Example A.0.4 (P1a and the results of Chapter 3 and Section 4.1)**

1. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$, $H = \{h_1, h_2, h_3\}$ and $\kappa_1 = 1$, $\kappa_2 = \kappa_3 = 2$. The agents' preferences are given by:

$(h_3,\{s_4\}) \succ^S_{s_1} (h_2\{s_2\}) \succ^S_{s_1} (h_3,\{s_3\}) \succ^S_{s_1} (h_3,\{s_2\}) \succ^S_{s_1} (h_1,\emptyset)$
$\succ^S_{s_1} (h_2,\{s_5\}) \succ^S_{s_1} (h_3,\{s_5\}) \succ^S_{s_1} (h_2,\{s_3\}) \succ^S_{s_1} (h_2,\{s_4\})$

$(h_1,\emptyset) \succ^S_{s_2} (h_3,\{s_5\}) \succ^S_{s_2} (h_3,\{s_4\}) \succ^S_{s_2} (h_2,\{s_1\}) \succ^S_{s_2} (h_3,\{s_1\})$
$\succ^S_{s_2} (h_2,\{s_5\}) \succ^S_{s_2} (h_2,\{s_4\}) \succ^S_{s_2} (h_2,\{s_3\}) \succ^S_{s_2} (h_3,\{s_3\})$

$(h_3,\{s_1\}) \succ^S_{s_3} (h_3,\{s_5\}) \succ^S_{s_3} (h_1,\emptyset) \succ^S_{s_3} (h_2,\{s_4\}) \succ^S_{s_3} (h_3,\{s_2\})$
$\succ^S_{s_3} (h_2,\{s_1\}) \succ^S_{s_3} (h_2,\{s_5\}) \succ^S_{s_3} (h_2,\{s_2\}) \succ^S_{s_3} (h_3,\{s_4\})$

$(h_3,\{s_3\}) \succ^S_{s_4} (h_3,\{s_1\}) \succ^S_{s_4} (h_2,\{s_2\}) \succ^S_{s_4} (h_3,\{s_5\}) \succ^S_{s_4} (h_2,\{s_5\})$
$\succ^S_{s_4} (h_1,\emptyset) \succ^S_{s_4} (h_2,\{s_1\}) \succ^S_{s_4} (h_2,\{s_3\}) \succ^S_{s_4} (h_3,\{s_2\})$

$(h_2,\{s_4\}) \succ^S_{s_5} (h_3,\{s_4\}) \succ^S_{s_5} (h_2,\{s_3\}) \succ^S_{s_5} (h_2,\{s_2\}) \succ^S_{s_5} (h_3,\{s_1\})$
$\succ^S_{s_5} (h_3,\{s_3\}) \succ^s_{s_5} (h_2,\{s_1\}) \succ^S_{s_5} (h_1,\emptyset) \succ^S_{s_5} (h_3,\{s_2\})$

2. Let $P = \{h_1, h_2, h_3, s_1, s_2, s_3, s_4, s_5\}$ and $\tilde{\kappa}_1 = 2$, $\tilde{\kappa}_2 = \tilde{\kappa}_3 = 3$. The participants' admissible preferences are given by:
$(h_3,\{s_4,h_3\}) \succ^P_{s_1} (h_2\{s_2,h_2\}) \succ^P_{s_1} (h_3,\{s_3,h_3\}) \succ^P_{s_1} (h_3,\{s_2,h_3\})$
$\succ^P_{s_1} (h_1,\{h_1\}) \succ^P_{s_1} (h_2,\{s_5,h_2\}) \succ^P_{s_1} (h_3,\{s_5,h_3\}) \succ^P_{s_1} (h_2,\{s_3,h_2\})$
$\succ^P_{s_1} (h_2,\{s_4,h_2\}) \succ^P_{s_1} \dots$

$(h_1,\{h_1\}) \succ^P_{s_2} (h_3,\{s_5,h_3\}) \succ^P_{s_2} (h_3,\{s_4,h_3\}) \succ^P_{s_2} (h_2,\{s_1,h_2\})$
$\succ^P_{s_2} (h_3,\{s_1,h_3\}) \succ^P_{s_2} (h_2,\{s_5,h_2\}) \succ^P_{s_2} (h_2,\{s_4,h_2\}) \succ^P_{s_2} (h_2,\{s_3,h_2\})$
$\succ^P_{s_2} (h_3,\{s_3,h_3\}) \succ^P_{s_2} \dots$

$(h_3,\{s_1,h_3\}) \succ^P_{s_3} (h_3,\{s_5,h_3\}) \succ^P_{s_3} (h_1,\{h_1\}) \succ^P_{s_3} (h_2,\{s_4,h_2\})$
$\succ^P_{s_3} (h_3,\{s_2,h_3\}) \succ^P_{s_3} (h_2,\{s_1,h_2\}) \succ^P_{s_3} (h_2,\{s_5,h_2\}) \succ^P_{s_3} (h_2,\{s_2,h_2\})$
$\succ^P_{s_3} (h_3,\{s_4,h_3\}) \succ^P_{s_3} \dots$

$(h_3,\{s_3,h_3\}) \succ^P_{s_4} (h_3,\{s_1,h_3\}) \succ^P_{s_4} (h_2,\{s_2,h_2\}) \succ^P_{s_4} (h_3,\{s_5,h_3\})$
$\succ^P_{s_4} (h_2,\{s_5,h_2\}) \succ^P_{s_4} (h_1,\{h_1\}) \succ^P_{s_4} (h_2,\{s_1,h_2\}) \succ^P_{s_4} (h_2,\{s_3,h_2\})$
$\succ^P_{s_4} (h_3,\{s_2,h_3\}) \succ^P_{s_4} \dots$

$(h_2, \{s_4, h_2\}) \succ^P_{s_5} (h_3, \{s_4, h_3\}) \succ^P_{s_5} (h_2, \{s_3, h_2\}) \succ^P_{s_5} (h_2, \{s_2, h_2\})$
$\succ^P_{s_5} (h_3, \{s_1, h_3\}) \succ^P_{s_5} (h_3, \{s_3, h_3\}) \succ^P_{s_5} (h_2, \{s_1, h_2\}) \succ^P_{s_5} (h_1, \{h_1\})$
$\succ^P_{s_5} (h_3, \{s_2, h_3\}) \succ^P_{s_5} \ldots$

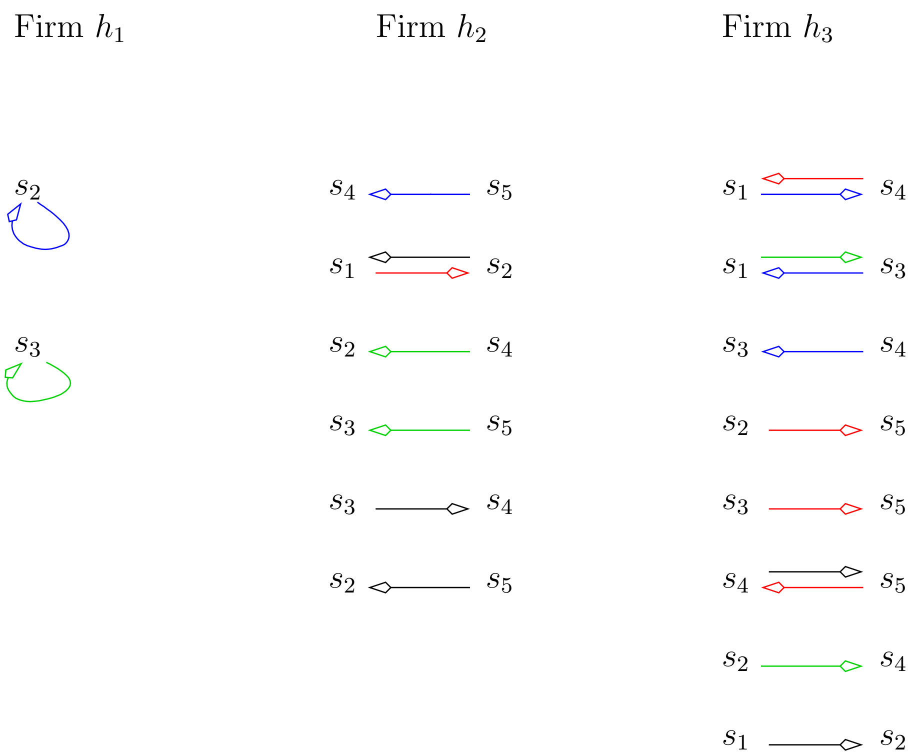
$(h_1, \{s_2\}) \succ^P_{h_1} (h_1, \{s_1\}) \succ^P_{h_1} (h_1, \{s_3\}) \succ^P_{h_1} (h_1, \{s_4\}) \succ^P_{h_1} (h_1, \{s_5\}) \succ^P_{h_1} \ldots$

$(h_2, \{s_1, s_2\}) \succ^P_{h_2} (h_2, \{s_1, s_3\}) \succ^P_{h_2} (h_2, \{s_1, s_4\}) \succ^P_{h_2} (h_2, \{s_1, s_5\})$
$\succ^P_{h_2} (h_2, \{s_2, s_3\}) \succ^P_{h_2} (h_2, \{s_2, s_4\}) \succ^P_{h_2} (h_2, \{s_2, s_5\}) \succ^P_{h_2} (h_2, \{s_3, s_4\})$
$\succ^P_{h_2} (h_2, \{s_3, s_5\}) \succ^P_{h_2} (h_2, \{s_4, s_5\}) \succ^P_{h_2} \ldots$

$(h_3, \{s_1, s_3\}) \succ^P_{h_3} (h_3, \{s_3, s_5\}) \succ^P_{h_3} (h_3, \{s_2, s_3\}) \succ^P_{h_3} (h_3, \{s_2, s_5\})$
$\succ^P_{h_3} (h_3, \{s_3, s_4\}) \succ^P_{h_3} (h_3, \{s_1, s_2\}) \succ^P_{h_3} (h_3, \{s_4, s_5\}) \succ^P_{h_3} (h_3, \{s_2, s_4\})$
$\succ^P_{h_3} (h_3, \{s_1, s_5\}) \succ^P_{h_3} (h_3, \{s_1, s_4\}) \succ^P_{h_3} \ldots$

Graph theoretic illustration of **P1a** given setting 1.:

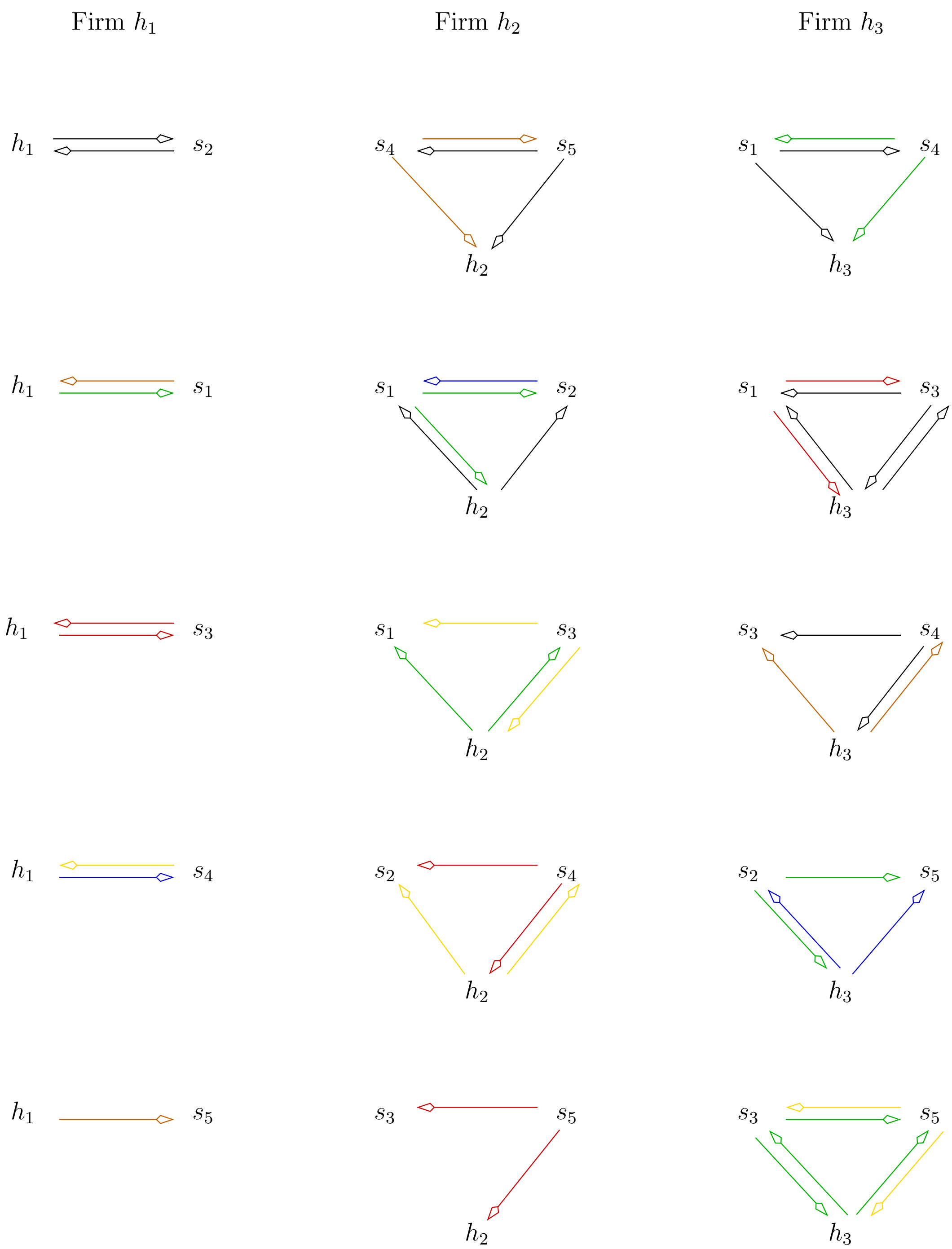
Agents only have preferences over agents. They do not ask firms for joint work and the firms have no influence on the course of the procedure. This exactly reflects the basic model introduced in Chapter 2. Illustrating the offers until procedure **P1a** stops, we get the next figure:

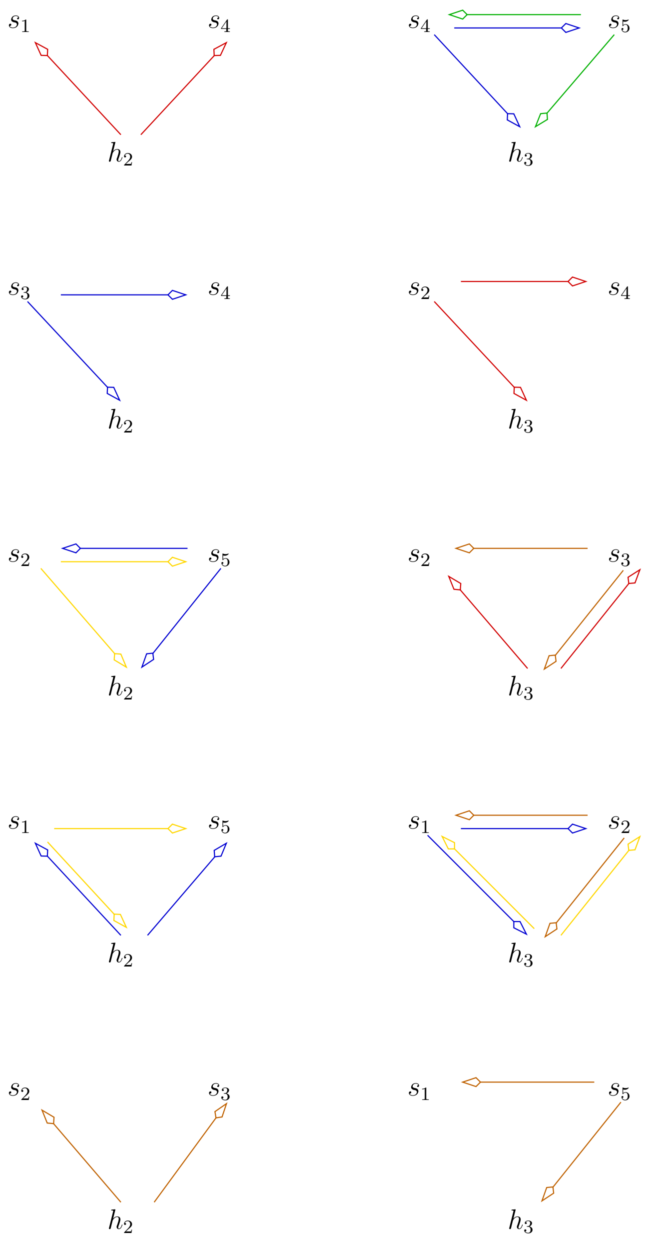



Each agent makes four offers until the procedure **P1a** stops. We observe two complete graphs with loops in firm $h_1$, one complete graph without loops in firm $h_2$ and three complete graphs without loops in firm $h_3$. Each complete graph in the second and the third firm includes two vertices/agents. Checking the conditions 2.1 and 2.2, we immediately realize that **P1a** stopped, since the matching $M_1 = (\{s_3\}, \{s_1, s_2\}, \{s_4, s_5\})$ is given.

Graph theoretic illustration of **P1a** given setting 2.:

Next we again want to illustrate the offers until **P1a** stops. After six rounds the procedure stops. We get the matching $M_2 = (\{s_4, h_1\}, \{s_1, s_2, h_2\}, \{s_3, s_5, h_3\})$.


Firm $h_1$
Firm $h_2$
Firm $h_3$
$h_1$ $s_2$
$s_4$ $s_5$ $h_2$
$s_1$ $s_4$ $h_3$
$h_1$ $s_1$
$s_1$ $s_2$ $h_2$
$s_1$ $s_3$ $h_3$
$h_1$ $s_3$
$s_1$ $s_3$ $h_2$
$s_3$ $s_4$ $h_3$
$h_1$ $s_4$
$s_2$ $s_4$ $h_2$
$s_2$ $s_5$ $h_3$
$h_1$ $s_5$
$s_3$ $s_5$ $h_2$
$s_3$ $s_5$ $h_3$




□

**Example A.0.5 (P1a; P2 and the results of Chapter 3 and Section 4.1)**

1. Let $H = \{h_1, h_2, h_3\}$ with $\kappa_1 = \kappa_2 = 2, \kappa_3 = 1$ and let $S = \{s_1, s_2, s_3, s_4, s_5\}$. The agents' lexicographic preferences are given by[1]:

[1]A bundle with different colors belongs to different matchings.

$(h_1, \{s_2\}) \succ_1^S (h_1, \{s_4\}) \succ_1^S (h_1, \{s_3\}) \succ_1^S (h_1, \{s_5\}) \succ_1^S (h_2, \{s_5\})$
$\succ_1^S (h_2, \{s_4\}) \succ_1^S (h_2, \{s_3\}) \succ_1^S (h_2, \{s_2\}) \succ_1^S (h_3, \emptyset)$

$(h_2, \{s_1\}) \succ_2^S (h_2, \{s_3\}) \succ_2^S (h_2, \{s_4\}) \succ_2^S (h_2, \{s_5\}) \succ_2^S (h_1, \{s_1\})$
$\succ_2^S (h_1, \{s_4\}) \succ_2^S (h_1, \{s_5\}) \succ_2^S (h_1, \{s_3\}) \succ_2^S (h_3, \emptyset)$

$(h_1, \{s_1\}) \succ_3^S (h_1, \{s_2\}) \succ_3^S (h_1, \{s_5\}) \succ_3^S (h_1, \{s_4\}) \succ_3^S (h_3, \emptyset)$
$\succ_3^S (h_2, \{s_1\}) \succ_3^S (h_2, \{s_5\}) \succ_3^S (h_2, \{s_4\} \succ_3^S (h_2, \{s_2\}))$

$(h_1, \{s_1\}) \succ_4^S (h_1, \{s_5\}) \succ_4^S (h_1, \{s_3\}) \succ_4^S (h_1, \{s_2\}) \succ_4^S (h_2, \{s_5\})$
$\succ_4^S (h_2, \{s_1\}) \succ_4^S (h_2, \{s_2\}) \succ_4^S (h_2, \{s_3\}) \succ_4^S (h_3, \emptyset)$

$(h_1, \{s_2\}) \succ_5^S (h_1, \{s_1\}) \succ_5^S (h_1, \{s_4\}) \succ_5^S (h_1, \{s_3\}) \succ_5^S (h_2, \{s_4\})$
$\succ_5^S (h_2, \{s_1\}) \succ_5^S (h_2, \{s_2\}) \succ_5^S (h_2, \{s_3\}) \succ_5^S (h_3, \emptyset)$

2. We have $P = \{s_1, s_2, s_3, s_4, s_5, h_1, h_2, h_3\}$ participants and $\tilde{\kappa}_1 = \tilde{\kappa}_2 = 3, \tilde{\kappa}_3 = 2$ jobs. The participants' lexicographic admissible preferences are given by:

   $(h_1, \{s_2, h_1\}) \succ_{s_1}^P (h_1, \{s_4, h_1\}) \succ_{s_1}^P (h_1, \{s_3, h_1\}) \succ_{s_1}^P (h_1, \{s_5, h_1\})$
   $\succ_{s_1}^P (h_2, \{s_5, h_2\}) \succ_{s_1}^P (h_2, \{s_4, h_2\}) \succ_{s_1}^P (h_2, \{s_3, h_2\}) \succ_{s_1}^P (h_2, \{s_2, h_2\})$
   $\succ_{s_1}^P (h_3, \{h_3\}) \ldots$

   $(h_2, \{s_1, h_2\}) \succ_{s_2}^P (h_2, \{s_3, h_2\}) \succ_{s_2}^P (h_2, \{s_4, h_2\}) \succ_{s_2}^P (h_2, \{s_5, h_2\})$
   $\succ_{s_2}^P (h_1, \{s_1, h_1\}) \succ_{s_2}^P (h_1, \{s_4, h_1\}) \succ_{s_2}^P (h_1, \{s_5, h_1\}) \succ_{s_2}^P (h_1, \{s_3, h_1\})$
   $\succ_{s_2}^P (h_3, \{h_3\}) \ldots$

   $(h_1, \{s_1, h_1\}) \succ_{s_3}^P (h_1, \{s_2, h_1\}) \succ_{s_3}^P (h_1, \{s_5, h_1\}) \succ_{s_3}^P (h_1, \{s_4, h_1\})$
   $\succ_{s_3}^P (h_3, \{h_3\}) \succ_{s_3}^P (h_2, \{s_1, h_2\}) \succ_{s_3}^P (h_2, \{s_5, h_2\}) \succ_{s_3}^P (h_2, \{s_4, h_2\}$
   $\succ_{s_3}^P (h_2, \{s_2, h_2\})) \ldots$

   $(h_1, \{s_1, h_1\}) \succ_{s_4}^P (h_1, \{s_5, h_1\}) \succ_{s_4}^P (h_1, \{s_3, h_1\}) \succ_{s_4}^P (h_1, \{s_2, h_1\})$
   $\succ_{s_4}^P (h_2, \{s_5, h_2\}) \succ_{s_4}^P (h_2, \{s_1, h_2\}) \succ_{s_4}^P (h_2, \{s_2, h_2\}) \succ_{s_4}^P (h_2, \{s_3, h_2\})$
   $\succ_{s_4}^P (h_3, \{h_3\}) \ldots$

$$(h_1, \{s_2, h_1\}) \succ^P_{s_5} (h_1, \{s_1, h_1\}) \succ^P_{s_5} (h_1, \{s_4, h_1\}) \succ^P_{s_5} (h_1, \{s_3, h_1\}) \succ^P_{s_5} (h_2, \{s_4, h_2\}) \succ^P_{s_5} (h_2, \{s_1, h_2\}) \succ^P_{s_5} (h_2, \{s_2, h_2\}) \succ^P_{s_5} (h_2, \{s_3, h_2\}) \succ^P_{s_5} (h_3, \{h_3\}) \ldots$$

$$(h_1, \{s_1, s_2\}) \succ^P_{h_1} (h_1, \{s_1, s_3\}) \succ^P_{h_1} (h_1, \{s_1, s_4\}) \succ^P_{h_1} (h_1, \{s_1, s_5\}) \succ^P_{h_1} (h_1, \{s_2, s_3\}) \succ^P_{h_1} (h_1, \{s_2, s_4\}) \succ^P_{h_1} (h_1, \{s_2, s_5\}) \succ^P_{h_1} (h_1, \{s_3, s_4\}) \succ^P_{h_1} (h_1, \{s_3, s_5\}) \succ^P_{h_1} (h_1, \{s_4, s_5\}) \ldots$$

$$(h_2, \{s_1, s_3\}) \succ^P_{h_2} (h_2, \{s_3, s_5\}) \succ^P_{h_2} (h_2, \{s_2, s_3\}) \succ^P_{h_2} (h_2, \{s_2, s_5\}) \succ^P_{h_2} (h_2, \{s_3, s_4\}) \succ^P_{h_2} (h_2, \{s_1, s_2\}) \succ^P_{h_2} (h_2, \{s_4, s_5\}) \succ^P_{h_2} (h_2, \{s_2, s_4\}) \succ^P_{h_2} (h_2, \{s_1, s_5\}) \succ^P_{h_2} (h_2, \{s_1, s_4\}) \ldots$$

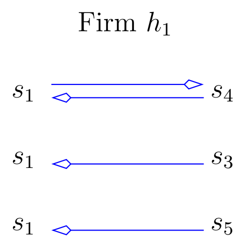
$$(h_3, \{s_2\}) \succ^P_{h_3} (h_3, \{s_1\}) \succ^P_{h_3} (h_3, \{s_3\}) \succ^P_{h_3} (h_3, \{s_4\}) \succ^P_{h_3} (h_3, \{s_5\}) \ldots$$

Graph theoretic illustration of **P1a** given setting 1.:

First, we analyze procedure **P1a**. All agents make their offers as long as a matching occurs. We get the matching $M = (\{s_1, s_2\}, \{s_4, s_5\}, \{s_3\})$. It satisfies the agent-max-min property, is Pareto efficient, but it is not $t_a$-stable. Either the agents $s_2$ and $s_4$ or the agents $s_2$ and $s_5$ constitute $t_a$-blocking coalitions.

Graph theoretic illustration of **P2** given setting 1.:

Apart from agent $s_2$, all agents mostly prefer firm $h_1$. Therefore, the agents $s_1, s_3, s_4, s_5$ and firm $h_1$ are selected by the procedure. The agents start to make their offers and it turns out that the first complete graph consists of agents $s_1$ and $s_4$. Therefore, see the next picture:



The agents $s_1$ and $s_4$ are matched with the firm $h_1$. The firm and the agents leave the market. We get a reduced set of preferences.

$(h_2, \{s_3\}) \succ_2^S (h_2, \{s_5\}) \succ_2^S (h_3, \emptyset)$

$(h_3, \emptyset) \succ_3^S (h_2, \{s_5\}) \succ_3^S (h_2, \{s_2\})$

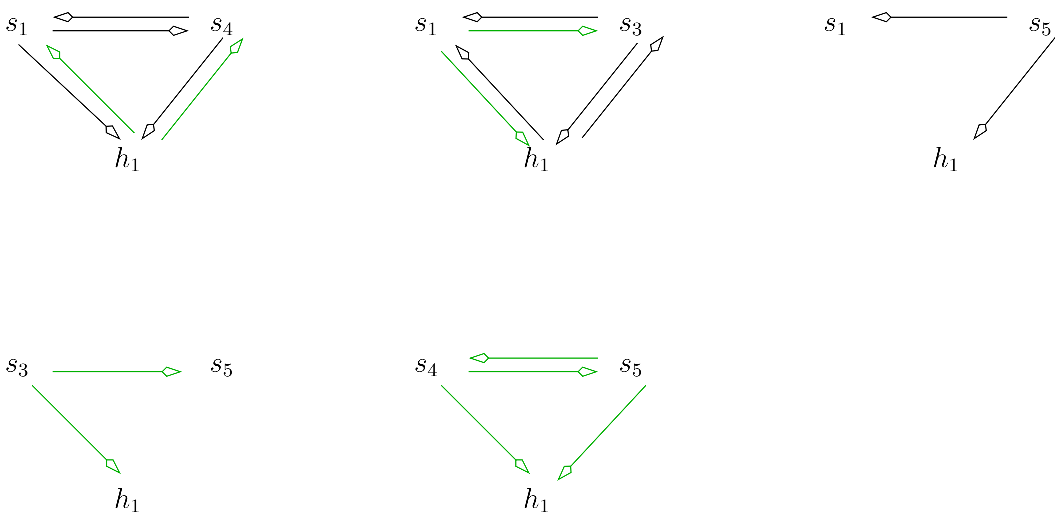
$(h_2, \{s_2\}) \succ_5^S (h_2, \{s_3\}) \succ_5^S (h_3, \emptyset)$

On the second level the agents $s_2$ and $s_5$ are selected together with firm $h_2$ and in addition, agent $s_3$ is selected with firm $h_3$. This immediately yields to the solution of the procedure. We get the matching $M^1 = (\{s_1, s_4\}, \{s_2, s_5\}, \{s_3\})$. The solution $g^{\mathbf{P2}}(\succ^S)$ does not satisfy the agent-max-min property, because agent $s_5$ gets a lower rank than in the matching $M$, selected by **P1a** given the setting 1.

<u>Graph theoretic illustration of **P2** given setting 2.</u>:

We select the agents $s_1, s_3, s_4, s_5$ and the firm $h_1$. Now the agents start to make their offers. In the next figure, we illustrate the offers, selected participants are making among each other.

Firm $h_1$



After two rounds of offers, we get two complete graphs. The first one consists of the agents $s_1, s_4$ and firm $h_1$, the second one of the agents $s_1, s_3$ and again firm $h_1$. We match $s_1, s_4, h_1$. They leave the market. Hence, we get the following reduced set of preferences on the one hand:

$(h_2, \{s_3, h_2\}) \succ^P_{s_2} (h_2, \{s_5, h_2\}) \succ^P_{s_2} (h_3, \{h_3\}) \ldots$

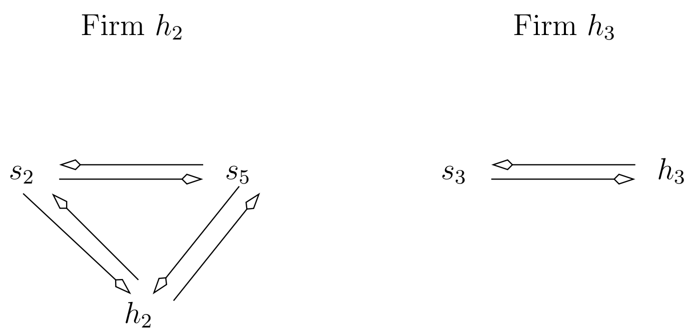
$(h_3, \{h_3\}) \succ^P_{s_3} (h_2, \{s_5, h_2\}) \succ^P_{s_3} (h_2, \{s_2, h_2\})) \ldots$

$(h_2, \{s_2, h_2\}) \succ^P_{s_5} (h_2, \{s_3, h_2\}) \succ^P_{s_5} (h_3, \{h_3\}) \ldots$

$(h_2, \{s_3, s_5\}) \succ^P_{h_2} (h_2, \{s_2, s_3\}) \succ^P_{h_2} (h_2, \{s_2, s_5\}) \ldots$

$(h_3, \{s_2\}) \succ^P_{h_3} (h_3, \{s_3\}) \succ^P_{h_3} (h_3, \{s_5\}) \ldots$

Here the $s_2, s_5, h_2$ are selected on the one hand, $s_3, h_3$ on the other hand. It immediately follows that after one round of offers we get the corresponding complete graphs. The matching $M^1 = (\{s_1, s_4, h_1\}, \{s_2, s_5, h_2\}, \{s_3, h_3\})$ is composed. Next, we briefly illustrate the round of offers.




On the other hand, if we match $s_1, s_3, h_1$ we get the following scenario. They leave the market. Therefore, we get another reduced set of preferences:

$(h_2, \{s_4, h_2\}) \succ^P_{s_2} (h_2, \{s_5, h_2\}) \succ^P_{s_2} (h_3, \{h_3\}) \ldots$

$(h_2, \{s_5, h_2\}) \succ^P_{s_4} (h_2, \{s_2, h_2\}) \succ^P_{s_4} (h_3, \{h_3\}) \ldots$

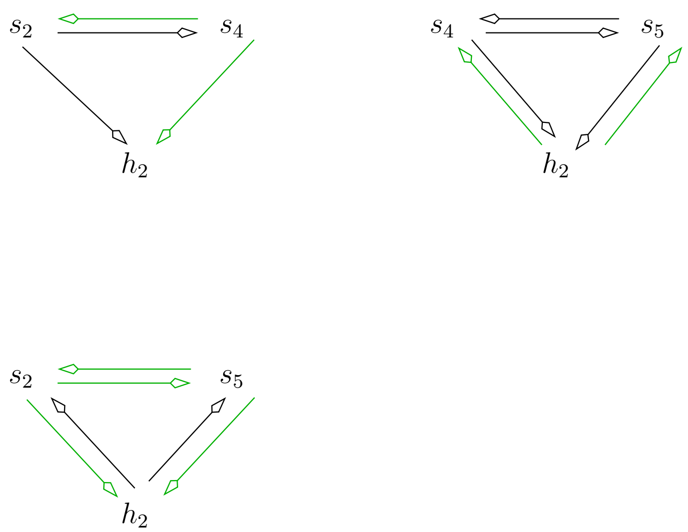
$(h_2, \{s_4, h_2\}) \succ^P_{s_5} (h_2, \{s_2, h_2\}) \succ^P_{s_5} (h_3, \{h_3\}) \ldots$

$(h_2, \{s_2, s_5\}) \succ^P_{h_2} (h_2, \{s_4, s_5\}) \succ^P_{h_2} (h_2, \{s_2, s_4\}) \ldots$

$(h_3, \{s_2\}) \succ^P_{h_3} (h_3, \{s_4\}) \succ^P_{h_3} (h_3, \{s_5\}) \ldots$

The procedure **P2** selects the agents $s_2, s_4, s_5$ and firm $h_2$. These participants start to announce the offers among each other. This is illustrated in the last figure. We get two more matchings, $M^2 = (\{s_1, s_3, h_1\}, \{s_4, s_5, h_2\}, \{s_2, h_3\})$ or $M^3 = (\{s_1, s_3, h_1\}, \{s_2, s_5, h_2\}, \{s_4, h_3\})$.

Firm $h_2$




□

# Appendix B

# Existing Procedures

We briefly want to give short overviews of procedures, already existing in the literature, to get a good intuition of their proceedings. The selection is based on the comparisons, which we make in the thesis. The purpose of this appendix is to facilitate comprehension, if we refer in former chapters to the procedures outlined here. For a deeper understanding or a more precise description, we kindly ask the reader to use the references.

### B.0.1 The Deferred Acceptance Procedure

The procedure is first given in Gale & Shapley (1962). For deepening see also Roth & Sotomayor (1990).

We have two finite disjoint sets, men and women, on the market. Each man has strict preferences over the women, and each woman has strict preferences over the men. A woman is *acceptable* to a man, if he likes her at least as well as remaining single, analogously for the women.
We cite the procedure from Gale & Shapley (1962):
"To start, each man proposes to his favorite woman, that is, to the first woman

on his preference list of acceptable women. Each woman rejects the proposal of any man who is unacceptable to her and each woman who receives more than one proposal rejects all but her most preferred of these. Any man whose proposal is not rejected at this point is kept "engaged".

At any step any man who was rejected at the previous step proposes to his next choice (i.e. to his most preferred woman among those who have not yet rejected him), so long as there remains an acceptable woman to whom he has not yet proposed. (If at any step of the procedure a man has already proposed to and been rejected by, all of the women he finds acceptable, then he issues no further proposals). Each woman receiving proposals rejects any from unacceptable men, and also rejects all but her most preferred among the group consisting of the new proposers together with any man she may kept engaged from the previous step.

The algorithm stops after any step in which no man is rejected. At this point, every man is either engaged to some woman or has been rejected by every woman on his list of acceptable women. The marriages are now consummated, with each man being matched to the woman to whom he is engaged. Women who did not receive any acceptable proposal, and men who were rejected by all women acceptable to them, will stay single."

### B.0.2 The Reduction Mechanism

Balinski & Ratier (1997) and Balinski & Ratier (1998) specify the marriage problem by a directed graph defined over a grid. We have two finite disjoint sets, men and women. Each man has strict preferences over the women, and each woman has strict preferences over the men. A woman is *acceptable* to a man, if he likes her at least as well as remaining single, analogously for the women.

The nodes of the marriage graphs are pairs of men and women, acceptable to each other. Each row in the grid corresponds to a man, each column to a woman. The directed arcs of the directed graph are horizontal or vertical arcs. A horizontal arc expresses a men's preference for a woman over another woman. The same holds for an vertical arc for a women.

We cite the procedure from Balinski & Ratier (1997):

"For each row and column in any order find in the row of a man the best-man node. Eliminate all nodes preceding this node in the woman's column; find in the column of each woman, the woman-best node, eliminate all nodes preceding this node in the man's row; until every man-best node is women-worst, and every woman-best is man-worst."

### B.0.3 Multi-Stage Deferred Acceptance Algorithm

We want to give a rough overview of the algorithm. For a detailed description of the various restrictions on preferences see Dutta & Masso (1997).

The algorithm consists of two main stages. In *stage 1* couples are treated as single individuals. all agents (workers, single individuals=couples) have a strict preference ordering over the set of firms. Firms have *group substitutable preferences*. Now, the deferred-acceptance algorithm is used with agents proposing. We get a matching. If all couples are matched, the procedure stops, otherwise it goes to *stage 2*. The couples not matched in the former stage now have preferences over firms and their partner. The couples matched in the former stage are again treated as single individuals. The preference structure of the in *stage 1* matched couples, of the workers and of the firms did not change at all. Again the classical deferred-acceptance algorithm with agents proposing is applied. If the same set of couples is matched as in *stage 1*, the algorithm stops.

# Basic Symbols

| | |
|---|---|
| $H$ | set of firms |
| $m$ | number of firms |
| $S$ | set of agents |
| $n$ | number of agents |
| $\kappa_k$ | set of all offered jobs in firm $k$ |
| $\kappa$ | total number of offered jobs |
| $\mathbf{C}_q$ | set of subsets with $q$ agents |
| $\mathbf{C}_{q-1}^{-i}$ | set of subsets of $q-1$ agents and without agent $i$ |
| $b_i = (k, C)$ | with $C \in \mathbf{C}_{q-1}^{-i}$ bundle of agent $i$ |
| $B_i^S$ | set of all bundles of agent $i$ |
| $\succ^S$ | agents' preference profile |
| $\succ_i^S$ | preference relation of agent $i$ |
| $\mathcal{P}^S$ | set of all possible preference profiles |
| $\mathcal{P}_i^S$ | set of all possible preferences of agent $i$ |
| $M(\succ_i^S)$ | preference relation of agent $i$ on $\mathcal{M}$ |
| $M_P(\succ_i^S)$ | strict preference relation of agent $i$ on $\mathcal{M}$ |
| $M_I(\succ_i^S)$ | indifferent preference relation of agent $i$ on $\mathcal{M}$ |
| $\mathcal{P}^{lex}$ | set of all possible lexicographic preference profiles |
| $\mathcal{P}_i^{lex}$ | set of all possible lexicographic preferences of agent $i$ |
| $H(\succ_i^S)$ | preference relation of agent $i$ on $H$ |
| $\mu$ | matching function |
| $C_{\mu,k}$ | $\kappa_k$-clique under $\mu$ for each firm $k \in H$ |
| $M_\mu$ | a matching under $\mu$ |
| $\mathcal{M}$ | set of all matchings |
| $v_i^S$ | agent i's ranking function |
| $g$ | outcome correspondence |

| | |
|---|---|
| $P$ | set of participants |
| $\tilde{\kappa}_k$ | number of offered jobs per firm |
| $\mathbf{D}_{q+1}$ | set of subsets of $q+1$ participants |
| $\mathbf{D}_q^{-p}$ | set of subsets of $q$ colleagues without participant $p$ |
| $\succ^P$ | participants' preference profile |
| $\succ_p^P$ | preference relation of participant $p$ |
| $\mathcal{P}^P$ | set of all possible preference profiles |
| $\mathcal{P}_p^P$ | set of all possible preferences of participant $p$ |
| $b_p = (k, D)$ | with $D \in \mathbf{D}_{\kappa_k}^{-p}$ bundle of participant $p$ |
| $B_p^P$ | set of all bundles of participant $p$ |
| $\mathcal{P}^{adm}$ | set of all possible preference profiles |
| $\mathbf{D}^k$ | set of admissible $\tilde{\kappa}_k$-cliques |
| $\tilde{\mu}$ | (admissible) matching function |
| $D_{\tilde{\mu},k}$ | $\tilde{\kappa}_k$-clique under $\tilde{\mu}$ |
| $M_{\tilde{\mu}}$ | a matching under $\mu$ |
| $\tilde{\mathcal{M}}$ | set of all matchings under $\tilde{\mu}$ |
| $v_p^P$ | participant $p$'s ranking function $p$ |
| $\mathcal{P}^{lexad}$ | set of all possible lexicographic admissible preference profiles |
| $\succ^H$ | firms' preference profile |
| $\succ_k^H$ | preference relation of firm $k$ |
| $\mathcal{P}^H$ | set of all possible preference profiles |
| $\mathcal{P}_k^H$ | set of all possible preferences of firm $k$ |
| $\hat{\mu}$ | inverse matching function |
| $\hat{\mathcal{M}}$ | set of all matchings under $\tilde{\mu}$ |
| $M(\succ_k^H)$ | firm $k$'s preference relation over matchings |
| $v_k^H$ | firm $k$'s preference relation over matchings |

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