On Solution Concepts of Assignment Games

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Contents

Chapter 1

Introduction

The origins of game theory date back to the year 1944 when the authors von Neumann and Morgenstern published their fundamental book [9] about games and economic behavior. Traditionally, this theory deals with situations of conflicts or cooperation which are described by certain mathematical models. In this context typically three different forms of models are considered: the models in extensive form, in normal form and in characteristic function (or coalitional) form. The first two categories of games are the central topics of non-cooperative game theory. Here, games with no binding agreements between the players are treated. In contrast to this kind of models, games in coalitional form permit cooperation and contracts. Hence, the main focus of cooperative game theory is the examination of groups of players (coalitions) who coordinate their actions and pool their winnings. On closer examination, we can observe that these games can be separated into games with or without transferable utility or more commonly with or without side-payments. In coalitional games without side-payments players cannot distribute their collective payoffs. Instead, each coalition is allocated a feasible set of outcomes and a status quo point, which is the payoff, if the players do not find an agreement on some feasible outcome. Different solution concepts make suggestions for potential outcomes. In contrast to coalitional games without side-payments, each coalition of a cooperative game with side-payments gets a payoff that can be distributed among its members. The pertinent question how to divide the payoffs among the members of a formed coalition can be answered with the aid of different solution concepts. Each concept proposes a set of distributions of the total payoffs to the players.

One of the most famous solution concepts in cooperative games with side-payments is the core which was introduced by Gillies [4] in 1953 as an adjunct to the studies of stable sets. This concept describes the set of Pareto optimal payoff vectors such that no coalition can raise a warrantable plea. The central issue is the fact that this concept suggests an empty set or a set that does not have to be a singleton. A second important solution concept beside the core, is the Shapley value which was introduced by Shapley [18] in 1953. This concept characterizes the expected marginal payoffs of the players. In contrast to the core, the Shapley value always exists and is a singleton. Besides the two solution concepts above, we are also interested in two further concepts which are strongly related. On the one hand we consider the nucleolus which was introduced by Schmeidler [16] in 1969. This concept minimizes highest dissatisfactions of coalitions. In our context, we will call a dissatisfaction of a coalition an excess. In addition, we also look at the modified nucleolus or modiclus which was introduced by Sudhölter in a series of papers [27, 29, 30]. In contrast to the nucleolus, the modified nucleolus minimizes the highest difference of excesses, meaning that the power and the blocking power of a coalition are taken into account in the same way. By definition, existence and uniqueness of both solution concepts can be shown.

In this thesis we concentrate on statements of different solution concepts for special classes of cooperative games with transferable utility. In particular, we are interested in the class of assignment games which was introduced by Shapley and Shubik [23] in 1972. Typically, an assignment game describes a one-to-one matching problem or a model for a two-sided market. Special features of these games are that they can be described by non-negative matrices and that the player set is separated into two different types. These properties imply a special structure of assignment games, which allows many novel results in the context of different solution concepts. For example, Shapley and Shubik [23] introduced the core of an assignment game as a non-empty polyhedron of optimal solutions of a corresponding linear program. It can be shown that this linear program only depends on the assignment matrix. A second fundamental and important result for us is about the nucleolus. In their paper, Solymosi and Raghavan [25] found an algorithm which calculates the nucleolus of an assignment game. Here, the maximum number of steps depends on the minimum number of players of both types. In a second paper [26], the authors characterize the assignment matrix in the case of convex assignment games, exact assignment games and in the case of assignment games with a stable core. With the help of these characterizations Raghavan and Sudhölter [12] were able to prove that if the core is stable, then the modified nucleolus is an element of the core.

A relevant task that we tackle in this thesis is to establish a connection between different solution concepts for assignment games. More precisely, our main interest is about general results for special classes of games. First of all, we consider the Shapley value of exact assignment games, before we try to extend our studies to a more general case. In the next step, we analyze some properties of the modified nucleolus. In order to do so, we formulate different properties of the least core of the dual assignment game. Finally, we examine our general results in the case of assignment games which are induced by a 2×2 assignment matrix.

The dissertation is organized as follows. Chapter 2 provides the basic definitions and solution concepts in the context of cooperative game theory. Before we consider special classes of games, we start with the formal definition of a cooperative game with transferable utility. Next, we introduce different solution concepts, which suggest possible distributions of the total payoffs. For different classes of games we verify specific properties of the solution concepts under consideration. In particular, we find many important results for assignment games. The special structure of these games reveals some interesting connections between the different solution concepts. Finally, we discuss the strong null player property of different solution concepts. This property allows us to deal with nullplayers so that we can restrict our attention to assignment games with the same number of players of each type.

In Chapter 3 we concentrate on the Shapley value, on the core, and on the relationship between both concepts. After introducing partially average convex games we present our most relevant result of this chapter; the Shapley value of an exact assignment game is an element of the core. For the direct proof we establish an alternative description of exact assignment games, which proposes a partition of the worth of the different coalitions. It is illustrated by some examples that exactness of an assignment game is a necessary condition for our result. Finally, we focus on non-exact assignment games for which the Shapley value is in the core.

In Chapter 4 we discuss some properties of the least core of the dual assignment game. After considering some general properties of the least core, we specify these results in the case of convex assignment games. In this case a delineation of the least core of the dual assignment game, the Shapley value, and the modiclus is possible. In the next step we generalize some results for convex assignment games to assignment games with a stable core. Another absorbing class is the set of exact assignment games which are induced by a symmetric matrix. Similar to the case of convex assignment games the calculation of the Shapley value and the modiclus is possible in a very simple way.

Chapter 5 deals with assignment games with a stable core which are induced by a 2×2 matrix. Throughout this chapter we pay attention to the intersection of the core and the least core of the dual assignment game. Therefore, we emphasize in the first section the results of the least core of the dual assignment game. We find a way to compute the extreme points of the least core of the dual assignment game only with the aid of the assignment matrix. As it turns out, there exists a geometric presentation of the least core of the dual assignment game. Since the least core is a subset of a hyperplane, we will see that it is simple to draw this polyhedron. In the following section, we discuss some properties of the core of assignment games. In this case, it is also possible to compute the extreme points of the core only with the aid of the assignment matrix. As in the case of the least core, the core is also a subset of a hyperplane, so that it is very simple to consider the intersection of the two polyhedrons. More precisely, we find a connection between the extreme points of the intersection of core and least core and the assignment matrix.

Finally in Chapter 6, we give a short summary of the most important results of the preceding chapters.

Chapter 2

Cooperative Games and Solution Concepts

2.1 Introduction

This chapter introduces the basic definitions of cooperative game theory, in particular the definition of assignment games and some well-known solution concepts like the Shapley value and the core. Furthermore, important connections between the solution concepts and the different classes of games are pointed out. In the first section, we start with the formal definition of a finite n-person cooperative game with transferable utility. In the next step we look at different classes of cooperative games, which satisfy some further conditions. Important classes among the additive games are, the superadditive and the convex games. The second section deals mainly with different solution concepts for cooperative games. These concepts describe different distributions of the total payoff to the players of the player set. The main focus of this section is the core, the Shapley value, the nucleolus and the modified nucleolus. These different concepts supply the different results for different classes of games. In the next section we consider assignment games, which are models for two-sided markets. These games were introduced by Shapley and Shubik [23] and they describe the interaction of two different types of players. With the aid of a non-negative matrix the worth of the potential interaction of different players can be computed. We finish this chapter by discussing the strong nullplayer property of different solution concepts. This property allows us to reduce our attention in the following chapters to assignment games which are induced by a $p \times p$ matrix.

2.2 Cooperative Games with Transferable Utility

We start this section with some basic definitions in the context of cooperative games with finitely many players. For this reason, we define for a finite set I, the power set $\mathcal{P}(I)$ and the complement of a finite subset $S \subseteq I$. In the following definition we will consider the power set.

Definition 2.2.1 *Let I be a finite set with $|I| = n$. Then, the set*

$$\mathcal{P}(I) = \mathcal{P} = \{S \subseteq I\}$$

is the **power set** *of I.*

The power set of I denotes the set of all subsets of I. One should remark that the power set $\mathcal{P}(I)$ has 2^n elements. Another important set is the complement S^c of a subset $S \subseteq I$. More formally, we have the following definition.

Definition 2.2.2 *Let $S \subseteq I$, $|S| = s$, be a subset of a finite set I with $|I| = n$. Then, the* **complement** *S^c of S is defined by*

$$S^c = \{i \in I \mid i \notin S\}.$$

The complement S^c consists all element of I which are not in the set S. As it can be shown in a very simple way, the complement S^c has $n - s$ elements.
With the aid of these useful definitions, we are able to define and to explain cooperative games with transferable utility. A cooperative game assigns a value for every non-empty coalition, that means, we consider a function $\boldsymbol{v}$ from the set of coalitions to the set of payoffs. This function describes how much collective payoff a set S of players can gain by forming a coalition. Additionally, it is assumed that the empty coalition can gain nothing. Summarizing we have the following definition.

Definition 2.2.3 *A* **cooperative game with transferable utility** *is a triple $(I, \mathcal{P}, \boldsymbol{v})$ such that I is a finite set and $\boldsymbol{v} : \mathcal{P}(I) \to \mathbb{R}$ is a real-valued mapping with*

$$\boldsymbol{v}(\emptyset) = 0.$$

The elements of the set I are called players and its power set $\mathcal{P}(I)$ denotes the set of coalitions. Note, that normally the players are named by numbers $1, \dots, n$ or by some abstract index set I. The mapping $\boldsymbol{v}$ is called characteristic function. If the players $S \subseteq I$ agree to cooperate, the profit $\boldsymbol{v}(S)$ from this cooperation is independent of what the players of coalition S^c can do. The idea of cooperative games with transferable utility is that the players $i \in S$ can split up the payoff $\boldsymbol{v}(S)$ among the members of the coalition S.

Remark 2.2.4 *Sometimes we only use the term cooperative game with transferable utility for the characteristic function $\boldsymbol{v}$ and not for the triple $(I, \mathcal{P}, \boldsymbol{v})$.*

Furthermore, we have in this thesis a short notation for the set of cooperative games. More formally we have:

Notation 2.2.5 *The set of cooperative games with player set I is given by*

$$\mathbb{V} = \big\{\boldsymbol{v} \mid \boldsymbol{v} : \mathcal{P}(I) \to \mathbb{R},\ \ \boldsymbol{v}(\emptyset) = 0\big\}.$$

Since we need to know the payoffs of $2^n - 1$ coalitions to identify a cooperative game, it is possible to describe a game $\boldsymbol{v} \in \mathbb{V}$ by a vector $\boldsymbol{z} \in \mathbb{R}^{2^n-1}$.
Sometimes, there exists some players in the game $(I, \mathcal{P}, \boldsymbol{v})$, who are not really important for the payoffs of the coalitions. In other words, we can say that the value $\boldsymbol{v}(S)$ is independent of the membership of this player.

Definition 2.2.6 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game. A player $i \in I$ is a* **nullplayer***, if we have*

$$\boldsymbol{v}(S \cup \{i\}) = \boldsymbol{v}(S) \quad \forall S \subseteq I.$$

The profit of every nullplayer $i \in I$ is zero, that means, it is the same for the coalition S if player i joins to the coalition or not.
Before we consider special classes of cooperative games, we define for every game $\boldsymbol{v}$ a game $\boldsymbol{v}^\star$ which reflects the preventive power. This game is called dual game and it assigns to coalition S the complementary worth of the complementary coalition.

Definition 2.2.7 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game. The* **dual game** $\boldsymbol{v}^{\star}$ *of game* $\boldsymbol{v}$ *is defined by*

$$\boldsymbol{v}^{\star}(S) = \boldsymbol{v}(I) - \boldsymbol{v}(S^c) \quad \forall S \in \mathcal{P}.$$

By definition, the dual game $\boldsymbol{v}^{\star}$ assigns a little value to a coalition $S \subseteq I$, if the complementary coalition $S^c \subseteq I$ is powerful, and vice versa.

2.2.1 Classes of Cooperative Games

Until now we did not specify cooperative games, this means that $\boldsymbol{v}$ does not need to satisfy further conditions. But sometimes we are interested in characteristic functions $\boldsymbol{v} \in \mathbb{V}$, which have certain properties and special structures. First of all, we look at the additive games, which are defined as follows.

Definition 2.2.8 *A cooperative game* $\boldsymbol{v}$ *is* **additive** *if*

$$\boldsymbol{v}(S) + \boldsymbol{v}(T) = \boldsymbol{v}(S \cup T) \quad \forall S, T \subseteq I,\ S \cap T = \emptyset.$$

In the case of additive games we only need to know the individual payoff $\boldsymbol{v}(\{i\})$ of each single player $i \in I$. Then, the worth of any coalition $S \in \mathcal{P}$ with more than one player can be calculated in a simple way by the summing up the individual payoffs of the single players $i \in S$. More formally, we have in this case for every coalition S the following equality:

$$\boldsymbol{v}(S) = \sum_{i \in S} \boldsymbol{v}(\{i\}).$$

Thus, it is possible to identify additive games with elements $\boldsymbol{x} \in \mathbb{R}^n$ and one can interpret them as distributions of utility. Sometimes additive games are even called (payoff) allocations or vectors of payoff.
In the following we distinguish between the class of all additive games and the class of additive games which are non-negative. Therefore, we have the following notation.

Notation 2.2.9 *The set of additive games is denoted by* $\mathbb{A}$*, this means that*

$$\mathbb{A} = \big\{ \boldsymbol{v} \in \mathbb{V} \mid \boldsymbol{v} \text{ is additive} \big\}.$$

The set of additive games which are non-negative are denoted by $\mathbb{A}_+$*, this means that*

$$\mathbb{A}_+ = \{\boldsymbol{x} \in \mathbb{A} \mid \boldsymbol{x} \geq 0\}.$$

A second class of games is the class of monotonic games. In this case we have the following definition.

Definition 2.2.10 *Let* $(I, \mathcal{P}, \boldsymbol{v})$ *be a game. The game* $\boldsymbol{v}$ *is* **monotonic** *if*

$$\boldsymbol{v}(S) \leq \boldsymbol{v}(T) \quad \forall S \subseteq T \subseteq I.$$

Thus, bigger coalitions get at least the same payoff as smaller subcoalitions.
In the next step we want to extend the additive games to a class of games which yields an incentive of forming coalitions.

Definition 2.2.11 *A cooperative game* $\boldsymbol{v}$ *is* **superadditive***, if*

$$\boldsymbol{v}(S) + \boldsymbol{v}(T) \leq \boldsymbol{v}(S \cup T) \quad \forall S, T \subseteq I, \ S \cap T = \emptyset$$

holds true.

In the case of superadditive games collaboration can only help but never hurt. That means in terms of savings, it is advantageous for disjoint coalitions S and T to form the union $S \cup T$. One of the most important subclasses of superadditive games are the convex games which are due to Shapley [20] in 1971.

Definition 2.2.12 *(Shapley [20])*
A cooperative game $\boldsymbol{v}$ *is* **convex** *if*

$$\boldsymbol{v}(S) + \boldsymbol{v}(T) \leq \boldsymbol{v}(S \cap T) + \boldsymbol{v}(S \cup T) \quad \forall S, T \subseteq I$$

holds true.

In order to interpret the term of convexness, we examine a second characterization of convex games. Therefore, we cite the next theorem which can be found in most of the books about cooperative game theory.

Theorem 2.2.13 *Let $\boldsymbol{v}$ be a cooperative game. Then, the following statements are equivalent:*

1. *$\boldsymbol{v}$ is convex*

2. *For $i \in I$ we have the following inequality:*
   $\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S) \leq \boldsymbol{v}(T \cup \{i\}) - \boldsymbol{v}(T) \quad \forall S \subseteq T \subseteq I \setminus \{i\}.$

The second statement means that the marginal contribution of player $i \in I$ to a coalition S is monotone nondecreasing with respect to set-theoretic inclusion. This interpretation explains the term convex. During this thesis, it is useful to have a short notation for convex games.

Notation 2.2.14 *We denote the set of convex games with $\mathbb{C}$. More formally, we have:*

$$\mathbb{C} = \{\boldsymbol{v} \in \mathbb{V} \mid \ \boldsymbol{v} \text{ is convex} \}.$$

2.3 Solution Concepts

In this section we study some well-known solution concepts for cooperative games with transferable utility. The most important concepts are among the core, the Shapley value, the nucleolus, and the modified nucleolus. Each concept assigns a set of allocations to a game. So, we can summarize that solution concepts recommend how $\boldsymbol{v}(I)$ should be divided among the players. In particular, throughout this section we are interested in solution concepts of convex games. But before introducing the first solution concept, we start with some special payoff vectors $\boldsymbol{x} \in \mathbb{A}$.

Definition 2.3.1 *Let $\boldsymbol{v}$ be a game and let $\boldsymbol{x} \in \mathbb{A}$ be a payoff vector.*

1. *$\boldsymbol{x}$ is* **Pareto optimal** *if $\boldsymbol{x}(I) = \boldsymbol{v}(I)$,*

2. *$\boldsymbol{x}$ is* **individually rational** *(w.r.t. $\boldsymbol{v}$), if for all $i \in I$ we have:*

$$x_i \geq \boldsymbol{v}(\{i\}),$$

3. $\boldsymbol{x}$ *is* **coalitional rational** *(w.r.t.* $\boldsymbol{v}$*), if for all* $S \in \mathcal{P}$ *we have:*

$$\boldsymbol{x}(S) \geq \boldsymbol{v}(S).$$

A Pareto optimal payoff vector $\boldsymbol{x}$ is a payoff vector, such that the total payoff $\boldsymbol{v}(I)$ is distributed to the players $i \in I$. In the case of an individual rational payoff vector $\boldsymbol{x}$, every single player $i \in I$ prefers the payoff x_i instead of the payoff $\boldsymbol{v}(\{i\})$. If every coalition prefers the payoff vector $\boldsymbol{x}$, the payoff vector is called coalitionally rational.
Now, we want to define some solution concepts. This means that we are looking at mappings which assign to each game a single vector or a set of feasible payoff vectors. We can conclude, that in our context, a solution concept is a correspondence

$$\sigma : \mathbb{V} \Rightarrow \mathbb{A}.$$

First, we will start with the non-empty set of Pareto optimal allocations, the so-called set of preimputations. More formally, we have:

Definition 2.3.2 *For any cooperative game* $\boldsymbol{v}$ *the* **set of preimputations** *is defined by*

$$X(\boldsymbol{v}) = \Big\{\boldsymbol{x} \in \mathbb{A} \,|\, \boldsymbol{x}(I) = \boldsymbol{v}(I)\Big\}.$$

In every preimputation the sum of all individual payoffs x_i should be equal to the payoff of the grand coalition $\boldsymbol{v}(I)$. In other words we can say that everything is divided among the players.
Most solution concepts are special subsets of the set of preimputations. For example if no player can be forced to accept less than $\boldsymbol{v}(\{i\})$, we have an imputation.

Definition 2.3.3 *The* **imputation set** *of a game* $\boldsymbol{v}$ *is defined by*

$$\mathcal{I}(\boldsymbol{v}) = \Big\{\boldsymbol{x} \in X(\boldsymbol{v}) \mid x_i \geq \boldsymbol{v}(\{i\}) \quad \forall i \in I\Big\}.$$

The imputation set is the set of Pareto optimal and individually rational payoff allocations. It is possible to check in a simple way, if this set is empty or non-empty. Therefore, one can consider our next remark.

Remark 2.3.4 *The imputation set $\mathcal{I}(\boldsymbol{v})$ is non-empty if and only if*

$$\boldsymbol{v}(I) \geq \sum_{i=1}^{n} \boldsymbol{v}(\{i\}).$$

2.3.1 Some Properties of Solution Concepts

In this section we want to consider some properties of solution concepts. After that, we continue our research about some further solution concepts like the core and the least core. First, we consider solution concepts in which existence and uniqueness are no problem. This property is called single valued. More formally, we have the following definition:

Definition 2.3.5 *A solution concept σ on a set $\Gamma \subseteq \mathbb{V}$ is* **single valued** *if*

$$|\sigma(\boldsymbol{v})| = 1 \quad \forall \boldsymbol{v} \in \Gamma.$$

In the case of a single valued solution concept, we can think of a function $\sigma : \mathbb{V} \rightarrow \mathbb{A}$ instead of a correspondence. Special single valued solution concepts are the additive concepts.

Definition 2.3.6 *A single valued solution concept σ on $\Gamma \subseteq \mathbb{V}$ is* **additive** *if*

$$\sigma(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \sigma(\boldsymbol{v}_1) + \sigma(\boldsymbol{v}_2) \quad \textit{whenever} \quad \boldsymbol{v}_1,\, \boldsymbol{v}_2,\, \boldsymbol{v}_1 + \boldsymbol{v}_2 \in \Gamma.$$

Here, it is irrelevant, if one considers the solution concept of the sum of two games or the sum of the solutions of two games.
Furthermore, it is useful to consider solution concepts which allocate the whole payoff of the grand coalition to the players.

Definition 2.3.7 *A solution concept σ on a set $\Gamma \subseteq \mathbb{V}$ is* **Pareto optimal** *if*

$$\sigma(\boldsymbol{v}) \subseteq X(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \Gamma.$$

Now, we want to think about the names of the players. In order to do so, we consider two games $(I, \mathcal{P}, \boldsymbol{v})$ and $(I, \mathcal{P}, \pi\boldsymbol{v})$ such that there exists a bijective mapping

$$\pi : I \rightarrow I.$$

Furthermore, we define with the aid of π a mapping $\pi : \mathbb{V} \to \mathbb{V}$ by

$$(\pi(\boldsymbol{v}))(S) = \boldsymbol{v}(\pi^{-1}(S)) \quad \forall S \subseteq I.$$

In particular, we get in the case of additive games $\boldsymbol{x} \in \mathbb{A}$ the following equality:

$$(\pi\boldsymbol{x})_i = x_{\pi^{-1}(i)} \quad \forall i \in I.$$

With the help of these definitions, we are able to define our next property of solution concepts.

Definition 2.3.8 *Let $\Gamma \subseteq \mathbb{V}$ be a set of games and let σ be a solution concept on Γ. The solution concept σ is* **anonymous** *if for each game $(I, \mathcal{P}, \boldsymbol{v}) \in \Gamma$ and each bijective mapping $\pi : I \to I$ the equality*

$$\sigma(\pi\boldsymbol{v}) = \pi(\sigma(\boldsymbol{v}))$$

holds.

Finally, it can be said, that an anonymous solution concept is permutation invariant. This property means, that the solution concept is independent of the name of the players such that a renumbering of the players does not change the payoffs of the players.

2.3.2 Core and Least Core

In this section we concentrate on two related solution concepts: the core and the least core. The core was first introduced by Gillies [4,5] as an adjunct to the studies of the stable sets. It is defined to be the set of efficient and coalitionally rational payoff allocations. More formally, we have the following definition:

Definition 2.3.9 *(Gillies [4,5])*
The **core** *of a cooperative game $\boldsymbol{v}$ is defined by*

$$\mathcal{C}(\boldsymbol{v}) = \left\{ \boldsymbol{x} \in X(\boldsymbol{v}) \;\middle|\; \boldsymbol{x}(S) \geq \boldsymbol{v}(S) \quad \forall S \in \mathcal{P} \right\}.$$

This means that, there is no core element $\boldsymbol{x}$ such that there exists a coalition $S \neq I$ which has an incentive to split off, if $\boldsymbol{x}$ is the proposed payoff allocation for the players. Here, the total payoffs $\boldsymbol{x}(S)$ allocated to coalition S are not smaller than the amount $\boldsymbol{v}(S)$ which coalition S can obtain by forming the coalition.
Note, that the core of a game may be empty and that it does not have to be single valued. Furthermore, we note that the core is always a polyhedron. An alternative definition of the core can be given with aid of the excess which is defined as follows.

Definition 2.3.10 *Let $\boldsymbol{v}$ be a cooperative game. For an allocation $\boldsymbol{x} \in \mathbb{A}$ the* **excess** *of a coalition $S \in \mathcal{P}$ at $\boldsymbol{x}$ with respect to game $\boldsymbol{v}$ is defined by*

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S).$$

A non-negative (non-positive) excess of S at $\boldsymbol{x}$ in the game $\boldsymbol{v}$ represents the gain (loss) to the coalition S if its members withdraw from the payoff vector $\boldsymbol{x}$ in order to form their own coalition. By definition we have $e(\emptyset, \boldsymbol{x}, \boldsymbol{v}) = 0$.
Now, we want to come back to the second possibility to define the core of a cooperative game $\boldsymbol{v}$. One can easily check, that the following definition of the core is also possible:

$$\mathcal{C}(\boldsymbol{v}) = \Big\{ \boldsymbol{x} \in X(\boldsymbol{v}) \;\Big|\; e(S, \boldsymbol{x}, \boldsymbol{v}) \leq 0 \quad \forall S \in \mathcal{P} \Big\}.$$

Before going on with some single valued solution concepts, we make the following additional remark about the connection of the above solution concepts.

Remark 2.3.11 *Since coalitionally rational payoff allocations are individually rational, we have*

$$\mathcal{C}(\boldsymbol{v}) \subseteq \mathcal{I}(\boldsymbol{v}) \subseteq X(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}.$$

In the next step we are interested in classes of games which have a non-empty core. The most important and most well-known games in this context are the convex games.

Theorem 2.3.12 *(Shapley [20])*
Let $\boldsymbol{v} \in \mathbb{C}$ be a convex game. Then, we have

$$\mathcal{C}(\boldsymbol{v}) \neq \emptyset.$$

Thus, we have found a subclass of games with non-empty core. For reason of completeness note that Shapley found a simple way to compute the extreme points of the core of convex games. For more details one can look in nearly all standard books on cooperative game theory, for example in Rosenmüller [13].
Next, we want to consider classes of games such that the core is non-empty. In this context, it is useful to have the definition of games, which are restricted on a coalition $T \in \mathcal{P}$.

Definition 2.3.13 *Let $(I, \mathcal{P}(I), \boldsymbol{v})$ be a game and let $T \in \mathcal{P}$. The* **restriction $\boldsymbol{v}^T$** *of $\boldsymbol{v}$ is defined by*

$$\boldsymbol{v}^T(S) = \boldsymbol{v}(S \cap T) \quad \forall S \in \mathcal{P}.$$

Bondareva [2] and Shapley [19] gave, independently, a characterization of games with a non-empty core. In our context, we restrict our attention only on the next definition.

Definition 2.3.14 *(Bondareva [2], Shapley [19])*
A cooperative game $\boldsymbol{v}$ is **balanced** *if its core $\mathcal{C}(\boldsymbol{v})$ is non-empty.*
A cooperative game $\boldsymbol{v}$ is **totally balanced** *if for all non-empty $T \in \mathcal{P}$ the game $\boldsymbol{v}^T$ is balanced.*

Thus, we have found a name for the class of games with a non-empty core. Note, that there exists a second possibility to define balanced games which can be found in nearly all standard books of cooperative game theory.
Due to the fact that the core of a game may be empty, Shapley and Shubik [21,22] have introduced the strong ϵ-core as a generation of the core.

Definition 2.3.15 *(Shapley and Shubik [21,22])*
For $\epsilon \in \mathbb{R}$, the **strong ϵ-core** *of a cooperative game $\boldsymbol{v}$ is defined by*

$$\mathcal{C}_\epsilon(\boldsymbol{v}) = \Big\{ \boldsymbol{x} \in X(\boldsymbol{v}) \;\Big|\; \boldsymbol{v}(S) - \boldsymbol{x}(S) \leq \epsilon \quad \forall\, \emptyset \neq S \subsetneqq I \Big\}.$$

The strong ϵ-core is the set of all preimputations that can not be improved upon by any coalition if one imposes a cost of ϵ (or a bonus of ϵ, if ϵ is negative) in all cases where a non-trivial coalition is formed.
A particular non-empty strong ϵ-core is the least core which was introduced by Maschler, Peleg and Shapley [10] in 1979. For its formal definition the following definition is helpful.

Definition 2.3.16 *Let*

$$\mu_0(\boldsymbol{x}, \boldsymbol{v}) = \max\{e(S, \boldsymbol{x}, \boldsymbol{v}) \,|\, \emptyset \neq S \subsetneqq I\}$$

denote the **maximal non-trivial excess** *of $\boldsymbol{v}$ at $\boldsymbol{x}$. Furthermore, let*

$$\mu(\boldsymbol{x}, \boldsymbol{v}) = \max\{e(S, \boldsymbol{x}, \boldsymbol{v}) \,|\, S \subseteq I\}$$

denote the **maximal excess** *of $\boldsymbol{v}$ at $\boldsymbol{x}$.*

The least core of $\boldsymbol{v}$ is defined to be a non-empty ϵ-core such that ϵ is as small as possible. More precisely, we have the following definition.

Definition 2.3.17 *(Maschler, Peleg and Shapley [10])*
The **least core** *of a cooperative game $\boldsymbol{v}$ is defined by*

$$\mathcal{LC}(\boldsymbol{v}) = \Big\{ \boldsymbol{x} \in X(\boldsymbol{v}) \,|\, e(S, \boldsymbol{x}, \boldsymbol{v}) \leq \mu_0(\boldsymbol{y}, \boldsymbol{v}) \quad \forall \boldsymbol{y} \in X(\boldsymbol{v}),\ \emptyset \neq S \subsetneqq I \Big\}.$$

The least core is by definition non-empty and it describes the set of all preimputations that minimize the maximum excess of non-trivial coalitions.
Before going on, we consider a simple example of a game such that the core is empty. In this case the least core is a strong ϵ-core with strictly positive ϵ.

Example 2.3.18 *Let $I = \{1, 2\}$ and let $\boldsymbol{v}(\emptyset) = 0$, $\boldsymbol{v}(S) = 1$ for all $S \in \mathcal{P}, S \neq \emptyset$. The core of this game is empty. Furthermore, we have:*

$$\mathcal{LC}(\boldsymbol{v}) = \mathcal{C}_{\frac{1}{2}}(\boldsymbol{v}) = \Big\{ \Big(\frac{1}{2}, \frac{1}{2}\Big) \Big\}.$$

Note, that in the case of games with an empty core, the least core is a particular strong ϵ such that ϵ is strictly positive.
Another class of cooperative games dues to Schmeidler [17] in 1972. These games are a subclass of the totally balanced games and they are defined as follows:

Definition 2.3.19 *(Schmeidler [17])*
A cooperative game $\boldsymbol{v}$ is **exact** *if for any $S \subseteq I$ there exists an allocation $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{v})$ such that $\boldsymbol{x}(S) = \boldsymbol{v}(S)$.*

In the case of exact games there exists an element of the core such that for any coalition the sum of the individual payoffs $\boldsymbol{x}(S)$ is not more than the value $\boldsymbol{v}(S)$ the players can get in the game by forming the coalition.

Notation 2.3.20 *We denote the set of exact games by $\mathbb{E}$. Furthermore, we have the set of balanced games $\mathbb{B}$ and the set of totally balanced games $\mathbb{T}$. More formally, we have*

$$\begin{aligned}
\mathbb{E} &= \{\boldsymbol{v} \in \mathbb{V} \mid \boldsymbol{v} \ \textit{ is exact}\,\}, \\
\mathbb{B} &= \{\boldsymbol{v} \in \mathbb{V} \mid \boldsymbol{v} \ \textit{ is balanced}\,\}, \\
\mathbb{T} &= \{\boldsymbol{v} \in \mathbb{V} \mid \boldsymbol{v} \ \textit{ is totally balanced}\,\}.
\end{aligned}$$

One of the most important examples of exact games is the class of convex games. This result dues to Shapley [20] and Schmeidler [17]. Thus, we have the following inclusions:

$$\mathbb{C} \subset \mathbb{E} \subset \mathbb{T}.$$

At the end of this section we concentrate on some further aspects of the core of cooperative games. In order to do so, we start with two definitions about the domination of an allocation. This definition dues to von Neumann and Morgenstern [9] in 1944.

Definition 2.3.21 *(von Neumann and Morgenstern [9])*
Let $(I, \mathcal{P}, \boldsymbol{v})$ a game. An allocation $\boldsymbol{y} \in \mathcal{I}(\boldsymbol{v})$ **dominates an allocation** *$\boldsymbol{x} \in \mathcal{I}(\boldsymbol{v})$* **via coalition** *$S \neq \emptyset$ if $\boldsymbol{y}(S) \leq \boldsymbol{v}(S)$ and $y_k > x_k$ for all $k \in S$.*

If $\boldsymbol{y}$ dominates $\boldsymbol{x}$ via coalition S, every player $i \in S$ prefer $\boldsymbol{y}$ for $\boldsymbol{x}$. The payoffs y_i are feasible because $\boldsymbol{y}(S) \leq \boldsymbol{v}(S)$.

Definition 2.3.22 *An allocation* $\boldsymbol{y}$ **dominates an allocation** $\boldsymbol{x}$ *if there exists a non-empty coalition* $S \in \mathcal{P}$ *such that* $\boldsymbol{y}$ *dominates* $\boldsymbol{x}$ *via* S.

Note, that an allocation can be dominated only via coalitions having non-negative excess at that allocation.

Definition 2.3.23 *The* **core** $\mathcal{C}(\boldsymbol{v})$ *of a game* $\boldsymbol{v}$ *is* **stable** *if for every imputation* $\boldsymbol{x} \in \mathcal{I}(\boldsymbol{v}) \setminus \mathcal{C}(\boldsymbol{v})$ *there exists a core allocation* $\boldsymbol{y} \in \mathcal{C}(\boldsymbol{v})$ *and a coalition* S *such that* $\boldsymbol{y}$ *dominates* $\boldsymbol{x}$ *via* S.

Several sufficient conditions for stability of the core have been discussed in the literature. Shapley [20] proved in 1971 that convexity of the game is a well-known one. Note, that core stability is invariant under adding nullplayers.
In our next definition, we introduce a further well-known property of the core, which dues to Sharkey in 1982.

Definition 2.3.24 *(Sharkey [24])*
Let $\boldsymbol{v}$ *be a cooperative game. The core is* **large** *if for any* $\boldsymbol{y} \in \mathbb{R}^n$ *satisfing* $\boldsymbol{y}(S) \geq \boldsymbol{v}(S)$ *for all* $S \subseteq I$ *there exists* $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{v})$ *such that* $\boldsymbol{x} \leq \boldsymbol{y}$.

Sharkey proved that largeness of the core implies core stability.

2.3.3 Nucleolus and Modified Nucleolus

In this section we continue our research with two further single valued solution concepts, which are strongly related. These concepts are the nucleolus, which minimizes highest excesses and the modified nucleolus (modiclus), which minimizes highest differences of excesses. Having in mind, that differences of excesses can be interpreted as a sum of excesses of the primal and the dual game, the modiclus represents constructive and preventive (blocking) powers of coalitions alike. In order to define the two concepts, we have to look at the definition of the lexicographic relation.

Definition 2.3.25 *For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ we say a vector $\boldsymbol{a}$ is* **lexicographically smaller** *than a vector $\boldsymbol{b}$, $\boldsymbol{a} \leq_{lex} \boldsymbol{b}$, if $\boldsymbol{a} = \boldsymbol{b}$ or if there exists an element $s \in \{1, \ldots, n\}$ such that $a_i = b_i$ for all $i < s$ and $a_s < b_s$.*

In this context one can consider a special element of the subset of $\mathbb{R}^n$.

Definition 2.3.26 *Let $C \subseteq \mathbb{R}^n$. A* **lexicographic minimum** *is an element $\boldsymbol{d} \in C$ such that*

$$\boldsymbol{d} \leq_{lex} \boldsymbol{c} \quad \forall \boldsymbol{c} \in C.$$

Note, that a compact subset $C \subseteq \mathbb{R}^n$ always has a unique lexicographic minimum. Here is an example of the lexicographic relation and the unique lexicographic minimum.

Example 2.3.27 *Consider the following elements of $\mathbb{R}^3$: $(1, 5, 7), (2, -1, 2), (5, 2, 3)$ and $(5, 3, 2)$. In this case, we have*

$$(1, 5, 7) \leq_{lex} (2, -1, 2)$$

and

$$(5, 2, 3) \leq_{lex} (5, 3, 1).$$

In order to compute the lexicographic minimum, we consider in the first step a compact and convex set:

$$C = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \subseteq \mathbb{R}^2.$$

Then, the lexicographic minimum of C is $\boldsymbol{c} = (-1, 0)$.

With the aid of the lexicographic relation, we are able to understand the following definition which dues to Schmeidler [16] in 1969. In our context we will use the short definition, which can be found in different standard books of cooperative game theory. For example, one can have a look at Driessen [3].

Definition 2.3.28 *(Schmeidler [16])*
Let $\boldsymbol{v}$ be a game. For $\boldsymbol{x} \in \mathbb{R}^n$ we define

$$\tilde{\theta}(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S) \quad S \in \mathcal{P}$$

to be the vector of excesses in a non increasing order. Then, the **nucleolus** *of $\boldsymbol{v}$ is defined by*

$$\nu(\boldsymbol{v}) = \Big\{\boldsymbol{x} \in \mathcal{I}(\boldsymbol{v}) \mid \tilde{\theta}(\boldsymbol{x}, \boldsymbol{v}) \leq_{lex} \tilde{\theta}(\boldsymbol{y}, \boldsymbol{v}) \quad \textit{for } \boldsymbol{y} \in \mathcal{I}(\boldsymbol{v})\Big\}.$$

The coordinates of $\tilde{\theta}(\boldsymbol{x}, \boldsymbol{v})$ measures the dissatisfaction of the coalitions at outcome $\boldsymbol{x}$. The coordinates are ordered in the way that the highest complaint comes first, then the second and so on, such that the total dissatisfaction is minimized in the nucleolus. Note, that the nucleolus always exists and that it is always unique. Furthermore, it is an element of the core, if the core is non-empty.
In a second step, we want to define a strongly related solution concept. This concept is the modified nucleolus or modiclus and it dues to Sudhölter, who studied this concept in a series of papers [27, 29, 30]. Instead of minimizing excesses, the modiclus minimizes differences of excesses. More formally, we have the following definition:

Definition 2.3.29 *(Sudhölter [27, 29, 30])*
Let $\boldsymbol{v}$ be a cooperative game. For $\boldsymbol{x} \in \mathbb{R}^n$ we define

$$\begin{aligned} \theta(\boldsymbol{x}, \boldsymbol{v}) &= ((\boldsymbol{v}(S) - \boldsymbol{x}(S)) - (\boldsymbol{v}(T) - \boldsymbol{x}(T))) \quad (S,T) \in \mathcal{P} \times \mathcal{P} \\ &= ((\boldsymbol{v}(S) - \boldsymbol{x}(S)) + (\boldsymbol{v}^\star(T^c) - \boldsymbol{x}(T^c))) \quad (S,T) \in \mathcal{P} \times \mathcal{P} \end{aligned}$$

to be the vector of differences of excesses in a non increasing order. Then,

$$\Psi(\boldsymbol{v}) = \{\boldsymbol{x} \in X(\boldsymbol{v}) \,|\, \theta(\boldsymbol{x}, \boldsymbol{v}) \leq_{lex} \theta(\boldsymbol{y}, \boldsymbol{v}) \quad \textit{for } \boldsymbol{y} \in X(\boldsymbol{v})\}$$

is the **modified nucleolus** *or* **modiclus** *of the game $\boldsymbol{v}$.*

In contrast to other solution concepts, the achievement power $\boldsymbol{v}(S)$ and the preventive power $\boldsymbol{v}^\star(S)$ play a totally symmetric role in general. Note, that $\theta(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{2^{2n}}$.
In the next step we want to consider a connection between the nucleolus and the modiclus. Therefore, we define a special game which is called the dual cover.

Definition 2.3.30 *Consider a game* $(I, \mathcal{P}, \boldsymbol{v})$*and let* $I^{1,2} = I \times \{0, 1\}$*. We define a second game* $(I^{1,2}, \mathcal{P}^{1,2}, \boldsymbol{v}^{1,2})$ *by*

$$\boldsymbol{v}^{1,2}(S, T) = \max \left\{ \boldsymbol{v}(S) + \boldsymbol{v}^\star(T), \boldsymbol{v}(T) + \boldsymbol{v}^\star(S) \right\} \quad \forall S, T \in \mathcal{P}.$$

This game is the **dual cover***.*

Here, we consider two copies of the player set. Then, we define on this new player set $I^{1,2}$ a new game $\boldsymbol{v}^{1,2}$ with the aid of the game $\boldsymbol{v}$ and the dual game $\boldsymbol{v}^\star$.
Sudhölter discusses in the following proposition the relation of the nucleolus of the dual cover and the modiclus of the game $\boldsymbol{v}$.

Proposition 2.3.31 *(Sudhölter [30])*
The modified nucleolus of a game $(I, \mathcal{P}, \boldsymbol{v})$ *is the restriction of the nucleolus of the game* $(I^{1,2}, \mathcal{P}^{1,2}, \boldsymbol{v}^{1,2})$ *to* I*, i.e.*

$$\Psi(\boldsymbol{v}) = \nu(\boldsymbol{v}^{1,2})_I.$$

This proposition implies that we can compute the modiclus with the aid of the nucleolus of the dual cover. Furthermore, one can justify this concept with the well-known arguments of the nucleolus in this special game.
In our next remark, we consider some properties of the modified nucleolus. Sudhölter [30] proved this results in 1997.

Remark 2.3.32 *(Sudhölter [29,30])*
The modified nucleolus is a singleton and it is self dual, i.e. we have for all games the following equality:

$$\Psi(\boldsymbol{v}) = \Psi(\boldsymbol{v}^\star).$$

Furthermore, the modified nucleolus satisfies anonymity.

That means, the modiclus is single valued and it is unimportant, if one consider the modiclus of game $\boldsymbol{v}$ or the modiclus of the dual game $\boldsymbol{v}^\star$.
In the case of convex games, the modified nucleolus is, as the Shapley value and the nucleolus, an element of the core. This result dues also to Sudhölter [30].

Theorem 2.3.33 *(Sudhölter [30])*
Let $\boldsymbol{v} \in \mathbb{C}$ be a convex game. Then, we have

$$\Psi(\boldsymbol{v}) \in \mathcal{C}(\boldsymbol{v}).$$

For completeness reasons, we define another non-empty set, which is called modified least core. In the following definition we have a look at this set.

Definition 2.3.34 *Let $\boldsymbol{v}$ be a cooperative game. The* **modified least core** *of $\boldsymbol{v}$ is defined by*

$$\mathcal{MLC}(\boldsymbol{v}) = \Big\{ \boldsymbol{x} \in X(\boldsymbol{v}) \,\Big|\, \mu(\boldsymbol{x}, \boldsymbol{v}) + \mu(\boldsymbol{x}, \boldsymbol{v}^\star) \leq \mu(\boldsymbol{y}, \boldsymbol{v}) + \mu(\boldsymbol{y}, \boldsymbol{v}^\star) \quad \forall \boldsymbol{y} \in X(\boldsymbol{v}) \Big\}.$$

The modified least core of the game $\boldsymbol{v}$ consists of all preimputations minimizing the sum of maximal excesses with respect to a game $\boldsymbol{v}$ and the dual game $\boldsymbol{v}^\star$. Note, that for every game the modified least core $\mathcal{MLC}(\boldsymbol{v})$ is a compact, convex subset of the set of preimputations $X(\boldsymbol{v})$.

Remark 2.3.35 *By definition, the modified nucleolus $\Psi(\boldsymbol{v})$ is an element of the modified least core. More formally, we have for all games the following correlation:*

$$\Psi(\boldsymbol{v}) \in \mathcal{MLC}(\boldsymbol{v})$$

2.3.4 The Shapley Value

Another way to distribute the payoff $\boldsymbol{v}(I)$ of the grand coalition to the players was introduced by Shapley [18] in 1953. This solution concept assigns a unique vector of payoffs to each game. The payoff vector can be thought of as a sort of expected payoff or an a priori measure of power. As we will see later on, the existence of this value can easily derived. For the formal definition of the Shapley value, we define for all subsets $T \subsetneqq I,\ |T| = t$ real numbers by

$$\gamma(t) = \frac{t!\,(n-t-1)!}{n!} = \frac{1}{n}\frac{1}{\binom{n-1}{t}}.$$

We will see in our next remark, that this numbers satisfy some further conditions.

Remark 2.3.36 *We have for all $i \in I$ the following equality:*

$$\sum_{S \subseteq I \setminus \{i\}} \gamma(s) = 1.$$

Thus, $\gamma : \mathcal{P}(|I \setminus \{i\}|) \to [0,1]$ can be regarded as a probability measure over the collection of subsets of $I \setminus \{i\}$.

One well-known possible definition of the Shapley value is the following one, which uses the above probability measure.

Definition 2.3.37 *(Shapley [18])*

Let $\boldsymbol{v}$ be a cooperative game. The **Shapley value** *of game $\boldsymbol{v}$ is a mapping*

$$\Phi : \mathbb{V} \to \mathbb{A}$$

$$\boldsymbol{v} \mapsto \Phi(\boldsymbol{v}),$$

such that

$$(\Phi(\boldsymbol{v}))_i := \phi_i(\boldsymbol{v}) = \sum_{S \subseteq I \setminus \{i\}} \gamma(s)(\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S)) \quad \forall\, i \in I.$$

In order to explain the meaning of the Shapley value we note that there exists $n!$ orderings of the player set I. Furthermore, we have $s!$ orderings of the players of coalition S and $(n-s-1)!$ orderings of the players of coalition $I \setminus (S \cup \{i\})$. Thus, $\gamma(s) = \frac{s!\,(n-s-1)!}{n!}$ can be seen as probability that player $i \in I$ joints the coalition S as last player, if all $n!$ orderings have the same probability. Since player i's marginal contribution is given by $\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S)$, one can interpret $\phi_i(\boldsymbol{v})$ as expected payoff of player i in the game $\boldsymbol{v}$ or as the power of player $i \in I$ in game $\boldsymbol{v}$.

In the case of convex games, we have more structure such that we have a more detailed result of the Shapley value in this case.

Theorem 2.3.38 *Let $\boldsymbol{v} \in \mathbb{C}$ be a convex game. Then, the Shapley value is the center of gravity of the extreme points of the the core. In particular, we have*

$$\Phi(\boldsymbol{v}) \in \mathcal{C}(\boldsymbol{v}).$$

This means that, the Shapley value is a special member of the core in convex games. If we consider a superadditive game, we only know that the Shapley value is an element of the imputation set.

Compared with the formal definition, Shapley [18] gives an elegant axiomatic characterization. In order to formulate these axioms, we need some further definitions.

Definition 2.3.39 *Let $\boldsymbol{v} \in \mathbb{V}$ be a game and let $T \in \mathcal{P}$. The* **carrier** *of $\boldsymbol{v}$ is the set*

$$C(\boldsymbol{v}) = \bigcap_{\substack{T \in \mathcal{P} \\ \boldsymbol{v}^T = \boldsymbol{v}}} T.$$

With the aid of the carrier of game $\boldsymbol{v}$ we can define our next property of solution concepts.

Definition 2.3.40 *A single valued solution concept σ on $\Gamma \subseteq \mathbb{V}$ satisfies the* **dummy property***, if*

$$C(\sigma(\boldsymbol{v})) \subseteq C(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \Gamma.$$

Here, nullplayers get a payoff of zero.

In the next theorem, which is due to Shapley [18] in 1953, we characterize the Shapley value with the aid of some axioms.

Theorem 2.3.41 *(Shapley [18])*

The Shapley value is the unique solution concept which satisfies

1. *additivity,*
2. *Pareto optimality,*
3. *anonymity,*
4. *dummy property.*

That means, the Shapley value is the unique solution of a system of four axioms.

2.4 Assignment Games

In this section we concentrate on the so-called assignment games which are models of two-sided matching markets with transferable utility. One of the most well-known examples is the house-market. Here, every seller offers one house and each buyer wants to buy at most one house. The worth of the transaction is defined by a non-negative payoff. Groups of players that depends only on buyers or sellers get no payoff because there is no transaction in this case. Groups consisting of buyers and sellers get the maximal payoff, that is possible, if one sums up the possible payoff of pairs of players. Assignment games were introduced in 1972 by Shapley and Shubik [23]. An important property of this class of games is that, these games can be described by a non-negative matrix in a very simple way. Furthermore, Shapley and Shubik proved in their paper that the core of these games is the non-empty set of optimal solutions of a problem which corresponds to the assignment problem. As a result of Balinski and Gale [1], one knows that the maximal number of extreme points of the core depends on the number of players of the two different types of players.

2.4.1 The Definition

There exists different ways to define an assignment game. In the following we will use the definition which uses an assignment of two finite set. Therefore, we need the next definition.

Definition 2.4.1 *Let S and T be finite sets. An* **assignment of (S,T)** *is a bijection $b : S^* \to T^*$ such that $S^* \subseteq S, T^* \subseteq T$ and $|S^*| = |T^*| = \min\{|S|, |T|\}$. We identify b with $\{(i, b(i)) \mid i \in S^*\}$.*

During this thesis, it is useful to have a short notation for the set of assignments of two finite sets. Therefore, we have:

Notation 2.4.2 *We denote the set of assignments of two finite sets (S, T) by $\mathcal{B}(S, T)$.*

With the help of assignments of two finites sets, we can give one possible definition of an assignment game, which was introduced by Shapley and Shubik [23] in 1972.

Definition 2.4.3 *(Shapley and Shubik [23])*
A cooperative game is an **assignment game** $\boldsymbol{v}_A$ *if there exist two non-empty, disjoint, finite sets* P *and* Q *and a non-negative matrix* $A = (a_{ij})_{(i,j)\in P\times Q}$ *such that* $I = P \cup Q$ *and*

$$\boldsymbol{v}_A(S) = \max_{b\in\mathcal{B}(S\cap P, S\cap Q)} \sum_{(i,j)\in b} a_{ij} \quad \forall\, S \subseteq I.$$

In the case of these games, the player sets contain two disjoint sets of agents P and Q. The assignment game describes the worth of the transaction between the two types of players. It should be mentioned that coalitions of different players of one type get no payoff, that means $\boldsymbol{v}_A(S) = 0$ for $S \subseteq P$ or $S \subseteq Q$.

Remark 2.4.4 *By definition, assignment games are superadditive.*

Note, that other definitions of assignment games are also possible, for example a definition as a linear programming game. A collection of different definitions of assignment games is given in Hoffmann [6]. The advantage of our present definition can be seen in the different proofs in the next chapters, because this definition allows for a simple treatment of certain results.
Before beginning with the first results and properties of assignment games, we start this section with one of the most important examples of assignment games: the glove games.

Definition 2.4.5 *Consider two disjoint non-empty sets* P *and* Q *containing* p *and* q *agents. Furthermore, let* $I = P \cup Q$*. The game* $(I, \mathcal{P}, \boldsymbol{v})$ *is a* **glove game** *if*

$$\boldsymbol{v}(S) = \min\{|P \cap S|, |Q \cap S|\} \quad \forall S \in \mathcal{P}.$$

This means that, a single glove is worth nothing and each pair of gloves has a value of one. Note, that this class of games is a subclass of assignment games. To see this, consider the $p \times q$ assignment matrix

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

The assignment game $\boldsymbol{v}_A$ describes the glove game with p and q agents.

Now, let us return to the player set $I = P \cup Q$ of an assignment game. During this thesis we have a special notation for the players and the diagonal pairs.

Notation 2.4.6 *In this thesis we denote w.l.o.g. the players of the two sets by*

$$P = \{1, \ldots, p\} \quad \textit{and} \quad Q = \{1', \ldots, q'\}.$$

Furthermore, let w.l.o.g. $p \leq q$. In this case, we denote the set of diagonal pairs by

$$D = \{(1, 1'), \ldots, (p, p')\} \subseteq P \times Q.$$

In the context of assignment games, we denote any payoff vector $\boldsymbol{x} \in \mathbb{A}$ by

$$\boldsymbol{x} = (u, v) \in \mathbb{R}^{p+q}.$$

This notation of the payoff vectors allows us a differentiation of the players of the two different types.

In the next step, we introduce a non-restrictive property of assignment games which is very useful for our proofs in the next chapters.

Definition 2.4.7 *Let $\boldsymbol{v}_A$ be an assignment game introduced by the $p \times q$ assignment matrix A. The main diagonal is an* **optimal assignment** *if*

$$\boldsymbol{v}_A(I) = \sum_{i \in P} a_{ii'}.$$

The assignment b is an **optimal assignment for S***, $S \in \mathcal{P}$, if $b \in \mathcal{B}(S \cap P, S \cap Q)$ and*

$$\boldsymbol{v}_A(S) = \sum_{(i,j') \in b} a_{ij'} \geq \sum_{(i,j') \in d} a_{ij'} \quad \forall\, d \in \mathcal{B}(S \cap P, S \cap Q).$$

In the case where the main diagonal is an optimal assignment, player $i \in P$ is matched with player $i' \in Q$. If there are more players of one type, some players are unmatched. In order to facilitate simpler proofs, we will consider during this thesis only assignment games such that the main diagonal is an optimal assignment.

Remark 2.4.8 *We assume w.l.o.g. that the main diagonal is an optimal assignment. Otherwise we reorder the matrix.*

Before looking at special classes of assignment games we start with a more simple description of the core, which was given by Shapley and Shubik [23] in 1972.

Theorem 2.4.9 *(Shapley and Shubik [23])*
Let $\boldsymbol{v}_A$ be an assignment game. Then, we have

$$\mathcal{C}(\boldsymbol{v}_A) = \Big\{\boldsymbol{x} \in \mathbb{A}_+ \mid \boldsymbol{x}(I) = \boldsymbol{v}_A(I)\ ,\ \ u_i + v_{j'} \geq a_{ij'} \quad \forall\, i \in P, j' \in Q\Big\}.$$

In particular we have $u_i + v_{j'} = a_{ij'}$ for all $(i, j') \in b$, where b is an optimal assignment for the player set I. Furthermore, players who are unmatched in an optimal assignment $b \in \mathcal{B}(P, Q)$ of the grand coalition get nothing in any core allocation $(u, v) \in \mathcal{C}(\boldsymbol{v}_A)$.

The above theorem allows us to restrict our attention in the context of the core only on the pairs $(i, j') \in P \times Q$ and the non-negative restriction. Other coalitions are, in the case of the core of assignment games, not important.
Now, we conclude that it is also possible to describe the core of an assignment game as a set of optimal solutions of a linear program which depends only on the assignment matrix A.

Corollary 2.4.10 *(Shapley and Shubik [23])*
The core of an assignment game $\boldsymbol{v}_A$ is equivalent to the set of optimal solutions of the following linear program:

$$\begin{aligned} u_i + v_{j'} &\geq a_{ij'} && && (2.1)\\ u_i, v_{j'} &\geq 0 && i \in P,\ j' \in Q\\ \sum_{i\in P} u_i + \sum_{j'\in Q} v_{j'} &\rightarrow \min\,. \end{aligned}$$

According to the Duality Theorem, one can see that the core of an assignment game is non-empty. To see this consider the dual linear program of (2.1). Note, that both linear programs have feasible solutions such that each linear program has at least one optimal solution. For a more detailed discussion, the reader is referred to Hoffmann [6].

Another interesting question is, how many extreme points describe the core of an assignment game. In their paper Balinski and Gale [1] give an answer to this question by computing the maximal number of extreme points of the core.

Theorem 2.4.11 *(Balinski and Gale [1])*
Let $\boldsymbol{v}_A$ be an assignment game which is induced by a $p \times q$ assignment matrix A. Furthermore, let $m = \min\{p, q\}$. Then, there are at most $\binom{2m}{m}$ extreme points of the core.

Before we continue with some further properties of assignment games, we make an additional comment about the dimension of the core. This result given in our next corollary is based on the properties of Theorem 2.4.9 of Shapley and Shubik [23].

Corollary 2.4.12 *(Shapley and Shubik [23])*
The dimension of the core is never greater than $\min(p, q)$. If the u-components are given, the v-components are completely determined and vice versa. Summarizing, it may be said that there are at least $\min(p, q)$ degrees of freedom in the core.

2.4.2 Special Classes of Assignment Games

During this section we restrict our attention on the connection of different characteristics between the assignment matrix and the properties of the introduced assignment game. These connections are the main results of Solymosi and Raghavan [26] in 2001. Here, we cite the three main theorems of their paper. Furthermore, we give some definitions which are important in this context. We start with some possible properties of a $m \times m$ matrix A.

Definition 2.4.13 *Let A be a $m \times m$ matrix. The matrix A has a* **dominant diagonal** *if $a_{ii} \geq a_{ij}$ and $a_{ii} \geq a_{ji}$ for all $i, j \in \{1, \ldots, m\}$.*

In this case the diagonal entry a_{ii} is not smaller that the others entries in the i-th row and i-th column. A second property of a $m \times m$ matrix is called doubly dominant diagonal and it is defined as follows:

Definition 2.4.14 *Let A be a $m \times m$ matrix. The matrix A has a* **doubly dominant diagonal** *if $a_{ii} + a_{kj} \geq a_{ij} + a_{ki}$ for all $i, j, k \in \{1, \ldots, m\}$.*

The two above properties are independent if m is bigger or equal than three. This means that, a matrix can have a dominant diagonal but not a doubly dominant diagonal and vice versa. In the next example we consider two matrices such that one has only a dominant diagonal and the other one has only a doubly dominant diagonal.

Example 2.4.15 *Consider the matrices*

$$A = \begin{pmatrix} 8 & 4 & 8 \\ 4 & 4 & 1 \\ 8 & 1 & 8 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The matrix A has a dominant diagonal but not a doubly dominant diagonal. The matrix B has a doubly dominant diagonal but not a dominant diagonal.

An exception is the 2×2 case. In this case every matrix with a dominant diagonal has a doubly dominant diagonal and vice versa.

Remark 2.4.16 *Let A be a 2×2 matrix. Then, the following statements are equivalent:*

1. *A has a dominant diagonal,*
2. *A has a doubly dominant diagonal.*

This means that, in the case of 2×2 matrices both properties are equivalent.
In the next steps we consider the relationship between properties of assignment games and the assignment matrix A. In the next theorem, Solymosi and Raghavan [26] discuss assignment games with a stable core. The relevant result in this context is the following theorem.

Theorem 2.4.17 *(Solymosi and Raghavan [26])*
Let A be a $p \times p$ assignment matrix such that its main diagonal is an optimal assignment. Furthermore, let $\boldsymbol{v}_A$ be the assignment game induced by matrix A. Then, the following statements are equivalent:

1. *$\mathcal{C}(\boldsymbol{v}_A)$ is stable,*

2. *A has a dominant diagonal.*

Thus, with the aid of this theorem, it is very simple to check whether an assignment game has a stable core or not.
In a second step, we consider exact assignment games. As in the case of assignment games with a stable core, Solymosi and Raghavan [26] have found some conditions on the assignment matrix. Furthermore, the authors proved that an assignment game is exact if and only if the assignment game has a large core. The results are summarized in the next theorem.

Theorem 2.4.18 *(Solymosi and Raghavan [26])*
Let A be a $p \times p$ assignment matrix such that its main diagonal is an optimal assignment. Furthermore, let $\boldsymbol{v}_A$ be the assignment game induced by matrix A. Then, the following statements are equivalent:

1. *$\boldsymbol{v}_A$ is exact,*

2. *$\mathcal{C}(\boldsymbol{v}_A)$ is large,*

3. *A has a dominant and a doubly dominant diagonal.*

This theorem permits the possibility to check if an assignment game is exact or not. In particular, this theorem implies that exact assignment games have a stable core. Before we are able to present convex assignment games, we need a further definition of $m \times m$ matrices.

Definition 2.4.19 *Let A be a $m \times m$ matrix. The matrix A is a* **diagonal matrix** *if $a_{ij} = 0$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.*

A diagonal matrix is the most restrictive property in the context of $m \times m$ matrices. By definition a diagonal matrix has a dominant and a doubly dominant diagonal.
The next theorem treats convex assignment games. Solymosi and Raghavan [26] showed that in this case the assignment matrix has a very special structure such that it is very simple to identify these games.

Theorem 2.4.20 *(Solymosi and Raghavan [26])*
Let A be a $p \times p$ assignment matrix such that its main diagonal is an optimal assignment. Furthermore, let $\boldsymbol{v}_A$ be the assignment game induced by matrix A. Then, the following statements are equivalent:

1. *$\boldsymbol{v}_A$ is convex,*

2. *A is a diagonal matrix.*

Since a diagonal matrix has a dominant and a doubly dominant diagonal, one immediately sees that convex assignment games are exact and have a stable core. Furthermore, we note that convexness is invariant under adding nullplayers such that we have the following result:

Corollary 2.4.21 *Let A be a $p \times q$ assignment matrix such that the diagonal is an optimal assignment. Then, the following statements are equivalent:*

1. *$\boldsymbol{v}_A$ is convex,*

2. *$a_{ij'} = 0 \quad \forall (i, j') \in P \times Q, (i, j') \notin D$.*

2.4.3 Modified Nucleolus of Assignment Games

In this section we are looking at the relationship between the core, the modified least core of an assignment game and the least core of the dual assignment game. These relationships are very useful since the modified nucleolus is an element of the modified least core. If we find some results in this case, we can restrict our research of the modiclus in some special cases on easier problems. In the next theorem, Sudhölter [31] proves that the modiclus of an assignment game is an element of the least core of the dual game.

Theorem 2.4.22 *(Sudhölter [31])*
The modified least core of an assignment game is a subset of the least core of the dual game, this means, we have for all assignment games:

$$\mathcal{MLC}(\boldsymbol{v}_A) \subseteq \mathcal{LC}(\boldsymbol{v}_A^\star).$$

Thus, the least core of the dual assignment game is an important set in the context of the modified nucleolus of assignment games because the modified nucleolus is an element of the modified least core. In the case of an assignment matrix with a dominant diagonal, the modified nucleolus of the induced assignment game is also an element of the core. This result is due to the following theorem of Raghavan and Sudhölter [12].

Theorem 2.4.23 *(Raghavan and Sudhölter [12])*
If $\boldsymbol{v}_A$ is an assignment game with a stable core, then the modified least core is a subset of the core. Thus, we have for all assignment games with a stable core

$$\mathcal{MLC}(\boldsymbol{v}_A) \subseteq \mathcal{C}(\boldsymbol{v}_A).$$

This means that, the modiclus of an assignment game is an element of the least core of the dual game and of the core, if it is stable. In particular we have in this case:

$$\Psi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A) \cap \mathcal{LC}(\boldsymbol{v}_A^\star) \neq \emptyset.$$

We conclude that Theorem 2.4.22 and Theorem 2.4.23 motivate a more detailed discussion of the core and the least core of the dual assignment game.

2.5 Strong Nullplayer Property

In this section we are looking at some solution concepts which satisfy the strong nullplayer property. With the aid of this property we can restrict our attention in the next chapters on assignment games which are induced by a $p \times p$ assignment matrix. In the case of assignment games with an unequal number of P and Q players this property allows us to add some nullplayers.
When we define the strong nullplayer property, we look at two games such that the first game is induced by the second one by adding a nullplayer. With the aid of these two games, we are able to define this property.

Definition 2.5.1 *Let $(I, \mathcal{P}(I), \boldsymbol{v})$ be a game and let $\tilde{I} = I \cup \{\tilde{n}\}$. Consider the game $(\tilde{I}, \mathcal{P}(\tilde{I}), \tilde{\boldsymbol{v}})$ which is defined by*

$$\tilde{\boldsymbol{v}}(S) = \boldsymbol{v}(I \cap S) \quad \forall S \in \mathcal{P}(\tilde{I}).$$

A solution concept δ satisfies the **strong nullplayer property** *if*

$$\delta(\tilde{\boldsymbol{v}}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}_I \in \delta(\boldsymbol{v}), x_{\tilde{n}} = 0 \right\} \tag{2.2}$$

holds true.

The strong nullplayer property implies that any nullplayer $i \in I$ gets the payoff zero. Additionally, the solution $\delta(\tilde{\boldsymbol{v}})$ of game $\tilde{\boldsymbol{v}}$ arises from the solution $\delta(\boldsymbol{v})$ of game $\boldsymbol{v}$ by adding a zero coordinate for the nullplayer to any element of the solution $\delta(\boldsymbol{v})$. Since the following chapter is about the Shapley value and the core of assignment games, it is useful to check that these solution concepts satisfy the strong nullplayer property. Thus, our next results are a preparation of the next chapter. In our first lemma, we show that the core satisfies the strong nullplayer property.

Lemma 2.5.2 *The core satisfies the strong nullplayer property.*

Proof.
We show the equality (2.2) in two steps.
" $\supseteq$ "
Let $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}_I \in \mathcal{C}(\boldsymbol{v}), x_{\tilde{n}} = 0\}$.
Since $\boldsymbol{x}_I \in \mathcal{C}(\boldsymbol{v})$, we can conclude that

$$\boldsymbol{x}(S) \geq \boldsymbol{v}(S \cap I) = \tilde{\boldsymbol{v}}(S) \quad \forall\, S \subseteq \tilde{I}.$$

Knowing that

$$\tilde{\boldsymbol{v}}(\tilde{I}) = \boldsymbol{v}(I),$$

one immediately sees that $\boldsymbol{x}(\tilde{I}) = \tilde{\boldsymbol{v}}(\tilde{I})$, such that we can conclude that $\boldsymbol{x} \in \mathcal{C}(\tilde{\boldsymbol{v}})$.
" $\subseteq$ "
Now, let $\boldsymbol{x} \in \mathcal{C}(\tilde{\boldsymbol{v}}) = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}(\tilde{I}) = \tilde{\boldsymbol{v}}(\tilde{I}),\ \boldsymbol{x}(S) \geq \tilde{\boldsymbol{v}}(S) \quad \forall S \in \mathcal{P}(\tilde{I})\}$.
By definition it follows immediately that

$$\boldsymbol{x}(\tilde{I}) \geq \boldsymbol{x}(I) \geq \boldsymbol{v}(I) = \tilde{\boldsymbol{v}}(\tilde{I}).$$

Thus, all inequalities are equalities, and in view of $\boldsymbol{x}(\tilde{I}) = \boldsymbol{x}(I)$, it is clear that $x_{\tilde{n}} = 0$. Since $\tilde{\boldsymbol{v}}(S) = \boldsymbol{v}(S)$ for all $S \in \mathcal{P}(I)$, we can conclude that $\boldsymbol{x}_I \in \mathcal{C}(\boldsymbol{v})$. □

Thus, if we look at the core, it is possible to add nullplayers without changing the results.

Another solution concept which satisfies the strong nullplayer property is the Shapley value. This result is topic of the following lemma.

Lemma 2.5.3 *The Shapley value satisfies the strong nullplayer property.*

Proof.

In order to prove the strong nullplayer property, we have to show that

$$\phi_i(\tilde{\boldsymbol{v}}) = \phi_i(\boldsymbol{v}) \quad \forall i \in \{1, \dots, n\}, \tag{2.3}$$

$$\phi_{\tilde{n}}(\tilde{\boldsymbol{v}}) = 0. \tag{2.4}$$

Note, that if the equality (2.3) holds, equality (2.4) follows immediately by the definition of the Shapley value. Now, let $i \in \{1, \dots n\}$ and let $S \subseteq I \setminus \{i\}$. For each $S = \{i_1, \dots, i_s\} \subseteq I \setminus \{i\}$ we define two subsets $K_s, L_s \subseteq \tilde{I} \setminus \{i\}$ by

1. $K_s = K = S = \{i_1 \dots, i_s\}$
2. $L_s = L = S \cup \{\tilde{n}\} = \{i_1 \dots, i_s, \tilde{n}\}$.

By definition of the game $\tilde{\boldsymbol{v}}$, we have for every sets $S \subseteq I \setminus \{i\}$ and the induced coalitions $K_s = K$ and $L_s = L$ the following result:

$$\tilde{\boldsymbol{v}}(K \cup \{i\}) = \tilde{\boldsymbol{v}}(L \cup \{i\}) = \boldsymbol{v}(S \cup \{i\}) \quad \text{and} \quad \tilde{\boldsymbol{v}}(K) = \tilde{\boldsymbol{v}}(L) = \boldsymbol{v}(S).$$

We have finished the proof, if we can show that

$$\gamma(s)(\boldsymbol{v}(S \cup \{i\} - \boldsymbol{v}(S)) = \gamma(k)(\tilde{\boldsymbol{v}}(K \cup \{i\}) - \tilde{\boldsymbol{v}}(K)) + \gamma(l)(\tilde{\boldsymbol{v}}(L \cup \{i\}) - \tilde{\boldsymbol{v}}(L)).$$

This equality holds true if and only if

$$\gamma(s) = \gamma(k) + \gamma(l)$$

holds true. To show this equality, we denote the number of players of coalition K and S with $t = |K| = |S|$.

Then, we have:

$$\begin{aligned}
\gamma(s) &= \frac{1}{n}\frac{1}{\binom{n-1}{t}} \\
&= \frac{t!\cdot(n-1-t)!}{n\cdot(n-1)!} \\
&= \frac{t!\cdot(n+1)\cdot(n-1-t)!}{(n+1)!} \\
&= \frac{t!\cdot(n-1-t)![(n-t)+(t+1)]}{(n+1)!} \\
&= \frac{t!\cdot(n-t)!+(t+1)!\cdot(n-t-1)!}{(n+1)!} \\
&= \frac{1}{(n+1)}\frac{1}{\binom{n}{t}}+\frac{1}{(n+1)}\frac{1}{\binom{n}{t+1}} \\
&= \gamma(k)+\gamma(l)
\end{aligned}$$

This equality proves our lemma. □

Thus, we have shown that the Shapley value satisfies the strong nullplayer property. In this case results about the Shapley value are independent of nullplayers.

For reasons of completeness let us remark that Sudhölter proved in his papers [29, 30] that the modiclus also satisfies the strong nullplayer property.

Since we have seen that there exists different solution concepts which satisfy the strong nullplayer property, we will see that there exists solution concepts which do not satisfy this property.

Remark 2.5.4 *Typically, the least core does not satisfy the strong nullplayer property.*

To see this, we look at an example of a cooperative game such that its least core does not satisfy the strong nullplayer property.

Example 2.5.5 *Consider the following game $(I, \mathcal{P}(I), \boldsymbol{v})$ which is defined by:*

$$\begin{aligned}
I &= \{1,2,3\}, \\
\boldsymbol{v}(I) &= 1, \\
\boldsymbol{v}(\{1,2\}) = \boldsymbol{v}(\{1,3\}) &= \frac{3}{4}, \\
\boldsymbol{v}(S) &= 0 \quad \forall S \in \mathcal{P}(I).
\end{aligned}$$

Let $\tilde{I} = \{1, 2, 3, 4\}$ *and let* $(\tilde{I}, \mathcal{P}(\tilde{I}), \tilde{\boldsymbol{v}})$ *be a game defined by*

$$\tilde{\boldsymbol{v}}(S) = \boldsymbol{v}(I \cap S) \text{ for all } S \in \mathcal{P}(\tilde{I}).$$

In this case we have

$$\left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right) \in \mathcal{LC}(\boldsymbol{v}) = \mathcal{C}_{-\frac{1}{8}}(\boldsymbol{v})$$

and

$$\left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}, 0\right) \in \mathcal{LC}(\tilde{\boldsymbol{v}}) = \mathcal{C}_0(\tilde{\boldsymbol{v}}).$$

But, there exists an element

$$(1, 0, 0, 0) \in \mathcal{LC}(\tilde{\boldsymbol{v}}) = \mathcal{C}_0(\tilde{\boldsymbol{v}})$$

with

$$(1, 0, 0) \notin \mathcal{LC}(\boldsymbol{v}).$$

Thus, the least core does not satisfy the strong nullplayer property.

As we will see in a later theorem, the least core of the dual assignment game satisfies the strong nullplayer property. This is possible because, as we will see later, the least core of these games has more structure.

Chapter 3

The Shapley Value and the Core

3.1 Introduction

Both, the Shapley value and the core are well-known solution concepts in cooperative game theory. Since these two concepts are characterized differently, it is not surprising that the Shapley value of a game need not be an element of the core, even if the core is non-empty. But there exist subclasses of balanced games such that the Shapley value is in the core. One of the most important subclass is the subclass of convex games. The main focus of this chapter is the well-studied subclass of assignment games. Until now, very little is known about their Shapley value. If one considers assignment games with an unequal numbers of P and Q players, the Shapley value cannot be in the core. To see this, note that the players who are not matched in an optimal assignment of the grand coalition get nothing in any core allocation. But the expected payoff of every non-nullplayer is however positive. During this chapter we are looking for some necessary conditions so that the Shapley value is an element of the core. Therefore, we will start in the first section with partially average convex games and with partially average convex assignment games. In the second section we concentrate on some properties of exact assignment games. The main result of this section is the connection between the Shapley value and the core in the case of exact assignment games. Next, we look at an example of a game with a large core such that the Shapley value is not an element of the core. In the last step we will see that there exist non-exact assignment games such that the Shapley value is in the core.

3.2 Partially Average Convex Games

In this section we concentrate on partially average convex games, which include the class of convex games. In order to define these games, we have a look at the following function.

Definition 3.2.1 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game. A function*

$$g^{\boldsymbol{v}} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$$

is defined by

$$g^{\boldsymbol{v}}(A, B) = \boldsymbol{v}(A \cup B) - \boldsymbol{v}(A) - \boldsymbol{v}(B) \quad \forall A, B \subseteq I, A \cap B = \emptyset.$$

Note, that by definition, we have for all $A, B \subseteq I, A \cap B = \emptyset$ the following equality:

$$g^{\boldsymbol{v}}(A, B) = g^{\boldsymbol{v}}(B, A).$$

That means, we have a kind of symmetry. The value $g^{\boldsymbol{v}}(A, B)$ can be interpreted as gain (or loss) if two disjoint coalitions cooperate instead of stayed in the two coalitions A and B. With the aid of this function, we can define our next class of cooperative games, which dues to Iñarra and Usategui [8] in 1993.

Definition 3.2.2 *(Iñarra and Usategui [8])*
A cooperative game $\boldsymbol{v}$ is **partially average convex** *if*

$$\binom{b}{r}^{-1} \sum_{R \subseteq B, |R| = r} g^{\boldsymbol{v}}(A, R) \geq \binom{b}{c}^{-1} \sum_{C \subset B, |C| = c} g^{\boldsymbol{v}}(A, B \setminus C)$$

for all $A, B \subseteq I,\ B \subseteq I \setminus A$ and for any r and c such that

$$\begin{aligned} r > \frac{ab}{n-b} > b - c \qquad & \text{if } A \cup B \subset I, \text{ and} \\ r = b > c \qquad & \text{if } A \cup B = I. \end{aligned}$$

Note, that every convex game is partially average convex. Furthermore, by Theorem 5 of Iñarra and Usategui [8] the Shapley value of partially average convex games is an element of the core.

Theorem 3.2.3 *(Iñarra and Usategui [8])*
The Shapley value of a partially average convex game is in the core.

In particular, we know that the core of these games is non-empty.

3.2.1 Partially Average Convex Assignment Games

In this section we are looking at partially average convex assignment games. We will see that in the case of assignment games Theorem 3.2.3 is no gain, because an assignment game $\boldsymbol{v}_A$ is convex if and only if it is partially average convex. This result is subject of the following lemma of Hoffmann and Sudhölter [7] in 2007.

Lemma 3.2.4 *(Hoffmann and Sudhölter [7])*
Let A be a non-negative $p \times p$ assignment matrix such that the main diagonal is an optimal assignment. If $p \geq 3$ holds true, the following statements are equivalent:

1. *$\boldsymbol{v}_A$ is partially average convex,*
2. *$\boldsymbol{v}_A$ is convex.*

Proof.
"1. $\Rightarrow$ 2."
Let $\boldsymbol{v}_A = \boldsymbol{v}$ be a partially average convex assignment game and let $i, j \in P$ such that $i \neq j$. Furthermore, let $A = P \cup \{j'\}$ and $B = \{i, i'\}$. For $r = b = 2$ and $c = 1$ we have

$$r = 2 > \frac{2p}{2p-2} = \frac{ab}{n-2} > 1 = b - c$$

and

$$\begin{aligned} \binom{b}{r}^{-1} \sum_{R \subseteq B, |R| = r} g^{\boldsymbol{v}}(A, R) &= 0, \\ \binom{b}{c}^{-1} \sum_{C \subset B, |C| = c} g^{\boldsymbol{v}}(A, B \setminus C) &= \frac{1}{2} \max_{k \in P \setminus \{i,j\}} a_{ki'}. \end{aligned}$$

Thus,

$$0 \geq \frac{1}{2} \max_{k \in P \setminus \{i,j\}} a_{ki'}$$

holds true if A is a diagonal matrix. By Theorem 2.4.17 of Solymosi and Raghavan [26] the induced assignment game $\boldsymbol{v}_A$ is a convex game.
"2. $\Rightarrow$ 1."
By Iñarra and Usategui [8], every convex game is partially average convex. $\square$
Thus, we have seen that an assignment game which is induced by a $p \times p$, $p \geq 3$, assignment matrix is convex if and only if it is partially average convex.

3.3 The Shapley Value as an Element of the Core

In this section we concentrate on some aspects which imply that the Shapley value is an element of the core. Iñarra and Usategui [8] have found an interesting result in this context. In order to cite their result, we have to define a real-valued function on the set of coalitions $\mathcal{P}$.

Definition 3.3.1 *(Iñarra and Usategui [8])*
Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game. A function

$$\alpha_T : \mathcal{P} \to \mathbb{R}$$

is defined by

$$\begin{aligned} \alpha_T(S) &= \frac{|S \cap T|n - st}{n - s} = s\left(\frac{|S \cap T|}{s} - \frac{|T \setminus S|}{n - s}\right) \quad \forall S, T \subseteq I,\ S \neq I \\ \alpha_T(I) &= t \quad \forall T \subseteq I. \end{aligned}$$

Note, that $\alpha_T(S)$ may be negative even if $S \cap T \neq \emptyset$.
In the next theorem of Iñarra and Usategui, we will see that there exists a property which is equivalent to the fact, that the Shapley value is an element of the core.

Theorem 3.3.2 *(Iñarra and Usategui [8])*
Let $(I, \mathcal{P}, \boldsymbol{v})$ a cooperative game. Then, the following statements are equivalent:

1. $\Phi(\boldsymbol{v}) \in \mathcal{C}(\boldsymbol{v})$,

2. $\sum_{\emptyset \neq S \subseteq I} \gamma(S)\, \alpha_T(S)\, g^{\boldsymbol{v}}(S \setminus T, S \cap T) \geq 0 \quad \forall T \subset I,\ T \neq \emptyset.$

Thus, we have a possibility to check, if the Shapley value is in the core or not. In the following sections, we are looking for other possibilities to characterize assignment games such that the Shapley Value is in the core. We will see that in the case of assignment games, we do not have to use the above results.

3.4 Exact Assignment Games

During this section we consider classes of assignment games such that the Shapley value is an element of the core. Based on the idea that both, the Shapley value and the core satisfy the strong nullplayer property, we assume throughout this section that $|P| = |Q|$. Otherwise we are allowed to add some nullplayers, i.e. we add some zero rows or columns to the assignment matrix. Furthermore, note that
$P = \{1, \ldots, p\}$, $Q = \{1', \ldots, p'\}$ and

$$\boldsymbol{v}_A(I) = \sum_{i \in P} a_{ii'}, \tag{3.1}$$

i.e. the main diagonal is an optimal assignment of the grand coalition. For the next steps it is useful to have a notation for the set of pairs in coalition S. More formally, we have the following definition.

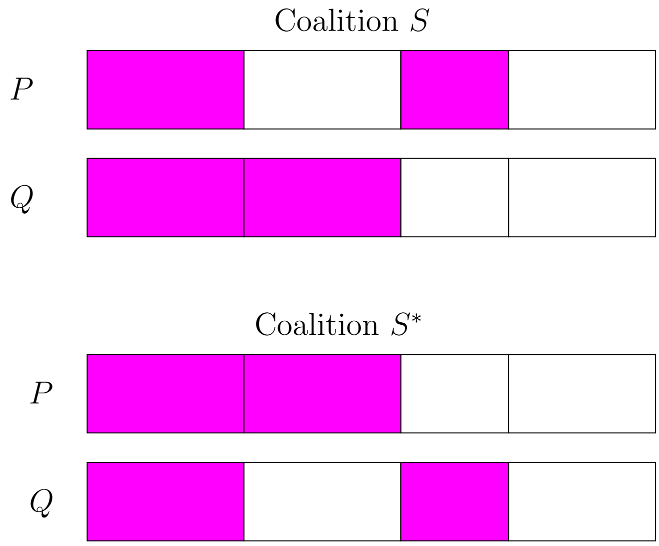

Figure 3.1: Coalition S and Coalition S^*

Definition 3.4.1 *Let $S \subseteq I$. Define*

$$S^* = \{i' \in Q \mid i \in S\} \cup \{i \in P \mid i' \in S\}.$$

Consequently, the set $T(S)$ which contains the pairs of coalition S is defined by

$$T(S) = S \cap S^*.$$

In Figure 3.1 we see a geometric view of the sets S and S^*.

Note, that S^* is the set of all partners of players in S according to an optimal assignment of I. Furthermore, for all $S \subseteq I$ we have $|S^*| = |S|$ and $(S^*)^* = S$. In our next remark we consider the worth of the set $T(S)$ of an assignment game.

Remark 3.4.2 *Let $\boldsymbol{v}_A$ be an assignment game. By Definition 2.4.3 and (3.1), we have for all $S \subseteq I$ the following equality:*

$$\boldsymbol{v}_A(T(S)) = \boldsymbol{v}_A(S \cap S^*) = \sum_{i \in P \cap S \cap S^*} a_{ii'}. \tag{3.2}$$

Thus, the worth of coalitions, which only contains pairs, is equal to the sum of the corresponding diagonal elements.

Our next lemma characterizes a partition of the payoff $\boldsymbol{v}_A(S)$ of an exact assignment game. We will see that in the case of exact assignment games we can compute this value by adding the payoffs $\boldsymbol{v}_A(T(S))$ and $\boldsymbol{v}_A(S \setminus T(S))$. With the aid of this partition, it is possible to prove our main Theorem 3.4.4.

Lemma 3.4.3 *(Hoffmann and Sudhölter [7])*

Let $\boldsymbol{v}_A$ be an assignment game such that the main diagonal of the assignment matrix is an optimal assignment. Then, the following statements are equivalent:

1. *$\boldsymbol{v}_A(S)$ satisfies*

$$\boldsymbol{v}_A(S) = \boldsymbol{v}_A(T(S)) + \boldsymbol{v}_A(S \setminus T(S)) \quad \forall S \subseteq I. \tag{3.3}$$

2. *$\boldsymbol{v}_A$ is exact.*

Proof.

We prove the above statement in two steps.

“1. $\Rightarrow$ 2.“

Assume that (3.3) is satisfied. Let $i \in P$ and $j, k \in P \setminus \{i\}$. By Definition 2.4.3, we have:

$$\begin{aligned} a_{ii'} + a_{jk'} &= \boldsymbol{v}_A(\{i, j, i', k'\}) \geq a_{ik'} + a_{ji'}, \\ a_{ii'} &= \boldsymbol{v}_A(\{i, j, i'\}) \geq a_{ji'}, \\ a_{ii'} &= \boldsymbol{v}_A(\{i, i', k'\}) \geq a_{ik'}. \end{aligned}$$

Hence, the assignment matrix A has a doubly dominant and a dominant diagonal such that the induced assignment game $\boldsymbol{v}_A$ is exact.

"2. $\Rightarrow$ 1."

Let $\boldsymbol{v}_A$ be an exact assignment game such that the assignment matrix A has a dominant and doubly dominant diagonal. Let $S \subseteq I$ and define $T = T(S) = S \cap S^*$. We shall prove (3.3) by induction on $|T| = t$. Indeed, if $t = 0$, then $T = \emptyset$ and (3.3) is true. Assume now that $t > 0$. Let b be an optimal assignment for S and let $i \in T$. By the inductive hypothesis and (3.2) it remains to show that

$$\boldsymbol{v}_A(S) \leq a_{ii'} + \boldsymbol{v}_A(S \setminus \{i, i'\}).$$

If $(i, i') \in b$, then the proof is complete. Hence, we may assume that $(i, i') \notin b$. The following three cases may occur.

Case 1:

There exists $j' \in S \cap Q, j \neq i$, such that $(i, j') \in b$ and i' is not matched in b, that is,

$$\{(j, i') \mid j \in P\} \cap b = \emptyset.$$

As the assignment matrix A has a dominant diagonal,

$$\boldsymbol{v}_A(S) = \boldsymbol{v}_A(S \setminus \{i, i', j'\}) + a_{ij'} \leq \boldsymbol{v}_A(S \setminus \{i, i', j'\}) + a_{ii'} \leq \boldsymbol{v}_A(S \setminus \{i, i'\}) + a_{ii'}.$$

Case 2:

The case where there exists $j \in S \cap P, j \neq i$, such that $(j, i') \in b$ and i is not matched in b, that is,

$$\{(i, j') \mid j \in P\} \cap b = \emptyset,$$

may be treated analogously to (1).

Case 3:

If $(i, k'), (j, i') \in b$ for some $j, k \in P \setminus \{i\}$, then we have

$$\begin{aligned}
\boldsymbol{v}_A(S) &= \boldsymbol{v}_A(S \setminus \{i, i', j, k'\}) + a_{ik'} + a_{ji'} \\
&\leq \boldsymbol{v}_A(S \setminus \{i, i', j, k'\}) + a_{ii'} + a_{jk'} \\
&\leq \boldsymbol{v}_A(S \setminus \{i, i'\}) + a_{ii'},
\end{aligned}$$

because A has a doubly dominant diagonal. □

Note, that Lemma 3.4.3 gives a further possible description of exact assignment games. With the aid of this result, it is possible to prove our main result in the next section.

3.4.1 The Shapley Value of Exact Assignment Games

In the previous sections of this chapter we looked for characteristics of partially average convex and of exact assignment games. The problem we now wish to study is the connection between the Shapley value and the core of exact assignment games. The following theorem is the main result of Hoffmann and Sudhölter [7] in 2007. This result can be proven in a direct way with the aid of Theorem 2.4.9 and Lemma 3.4.3. The results of Iñarra and Usategui [8] are not used in the proof.

Theorem 3.4.4 *(Hoffmann and Sudhölter [7])*
Let $\boldsymbol{v}_A$ be an assignment game induced by the $p \times p$ assignment matrix A. Furthermore, let the main diagonal of the matrix A be an optimal assignment. In the case that $\boldsymbol{v}_A$ is an exact assignment game, we have

$$\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A).$$

Proof.
Let $\Phi(\boldsymbol{v}_A) = \boldsymbol{x} = (u, v)$. By Definition 2.3.37, we have $\Phi(\boldsymbol{v}_A) \geq 0$. In view of Theorem 2.4.9 and (3.1) it suffices to show that

$$\begin{aligned} u_i + v_{i'} &= a_{ii'} \text{ for all } i \in P, & (3.4)\\ u_i + v_{j'} &\geq a_{ij'} \text{ for all } i, j \in P. & (3.5) \end{aligned}$$

Note, that (3.4) implies that $\boldsymbol{x}(I) = \boldsymbol{v}_A(I) = \sum_{i \in P} a_{ii'}$. Now, let $i \in P$.

Step 1:
We shall show that for $i \in P$ we have $u_i + v_{i'} = a_{ii'}$. Let $|I| = n$ and note that, by the definition of $\gamma(\cdot)$, we have

$$\gamma(s) = \gamma(n - s - 1) \text{ for all } s = 0, \ldots, n - 1. \quad (3.6)$$

By (3.6) and Definition 2.3.37 it suffices to find a bijection

$$f : \mathcal{P}(I \setminus \{i\}) \to \mathcal{P}(I \setminus \{i'\})$$

such that, for all $S \subseteq I \setminus \{i\}$, we have

$$\begin{aligned} n - 1 &= |S| + |f(S)|, && (3.7) \\ a_{ii'} &= (\boldsymbol{v}_A(S \cup \{i\}) - \boldsymbol{v}_A(S)) + (\boldsymbol{v}_A(f(S) \cup \{i'\}) - \boldsymbol{v}_A(f(S))). && (3.8) \end{aligned}$$

For each $S \subseteq I \setminus \{i\}$ let

$$f(S) = I \setminus (S^* \cup \{i'\}).$$

Note, that a more geometric view of the sets S and $f(S)$ can be seen in Figure 3.2 and in Figure 3.3.

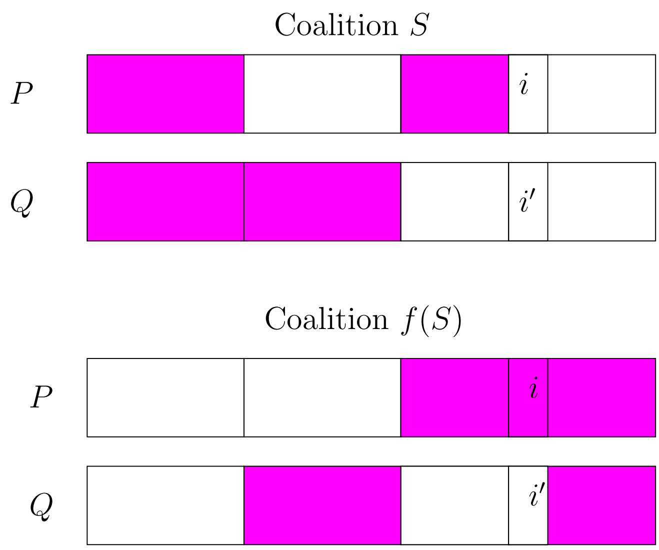

Figure 3.2: Coalition S and $f(S)$ with $i' \notin S$

We shall now verify that f has the desired properties. Let $S \subseteq I \setminus \{i\}$. By definition $i' \notin f(S)$ and

$$f(S)^* = I \setminus (S \cup \{i\}). \tag{3.9}$$

Hence $S = I \setminus (f(S)^* \cup \{i\})$. So, f is a bijection with the desired range. As $|S| = |S^*|$ and $i' \notin S^*$, (3.7) is satisfied. In order to show (3.8), we distinguish two cases.

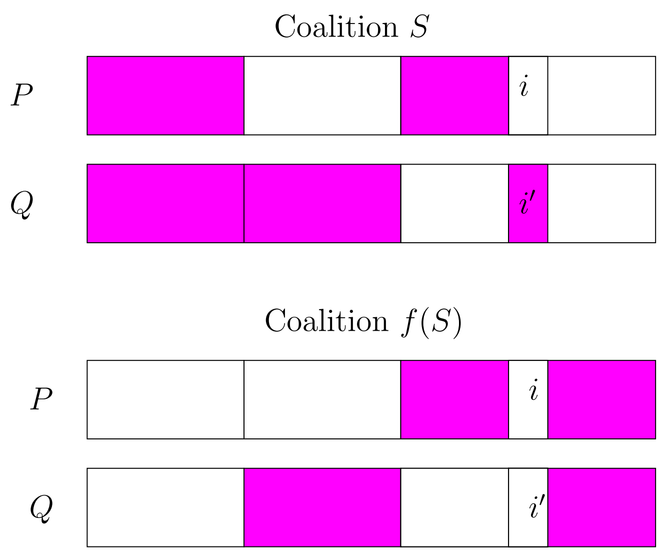

Figure 3.3: Coalition S and $f(S)$ with $i' \in S$

Case 1:

Let $i' \in S$ (that is, $i \notin f(S)$). By Lemma 3.4.3 and (3.2), we have

$$\boldsymbol{v}_A(S \cup \{i\}) - \boldsymbol{v}_A(S) = a_{ii'} + \boldsymbol{v}_A\Big((S \setminus \{i'\}) \setminus S^*\Big) - \boldsymbol{v}_A\Big(S \setminus S^*\Big),$$
$$\boldsymbol{v}_A(f(S) \cup \{i'\}) - \boldsymbol{v}_A(f(S)) = \boldsymbol{v}_A\Big((f(S) \cup \{i'\}) \setminus f(S)^*\Big) - \boldsymbol{v}_A\Big(f(S) \setminus f(S)^*\Big)$$

Now, (3.8) follows because by (3.9), we have

$$\begin{aligned} f(S) \setminus f(S)^* = (S \setminus \{i\}) \setminus (S^* \cup \{i'\}) &= (S \setminus \{i'\} \setminus S^*) \\ (f(S) \cup \{i'\}) \setminus f(S)^* = (S \cup \{i\}) \setminus S^* &= S \setminus S^*. \end{aligned}$$

Case 2:

In the case that $i' \notin S$ (that is, $i \in f(S)$) we may proceed analogously. Just the roles of S and $f(S)$ have to be exchanged.

Step 2:

Let $j \in P$. We shall now show that $u_i + v_{j'} \geq a_{ij'}$ for all $(i, j') \in P \times Q$. By the first step we may assume that $j \neq i$. By Remark 2.3.36 and (3.6), we have

$$1 = \sum_{S \subseteq I \setminus \{i\}} \gamma(|S|) = \sum_{S \subseteq I \setminus \{i,i'\}} \Big(\gamma(|S|) + \gamma(|I \setminus (S \cup \{i\})|)\Big) = 2 \sum_{S \subseteq I \setminus \{i,i'\}} \gamma(|S|). \quad (3.10)$$

Let $\mathcal{R} = \mathcal{P}(I \setminus \{i, i', j, j'\})$. By (3.6) applied to any $R \cup \{i'\}$, $R \in \mathcal{R}$ and by (3.10),

$$\begin{aligned} 1 &= 2 \sum_{R \subseteq I \setminus \{i,i',j,j'\}} \Big(\gamma(|R|) + \gamma(|R \cup \{i'\}|) + \gamma(|R \cup \{j\}|) + \gamma(|R \cup \{j'\}|) \Big) \\ &= \sum_{R \subseteq I \setminus \{i,i',j,j'\}} \Big(2\gamma(|R|) + 6\gamma(|R| + 1) \Big). \end{aligned} \tag{3.11}$$

For $R \subseteq I$, let $r = |R|$. By (3.11) it suffices to show that for all $R \in \mathcal{R}$ there are $c(R), d(R) \in \mathbb{R}$, such that

$$u_i + v_{j'} = \textstyle\sum_{R \in \mathcal{R}} (c(R)\gamma(r) + d(R)\gamma(r+1)), \tag{3.12}$$

$$c(R) \geq 2a_{ij'} \text{ and } d(R) \geq 6a_{ij'}. \tag{3.13}$$

By Definition 2.3.37 and using $s = |S|$, we have

$$\begin{aligned} u_i + v_{j'} = \sum_{S \subseteq I \setminus \{i,j'\}} & \Big(\gamma(s) \Big(\boldsymbol{v}_A(S \cup \{i\}) + \boldsymbol{v}_A(S \cup \{j'\}) - 2\boldsymbol{v}_A(S) \Big) + \\ & \gamma(s+1) \Big(2\boldsymbol{v}_A(S \cup \{i, j'\}) - \boldsymbol{v}_A(S \cup \{j'\}) - \boldsymbol{v}_A(S \cup \{i\}) \Big) \Big). \end{aligned} \tag{3.14}$$

To every $R \in \mathcal{R}$ we assign three coalitions defined by

$$\begin{aligned} f_j(R) &= (I \setminus R^*) \setminus \{j\}, \\ f_{i'}(R) &= (I \setminus R^*) \setminus \{i'\}, \\ f_{ij'}(R) &= (I \setminus R^*) \setminus \{i, j'\}. \end{aligned}$$

Note, that $f_j(\cdot)$, $f_{i'}(\cdot)$ and $f_{ij'}(\cdot)$ are injective mappings in $\mathcal{R}$. Observe that for any $S \subseteq I \setminus \{i, j'\}$ there exists a unique $R \in \mathcal{R}$ such that S coincides with one of the sets R, $R \cup \{j\}$, $R \cup \{i'\}$ and $f_{ij'}(R)$. Similarly, note that there exists a unique $R \in \mathcal{R}$ such that $S \cup \{i, j'\}$ coincides with one of the sets $I \setminus R^*$, $f_j(R)$, $f_{i'}(R)$ and $R \cup \{i, j'\}$. Let $R \in \mathcal{R}$ and let

$$\begin{aligned} c(R) &= \boldsymbol{v}_A(R \cup \{i\}) + \boldsymbol{v}_A(R \cup \{j'\}) - 2\boldsymbol{v}_A(R) + 2\boldsymbol{v}_A(I \setminus R^*) \\ &\quad - \boldsymbol{v}_A((I \setminus R^*) \setminus \{i\}) - \boldsymbol{v}_A((I \setminus R^*) \setminus \{j'\}, \end{aligned}$$

let

$$\begin{aligned} d_j(R) &= \boldsymbol{v}_A(R \cup \{i,j\}) + \boldsymbol{v}_A(R \cup \{j,j'\}) - 2\boldsymbol{v}_A(R \cup \{j\}) \\ &\quad +2\boldsymbol{v}_A(f_j(R)) - \boldsymbol{v}_A(f_j(R) \setminus \{i\}) - \boldsymbol{v}_A(f_j(R) \setminus \{j'\}), \\ d_{i'}(R) &= \boldsymbol{v}_A(R \cup \{i,i'\}) + \boldsymbol{v}_A(R \cup \{i',j'\}) - 2\boldsymbol{v}_A(R \cup \{i'\}) \\ &\quad +2\boldsymbol{v}_A(f_{i'}(R)) - \boldsymbol{v}_A(f_{i'}(R) \setminus \{i\}) - \boldsymbol{v}_A(f_{i'}(R) \setminus \{j'\}), \\ d_{ij'}(R) &= \boldsymbol{v}_A(f_{ij'}(R) \cup \{i\}) + \boldsymbol{v}_A(f_{ij'}(R) \cup \{j'\}) - 2\boldsymbol{v}_A(f_{ij'}(R)) \\ &\quad +2\boldsymbol{v}_A(R \cup \{i,j'\}) - \boldsymbol{v}_A(R \cup \{j'\}) - \boldsymbol{v}_A(R \cup \{i\}), \end{aligned}$$

and put $d(R) = d_j(R) + d_{i'}(R) + d_{ij'}(R)$. Thus, (3.12) is implied by (3.14) and (3.6). Let $\tilde{R} = R \setminus R^*$. Lemma 3.4.3 together with (3.2) yield

$$\begin{aligned} c(R) &= a_{ii'} + a_{jj'} + \boldsymbol{v}_A(\tilde{R} \cup \{i\}) - \boldsymbol{v}_A(\tilde{R} \cup \{i'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j\}) + \boldsymbol{v}_A(\tilde{R} \cup \{j'\}), \\ d_j(R) &= a_{ii'} + a_{jj'} - 2\boldsymbol{v}_A(\tilde{R} \cup \{j\}) + 2\boldsymbol{v}_A(\tilde{R} \cup \{j'\}) + \boldsymbol{v}_A(\tilde{R} \cup \{i,j\}) \\ &\quad -\boldsymbol{v}_A(\tilde{R} \cup \{i',j'\}), \\ d_{i'}(R) &= a_{ii'} + a_{jj'} + 2\boldsymbol{v}_A(\tilde{R} \cup \{i\}) - 2\boldsymbol{v}_A(\tilde{R} \cup \{i'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{i,j\}) \\ &\quad +\boldsymbol{v}_A(\tilde{R} \cup \{i',j'\}), \\ d_{ij'}(R) &= a_{ii'} + a_{jj'} - \boldsymbol{v}_A(\tilde{R} \cup \{i\}) + \boldsymbol{v}_A(\tilde{R} \cup \{i'\}) + \boldsymbol{v}_A(\tilde{R} \cup \{j\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j'\}) \\ &\quad +2\boldsymbol{v}_A(\tilde{R} \cup \{i,j'\}) - 2\boldsymbol{v}_A(\tilde{R} \cup \{i',j\}). \end{aligned}$$

We conclude that

$$\begin{aligned} d(R) = 3a_{ii'} + 3a_{jj'} &+ \boldsymbol{v}_A(\tilde{R} \cup \{i\}) - \boldsymbol{v}_A(\tilde{R} \cup \{i'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j\}) \\ &+\boldsymbol{v}_A(\tilde{R} \cup \{j'\}) + 2\boldsymbol{v}_A(\tilde{R} \cup \{i,j'\}) - 2\boldsymbol{v}_A(\tilde{R} \cup \{i',j\}). \end{aligned}$$

Therefore, in order to verify (3.13) it suffices to prove that

$$a_{ij'} \leq a_{ii'} + \boldsymbol{v}_A(\tilde{R} \cup \{j'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{i'\}), \tag{3.15}$$

$$a_{ij'} \leq a_{jj'} + \boldsymbol{v}_A(\tilde{R} \cup \{i\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j\}), \tag{3.16}$$

$$2a_{ij'} \leq a_{ii'} + a_{jj'} + \boldsymbol{v}_A(\tilde{R} \cup \{i,j'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j,i'\}). \tag{3.17}$$

In order to show (3.15) we distinguish two cases. If i' is not matched in an optimal assignment of $\tilde{R} \cup \{i'\}$, that is, if $\boldsymbol{v}_A(\tilde{R} \cup \{i'\}) = \boldsymbol{v}_A(\tilde{R})$, then the desired inequality is immediately implied, because A has a dominant diagonal. If i' is matched to some $k \in P \cap \tilde{R}$ in an optimal assignment, that is, if $\boldsymbol{v}_A(\tilde{R} \cup \{i'\}) = a_{ki'} + \boldsymbol{v}_A(\tilde{R} \setminus \{k\})$, then $\boldsymbol{v}_A(\tilde{R} \cup \{j'\}) \geq a_{kj'} + \boldsymbol{v}_A(\tilde{R} \setminus \{k\})$ implies that

$$a_{ii'} + \boldsymbol{v}_A(\tilde{R} \cup \{j'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{i'\}) \geq a_{ii'} + a_{kj'} - a_{ki'}$$

and, so (3.15) is valid, because A has a doubly dominant diagonal. In a completely analogous way we may show (3.16).

In order to prove (3.17), put

$$\beta = a_{ii'} + a_{jj'} + \boldsymbol{v}_A(\tilde{R} \cup \{i, j'\}) - \boldsymbol{v}_A(\tilde{R} \cup \{j, i'\}).$$

Let $b \in \mathcal{B}(P \cap (\tilde{R} \cup \{j\}), Q \cap (\tilde{R} \cup \{i'\}))$ be an optimal assignment for $\tilde{R} \cup \{j, i'\}$. Four cases may occur.

Case 1:

$(j, i') \in b$. So, $\boldsymbol{v}_A(\tilde{R} \cup \{j, i'\}) = \boldsymbol{v}_A(\tilde{R}) + a_{ji'}$. As $\boldsymbol{v}_A(\tilde{R} \cup \{i, j'\}) \geq \boldsymbol{v}_A(\tilde{R}) + a_{ij'}$, we may conclude that

$$\beta \geq a_{ii'} + a_{jj'} + a_{ij'} - a_{ji'} \geq 2a_{ij'},$$

because A has a dominant diagonal.

Case 2:

There exists some $\ell \in P \cap \tilde{R}$ such that $(\ell, i') \in b$ and j is not matched in b. Then, $\boldsymbol{v}_A(\tilde{R} \cup \{j, i'\}) = a_{\ell i'} + \boldsymbol{v}_A(\tilde{R} \setminus \{\ell\})$ and $\boldsymbol{v}_A(\tilde{R} \cup \{i, j'\}) \geq a_{\ell j'} + \boldsymbol{v}_A(\tilde{R} \setminus \{\ell\})$. So,

$$\beta \geq a_{ii'} + a_{jj'} + a_{\ell j'} - a_{\ell i'} \geq a_{jj'} + a_{ij'} + a_{\ell i'} - a_{\ell i'} \geq 2a_{ij'},$$

where the second inequality is true, because A has a doubly dominant diagonal, and the third inequality holds, because A has a dominant diagonal and $\boldsymbol{v}_A$ is superadditive.

Case 3:

There exists $k' \in Q \cap \tilde{R}$ such that $(j, k') \in b$ and i' is not matched in b. Then, $\boldsymbol{v}_A(\tilde{R} \cup \{j, i'\}) = a_{jk'} + \boldsymbol{v}_A(\tilde{R} \setminus \{k'\})$, $\boldsymbol{v}_A(\tilde{R} \cup \{i, j'\}) \geq a_{ik'} + \boldsymbol{v}_A(\tilde{R} \setminus \{k'\})$ and, analogously to Case 2, we have

$$\beta \geq a_{ii'} + a_{jj'} + a_{ik'} - a_{jk'} \geq a_{ii'} + a_{jk'} + a_{ij'} - a_{jk'} \geq 2a_{ij'}.$$

Case 4:

There exist $k' \in Q \cap \tilde{R}$ and $\ell \in P \cap \tilde{R}$ such that $(j,k'),(\ell,i') \in b$. Then,

$$\boldsymbol{v}_A(\tilde{R} \cup \{j,i'\}) = a_{jk'} + a_{\ell i'} + \boldsymbol{v}_A(\tilde{R} \setminus \{\ell,k'\})$$

and

$$\boldsymbol{v}_A(\tilde{R} \cup \{i,j'\}) \geq a_{ik'} + a_{\ell j'} + \boldsymbol{v}_A(\tilde{R} \setminus \{\ell,k'\}).$$

Applying the fact that A has a doubly dominant diagonal twice yields,

$$\begin{aligned} \beta &\geq a_{ii'} + a_{jj'} + a_{ik'} + a_{\ell j'} - a_{jk'} - a_{\ell i'} \\ &\geq a_{ij'} + a_{\ell i'} + a_{jk'} + a_{ij'} - a_{jk'} - a_{\ell i'} \\ &= 2a_{ij'}. \end{aligned}$$

Thus, we have shown that the Shapley value of exact assignment games lies in the core. $\square$

Before we consider cooperative games with a large core, we summarize by saying that there exists a further class of games such that the Shapley value is an element of the core. We have seen that convex games, partially average convex games and exact assignment games are such games.

3.4.2 Games with a Large Core

In this section we introduce an example of an exact cooperative game with transferable utility $(I, \mathcal{P}, \boldsymbol{v})$ such that the core is large and does not contain the Shapley value. Since assignment games are exact if and only if they have a large core, this example completes our Theorem 3.4.4 because the example shows that the structure of the assignment games is necessary for the above result. In the following example we present a cooperative game with five players, which is defined with the aid of a minimum of three non-negative additive functions.

Example 3.4.5 *(Hoffmann and Sudhölter [7])*
Let $I = \{1, \ldots, 5\}$ *and let* $\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\lambda}^3 \in \mathbb{A}$ *be given by*

$$\boldsymbol{\lambda}^1 = (1,1,1,0,0), \boldsymbol{\lambda}^2 = (1,1,0,1,0), \boldsymbol{\lambda}^3 = (0,0,1,1,1).$$

We define a cooperative game $(I, \mathcal{P}, \boldsymbol{v})$ *by*

$$\boldsymbol{v}(S) = \min_{r=1,2,3} \boldsymbol{\lambda}^r(S) \quad \forall S \subseteq I.$$

In the next step we think about exactness and the Shapley value of this game. We will see, that in the case of the above example, the Shapley value is an element of the core. For more details, consider the following remark.

Remark 3.4.6 *(Hoffmann and Sudhölter [7])*
Since $\boldsymbol{\lambda}^r(I) = 3$ *for all* $r = 1, 2, 3$*, we can conclude that the game* $(I, \mathcal{P}, \boldsymbol{v})$ *of Example 3.4.5 is exact. By Definition 2.3.37, we have*

$$\Phi(\boldsymbol{v}) = \Phi = \frac{1}{60}(36, 36, 41, 41, 26).$$

In order to check that the Shapley value is not an element of the core, consider the following inequality

$$\boldsymbol{v}(\{2,3,4\}) = 2 > \frac{118}{60} = \Phi(\{2,3,4\}),$$

such that we can conclude that

$$\Phi(\boldsymbol{v}) \notin \mathcal{C}(\boldsymbol{v}).$$

In the last step, we are left to prove the following lemma in which we show that the above game has a large core.

Lemma 3.4.7 *(Hoffmann and Sudhölter [7])*
The game $(I, \mathcal{P}, \boldsymbol{v})$ *of Example 3.4.5 has a large core.*

Proof:

Let

$$X = \{\boldsymbol{x} \in \mathbb{A} \mid \boldsymbol{x}(S) \geq \boldsymbol{v}(S) \text{ for all } S \subseteq I\}.$$

Taking into account that $\boldsymbol{v}$ is monotonic, that 1 and 2 as well as 3 and 4 are substitutes, and a careful inspection of the coalition function $\boldsymbol{v}$ yields that $\boldsymbol{x} \in X$ if and only if

$$\boldsymbol{x} \geq 0; \tag{3.18}$$

$$x_3 + x_4 \geq 1; \tag{3.19}$$

$$x_i + x_j \geq 1 \qquad \text{for all } i \in \{1,2\} \text{ and } j \in \{3,4,5\}; \tag{3.20}$$

$$x_i + x_3 + x_4 \geq 2 \quad \text{for all } i \in \{1,2\}. \tag{3.21}$$

Note, that (3.20) and (3.21) imply $\boldsymbol{x}(I) \geq 3 = \boldsymbol{v}(I)$. Let $\boldsymbol{x} \in X$ and

$$Y = \{\boldsymbol{y} \in X \mid \boldsymbol{y} \leq \boldsymbol{x}\}.$$

Then, Y is a non-empty compact (convex polyhedral) set. As $\boldsymbol{y} \mapsto \boldsymbol{y}(I)$, $\boldsymbol{y} \in Y$, is continuous, there exists $\boldsymbol{z} \in Y$ such that $\boldsymbol{z}(I) \leq \boldsymbol{y}(I)$ for all $\boldsymbol{y} \in Y$. It remains to show that $\boldsymbol{z}(I) = \boldsymbol{v}(I) = 3$. By (3.18) one of the following 4 cases occurs.

Case 1:

$z_1 = 0$. Then, by (3.20), $z_i \geq 1$ for all $i = 3,4,5$, and, by (3.18), $\boldsymbol{z} \geq \boldsymbol{\lambda}^3$. As $\boldsymbol{\lambda}^3 \in X$, $\boldsymbol{z} = \boldsymbol{\lambda}^3$ and the proof is complete.

Case 2:

The case $z_2 = 0$ may be treated analogously.

Case 3:

$z_5 = 0$. Then, (3.20) implies $z_1, z_2 \geq 1$ and (3.19) implies that $z_3 + z_4 \geq 1$. As $\boldsymbol{z}(I)$ is minimal, $\boldsymbol{z} \geq 0$ implies that $z_3 + z_4 = 1 = z_1 = z_2$. Hence, $\boldsymbol{z}(I) = 3$.

Case 4:

$z_1, z_2, z_5 > 0$. Then, $z_1 = z_2 \leq 2$ by minimality of $\boldsymbol{z}(I)$. In view of (3.19), $z_1 = z_2 \leq 1$. Again minimality implies that $z_1 + z_5 = z_2 + z_5 = 1$. By (3.21), $z_3 + z_4 \geq 2 - z_1$. So it remains to show that $z_3 + z_4 = 2 - z_1$. Assume the contrary. As 3 and 4 are substitutes, we may assume that $z_3 \geq z_4$. Then, $z_3 > 1 - \frac{z_1}{2}$. As $z_1 \leq 2$, there exists $\epsilon \geq 0$ such that $z_3 - \epsilon \geq 0$, $z_1 + z_3 - \epsilon \geq 1$ and $z_3 + z_4 - \epsilon \geq 2 - z_1$ and a contradiction to the minimality of $\boldsymbol{z}(I)$ has been obtained. $\square$

Thus, we have found an example of an exact game with a large core, such that the Shapley value is not an element of the core. This means, the special structure of assignment games is necessary for our Theorem 3.4.4. Without this structure, our theorem is invalid.

3.4.3 Some Examples

Before closing this chapter we will consider some examples of non-exact assignment games. We will see that all conditions of Theorem 3.4.4 are necessary. First, we show that our Theorem 3.4.4 is not valid if the condition of exactness is replaced by core stability, that is, if the matrix is without a doubly dominant diagonal. To check this, consider the next example which presents a non-exact assignment game.

Example 3.4.8 *(Hoffmann and Sudhölter [7])*
Consider the following assignment matrix

$$A = \begin{pmatrix} 8 & 4 & 8 \\ 4 & 4 & 1 \\ 8 & 1 & 8 \end{pmatrix}.$$

It is easy to see that this matrix A has a dominant diagonal. But since

$$a_{11'} + a_{32'} \leq a_{12'} + a_{31'},$$

we may conclude that the matrix A has not a doubly dominant diagonal. In the case of the induced assignment game $\boldsymbol{v}_A$, the Shapley value in not an element of the core. Here, we have

$$\Phi(\boldsymbol{v}_A) = \left(\frac{82}{20}, \frac{39}{20}, \frac{79}{20}, \frac{82}{20}, \frac{39}{20}, \frac{79}{20}\right) = (u, v) \notin \mathcal{C}(\boldsymbol{v}_A),$$

because $u_i + v_{i'} \neq a_{ii'}$ for all $i \in P$. This is in contrast to the property of the allocations of the core.

Thus, the dominant diagonal is necessary. In the following example one can easily verify the result of Theorem 3.4.4, if the assignment matrix has doubly dominant diagonal but not a dominant diagonal.

Example 3.4.9 *Consider the following assignment matrix*

$$B = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

This matrix has a doubly dominant diagonal but not a dominant diagonal. The Shapley value is not an element of the core:

$$\Phi(\boldsymbol{v}_B) = \left(\frac{53}{60}, \frac{14}{60}, \frac{83}{60}, \frac{53}{60}, \frac{14}{60}, \frac{83}{60}\right) = (u, v) \notin \mathcal{C}(\boldsymbol{v}_B),$$

because $u_i + v_{i'} \neq b_{ii'}$ *for all* $i \in P$.

Summarizing, it may be said that exactness of the assignment game is necessary condition for the above theorem.

3.5 Non-exact Assignment Games

In this section we want to check, if the core membership of the Shapley value of assignment games is necessary for exactness. Therefore, we are looking for non-exact assignment games such that the Shapley value is in the core. If there exists such games, we know that our Theorem 3.4.4 is complete, that means, the other direction is not valid. In order to do so, we look at our next example of a non-exact assignment game.

Example 3.5.1 *(Hoffmann and Sudhölter [7])*
Consider the 3×3 *assignment matrix*

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix}.$$

Let $\boldsymbol{v}_A$ *be the induced assignment game. Note, that on the one hand* $\boldsymbol{v}_A$ *is not exact, because*

$$a_{11'} + a_{32'} = 2 < 4 = a_{12'} + a_{31'}$$

and that on the other hand the matrix has a dominant diagonal such that the core is stable. We conclude that

$$\Phi(\boldsymbol{v}_A) = (1, 1, 1, 1, 1, 1)$$

and that by Theorem 2.4.9 we have: $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$.

This example implies that the core membership of the Shapley value is not necessary for exactness of the assignment game.

Next, we want to characterize the set of assignment games which are induced by 2×2 matrices such that the Shapley value is an element of the core. Therefore, it is useful to check the following property of the Shapley value of assignment games which are induced by a $p \times p$ matrix A.

Lemma 3.5.2 *(Hoffmann and Sudhölter [7])*
Let $\boldsymbol{v}_A$ be an assignment game which is induced by the $p \times p$ assignment matrix A such that (3.1) is valid. Then, for any

$$t \geq \max_{(i,j') \in P \times Q} (-a_{ij'})$$

we claim that with $T = A + t = (a_{ij'} + t)_{i,j \in P}$ the following equality holds:

$$\Phi(\boldsymbol{v}_T) = \frac{t}{2} + \Phi(\boldsymbol{v}_A). \tag{3.22}$$

Proof.
To see (3.22), observe that

$$\boldsymbol{v}_T(S) = \boldsymbol{v}_A(S) + t \min\{|S \cap P|, |S \cap Q|\} \text{ for any } S \subseteq I.$$

By $p = q$, we may conclude that for each $i \in I$ and $S \subseteq I \setminus \{i\}$, we get

$$\begin{aligned} & \Big(\boldsymbol{v}_T(S \cup \{i\}) - \boldsymbol{v}_T(S)\Big) + \Big(\boldsymbol{v}_T(I \setminus S) - \boldsymbol{v}_T(I \setminus (S \cup \{i\}))\Big) \\ = \quad & t + \Big(\boldsymbol{v}_A(S \cup \{i\}) - \boldsymbol{v}_A(S)\Big) + \Big(\boldsymbol{v}_A(I \setminus S) - \boldsymbol{v}_A(I \setminus (S \cup \{i\}))\Big). \end{aligned}$$

As $\gamma(s) = \gamma(n - s - 1)$, our claim follows directly from Definition 2.3.37. $\square$

That means, we can compute the Shapley value of the new assignment game $\boldsymbol{v}_T$ in a very simple way with the aid of the Shapley value of the game $\boldsymbol{v}_A$. Note, that the cardinality of P and Q have to conform. Otherwise, the above result is invalid.

In the next proposition we characterize the set of 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the Shapley value $\Phi(\boldsymbol{v}_A)$ is an element the core $\mathcal{C}(\boldsymbol{v}_A)$. Here, we complete the proof of Hoffmann and Sudhölter [7], who give only an idea of the proof.

Proposition 3.5.3 *(Hoffmann and Sudhölter [7])*
Consider a non-negative 2×2 *assignment matrix* $A = (a_{ij'})_{i,j \in P}$ *such that (3.1) is valid. Put* $a = \min\{a_{11'}, a_{22'}\}, b = \max\{a_{12'}, a_{21'}\}$, *and* $c = \min\{a_{12'}, a_{21'}\}$. *Then, the following statements are equivalent:*

1. $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$,

2. $a \geq b$ *or* $\Big(a_{11'} = a_{22'}$ *and* $4a \geq 3b + c\Big)$.

Proof:
First of all, we compute the Shapley value $\Phi(\boldsymbol{v}_A) = (u, v)$. Here, we have by Definition 2.3.37 the following values:

$$\begin{aligned} u_1 &= \frac{1}{12}\max\{b,d\} + \frac{5}{12}a + \frac{2}{12}b - \frac{1}{12}c - \frac{1}{12}d, \\ u_2 &= \frac{1}{12}\max\{b,d\} + \frac{3}{12}a - \frac{4}{12}b + \frac{1}{12}c + \frac{5}{12}d, \\ v_{1'} &= -\frac{3}{12}\max\{b,d\} + \frac{5}{12}a + \frac{1}{12}c + \frac{3}{12}d, \\ v_{2'} &= \frac{1}{12}\max\{b,d\} - \frac{1}{12}a + \frac{2}{12}b - \frac{1}{12}c + \frac{5}{12}d. \end{aligned}$$

In order to prove our claim, we consider a non-exact assignment game $\boldsymbol{v}_A$ such that $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$. By (3.1), Theorem 2.4.18, and anonymity we may assume that

$$b > a \text{ and } c < d.$$

As $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$, we know by Theorem 2.4.9 that

$$u_1 + v_{1'} = a. \tag{3.23}$$

To check the inequality $4a \geq 3b + c$, we show in the next two steps, that

$$c < a = d.$$

Step 1:
In the first step we will show that

$$c < a$$

holds true. In order to prove this, we assume that $c \geq a$ and consider the assignment game $\boldsymbol{v}_R$ which is induced by

$$R = A - a = \begin{pmatrix} 0 & b-a \\ c-a & d-a \end{pmatrix}.$$

By Lemma 3.5.2 we have in the case of assignment game $\boldsymbol{v}_R$ the following equality:

$$\Phi(\boldsymbol{v}_R) = (\tilde{u}, \tilde{v}) = \Phi(\boldsymbol{v}_A) - \frac{a}{2}. \tag{3.24}$$

As $b > a$, Definition 2.3.37 implies that $\tilde{u}_1 > 0$ and $\tilde{v}_{1'} \geq 0$, such that we can conclude that $\tilde{u}_1 + \tilde{v}_{1'} > 0$. By (3.24) we have

$$u_1 + v_{1'} > a,$$

which contradicts to equality (3.23). Thus, we have $c < a$.

Step 2:

In the second step we will show that $a = d$. Therefore, we assume that $d > b$. In this case we have

$$u_1 + v_{1'} = \frac{10}{12}a + \frac{2}{12}b > a,$$

such that we can conclude that $\Phi(\boldsymbol{v}_A) \notin \mathcal{C}(\boldsymbol{v}_A)$.

Since $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$, we have with (3.23) and $d \leq b$ the following equality:

$$u_1 + v_{1'} = \frac{10}{12}a + \frac{2}{12}d = a.$$

One immediately sees that $a = d$.

By Theorem 2.4.9 and $a = d$, we finally have:

$$\begin{aligned} u_1 + v_{2'} &= \frac{4}{12}a + \frac{6}{12}b - \frac{2}{12}c + \frac{4}{12}d \\ &= \frac{8}{12}a + \frac{6}{12}b - \frac{2}{12}c \\ &\geq b. \end{aligned}$$

Summarizing it may be said that this inequality holds true if and only if $4a \geq 3b + c$ holds true. $\square$

Thus, we have characterized the set of assignment games such that the Shapley value is in the core. We have seen that in the case of 2×2 games there exists a second class of games beside the exact assignment games. Here, the assignment matrix satisfies some special conditions.

The natural question that arises from Proposition 3.5.3 is whether the set of assignment matrices satisfying $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$ is convex or not. If this is not the case, it is enough to find an example.

Remark 3.5.4 *The set of matrices, such that the induced assignment games satisfy $\Phi(\boldsymbol{v}) \in \mathcal{C}(\boldsymbol{v})$ is not convex. To see this, consider*

$$A = \begin{pmatrix} 20 & 24 \\ 0 & 20 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 8 \\ 0 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 15 & 16 \\ 0 & 14 \end{pmatrix}.$$

One immediately sees that $C := \frac{1}{2}(A+B)$. Furthermore, we have the following Shapley values:

$$\begin{aligned} \Phi(\boldsymbol{v}_A) &= \frac{1}{3}(38, 22, 22, 38), \\ \Phi(\boldsymbol{v}_B) &= \frac{1}{3}(17, 10, 13, 14), \\ \Phi(\boldsymbol{v}_C) &= \frac{1}{12}(109, 65, 69, 103). \end{aligned}$$

In this case we have $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$ and $\Phi(\boldsymbol{v}_B) \in \mathcal{C}(\boldsymbol{v}_B)$, but $\Phi(\boldsymbol{v}_C) \notin \mathcal{C}(\boldsymbol{v}_C)$.

Thus, we can conclude that the set of matrices, which induce assignment games such that the Shapley value is in the core, is not convex. The above example introduces an example in which a convex combination of two assignment matrices does not satisfy this property.

In the next example, we consider the connection between the convex combination of assignment matrices and the convex combination of assignment games.

Example 3.5.5 *Consider*

$$A = \begin{pmatrix} 9 & 11 & 1 \\ 3 & 9 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 9 & 11 \\ 1 & 1 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 6 & 1 \\ 2 & 9 & 6 \\ 1 & 1 & 6 \end{pmatrix}$$

and let $\boldsymbol{v}_A$, $\boldsymbol{v}_B$ and $\boldsymbol{v}_C$ be the induced assignment games. In this case, we have

$$C = \frac{1}{2}(A + B).$$

But the induced assignment games does not satisfy

$$\boldsymbol{v}_C(S) = \frac{1}{2}(\boldsymbol{v}_A(S) + \boldsymbol{v}_B(S)) \quad \forall S \in \mathcal{P}.$$

To see this, consider the value of coalition $S = \{1, 2, 1', 2', 3'\}$. Here, we have

$$\begin{aligned} \boldsymbol{v}_A(S) &= 18 \\ \boldsymbol{v}_B(S) &= 14 \\ \boldsymbol{v}_C(S) &= 15 \neq \frac{1}{2}(\boldsymbol{v}_A(S) + \boldsymbol{v}_B(S)). \end{aligned}$$

This example illustrates that it is not the same, if one considers the convex combination of assignment games or if one considers the convex combination of assignment matrices of the corresponding assignment games.
Finally, we want to find a way to construct non-exact assignment games which are induced by a $p \times p$ matrix such that the Shapley value is an element of the core. This construction is the result of our following lemma.

Lemma 3.5.6 *Consider the two assignment games which are induced by the following two assignment matrices*

$$A = \begin{pmatrix} a_{11'} & a_{12'} & 0 & \cdots & 0 \\ a_{21'} & a_{22'} & 0 & \cdots & \vdots \\ 0 & 0 & a_{33'} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{pp'} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_{11'} & a_{12'} \\ a_{21'} & a_{22'} \end{pmatrix}.$$

Then, the Shapley value $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$ if and only if $\Phi(\boldsymbol{v}_B) \in \mathcal{C}(\boldsymbol{v}_B)$.

Proof.

In order to prove our claim, we define a second $p \times p$ assignment matrix $\tilde{B}$ by adding zero rows and zero columns the the matrix B. By Definition 2.3.37, we can see in a simple way that we have the following connections between the Shapley values of the different assignment games:

$$\begin{aligned} \phi_i(\boldsymbol{v}_A) &= \phi_i(\boldsymbol{v}_B) = \phi_i(\boldsymbol{v}_{\tilde{B}}), \\ \phi_{j'}(\boldsymbol{v}_A) &= \phi_{j'}(\boldsymbol{v}_B) = \phi_{j'}(\boldsymbol{v}_{\tilde{B}}) \quad \forall\, i, j \in \{1,2\} \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \phi_i(\boldsymbol{v}_A) &= \phi_i(\boldsymbol{v}_{\tilde{B}}) + \frac{a_{ii'}}{2}, \\ \phi_{j'}(\boldsymbol{v}_A) &= \phi_{j'}(\boldsymbol{v}_{\tilde{B}}) + \frac{a_{jj'}}{2} \quad \forall\, i, j \in P \setminus \{1,2\}. \end{aligned} \tag{3.26}$$

Since the Shapley value and the core satisfy the strong nullplayer property, we may conclude that $\Phi(\boldsymbol{v}_{\tilde{B}}) \in \mathcal{C}(\boldsymbol{v}_{\tilde{B}})$ if and only if $\Phi(\boldsymbol{v}_B) \in \mathcal{C}(\boldsymbol{v}_B)$.
Furthermore, note that by definition, we have

$$\sum_{i=1}^{2p} \phi_i(\boldsymbol{v}_A) = \boldsymbol{v}_A(I) = \boldsymbol{v}_{\tilde{B}}(\{1,2,1',2'\}) + \sum_{i=3}^{p} a_{ii'}.$$

One can check in a simple way that we have $\Phi(\boldsymbol{v}_{\tilde{B}}) \in \mathcal{C}(\boldsymbol{v}_{\tilde{B}})$ if and only if $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$. This follows immediately from the fact that $\Phi(\boldsymbol{v}_A) = (u, v)$ satisfies the inequalities of Corollary 2.4.9 if and only of $\Phi(\boldsymbol{v}_{\tilde{B}}) = (\tilde{u}, \tilde{v})$ satisfies it. This fact follows directly from the equalities (3.25) and (3.26). □

Chapter 4

The Least Core of the Dual Assignment Game

4.1 Introduction

Since the modified nucleolus is an element of the least core of the dual assignment game, we concentrate our attention on aspects of this set. In order to consider the least core of the dual assignment game, we look at some general results of c-convex games. C-convex games are very interesting in our context since the dual assignment game is such a game. The most important facts in our case are the definition of these games and some results about their least core. After having introduced c-convex games, we continue our research with some results about the least core of a general dual assignment game. Here, we find conditions that are satisfied by every element of the least core. With the aid of these conditions an important result, that the least core of dual assignment games satisfies the strong nullplayer property, can be proved. This property allows the restriction to assignment games which are induced by $p \times p$ matrices. A more detailed result is possible in the case of convex assignment games. Here, we find a simple way to compute all extreme points of the least core with the aid of the assignment matrix. Furthermore, it is possible to compute some other solution concepts in a very simple way. In particular, we show that some of these concepts coincide in this special case. In a next step we look for a correlation between the least core of the dual assignment game in the case of convex games and in the case of games with a stable core. The surprising result is that the extreme points in the case of convex assignment games stay extreme in the more general case.

Our last section is devoted to assignment games which are induced by a symmetric $p \times p$ matrix. As in the case of convex assignment games, it is possible to find some special properties of the least core of the dual assignment game and the modiclus.

4.2 The Least Core of C-convex Games

This section is devoted to the least core of c-convex games. It is a preparation for the next sections where we consider the least core of dual assignment games. In the first part of this section we look at the definition of c-convex games and some properties. Next, we summarize some basic results of c-convex games. But now, we start with the formal definition of c-convex games which can be found for example in Sudhölter [31].

Definition 4.2.1 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game and let (A, B), $A, B \neq \emptyset$, $A \cap B = \emptyset$ be a partition of I, this means that $I = A \cup B$. The game $\boldsymbol{v}$ is* **complementary convex** *or* **c-convex** *with respect to (A, B) if and only if*

$$\begin{aligned} \boldsymbol{v}(S) + \boldsymbol{v}(T) \quad \leq \quad & \boldsymbol{v}\big(((S \cap T) \cap A) \cup ((S \cup T) \cap B)\big) \\ & + \boldsymbol{v}\big(((S \cup T) \cap A) \cup ((S \cap T) \cap B)\big) \quad \forall S, T \in \mathcal{P}. \end{aligned}$$

A well-known class of c-convex games is the class of convex games. By definition every convex game is c-convex w.r.t. $(I, \emptyset)$. As in the case of convex games, note that complementary convexness can be expressed in terms of increasing marginal contributions of the players. To see this, we can look at the following lemma, proven in Sudhölter [31].

Lemma 4.2.2 *(Sudhölter [31])*
Let $(I, \mathcal{P}, \boldsymbol{v})$ be a game and let (A, B) be a partition of I. Then, the following statements are equivalent.

1. *$\boldsymbol{v}$ is c-convex with respect to (A, B).*

2.

$$\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S) \quad \leq \quad \boldsymbol{v}(T \cup \{i\}) - \boldsymbol{v}(T) \quad \textit{for } i \in A \setminus T, \tag{4.1}$$

$$\boldsymbol{v}(T \cup \{j\}) - \boldsymbol{v}(T) \quad \leq \quad \boldsymbol{v}(S \cup \{j\}) - \boldsymbol{v}(S) \quad \textit{for } j \in B \setminus S, \tag{4.2}$$

holds true for $S, T \subseteq I$ *with* $S \cap A \subseteq T$, $T \cap B \subseteq S$.

3. *For* $S \subseteq I$, $i, i_0 \in A \setminus S$, $j, j_0 \in B \setminus S, i \neq i_0, j \neq j_0$ *the following properties hold:*

$$\boldsymbol{v}(S \cup \{i,j\}) - \boldsymbol{v}(S \cup \{j\}) \leq \boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S), \tag{4.3}$$

$$\boldsymbol{v}(S \cup \{i,j\}) - \boldsymbol{v}(S \cup \{i\}) \leq \boldsymbol{v}(S \cup \{j\}) - \boldsymbol{v}(S), \tag{4.4}$$

$$\boldsymbol{v}(S \cup \{i,i_0\}) - \boldsymbol{v}(S \cup \{i\}) \geq \boldsymbol{v}(S \cup \{i_0\}) - \boldsymbol{v}(S), \tag{4.5}$$

$$\boldsymbol{v}(S \cup \{j,j_0\}) - \boldsymbol{v}(S \cup \{j\}) \geq \boldsymbol{v}(S \cup \{j_0\}) - \boldsymbol{v}(S). \tag{4.6}$$

In the case of convex games, the second statement of the above lemma implies the same equalities as Theorem 2.2.13.
In the next step we consider a second well-known class of c-convex games, namely the class of dual assignment games. This result dues to Sudhölter [31] in 2002 and it can be proven with the aid of Lemma 4.2.2.

Remark 4.2.3 *(Sudhölter [31])*
The dual assignment game $\boldsymbol{v}_A^\star$ *is c-convex with respect to* (P, Q).

The above remark shows the connection between c-convex games and assignment games. In our case, it gives an incentive to pay attention on the least core of c-convex games because the modified nucleolus of an assignment game is an element of the least core of the dual game. Next, we are interested in a description of the extreme points of the least core of c-convex games. For this, it is useful to have the following definition.

Definition 4.2.4 *Let* $(I, \mathcal{P}, \boldsymbol{v})$ *be a c-convex game with respect to* (A, B). *We define a function* $\gamma : \mathbb{V} \to \mathbb{R}$ *by*

$$\gamma(\boldsymbol{v}) = \frac{\boldsymbol{v}(A) + \boldsymbol{v}(B) - \boldsymbol{v}(I)}{2}.$$

Note, that in the case of dual assignment games, we have

$$\gamma(\boldsymbol{v}_A^\star) = \frac{1}{2}\boldsymbol{v}_A(I) = \frac{1}{2}\boldsymbol{v}_A^\star(I).$$

Remark 4.2.5 *Sometimes we will only write γ instead of $\gamma(\boldsymbol{v})$.*

With the aid of Definition 4.2.4 it is possible to give a short description of a lower bound of the maximal excess $\mu(\boldsymbol{x}, \boldsymbol{v})$. In order to do so, let $\boldsymbol{v}$ be a c-convex game w.r.t. (A, B) for some non-empty sets A and B. Furthermore, let $\boldsymbol{x} \in X(\boldsymbol{v})$. In view of

$$\begin{aligned} 2\mu(\boldsymbol{x}, \boldsymbol{v}) &\geq \boldsymbol{v}(A) - \boldsymbol{x}(A) + \boldsymbol{v}(B) - \boldsymbol{x}(B), \\ &= \boldsymbol{v}(A) + \boldsymbol{v}(B) - \boldsymbol{x}(I) \\ &= \boldsymbol{v}(A) + \boldsymbol{v}(B) - \boldsymbol{v}(I) \\ &= 2\gamma(\boldsymbol{v}) \end{aligned}$$

we have found a lower bound for the maximal excess of $\boldsymbol{v}$ at an arbitrary preimputation $\boldsymbol{x} \in X(\boldsymbol{v})$. In particular implies the definition of c-convex games that

$$\boldsymbol{v}(I) = \boldsymbol{v}(I) + \boldsymbol{v}(\emptyset) \leq \boldsymbol{v}(A) + \boldsymbol{v}(B),$$

such that we can conclude that

$$\gamma(\boldsymbol{v}) \geq 0.$$

In the case of the dual assignment game $\boldsymbol{v}_A^\star$ we have for any $\boldsymbol{x} \in X(\boldsymbol{v}_A)$ the following inequality:

$$\mu(\boldsymbol{x}, \boldsymbol{v}_A^\star) \geq \gamma(\boldsymbol{v}_A^\star) = \frac{1}{2}\boldsymbol{v}_A(I) \geq 0.$$

Before we consider the extreme points of the least core of c-convex games, we have a look at the following lemma of Sudhölter [31]. In this lemma a more simple definition of the least core of c-convex games is given.

Lemma 4.2.6 *(Sudhölter[31])*
Let $\boldsymbol{v}$ be a c-convex game with respect to (A, B). Then, we have

$$\mathcal{LC}(\boldsymbol{v}) = \{\boldsymbol{x} \in X(\boldsymbol{v}) \mid \mu(\boldsymbol{x}, \boldsymbol{v}) = \gamma(\boldsymbol{v})\}.$$

Thus, there exist preimputations which guarantee that $\gamma(\boldsymbol{v})$ is the highest possible excess.

4.2.1 Extreme Points of the Least Core

In this section we will see that there is also a simple way to compute the extreme points of the least core of c-convex games. In order to do so, we have a look at (A,B)-tight sequences.

Definition 4.2.7 *Let $\boldsymbol{v}$ be a c-convex game w.r.t. (A, B). A sequence $S^1, \ldots, S^n$ is* **(A,B)-tight***, if*

1. $S^1 = A, \quad S^n = B,$
2. $(S^{i+1} \cap A) \subseteq (S^i \cap A), \quad (S^i \cap B) \subseteq (S^{i+1} \cap B), \quad S^i \neq S^{i+1} \quad \forall i \in \{1, \ldots, n-1\},$
3. $|(S^i \cap A) \backslash (S^{i+1} \cap A)| \leq 1, \quad |(S^{i+1} \cap B) \backslash (S^i \cap B)| \leq 1 \quad \forall i \in \{1, \ldots, n-1\}.$

The first condition implies that the (A, B)-tight sequence starts with set A and ends with set B. The second condition means that the number of elements of set $(S^i \cap A)$ is decreasing in i and increasing in the case of $(S^i \cap B)$. Furthermore, the sets S^i are different in each step. The last condition implies that in each step the sets $(S^i \cap A)$ and $(S^i \cap B)$ change at most one element.
Before we continue our analysis of (A, B)-tight sequences, we have a look at some useful notations.

Notation 4.2.8 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a cooperative game. Consider a coalition $S \in \mathcal{P}$. Sometimes we will write instead of coalition $S = \{s_1, \ldots, s_k\}$ a vector*

$$1_S := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} .$$

The entry 1 means that player i is an element of coalition S, the entry 0 implies that player i is not an element of the coalition.

With the help of this notation, we are able to present an (A, B)-tight sequence in a more simple way. Our next example describes a possible (A, B)-tight sequence for a c-convex game with four players.

Example 4.2.9 *Let $(I, \mathcal{P}, \boldsymbol{v})$ be a c-convex w.r.t. $A = \{1, 2\}$ and $B = \{3, 4\}$. Then,*

$$\begin{array}{rcl} S^1 & = & 1\ \ 1\ \ 0\ \ 0 \\ S^2 & = & 0\ \ 1\ \ 0\ \ 0 \\ S^3 & = & 0\ \ 1\ \ 1\ \ 0 \\ S^4 & = & 0\ \ 0\ \ 1\ \ 1 \end{array}$$

is one possible (A, B)-tight sequence.

Note, that we can check in a simple way, that this sequence satisfies all conditions of Definition 4.2.7 and that there exist also some other (A, B)-tight sequences. In this context the natural question that arises is how many (A, B)-tight sequences exist. This question is answered in the next lemma of Sudhölter [28].

Lemma 4.2.10 *(Sudhölter [28])*
Let $(I, \mathcal{P}, \boldsymbol{v})$ be a c-convex game w.r.t. (A, B). Then, there exist exactly $(a+b-1)!\, ab$ (A,B)-tight sequences.

Thus, the maximal number of (A,B)-tight sequences is well-known. It depends only on the number of elements of the sets A and B.
In the next step we concentrate on the relationship between (A, B)-tight sequences and extreme points of the least core of c-convex games. The next theorem of Sudhölter [28] shows a possibility to compute extreme points of the least core with the aid of (A, B)-tight sequences. Thus, we will see that Lemma 4.2.10 is very useful in the context of extreme points of the least core.

Theorem 4.2.11 *(Sudhölter [28])*
Let $\boldsymbol{v}$ be a c-convex game w.r.t. (A,B).

1. *If $\boldsymbol{x}$ is an extreme point of $\mathcal{LC}(\boldsymbol{v})$, then there is a (A,B)-tight sequence $(S^1, \ldots, S^n)$ satisfying $e(S^i, \boldsymbol{x}, \boldsymbol{v}) = \gamma(\boldsymbol{v})$ for all $i \in \{1, \ldots, n\}$.*

2. *An (A,B)-tight sequence $(S^1, \ldots, S^n)$ uniquely determines a preimputation $\boldsymbol{x} \in X(\boldsymbol{v})$ satisfying $e(S^i, \boldsymbol{x}, \boldsymbol{v}) = \gamma(\boldsymbol{v})$ for all $i \in \{1, \ldots, n\}$.*

Summarizing, the extreme points of the least core of c-convex games can be computed by the following procedure:

1. Compute all (A,B)-tight sequences $S^1, \ldots, S^n$ and to each one the unique preimputation $\boldsymbol{x} \in X(\boldsymbol{v})$ satisfying $e(S^i, \boldsymbol{x}, \boldsymbol{v}) = \gamma(\boldsymbol{v})$ for all $i \in \{1, \ldots, n\}$.

2. Eliminate those $\boldsymbol{x} \in X(\boldsymbol{v})$ for which there exists a coalition $S \in \mathcal{P}$ with

$$e(S, \boldsymbol{x}, \boldsymbol{v}) > \gamma(\boldsymbol{v}).$$

Lemma 4.2.10 implies together with this procedure that there are at most

$$(a + b - 1)!\, a\, b$$

extreme points of the least core of c-convex games. That means, we have an upper boundary for the number of extreme points.

4.3 Properties of the Least Core of the Dual Assignment Game

In this section we restrict our attention on a special class of c-convex games, the class of the dual assignment games. We will see, that elements of the least core of the dual assignment game, in particular the modiclus, satisfy some special conditions. The following lemma describes this helpful conditions, which are important for further results in the next chapters and sections. For example, we can use these results in order to prove the strong nullplayer property of the least core of the dual assignment game.

Lemma 4.3.1 *Let $\boldsymbol{v}_A$ be an assignment game. Any element of the least core of the dual assignment game $\boldsymbol{x} = (u, v) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ satisfies the following conditions:*

1. $\boldsymbol{x}(P) = \boldsymbol{x}(Q) = \sum\limits_{i \in P} u_i = \sum\limits_{j' \in Q} v_{j'} = \frac{1}{2}\boldsymbol{v}_A(I) = \gamma(\boldsymbol{v}_A)$,

2. $\boldsymbol{x} = (u, v) \geq 0$,

3. $u_i \leq \max\limits_{j' \in Q} a_{ij'}$ *and* $v_{i'} \leq \max\limits_{j \in P} a_{ji'}$ $\forall i \in P, \forall j' \in Q$.

Note, that by 1. we have $u_i, v_{j'} \leq \gamma(\boldsymbol{v}_A)$ for all $i \in P$ and all $j' \in Q$.

Proof.

We prove this lemma in three steps, one step for each statement.

Step 1:

By definition of the dual assignment game $\boldsymbol{v}_A^\star$, we have:

$$\boldsymbol{v}_A^\star(P) = \boldsymbol{v}_A^\star(Q) = \boldsymbol{v}_A(I).$$

Furthermore, for every element $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$, Lemma 4.2.6 implies

$$\begin{aligned} \boldsymbol{v}_A^\star(P) - \boldsymbol{x}(P) = \boldsymbol{v}_A(I) - \boldsymbol{x}(P) \leq \frac{1}{2}\boldsymbol{v}_A(I), \\ \boldsymbol{v}_A^\star(Q) - \boldsymbol{x}(Q) = \boldsymbol{v}_A(I) - \boldsymbol{x}(Q) \leq \frac{1}{2}\boldsymbol{v}_A(I), \end{aligned}$$

such that

$$\frac{1}{2}\boldsymbol{v}_A(I) \leq \boldsymbol{x}(P) \quad \text{and} \quad \frac{1}{2}\boldsymbol{v}_A(I) \leq \boldsymbol{x}(Q).$$

Using the fact that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star) \subseteq X(\boldsymbol{v}_A)$ is an element of the preimputation set, we know that

$$\boldsymbol{x}(P) + \boldsymbol{x}(Q) = \boldsymbol{v}_A(I),$$

such that we can conclude

$$\boldsymbol{x}(P) = \boldsymbol{x}(Q) = \sum_{i \in P} u_i = \sum_{i' \in Q} v_{i'} = \frac{1}{2}\boldsymbol{v}_A(I).$$

Step 2:

For $i \in P$ and $i' \in Q$ we define two sets $S = P \cup \{i'\}$ and $T = Q \cup \{i\}$ which satisfy

$$\boldsymbol{v}_A^\star(S) = \boldsymbol{v}_A^\star(T) = \boldsymbol{v}_A(I).$$

Using the first statement of this lemma we can conclude that

$$\begin{aligned} &1.\ \boldsymbol{v}_A^\star(S) - \left(\sum_{i \in P} u_i + v_{i'}\right) = \frac{1}{2}\boldsymbol{v}_A(I) - v_{i'} \leq \frac{1}{2}\boldsymbol{v}_A(I), \\ &2.\ \boldsymbol{v}_A^\star(T) - \left(\sum_{i' \in Q} v_{i'} + u_i\right) = \frac{1}{2}\boldsymbol{v}_A(I) - u_i \leq \frac{1}{2}\boldsymbol{v}_A(I). \end{aligned}$$

Thus, every element $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ satisfies

$$\boldsymbol{x} = (u, v) \geq 0.$$

Step 3:

Let $i \in P$ and consider $S = P \backslash \{i\}$ such that

$$\boldsymbol{v}_A^\star(S) = \boldsymbol{v}_A(I) - \max_{j' \in Q} a_{ij'}.$$

Every element $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ satisfies:

$$\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S) = \boldsymbol{v}_A^\star(I) - \max_{i' \in Q} a_{ij'} - \sum_{j \neq i} u_i \leq \frac{1}{2}\boldsymbol{v}_A(I),$$

$$\text{i.e.} \quad \frac{1}{2}\boldsymbol{v}_A(I) - \max_{j' \in Q} a_{ij'} \leq \sum_{j \neq i} u_i = \frac{1}{2}\boldsymbol{v}_A(I) - u_i.$$

The above inequality implies that

$$\max_{j' \in Q} a_{ij'} \geq u_i.$$

Since a symmetric argument validates the case of $j' \in Q$ and $T = Q \setminus \{j'\}$. This completes the proof. $\square$

This means that we have shown in our Lemma 4.3.1, that each element of the least core of the dual assignment game treats the players of the two types P and Q equally. Furthermore, we proved that all coordinates of the elements are non-negative. In the last step we showed, that there exists an upper boundary for each coordinate of the element of the least core of the dual assignment game.

Now, we proceed with an argumentation that it is possible to restrict our discussion on assignment games which are introduced by a $p \times p$ assignment matrix A. Therefore, we show that the least core of the dual assignment game satisfies the strong nullplayer property.

Lemma 4.3.2 *The least core of the dual assignment game satisfies the strong nullplayer property.*

Proof.

Let $(I, \mathcal{P}(I), \boldsymbol{v}_A)$ be an assignment game which is induced by the $p \times q$ assignment matrix A. In order to prove the strong nullplayer property, we define a second assignment game $(\tilde{I}, \mathcal{P}(\tilde{I}), \boldsymbol{v}_{\tilde{A}})$ by adding a nullplayer to the game $\boldsymbol{v}_A$. W.l.o.g. we get the corresponding assignment matrix $\tilde{A}$ from the assignment matrix A by adding a zero row to the matrix A.[1] In this case the assignment matrix $\tilde{A}$ is a $(p+1) \times q$ matrix. Furthermore, we have the following player sets $P \subset \tilde{P} = P \cup \{\tilde{p}\}$ and $Q = \tilde{Q}$. In particular, we have by definition

$$\tilde{\boldsymbol{v}}_A(S) := \boldsymbol{v}_{\tilde{A}}(S) = \boldsymbol{v}_A(S \cap I) \quad \forall S \in \mathcal{P}(\tilde{I})$$

such that

$$\tilde{\boldsymbol{v}}^\star_A(S) := \boldsymbol{v}^\star_{\tilde{A}}(S) = \boldsymbol{v}^\star_A(S \cap I) \quad \forall\, S \in \mathcal{P}(\tilde{I}).$$

To show our claim, we have to prove that the following equality holds:

$$\mathcal{LC}(\tilde{\boldsymbol{v}}^\star_A) = \Big\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}_I \in \mathcal{LC}(\boldsymbol{v}^\star_A), x_{\tilde{p}} = 0\Big\}.$$

We show the above equality in two steps.

" $\subseteq$ "

In order to prove the first step, let $\boldsymbol{x} \in \mathcal{LC}(\tilde{\boldsymbol{v}}^\star_A)$. By Lemma 4.3.1, every element of the least core of the dual assignment game is non-negative and we have

$$\boldsymbol{x}(\tilde{P}) = \boldsymbol{x}(\tilde{Q}) = \frac{1}{2}\boldsymbol{v}_A(I).$$

Now, remember that by definition, we have

$$\boldsymbol{v}^\star_A(I) = 2\,\gamma(\tilde{\boldsymbol{v}}^\star_A) = 2\,\gamma(\boldsymbol{v}^\star_A) = \tilde{\boldsymbol{v}}^\star_A(\tilde{P}) = \tilde{\boldsymbol{v}}^\star_A(\tilde{Q}) = \tilde{\boldsymbol{v}}^\star_A(P) = \tilde{\boldsymbol{v}}^\star_A(Q).$$

After these general thoughts we can show that $x_{\tilde{p}} = 0$. Since $\boldsymbol{x}$ is non-negative we have to verify that $x_{\tilde{p}} > 0$. Therefore, we assume that $x_{\tilde{p}} > 0$. Then, it follows immediately that

$$\boldsymbol{x}(\tilde{P}) > \boldsymbol{x}(P),$$

such that

$$\begin{aligned}
\gamma(\tilde{\boldsymbol{v}}^\star_A) &= 2\,\gamma(\tilde{\boldsymbol{v}}^\star_A) - \gamma(\tilde{\boldsymbol{v}}^\star_A) \\
&= \tilde{\boldsymbol{v}}^\star_A(\tilde{P}) - \boldsymbol{x}(\tilde{P}) \\
&< \tilde{\boldsymbol{v}}^\star_A(\tilde{P}) - \boldsymbol{x}(P) \\
&= \tilde{\boldsymbol{v}}^\star_A(P) - \boldsymbol{x}(P).
\end{aligned}$$

[1] Note, that we use a symmetric argument if we add a zero column instead of a zero row to the matrix.

Knowing that by Lemma 4.2.6 we have

$$\gamma(\tilde{\boldsymbol{v}}_A^\star) = \max_{S \in \tilde{I}} \{\tilde{\boldsymbol{v}}_A^\star(S) - \boldsymbol{x}(S)\},$$

the contradiction follows, such that we can conclude that $x_{\tilde{p}} = 0$.

Since $x_{\tilde{p}} = 0$, we know that $\boldsymbol{x}_I \in X(\boldsymbol{v}_A^\star)$. Furthermore, we have

$$\begin{aligned} \gamma(\tilde{\boldsymbol{v}}_A^\star) &= \max_{S \in \tilde{I}} \{\tilde{\boldsymbol{v}}_A^\star(S) - \boldsymbol{x}(S)\} \\ &= \max_{S \in \tilde{I}} \{\tilde{\boldsymbol{v}}_A^\star(S \cap I) - \boldsymbol{x}(S \cap I)\} \\ &= \max_{S \in I} \{\boldsymbol{v}_A^\star(S \cap I) - \boldsymbol{x}_I(S)\} \\ &= \max_{S \in I} \{\boldsymbol{v}_A^\star(S) - \boldsymbol{x}_I(S)\}. \end{aligned}$$

These properties imply that $\boldsymbol{x}_I \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. Together with $x_{\tilde{p}} = 0$ we have finished our first step of the proof.

" $\supseteq$ "

Now, let $\boldsymbol{x}_I \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ and let $x_{\tilde{p}} = 0$. By definition, it follows immediately that $\boldsymbol{x} \in X(\tilde{\boldsymbol{v}}_A^\star)$. Similar to the above part of this proof, we have

$$\begin{aligned} \gamma(\boldsymbol{v}_A^\star) &= \max_{S \in I} \{\boldsymbol{v}_A^\star(S) - \boldsymbol{x}_I(S)\} \\ &= \max_{S \in \tilde{I}} \{\tilde{\boldsymbol{v}}_A^\star(S) - \boldsymbol{x}(S)\} \\ &= \gamma(\tilde{\boldsymbol{v}}_A^\star), \end{aligned}$$

such that we may conclude that $\boldsymbol{x} \in \mathcal{LC}(\tilde{\boldsymbol{v}}_A^\star)$. $\square$

Thus, in the context of the least core of the dual assignment game, we can assume that the assignment matrix is a $p \times p$ matrix. If this is not the case, we can add some nullplayers until the corresponding assignment matrix is a square one. Note, that the least core of normal games does not have to satisfy the strong nullplayer property. In this case we have found a counter-example.

4.4 Convex Assignment Games

In this section we restrict our attention on convex assignment games and on their least core of the dual game. Beside the result that the least core of dual assignment games satisfies the strong nullplayer property, we concentrate on the calculation of elements of the least core of dual assignment game. In particular we check that it is possible to find a simple way to calculate all extreme points of this polyhedron. Another interesting point in this context is the connection between assignment games with a stable core and convex assignment games, if the two assignment matrices have the same diagonal: every extreme point of the dual least core of the convex assignment game keeps extreme in the least core of the corresponding game.

4.4.1 Properties of the Least Core

In the case of convex assignment games it is possible to form the least core of the dual assignment game directly from the assignment matrix. Note, that Solymosi and Raghavan [26] proved that the assignment matrix is a diagonal matrix if and only if the assignment game is convex. First, we will introduce some conditions which are satisfied if we are looking at an element of the least core of the dual game. In a next step we are looking at the extreme points. In the following theorem we identify the least core of dual assignment game with a subset of $\mathbb{R}^n$ which satisfies three conditions.

Theorem 4.4.1 *Let $\boldsymbol{v}_A$ be a convex assignment game which is induced by the $p \times p$ diagonal matrix A. Then, the following statements are equivalent:*

1. $\boldsymbol{x} = (u, v) \in \mathcal{LC}(\boldsymbol{v}_A^\star) \subseteq \mathbb{R}^{p+p}$,
2. $\boldsymbol{x} = (u, v) \in \mathbb{R}^{p+p}$ *meets the three conditions:*
   (a) $\boldsymbol{v}_A(I) = \boldsymbol{x}(I)$, *i.e.* $\boldsymbol{x} \in X(\boldsymbol{v}_A)$
   (b) $u_i, v_{i'} \in [0, a_{ii'}] \quad \forall i \in P$,
   (c) $u_i = v_{i'} \quad \forall i \in P$.

Proof.

We prove the statement of the above theorem in two steps.

" $\subseteq$ ":

First of all, consider an element $\boldsymbol{x} = (u, v) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. In order to prove our claim we have to show that the statements $(a), (b)$ and (c) hold true.

It is clear that in view of Lemma 4.3.1, the first two statements (a) and (b) hold true. In order to prove statement (c) we assume for the contrary, that $u_i \neq v_{i'}$ for some $i \in P$. This assumption implies that we have w.l.o.g. the following inequality:

$$a_{ii'} \geq u_i > v_{i'}. \tag{4.7}$$

Now, let $S = P \setminus i \cup \{i'\}$ such that

$$\boldsymbol{v}_A^\star(S) = \boldsymbol{v}_A(I) - 0 = \boldsymbol{v}_A(I).$$

By Lemma 4.3.1 and (4.7), we have the following inequality:

$$\frac{1}{2}\boldsymbol{v}_A(I) = \sum_{i \in P} u_i > \sum_{j \neq i, j \in P} u_j + v_{i'} = \boldsymbol{x}(S),$$

such that

$$\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S) = \boldsymbol{v}_A(I) - \sum_{j \neq i, j \in P} u_j + v_{i'} > \frac{1}{2}\boldsymbol{v}_A(I).$$

This contradicts the fact that $\boldsymbol{x}$ is an element of the least core of the dual assignment game which satisfies in particular:

$$\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S) \leq \frac{1}{2}\boldsymbol{v}_A(I).$$

" $\supseteq$ ":

Now, let $\boldsymbol{x} = (u, v) \in \mathbb{R}^{p+p}$ such that $(a), (b)$ and (c) hold true. In order to prove that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$, we define for $S \subseteq I$ the following set:

$$Q(S) = \{j' \in S \cap Q \mid j \in S\}.$$

In order to get a more geometric view of set $Q(S)$, consider Figure 4.1. For any $j' \in (Q \cap S) \backslash Q(S)$, we have $u_j = v_{j'}$ and

$$\boldsymbol{v}_A(S) = \boldsymbol{v}_A((S \cup \{j\}) \backslash \{j'\}). \tag{4.8}$$

This means that we can exchange the players j and j' without revision of $\boldsymbol{v}_A(S)$. Furthermore, we should note that we have for $i' \in Q \backslash S$, $i \notin S$ the following equality:

$$\boldsymbol{v}_A(S \cup \{i\}) = \boldsymbol{v}_A(S). \tag{4.9}$$

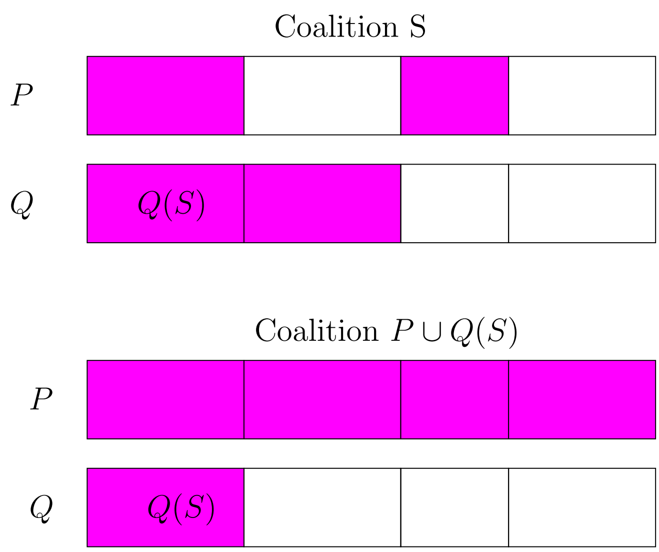

Figure 4.1: Coalition S and $P \cup Q(S)$

This means that we can add some players $i \in P \backslash S$ without revision of the value $\boldsymbol{v}_A(S)$. In view of our two equalities (4.8) and (4.9), it is clear that

$$\boldsymbol{v}_A(S) = \boldsymbol{v}_A(P \cup Q(S))$$

holds true. Furthermore, we have for all players $i \in P$ the equalities $u_i = v_{i'} \geq 0$, such that we have the following inequality:

$$\boldsymbol{x}(S) \leq \boldsymbol{x}(P \cup Q(S)).$$

After these general thoughts we can conclude that:

$$\begin{aligned} \boldsymbol{v}_A(S) - \boldsymbol{x}(S) &\geq \boldsymbol{v}_A(P \cup Q(S)) - \boldsymbol{x}(P \cup Q(S)) \\ &= \sum_{i' \in Q(S)} a_{ii'} - \boldsymbol{x}(P) - \boldsymbol{x}(Q(S)) \\ &= \sum_{i' \in Q(S)} a_{ii'} - \frac{1}{2}\boldsymbol{v}_A(I) - \boldsymbol{x}(Q(S)) \\ &\geq -\frac{1}{2}\boldsymbol{v}_A(I). \end{aligned}$$

Since the definition of $\boldsymbol{v}_A^\star$ implies that

$$\boldsymbol{v}_A^\star(S^c) - \boldsymbol{x}(S^c) = \boldsymbol{x}(S) - \boldsymbol{v}_A(S),$$

we can conclude that

$$\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star). \quad \square$$

Thus, we have proved that all elements of the least core of the dual assignment game satisfy three conditions and that all elements which satisfy these conditions are elements of the least core of the dual assignment game. As a second step we consider the extreme points of the least core in this special case. In the following corollary, we show that it is possible to compute the extreme points directly from the assignment matrix.

Corollary 4.4.2 *Let A be a $p \times p$ diagonal matrix. Futhermore let $\boldsymbol{v}_A$ be the induced assignment game and let $\boldsymbol{x} = (u, v) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. The element (u, v) is an extreme point of the least core of the dual assignment game if and only if (u, v) satisfies at least $p - 1$ of the following $2p$ conditions:*

1. $u_i = 0 \quad \forall i \in P$
2. $u_i = a_{ii'} \quad \forall i \in P.$

Our corollary shows that in the case of convex assignment games it is possible to calculate the extreme points of the least core of the dual assignment game directly from the assignment matrix. The following example demonstrates the above properties of the extreme points of the least core of the dual assignment game.

Example 4.4.3 *Consider a convex assignment game $\boldsymbol{v}_A$ which is introduced by the 3×3 assignment matrix*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

The extreme points of the least core of the dual game are

$$\boldsymbol{y}_1 = (0, 0, 7, 0, 0, 7), \quad \boldsymbol{y}_2 = (0, 4, 3, 0, 4, 3),$$

$$\boldsymbol{y}_3 = (2, 0, 5, 2, 0, 5), \quad \boldsymbol{y}_4 = (2, 4, 1, 2, 4, 1).$$

Thus, this example demonstrates how to compute the extreme points of the least core of the dual assignment game in the case of convex games.

4.4.2 Assignment Games with a Stable Core

This section is about assignment games with a stable core. In this case the diagonal of the assignment matrix supplies some extreme points of the least core of the induced dual assignment game. We will show that the least core of the dual assignment game in the case of convex assignment games is a subset of the least core in the case of assignment games with a stable core. More formally, we have our next lemma.

Lemma 4.4.4 *Let $\boldsymbol{v}_A$ be an assignment game which is induced by a $p \times p$ matrix A such that the assignment game $\boldsymbol{v}_A$ has a stable core. Furthermore, let B be the $p \times p$ the diagonal matrix satisfying $b_{ii'} = a_{ii'}$ for all $i \in P$. Then, we have*

$$\mathcal{LC}(\boldsymbol{v}_B^\star) \;\subseteq\; \mathcal{LC}(\boldsymbol{v}_A^\star).$$

In particular, we have

$$EXT \;\; \mathcal{LC}(\boldsymbol{v}_B^\star) \;\subseteq\; EXT \;\; \mathcal{LC}(\boldsymbol{v}_A^\star).$$

Proof.

We prove our statements in two steps.

Step 1:

In the first step of the proof, we show that any element of $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_B^\star)$ is an element of the least core $\mathcal{LC}(\boldsymbol{v}_A^\star)$. Therefore, let $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_B^\star) \subseteq X(\boldsymbol{v}_B^\star) = X(\boldsymbol{v}_A^\star)$. In order to show that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ we have to demonstrate that

$$\max_{S \in \mathcal{P}} \{\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S)\} = \frac{1}{2}\boldsymbol{v}_A(I).$$

By definition, we have for each $S \in \mathcal{P}$ the following inequality

$$\boldsymbol{v}_A^\star(S) \leq \boldsymbol{v}_B^\star(S),$$

such that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_B^\star)$ implies

$$\max_{S \in \mathcal{P}} \{\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S)\} \leq \frac{1}{2}\boldsymbol{v}_A(I). \tag{4.10}$$

Furthermore, we have for any $\boldsymbol{x} \in X(\boldsymbol{v}_B^\star)$ the following inequality

$$\max_{S \in \mathcal{P}} \{\boldsymbol{v}_A^\star(S) - \boldsymbol{x}(S)\} \geq \frac{1}{2}\boldsymbol{v}_A(I). \tag{4.11}$$

Consequently, we may conclude from (4.10) and (4.11) that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ such that

$$\mathcal{LC}(\boldsymbol{v}_B^\star) \subseteq \mathcal{LC}(\boldsymbol{v}_A^\star).$$

Step 2:

As a second step, we want to look at the extreme points of these two polyhedrons. One can easily see that the extreme points of $\mathcal{LC}(\boldsymbol{v}_B^\star)$ keeps extreme in $\mathcal{LC}(\boldsymbol{v}_A^\star)$. All extreme points $\boldsymbol{y} \in \mathcal{LC}(\boldsymbol{v}_B^\star)$ satisfy the conditions of Corollary 4.4.2 and it is not possible to find elements $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ which satisfy the conditions of Lemma 4.3.1 and

$$\boldsymbol{y} = \lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2 \quad \lambda \in (0,1). \quad \square$$

Our next example demonstrates that the dominant diagonal is a necessary condition for the above lemma. In the case of assignment games without dominant diagonal it could happen that an extreme point of the least core of the dual assignment game does not keep extreme. Fore more details see the next example.

Example 4.4.5 *Consider the following two assignment matrices* $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. *Furthermore let* $\boldsymbol{v}_A$ *and* $\boldsymbol{v}_B$ *be the induced assignment games. The least core* $\mathcal{LC}(\boldsymbol{v}_A^\star)$ *is defined by:*

$$\begin{aligned} \mathcal{LC}(\boldsymbol{v}_A^\star) = CnvH\Big\{ & (0,\ 3.5,\ 0,\ 3.5),\ (3.5,\ 0,\ 0,\ 3.5), \\ & (0,\ 3.5,\ 3,\ 0.5),\ (3.5,\ 0,\ 2,\ 1.5), \\ & (2.5,\ 1,\ 3,\ 0.5) \ \Big\} \end{aligned}$$

In this case only one of the two extreme points $\boldsymbol{x}_1$ *and* $\boldsymbol{x}_2$ *of* $\mathcal{LC}(\boldsymbol{v}_B^\star)$ *keeps extreme in the least core* $\mathcal{LC}(\boldsymbol{v}_A^\star)$. *We have*

$$\boldsymbol{x}_1 = (2, 1.5, 2, 1.5) \notin EXT\mathcal{LC}(\boldsymbol{v}_A^\star),$$
$$\boldsymbol{x}_2 = (0, 3.5, 0, 3.5) \in EXT\mathcal{LC}(\boldsymbol{v}_A^\star).$$

This result is possible because $u_i \in [0, \max_{j' \in Q} a_{ij'}] \not\subset [0, a_{ii'}]$.

4.5 Symmetric and Exact Assignment Games

Throughout the previous section we have studied convex assignment games. By Solymosi and Raghavan [26] these games are induced by diagonal matrices which are particularly symmetric. In this section we restrict our attention on assignment games which are induced by a symmetric matrix. As a second step, we consider such games, which are also exact. But in the first step we start with the formal definition of a symmetric matrix.

Definition 4.5.1 *Let A be a $m \times m$ matrix. The matrix A is* **symmetric**, *if*

$$a_{ij} = a_{ji} \quad \forall i, j \in \{1, \dots, m\}.$$

This means, that a matrix is symmetric if and only if the matrix A and the transposed matrix A^t are equal.

4.5.1 On some Solution Concepts

Now, we make a first attempt to compute the modiclus of assignment games. Earlier we saw that if the assignment game has a stable core the modiclus is an element of the core and the least core of the dual game. We show in our next theorem that this holds true only if the assignment matrix is symmetric. In this case one can compute the modiclus of this games in a very simple way. Furthermore, we should mention that the modiclus is the center of gravity of the core.

Theorem 4.5.2 *Let A be a symmetric $p \times p$ assignment matrix such that its main diagonal is an optimal assignment. Furthermore let $\boldsymbol{v}_A$ be the induced assignment game. In this case we have*

$$\mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A) \neq \emptyset.$$

In particular, we have:

$$\Psi(\boldsymbol{v}_A) = \frac{1}{2}(a_{11'}, \dots, a_{pp'}, a_{11'}, \dots, a_{pp'}) \in \mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A).$$

Note, that the second part of the proof is analogue to the proof of Raghavan and Sudhölter [12].

Proof.

We show the correctness of the statements in three steps.

Step 1:

At the beginning of the proof, we show that

$$\boldsymbol{m} = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}) \in \mathcal{C}(\boldsymbol{v}_A).$$

Using the fact that the matrix A is symmetric, we know that $(u, v) \in \mathcal{C}(\boldsymbol{v}_A)$ if and only if $(v, u) \in \mathcal{C}(\boldsymbol{v}_A)$. Since the main diagonal is an optimal assignment we can conclude by Theorem 2.4.9 that

$$u_i + v_{i'} = a_{ii'} \quad \forall i \in P.$$

Summarizing, we have

$$\boldsymbol{m} = \frac{1}{2}(u, v) + \frac{1}{2}(v, u) = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}) \in \mathcal{C}(\boldsymbol{v}_A). \tag{4.12}$$

Step 2:

In the second step of the proof we show that

$$\boldsymbol{m} = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}) \in \mathcal{MLC}(\boldsymbol{v}_A). \tag{4.13}$$

Therefore, we conclude that every preimputation $\boldsymbol{x} \in X(\boldsymbol{v}_A)$ implies the following two inequalities:

1. $\mu(\boldsymbol{x}, \boldsymbol{v}_A) \geq 0$
2. $\mu(\boldsymbol{x}, \boldsymbol{v}_A^\star) \geq \frac{1}{2}\boldsymbol{v}_A(I)$.

In order to prove (4.13) we have to show that

$$\mathcal{MLC}(\boldsymbol{v}_A) = \left\{ \boldsymbol{x} \in \mathcal{C}(\boldsymbol{v}_A) \,\middle|\, \mu(\boldsymbol{x}, \boldsymbol{v}_A^\star) = \frac{1}{2}\boldsymbol{v}_A(I) \right\}. \tag{4.14}$$

Equation (4.14) holds true if we can find a single element of the core that satisfies

$$\mu(\boldsymbol{x}, \boldsymbol{v}_A^\star) = \frac{1}{2}\boldsymbol{v}_A(I).$$

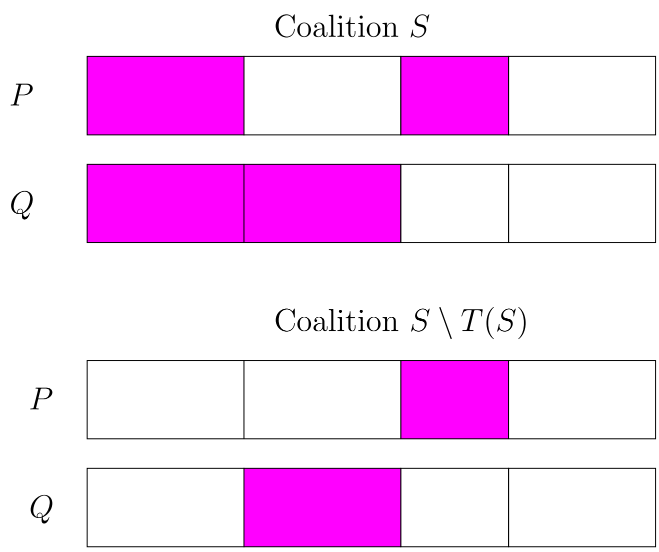

Figure 4.2: Coalition S and $S \setminus T(S)$

Later, we will see that this element is our allocation $\boldsymbol{m}$. By Definition 3.4.1 the set of pairs $(i, i') \in S$ is denoted by $T(S) = S \cap S^*$. For a more geometric view, consider Figure 4.2 in which the sets S and $S \setminus T(S)$ are illustrated. For $S \subseteq I$ we denote $T = T(S)$. Using the fact that assignment games are superadditive, we can conclude that

$$\boldsymbol{v}_A(S) \geq \boldsymbol{v}_A(S \setminus T) + \boldsymbol{v}_A(T) \geq \boldsymbol{v}_A(S \setminus T) + \sum_{i \in T \cap P} a_{ii'}$$

and hence,

$$\begin{aligned}
\boldsymbol{v}_A(S) - \boldsymbol{m}(S) &\geq \boldsymbol{v}_A(S \setminus T) + \sum_{i \in T \cap P} a_{ii'} - \boldsymbol{m}(S \setminus T) - \boldsymbol{m}(T) \\
&\geq \boldsymbol{v}_A(S \setminus T) - \boldsymbol{m}(S \setminus T) \\
&\geq -\boldsymbol{m}(S \setminus T) \\
&\geq -\sum_{i \in P} \frac{1}{2} a_{ii'} \\
&= -\frac{1}{2} \boldsymbol{v}_A(I).
\end{aligned}$$

One immediately sees, by definition of the dual assignment game, that

$$\mu(\boldsymbol{m}, \boldsymbol{v}_A^\star) = \frac{1}{2} \boldsymbol{v}_A(I)$$

such that

$$\boldsymbol{m} \in \mathcal{MLC}(\boldsymbol{v}_A) \subseteq \mathcal{C}(\boldsymbol{v}_A) \cap \mathcal{LC}(\boldsymbol{v}_A^\star).$$

Step 3:

Finally, we can conclude that in the case of a symmetric assignment matrix, we have

$$\Psi(\boldsymbol{v}_A) = (u, v) = (v, u) \in \mathcal{MLC}(\boldsymbol{v}_A) \subseteq \mathcal{C}(\boldsymbol{v}_A),$$

such that

$$u_i + u_{i'} = a_{ii'} \quad \forall\, i \in P.$$

All together we can conclude that

$$\boldsymbol{m} = \Psi(\boldsymbol{v}_A) = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}) \in \mathcal{C}(\boldsymbol{v}_A) \cup \mathcal{LC}(\boldsymbol{v}_A^{\star}). \quad \square$$

With the aid of some arguments of this proof, we are able to make some thoughts about further relations between the core and the modiclus in the case of assignment games which are induced by a symmetric matrix. We will note in our next remark that the modiclus is the center of gravity of the core. More precisely, we have the following remark.

Remark 4.5.3 *In the case of Theorem 4.5.2, the modiclus $\Psi(\boldsymbol{v}_A)$ is the center of gravity of the extreme points of the core and respectively to the core. This follows from the fact that $(u, v) \in \mathcal{C}(\boldsymbol{v}_A)$ if and only if $(v, u) \in \mathcal{C}(\boldsymbol{v}_A)$.*

The following lemma is concerned with different single valued solutions concepts which are an element of the core. We will see that all these solution concepts conform with center of gravity of the core in the case of assignment games which are induced by symmetric assignment matrices.

Lemma 4.5.4 *Let A be a symmetric $p \times p$ assignment matrix such that the main diagonal is an optimal assignment. Furthermore, consider the assignment game $\boldsymbol{v}_A$ which is induced by A. Then, every single valued solution concept $\Upsilon(\boldsymbol{v}_A)$, which is an element of the core $\mathcal{C}(\boldsymbol{v}_A)$, satisfies*

$$\Upsilon(\boldsymbol{v}_A) = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}).$$

In particular, $\Upsilon(\boldsymbol{v}_A)$ is the center of gravity of the core.

Proof.

Since the assignment matrix is symmetric, it follows that the solution concept is also symmetric, i.e.

$$\Upsilon(\boldsymbol{v}_A) = (u, v) = (v, u).$$

On the other hand $\Upsilon(\boldsymbol{v}_A)$ is an element of the core such that we have by Theorem 2.4.9 the following equalities:

$$u_i + u_{i'} = a_{ii'} \quad \forall i \in P.$$

Combining these facts, we can conclude

$$\Upsilon(\boldsymbol{v}_A) = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}). \quad \square$$

This means that, in the case of assignment games which are induced by a symmetric assignment matrix, we know something about single valued solution concepts which are elements of the core. These concepts conform and they can be computed in a very simple way. Furthermore, we have found a more geometric way to compute these concepts. To demonstrate the statement of the above lemma, consider the following example.

Example 4.5.5 *In the case of exact assignment games, which are introduced by a symmetric matrix, particularly convex assignment games, the modiclus and the Shapley value conform, that means, we have:*

$$\Phi(\boldsymbol{v}_A) = \Psi(\boldsymbol{v}_A) = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}).$$

We mention only for reasons of completeness that the nucleolus also agree with the Shapley value and the modiclus in this special case.
Next, we note that solution concepts of more general assignment games do not have to conform.

Remark 4.5.6 *In the case of Theorem 4.5.2 it is not necessary that the Shapley value and the modiclus are the same, because the Shapley value is not always in the core if the game is not exact.*

For example, we can consider the following assignment game:

Example 4.5.7 *Consider the following assignment matrix* $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$. *This matrix is symmetric but the induced assignment game* $\boldsymbol{v}_A$ *is not exact. The Shapley value and the modiclus are*

$$\begin{aligned} \Phi(\boldsymbol{v}_A) &= \frac{1}{3}(7,2,7,2), \\ \Psi(\boldsymbol{v}_A) &= \frac{1}{2}(5,1,5,1). \end{aligned}$$

In particular, we have:

$$\Phi(\boldsymbol{v}_A) \neq \Psi(\boldsymbol{v}_A) \quad \textit{and} \quad \Phi(\boldsymbol{v}_A) \notin \mathcal{C}(\boldsymbol{v}_A).$$

In the next step, we consider an exact assignment game, such that the assignment matrix is not symmetric. Here, the Shapley value, the nucleolus, and the modified nucleolus are elements of the core and of the least core of the dual assignment game but the solution concepts do not conform.

Example 4.5.8 *Consider the following assignment matrix* $A = \begin{pmatrix} 4 & 0 \\ 2 & 8 \end{pmatrix}$. *Here, the core and the least core of the induced assignment game* $\boldsymbol{v}_A$ *are defined by:*

$$\begin{aligned} \mathcal{C}(\boldsymbol{v}_A) = CnvH\Big\{ &(0,\ 0,\ 4,\ 8),\ (0,\ 8,\ 4,\ 0), \\ &(2,\ 0,\ 2,\ 8),\ (4,\ 2,\ 0,\ 6),\ (4,\ 8,\ 0,\ 0)\Big\}, \\ \mathcal{LC}(\boldsymbol{v}_A^{\star}) = CnvH\Big\{ &(0,\ 6,\ 0,\ 6),\ (0,\ 6,\ 2,\ 4), \\ &(2,\ 4,\ 4,\ 2),\ (4,\ 2,\ 4,\ 2)\Big\} \end{aligned}$$

Furthermore, we have for the nucleolus, the modified nucleolus and the Shapley value:

$$\begin{aligned} \nu(\boldsymbol{v}_A) &= (2,4,2,4)\,, \\ \Psi(\boldsymbol{v}_A) &= \frac{1}{2}\,(3,9,5,7)\,, \\ \Phi(\boldsymbol{v}_A) &= \frac{1}{6}\,(11,25,13,23)\,. \end{aligned}$$

The correlations of this example are as follows:

1. $\Psi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$,

2. $\Phi(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$,

3. $\nu(\boldsymbol{v}_A) \in \mathcal{C}(\boldsymbol{v}_A)$,

4. $\Phi(\boldsymbol{v}_A) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$,

5. $\nu(\boldsymbol{v}_A) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$,

6. $\Psi(\boldsymbol{v}_A) \neq \nu(\boldsymbol{v}_A) \neq \Phi(\boldsymbol{v}_A)$.

Thus, we can conclude that exactness is not sufficent for different solution concepts to conform.

4.5.2 Characteristics of the Least Core

In the above section we have considered the connection between the two elements (u, v) and (v, u) of the core in the case of assignment games which are induced by a symmetric matrix. Immediately, we have checked that

$$\boldsymbol{m} = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}) = \frac{1}{2}((u, v) + (v, u))$$

is the center of gravity of the core. Now, we want to consider the center of gravity of the least core of the dual assignment game. In the next lemma we show that there exists for any $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ a second element $\boldsymbol{y} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. With the aid of these two elements we can compute the center of gravity of the least core of the dual assignment game.

Lemma 4.5.9 *Let $\boldsymbol{v}_A$ be an exact assignment game such that the corresponding assignment matrix A is symmetric. Then, the following statements are equivalent:*

1. $\boldsymbol{x} = (u, v) \in EXT\,\mathcal{LC}(\boldsymbol{v}_A^\star)$,

2. $\boldsymbol{y} = ((a_{11'} - u_1), \ldots, (a_{pp'} - u_p), (a_{11'} - v_{1'}), \ldots, (a_{pp'} - v_{p'})) \in EXT\,\mathcal{LC}(\boldsymbol{v}_A^\star)$.

Proof.

In order to prove our claim, we show that $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$ implies that $\boldsymbol{y} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$.

The other direction follows in a similar way. Now, let $\boldsymbol{x} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. In order to show that $\boldsymbol{y} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$, we have to check the following equality:

$$\boldsymbol{v}_A(S) - \boldsymbol{y}(S) = \boldsymbol{v}_A(S^c) - \boldsymbol{x}(S^c) \quad \forall S \in \mathcal{P}.$$

For $S \subseteq I$ define T and $\tilde{T}$ as $T = T(S) = S \cap S^\star$ and $\tilde{T} = \tilde{T}(S^c) = S^c \cap (S^c)^\star$. By Lemma 3.4.3, we have for every coalition S the following equalities:

$$\begin{aligned}
\boldsymbol{v}_A(S) - \boldsymbol{y}(S) &= \boldsymbol{v}_A(S) - \sum_{i \in S\cap P} a_{ii'} - \sum_{i' \in S\cap Q} a_{ii'} + \boldsymbol{x}(S) \\
&= \boldsymbol{v}_A(S \setminus T) + \boldsymbol{v}_A(T) - \sum_{i \in S\cap P} a_{ii'} - \sum_{i' \in S\cap Q} a_{ii'} + \sum_{i \in P} a_{ii'} - \boldsymbol{x}(S^c) \\
&= \boldsymbol{v}_A(S \setminus T) + \sum_{i \in T} a_{ii'} - \sum_{i \in S\cap P} a_{ii'} - \sum_{i' \in S\cap Q} a_{ii'} + \sum_{i \in P} a_{ii'} - \boldsymbol{x}(S^c) \\
&= \boldsymbol{v}_A(S \setminus T) - \sum_{i \in S\cap P} a_{ii'} - \sum_{i' \in (S\setminus T)\cap Q} a_{ii'} + \sum_{i \in P} a_{ii'} - \boldsymbol{x}(S^c) \\
&= \boldsymbol{v}_A(S \setminus T) - \boldsymbol{v}_A(\tilde{T}) - \boldsymbol{x}(S^c) \\
&= \boldsymbol{v}_A(S^c) - \boldsymbol{x}(S^c).
\end{aligned}$$

These equalities imply that $\boldsymbol{y} \in \mathcal{LC}(\boldsymbol{v}_A^\star)$. So, we can conclude that $\boldsymbol{y} \in EXT\,\mathcal{LC}(\boldsymbol{v}_A^\star)$ if and only if $\boldsymbol{x} \in EXT\,\mathcal{LC}(\boldsymbol{v}_A^\star)$. □

Thus, if we know one element of the least core of the dual assignment game, we can compute a second element in a very simple way. According to Remark 4.5.3 we have the following remark in the context of the least core of the dual assignment game.

Remark 4.5.10 *In the case of Lemma 4.5.9 the following statements hold true:*

1. *The center of gravity of the least core of the dual assignment game is given by*

$$\boldsymbol{m} = \frac{1}{2}(a_{11'}, \ldots, a_{pp'}, a_{11'}, \ldots, a_{pp'}).$$

2. *The center of gravity of the core agrees with the center of gravity of the least core of the dual assignment game.*

In the case of exact assignment games which are induced by a non-symmetric matrix, the center of gravity of the core and the center of gravity of the least core of the dual game are not necessarily the same. To demonstrate this certainty, we consider the following example.

Example 4.5.11 *Consider the assignment game $\boldsymbol{v}_A$ which is induced by the assignment matrix*

$$A = \begin{pmatrix} 4 & 2 \\ 4 & 6 \end{pmatrix}.$$

Then, the center of gravity of $\mathcal{C}(\boldsymbol{v}_A)$ is $(2,4,2,2)$ and the center of gravity of $\mathcal{LC}(\boldsymbol{v}^{\star}_A)$ is $(2,3,2,3)$.

This means that, in the case of assignment games which are induced by a non-symmetric assignment matrix, we can not say anything about the connection of the two centers of gravity.

Chapter 5

Assignment Games induced by 2×2 matrices

5.1 Introduction

In this chapter we consider assignment games with a stable core which are induced by 2×2 matrices. We mainly focus on the core and the least core of the dual game. Before we compute the extreme points of the two polyhedrons we start with a mixture of an analytic and geometric consideration of both sets. The geometric results are very helpful for computing the extreme points of the non-empty intersection of the two sets. At the end of this chapter we take a short look at two examples in which the intersection is only a single point. One can easily check that this point is the modified nucleolus of the assignment game. During this chapter we restrict our attention to assignment games which are induced by 2×2 matrices with a dominant diagonal. Furthermore, we make the non-restrictive assumption that $a_{11'} \leq a_{22'}$ holds true.

5.2 The Least Core of Dual Assignment Games

This section is about the least core of the dual assignment game in the case that the assignment matrix is a 2×2 matrix with a dominant diagonal. But before we start with our description of the least core of the dual assignment game, we consider a special kind of polyhedrons. First of all, we give the formal definition of these subsets of $\mathbb{R}^n$.

Definition 5.2.1 *For $n \in \mathbb{N}$ the* **unit simplex** *of $\mathbb{R}^n$ is defined by:*

$$\mathbf{X}^n = \Big\{ \boldsymbol{x} \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1 \Big\}.$$

Let $\alpha \in \mathbb{R}_+$. Another simplex is defined by:

$$\alpha \cdot \mathbf{X}^n = \Big\{ \boldsymbol{x} \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = \alpha \Big\}.$$

Summarizing we can say that the unit simplex of $\mathbb{R}^n$ denotes the set of non-negative vectors $\boldsymbol{x} \in \mathbb{R}^n$, which have the property that the sum of the coordinates is equal to one.

Now, let $\boldsymbol{v}_A$ be the assignment game which is induced by the assignment matrix A. Lemma 4.2.6 and Lemma 4.3.1 imply that the least core of the dual assignment game is a subset of the simplex $\boldsymbol{v}_A(I) \cdot \mathbf{X}^n \subset \mathbb{R}^n$. In particular, we have the following two equalities for the case of the least core of the dual assignment game:

$$u_1 + u_2 = \gamma \quad \text{and} \quad v_{1'} + v_{2'} = \gamma.$$

Thus, w.l.o.g. it is sufficient to know the two coordinates $u_1 \in [0, \gamma]$ and $v_{1'} \in [0, \gamma]$. In order to form a geometric idea of the least core of the dual assignment game, we look at the subset

$$F = \{(u_1, v_{1'}) \mid u_1, v_{1'} \in [0, \gamma]\} \subseteq \mathbb{R}^2.$$

Next, we define a mapping

$$\begin{aligned} M : F &\rightarrow \mathbb{R}^4 \\ (u_1, v_{1'}) &\mapsto (u_1, \gamma - u_1, v_{1'}, \gamma - v_{1'}). \end{aligned}$$

The motivation of the above mapping and the set F becomes completely evident when we note that the least core of the dual assignment game is by definition a subset of

$$M(F) = \{M(u_1, v_{1'}) \mid (u_1, v_{1'}) \in F\}.$$

This means that we have more formally:

$$\begin{aligned} \mathcal{LC}(\boldsymbol{v}_A^\star) &\subseteq M(F) = CnvH\Big\{(\gamma, 0, 0, \gamma), (0, \gamma, 0, \gamma), (0, \gamma, \gamma, 0), (\gamma, 0, \gamma, 0)\Big\} \\ &\subset \boldsymbol{v}_A(I) \cdot \mathbf{X}^4. \end{aligned} \tag{5.1}$$

Since the set $M(F)$ has dimension two, the least core of the dual assignment game has at most dimension two. But, before we continue our research on the least core of the dual assignment game, we compute the set $M(F)$ for an example of a special assignment game.

Example 5.2.2 *Consider the assignment matrix* $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$. *In the case of the induced assignment game* $\boldsymbol{v}_A$, *we have*

$$M(F) = CnvH\Big\{(7,0,0,7),(0,7,0,7),(0,7,7,0),(7,0,7,0)\Big\} \subset 14 \cdot \mathbf{X}^4.$$

Thus, it is possible to get a geometric point of view of the least core of the dual assignment game in a very simple way. For more details, one can consider Figure 5.1, in which the set $M(F)$ is illustrated.

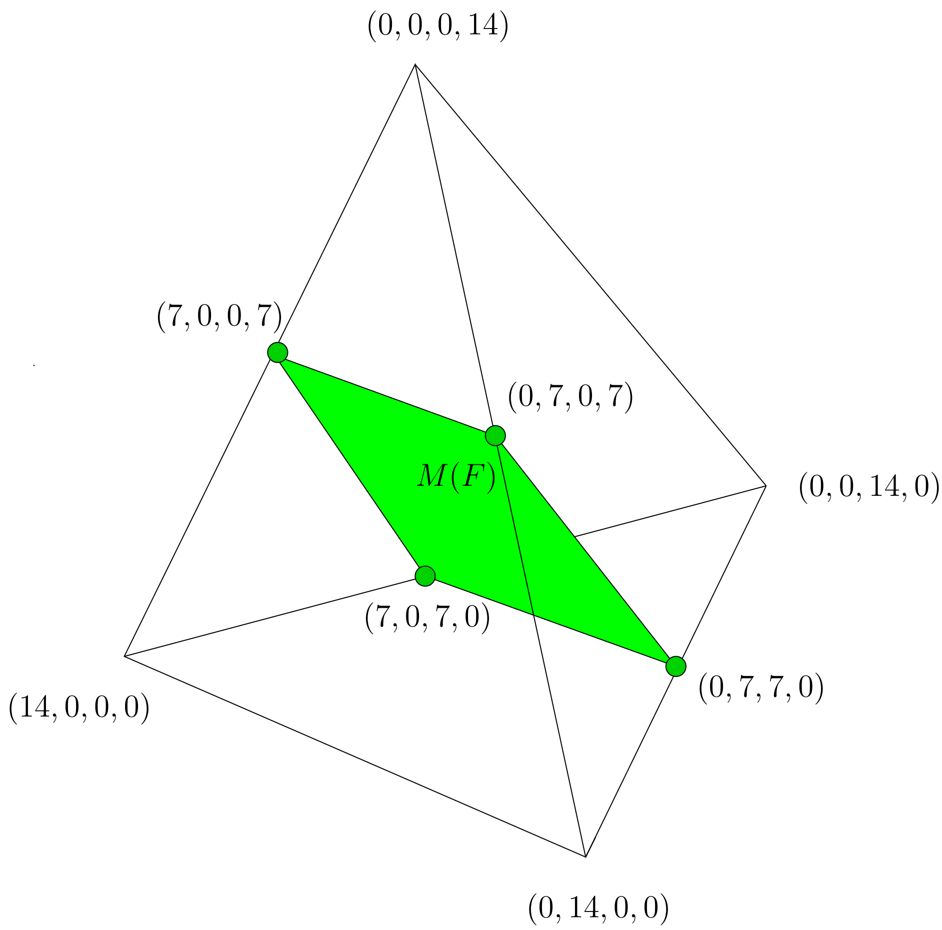

Figure 5.1: Set $M(F)$ of Example 5.2.2

5.2.1 Extreme Points of the Least Core

In this section we want to compute the extreme points of the least core of the dual assignment game with the help of the procedure introduced in the case of the least core of c-convex games. Note, that it is useful to restrict our attention to assignment games induced by an assignment matrix which satisfies, w.l.o.g., $a_{11'} \leq a_{22'}$. The other case follows in a symmetric way or by changing the order of the assignment matrix.

Corollary 5.2.3 *Let A be a 2×2 assignment matrix with dominant diagonal such that $a_{11'} \leq a_{22'}$ holds true. Furthermore, let $\boldsymbol{v}_A$ be the induced assignment game. Then, there exists at most six extreme points of the least core $\mathcal{LC}(\boldsymbol{v}_A^\star)$, which are defined by:*

$$\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ \gamma \\ 0 \\ \gamma \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} a_{11'} \\ a_{22'} - \gamma \\ a_{11'} \\ a_{22'} - \gamma \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} a_{11'} - a_{21'} \\ \gamma - a_{11'} + a_{21'} \\ a_{11'} \\ a_{22'} - \gamma \end{pmatrix},$$

$$\boldsymbol{x}_4 = \begin{pmatrix} a_{12'} \\ \gamma - a_{12'} \\ 0 \\ \gamma \end{pmatrix}, \boldsymbol{x}_5 = \begin{pmatrix} a_{11'} \\ a_{22'} - \gamma \\ a_{11'} - a_{12'} \\ \gamma - a_{11'} + a_{12'} \end{pmatrix}, \quad \boldsymbol{x}_6 = \begin{pmatrix} 0 \\ \gamma \\ a_{21'} \\ \gamma - a_{21'} \end{pmatrix}.$$

Proof.

In order to prove our claim, we consider for each extreme point one (P, Q)-tight sequence, which yields the corresponding extreme point.

For $\boldsymbol{x}_1$ we have:

$$\begin{array}{rcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 1 & 1 & 1 & 0 \\ S^3 & = & 1 & 0 & 1 & 1 \\ S^4 & = & 0 & 0 & 1 & 1, \end{array}$$

for $\boldsymbol{x}_2$ we have:

$$\begin{array}{rcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 0 & 1 & 0 & 0 \\ S^3 & = & 0 & 0 & 0 & 1 \\ S^4 & = & 0 & 0 & 1 & 1, \end{array}$$

for $\boldsymbol{x}_3$ we have:

$$\begin{array}{lcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 1 & 0 & 0 & 1 \\ S^3 & = & 0 & 0 & 0 & 1 \\ S^4 & = & 0 & 0 & 1 & 1, \end{array}$$

for $\boldsymbol{x}_4$ we have:

$$\begin{array}{lcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 1 & 1 & 1 & 0 \\ S^3 & = & 0 & 1 & 1 & 0 \\ S^4 & = & 0 & 0 & 1 & 1, \end{array}$$

for $\boldsymbol{x}_5$ we have:

$$\begin{array}{lcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 0 & 1 & 0 & 0 \\ S^3 & = & 0 & 1 & 1 & 0 \\ S^4 & = & 0 & 0 & 1 & 1. \end{array}$$

Last but not least, we have for $\boldsymbol{x}_6$:

$$\begin{array}{lcllll} S^1 & = & 1 & 1 & 0 & 0 \\ S^2 & = & 1 & 0 & 0 & 1 \\ S^3 & = & 1 & 0 & 1 & 1 \\ S^4 & = & 0 & 0 & 1 & 1. \end{array}$$

With the aid of these (P,Q)-tight sequences, we can implement the procedure of the extreme points of the least core of c-convex games. This proceeding yields the extreme points of our corollary. In the last step, we have to check that the other possible (P,Q)-tight sequences do not yield further extreme points. So, we can conclude that we have at most six extreme points of the above design. □

Thus, it is possible to compute all extreme points of the least core of the dual assignment game with the aid of the assignment matrix. The fundamental idea of our proof is the fact that the dual assignment game is c-convex w.r.t. (P,Q).

Before we continue our research on the core and the least core of the dual assignment game, we look at an example of an assignment game without a stable core. This example illustrates the fact that the dominant diagonal in Corollary 5.2.3 is a necessary condition.

Example 5.2.4 *Consider the assignment matrix* $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$. *Note, that this matrix does not have a dominant diagonal. The least core of the dual assignment* *game $\boldsymbol{v}_A^\star$ is defined by:*

$$\mathcal{LC}(\boldsymbol{v}_A^\star) = CnvH\Big\{ (0,\ 3.5,\ 0,\ 3.5),\ (3.5,\ 0,\ 0,\ 3.5),\ (0,\ 3.5,\ 3,\ 0.5),\ (3.5,\ 0,\ 2,\ 1.5),\ (2.5,\ 1,\ 3,\ 0.5) \Big\}.$$

Thus, in this case we have other extreme points different from the case of assignment games with a stable core.

In the following example, we consider an assignment game with a stable core. In this case, we have a look at the corresponding least core of the dual assignment game.

Example 5.2.5 *Consider the assignment matrix $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$ and let $\boldsymbol{v}_A$ be the induced assignment game. The least core of the dual assignment game is defined by*

$$\mathcal{LC}(\boldsymbol{v}_A^\star) = CnvH\Big\{(0,7,0,7),(6,1,6,1),(4,3,0,7),(5,2,6,1),(6,1,2,5),(0,7,1,6)\Big\}.$$

In the next step, we want to get a clearer picture of the least core of the dual assignment game. Therefore, we define a projection

$$\begin{aligned} P : \mathbb{R}^4 &\ \rightarrow\ \mathbb{R}^3 \\ (u,v) &\ \mapsto\ (u_1, u_2, v_{1'}). \end{aligned} \tag{5.2}$$

Note, that this proceeding is feasible since the sum of the coordinates of each element of the least core of the dual assignment game is constant and equal to $\boldsymbol{v}_A(I)$. The characteristics of the set $\mathcal{LC}(\boldsymbol{v}_A^\star)$ remains unmodified under our projection defined in (5.2).

In Figure 5.2, the projection $P(\mathcal{LC}(\boldsymbol{v}_A^\star))$ of the least core of the dual assignment game is illustrated.

5.3 The Core

In this section we concentrate on some aspects of the core of exact assignment games which are induced by 2×2 matrices. Analog to the above section, we start with

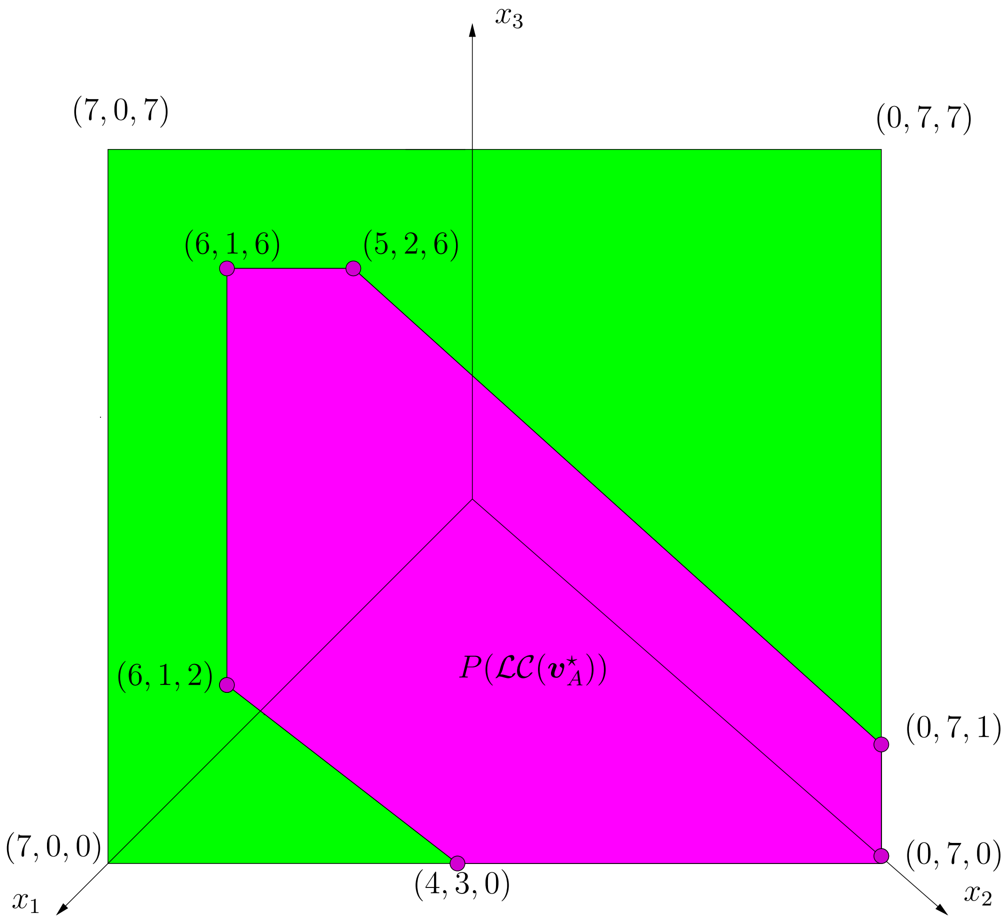

Figure 5.2: Projection of $\mathcal{LC}(\boldsymbol{v}_A^\star)$ of Example 5.2.5

some geometric results before looking at the extreme points of the core. First of all, we define a polyhedron which contains the core. In order to do so, one should note that by Theorem 2.4.9 we have for all $i \in \{1,2\}$ the following equality:

$$u_i + v_{i'} = a_{ii'}.$$

Thus, w.l.o.g., it is sufficient to know u_i for all $i \in \{1,2\}$.

Similar to the first part of this chapter, we consider the set

$$E = \{(u_1,u_2) \mid u_i \in [0, a_{ii'}]\} \subseteq \mathbb{R}^2$$

and we define a mapping

$$\begin{aligned} K : E &\rightarrow \mathbb{R}^4 \\ (u_1,u_2) &\mapsto (u_1,u_2,a_{11'}-u_1,a_{22'}-u_2). \end{aligned}$$

By definition, the core of an assignment game is a subset of the polyhedron

$$K(E) = \{K(u_1, u_2) \mid (u_1, u_2) \in E\}.$$

The set $K(E)$ can also be described by a convex hull of four extreme points:

$$\begin{aligned} K(E) = & CnvH\Big\{(0, 0, a_{11'}, a_{22'}), (a_{11'}, a_{22'}, 0, 0), (a_{11'}, 0, 0, a_{22'}), (0, a_{22'}, a_{11'}, 0)\Big\} \\ \subset & \, \boldsymbol{v}_A(I) \cdot \mathbf{X}^4. \end{aligned} \quad (5.3)$$

Before we describe the core in a more detailed way, we look at an example in which the set $K(E)$ is considered.

Example 5.3.1 *Consider the assignment matrix* $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$ *and let* $\boldsymbol{v}_A$ *be the induced assignment game. In this case we have:*

$$K(E) = CnvH\Big\{(0, 0, 6, 8), (6, 8, 0, 0), (6, 0, 0, 8), (0, 8, 6, 0)\Big\}.$$

Obviously, the set $K(E)$ has at most dimension two, such that the core of the corresponding assignment game has at most dimension two. This means that we have the geometric argument of Corollary 2.4.12. In order to demonstrate our geometric result, we can look at Figure 5.3, where the set $K(E)$ of Example 5.3.1 can be seen. This set is the intersection of a simplex and a hyperplane.

5.3.1 Extreme Points of the Core

As in the case of the least core of the dual assignment game, it is possible to calculate the extreme points of the core in a very simple way. This result is based on the idea that the core-correspondence of balanced games is continuous. The detailed proof of the following lemma can be found in Hoffmann [6] in 2004. Note, that the claim of our next lemma is independent of the relation between $a_{11'}$ and $a_{22'}$.

Lemma 5.3.2 *(Hoffmann [6])*
Let A be a 2×2 assignment matrix with a dominant diagonal. Furthermore, let $\boldsymbol{v}_A$ be the induced assignment game. Then, the at most six extreme points of core $\mathcal{C}(\boldsymbol{v}_A)$ are defined as follows:

$$\boldsymbol{d}_1 = \begin{pmatrix} 0 \\ 0 \\ a_{11'} \\ a_{22'} \end{pmatrix}, \quad \boldsymbol{d}_3 = \begin{pmatrix} a_{11'} - a_{21'} \\ 0 \\ a_{21'} \\ a_{22'} \end{pmatrix}, \quad \boldsymbol{d}_5 = \begin{pmatrix} 0 \\ a_{22'} - a_{12'} \\ a_{11'} \\ a_{12'} \end{pmatrix},$$

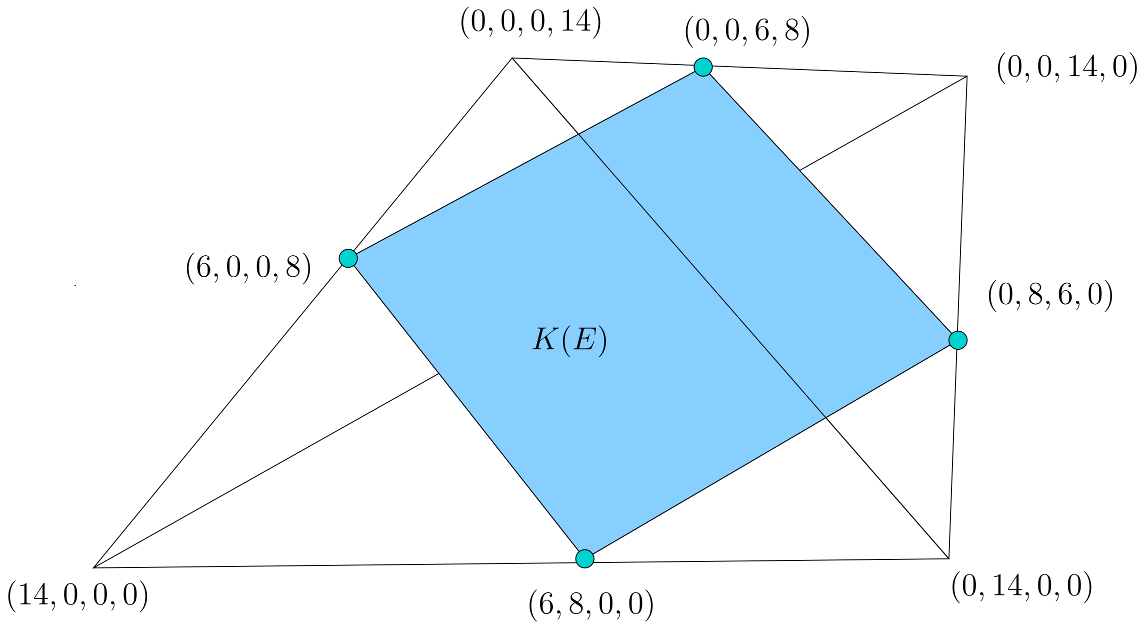

Figure 5.3: Set $K(E)$ of Example 5.3.1

$$\boldsymbol{d}_2 = \begin{pmatrix} a_{11'} \\ a_{22'} \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{d}_4 = \begin{pmatrix} a_{12'} \\ a_{22'} \\ a_{11'} - a_{12'} \\ 0 \end{pmatrix}, \quad \boldsymbol{d}_6 = \begin{pmatrix} a_{11'} \\ a_{21'} \\ 0 \\ a_{22'} - a_{21'} \end{pmatrix}.$$

This lemma implies that it is sufficient to know the entries of the assignment matrix if we want to calculate the core of an assignment game. Note, that we need to know the entries of the diagonal if we want to compute the extreme points $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. In the case of $\boldsymbol{d}_3$ and $\boldsymbol{d}_5$, the rows of the matrix are important. Finally, we consider the columns of the matrix in order to calculate $\boldsymbol{d}_4$ and $\boldsymbol{d}_6$. As in the case of the least core, we consider our well-known standard example to point out the connection between the core and the corresponding assignment matrix.

Example 5.3.3 *Consider the assignment matrix* $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$. *The core of the induced assignment game* $\boldsymbol{v}_A$ *is given by*

$$\mathcal{C}(\boldsymbol{v}_A) = CnvH\Big\{(0,0,6,8), (0,4,6,4), (4,8,2,0), (5,0,1,8), (6,1,0,7), (6,8,0,0)\Big\}.$$

In order to get a better picture of the core, we use the same projection as in (5.2). That means, we consider

$$\begin{aligned} P : \mathbb{R}^4 &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto (u_1, u_2, v_{1'}). \end{aligned}$$

This is possible because the sum of all coordinates of each element of the core is constant and equal to $\boldsymbol{v}_A(I)$. Thus, it is sufficient to know only the first three coordinates. In Figure 5.4 the projection of the core as a subset of $K(E)$ is illustrated.

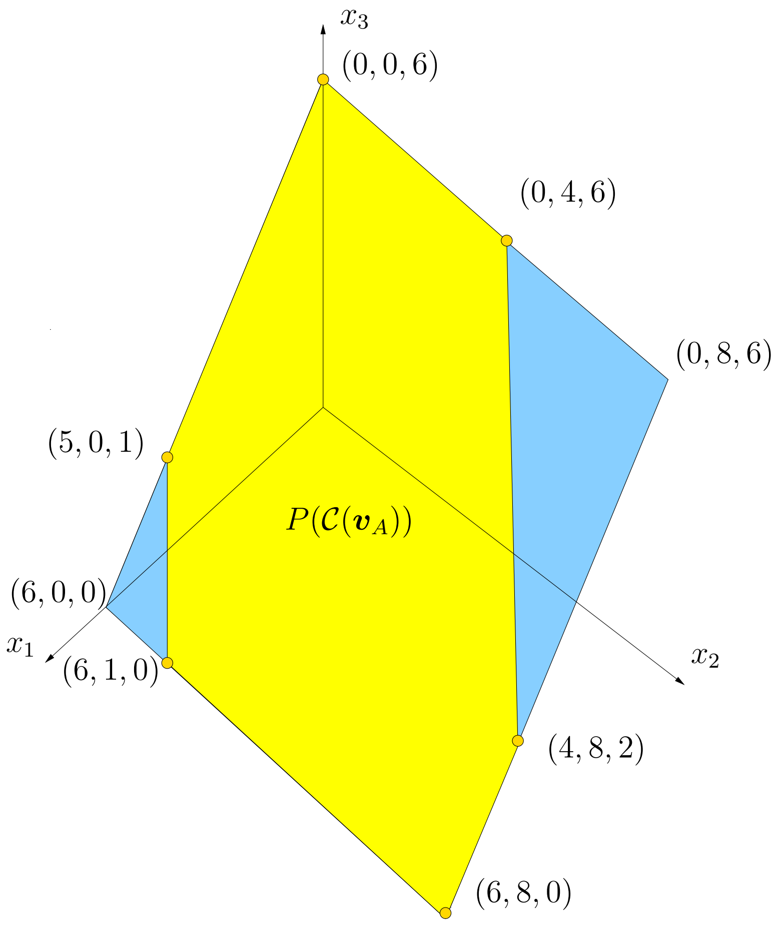

Figure 5.4: Projection of the core $\mathcal{C}(\boldsymbol{v}_A)$ of Example 5.3.3

5.4 The Intersection of Core and Least Core

Throughout the previous sections, we have considered the core and the least core of the dual assignment game. In this section we want to analyze the intersection of these two polyhedrons. Remember that by Theorem 2.4.22 and Theorem 2.4.23 this intersection is non-empty because, for example, the modiclus is an element of these two sets. Based on our observations of the above section, we can conclude that the dimension of the intersection of the two polyhedrons is at least one.

At the beginning of this section we restrict our considerations to the non-empty intersection of $M(F)$ and $K(E)$. By (5.1) and (5.3) we have shown that the intersection $M(F) \cap K(E)$ is a compact interval $I(A)$ which is defined by:

$$M(F) \cap K(E) = I(A) := [(a_{11'}, \gamma - a_{11'}, 0, \gamma), (0, \gamma, a_{11'}, \gamma - a_{11'})].$$

Since the least core of the dual assignment game is a subset of $M(F)$ and the core is a subset of $K(E)$, we can conclude

$$\mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A) \subseteq I(A).$$

In the next example we consider our well-known assignment game from the above sections. Here, we consider the non-empty intersection $M(F) \cap K(E)$.

Example 5.4.1 *Consider the assignment matrix* $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$. *Then, the intersection of* $M(F)$ *and* $K(E)$ *is given by*

$$M(F) \cap K(E) = I(A) = [(6, 1, 0, 7), (0, 7, 6, 1)].$$

Thus, in order to compute the intersection of $M(F)$ and $K(E)$, it is sufficient to know the assignment matrix. For a better overview of the geometry, the reader is referred to Figure 5.5.

5.4.1 Extreme Points of the Intersection

In this section we are interested in the calculation of the extreme points of the intersection of core and least core of the dual assignment game. Knowing that this intersection is a subset of the interval $I(A)$, we can conclude that there exist at most

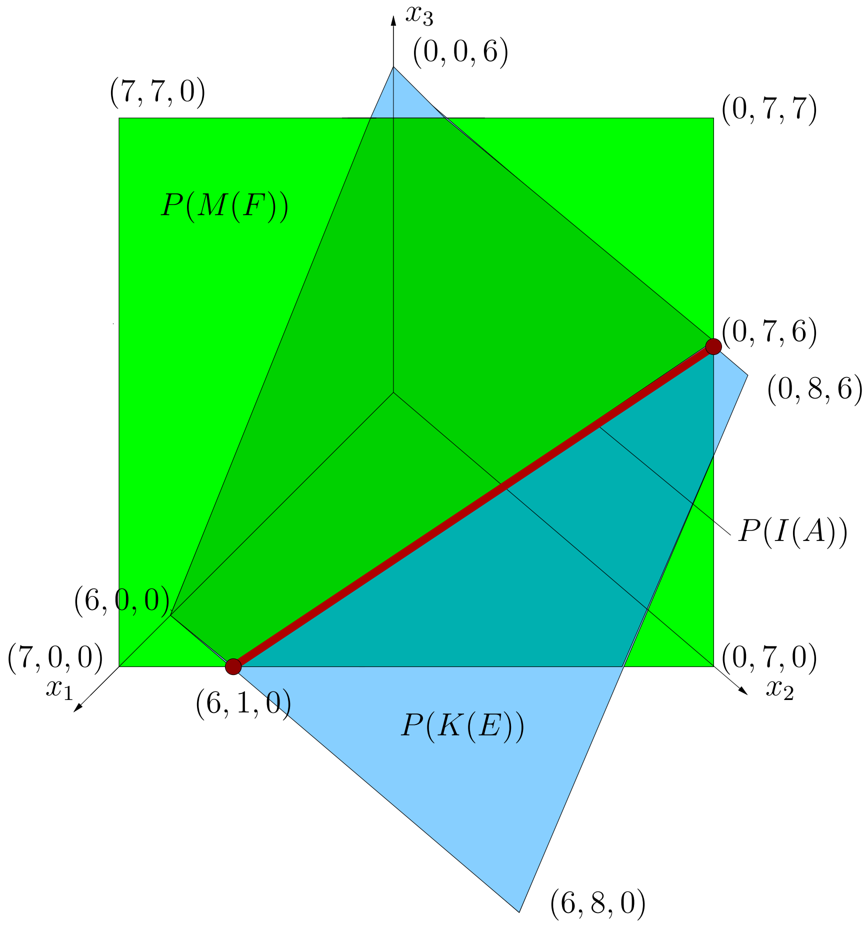

Figure 5.5: Projection of $K(E) \cap M(F)$ of Example 5.2.2

two extreme points. During this section, we have to distinguish between two cases: $\gamma > a_{12'} + a_{21'}$ and $\gamma \leq a_{12'} + a_{21'}$.

In our next lemma, we compute the extreme points of the intersection of core and least core for the case $\gamma > a_{12'} + a_{21'}$. The idea of the proof is based on the procedure of computing extreme points of a convex polyhedron and on our geometric arguments.

Lemma 5.4.2 *Let A be a 2×2 assignment matrix such that the corresponding assignment game $\boldsymbol{v}_A$ has a stable core. Furthermore, let $a_{11'} \leq a_{22'}$ and $\gamma > a_{21'} + a_{12'}$. Then, the two extreme points (u, v) and $(\tilde{u}, \tilde{v})$ of the intersection $\mathcal{LC}(\boldsymbol{v}_A^{\star}) \cap \mathcal{C}(\boldsymbol{v}_A)$*

are defined as follows:

$$\begin{aligned} u_1 &= \frac{a_{12'}}{2} + \frac{a_{11'}}{2}, \\ u_2 &= \gamma - \frac{a_{12'}}{2} - \frac{a_{11'}}{2}, \\ v_{1'} &= \frac{a_{11'}}{2} - \frac{a_{12'}}{2}, \\ v_{2'} &= \gamma + \frac{a_{12'}}{2} - \frac{a_{11'}}{2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{u}_1 &= \frac{a_{11'}}{2} - \frac{a_{21'}}{2}, \\ \tilde{u}_2 &= \gamma - \frac{a_{11'}}{2} + \frac{a_{21'}}{2}, \\ \tilde{v}_{1'} &= \frac{a_{11'}}{2} + \frac{a_{21'}}{2}, \\ \tilde{v}_{2'} &= \gamma - \frac{a_{21'}}{2} - \frac{a_{11'}}{2}. \end{aligned}$$

Proof.

We show that (u, v) and $(\tilde{u}, \tilde{v})$ are elements of the core $\mathcal{C}(\boldsymbol{v}_A)$ and of the least core of the dual assignment game $\mathcal{LC}(\boldsymbol{v}_A^\star)$. Therefore, we have to check the inequalities which describe the two polyhedrons. Last but not least we have to find for each extreme point (u, v) and $(\tilde{u}, \tilde{v})$ four inequalities such that

1. the corresponding inequalities are equalities,

2. the extreme point is the unique solution of a system of equalities.

In the steps 1a and 2a of the proof, we show that the point (u, v) is an element of $\mathcal{C}(\boldsymbol{v}_A)$ and $\mathcal{LC}(\boldsymbol{v}_A^\star)$. In step 3a, we point out a system of four equalities with unique solution (u, v). Then, we repeat the procedure in step 1b, 2b and 3b for the second point $(\tilde{u}, \tilde{v})$. Remember that it is not possible to have more than two extreme points, because the dimension of the intersection is at most one.

Step 1a:

We show in our first step that

$$\boldsymbol{x} = (u, v) \in \mathcal{C}(\boldsymbol{v}_A)$$

holds.

By Theorem 2.4.9 we have to check the following three inequalities:

1. $(u, v) \geq 0$,
2. $\sum_{i \in P} u_i + \sum_{j' \in Q} v_{j'} = \sum_{i \in P} a_{ii'}$,
3. $u_i + v_{j'} \geq a_{ij'} \quad \forall\, i, j \in P$.

Since the assignment matrix A has a dominant diagonal, the inequality

$$(u, v) \geq 0$$

follows by definition. Furthermore, we can conclude

$$\sum_{i \in P} u_i + \sum_{j' \in Q} v_{j'} = 2\gamma = \sum_{i \in P} a_{ii'}.$$

Since $\gamma > a_{12'} + a_{21'}$, we have

$$\begin{aligned} u_1 + v_{1'} &= a_{11'}, \\ u_1 + v_{2'} &= \gamma + a_{12'} \geq a_{12'}, \\ u_2 + v_{1'} &= \gamma - a_{12'} > a_{21'}, \\ u_2 + v_{2'} &= a_{11'} + a_{22'} - a_{11'} = a_{22'}, \end{aligned}$$

hence we can conclude $(u, v) \in \mathcal{C}(\boldsymbol{v}_A)$.

Step 2a:

In order to show that (u, v) is an element of the least core of the dual assignment game, we have to check that for $\boldsymbol{x} = (u, v)$ the following equality

$$\max_{S \in \mathcal{P}} e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star) = \gamma$$

holds true.

To check this equality, we compute $e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star)$ for every coalition $S \in \mathcal{P}$:

$$\begin{aligned}
e(\emptyset, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} - a_{12'}}{2} \leq \gamma, \\
e(\{1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{12'}}{2} \leq \gamma, \\
e(\{1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} + a_{12'}}{2} \leq \gamma, \\
e(\{2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{12'} + a_{11'}}{2} \leq \gamma, \\
e(\{1\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{12'}}{2} \leq \gamma, \\
e(\{1, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \gamma - a_{12'} - a_{21'} \leq \gamma, \\
e(\{1, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{1, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \gamma - \frac{a_{12'}}{2} - \frac{a_{11'}}{2} \leq \gamma, \\
e(\{1, 2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{1, 2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{12'}}{2} \leq \gamma, \\
e(\{1, 2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} + a_{12'}}{2} \leq \gamma, \\
e(\{1, 2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma.
\end{aligned}$$

Since $(u, v) \in X(\boldsymbol{v}_A)$, we know that (u, v) is an element of the least core of the dual assignment game.

Step 3a:

In the last step, we have to show that (u, v) is an extreme point of the intersection. Therefore, we have to find four equalities among the inequalities of step 1a and step 2a such that (u, v) is the unique solution of the induced system of equalities. Here, we consider the following system of equalities:

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_{1'} \\ x_{2'} \end{pmatrix} = \begin{pmatrix} -a_{11'} - a_{22'} \\ -\gamma + a_{12'} \\ -\gamma \\ -a_{11'} \end{pmatrix}.$$

We can easily check that (u, v) is the unique solution of this system of equalities, such that we can conclude that (u, v) is an extreme points of the intersection of $\mathcal{C}(\boldsymbol{v}_A) \cap \mathcal{LC}(\boldsymbol{v}_A^\star)$.

In order to prove that $(\tilde{u}, \tilde{v})$ is the second extreme point, we have to do the same three steps.

Step 1b:

As in the case of the first extreme point (u, v), we have

$$(\tilde{u}, \tilde{v}) \geq 0$$

and

$$\sum_{i \in P} \tilde{u}_i + \sum_{j' \in Q} \tilde{v}_{j'} = 2\gamma = \sum_{i \in P} a_{ii'}.$$

Furthermore, we have the following four equalities:

$$\begin{aligned}
\tilde{u}_1 + \tilde{v}_{1'} &= a_{11'}, \\
\tilde{u}_1 + \tilde{v}_{2'} &= \gamma - a_{21'} > a_{12'}, \\
\tilde{u}_2 + \tilde{v}_{1'} &= \gamma + a_{21'} \geq a_{21'}, \\
\tilde{u}_2 + \tilde{v}_{2'} &= a_{11'} + a_{22'} - a_{11'} = a_{22'},
\end{aligned}$$

such that we can conclude $(\tilde{u}, \tilde{v}) \in \mathcal{C}(\boldsymbol{v}_A)$.

Step 2b:

In this step we prove that the point $\boldsymbol{x} = (\tilde{u}, \tilde{v})$ is an element of the least core of the dual assignment game. Therefore, we have to show that

$$\max_{S \in \mathcal{P}} e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star) = \gamma$$

holds true.

Again, we check this by computing $e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star)$ for all coalitions $S \in \mathcal{P}$:

$$\begin{aligned}
e(\emptyset, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} + a_{21'}}{2} \leq \gamma, \\
e(\{1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{21'}}{2} \leq \gamma, \\
e(\{1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} - a_{21'}}{2} \leq \gamma, \\
e(\{2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \gamma - a_{12'} - a_{21'} \leq \gamma, \\
e(\{2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{21'}}{2} \leq \gamma, \\
e(\{1\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} + a_{21'}}{2} \leq \gamma, \\
e(\{1, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \gamma, \\
e(\{1, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{1, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} + a_{21'}}{2} \leq \gamma, \\
e(\{1, 2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{1, 2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{21'}}{2} \leq \gamma, \\
e(\{1, 2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} - a_{21'}}{2} \leq \gamma, \\
e(\{1, 2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma.
\end{aligned}$$

Since $\boldsymbol{x} = (\tilde{u}, \tilde{v}) \in X(\boldsymbol{v}_A)$, we have proved our claim that $(\tilde{u}, \tilde{v}) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$.

Step 3b:

In the last step, we have to find a system of equalities with unique solution $(\tilde{u}, \tilde{v})$. Here, we consider

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_{1'} \\ x_{2'} \end{pmatrix} = \begin{pmatrix} -a_{11'} - a_{22'} \\ -\gamma + a_{21'} \\ -\gamma \\ -a_{11'} \end{pmatrix}.$$

So, we can conclude that (u, v) and $(\tilde{u}, \tilde{v})$ are the two extreme points of the intersection of core and least core of the dual assignment game. $\square$

Note, that in the case in which we do not know anything about the dimension of the intersection, it could happen that there exist more than two extreme points. Thus, our procedure to start with some geometric arguments is very useful.

Now, we can compute the extreme points of the intersection in a simple way. In the follwing example, we consider our well-known assignment game of the above sections.

Example 5.4.3 *Let* $A = \begin{pmatrix} 6 & 4 \\ 1 & 8 \end{pmatrix}$. *Furthermore, let* $\boldsymbol{v}_A$ *be the induced assignment game. The extreme points of the intersection of core and least core are*

$$\begin{aligned} (u, v) &= (5, 2, 1, 6) \\ (\tilde{u}, \tilde{v}) &= (2.5, 4.5, 3.5, 3.5). \end{aligned}$$

Thus, we get the extreme points in a very simple way.

In the next step we conclude our research in the case of $\gamma > a_{12'} + a_{21'}$ with a special class of assignment games: the convex assignment games. We show that in this case the intersection of both polyhedrons is only one single point, which is the modified nucleolus. This result is the topic of the next corollary.

Corollary 5.4.4 *Let A be a non-negative 2×2 diagonal matrix and let $\boldsymbol{v}_A$ be the induced assignment game. Then, the intersection $\mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A)$ is only one single point. More precisely, we have in this case:*

$$(u, v) = (\tilde{u}, \tilde{v}) = (\frac{a_{11'}}{2}, \gamma - \frac{a_{11'}}{2}, \frac{a_{11'}}{2}, \gamma - \frac{a_{11'}}{2}).$$

Proof.

We know by our Lemma 5.4.2 that (u, v) and $(\tilde{u}, \tilde{v})$ are extreme points of the intersection. In our case the two points conform. But it is not possible to get a second extreme point, because the least core of the dual assignment game

$$\mathcal{LC}(\boldsymbol{v}_A^\star) = CnvH\Big\{(0, \gamma, 0, \gamma), (a_{11'}, a_{22'} - a_{11'}, a_{11'}, a_{22'} - a_{11'})\Big\}$$

has dimension one such that the intersection $\mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A)$ is only one single point.

□

Thus, we have seen that there exits classes of assignment games such that the intersection of core and least core of the dual assignment game is only one single point.

In our next example, we consider a convex assignment game. Here, we compute the core, the least core of the dual assignment game and the intersection of both sets.

Example 5.4.5 *Let* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *be a 2×2 assignment matrix. In the case of the induced assignment game $\boldsymbol{v}_A$, we have the following sets:*

$$\begin{aligned} \mathcal{C}(\boldsymbol{v}_A) &= \Big\{(1,1,0,0),(0,0,1,1),(1,0,1,0),(0,1,0,1)\Big\}, \\ \mathcal{LC}(\boldsymbol{v}_A^{\star}) &= \Big\{(1,0,1,0),(0,1,0,1)\Big\}. \end{aligned}$$

Thus, the intersection of both sets is

$$\mathcal{LC}(\boldsymbol{v}_A^{\star}) \cap \mathcal{C}(\boldsymbol{v}_A) = \frac{1}{2}(1,1,1,1) = (u,v) = (\tilde{u},\tilde{v}).$$

The above example is illustrated in Figure 5.6. Here, one can see the core and the least core of the dual assignment game in the case of convex games.

Since the modified nucleolus of the convex game of our example is the center of the intersection, one might think that this holds true also for all assignment games in general. In the next example, we consider an assignment game such that the modified nucleolus is not the center of the intersection of core and least core of the dual game. This means that for computing the modiclus it is not sufficient to know the intersection of both sets.

Example 5.4.6 *Let* $A = \begin{pmatrix} 6 & 4 \\ 2 & 8 \end{pmatrix}$ *be a 2×2 assignment matrix. The modified nucleolus of the induced assignment game $\boldsymbol{v}_A$ is:*

$$\Psi(\boldsymbol{v}_A) = \frac{1}{3}(10,11,8,13).$$

The extreme points of the intersection of core and least core of the dual assignment game are:

$$(u,v) = (5,2,1,6),$$

$$(\tilde{u},\tilde{v}) = (2,5,4,3).$$

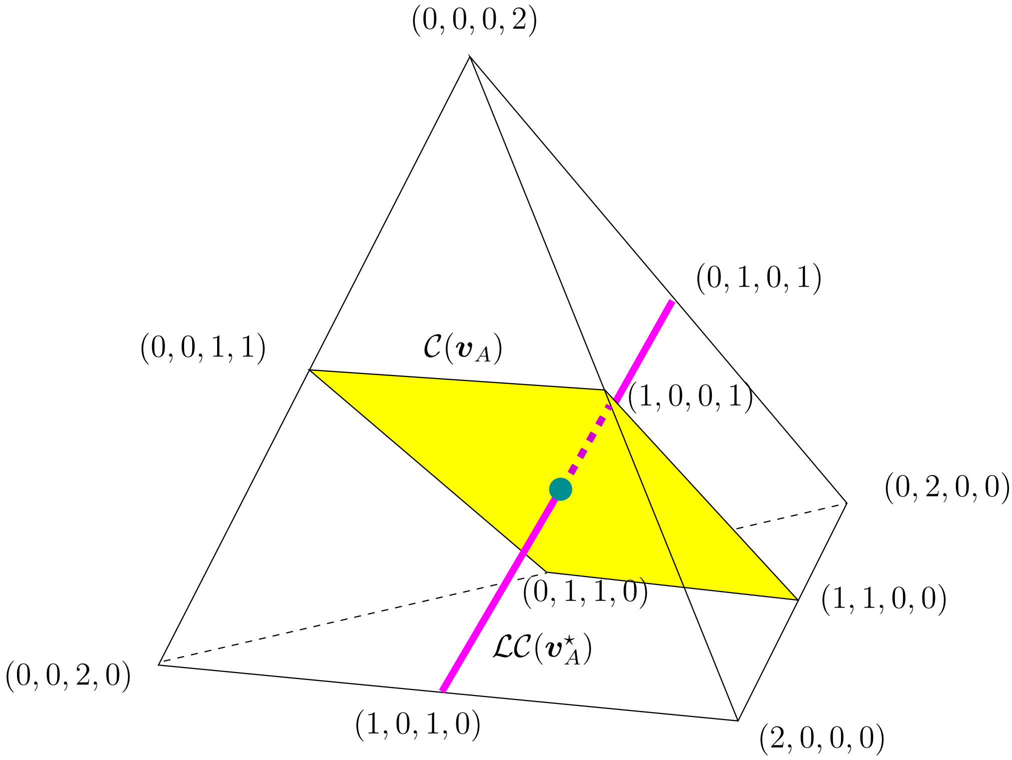

Figure 5.6: Convex Assignment Game induced by a 2 × 2 Matrix

In particular, we have:

$$\Psi(\boldsymbol{v}_A) \neq \frac{1}{2}((u, v) + (\tilde{u}, \tilde{v})).$$

But there exist also assignment games in which the modified nucleolus is the center of the intersection. In order to see this, we have a look at the following example.

Example 5.4.7 *Let* $A = \begin{pmatrix} 6 & 0 \\ 2 & 8 \end{pmatrix}$ *be a* 2×2 *assignment matrix. The modified nucleolus of the induced assignment game* $\boldsymbol{v}_A$ *is*

$$\Psi(\boldsymbol{v}_A) = \frac{1}{2}(5, 9, 7, 7).$$

The two extreme points of the intersection of core and least core of the dual assignment game are:

$$\begin{aligned} (u, v) &= (3, 4, 3, 4) \\ (\tilde{u}, \tilde{v}) &= (2, 5, 4, 3). \end{aligned}$$

In this case, we can conclude that

$$\Psi(\boldsymbol{v}_A) = \frac{1}{2}((u, v) + (\tilde{u}, \tilde{v})).$$

So, we can conclude that it is not sufficient to konw the etreme points of the intersection of core and least core if we want to compute the modfied nucleolus. Sometimes the modified nucleolus is the center of the intersection and sometimes it is not. Only in the case of convex assignment games our procceding yields the modified nucleolus.

Until now, we have focused only on the extreme points for the case $\gamma > a_{12'}+a_{21'}$. In the next lemma, we have a look at these extreme points in the case that $\gamma \leq a_{12'}+a_{21'}$ holds true. As in the first case, we find two extreme points of the intersection of both sets. More precisely, we have:

Lemma 5.4.8 *Let A be a non-negative 2×2 matrix, such that the induced assignment game $\boldsymbol{v}_A$ has a stable core. Furthermore, let $a_{11'} \leq a_{22'}$ and $\gamma \leq a_{21'} + a_{12'}$. Then, we have the two extreme points (u,v) and $(\tilde{u},\tilde{v})$ of the intersection of the core and the least core of the dual assignment game:*

$$\begin{aligned}
u_1 &= \frac{a_{11'}}{2} - \frac{\gamma}{2} + \frac{a_{12'}}{2},\\
u_2 &= \frac{3}{2}\gamma - \frac{a_{11'}}{2} - \frac{a_{12'}}{2},\\
v_{1'} &= \frac{\gamma}{2} - \frac{a_{12'}}{2} + \frac{a_{11'}}{2},\\
v_{2'} &= \frac{\gamma}{2} + \frac{a_{12'}}{2} - \frac{a_{11'}}{2},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{u}_1 &= \frac{\gamma}{2} - \frac{a_{21'}}{2} + \frac{a_{11'}}{2},\\
\tilde{u}_2 &= \frac{\gamma}{2} + \frac{a_{21'}}{2} - \frac{a_{11'}}{2},\\
\tilde{v}_{1'} &= \frac{a_{11'}}{2} - \frac{\gamma}{2} + \frac{a_{21'}}{2},\\
\tilde{v}_{2'} &= \frac{3}{2}\gamma - \frac{a_{11'}}{2} - \frac{a_{21'}}{2}.
\end{aligned}$$

Proof.

The idea of the proof is the same idea as in Lemma 5.4.2. So we have for each extreme point the following three steps. First, we show that our point (u,v) is an element of the core. In the second step, we prove that this point is also an element of the least core of the dual assignment game. Last but not least, we show that

(u, v) is an extreme point, that is, it is a solution of a special system of equalities. After this, we repeat the same steps for the point $(\tilde{u}, \tilde{v})$.

Step 1a:

By definition,

$$(u, v) \geq 0$$

follows immediately. Furthermore, we have

$$\sum_{i \in P} u_i + \sum_{j' \in Q} v_{j'} = 2\gamma = \sum_{i \in P} a_{ii'}.$$

In order to show that (u, v) is an element of the core, we only have to check that

$$u_i + v_{j'} \geq a_{ij'}$$

holds for all $(i, j') \in P \times Q$. More precisely, we have the following inequalities:

$$\begin{aligned} u_1 + v_{1'} &= a_{11'}, \\ u_1 + v_{2'} &= a_{12'}, \\ u_2 + v_{1'} &= 2\gamma - a_{12'} \geq \gamma - a_{12'} \geq a_{21'}, \\ u_2 + v_{2'} &= 2\gamma - a_{11'} = a_{22'}, \end{aligned}$$

such that we can conclude that (u, v) is an element of the core.

Step 2a:

In this step, we have to check that (u, v) is an element of the least core of the dual assignment game. Therefore, we have to check that

$$\boldsymbol{x} = (u, v) \in X(\boldsymbol{v}_A)$$

and that

$$\max_{S \in \mathcal{P}} e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star) = \gamma \tag{5.4}$$

holds true. By definition, $\boldsymbol{x} \in X(\boldsymbol{v}_A)$ follows immediately. The equality (5.4) follows from the fact that $\gamma \leq a_{12'} + a_{21'}$ holds.

To see this, consider the following excesses:

$$\begin{aligned}
e(\emptyset, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 0 \leq \gamma, \\
e(\{2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{22'} - a_{12'}}{2} \leq \gamma, \\
e(\{1'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{22'} - a_{11'} + a_{12'}}{2} \leq \gamma, \\
e(\{1', 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 2\gamma - \gamma = \gamma, \\
e(\{2\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{22'} + a_{12'}}{2} - \frac{\gamma}{2} \leq \gamma, \\
e(\{2, 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= a_{11'} + a_{22'} - 2\gamma = 0 \leq \gamma, \\
e(\{2, 1'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= a_{11'} - a_{12'} \leq a_{11'} + a_{21'} - \gamma \leq \gamma, \\
e(\{2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{11'} - a_{12'}}{2} - \frac{\gamma}{2} \leq \gamma, \\
e(\{1\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{11'} - a_{12'}}{2} + \frac{\gamma}{2} \leq \frac{a_{11'} + a_{21'}}{2} \leq \gamma, \\
e(\{1, 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 2\gamma - a_{12'} - a_{21'} \leq 2\gamma - \gamma = \gamma, \\
e(\{1, 1'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 0 \leq \gamma, \\
e(\{1, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{22'} - a_{12'}}{2} + \frac{\gamma}{2} \leq \gamma, \\
e(\{1, 2\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 2\gamma - \gamma = \gamma, \\
e(\{1, 2, 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \frac{a_{22'} - a_{12'}}{2} + \frac{\gamma}{2} \leq \frac{a_{22'} + a_{21'}}{2} \leq \gamma, \\
e(\{1, 2, 1'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= \gamma + \frac{\gamma}{2} - \frac{a_{11'} + a_{12'}}{2} \leq \gamma + \frac{a_{21'} - a_{11'}}{2} \leq \gamma, \\
e(\{1, 2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}^\star_A) &= 0 \leq \gamma.
\end{aligned}$$

So, we can conclude $\boldsymbol{x} = (u, v) \in \mathcal{LC}(\boldsymbol{v}^\star_A)$.

Step 3a:

In the last step, we have to find a system of equalities with unique solution (u, v). One can easily check that the following equalities satisfy this condition:

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_{1'} \\ z_{2'} \end{pmatrix} = \begin{pmatrix} -a_{11'} - a_{22'} \\ -\gamma \\ -a_{12'} \\ -a_{11'} \end{pmatrix}.$$

All together, we can sum up that (u, v) is an extreme point of the intersection of core and least core of dual assignment game.

In the case of $(\tilde{u}, \tilde{v})$, we have to go through the same three steps as in the case of (u, v).

Step 1b:

First of all, we check that $(\tilde{u}, \tilde{v})$ is an element of the core. By definition we have

$$(\tilde{u}, \tilde{v}) \geq 0$$

and

$$\sum_{i \in P} \tilde{u}_i + \sum_{j' \in Q} \tilde{v}_{j'} = 2\gamma = \sum_{i \in P} a_{ii'}.$$

Furthermore, we have

$$\begin{aligned}
\tilde{u}_1 + \tilde{v}_{1'} &= a_{11'}, \\
\tilde{u}_1 + \tilde{v}_{2'} &= 2\gamma - a_{21'} \geq a_{12'}, \\
\tilde{u}_2 + \tilde{v}_{1'} &= a_{21'}, \\
\tilde{u}_2 + \tilde{v}_{2'} &= 2\gamma - a_{11'} = a_{22'},
\end{aligned}$$

such that we can conclude that $(\tilde{u}, \tilde{v}) \in \mathcal{C}(\boldsymbol{v}_A)$.

Step 2b:

As in the case of (u, v), we have to check that $(\tilde{u}, \tilde{v})$ is an element of the least core of the dual assignment game. Since

$$\boldsymbol{x} = (\tilde{u}, \tilde{v}) \in X(\boldsymbol{v}_A),$$

we have to check that

$$\max_{S \in \mathcal{P}} e(S, \boldsymbol{x}, \boldsymbol{v}_A^\star) = \gamma$$

holds true.

With the aid of $\gamma \leq a_{12'} + a_{21'}$, we get the following inequalities:

$$\begin{aligned}
e(\emptyset, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} - a_{11'}}{4} + \frac{a_{21'}}{2} \leq \gamma, \\
e(\{1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{21'}}{2} + \frac{\gamma}{2} \leq \frac{a_{11'} + a_{12'}}{2} \leq \gamma, \\
e(\{1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{22'} - a_{21'}}{2} + \frac{\gamma}{2} \leq \frac{a_{22'} + a_{12'}}{2} \leq \gamma, \\
e(\{2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= -2\gamma + a_{11'} = -a_{22'} \leq \gamma, \\
e(\{2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - a_{12'} - a_{21'} \leq \gamma, \\
e(\{2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} - a_{21'}}{2} + \frac{\gamma}{2} \leq \gamma, \\
e(\{1\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} + a_{21'}}{2} - \frac{\gamma}{2} \leq \frac{a_{11'} - a_{12'}}{2} \leq \gamma, \\
e(\{1, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{1, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma, \\
e(\{1, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{21'} - a_{11'}}{2} + \frac{\gamma}{2} \leq \gamma, \\
e(\{1, 2\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 2\gamma - \gamma = \gamma, \\
e(\{1, 2, 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \frac{a_{11'} + a_{21'}}{2} - \frac{\gamma}{2} \leq \gamma, \\
e(\{1, 2, 1'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= \gamma + \frac{\gamma}{2} - \frac{a_{11'} + a_{21'}}{2} \leq \gamma + \frac{a_{12'} - a_{11'}}{2} \leq \gamma, \\
e(\{1, 2, 1', 2'\}, \boldsymbol{x}, \boldsymbol{v}_A^\star) &= 0 \leq \gamma,
\end{aligned}$$

such that we can conclude $(\tilde{u}, \tilde{v}) \in \mathcal{LC}(\boldsymbol{v}_A^\star)$.

Step 3b:

In order to prove that $(\tilde{u}, \tilde{v})$ is an extreme point of the intersection of both sets, we have to find four equalities such that $(\tilde{u}, \tilde{v})$ is the unique solution. Therefore, we consider

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_{1'} \\ z_{2'} \end{pmatrix} = \begin{pmatrix} -a_{11'} - a_{22'} \\ -\gamma \\ -a_{21'} \\ -a_{11'} \end{pmatrix}.$$

Thus, we can conclude that (u, v) and $(\tilde{u}, \tilde{v})$ are extreme points of the intersection of the core and the least core of the dual assignment game. Furthermore, we note that there are no other extreme points. In order to check this, we have to consider our geometric arguments. $\square$

In fact, we have shown in the proofs of the above two lemmas, that it is possible to calculate the extreme points of the intersection of core and least core of the dual assignment game in a very simple way. It should be mentioned that these calculations depend only on the relation between γ and $a_{12'} + a_{21'}$.

In the next example we compute the extreme points of the intersection with the aid of the lemma above.

Example 5.4.9 *Let* $A = \begin{pmatrix} 4 & 4 \\ 2 & 6 \end{pmatrix}$ *be a* 2×2 *assignment matrix and let* $\boldsymbol{v}_A$ *be the induced assignment game. The extreme points of the intersection* $\mathcal{C}(\boldsymbol{v}_A) \cap \mathcal{LC}(\boldsymbol{v}_A^\star)$ *are defined by*

$$\begin{aligned} (u, v) &= (1.5, 3.5, 2.5, 2.5), \\ (\tilde{u}, \tilde{v}) &= (3.5, 1.5, 0.5, 4.5). \end{aligned}$$

After this example, let us consider an example in which the intersection of the two sets is only one single point. The most well-known example is the glove game which is induced by a 2×2 assignment matrix. Formally, we have:

Example 5.4.10 *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *be a* 2×2 *matrix. Furthermore, let* $\boldsymbol{v}_A$ *be the assignment game which is induced by the assignment matrix* A*. In this case we have:*

$$\begin{aligned} \mathcal{C}(\boldsymbol{v}_A) &= CnvH\Big\{(1,1,0,0),(0,0,1,1)\Big\}, \\ \mathcal{LC}(\boldsymbol{v}_A^\star) &= CnvH\Big\{(1,0,1,0),(0,1,0,1),(1,0,0,1),(0,1,1,0)\Big\}. \end{aligned}$$

The intersection of both sets is only one single point:

$$\mathcal{C}(\boldsymbol{v}_A) \cap \mathcal{LC}(\boldsymbol{v}_A^\star) = \frac{1}{2}(1,1,1,1).$$

In order to get a better geometric idea, we can consider Figure 5.7. This figure demonstrates the core, the least core of the dual assignment game and the intersection of both sets.

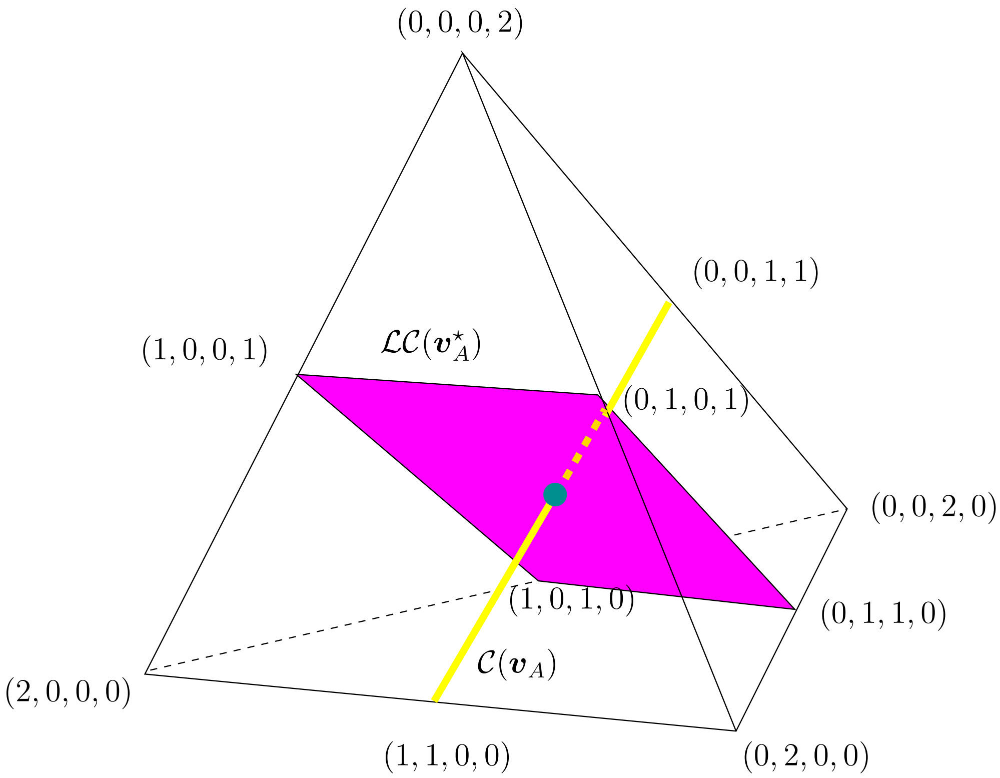

Figure 5.7: The glove game

After having considered the standard glove game, we want to note, that there exists a more general class of these games. These games are induced by an assignment matrix which is not normalized.

Corollary 5.4.11 *Let $c \in \mathbb{R}_+$. Consider the assignment matrix*

$$A = \begin{pmatrix} c & c \\ c & c \end{pmatrix}$$

and let $\boldsymbol{v}_A$ be the induced assignment game. Then, we have

$$\begin{aligned} \mathcal{LC}(\boldsymbol{v}_A^\star) &= CnvH\Big\{(c,0,0,c),(c,0,c,0),(0,c,0,c),(0,c,c,0)\Big\}, \\ \mathcal{C}(\boldsymbol{v}_A) &= CnvH\Big\{(c,c,0,0),(0,0,c,c)\Big\}, \end{aligned}$$

such that the dimension of the intersection of core and least core of the dual assignment game is only one single point:

$$\mathcal{LC}(\boldsymbol{v}_A^\star) \cap \mathcal{C}(\boldsymbol{v}_A) = (u,v) = (\tilde{u},\tilde{v}) = \frac{1}{2}(c,c,c,c).$$

Since the modified nucleolus of the game of our example is the center of the intersection, one might think that this holds true also for all assignment games in general. In the next example, we consider an assignment game such that the modified nucleolus is not in the center of the intersection of core and least core of the dual game. This means that for computing the modiclus it is not sufficient to know the intersection of both sets.

Example 5.4.12 *Let* $A = \begin{pmatrix} 6 & 5 \\ 3 & 8 \end{pmatrix}$ *be a* 2×2 *assignment matrix. The modified nucleolus of the induced assignment game* $\boldsymbol{v}_A$ *is*

$$\Psi(\boldsymbol{v}_A) = \frac{1}{3}(10, 11, 8, 13).$$

The two extreme points of the intersection of core and least core of the dual assignment game are:

$$\begin{aligned}(u, v) &= (2, 5, 4, 3)\\ (\tilde{u}, \tilde{v}) &= (5, 2, 1, 6).\end{aligned}$$

In this case, we can conclude that

$$\Psi(\boldsymbol{v}_A) \neq \frac{1}{2}((u, v) + (\tilde{u}, \tilde{v})).$$

But there are also assignment games in which the modified nucleolus is the center of the intersection. In order to see this, we have a look at the following example.

Example 5.4.13 *Let* $A = \begin{pmatrix} 8 & 8 \\ 4 & 10 \end{pmatrix}$ *be a* 2×2 *assignment matrix. The modified nucleolus of the induced assignment game* $\boldsymbol{v}_A$ *is:*

$$\Psi(\boldsymbol{v}_A) = (5, 4, 3, 6).$$

The extreme points of the intersection of core and least core of the dual assignment game are:

$$\begin{aligned}(u, v) &= \frac{1}{2}(7, 11, 9, 9),\\ (\tilde{u}, \tilde{v}) &= \frac{1}{2}(13, 5, 3, 15).\end{aligned}$$

In particular, we have:

$$\Psi(\boldsymbol{v}_A) = \frac{1}{2}((u, v) + (\tilde{u}, \tilde{v})).$$

Thus, if we want to calculate modiclus we have to find another way.

Chapter 6

Conclusions

In this thesis we provided a basic definition of cooperative game theory with transferable utility including the most well-known solution concepts. Important conditions, like the strong nullplayer property were shown. In the next step our main interest was on the connections between different solution concepts in the case of the so-called assignment games. Here, we particularly analyzed the connection between the Shapley value and the core in the case of exact assignment games. Our main result in this context is that the Shapley value is an element of the core. For the direct proof, we needed an alternative description of exact assignment games. In order to show that exactness is a necessary condition for our theorem, we considered non-exact assignment games such that the Shapley value is not an element of the core. But we have also seen that there exists non-exact assignment games such that the Shapley value is in the core. Next, we made some statements about the modified nucleolus of an assignment game. In this context, we started by considering the class of c-convex games which yield the class of dual assignment games. Our results are very useful since the modified nucleolus is self dual and it is an element of the least core of the dual assignment game. We found some properties of the least core of the dual assignment game which can be extended to some special subclasses. For example in the case of convex assignment games, it is possible to compute the most important solution concepts, like the core, the Shapley value, the nucleolus, modiclus and the least core of the dual game. Note that for the calculations, we only need to use the assignment matrix. Chapter 5 gave a more detailed analysis of the core and the least core of the dual assignment game in the case of exact assignment games which are induced by a 2×2 matrix. Since the two polyhedrons have dimension two, it is possible to gain a more geometrical perspective. In particular, we generated an image of the intersection of both sets such that we were able to compute this intersection.

Chapter 7

Index of Notations

e^i	i-th unit vector
$\mathbb{R}^n_S$	subspace of $\mathbb{R}^n$ spanned by $(e^i)_{i \in S}$
$\mathbf{X}^n$	unit simplex of $\mathbb{R}^n$
$\boldsymbol{x}_S$	projection of $\boldsymbol{x} \in \mathbb{R}^n$ to $\mathbb{R}^n_S$
$a \leq_{lex} b$	lexicographically smaller
$CnvH\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$	convex hull of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$
$EXT\ \mathcal{S}$	extreme points of the set $\mathcal{S}$
$\mathcal{P}(I),\ \mathcal{P}$	power set (of I)
$\boldsymbol{v}^\star$	dual game of game $\boldsymbol{v}$
$\boldsymbol{v}^T$	game which is restricted on T
$\boldsymbol{v}_A$	assignment game induced by matrix A
$e(S, \boldsymbol{x}, \boldsymbol{v})$	excess of S at $\boldsymbol{x}$ w.r.t. $\boldsymbol{v}$
$\mu(\boldsymbol{x}, \boldsymbol{v})$	maximal excess of $\boldsymbol{v}$ at $\boldsymbol{x}$
$\mu_0(\boldsymbol{x}, \boldsymbol{v})$	maximal non-trivial excess of $\boldsymbol{v}$ at $\boldsymbol{x}$
$X(\boldsymbol{v})$	set of preimputations of $\boldsymbol{v}$
$\mathcal{I}(\boldsymbol{v})$	set of imputations of $\boldsymbol{v}$
$\mathcal{C}(\boldsymbol{v})$	core of game $\boldsymbol{v}$
$\mathcal{C}_\epsilon(\boldsymbol{v})$	ϵ-core of game $\boldsymbol{v}$
$\mathcal{LC}(\boldsymbol{v})$	least core of game $\boldsymbol{v}$
$\mathcal{MLC}(\boldsymbol{v})$	modified least core of game $\boldsymbol{v}$
$\Phi(\boldsymbol{v})$	Shapley value of $\boldsymbol{v}$
$\nu(\boldsymbol{v})$	nucleolus of $\boldsymbol{v}$
$\Psi(\boldsymbol{v})$	modified nucleolus (modiclus) of $\boldsymbol{v}$
$\mathbb{V}$	set of cooperative games
$\mathbb{A}$	set of additive games
$\mathbb{C}$	set of convex games
$\mathbb{E}$	set of exact games
$\mathbb{B}$	set of balanced games
$\mathbb{T}$	set of totally balanced games

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