# Asymptotic Behavior of some Stochastic Evolutions in Continuum

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# Introduction

The study of interacting particle systems is motivated by a large class of applications in various disciplines. As a sub-class of the huge field of complex systems, they originally appeared in statistical physics as mathematical descriptions of physical phenomena. In particular, one has to mention here microscopic descriptions of the behavior of gases or, in the case of the famous Ising model, the magnetization of ferromagnets. They are also used as models for sociological and enonomical behavior as well as for biological and ecological systems. Important examples here, aside from the classical physical models, are infection spreading and agent behavior models or birth-and-death processes and population dynamics. Thus, depending on the area, one also speaks of agents or individuals instead of particles and hence of individual based models.

One can basically distinguish between two classes of interacting particle systems. The first one consists of the so called *lattice models*. Here particles live on a discrete space like the lattice $\mathbb{Z}^d$, $d \geq 1$, or, more generally, an infinite graph. Note that $\mathbb{Z}^d$ has a natural graph structure, where the vertices are the points of $\mathbb{Z}^d$ and $x, y \in \mathbb{Z}^d$ are adjacent if and only if their Euclidean distance equals to one. The second type of models are the *continuous* ones, where the underlying space is the Euclidean space $\mathbb{R}^d$ or a Riemannian manifold.

For many problems from applications, lattice models are an appropriate description of the situation, e.g., for network models (internet, social networks) or models based on crystallic structures. But often it is much more reasonable to consider the continuous case and lattice models are only used as a technically easier alternative. Therefore, continuous versions of many classical lattice models like lattice gas or birth-and-death dynamics (for further examples of lattice models see, e.g., [Lig85]) have been developed and studied.

The general framework for continuous infinite particle models is configuration space analysis. The *configuration space* $\Gamma$ over a Riemannian manifold $X$ consists of all locally finite subsets of $X$, i.e., of all sets of points without accumulation points in $X$. The points of a configuration reflect the particles or individuals in a model. Thus, a configuration describes (infinitely many) indistinguishable particles. Each configuration $\gamma \in \Gamma$ can be identified with a positive integer-valued Radon measure by identifying each of its points $x \in \gamma$ with the Dirac measure with unite mass in concentrated in $x$. Thus, $\Gamma$ obtains a topological structure induced by the vague topology on the space of all Radon measures on $X$. Furthermore, $\Gamma$ has a natural differentiable geometry obtained as a lifting of the geometry of the underlying manifold, cf., e.g., [AKR98a, AKR98b]). Even for $X = \mathbb{R}^d$, this geometry is non-flat. Measures on $\Gamma$ (with the Borel $\sigma$-algebra), so called *point processes*, describe states of particle systems. The state of a system without interaction is described by the *Poisson measure*. Introduction of interaction via potentials then leads to the notion of *Gibbs states*. Together with the gradient from the differential geometry, this allows to construct Dirichlet forms describing stochastic processes on $\Gamma$, e.g., free equilibrium diffusion in the case of the Poisson measure or diffusions of infinitely many particles interacting via pair potentials in case of the corresponding Gibbs measures. See, e.g., [MR92] for the general theory of Dirichlet forms and associated stochastic processes.

In this work, we study three infinite particle resp. individual based models on $\mathbb{R}^d$. The first model is the diffusion $(X_t)_{t\geq 0}$ of a particle in $\mathbb{R}^d$ interacting with the points of a random configuration via a pair potential. We will apply a general technique developed by C. Kipnis and S.R.S. Varadhan [KV86] and A. De Masi et al. [DFGW89] to obtain an invariance principle for this process, i.e., the convergence to a Brownian motion under a space-time scaling. Note that $X_t$ describes a stochastic dynamics on $\mathbb{R}^d$. But for application of the invariance principle scheme, we have to work with the corresponding environment process, i.e., the motion of the environment as seen from the point of view of the moving particle. This gives a stochastic process on the configuration space $\Gamma$.

The second model we discuss is a tagged particle dynamics, i.e., the motion of one marked particle in an infinite interacting system of identical particles. This process has been constructed recently by T. Fattler and M. Grothaus [FG08]. We will apply the same technique as for the diffusion in random environment to obtain an invariance principle here, too.

For both models, the diffusion in random environment and the tagged particle process, invariance principles have been discussed in [DFGW89], but only on a heuristic level. And nevertheless, the authors have to make strong assumptions on the interaction potentials like boundedness, positivity, and finite range, which are hardly ever satisfied in realistic models. In contrast to this, we can show all necessary results rigorously for a wide class of potentials as appearing in applications from statistical physics.

While the first two models are of a physical nature, the third model studied in this work describes a population dynamics, namely a continuous birth-and-death process with jumps. We will prove the existence of invariant measures for this dynamics and a corresponding ergodicity result.

In the following we will give a more detailed overview over the contents and results of this work.

## Configuration spaces

In Chapter 1, we recall basic definitions and general results from configuration space analysis. Although one can basically consider configuration spaces over Riemannian manifolds or even more general topological spaces, we restrict ourselves in the presentation to the case where the underlying space is $\mathbb{R}^d$, since all models considered in this work are based on this space.

After an introduction containing general notations used in this work, we recall the definitions of finite, of simple, and of multiple configurations, and we introduce the topologies on the corresponding spaces $\Gamma_0$, $\Gamma$, and $\ddot{\Gamma}$, resp. Of course, by having a topological structure on the configuration spaces, we can consider them as measurable spaces by introducing the corresponding Borel $\sigma$-algebras. The basic measures on $\Gamma_0$ and $\Gamma$ are the Lebesgue-Poisson measure and the Poisson measure, resp., which are introduced in Section 1.3. As mentioned above, $\Gamma$ has a natural geometric structure obtained by lifting of the underlying manifold geometry. It can be shown (see [AKR98a]), that the mixed Poisson measures are the proper volume elements for this geometry.

The Poisson measure on $\Gamma$ corresponds to the "free" case, i.e., it describes a state of a system of non-interacting particles. If one introduces interaction of particles via potentials, this leads to the notion of Gibbs measures. In Section 1.4 we recall the definitions (via Dobrushin-Lanford-Ruelle equations) of grand canonical and canonical Gibbs measures for general potentials. Afterwards we discuss the important case of two-body interactions via (symmetric) pair potentials. We recall the well-known characterization of such Gibbs measures via the Georgii-Nguyen-Zessin identity, cf. Proposition 1.4.8. Afterwards, we recall an existence result for grand canonical Gibbs measures w.r.t. pair potentials, more precisely for so-called Ruelle measures (see [Rue70]), where the basic intensity measure is the Lebesgue measure $dx$, and, more generally, for Gibbs measures w.r.t. an intensity measure absolutely continuous w.r.t. $dx$ with bounded Radon-Nikodym derivative (see [FG08]). This result holds, for instance, for the Lennard-Jones potential, which has a singularity at zero, a negative (i.e. attractive) part, and an infinite range. For the models considered in Chapter 2 and Chapter 3 we always have such potentials in mind.

In Section 1.5, we present the mentioned lifting procedure of the geometry of the underlying space to the configuration space. We introduce the intrinsic gradient $\nabla^\Gamma$ on $\Gamma$ and show how it acts on so called smooth bounded cylinder functions $\mathcal{F}C_b^\infty$. The pre-Dirichlet forms in the later chapters will be defined on this class of functions. Furthermore, we introduce another flat gradient $\mathbb{D}$ corresponding to spatial shifts of configurations.

The last section of this chapter is on harmonic analysis on configuration spaces as developed in [KK02]. The basic object here is the so-called $K$-transform, which maps functions on the finite configuration space $\Gamma_0$, so-called quasi-observables, into functions on $\Gamma$, so-called observables. The $\star$-convolution, a combinatorial convolution of quasi-observables, is mapped into a product under the $K$-transform, i.e., $K(G_1 \star G_2) = KG_1 \cdot KG_2$. Thus, the $K$-transform can be considered as a combinatorial Fourier transform for configuration space analysis. Via dualization, one obtains a transform $K^*$ mapping a probability measure on $\Gamma$ into a measure on $\Gamma_0$, the corresponding correlation measure. If the latter one is absolutely continuous w.r.t. the Lebesgue-Poisson measure, then its Radon-Nikodym derivative, or more precisely the corresponding system of densities w.r.t. the underlying intensity measure, is just the well-known system of correlation functions from statistical physics.

## Invariance principle for a diffusion in random environment

The first model studied in this work is the diffusive motion of a particle in a random environment consisting of a configuration of infinitely many other (frozen) particles. The interaction of the diffusing particle with the environment is described by a symmetric pair potential $V_I$, which may be of Lennard-Jones type. The randomness of the environment is given by a Ruelle measure $\mu_E$ w.r.t a symmetric pair potential $V_E$ (not necessarily coinciding with $V_I$). In physical language, this means that the diffusing particle and the environment particles may be of different nature.

We apply a general scheme (see [DFGW89]) to this model to prove an invariance principle for the corresponding process. We recall this framework and the necessary conditions for its application in Section 2.1. This also includes the precise definition of an invariance principle in the first subsection as well as the most famous example, namely Donsker's invariance principle, i.e., convergence of a simple random walk to a standard Brownian motion. The aforementioned approach applies to a process $(X_t)_{t\geq 0}$, which can be written in the so-called standard decomposition consisting of a square-integrable martingale and an integral over the mean forward velocity of a Markov process. In applications, this Markov process is the environment process of $(X_t)_{t\geq 0}$.

In Section 2.2, we give the precise definition of the studied model in terms of the corresponding stochastic differential equation. Usually, one would solve this equation for a fixed environment and any possible starting point, thus get the corresponding diffusion process $(X_t)_{t\geq 0}$, and then obtain the environment process by shifting the environment by $-X_t$, $t \geq 0$. But we will construct the environment process directly (see Subsection 2.3.3) as a process on the configuration space and then obtain the one-particle process via the standard decomposition. But for the sake of completeness, we have included some known results on weak and even strong solutions of the stochastic differential equation (see [KKR04, KR05]) in this section.

In Section 2.3, we construct the environment process using the general theory of Dirichlet forms, cf., e.g., [MR92]. The pre-Dirichlet form associated to this process has the following representation

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(F,G) := \int_\Gamma \big(\mathbb{D}F(\gamma), \mathbb{D}G(\gamma)\big)\, d\mu^*(\gamma), \quad F, G \in \mathcal{F}C_b^\infty.$$

Here, $\mu^*$ is absolutely continuous w.r.t the Ruelle measure $\mu_E$ and the density expression includes the interaction $V_I$. Via an integration by parts formula, we obtain the corresponding pre-generator and show that the form is closable and its closure is a symmetric, conservative Dirichlet form. Afterwards, we prove quasi-regularity of the form on the bigger space $\ddot{\Gamma}$ of multiple configurations as well as locality. Thus, by the general theory, we obtain the existence of a corresponding diffusion on $\ddot{\Gamma}$. By proving that the set $\ddot{\Gamma} \setminus \Gamma$ is exceptional for this Dirichlet form, we show that the diffusion is, in fact, supported by the smaller space $\Gamma$.

In the last section we show how to apply the general approach in the considered model. We get the diffusion in random environment via the standard decomposition and obtain an invariance principle for this process. To this end, we also have to assume ergodicity of the environment process. We recall a general result on ergodicity of processes in terms of irreducibility of the corresponding Dirichlet forms.

All results in this section are valid for a large class of interaction potentials including the Lennard-Jones potential.

## Invariance principle for a tagged particle process

In Chapter 3, an invariance principle for another model is discussed, namely for a tagged particle process. Consider an equilibrium diffusion of infinitely many (indistinguishable) particles interacting with each other via a symmetric pair potential. Then the tagged particle process describes the motion of one of these particles in the environment of the others. We explain this model with more details in terms of a system of stochastic differential equations at the beginning of Chapter 3.

In a recent article by T. Fattler and M. Grothaus [FG08], a tagged particle process was constructed rigorously for a wide class of interaction potentials including Lennard-Jones type ones. They used a Dirichlet form approach similar to the one in the previous chapter. In particular, the corresponding environment process was constructed directly. We will recall their construction results in Section 3.1. In particular, we will explain in all details, how to construct the environment process, to apply the invariance principle scheme later.

Afterwards in Section 3.2 we will prove the remaining conditions to apply the general scheme for invariance principles explained in Section 2.1. As well as the model in Chapter 1, also this model had been treated in [DFGW89] only on a heuristical level and under strong assumptions on the interaction. Here, the invariance principle is proven for general potetials like Lennard-Jones, and the proof is done in all necessary rigor.

## Continuous contact model with jumps

In the final chapter, we will discuss a continuous contact model with jumps.

The lattice version of the contact model is well-studied. The name is due to its interpretation as a model for infection spreading. Namely, consider the lattice $\mathbb{Z}^d$ with the abovementioned graph structure and introduce a spin $\sigma(x) \in \{0, 1\}$ for each vertex $x \in \mathbb{Z}^d$. The interpretation is, that the vertices represent the individuals of a society or a population and the edges of the lattice stand for contacts between the individuals. A spin $\sigma(x) = 0$ stands for a healthy individual $x$, and $\sigma(x) = 1$ says that $x$ is infected with a certain disease. The dynamics then is the following: infected individuals become healthy after an exponentially distributed random time with fixed rate, and healty ones become infected with a rate proportional to the number of infected neighbors. Another interpretation of this model is a birth-and-death dynamics on the lattice. Here $\sigma(x) = 0, 1$ represent free sites and those ones occupied by individuals, resp. Then individuals die ("become healthy") with a fixed rate or produce offspring on a free neighbor site with a certain rate. Note that this is a model of biological evolution, since new individuals only arise from existing ones.

In recent years, a continuous version of the contact model has been developed, see, e.g., [KS06, GK06], where individuals do not only live on lattice points but on the whole continuum $\mathbb{R}^d$. More precisely, the individuals are represented by the points of a configuration $\gamma \in \Gamma$. This case is essentially different from the lattice case. Namely, since there is no notion of neighbors anymore, one has to change the mechanism of giving birth to new individuals. This is done by introducing a probability distribution $a(x)\,dx$, $a \geq 0$, even, with $\langle a \rangle := \int a(x)dx = 1$. Thus, an individual at point $x$ produces offspring with a fixed rate $\varkappa > 0$ and spatially distributed w.r.t. $a(x-y)\,dy$. On the other hand, this mechanism simplifies the model compared to the lattice case, since the problem of free neighbor sites for the offspring disappears. Namely, for a given configuration $\gamma \in \Gamma$ of individuals, almost every (w.r.t. the Lebesgue measure) points in $\mathbb{R}^d$ are free.

Note that, in the contact model, the motion of individuals is not considered. The model only applies to immobile individuals like plants. In this work we modify the continuous contact model by allowing motion in the form of jumps of individuals.

Individual based models on the configuration space can be described in terms of corresponding (formal) generators describing the mechanism of the dynamics. The one for the continuous contact model has the form

$$(L_{\mathrm{C}}F)(\gamma) := \sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$$

for proper functions $F$ on $\Gamma$. Here $\varkappa$ and $a$ are as above. The first summand describes the death of individuals (the model is normalized to a death rate equal to 1), and the second one the birth mechanism. In the model studied in this work, we will add a jump generator to $L_C$, the generator of a so-called free Kawasaki dynamics. So, the continuous contact model with jumps is described by

$$(L_{\mathrm{CJ}}F)(\gamma) := (L_{\mathrm{C}}F)(\gamma) + \sum_{y\in\gamma}\int_{\mathbb{R}^d} w(x-y)[F(\gamma\setminus y\cup x) - F(\gamma)]\,dx.$$

Here $w > 0$ is an even density function, but not necessarily normalized. In particular, the usual contact model ($w = 0$) is included in our considerations.

In the second section of Chapter 4, we will construct a continuous contact model for a given initial configuration from a certain set of admissible ones. The basic assumption is a fast enough decay of the birth and the jump distributions $a, w$ at infinity, namely we assume a decay faster than polynomially, see Theorem 4.2.2.

In Section 4.3, the model is studied in terms of corresponding correlation functions of the states. Via the framework of harmonic analysis, the mechanism of the dynamics is transported to the level of correlation functionals and correlation functions. This gives a Focker-Planck equation and, for a given initial system of functions $(k_0^{(n)})_n$ at time zero, a corresponding Cauchy problem, which we will solve, see Proposition 4.3. We will prove a priori bounds for the solution, namely if the initial system has a factorial growth, i.e., $k_0^{(n)} \leq C^n n!$ for some $C > 0$, then this property is preserved under time evolution (but with a constant $C(t)$ depending on the time). With the help of this, we can also prove, that if the initial system is a system of correlation functions of a measure $\mu_0$ on $\Gamma$, then for any $t > 0$, the solution $(k_t^{(n)})_n$ is a system of correlation functions as well for some measure $\mu_t$. Note that this is not a priori clear. With this we obtain a time evolution of the state of the dynamics and a corresponding Markov function $X_t^\mu$, see Theorem 4.3.

In Section 4.4, in the translation invariant, critical (i.e., $\varkappa = 1$) case, for dimension $d \geq 2$, we prove the existence of a continuum of invariant measures for the model parametrized by the corresponding density $\rho > 0$, i.e., the first correlation function, under some conditions on $a, w$. Given an initial state, we prove the convergence of the corresponding non-equlibrium evolution of states to the equilibrium measure with the density of the initial state. (See Theorem 4.4.2 for both results.) This theorem is the main result in this chapter. Note that, in the case of the usual contact model, a similar result was proven by Yu. Kondratiev, O. Kutoviy, and S. Pirogov [KKP08], but their result is only valid for $d \geq 3$, not in the biologically important case $d = 2$. We show that, by allowing long jumps for the individuals, one obtains invariant measures also in the latter case.

In the final section we discuss a result on clustering of the system in the subcritical case $\varkappa < 1$.

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# Chapter 1

# Configuration Spaces

In this chapter we will recall the basic definitions and results about continuous configurations.

## 1.1 General notations

Throughout this work will use the following notations:

For a topological space $X$ we will use the following notations:

$\mathcal{O}(X)$: the set of all open subsets of $X$;

$\mathcal{B}(X)$: the corresponding Borel $\sigma$-algebra on $X$;

$\mathcal{O}_c(X)$, $\mathcal{B}_c(X)$: all open resp. Borel-measurable sets in $X$ with compact closure;

$L^0(X)$: all (Borel-)measurable functions on $X$;

$B(X)$: the set of all bounded measurable functions on $X$;

$C(X)$: the set of all continuous functions on $X$;

$C_0(X)$: the set of all continuous functions on $X$ with compact support.

On the space $\mathbb{R}^d$, we will denote the Euclidean norm by $|\cdot|$ and the corresponding inner product by $(\cdot,\cdot)$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by $dx$.

## 1.2 The space of configurations

One can define the configuration space $\Gamma(X)$ over a general locally compact, Polish space. Here we will just recall the case $X = \mathbb{R}^d$. For the general case and further details we refer, e.g., to [AKR98a].

For $\Lambda \subset \mathbb{R}^d$ let $\Gamma(\Lambda)$ denote the set of all locally finite subsets of $\Lambda$, i.e.,

$$\Gamma(\Lambda) := \{\gamma \subset \Lambda \,:\, |\gamma \cap K| < \infty \text{ for all } K \text{ compact}\}, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality. The sets $\gamma \in \Gamma(\Lambda)$ are called *configurations*. $\Gamma(\Lambda)$ is called the *(continuous) configuration space* over $\Lambda$.

Furthermore, for $n \in \mathbb{N}$, set

$$\Gamma_0^{(n)}(\Lambda) := \{\gamma \in \Gamma(\Lambda) \,:\, |\gamma| = n\}$$

and

$$\Gamma_0^{(0)}(\Lambda) := \{\emptyset\}.$$

Then

$$\begin{aligned}\Gamma_0(\Lambda) :=& \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}(\Lambda) \\ =&\{\gamma \in \Gamma(\Lambda) \,:\, |\gamma| < \infty\}\end{aligned} \tag{1.2}$$

denotes the *space of finite configurations* over $\Lambda$ .

We write $\Gamma := \Gamma(\mathbb{R}^d)$ and $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$.

For $\Lambda \subset \mathbb{R}^d$ let

$$\widetilde{\Lambda^n} := \{(x_1, \ldots, x_n) \in \Lambda^n \,:\, x_i \neq x_j \text{ if } i \neq j\} \subset (\mathbb{R}^d)^n.$$

Define the symmetrization mapping

$$\begin{aligned}\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} &\to \Gamma_0^{(n)}, \\ (x_1, \ldots, x_n) &\mapsto \{x_1, \ldots, x_n\}.\end{aligned} \tag{1.3}$$

This gives a bijection

$$\widetilde{\Lambda^n}/S_n \to \Gamma_0^{(n)}(\Lambda),$$

where $S_n$ denotes the permutation group over $\{1, \ldots, n\}$, and hence induces a topology $\mathcal{O}(\Gamma_0^{(n)}(\Lambda))$. The space $\Gamma_0(\Lambda)$ is then equipped with the topology $\mathcal{O}(\Gamma_0(\Lambda))$ of the disjoint union of the spaces $\big(\Gamma_0^{(n)}(\Lambda), \mathcal{O}(\Gamma_0^{(n)}(\Lambda))\big)$, $n \geq 0$.

The space $\Gamma(\Lambda)$ can be considered as a subset of the space of all positive Radon measures over $\mathbb{R}^d$ via the following identification:

$$\gamma \equiv \sum_{x\in\gamma} \delta_x, \tag{1.4}$$

where $\delta_x$ denotes the Dirac measure with mass in $x$. $\emptyset$ is identified with the zero-measure. Thus, $\Gamma(\Lambda)$ is topologized by the vague topology on the space of Radon measures, i.e., the weakest topology such that all mappings

$$\Gamma(\Lambda) \ni \gamma \mapsto \langle f, \gamma\rangle := \int_\Lambda f(x)\, d\gamma(x) = \sum_{x\in\gamma} f(\gamma), \quad f \in C_0(\Lambda),$$

are continuous. In the following, we will use the identification (1.4) without further notice and the definition

$$\langle f, \gamma\rangle := \int_\Lambda f(x)\, d\gamma(x) = \sum_{x\in\gamma} f(\gamma)$$

for all functions $f$ for which it makes sense.

**Remark 1.2.1.** One can show that $\Gamma$ with the vague topology can be metrized in such a way that it becomes a complete, separable metric space, i.e., $\Gamma$ is a Polish space (cf., e.g., [Kut03, Remark 3.11]).

The space $\ddot{\Gamma}$ of *multiple configurations* is defined as the set of all locally finite Radon measures taking values in $\mathbb{N}\cup\{0,+\infty\}$, equipped with the vague topology as well. We have $\Gamma \subset \ddot{\Gamma}$ since

$$\Gamma = \{\gamma \in \ddot{\Gamma} \,:\, \gamma(\{x\}) \le 1 \text{ for all } x \in \mathbb{R}^d\}.$$

For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ define

$$N_\Lambda(\gamma) := \gamma(\Lambda), \quad \gamma \in \ddot{\Gamma}, \tag{1.5}$$

then

$$\mathcal{B}(\ddot{\Gamma}) = \{N_\Lambda \,:\, \Lambda \in \mathcal{O}_c(\mathbb{R}^d)\}, \tag{1.6}$$

$$\mathcal{B}(\Gamma) = \{N_\Lambda \restriction_\Gamma :\, \Lambda \in \mathcal{O}_c(\mathbb{R}^d)\}. \tag{1.7}$$

For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, set

$$\mathcal{B}_\Lambda(\ddot{\Gamma}) := \{N_{\Lambda'} \,:\, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d), \Lambda' \subset \Lambda\}, \tag{1.8}$$

$$\mathcal{B}_\Lambda(\Gamma) := \{N_{\Lambda'} \restriction_\Gamma :\, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d), \Lambda' \subset \Lambda\}. \tag{1.9}$$

For $\Lambda \subset \mathbb{R}^d$ and $\gamma \in \ddot{\Gamma}$, we will write $\gamma_\Lambda := \gamma \restriction_\Lambda$ ($\equiv \gamma \cap \Lambda$ if $\gamma \in \Gamma$).

## 1.3 Lebesgue-Poisson and Poisson measure

The basic measures on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ and $(\Gamma, \mathcal{B}(\Gamma)$ are the Lebesgue-Poisson measure and the Poisson measure, respectively.

Fix a non-atomic, locally finite measure $\sigma > 0$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and a constant $z > 0$. $\sigma$ is called *intensity measure*, and $z$ is called *activity parameter*.

The $n$-product measure $\sigma^{\otimes n}$ of $\sigma$ can be considered as a measure on $\widetilde{(\mathbb{R}^d)^n}$, since $\sigma^{\otimes n}\big((\mathbb{R}^d)^n \setminus \widetilde{(\mathbb{R}^d)^n}\big) = 0$. Then the *Lebesgue-Poisson measure* on $\Gamma_0$ is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{\otimes n} \circ \mathrm{sym}_n^{-1}. \tag{1.10}$$

Here $\sigma^{\otimes n} \circ \mathrm{sym}_n^{-1}$, $n \geq 1$, denotes the image measure of $\sigma^{\otimes n}$ under the mapping $\mathrm{sym}_n$ on $\Gamma_0^{(n)}$, and for $n = 0$ it is defined as $\delta_\emptyset$.

If $\sigma(dx) = dx$, we just write $\lambda_z$ instead of $\lambda_{z\,dx}$.

For $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ let

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma} \restriction_\Lambda .$$

Note that this defines a probability measure. The *Poisson measure* on $\Gamma$ is then defined as

$$\pi_{z\sigma} := \operatorname*{proj\,lim}_{\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)} \pi_{z\sigma}^\Lambda. \tag{1.11}$$

As above, we write $\pi_z := \pi_{z\,dx}$.

One can also characterize the Poisson measure via its Laplace transform. $\pi_{z\sigma}$ is the probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ which satisfies

$$\int_\Gamma \exp\left(\sum_{x\in\gamma} \varphi(x)\right) d\pi_{z\sigma}(\gamma) = \exp\left(z \int_{\mathbb{R}^d} (e^{\varphi(x)} - 1)\, d\sigma(x)\right) \tag{1.12}$$

for every $\varphi \in \mathcal{D}$, the Schwartz space of all infinitely differentiable functions with compact support.

The following result is useful in applications:

**Proposition 1.3.1** (Minlos-Lemma)**.** *Let $n \in \mathbb{N}$, $n \geq 2$, and $z > 0$ be given. Then*

$$\int_{\Gamma_0} \cdots \int_{\Gamma_0} G(\eta_1 \cup \cdots \cup \eta_n) H(\eta_1, \ldots, \eta_n)\, d\lambda_{z\sigma}(\eta_1) \cdots d\lambda_{z\sigma}(\eta_n)$$
$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1,\ldots,\eta_n)\in\mathcal{P}_\emptyset^n(\eta)} H(\eta_1, \ldots, \eta_n)\, d\lambda_{z\sigma}(\eta) \tag{1.13}$$

*for all measurable functions $G : \Gamma_0 \to \mathbb{R}$ and $H : \Gamma_0 \times \cdots \times \Gamma_0 \to \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}^n_\emptyset(\eta)$ denotes the set of all ordered partitions of $\eta$ in $n$ parts, which may be empty.*

For the proof we refer, e.g., to [Oli02].

Note that for a function $H : \Gamma_0 \times \mathbb{R}^d \to \mathbb{R}$ the Minlos-Lemma has the form

$$\int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta \cup x) H(\eta, x)\, z d\sigma(x)\, d\lambda_{z\sigma}(\eta) = \int_{\Gamma_0} G(\eta) \sum_{y \in \eta} H(\eta \setminus y, y)\, d\lambda_{z\sigma}(\eta). \quad (1.14)$$

## 1.4 Gibbs measures

Fix an intensity measure $\sigma$ on $\mathbb{R}^d$ and an activity parameter $z > 0$ as in Section 1.3.

Let $\Phi$ be a *potential*, i.e., a function $\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ such that $\Phi = \Phi \upharpoonright_{\Gamma_0}$, $\Phi(\emptyset) = 0$, and $\gamma \mapsto \Phi(\gamma_\Lambda)$ is $\mathcal{B}_\Lambda(\Gamma)$-measurable for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$.

### 1.4.1 Grand canonical and canonical Gibbs measures

For any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ the *conditional energy* $E^\Phi_\Lambda : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E^\Phi_\Lambda(\gamma) := \begin{cases} \sum_{\substack{\gamma' \subset \gamma, \\ \gamma'(\Lambda) > 0}} \Phi(\gamma'), & \text{if } \sum_{\substack{\gamma' \subset \gamma, \\ \gamma'(\Lambda) > 0}} |\Phi(\gamma')| < \infty, \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.15)$$

#### Grand canonical Gibbs measures

**Definition 1.4.1.** For any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, $\gamma \in \Gamma$ (*boundary condition*), and $\Delta \in \mathcal{B}(\Gamma)$ define the *grand canonical specification* as

$$\Pi^\Phi_\Lambda(\gamma, \Delta) := 1\!\!1_{Z^\Phi_\Lambda < \infty}(\gamma) (Z^\Phi_\Lambda(\gamma))^{-1} \int_\Gamma 1\!\!1_\Delta(\gamma'_\Lambda \cup \gamma_{\Lambda^c}) e^{-E^\Phi_\Lambda(\gamma'_\Lambda \cup \gamma_{\Lambda^c})}\, d\pi_{z\sigma}(\gamma'). \quad (1.16)$$

Here $\Lambda^c := \mathbb{R}^d \setminus \Lambda$, and

$$Z^\Phi_\Lambda(\gamma) := \int_\Gamma e^{-E^\Phi_\Lambda(\gamma'_\Lambda \cup \gamma_{\Lambda^c})}\, d\pi_{z\sigma}(\gamma')$$

denotes the corresponding *partition function*.

It is well-known (cf., e.g., [Pre76]) that $(\Pi_\Lambda^\Phi)_{\Lambda\in\mathcal{O}_c(\mathbb{R}^d)}$ is a *specification*, i.e., for all $\Lambda, \Lambda' \in \mathcal{O}_c(\mathbb{R}^d)$

(S1) $\Pi_\Lambda^\Phi(\gamma, \Gamma) \in \{0, 1\}$ for all $\gamma \in \Gamma$;

(S2) $\Pi_\Lambda^\Phi(\cdot, \Delta)$ is $\mathcal{B}_{\mathbb{R}^d\setminus\Lambda}(\Gamma)$-measurable for all $\Delta \in \mathcal{B}(\Gamma)$;

(S3) $\Pi_\Lambda^\Phi(\cdot, \Delta' \cap \Delta) = 1\!\!1_{\Delta'}\Pi_\Lambda^\Phi(\cdot, \Delta)$ for all $\Delta \in \mathcal{B}(\Gamma)$, $\Delta' \in \mathcal{B}_{\mathbb{R}^d\setminus\Lambda}(\Gamma)$;

(S4) $\Pi_{\Lambda'}^\Phi = \Pi_{\Lambda'}^\Phi \Pi_\Lambda^\Phi$ if $\Lambda \subset \Lambda'$.

Here for $\gamma \in \Gamma$, $\Delta \in \mathcal{B}(\Gamma)$

$$\big(\Pi_{\Lambda'}^\Phi \Pi_\Lambda^\Phi\big)(\gamma, \Delta) := \int_\Gamma \Pi_\Lambda^\Phi(\gamma', \Delta)\, \Pi_{\Lambda'}^\Phi(\gamma,\, d\gamma').$$

**Definition 1.4.2.** A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a *grand canonical Gibbs measure with interaction potential* $\Phi$ iff it satisfies the *Dobrushin-Lanford-Ruelle equations (DLR)*

$$\mu \Pi_\Lambda^\Phi = \mu \tag{1.17}$$

for all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. Here

$$\mu\Pi_\Lambda^\Phi(\Delta) := \int_\Gamma \Pi_\Lambda^\Phi(\gamma', \Delta)\, d\mu(\gamma').$$

Let $\mathcal{G}^{\mathrm{gc}}(z\sigma, \Phi)$ denote the set of all such measures $\mu$.

**Remark 1.4.3.** (i) We have $\mu(\{Z_\Lambda^{\sigma,\Phi} < \infty\}) = 1$ for all $\mu \in \mathcal{G}^{\mathrm{gc}}(\sigma, \Phi)$.

(ii) For a sub-$\sigma$-algebra $\Sigma \subset \mathcal{B}(\Gamma)$ and a probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ let $\mathbb{E}_\mu[\cdot \mid \Sigma]$ denote the conditional expectation with respect to $\Sigma$. For $G \in \mathcal{B}_b(\Gamma)$ (= bounded $\mathcal{B}(\Gamma)$-measurable functions) we set

$$(\Pi_\Lambda^\Phi G)(\gamma) := \int_\Gamma G(\gamma')\, \Pi_\Lambda^\Phi(\gamma,\, d\gamma').$$

Then a probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ is a grand canonical Gibbs measure if and only if for all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and all $G \in \mathcal{B}_b(\Gamma)$

$$\mathbb{E}_\mu[G \mid \mathcal{B}_{\Lambda^c}(\Gamma)] = \Pi_\Lambda^\Phi G \quad \mu\text{-a.e.}$$

(iii) We may always replace $\mathcal{O}_c(\mathbb{R}^d)$ by $\mathcal{B}_c(\mathbb{R}^d)$ without any changes. In particular, we obtain the same set $\mathcal{G}^{\mathrm{gc}}(z\sigma, \Phi)$.

**Canonical Gibbs measures**

**Definition 1.4.4.** For any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, $\gamma \in \Gamma$, and $\Delta \in \mathcal{B}(\Gamma)$ define the *canonical specification* as

$$\hat{\Pi}^{\Phi}_{\Lambda}(\gamma, \Delta) := \begin{cases} \frac{\Pi^{\Phi}_{\Lambda}(\gamma, \Delta \cap \{N_\Lambda = \gamma(\Lambda)\})}{\Pi^{\Phi}_{\Lambda}(\gamma, \{N_\Lambda = \gamma(\Lambda)\})}, & \text{if } \Pi^{\Phi}_{\Lambda}(\gamma, \{N_\Lambda = \gamma(\Lambda)\}) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.18}$$

Then analogously to grand canonical Gibbs measures, also canonical Gibbs measures are defined via the Dobrushin-Lanford-Ruelle equation:

**Definition 1.4.5.** A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a *canonical Gibbs measure with interaction potential* $\Phi$ iff it satisfies

$$\mu \hat{\Pi}^{\Phi}_{\Lambda} = \mu \tag{1.19}$$

for all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$. Let $\mathcal{G}^{c}(z\sigma, \Phi)$ denote the set of all such measures $\mu$.

**Remark 1.4.6.** (i) By [Pre79, Proposition 2.1] the following inclusion holds:

$$\mathcal{G}^{gc}(z\sigma, \Phi) \subset \mathcal{G}^{c}(z\sigma, \Phi). \tag{1.20}$$

(ii) $\mathcal{G}^{gc}(z\sigma, \Phi), \mathcal{G}^{c}(z\sigma, \Phi)$ are convex sets. Let $\mathrm{ex}\mathcal{G}^{gc}(z\sigma, \Phi), \mathrm{ex}\mathcal{G}^{c}(z\sigma, \Phi)$ denote their respective sets of extremal points. Then, e.g., by [Pre76, Theorem 2.2], it holds that

$$\mathrm{ex}\mathcal{G}^{gc}(z\sigma, \Phi) \neq \emptyset, \quad \mathrm{ex}\mathcal{G}^{c}(z\sigma, \Phi) \neq \emptyset,$$

provided

$$\mathcal{G}^{gc}(z\sigma, \Phi) \neq \emptyset, \quad \mathcal{G}^{c}(z\sigma, \Phi) \neq \emptyset, \quad \text{resp},$$

and that any canonical or grand canonical $\mu$ has an integral representation in terms of the respective extremal Gibbs measures.

### 1.4.2 Gibbs measures for pair potentials

A special class of potentials are pair potentials describing the interaction of two particles.

**Definition 1.4.7.** A *pair potential* is a Lebesgue-measurable, even function $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, i.e., $V(x) = V(-x)$, $x \in \mathbb{R}^d$.

Let $V$ be a pair potential. $V$ defines a potential via

$$\Phi_V(\gamma) := \begin{cases} V(x-y), & \text{if } \gamma = \{x, y\}, \\ 0, & \text{if } |\gamma| \neq 2. \end{cases}$$

Let $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$. It is useful, to decompose the conditional energy $E_\Lambda^V := E_\Lambda^{\Phi_V}$ in the following way:

$$E_\Lambda^V(\gamma) = E_\Lambda^V(\gamma_\Lambda) + W(\gamma_\Lambda \mid \gamma_{\Lambda^c}), \tag{1.21}$$

where

$$W^V(\gamma \mid \gamma') := \begin{cases} \sum_{\substack{x\in\gamma \\ y\in\gamma'}} V(x-y), & \text{if } \sum_{\substack{x\in\gamma \\ y\in\gamma'}} |V(x-y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

$\gamma, \gamma' \in \Gamma$, denotes the *interaction energy* of $\gamma$ and $\gamma'$. (Usually, we have $\gamma \cap \gamma' = \emptyset$.)

For $x \in \mathbb{R}^d$ and $\gamma \in \Gamma$ we call

$$E^V(x, \gamma) := W^V(\{x\} \mid \gamma)$$

the *relative energy* of a particle at $x$ w.r.t. $\gamma$.

Consider an intensity measure $\sigma$ and an activity parameter $z > 0$. We write

$$\mathcal{G}^{\mathrm{gc}}(z\sigma, V), \quad \mathcal{G}^{\mathrm{c}}(z\sigma, V),$$

for the Gibbs measures w.r.t. the corresponding potential $\Phi_V$. It is well-known (cf., e.g., [KK03, Theorem 3.12]), that the grand canonical Gibbs measures can be characterized by the following identity:

**Proposition 1.4.8.** *Let $\mu$ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. Then $\mu \in \mathcal{G}^{gc}(z\sigma, V)$ if and only if it satisfies*

$$\int_\Gamma \sum_{x\in\gamma} H(x, \gamma)\, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} H(x, \gamma \cup x) e^{-E^V(x,\gamma)} z\, d\sigma(x)\, d\mu(\gamma) \tag{1.22}$$

*for any positive $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Gamma)$-measurable function $H$.* (1.22) *is called the* Georgii-Nguyen-Zessin identity (GNZ)*. In the free case ($V \equiv 0$), it is known as* Mecke identity *and holds only for the Poisson measure $\pi_{z\sigma}$.*

### Standard properties of pair potentials and existence of corresponding grand canonical Gibbs measures

In Definition 1.4.9 below, we will formulate some standard conditions on the potential $V$ that ensure the existence of corresponding Gibbs measures (for appropriate intensity measures $\sigma$). Therefore, we introduce the following notations.

For $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ define the cube

$$Q_r := \{x \in \mathbb{R}^d \mid r_i - \tfrac{1}{2} \le x_i < r_i + \tfrac{1}{2}\}.$$

Furthermore, set $\Lambda_N := [-N + \frac{1}{2}, N - \frac{1}{2})^d$.

Let $\Omega_1 := B_1(0)$ and $\Omega_n := B_n(0) \setminus B_{n-1}(0)$, $n \ge 2$, where $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ denotes the open ball with center in $x \in \mathbb{R}^d$ and radius $r > 0$. Set

$$\Gamma_{\mathrm{fd}}(\{\Omega_n\}_n) := \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \{\gamma \in \Gamma \,:\, |\gamma_{\Omega_n}| \le M\sigma(\Omega_n)\}, \tag{1.23}$$

the set of *configurations of finite density.* A probability measure $\mu$ on $\Gamma$ with

$$\mu\big(\Gamma_{\mathrm{fd}}(\{\Omega_n\}_n)\big) = 1$$

is called *tempered.*

**Definition 1.4.9.** For a pair potential $V$ define the following properties:

(S) (*Stability*) There exists $B \ge 0$ such that for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma(\Lambda)$

$$E^V_\Lambda(\gamma) \ge -B\,|\gamma|\,. \tag{1.24}$$

(SS) (*Superstability*) There exist $A > 0$, $B \ge 0$ such that if $\gamma \in \Gamma(\Lambda_N)$ for some $N$ then

$$E^V_{\Lambda_N}(\gamma) \ge \sum_{r \in \mathbb{Z}^d} (A\,|\gamma_{Q_r}|^2 - B\,|\gamma_{Q_r}|). \tag{1.25}$$

(LR) (*Lower regularity*) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\|r\|_\infty) < \infty$$

and for any disjoint $\Lambda', \Lambda''$, which are finite unions of cubes of the form $Q_r$, and any $\gamma' \in \Gamma(\Lambda'), \gamma'' \in \Gamma(\Lambda'')$

$$W^V(\gamma' \mid \gamma'') \ge -\sum_{r', r'' \in \mathbb{Z}^d} a(\|r' - r''\|_\infty) \left|\gamma'_{Q_{r'}}\right| \left|\gamma''_{Q_{r''}}\right|. \tag{1.26}$$

(Here $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^d$.)

(I) (*Integrability*)

$$\int_{\mathbb{R}^d} \left|1 - e^{-V(x)}\right| dx < \infty. \tag{1.27}$$

(D) (*Differentiability*) The function $e^{-V}$ is weakly differentiable on $\mathbb{R}^d$, $V$ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$. The gradient $\nabla V$, considered as a Lebesgue-a.e. defined function on $\mathbb{R}^d$, satisfies

$$|\nabla V| \in L^1(\mathbb{R}^d, e^{-V(x)}\, dx) \cap L^2(\mathbb{R}^d, e^{-V(x)}\, dx).$$

(LS) (*Local summability*) Suppose that $\sigma(\Omega_n) \geq \kappa(n+1)$ for some $\kappa > 0$ and all $n \in \mathbb{N}$. For all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and all $\gamma \in \Gamma_{\mathrm{fd}}(\{\Omega_n\}_n)$ it holds that

$$\lim_{n\to\infty} \sum_{y \in \gamma_{B_n(0)\setminus\Lambda}} \nabla V(\cdot - y) \text{ exists in } L^1_{\mathrm{loc}}(\Lambda, \sigma).$$

**Remark 1.4.10.** (i) (SS) implies (S), and (S) implies that $V$ is bounded from below.

(ii) In applications, one often has $V \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Nevertheless, condition (D) does not exclude a singularity at point 0 here. (A singularity at zero reflects repulsion of particles at small distances.)

**Example 1.4.11.** A concrete example of a potential satisfying all conditions (SS), (I), (LR), (D), and (LS) (see [AKR98b, Example 4.1]) is the well-known *Lennard-Jones potential* $V_{\mathrm{LJ}}$ from atomic and molecular physics. Let $d = 3$. For fixed parameters $a, b > 0$

$$V_{\mathrm{LJ}}(x) := \frac{a}{|x|^{12}} - \frac{b}{|x|^6}, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{1.28}$$

Note that $V_{\mathrm{LJ}}$ has a singularity at the point 0, a negative part, and an infinite range, i.e., its support is not compact.

Consider now the case $\sigma(dx) := \rho(x)\, dx$ with a bounded, nonnegative density $\rho$. Fix a pair potential $V$ and an activity parameter $z > 0$.

Then the following result holds, see [FG08, Theorem 2.12] and, for the case $\rho \equiv 1$, [Rue70]. Let $\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z\rho, V)$ denote the set of grand canonical Gibbs measures which have correlation functions, that satisfy a Ruelle bound, cf. Subsection 1.6.2 below.

**Theorem 1.4.12.** *Suppose that $V$ satisfies (SS), (I) and (LR). Then*

$$\mathcal{G}^{gc}_{Rb}(z\rho, V) \neq \emptyset.$$

In the case $\sigma(\Omega_n) \geq \kappa(n+1)$ as in the definition of (LS), one has that every $\mu \in \mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z\rho, V)$ is tempered. In the case $\rho \equiv 1$, those Gibbs measures are also called *Ruelle measures*. We will just write $\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z, V)$ for Ruelle measures.

## 1.5 Geometry on configuration spaces

In this section, we will discuss two types of geometries on $\Gamma$ resp. the corresponding gradients. First we will recall the definition of the intrinsic gradient. For further details we refer to [AKR98a, AKR98b]. Afterwards, we will discuss the shift gradient.

The gradients (and later the corresponding pre-Dirichlet forms, see Sections 2.3.1, 3.1) are defined for bounded smooth cylinder functions.

**Definition 1.5.1.** The bounded smooth *cylinder functions*

$$\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(C_0^\infty(\mathbb{R}^d), \Gamma)$$

on $\Gamma$ are defined as all functions $F : \Gamma \to \mathbb{R}^d$ having a (non-unique!) representation of the form

$$F(\gamma) = g_F\big(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle\big), \quad \gamma \in \Gamma, \tag{1.29}$$

with $N \in \mathbb{N}$, $f_1, \ldots, f_N \in C_0^\infty(\mathbb{R}^d)$ (smooth functions with compact support), and $g_F \in C_b^\infty(\mathbb{R}^N)$ (smooth functions with bounded derivatives of all orders).

Since the functions $f_j$, $j = 1, \ldots, N$, have compact supports, $F(\gamma) = F(\gamma \cap \Lambda)$, $\Lambda := \bigcup_j \operatorname{supp}(f_j)$, for any $\gamma \in \Gamma$. This justifies the notion of cylinder functions. Note, that $\mathcal{F}C_b^\infty$ is an algebra of functions.

### 1.5.1 The intrinsic gradient

Consider the group $\operatorname{Diff}_0(\mathbb{R}^d)$ of diffeomorphisms of $\mathbb{R}^d$ which coincide with the identity outside of some compact set. Any $\psi \in \operatorname{Diff}_0(\mathbb{R}^d)$ defines a transformation on $\Gamma$ via

$$\gamma \mapsto \psi(\gamma) := \{\psi(y) \mid y \in \gamma\}.$$

Let $v \in V_0(\mathbb{R}^d)$, i.e., $v$ is a smooth vector field on $\mathbb{R}^d$ with compact support, and let $\psi_t^v$, $t \in \mathbb{R}$, be the corresponding flow, i.e., $\frac{d}{dt}\psi_t^v(x) = v(\psi_t^v(x))$, $\psi_0^v(x) = x$. Thus, for $v \in V_0(\mathbb{R}^d)$ and for any $\gamma \in \Gamma$ we obtain a curve in $\Gamma$ via

$$t \mapsto \psi_t^v(\gamma) = \{\psi_t^v(y) \mid y \in \gamma\}, \quad t \in \mathbb{R}.$$

Then the *directional derivative of* $F \in \mathcal{F}C_b^\infty$ *along the vector field* $v \in V_0(\mathbb{R}^d)$ is defined as

$$(\nabla_v^\Gamma F)(\gamma) := \frac{d}{dt} F(\psi_t^v(\gamma))\Big|_{t=0}. \tag{1.30}$$

For $F = g_F\big(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle\big)$, this gives

$$(\nabla_v^\Gamma F)(\gamma) = \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle\big)\langle \nabla_v f_j, \gamma\rangle$$

Here, $\nabla_v$ denotes the usual directional derivative along the vector field $v$ on $\mathbb{R}^d$, i.e.,

$$(\nabla_v f)(x) := (\nabla f(x), v(x)), \quad f \in C_0^\infty(\mathbb{R}^d), x \in \mathbb{R}^d.$$

**Definition 1.5.2.** The *tangent space* $T_\gamma\Gamma$ of the configuration space $\Gamma$ at the point $\gamma \in \Gamma$ is defined as the Hilbert space $L^2(\mathbb{R}^d \to \mathbb{R}^d, \gamma)$ of measurable, $\gamma$-square-integrable vector fields $V_\gamma : \mathbb{R}^d \to \mathbb{R}^d$ with the inner product

$$(V_\gamma^1, V_\gamma^2)_{T_\gamma\Gamma} := \int (V_\gamma^1(x), V_\gamma^2(x))\, d\gamma(x) \tag{1.31}$$

The corresponding tangent bundle is given by

$$T\Gamma := \bigcup_{\gamma\in\Gamma} T_\gamma\Gamma. \tag{1.32}$$

Note that any $v \in V_0(\mathbb{R}^d)$ can be considered as a "constant" vector field on $\Gamma$ via

$$\Gamma \ni \gamma \mapsto V_\gamma(\cdot) := v(\cdot) \in T_\gamma\Gamma, \tag{1.33}$$

since

$$(v, v)_{T_\gamma\Gamma} = \int_{\mathbb{R}^d} |v(x)|^2 \, d\gamma(x) = \sum_{y\in\gamma} |v(y)|^2 < \infty.$$

**Definition 1.5.3.** The *intrinsic gradient* $\nabla^\Gamma F$ of a function $F \in \mathcal{F}C_b^\infty$ is defined as the mapping

$$\Gamma \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma\Gamma$$

such that for any $v \in V_0(\mathbb{R}^d)$

$$(\nabla_v^\Gamma F)(\gamma) = \big((\nabla^\Gamma F)(\gamma), v\big)_{T_\gamma\Gamma}. \tag{1.34}$$

This leads to

$$\nabla^\Gamma F(\gamma) = \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle\big)\nabla f_j.$$

Note that $\nabla^\Gamma F(\gamma) \in V_0(\mathbb{R}^d)$.

### 1.5.2 The shift gradient

We want to define a second gradient based on shifts of configurations in space. To do this, for $x_0 \in \mathbb{R}^d$, define the *space shift* by $x_0$ on $\Gamma$ as

$$\Theta_{x_0} : \Gamma \to \Gamma,$$
$$\gamma \mapsto \gamma + x_0 := \{y + x_0 \mid y \in \gamma\}.$$

**Definition 1.5.4.** For $F : \Gamma \to \mathbb{R}$ and $h \in \mathbb{R}^d$, $h \neq 0$, we define the *shift-directional derivative of $F$ in direction $h$* via

$$\mathbb{D}_h F(\gamma) := \frac{d}{dt} F(\Theta_{th}\gamma)\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\big(F(\gamma + th) - F(\gamma)\big), \tag{1.35}$$

provided the right hand side is well-defined.

For $F \in \mathcal{F}C_b^\infty$, $F = g\big(\langle f_1, \cdot\rangle, \ldots, \langle f_N, \cdot\rangle\big)$, we obtain by the chain rule

$$\mathbb{D}_h F(\gamma) = \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle\big)\langle \nabla_h f_j, \gamma\rangle.$$

So, the shift-directional derivative coincides with the intrinsic directional derivative on $\Gamma$ along a vector field $v \in V_0(\mathbb{R}^d)$ with $v \equiv h$ on a neighborhood of the supports of the functions $f_i$, $1 \leq i \leq n$.

Define the tangent space at point $\gamma \in \Gamma$ corresponding to $\mathbb{D}$ by $T_\gamma^{\mathbb{D}}\Gamma := \mathbb{R}^d$.

**Definition 1.5.5.** We define the *shift-gradient* $\mathbb{D}F$ of a function $F : \Gamma \to \mathbb{R}$ as the mapping

$$\Gamma \ni \gamma \mapsto (\mathbb{D}F)(\gamma) \in \mathbb{R}^d$$

such that for any $h \in \mathbb{R}^d \setminus \{0\}$

$$(\mathbb{D}_h F)(\gamma) = \big((\mathbb{D}F)(\gamma), h\big). \tag{1.36}$$

For a vector field $v \in V_0(\mathbb{R}^d)$, define (analogously as for functions $f \in C_0^\infty(\mathbb{R}^d)$)

$$\langle v, \gamma\rangle := \int_{\mathbb{R}^d} v(x)\, d\gamma(x) = \sum_{y\in\gamma} v(y). \tag{1.37}$$

Then, for $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\mathbb{D}\langle f, \cdot\rangle(\gamma) = \langle \nabla f, \gamma\rangle,$$

and for $F \in \mathcal{F}C_b^\infty$ we obtain

$$\mathbb{D}F(\gamma) = \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle\big)\langle \nabla f_j, \gamma\rangle \in \mathbb{R}^d. \tag{1.38}$$

hence

$$\mathbb{D}F(\gamma) = \langle \nabla^\Gamma F(\gamma), \gamma\rangle. \tag{1.39}$$

## 1.6 Harmonic analysis and $K$-transform

In this section we will recall some facts from harmonic analysis on configuration spaces, cf. [KK02].

We will use the following notations:

$\mathcal{F}L^0(\Gamma)$: the set of *cylinder functions* on $\Gamma$, i.e., all measurable functions $F$ on $\Gamma$ with

$$F(\gamma) = F \restriction_{\Gamma(\Lambda)} (\gamma_\Lambda)$$

for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$;

$L^0_{\mathrm{ls}}(\Gamma_0)$: all measurable functions on $\Gamma_0$ with *local support*, i.e., all measurable functions $G$ with

$$G \restriction_{\Gamma_0 \setminus \Gamma_0(\Lambda)} = 0$$

for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$;

$L^0_{\mathrm{bs}}(\Gamma_0)$: all measurable functions on $\Gamma_0$ with *bounded support*, i.e., all measurable functions $G$ with

$$G \restriction_{\Gamma_0 \setminus \bigsqcup_{n=0}^{N} \Gamma_0^{(n)}(\Lambda)} = 0$$

for some $N \in \mathbb{N}$ and some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$;

$B_{\mathrm{bs}}(\Gamma_0)$: all bounded measurable functions on $\Gamma_0$ with bounded support;

$\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$: all probability measures on $\Gamma$ with *finite local moments* of all orders, i.e., all $\mu$ with

$$\int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < +\infty \quad \forall \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \geq 0;$$

$\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ : all *locally finite* measures on $\Gamma_0$, i.e., all $\rho$ with $\rho(A) < +\infty$ for all bounded sets $A$ from $\mathcal{B}(\Gamma_0)$.

### 1.6.1 The $K$-transform

Functions on $\Gamma$, $\Gamma_0$ are also called *observables*, *quasi-observables*, respectively. The $K$*-transform* maps quasi-observables into observables:

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad G \in L^0_{\mathrm{ls}}(\Gamma_0), \gamma \in \Gamma. \tag{1.40}$$

Here $\xi \Subset \gamma$ means that $\xi$ is a finite subset of $\gamma$. Note that $KG \in \mathcal{F}L^0(\Gamma)$ for every $G \in L^0_{\mathrm{ls}}(\Gamma_0)$.

The $K$-transform is linear, positivity-preserving, and invertible with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad F \in \mathcal{F}L^0(\Gamma), \eta \in \Gamma_0. \tag{1.41}$$

One can introduce a convolution of quasi-observables

$$(G_1 \star G_2)(\eta) := \sum_{(\xi_1,\xi_2,\xi_3) \in \mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2) G_2(\xi_2 \cup \xi_3), \quad G_1, G_2 \in L^0(\Gamma_0), \eta \in \Gamma_0. \tag{1.42}$$

Here $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions of $\eta$ into three (not necessarily non-empty) subsets. Under the $K$-transform, this convolution is mapped into a product, i.e.,

$$K(G_1 \star G_2)(\eta) = KG_1(\eta) \cdot KG_2(\eta) \quad \forall G_1, G_2 \in L^0(\Gamma_0), \eta \in \Gamma_0.$$

Therefore, the $K$-transform can be considered as a Fourier transform in configuration space analysis.

### 1.6.2 Correlation measure and correlation functions

The transformation $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the $K$-transform, is defined via

$$\int_\Gamma KG(\gamma)\, d\mu(\gamma) = \int_{\Gamma_0} G(\eta)\, d(K^*\mu)(\eta), \quad \forall G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0), \mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma). \tag{1.43}$$

The measure $\rho_\mu := K^*\mu$ is called the *correlation measure* of $\mu$. It has been shown in [KK02] that for any $G \in L^1(\Gamma_0, \rho_\mu)$ the series

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi)$$

is $\mu$-a.s. absolutely convergent, $KG \in L^1(\Gamma, \mu)$, and (1.43) still holds.

**Example 1.6.1.** The correlation measure of the Poisson measure $\pi_{z\sigma}$ is the Lebesgue-Poisson measure $\lambda_{z\sigma}$.

Assume that for some $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ the corresponding correlation measure $\rho_\mu$ is absolutely continuous w.r.t. the Lebesgue-Poisson measure $\lambda_{z\sigma}$ with Radon-Nikodym derivative

$$k_\mu : \Gamma_0 \to \mathbb{R}_+,$$
$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_{z\sigma}}(\eta), \quad \eta \in \Gamma_0.$$

$k_\mu$ is called *correlation functional* of $\mu$ The corresponding functions

$$k^{(n)}_\mu : (\mathbb{R}^d)^n \to \mathbb{R}_+, \quad n \in \mathbb{N}, \qquad (1.44)$$
$$k^{(n)}_\mu(x_1, \dots, x_n) := \mathbb{1}_{(x_1,\dots,x_n)\in\widetilde{(\mathbb{R}^d)^n}} k^{(n)}_\mu(\{x_1, \dots, x_n\}),$$

are the well-known *correlation functions* from statistical physics, cf., e.g., [Rue69, Rue70]. They are also characterized by the following formula:

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1, \dots, x_n)\, d\mu(\gamma)$$
$$= \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} f^{(n)}(x_1, \dots, x_n) k^{(n)}_\mu(x_1, \dots, x_n)\, d\sigma(x_1) \cdots d\sigma(x_n) \quad (1.45)$$

for any $n \in \mathbb{N}$ and any measurable, symmetric function $f^{(n)} : \mathbb{R}^d \to [0, +\infty]$.

Note that there is a one-to-one relation between a correlation functional $k_\mu$ of a measure $\mu$ and the corresponding *Ursell functional* $u_\mu$ given by

$$k_\mu(\eta) = \sum_{(\eta_1,\dots,\eta_n)\in\mathcal{P}(\eta)} u_\mu(\eta_1) \cdots u_\mu(\eta_n), \quad \eta \in \Gamma_0, \qquad (1.46)$$

where $\mathcal{P}(\eta)$ is the set of all partitions of $\eta$, cf., e.g., [Rue69].

# Chapter 2

# Invariance Principle for a Diffusion in Random Environment

In this chapter and the following one, we will discuss invariance principles, i.e., convergence to Brownian motion, for two models of stochastic evolutions in random environments. First we will consider the motion of a diffusing particle in $\mathbb{R}^d$ interacting with a with frozen configuration of particles, which are randomly distributed over the space. The interaction is described by a symmetric pair potential. Afterwards, in Chapter 3, we will study a tagged particle process, i.e., the motion of one particle in an equilibrium infinite particle diffusion.

The problem of an invariance principle for a diffusion interacting with a random configuration has been studied in several works. In [DFGW89], the authors proved a general result for invariance principles, cf. also Subsection 2.1.2, and applied it to this situation. But the application part in this article was only on a rather heuristic level, and even on this level they had to assume that the interaction potential is smooth, positive, and compactly supported. These assumptions are hardly ever satisfied in physical models. Furthermore, some integrability properties neccessary in their general framework were only assumed to be satisfied. In [Str05], this result was generalized to the case of potentials with a singularity at the origin, a negative part, and an infinite range, e.g. the Lennard-Jones potential. The integrability conditions were proven under the general assumptions on the potential. In both works, the environment was a sample of a grand canonical Gibbs measure w.r.t. the interaction potential.

In this work, we will consider a similar class of potentials as in [Str05], in particular the Lennard-Jones potential (1.28) is included. But we will allow that the interaction potential $V_I$ between the diffusion and the configuration differs from the potential $V_E$ for the Gibbs measure of the environment. From the physical point of view this means that the diffusing particle and the particles of the environment are of a different type.

For the application of the general scheme, we will have to work with the environment process. This describes the motion of the environment as seen from the diffusing particle. Since the environment is a configuration, the environment process will be a stochastic process on the configuration space $\Gamma$. In both [DFGW89] and [Str05], this process was obtained by first constructing the diffusion $X_t$ and afterwards obtaining the environment process $\xi_t$ as a shift of the points of the environment configuration $\gamma$ by this diffusion, i.e

$$\xi_t := \{y - X_t \,:\, y \in \gamma\}.$$

But in this work we will construct the environment process directly using the general theory of Dirichlet forms, cf., e.g., [MR92].

Throughout this chapter, we will consider $\mathbb{R}^d$ with $d \geq 2$ as underlying space. Then for any configuration $\gamma \in \Gamma$, $\mathbb{R}^d \setminus \gamma$ is connected. Note that in the one-dimensional case, for a repulsive interaction potential, the diffusing particle would be trapped between to points of the configuration.

## 2.1 General theory of invariance principles

In this section, we will recall the general scheme of De Masi et al. [DFGW89] (see also [Gol95]), which we will use to prove an invariance principle later. This framework deals with antisymmetric functionals of reversible, ergodic Markov processes. In the application, the latter one is the environment process, and the diffusion is treated as a functional of this.

We will start with the definition of an invariance principle.

### 2.1.1 Central limit theorem and invariance principle

Consider a stochastic process $X_t \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ or $t \in \mathbb{N}_0$. $X_t$ satisfies the *central limit theorem (CLT)* if

$$\frac{X_t}{\sqrt{t}} \to N(0, D) \tag{2.1}$$

in the sense of weak convergence of distributions, where $N(0, D)$ denotes the Gaussian distribution with mean 0 and variance $D \geq 0$. Let

$$X_t^{(\varepsilon)} := \varepsilon X_{\varepsilon^{-2}t}, \quad \varepsilon > 0. \tag{2.2}$$

Closely connected to the (CLT)-property is the convergence

$$X_t^{\varepsilon} \to W_D(t) \tag{2.3}$$

in the sense of finite dimensional distributions, where $W_D$ denotes a Brownian motion with covariance $Dt$. In fact, let us denote both conditions together by (CLT).

Stronger than (CLT) is the *invariance principle (IP)*:

**Definition 2.1.1.** A process $(X_t)_t$ satisfies an invariance principle, if

$$X^{\varepsilon} \to W_D \tag{2.4}$$

in the sense of weak-convergence of the corresponding distributions on the paths-space.

The (IP) amounts to (CLT) plus tightness of the family of paths-space distributions.

**Example 2.1.2.** A well-known example for convergence to Brownian motion is Donsker's invariance principle. Consider a sequence $(Y_n)_{n\in\mathbb{N}}$ of independent, identically distributed (i.i.d.) $\mathbb{R}^d$-valued random variables with mean vector 0 and covariance matrix $I$, the identity matrix. Define

$$X_t^n := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} Y_k, \quad t \geq 0.$$

Then, as $n \to \infty$, $X_t^n$ converges to a standard Brownian motion on $\mathbb{R}^d$ in the sense as in (2.4), see, e.g., [EK86, Chapter 5, Theorem 1.2(c)]. In particular, a simple random walk on $\mathbb{Z}$ obtained from coin tossing converges to one-dimensional Brownian motion.

### 2.1.2 The standard decomposition

The above-mentioned scheme from [DFGW89] applies to processes $X_t$ which are antisymmetric functionals of ergodic, reversible Markov processes. More precisely, $X_t$ satisfies the *standard decomposition*

$$X_t = \int_0^t \Phi(\xi_s)\, ds + M_t. \tag{2.5}$$

Here $\xi_t$ is an ergodic, reversible Markov process, and $M_t$ is a square-integrable martingale. $\Phi$ is called *mean forward velocity*. In applications, we consider motions $X_t$ in random environments. Then $\xi_t$ is the *environment process* corresponding to $X_t$. Reversibility of $\xi_t$ reflects the symmetry of the motion together with translation invariance of the environment. Basically, we will have to show $L^1$- and $L^2$-integrability of $\Phi$ w.r.t. the reversible measure of $\xi_t$ to obtain (CLT) and (IP), resp., for $X_t$.

Ergodic, square-integrable martingales $M_t$ with stationary increments satisfy the invariance principle, see, e.g., [Hel82]. Therefore, proving (IP) for $X_t$ of the form 2.5 can be reduced to the proving an (IP) for the integral term

$$S_t := \int_0^t \Phi(\xi_s)\, ds.$$

Kipnis and Varadhan [KV86] have done this by decomposing $S_t$ into another square-integrable martingale plus an asymptotically negligible term. In [DFGW89], it is shown that some conditions on $\Phi$ in [KV86] are automatically satisfied due to symmetry and antisymmetry properties of the involved processes, and thus that the conditions are reduced to the above-mentioned integrability properties.

### 2.1.3 The IP-scheme

As explained before, we will discuss a diffusion interacting with a frozen configuration and a tagged particle process. In both cases, the environment process is a process on the configuration space $\Gamma$.

Based on [DFGW89], one can apply the following procedure to these models:

(i) Construct the process $X_t$.

(ii) Construct the corresponding environment process $\xi_t$.

(iii) Identify a probability measure $\mu$ on $\Gamma$, such that $\xi$ is ergodic and reversible w.r.t. $\mu$.

(iv) Rewrite $X_t$ in form of the standard decomposition (2.5).

(v) Prove $\Phi \in L^2(\mu)$.

Then one can apply [DFGW89, Theorem 2.2] to obtain an invariance principle.

As mentioned at the beginning, we can skip (i) and construct the environment process directly, see Section 2.3. Then we obtain the diffusion later as the functional from the standard decomposition (2.5).

But nevertheless, we have to describe the model in details. This is done in the following section.

## 2.2 The diffusion process

In this section, fix a configuration $\gamma \in \Gamma(\mathbb{R}^d)$. $\gamma$ should describe the positions of infinitely many indistinguishable particles. We will describe a diffusion interacting with these particles via a pair potential $V_I$. Since we do not consider the randomness of the environment here and hence neither the potential $V_E$ for the environment Gibbs measure, we will just write $V$ instead of $V_I$.

So, let $V$ be a symmetric, translation-invariant pair potential, i.e., $V(x-y) = \tilde{V}(|x-y|)$, $x, y \in \mathbb{R}^d$, $x \neq y$, for some proper function $\tilde{V} : (0, +\infty) \to \mathbb{R}$. Assume repulsion at small distances, i.e., $\lim_{x\to 0} V(x) = +\infty$, and hence set $V(0) = +\infty$. Furthermore, assume decay of the potential at infinity, i.e., $\lim_{|x|\to+\infty} V(x) = 0$.

### 2.2.1 The corresponding evolution equation

Assume that the relative energy function

$$E(x) := E_V(x,\gamma) := \begin{cases} \sum_{y\in\gamma} V(x-y), & \text{if } \sum_{y\in\gamma} |V(x-y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

is finite on $\mathbb{R}^d \setminus \gamma$. This is obviously satisfied, if $V$ is a finite range potential, i.e., its support is bounded, since in this case there are only finitely many summands nonequal zero. But under the more general assumptions from Section 2.3 it is still fulfilled.

Let $\rho(x) := \rho(x,\gamma) := \exp(-E(x,\gamma))$. Then $\rho > 0$ outside of $\gamma$, and, since $\gamma$ is a Lebesgue nullset, the logarithmic derivative $\beta(x) := \beta(x,\gamma) := \frac{\nabla\rho(x)}{\rho(x)} = -\sum_{y\in\gamma} \nabla V(x-y)$ is well-defined Lebesgue-a.e.

Then, the motion of the diffusing particle can be described by the following stochastic differential equation

$$\begin{cases} dX_t = \beta(X_t)\,dt + \sqrt{2}\,dW_t, \\ X_0 = x_0\,(\in \mathbb{R}^d \setminus \gamma). \end{cases} \tag{2.6}$$

Here $W_t$ is a standard Brownian motion on $\mathbb{R}^d$. (2.6) is a stochastic differential equation describing a symmetric distorted Brownian motion on $\mathbb{R}^d$ with singular drift.

### 2.2.2 On solutions for the evolution equation

The stochastic equation (2.6) has been studied in several works. We want to summarize some results in this subsection.

In [KKR04], the problem has been treated by a Dirichlet forms approach, and existence of a weak solution is proved. For this, the authors consider the following set of admissible configurations for the environment:

$$\Gamma_{\text{ad}} := \{\gamma \in \Gamma \,:\, \forall r > 0 \exists c(\gamma, r) > 0\, |\gamma \cap B(x,r)| \le c(\gamma, r) \log(2 + |x|)\}. \tag{2.7}$$

For applications, the restriction to configurations $\gamma \in \Gamma_{\text{ad}}$ is not too strong, since, e.g. by [KKK04], it follows that $\mu(\Gamma_{\text{ad}}) = 1$ for any Ruelle measure $\mu$ w.r.t. to a superstable pair potential. Equation (2.6) corresponds to the pre-Dirichlet form

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} \big(\nabla u, \nabla v\big) \rho(x)\,dx, \quad u, v \in C_0^\infty(\mathbb{R}^d),$$

with pre-generator

$$Lu = \Delta u + \big(\beta, \nabla u\big), \quad u \in C_0^\infty(\mathbb{R}^d).$$

In [KKR04] it is shown that, for $\gamma \in \Gamma_{\text{ad}}$ and an interaction potential $V$ satisfying some integrability condition for the singularity and with proper decay at infinity, the form $\big(\mathcal{E}, C_0^\infty\big)$ is closable and its closure is associated to a stochastic process solving (2.6). Furthermore, $L^1$- and $L^2$-uniqueness of the generator is proven, i.e., conservativity of the form and the corresponding process and essential self-adjointness of the generator, resp.

In [KR05], the authors apply their general result to (2.6) to even prove existence and uniqueness of a strong solution. They treat the same set of admissible configurations $\Gamma_{\text{ad}}$ as in [KKR04] and comparable interaction potentials.

## 2.3 The environment process

As mentioned before, from a solution $X_t$ to (2.6) on can construct the environment process by shifting every point of the environment configuration by $-X_t$. But we will construct the environment process directly as a stochastic process on the configuration space, ergodic and reversible w.r.t. a perturbed Gibbs measure, see (2.8). From now on, we have to distinguish between the interaction potential $V_I$ and the potential $V_E$ for the Gibbs measure for the environment.

We make the following assumptions:

**Assumption 2.3.1.** $V_I$: $V_I$ satisfies (I), (D) from Definition 1.4.9, and $V_I$ is bounded from below.

$V_E$: $V_I$ satisfies (SS), (LR), (I), (D), and (LS) from Definition 1.4.9. Hence, by Theorem 1.4.12, there exist corresponding grand canonical Gibbs measures.

$\mu_E$ Let $z > 0$ an activity parameter, and let $\mu_E \in \mathcal{G}_{\mathrm{Rb}}^{\mathrm{gc}}(z, V_E)$. Suppose furthermore, that $\mu_E$ is translation invariant, i.e., $\mu_E(A) = \mu_E(A+h)$, $A \in \mathcal{B}(\Gamma)$, $h \in \mathbb{R}^d$, where

$$A + h := \{\gamma + h \,:\, \gamma \in A\}.$$

Let $E_I$, $E_E$ denote the relative energy functions w.r.t. $V_I$, $V_E$, resp.

**Remark 2.3.2.** (i) Below, from Subsection 2.3.4 on, we will additionally assume that $\mu_E$ is such that the Dirichlet form $\big(\mathcal{E}_{\mu^*}^{\mathbb{D}}, D(\mathcal{E}_{\mu^*}^{\mathbb{D}})\big)$ (see (2.12) and Proposition 2.3.7) is irreducible.

(ii) Let $\mu \in \mathcal{G}_{\mathrm{Rb}}^{\mathrm{gc}}(z, V)$ for some $z > 0$ and a symmetric pair potential $V$. Fix some $h \in \mathbb{R}^d$ and define

$$\tilde{\mu}(A) := \mu(A+h), \quad A \in \mathcal{B}(\Gamma).$$

Then, by Georgii-Nguyen-Zessin identity (1.22) and by translation invariance of the Lebesgue measure, for any positive $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Gamma)$-

measurable function $H$

$$\begin{aligned}
&\int_\Gamma \sum_{x\in\gamma} H(x,\gamma)\,d\tilde{\mu}(\gamma)\\
&\quad= \int_\Gamma \sum_{x\in\gamma} H(x+h,\gamma+h)\,d\mu(\gamma)\\
&\quad= \int_{\mathbb{R}^d}\int_\Gamma H(x+h,(\gamma\cup x)+h) z e^{-E^V(x,\gamma)}\,d\mu(\gamma)\,dx\\
&\quad= \int_{\mathbb{R}^d}\int_\Gamma H(x+h,(\gamma+h)\cup(x+h)) z e^{-E^V(x+h,\gamma+h)}\,d\mu(\gamma)\,dx\\
&\quad= \int_{\mathbb{R}^d}\int_\Gamma H(x+h,\gamma\cup(x+h)) z e^{-E^V(x+h,\gamma)}\,d\tilde{\mu}(\gamma)\,dx\\
&\quad= \int_\Gamma\int_{\mathbb{R}^d} H(x,\gamma\cup x) z e^{-E^V(x,\gamma+h)}\,dx\,d\tilde{\mu}(\gamma).
\end{aligned}$$

Thus, $\tilde{\mu}$ also satisfies (1.22), and hence $\tilde{\mu}\in\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z,V)$, too.

In particular, if $|\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z,V)| = 1$, which is always satisfied for small enough $z$ (cf., e.g., [Rue70]), then $\mu=\tilde{\mu}$, i.e., $\mu$ is translation invariant.

Define

$$d\mu^*(\gamma) := \frac{1}{Z} e^{-E_I(0,\gamma)}\,d\mu_E(\gamma), \tag{2.8}$$

where $Z := \int_\Gamma e^{-E_I(0,\cdot)}\,d\mu_E < +\infty$ by Lemma 2.3.3 below. Since $e^{-E_I(0,\gamma)} = 0$ only if $0\in\gamma$ and $\mu_E(\{\gamma : 0\in\gamma\}) = 0$, $\mu^*$ and $\mu_E$ are equivalent measures. $\mu^*$ will be the symmetrizing measure for the environment process.

**Lemma 2.3.3.** *Let $p\geq 1$. Then $e^{-E_I(x_0,\cdot)}\in L^p(\Gamma,\mu_E)$ for every $x_0\in\mathbb{R}^d$. Moreover,*

$$\sup_{x_0\in\mathbb{R}^d} \|e^{-E_I(x_0,\cdot)}\|_{L^p(\Gamma,\mu_E)} < \infty.$$

*The same holds for $e^{-E_E(\cdot,\cdot)}$.*

*Proof.* We will only show the result for $e^{-E_I(\cdot,\cdot)}$. The proof is completely the same for $e^{-E_E(\cdot,\cdot)}$.

For $x_0\in\mathbb{R}^d$ define $\theta := \theta_{x_0} : \mathbb{R}^d\to\mathbb{R}$, $\theta_{x_0}(x) := \left|1 - e^{-pV_I(x-x_0)}\right| \geq 0$. Because of the integrability assumption (I), we have that $\theta_{x_0}\in L^1(\mathbb{R}^d, dx)$. For any $\gamma\in\Gamma$ it holds that

$$\prod_{y\in\gamma}(1+\theta(y)) = \sum_{n=0}^{\infty}\sum_{\{y_1,\dots,y_n\}\subset\gamma} \theta(y_1)\cdots\theta(y_n). \tag{2.9}$$

Since $f^{(n)}(x_1,\ldots,x_n) := \theta(x_1)\cdots\theta(x_n)$, $x_1,\ldots,x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, is a non-negative symmetric function on $(\mathbb{R}^d)^n$ for any $n$, we have by (1.45) that

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} f^{(n)}(x_1,\ldots,x_n)\, d\mu_E(\gamma) \\ = \frac{z^n}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\ldots,x_n) k^{(n)}_{\mu_E}(x_1,\ldots,x_n)\, dx_1\cdots dx_n, \quad (2.10)$$

$n \in \mathbb{N}$, where $k^{(n)}_{\mu_E}$, $n \in \mathbb{N}$, denote the correlation functions of $\mu_E$. By assumption, $\big(k^{(n)}_{\mu_E}\big)_n$ has a Ruelle bound. So there exists a $\xi > 0$ such that $k^{(n)}_{\mu_E} \le \xi^n$ for all $n \in \mathbb{N}$. From this it follows

$$\begin{aligned}
&\int_\Gamma \left|e^{-E_I(x_0,\gamma)}\right|^p \, d\mu_E(\gamma) \\
&\quad = \int_\Gamma e^{-p\sum_{y\in\gamma} V_I(x_0-y)}\, d\mu_E(\gamma) \\
&\quad = \int_\Gamma \prod_{y\in\gamma}\big(1+(e^{-pV_I(x_0-y)}-1)\big)\, d\mu_E(\gamma) \\
&\quad \le \int_\Gamma \prod_{y\in\gamma}\big(1+\theta(y)\big)\, d\mu_E(\gamma) \\
&\quad \overset{(2.9)}{=} \int_\Gamma \sum_{n=0}^\infty \sum_{\{y_1,\ldots,y_n\}\subset\gamma} \theta(y_1)\cdots\theta(y_n)\, d\mu_E(\gamma) \\
&\quad \overset{(2.10)}{=} \sum_{n=0}^\infty \frac{z^n}{n!}\int_{(\mathbb{R}^d)^n} \theta(x_1)\cdots\theta(x_n)\underbrace{k^{(n)}_{\mu_E}(x_1,\ldots,x_n)}_{\le \xi^n}\, dx_1\cdots dx_n \\
&\quad \le \sum_{n=0}^\infty \frac{z^n}{n!}\xi^n \|\theta\|^n_{L^1} = e^{z\xi\|\theta\|_{L^1}} < \infty.
\end{aligned}$$

This proves the first part of the assertion.

But the term $C := e^{z\xi\|\theta_{x_0}\|_{L^1}}$ in the last equation is independent of $x_0$, since, by translation invariance of the Lebesgue measure, $\|\theta_{x_0}\|_{L^1} = \|\theta_0\|_{L^1}$ for all $x_0$. Thus,

$$\sup_{x_0\in\mathbb{R}^d}\int_\Gamma \left|e^{-E(x_0,\gamma)}\right|^p \, d\mu_E(\gamma) \le C < \infty.$$

□

The next result will be helpful for several of the following proofs in this chapter.

**Corollary 2.3.4.** *There exist a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$*

$$\int_\Gamma e^{-E_I(0,\gamma)} e^{-E_E(y,\gamma)}\, d\mu_E(\gamma) + \int_\Gamma e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} e^{-E_E(y,\gamma)}\, d\mu_E(\gamma) \le C. \tag{2.11}$$

*Proof.* The assertion follows directly form Lemma 1.45 and the Cauchy-Schwarz inequality. □

## 2.3.1 Corresponding Dirichlet form, generator and closability

The environment process will be associated to (the closure of) the following pre-Dirichlet form:

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(F, G) := \int_\Gamma \big(\mathbb{D}F(\gamma), \mathbb{D}G(\gamma)\big)\, d\mu^*(\gamma), \quad F, G \in \mathcal{F}C_b^\infty. \tag{2.12}$$

We want to compute the pre-generator $(H^{\mathbb{D}}_{\mu^*}, \mathcal{F}C_b^\infty)$ of $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, \mathcal{F}C_b^\infty)$.

For $h \in \mathbb{R}^d \setminus \{0\}$ we have

$$\begin{aligned} \mathbb{D}_h e^{-E_I(0,\gamma)} &= \lim_{t\to 0} \frac{1}{t}\big(\underbrace{e^{-E_I(0,\gamma+th)}}_{=e^{-E_I(-th,\gamma)}} - e^{-E_I(0,\gamma)}\big) \\ &= -\nabla_h e^{-E_I(0,\gamma)}. \end{aligned} \tag{2.13}$$

With this we can compute the adjoint operator $\mathbb{D}^*_{h,\mu^*}$ of $\mathbb{D}_h$ on $\mathcal{F}C_b^\infty$ with respect to $L^2(\Gamma, \mu^*)$. Let $F, G \in \mathcal{F}C_b^\infty$, then, using the invariance of $\mu_E$ w.r.t. space shifts

$$\begin{aligned} \big(\mathbb{D}_h F, G\big)_{L^2(\Gamma,\mu^*)} &= \int_\Gamma \mathbb{D}_h F(\gamma) G(\gamma)\, d\mu^*(\gamma) \\ &= \lim_{t\to 0}\frac{1}{t}\bigg(\int_\Gamma F(\gamma+th)G(\gamma)\frac{1}{Z}e^{-E_I(0,\gamma)}\, d\mu_E(\gamma) \\ &\qquad - \int_\Gamma F(\gamma)G(\gamma)\frac{1}{Z}e^{-E_I(0,\gamma)}\, d\mu_E(\gamma)\bigg) \\ &= \lim_{t\to 0}\frac{1}{t}\bigg(\int_\Gamma F(\gamma)G(\gamma-th)\frac{1}{Z}e^{-E_I(0,\gamma-th)}\, d\mu_E(\gamma) \\ &\qquad - \int_\Gamma F(\gamma)G(\gamma)\frac{1}{Z}e^{-E_I(0,\gamma)}\, d\mu_E(\gamma)\bigg) \end{aligned}$$

$$
\begin{aligned}
&= \int_\Gamma F(\gamma) \lim_{t\to 0} \frac{1}{t}\Big( G(\gamma - th)\frac{1}{Z}e^{-E_I(0,\gamma-th)} \\
&\qquad - G(\gamma)\frac{1}{Z}e^{-E_I(0,\gamma)}\Big)\, d\mu_E(\gamma) \\
&= \int_\Gamma F(\gamma)\mathbb{D}_{-h}G(\gamma)\frac{1}{Z}e^{-E_I(0,\gamma)}\, d\mu_E(\gamma) \\
&\qquad + \int_\Gamma F(\gamma)G(\gamma)\nabla_{-h}\big(\frac{1}{Z}e^{-E_I(0,\gamma)}\big)\, d\mu_E(\gamma) \\
&= \int_\Gamma F(\gamma)\mathbb{D}_{-h}G(\gamma)\, d\mu^*(\gamma) \\
&\qquad + \int_\Gamma F(\gamma)G(\gamma)\Big(\frac{\nabla \frac{1}{Z}e^{-E_I(0,\gamma)}}{\frac{1}{Z}e^{-E_I(0,\gamma)}}, -h\Big)\, d\mu^*(\gamma).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{D}^*_{h,\mu^*} &= \mathbb{D}_{-h} + \Big(\frac{\nabla \frac{1}{Z}e^{-E_I(0,\cdot)}}{\frac{1}{Z}e^{-E_I(0,\cdot)}}, -h\Big) \\
&= -\mathbb{D}_h - \Big(\sum_{y\in\cdot} \nabla V_I(y), h\Big) \qquad (2.14)
\end{aligned}
$$

on $\mathcal{F}C_b^\infty$.

**Definition 2.3.5.** For a vector field $V : \Gamma \to \mathbb{R}^d$, $\operatorname{div}^{\mathbb{D},\mu^*} V$ is defined via the duality relation

$$
\int_\Gamma \big(\mathbb{D}F(\gamma), V(\gamma)\big)\, d\mu^*(\gamma) = -\int_\Gamma F(\gamma)(\operatorname{div}^{\mathbb{D},\mu^*} V)(\gamma)\, d\mu^*(\gamma) \qquad (2.15)
$$

for all $F \in \mathcal{F}C_b^\infty$, provided it exists.

We want to compute $\operatorname{div}^{\mathbb{D},\mu^*} \mathbb{D}F$ for $F \in \mathcal{F}C_b^\infty$. Therefore we introduce a larger set of functions on $\Gamma$:

$$
\begin{aligned}
\mathcal{F}C_{pb}^\infty := \mathcal{F}C_{pb}^\infty(C_0^\infty(\mathbb{R}^d), \Gamma) := \{g\big(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle\big) \, : \\
N \in \mathbb{N},\, g \in C_{pb}^\infty(\mathbb{R}^N),\, f_1, \dots, f_N \in C_0^\infty(\mathbb{R}^d)\}. \quad (2.16)
\end{aligned}
$$

Here $C_{pb}^\infty(\mathbb{R}^N)$ denotes the set of all smooth functions on $\mathbb{R}^d$ for which all derivatives of any order are bounded by polynomials. For each $F \in \mathcal{F}C_{pb}^\infty$, $\mathbb{D}F$ is defined, and the same equations as for $F \in \mathcal{F}C_b^\infty$ hold. Note that $\mathcal{F}C_b^\infty \subset \mathcal{F}C_{pb}^\infty$ and $\langle f, \cdot\rangle \in \mathcal{F}C_{pb}^\infty$ for any $f \in C_0^\infty(\mathbb{R}^d)$.

Consider a vector field

$$V(\gamma) := \sum_{j=1}^{N} G_j(\gamma) h_j,$$

where $G_j \in \mathcal{F}C_{pb}^{\infty}$ and $h_j \in \mathbb{R}^d$, $1 \le j \le N$. We denote the set of these vector fields by $\mathcal{V}^{\mathbb{D}}\mathcal{F}C_{pb}^{\infty}$. Then

$$\begin{aligned}
&\int_\Gamma \big(\mathbb{D}F(\gamma), V(\gamma)\big)\, d\mu^*(\gamma) \\
&\quad = \sum_{j=1}^{N} \int_\Gamma G_j(\gamma)\big(\mathbb{D}F(\gamma), h_j\big)\, d\mu^*(\gamma) \\
&\quad = \sum_{j=1}^{N} \int_\Gamma G_j(\gamma) \mathbb{D}_{h_j} F(\gamma)\, d\mu^*(\gamma) \\
&\quad \overset{(2.14)}{=} -\sum_{j=1}^{N} \int_\Gamma F(\gamma) \mathbb{D}_{h_j} G(\gamma)\, d\mu^*(\gamma) \\
&\qquad\quad - \sum_{j=1}^{N} \int_\Gamma F(\gamma) G(\gamma) \big(\sum_{y\in\gamma} \nabla V_I(y), h_j\big)\, d\mu^*(\gamma).
\end{aligned}$$

Hence

$$\operatorname{div}^{\mathbb{D},\mu^*} V(\gamma) = \sum_{j=1}^{N} \mathbb{D}_{h_j} G_j(\gamma) + \sum_{j=1}^{N} \big(\sum_{y\in\gamma} \nabla V_I(y), h_j\big) G_j(\gamma). \tag{2.17}$$

**Lemma 2.3.6.** *Let $F \in \mathcal{F}C_b^{\infty}$. Then $\mathbb{D}F \in \mathcal{V}^{\mathbb{D}}\mathcal{F}C_{pb}^{\infty}$.*

*Proof.* Consider a representation $F = g_F\big(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle\big)$. Then by (1.38)

$$\begin{aligned}
\mathbb{D}F(\gamma) &= \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle\big) \langle \nabla f_j, \gamma\rangle \\
&= \sum_{i=1}^{d} \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle\big) \langle \partial_{x_i} f_j, \gamma\rangle e_i,
\end{aligned}$$

where $e_i$, $1 \le i \le d$, denote the canonical unit vectors in $\mathbb{R}^d$. Since

$$\partial_j g_F\big(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle\big) \langle \partial_{x_i} f_j, \cdot\rangle \in \mathcal{F}C_{pb}^{\infty}$$

the assertion follows. □

By the previous lemma, $\operatorname{div}^{\mathbb{D},\mu^*}\mathbb{D}F$ is defined for any $F \in \mathcal{F}C_b^\infty$, and, for $F = g_F\big(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle\big)$, we have by (2.17) and the proof of the lemma

$$\begin{aligned}
\operatorname{div}^{\mathbb{D},\mu^*}\mathbb{D}F &= \sum_{i=1}^d \mathbb{D}_{e_i}\left(\sum_{j=1}^N \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle \partial_{x_i} f_j,\gamma\rangle\right)\\
&\quad + \sum_{i=1}^d\left(\sum_{j=1}^N \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle \partial_{x_i} f,\gamma\rangle\right)\langle \partial_{x_i} V_I,\gamma\rangle\\
&= \sum_{i=1}^d \sum_{j,k=1}^N \partial_k\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle \partial_{x_i} f_k,\gamma\rangle\langle \partial_{x_i} f_j,\gamma\rangle\\
&\quad + \sum_{i=1}^d\left(\sum_{j=1}^N \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\right)\langle \partial^2_{x_i} f_j,\gamma\rangle\\
&\quad + \sum_{i=1}^d\left(\sum_{j=1}^N \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle \partial_{x_i} f,\gamma\rangle\right)\langle \partial_{x_i} V_I,\gamma\rangle\\
&= \sum_{i,j=1}^N \partial_i\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\big(\langle \nabla f_i,\gamma\rangle,\langle \nabla f_j,\gamma\rangle\big)\\
&\quad + \sum_{j=1}^N \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\times\\
&\quad\quad \times\big(\langle \Delta f_j,\gamma\rangle + (\langle \nabla f_j,\gamma\rangle,\langle \nabla V_I,\gamma\rangle)\big). \qquad (2.18)
\end{aligned}$$

It follows, that

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(F,G) = -\langle \operatorname{div}^{\mathbb{D},\mu^*}\mathbb{D}F, G\rangle_{L^2(\mu^*)} \qquad (2.19)$$

for $F, G \in \mathcal{F}C_b^\infty$, hence set $H^{\mathbb{D}}_{\mu^*} := -L^{\mathbb{D}}_{\mu^*} := -\operatorname{div}^{\mathbb{D},\mu^*}\mathbb{D}$ on $\mathcal{F}C_b^\infty$. Furthermore, conservativity obviously holds, i.e., $1 \in \mathcal{F}C_b^\infty$ and $\mathcal{E}^{\mathbb{D}}_{\mu^*}(1,1) = 0$. Therefore we have the following proposition:

**Proposition 2.3.7.** *The form $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, \mathcal{F}C_b^\infty)$ is closable on $L^2(\mu^*)$, and its closure $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$ is a symmetric, conservative Dirichlet form. It is generated by the Friedrichs extension of $(H^{\mathbb{D}}_{\mu^*}, \mathcal{F}C_b^\infty)$, which we also denote by $H^{\mathbb{D}}_{\mu^*}$.*

The gradient $\mathbb{D}$ extends to a linear operator on $D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$, which we also denote by $\mathbb{D}$. We will write $\mathcal{E}^{\mathbb{D}}_{\mu^*}$ now in terms of the corresponding *square field operator*:

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(F,G) = \int_\Gamma S^{\mathbb{D}}(F,G)\, d\mu^*(\gamma), \quad F, G \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*}),$$

with $S^{\mathbb{D}}(F,G) = (\mathbb{D}F(\gamma), \mathbb{D}G(\gamma))$. Furthermore, we will write $S^{\mathbb{D}}(F) := S^{\mathbb{D}}(F,F)$.

### 2.3.2 Quasi-regularity

Now consider the bigger space $\ddot{\Gamma}$ of multiple configurations. Since $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider $\mu^*$ as a measure on $\ddot{\Gamma}$. Then $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$ is a Dirichlet form on $L^2(\ddot{\Gamma}, \mu^*)$.

To construct a process associated to $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$, we have to show quasi-regularity of the form, see, e.g [MR92, Chapter IV, Definition 3.1 and Theorem 3.5]. We will prove quasi-regularity on $\ddot{\Gamma}$. So we will obtain a process on $\ddot{\Gamma}$ as well. Then we will show, that under the assumption $d \geq 2$, $\ddot{\Gamma} \setminus \Gamma$ is an $\mathcal{E}^{\mathbb{D}}_{\mu^*}$-exceptional set. So the process is actually supported on the simple configurations $\Gamma$.

**Lemma 2.3.8.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a positive, measurable function with $\operatorname{supp} f$ compact, and $B \subset \mathbb{R}^d$ compact. Then*

$$\int_{\ddot{\Gamma}} \langle f, \gamma \rangle N_B(\gamma)\, d\mu^*(\gamma) < +\infty.$$

*Proof.* Note that $\mu^*$ and $\mu_E$ have full support on $\Gamma \subset \ddot{\Gamma}$, so we only have to prove the assertion for the integral over $\Gamma$. Therefore, $N_B = \sum_{x \in \cdot} \mathbb{1}_B(x)$.

Iterated application of the Georgii-Nguyen-Zessin identity (1.22) gives

$$\begin{aligned}
&\int_{\Gamma} \langle f, \gamma \rangle N_B(\gamma)\, d\mu^*(\gamma) \\
&\quad = \frac{z}{Z} \int_{\mathbb{R}^d} f(x) e^{V_I(x)} \int_{\Gamma} [N_B(\gamma) + \mathbb{1}_B(x)] e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)}\, d\mu_E(\gamma)\, dx \\
&\quad = \frac{z}{Z} \int_{\mathbb{R}^d} f(x) e^{V_I(x)} \times \\
&\qquad \times \Bigg[ \int_{\mathbb{R}^d} \mathbb{1}_B(y) e^{-V_I(y)} e^{-V_E(x-y)} z \int_{\Gamma} e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} e^{-E_E(y,\gamma)}\, d\mu_E(\gamma)\, dy \\
&\qquad\quad + \mathbb{1}_B(x) \int_{\Gamma} e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)}\, d\mu_E(\gamma) \Bigg] dx \\
&\quad < +\infty.
\end{aligned}$$

Here we have used Corollary 2.3.4 and the fact that $f$ and $\mathbb{1}_B$ have bounded support while $e^{-V_I}$ and $e^{-V_E}$ are bounded. $\square$

**Proposition 2.3.9.** *$(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$ is quasi-regular on $L^2(\ddot{\Gamma}, \mu^*)$.*

*Proof.* This proof is a modification of the ones for [MR00, Proposition 4.1] and [FG08, Lemma 5.10]

To prove quasi-regularity, it suffices to show that there exists a bounded, complete metric $\bar{\rho}$ on $\ddot{\Gamma}$, which generates the vague topology on $\ddot{\Gamma}$ and fulfills the following condition: for all $\gamma_0 \in \Gamma$

$$\bar{\rho}(\cdot, \gamma_0) \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*}) \quad \text{and} \quad S^{\mathbb{D}}(\bar{\rho}(\cdot, \gamma_0)) \le \eta \ \mu^*\text{-a.e.}$$

for some function $\eta \in L^1(\ddot{\Gamma}, \mu^*)$ independent of $\gamma_0$.

Consider the sequence $(B_k)_{k\in\mathbb{N}}$ of open balls in $\mathbb{R}^d$ with center 0 and radius $k$. It is well-exhausting in the sense of [MR00] with $\delta_k := \frac{1}{2}$ for all $k$, i.e.,

$$B_k \nearrow, \quad \bigcup_k B_k = \mathbb{R}^d, \quad \text{and} \quad B_k^{\delta_k} \subset B_{k+1}.$$

Here $B_k^{\delta_k} := \{x \in \mathbb{R}^d \,:\, \operatorname{dist}(x, B_k) < \delta_k\} = B_{k+\delta_k}$. For $k \in \mathbb{N}$, set

$$g_k(x) := \tfrac{2}{3}\big(\tfrac{1}{2} - \operatorname{dist}(x, B_k) \wedge \tfrac{1}{2}\big), \quad x \in \mathbb{R}^d,$$

and $\phi_k := 3g_k$.

Let $S(f,g) := (\nabla f, \nabla g)$, $f, g \in H^{1,2}(\mathbb{R}^d)$, denote the standard square field operator on $L^2(\mathbb{R}^d, dx)$. Here

$$H^{1,2}(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d, dx) \,:\, \partial_i u \in L^2(\mathbb{R}^d, dx), 1 \le i \le n\},$$

with derivatives in the sense of Schwartz distributions, denotes the $(1,2)$-Sobolev space on $\mathbb{R}^d$ (with von Neumann or Dirichlet boundary conditions; both spaces coincide on $\mathbb{R}^d$). Again, we write $S(f) := S(f,f)$. The form $\big(\int S(\cdot,\cdot)\,dx, H^{1,2}(\mathbb{R}^d)\big)$ is the closure of $\big(\int S(\cdot,\cdot)\,dx, C_0^\infty(\mathbb{R}^d)\big)$. By [MR00, Example 4.5.1], we have that $(S, H^{1,2}(\mathbb{R}^d))$ satisfies the following condition:

(Q) There exist $\chi_j \in C_0^\infty(\mathbb{R}^d)$, $\chi_j > 0$, $j \in \mathbb{N}$, and $f_{ln} \in C(\mathbb{R}^d)$, $l, n \in \mathbb{N}$, such that

  (i) $\sup_{l\in\mathbb{N}} f_{ln} = |y_n - \cdot|$ for all $n \in \mathbb{N}$ and some $\{y_n \,:\, n \in \mathbb{N}\} \subset \mathbb{R}^d$ dense;

  (ii) there exists $C > 0$ such that, for all $j, l, n \in \mathbb{N}$ and all $\phi \in C_b^\infty(\mathbb{R}^d)$, $\chi_j(\phi \circ f_{ln}) \in C_0^\infty(\mathbb{R}^d)$ and

$$S(\chi_j(\phi \circ f_{ln})) \le C \sup\{\|\phi'\|_\infty, \|\phi\|_\infty\}^2 (\chi_j + S(\chi_j)^{\frac{1}{2}})^2;$$

(iii) for all $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $\chi_j \equiv 1$ on $B_k$.

Choose $(j_k)_{k\in\mathbb{N}}$ such that $\chi_{j_k} \equiv 1$ on $B_{k+1}$. Then, by [MR00, Lemma 4.10], for all $k, j \in \mathbb{N}$, $\phi_k g_j \in H^{1,2}(\mathbb{R}^d)$ and

$$S(\phi_k g_j) \le \tilde{\chi}_{j_k}^2, \tag{2.20}$$

with $\tilde{\chi}_{j_k} = 4\chi_{j_k}\big(S(\chi_{j_k})^{\frac{1}{2}} + C(\chi_{j_k} + S(\chi_{j_k})^{\frac{1}{2}})\big)^2$.

Let $f \in H^{1,2}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, then $\langle f, \cdot\rangle \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$, and

$$S^{\mathbb{D}}(\langle f, \cdot\rangle) = \big(\langle \nabla f, \cdot\rangle, \langle \nabla f, \cdot\rangle\big).$$

If $\operatorname{supp} f \subset B_{k+1}$, then, by the Cauchy-Schwarz inequality and the Jensen inequality,

$$\begin{aligned}
S^{\mathbb{D}}(\langle f, \cdot\rangle)(\gamma) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\nabla f(x), \nabla f(y)\big)\, d\gamma(x)\, d\gamma(y) \\
&\le \int_{\mathbb{R}^d} \big(\nabla f(x), \nabla f(x)\big)^{1/2} d\gamma(x) \int_{\mathbb{R}^d} \big(\nabla f(y), \nabla f(y)\big)^{1/2} d\gamma(y) \\
&= N_{B_{k+1}}(\gamma)^2 \bigg(\int_{\mathbb{R}^d} S(f)(x)^{1/2} \frac{1}{N_{B_{k+1}}(\gamma)}\, d\gamma(x)\bigg)^2 \\
&\le N_{B_{k+1}}(\gamma) \int_{\mathbb{R}^d} S(f)(x)\, d\gamma(x) \\
&= N_{B_{k+1}}(\gamma)\langle S(f), \gamma\rangle.
\end{aligned} \tag{2.21}$$

In particular, this holds for $f = \phi_k g_j$, and with (2.20) we have

$$S^{\mathbb{D}}(\langle \phi_k g_j, \cdot\rangle) \le N_{B_{k+1}}(\cdot)\langle \tilde{\chi}_{j_k}^2, \cdot\rangle. \tag{2.22}$$

Let $\zeta \in C_b^\infty(\mathbb{R})$ (i.e., $\zeta$ is a bounded, smooth function on $\mathbb{R}$), such that $0 \le \zeta \le 1$ on $[0, \infty)$, $\zeta(t) = t$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\zeta' > 0$, and $\zeta'' \le 0$. Then similarly as in [RS95, Lemma 3.2] we obtain that for any fixed $\gamma_0 \in \ddot{\Gamma}$ and for any $k, n \in \mathbb{N}$

$$\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big) \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*}).$$

Furthermore,

$$S^{\mathbb{D}}\bigg(\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big)\bigg) \le N_{B_{k+1}}(\cdot)\langle \tilde{\chi}_{j_k}^2, \cdot\rangle \quad \mu^*\text{-a.e.} \tag{2.23}$$

Set

$$F_k(\gamma, \gamma_0) := \zeta\big(\sup_{j\in\mathbb{N}} |\langle \phi_k g_j, \gamma\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big),$$

then for fixed $\gamma_0 \in \ddot{\Gamma}$

$$\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big) \longrightarrow F_k(\cdot, \gamma_0), \quad n \to \infty,$$

pointwisely and in $L^2(\ddot{\Gamma}, \mu^*)$. Hence, by (2.23), the Banach-Alaoglu theorem, and the Banach-Saks theorem (see, e.g., [MR92, Appendix A.2]),

$$\begin{gathered} F_k(\cdot, \gamma_0) \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*}), \\ S^{\mathbb{D}}(F_k(\cdot, \gamma_0)) \le N_{B_{k+1}}(\cdot)\langle \tilde{\chi}^2_{j_k}, \cdot\rangle \quad \mu^*\text{-a.e.} \end{gathered} \tag{2.24}$$

Set

$$c_k := \left(1 + \int_{\ddot{\Gamma}} N_{B_{k+1}}(\gamma)\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu^*(\gamma)\right)^{-\frac{1}{2}} 2^{-\frac{k}{2}}$$

Then, by Lemma 2.3.8, $c_k \in (0, \infty)$ for all $k$, and $c_k \to 0$, $k \to \infty$. Define

$$\bar{\rho}(\gamma_1, \gamma_2) := \sup_{k\in\mathbb{N}} c_k F_k(\gamma_1, \gamma_2). \tag{2.25}$$

Then, by [MR00, Theorem 3.6], $\bar{\rho}$ is a bounded, complete metric on $\ddot{\Gamma}$, which generates the vague topology.

By (2.24), we have that

$$\begin{aligned} S^{\mathbb{D}}(c_k F_k(\cdot, \gamma_0)) &= c_k^2 S^{\mathbb{D}}(F_k(\cdot, \gamma_0)) \\ &\le 2^{-k}\left(1 + \int_{\ddot{\Gamma}} N_{B_{k+1}}(\gamma)\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu^*(\gamma)\right)^{-1} N_{B_{k+1}}(\cdot)\langle \tilde{\chi}^2_{j_k}, \cdot\rangle \quad \mu^*\text{-a.e.} \end{aligned}$$

Set

$$\eta := \sup_k \left[2^{-k}\left(1 + \int_{\ddot{\Gamma}} N_{B_{k+1}}(\gamma)\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu^*(\gamma)\right)^{-1} N_{B_{k+1}}(\cdot)\langle \tilde{\chi}^2_{j_k}, \cdot\rangle\right],$$

then

$$\begin{aligned} \int_{\ddot{\Gamma}} \eta(\gamma')\, d\mu^*(\gamma') &\le \sum_{k=1}^{\infty} 2^{-k}\left(1 + \int_{\ddot{\Gamma}} N_{B_{k+1}}(\gamma)\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu^*(\gamma)\right)^{-1} \times \\ &\qquad \times \int_{\ddot{\Gamma}} N_{B_{k+1}}(\gamma')\langle \tilde{\chi}^2_{j_k}, \gamma'\rangle \, d\mu^*(\gamma') \\ &\le \sum_{k=1}^{\infty} 2^{-k} = 1 < +\infty, \end{aligned}$$

so $\eta \in L^1(\mu^*)$, and as in [RS95, Lemma 3.2] we have for all $n \in \mathbb{N}$

$$S^{\mathbb{D}}\big(\sup_{k\le n} c_k F_k(\cdot,\gamma_0)\big) \le \eta \quad \mu^*\text{-a.e.}$$

But $\sup_{k\le n} c_k F_k(\cdot,\gamma_0) \to \bar\rho(\cdot,\gamma_0)$ as $n \to \infty$ pointwisely and in $L^2(\ddot\Gamma,\mu^*)$. Hence, by the Banach-Alaoglu theorem and the Banach-Saks theorem,

$$\bar\rho(\cdot,\gamma_0) \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*}) \quad \text{and} \quad S^{\mathbb{D}}(\bar\rho(\cdot,\gamma_0)) \le \eta.$$

Thus, the assertion is proved. □

### 2.3.3 Corresponding process

**Lemma 2.3.10.** *$(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$ is local, i.e., $\mathcal{E}^{\mathbb{D}}_{\mu^*}(F,G) = 0$ provided $F, G \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$ and $\operatorname{supp}(|F|\,\mu^*)$ and $\operatorname{supp}(|G|\,\mu^*)$ are disjoint sets.*

*Proof.* Since $\mathbb{D}$ satisfies the product rule, the proof is the same as in [MR00, Proposition 4.12]. We recall it here just for completeness.

It is enough to show that for every $F \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$

$$S^{\mathbb{D}}(F) = 0 \ \mu^*\text{-a.e. on } \ddot\Gamma \setminus \operatorname{supp}(|F|\,\mu^*).$$

Since $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$ is quasi-regular, by [MR92, Chapter V, Proposition 1.7], there exists $G \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$ with $0 \le G \le 1\!\!1_{\ddot\Gamma\setminus\operatorname{supp}(|F|\mu^*)}$ and $G > 0$ $\mu^*$-a.e. on $\ddot\Gamma \setminus \operatorname{supp}(|F|\,\mu^*)$. Thus $GF = 0$ and hence

$$0 = S^{\mathbb{D}}(GF,F) = GS^{\mathbb{D}}(F,F) + FS^{\mathbb{D}}(G,F).$$

Therefore, $S^{\mathbb{D}}(F) = 0$ on $\ddot\Gamma \setminus \operatorname{supp}(|F|\,\mu^*)$. □

As a consequence of Proposition 2.3.9 and Lemma 2.3.10 we obtain the following result:

**Theorem 2.3.11.** *(i) There exists a conservative diffusion process*

$$\mathbb{M}^{\mathbb{D},\mu^*} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, (\xi_t)_{t\ge 0}, (\mathbb{P}_\gamma)_{\gamma\in\ddot\Gamma}\big)$$

*on $\ddot\Gamma$, which is properly associated with $(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*}))$, i.e., for all $F \in L^2(\ddot\Gamma,\mu^*)$ and all $t > 0$ the function*

$$\gamma \mapsto p_t F(\gamma) := \int_\Omega F(\xi_t)\, d\mathbb{P}_\gamma, \quad \gamma \in \ddot\Gamma, \tag{2.26}$$

*is an* $\mathcal{E}^{\mathbb{D}}_{\mu^*}$*-quasi-continuous version of* $\exp(-tH^{\mathbb{D}}_{\mu^*})F$. $\mathbb{M}^{\mathbb{D},\mu^*}$ *is up to* $\mu^*$*-equivalence unique. In particular,* $\mathbb{M}^{\mathbb{D},\mu^*}$ *is* $\mu^*$*-symmetric, i.e.,*

$$\int_{\ddot{\Gamma}} G p_t F \, d\mu^* = \int_{\ddot{\Gamma}} F p_t G \, d\mu^*, \quad F, G \in L^0(\ddot{\Gamma}, \mu^*), \geq 0,$$

*and has* $\mu^*$ *as invariant measure.*

*(ii) The process* $\mathbb{M}^{\mathbb{D},\mu^*}$ *from (i) is the (up to* $\mu^*$*-equivalence) unique diffusion process having* $\mu^*$ *as invariant measure and solving the martingale problem for* $(H^{\mathbb{D}}_{\mu^*}, D(H^{\mathbb{D}}_{\mu^*}))$*, i.e., for all* $G \in D(H^{\mathbb{D}}_{\mu^*})$

$$\tilde{G}(\xi_t) - \tilde{G}(\xi_0) + \int_0^t H^{\mathbb{D}}_{\mu^*} G(\xi_t) \, ds, \quad t \geq 0,$$

*is an* $(\mathcal{F}_t)_t$*-martingale under* $\mathbb{P}_\gamma$ *for* $\mathcal{E}^{\mathbb{D}}_{\mu^*}$*-q.a.* $\gamma \in \ddot{\Gamma}$*. (Here* $\tilde{G}$ *denotes an* $\mathcal{E}^{\mathbb{D}}_{\mu^*}$*-quasi-continuous version of G, cf. [MR92, Chapter IV, Proposition 3.3].)*

*Proof.* (i) follows directly from [MR92, Chapter IV, Theorem 3.5, and Chapter V, Theorem 1.11]. (ii) follows from [AR95, Theorem 3.5]. □

**Remark 2.3.12.** The process $\mathbb{M}^{\mathbb{D},\mu^*}$ can be taken to be canonical, i.e., $\Omega := D([0,\infty) \to \Gamma)$, the space of càdlàg functions $\omega : [0,\infty) \to \Gamma$, $\xi_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \Omega$, and $\mathcal{F}, (\mathcal{F}_t)_{t\geq 0}$ is the corresponding minimum completed admissible family.

The following lemma shows, that the process $\mathbb{M}^{\mathbb{D},\mu^*}$ actually lives on the smaller space $\Gamma$.

**Lemma 2.3.13.** *The set* $\ddot{\Gamma} \setminus \Gamma$ *is* $\mathcal{E}^{\mathbb{D}}_{\mu^*}$*-exceptional.*

*Proof.* The proof is a modification of the ones for [RS98, Proposition 1 and Corollary 1].

Let $N := \ddot{\Gamma} \setminus \Gamma$. By construction, we have that $\mu^*(N) = 0$. Thus it is sufficient to show that $\mathbb{1}_N$ is $\mathcal{E}^{\mathbb{D}}_{\mu^*}$-quasi-continuous. It is even sufficient to prove this locally, i.e. for any $a \in \mathbb{N}$ the function $\mathbb{1}_{N_a}$ is quasi-continuous, where

$$N_a := \{\gamma \in \ddot{\Gamma} : \sup_{x\in[-a,a]^d} \gamma(\{x\}) \geq 2\}.$$

For this we have to find a sequence of functions $U_n \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$ with

$$\sup_n \mathcal{E}^{\mathbb{D}}_{\mu^*}(U_n, U_n) < +\infty \quad \text{and} \quad U_n \to \mathbb{1}_{N_a} \text{ pointwisely as } n \to \infty.$$

Let $\phi \in C_0^\infty(\mathbb{R})$ with $\mathbb{1}_{[0,1]} \le \phi \le \mathbb{1}_{[-\frac{1}{2},\frac{3}{2})}$ and $|\phi'| \le 3 \cdot \mathbb{1}_{[-\frac{1}{2},\frac{3}{2})}$. For any $n \in \mathbb{N}$, $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, set

$$\phi_{n,i}(x) := \prod_{k=1}^d \phi(nx_k - i_k).$$

Then $\phi_{n,i} \in C_0^\infty(\mathbb{R}^d)$. Set $\mathbb{1}_{n,i}(x) := \prod_{k=1}^d \mathbb{1}_{[-\frac{1}{2},\frac{3}{2})}(nx_k - i_k)$, then $\phi_{n,i} \le \mathbb{1}_{n,i}$.

$$\partial_j \phi_{n,i}(x) = n\phi'(nx_k - i_k) \prod_{k=1}^d \phi(nx_k - i_k),$$

so $\big(\partial_j \phi_{n,i}(x)\big)^2 \le 9n^2 \mathbb{1}_{n,i}(x)$ and hence

$$|\nabla \phi_{n,i}(x)|^2 \le 9n^2 d \mathbb{1}_{n,i}(x). \tag{2.27}$$

Let $\psi \in C^\infty(\mathbb{R})$ with $\mathbb{1}_{[2,\infty)} \le \psi \le \mathbb{1}_{[1,\infty)}$ and $|\psi'| \le 2 \cdot \mathbb{1}_{[1,\infty)}$. Let $A_n := [-na, na]^d \cap \mathbb{Z}^d$, then

$$U_n := \psi\big(\sup_{i \in A_n} \langle \phi_{n,i}, \cdot \rangle\big) \in D(\mathcal{E}_{\mu^*}^{\mathbb{D}}),$$

$U_n$ is continuous, and $U_n \to \mathbb{1}_{N_a}$ pointwisely. So it remains to show that $\mathcal{E}_{\mu^*}^{\mathbb{D}}(U_n, U_n)$ is bounded in $n$.

We have

$$\begin{aligned}\big(\psi'\big(\sup_{i \in A_n} \langle \phi_{n,i}, \gamma \rangle\big)\big)^2 &\le 4 \cdot \mathbb{1}_{\{\sup_{i \in A_n} \langle \phi_{n,i}, \gamma \rangle > 1\}} \\ &\le 4 \cdot \mathbb{1}_{\{\sup_{i \in A_n} \langle \mathbb{1}_{n,i}, \gamma \rangle \ge 2\}}.\end{aligned} \tag{2.28}$$

$$\begin{aligned}S^{\mathbb{D}}(U_n)(\gamma) &= \big(\psi'\big(\sup_{i \in A_n} \langle \phi_{n,i}, \gamma \rangle\big)\big)^2 S^{\mathbb{D}}\big(\sup_{i \in A_n} \langle \phi_{n,i}, \cdot \rangle\big)(\gamma) \\ &\le \big(\psi'\big(\sup_{i \in A_n} \langle \phi_{n,i}, \gamma \rangle\big)\big)^2 \sup_{i \in A_n} S^{\mathbb{D}}\big(\langle \phi_{n,i}, \cdot \rangle\big)(\gamma) \\ &\overset{(2.21)}{\le} \big(\psi'\big(\sup_{i \in A_n} \langle \phi_{n,i}, \gamma \rangle\big)\big)^2 \sup_{i \in A_n} \langle \mathbb{1}_{n,i}, \gamma \rangle \langle S(\phi_{n,i}), \gamma \rangle \\ &\overset{(2.27),(2.28)}{\le} 4 \cdot \mathbb{1}_{\{\sup_{i \in A_n} \langle \mathbb{1}_{n,i}, \gamma \rangle \ge 2\}} \sup_{i \in A_n} \langle \mathbb{1}_{n,i}, \gamma \rangle 9n^2 d \langle \mathbb{1}_{n,i}, \gamma \rangle \\ &\le \text{const} \cdot n^2 \sum_{i \in A_n} \mathbb{1}_{\{\langle \mathbb{1}_{n,i}, \gamma \rangle \ge 2\}} \langle \mathbb{1}_{n,i}, \gamma \rangle^2.\end{aligned} \tag{2.29}$$

But, by the Georgii-Nguyen-Zessin identity (1.22), we have

$$\begin{aligned}
&\int_{\{\langle \mathbb{1}_{n,i},\gamma\rangle \geq 2\}} \langle \mathbb{1}_{n,i},\gamma\rangle^2 \, d\mu^*(\gamma) \\
&\quad \leq \tfrac{z}{Z} \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x) e^{-V_I(x)} \times \\
&\quad \times \int_{\ddot{\Gamma}} \left[\langle \mathbb{1}_{n,i},\gamma\rangle + \mathbb{1}_{n,i}(x)\right] e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} \, d\mu_E(\gamma)\, dx \\
&\quad = \tfrac{z}{Z} \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x) e^{-V_I(x)} \Bigg[ z \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(y) e^{-V_I(y)} e^{-V_E(x-y)} \times \\
&\quad \times \int_{\ddot{\Gamma}} e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} e^{-E_E(y,\gamma)} \, d\mu_E(\gamma)\, dy \\
&\qquad + \mathbb{1}_{n,i}(x) \int_{\ddot{\Gamma}} e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} \, d\mu_E(\gamma) \Bigg] dx \\
&\quad \leq \text{const} \cdot \underbrace{\left(\int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x)\, dx\right)^2}_{\substack{\leq \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x)\,dx \text{ for } n>2, \\ \text{since then } \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x)\,dx = (\frac{2}{n})^d < 1}} + \text{const} \cdot \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x)\, dx,
\end{aligned}$$

where we have used Corollary 2.3.4 and boundedness of $e^{-V_I}$, $e^{-V_E}$. So with (2.29) we obtain for $n > 2$

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(U_n, U_n) \leq \text{const} \cdot n^2 \sum_{i \in A_n} \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(x)\, dx.$$

But for each $x \in \mathbb{R}^d$, $\mathbb{1}_{n,i}(x) \neq 0$ for at most $2^d$ ones of the points $i \in A_n$. Hence

$$\mathcal{E}^{\mathbb{D}}_{\mu^*}(U_n, U_n) \leq \text{const} \cdot n^2 2^d (\tfrac{2}{n})^d = \text{const} \cdot n^{2-d}. \tag{2.30}$$

Thus, since $d \geq 2$,

$$\sup_{n \in \mathbb{N}} \mathcal{E}^{\mathbb{D}}_{\mu^*}(U_n, U_n) < +\infty.$$

This proves the assertion. □

### 2.3.4 On ergodicity

For the application of the invariance principle scheme from Section 2.1 we need ergodicity of the environment process. In this subsection we will give conditions on the corresponding Dirichlet form, which ensure this.

Consider the process

$$\mathbb{M}^{\mathbb{D},\mu^*} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (\xi_t)_{t\geq 0}, (\mathbb{P}_\gamma)_{\gamma\in\Gamma}\big)$$

from Theorem 2.3.11, associated to $\big(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*})\big)$.

Set

$$\mathbb{P}_{\mu^*} := \int_\Gamma \mathbb{P}_\gamma \, d\mu^*(\gamma).$$

We recall the following well-known result on ergodicity, cf., e.g., [AKR98b, Theorem 6.1]:

**Theorem 2.3.14.** *The following assertions are equivalent:*

*(i)* $\big(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*})\big)$ *is* irreducible, *i.e., for all (bounded and hence for all)* $F \in D(\mathcal{E}^{\mathbb{D}}_{\mu^*})$ *with* $\mathcal{E}^{\mathbb{D}}_{\mu^*}(F,F) = 0$ *it follows that* $F$ *is constant.*

*(ii)* $\big(e^{-tH^{\mathbb{D}}_{\mu^*}}\big)_{t>0}$ *is* irreducible, *i.e., if* $G \in L^2(\Gamma,\mu^*)$ *with* $e^{-tH^{\mathbb{D}}_{\mu^*}}(GF) = Ge^{-tH^{\mathbb{D}}_{\mu^*}}F$ *for all* $F \in L^\infty(\Gamma,\mu^*)$*,* $t > 0$*, then* $G$ *is constant.*

*(iii) If* $F \in L^2(\Gamma,\mu^*)$ *with* $e^{-tH^{\mathbb{D}}_{\mu^*}}F = F$ *for all* $t > 0$*, then* $F$ *is constant.*

*(iv)* $\big(e^{-tH^{\mathbb{D}}_{\mu^*}}\big)_{t>0}$ *is* ergodic, *i.e., for all* $F \in L^2(\Gamma,\mu^*)$

$$\int \left[ e^{-tH^{\mathbb{D}}_{\mu^*}}F - \int F\, d\mu^* \right]^2 d\mu^* \to 0, \quad t \to \infty.$$

*(v) If* $F \in D(H^{\mathbb{D}}_{\mu^*})$ *with* $H^{\mathbb{D}}_{\mu^*}F = 0$*, then* $F$ *is constant.*

*(vi)* $\mathbb{P}_{\mu^*}$ *is* time-ergodic*, i.e., every bounded* $\mathcal{F}$*-measurable function* $G : \Omega \to \mathbb{R}$*, which is invariant under time-shifts, is constant* $\mathbb{P}_{\mu^*}$*-a.e.*

*In this case,* $(p_t)_{t>0}$ *as defined in* (2.26) *satisfies*

$$\lim_{t\to\infty} p_t F = \int F\, d\mu^* \quad \mathcal{E}^{\mathbb{D}}_{\mu^*}\text{-q.e.}$$

*for all bounded* $\mathcal{B}(\Gamma)$*-measurable functions* $F : \Gamma \to \mathbb{R}$*.*

## 2.4 The invariance principle

The mean forward velocity in the standard decomposition for the diffusion in a random environment has the following form:

$$\Phi(\gamma) := \nabla_x E_I(0,\gamma) = \sum_{y\in\gamma} \nabla V_I(y) = -\beta_{\gamma,I}(0). \tag{2.31}$$

**Lemma 2.4.1.** $\Phi \in L^2(\Gamma, \mu^*)$.

*Proof.* By Corollary 1.45 and iterated application of the Georgii-Ngyuen-Zessin identity (1.22), we obtain that

$$\begin{aligned}
&\int_\Gamma |\Phi(\gamma)|^2 \, d\mu^*(\gamma)\\
&\quad\le \int_\Gamma \sum_{x\in\gamma} |\nabla V_I(x)| \sum_{y\in\gamma} |\nabla V_I(y)| \,\frac{1}{Z} e^{-E_I(0,\gamma)}\, d\mu_E(\gamma)\\
&\quad= \frac{z}{Z}\int_\Gamma \int_{\mathbb{R}^d} |\nabla V_I(x)| \sum_{y\in\gamma\cup x} |\nabla V_I(y)|\, e^{-E_I(0,\gamma\cup x)} e^{-E_E(x,\gamma)}\, dx\, d\mu_E(\gamma)\\
&\quad= \frac{z}{Z}\int_{\mathbb{R}^d} |\nabla V_I(x)| \int_\Gamma \Big[ |\nabla V_I(x)|\, e^{-E_I(0,\gamma\cup x)} e^{-E_E(x,\gamma)}\\
&\qquad + \sum_{y\in\gamma} |\nabla V_I(y)|\, e^{-E_I(0,\gamma\cup x)} e^{-E_E(x,\gamma)} \Big]\, d\mu_E(\gamma)\, dx\\
&\quad= \frac{z}{Z}\int_{\mathbb{R}^d} |\nabla V_I(x)| \Big[ |\nabla V_I(x)|\, e^{-V_I(x)} \int_\Gamma e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)}\, d\mu_E(\gamma)\\
&\qquad + e^{-V_I(x)} \int_\Gamma \sum_{y\in\gamma} |\nabla V_I(y)|\, e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)}\, d\mu_E(\gamma) \Big]\, dx\\
&\quad\le \frac{z}{Z}\int_{\mathbb{R}^d} |\nabla V_I(x)| \Big[ C\, |\nabla V_I(x)|\, e^{-V_I(x)}\\
&\qquad + e^{-V_I(x)} z \int_{\mathbb{R}^d} |\nabla V_I(y)|\, e^{-V_I(y)} e^{-V_E(y-x)} \times\\
&\qquad\quad \times \int_\Gamma e^{-E_I(0,\gamma)} e^{-E_E(x,\gamma)} e^{-E_E(y,\gamma)}\, d\mu_E(\gamma)\, dy \Big]\, dx\\
&\quad\le \frac{zC}{Z}\int_{\mathbb{R}^d} |\nabla V_I(x)|^2\, e^{-V_I(x)}\\
&\qquad + z\, |\nabla V_I(x)|\, e^{-V_I(x)} \int_{\mathbb{R}^d} |\nabla V_I(y)|\, e^{-V_I(y)} e^{-V_E(y-x)}\, dy\, dx\\
&\quad< +\infty,
\end{aligned}$$

where $C > 0$ denotes the constant from Corollary 1.45. Here the last line follows from the differentiability assumption (D) for $V_I$ and boundedness of $e^{-V_E}$. □

Thus, we can apply [DFGW89, Theorem 2.2.(ii)]:

**Theorem 2.4.2.** *Let $V_I$,$V_E$, and $\mu_E$ be as in Assumptions 2.3.1. Suppose that the corresponding Dirichlet form $\left(\mathcal{E}^{\mathbb{D}}_{\mu^*}, D(\mathcal{E}^{\mathbb{D}}_{\mu^*})\right)$ is irreducible. Consider the associated process*

$$\mathbb{M}^{\mathbb{D},\mu^*} = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (\xi_t)_{t\geq 0}, (\mathbb{P}_\gamma)_{\gamma\in\Gamma}\right)$$

*from Theorem 2.3.11 in the canonical case. Let*

$$X_t := -\int_0^t \Phi(\xi_s)\, ds + \sqrt{2} W_t, \tag{2.32}$$

*where $W_t$ is a standard Brownian motion on $\mathbb{R}^d$. Then $X_t$ solves (2.6). Let $X^{(\varepsilon)}_t := \varepsilon X_{\varepsilon^{-2}t}$, $\varepsilon > 0$. Then*

$$X^\varepsilon \to W_D \tag{2.33}$$

*in the sense of weak-convergence of the corresponding distributions on the paths-space. Here $W_D$ denotes a Brownian motion with covariance $Dt$, and $D$ is given by*

$$D_{ij} = 2\delta_{ij} + 2(\Phi_i, (L^{\mathbb{D}}_{\mu^*})^{-1}\Phi_j)_{L^2(\mu^*)}.$$

# Chapter 3

# Invariance Principle for a Tagged Particle Process

As in the previous chapter we will always consider $\mathbb{R}^d$ with $d \geq 2$. Thus $\mathbb{R}^d \setminus \gamma$ is connected for every configuration $\gamma \in \Gamma$.

The tagged particle process describes the motion of a fixed particle in an equilibrium motion of infinitely many interacting Brownian particles in $\mathbb{R}^d$. The latter dynamics can be desribed heuristically by the following infinite system of stochastic differential equations:

$$dY_t^i = -\sum_{\substack{j\in\mathbb{N}\\ j\neq i}} \nabla V(Y_t^i - Y_t^j)\, dt + \sqrt{2}\, dW_t^i, \quad t \geq 0, i \in \mathbb{N}. \tag{3.1}$$

Here $V$ denotes a symmetric pair potential, and $W_t^i$, $i \in \mathbb{N}$, are independent Brownian motions. Such dynamics are called *gradient stochastic dynamics*. The informal generator corresponding to (3.1) is given by

$$L_{\mathrm{gsd}} = \sum_{i\in\mathbb{N}}^{\infty} \partial_{y^i}^2 - \sum_{i\in\mathbb{N}} \Bigg( \sum_{\substack{j\in\mathbb{N}\\ j\neq i}} \nabla V(y^i - y^j) \Bigg) \partial_{y^j}.$$

Gradient stochastic dynamics have been constructed by Lang [Lan77] for $V \in C_0^3(\mathbb{R}^d)$ using finite-dimensional approximations and stochastic differential equations. Osada [Osa96] and Yoshida [Yos96] have also discussed singular potentials like Lennard-Jones by using Dirichlet form techniques, but they could not show that their processes are actually weak solutions of (3.1). This was proved by Albeverio, Kondratiev, and Röckner [AKR98b] by showing an integration by parts formula for the corresponding Gibbs measures. Ma and

Röckner [MR00] proved existence of a solution by another approach than integration by parts. Finally, Grothaus, Kondratiev, and Röckner [GKR07] discussed an $N/V$-limit. Their work also includes the case $d = 1$.

The tagged particle process looks the following. Consider a solution $Y_t$ of (3.1). Now fix one particle and consider its motion in the sea of the other particles. Therefore, consider the following coordinate transformation:

$$X_t := Y_t^1, \quad \xi_t^i := Y_t^{i+1} - Y_t^1,\ i \in \mathbb{N}. \tag{3.2}$$

Then we can rewrite (3.1) as

$$dX_t = \sum_{i\in\mathbb{N}} \nabla V(\xi_t^i)\,dt + \sqrt{2}\,dW_t^1 \tag{3.3}$$

$$\begin{aligned} d\xi_t^i = &-\Big(\sum_{\substack{j\in\mathbb{N}\\ j\neq i}} \nabla V(\xi_t^i - \xi_t^j) + \nabla V(\xi_t^i) + \sum_{j\in\mathbb{N}} \nabla V(\xi_t^j)\Big)\,dt \\ &+ \sqrt{2}\,d\big(W_t^{i+1} - W_t^1\big). \end{aligned} \tag{3.4}$$

$X_t$ and $\xi_t = (\xi_t^i)_i$ denote now the tagged particle process and the environment process, resp. The coordinate transform (3.2) gives

$$\partial_{y^1} = \partial_x - \sum_{i\in\mathbb{N}} \partial_{\xi^i}, \quad \partial_{y^{i+1}} = \partial_{\xi^i}.$$

Thus, the informal generator for the coupled process $(X_t, \xi_t^1, \xi_t^2, \dots)$ has the form

$$\begin{aligned} L_{\text{coup}} = &\sum_{i\in\mathbb{N}}^{\infty} \partial_{\xi^i}^2 + \Big(\partial_x - \sum_{i\in\mathbb{N}}^{\infty} \partial_{\xi^i}\Big)^2 + \sum_{i\in\mathbb{N}} \nabla V(\xi^i)\Big(\partial_x - \sum_{i\in\mathbb{N}}^{\infty} \partial_{\xi^i}\Big) \\ &- \sum_{i\in\mathbb{N}} \Big(\nabla V(\xi^i) + \sum_{\substack{j\in\mathbb{N}\\ j\neq i}} \nabla V(\xi^i - \xi^j)\Big)\partial_{\xi^i}. \end{aligned} \tag{3.5}$$

We will show an invariance principle for the tagged particle dynamics. For this we will again apply the general scheme described in Section 2.1. First, let us recall the construction of the tagged particle process.

## 3.1 Construction of a tagged particle process

A tagged particle process has been discussed, e.g., by Guo and Papanicolaou [GP87] and De Masi et al. [DFGW89]. But both works only treat the case of bounded, positive interaction potentials with compact support. Fattler and Grothaus [FG08] have The constructed rigorously the process for a wide class of interaction potentials $V$ including the Lennard-Jones potential (1.28). They use a Dirichlet forms approach on the configuration space. For completeness, we will recall their construction here. Note that, as in the previous chapter, the environment process is constructed directly here, see Theorem 3.1.5, which is useful for the application of the invariance principle scheme.

The generator for the environment process $\xi_t$ consists of a generator of gradient stochastic dynamics with additional drift plus extra terms. Finally, the generator of the coupled process $(X_t, \xi_t)$ contains the environment process genrator. Therefore, the construction is divided into four steps. We recall the Dirichlet form and the generator of gradient stochastic dynamics (gsd), and afterwards we will add an additional drift (gsdad). Then, we will discuss the environment process (env) and finally the coupled process (coup).

**Assumption 3.1.1.** Throughout this chapter, let $V$ be a symmetric pair potential which satisfies (SS), (I), (LR), (D) and (LS) (see Subsection 1.4.2), and let $z > 0$ be an activity parameter.

**Remark 3.1.2.** In contrast to the situation of a diffusing particle in a frozen environment, as described in the previous chapter, here it is not reasonable to consider different potentials for the environment Gibbs measure and the interaction of the tagged particle with the environment.

## 3.1.1 Dynamics w.r.t. the intrinsic gradient

### Gradient stochastic dynamics

Consider a grand canonical Gibbs measure $\mu_0 \in \mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z, V)$. In [AKR98b], the gradient stochastic dynamics is described by the closure of the pre-Dirichlet form

$$\mathcal{E}^{\mu_0}_{\mathrm{gsd}}(F, G) := \int_\Gamma (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma \Gamma}\, d\mu_0(\gamma), \tag{3.6}$$

$F, G \in \mathcal{F}C_b^\infty$. Via integration by parts, its generator is obtained as the Friedrichs extension of

$$L^{\mu_0}_{\mathrm{gsd}} F(\gamma) = \sum_{i,j=1}^N \partial_i \partial_j g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_N, \gamma\rangle\big) \langle (\nabla f_i, \nabla f_j), \gamma\rangle$$

$$
\begin{aligned}
&+\sum_{j=1}^{N} \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\times \\
&\times\left(\langle \Delta f_j,\gamma\rangle - \sum_{\{x,y\}\subset\gamma}\big(\nabla V(x-y),\nabla f_j(x)-\nabla f_j(y)\big)\right),
\end{aligned}
$$

$F\in\mathcal{F}C_b^\infty$, $F(\gamma)=g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)$.

### Gradient stochastic dynamics with additional drift

Now consider the Dirichlet form (3.6) but with a measure $\mu\in\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(ze^{-V},V)$ instead of $\mu_0\in\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(z,V)$.

$$
\mathcal{E}^{\mu}_{\mathrm{gsdad}}(F,G):=\int_\Gamma(\nabla^\Gamma F(\gamma),\nabla^\Gamma G(\gamma))_{T_\gamma\Gamma}\,d\mu(\gamma), \tag{3.7}
$$

$F,G\in\mathcal{F}C_b^\infty$, is related to a gradient stochastic dynamics with additional drift term. Again, via integration by parts, its generator is obtained as

$$
\begin{aligned}
L^{\mu}_{\mathrm{gsdad}}F(\gamma)=&\sum_{i,j=1}^{N}\partial_i\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle(\nabla f_i,\nabla f_j),\gamma\rangle \\
&+\sum_{j=1}^{N}\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\times \\
&\times\Bigg(\langle\Delta f_j,\gamma\rangle+\langle(\nabla V,\nabla f_j),\gamma\rangle \\
&\quad-\sum_{\{x,y\}\subset\gamma}\big(\nabla V(x-y),\nabla f_j(x)-\nabla f_j(y)\big)\Bigg),
\end{aligned}
$$

$F\in\mathcal{F}C_b^\infty$.

In [FG08, Theorem 5.3] it is shown, that $\big(\mathcal{E}^{\mu}_{\mathrm{gsdad}},\mathcal{F}C_b^\infty\big)$ is closable, and its closure $\big(\mathcal{E}^{\mu}_{\mathrm{gsdad}},D(\mathcal{E}^{\mu}_{\mathrm{gsdad}})\big)$ is a conservative, symmetric Dirichlet form, which is quasi-regular and local on the multiple configuration space $\ddot{\Gamma}$. The arguments here are similar to the classical case, i.e., the ones for $\mathcal{E}^{\mu_0}_{\mathrm{gsd}}$ treated in [AKR98b]. Therefore we only sketch the line of argumentation here.

Closability follows immediately from the representation of $\mathcal{E}^{\mu}_{\mathrm{gsdad}}$ via its generator $L^{\mu}_{\mathrm{gsdad}}$, and conservativity is obvious.

Quasi-regularity on $\ddot{\Gamma}$ follows as a special case of the results in [MR00]. Note that the square field operator

$$
S^\Gamma(F,G):=(\nabla^\Gamma F,\nabla^\Gamma G)_{T\Gamma},\quad F,G\in\mathcal{F}C_b^\infty,
$$

is given by lifting the standard square field operator

$$S(f,g) := (\nabla f, \nabla g), \quad f,g \in C_0^\infty(\mathbb{R}^d),$$

to the configuration space, i.e., it holds that

$$S^\Gamma(F,G)(\gamma) = \sum_{i=1}^N \sum_{j=1}^M \partial_i g_F\big(\langle f_1,\gamma\rangle,\ldots,\langle f_N,\gamma\rangle\big)\times \\ \times\, \partial_j g_G\big(\langle g_1,\gamma\rangle,\ldots,\langle g_M,\gamma\rangle\big)\langle S(f_i,g_j),\gamma\rangle, \quad \gamma\in\ddot{\Gamma}. \quad (3.8)$$

Thus, $S^\Gamma$ is as in the general framework of [MR00]. In Proposition 2.3.9 in the previous chapter and in Proposition 3.1.4 below, this framework is not directly applicable, since

$$S^{\mathbb{D}}(F,G) := (\mathbb{D}F, \mathbb{D}G), \quad F,G \in \mathcal{F}C_b^\infty,$$

is not given as a lifting of some underlying square field operator.

Finally, locality of $\big(\mathcal{E}^\mu_{\mathrm{gsdad}}, \mathcal{F}C_b^\infty\big)$ is satisfied, since $\nabla^\Gamma$ satisfies the product rule (cf. also Lemma 2.3.10). Hence, by the general theory of Dirichlet forms (see, e.g., [MR92]), there exists a conservative diffusion process on $\ddot{\Gamma}$ which is properly associated to $\big(\mathcal{E}^\mu_{\mathrm{gsdad}}, D(\mathcal{E}^\mu_{\mathrm{gsdad}})\big)$, has $\mu$ as invariant measure, and solves the corresponding martingale problem. Since we have assumed that $d\geq 2$, by the results from [RS98], we have that $\ddot{\Gamma}\setminus\Gamma$ is $\mathcal{E}^\mu_{\mathrm{gsdad}}$-exceptional. Hence the diffusion process is actually supported on the smaller space $\Gamma$.

### 3.1.2 The environment process

Since we will need the environment process for the application of the invariance principle scheme, we will recall its construction.

The environment process for the tagged particle is described by the following pre-Dirichlet form on $L^2(\Gamma,\mu)$:

$$\mathcal{E}^\mu_{\mathrm{env}}(F,G) := \int_\Gamma (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma\Gamma} + \big(\mathbb{D}F(\gamma), \mathbb{D}G(\gamma)\big)\, d\mu(\gamma), \quad (3.9)$$

$F\in\mathcal{F}C_b^\infty$. Recall that here

$$\mathbb{D}F(\gamma) = \langle \nabla^\Gamma F(\gamma), \gamma\rangle.$$

By similar arguments as in the proof of Proposition 2.3.7, we have that $\big(\mathcal{E}^{\mu}_{\mathrm{env}}, \mathcal{F}C_b^{\infty}\big)$ is generated on $L^2(\mu)$ by

$$\begin{aligned} L^{\mu}_{\mathrm{env}}F(\gamma) = L^{\mu}_{\mathrm{gsdad}}F(\gamma) &\\ + \sum_{i,j=1}^{N} \partial_i\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\big(\langle\nabla f_i,\gamma\rangle,\langle\nabla f_j,\gamma\rangle\big) &\\ + \sum_{j=1}^{N} \partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\times &\\ \times\Bigg(\langle\Delta f_j,\gamma\rangle + \big(\langle\nabla V,\gamma\rangle,\langle\nabla f_j,\gamma\rangle\big) &\\ - \Big(\sum_{\{x,y\}\subset\gamma}\nabla V(x-y), \sum_{\{x,y\}\subset\gamma}(\nabla f_j(x)-\nabla f_j(y))\Big)\Bigg). & \end{aligned} \tag{3.10}$$

The conservativity, i.e., $1 \in \mathcal{F}C_b^{\infty}$ and $\mathcal{E}^{\mu}_{\mathrm{env}}(1,1) = 0$, is again obvious. Hence we obtain the following result:

**Proposition 3.1.3.** *The form $(\mathcal{E}^{\mu}_{env}, \mathcal{F}C_b^{\infty})$ is closable on $L^2(\mu)$, and its closure $(\mathcal{E}^{\mu}_{env}, D(\mathcal{E}^{\mu}_{env}))$ is a symmetric, conservative Dirichlet form. It is generated by the Friedrichs extension of $(H^{\mu}_{env} := -L^{\mu}_{env}, \mathcal{F}C_b^{\infty})$, which we also denote by $H^{\mu}_{env}$.*

Now we will show the quasi-regularity of $\big(\mathcal{E}^{\mu}_{\mathrm{env}}, D(\mathcal{E}^{\mu}_{\mathrm{env}})\big)$ on the multiple configuration space $\ddot{\Gamma}$.

Consider $\mu$ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and $(\mathcal{E}^{\mu}_{\mathrm{env}}, D(\mathcal{E}^{\mu}_{\mathrm{env}}))$ as a Dirichlet form on $L^2(\ddot{\Gamma},\mu)$. The gradients $\nabla^{\Gamma}$ and $\mathbb{D}$ extend to linear operators on $D(\mathcal{E}^{\mu}_{\mathrm{env}})$, which we denote with the same symbols. The same holds for the corresponding square field operators $S^{\Gamma}$ and $S^{\mathbb{D}}$, resp. Also recall the notation $S(F) := S(F,F)$ for any square field operator.

**Proposition 3.1.4.** *$(\mathcal{E}^{\mu}_{env}, D(\mathcal{E}^{\mu}_{env}))$ is quasi-regular on $L^2(\ddot{\Gamma},\mu)$.*

*Proof.* The proof is analogous to the one for Proposition 2.3.9. It suffices to show that there exists a bounded, complete metric $\bar{\rho}$ on $\ddot{\Gamma}$, which generates the vague topology on $\ddot{\Gamma}$ and fulfills the following condition: for all $\gamma_0 \in \Gamma$

$$\bar{\rho}(\cdot,\gamma_0) \in D(\mathcal{E}^{\mu}_{\mathrm{env}}) \quad \text{and} \quad (S^{\Gamma} + S^{\mathbb{D}})(\bar{\rho}(\cdot,\gamma_0)) \le \eta \ \mu\text{-a.e.}$$

for some function $\eta \in L^1(\ddot{\Gamma},\mu)$ independent of $\gamma_0$.

For $k \in \mathbb{N}$, let $B_k$ denote the open ball in $\mathbb{R}^d$ with center 0 and radius $k$. Set

$$g_k(x) := \tfrac{2}{3}\big(\tfrac{1}{2} - \mathrm{dist}(x, B_k) \wedge \tfrac{1}{2}\big), \quad x \in \mathbb{R}^d,$$

and $\phi_k := 3g_k$.

By [MR00, Example 4.5.1], we have that the standard square field operator $(S, H^{1,2}(\mathbb{R}^d))$, where $H^{1,2}(\mathbb{R}^d)$ denotes the $(1,2)$-Sobolev space on $\mathbb{R}^d$, satisfies the following condition:

(Q) There exist $\chi_j \in C_0^\infty(\mathbb{R}^d)$, $\chi_j > 0$, $j \in \mathbb{N}$, and $f_{ln} \in C(\mathbb{R}^d)$, $l, n \in \mathbb{N}$, such that

- (i) $\sup_{l\in\mathbb{N}} f_{ln} = |y_n - \cdot|$ for all $n \in \mathbb{N}$ and some $\{y_n : n \in \mathbb{N}\} \subset \mathbb{R}^d$ dense;
- (ii) there exists $C > 0$ such that, for all $j, l, n \in \mathbb{N}$ and all $\phi \in C_b^\infty(\mathbb{R}^d)$, $\chi_j(\phi \circ f_{ln}) \in C_0^\infty(\mathbb{R}^d)$ and

$$S(\chi_j(\phi \circ f_{ln})) \le C \sup\{\|\phi'\|_\infty, \|\phi\|_\infty\}^2 (\chi_j + S(\chi_j)^{\frac{1}{2}})^2;$$

- (iii) for all $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $\chi_j \equiv 1$ on $B_k$.

Choose $(j_k)_{k\in\mathbb{N}}$ such that $\chi_{j_k} \equiv 1$ on $B_{k+1}$. Then, by [MR00, Lemma 4.10], for all $k, j \in \mathbb{N}$, $\phi_k g_j \in H^{1,2}(\mathbb{R}^d)$ and

$$S(\phi_k g_j) \le \tilde{\chi}_{j_k}^2, \tag{3.11}$$

with $\tilde{\chi}_{j_k} = 4\chi_{j_k}\big(S(\chi_{j_k})^{\frac{1}{2}} + C(\chi_{j_k} + S(\chi_{j_k})^{\frac{1}{2}})\big)^2$.

Let $f \in H^{1,2}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, then $\langle f, \cdot\rangle \in D(\mathcal{E}^\mu_{\mathrm{env}})$, and

$$(S^\Gamma + S^{\mathbb{D}})(\langle f, \cdot\rangle) = \langle (\nabla f, \nabla f), \cdot\rangle + \big(\langle \nabla f, \cdot\rangle, \langle \nabla f, \cdot\rangle\big).$$

If $\operatorname{supp} f \subset B_{k+1}$, then, by (2.21),

$$S^{\mathbb{D}}(\langle f, \cdot\rangle)(\gamma) = N_{B_{k+1}}(\gamma)\langle S(f), \gamma\rangle.$$

In particular, this holds for $f = \phi_k g_j$. Furthermore, for any $f \in H^{1,2}(\mathbb{R}^d)$ it holds that

$$S^\Gamma(\langle f, \cdot\rangle)(\gamma) = \langle S(f), \gamma\rangle, \tag{3.12}$$

since $S^\Gamma$ is the lifting of $S$ to $\ddot{\Gamma}$. Thus, with (3.11), we have

$$(S^\Gamma + S^{\mathbb{D}})(\langle \phi_k g_j, \cdot\rangle) \le (1 + N_{B_{k+1}}(\cdot))\langle \tilde{\chi}_{j_k}^2, \cdot\rangle. \tag{3.13}$$

Let $\zeta \in C_b^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $[0,\infty)$, $\zeta(t) = t$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\zeta' > 0$, and $\zeta'' \le 0$. Then similarly as in [RS95, Lemma 3.2] we obtain that for any fixed $\gamma_0 \in \ddot{\Gamma}$ and for any $k, n \in \mathbb{N}$

$$\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big) \in D(\mathcal{E}^\mu_{\mathrm{env}}).$$

Furthermore,

$$(S^\Gamma + S^{\mathbb{D}})\bigg(\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big)\bigg) \le (1 + N_{B_{k+1}}(\cdot))\langle \tilde{\chi}^2_{j_k}, \cdot\rangle \quad \mu\text{-a.e.} \tag{3.14}$$

Set

$$F_k(\gamma, \gamma_0) := \zeta\big(\sup_{j\in\mathbb{N}} |\langle \phi_k g_j, \gamma\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big),$$

then for fixed $\gamma_0 \in \ddot{\Gamma}$

$$\zeta\big(\sup_{j\le n} |\langle \phi_k g_j, \cdot\rangle - \langle \phi_k g_j, \gamma_0\rangle|\big) \longrightarrow F_k(\cdot, \gamma_0), \quad n \to \infty,$$

pointwisely and in $L^2(\ddot{\Gamma}, \mu)$. Hence, by (3.14), the Banach-Alaoglu theorem, and the Banach-Saks theorem (see, e.g., [MR92, Appendix A.2]),

$$\begin{gathered} F_k(\cdot, \gamma_0) \in D(\mathcal{E}^\mu_{\mathrm{env}}), \\ (S^\Gamma + S^{\mathbb{D}})(F_k(\cdot, \gamma_0)) \le (1 + N_{B_{k+1}}(\cdot))\langle \tilde{\chi}^2_{j_k}, \cdot\rangle \quad \mu\text{-a.e.} \end{gathered} \tag{3.15}$$

Set

$$c_k := \left(1 + \int_{\ddot{\Gamma}} (1 + N_{B_{k+1}}(\gamma))\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu(\gamma)\right)^{-\frac{1}{2}} 2^{-\frac{k}{2}}.$$

Then the Ruelle bound property of $\mu$ implies $c_k \in (0,\infty)$ for all $k$, and $c_k \to 0$, $k \to \infty$. Define

$$\bar{\rho}(\gamma_1, \gamma_2) := \sup_{k\in\mathbb{N}} c_k F_k(\gamma_1, \gamma_2). \tag{3.16}$$

Then, by [MR00, Theorem 3.6], $\bar{\rho}$ is a bounded, complete metric on $\ddot{\Gamma}$, which generates the vague topology.

By (3.15), we have that

$$\begin{aligned} (S^\Gamma + S^{\mathbb{D}})(c_k F_k(\cdot, \gamma_0)) &= c_k^2 (S^\Gamma + S^{\mathbb{D}})(F_k(\cdot, \gamma_0)) \\ &\le 2^{-k}\left(1 + \int_{\ddot{\Gamma}} (1 + N_{B_{k+1}}(\gamma))\langle \tilde{\chi}^2_{j_k}, \gamma\rangle \, d\mu(\gamma)\right)^{-1} \times \\ &\quad \times (1 + N_{B_{k+1}}(\cdot))\langle \tilde{\chi}^2_{j_k}, \cdot\rangle \quad \mu\text{-a.e.} \end{aligned}$$

Set

$$\eta := \sup_k \left[ 2^{-k} \left( 1 + \int_{\ddot{\Gamma}} (1 + N_{B_{k+1}}(\gamma)) \langle \tilde{\chi}_{j_k}^2, \gamma \rangle \, d\mu(\gamma) \right)^{-1} (1 + N_{B_{k+1}}(\cdot)) \langle \tilde{\chi}_{j_k}^2, \cdot \rangle \right],$$

then

$$\begin{aligned} \int_{\ddot{\Gamma}} \eta(\gamma') \, d\mu(\gamma') &\leq \sum_{k=1}^{\infty} 2^{-k} \left( 1 + \int_{\ddot{\Gamma}} (1 + N_{B_{k+1}}(\gamma)) \langle \tilde{\chi}_{j_k}^2, \gamma \rangle \, d\mu(\gamma) \right)^{-1} \times \\ &\qquad \times \int_{\ddot{\Gamma}} (1 + N_{B_{k+1}}(\gamma')) \langle \tilde{\chi}_{j_k}^2, \gamma' \rangle \, d\mu(\gamma') \\ &\leq \sum_{k=1}^{\infty} 2^{-k} = 1 < +\infty, \end{aligned}$$

so $\eta \in L^1(\mu)$, and as in [RS95, Lemma 3.2] we have for all $n \in \mathbb{N}$

$$(S^{\Gamma} + S^{\mathbb{D}}) \big( \sup_{k \leq n} c_k F_k(\cdot, \gamma_0) \big) \leq \eta \quad \mu\text{-a.e.}.$$

But $\sup_{k \leq n} c_k F_k(\cdot, \gamma_0) \to \bar{\rho}(\cdot, \gamma_0)$ as $n \to \infty$ pointwisely and in $L^2(\ddot{\Gamma}, \mu)$. Hence, by the Banach-Alaoglu theorem and the Banach-Saks theorem,

$$\bar{\rho}(\cdot, \gamma_0) \in D(\mathcal{E}^{\mu}_{\text{env}}) \quad \text{and} \quad (S^{\Gamma} + S^{\mathbb{D}})(\bar{\rho}(\cdot, \gamma_0)) \leq \eta.$$

Thus, the assertion is proved. □

Locality of $\big(\mathcal{E}^{\mu}_{\text{env}}, D(\mathcal{E}^{\mu}_{\text{env}})\big)$ follows again from the product rule for $\nabla^{\Gamma}$ and $\mathbb{D}$, see also the proof of Lemma 2.3.10.

Thus, by the general theory of Dirichlet forms, we have the following result:

**Theorem 3.1.5.** *(i) There exists a conservative diffusion process*

$$\mathbb{M}^{\mu}_{env} = \big( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\xi_t)_{t \geq 0}, (\mathbb{P}_{\gamma})_{\gamma \in \ddot{\Gamma}} \big)$$

*on $\ddot{\Gamma}$, which is properly associated with $(\mathcal{E}^{\mu}_{env}, D(\mathcal{E}^{\mu}_{env}))$, i.e., for all $F \in L^2(\ddot{\Gamma}, \mu)$ and all $t > 0$ the function*

$$\gamma \mapsto p_t F(\gamma) := \int_{\Omega} F(\xi_t) \, d\mathbb{P}_{\gamma}, \quad \gamma \in \ddot{\Gamma}, \tag{3.17}$$

*is an $\mathcal{E}^{\mu}_{env}$-quasi-continuous version of $\exp(-tH^{\mu}_{env})F$. $\mathbb{M}^{\mu}_{env}$ is up to $\mu$-equivalence unique. In particular, $\mathbb{M}^{\mu}_{env}$ is $\mu$-symmetric, i.e.,*

$$\int_{\ddot{\Gamma}} G p_t F \, d\mu = \int_{\ddot{\Gamma}} F p_t G \, d\mu, \quad F, G \in L^0(\ddot{\Gamma}, \mu), \geq 0,$$

*and has $\mu$ as invariant measure.*

*(ii) The process $\mathbb{M}^{\mu}_{env}$ from (i) is the (up to $\mu$-equivalence) unique diffusion process having $\mu$ as invariant measure and solving the martingale problem for $(H^{\mu}_{env}, D(H^{\mu}_{env}))$, i.e., for all $G \in D(H^{\mu}_{env})$*

$$\tilde{G}(\xi_t) - \tilde{G}(\xi_0) + \int_0^t H^{\mu}_{env} G(\xi_t)\, ds, \quad t \geq 0,$$

*is an $(\mathcal{F}_t)_t$-martingale under $\mathbb{P}_\gamma$ for $\mathcal{E}^{\mu}_{env}$-q.a. $\gamma \in \ddot{\Gamma}$. (Here $\tilde{G}$ denotes an $\mathcal{E}^{\mu}_{env}$-quasi-continuous version of $G$, cf. [MR92, Chapter IV, Proposition 3.3].)*

*Proof.* (i) follows directly from [MR92, Chapter IV, Theorem 3.5, and Chapter V, Theorem 1.11]. (ii) follows from [AR95, Theorem 3.5]. □

**Remark 3.1.6.** The process $\mathbb{M}^{\mu}_{\text{env}}$ can be taken to be canonical, i.e., $\Omega := D([0,\infty) \to \Gamma)$, the space of càdlàg functions $\omega : [0,\infty) \to \Gamma$, $\xi_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \Omega$, and $\mathcal{F}, (\mathcal{F}_t)_{t\geq 0}$ is the corresponding minimum completed admissible family.

The following lemma shows, that the process $\mathbb{M}^{\mu}_{\text{env}}$ actually lives on the smaller space $\Gamma$.

**Lemma 3.1.7.** *The set $\ddot{\Gamma} \setminus \Gamma$ is $\mathcal{E}^{\mu}_{env}$-exceptional.*

*Proof.* The proof is analogous to the one for Lemma 2.3.13.

It is sufficient to prove that for any $a \in \mathbb{N}$ the function $\mathbb{1}_{N_a}$ is $\mathcal{E}^{\mu}_{\text{env}}$-quasi-continuous, where

$$N_a := \{\gamma \in \ddot{\Gamma} : \sup_{x\in[-a,a]^d} \gamma(\{x\}) \geq 2\}.$$

For this we have to find a sequence of functions $U_n \in D(\mathcal{E}^{\mu}_{\text{env}})$ with

$$\sup_n \mathcal{E}^{\mu}_{\text{env}}(U_n, U_n) < +\infty \quad \text{and} \quad U_n \to \mathbb{1}_{N_a} \text{ pointwisely as } n \to \infty.$$

Let $\phi$, $\phi_{n,i}$, $\mathbb{1}_{n,i}$, and $\psi$ be as in the proof of Lemma 2.3.13. Let $A_n := [-na, na]^d \cap \mathbb{Z}^d$, then

$$U_n := \psi\big(\sup_{i\in A_n} \langle \phi_{n,i}, \cdot\rangle\big) \in D(\mathcal{E}^{\mu}_{\text{env}}),$$

$U_n$ is continuous, and $U_n \to \mathbb{1}_{N_a}$ pointwisely. So it remains to show that $\mathcal{E}^{\mu}_{\text{env}}(U_n, U_n)$ is bounded in $n$.

We have

$$
\begin{aligned}
&(S^\Gamma + S^{\mathbb{D}})(U_n)(\gamma)\\
&\quad = \big(\psi'\big(\sup_{i\in A_n}\langle \phi_{n,i},\gamma\rangle\big)\big)^2 (S^\Gamma + S^{\mathbb{D}})\big(\sup_{i\in A_n}\langle \phi_{n,i},\cdot\rangle\big)(\gamma)\\
&\quad \le \big(\psi'\big(\sup_{i\in A_n}\langle \phi_{n,i},\gamma\rangle\big)\big)^2 \sup_{i\in A_n}(S^\Gamma + S^{\mathbb{D}})\big(\langle \phi_{n,i},\cdot\rangle\big)(\gamma)\\
&\quad \overset{(2.21),(3.12)}{\le} \big(\psi'\big(\sup_{i\in A_n}\langle \phi_{n,i},\gamma\rangle\big)\big)^2 \sup_{i\in A_n}[(1+\langle \mathbb{1}_{n,i},\gamma\rangle)\langle S(\phi_{n,i}),\gamma\rangle]\\
&\quad \overset{(2.27),(2.28)}{\le} 4\cdot \mathbb{1}_{\{\sup_{i\in A_n}\langle \mathbb{1}_{n,i},\gamma\rangle\ge 2\}} \sup_{i\in A_n}[(1+\langle \mathbb{1}_{n,i},\gamma\rangle)9n^2 d\langle \mathbb{1}_{n,i},\gamma\rangle]\\
&\quad \le \mathrm{const}\cdot n^2 \sum_{i\in A_n} \mathbb{1}_{\{\langle \mathbb{1}_{n,i},\gamma\rangle\ge 2\}}\langle \mathbb{1}_{n,i},\gamma\rangle^2.
\end{aligned}
\tag{3.18}
$$

Let $(k_\mu^{(n)})_n$ denote the system of correlation functions of $\mu$. By assumption there exists a constant $C_\mu > 0$ such that

$$k_\mu^{(n)} \le C_\mu^n \quad \forall n\in\mathbb{N}$$

(Ruelle bound). Let

$$G(\eta) := \mathbb{1}_{n,i}(y)^2 \mathbb{1}_{\{y\}}(\eta) + 2\mathbb{1}_{n,i}(x)\mathbb{1}_{n,i}(y)\mathbb{1}_{\{x,y\}}(\eta),$$

then

$$
\begin{aligned}
&\int_{\{\langle \mathbb{1}_{n,i},\gamma\rangle\ge 2\}} \langle \mathbb{1}_{n,i},\gamma\rangle^2\, d\mu(\gamma)\\
&\qquad \le \int_\Gamma KG(\gamma)\\
&\qquad = \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(y)^2 k_\mu^{(1)}(y) z e^{-V(y)}\, dy\\
&\qquad\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 2\mathbb{1}_{n,i}(x)\mathbb{1}_{n,i}(y) k_\mu^{(2)}(x,y)\tfrac{z^2}{2!} e^{-V(x)}e^{-V(y)}\, dx\, dy\\
&\qquad \le C_\mu z \int_{\mathbb{R}^d} \mathbb{1}_{n,i}(y)e^{-V(y)}\, dy + C_\mu^2 z^2\left(\int_{\mathbb{R}^d} \mathbb{1}_{n,i}(y)e^{-V(y)}\, dy\right)^2\\
&\qquad \le \mathrm{const}\int_{\mathbb{R}^d} \mathbb{1}_{n,i}(y)\, dy,
\end{aligned}
$$

since $e^{-V}$ is bounded. Then with the same argument as in the proof of Lemma 2.3.13, we obtain

$$\mathcal{E}^\mu_{\mathrm{env}}(U_n,U_n) \le \mathrm{const}\cdot n^{2-d}. \tag{3.19}$$

Thus, since $d \geq 2$,

$$\sup_{n \in \mathbb{N}} \mathcal{E}^{\mu}_{\mathrm{env}}(U_n, U_n) < +\infty.$$

This proves the assertion. □

### 3.1.3 The coupled process

The coupled process, which describes the motion of the tagged particle together with the environment, lives on $\mathbb{R}^d \times \Gamma$, and the corresponding pre-Dirichlet form acts on functions $\mathfrak{F} \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}C_b^\infty$, i.e.,

$$\mathfrak{F}(x, \gamma) = f(x)F(\gamma), \quad x \in \mathbb{R}^d, \gamma \in \Gamma,$$

for some $f \in C_0^\infty(\mathbb{R}^d), F \in \mathcal{F}C_b^\infty$. For brevity, we write $\mathfrak{F} = fF$ instead of $\mathfrak{F} = f \otimes F$. Furthermore, introduce the following gradients on $\mathbb{R}^d \times \Gamma$:

$$(\mathbb{D} - \nabla)\mathfrak{F} := f\mathbb{D}F - \nabla f F \quad \text{and} \quad \nabla^\Gamma \mathfrak{F} := f \nabla^\Gamma F.$$

Then the pre-Dirichlet form on $L^2(\mathbb{R}^d \times \Gamma, dx \otimes \mu)$ for the coupled process has the form

$$\begin{aligned}
&\mathcal{E}^{dx\otimes\mu}_{\mathrm{coup}}(\mathfrak{F}, \mathfrak{G}) \\
&\quad = \int_{\mathbb{R}^d \times \Gamma} \big((\mathbb{D} - \nabla)\mathfrak{F}(x, \gamma), (\mathbb{D} - \nabla)\mathfrak{G}(x, \gamma)\big) \\
&\qquad + (\nabla^\Gamma \mathfrak{F}(x, \gamma), \nabla^\Gamma \mathfrak{G}(x, \gamma))_{T_\gamma\Gamma}\, dx d\mu(\gamma) \qquad (3.20) \\
&\quad = \int_{\mathbb{R}^d} f(x)g(x)\, dx \int_\Gamma \big[(\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma\Gamma} + \big(\mathbb{D}F(\gamma), \mathbb{D}G(\gamma)\big)\big]\, d\mu(\gamma) \\
&\qquad - \int_{\mathbb{R}^d \times \Gamma} \big[f(x)G(\gamma)(\mathbb{D}F(\gamma), \nabla g(x)) \\
&\qquad\quad + F(\gamma)g(x)(\nabla f(x), \mathbb{D}G(\gamma))\big]\, dx d\mu(\gamma) \\
&\qquad + \int_\Gamma F(\gamma)G(\gamma)\, d\mu(\gamma) \int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x))\, dx, \qquad (3.21)
\end{aligned}$$

$\mathfrak{F} = fF, \mathfrak{G} = gG \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}C_b^\infty$.

$\mathcal{E}^{dx\otimes\mu}_{\text{coup}}$ is generated on $L^2(\mathbb{R}^d\times\Gamma, dx\otimes\mu)$ by

$$
\begin{aligned}
&L^{dx\otimes\mu}_{\text{coup}}\mathfrak{F}(x,\gamma) && (3.22)\\
&\quad = f(x)L^{\mu}_{\text{env}}F(\gamma)\\
&\qquad - 2\big(\mathbb{D}F(\gamma),\nabla f(x)\big) - \sum_{j=1}^{N}\partial_j g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)\langle\Delta f_k,\gamma\rangle f(x)\\
&\qquad\quad + F(\gamma)\sum_{y,y'\in\gamma}\big(\nabla V(y-y'),\nabla f(y)-\nabla f(y')\big)\\
&\qquad\quad - F(\gamma)\sum_{y\in\gamma}\big(\nabla V(y),\nabla f(y)\big)\\
&\qquad + F(\gamma)\Delta f(x), && (3.23)
\end{aligned}
$$

$\mathfrak{F}(x,\gamma) = f(x)g_F\big(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle\big)$.

Also $\mathcal{E}^{dx\otimes\mu}_{\text{coup}}$ gives a corresponding diffusion:

**Theorem 3.1.8.** *$\big(\mathcal{E}^{dx\otimes\mu}_{coup}, C_0^\infty(\mathbb{R}^d)\otimes\mathcal{F}C_b^\infty\big)$ is closable in $L^2(\mathbb{R}^d\times\Gamma, dx\otimes\mu)$. Its closure $\big(\mathcal{E}^{dx\otimes\mu}_{coup}, D(\mathcal{E}^{dx\otimes\mu}_{coup})\big)$ is a conservative, local, quasi-regular, symmetric Dirichlet form, which is generated by the Friedrichs extension of $L^{dx\otimes\mu}_{coup}$. $\big(\mathcal{E}^{dx\otimes\mu}_{coup}, D(\mathcal{E}^{dx\otimes\mu}_{coup})\big)$ is properly associated with a conservative diffusion process*

$$\mathbb{M}^{\mathbb{R}^d\times\Gamma,dx\otimes\mu}_{coup} = \big(\Omega^{coup},\mathcal{F}^{coup},(\mathcal{F}^{coup}_t)_{t\geq 0},((X_t,\xi_t)^{coup})_{t\geq 0}.(\mathbb{P}^{coup}_{(x,\gamma)})_{(x,\gamma)\in\mathbb{R}^d\times\Gamma}\big),$$

*which has $dx\otimes\mu$ as invariant measure and solves the associated martingale problem.*

*Proof.* See [FG08, Theorems 5.15, 5.16, Remark 5.17]. The ideas of the proof are the following: The arguments for closability, conservativity, and locality are as before, i.e., the properties are obvious respectively follow directly from the product rule for the gradients $(\mathbb{D}-\nabla)$ and $\nabla^\Gamma$.

Quasi-regularity follows from the quasi-regularity of the components of $\mathcal{E}^{dx\otimes\mu}_{\text{coup}}$. Namely, one has to construct an $\mathcal{E}^{dx\otimes\mu}_{\text{coup}}$-nest of compact sets on $L^2(\mathbb{R}^d\times\ddot{\Gamma}, dx\otimes\mu)$. Therefor let $\mathcal{E}(f,g) := \int_{\mathbb{R}^d}(\nabla f,\nabla g)\,dx$, $f,g\in H^{1,2}(\mathbb{R}^d)$, denote the standard Dirichlet form on $L^2(\mathbb{R}^d, dx)$. Both $(\mathcal{E}, H^{1,2}(\mathbb{R}^d))$ and $(\mathcal{E}^\mu_{\text{env}}, D(\mathcal{E}^\mu_{\text{env}}))$ are quasi-regular on $L^2(\mathbb{R}^d,dx)$ and $L^2(\ddot{\Gamma},\mu)$, resp. Thus there exists an $\mathcal{E}^\mu_{\text{env}}$-nest $(E_k)_k$ of compact sets in $\ddot{\Gamma}$, and the closed balls $(\overline{B_k})_k$ form an $\mathcal{E}$-nest of compact sets on $\mathbb{R}^d$. Then $(F_k)_k := (\overline{B_k}\times E_k)_k$ is an $\mathcal{E}^{dx\otimes\mu}_{\text{coup}}$-nest of compact sets. □

The tagged particle process is then obtained as a projection of the coupled process to its first component. Note that, in general, this is no longer a Markov process.

## 3.2 Application of IP-scheme

Fix an activity parameter $z > 0$. Let $\mu \in \mathrm{ex}\mathcal{G}^{\mathrm{gc}}_{\mathrm{Rb}}(ze^{-V}, V)$ such that the Dirichlet form $\big(\mathcal{E}^{\mu}_{\mathrm{env}}, D(\mathcal{E}^{\mu}_{\mathrm{env}})\big)$ is irreducible (see Theorem 2.3.14). Consider the environment process

$$\mathbb{M}^{\mu}_{\mathrm{env}} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\xi_t)_{t\geq 0}, (\mathbb{P}_\gamma)_{\gamma\in\ddot{\Gamma}}\big)$$

from Theorem 3.1.5. By the assumptions on $\mu$ and the construction of the process, $\xi$ is reversible and ergodic w.r.t. $\mu$.

We can write the tagged particle process in the form of the standard decomposition (2.5) as

$$X_t = \int_0^t \Phi(\xi_s)\, ds + \sqrt{2}W_t \tag{3.24}$$

with

$$\Phi(\gamma) := \sum_{y\in\Gamma} \nabla V(y). \tag{3.25}$$

**Lemma 3.2.1.** $\Phi \in L^2(\mu)$.

*Proof.* Let $(k_\mu^{(n)})_n$ denote the system of correlation functions of $\mu$. By assumption there exists a constant $C_\mu > 0$ such that

$$k_\mu^{(n)} \leq C_\mu^n \quad \forall n \in \mathbb{N}$$

(Ruelle bound).

Set

$$G(\eta) := |\nabla V(y)|^2\, \mathbb{1}_{\{y\}}(\eta) + 2\,|\nabla V(x)|\,|\nabla V(y)|\, \mathbb{1}_{\{x,y\}}(\eta).$$

Then

$$\begin{aligned}
&\int_\Gamma \Phi(\gamma)^2\, d\mu(\gamma)\\
&\quad\leq \int_\Gamma KG(\gamma)\, d\mu(\gamma)\\
&\quad= \int_{\mathbb{R}^d} |\nabla V(y)|^2\, k_\mu^{(1)}(y) z e^{-V(y)}\, dy\\
&\qquad+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 2\,|\nabla V(x)|\,|\nabla V(y)|\, k_\mu^{(2)}(x,y) \tfrac{z^2}{2!} e^{-V(x)} e^{-V(y)}\, dxdy\\
&\quad\leq C_\mu z \int_{\mathbb{R}^d} |\nabla V(y)|^2 e^{-V(y)}\, dy + C_\mu^2 z^2 \bigg(\int_{\mathbb{R}^d} |\nabla V(y)|\, e^{-V(y)}\, dy\bigg)^2 < +\infty
\end{aligned}$$

since, by the assumption (D), $|\nabla V| \in L^1(\mathbb{R}^d, e^{-V(x)}\,dx) \cap L^2(\mathbb{R}^d, e^{-V(x)}\,dx)$.
□

Thus, we can apply [DFGW89, Theorem 2.2.(ii)] to obtain an invariance principle for $X_t$:

**Theorem 3.2.2.** *Let $V$ be a symmetric pair potential satisfying (SS), (I), (LR), (D) and (LS), and let $z > 0$. Let $\mu \in \mathcal{G}^{gc}_{Rb}(ze^{-V}, V)$. Suppose that the corresponding Dirichlet form $\big(\mathcal{E}^{\mu}_{env}, D(\mathcal{E}^{\mu}_{env})\big)$ is irreducible. Let $X^{(\varepsilon)}_t := \varepsilon X_{\varepsilon^{-2}t}$, $\varepsilon > 0$. Then*

$$X^{\varepsilon} \to W_D \tag{3.26}$$

*in the sense of weak-convergence of the corresponding distributions on the paths-space. Here $W_D$ denotes a Brownian motion with covariance $Dt$, and $D$ is given by*

$$D_{ij} = 2\delta_{ij} + 2(\Phi_i, (L^{\mu}_{env})^{-1}\Phi_j)_{L^2(\mu)}.$$

# Chapter 4

# Continuous Contact Model with Jumps

## 4.1 Description of the model

### 4.1.1 Generator

A continuous contact process has been studied recently in [KS06, KKP08, FKS09]. It is a special type of spatial birth-and-death processes, cf. eg. [Pre75, HS78, KL05]. The mechanism of the dynamics is described by the following formal generator acting on functions on $\Gamma$:

$$(L_{\mathrm{C}}F)(\gamma) := \sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x-y)[F(\gamma\cup x) - F(\gamma)]\, dx \tag{4.1}$$

with $\varkappa > 0$ and $0 \le a \in L^1$ an even probability density function. The first term describes the death part. Points of the configuration die independently after an exponentially distributed life time. The second summand describes the birth of particles. In the contact model, particles of the configuration generate independently from each other new particles distributed in space according to $a$.

In this chapter we will study a modification of the contact process by allowing the points also to perform jumps in the space, i.e., we add a jump part to the generator (4.1). Let $\varkappa > 0$, and let $0 \le a, w \in L^1(\mathbb{R}^d)$ be arbitrary even functions with $\langle a\rangle := \int_{\mathbb{R}^d} a(x)\, dx = 1$. We consider the following Markov pre-generator on the configuration space $\Gamma$:

$$(LF)(\gamma) := \sum_{x\in\gamma}[F(\gamma\setminus x) - F(\gamma)] + \varkappa\int_{\mathbb{R}^d}\sum_{y\in\gamma} a(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$$
$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} w(x-y)[F(\gamma\setminus y\cup x) - F(\gamma)]\,dx. \tag{4.2}$$

The last term is a generator of so-called free Kawasaki dynamics, cf., e.g., [KLR07, KLR08]. Note that, in contrast to birth-and-death dynamics, Kawasaki dynamics is conservative, i.e., the number of particles (if it is finite) does not vary in time.

Note that the case $w \equiv 0$, i.e., the usual contact model, is included in all further considerations.

To give meaning to the operator $L$, let $\mathcal{F}C_b := \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma)$ denote the set of all continuous bounded cylinder functions on $\Gamma$, i.e., all $F$ which have a (non-unique) representation as

$$F(\gamma) = g_F\big(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle\big)$$

with $N \in \mathbb{N}$, $g_F \in C_b(\mathbb{R}^N)$, and $\varphi \in C_0(\mathbb{R}^d)$, $i = 1, \ldots, N$. Then for each $F \in \mathcal{F}C_b$, $(LF)(\gamma)$ is well-defined at least pointwisely.

### 4.1.2 Application: plankton dynamics

The model described by (4.2) serves as a model for a plankton dynamics. The points of a configuration $\gamma$ represent individuals of the plankton. Then the contact model part of (4.2) describes the birth-and-death of individuals, and the jump part describes their motion.

In the literature, see, e.g., [YRS01], the motion of plankton is often modelled by diffusion. But motion and birth-and-death happen on different time scales, e.g., motion in terms of minutes and birth-and-death in terms of days. Therefore, it is appropriate to think of motion on the bio-time scale as (long) jumps.

## 4.2 Construction of a continuous contact process with jumps

We will construct now a Markov process with generator $L$. The arguments are a modification of those for the construction of a usual continuous contact process (cf. [KS06, FKS09]) to our situation.

For any $\beta > d$ introduce

$$e_\beta(x) := \frac{1}{(1+|x|)^\beta}, \quad x \in \mathbb{R}^d, \tag{4.3}$$

$$\Psi_\beta(x,y) := e_\beta(x)e_\beta(y)\frac{|x-y|+1}{|x-y|}1\!\!1_{\{x\neq y\}}, \quad x,y \in \mathbb{R}^d. \tag{4.4}$$

Furthermore, define the following functions on $\Gamma$:

$$\mathbb{L}_\beta(\gamma) := \langle e_\beta, \gamma\rangle = \sum_{x\in\gamma} e_\beta(x), \tag{4.5}$$

$$\mathbb{E}_\beta(\gamma) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi_\beta(x,y)\, d\gamma(x)d\gamma(y) = \sum_{\{x,y\}\subset\gamma} \Psi_\beta(x,y), \tag{4.6}$$

and

$$\mathbb{V}_\beta(\gamma) := \mathbb{L}_\beta(\gamma) + \mathbb{E}_\beta(\gamma). \tag{4.7}$$

Let $\mathcal{K}(\Gamma)$ denote the set of all relatively compact subsets of $\Gamma$. In [KK06, Proposition 3.1 and Remark 3.8] it is shown that for any $C > 0$

$$\{\gamma \in \Gamma \,:\, \mathbb{V}_\beta(\gamma) \le C\} \in \mathcal{K}(\Gamma).$$

Hence, for any $\beta > d$, the set

$$\Gamma_\beta := \{\gamma \in \Gamma \,:\, \mathbb{V}_\beta(\gamma) < +\infty\} \tag{4.8}$$

is $\sigma$-compact. For $\beta_1 < \beta_2$, $\Gamma_{\beta_1} \subset \Gamma_{\beta_2}$, and define

$$\Gamma_\infty := \bigcup_{\beta > d} \Gamma_\beta \subset \Gamma. \tag{4.9}$$

The space $\Gamma_\infty$ gives support to a large class of probability measures on $\Gamma$, cf., e.g., [FKS09].

The next lemma shows that $\mathbb{V}_\beta$ can be considered as a Lyapunov function for $L$ under the assumption of polynomial decay of $a, w$. Note that, since $\mathbb{V}_\beta$ is unbounded, the inequality (4.10) below has sense only pointwisely.

**Lemma 4.2.1.** *Let $d \ge 2$, and assume that*

$$(\varkappa a + w)(x) \le \frac{A}{(1+|x|)^{\beta+\delta}}, \quad x \in \mathbb{R}^d,$$

*for some $A > 0$ and some $\beta, \delta > d$. Then there exists a constant $C > 0$ such that (pointwisely)*

$$L\mathbb{V}_\beta(\gamma) \le C\mathbb{V}_\beta(\gamma). \tag{4.10}$$

*Proof.* The assumption on the polynomial bound of $a, w$ leads to the following estimate:

$$(\varkappa a + w)(x - y)e_\beta(x) \leq \frac{A}{(1 + |x - y|)^\delta} e_\beta(y). \tag{4.11}$$

Namely, since $|x - y| + |x| \geq |y|$, $x, y \in \mathbb{R}^d$, we have

$$(1 + |x - y|)^\beta (1 + |x|)^\beta \geq (1 + |y|)^\beta, \quad x, y \in \mathbb{R}^d,$$

and this implies (4.11).

First consider $\gamma \in \Gamma_0$. Then, with (4.11),

$$\begin{aligned}
L\mathbb{L}_\beta(\gamma) =& \sum_{x\in\gamma} [\langle e_\beta, \gamma \setminus x\rangle - \langle e_\beta, \gamma\rangle] \\
&+ \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x - y)[\langle e_\beta, \gamma \cup x\rangle - \langle e_\beta, \gamma\rangle]\, dx \\
&+ \sum_{y\in\gamma} \int_{\mathbb{R}^d} w(x - y)[\langle e_\beta, \gamma \setminus y \cup x\rangle - \langle e_\beta, \gamma\rangle]\, dx \\
=& - \sum_{x\in\gamma} e_\beta(x) + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a(x - y) e_\beta(x)\, dx \\
&+ \sum_{y\in\gamma} \int_{\mathbb{R}^d} w(x - y)[e_\beta(x) - e_\beta(y)]\, dx \\
=& - \mathbb{L}_\beta(\gamma) + \sum_{y\in\gamma} \int_{\mathbb{R}^d} (\varkappa a + w)(x - y) e_\beta(x)\, dx \\
&- \sum_{y\in\gamma} e_\beta(y)\langle w\rangle \\
\leq& - (1 + \langle w\rangle)\mathbb{L}_\beta(\gamma) + \sum_{y\in\gamma} e_\beta(y) \int_{\mathbb{R}^d} \frac{A}{(1 + |x - y|)^\delta}\, dx \\
=& C'\mathbb{L}_\beta(\gamma),
\end{aligned}$$

with $C' := AC_\delta - (1 + \langle w\rangle)$, $C_\delta := \int_{\mathbb{R}^d} \frac{dx}{(1+|x|)^\delta} < +\infty$.

Similarly,

$$
\begin{aligned}
&L\mathbb{E}_\beta(\gamma)\\
&\quad= \sum_{x\in\gamma}\Big[\sum_{\{z_1,z_2\}\subset(\gamma\setminus x)}\Psi_\beta(z_1,z_2) - \sum_{\{z_1,z_2\}\subset\gamma}\Psi_\beta(z_1,z_2)\Big]\\
&\qquad + \varkappa\int_{\mathbb{R}^d}\sum_{y\in\gamma}a(x-y)\Big[\sum_{\{z_1,z_2\}\subset(\gamma\cup x)}\Psi_\beta(z_1,z_2) - \sum_{\{z_1,z_2\}\subset\gamma}\Psi_\beta(z_1,z_2)\Big]\,dx\\
&\qquad + \sum_{y\in\gamma}\int_{\mathbb{R}^d}w(x-y)\Big[\sum_{\{z_1,z_2\}\subset(\gamma\setminus y\cup x)}\Psi_\beta(z_1,z_2) - \sum_{\{z_1,z_2\}\subset\gamma}\Psi_\beta(z_1,z_2)\Big]\,dx\\
&\quad= -\sum_{x\in\gamma}\sum_{y\in(\gamma\setminus x)}\Psi_\beta(x,y)\\
&\qquad + \varkappa\int_{\mathbb{R}^d}\sum_{y\in\gamma}a(x-y)\Big[\sum_{z\in\gamma}\Psi_\beta(x,z)\Big]\,dx\\
&\qquad + \sum_{y\in\gamma}\int_{\mathbb{R}^d}w(x-y)\Big[\sum_{z\in(\gamma\setminus y)}\Psi_\beta(x,z) - \sum_{z\in(\gamma\setminus y)}\Psi_\beta(y,z)\Big]\,dx\\
&\quad= -\mathbb{E}_\beta(\gamma) + \varkappa\sum_{y\in\gamma}\sum_{z\in\gamma}\int_{\mathbb{R}^d}a(x-y)\Psi_\beta(x,z)\,dx\\
&\qquad + \sum_{y\in\gamma}\sum_{z\in(\gamma\setminus y)}\int_{\mathbb{R}^d}w(x-y)\Psi_\beta(x,z)\,dx - \langle w\rangle\sum_{y\in\gamma}\sum_{z\in(\gamma\setminus y)}\Psi_\beta(y,z)\\
&\quad= -(1+\langle w\rangle)\mathbb{E}_\beta(\gamma) + \sum_{y\in\gamma}\sum_{z\in\gamma}\int_{\mathbb{R}^d}(\varkappa a + w)(x-y)\Psi_\beta(x,z)\,dx\\
&\quad= -(1+\langle w\rangle)\mathbb{E}_\beta(\gamma)\\
&\qquad + \sum_{y\in\gamma}\sum_{z\in\gamma}e_\beta(z)\int_{\mathbb{R}^d}(\varkappa a + w)(x-y)e_\beta(x)\frac{|x-z|+1}{|x-z|}\mathbb{1}_{\{x\neq z\}}\,dx\\
&\quad\le -(1+\langle w\rangle)\mathbb{E}_\beta(\gamma)\\
&\qquad + \sum_{y\in\gamma}\sum_{z\in\gamma}e_\beta(y)e_\beta(z)\int_{\mathbb{R}^d}\frac{A}{(1+|x-y|)^\delta}\frac{|x-z|+1}{|x-z|}\mathbb{1}_{\{x\neq z\}}\,dx.
\end{aligned}
$$

But we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{1}{(1+|x-y|)^\delta}\frac{|x-z|+1}{|x-z|}\,dx\\
&\qquad = \int_{\mathbb{R}^d}\frac{1}{(1+|x|)^\delta}\frac{|x-(z-y)|+1}{|x-(z-y)|}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= C_\delta + \int_{\mathbb{R}^d} \frac{1}{(1+|x|)^\delta} \frac{1}{|x-(z-y)|}\, dx \\
&= C_\delta + \int_{B_1(z-y)} \dots\, dx + \int_{B_1(z-y)^c} \dots\, dx \\
&\le C_\delta + \int_{B_1(z-y)} \frac{1}{|x-(z-y)|}\, dx + \int_{B_1(z-y)^c} \frac{1}{(1+|x|)^\delta} \frac{1}{1}\, dx \\
&= 2C_\delta + \tilde{C},
\end{aligned}
$$

where $\tilde{C} := \int_{B_1(0)} \frac{dx}{|x|} < \infty$ since $d \ge 2$. Therefore, since $e_\beta(y)e_\beta(z) \le \Psi_\beta(y,z)$, $y \neq z$, and $e_\beta(y)^2 \le e_\beta(y)$,

$$
\begin{aligned}
L\mathbb{E}_\beta(\gamma) &\le -(1+\langle w\rangle)\mathbb{E}_\beta(\gamma) + C'' \sum_{y\in\gamma}\sum_{z\in\gamma} e_\beta(y)e_\beta(z) \\
&\le \big(C'' - (1+\langle w\rangle)\big)\mathbb{E}_\beta(\gamma) + C''\mathbb{L}_\beta(\gamma),
\end{aligned}
$$

where $C'' := A(2C_\delta + \tilde{C})$.

Since $C' \le C'' - (1+\langle w\rangle)$, we thus obtain

$$
\begin{aligned}
L\mathbb{V}_\beta(\gamma) &\le (C' + C'')\mathbb{L}_\beta(\gamma) + \big(C'' - (1+\langle w\rangle)\big)\mathbb{E}_\beta(\gamma) \\
&\le \big(2C'' - (1+\langle w\rangle)\big)\mathbb{V}_\beta(\gamma) \quad (< +\infty).
\end{aligned}
$$

Thus we have proved the assertion of the lemma for $\gamma \in \Gamma_0$.

But from the above calculations we can obtain the assertion also for a general $\gamma \in \Gamma_\beta$, because $\mathbb{V}_\beta(\gamma) < +\infty$, hence $L\mathbb{V}_\beta(\gamma) < +\infty$, and $L\mathbb{V}_\beta(\gamma) \ge -(1+\langle w\rangle)\mathbb{V}_\beta(\gamma) > -\infty$. (Note that we can also here interchange infinite sums with integrals because all limits appearing are monotone ones.)

□

**Theorem 4.2.2.** *Under the conditions of Lemma 4.2.1, for any initial configuration $\gamma \in \Gamma_\beta$, there exists a Markov process $(X_t^\gamma)_{t\ge 0}$ with generator $(L, B(\Gamma))$, which starts in $\gamma$ and satisfies*

$$\forall t \ge 0 \quad X_t^\gamma \in \Gamma_\beta \quad a.s.$$

*In particular, if $a(x), w(x) \to 0$ rapidly as $|x| \to \infty$, i.e., faster than any polynomial, then the corresponding process exists for every starting configuration $\gamma \in \Gamma_\infty$ and stays in $\Gamma_\infty$ for all times.*

*Proof.* The proof is similar to the proof of Theorem 3.1 in [KS06].

First we consider the situation of a finite initial configuration $\gamma \in \Gamma_0$. We can rewrite the generator $L$ in the standard form of a pure jump Markov generator: For any $\eta \in \Gamma_0$ we have

$$LF(\eta) = \lambda(\eta) \int_{\Gamma_0} \big[F(\eta') - F(\eta)\big]\, Q(\eta, d\eta'), \tag{4.12}$$

where

$$\lambda(\eta) = \big(1 + \varkappa + \langle w \rangle\big)\, |\eta|$$

and

$$Q(\eta, d\eta') = \frac{1}{\lambda(\eta)} \Bigg[\sum_{x\in\eta} \delta_{\eta\setminus x}(d\eta') + \varkappa \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y)\, \delta_{\eta\cup x}(d\eta')\, dx + \sum_{y\in\eta} \int_{\mathbb{R}^d} w(x-y)\, \delta_{\eta\setminus y\cup x}(d\eta')\, dx\Bigg].$$

Then the theory of pure jump Markov processes from [GS75] gives, for each starting point $\eta \in \Gamma_0$, the existence of a pure jump Markov process

$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t^\eta)_{0\leq t<\zeta}, \mathbb{P}_\eta\big) \tag{4.13}$$

on $\Gamma_0$ with generator $(L, B(\Gamma)$, where $\zeta(\omega)$ denotes the life time of the process, .

We want to show, that for any starting configuration $\eta \in \Gamma_0$

$$\mathbb{P}_\eta(\zeta = +\infty) = 1. \tag{4.14}$$

Let $k \in \mathbb{N}$, then $|\cdot| \wedge k \in B(\Gamma_0)$. Fix $\eta \in \Gamma_0$.

$$\begin{aligned} L\,|\eta| &= \sum_{x\in\eta} \underbrace{[|\eta\setminus x| \wedge k - |\eta| \wedge k]}_{\leq 0} + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)[|\eta\cup x| \wedge k - |\eta|\wedge k]\, dx \\ &\quad + \sum_{y\in\gamma} \int_{\mathbb{R}^d} w(x-y)\, \underbrace{[|\eta\setminus y\cup x| \wedge k - |\eta|\wedge k]}_{=0}\, dx \\ &\leq \varkappa\, |\eta|\, \underbrace{[(|\eta|+1)\wedge k - |\eta|\wedge k]}_{=\mathbb{1}_{|\eta|<k}} \\ &\leq \varkappa\, |\eta| \wedge k. \end{aligned}$$

Hence, by martingale representation, we obtain

$$\begin{aligned} \mathbb{E}_\eta[|X_t| \wedge k] &= |\eta| \wedge k + \varkappa \mathbb{E}_\eta\Big[\int_0^t (L\,|\cdot| \wedge k)(X_s)\, ds\Big] \\ &\leq |\eta| \wedge k + \varkappa \int_0^t \mathbb{E}_\eta[|X_s| \wedge k]\, ds. \end{aligned}$$

For $k \to \infty$ this gives

$$\mathbb{E}_\eta[|X_t^\eta|] \le |\eta| + \varkappa \int_0^t \mathbb{E}_\eta[|X_s^\eta|]\, ds,$$

and thus, by the Gronwall inequality,

$$\mathbb{E}_\eta[|X_t^\eta|] \le |\eta|\, e^{\varkappa t}. \tag{4.15}$$

This implies (4.14).

Now we can construct the process for an infinite initial configuration $\gamma \in \Gamma_\beta$, $\beta > d$. Let $\gamma_n := \gamma \cap B_n(0) \in \Gamma_0$, $n \in \mathbb{N}$, and consider the corresponding sequence of Markov processes

$$\big((X_t^{\gamma_n})_{t\ge 0}\big)_{n\in\mathbb{N}}.$$

As well as for the usual contact process, we also have in our situation a monotonic structure of these processes:

$$\forall n \in \mathbb{N}, \forall t \ge 0 \qquad X_t^{\gamma_n} \subset X_t^{\gamma_{n+1}} \quad \text{a.s.} \tag{4.16}$$

Namely, for $\eta_n := \gamma_{n+1} \setminus \gamma_n$, because of the additive structure of the generator $L$, we have that $X_t^{\gamma_n}$ and $X_t^{\eta_n}$ are independent Markov processes with $X_t^{\gamma_n} \cap X_t^{\eta_n} = \emptyset$ a.s., hence $X_t^{\gamma_{n+1}} = X_t^{\gamma_n} \cup X_t^{\eta_n}$ a.s. Introduce the limiting process

$$X_t^\gamma(\omega) := \bigcup_{n\in\mathbb{N}} X_t^{\gamma_n}(\omega). \tag{4.17}$$

Let $k \in \mathbb{N}$, then $\mathbb{V}_\beta(\cdot) \wedge k \in B(\Gamma_0)$. Fix $\eta \in \Gamma_0$. Note that

$$a \wedge k - b \wedge k \le (a - b \wedge k) \wedge k \tag{4.18}$$

for all $a, b \ge 0$. Namely,

$$\begin{cases} a \wedge k - b \wedge k \le (a-b) \wedge k \le (a - b \wedge k) \wedge k, & \text{if } a \ge b, \\ a \wedge k - b \wedge k = (a \wedge k - b \wedge k) \wedge k \le (a - b \wedge k) \wedge k, & \text{if } b \ge a. \end{cases}$$

By (4.18), the Jensen inequality ($\cdot \wedge k$ is concave), and Lemma 4.2.1 we get

$$
\begin{aligned}
L(\mathbb{V}_\beta \wedge k)(\eta) &= \sum_{y\in\eta}[\mathbb{V}_\beta(\eta\setminus y)\wedge k - \mathbb{V}_\beta(\eta)\wedge k] \\
&\quad + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)[\mathbb{V}_\beta(\eta\cup x)\wedge k - \mathbb{V}_\beta(\eta)\wedge k]\, dx \\
&\quad + \sum_{y\in\gamma} \int_{\mathbb{R}^d} w(x-y)[\mathbb{V}_\beta(\eta\setminus y\cup x)\wedge k - \mathbb{V}_\beta(\eta)\wedge k]\, dx \\
&\le \sum_{y\in\eta}\Bigg[[\mathbb{V}_\beta(\eta\setminus y) - \mathbb{V}_\beta(\eta)\wedge k]\wedge k \\
&\quad + \varkappa \int_{\mathbb{R}^d} a(x-y)[\mathbb{V}_\beta(\eta\cup x) - \mathbb{V}_\beta(\eta)\wedge k]\, dx \wedge k \\
&\quad + \langle w\rangle \int_{\mathbb{R}^d} \tfrac{w}{\langle w\rangle}(x-y)[\mathbb{V}_\beta(\eta\setminus y\cup x) - \mathbb{V}_\beta(\eta)\wedge k]\, dx\wedge k\Bigg] \\
&\le \underbrace{(1+\varkappa+\langle w\rangle)}_{=:C'} |\eta| \Bigg[\frac{1}{C'|\eta|}\times \\
&\quad \times \Bigg(\sum_{y\in\eta}[\mathbb{V}_\beta(\eta\setminus y) - \mathbb{V}_\beta(\eta)] + |\eta|\,[\mathbb{V}_\beta(\eta) - \mathbb{V}_\beta(\eta)\wedge k] \\
&\qquad + \sum_{y\in\eta}\varkappa \int_{\mathbb{R}^d} a(x-y)[\mathbb{V}_\beta(\eta\cup x) - \mathbb{V}_\beta(\eta)]\, dx \\
&\qquad + \varkappa\,|\eta|\,[\mathbb{V}_\beta(\eta) - \mathbb{V}_\beta(\eta)\wedge k] \\
&\qquad + \langle w\rangle \int_{\mathbb{R}^d} \tfrac{w}{\langle w\rangle}(x-y)[\mathbb{V}_\beta(\eta\setminus y\cup x) - \mathbb{V}_\beta(\eta)]\, dx \\
&\qquad + \langle w\rangle\,|\eta|\,[\mathbb{V}_\beta(\eta) - \mathbb{V}_\beta(\eta)\wedge k]\Bigg)\Bigg]\wedge k \\
&= C'|\eta|\left[\frac{1}{C'|\eta|} L\mathbb{V}_\beta(\eta) + [\mathbb{V}_\beta(\eta) - \mathbb{V}_\beta(\eta)\wedge k]\right]\wedge k \\
&\le C'|\eta|\left[\frac{C}{C'|\eta|}\mathbb{V}_\beta(\eta) + [\mathbb{V}_\beta(\eta) - \mathbb{V}_\beta(\eta)\wedge k]\right]\wedge k,
\end{aligned}
$$

where $C$ is the constant from Lemma 4.2.1.

Consider now $\gamma_n$, $n \in \mathbb{N}$. Then by martingale representation

$$
\begin{aligned}
&\mathbb{E}_{\gamma_n}\big[(\mathbb{V}_\beta\wedge k)(X_t^{\gamma_n})\big] \\
&\qquad = (\mathbb{V}_\beta\wedge k)(\gamma_n) + \mathbb{E}_{\gamma_n}\int_0^t L(\mathbb{V}_\beta\wedge k)(X_s^{\gamma_n})\, ds
\end{aligned}
$$

$$\leq (\mathbb{V}_\beta \wedge k)(\gamma_n) + \mathbb{E}_{\gamma_n} \int_0^t C' \left|X_s^{\gamma_n}\right| \times$$
$$\times \left[ \frac{C}{C' \left|X_s^{\gamma_n}\right|} \mathbb{V}_\beta(X_s^{\gamma_n}) + [\mathbb{V}_\beta(X_s^{\gamma_n}) - \mathbb{V}_\beta(X_s^{\gamma_n}) \wedge k] \right] \wedge k \, ds.$$

Note that we can interchange $\mathbb{E}_{\gamma_n}$ and $\int_0^t$ here by (4.15). For $k \to \infty$ this gives

$$\mathbb{E}_{\gamma_n}\left[\mathbb{V}_\beta(X_t^{\gamma_n})\right] \leq \mathbb{V}_\beta(\gamma_n) + C \int_0^t \mathbb{E}_{\gamma_n}[\mathbb{V}_\beta(X_s^{\gamma_n})] \, ds,$$

and thus, by the Gronwall inequality, we have for all $n \in \mathbb{N}$ and all $t \geq 0$

$$\mathbb{E}_{\gamma_n}\left[\mathbb{V}_\beta(X_t^{\gamma_n})\right] \leq \mathbb{V}_\beta(\gamma) e^{Ct}. \tag{4.19}$$

Using the monotonicity property (4.16) of the sequence of processes $(X_t^{\gamma_n})_{n \in \mathbb{N}}$, we obtain the estimate (4.19) also for the limiting process $X_t^\gamma$, and hence

$$X_t^\gamma \in \Gamma_\beta \quad \text{a.s.}$$

By construction, the process $X_t^\gamma$ has the generator $L$ and $X_0^\gamma = \gamma$. Thus, the proof is finished. □

## 4.3 Time evolution of correlation functions

Now we will study the dynamics of the correlation functions corresponding to generator (4.2). Via the theory of harmonic analysis on configuration spaces (cf. Section 1.6) we will get the correlation functions in terms of solutions of a hierachical system of equations, see (4.26). We will solve this system of equations and show that the solutions are indeed correlation functions of a corresponding measure on $\Gamma$.

### 4.3.1 Symbol of the generator on the space of finite configurations

The image of the generator $L$ under the $K$-transform $\hat{L} := K^{-1} L K$ on quasi-observables is called the *symbol* of $L$.

To compute the symbol of $L$ we need the following lemma about the $K$-transform:

**Lemma 4.3.1.** *For $G \in B_{bs}(\Gamma_0)$ and $y \in \mathbb{R}^d$ it holds that*

$$K\big(G(\cdot \setminus y)\big)(\xi) = \begin{cases} (KG)(\xi \setminus y), & \text{if } y \notin \xi \\ 2(KG)(\xi \setminus y), & \text{if } y \in \xi. \end{cases}$$

*Proof.*

$$\begin{aligned} K\big(G(\cdot \setminus y)\big)(\xi) &= \sum_{\rho \subset \xi} G(\rho \setminus y) \\ &= \begin{cases} \sum_{\rho \subset (\xi \setminus y)} G(\rho), & \text{if } y \notin \xi, \\ \sum_{\rho \subset (\xi \setminus y)} G(\rho) + \sum_{\rho \subset (\xi \setminus y)} G((\rho \cup y) \setminus y), & \text{if } y \in \xi \end{cases} \\ &= \begin{cases} (KG)(\xi \setminus y), & \text{if } y \notin \xi, \\ 2(KG)(\xi \setminus y), & \text{if } y \in \xi. \end{cases} \end{aligned}$$

□

**Proposition 4.3.2.** *For functions $G \in B_{bs}(\Gamma_0)$ the symbol of $L$ has the following form:*

$$\begin{aligned} (\hat{L}G)(\eta) = &- |\eta|\, G(\eta) \\ &+ \varkappa \int_{\mathbb{R}^d} \sum_{y \in \eta} a(x-y) G(\eta \setminus y \cup x)\, dx \\ &+ \varkappa \int_{\mathbb{R}^d} \sum_{y \in \eta} a(x-y) G(\eta \cup x)\, dx \\ &+ \int_{\mathbb{R}^d} \sum_{y \in \eta} w(x-y) G(\eta \setminus y \cup x)\, dx - \langle w \rangle\, |\eta|\, G(\eta). \end{aligned} \tag{4.20}$$

*Proof.* Write

$$(\hat{L}G)(\eta) =: I_{\mathrm{D}}(\eta) + I_{\mathrm{B}}(\eta) + I_{\mathrm{J}}(\eta)$$

with

$$\begin{aligned} I_{\mathrm{D}}(\eta) =& K^{-1}\left( \sum_{x \in \cdot} [(KG)(\cdot \setminus x) - (KG)(\cdot)] \right)(\eta), \\ I_{\mathrm{B}}(\eta) =& K^{-1}\left( \varkappa \int_{\mathbb{R}^d} \sum_{y \in \cdot} a(x-y)[(KG)(\cdot \cup x) - (KG)(\cdot)]\, dx \right)(\eta), \\ I_{\mathrm{J}}(\eta) =& K^{-1}\left( \sum_{y \in \cdot} \int_{\mathbb{R}^d} w(x-y)[(KG)(\cdot \setminus y \cup x) - (KG)(\cdot)]\, dx \right)(\eta). \end{aligned}$$

From [KKP08, Proposition 3.1] it follows that

$$
\begin{aligned}
I_{\mathrm{D}}(\eta) =& - |\eta| \, G(\eta), \\
I_{\mathrm{B}}(\eta) =& \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G(\eta\setminus y \cup x)\, dx + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G(\eta \cup x)\, dx,
\end{aligned}
$$

so it remains to show that

$$
I_{\mathrm{J}}(\eta) = \int_{\mathbb{R}^d} \sum_{y\in\eta} w(x-y) G(\eta \setminus y \cup x)\, dx - \langle w\rangle \, |\eta| \, G(\eta).
$$

$$
I_{\mathrm{J}}(\eta) = \sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \sum_{y\in\zeta} \int_{\mathbb{R}^d} w(x-y) \big[ \sum_{\rho\subset(\zeta\setminus y\cup x)} G(\rho) - \sum_{\rho\subset\zeta} G(\rho) \big] \, dx.
$$

For $y\in\zeta$ and $x \notin (\zeta\setminus y)$ one has

$$
\begin{aligned}
\sum_{\rho\subset(\zeta\setminus y\cup x)} G(\rho) - \sum_{\rho\subset\zeta} G(\rho) &= \sum_{\rho\subset(\zeta\setminus y\cup x)} G(\rho) - \sum_{\rho\subset(\zeta\setminus y)} G(\rho) - \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup y) \\
&= \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup x) - \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_{\mathrm{J}}(\eta) =& \sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \int_{\mathbb{R}^d} \sum_{y\in\zeta} w(x-y) \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup x)\, dx \\
&- \sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \int_{\mathbb{R}^d} \sum_{y\in\zeta} w(x-y) \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup y)\, dx \\
\overset{\text{Lemma 4.3.1}}{=}& \sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \int_{\mathbb{R}^d} \sum_{y\in\zeta} w(x-y) \tfrac{1}{2} K(G(\cdot\setminus y\cup x))(\zeta)\, dx \\
&- \sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \sum_{y\in\zeta} \sum_{\rho\subset(\zeta\setminus y)} G(\rho\cup y) \langle w\rangle.
\end{aligned}
$$

Since

$$
\sum_{y\in\zeta} w(x-y) = K\big(w(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\big)(\zeta),
$$

we obtain

$$\begin{aligned}
I_{\mathrm{j}}(\eta) =&\tfrac{1}{2}\int_{\mathbb{R}^d} K^{-1}\Big[K\Big(\big(w(x-\cdot)\mathbb{1}_{\{|\cdot|=1\}}(\cdot)\big)\star G(\cdot\setminus y\cup x)\Big)\Big](\eta)\,dx\\
&-\langle w\rangle\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{y\in\zeta}K(G(\cdot\cup y))(\zeta\setminus y)\\
=&\tfrac{1}{2}\int_{\mathbb{R}^d}\Big(\big(w(x-\cdot)\mathbb{1}_{\{|\cdot|=1\}}(\cdot)\big)\star G(\cdot\setminus y\cup x)\Big)(\eta)\,dx\\
&-\langle w\rangle\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{y\in\zeta}K(G(\cdot\cup y))(\zeta\setminus y).
\end{aligned}\tag{4.21}$$

Changing the order of summation in the second term of (4.21) we obtain

$$\begin{aligned}
-\langle w\rangle&\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{y\in\zeta}K(G(\cdot\cup y))(\zeta\setminus y)\\
&=-\langle w\rangle\sum_{y\in\eta}\sum_{\zeta\subset(\eta\setminus y)}(-1)^{|\eta\setminus(\zeta\cup y)|}K(G(\cdot\cup y))(\zeta)\\
&=-\langle w\rangle\sum_{y\in\eta}K^{-1}\big(K(G(\cdot\cup y))\big)(\zeta\setminus y)\\
&=-\langle w\rangle\,|\eta|\,G(\eta).
\end{aligned}$$

For the first term in (4.21) we have

$$\begin{aligned}
&\tfrac{1}{2}\int_{\mathbb{R}^d}\Big(\big(w(x-\cdot)\mathbb{1}_{\{|\cdot|=1\}}(\cdot)\big)\star G(\cdot\setminus y\cup x)\Big)(\eta)\,dx\\
&=\tfrac{1}{2}\int_{\mathbb{R}^d}\sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)}\underbrace{\big(w(x-\cdot)\mathbb{1}_{\{|\cdot|=1\}}(\cdot)\big)(\xi_1\cup\xi_2)}_{(*)}G((\xi_2\cup\xi_3)\setminus y\cup x)\,dx.
\end{aligned}$$

There are only two cases in which $(*)\neq 0$, namely $(|\xi_1|=1,\xi_2=\emptyset)$ and $(\xi_1=\emptyset,|\xi_2|=1)$. Hence

$$\begin{aligned}
\tfrac{1}{2}&\int_{\mathbb{R}^d}\sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)}\big(w(x-\cdot)\mathbb{1}_{\{|\cdot|=1\}}(\cdot)\big)(\xi_1\cup\xi_2)G((\xi_2\cup\xi_3)\setminus y\cup x)\,dx\\
&=\tfrac{1}{2}\int_{\mathbb{R}^d}\sum_{y\in\eta}w(x-y)G((\eta\setminus y)\setminus y\cup x)\,dx\\
&\quad+\tfrac{1}{2}\int_{\mathbb{R}^d}\sum_{y\in\eta}w(x-y)G((y\cup(\eta\setminus y))\setminus y\cup x)\,dx\\
&=\int_{\mathbb{R}^d}\sum_{y\in\eta}w(x-y)G(\eta\setminus y\cup x)\,dx.
\end{aligned}$$

Thus, the assertion follows. □

### 4.3.2 The adjoint operator to the symbol $\hat{L}$

Let $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ be absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda$ (with activity parameter $z = 1$). Let $k(\eta) := \frac{d\rho}{d\lambda}(\eta)$, $\eta \in \Gamma_0$, denote the corresponding Radon-Nikodym derivative.

**Proposition 4.3.3.** *Assume that*

$$k(\eta) \le C^{|\eta|}\,|\eta|!, \quad \eta \in \Gamma_0 \tag{4.22}$$

*for some $C > 0$. Then $\hat{L}(B_{bs}(\Gamma_0)) \subset L^1(\Gamma_0, \rho)$.*

*Proof.* Let $G \in B_{\mathrm{bs}}(\Gamma_0)$. Then by Proposition 4.3.2 and the assumption we have

$$\begin{aligned}
&\int_{\Gamma_0} \left|\hat{L}G(\eta)\right| \, d\rho(\gamma) \\
&\qquad = \int_{\Gamma_0} \left|\hat{L}G(\eta)\right| k(\eta)\, d\lambda(\eta) \\
&\qquad \le \int_{\Gamma_0} (1 + \langle w\rangle)\, |\eta|\, |G(\eta)|\, C^{|\eta|}\, |\eta|!\, d\lambda(\eta) && \text{(a)} \\
&\qquad\quad + \int_{\Gamma_0}\int_{\mathbb{R}^d} \sum_{y\in\eta} (\varkappa a + w)(x-y)\, |G(\eta \setminus y \cup x)|\, C^{|\eta|}\, |\eta|!\, dx\, d\lambda(\eta) && \text{(b)} \\
&\qquad\quad + \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)\, |G(\eta \cup x)|\, C^{|\eta|}\, |\eta|!\, dx\, d\lambda(\eta). && \text{(c)}
\end{aligned}$$

We will show that all the terms (a), (b), and (c) are finite.

Since $G \in B_{\mathrm{bs}}(\Gamma_0)$ there exist $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that

$$G \restriction_{\Gamma_0 \setminus \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}} = 0.$$

Term (a):

$$\begin{aligned}
&\int_{\Gamma_0} (1 + \langle w\rangle)\, |\eta|\, |G(\eta)|\, C^{|\eta|}\, |\eta|!\, d\lambda(\eta) \\
&\qquad = (1 + \langle w\rangle) N C^N N! \int_{\Gamma_\Lambda} |G(\eta)|\, d\lambda(\eta) < \infty
\end{aligned}$$

since $G$ is bounded.

Term (b) (and term (c)): Here finiteness can be shown by application of the Minlos lemma, cf. (1.13) and (1.14):

$$\begin{aligned}
&\int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{y\in\eta}(\varkappa a+w)(x-y)\,|G(\eta\setminus y\cup x)|\,C^{|\eta|}\,|\eta|!\,dx\,d\lambda(\eta)\\
&\quad=\int_{\mathbb{R}^d}\int_{\Gamma_0}C^{|\eta|}\,|\eta|!\sum_{y\in\eta}(\varkappa a+w)(x-y)\,|G(\eta\setminus y\cup x)|\;d\lambda(\eta)\,dx\\
&\quad\overset{(1.14)}{=}\int_{\mathbb{R}^d}\int_{\Gamma_0}\int_{\mathbb{R}^d}C^{|\eta\cup z|}\,|\eta\cup z|!(\varkappa a+w)(x-z)\,|G(\eta\cup x)|\;dz\,d\lambda(\eta)\,dx\\
&\quad\overset{(1.14)}{=}\int_{\mathbb{R}^d}\int_{\Gamma_0}|G(\eta)|\sum_{y\in\eta}C^{|\eta\setminus y\cup z|}\,|\eta\setminus y\cup z|!(\varkappa a+w)(y-z)\,d\lambda(\eta)\,dz\\
&\quad\le C^N N!\int_{\Gamma_\Lambda}|G(\eta)|\sum_{y\in\eta}\int_{\mathbb{R}^d}(\varkappa a+w)(y-z)\,dz\,d\lambda(\eta)\\
&\quad\le NC^N N!(\varkappa+\langle w\rangle)\int_{\Gamma_\Lambda}|G(\eta)|\,d\lambda(\eta)<\infty.
\end{aligned}$$

Term (c):

$$\begin{aligned}
&\varkappa\int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{y\in\eta}a(x-y)\,|G(\eta\cup x)|\,C^{|\eta|}\,|\eta|!\,dx\,d\lambda(\eta)\\
&\qquad\overset{(1.14)}{=}\varkappa\int_{\Gamma_0}|G(\eta)|\sum_{z\in\eta}C^{|\eta\setminus z|}\,|\eta\setminus z|!\sum_{y\in\eta\setminus z}a(z-y)\,d\lambda(\eta)\\
&\qquad=\varkappa C^{N-1}(N-1)!\int_{\Gamma_\Lambda}|G(\eta)|\sum_{z\in\eta}\sum_{y\in\eta\setminus z}a(z-y)\,d\lambda(\eta)<\infty.
\end{aligned}$$

Thus, the assertion is proved. □

The adjoint operator $\hat{L}^*$ to the symbol $\hat{L}$ is defined via the duality relation given by the scalar product in $L^2(\Gamma_0,\lambda)$,

$$\begin{aligned}
&\int_{\Gamma_0}\hat{L}G(\eta)\,\rho(d\eta)=(\hat{L}G,k)_{L^2(\Gamma_0,\lambda)}\\
&\qquad\qquad=(G,\hat{L}^*k)_{L^2(\Gamma_0,\lambda)}=\int_{\Gamma_0}G(\eta)(\hat{L}^*k)(\eta)\,\lambda(d\eta). \quad (4.23)
\end{aligned}$$

**Proposition 4.3.4.** *The adjoint operator $\hat{L}^*$ to the symbol $\hat{L}$ on the space* *of functions that satisfy* (4.22) *has the following form:*

$$\begin{aligned}(\hat{L}^* k)(\eta) = &- |\eta|\, k(\eta)\\ &+ \varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k(\eta\setminus x \cup y)\, dy\\ &+ \varkappa \sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in(\eta\setminus x)} a(x-y)\\ &+ \sum_{x\in\eta} \int_{\mathbb{R}^d} w(x-y) k(\eta\setminus x\cup y)\, dy - \langle w\rangle\, |\eta|\, k(\eta). \end{aligned} \tag{4.24}$$

*Proof.* We use the same notation as in the proof of Proposition 4.3.2. Furthermore, write $I_{\mathrm{B}} =: I_{\mathrm{B}}^{(1)} + I_{\mathrm{B}}^{(2)}$ and $I_{\mathrm{J}} =: I_{\mathrm{J}}^{(1)} + I_{\mathrm{J}}^{(2)}$ with

$$\begin{aligned} I_{\mathrm{B}}^{(1)}(\eta) &:= \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G(\eta\setminus y\cup x)\, dx,\\ I_{\mathrm{B}}^{(2)}(\eta) &:= \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G(\eta\cup x)\, dx,\\ I_{\mathrm{J}}^{(1)}(\eta) &:= \int_{\mathbb{R}^d} \sum_{y\in\eta} w(x-y) G(\eta\setminus y\cup x)\, dx,\\ I_{\mathrm{J}}^{(2)}(\eta) &:= -\langle w\rangle\, |\eta|\, G(\eta). \end{aligned}$$

Then, from [KKP08, Proposition 4.2] we obtain for the death part and the birth part

$$\begin{aligned} \int_{\Gamma_0} I_{\mathrm{D}}(\eta) k(\eta)\, \lambda(d\eta) &= \int_{\Gamma_0} G(\eta)[-|\eta|\, k(\eta)]\, \lambda(d\eta),\\ \int_{\Gamma_0} I_{\mathrm{B}}^{(1)}(\eta) k(\eta)\, \lambda(d\eta) &= \varkappa \int_{\Gamma_0} G(\eta) \left[\sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k(\eta\setminus x\cup y)\, dy\right] \lambda(d\eta),\\ \int_{\Gamma_0} I_{\mathrm{B}}^{(2)}(\eta) k(\eta)\, \lambda(d\eta) &= \varkappa \int_{\Gamma_0} G(\eta) \left[\sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in(\eta\setminus x)} a(x-y)\right] \lambda(d\eta). \end{aligned}$$

Since the expressions $I_{\mathrm{J}}^{(1)}(\eta)$ and $I_{\mathrm{J}}^{(2)}(\eta)$ corresponding to the jump part are of the same form as $I_{\mathrm{B}}^{(1)}(\eta)$ and $I_{\mathrm{D}}(\eta)$, respectively, we also obtain

$$\begin{aligned} \int_{\Gamma_0} I_{\mathrm{J}}^{(1)}(\eta) k(\eta)\, \lambda(d\eta) &= \int_{\Gamma_0} G(\eta) \left[\sum_{x\in\eta} \int_{\mathbb{R}^d} w(x-y) k(\eta\setminus x\cup y)\, dy\right] \lambda(d\eta),\\ \int_{\Gamma_0} I_{\mathrm{J}}^{(2)}(\eta) k(\eta)\, \lambda(d\eta) &= \int_{\Gamma_0} G(\eta)[-\langle w\rangle\, |\eta|\, k(\eta)]\, \lambda(d\eta). \end{aligned}$$

This proves the assertion. □

### 4.3.3 Evolution equation associated to $\hat{L}^*$

The evolutional equation associated with the operator $\hat{L}^*$ has the form

$$\begin{aligned}\frac{\partial k_t}{\partial t}(\eta) =& \hat{L}^* k_t(\eta) \\ =& -(1+\langle w\rangle)\,|\eta|\,k_t(\eta) \\ &+ \sum_{x\in\eta}\int_{\mathbb{R}^d}(\varkappa a + w)(x-y)k_t(\eta\setminus x\cup y)\,dy \\ &+ \varkappa\sum_{x\in\eta}k_t(\eta\setminus x)\sum_{y\in(\eta\setminus x)}a(x-y). \end{aligned} \tag{4.25}$$

Since a function $k$ on $\Gamma_0$ corresponds to a family of functions $(k^{(n)})_{n\in\mathbb{N}}$, $k^{(n)}$ a function on the $n$-point configurations $\Gamma_0^{(n)}$, we can rewrite (4.25) as a system of equations:

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) = \hat{L}_n^* k_t^{(n)}(x_1,\dots,x_n) + f_t^{(n)}(x_1\dots,x_n), \quad n\in\mathbb{N}, \tag{4.26}$$

with

$$\begin{aligned}&\hat{L}_n^* k_t^{(n)}(x_1,\dots,x_n) \\ &\quad := -(1+\langle w\rangle)n k_t^{(n)}(x_1,\dots,x_n) \\ &\qquad + \sum_{i=1}^n\int_{\mathbb{R}^d}(\varkappa a+w)(x_i-y)k_t^{(n)}(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)\,dy, \\ &f_t^{(n)}(x_1,\dots,x_n) \\ &\quad := \begin{cases}\varkappa\sum_{i=1}^n k_t^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n)\sum_{j\neq i}a(x_i-x_j), & n\geq 2\\ 0, & n=1.\end{cases}\end{aligned}$$

Note that $\hat{L}_n^* k_t^{(n)}$ only depends on the $n$-point function $k_t^{(n)}$ and $f_t^{(n)}$ only on the $(n-1)$-point function $k_t^{(n-1)}$.

Now fix an $n\in\mathbb{N}$. We consider the Cauchy problem

$$\begin{cases}\dfrac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) = \hat{L}_n^* k_t^{(n)}(x_1,\dots,x_n) + f_t^{(n)}(x_1\dots,x_n), \quad t\geq 0,\\ k_t^{(n)}(x_1,\dots,x_n)\Big|_{t=0} = k_0^{(n)}(x_1,\dots,x_n)\end{cases} \tag{4.27}$$

in the Banach space $X_n := L^\infty((\mathbb{R}^d)^n, dx^{\otimes n})$.

**Remark 4.3.5.** The operator $\hat{L}_n^*$ in $X_n$ can also be written in the following way:

$$\hat{L}_n^* k^{(n)}(x_1,\dots,x_n) = n(\varkappa-1)k^{(n)}(x_1,\dots,x_n) + \sum_{i=1}^n L^i_{\varkappa a+w} k^{(n)}(x_1,\dots,x_n) \tag{4.28}$$

with

$$\begin{aligned} L^i_{\varkappa a+w} k^{(n)}(x_1,\dots,x_n) = \int_{\mathbb{R}^d} &(\varkappa a+w)(x_i-y)\times \\ &\times \left[k^{(n)}(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) - k^{(n)}(x_1,\dots,x_n)\right] dy. \end{aligned}$$

For each $i$, $L^i_{\varkappa a+w}$ is a generator of a Markov process on $(\mathbb{R}^d)^n$ (see [GS75]), which describes the jump of a particle from the point $(x_1,\dots,x_i,,\dots,x_n)$ to $(x_1,\dots,y,\dots,x_n)$. The generator of a pure-jump Markov process has the form

$$\lambda(x)\int [f(y)-f(x)]Q(x,dy),$$

where $Q$ is a probability kernel describing the transition probabilities of the embedded Markov chain, and $\lambda(x)$ describes the jump-rate, i.e., the first exit time from the state $x$ is distributed exponentially with rate $\lambda(x)$. In this case, we have

$$\lambda(x) \equiv \varkappa + \langle w\rangle, \quad \pi(x,dy) = \frac{\varkappa a+w}{\varkappa+\langle w\rangle}(x-y)\,dy.$$

**Lemma 4.3.6.** *Let $a, w \in L^1(\mathbb{R}^d)$ be nonnegative even functions. Then, for any $n \geq 1$, the operator $\hat{L}_n^*$ is a bounded linear operator both in $X_n$ and in $L^1((\mathbb{R}^d)^n)$.*

*Moreover, for each $1 \leq i \leq n$, the operator $L^i_{\varkappa a+w}$ generates a contraction semigroup on $X_n$ and on $L^1((\mathbb{R}^d)^n)$.*

*Proof.* The first part of this lemma is trivial.

The second part follows in the case of the space $X_n$ directly from Remark 4.3.5, and in the case of $L^1((\mathbb{R}^d)^n)$ it is a consequence of the Beurling-Deny criterion, cf., e.g., [RS78]. □

E.g. by [IK02, Theorem 2.13], the previous lemma implies the following result:

**Proposition 4.3.7.** *Let $n \geq 1$ be arbitrary and fixed. The solution to the Cauchy problem* (4.27) *in the Banach space $X_n$ is given by*

$$\begin{aligned} k_t^{(n)}(x_1, \ldots, x_n) =& e^{n(\varkappa-1)t}\left[\bigotimes_{i=1}^{n} e^{tL^i_{\varkappa a+w}}\right] k_0^{(n)}(x_1, \ldots, x_n) \\ &+ \varkappa e^{n(\varkappa-1)t}\int_0^t e^{-n(\varkappa-1)s}\left[\bigotimes_{i=1}^{n} e^{(t-s)L^i_{\varkappa a+w}}\right] \times \\ &\times \sum_{i=1}^{n} k_s^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j\neq i} a(x_i - x_j)\, ds. \end{aligned} \tag{4.29}$$

The next proposition establishes a priori estimates for the evolution of the correlation functions in time.

**Proposition 4.3.8.** *Let $a, w \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be non-negative, even functions. Assume that there exists a constant $C > 0$, such that*

$$k_0^{(n)}(x_1, \ldots, x_n) \leq n!C^n \quad \forall n \geq 1, \forall (x_1, \ldots, x_n) \in \mathbb{R}^d.$$

*Then, for any $t \geq 0$ and Lebesgue-a.a. $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, $n \geq 1$,*

$$k_t^{(n)}(x_1, \ldots, x_n) \leq \varkappa(t)^n(1 + \|a\|_{L^\infty})^n e^{n(\varkappa-1)t}(C + t)^n n!, \tag{4.30}$$

*where $\varkappa(t) := \max\{1, \varkappa, \varkappa e^{-(\varkappa-1)t}\}$.*

Later, in Proposition 4.5.1, we will discuss a situation, where one also has a lower factorial bound for the correlation functions and hence clustering of the points.

*Proof of Proposition 4.3.8.* The assertion is proved via induction over $n$.

In the case $n = 1$, (4.30) follows directly from Proposition 4.3.7 and the assumption on $k_0^{(1)}$: For a.a. $x_1 \in \mathbb{R}^d$ and $t \geq 0$ we have

$$\begin{aligned} k_t^{(1)}(x_1) &= e^{1(\varkappa-1)t} e^{tL^1_{\varkappa a+w}} k_0^{(1)}(x_1) \\ &\leq \varkappa(t)^1(1 + \|a\|_{L^\infty})^1 e^{1(\varkappa-1)t}(C + t)^1 1!. \end{aligned}$$

Now assume, that, for any $t \geq 0$, the estimate (4.30) holds for $k_t^{(n-1)}$. Then, again by Proposition 4.3.7, we get

$$
\begin{aligned}
&k_t^{(n)}(x_1, \ldots, x_n) \\
&\quad = e^{n(\varkappa-1)t} \left[ \bigotimes_{i=1}^{n} e^{tL^i_{\varkappa a+w}} \right] k_0^{(n)}(x_1, \ldots, x_n) \\
&\qquad + \varkappa e^{n(\varkappa-1)t} \int_0^t e^{-n(\varkappa-1)s} \left[ \bigotimes_{i=1}^{n} e^{(t-s)L^i_{\varkappa a+w}} \right] \times \\
&\qquad \times \sum_{i=1}^{n} \underbrace{k_s^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n)}_{\substack{\leq \varkappa(s)^{n-1}(1+\|a\|_{L^\infty})^{n-1} \\ e^{(n-1)(\varkappa-1)s}(C+s)^{n-1}(n-1)!}} \underbrace{\sum_{j \neq i} a(x_i - x_j)}_{\leq (n-1)(1+\|a\|_{L^\infty})} \, ds \\
&\quad \leq e^{n(\varkappa-1)t} C^n n! \\
&\qquad + \varkappa e^{n(\varkappa-1)t} \int_0^t e^{-n(\varkappa-1)s} n!(n-1)\varkappa(s)^{n-1} e^{(n-1)(\varkappa-1)s} \times \\
&\qquad \times (1+\|a\|_{L^\infty})^n (C+s)^{n-1} \, ds.
\end{aligned}
$$

Noticing that $(1 \leq)\varkappa(s) \leq \varkappa(t)$ for $s \leq t$ and $\varkappa e^{-(\varkappa-1)s} \leq \varkappa(s)$, we obtain

$$
\begin{aligned}
k_t^{(n)}(x_1, \ldots, x_n) &\leq e^{n(\varkappa-1)t} C^n n! \\
&\quad + e^{n(\varkappa-1)t} n! n \varkappa(t)^n (1+\|a\|_{L^\infty})^n \int_0^t (C+s)^{n-1} \, ds \\
&\leq \varkappa(t)^n (1+\|a\|_{L^\infty})^n e^{n(\varkappa-1)t} C^n n! \\
&\quad + \varkappa(t)^n (1+\|a\|_{L^\infty})^n e^{n(\varkappa-1)t} \big( (C+t)^n - C^n \big) n! \\
&= \varkappa(t)^n (1+\|a\|_{L^\infty})^n e^{n(\varkappa-1)t} (C+t)^n n!.
\end{aligned}
$$

This proves the assertion. □

With the help of the previous proposition, we can approximate solutions of the Cauchy problem (4.27) for $a, w$ with unbounded support by solutions of (4.27) for $a, w$ with bounded support:

**Corollary 4.3.9.** *Let $0 \leq a, w \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be arbitrary even functions with $\|a\|_{L^1} = 1$ and $a(x) + w(x) \to 0$ as $|x| \to \infty$. Let $k^{(n)}_{t,\varkappa a+w}$ be the solution to (4.27) in $X_n$. Suppose that the conditions of Proposition 4.3.8 are fulfilled. Then for all sequences $\{a_l\}_l, \{w_l\}_l \subset C_0(\mathbb{R}^d)$ such that*

$$a_l \to a,\ w_l \to w \quad \text{in } L^1(\mathbb{R}^d) \text{ and } X_1 = L^\infty(\mathbb{R}^d) \tag{4.31}$$

*(and such sequences exist)*

$$k^{(n)}_{t,\varkappa a_l + w_l} \to k^{(n)}_{t,\varkappa a + w}$$

*in* $X_n$ *as* $l \to \infty$.

*Proof.* Fix $\{a_l\}_l, \{w_l\}_l \subset C_0(\mathbb{R}^d)$ satisfying (4.31).

The assertion is again proved by induction over $n$. In the case $n = 1$, by Proposition 4.3.7, we have

$$k^{(1)}_{t,\varkappa a_l + w_l} - k^{(1)}_{t,\varkappa a + w} = e^{(\varkappa - 1)t}\Big( e^{tL^1_{\varkappa a_l + w_l}} k^{(1)}_0 - e^{tL^1_{\varkappa a + w}} k^{(1)}_0 \Big).$$

But this converges to 0 in $X_1$, since (4.31) implies strong convergence of the bounded generators $L^1_{\varkappa a_l + w_l}$ in $L^\infty$, and hence strong convergence of the corresponding semigroups.

Now assume the assertion holds for $n-1$. Again by Proposition 4.3.7 we have

$$\begin{aligned}
&k^{(n)}_{t,\varkappa a_l + w_l}(x_1, \dots, x_n) - k^{(n)}_{t,\varkappa a + w}(x_1, \dots, x_n) \\
&\quad = e^{n(\varkappa - 1)t} \Bigg( \Big[ \bigotimes_{i=1}^n e^{tL^i_{\varkappa a_l + w_l}} \Big] k^{(n)}_0(x_1, \dots, x_n) \\
&\qquad - \Big[ \bigotimes_{i=1}^n e^{tL^i_{\varkappa a + w}} \Big] k^{(n)}_0(x_1, \dots, x_n) \Bigg) \\
&\qquad + \varkappa e^{n(\varkappa - 1)t} \int_0^t e^{-n(\varkappa - 1)s} \Bigg( \Big[ \bigotimes_{i=1}^n e^{(t-s)L^i_{\varkappa a_l + w_l}} \Big] \times \\
&\qquad \times \sum_{i=1}^n k^{(n-1)}_{s,\varkappa a_l + w_l}(x_1, \dots, \check{x}_i, \dots, x_n) \sum_{j \neq i} a_l(x_i - x_j) \\
&\qquad - \Big[ \bigotimes_{i=1}^n e^{(t-s)L^i_{\varkappa a + w}} \Big] \sum_{i=1}^n k^{(n-1)}_{s,\varkappa a + w}(x_1, \dots, \check{x}_i, \dots, x_n) \times \\
&\qquad \times \sum_{j \neq i} a(x_i - x_j) \Bigg) \, ds
\end{aligned} \tag{4.32}$$

The first summand in (4.32) converges to 0 in $X_n$ since (4.31) implies strong convergence of the generators $L^i_{\varkappa a_l + w_l}$ to $L^i_{\varkappa a + w}$ and hence strong convergence of the corresponding semigroups.

For the second summand in (4.32), we have that

$$\sum_{i=1}^{n} k^{(n-1)}_{s,\varkappa a_l+w_l}(x_1,\dots,\check{x}_i,\dots,x_n)\sum_{j\neq i} a_l(x_i-x_j)$$

converges to

$$\sum_{i=1}^{n} k^{(n-1)}_{s,\varkappa a+w}(x_1,\dots,\check{x}_i,\dots,x_n)\sum_{j\neq i} a(x_i-x_j)$$

in $X_n$, since $k^{(n-1)}_{s,\varkappa a_l+w_l} \to k^{(n-1)}_{s,\varkappa a+w}$, $l\to\infty$, in $X_{n-1}$ by induction assumption and $a_l \to a$ in $X_1$. It follows from Proposition 4.3.8 that

$$\| \left[\bigotimes_{i=1}^{n} e^{(t-s)L^i_{\varkappa a_l+w_l}}\right] \sum_{i=1}^{n} k^{(n-1)}_{s,\varkappa a_l+w_l}(x_1,\dots,\check{x}_i,\dots,x_n)\sum_{j\neq i} a_l(x_i-x_j)\|_{X_n}$$

and the same expression with $a$ and $w$ instead of $a_l$ and $w_l$, respectively, are uniformly bounded in $s\in[0,t]$. Therefore, also the second summand in (4.32) converges to 0 in $X_n$. □

### 4.3.4 Solutions of (4.27) as correlation functions

Now we consider the following question: suppose that the system of initial conditions $(k_0^{(n)})_n$ for the Cauchy problem (4.27) is a system of correlation functions, i.e., there exists a probability measure $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, whose correlation measure is absolutely continuous w.r.t. the Lebesgue-Poisson measure and whose correlation functions are $(k_0^{(n)})_n$. Is this property preserved under time evolution? For every $t\geq 0$, does there exist a corresponding probability measure $\mu_t\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ with correlation functions $(k_t^{(n)})_n$, where $(k_t^{(n)})_n$ are the solutions of (4.27)?

To answer this question we will apply A. Lenard's result about construction of corresponding measures for given correlation functions, cf. [Len73, KK02]. Let $\rho\in\mathcal{M}(\Gamma_0)$ with corresponding system $(k^{(n)})_n$. Assume that $\rho$ is locally finite and normalized, i.e., $\rho(\{\emptyset\})=1$. In order to show that $\rho$ is a correlation measure of some $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ one has to check the following two conditions:

- (Lenard positivity)
  For any $G\in B_{\mathrm{bs}}(\Gamma_0)$ with $KG\geq 0$ it holds that

$$\int_{\Gamma_0} G(\eta)\,d\rho(\eta)\geq 0. \tag{4.33}$$

- (Moment growth condition)
  For any bounded set $\Lambda \subset \mathbb{R}^d$ and $j \geq 0$

$$\sum_{n=0}^{\infty} (m_{n+j}^{\Lambda})^{-\frac{1}{n}} = +\infty, \tag{4.34}$$

  where

$$m_n^{\Lambda} := \frac{1}{n!} \int_{\Lambda} \cdots \int_{\Lambda} k^{(n)}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

**Remark 4.3.10.** (i) Lenard positivity ensures existence of such a measure $\mu$, and the moment growth condition ensures its uniqueness.

(ii) If a system of functions $(k^{(n)})_{n\geq 0}$ satisfies, for a constant $C > 0$ independent of $n$,

$$k^{(n)}(x_1, \ldots, x_n) \leq n! C^n \quad \text{for all } n,$$

(cf. also (4.22)) then it also satisfies the moment growth condition.

Denote the set of all probability measure $\mu$ whose correlation functions satisfy the assumption from Remark 4.3.10 (ii) by $\mathcal{M}^1_{C,\mathrm{fac}}(\Gamma)$.

**Lemma 4.3.11.** *Let $0 \leq a, w \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ with $\|a\|_{L^1} = 1$ and $a(x) + w(x) \to 0$ as $|x| \to \infty$, and suppose that the assumptions of Proposition 4.3.8 are satisfied. Then for any $t \geq 0$, the system of solutions $(k_t^{(n)})_n$ of* (4.27) *is positive in the sense of* (4.33).

*Proof.* By Corollary 4.3.9, it suffices to prove the assertion under the assumption that $a, w$ have polynomial decay at infinity:

$$(\varkappa a + w)(x) \leq \frac{A}{(1 + |x|)^{\beta+\delta}}, \quad x \in \mathbb{R}^d,$$

for some $A > 0$ and some $\beta, \delta > d$

We have to show

$$\sum_{n\geq 0} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G^{(n)}(x_1, \ldots, x_n) k_t^{(n)}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \geq 0 \tag{4.35}$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$ with $KG \geq 0$.

As in Section 4.2, let

$$e_\beta(x) := \frac{1}{(1 + |x|)^\beta}, \quad x \in \mathbb{R}^d.$$

Furthermore set

$$F^{(n)}(\gamma) := \sum_{\{x_1,\ldots,x_n\}\subset\gamma} e_\beta(x_1)\cdots e_\beta(x_n), \quad n\in\mathbb{N}, |\gamma|\geq n.$$

If $|\gamma| < N$, then set $F^{(n)}(\gamma) := 0$. Define a function $G_n$ on $\Gamma_0$ by the corresponding family of functions

$$\left(G_n^{(k)} : \widetilde{(\mathbb{R}^d)^k} \to \mathbb{R}_+\right)_{k\in\mathbb{N}},$$
$$G_n^{(k)}(x_1,\cdots,x_k) := e_\beta(x_1)\cdots e_\beta(x_k)\mathbb{1}_{k=n}.$$

Then $KG_n(\gamma) = F^{(n)}(\gamma)$.

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be such that its correlation measure $\rho_\mu$ is absolutely continuous w.r.t. the Lebesgue-Poisson measure and its correlation functions $(k_\mu^{(m)})_{m\in\mathbb{N}}$ are bounded. Then we have that

$$\begin{aligned}\int_\Gamma F^{(n)}(\gamma)\,\mu(d\gamma) &= \int_{\Gamma_0} G_n(\eta)\,d\rho_\mu(\eta)\\ &= \frac{1}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e_\beta(x_1)\cdots e_\beta(x_n)k^{(n)}(x_1,\ldots,x_n)\,dx_1\cdots dx_n < \infty.\end{aligned} \tag{4.36}$$

In particular, we obtain that $\mu(\Gamma_\beta)=1$, cf. Section 4.2. Hence, by Theorem 4.2.2, there exists a corresponding Markov process $(X_t^\gamma)_{t\geq 0}$ with generator $L$, which is a.s. in $\Gamma_\beta$.

Similarly to the computation in the proof of Lemma 4.2.1 we have for $n\geq 2$ (pointwisely)

$$\begin{aligned}&LF^{(n)}(\gamma)\\ &\quad= -\sum_{x\in\gamma}\sum_{\{z_1,\ldots,z_{n-1}\}\in(\gamma\setminus x)} e_\beta(x)e_\beta(z_1)\cdots e_\beta(z_{n-1})\\ &\qquad+\varkappa\int_{\mathbb{R}^d}\sum_{y\in\gamma}a(x-y)\Bigg[\sum_{\{z_1,\ldots,z_{n-1}\}\in\gamma} e_\beta(x)e_\beta(z_1)\cdots e_\beta(z_{n-1})\Bigg]dx\\ &\qquad+\sum_{y\in\gamma}\int_{\mathbb{R}^d} w(x-y)\Bigg[\sum_{\{z_1,\ldots,z_{n-1}\}\in(\gamma\setminus y)} e_\beta(x)e_\beta(z_1)\cdots e_\beta(z_{n-1})\\ &\qquad\quad-\sum_{\{z_1,\ldots,z_{n-1}\}\in(\gamma\setminus y)} e_\beta(y)e_\beta(z_1)\cdots e_\beta(z_{n-1})\Bigg]dx\\ &\quad= -(1+\langle w\rangle)F^{(n)}(\gamma)\\ &\qquad+\sum_{y\in\gamma}\sum_{\{z_1,\cdots,z_{n-1}\}\subset\gamma} e_\beta(z_1)\ldots e_\beta(z_{n-1})\int_{\mathbb{R}^d}(\varkappa a+w)(x-y)e_\beta(x)\,dx\end{aligned}$$

$$\begin{aligned}
&\le -(1+\langle w\rangle)F^{(n)}(\gamma) \\
&\quad+\sum_{y\in\gamma}\sum_{\{z_1,\cdots,z_{n-1}\}\subset\gamma} e_\beta(z_1)\dots e_\beta(z_{n-1})\int_{\mathbb{R}^d}\frac{A}{(1+|x-y|)^\delta}e_\beta(y)\,dx \\
&= -(1+\langle w\rangle)F^{(n)}(\gamma) + AC_\delta\sum_{y\in\gamma}\sum_{\{z_1,\cdots,z_{n-1}\}\subset\gamma} e_\beta(y)e_\beta(z_1)\dots e_\beta(z_{n-1}) \\
&\le \big(AC_\delta-(1+|w|)\big)F^{(n)}(\gamma)+AC_\delta F^{(n-1)}(\gamma),
\end{aligned}$$

and hence we obtain

$$L\mathbb{L}^{(N)}(\gamma)\le C\mathbb{L}^{(N)}(\gamma) \tag{4.37}$$

for some $C>0$. So, similarly as in the proof of Theorem 4.2.2, we have

$$\mathbb{E}\big[\mathbb{L}^{(N)}(X_t^\gamma)\big]\le \mathbb{L}^{(N)}(\gamma)e^{Ct}. \tag{4.38}$$

Let $(\mu_t)_{t\ge0}$ denote the corresponding evolution of $\mu_0$ described by the dual Kolmogorov equation

$$\begin{cases}\dfrac{\partial\mu_t}{\partial t}=L^*\mu_t, \quad t\ge0,\\ \mu_t\big|_{t=0}=\mu_0.\end{cases}$$

We want to show that $\mu_t\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, so we have to show the finiteness of all local moments. Therefore, let $\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $\Lambda$ compact, and $N\in\mathbb{N}$. Let

$$\varepsilon:=\varepsilon_{\beta,\Lambda}:=\min_{x\in\Lambda}(e_\beta(x))\,(>0).$$

Then, if $|\gamma_\Lambda|\le N$,

$$\begin{aligned}
\mathbb{L}^{(N)}(\gamma)&\ge\mathbb{L}^{(N)}(\gamma_\Lambda)\\
&=\sum_{n=1}^N\sum_{\{x_1,\dots,x_n\}\subset\gamma\Lambda} e_\beta(x_1)\cdots e_\beta(x_n)\\
&\ge\sum_{n=1}^N\varepsilon^n\binom{|\gamma_\Lambda|}{n}\\
&\ge(1+\varepsilon)^{|\gamma_\Lambda|}-1=:C'>0.
\end{aligned}$$

Recall Stirling's formula:

$$1\le\frac{n!}{\sqrt{2\pi n}e^{-n}n^n}\le e^{\frac{1}{12n}}, \quad n\in\mathbb{N}.$$

In the case $\ell := |\gamma_\Lambda| > N$, this gives

$$
\begin{aligned}
\mathbb{L}^{(N)}(\gamma) &\geq \mathbb{L}^{(N)}(\gamma_\Lambda) \\
&\geq \varepsilon^N \binom{|\gamma_\Lambda|}{N} \\
&\geq \varepsilon^N \frac{\sqrt{2\pi\ell}e^{-\ell}\ell^\ell}{\sqrt{2\pi N}e^{-N}N^N e^{\frac{1}{12N}}\sqrt{2\pi(\ell-N)}e^{-(\ell-N)}(\ell-N)^{(\ell-N)}e^{\frac{1}{12(\ell-N)}}} \\
&\geq \varepsilon^N \frac{1}{\sqrt{2\pi N}} \frac{1}{e^{\frac{1}{12N}+1}} \frac{1}{N^N}\ell^N =: \varepsilon^N C_N \left|\gamma_\Lambda\right|^N .
\end{aligned}
$$

Therefore, by (4.36) with $\mu := \mu_0$ and by (4.38) ,

$$
\begin{aligned}
\int_\Gamma \left|\gamma_\Lambda\right|^N \, \mu_t(d\gamma) &= \int_{\{|\gamma_\Lambda|\leq N\}} \cdots + \int_{\{|\gamma_\Lambda|>N\}} \cdots \\
&\leq \int_{\{|\gamma_\Lambda|\leq N\}} N^N \, \mu_t(d\gamma) + \frac{1}{\varepsilon^N C_N}\int_\Gamma \mathbb{E}\big[\mathbb{L}^{(N)}(X_t^\gamma)\big] \, \mu_0(d\gamma) \\
&\leq N^N + \frac{e^C t}{\varepsilon^N C_N}\int_\Gamma \mathbb{L}^{(N)}(\gamma) \, \mu_0(d\gamma) < \infty.
\end{aligned}
$$

This proves that $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$.

Hence, there exists a Markov evolution of the corresponding correlation measures on $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ associated with the generator $L$. Thus, (4.35) is obviously fulfilled because of the Markov property of the semigroup corresponding to the evolution of states. □

For any system of functions $(k_t^{(n)})_{n\geq 0}$ define $\mathcal{L}_k : \mathcal{F}_k \to \mathbb{C}$,

$$
\mathcal{L}_k(\theta) := \sum_{n=0}^\infty \frac{1}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \theta(x_1)\cdots\theta(x_n)k^{(n)}(x_1,\ldots,x_n)\,dx_1\cdots dx_n, \quad (4.39)
$$

where $\mathcal{F}_k$ denotes the set of all functions $\theta$ such that (4.39) makes sense.

**Remark 4.3.12.** (i) For $\delta > 0$ define

$$
U_\delta^1 := \{\theta \in L^1(\mathbb{R}^d) \, : \, \|\theta\|_{L^1(\mathbb{R}^d)} \leq \delta\}.
$$

If the system $(k_t^{(n)})_{n\geq 0}$ satisfies the assumptions of Remark 4.3.10 (ii), then $\mathcal{L}_k$ is holomorphic in $U_\delta^1$ for some $\delta > 0$.

(ii) Suppose that $(k_t^{(n)})_{n\geq 0}$ is the system of correlation functions of some measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. Then, e.g. by [KK02], the functional $\mathcal{L}_k$ is connected with $\mu$ via

$$\begin{aligned}\mathcal{L}_k(\theta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \theta(x_1)\cdots\theta(x_n) k^{(n)}(x_1,\ldots,x_n)\, dx_1 \cdots dx_n \\ &= \int_\Gamma \Big( K \prod_{x\in\cdot} \theta(x) \Big)(\gamma)\, \mu(d\gamma) \\ &= \int_\Gamma \prod_{x\in\gamma} (1+\theta(x))\, \mu(d\gamma), \quad \theta \in \mathcal{F}_k.\end{aligned}$$

The latter term

$$\mathcal{L}_\mu(\theta) := \int_\Gamma \prod_{x\in\gamma} (1+\theta(x))\, \mu(d\gamma) \tag{4.40}$$

is called *Bogoliubov functional* of $\mu$. Set

$$\mathcal{M}^1_{\mathrm{hol}}(\Gamma) := \{\mu \in \mathcal{M}^1(\Gamma) \,:\, \mathcal{L}_\mu \text{ is holomorphic in } U^1_\delta \text{ for some } \delta > 0\}. \tag{4.41}$$

So, by (i),

$$\mathcal{M}^1_{C,\mathrm{fac}}(\Gamma) \subset \mathcal{M}^1_{\mathrm{hol}}(\Gamma).$$

**Theorem 4.3.13.** *Let $0 \leq a, w \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be even functions with $\|a\|_{L^1} = 1$ and $a(x) + w(x) \to 0$ as $|x| \to \infty$. Then for any $\mu \in \mathcal{M}^1_{C,fac}(\Gamma)$ there exists a Markov function $X_t^\mu$ on $\Gamma$ associated to the generator $L$ with initial distribution $\mu$, such that for any $t \geq 0$ the corresponding distribution $\mu_t$ of $X_t^\mu$ lies in $\mathcal{M}^1_{C,fac}(\Gamma)$.*

*Proof.* By Proposition 4.3.8, Lemma 4.3.11, and Remark 4.3.10 (ii) we obtain that the corresponding solutions of the Cauchy problem (4.27) are correlation functions for all times $t \geq 0$. This gives the corresponding evolution $\mu_t$ on $\mathcal{M}^1(\Gamma)$ and thus all finite dimensional distributions of $X_t^\mu$. □

## 4.4 Invariant measures

Consider the translation invariant case, so the first correlation function $k_t^{(1)}$ is independent of $x \in \mathbb{R}^d$:

$$k_t^{(1)}(x) =: \rho_t \quad \forall t \geq 0.$$

The function $\rho$ is called *density.*

From equation (4.26) describing the time evolution of the correlation functions we obtain

$$\begin{aligned}\frac{\partial \rho_t}{\partial t} &= -(1+\langle w\rangle)\rho_t + \int_{\mathbb{R}^d} (\varkappa a + w)(-y)\rho_t\, dy \\ &= (\varkappa - 1)\rho_t, \\ \rho_t\big|_{t=0} &= \rho_0,\end{aligned}$$

and hence

$$\rho_t = \exp((\varkappa - 1)t)\rho_0. \tag{4.42}$$

Thus one has three cases:

subcritical ($\varkappa < 1$): $\rho_t \to 0$, as $t \to \infty$;

supercritical ($\varkappa > 1$): $\rho_t \to \infty$, as $t \to \infty$;

critical ($\varkappa = 1$): $\rho_t = \rho_0 = \rho$.

Invariant measures can exist only in the critical case. So, from now on, we assume $\varkappa = 1$.

Due to Theorem 4.3.13, all invariant measures can be described in terms of the corresponding system of correlation functions as positive solutions of

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \dots, x_n) = \hat{L}_n^* k_t^{(n)}(x_1, \dots, x_n) + f_t^{(n)}(x_1 \dots, x_n) = 0, \quad n \geq 1.$$

With (4.26) this can be formulated the following way:

**Proposition 4.4.1.** *If a measure $\mu \in \mathcal{M}^1(\Gamma)$ is an invariant measure for $X_t^\mu \in \Gamma$, then the system of the corresponding correlation functions of $\mu$ is a solution to the following recurrent system of equations: for $n \geq 1$*

$$\begin{aligned}&(1+\langle w\rangle)nk^{(n)} \\ &\quad= \sum_{i=1}^n k^{(n-1)}(x_1, \dots, \check{x}_i, \dots, x_n) \sum_{j\neq i} a(x_i - x_j) \\ &\qquad+ \sum_{i=1}^n \int_{\mathbb{R}^d} (a+w)(x_i - y) k^{(n)}(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n)\, dy.\end{aligned} \tag{4.43}$$

The next theorem deals with the inverse problem. Under certain conditions on $a$ and $w$ it shows existence of a continuum of invariant measures.

**Theorem 4.4.2.** *Let $d \geq 2$. Let $0 \leq a, w \in L^1(\mathbb{R}^d)$ be even continuous functions such that*

*(i)* $\|a\|_{L^1} = 1$,

*(ii)* $\int_{\mathbb{R}^d} |x|^2 a(x)\, dx < \infty$,

*(iii)* $\hat{a} := \int_{\mathbb{R}^d} e^{-i(\cdot,x)} a(x)\, dx \in L^1(\mathbb{R}^d)$,

*(iv)* $\int_{|p|\leq 1} \frac{\hat{a}(p)}{1+\langle w\rangle - (\hat{a}+\hat{w})(p)}\, dp < \infty$.

*Then the following assertions hold: (I) For any $\rho \in \mathbb{R}_+$, there exists a unique measure $\mu^\rho \in \mathcal{M}^1(\Gamma)$ whose correlation functions $\{k^{(n),\rho}\}_{n\geq 0}$ are translation invariant, solve equation* (4.43) *and satisfy*

$$\|k^{(n),\rho}\|_{X_n} \leq C(\rho)^n (n!)^2, \quad n \geq 1 \tag{4.44}$$

*for some positive constant $C(\rho)$. Moreover, $k^{(1),\rho} \equiv \rho$.*

*(II) Let $\mu_t$ be the distribution of $X_t^{\mu_0}$, $\mu_0 \in \mathcal{M}^1_{C,fac}(\Gamma)$ , at time $t \geq 0$, and let $(k_t^{(n)})_{n\geq 0}$ denote the corresponding system of correlation functions of $\mu_t$. Then in the critical case:*

*(i)* $k_t^{(1)} = k_0^{(1)} =: \rho$ *for all* $t \geq 0$;

*(ii) for any $n \geq 2$ and any $\varphi \in L^1\big((\mathbb{R}^d)^n\big)$,*

$$\big(k_t^{(n)}, \varphi\big) \to \big(k^{(n),\rho}, \varphi\big), \quad t \to \infty,$$

*where $\big(k_t^{(n)}, \cdot\big)$ and $\big(k^{(n),\rho}, \cdot\big)$ denote the corresponding functionals on $L^1\big((\mathbb{R}^d)^n\big)$.*

**Remark.** The integrability condition (iv) in the previous theorem is satisfied e.g. for $\hat{w}(p) = e^{-|p|^\alpha}$, $1 \leq \alpha < 2$, ($\alpha$-stable distribution), and any $a$ satisfying the other conditions. For $w \equiv 0$, i.e., the usual contact model, condition (iv) is automatically satisfied for dimension $d \geq 3$, cf. [KKP08].

*Proof of Theorem 4.4.2.* (I): For a given $\rho > 0$, we will construct inductively a system of solutions $k^{(n),\rho}$ of (4.43) in the Banach space $X_n$, which satisfies (4.44) and hence the moment growth condition (4.34). Lenard positivity

(4.33) of the system follows from part (II), since, for given $\rho$, we can always find a measure $\mu_0$ with $k_0^{(1)} = \rho$. Together this proves part (I).

Set $k^{(1),\rho} := \rho$. First, consider the case $n = 2$. Since we are in the translation invariant case, a proper solution $k^{(2),\rho}$ must be of the form

$$k^{(2)}(x_1, x_2) = k^{(2),\rho}(x_1 - x_2, 0) =: k(x_1 - x_2),$$

where $k$ is an even function on $\mathbb{R}^d$. Then equation (4.43) reads as

$$\begin{aligned}(1 + \langle w\rangle)2k(x_1 - x_2) =& \sum_{i=1}^{2} \rho \sum_{j\neq i} a(x_i - x_j)\\ &+ \sum_{i=1}^{2} \sum_{j\neq i} \int_{\mathbb{R}^d} (a + w)(x_i - y)k(x_j - y)\, dy\\ =&2\rho a(x_1 - x_2) + 2\int_{\mathbb{R}^d} (a + w)(x_1 - x_2 - y)k(y)\, dy,\end{aligned}$$

hence

$$\big((a + w) * k\big)(x_1 - x_2) - (1 + \langle w\rangle)k(x_1 - x_2) = -\rho a(x_1 - x_2), \qquad (4.45)$$

where $(a * k)(x) := \int_{\mathbb{R}^d} a(x - y)k(y)\, dy$ denotes the usual convolution. Suppose, (4.45) has a solution $v \in L^1(\mathbb{R}^d)$. Then one gets for the Fourier transform $\hat{v}$

$$(\hat{a} + \hat{w})(p)\hat{v}(p) - (1 + \langle w\rangle)\hat{v}(p) = -\rho\hat{a}(p),$$

and thus

$$\hat{v}(p) = \frac{\rho\hat{a}(p)}{1 + \langle w\rangle - (\hat{a} + \hat{w})(p)}. \qquad (4.46)$$

Under the assumptions on $a$ and $w$, $\frac{\rho\hat{a}(p)}{1+\langle w\rangle-(\hat{a}+\hat{w})(p)}$ is integrable at $p = 0$ for dimension $d \geq 2$, and hence in $L^1(\mathbb{R}^d)$. Therefore,

$$v(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(p,x)} \frac{\rho\hat{a}(p)}{1 + \langle w\rangle - (\hat{a} + \hat{w})(p)}\, dp \in L^\infty(\mathbb{R}^d). \qquad (4.47)$$

**Remark.** Suppose that the solution to (4.45) is a second correlation function. Then, application of Fourier transform does not have any physical sense, since, in general, second correlation functions are not integrable. But the second Ursell function $u^{(2)}(x_1, x_2) := k^{(2)}(x_1, x_2) - k^{(1)}(x_1)k^{(1)}(x_2)$ is in many applications integrable in one coordinate. In our case we have

$$u(x_1 - x_2) := u^{(2)}(x_1 - x_2, 0) = k(x_1 - x_2) - \rho^2.$$

It is an easy computation to show that the equation for $u$ has exactly the same form as (4.45) for $k$, namely

$$(a+w)*u(x_1-x_2)-(1+\langle w\rangle)u(x_1-x_2)=-\rho a(x_1-x_2).$$

With this remark and (4.47) one can then easily check that

$$k^{(2),\rho}(x_1,x_2):=v(x_1-x_2)+\rho^2$$

is a solution to (4.45) in $X_2$.

Let

$$A:=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{|\hat{a}(p)|}{1+\langle w\rangle-(\hat{a}+\hat{w})(p)}\,dp.$$

Then

$$k^{(2),\rho}(x_1,x_2)\le\rho A+\rho^2\le C^2(2!)^2$$

for any constant $C\ge\frac{1}{2}\sqrt{\rho A+\rho^2}$. Choose

$$C:=C(\rho):=\max\{A,\tfrac{1}{2}\sqrt{\rho A+\rho^2}\}.$$

Now consider the case $n\ge3$. Assume that $k^{(n-1),\rho}$ is already constructed and satisfies the estimate $\|k^{(n-1),\rho}\|_{X_{n-1}}\le C^{n-1}((n-1)!)^2$. Then, equation (4.43) gives

$$\begin{aligned}\hat{L}_n^*k^{(n)}(x_1,\ldots,x_n)&=-\sum_{i=1}^{n}k^{(n-1),\rho}(x_1,\ldots,\check{x}_i,\ldots,x_n)\sum_{j\neq i}a(x_i-x_j)\\&:=-f^{(n),\rho}(x_1,\ldots,x_n).\end{aligned}\tag{4.48}$$

The function

$$k^{(n),\rho}(x_1,\ldots,x_n):=\int_0^\infty\left(e^{t\hat{L}_n^*}f^{(n),\rho}\right)(x_1,\ldots,x_n)\,dt\tag{4.49}$$

is a solution to (4.48) in the Banach space $X_n$ provided

$$\int_0^\infty\left(e^{t\hat{L}_n^*}f^{(n),\rho}\right)(x_1,\ldots,x_n)\,dt<\infty\quad\text{for a.a. }(x_1,\ldots,x_n)\in(\mathbb{R}^d)^n$$

and

$$e^{t\hat{L}_n^*}f^{(n),\rho}\to0,\quad t\to\infty.$$

Therefore, to prove the existence of a solution to (4.48) we have to show that the right hand side of (4.49) has sense in $X_n$. By the induction assumption for $(n-1)$ and the Markov property of $e^{t\hat{L}_n^*}$ we obtain

$$\begin{aligned}
&\int_0^\infty \left(e^{t\hat{L}_n^*} f^{(n),\rho}\right)(x_1,\dots,x_n)\,dt \\
&\quad\le \int_0^\infty \left(e^{t\hat{L}_n^*} C^{n-1}((n-1)!)^2 \sum_{i=1}^n \sum_{j\neq i} a(\cdot_i - \cdot_j)\right)(x_1,\dots,x_n)\,dt \\
&\quad= C^{n-1}((n-1)!)^2 \sum_{i=1}^n \sum_{j\neq i} \int_0^\infty \left(e^{t(L^i_{a+w}+L^j_{a+w})} a(\cdot_i - \cdot_j)\right)(x_i,x_j)\,dt.
\end{aligned}$$

Since $e^{tL^j_{a+w}}$ is a contraction semigroup on $X_n$, there exists a Lebesgue nullset $N$ such that, since $\|\cdot\|_{L^\infty} \le \|\hat{\cdot}\|_{L^1}$,

$$\begin{aligned}
&\int_0^\infty \left(e^{t(L^i_{a+w}+L^j_{a+w})} a(\cdot_i - \cdot_j)\right)(x_i,x_j)\,dt \\
&\quad\le \int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N} \left(e^{tL^i_{a+w}} a(\cdot_i - x_j)\right)(x_i)\,dt \\
&\quad\le \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N} \int_{\mathbb{R}^d} \left| \left(\widehat{e^{tL^i_{a+w}} a(\cdot_i - x_j)}\right)(p)\right| dp\,dt \\
&\quad= \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N} \int_{\mathbb{R}^d} e^{t\left((\hat{a}+\hat{w})(p)-1-\langle w\rangle\right)} \times \\
&\qquad\times \left|\int_{\mathbb{R}^d} e^{-i(p,x)} a(x-x_j)\,dx\right| dp\,dt \\
&\quad= \frac{1}{(2\pi)^d}\int_0^\infty \sup_{x_j\in\mathbb{R}^d\setminus N} \int_{\mathbb{R}^d} e^{t\left((\hat{a}+\hat{w})(p)-1-\langle w\rangle\right)} \left|e^{-i(p,x_j)}\hat{a}(p)\right| dp\,dt \\
&\quad\le \frac{1}{(2\pi)^d}\int_0^\infty \int_{\mathbb{R}^d} e^{t\left((\hat{a}+\hat{w})(p)-1-\langle w\rangle\right)} |\hat{a}(p)|\,dp\,dt. \qquad (4.50)
\end{aligned}$$

For any $p \neq 0$

$$\int_0^\infty e^{t\left((\hat{a}+\hat{w})(p)-1-\langle w\rangle\right)}\,dt = \frac{1}{1+\langle w\rangle - (\hat{a}+\hat{w})(p)}.$$

Therefore, because of the Fubini theorem and (4.47), we have that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} e^{t\left((\hat{a}+\hat{w})(p)-1-\langle w\rangle\right)} |\hat{a}(p)|\,dp\,dt \\
&\quad= \int_{\mathbb{R}^d} \frac{|\hat{a}(p)|}{1+\langle w\rangle - (\hat{a}+\hat{w})(p)}\,dp < \infty.
\end{aligned}$$

Therefore, with the results from the case $n = 2$, we obtain that for Lebesgue-a.a. $(x_1, \dots, x_n)$

$$\int_0^\infty \left( e^{t\hat{L}_n^*} f^{(n),\rho} \right)(x_1, \dots, x_n)\, dt \le C^{n-1}((n-1)!)^2 n(n-1)A \le C^n (n!)^2.$$

This finishes the proof of part (I) of Theorem 4.4.2.

(II): The first statement of this part is trivial.

Second assertion: consider the difference between the correlation functions $k_t^{(n)}$ and $k^{(n),\rho}$, where the latter one is the one constructed in part (I):

$$\begin{aligned} &k_t^{(n)}(x_1, \dots, x_n) - k^{(n),\rho}(x_1, \dots, x_n) \\ &= \big[e^{t\hat{L}_n^*} - \mathbb{1}\big] k^{(n),\rho}(x_1, \dots, x_n) + e^{t\hat{L}_n^*}\big[k_0^{(n)}(x_1, \dots, x_n) - k^{(n),\rho}(x_1, \dots, x_n)\big] \\ &\quad + \int_0^t e^{s\hat{L}_n^*} f_{t-s}^{(n)}(x_1, \dots, x_n)\, ds. \end{aligned}$$

Since

$$\begin{aligned} \big[e^{t\hat{L}_n^*} - \mathbb{1}\big] k^{(n),\rho}(x_1, \dots, x_n) &= \int_0^t e^{s\hat{L}_n^*} \hat{L}_n^* k^{(n),\rho}(x_1, \dots, x_n)\, ds \\ &= -\int_0^t e^{s\hat{L}_n^*} f^{(n),\rho}(x_1, \dots, x_n)\, ds, \end{aligned}$$

we have

$$\begin{aligned} &k_t^{(n)}(x_1, \dots, x_n) - k^{(n),\rho}(x_1, \dots, x_n) \\ &= e^{t\hat{L}_n^*}\big[k_0^{(n)}(x_1, \dots, x_n) - k^{(n),\rho}(x_1, \dots, x_n)\big] && \text{(a)} \\ &\quad + \int_0^t e^{s\hat{L}_n^*}\big[f_{t-s}^{(n)}(x_1, \dots, x_n) - f^{(n),\rho}(x_1, \dots, x_n)\big]\, ds. && \text{(b)} \end{aligned}$$

Ad (b):

Similar to the computations for (4.49) and by Proposition 4.3.8 one can show

$$\begin{aligned} &\int_0^t e^{s\hat{L}_n^*} f_{t-s}^{(n)}(x_1, \dots, x_n)\, ds \in X_n, \\ &\int_0^\infty e^{s\hat{L}_n^*} f^{(n),\rho}(x_1, \dots, x_n)\, ds \in X_n. \end{aligned}$$

We use induction over $n$ to show that (b) tends to zero. For $n = 1$ this is trivial. So assume now, that

$$k_t^{(n-1)} \to k^{(n-1),\rho} \quad \text{in } X_{n-1},\ t \to \infty. \tag{4.51}$$

This immediately implies

$$f_{t-s}^{(n)} \to f^{(n),\rho} \quad \text{in } X_n,\ t \to \infty. \tag{4.52}$$

Therefore, by Proposition 4.3.8, there exists a constant $K > 0$ such that for any $t \geq 0$

$$\|k_t^{(n-1)}\|_{X_{n-1}} \leq K \|k^{(n-1),\rho}\|_{X_{n-1}}.$$

Let $0 < T \leq t$. Then

$$\begin{aligned}
&\int_T^t e^{s\hat{L}_n^*} \left[ f_{t-s}^{(n)}(x_1, \ldots, x_n) - f^{(n),\rho}(x_1, \ldots, x_n) \right] ds \\
&\quad \leq \int_T^t e^{s\hat{L}_n^*} \left[ \left| f_{t-s}^{(n)}(x_1, \ldots, x_n) \right| - \left| f^{(n),\rho}(x_1, \ldots, x_n) \right| \right] ds \\
&\quad \leq (1+K) \int_T^t \|k^{(n-1),\rho}\|_{X_{n-1}} \sum_{i=1}^n \sum_{j \neq i} \left( e^{s(L_{a+w}^i + L_{a+w}^j)} a(\cdot_i - \cdot_j) \right)(x_i, x_j)\, ds \\
&\quad \leq (1+K) \int_T^\infty \ldots\, ds.
\end{aligned} \tag{4.53}$$

By (4.50), this expression becomes abitrarily small for large enough $T$. But, by (4.52) and the contraction property of the semigroup $e^{t\hat{L}_n^*}$, also

$$\int_0^T e^{s\hat{L}_n^*} \left[ f_{t-s}^{(n)}(x_1, \ldots, x_n) - f^{(n),\rho}(x_1, \ldots, x_n) \right] ds \to 0 \quad \text{in } X_n,\ t \to \infty.$$

Thus we have shown the convergence of expression (b) to zero.

Ad (a):

The convergence of expression (a) to zero means that the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = \hat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n), \quad t \geq 0, \\ k_t^{(n)}(x_1, \ldots, x_n)\Big|_{t=0} = k_0^{(n)}(x_1, \ldots, x_n) \in X_n, \end{cases} \tag{4.54}$$

asymptotically does not depend on the initial conditions. The boundedness of the operator $\hat{L}_n^*$ implies the existence of the solution $k_t^{(n)} = e^{t\hat{L}_n^*} k_0^{(n)}$ as a function from $X_n$.

The latter fact allows us to look at the solution of (4.54) in the class of generalized functions $\big(L^1((\mathbb{R}^d)^n)\big)' \subset \mathcal{S}'((\mathbb{R}^d)^n)$, where $\mathcal{S}'((\mathbb{R}^d)^n)$ denotes the continuous linear functionals on the space $\mathcal{S}((\mathbb{R}^d)^n)$ of rapidly decreasing functions on $(\mathbb{R}^d)^n$. Since the Fourier transform is well-defined on $\mathcal{S}'((\mathbb{R}^d)^n)$, we can consider the following functional: for $\varphi \in \mathcal{S}((\mathbb{R}^d)^n)$

$$\begin{aligned}\big(\widehat{k_t^{(n)}}, \varphi\big) &= \big(k_t^{(n)}, \hat{\varphi}\big)\\ &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{t\hat{L}_n^*} k_0^{(n)}(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.\end{aligned}$$

Since

$$\|\hat{L}_n^* k\|_{L^\infty((\mathbb{R}^d)^n)} \le n\big(1 + \langle a\rangle + 2\langle w\rangle\big)\|k\|_{L^\infty((\mathbb{R}^d)^n)},$$

we get for all $N \in \mathbb{N}$

$$\begin{aligned}&\left|\sum_{l=0}^{N} \frac{t^l}{l!}\big(\big(\hat{L}_n^*\big)^l k_0^{(n)}\big)(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\right|\\ &\qquad\le \sum_{l=0}^{N} \frac{t^l}{l!} n^l \big(1 + \langle a\rangle + 2\langle w\rangle\big)^l \|k_0^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}\, |\hat{\varphi}(x_1, \ldots, x_n)|\\ &\qquad\le e^{tn\left(1+\langle a\rangle+2\langle w\rangle\right)} \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}\, |\hat{\varphi}(x_1, \ldots, x_n)| \in L^1((\mathbb{R}^d)^n).\end{aligned}$$

Hence, by Lebesgue's dominated convergence theorem,

$$\big(\widehat{k_t^{(n)}}, \varphi\big) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\big(\hat{L}_n^*\big)^l k_0^{(n)}\big)(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \tag{4.55}$$

For $l = 1$, the integral in the last expression gives

$$\begin{aligned}&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\hat{L}_n^* k_0^{(n)}\big)(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\\ &\quad= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} (-(1 + \langle w\rangle)n) k_0^{(n)}(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\\ &\qquad+ \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i=1}^{n} (a + w)(x_i - y) k_0^{(n)}(x_1, \ldots, y, \ldots, x_n) \times\\ &\qquad\quad \times \hat{\varphi}(x_1, \ldots, x_n)\, dy\, dx_1 \ldots dx_n\\ &\quad= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_n)(-(1 + \langle w\rangle)n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\\ &\qquad+ \sum_{i=1}^{n} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, y, \ldots, x_n) \times\\ &\qquad\quad \times \int_{\mathbb{R}^d} (a + w)(y - x_i)\hat{\varphi}(x_1, \ldots, x_n)\, dx_i\, dy\, dx_1 \cdots \check{dx_i} \cdots dx_n\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_n)(-(1 + \langle w \rangle)n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \\
&\quad + \sum_{i=1}^{n} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_i, \ldots, x_n) \times \\
&\qquad \times \int_{\mathbb{R}^d} (a + w)(x_i - y)\hat{\varphi}(x_1, \ldots, y, \ldots, x_n)\, dy\, dx_i\, dx_1 \cdots \check{dx_i} \cdots dx_n \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_n)\big(\hat{L}_n^* \hat{\varphi}\big)(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.
\end{aligned}
$$

From this it follows

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\big(\hat{L}_n^*\big)^l k_0^{(n)}\big)(x_1, \ldots, x_n)\hat{\varphi}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \\
&\qquad = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_n)\big(\big(\hat{L}_n^*\big)^l \hat{\varphi}\big)(x_1, \ldots, x_n)\, dx_1 \cdots dx_n
\end{aligned}
$$

for each $l \in \mathbb{N}$, and hence

$$
\big(\widehat{k_t^{(n)}}, \varphi\big) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \ldots, x_n)\big(\big(\hat{L}_n^*\big)^l \hat{\varphi}\big)(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \tag{4.56}
$$

Furthermore,

$$
\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{R}^d} (a + w)(x_i - y)\hat{\varphi}(x_1, \ldots, y, \ldots, x_n)\, dy \\
&\qquad = \sum_{i=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} (a + w)(y - x_i) e^{-i\langle (x_1, \ldots, y, \ldots, x_n), (p_1, \ldots, p_n)\rangle} \times \\
&\qquad\qquad \times \varphi(p_1, \ldots, p_n)\, dp_1 \cdots dp_n\, dy \\
&\qquad = \sum_{i=1}^{n} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (a + w)(z) e^{-i\langle (x_1, \ldots, x_n), (p_1, \ldots, p_n)\rangle} e^{-i\langle z, p_i\rangle} \times \\
&\qquad\qquad \times \varphi(p_1, \ldots, p_n)\, dz\, dp_1 \cdots dp_n \\
&\qquad = \mathcal{F}^{(n)}\left[\sum_{i=1}^{n} (\hat{a} + \hat{w})(\cdot_i)\varphi\right](x_1, \ldots, x_n),
\end{aligned}
$$

hence

$$
\big(\hat{L}_n^* \hat{\varphi}\big)(x_1, \ldots, x_n) = \mathcal{F}^{(n)}\left[\left(\sum_{i=1}^{n} (\hat{a} + \hat{w})(\cdot_i) - (1 + \langle w \rangle)n\right)\varphi\right](x_1, \ldots, x_n),
$$

and, inductively, for every $l \in \mathbb{N}$,

$$\left(\left(\hat{L}_n^*\right)^l \hat{\varphi}\right)(x_1,\dots,x_n) = \mathcal{F}^{(n)}\left[\left(\sum_{i=1}^n (\hat{a}+\hat{w})(\cdot_i) - (1+\langle w\rangle)n\right)^l \varphi\right](x_1,\dots,x_n). \tag{4.57}$$

Therefore,

$$\begin{aligned}\left(\widehat{k_t^{(n)}},\varphi\right) &= \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} k_0^{(n)}(x_1,\dots,x_n)\left(e^{t\hat{L}_n^*}\hat{\varphi}\right)(x_1,\dots,x_n)\,dx_1\dots dx_n \\ &= \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} k_0^{(n)}(x_1,\dots,x_n)\times \qquad (4.58)\\ &\qquad \times \mathcal{F}^{(n)}\left[e^{t\left(\sum_{i=1}^n(\hat{a}+\hat{w})(\cdot_i)-(1+\langle w\rangle)n\right)}\varphi\right](x_1,\dots,x_n)\,dx_1\dots dx_n. \qquad (4.59)\end{aligned}$$

By Remark 4.3.5 and Lemma 4.3.6, in the critical case $\varkappa = 1$, $e^{t\hat{L}_n^*}$ is a contraction semigroup in $L^1((\mathbb{R}^d)^n)$. Furthermore, one can show that

$$\mathcal{F}^{(n)}\left[e^{t\left(\sum_{i=1}^n(\hat{a}+\hat{w})(\cdot_i)-(1+\langle w\rangle)n\right)}\varphi\right](x_1,\dots,x_n) \to 0, \quad t\to\infty,$$

pointwisely. These two facts imply

$$e^{t\hat{L}_n^*}\hat{\varphi} \to 0 \text{ in } L^1((\mathbb{R}^d)^n), \quad t\to\infty,$$

thus

$$\left(\widehat{k_t^{(n)}},\varphi\right) \to 0, \quad t\to\infty,$$

and hence

$$\left(k_t^{(n)},\varphi\right) \to 0, \quad t\to\infty. \tag{4.60}$$

Since $\mathcal{S}((\mathbb{R}^d)^n)$ is dense in $L^1((\mathbb{R}^d)^n)$ and

$$\|k_t^{(n)}\|_{(L^1((\mathbb{R}^d)^n)'} = \|k_t^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \le \|k_0^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)},$$

we have (4.60) for every $\varphi \in L^1((\mathbb{R}^d)^n)$. This finishes the proof. □

## 4.5 Clustering

Consider the translation invariant, subcritical case, i.e.,

$$k_t^{(1)} \equiv \rho_t,\ t\ge 0, \text{ and } \varkappa < 1.$$

By (4.42), we have that the density $\rho_t$ converges to zero as $t \to \infty$. In fact, from Proposition 4.3.8 it follows, that the correlation functions of all orders tend to zero as $t \to \infty$.

On the other hand, Proposition 4.3.8 implies, for fixed $t$, a factorial bound for $k_t^{(n)}$. Thus one can expect clustering of the system. We will prove this in the next proposition. Starting from Poisson distribution of the particles we obtain a lower bound, factorial in the order $n$, for the correlation functions in a small region.

So, let $k_0^{(n)} = C^n$. Let $B \subset \mathbb{R}^d$ a bounded domain such that

$$\alpha := \inf_{x,y \in B} a(x-y) > 0.$$

Set $\beta := \min(\alpha\varkappa, C)$.

**Proposition 4.5.1.** *Let $t \geq 1$. Then for any $\{x_1, \ldots, x_n\} \subset B$, $n \geq 1$, one has*

$$k_t^{(n)}(x_1, \ldots, x_n) \geq \beta^n e^{n(\varkappa-1)t} n!. \tag{4.61}$$

*Proof.* We will prove the assertion by induction over $n$. Let $n = 1$. Then by (4.42)

$$k_t^1(x_1) = \rho_t = e^{(\varkappa-1)t} C \geq \beta^1 e^{1(\varkappa-1)t} 1!.$$

Now let $n \geq 2$ and assume that the assertion of the proposition is true for $n-1$. Then by Proposition 4.3.7

$$\begin{aligned} k_t^{(n)}(x_1, \ldots, x_n) \geq \varkappa e^{n(\varkappa-1)t} \int_0^t & e^{-n(\varkappa-1)s} \left[ \bigotimes_{i=n}^{n} e^{(t-s)L^i_{\varkappa a + w}} \right] \times \\ & \times \sum_{i=1}^n \underbrace{k_s^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n)}_{\geq \beta^{n-1} e^{(n-1)(\varkappa-1)s}(n-1)!} \sum_{j \neq i} a(x_i - x_j)\, ds \\ & \geq \varkappa e^{n(\varkappa-1)t} n \beta^{n-1} (n-1)!(n-1)\alpha \int_0^t \underbrace{e^{(-1)(\varkappa-1)s}}_{\geq e^0 = 1} ds \\ & \geq \beta^n e^{n(\varkappa-1)t} n!. \end{aligned}$$

In the last line we have used that $t \geq 1$.

Thus, the assertion is proved. $\square$

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