PART 2.

ON REDUCED K-THEORY FOR CENTRAL SIMPLE ALGEBRAS

The incompatibility of life's elements: Everest silences you. When you come down, nothing seems worth saying, nothing at all. You find the nothingness wrapping you up, like a sound. Non being. You can't keep it up, of course. The world rushes in soon enough. What shuts you up is, I think, the sight you've had of perfection. Why speak if you can't manage perfect thoughts, perfect sentences. It feels like a betrayal of what you've been through. But it fades, you accept that certian compromises, closures, are required if you're to continue...

Section 1. Wedderburn's factorization theorem

We begin this part by providing a short and elementary proof of the key theorem of reduced K-theory, namely Platonov's Congruence theorem. Our proof is based on Wedderburn's factorization theorem. We then use this approach to give an explicit formula for the reduced Whitehead group in certain cases. But then we postpone this line of research until Section 3 which concentrates on computational aspects of the group $SK_1(D)$. The results that we obtain in this section leads us to study the descending central series of the multiplicative group of a division ring which we explore in Section 2. We assume that the reader is familiar and comfortable with the theory of central simple algebras. A very nice source for the theory is the book of Draxl [2].

Let D be a division algebra with center F. If $a \in D$ is algebraic over F of degree m, then by Wedderburn's factorization theorem, one can find m conjugates of a such that the sum and the product of them are in F. This observation has been used in many different circumstances to give a short proof of known theorems of central simple algebras. (See [25 for a list of these theorems.) Here we will use this fact to prove Platonov's congruence theorem.

The non-triviality of the reduced Whitehead group $SK_1(D)$ was first shown by V. P. Platonov who developed a so-called *reduced K-theory* to compute $SK_1(D)$ for certain division algebras. The key step in his theory is the "congruence theorem" which is used to connect $SK_1(\overline{D})$ where $\overline{D}$ is a residue division algebra of D to $SK_1(D)$. This in effect enables one to compute the group $SK_1(D)$ for certain division algebras. (See [18] and [21].)

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Before we describe the congruence theorem, we employ Wedderburn's factorization theorem to obtain a result regarding normal subgroups of division algebras.

We fix some notation. Let D be a division algebra over its center F with index $i(D) = n$. Then $Nrd_{D/F} : D^* \longrightarrow F^*$ is the reduced norm function and $SK_1(D) = D^{(1)}/D'$ is the reduced Whitehead group where $D^{(1)}$ is the kernel of $Nrd_{D/F}$. Put $SH^0(D)$ for the cokernel of $Nrd_{D/F}$. we take $\mu_n(F)$ for the group of $n-th$ roots of unity in F, and $Z(D')$ for the center of the group D'. Observe that $\mu_n(F) = F^* \cap D^{(1)}$ and $Z(D') = F^* \cap D'$. If G is a group, denote by G^n the subgroup of G generated by the $n-th$ powers of elements of G. Let $exp(G)$ stands for the exponent of the group G. If H and K are subgroups of G, denote by $[H, K]$ the subgroup of G generated by mixed-commutators $[h, k] = hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$. For convenience we denote $[D^*, D^*]$ by D'. Denote by $det : GL_n(D)/SL_n(D) \longrightarrow D^*/D'$ the Dieudonne determinant, where $GL_n(D)$ is the general linear group and $SL_n(D)$ is its commutator subgroup (See [2]).

We are now in a position to state our main lemma which is interesting in its own right.

Lemma 1.1. *Let D be a division algebra with center F, of index n. Let N be a normal subgroup of D^*. Then $N^n \subseteq Nrd_{D/F}(N)[D^*, N]$.*

Proof. Let $a \in N$ whose minimal polynomial $f(x) \in F[x]$ is of degree m. From the theory of central simple algebras (cf. [23], §9), we have,

$$f(x)^{n/m} = x^n - Trd_{D/F}(a)x^{n-1} + \cdots + (-1)^n Nrd_{D/F}(a), \tag{1}$$

where $Nrd_{D/F} : D^* \longrightarrow F^*$ is the reduced norm, $Trd_{D/F}$ is the reduced trace and the right hand side of the equality (1) is the reduced characteristic polynomial of a. Using Wedderburn's factorization theorem for the minimal polynomial $f(x)$ of a, one obtains $f(x) = (x - d_1ad_1^{-1}) \cdots (x - d_mad_m^{-1})$ where $d_i \in D$. From the equality (1), it follows now that

$$Nrd_{D/F}(a) = (d_1ad_1{}^{-1} \cdots d_mad_m{}^{-1})^{n/m}.$$

Since N is a normal subgroup of D^*, it follows that $Nrd_{D/F}(a) \in N$. But

$$d_1ad_1{}^{-1} \cdots d_mad_m{}^{-1} = [d_1, a]a[d_2, a]a \cdots [d_m, a]a = a^m d_a$$

for some $d_a \in [D^*, N]$. Therefore $a^n = Nrd_{D/F}(a)d'_a$ where $d'_a \in [D^*, N]$. Thus $N^n \subseteq Nrd_{D/F}(N)[D^*, N]$. □

Note that $Nrd_{D/F}(N) \subseteq D^* \cap N$. Let $N = D^*$. Then by above Lemma, for any $x \in D^*$, $x^n = Nrd_{D/F}(x)d_x$ where $d_x \in D'$. This shows that the group $G(D) = D^*/F^*D'$ is a torsion group of bounded exponent n. Some algebraic properties of this group are studied in Section 3.

In order to describe Platonov's congruence theorem, we need to recall some concepts from valued division algebras.

Let D be a finite dimensional division algebra with center a Henselian field F.

Recall that a valuation v on a field F is called *Henselian* if and only if v has a unique extension to each field algebraic over F. Therefore v has a unique extension denoted also by v to D ([28]). Denote by V_D, V_F the valuation rings of v on D and F respectively and let M_D, M_F denote their maximal ideals and $\overline{D}, \overline{F}$ their residue division algebra and residue field, respectively. We let Γ_D, Γ_F denote the value groups of v on D and F, respectively and U_D, U_F the groups of units of V_D, V_F respectively. Furthermore, we assume that D is a *tame* division algebra, i.e., $Char\overline{F}$ does not divide $i(D)$, the index of D. The quotient group Γ_D/Γ_F is called the *relative value group* of the valuation. In this setting it turns out that D is *defectless*, namely we have $[\overline{D} : \overline{F}][\Gamma_D : \Gamma_F] = [D : F]$. D is said to be *unramified* over F if $[\Gamma_D : \Gamma_F] = 1$. At the other extreme D is said to be *totally ramified* if $[D : F] = [\Gamma_D : \Gamma_F]$. D is called *semiramified* if $\overline{D}$ is a field and $[\overline{D} : \overline{F}] = [\Gamma_D : \Gamma_F] = i(D)$. Since the valuation is Henselian, Hensel's lemma can be used to obtain a relation between the reduced norm of D and that of its residue algebra, i.e.

$$\overline{Nrd_D(a)} = N_{Z(\overline{D})/\overline{F}} Nrd_{\overline{D}}(\overline{a})^{n/mm'}, \tag{1.*}$$

where $a \in U_D$ and $m = i(\overline{D})$ and $m' = [Z(\overline{D}) : \overline{F}]$ (see [4]). For a recent account of the theory of Henselian valued division algebras see [9].

Platonov's congruence theorem asserts that if D is a tame division algebra over a Henselian field F then $(1 + M_D) \cap D^{(1)} \subseteq D'$. This is the crucial theorem of reduced K-theory which is proved in [18] (Note that [18] provides a lengthy and complicated proof for the special case of a complete discrete valuation of rank 1, and [4] notes that the same proof works for general case of tame Henselian valued division algebras). Here we give a short and elementary proof of this fact.

Theorem 1.2 *(Congruence Theorem). Let D be a tame division algebra over a Henselian field $F = Z(D)$, of index n. Then $(1 + M_D) \cap D^{(1)} = [D^*, 1 + M_D]$.*

Proof. First we show that $(1+M_F) \cap D^{(1)} = 1$. Let $1+f \in 1+M_F$. If $1+f \in D^{(1)}$, then $(1 + f)^n = 1$. But $v((1 + f)^n - 1) = v(f)$. This shows that $f = 0$ and so our claim. Now take $N = 1 + M_D$. By Lemma 1,

$$(1 + M_D)^n \subseteq \Big((1 + M_D) \cap F^*\Big)\Big[D^*, (1 + M_D)\Big].$$

Since the valuation is tame and Henselian, Hensel's lemma shows that $(1+M_D)^n = 1 + M_D$. Therefore $1 + M_D = (1 + M_F)\Big[D^*, (1 + M_D)\Big]$. Now using the fact that $(1 + M_F) \cap D^{(1)} = 1$, the theorem follows. □

Remark 1.3. There is an elegant proof of the congruence theorem by A. Suslin in [26], in the case of a discrete valuation of rank 1. This proof uses substantial results from valuation theory and the fact that the group $SK_1(D)$ is torsion of bounded exponent $n = i(D)$. Using results of Ershov in [4], Suslin's proof can be written for arbitrary tame Henselian division algebras.

Having the congruence theorem, it is easy to see, in the case of discrete valuation of rank 1, that the sequence,

$$SK_1(\overline{D}) \to SK_1(D) \to L_1/L_{\sigma-1} \to 1,$$

is exact where $L = Nrd(\overline{D})$, $L_1 = L \cap N^{-1}_{Z(\overline{D})/\overline{F}}(1)$ and $L_{\sigma-1}$ = the image of L under the homomorphism $a \mapsto \sigma(a)a^{-1}$, where $\langle\sigma\rangle = Gal(Z(\overline{D})/\overline{F})$. This leads to computations of $SK_1(D)$ for certain division algebras. (See [18], [21] and [26].)

Another look at the proof of Theorem 2 shows that $1 + M_D \subseteq (1 + M_F)D'$ and therefore $1 + M_D \subseteq U_F D'$. Put $G(\overline{D}) = \overline{D}^*/\overline{F}^*\overline{D'}$. In many applications, it is easy to obtain information about the residue data of division algebras. The following theorem gives an explicit formula for the group $SK_1(D)$ when the group $G(\overline{D})$ is trivial.

Theorem 1.4. *Let D be a tame division algebra over a Henselian field $F = Z(D)$, of index n. If $G(\overline{D}) = 1$ then $SK_1(D) = \mu_n(F)/Z(D')$.*

Proof. The reduction map $U_D \longrightarrow \overline{D}^*$ induces an isomorphism $\overline{D}^* \longrightarrow U_D/1+M_D$, $\overline{a} \mapsto (1 + M_D)a$. Since $1 + M_D \subseteq U_F D'$, it follows that

$$\overline{D}^*/\overline{F}^*\overline{D'} \xrightarrow{\simeq} U_D/U_F D'.$$

Now if $G(\overline{D}) = \overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $U_D = U_F D'$. But $D^{(1)} \subseteq U_D$. This shows that $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$, the theorem follows. □

Note that Hensel's lemma implies that $\mu_n(F) \simeq \mu_n(\overline{F})$. In particular if D is a totally ramified division algebra, i.e. $\overline{D} = \overline{F}$, then $G(\overline{D}) = 1$. We close this section with an example but will be back to this topic again in Section 3 where we systematically study the *SK_1-like functor* $G(D)$. We will show that $G(D)$ has the most important functorial properties of the reduced Whitehead group SK_1. The structure of $G(D)$ turns out to carry significant information about the arithmetic of D. Along these lines, we employ $G(D)$ to compute the group $SK_1(D)$.

Example 1.5. Let $\mathbb{C}$ be the field of complex numbers and r be a nonnegative integer. Let $D_1 = \mathbb{C}((x_1))$ and define $\sigma_1 : D_1 \to D_1$ by the rule $\sigma_1(x_1) = -x_1$. Now let $D_2 = D_1((x_2, \sigma_1))$ and set $D_3 = D_2((x_3))$. Again define $\sigma_3 : D_3 \to D_3$ by $\sigma_3(x_3) = -x_3$. In general, if i is even, set $D_{i+1} = D_i((x_{i+1}))$ and if i is odd define $\sigma_i : D_i \to D_i$ by $\sigma_i(x_i) = -x_i$ and $D_{i+1} = D_i((x_{i+1}, \sigma_i))$. By Hilbert's construction (see [3], §1 and §24), $D = D_{2r} = \mathbb{C}((x_1, \cdots, x_{2r}, \sigma_1, \cdots, \sigma_{2r-1}))$ is a division algebra with center $F = \mathbb{C}((x_1^2, x_2^2, \cdots, x_{2r-1}^2, x_{2r}^2))$ and $n = i(D) = 2^r$. Finally define $v : D^* \to \Gamma_D = Z^{2r}$ by the rule $v(\sum c_i x_1^{i_1} \cdots x_{2r}^{i_{2r}}) = (i_1, \cdots, i_{2r})$ where $i_1, \cdots, i_{2r}$ are the smallest powers of the x_i's in the lexicographic order. It can be observed that v is a tame valuation and $\overline{D} = \mathbb{C}$ and $\overline{F} = \mathbb{C}$. Therefore $G(\overline{D}) = 1$. Theorem 3 implies that $SK_1(D) = \mu_n(F)/Z(D')$. From the multiplication rule in D, it follows that

$$D' \subseteq \Big\{ \pm 1 + \sum_{i>0} c_i x_1^{i_1} \cdots x_{2r}^{i_{2r}} \Big\}.$$

Since $Z(D') \subseteq \mu_n(F)$, it follows that $Z(D') = \{1, -1\}$. But $\mu_n(F) = \mu_n(\overline{F}) = \mathbb{Z}_{2^r}$, hence $SK_1(D) = \mathbb{Z}_{2^{r-1}}$.

In Section 3 as another application of Lemma 1.1, we obtain theorems of reduced K-theory which previously required heavy machinery, as simple consequence of this approach.

Section 2. On central series of the multiplicative group of division rings

From Lemma 1.1 it follows a very curios result on the descending central series of a division algebra (see Lemma 2.1) which motivates us to study this series. In this Section we show that certain properties which a term in the descending central series may have, can be lifted to the full multiplicative group and it determines quotients of consecutive terms in the descending central series, in tame Henselian unramified or totally ramified cases.

We follow the convention that if D is a division ring with center F then D is called a *division algebra* if $[D : F]$ is finite.

Let D be a division ring and D^* the multiplicative group of D. Put $G^0(D) = D^*$ and for any natural number i, define $G^i(D) = [D^*, G^{i-1}(D)]$, i.e. the subgroup generated by the mix-commutators of D^* and $G^{i-1}(D)$. The sequence

$$\cdots \subseteq G^2(D) \subseteq G^1(D) \subseteq G^0(D) = D^*$$

is called *the descending central series* of D^*. It is a classical result that if D is noncommutative then the multiplicative group of D^* is not nilpotent, that is, no term in the above series is 1 [11, p.223].

In this note we study the subgroups $G^i(D)$ above. We shall show that several properties they may have, can actually be lifted in a natural way to the group D^* and we shall compute the consecutive quotients $G^i(D)/G^{i+1}(D)$ where $i \geq 1$ in several cases.

The results divide into two parts. The main result in subsection 1 is that if some $G^i(D)$ is algebraic over the center F of D then the F-subalgebra of D generated by $G^i(D)$ is all of D, in particular D is an algebraic division ring. This Theorem is used to generalize results of Kaplansky and Jacobson which provide conditions when a division ring is commutative. The work above shows that the subgroups $G^i(D)$ are big in D^*.

The results in subsection 2 determine the consecutive quotients $G^i(D)/G^{i+1}(D)$ for $i \geq 1$ when D is a tame Henselian division algebra which is either totally ramified or unramified. This extends previous work of U. Rehmann [22] and P. Draxl [3, Vortrag 7] over local division algebras (also see C. Riehm [24].) where it is shown that the descending central series of D^* becomes stationary and the quotients $G^i(D)/G^{i+1}(D)$ are calculated.

1. Descending Central Series in Division Rings

Before stating our first Lemma, we fix some notation. If G is a group, denote by G^n the subgroup of G generated by all $n-th$ powers of elements of G. As in Section 1, if H and K are subgroups of G, denote by $[H, K]$ the subgroup of G generated by mix-commutators $[h, k] = hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$. For convenience we sometimes denote $[D^*, D^*]$ by D'. We say that a subset S of D is *algebraic* over F if each element of S is algebraic over F. Also if S and T are subsets of D, then S is said to be *radical* over T, if for any element $x \in S$, there is a natural number r such that $x^r \in T$.

As we mentioned in the beginning of this section, the first result is the immediate consequence of the Lemma 1.1. Remember that if N is a normal subgroup of a division ring D of index n then $N^n \subseteq Nrd_{D/F}(N)[D^*, N]$.

Now let $N = G^1(D)$. Since for $a \in [D^*, D^*]$, $Nrd_{D/F}(a) = 1$, the above lemma shows that $G^1(D)^n \subseteq G^2(D)$. Letting in general $N = G^i(D)$ where $i \geq 1$, we obtain by the same argument that $G^i(D)^n \subseteq G^{i+1}(D)$. As a consequence, we get the following corollary.

Corollary 2.1. *Let D be a division algebra of index n. For any $i > 0$, the quotient $G^i(D)/G^{i+1}(D)$ is a torsion abelian group of bounded exponent n.*

The corollary says nothing about the quotient $G^0(D)/G^1(D)$. But using a theorem of Jacobson [11, p. 219], it is an easy exercise to see that the group $G^0(D)/G^1(D)$, namely $K_1(D) = D^*/D'$, is not a torsion group.

In the rest of this section we concentrate on the general case of a division ring. The main result is to show that the algebraicity of a subgroup of D^* which contains a term of the descending central series of division ring, gives rise to the algebraicity of D.

Lemma 2.2. *Let D be a division ring with center F and N be a subgroup of D^* which contains some $G^i(D)$. If $a \in D$ is algebraic over F, then a is radical over F^*N.*

Proof. Clearly we can replace N by $G^i(D)$. The proof will be by induction on i. For $i = 0$, there is nothing to prove. Suppose there is a nonnegative integer r such that $a^r = fb$ for some $f \in F$ and $b \in G^{i-1}(D)$. It suffices to show that there is a nonnegative integer m such that $b^m = ec$ for some $e \in F$ and $c \in G^i(D)$. Since a is algebraic over F, so is b. From field theory, we have

$$f(x) = x^m - Tr_{F(b)/F}(b)x^{m-1} + \cdots + (-1)^m N_{F(b)/F}(b), \tag{1}$$

where $f(x)$ is the minimal polynomial of b over F. Now using Wedderburn's factorization theorem for $f(x)$ as in the Lemma 1.1, it follows that

$$N_{F(b)/F}(b) = [d_1, b]b[d_2, b]b \cdots [d_m, b]b = b^m d_b$$

where $d_b \in [D^*, G^{i-1}(D)] = G^i(D)$. Let $e = N_{F(b)/F}$. Thus $b^m = ec$ where $c = d_b^{-1}$. □

We are now in a position to show how the properties of a subgroup which appear in the descending central series of D^* can be lifted to D^*. The following Theorem shows that the algebraicity of a division ring is inherited from the algebraicity of any subgroup containing some $G^i(D)$. In contrast, note that a division ring can be transcendental over its center and yet have maximal subfields which are algebraic (Example 2.4).

Theorem 2.3. *Let D be a division ring with center F and N a subgroup of D^* containing some $G^i(D)$. If N is algebraic over F then the F-subalgebra generated by N is D. In particular D is an algebraic division ring.*

Proof. Clearly we can replace N by $G^i(D)$. The conclusions of the theorem are trivial if $D = F$. So we can assume $D \neq F$. Let $A = F^*G^i(D)$. Since D^* is not a nilpotent group, $G^i(D) \nsubseteq F$. So $F \subsetneq A$. Suppose $a \in A$ and b is an algebraic element of D^*. Denote by $\bar{a}$ and $\bar{b}$ the images of a and b in the quotient group $D^*/F^*G^{i+1}(D)$. Since $a \in F^*G^i(D)$, $\bar{a}$ commutes with $\bar{b}$. By Lemma 2.2, $\bar{a}$ and $\bar{b}$ are torsion elements. Therefore $\overline{ab}$ is torsion. It follows that ab is algebraic over F. Next consider the element $a + b = a(1 + a^{-1}b)$. Since $a \in F^*G^i(D)$ and b are algebraic, $1 + a^{-1}b$ is algebraic. It follows that $a + b$ is algebraic. Consider the ring $\langle A \rangle$ generated by elements of A. From the above it follows that $\langle A \rangle$ is algebraic over F. Therefore $\langle A \rangle$ is a division ring. Obviously $\langle A \rangle^*$ is a normal subgroup of D^* and so by Cartan-Brauer-Hua [11, p.222], $\langle A \rangle = D$. Hence D is generated as a F-subalgebra by the elements of $G^i(D)$ and D is algebraic over F. □

For the sake of completeness, we give an example showing that the algebraicity of a maximal subfield of a division ring D, does not give rise to the algebraicity of D. (Also see [5], p. 280.)

Example 2.4. Let L be a field which is algebraic over its prime subfield and $\sigma \in Aut(L)$ such that $ord(\sigma) = \infty$, e.g., take $L = \overline{\mathbb{Z}_p}$ and $\sigma(x) = x^{p^r}$ or $L = \bigcup_{i=1}^{\infty} F_{p^{2^i}}$ and $\sigma(x) = x^{p^r}$ where p is a prime number and r is a natural number. Let K denote the fixed field of σ. Now let $D = L((X, \sigma))$ denote the formal twisted Laurent series in the indeterminant X with twisting $Xl = \sigma(l)X$. By Hilbert classical construction (cf. [2]), D is a division algebra with center $Z(D) = K$. We show that L is a maximal subfield of D. Suppose $L \subsetneq M \subseteq D$ and M is a field. Take $\lambda \in M \backslash L$. Clearly $\lambda = \sum_{i=r}^{\infty} a_i X^i$ where r is an integer. Since $M \neq L$, there is $0 \neq n \in \mathbb{Z}$ such that $a_n \neq 0$. Now for all $l \in L$, $l\lambda = \lambda l$. Therefore $\sum l a_i X^i = \sum a_i \sigma^i(l) X^i$. In particular, for all $l \in L$ we have $\sigma^n(l) = l$. This means that $o(\sigma)$ is finite, which is a contradiction, and therefore $L = M$. Therefore L is a maximal subfield of D which is algebraic over $Z(D)$. But clearly D is not algebraic. We remark that L and $K((X))$ are two maximal subfields of D such that L is algebraic over K whereas $K((X))$ is transcendental over K.

We are now ready to generalize some commutativity theorems for a division ring. The following is a generalization of Kaplansky's Theorem (See [11, p. 259] and [14]).

Corollary 2.5. *Let D be a division ring with center F. If a subgroup N containing some $G^i(D)$ is radical over F then D is commutative.*

Proof. Since N is radical over F, we conclude by Theorem 2.3 that D is algebraic division ring. Thus by Lemma 2.2, D is radical over F^*N. Since N is radical over F, it follows that D is radical over F. Now applying Kaplansky's Theorem, the proof is complete. □

Next we generalize the Noether-Jacobson Theorem asserting that any noncommutative algebraic division ring D contains an element in $D \setminus F$ which is separable over F. (See [11], p.257.)

Corollary 2.6. *Let D be non-commutative algebraic division ring with center F. Then for any subgroup N containing some $G^i(D)$ there exists an element of $N \setminus F$ which is separable over F.*

Proof. Suppose this is not the case. Then all elements in N are purely inseparable over F. This means that N becomes radical over F. Now apply Corollary 2.5 to get a contradiction. □

The following can be viewed as a generalization of a Jacobson's Theorem. (See [11], p.219.)

Corollary 2.7. *Let D be algebraic division ring with center F. If a subgroup N containing some $G^i(D)$ is algebraic over a finite subfield of F, then D is commutative.*

Proof. Exercise. □

2. Descending Central Series in Valued Division Algebras

In this section we study the descending central series of a Henselian valued division algebra. Theorem 2.9 determines completely this series in the tame totally ramified case. At the other extreme, namely in the tame unramified case we show that the quotient group $G^i(D)/G^{i+1}(D)$ is *stable under reduction*, namely

$$\frac{G^i(D)}{G^{i+1}(D)} \simeq \frac{G^i(\overline{D})}{G^{i+1}(\overline{D})}.$$

Recall that a subgroup H of a group G is called *G-perfect*, if $[G, H] = H$. The following Corollary is an immediate consequence of the Theorem 1.2.

Corollary 2.8. *Let D be a tame, Henselian division algebra. Then $[D^*, 1 + M_D]$ is D^*-perfect. In particular $[D^*, 1 + M_D] \subseteq G^i(D)$ for all $i \geq 0$.*

Theorem 2.9. *Let D be a tame, totally ramified Henselian division algebra with center F and index n. Then*

(i) $G^1(D)/G^2(D) = \mathbb{Z}_e$, where $e = exp(\Gamma_D/\Gamma_F)$.

(ii) $G^i(D) = G^{i+1}(D)$ where $i \geq 2$.

Proof. Since D is totally ramified, $\overline{D} = \overline{F}$. Thus $U_D = U_F(1 + M_D)$. Thus $[D^*, D^*] \subseteq U_F(1 + M_D)$. Therefore $G^2(D) \subseteq [D^*, 1 + M_D]$. From Corollary 2.8, it follows $G^2(D) = [D^*, 1 + M_D]$ and $G^2(D)$ is D^*-perfect. Thus $G^2(D) = G^i(D)$ for all $i \geq 2$. This proves (ii).

Now consider the reduction map $U_D \longrightarrow \overline{D}^*$. Restriction of this map to D' gives rise to an isomorphism

$$\frac{D'}{D' \cap (1 + M_D)} \xrightarrow{\simeq} \overline{D'}.$$

From the equality $G^2(D) = [D^*, 1 + M_D]$ above and THeorem 1.2, we get $G^2(D) = D' \cap (1 + M_D)$. Thus $D'/G^2(D) \simeq \overline{D'}$. On the other hand $\overline{D'} \simeq \mathbb{Z}_e$ where $e = exp(\Gamma_D/\Gamma_F)$. (See the proof of Theorem 3.1 in [27].) Therefore

$$\frac{D'}{G^2(D)} = \mathbb{Z}_e$$

and the proof is complete. □

The calculation of $G^1(D)/G^2(D)$ in the above theorem was possible because we were able to identify $(1 + M_D) \cap D'$ with $[D^*, 1 + M_D]$, thanks to Theorem 1.2.

Theorem 2.10. *Let D be a tame, unramified Henselian division algebra. Then*

(i) $[D^*, 1 + M_D] \subsetneq G^i(D)$, for any $i \geq 1$.

(ii) $G^i(D)/G^{i+1}(D) \simeq G^i(\overline{D})/G^{i+1}(\overline{D})$, for any $i \geq 1$.

Proof. As in the proof of Theorem 2.9, the restriction of reduction map $U_D \longrightarrow \overline{D}$ to $[D^*, D^*]$ gives rise to an isomorphism

$$\frac{G^1(D)}{[D^*, 1 + M_D]} \xrightarrow{\simeq} \overline{[D^*, D^*]}.$$

Since D is unramified, $D^* = F^* U_D$. Therefore for $a, b \in D^*$, the element $c = aba^{-1}b^{-1}$ may be written in the form $c = \alpha\beta\alpha^{-1}\beta^{-1}$ where α and $\beta \in U_D$. This shows that $\overline{[D^*, D^*]} = [\overline{D}^*, \overline{D}^*]$. By Corollary 2.5, $[\overline{D}^*, \overline{D}^*]$ is not a torsion group. Therefore $G^1(D)/[D^*, 1 + M_D]$ is not torsion. On the other hand by Corollary 2.1, $G^1(D)/G^i(D)$ is a torsion group and by Corollary 2.8, $[D^*, 1 + M_D] \subseteq G^i(D)$. This shows that $[D^*, 1 + M_D] \subsetneq G^i(D)$.

(ii) Since the valuation is unramified, it can be shown as above that, $G^i(\overline{D}) = \overline{G^i(D)}$. As above the restriction of reduction map to the subgroup $G^i(D)$ give rises to an isomorphism

$$\frac{G^i(D)}{[D^*, 1 + M_D]} \xrightarrow{\simeq} G^i(\overline{D}).$$

Therefore

$$\frac{G^i(D)}{G^{i+1}(D)} \simeq \frac{G^i(\overline{D})}{G^{i+1}(\overline{D})}$$

and we are done. □

Dieudonne has shown that the projective special linear group

$$PSL_n(D) = \frac{SL_n(D)}{Z(SL_n(D))}$$

is a simple group where $n = 2$ and D has more than 3 elements or $n > 2$ [2, §21] and [10, p.191]. The following theorem shows that if a noncommutative division algebra enjoys a tame and discrete valuation then

$$PSL_1(D) = \frac{D'}{Z(D')}$$

is not a simple group.

Theorem 2.11. *Let D be a tame and discrete valued division algebra. Then $PSL_1(D)$ is not a simple group.*

Proof. We consider two cases. Suppose $\overline{D}$ is commutative. It follows that $D'' \subseteq 1 + M_D$. By induction one shows that if $D^{(i)}$ denotes the $i-th$ derived subgroup of D and $i \geq 2$, then $D^{(i)} \subseteq 1 + {M_D}^{2^{(i-2)}}$. Suppose $D' = D''$. Obviously $D' = D^{(i)}$ for all $i \geq 1$. Thus $D' \subseteq \bigcap_{i=2}^{\infty} 1 + {M_D}^{2^{(i-2)}} = 1$. Thus D is commutative, which is a contradiction. Thus $D'' \subsetneq D'$. Thus $U_F D''/U_F \lhd U_F D'/U_F$. But $PSL_1(D) = U_F D'/U_F$. This shows that $PSL_1(D)$ is not a simple group.

We are left with the case when $\overline{D}$ is not commutative. In this case consider the normal subgroup $N = Z(D')(1 + M_D \cap D')$ of D'. It is easy to show that N does not coincide with D'. Suppose $N = Z(D')$. Since the valuation is tame, it follows that $(1 + M_D) \cap D' = 1$. Thus $[D^*, 1 + M_D] = 1$. Thus $1 + M_D \subseteq F$. That is, D is commutative and therefore $\overline{D}$ is commutative, which is a contradiction. Therefore $PSL_1(D)$ has a non-trivial normal subgroup and the proof is complete. □

Section 3. SK_1-like Functors for division algebras

Let D be a division algebra with center F. The non-triviality of the important group $SK_1(D)$ is shown by V. P. Platonov who developed a so-called *Reduced K-Theory* to compute $SK_1(D)$ for certain division algebras. The group $SK_1(D)$ enjoys some interesting properties which distinguish it from the *K-Theory* functor $K_1(D)$. An interesting characteristic of the group $SK_1(D)$ is its behavior under extension of the ground field. Namely for any field extension L/F one has a homomorphism $SK_1(D) \to SK_1(D \otimes_F L)$. On the other hand SK_1 enjoys a transfer map, that is, if L/F is a finite extension, then there exist a norm homomorphism $SK_1(D \otimes_F L) \to SK_1(D)$. Since $SK_1(M_n(L)) = 1$, one can then deduce that SK_1 is a torsion abelian group of bounded exponent $i(D)$ and if the degree $[L : F]$ is relatively prime to index of D, then $SK_1(D) \hookrightarrow SK_1(D \otimes_F L)$. Moreover the primary decomposition of a division algebra induces a corresponding decomposition of $SK_1(D)$. Furthermore in the case of a valued division algebra SK_1 is stable, namely $SK_1(D) = SK_1(\overline{D})$, where D is unramified division algebra. (See [15] for the complete list of the properties of SK_1 and [2] for the proofs).

In this section we investigate the group $G(D) = D^*/F^*D'$ where D is a division algebra with center F and D' the commutator subgroup of D^*. We shall show that G enjoys most important functorial properties of the reduced Whitehead group SK_1. We show that the functor G may grow "pathologically" for an algebraic extension of the ground field whose degree is prime to the index of D. It is then shown that this functor satisfies a decomposition property analogous to one for $SK_1(D)$. To be more precise, we will show the following properties:

i. For any field extension L/F one has a homomorphism $G(D) \to G(D \otimes_F L)$.

ii. If L/F is a finite extension, then there exist a transfer homomorphism $G(D \otimes_F L) \to G(D)$. (Proposition 3.3)

iii. $G(D)$ is a torsion group of bounded exponent $i(D)$ (Corollary 3.4, Lemma 1.1)

iv. If $[L : F]$ is relatively prime to $i(D)$, then $G(D) \hookrightarrow G(D \otimes_F L)$. (Corollary 3.5).

v. If $D = D_1 \otimes_F D_2 \otimes_F \cdots \otimes_F D_K$ and the $i(D_i)$ are relatively prime, then $G(D) \simeq \coprod G(D_i)$. (Theorem 3.8)

vi. If D is unramfied tame Henselian division algebra, then $G(D) \simeq G(\overline{D})$. (Theorem 3.11 i)

It turns out that there is a close connection between the group structure of $G(D)$ and algebraic structure of D. For example in subsection 3, after establishing a fundemental connection between $G(D)$, its residue version and relative value group when D admits a Henselian valuation, we show that if D is a totally ramified division algebra, then there is a one to one correspondence between the isomorphism classes of F-subalgebras of D and the subgroups of $G(D)$.

We then use $G(D)$ to compute $SK_1(D)$ for certain division algebras. We show that if $G(D)$ canonically coincides with the relative value group, then there is an explicit formula for the group $SK_1(D)$ (Theorem 3.17). It turns out that some theorems and examples of reduced K-theory which require heavy machinery can all be viewed as simple examples of our case (Example 3.19, 3.20 and 3.21). Section 4 is devoted to the unitary version of the group $G(D)$.

1. Functor $G(D) = D^*/F^*D'$

Let $\mathcal{C}$ be the category of all central simple algebras and $G : \mathcal{C} \longrightarrow \mathcal{A}b$ be a covariant functor from $\mathcal{C}$ to the category of abelian groups such that for any central simple algebra A with center F, $G(A) = A^*/F^*A'$.

It is easy to observe that the functor G has the following properties:

D1. There is a collection of homomorphisms $d_n : G(M_n(D)) \longrightarrow G(D)$ for each division algebra D and each positive integer n such that for each $x \in G(D)$, $d_n i_n(x) = x^n$ where $i_n : G(D) \longrightarrow G(M_n(D))$ is the homomorphism induced by the natural embedding $D \longrightarrow M_n(D)$ and d_n induced by Dieudonne determinant [, §20].

D2. For any field F, $G(F)$ is trivial.

D3. If $x \in Ker(G(M_n(D)) \xrightarrow{d_n} G(D))$, where D is a division algebra and $n \in \mathbb{N}$, then $x^n = 1$.

On the other hand there have been other groups associated with a division ring D which have been used to study the arithmetic and algebraic structure of D. For example the square class group D^*/D^{*2} in [12] in connection with Witt ring of a division algebra or the group $D^*/Nrd(D^*)D'$. The following examples show that some important groups already associated to D share the three conditions above.

Example 3.1. Let $A \in \mathcal{C}$ with center F, then it is easy to observe that functors $\mathfrak{G}(A) = (A^*)^2/(F^*)^2A'$, and $\mathfrak{G}(A) = A^*/F^*A'_r$ where $A'_r = \{x \in A^* | x^r \in A'\}$ and $r \in \mathbb{N}$ also satisfy the properties **D1,D2** and **D3** above.

Example 3.2. Let $A \in \mathcal{C}$ be a central simple algebra finite over its center. The following commutative diagram with exact rows shows that $SK_1(D) = D^{(1)}/D'$ and $SH^0(D) = F^*/Nrd_{D/F}(D^*)$ satisfy the three conditions above,

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & SK_1(D) & \longrightarrow & K_1(D) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \eta_n} & & \downarrow & & \\
1 & \longrightarrow & SK_1(M_n(D)) & \longrightarrow & K_1(M_n(D)) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(M_n(D)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle det} & & \downarrow{\scriptstyle 1} & & \downarrow & & \\
1 & \longrightarrow & SK_1(D) & \longrightarrow & K_1(D) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(D) & \longrightarrow & 1
\end{array}
$$

where $\eta_n(x) = x^n$ for any $x \in F^*$ and D is a division algebra with center F. Note that in order to consider SK_1 and SH^0 as functors, we should limit the objects of our category (See §22, §23 in [2]).

In the same way, it can be seen that $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ and $\mathfrak{G}(A) = A^*/F^*A^{(1)} \simeq Nrd(A^*)/{F^*}^{DegA}$ also satisfy **D1,D2** and **D3**.

For the rest of this section we restrict our attention to the functor $G(A) = A^*/F^*A'$, although the results we get can be formulated and proved *mutatis mutandis* for all the functors above.

Our primary aim in this section is to show that the functor G shares almost all important functorial properties of SK_1. Clearly the natural embedding of D in $D \otimes_F L$ where L is a finite field extension of F, induces a group homomorphism $\mathcal{I} : G(D) \longrightarrow G(D \otimes_F L)$. The following proposition provides us with a homomorphism in the opposite direction.

Proposition 3.3. (Transfer map) *Let D be a division ring with center F and L be a finite extension of F such that $[L : F] = m$. Then there is a homomorphism $\mathcal{P} : G(D \otimes_F L) \longrightarrow G(D)$ such that $\mathcal{P}\mathcal{I} = \eta_m$, where $\eta_m(x) = x^m$.*

Proof. Consider the regular representation $L \xrightarrow{\iota} M_m(F)$ and the corresponding sequence when we tensor over F with D:

$$D \longrightarrow D \otimes_F L \xrightarrow{1\otimes\iota} D \otimes_F M_m(F) \longrightarrow M_m(D)$$

$$a \longmapsto a \otimes 1 \longmapsto a \otimes 1 \longmapsto aI_m \tag{3.1}$$

$$1 \otimes \ell \longmapsto 1 \otimes \iota(\ell) \longmapsto \iota(\ell).$$

Thanks to the Dieudonne determinant, there is a homomorphism $K_1(D \otimes_F L) \to K_1(D)$ which maps the center of $D \otimes_F L$ into the center of D. Therefore $G(D \otimes_F L) \to G(D)$. Again the sequence (3.1) shows that $\mathcal{P}\mathcal{I}(x) = x^m$. $\square$

Note that in the above proposition D could be an infinite dimensional division algebra. If D is finite dimension over its center F, then it turns out that $G(D)$ is a torsion group.

Corollary 3.4. *Let D be a division algebra of index n. Then $G(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$.*

Proof. Thanks to Proposition 3.3, for any finite field extension L of $F = Z(D)$, we have the sequence of homomorphisms $G(D) \xrightarrow{\mathcal{I}} G(D \otimes_F L) \xrightarrow{\mathcal{P}} G(D)$, such that $\mathcal{P}\mathcal{I}(x) = x^m$, where $x \in \mathfrak{G}(D)$ and $[L : F] = m$. Now let L be a maximal subfield of D. Since L is a splitting field for D, we get the sequence of homomorphisms $G(D) \xrightarrow{\mathcal{I}} G(M_n(L)) \xrightarrow{\mathcal{P}} G(D)$. From **D2** and **D3** it follows that $G(M_n(L))$ is a torsion group of bounded exponent n. Now the fact that for any $x \in G(D)$, $\mathcal{PI}(x) = x^n$, shows that $G(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$. $\square$

It is now immediate that if A is a central simple algebra, then $G(A)$ is also torsion. Later in this subsection we show that the bound can be reduced to n, the index of D.

The following corollary shows that the analogous result for the behavior of SK_1 under extension of the ground field holds for G too. Namely, we show that $G(D)$ embeds in $G(D \otimes_F L)$ when the index of D and $[L : F]$ are relatively prime.

Corollary 3.5. *Let D be a division ring over its center F and L/F be a finite field extension such that $[L : F]$ is relatively prime to the index of D. Then the canonical homomorphism $G(D) \xrightarrow{\mathcal{I}} G(D \otimes_F L)$ is injective.*

Proof. Let $i(D) = n$ and $[L : F] = m$. Suppose $\mathcal{I}(x) = 1$ for some $x \in G(D)$. By Proposition 3.3, $\mathcal{PI}(x) = x^m = 1$. But by Corollary 3.4, $G(D)$ is torsion of bounded exponent n^2. Hence $x^{n^2} = 1$. Since m and n are relatively prime, $x = 1$ and the proof is complete. $\square$

In the next section we compute the functor G for certain division algebras. But before we continue with the functorial properties of G, let us consider the case when the group $G(D)$ is trivial. Besides **D1, D2** and **D3**, the functor G enjoys an additional property, namely there is a natural transformation $\tau : K_1 \longrightarrow G$ such that,

(1) For any object A in $\mathcal{C}$, $\tau_A : K_1(A) \longrightarrow G(A)$ is an epimorphism.

(2) For any division algebra D and any positive integer n, the following diagram commutes,

$$\begin{array}{ccc} K_1(M_n(D)) & \xrightarrow{\tau} & G(M_n(D)) \\ \downarrow{\scriptstyle det} & & \downarrow{\scriptstyle d} \\ K_1(D) & \xrightarrow{\tau} & G(D). \end{array}$$

Note that the functors of Example 3.1, or $\mathfrak{G}(D) = Nrd_{D/F}(D^*)/F^{*i(D)}$ and $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ satisfy the above property as well.

The following is almost the only known example where $G(D)$ (and above functors) is trivial.

Corollary 3.6. *Let D be a division algebra of quaternions over a real-closed field. Then $G(D) = 1$.*

Proof. For any finite field extension L of $F = Z(D)$, the following diagram is commutative,

$$\begin{array}{ccc} K_1(D \otimes_F L) & \xrightarrow{\mathcal{P}} & K_1(D) \\ \downarrow \tau & & \downarrow \tau \\ G(D \otimes_F L) & \xrightarrow{\mathcal{P}} & G(D). \end{array}$$

Now since D is algebraically closed (See [11], Section 16), thanks to Proposition 3.3 and above diagram, $G(D \otimes_F L) \xrightarrow{\mathcal{P}} G(D)$ is an epimorphism. Replace L by $\overline{F}$, the algebraic closure of F. Because $\overline{F}$ is a splitting field for D, $G(D \otimes_F \overline{F}) = G(M_2(\overline{F}))$. We show that $G(M_2(\overline{F}))$ is a trivial group and hence the corollary follows. Since $\tau : K_1 \longrightarrow G$ is a natural epimorphism, there is a composite homomorphism

$$\psi : K_1(\overline{F}) = \overline{F}^* \xrightarrow{\simeq} K_1(M_2(\overline{F})) \xrightarrow{epi} G(M_2(\overline{F})).$$

Take $x \in G(M_2(\overline{F}))$. Since $\overline{F}$ is algebraically closed, there exist $y \in K_1(\overline{F}) = \overline{F}^*$ such that $\psi(y^2) = x$. But $G(M_2(\overline{F}))$ is a torsion group of bounded exponent 2, hence $x = 1$. This shows that $G(M_2(\overline{F}))$ is trivial and the proof is complete. □

Back to the functorial properties of G, the next step is to replace the field L in Proposition 3.3 by a division ring. The following proposition shows that the same result holds here too.

Proposition 3.7. *Let A and B be division algebras with center F such that $[B : F]$ is finite. Then there is a homomorphism $\mathcal{P} : G(A \otimes_F B) \longrightarrow G(A)$ such that $\mathcal{P}\mathcal{I} = \eta_{[B:F]}$.*

Proof. Let $[B : F] = m$. We have the following sequence of F-algebra homomorphisms,

$$A \longrightarrow A \otimes_F B \longrightarrow A \otimes_F B \otimes_F B^{op} \longrightarrow A \otimes_F M_m(F) \longrightarrow M_m(A).$$

This implies the group homomorphism $\mathcal{P} : G(A \otimes_F B) \longrightarrow G(M_m(A)) \xrightarrow{d} G(A)$. The rest of the proof follows from **D1**. □

Note that in the above proposition A could be of infinite dimension over its center F. A same statement as Corollary 3.5 could be obtained here too. In particular if $(i(A), i(B)) = 1$ then $G(A)$ embeds in $G(A \otimes_F B)$ and similarly for B. Employing torsion theory of groups and sequences which appeared in the above propositions, we can write the primary decomposition for $G(D)$. The proof follows more or less the same pattern as for $SK_1(D)$.

Theorem 3.8. *Let A and B be division algebras with center F such that $(i(A), i(B)) = 1$. Then $G(A \otimes_F B) = G(A) \times G(B)$.*

Proof. By Corollary 3.4, $G(A \otimes_F B)$ is a torsion group of bounded exponent m^2n^2 where $m = i(A)$ and $n = i(B)$. Therefore $G(A \otimes_F B) \simeq \mathcal{G} \times \mathcal{H}$, where $exp(\mathcal{G})|m^2$ and $exp(\mathcal{H})|n^2$. By Proposition 3.7, we have the sequence:

$$G(A) \xrightarrow{\phi} G(A \otimes_F B) \xrightarrow{\psi} G(A \otimes B \otimes B^{op}) \xrightarrow{\theta} G(A) \tag{3.2}$$

such that $\theta\psi\phi = \eta_{n^2}$. Hence $G(A) = \eta_{n^2}\eta_{n^2}(G(A)) = \eta_{n^2}\theta\psi\phi(G(A)) \subseteq \theta\psi\eta_{n^2}(\mathcal{G} \times \mathcal{H}) = \theta\psi(\mathcal{G}) \subseteq G(A)$. This shows that $\theta\psi|_{\mathcal{G}} : \mathcal{G} \longrightarrow G(A)$ is surjective. Next we show that $\theta\psi|_{\mathcal{G}}$ is injective. Considering the regular representation $B^{op} \longrightarrow M_{n^2}(F)$. As Proposition 3.3, we have the following sequence

$$G(A \otimes_F B) \xrightarrow{\psi} G(A \otimes B \otimes B^{op}) \xrightarrow{\psi'} G(A \otimes B \otimes M_{n^2}(F)) \xrightarrow{\theta'} G(A \otimes_F B)$$

such that $\theta'\psi'\psi = \eta_{n^2}$. Now if $1 \neq w \in \mathcal{G}$, then $\theta'\psi'\psi(w) = \eta_{n^2}(w) = w^{n^2} \neq 1$. Therefore $\psi|_{\mathcal{G}}$ is injective. Rewrite the sequence (3.2) as follows:

$$G(A \otimes_F B) \xrightarrow{\psi} G(A \otimes B \otimes B^{op}) \xrightarrow{iso} G(M_{n^2}(A)) \xrightarrow{d} G(A).$$

Suppose $x \in \mathcal{G}$ such that $\theta\psi(x) = 1$. The above sequence and **D3** shows that $\psi(x)^{n^2} = 1$. Since $\psi|_{\mathcal{G}}$ is injective, $x^{n^2} = 1$. On the other hand because $exp(\mathcal{G})|m^2$ then $x^{m^2} = 1$. Since m and n are relatively prime, $x = 1$. This shows that $\theta\psi$ is an isomorphism and so $G(A) \simeq \mathcal{G}$. In the similar way it can be shown that $G(B) \simeq \mathcal{H}$. Therefore the proof is complete. □

Let $A = M_m(D)$ be a central simple algebra. From Corollary 3.4 and **D3** it is immediate that $G(A)$ is a torsion group of bounded exponent $m[D : Z(D)]$. Recall Lemma 1.1, which says that for any normal subgroup N of a division algebra D, we have $N^n \subseteq Z(N)[D^*, N]$. If we take $N = D^*$, then for any $x \in D^*$, $x^n \in F^*D'$. This in effect shows that $G(D) = D^*/F^*D'$ is a torsion group of bounded exponent n.

In the next subsection we will show yet another SK_1-like property for the group $G(D)$. Namely $G(D)$ satisfy the following stability, $G(D) \simeq G\big(D((x))\big)$ where $D((x))$ is the division ring of formal Laurent series (Corollary 3.16). We close this section by the following theorem, which shows that the group $G(D) = D^*/F^*D'$ does not always follow the same pattern as the reduced Whitehead group $SK_1(D)$. Namely $G(D)$ is not “homotopy invariant”.

Theorem 3.9 (J. -P. Tignol). *Let D be a division algebra over its center F with index n. Then the following sequence where $\wp$ runs over the irreducible monic polynomials of $F[x]$ and $n_\wp$ is the index of $D \otimes_F F[x]/\wp$, is split exact.*

$$1 \longrightarrow G(D) \longrightarrow G(D(x)) \longrightarrow \bigoplus_{\wp} \frac{\mathbb{Z}}{n/n_\wp\mathbb{Z}} \longrightarrow 1.$$

Proof. By Proposition 7 in [12], the sequence

$$1 \longrightarrow K_1(D) \longrightarrow K_1(D(x)) \longrightarrow \bigoplus_{\wp} n_{\wp}/n\mathbb{Z} \longrightarrow 1$$

which is obtained from the localization exact sequence of algebraic K-theory is split exact. Now since the group $G(D)$ is the cokernel of the natural map $K_1(F) \longrightarrow K_1(D)$, applying the snake lemma to the commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_1(F) & \longrightarrow & K_1(F(x)) & \longrightarrow & \bigoplus_{\wp} \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_1(D) & \longrightarrow & K_1(D(x)) & \longrightarrow & \bigoplus_{\wp} n_{\wp}/n\mathbb{Z} & \longrightarrow & 1 \end{array}$$

the result follows. □

2. On The Group $G(D)$ Over Henselian Division Algebras

We start with the following theorem which describes a fundemental connection between the group $G(D)$ and its residue version.

Theorem 3.10. *Let D be a tame division algebra over a Henselian field $F = Z(D)$ with index n. Let L/F be a subfield of D. Then the following sequence is exact.*

$$1 \longrightarrow \overline{D}^*/\overline{L}^*\overline{D'} \longrightarrow D^*/L^*D' \longrightarrow \Gamma_D/\Gamma_L \longrightarrow 1. \tag{3.3}$$

Proof. Consider the normal subgroup $1 + M_D$ of D^*. Thanks to Lemma 1.1, we have

$$(1 + M_D)^n \subseteq ((1 + M_D) \cap F^*)[D^*, 1 + M_D]. \tag{3.4}$$

We will show that $(1 + M_D) = (1 + M_D)^n$. Let $a \in 1 + M_D$. Consider the field $F(a)$ and $a \in 1 + M_{F(a)}$. Since F is a Henselian field, so is $F(a)$. The polynomial $f(x) = x^n - a$ has 1 as a simple root modulo $M_{F(a)}$, because $Char\overline{F(a)}$ does not divide n. Applying Hensel's lemma to the polynomial $f(x) = x^n - a$, we obtain an element $b \in 1 + M_{F(a)}$ such that $b^n = a$. This shows that $a \in (1 + M_D)^n$. Thus $1 + M_D$ is *n-divisible*, namely, $(1 + M_D) = (1 + M_D)^n$. Hence from (3.4) it follows that $1 + M_D \subseteq (1 + M_F)D'$. Now consider the reduction map $U_D \longrightarrow \overline{D}^*$. We have the following sequence:

$$\overline{D}^* \xrightarrow{\simeq} U_D/1 + M_D \xrightarrow{nat.} U_D/(1 + M_F)D' \xrightarrow{nat.} U_D/(1 + M_L)D' \xrightarrow{nat.}$$
$$\xrightarrow{nat.} U_D/U_LD' \xrightarrow{\simeq} L^*U_D/L^*D'.$$

Therefore $\psi : \overline{D}^*/(\overline{L}^*)\overline{D'} \longrightarrow L^*U_D/L^*D'$ is an isomorphism. Considering the fact that $D^*/L^*U_D \simeq \Gamma_D/\Gamma_L$, the theorem follows. □

Now we are ready to compute $G(D)$ for some certain cases.

Theorem 3.11. *Let D be a Henselian division algebra tame over its center F with index n. Then*

i. If D is unramified over F then $G(D) \simeq G(\overline{D})$.

ii. If D is totally ramified over F then $G(D) = \Gamma_D/\Gamma_F$.

iii. If D is semiramified and $\overline{D}$ is cyclic over $\overline{F}$ then the following sequence where $N_{\overline{D}/\overline{F}}$ is the norm function, is exact.

$$1 \longrightarrow N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1.$$

Proof. i. Writing (3.3) for $L = F$, we have:

$$1 \longrightarrow \overline{D}^*/\overline{F}^*\overline{D'} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1.$$

Now if (D, v) is unramified, namely $[\Gamma_D : \Gamma_F] = 1$, then $\overline{D}^*/\overline{F}^*\overline{D'} \simeq D^*/F^*D'$. On the other hand $Z(\overline{D}) = \overline{F}$ and $D^* = F^*U_D$. Therefore, for $a, b \in D^*$, the element $c = aba^{-1}b^{-1}$ may be written in the form $c = \alpha\beta\alpha^{-1}\beta^{-1}$, where α and $\beta \in U_D$. This shows $\overline{D'} = \overline{D}'$, so $G(D) \simeq G(\overline{D})$.

ii. If D is totally ramified over F then $\overline{D} = \overline{F}$. Writing (3.3) for $L = F$, since the group $\overline{D}^*/\overline{F}^*\overline{D'}$ is trivial $G(D) = \Gamma_D/\Gamma_F$.

iii. Let D be semiramified and $\overline{D}$ be cyclic over $\overline{F}$. Consider the norm function $N_{\overline{D}/\overline{F}} : \overline{D}^* \longrightarrow \overline{F}^*$. Moreover for any $x \in U_D$, from (1.*) it follows that, $\overline{Nrd_{D/F}(x)} = N_{\overline{D}/\overline{F}}(\overline{x})$. This shows that $\overline{D'} \subseteq KerN_{\overline{D}/\overline{F}}$. But if $x \in KerN_{\overline{D}/\overline{F}}$ then by Hilbert theorem 90, there exists $\overline{a}$ such that $x = \overline{a}\sigma(\overline{a})^{-1}$, where σ is the generator of $Gal(\overline{D}/\overline{F})$. It is well known that the *fundemental homomorphism* $D^* \longrightarrow Gal(Z(\overline{D})/\overline{F})$ is surjective [9]. Therefore $\sigma : \overline{D} \longrightarrow \overline{D}$ is of the form $\sigma(\overline{a}) = \overline{cac^{-1}}$, for some $c \in D^*$. This shows that $x \in \overline{D'}$. Therefore $KerN_{\overline{D}/\overline{F}} = \overline{D'}$. Now it is easy to see that $\overline{D}^*/\overline{F}^*\overline{D'} \simeq N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n}$. So thanks to (3.3), $1 \longrightarrow N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1$ is exact. □

Remark 3.12. If $\overline{D}$ is a cyclic field extension of $\overline{F}$, a similar proof as *iii.* above shows that $KerN_{\overline{D}/\overline{F}} \subseteq \overline{D'}$. In particular it follows that $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} \longrightarrow \overline{D}^*/\overline{F}^*\overline{D'}$, where $[\overline{D} : \overline{F}] = f$, is always surjective. Therefore if $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} = 1$ then $G(D) = \Gamma_D/\Gamma_F$. This will be used in Example 3.19.

Using Theorem 3.2 *iii.*, we construct division algebras such that the group $G(D)$ is cyclic.

Example 3.13. Let $\mathbb{C}$ be the field of complex numbers. Let $1 \neq \sigma \in Gal(\mathbb{C}/\mathbb{R})$ where $\mathbb{R}$ is real numbers. Then by Hilbert construction (See [2], §1), $D = \mathbb{C}((x, \sigma))$ is a division ring with center $F = \mathbb{R}((x^2))$. We show that $G(D) = \mathbb{Z}_2$. D has a natural valuation such that $\Gamma_D/\Gamma_F = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Clearly $\overline{D} = \mathbb{C}$ and $\overline{F} = \mathbb{R}$. Since $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{R}^2$ by Theorem 3.2 *iii.*, $G(D) = \Gamma_D/\Gamma_F = \mathbb{Z}_2$.

Example 3.14. Let $q \geq 3$ be a prime number. Take a prime number $p \neq q$ such that $(p-1, q) = 1$. Consider the cyclic extension $\mathbb{F}_{p^q}/\mathbb{F}_p$ where $\mathbb{F}_p$ and $\mathbb{F}_{p^q}$ are fields with p and p^q elements respectively. Let σ be a generator of the cyclic group $Gal(\mathbb{F}_{p^q}/\mathbb{F}_p)$. By Hilbert construction $D = \mathbb{F}_{p^q}((x, \sigma))$ is a division algebra with center $F = Z(D) = \mathbb{F}_p((x^q))$ and $i(D) = ord(\sigma) = q$. D has a natural valuation which is tame and Henselian. It is easy to see that with this valuation D is semiramified, $\overline{D} = \mathbb{F}_{p^q}$ and $\overline{F} = \mathbb{F}_p$. Since $N_{\overline{D}/\overline{F}}$ is surjective, by Theorem 3.11 *iii*, it follows that $G(D) = \mathbb{Z}_q$.

There have been significant results on the structure of the relative value group in the case of a totally ramified algebra. Using Theorem 3.11 we can write interesting statements relating the group structure of $G(D)$ to the algebraic structure of D. Recall that the group $G(D)$ is torsion of bounded exponent n.

Theorem 3.15. *Let D be a valued division algebra tame and totally ramified over a Henselian field $F = Z(D)$ of index n. Then,*

i. There is a one to one correspondence between the isomorphism classes of F-subalgebras of D and the subgroups of $G(D)$.

ii. $exp(G(D))$ divides the exponent of D, i.e., the order of $[D]$ in $Br(F)$, the Brauer group of F.

iii. D is a cyclic division algebra if and only if $exp(G(D)) = n$.

Proof. The theorem follows by comparing Theorem 3.11 *ii.*, with the results on the relative value group in the case of a totally ramified valuation (See for example [27]). □

Corollary 3.16. *Let D be a finite dimensional division algebra over its center F. If $Char F \nmid i(D)$ then $G(D) \backsimeq G(D((x)))$.*

□

Now we are in a position to use the group $G(D)$ to compute $SK_1(D)$. The following theorem enables us to compute $SK_1(D)$ when, roughly speaking, $G(\overline{D})$ is trivial. Note that we do not use any results from reduced K-theory.

Theorem 3.17. *Let D be a tame division algebra over a Henselian field $F = Z(D)$ of index n.*

i. If $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $SK_1(D) = \mu_n(\overline{F})/(\overline{D'} \cap \overline{F})$.

ii. If D is a cyclic division algebra with a maximal cyclic extension L/F such that $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $SK_1(D) = 1$.

Proof. *i.* As the proof of Theorem 3.10 shows, we have a natural isomorphism,

$$\psi : \overline{D}^*/(\overline{F}^*)\overline{D'} \longrightarrow U_D/U_F D'.$$

Now if $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $U_D = U_F D'$. But $D^{(1)} \subseteq U_D$. This shows that $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$, it follows that $SK_1(D) = \mu_n(F)/Z(D')$. Since D is tame and Henselian over its center F, using Hensel's lemma, it is easy to see that $\mu_n(F) \to \mu_n(\overline{F}), a \mapsto \overline{a}$ is an isomorphism. Also it is not difficult to show that $Z(D') \simeq \overline{Z(D')} = \overline{D'} \cap \overline{F}$. Therefore $SK_1(D) = \mu_n(\overline{F})/(\overline{D'} \cap \overline{F})$.

ii. The same proof as *i.* shows that if $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $U_D = U_L D'$. Therefore $D^{(1)} \subseteq U_L D'$. Let $x \in D^{(1)}$. Then $x = ld$ where $l \in L$ and $d \in D'$. So $Nrd_{D/F}(x) = N_{L/F}(l) = 1$. Hilbert theorem 90 for the cyclic extension L/F guarantee that $l = a\sigma(a)^{-1}$, where σ is a generator of $Gal(L/F)$. Now the Skolem-Noether theorem implies that $\sigma(a) = cac^{-1}$ where $c \in D^*$. Therefore $l = aca^{-1}c^{-1}$. This shows that $D^{(1)} = D'$. □

Remark 3.18. Part *i.* of the above theorem shows that if D is totally ramified, then $SK_1(D) = \mu_n(\overline{F})/\overline{D'}$. This shows that Tignol's formula for $SK_1(D)$ is a special case of Theorem 3.17 *i.* (See [12]).

We deduce both theorems of Lipnitskii [13] which are obtained by using heavy machinery of reduced K- theory, as natural examples of the above theorem.

Example 3.19. *For any division algebra D with center $F = \mathbb{R}((x_1, \cdots, x_m))$ where $\mathbb{R}$ is the real numbers, $SK_1(D)$ is trivial.*

Proof. From number theory, it is well known that $[D : F] = 2^s$ where $s \leq m$. Since the complete field $F = \mathbb{R}((x_1, \cdots, x_m))$ has a natural valuation, then D admits a valuation which is obviously tame. It is clear that $\overline{F} = \mathbb{R}$. Because the only division algebras over real numbers are either the quaternion $\mathbb{H}_{\mathbb{R}}$ or the field $\mathbb{C}$ of complex numbers, therefore $\overline{D} = \mathbb{H}_{\mathbb{R}}$ or $\overline{D} = \mathbb{C}$. Now Corollary 3.6 and Remark 3.12 show that in either case $\overline{D}^*/\overline{F}^*\overline{D'} = 1$. Now by Theorem 3.17, $SK_1(D) \simeq \mu_{i(D)}(\mathbb{R})/(\overline{D'} \cap \mathbb{R})$. But clearly $\mu_{i(D)}(\mathbb{R}) = \{1, -1\}$. On the other hand if $i(D)$ is even, then $-1 \in D'$ (See [29]). Thus $SK_1(D) = 1$.

Example 3.20. *For any division algebra with center $F = C((x_1, \cdots, x_m))$ where C is an algebraically closed field, and $charC \nmid i(D)$, $SK_1(D)$ is cyclic.*

Example 3.21. *Hilbert classical construction of division algebras.* Let L be a field and $\sigma \in Aut(L)$ with $o(\sigma) = n$ such that $charL \nmid n$. Let $F = Fix(\sigma)$ be the fixed field of σ. Hence $Gal(L/F)$ is a cyclic group with the generator σ. Let $D = L((x, \sigma))$ be the division ring of formal Laurent series. It follows that $Z(D) = F((x^n))$ and $i(D) = n$. D has a natural valuation, and it is easy to see that with this valuation

D is semiramified and $L((x^n))$ is a maximal subfield of D. Now by Theorem 3.17 *ii.*, $SK_1(D)$ is trivial.

Example 3.22. From Theorem 3.17 *ii.*, it is immediate that reduced Whitehead group of a tame division algebra over a local field is trivial.

Because most of the interesting valued division algebras arise from the iterated formal power series fields, we may consider *r-iterated Henselian division algebras.* Following Platonov in [19], we define inductively an r-iterated Henselian field F if its residue field $\overline{F}$ is an $(r-1)$-iterated Henselian field.[1] Let $(D_i, v_i), 0 \leq i \leq r-1$ be an r iterated Henselian division algebra ($\overline{D_i} = D_{i+1}$). Let $\Phi_i : U_{D_{i-1}} \longrightarrow D_i$ be the i-th natural reduction map. Then $\Phi_i(\Phi_{i-1}(\cdots(\Phi_1(a))\cdots))$ is called an i-iterated reduction, if it is defined. Denote the r iterated Henselian division algebra by D, ($D = D_0, Z(D) = F = F_0$). We also need the following notations in order to state the following lemma. By $[U_D]_i$ and $[U_F]_i$ we denote the set of all elements of D and F respectively, such that i iterated reduction is defined. Also by $[1+M_D]_i$ and $[1+M_F]_i$, we denote the subsets of $[U_D]_i$ and $[U_F]_i$ such that the i iterated reduction equals one. Clearly $[1+M_F]_1 = 1+M_F$.

Lemma 3.23. *Let D be an i iterated tame division algebra of finite dimension over a Henselian field $F = Z(D)$ with index n.*

i. For each $a \in [1+M_D]_i$ there is $b \in [1+M_F]_i$ such that $ab \in D'$.

ii. $[1+M_D]_i \subsetneqq [1+M_F]_i D'$.

Proof. *i.* Let $a \in [1+M_D]_i$. Then a is contained in a maximal subfield of D, say L. Therefore $a \in [1+M_L]_i$. By lemma 3 of [19], we have $N_{L/F}([1+M_L]_i) = [1+M_F]_i$. So $Nrd_{D/F}(a) = N_{L/F}(a) \in [1+M_F]_i$. Let $t = Nrd_{D/F}(a)$. Using an inductive argument for Hensel's lemma, we will show that there exists $c \in [1+M_F]_i$ such that $c^n = t$. Let $s \in 1+M_F = [1+M_F]_1$. Applying Hensel's lemma for $f(x) = x^n - s$ gives $c \in 1+M_F$ such that $c^n = s$. Now it is not hard to see that $\Phi_1([1+M_F]_i) = [1+M_{\overline{F}}]_{i-1}$. Therefore $[1+M_F]_i/[1+M_F]_1 \simeq [1+M_{\overline{F}}]_{i-1}$. Now by induction, we conclude that $[1+M_F]_i$ is n-divisible. Therefore exist $c \in [1+M_F]_i$ such that $c^n = t$. Now $Nrd_{D/F}(a) = c^n$. So $Nrd_{D/F}(ac^{-1}) = 1$. Hence $ac^{-1} \in D^{(1)} \cap [1+M_D]_i$. Applying Platonov's generalized congruence theorem (cf. [19] and [4]) , we obtain $ac^{-1} \in D'$. Take $b = c^{-1}$ and the proof is complete.

ii. Applying the first part of the lemma for $i = 1$, in each step of reduction we have, $1 + M_{D_i} \subseteq (1 + M_{K_i}){D_i}'$ where $K_i = Z(D_i)$. First we show that in each step of reduction, ${D_i}' \nsubseteq 1 + M_{D_i}$. Consider the groups $\Delta = {D_i}^*/1 + M_{D_i}$ and $P(D_i) = (1 + M_{K_i}){D_i}'/(1 + M_{D_i})$. One can easily observe that $P(D_i) = \Delta'$ and as Theorem 2.11 of [1] shows, the center of Δ is ${K_i}^*(1 + M_{D_i})/(1 + M_{D_i})$. We

[1] See Ershov's comment in [4] on iterated valued field. Among other things, considering iterated valued field, enables us to have more insight in each step of reduction.

claim that Δ is not an abelian group, for otherwise $U_{D_i} = U_{K_i}(1+M_{D_i})$ which implies that D_i is totally ramified. Thus $\overline{D_i}' = \mu_e(\overline{K_i})$, where $e = exp(\Gamma_{D_i}/\Gamma_{K_i})$, (cf. the proof of Theorem 3.1 of [27]) which leads us to a contradiction. Therefore D_i' is not in $1+M_{D_i}$ and Δ is not abelian. But $\Phi^{-1}(1+M_{D_i}) = [1+M_D]_{i+1}$. If $D' \subseteq [1+M_D]_{i+1}$ then $\Phi(D') \subseteq \Phi([1+M_D]_{i+1}) = 1+M_{D_i}$. But $D_i' \subseteq \Phi(D')$ so $D_i' \subseteq 1+M_{D_i}$ a contradiction. □

Remark 3.24. In the proof of *i.* above, we could use Lemma 1.1 and avoid the Platonov congruence theorem.

Theorem 3.25. *Let D be an r iterated tame division algebra over a Henselian field F of index n. If there is an $0 \le \ell \le r-1$ such that $\overline{D}_\ell/\overline{F}_\ell\overline{D'}_\ell = 1$ then $SK_1(D) \simeq \mu_n(F)/Z(D')$.*

Proof. For any $0 \le k \le r-1$, consider the $k+1-th$ reduction map

$$[U_D]_{k+1} \overset{\Phi_{k+1}\Phi_k\cdots\Phi_1}{\longrightarrow} \overline{D_k}^*.$$

Thanks to Lemma 3.23 *i.*, we have:

$$\overline{D_k}^* \overset{\simeq}{\to} [U_D]_{k+1}/[1+M_D]_{k+1} \overset{nat.}{\to} [U_D]_{k+1}/[1+M_F]_{k+1}D' \overset{nat.}{\to} [U_D]_{k+1}/[U_F]_{k+1}D'.$$

Therefore,

$$\overline{D_k}^*/\overline{F_k}^*\overline{D_k'} \overset{\simeq}{\longrightarrow} [U_D]_{k+1}/[U_F]_{k+1}D'.$$

Hence if there is a ℓ such that $\overline{D_\ell}^*/\overline{F_\ell}^*\overline{D'_\ell}^* = 1$ then $[U_F]_{\ell+1}D' = [U_D]_{\ell+1}$. By lemma 1 in [19] $D^{(1)} \subseteq [U_D]_{\ell+1}$ so $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$ the theorem follows. □

Considering the fact that each Henselian division algebra is a 1-iterated division algebra, we recover Theorem 3.17 *i.* from the theorem above.

3. On The Unitary Setting

Let D be a division ring with an involution τ over its center F with index n. Let $S_\tau(D) = \{a \in D | a^\tau = a\}$ be the subspace of symmetric elements and $\Sigma_\tau(D)$ the subgroup of D^* generated by nonzero symmetric elements. Here we concentrate on involutions of the first kind, i.e. $\Sigma_\tau(D) \cap F^* = F^*$.

Definition 3.26. Let D be a division ring with an involution τ. Then the group $KU_1(D) = D^*/\Sigma_\tau(D)D'$ is called unitary Whitehead group and the $GU(D) = \Sigma_\tau(D)D'/F^*D'$ the unitary version of $G(D)$.

We will prove that there is a stability theorem for $GU(D)$ similar to one in Corollary 3.16. The first part of the following theorem was first proved by Platonov and Yanchevskii [20].

Theorem 3.27. *Let D be a finite dimensional tame and unramified division algebra with an involution of the first kind over a Henselian field $Z(D)=F$. Then $KU_1(\overline{D}) \backsimeq KU_1(D)$ and $GU(\overline{D}) \backsimeq GU(D)$.*

Proof. Consider the following sequence:

$$\overline{D}^* \longrightarrow U_D/1+M_D \longrightarrow F^*U_D/F^*(1+M_D) \longrightarrow D^*/\Sigma_\tau(D)D'.$$

Because the valuation is unramified, we have $\Sigma_{\overline{\tau}}(\overline{D}) = \overline{\Sigma_\tau(D)\cap U_D}$ (See [20]), and $\overline{D}' = \overline{D'}$ (See the proof of Theorem 3.11 *i.*). Therefore we have the following isomorphism: $\overline{D}^*/\Sigma_{\overline{\tau}}(\overline{D})\overline{D}' \xrightarrow{\backsimeq} D^*/\Sigma_\tau(D)D'$.

For the second part, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & GU(\overline{D}) & \longrightarrow & G(\overline{D}) & \longrightarrow & KU_1(\overline{D}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle iso.} & & \downarrow{\scriptstyle iso.} & & \\
1 & \longrightarrow & GU(D) & \longrightarrow & G(D) & \longrightarrow & KU_1(D) & \longrightarrow & 1.
\end{array}$$

The two of the vertical arrows are isomorphisms, thanks to the first part of this theorem and Theorem 3.11 *i.*. Therefore the third one is also an isomorphism which completes the proof. □

If D has an involution of the first kind, then $D((x))$ enjoys a natural involution which is induced by the one from D. Therefore if $Char F \neq 2$ then thanks to the above theorem, we have $GU(D) \backsimeq GU(D((x)))$ which is a stability theorem for $GU(D)$.

REFERENCES

1. M. Chacron, *c-valuations and normal c-ordering*, Canad. J. Math. **41** (1989), 14-67.
2. P. Draxl, *Skew Field*, London Math. Soc. Lecture Note Ser. Vol 81, Cambridge, Univ. Press. Cambridge, 1983.
3. P. Draxl, M. Kneser (eds.), *SK_1 von Schiefkörpern*, Lecture Notes in Math. Vol 778, Springer, Berlin, 1980.
4. Y. Ershov, *Henselian valuation of division rings and the group $SK_1(D)$*, Math USSR Sb. **45** (1983), 63-71.
5. K. Goodearl, R. Warfield, *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Student Texts. 16, Cambridge, Univ. Press. Cambridge, 1989.
6. R. Hazrat, *SK_1-like functors for division algebras*, J. Algebra **239** (2001), no. 2, 573-588.
7. R. Hazrat, *Wedderburn's factorization theorem, application to reduced K-theory*, Proc. Amer. Math. Soc. **130** (2002), no. 2, 311-314.
8. R. Hazrat, *On central series of the multiplicative group of division rings*, Algebra Colloq. **9** (2002), no. 1, 99-106.
9. B. Jacob, A. R. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), no. 1, 126-179.
10. N. Jacobson, *Structure of Rings*, Amer. Math. Soc., New York, 1991.
11. T. Y. Lam, *A first course in noncommutative rings*, Springer-Verlag, New York, 1991.
12. D. W. Lewis, J. -P. Tignol, *Square class groups and Witt rings of central simple algebras*, J. Algebra **154** (1993), no. 2, 360-376.
13. V. A. Lipnitskii, *On the Tannaka-Artin problem over special fields*, Soviet Math. Dokl. **17** (1976), no. 3, 639-642.
14. M. Mahdavi-Hezavehi, *On derived groups of division rings*, Comm. Algebra **23** (1995), no. 3, 913-926.
15. A. S. Merkurjev, *K-theory of simple algebras*, Proceedings of Symposia in Pure Mathematics **58** (1995), no. 1, 65-83.
16. R. S. Pierce, *Associative Algebras*, Springer-Verlag, New York, 1982.
18. V. P. Platonov, *The Tannaka-Artin problem and reduced K-theory*, Math USSR Izv. **10** (1976), 211-243.
19. V. P. Platonov, *On reduced K-theory for n-fold Hensel fields*, Soviet Math. Dokl. **20** (1979), no. 6, 1436-1439.
20. V. P. Platonov, V. I. Yanchevskii, *Dieudonne's conjecture on the structure of unitary groups over a division ring, and hermitian K-theory*, Math. USSR Izv. **25** (1985), no. 3, 573-598.
21. V. P. Platonov, V. Yanchevskii, *Algebra IX, Finite dimensional division algebras*, Encyclopaedia Math. Sci. 77, Springer, Berlin, 1995.
22. U. Rehmann, *Die kommutatorfaktorgruppe der normeinsgruppe einer p-adischen Divisionalgebra*, Arc. Math. **32** (1979), 318-322.
23. I. Reiner, *Maximal Orders*, Academic Press, London, 1975.
24. C. Riehm, *The norm 1 group of a p-adic division algebra*, Amer. J. Math. **92** (1970), 499-523.
25. L. Rowen, Y. Segev, *The multiplicative group of a division algebra of degree 5 and Wedderburn's factorization theorem* (to appear).
26. A. Suslin, *SK_1 of division algebras and Galois cohomology*, Advances in Soviet Math 4, Amer. Math. Soc. (1991), 75-99.
27. J. -P. Tignol, A. R. Wadsworth, *Totally ramified valuations on finite-dimensional division algebras*, Tran. Amer. Math. Soc. **302** (1987), no. 1, 223-250.
28. A. R. Wadsworth, *Extending valuations to finite dimensional division algebras*, Proc. Amer. Math. Soc. **98** (1986), 20-22.

29. A. R. Wadsworth, *In a division algebra of even degree -1 is a product of squares*, Bull. Soc. Math. Belg. SEr. B (1990), no. 42, 103-104.
30. V. Yanchevskii, *Reduced norms of simple algebras over function fields*, Proc. Steklov Inst. Math. **183** (1991), 261-269.