The Jacobson topology of $\mathrm{Prim}_* \, L^1(G)$ for exponential Lie groups

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Abstract

Much research has been done in order to understand the ideal theory of group algebras. In this context a locally compact group G is called $*$-regular if

$$(\star) \qquad \ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho \quad \text{implies} \quad \ker_{C^*(G)} \pi \subset \ker_{C^*(G)} \rho$$

for all unitary representations π and ρ of G. In [3] Boidol characterized the $*$-regular ones among all exponential solvable Lie groups by a purely algebraic condition. An interesting and open question is whether these groups satisfy the weaker property of primitive $*$-regularity: Does the implication $(\star)$ hold for all irreducible representations?

Our main result is that all exponential solvable Lie groups G up to dimension seven have this property. So far no counter example is known. In this work the non-$*$-regular exponential Lie groups in low dimensions are classified and investigated case by case. We give an explicit description of the closure of one-point sets $\{\pi\}$ in $\widehat{G}$ for representations π which are not induced from a nilpotent normal subgroup. Recall that G is a type I group and that $\widehat{G} = \operatorname{Prim} C^*(G)$ carries the Jacobson topology. In order to prove the contrapositive of $(\star)$ we establish a functional calculus for differential operators in $L^1(G)$ and make extensive use of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In this way the problem of producing suitable functions f in $\ker_{L^1(G)} \pi \setminus \ker_{L^1(G)} \rho$ is reduced to the solution of Fourier multiplier problems of commutative harmonic analysis. These methods enable us to deduce some information about the Jacobson topology of the primitive ideal space $\operatorname{Prim}_* L^1(G)$ which is yet poorly understood for non-$*$-regular groups.

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1 Introduction

The common term 'regularity' has quite different meanings in mathematics. In this paper we shall use it in the sense of a regular function algebra on a topological space, which separates points from closed subsets. If $\mathcal{A}$ is a Banach $*$-algebra and $C^*(\mathcal{A})$ is its C^*-completion, then the generalized Gelfand transform

$$\widehat{a}(P) = a + P \in C^*(\mathcal{A})/P$$

yields an algebra $\{\widehat{a} : a \in C^*(\mathcal{A})\}$ of functions on the primitive ideal space $\operatorname{Prim} C^*(\mathcal{A})$. By definition of the Jacobson topology on $\operatorname{Prim} C^*(\mathcal{A})$ this algebra separates points from closed sets. One may ask whether the subalgebra $\{\widehat{a} : a \in \mathcal{A}\}$ is large enough to have this separation property. If this is the case, then we say that $\mathcal{A}$ is $*$-regular. We point out that in case of a non-commutative Banach $*$-algebra $\mathcal{A}$ the topological space $\operatorname{Prim} C^*(\mathcal{A})$ is typically far from being a T_1-space. This fact motivates us to raise the following question: Does the function algebra $\{\widehat{a} : a \in \mathcal{A}\}$ separate points from the closure of one-point subsets of $\operatorname{Prim} C^*(\mathcal{A})$? If the answer is in the affirmative, then $\mathcal{A}$ is said to be primitive $*$-regular.

The investigation of the (primitive) $*$-regularity of $\mathcal{A}$ naturally involves the set $\operatorname{Prim}_* \mathcal{A}$ of kernels of irreducible $*$-representations of $\mathcal{A}$ provided with the Jacobson topology. If $\mathcal{A} = L^1(G)$ is the group algebra of a locally compact group, then a different interpretation of 'regularity' presents itself: L^1-functions on G are more regular (and more concrete) than arbitrary elements of the C^*-algebra $C^*(G)$. These considerations might have convinced the reader that the topology of $\operatorname{Prim}_* L^1(G)$ deserves further study.

In [2] Boidol and Leptin initiated the investigation of the class $[\Psi]$ of $*$-regular locally compact groups. Far reaching results have been obtained in this direction: First Boidol has characterized the $*$-regular groups among the exponential solvable Lie groups by a purely algebraic condition on the stabilizers $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ of linear functionals $f \in \mathfrak{g}^*$, see Theorem 5.4 of [3] and Lemma 2 of [28]. More generally Boidol has proved in [4] that a connected locally compact group G is $*$-regular if and only if all primitive ideals of $C^*(G)$ are (essentially) induced from a normal subgroup M whose Haar measure has polynomial growth.

Recall that a Banach $*$-algebra $\mathcal{A}$ is called symmetric if and only if elements of the form a^*a have positive spectra for all $a \in \mathcal{A}$. In [28] Poguntke has determined the simple modules of the group algebra $L^1(G)$ for exponential solvable Lie groups G. From this classification he deduced that an exponential Lie group is symmetric if and only if it is $*$-regular, see Theorem 10 of [28].

In my thesis I will investigate whether exponential solvable Lie groups are primitive $*$-regular. The main result is that all exponential Lie groups up to dimension seven and certain families of such groups in arbitrary dimensions have this property, see Subsection 15, in particular Theorem 15.7 and Proposition 15.2. Furthermore I will give several general results from which one can derive information about the Jacobson topology of $\operatorname{Prim}_* L^1(G)$, see Subsection 5.1 (in particular Theorem 5.1 and Theorem 5.18) and Subsection 7.3. Certainly I am led by the conjecture that all of these groups are primitive $*$-regular because no counter-example seems to be known.

The text books of Leptin and Ludwig [23], Folland [11], and Dixmier [8] cover many aspects of the representation theory of Lie groups. The subsequent results thereof will be used without further comment in the rest of this paper:

- the definition of induced representations for locally compact groups and their elementary properties: induction in stages, commutation with direct sums, and continuity with respect to the Fell topology. See Chapter 6 of [11].
- the basic theory of C^*-algebras and their representations as it is presented in the first five Chapters of [8].
- parts of the representation theory of exponential Lie groups: the definition of the Kirillov map and the construction of irreducible unitary representations via Pukanszky polarizations. These results can be found in Chapter 4 and 6 of [1], and Chapter 1 of [23].
- the concept of the adjoint algebra $\mathcal{A}^b$ of a Banach $*$-algebra $\mathcal{A}$ (also known as multiplier algebra in the literature), in particular in case of group algebras $L^1(G)$ and their C^*-completion $C^*(G)$. An early reference is [19].
- the fact that for exponential solvable Lie groups the Kirillov map

$$\mathcal{K} : \mathfrak{g}^* / \operatorname{Ad}^*(G) \longrightarrow \operatorname{Prim} C^*(G)$$

  is a homeomorphism with respect to the quotient topology on the orbit space and the Jacobson topology on the primitive ideal space. A proof of this fairly deep result can be found in Chapter 2 and 3 of [23]. Mostly we regard $\mathcal{K}$ as a map from $\mathfrak{g}^*$ onto $\operatorname{Prim} C^*(G)$ which is constant on $\operatorname{Ad}^*(G)$-orbits. Owing to the bicontinuity of the Kirillov map $\mathcal{K}$, the topological space $\operatorname{Prim} C^*(G)$ is well-understood whereas the Jacobson topology of $\operatorname{Prim}_* L^1(G)$ is unknown to a great extent, at least for non-$*$-regular groups.

Let us sketch the setup of this paper: In the first section we introduce the basic definition of $\mathcal{A}$-determined ideals of $C^*(\mathcal{A})$ in order to characterize the property of (primitive) $*$-regularity, see Definition 2.1. In Section 3.1 and 3.2 we prove sufficient criteria for ideals $\ker_{C^*(G)} \pi$ to be $L^1(G)$-determined. From these results we deduce a strategy for proving primitive $*$-regularity of exponential Lie groups G in Section 3.4. In this way we come up against the following problem:

Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Let $f \in \mathfrak{g}^*$ be in general position such that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$ and let $g \in \mathfrak{g}^*$ be critical for the orbit $\operatorname{Ad}^*(G)f$. (For the precise definition of critical orbits $\operatorname{Ad}^*(G)g$ see Definition 3.29 in Subsection 3.4. Compare also Definition 1.2.) Does the relation

$$(\sharp) \qquad \ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$$

hold for the irreducible representations $\pi = \mathcal{K}(f)$ and $\rho = \mathcal{K}(g)$? Producing suitable functions in $\ker_{L^1(G)} \pi$ turns out to be a great challenge. In Subsection 3.4 we will explain why such coadjoint orbits $\operatorname{Ad}^*(G)f$ and $\operatorname{Ad}^*(G)g$ are the only ones to be considered.

At least in low dimensions it frequently occurs that G contains the 3-dimensional Heisenberg group B as a normal subgroup. If Z denotes the center of B, then we must distinguish the non-central case $Z \not\subset Z(G)$ and the central case $Z \subset Z(G)$.

In Sections 5 and 7 we will develop the adequate tools for proving relation ($\sharp$). If $Z \not\subset Z(G)$, then we can apply the achievements of Section 5.1 which are partly inspired by the following question: To what extent can the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ be used to define suitable functions in the L^1-kernel of irreducible unitary representations π of G? It goes without saying that we are looking for an answer beyond the trivial inclusion

$$\ker_{\mathcal{U}(\mathfrak{g})} d\pi \; * \; \mathcal{C}_0^\infty(G) \; \subset \; \ker_{L^1(G)} \pi \; .$$

This approach leads us directly to the problem of establishing a functional calculus for elements of $\mathfrak{z}\mathfrak{m}$ acting as differential operators in $L^1(M)$, compare Theorem 5.1 and Lemma 5.4 as well as Theorem 5.17 and 5.18. Here M is the connected subgroup of G with Lie algebra $\mathfrak{m}$ and $\mathfrak{z}\mathfrak{m}$ denotes the center of the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$.

In the central case one can use the results of Section 7.1 in order to translate the original problem ($\sharp$) into a simpler one for representations of a certain subquotient of the group algebra, see Theorem 7.10. Iterating this procedure with the aid of Proposition 7.12 and 7.13 if necessary, one arrives in the commutative situation at last, i.e., one has to treat orbits of characters of a Beurling algebra $L^1(K, w)$ where K is a vector group and w an exponential weight function. Eventually one can resort to the results of Section 7.3.

The reader will realize that the proofs of the relevant results in Section 5.1 and 7.3 are based upon the same technique, namely the solution of Fourier multiplier problems

$$\widehat{c}(x, \xi) = \psi(\xi)\, \widehat{a}(x, \xi)$$

for given Schwartz functions a and certain continuous (not necessarily differentiable) multipliers ψ of polynomial growth. Here $\widehat{a}$ denotes the partial Fourier transform of a with respect to the second variable. For our purposes the solution c must be an L^1-function. Compare the proof of Theorem 5.1 and Remark 7.20.

In Sections 9 to 14 we prove relation ($\sharp$) for $\mathfrak{n}$ running through all nilpotent Lie algebras up to dimension 5 and all possible coabelian extensions $\mathfrak{g}$ of $\mathfrak{n}$ such that there exist linear functionals $f \in \mathfrak{g}^*$ in general position and $g \in \mathfrak{g}^*$ critical for the orbit $\mathrm{Ad}^*(G)f$ as above. Apparently, the determination of all these $\mathfrak{g}$ matches the classification of non-$*$-regular exponential Lie algebras in low dimensions. The proof of ($\sharp$) consists of four steps:

- Describe the algebraic structure of $\mathfrak{g}$
- Determine the closure of $\mathrm{Ad}^*(G)f$ in $\mathfrak{g}^*$
- Compute the representations $\pi = \mathcal{K}(f)$ and $\rho = \mathcal{K}(g)$
- Separate ρ from π by L^1-functions with the aid of the results of Section 4 or 6

We will see in Section 15 that these results yield our main theorem: All exponential solvable Lie groups up to dimension 7 are primitive $*$-regular.

The subsequent exposition should serve as a thread through Sections 9 to 14: As before, let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g},\mathfrak{g}]$. Assume that $f \in \mathfrak{g}^*$ is in general position such that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$. We define $r : \mathfrak{g}^* \longrightarrow \mathfrak{n}^*$ to be the linear projection given by restriction from $\mathfrak{g}$ to $\mathfrak{n}$. Let $f' = r(f) = f \,|\, \mathfrak{n}$. If Ω' denotes the closure of the orbit $\mathrm{Ad}^*(G)f'$ in $\mathfrak{n}^*$, then $\Omega = r^{-1}(\,\Omega'\,)$ is a closed, $\mathrm{Ad}^*(G)$-invariant subset of $\mathfrak{g}^*$ containing $\mathrm{Ad}^*(G)f$.

We point out that the leading idea of Sections 9 to 14 is to examine in how far the ideal situation described in Lemma 1.1 and Theorem 1.3 is present. Note that any polynomial p on $\mathfrak{m}^*$ can be regarded as a polynomial function on $\mathfrak{g}^*$. This fact is used tacitly in

Lemma 1.1 (characterization of the closure of orbits in general position)**.** *Let $\mathfrak{g}$, f, $\mathfrak{m}$, and Ω be as above. Further we assume: There exists a complex valued, polynomial function p on $\mathfrak{m}^*$ such that p is constant on the $\mathrm{Ad}^*(M)$-orbits contained in Ω, and there exist an ideal $\mathfrak{z}$ of $\mathfrak{g}$ such that $\mathfrak{z} \subset \mathfrak{z}\mathfrak{m}$ and a complex-valued, continuous function ψ on $\mathfrak{z}^*$ such that*

$$p\,(\,\mathrm{Ad}^*(x)f\,) = \psi\,(\,\mathrm{Ad}^*(x)f\,|\,\mathfrak{z}\,)$$

for all $x \in G$. Let $g \in \Omega$ be arbitrary. Then g is an element of the closure of $\mathrm{Ad}^*(G)f$ if and only if $p(g) = \psi(\,g\,|\,\mathfrak{z}\,)$.

In many concrete examples we will establish the validity of this lemma: Using explicit formulas for the coadjoint representation, we will show how to define functions p and ψ as above such that this characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$ holds true. It turns out that ψ is typically a continuous function of polynomial growth, but not a polynomial.

Since the Kirillov map is a homeomorphism, this lemma contains a description of the closure of one-point sets $\{\pi\}$ in $\widehat{G}$ for representations $\pi = \mathcal{K}(f)$ in general position.

Definition 1.2 (critical orbits)**.** We say that the orbit $\mathrm{Ad}^*(G)g$ is critical for the orbit $\mathrm{Ad}^*(G)f$ if $\mathrm{Ad}^*(G)g$ is contained in Ω, but not in the closure of $\mathrm{Ad}^*(G)f$.

Recall that symmetrization defines a linear isomorphism from the symmetric algebra $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ onto the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$. Equality 1.4 is well-known if W is in the center of $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ and corresponds to an $\mathrm{Ad}(M)$-invariant polynomial p under the Duflo isomorphism, which is a modification of symmetrization. In the next theorem $\tilde{g}$ denotes the restriction to $\mathfrak{m}$ of a linear functional $g \in \mathfrak{g}^*$.

Theorem 1.3 (separation of representations by $\boldsymbol{L^1}$-functions)**.** *Let $\mathfrak{g}$, f, $\mathfrak{m}$, Ω, p, and ψ be as above. Assume that there exists an element W in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ of the complexification of $\mathfrak{m}$ such that*

$$d\rho(W) = p(g)\cdot\mathrm{Id} \qquad (1.4)$$

is a scalar operator for all $g \in \Omega$ *where* $\rho = \mathcal{K}(\tilde{g})$ *is in* $\widehat{M}$*. If* g *is critical for the orbit* $\mathrm{Ad}^*(G)f$*, then the relation*

$$\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$$

holds for the kernels of the irreducible representations $\pi = \mathcal{K}(f)$ *and* $\rho = \mathcal{K}(g)$*.*

Sketch of the proof. Let M denote the closed, connected subgroup of G whose Lie algebra is given by $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. Restricting the representations π and ρ to the normal subgroup M, it suffices to prove that

$$\bigcap_{x \in G} \ker_{L^1(M)} \tilde{\pi}_x \not\subset \ker_{L^1(M)} \tilde{\rho}$$

holds for $\tilde{\pi}_x = \mathcal{K}(\mathrm{Ad}^*(x)\tilde{f})$ and $\tilde{\rho} = \mathcal{K}(\tilde{g})$. Let $a \in \mathcal{C}_0^\infty(M)$ such that $\tilde{\rho}(a) \neq 0$. We define $b = W * a$. **Assume** that there exists a smooth solution c in $L^1(M)$ of the Fourier multiplier problem

$$\widehat{c}(x, -\xi) = \psi(\xi)\, \widehat{a}(x, -\xi) \tag{1.5}$$

where $\widehat{a}$ denotes the partial Fourier transform of a with respect to $\mathfrak{z}$. Since g is critical for $\mathrm{Ad}^*(G)f$, the preceding lemma implies $p(g) \neq \psi(g \,|\, \mathfrak{z})$. Now it is easy to see that

$$\tilde{\pi}_x(b) = d\tilde{\pi}_x(W)\, \tilde{\pi}_x(a) = p\,(\,\mathrm{Ad}^*(x)f\,)\; \tilde{\pi}_x(a) = \psi\,(\,\mathrm{Ad}^*(x)f \,|\, \mathfrak{z}\,)\; \tilde{\pi}_x(a) = \tilde{\pi}_x(c)$$

and

$$\tilde{\rho}(b) = d\tilde{\rho}(W)\, \tilde{\rho}(a) = p(g)\, \tilde{\rho}(a) \neq \psi\,(\,g \,|\, \mathfrak{z}\,)\, \tilde{\rho}(a) = \tilde{\rho}(c)\ .$$

Thus we get $\tilde{\pi}_x(b-c) = 0$ for all $x \in G$ and $\tilde{\rho}(b-c) \neq 0$. This proves our theorem. □

To be precise, these considerations establish our theorem only under two additional assumptions: the solvability of the multiplier problem given by Equality 1.5 and the validity of the preceding lemma. In this sense the preceding theorem is actually a meta-theorem. In Sections 9 to 14 we will verify these two assertions for many non-$*$-regular exponential Lie groups and we will prove the existence of an element W in $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ such that Equality 1.4 holds.

However, if $\dim \mathfrak{z}\mathfrak{m} > 2$, a characterization of the closure of $\mathrm{Ad}^*(G)f$ by means of p and ψ as in Lemma 1.1 is not always possible, compare Remark 9.28 and 12.35. This observation leads us to the definition of the admissible part Ω_a of Ω which is an $\mathrm{Ad}(G)$-invariant subset of Ω containing $\mathrm{Ad}^*(G)f$, see Definition 9.17 and 12.14. For admissible g one can find p and ψ such that Lemma 1.1 holds true. The proof of a variant of Theorem 1.3 for non-admissible g is beyond the scope of this work and gives reason to further investigations.

We conclude this introduction with a few notational conventions: The Lie algebra of a Lie group G is always denoted by the corresponding German letter $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then H denotes the unique connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Recall that connected Lie subgroups of exponential Lie groups are always closed and simply connected. The nilradical $\mathfrak{n}$ of a solvable Lie algebra $\mathfrak{g}$ is the maximal nilpotent ideal of $\mathfrak{g}$. Throughout this paper $\mathcal{K} = \mathcal{K}_G$ denotes the Kirillov map of the exponential Lie group G. If the subscript G is omitted, then it should be clear from the context which group is meant.

2 Primitive regularity of Banach $*$-algebras

In this section we define and discuss the notion of primitive $*$-regularity in the abstract setting of Banach $*$-algebras. Let $\mathcal{A}$ be an involutive ($*$-semisimple) Banach algebra with a bounded approximate identity and let $C^*(\mathcal{A})$ denote the enveloping C^*-algebra of $\mathcal{A}$ in the sense of Dixmier [8]. The C^*-norm on $C^*(\mathcal{A})$ is given by

$$|a|_* = \sup_{\pi \in \widehat{\mathcal{A}}} |\pi(a)|$$

for all $a \in \mathcal{A}$ where $\widehat{\mathcal{A}}$ denotes the set of equivalence classes of topologically irreducible $*$-representations of $\mathcal{A}$. The inclusion $\mathcal{A} \longrightarrow C^*(\mathcal{A})$ is an (injective) continuous homomorphism of Banach $*$-algebras. All ideals in these Banach algebras are assumed to be two-sided and closed in the respective norm. Let us define $\operatorname{Prim} C^*(\mathcal{A})$ as the set of primitive ideals in $C^*(\mathcal{A})$ and $\operatorname{Prim}_* \mathcal{A}$ as the set of kernels of representations in $\widehat{\mathcal{A}}$. For ideals I of $C^*(\mathcal{A})$ we define the hull of I

$$h(I) = \{P \in \operatorname{Prim} C^*(\mathcal{A}) : P \supset I\}$$

and for subsets X of $\operatorname{Prim} C^*(\mathcal{A})$ we define the kernel of X

$$k(X) = \bigcap_{P \in X} P\,.$$

We point out that a closed ideal I of the C^*-algebra $C^*(\mathcal{A})$ is automatically involutive and that $I = k(h(I))$, see [8]. The dual $\operatorname{Prim} C^*(\mathcal{A})$ is regarded as a topological space with the Jacobson topology, i.e., a subset $X \subset \operatorname{Prim} C^*(\mathcal{A})$ is closed if and only if there exists an ideal I of $C^*(\mathcal{A})$ such that $X = h(I)$. Likewise we can state the according definitions of hulls and kernels for $\mathcal{A}$ and we provide $\operatorname{Prim}_* \mathcal{A}$ with the Jacobson topology as well. For ideals I of $C^*(\mathcal{A})$ we define the ideal $I' = I \cap \mathcal{A}$ of $\mathcal{A}$. The natural map

$$\Psi : \operatorname{Prim} C^*(\mathcal{A}) \longrightarrow \operatorname{Prim}_* \mathcal{A} \;\text{ given by }\; \Psi(P) = P' = P \cap \mathcal{A}$$

is continuous and surjective and evidently satisfies

$$k(\Psi(X)) = k(X) \cap \mathcal{A}$$

for subsets X of $\operatorname{Prim} C^*(\mathcal{A})$ and

$$h(I) \subset \Psi^{-1}(h(I'))$$

for ideals I of $C^*(\mathcal{A})$. The next definition is basic for the subsequent investigation.

Definition 2.1. Let I be a closed ideal of $C^*(\mathcal{A})$. Then I is called $\mathcal{A}$-determined if and only if the following equivalent conditions hold:

(i) For all ideals $J \lhd C^*(\mathcal{A})$ the inclusion $I' \subset J'$ implies $I \subset J$.

(ii) For all $P \in \operatorname{Prim} C^*(\mathcal{A})$ the inclusion $I' \subset P'$ implies $I \subset P$ which means $h(I) = \Psi^{-1}(h(I'))$.

(iii) I is the closure of I' with respect to the C^*-norm.

(iv) $C^*(\mathcal{A}/I') = C^*(\mathcal{A})/I$

We have to verify that these four conditions are equivalent. Obviously *(i)* implies *(ii)* and the closure I_0 of I' in the C^*-norm is contained in I. Let us suppose *(ii)*. If $P \in \operatorname{Prim} C^*(\mathcal{A})$ such that $P \supset I_0$, then $P' \supset I'$ and thus $P \supset I$. Consequently $I_0 = k(h(I_0)) \supset I$ and this proves *(iii)*. We know that $C^*(\mathcal{A})/I$ is a C^*-algebra and we have the following commutative diagram where Φ is a well-defined, continuous homomorphism of Banach $*$-algebras with a dense image in $C^*(\mathcal{A})/I$:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & C^*(\mathcal{A}) \arrow[d] \\ \mathcal{A}/I' \arrow[r, "\Phi"] & C^*(\mathcal{A})/I \end{tikzcd}$$

If condition *(iii)* holds, then $I' \subset \ker \pi'$ implies $I \subset \ker \pi$ for all $\pi \in C^*(\mathcal{A})^\frown$ and thus

$$\begin{aligned} |\,\Phi(\dot a)\,| &= \sup\{\,|\pi(a)| : \pi \in C^*(\mathcal{A})^\frown \text{ such that } I \subset \ker \pi\,\} \\ &= \sup\{\,|\pi'(a)| : \pi' \in \widehat{\mathcal{A}} \text{ such that } I' \subset \ker \pi'\,\} = |\,\dot a\,|_* \end{aligned}$$

for all $a \in \mathcal{A}$. This equation shows that Φ extends to an isomorphism of C^*-algebras from $C^*(\mathcal{A}/I')$ onto $C^*(\mathcal{A})/I$ which proves that *(iii)* implies *(iv)*. Finally from condition *(iv)* it follows that any non-degenerate $*$-representation of $\mathcal{A}/I'$ actually extends to a representation of $C^*(\mathcal{A})/I$ and this makes *(i)* evident.

Definition 2.2. A Banach $*$-algebra $\mathcal{A}$ is called $*$-regular if and only if every closed ideal I of $C^*(\mathcal{A})$ is $\mathcal{A}$-determined. The algebra $\mathcal{A}$ is called primitive $*$-regular if and only if every primitive ideal $P \in \operatorname{Prim} C^*(\mathcal{A})$ is $\mathcal{A}$-determined.

Part *(ii)* of the next lemma implies that our definition of $*$-regularity coincides with Boidol's original definition, a characterization which has already been proved in [2].

Lemma 2.3.

(i) If $\mathcal{A}$ is primitive $*$-regular, then $\Psi : \operatorname{Prim} C^*(\mathcal{A}) \longrightarrow \operatorname{Prim}_* \mathcal{A}$ is injective.

(ii) A Banach $*$-algebra $\mathcal{A}$ is $*$-regular if and only Ψ is a homeomorphism with respect to the Jacobson topology on $\operatorname{Prim} C^*(\mathcal{A})$ and $\operatorname{Prim}_* \mathcal{A}$.

Proof. If $\mathcal{A}$ is primitive $*$-regular, then $P = P_0 = \overline{\Psi(P)}$ is uniquely determined by $\Psi(P)$ for all $P \in \operatorname{Prim} C^*(\mathcal{A})$. This proves *(i)*. In order to prove *(ii)*, let us suppose that $\mathcal{A}$ is $*$-regular. Since Ψ is a continuous bijection, it suffices to prove that Ψ maps closed sets onto closed sets. But if X is a closed subset of $\operatorname{Prim} C^*(\mathcal{A})$, then there exists a closed ideal I of $C^*(\mathcal{A})$ such that $X = h(I)$ and we see that $\Psi(X) = h(I')$ is closed in $\operatorname{Prim}_* \mathcal{A}$ because I is $\mathcal{A}$-determined. Now we prove the opposite implication. Assume that Ψ is a homeomorphism, I is a closed ideal of $C^*(\mathcal{A})$, and $P \in \operatorname{Prim} C^*(\mathcal{A})$ such that $I' \subset P'$. Define $X = h(I)$. Since $I' = k(\Psi(X))$, it follows

$$h(I') = h(k(\Psi(X))) = \overline{\Psi(X)} = \Psi(X)$$

because Ψ maps closed sets onto closed sets. Now $P' \in \Psi(X)$ implies $P \in X$ so that $P \supset I$ because Ψ is injective. This proves the asserted equivalence. $\square$

Because of its technical importance we state the following easy fact as a lemma, but we omit the short proof.

Lemma 2.4. *Let I and J be closed ideals of $C^*(\mathcal{A})$ such that I is $\mathcal{A}$-determined and $I \subset J$. Then J is $\mathcal{A}$-determined if and only if the ideal J/I of $C^*(\mathcal{A})/I$ is $\mathcal{A}/I'$-determined.*

The next lemma will come in handy in Subsection 3.3.

Lemma 2.5. *Let $\mathcal{A}$ be a Banach $*$-algebra. Any finite intersection of $\mathcal{A}$-determined ideals of $C^*(\mathcal{A})$ is again $\mathcal{A}$-determined.*

Proof. Let I_1 and I_2 be $\mathcal{A}$-determined ideals of $C^*(\mathcal{A})$. Let $P \in \operatorname{Prim} C^*(\mathcal{A})$ such that $I_1' \cdot I_2' \subset I_1' \cap I_2' \subset P'$. Since P' is a prime ideal of $\mathcal{A}$, it follows $I_1' \subset P'$ or $I_2' \subset P'$. Since I_1 and I_2 are $\mathcal{A}$-determined, we obtain $I_1 \subset P$ or $I_2 \subset P$ and thus $I_1 \cap I_2 \subset P$. Consequently $I_1 \cap I_2$ is $\mathcal{A}$-determined and the assertion of this lemma follows by induction. $\square$

Recall that the group algebra $L^1(G)$ of a locally compact group G is a Banach $*$-algebra which contains approximate identities.

Definition 2.6. A locally compact group G is called (primitive) $*$-regular if and only if $L^1(G)$ is (primitive) $*$-regular. A Lie algebra $\mathfrak{g}$ is (primitive) $*$-regular if and only if the unique connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$ has this property.

Let us give a few examples of $*$-regular groups.

Remark 2.7. Let G be a locally compact group.

(i) If G has polynomial growth, then G is $*$-regular. This has been proved in Theorem 2 of [2] based on the ideas of [7].

(ii) If G is a connected nilpotent group, then G has polynomial growth. This result can be found in Theorem 1.4 of [17].

(iii) If G is connected and metabelian, then G is $*$-regular. See Theorem 3.5 of [3].

The next remark shows that we can pass to quotients of locally compact groups by Lemma 2.4.

Remark 2.8. Let A be a closed normal subgroup of the locally compact group G. Let $\dot{G}$ denote the quotient group G/A. Then

$$Tf\,(\dot{x}) = \int\limits_A f(xa)\,da$$

gives a quotient map of Banach $*$-algebras from $L^1(G)$ onto $L^1(\dot{G})$, see p. 68 of [30]. In particular $L^1(\dot{G})$ is isometrically isomorphic to $L^1(G)/\ker_{L^1(G)} T$. One easily verifies that T extends to a quotient map from $C^*(G)$ onto $C^*(\dot{G})$. Thus

$$C^*(G)/\ker_{C^*(G)} T \cong C^*(\dot{G}) = C^*(L^1(\dot{G})) \cong C^*(L^1(G))/\ker_{L^1(G)} T)$$

which means that $\ker_{C^*(G)} T$ is an $L^1(G)$-determined ideal of $C^*(G)$.

3 Primitive regularity of exponential Lie groups

This section consists of four subsections. Let G denote an exponential Lie group. In the first two parts we formulate sufficient criteria for ideals of $C^*(G)$ to be $L^1(G)$-determined, see Theorem 3.13 and Theorem 3.23. From these results we deduce a strategy for proving the primitive $*$-regularity of exponential Lie groups in the last subsection. This plan will be carried out in the Sections 9 to 14. The third part of this section contains only accessories: We will see that the ideal theory of a $*$-regular exponential Lie group is particularly simple.

3.1 Inducing primitive ideals from a stabilizer

The purpose of this subsection is to prove Proposition 3.12 from which we will deduce Theorem 3.13. Although the assertion of this theorem is well-known, we believe that it is justified to give a complete proof of it here. In the context of $*$-regularity the significance of Proposition 3.12 cannot be overestimated. To begin with, let us recall some basic results on (very particular) direct integral representations. More information on direct integrals can be found in Part II of [9] and Chapter 7.4 of [11].

Lemma 3.1. *Let G be a locally compact group, X a locally compact space, and μ a Radon measure on X such that* $\operatorname{supp}(\mu) = X$. *Let $\{\pi_x : x \in X\}$ be a family of unitary representations of G in some Hilbert space $\mathfrak{H}$ such that the map $G \times X \longrightarrow \mathfrak{H}$, $(g, x) \mapsto \pi_x(g)\xi$ is continuous for all $\xi \in \mathfrak{H}$.*

(i) Then the formula $\pi(g)\varphi\ (x) = \pi_x(g)\cdot\varphi(x)$ defines a strongly continuous, unitary representation of G in the Hilbert space $L^2(X, \mathfrak{H}, \mu)$ which is called the direct integral representation of the $\{\pi_x : x \in X\}$. Using the integrated form of the π_x, one obtains the bounded $*$-representation $\pi(a)\varphi\ (x) = \pi_x(a)\cdot\varphi(x)$ of $C^*(G)$ in $L^2(X, \mathfrak{H}, \mu)$. This is just the integrated form of the group representation π.

(ii) The unitary representation π is weakly equivalent to the set $\{\pi_x : x \in X\}$ of representations of G which means

$$\ker_{C^*(G)} \pi = \bigcap_{x\in X} \ker_{C^*(G)} \pi_x\ .$$

Sometimes one writes $L^2(X, \mathfrak{H}, \mu) = \int_X^{\oplus} \mathfrak{H}_x$ and $\pi = \int_X^{\oplus} \pi_x$.

Proof. It is obvious that $\pi(g)\varphi\ (x) = \pi_x(g)\cdot\varphi(x)$ defines a unitary representation of G in $L^2(X, \mathfrak{H}, \mu)$. Let us prove that π is strongly continuous: Let $\varphi \in \mathcal{C}_0(X, \mathfrak{H})$, $\varphi \neq 0$, and $\epsilon > 0$ be arbitrary. Define $K = \operatorname{supp}(\varphi)$. Since $\operatorname{supp}(\mu) = X$, it follows $\mu(K) > 0$. Since K is compact, $\pi_x(e) = \mathrm{Id}$ for all x, and $(g, x) \mapsto \pi_x(g)\cdot\varphi(x) - \varphi(x)$ is continuous on $G \times K$, there exists an open neighborhood U of $e \in G$ such that

$$|\,\pi_x(g)\cdot\varphi(x) - \varphi(x)\,| \le \epsilon\mu(K)^{-1/2}$$

for all $g \in U$ and $x \in X$. Now it follows

$$|\,\pi(g)\varphi - \varphi\,|_2^2 = \int_X |\,\pi_x(g)\cdot\varphi(x) - \varphi(x)\,|^2\, d\mu(x) \le \epsilon^2$$

for all $g \in U$ and this proves the strong continuity. Clearly the formula $\pi(a)\varphi\,(x) = \pi_x(a)\cdot\varphi(x)$ defines a bounded $*$-representation of $C^*(G)$ in $L^2(X,\mathfrak{H},\mu)$. We must show that this is the integrated form of the group representation π. Let us define $\tilde\pi(f)\varphi = \int_G f(g)\pi(g)\varphi\,dg$ for $f \in L^1(G)$. For $f \in \mathcal{C}_0(G)$ and $\varphi,\psi \in \mathcal{C}_0(X,\mathfrak{H})$ we obtain

$$\langle\,\pi(f)\varphi,\psi\,\rangle = \langle\,\tilde\pi(f)\varphi,\psi\,\rangle$$

by Fubini's theorem and this yields our claim. Finally we prove the weak equivalence of π and $\{\pi_x : x \in X\}$. It is obvious that $\ker_{C^*(G)}\pi \supset \bigcap_{x\in X}\ker_{C^*(G)}\pi_x$. Let $a \in C^*(G)$ such that $\pi(a) = 0$. Now it follows

$$|\,\pi(a)\varphi\,|_2^2 = \int_X |\,\pi_x(a)\cdot\varphi(x)\,|^2\,d\mu(x) = 0$$

and thus $\pi_x(a)\cdot\varphi(x) = 0$ for all $\varphi \in \mathcal{C}_0(X,\mathfrak{H})$ and almost all $x \in X$. Since the function $x \mapsto \pi_x(a)\xi$ is continuous, we see that $\pi_x(a) = 0$ for all x so that $a \in \bigcap_{x\in X}\ker_{C^*(G)}\pi_x$. This finishes the proof of our lemma. $\square$

Part *(i)* and *(ii)* of the following lemma have been proved on p. 32 of [23]. These assertions remain valid if one drops the additional assumption of the existence of relatively invariant measures on the homogeneous space G/H.

Lemma 3.2. *Let H be a closed subgroup of the locally compact group G. Let σ be a unitary representation of G in the Hilbert space $\mathfrak{H}_\sigma$ and σ_0 its restriction to H.*

(i) Let τ be a unitary representation of H in $\mathfrak{H}_\tau$. Then $\pi = \operatorname{ind}_H^G(\sigma_0\otimes\tau)$ and $\rho = \sigma\otimes\operatorname{ind}_H^G\tau$ are unitarily equivalent.

(ii) The representation $\pi = \operatorname{ind}_H^G\sigma_0$ is unitarily equivalent to the tensor product $\sigma\otimes\lambda$ where λ denotes the left regular representation of G in $L^2(G/H)$.

(iii) If H is a normal subgroup of G such that G/H is abelian, then $\pi = \operatorname{ind}_H^G\sigma_0$ is weakly equivalent to the set $\{\alpha\otimes\sigma : \alpha \in (G/H)^\frown\}$ which means

$$\ker_{C^*(G)}\pi = \bigcap_{\alpha\in(G/H)^\frown}\ker_{C^*(G)}\,\alpha\otimes\sigma\,.$$

Proof. First we prove *(i)*. Let $\mathcal{C}_0^\tau(G,\mathfrak{H}_\tau)$ denote the vector space of all continuous functions $\psi : G \longrightarrow \mathfrak{H}_\tau$ such that $\psi(xh) = \tau(h)^*\,\psi(x)$ for all $h \in H$ and $x \in G$, and such that the support of ψ is compact modulo H. By definition the representation space $L^2_\tau(G,\mathfrak{H}_\tau)$ of $\operatorname{ind}_H^G\tau$ is the closure of $\mathcal{C}_0^\tau(G,\mathfrak{H}_\tau)$ with respect to the L^2-norm given by integration with respect to a relatively G-invariant measure on G/H. It is easy to see that the linear map $U : \mathfrak{H}_\sigma\otimes\mathcal{C}_0^\tau(G,\mathfrak{H}_\tau) \longrightarrow \mathcal{C}_0^{\sigma_0\otimes\tau}(G,\mathfrak{H}_\sigma\otimes\mathfrak{H}_\tau)$ given by

$$U(\xi\otimes\psi)\,(x) = \sigma(x)^*\xi\otimes\psi(x)$$

extends to a unitary isomorphism of Hilbert spaces from $\mathfrak{H}_\rho = \mathfrak{H}_\sigma\otimes L^2_\tau(G,\mathfrak{H}_\tau)$ onto $\mathfrak{H}_\pi = L^2_{\sigma_0\otimes\tau}(G,\mathfrak{H}_\sigma\otimes\mathfrak{H}_\tau)$ and intertwines ρ and π. This proves part *(i)*. If we choose $\tau = 1$, then $\lambda = \operatorname{ind}_H^G 1$ is the left regular representation of G in $L^2(G/H)$ and the claim of *(ii)* becomes obvious. Finally we come to the proof of *(iii)*. Let us write

$A = G/H$. Using the Haar measure of A we can realize $\pi = \mathrm{ind}_H^G \sigma_0$ in the Hilbert space $L^2(A, \mathfrak{H}_\sigma)$ such that $\pi(g)\varphi\ (x) = \sigma(g)\cdot\varphi(g^{-1}x)$. The Fourier transformation $L^2(A, \mathfrak{H}_\sigma) \longrightarrow L^2(\widehat{A}, \mathfrak{H}_\sigma)$ defined by

$$\widehat{\varphi}(\alpha) = \int_A \varphi(a)\, \overline{\alpha(a)}\, da$$

is a unitary isomorphism and serves as an intertwining operator. In $L^2(\widehat{A}, \mathfrak{H}_\sigma)$ the representation π is given by

$$\pi(g)\varphi\ (\alpha) = \overline{\alpha}(g)\, \sigma(g)\cdot\varphi(\alpha)\ .$$

Thus we see that π is a direct integral of the representations $\{\overline{\alpha} \otimes \sigma : \alpha \in \widehat{A}\}$ in the sense of Lemma 3.1 so that the assertion of *(iii)* becomes apparent by part *(ii)* of that lemma. □

The following basic result has been proved on p. 23 of [1].

Remark 3.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{n}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, and $f \in \mathfrak{g}^*$. Let $\mathfrak{g}_f = \mathfrak{g}^{\perp_B}$ denote the stabilizer of f in $\mathfrak{g}$ which is equal to the radical of the skew, bilinear form $B(X, Y) = f([X, Y])$ on $\mathfrak{g}$. The ideal $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ of $\mathfrak{g}$ is a stabilizer in the sense that $X \in \mathfrak{m}$ if and only if $\mathrm{Ad}^*(\exp tX)f \in \mathrm{Ad}^*(N)f$ for all t. It is easy to see that the image of the linear map $\varphi : \mathfrak{g}_{f'} \longrightarrow \mathfrak{g}^*$, $\varphi(X) = \mathrm{ad}^*(X)f$ is equal to $\mathfrak{m}^\perp$. Here $f' = f \,|\, \mathfrak{n}$ and $\mathfrak{g}_{f'} = \mathfrak{n}^{\perp_B}$. If $G_{f'}$ denotes the connected subgroup of G with Lie algebra $\mathfrak{g}_{f'}$, then it follows $\mathrm{Ad}^*(G_{f'})f = f + \mathfrak{m}^\perp$. In particular we have $\mathrm{Ad}^*(G)f \supset f + \mathfrak{m}^\perp$.

The next proposition allows us to compute the C^*-kernel of induced representations. A more general version of this proposition can be found in Chapter 1, Section 5 of [23].

Proposition 3.4. *Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}]$. Let $f \in \mathfrak{g}^*$, $l = f \,|\, \mathfrak{h}$ in $\mathfrak{h}^*$, $\sigma = \mathcal{K}(l)$ in $\widehat{H}$, and $\pi = \mathrm{ind}_H^G \sigma$. Then*

$$\ker_{C^*(G)} \pi = \bigcap_{h \in f + \mathfrak{h}^\perp} \ker_{C^*(G)} \mathcal{K}(h)\ .$$

Proof. By induction in stages it suffices to verify the assertion of this theorem in the one-codimensional case. Here we use the fact that the process of inducing representations is continuous with respect to the Fell topology so that in particular it preserves the relation of weak containment. Now let us assume $\dim \mathfrak{g}/\mathfrak{h} = 1$. At first we treat the case $\mathfrak{g}_f \subset \mathfrak{h}$. Let us choose a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{h}$ at $l \in \mathfrak{h}^*$. It is easy to see that $\mathfrak{p} \subset \mathfrak{g}$ is also a Pukanszky polarization at $f \in \mathfrak{g}^*$. By induction in stages we obtain $\pi = \mathrm{ind}_H^G \sigma = \mathrm{ind}_P^G \chi_f$ and thus $\pi = \mathcal{K}(f)$ is irreducible. From Remark 3.3 it follows $\mathrm{Ad}^*(G)f \supset f + \mathfrak{h}^\perp$. This observation implies

$$\ker_{C^*(G)} \pi = \ker_{C^*(G)} \mathcal{K}(f) = \bigcap_{h \in f + \mathfrak{h}^\perp} \ker_{C^*(G)} \mathcal{K}(h)$$

in the case $\mathfrak{g}_f \subset \mathfrak{h}$ because the Kirillov map $\mathcal{K}$ is constant on $\mathrm{Ad}^*(G)$-orbits. Finally we assume $\mathfrak{g}_f \not\subset \mathfrak{h}$. Using the concept of Vergne polarizations passing through $\mathfrak{h}$ we see that there exists a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{g}$ at $f \in \mathfrak{g}^*$ such that $\mathfrak{q} = \mathfrak{g} \cap \mathfrak{h}$ is a Pukanszky polarization at $l \in \mathfrak{h}^*$. Recall that the representation space of $\rho = \mathcal{K}(f)$ is $L^2_{\chi_f}(G)$, that of $\sigma = \mathcal{K}(l)$ is $L^2_{\chi_l}(H)$. We point out that the restriction of functions from G to H gives a linear isomorphism $\mathcal{C}_0^{\chi_f}(G) \longrightarrow \mathcal{C}_0^{\chi_l}(H)$ which extends to a unitary isomorphism $L^2_{\chi_f}(G) \longrightarrow L^2_{\chi_l}(H)$ and intertwines $\rho\,|\,H$ and σ. This argument shows that without loss of generality we can suppose $\rho\,|\,H = \sigma$. Now we apply Lemma 3.2 to $\pi = \mathrm{ind}_H^G\,\sigma = \mathrm{ind}_H^G(\rho\,|\,H)$ and obtain

$$\ker_{C^*(G)} \pi = \bigcap_{\alpha \in (G/H)^\frown} \ker_{C^*(G)} \alpha \otimes \rho\,.$$

Since linear functionals $q \in \mathfrak{h}^\perp \subset \mathfrak{g}^*$ correspond to the characters $\alpha(\exp X) = e^{iq(X)}$ of G/H and since $\mathcal{K}(f+q) = \alpha \otimes \rho$, this yields the assertion of our proposition. □

Note that Equality 3.7 of the following theorem states that the kernel of the induced $*$-representation is the induced ideal. It is interesting to compare our method of inducing ideals of C^*-algebras to that of the so-called Rieffel correspondence, see Proposition 9 in Section 3 of [14] and Chapter 3.3 of [29].

Theorem 3.5. *Let H be a closed, normal subgroup of the locally compact group G such that G/H is amenable. Let σ be a unitary representation of H in some Hilbert space $\mathfrak{H}$ and $\pi = \mathrm{ind}_H^G\,\sigma$ the induced representation of G. Then it holds*

$$\ker_{C^*(H)}\ \pi = \bigcap_{x \in G} \ker_{C^*(H)}\ x\cdot\sigma \tag{3.6}$$

and

$$\ker_{C^*(G)}\ \pi = \left(\ker_{C^*(H)}\ \pi\ * C^*(G)\right)^{-}\ . \tag{3.7}$$

In particular $\pi\,|\,H$ is weakly equivalent to the G-orbit $G\cdot\sigma$. The analogous equalities hold for $\ker_{L^1(G)}\pi$ in $L^1(G)$.

Proof. The representation space $L^2_\sigma(G,\mathfrak{H})$ of the induced representation $\pi = \mathrm{ind}_H^G\,\sigma$ is the completion of $\mathcal{C}_0^\sigma(G,\mathfrak{H})$ with respect to the L^2-norm given by integration with respect to the Haar measure of the group G/H. For $h \in H$ we have $\pi(h)\varphi\ (x) = \varphi(h^{-1}x) = \sigma(h^x)\cdot\varphi(x)$. It follows that the restriction of the induced representation to $C^*(H)$ is given by $\pi(a)\varphi\ (x) = \sigma(a^x)\cdot\varphi(x)$ so that 3.6 becomes obvious. Intersecting with $L^1(H)$ we obtain

$$\ker_{L^1(H)} \pi = \bigcap_{x \in G} \ker_{L^1(H)}\ x\cdot\sigma\ .$$

Note that $\pi\,|\,H$ is the direct integral of $\{x\cdot\sigma : x \in G\}$. Now we prove Equality 3.7. The inclusion from the right to the left of 3.7 is obvious. In order to prove the opposite inclusion, by Theorem 2.9.7 of [8] it suffices to verify that if ρ is an (irreducible) representation of $C^*(G)$ such that

$$\ker_{C^*(G)} \rho \supset \left(\ker_{C^*(H)} \pi\ * C^*(G)\right)^{-}\ , \tag{3.8}$$

then it follows $\ker_{C^*(G)} \rho \supset \ker_{C^*(G)} \pi$. But Relation 3.8 implies $\ker_{C^*(H)} \rho \supset \ker_{C^*(H)} \pi$ so that $\rho\,|\,H \ll \pi\,|\,H$, i.e., $\rho\,|\,H$ is weakly contained in $\pi\,|\,H$. Furthermore the amenability of G/H yields $1_G \ll \mathrm{ind}_H^G\ 1_H$. Since inducing representations is continuous with respect to the Fell topology, we conclude that the weak equivalence $\pi \,|\, H \approx \{x{\cdot}\sigma : x \in G\}$ implies

$$\operatorname{ind}_H^G(\pi \,|\, H) \approx \{\operatorname{ind}_H^G(x{\cdot}\sigma) : x \in G\} \approx \operatorname{ind}_H^G \sigma$$

because the representations $\operatorname{ind}_H^G(x{\cdot}\sigma)$ are all unitarily equivalent. Again by the continuity of inducing representations and from Lemma 3.2 it follows that

$$\begin{aligned}\rho &\cong \rho \otimes 1_G \ll \rho \otimes \operatorname{ind}_H^G 1_H \cong \operatorname{ind}_H^G(\rho \,|\, H \otimes 1_H)\\ &\cong \operatorname{ind}_H^G(\rho \,|\, H) \ll \operatorname{ind}_H^G(\pi \,|\, H) \approx \operatorname{ind}_H^G \sigma = \pi\end{aligned}$$

so that $\ker_{C^*(G)} \rho \supset \ker_{C^*(G)} \pi$. These considerations prove 3.7. We point out that

$$\ker_{L^1(G)} \pi = \big(\ker_{L^1(H)} \pi * L^1(G)\big)^{-} \tag{3.9}$$

is **not** an immediate consequence of 3.7. Again the inclusion from the right to the left of 3.9 is trivial. In order to prove the opposite inclusion we invoke the machinery of twisted covariance algebras developed profoundly in [14] and [15]. It is known that the group algebra $L^1(G)$ is isomorphic to the twisted covariance algebra $L^1(G, L^1(H), \tau)$ with group action

$$a^x(h) = \delta(x^{-1})\, a(h^{x^{-1}})$$

of G on $L^1(H)$ and twist $\tau : H \longrightarrow \mathcal{U}(L^1(H)^b)$ given by

$$\tau(h)a\ (k) = a(h^{-1}k)\ .$$

Any $*$-representation π of $L^1(G, L^1(H), \tau)$ is given by the formula

$$\pi(f)\varphi = \int_{G/H} \pi_1(g)\pi_2(f(g))\varphi\, dg$$

for some covariance pair (π_1, π_2), i.e., representations π_1 of G and π_2 of $L^1(H)$ such that

$$\pi_2(a^x) = \pi_1(x)^*\pi_2(a)\pi_1(x) \quad \text{and} \quad \pi_2(\tau(n)) = \pi_1(n)\ .$$

Here π_1 is the representation π considered as a group representation of G and π_2 is the restriction of π to $L^1(H)$ and $L^1(H)^b$. If in particular we consider the induced representation $\pi = \operatorname{ind}_H^G \sigma$ as a representation of $L^1(G, L^1(H), \tau)$, then the covariance pair (π_1, π_2) defining π is given by

$$\pi_1(g)\varphi\ (x) = \varphi(g^{-1}x) \quad \text{and} \quad \pi_2(a)\varphi\ (x) = \sigma(a^x){\cdot}\varphi(x)$$

for $g, x \in G$, $a \in L^1(H)$, and $\varphi \in \mathcal{C}_0^\sigma(G, \mathfrak{H})$. Let us define the G- and τ-invariant ideal $I = \ker_{L^1(H)} \pi_2$ of $L^1(H)$. We claim that

$$\ker_{L^1(G, L^1(H), \tau)} \pi = L^1(G, I, \tau)\ ,$$

the inclusion from the right to the left being trivial. Let $f \in L^1(G, L^1(H), \tau)$ such that $\pi(f) = 0$. The pointwise formula

$$0 = \pi(f)\varphi\ (x) = \int_{G/H} \sigma(f(g)^{g^{-1}x}){\cdot}\varphi(g^{-1}x)\, dg$$

for all $\varphi \in \mathcal{C}_0^\sigma(G,\mathfrak{H})$ and $x \in G$ implies $\sigma(f(g)^x) = 0$ for all x so that $f \in L^1(G,I,\tau)$. In order to complete the proof of this lemma, it remains to verify the non-trivial inclusion of

$$L^1(G,I,\tau) = \big(\, I * L^1(G,L^1(H),\tau)\,\big)^{-} \ .$$

Clearly it suffices to prove this inclusion in the untwisted algebra $L^1(G,L^1(H))$. Note that $I * L^1(G,L^1(H))$ is invariant under multiplication by elements of $\mathcal{C}_0(G/H)$. Let $f \in \mathcal{C}_0(G,I)$, $f \neq 0$. Define $K = \operatorname{supp}(f)$ and fix an open, relatively compact subset W of G such that $K \subset W$. Let $\epsilon > 0$ be arbitrary. Lemma 3.10 implies that there exists a finite, open covering $\{U_\lambda : \lambda \in L\}$ of K such that $U_\lambda \subset W$ and functions $g_\lambda \in I * L^1(G,L^1(H),\tau)$ such that

$$|\, f(x) - g_\lambda(x)\,| < \epsilon/|W|$$

for all $x \in U_\lambda$ and all $\lambda \in L$. Next we choose a partition of unity subordinate to $\{U_\lambda : \lambda \in L\}$, i.e., functions $\varphi_\lambda \in \mathcal{C}_0(G)$ such that $0 \le \varphi_\lambda \le 1$, $\operatorname{supp}(\varphi_\lambda) \subset U_\lambda$, and $\sum_{\lambda\in L}\varphi_\lambda = 1$ on K. If we define $g = \sum_{\lambda\in L}\varphi_\lambda g_\lambda$ in $I * L^1(G,L^1(H),\tau)$, then we obtain

$$|\, f - g\,|_1 \le \sum_{\lambda\in L} \int\limits_G \varphi_\lambda(x)\,|\, f(x) - g_\lambda(x)\,| < \epsilon$$

which proves our claim. □

Lemma 3.10. *Let $f \in \mathcal{C}_0(G,I)$, $x_0 \in G$, and $\epsilon > 0$. Then there exists a function $g \in I * \mathcal{C}_0(G,I)$ such that $|\, f(x_0) - g(x_0)\,| < \epsilon$.*

Proof. Since $L^1(H)$ has an approximate identity, there exists an element $u \in L^1(H)$ such that $|\, f(x_0) - f(x_0) * u\,| < \epsilon$. Let us choose a function $\beta \in \mathcal{C}_0(G,L^1(H))$ such that $\beta(x_0) = u$ and define $g = f(x_0)^{x_0^{-1}} * \beta$. Then $|\, f(x_0) - g(x_0)\,| = |\, f(x_0) - f(x_0) * u\,| < \epsilon$ and the proof is complete. □

Definition 3.11. Let H be a closed normal subgroup of the locally compact group G. An ideal I of $C^*(G)$ is said to be induced from H if there exists an ideal J of $C^*(H)$ such that $I = (\, J * C^*(G)\,)^{-}$.

The next proposition enlightens the significance of the stabilizer M.

Proposition 3.12. *Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g},\mathfrak{g}]$. Let M denote the connected subgroup of G whose Lie algebra is given by $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. Let $\pi = \mathcal{K}(f)$ be in $\widehat{G}$. Then the ideal $\ker_{C^*(G)}\pi$ is induced from the stabilizer M in the sense that*

$$\ker_{C^*(G)}\pi = \big(\ker_{C^*(M)}\pi * C^*(G)\,\big)^{-} \ .$$

The analogous equality is valid in $L^1(G)$.

Proof. Let $l = f\,|\,\mathfrak{m}$ be in $\mathfrak{m}^*$ and $\sigma = \mathcal{K}(l)$ be in $\widehat{M}$. From Remark 3.3 it follows that $\mathrm{Ad}^*(G)f \supset f + \mathfrak{m}^\perp$. Now Proposition 3.4 implies

$$\ker_{C^*(G)}\pi = \ker_{C^*(G)}\operatorname{ind}_M^G\sigma \ .$$

This means that the C^*-kernel of the irreducible representation π is equal to the C^*-kernel of the (possibly reducible) representation $\operatorname{ind}_M^G\sigma$. But Theorem 3.5 states that the C^*-kernel of an induced representation is an induced ideal. Note that the assumption of G/M being amenable is satisfied because G/M is a connected, abelian Lie group. The same conclusions hold in $L^1(G)$ and the proof is complete. □

The assertion of the preceding theorem actually holds for any closed normal subgroup $\tilde{M}$ such that $M \subset \tilde{M}$. Now we come to the main result of this subsection which can also be found in Boidol [3].

Theorem 3.13. *Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Let $f \in \mathfrak{g}^*$ such that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is nilpotent. Let $\pi = \mathcal{K}(f)$ be in $\widehat{G}$. Then the primitive ideal $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined.*

Proof. Let $\rho \in \widehat{G}$ be arbitrary such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$. We restrict these representations to the normal subgroup M and obtain $\ker_{L^1(M)} \pi \subset \ker_{L^1(M)} \rho$. It is well-known that nilpotent Lie groups M are $*$-regular because the Haar measure of M has polynomial growth, see Theorem 2 of [2]. Thus we see $\ker_{C^*(M)} \pi \subset \ker_{C^*(M)} \rho$. Since the ideal $\ker_{C^*(G)} \pi$ is induced from M by the preceding theorem, it follows

$$\ker_{C^*(G)} \pi = \big(\ker_{C^*(M)} \pi \, * \, C^*(G)\big)^{-} \subset \big(\ker_{C^*(M)} \rho \, * \, C^*(G)\big)^{-} \subset \ker_{C^*(G)} \rho \, .$$

This finishes the proof of our theorem. □

Remark 3.14. More generally, one obtains the following: Let M be $*$-regular, closed normal subgroup of the locally compact group G. If the ideal I of $C^*(G)$ is induced from M, then I is $L^1(G)$-determined.

Lemma 3.15. *Let M be a closed normal subgroup of G such that G/M is amenable. Let $\{I_k : k \in X\}$ be a set of ideals of $C^*(G)$ which are induced from M. Then the intersection $\bigcap_{k \in X} I_k$ is also induced from M.*

Proof. Since I_k is induced from M, there exists an ideal J_k of $C^*(M)$ such that

$$I_k = (\, J_k \, * \, C^*(G)\,)^{-} \, .$$

Let σ_k be a unitary representation of M such that $J_k = \ker_{C^*(M)} \sigma_k$. Now we define $\pi_k = \operatorname{ind}_M^G \sigma_k$ so that $I_k = \ker_{C^*(G)} \pi_k$ by Theorem 3.5. Since

$$\pi = \sum_{k \in X}^{\oplus} \pi_k = \operatorname{ind}_M^G \left(\sum_{k \in X}^{\oplus} \sigma_k \right) ,$$

it follows again from Theorem 3.5 that $\bigcap_{k \in X} I_k = \ker_{C^*(G)} \pi$ is induced from M. □

3.2 Closed orbits in the unitary dual of the nilradical

Let G be an exponential Lie group and π in $\widehat{G}$. It is well-known that there exists a unique orbit $G \cdot \tau$ in the dual $\widehat{N}$ of the nilradical such that

$$\ker_{C^*(N)} \pi = k(G \cdot \tau) .$$

In this subsection we discuss the case of $G \cdot \tau$ being closed in $\widehat{N}$. Our aim is to illustrate the proof of Theorem 3.21 which is a consequence of the classification of simple $L^1(G)$-modules developed by Poguntke in [28]. From this we will deduce Theorem 3.23. Let us begin with the preparations.

Lemma 3.16. *Let $\mathcal{A}$ be Banach $*$-algebra and π an irreducible $*$-representation of $\mathcal{A}$ in a Hilbert space $\mathfrak{H}$.*

(i) Let $\xi \in \mathfrak{H}$ be non-zero. Then the subspace $\pi(\mathcal{A})\xi$ is non-zero and dense in $\mathfrak{H}$. If I is an ideal of $\mathcal{A}$ such that $I \not\subset \ker \pi$, then $\pi(I)\xi$ is also non-zero and dense.

(ii) Let us suppose that there exists an element $p \in \mathcal{A}$ such that $\pi(p) \neq 0$ has finite rank. Let I denote the ideal of $\mathcal{A}$ consisting of all $f \in \mathcal{A}$ such that $\pi(f)$ has finite rank and $E = \pi(I)\mathfrak{H}$ the $\pi(\mathcal{A})$-invariant subspace generated by the set $\{\pi(f)\eta : f \in I,\, \eta \in \mathfrak{H}\}$. Then E is a simple I-module and in particular a simple $\mathcal{A}$-module. We have $\mathrm{Ann}_{\mathcal{A}}(E) = \ker_{\mathcal{A}} \pi$.

Proof. First we prove *(i)*. Since $\pi \neq 0$ is irreducible, the subspace $\pi(\mathcal{A})\xi$ is non-zero and dense in $\mathfrak{H}$. From $\pi(I)\mathfrak{H} \neq 0$ it follows $\pi(I)\pi(\mathcal{A})\xi \neq 0$. Now $\pi(I)\pi(\mathcal{A})\xi \subset \pi(I)\xi$ implies $\pi(I)\xi \neq 0$. Hence the $\pi(\mathcal{A})$-invariant subspace $\pi(I)\xi$ is also dense. Now we prove *(ii)*. Let $\xi \in E$ be non-zero. We must show $\pi(I)\xi = E$. For every $f \in I$ the subspace $\pi(f)\pi(\mathcal{A})\xi \subset \pi(f)\mathfrak{H}$ is dense. This implies $\pi(f)\pi(\mathcal{A})\xi = \pi(f)\mathfrak{H}$ because $\pi(f)\mathfrak{H}$ is finite-dimensional. We have shown $\pi(f)\mathfrak{H} \subset \pi(I)\xi$ for every $f \in I$. Thus $E = \pi(I)\mathfrak{H} = \pi(I)\xi$. The rest is obvious. □

In the next proposition we combine a few results that are successively proved in [31], see p.45, pp. 61- 62, and p. 65. Our main interest lies in part *(v)* and its consequences. For a definition of the notions 'strictly irreducible' and '$\mathcal{B}$-admissible' see also [31].

Proposition 3.17. *Let E be a complex vector space and $\mathcal{B} \subset \mathrm{End}(E)$ a strictly irreducible, complex Banach algebra. Let us fix a $\mathcal{B}$-admissible norm on E. Then the following is true:*

(i) Let $T \in \mathrm{End}(E)$ be non-zero such that $AT = TA$ for all $A \in \mathcal{B}$. Then T is a linear isomorphism and $T, T^{-1} \in \mathcal{B}(E)$. For the commutant

$$\mathcal{B}' = \{T \in \mathrm{End}(E) : TA = AT \text{ for all } A \in \mathcal{B}\}$$

of $\mathcal{B}$ in $\mathrm{End}(E)$ we obtain $\mathcal{B}' = \mathbb{C}\cdot\mathrm{Id}_E$. This is a variant of Schur's lemma.

(ii) The Banach algebra $\mathcal{B}$ is two-fold transitive on E, i.e., for any linear independent $v, w \in E$ and any $a, b \in E$ there exists an element $A \in \mathcal{B}$ such that $Av = a$ and $Aw = b$. Moreover $\mathcal{B}$ is even strictly dense, i.e., for $n \geq 1$, $v_1, \ldots, v_n \in E$ linearly independent, and any $a_1, \ldots, a_n \in E$ there exists an operator $A \in \mathcal{B}$ such that $Av_j = a_j$ for $1 \leq j \leq n$. This is the Jacobson density theorem.

(iii) *If I is an ideal of $\mathcal{B}$ such that $I^2 = 0$, then $I = 0$.*

(iv) *If $P \in \mathcal{B}$ is a minimal idempotent (i.e., $P^2 = P$ and $P\mathcal{B}P = \mathbb{C}P$), then $\mathcal{B}P$ is a minimal left ideal in $\mathcal{B}$.*

(v) *An idempotent $P \in \mathcal{B}$ is minimal if and only if P is a projection of rank one.*

Part *(ii)* of Proposition 3.18 gives a first impression of the importance of minimal hermitian idempotents in Banach $*$-algebras.

Proposition 3.18. *Let $\mathcal{A}$ be Banach $*$-algebra with bounded approximate identity and enveloping C^*-algebra $C^*(\mathcal{A})$.*

(i) *Let π be an irreducible, faithful $*$-representation of $\mathcal{A}$ in a Hilbert space $\mathfrak{H}$ and $p \in \mathcal{A}$ such that $p^2 = p = p^*$. Then p is a minimal hermitian idempotent in $\mathcal{A}$ if and only if $\pi(p)$ is a one-dimensional orthogonal projection.*

(ii) *Assume that there exist minimal hermitian idempotents in $\mathcal{A}$. If π and ρ are faithful, irreducible $*$-representations of $\mathcal{A}$, then π and ρ are unitarily equivalent.*

Proof. First we prove *(i)*. Let $p \in \mathcal{A}$ such that $\pi(p)$ is a one-dimensional, orthogonal projection. Obviously

$$\pi(\mathbb{C}p) = \mathbb{C}\pi(p) = \pi(p)\,\pi(\mathcal{A})\,\pi(p) = \pi(p\mathcal{A}p)$$

and thus $p\mathcal{A}p = \mathbb{C}p$ because π is faithful. For the converse let $p \in \mathcal{A}$ be a minimal hermitian idempotent so that $p\mathcal{A}p = \mathbb{C}p$. Then it follows $pC^*(\mathcal{A})p = \mathbb{C}p$. Furthermore

$$\pi(p)\,\pi(C^*(\mathcal{A}))\,\pi(p) = \mathbb{C}\pi(p)$$

so that $P = \pi(p)$ is a minimal idempotent in the Banach algebra $\mathcal{B} = \pi(C^*(\mathcal{A}))$ which is strictly irreducible on $\mathfrak{H}$ by Kadison's theorem, see p. 253 of [31]. Now Proposition 3.17 implies that $\pi(p)$ is a one-dimensional projection. Finally we come to the proof of *(ii)*. Let $p \in \mathcal{A}$ be a minimal hermitian idempotent. Let π and ρ be faithful, irreducible $*$-representations of $\mathcal{A}$ in Hilbert spaces $\mathfrak{H}_\pi$ and $\mathfrak{H}_\rho$. Since $\pi(p)$ and $\rho(p)$ are one-dimensional, orthogonal projections by part *(i)*, there exist unit vectors $\xi \in \mathfrak{H}_\pi$ and $\eta \in \mathfrak{H}_\rho$ such that $\pi(p) = \langle -, \xi\rangle\, \xi$ and $\rho(p) = \langle -, \eta\rangle\, \eta$. Now let us consider the positive linear functionals $f_\pi, f_\rho : \mathcal{A} \longrightarrow \mathbb{C}$ given by

$$f_\pi(a) = \langle \pi(a)\xi, \xi\rangle \quad \text{and} \quad f_\rho(a) = \langle \rho(a)\eta, \eta\rangle\ .$$

Then $f_\pi(p) = 1 = f_\rho(p)$ so that $f_\pi = f_\rho$ on $p\mathcal{A}p$. Furthermore

$$f_\pi(a) = f_\pi(pap) = f_\rho(pap) = f_\rho(a)$$

which in particular implies

$$|\pi(a)\xi|^2 = f_\pi(a^*a) = f_\rho(a^*a) = |\rho(a)\eta|^2$$

for all $a \in \mathcal{A}$. This equation shows that there is a well-defined, linear map U from the dense subspace $\pi(\mathcal{A})\xi \subset \mathfrak{H}_\pi$ onto the dense subspace $\rho(\mathcal{A})\eta \subset \mathfrak{H}_\rho$ given by

$$U(\pi(a)\xi) = \rho(a)\eta\ .$$

Obviously U extends to a unitary isomorphism from $\mathfrak{H}_\pi$ onto $\mathfrak{H}_\rho$ which intertwines π and ρ. $\square$

Remark 3.19. Let G be an exponential solvable Lie group and $\pi, \rho \in \widehat{G}$ such that $I = \ker_{L^1(G)} \pi = \ker_{L^1(G)} \rho$. In [27] Poguntke proved the momentous result that there exists a $p \in L^1(G)$ such that $\pi(p)$ is a one-dimensional, orthogonal projection. Since the canonical image of p in $L^1(G)/I$ is a minimal hermitian idempotent, it follows that $\rho(p)$ is a one-dimensional, orthogonal projection, too. Now Proposition 3.18 implies that π and ρ are unitarily equivalent. In particular G is a type I group. Furthermore we see that the natural map $\Psi : \operatorname{Prim} C^*(G) \longrightarrow \operatorname{Prim}_* L^1(G)$, $\Psi(P) = P \cap L^1(G)$ is injective, which is necessary for G to be primitive $*$-regular by Lemma 2.3.

Lemma 3.20. *Let $\mathcal{A}$ be a (complex) Banach algebra and $p \in \mathcal{A}$ such that $p^2 = p$. If E is a simple $\mathcal{A}$-module such that $pE \neq 0$, then pE is a simple $p\mathcal{A}p$-module and*

$$\operatorname{Ann}_{p\mathcal{A}p}(pE) = p\mathcal{A}p \cap \operatorname{Ann}_{\mathcal{A}}(E) \ .$$

There is a canonical bijection between the set of isomorphism classes of simple $p\mathcal{A}p$-modules and the set of isomorphism classes of simple $\mathcal{A}$-modules E such that $pE \neq 0$. Further, if E and F are simple $\mathcal{A}$-modules such that $pE \neq 0$ and $pF \neq 0$, then $\operatorname{Ann}_{\mathcal{A}}(E) \subset \operatorname{Ann}_{\mathcal{A}}(F)$ implies $\operatorname{Ann}_{p\mathcal{A}p}(pE) \subset \operatorname{Ann}_{p\mathcal{A}p}(pF)$.

Proof. Clearly the non-trivial subspace pE is $p\mathcal{A}p$-invariant. For non-zero $\xi \in pE$ we have $p\mathcal{A}p\xi = p\mathcal{A}\xi = pE$ and thus pE is a simple $p\mathcal{A}p$-module. The existence of the asserted canonical bijection is proved in Theorem 1 of [28] and the statements about the annihilators are obvious. □

Theorem 3.21. *Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Let π, ρ be in $\widehat{G}$ such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$ and $\ker_{C^*(N)} \pi = \ker_{C^*(N)} \rho$. Then π and ρ are unitarily equivalent.*

Proof. In this proof we adopt the notation of the article [28] of Poguntke. We will see that this theorem is an immediate consequence of the results of [28]. Let E and F denote the simple $L^1(G)$-modules associated to the representations π and ρ in the sense of Proposition 3.16 so that

$$\operatorname{Ann}_{L^1(G)}(E) = \ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho = \operatorname{Ann}_{L^1(G)}(F) \ .$$

By Theorem 7 of [28] we know that there exists a unique G-orbit $G \cdot \tau$ in $\widehat{N}$ such that

$$\operatorname{Ann}_{L^1(N)} E = \ker_{L^1(N)} \pi = k(G \cdot \tau) = \ker_{L^1(N)} \rho = \operatorname{Ann}_{L^1(N)} F \ .$$

Following the considerations of Section 5 of [28] we choose a normal subgroup H of G such that $N \subset H$ and such that H/N is a vector space complement to K/N in G/N where K denotes the stabilizer of $\tau \in \widehat{N}$. Let $\gamma = \operatorname{ind}_N^H \tau$. One verifies easily that the quotient $L^1(G)/(\ker_{L^1(H)} \gamma * L^1(G))^{-}$ is isomorphic to the Leptin algebra

$$\mathcal{B} = L^1(G/H, L^1(H)/\ker_{L^1(H)} \gamma, T, P)$$

with induced G/H-action and multiplier. The definition of $\mathcal{B}$ depends on the choice of a cross section for the quotient map $G \twoheadrightarrow G/H$. Theorem 3.5 implies

$$\ker_{L^1(H)} \gamma = \big(k(G \cdot \tau) * L^1(H) \big)^{-}$$

so that both E and F can be regarded as $\mathcal{B}$-modules. Let us fix an element q in $L^1(H)/\ker_{L^1(H)} \gamma$ such that $\gamma(q)$ is a rank one projection, for a proof of its existence see [27]. One can prove that $E' = qE \neq 0$ and $F' = qF \neq 0$ are simple $q*\mathcal{B}*q$-modules and that the algebra $q*\mathcal{B}*q$ is isomorphic to the weighted twisted convolution algebra $\mathcal{J} = L^1(W, m, w)$ on the vector group $G/H = W = X \oplus Z$ where Z denotes the kernel of the bicharacter m in the sense of Section 2 of [28]. Clearly the inclusion of the annihilators is preserved, i.e.,

$$\operatorname{Ann}_{L^1(G)}(E) \subset \operatorname{Ann}_{L^1(G)}(F) \quad \text{implies} \quad \operatorname{Ann}_{\mathcal{J}}(E') \subset \operatorname{Ann}_{\mathcal{J}}(F'),$$

compare Proposition 3.20. Using the fact that $L^1(X, m, w)$ acts from both sides on $\mathcal{J}$ and that it contains a minimal hermitian idempotent p such that $(p \star \mathcal{J})E' \neq 0$, we restrict to the subalgebra $p \star \mathcal{J} \star p$ and obtain simple $p \star \mathcal{J} \star p$-modules pE' and pF' such that

$$\operatorname{Ann}_{p\star\mathcal{J}\star p}(pE') \subset \operatorname{Ann}_{p\star\mathcal{J}\star p}(pF').$$

But since $p \star \mathcal{J} \star p$ emerges as to be isomorphic to the complex commutative Banach algebra $L^1(Z, w_0)$, these two simple modules are one-dimensional, the inclusion of their annihilators is an equality, and they are isomorphic. By means of the bijection between $p\mathcal{A}p$-modules and certain $\mathcal{A}$-modules (Lemma 3.20), we conclude that E' and F', and also E and F are isomorphic which in particular implies

$$\ker_{L^1(G)} \pi = \operatorname{Ann}_{L^1(G)}(E) = \operatorname{Ann}_{L^1(G)}(F) = \ker_{L^1(G)} \rho .$$

Finally Remark 3.19 shows that π and ρ are unitarily equivalent so that the proof is complete. □

Remark 3.22. Here we give a sufficient criterion for the orbit $G \cdot \tau$ to be closed in $\widehat{N}$. Let G, $\mathfrak{g}$, f, $\mathfrak{n}$ be as usual and $f' = f \,|\, \mathfrak{n}$. Let $\pi = \mathcal{K}(f)$ be in $\widehat{G}$ and $\tau = \mathcal{K}(f')$ in $\widehat{N}$. Further let us suppose $\mathfrak{g} = \mathfrak{g}_{f'} + \mathfrak{n}$. Theorem 3.1.4 of [5] implies that $\operatorname{Ad}^*(G)f' = \operatorname{Ad}^*(N)f'$ is closed in $\mathfrak{n}^*$ because N acts unipotently on $\mathfrak{n}^*$. Since the Kirillov map $\mathcal{K} : \mathfrak{n}^*/\operatorname{Ad}^*(N) \longrightarrow \widehat{N}$ is a homeomorphism, it follows that $G \cdot \tau = \mathcal{K}(\operatorname{Ad}^*(G)f')$ is closed in $\widehat{N}$. On the other hand, it is well known that $\pi \,|\, N$ is weakly equivalent to the orbit $G \cdot \tau$ so that $\ker_{C^*(N)} \pi = k(G \cdot \tau)$.

Theorem 3.23. *Let G be an exponential solvable Lie group and let N be a connected nilpotent subgroup of G such that $N \supset (G, G)$. Let π be in $\widehat{G}$. There exists a unique G-orbit $G \cdot \tau$ in $\widehat{N}$ such that*

$$k(G \cdot \tau) = \ker_{C^*(N)} \pi .$$

If $G \cdot \tau$ is closed in $\widehat{N}$, then $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined.

Proof. The existence and uniqueness of $G \cdot \tau$ is well-known. Let $\rho \in \widehat{G}$ be such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$. Restricting to the normal subgroup N we obtain $\ker_{L^1(N)} \pi \subset \ker_{L^1(N)} \rho$. Since N is $*$-regular as a nilpotent group, it follows that

$$k(G \cdot \tau) = \ker_{C^*(N)} \pi \subset \ker_{C^*(N)} \rho .$$

Our assumption of $G \cdot \tau$ being closed in $\widehat{N}$ implies $\ker_{C^*(N)} \pi = \ker_{C^*(N)} \rho$. Now Theorem 3.21 shows that π and ρ are unitarily equivalent so that in particular $\ker_{C^*(G)} \pi = \ker_{C^*(G)} \rho$. This finishes our proof. □

3.3 The ideal theory of $*$-regular exponential Lie groups

The results of this subsection are not new. They can be found in Boidol's paper [3], and in a more general context in [4]. For the convenience of the reader we give a short proof for the if-part of Theorem 5.4 of [3]. The following definition has been adapted from the introduction of [4].

Definition 3.24. Let G be an exponential Lie group. If A is a closed normal subgroup of G, then T_A denotes the quotient map from $C^*(G)$ onto $C^*(G/A)$. We say that a closed ideal I of $C^*(G)$ is essentially induced from a nilpotent normal subgroup if there exist closed normal subgroups A and M of G such that $A \subset M$ and such that the following conditions are satisfied:

(i) $\ker_{C^*(G)} T_A \subset I$,

(ii) the group M/A is nilpotent (so that its Haar measure has polynomial growth),

(iii) the ideal I is induced from M in the sense of Definition 3.11.

It follows from Remark 3.14 and Remark 2.4 that ideals I of $C^*(G)$ which are essentially induced from a nilpotent normal subgroup are $L^1(G)$-determined.

Definition 3.25. Let $\mathfrak{g}$ be an exponential Lie algebra and $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ its commutator ideal. We say that $\mathfrak{g}$ satisfies condition (R) if the following is true: If $f \in \mathfrak{g}^*$ is arbitrary and $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is its stabilizer, then $f = 0$ on $\mathfrak{m}^\infty$. Here $\mathfrak{m}^\infty = \bigcap_{k=1}^{\infty} C^k\mathfrak{m}$ denotes the smallest ideal of $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{m}^\infty$ is nilpotent.

Note that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ depends only on the orbit $\mathrm{Ad}^*(G)f$. The following observation is extremely useful: Let $f \in \mathfrak{g}^*$ and $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ be its stabilizer such that $\mathfrak{m}/\mathfrak{m}^\infty$ is nilpotent. If $\gamma_1, \ldots, \gamma_r$ are the roots of $\mathfrak{g}$, then we define the ideal $\tilde{\mathfrak{m}} = \bigcap_{i \in S} \ker \gamma_i$ of $\mathfrak{g}$ where $S = \{i : \ker \gamma_i \supset \mathfrak{m}\}$. It is easy to see that $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ and that $\tilde{\mathfrak{m}}/\mathfrak{m}^\infty$ is nilpotent, too. Further there are only finitely many ideals $\tilde{\mathfrak{m}}$ of this kind.

Theorem 3.26. *Let G be an exponential Lie group such that its Lie algebra $\mathfrak{g}$ satisfies condition (R). Then any ideal I of $C^*(G)$ is a finite intersection of ideals which are essentially induced from a nilpotent normal subgroup. In particular G is $*$-regular.*

Proof. Since $I = k(h(I))$ by Theorem 2.9.7 of [8], there is a closed, $\mathrm{Ad}^*(G)$-invariant subset $X \subset \mathfrak{g}^*$ such that $I = \bigcap_{f \in X} \ker_{C^*(G)} \mathcal{K}(f)$. There exists a decomposition $X = \bigcup_{k=1}^r X_k$ of X and ideals $\{\tilde{\mathfrak{m}}_k : 1 \le k \le r\}$ as above such that $\mathfrak{g}_f + \mathfrak{n} \subset \tilde{\mathfrak{m}}_k$ for all $f \in X_k$. Induction in stages and Proposition 3.12 imply that $\ker_{C^*(G)} \mathcal{K}(f)$ is induced from $\tilde{M}_k$ for all $f \in X_k$. Now it follows from Lemma 3.15 that

$$I_k = \bigcap_{f \in X_k} \ker_{C^*(G)} \mathcal{K}(f)$$

is induced from $\tilde{M}_k$, too. This means that I_k is essentially induced from a nilpotent normal subgroup because $f = 0$ on $\tilde{\mathfrak{m}}_k^\infty$ by condition (R) and $\tilde{M}_k/\tilde{M}_k^\infty$ is nilpotent. Finally Lemma 2.5 implies that the ideal $I = \bigcap_{k=1}^{r} I_k$ is $L^1(G)$-determined. $\square$

3.4 A strategy for proving the primitive $*$-regularity of exponential solvable Lie groups

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. In order to prove that G is primitive $*$-regular, one must show that $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined for all $\pi \in \widehat{G}$, i.e., one must prove that

$$\ker_{C^*(G)} \pi \not\subset \ker_{C^*(G)} \rho \quad \text{implies} \quad \ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$$

for all $\rho \in \widehat{G}$. Let $f, g \in \mathfrak{g}^*$ such that $\pi = \mathcal{K}(f)$ and $\rho = \mathcal{K}(g)$. Since the Kirillov map of G is a homeomorphism, the relation for the C^*-kernels is equivalent to

$$\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-} .$$

From the preceding subsections we extract the following observations:

1. Let $\mathfrak{a}$ be a non-trivial ideal of $\mathfrak{g}$ such that $f = 0$ on $\mathfrak{a}$. Let A be the connected subgroup of G with Lie algebra $\mathfrak{a}$. Since $\pi = 1$ on A, we can pass over to a representation $\dot{\pi}$ of the quotient $\dot{G} = G/A$. It follows from Remark 2.8 that $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined if and only if $\ker_{C^*(\dot{G})} \dot{\pi}$ is $L^1(\dot{G})$-determined. Often $\dot{G}$ is known to be primitive $*$-regular by induction.

2. If the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is nilpotent, then $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined by Theorem 3.13.

3. If $\mathfrak{g} = \mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$, then $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined by Remark 3.22 and Theorem 3.23.

4. If $\mathrm{Ad}^*(G)g'$ is not contained in the closure of $\mathrm{Ad}^*(G)f'$, then it follows $\ker_{C^*(N)} \pi \not\subset \ker_{C^*(N)} \rho$ because the Kirillov map is an homeomorphism. Since N is $*$-regular, we obtain $\ker_{L^1(N)} \pi \not\subset \ker_{L^1(N)} \rho$ and hence $\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$.

Lemma 3.27. *Assume that there exists a one-codimensional nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$. Then G is primitive $*$-regular.*

Proof. Let $f \in \mathfrak{g}^*$ be arbitrary. The assumption $\dim \mathfrak{g}/\mathfrak{n} = 1$ implies that either $\mathfrak{n} = \mathfrak{m}$ is nilpotent or $\mathfrak{g} = \mathfrak{m}$. Clearly the preceding remarks show that $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined. □

These observations suggest the following definitions.

Definition 3.28. A linear functional $f \in \mathfrak{g}^*$ is said to be in general position if $f \neq 0$ on any non-trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$.

As usual let f' and g' denote the restrictions to $\mathfrak{n}$.

Definition 3.29. Let $f \in \mathfrak{g}^*$ be in general position. Then $g \in \mathfrak{g}^*$ is called critical for the orbit $\mathrm{Ad}^*(G)f$ if and only if the following conditions are satisfied:

$$\textit{(i)}\ \mathrm{Ad}^*(G)g \not\subset (\,\mathrm{Ad}^*(G)f\,)^{-}$$

$$\textit{(ii)}\ \mathrm{Ad}^*(G)g' \subset (\,\mathrm{Ad}^*(G)f'\,)^{-}$$

$$\textit{(iii)}\ \mathrm{Ad}^*(G)g' \neq \mathrm{Ad}^*(G)f'$$

Remark 3.30. From these considerations we conclude that in order to prove the primitive $*$-regularity of G it suffices to verify the following two assertions:

1. Any proper quotient $\dot{G}$ of G is primitive $*$-regular.

2. If $f \in \mathfrak{g}^*$ is in general position such that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$ and if $g \in \mathfrak{g}^*$ is critical for the orbit $\mathrm{Ad}^*(G)f$, then it follows
$$\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho\,.$$

Since $\ker \pi$ is induced from M as an ideal of $C^*(G)$ and $L^1(G)$ by Proposition 3.12, it follows that the inclusion $\ker \pi \subset \ker \rho$ in $C^*(G)$ or $L^1(G)$ is equivalent to the respective inclusion in $C^*(M)$ or $L^1(M)$.

Let $\tilde{f}$ denote the restriction of f to $\mathfrak{m}$. Note that $\tilde{f}$ is in general position in the following sense: If $\mathfrak{a} \subset \mathfrak{m}$ is a non-trivial ideal of $\mathfrak{g}$, then $f(\mathfrak{a}) \neq 0$. Furthermore we have $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n} = \mathfrak{m}_{\tilde{f}} + \mathfrak{n}$. In analogy to Definition 3.29 we say that $\tilde{g} \in \mathfrak{m}^*$ is critical w. r. t. the orbit $\mathrm{Ad}^*(G)\tilde{f}$ if $\mathrm{Ad}^*(G)\tilde{g}$ is not contained in the closure of $\mathrm{Ad}^*(G)\tilde{f}$ and if conditions *(ii)* and *(iii)* of Definition 3.29 are satisfied for $f' = \tilde{f}\,|\,\mathfrak{n}$ and $g' = \tilde{g}\,|\,\mathfrak{n}$. Since
$$\mathrm{Ad}^*(G)f = \mathrm{Ad}^*(G)f + \mathfrak{m}^\perp$$
by Remark 3.3, it follows that g is critical w. r. t. $\mathrm{Ad}^*(G)f$ if and only if $\tilde{g} = g\,|\,\mathfrak{m}$ is critical w. r. t. $\mathrm{Ad}^*(G)\tilde{f}$.

Let $d_1, \ldots, d_m$ be in $\mathfrak{g}$ such that their canonical images form a basis of $\mathfrak{g}/\mathfrak{m}$. Composing the smooth map
$$E(s) = \exp(s_1 d_1) \cdot \ldots \cdot \exp(s_m d_m)$$
with the quotient map $G \twoheadrightarrow G/M$, we obtain a diffeomorphism from $\mathbb{R}^m$ onto G/M. Further let $\tilde{f}_s = \mathrm{Ad}^*(E(s))\tilde{f}$ be in $\mathfrak{m}^*$ and $\tilde{\pi}_s = \mathcal{K}(\tilde{f}_s)$ in $\widehat{M}$. It is well-known that $\pi\,|\,M$ is weakly equivalent to the set $\{\tilde{\pi}_s : s \in \mathbb{R}^m\}$. Now it is easy to see that we can replace the second assertion by the equivalent condition

3. Let $\mathfrak{m}$ be a proper, non-nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{m} \supset \mathfrak{n}$. If $\tilde{f} \in \mathfrak{m}^*$ is in general position such that $\mathfrak{m} = \mathfrak{m}_{\tilde{f}} + \mathfrak{n}$ and if $\tilde{g} \in \mathfrak{m}^*$ is critical for the orbit $\mathrm{Ad}^*(G)\tilde{f}$, then the relation
$$\bigcap_{s \in \mathbb{R}^m} \ker_{L^1(M)} \tilde{\pi}_s \not\subset \ker_{L^1(M)} \tilde{\rho} \tag{3.31}$$
holds for the representations $\tilde{\pi}_s = \mathcal{K}(\tilde{f}_s)$ and $\tilde{\rho} = \mathcal{K}(\tilde{g})$.

In the rest of this paper we will carry out the following plan: In Sections 5 and 7 we will develop tools which are helpful for proving Relation 3.31 in various situations. For $\mathfrak{n}$ running through all nilpotent Lie algebras of dimension ≤ 5, we will verify the preceding condition for all possible coabelian extensions $\mathfrak{g}$ of $\mathfrak{n}$ in Sections 9 to 14, i.e., we will prove Relation 3.31 for all f in general position such that $\mathfrak{m} = \mathfrak{m}_{\tilde{f}} + \mathfrak{n}$ and all critical $\tilde{g}$. Finally we will see in Section 15 that these results suffice to prove the primitive $*$-regularity of all exponential solvable Lie groups of dimension ≤ 7.

Remark 3.32. Let $f \in \mathfrak{m}^*$ such that the stabilizer condition $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ holds. Then the ideal $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$ is contained in $\ker f$. If f is in general position, it follows $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = 0$ so that $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

4 Nilpotent Lie algebras

The classification of nilpotent Lie algebras (over the real field) is well-known in low dimensions. These results can be found e.g. in [6] for algebras of dimension ≤ 5, and in [24] for algebras of dimension ≤ 6. For the convenience of the reader, we provide a list of all non-commutative, nilpotent Lie algebras up to dimension 5.

For each algebra we write down the ideals $C^k\mathfrak{n}$ of the descending central series, which are inductively defined by

$$C^{k+1}\mathfrak{n} = [\mathfrak{n}, C^k\mathfrak{n}] \ ,$$

and their dimensions. If there are further characteristic ideals, then we mention their dimensions and commutator relations as well. The notion

$$\mathfrak{a} \underset{j}{\subset} \mathfrak{b}$$

indicates that the codimension (the dimension of the quotient $\mathfrak{b}/\mathfrak{a}$) equals j. Finally, we note the Lie brackets of a suitably chosen basis of $\mathfrak{n}$.

In this section, $f' \in \mathfrak{n}^*$ denotes an arbitrary linear functional in general position, i.e., $f' \neq 0$ on any non-zero characteristic ideal of $\mathfrak{n}$.

1. **3-dimensional Heisenberg algebra $\mathfrak{g}_{3,1}$**
   The descending central series of this 2-step nilpotent Lie algebra is given by
   $$\mathfrak{n} \underset{2}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \{0\} \ ,$$
   where $\mathfrak{z}\mathfrak{n} = C^1\mathfrak{n}$. There exists a basis $e_1, ..., e_3$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$. It holds $\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$.

2. **$\mathbb{R} \times$ 3-dimensional Heisenberg algebra**
   This algebra is 2-step nilpotent. It contains the following characteristic ideals:
   $$\mathfrak{n} \underset{2}{\supset} \mathfrak{z}\mathfrak{n} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \{0\} \ .$$
   There exists a basis $e_1, ..., e_4$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$. We have $\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$.

3. **4-dimensional filiform algebra $\mathfrak{g}_{4,3}$**
   A descending series of characteristic ideals of this 3-step nilpotent Lie algebra is given by
   $$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} \{0\} \ ,$$
   where $\mathfrak{c}$ is a commutative ideal, namely the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$. It holds $\mathfrak{z}\mathfrak{n} = C^2\mathfrak{n}$. There is a basis $e_1, ..., e_4$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_4$.

4. **5-dimensional Heisenberg algebra $\mathfrak{g}_{5,1}$**
The descending central series of this 2-step nilpotent Lie algebra is given by
$$\mathfrak{n} \underset{4}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \{0\} \; .$$
Its center is $\mathfrak{z}\mathfrak{n} = C^1\mathfrak{n}$. There is a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_5$ and $[e_3, e_4] = e_5$. It holds $\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$.

5. **$\mathbb{R}^2 \times$ 3-dimensional Heisenberg algebra**
This 2-step nilpotent algebra contains the following series of characteristic ideals:
$$\mathfrak{n} \underset{2}{\supset} \mathfrak{z}\mathfrak{n} \underset{2}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \{0\} \; .$$
There is a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$. We have $\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$.

6. **the algebra $\mathfrak{g}_{5,2}$**
The central series of this 5-dimensional, 2-step nilpotent algebra is given by
$$\mathfrak{n} \underset{3}{\supset} C^1\mathfrak{n} \underset{2}{\supset} \{0\} \; .$$
It holds $\mathfrak{z}\mathfrak{n} = C^1\mathfrak{n}$. There is a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_4$ and $[e_1, e_3] = e_5$.

7. **the algebra $\mathfrak{g}_{5,3}$**
In this 3-step nilpotent Lie algebra, we find the characteristic ideals
$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} \mathfrak{b} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} \{0\} \; ,$$
where $\mathfrak{c}$ is the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$ satisfying $[\mathfrak{c}, \mathfrak{c}] = C^2\mathfrak{n}$. In particular, $C^1\mathfrak{n}$ is commutative. Further $\mathfrak{b}$ is the preimage of $\mathfrak{z}(\mathfrak{n}/C^2\mathfrak{n})$ under the quotient map and $\mathfrak{z}\mathfrak{n} = C^2\mathfrak{n}$. There is a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, and $[e_2, e_3] = e_5$. We have $\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$.

8. **$\mathbb{R} \times$ 4-dimensional filiform algebra**
This Lie algebra is 3-step nilpotent. A series of characteristic ideals is given by
$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} \mathfrak{b} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} \{0\} \; .$$
Here $\mathfrak{c}$ is the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$, and $\mathfrak{b} = C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. The ideal $\mathfrak{c}$ is commutative, the center $\mathfrak{z}\mathfrak{n}$ is 2-dimensional. There exists a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_4$.

9. **the algebra $\mathfrak{g}_{5,4}$**
The descending central series of this 5-dimensional, 3-step nilpotent algebra is
$$\mathfrak{n} \underset{2}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{2}{\supset} \{0\} \; ,$$
where $\mathfrak{z}\mathfrak{n} = C^2\mathfrak{n}$. Further $C^1\mathfrak{n}$ is commutative and equal to its centralizer in $\mathfrak{n}$. There is a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, and $[e_2, e_3] = e_5$.

10. 5-dimensional filiform algebra $\mathfrak{g}_{5,5}$

In this 4-step nilpotent Lie algebra we find the series of characteristic ideals

$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} C^3\mathfrak{n} \underset{1}{\supset} \{0\} \ ,$$

where $\mathfrak{c}$, the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$, is commutative and $\mathfrak{z}\mathfrak{n} = C^3\mathfrak{n}$. Hence there exists a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, and $[e_1, e_4] = e_5$.

11. the algebra $\mathfrak{g}_{5,6}$

This 4-step nilpotent algebra contains the following series of characteristic ideals:

$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} C^3\mathfrak{n} \underset{1}{\supset} \{0\} \ .$$

Here $\mathfrak{c}$ is the centralizer of $C^2\mathfrak{n}$ in $\mathfrak{n}$ satisfying $[\mathfrak{c}, \mathfrak{c}] = C^3\mathfrak{n}$. In particular $C^1\mathfrak{n}$ is commutative. Further $\mathfrak{z}\mathfrak{n} = C^3\mathfrak{n}$. There exists a basis $e_1, ..., e_5$ of $\mathfrak{n}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, and $[e_2, e_3] = e_5$.

5 Functional calculus for central elements

The purpose of this section is to present Theorem 5.1 and Lemma 5.4, which can be regarded as a piece of information about the Jacobson topology of $\mathrm{Prim}_* L^1(M)$. More exactly, this theorem gives us a sufficient criterion, which is easy to check, for a point $\ker_{L^1(M)} \rho$ not to belong to the closure of the set $\{\ker_{L^1(M)} \pi_s : s \in S\}$. Eventually we are interested in the case where this set is the orbit of a group acting on M and hence on $\mathrm{Prim}_* L^1(M)$.

We anticipate that the crucial step in the proof of Theorem 5.1 is to establish a functional calculus for elements in the center of the Lie algebra of M, considered as differential operators in $L^1(M)$. Another interpretation of this procedure is that of solving the multiplier theorem given by Equation 5.3.

The technical part of the proof of Theorem 5.1 has been extracted from the first subsection. For the convenience of the reader, we provide the details of the proof in Subsection 5.2.

5.1 The main theorem and its corollaries

Let M be an exponential solvable Lie group and let Z be a closed, connected, central subgroup of M. Denote by $\mathfrak{m}$ and $\mathfrak{z}$ the Lie algebras of M and Z respectively. Let l be the dimension of Z, and k the dimension of M/Z.

We fix a coexponential basis $\mathcal{B} = \{b_1, \ldots, b_k\}$ for $\mathfrak{z}$ in $\mathfrak{m}$. For example, we can choose vectors $b_1, \ldots, b_k$ in $\mathfrak{m}$ whose canonical images in $\mathfrak{m}/\mathfrak{z}$ form a Malcev basis of $\mathfrak{m}/\mathfrak{z}$. We define a smooth map $\Phi_1 : \mathbb{R}^k \longrightarrow M$ by

$$\Phi_1(x) = \exp(x_1 b_1) \cdot \ldots \cdot \exp(x_k b_k) \ .$$

Let $q : M \longrightarrow M/Z$ be the quotient map. By definition $q \circ \Phi_1$ is a diffeomorphism from $\mathbb{R}^k$ onto M/Z. Equivalently, the map

$$\Phi : \mathbb{R}^k \times \mathfrak{z} \longrightarrow M,\ \Phi(x,z) = \Phi_1(x) \exp(z)$$

is a global diffeomorphism. This is a canonical coordinate system of the second kind.

Since the modular function $\Delta_{M,Z}$ is trivial, it follows

$$\int\limits_M f(m)\, dm = \int\limits_{\mathbb{R}^k} \int\limits_{\mathfrak{z}} f(\Phi(x,z))\, dz dx$$

for all $f \in \mathcal{C}_0(M)$. Here dm denotes the Haar measure of M, and dx and dz denote the Lebesgue measures on $\mathbb{R}^k$ and $\mathfrak{z}$ respectively.

Let π be a unitary representation of M such that its restriction to the central subgroup Z is character, i.e., there exists a $\zeta \in \hat{Z}$ such that $\pi(z) = \zeta(z) \cdot \mathrm{Id}$. Any character $\zeta \in \hat{Z}$ corresponds to a linear functional $\zeta^\infty \in \mathfrak{z}^*$ by

$$\zeta(\exp z) = e^{i\langle \zeta^\infty, z\rangle} \ .$$

In particular let π be irreducible. In this case it follows from Schur's lemma that $\pi|Z = \zeta\cdot\mathrm{Id}$.

We have $\pi = \mathcal{K}(u)$ for some $u \in \mathfrak{m}^*$, because the Kirillov map $\mathcal{K}$ yields a bijection from $\mathfrak{m}^*/\operatorname{Ad}^*(M)$ onto $\widehat{M}$. According to the definition of $\mathcal{K}$ we choose a Pukanszky polarization $\mathfrak{p}$ at u and obtain

$$\pi = \operatorname{ind}_P^M \chi_u \, .$$

Since $\mathfrak{z}$ is contained in $\mathfrak{p}$, we get $\zeta = \chi_u|Z$ and $\zeta^\infty = u|\mathfrak{z}$.

By abuse of notation, for any function f on M we denote the function $f \circ \Phi$ on $\mathbb{R}^k \times \mathfrak{z}$ again by f.

Now let $f \in L^1(M)$. We introduce the partial Fourier transformation with respect to the variable z. The Fubini theorem implies that $z \mapsto f(x,z)$ is in $L^1(\mathbb{R}^l)$ for almost all $x \in \mathbb{R}^k$. For these x and all $\xi \in \mathfrak{z}^*$ we define

$$\widehat{f}(x,\xi) = \int_{\mathfrak{z}} f(x,z) e^{-i\langle \xi, z\rangle} \, dz \, .$$

It is easy to see that for fixed $\xi \in \mathfrak{z}^*$ the function $x \mapsto \widehat{f}(x,\xi)$ is in $L^1(\mathbb{R}^k)$. We obtain

$$\begin{aligned} \pi(f)\varphi &= \int_M f(m)\,\pi(m)\varphi \, dm = \int_{\mathbb{R}^k}\int_{\mathfrak{z}} f(x,z)\,\pi(\Phi_1(x))\zeta(\exp z)\varphi \, dz \, dx \\ &= \int_{\mathbb{R}^k} \widehat{f}(x,-\zeta^\infty)\,\pi(\Phi_1(x))\varphi \, dx \end{aligned}$$

for every element φ in the representation space of π.

We recall that any unitary representation π of M in a Hilbert space $\mathfrak{H}_\pi$ gives rise to an infinitesimal representation π^∞ (sometimes also denoted by $d\pi$) of $\mathfrak{m}$ on the subspace of $\mathcal{C}^\infty$-vectors $\mathfrak{H}_\pi^\infty$ by

$$\pi^\infty(X)\varphi = \frac{d}{dt}_{|t=0} \pi(\exp tX)\varphi \, .$$

This representation can be extended to the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ of the complexification of $\mathfrak{m}$.

After these preparations we can now prove the following basic theorem.

Theorem 5.1. *Let M be an exponential solvable Lie group, and Z a closed, connected, l-dimensional subgroup which is contained in the center of M. Let $\{\pi_s : s \in S\}$ be a family of unitary representations of M such that the restriction of π_s to Z is a character ζ_s, and ρ a unitary representation of M whose restriction to Z equals the character η. Further we assume that $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ is a function whose derivatives up to order $l+1$ exist, are continuous and have polynomial growth. Let $h, f \in L^1(M)$ be smooth functions such that*

$$\pi_s(h) = \psi(\zeta_s^\infty)\pi_s(f)$$

for all $s \in S$ and

$$\rho(h) \neq \psi(\eta^\infty)\rho(f) \, .$$

More exactly, we suppose that $f \in \mathcal{C}_0^\infty(M)$ or that f is a Schwartz function in the coordinates from above. Then the relation

$$\bigcap_{s\in S} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho \tag{5.2}$$

holds for the L^1-kernels of these representations.

Proof. We fix canonical coordinates of the second kind as above and use the notation introduced before. It is a result of Euclidean Fourier analysis that the assumptions on ψ and f imply the existence of a smooth function $g \in L^1(M)$ such that

$$\widehat{g}(x, -\xi) = \psi(\xi)\ \widehat{f}(x, -\xi) \tag{5.3}$$

holds for all $x \in \mathbb{R}^k$ and $\xi \in \mathfrak{z}^*$. We sketch the proof of the existence of g and of its differentiability and integrability properties in the second subsection. Accepting this for the time being, we obtain

$$\rho(g) = \psi(\eta^\infty)\rho(f)$$

and

$$\pi_s(g) = \psi(\zeta_s^\infty)\pi_s(f)$$

for all $s \in S$. Finally, we see that the function $h - g \in L^1(M)$ satisfies $\pi_s(h - g) = 0$ for all s and $\rho(h - g) \neq 0$. This proves our theorem. □

The following lemma tackles the problem of finding smooth functions $h, f \in L^1(M)$ satisfying the conditions of Theorem 5.1.

Lemma 5.4. *Let M, Z, π_s and ρ be given as in the above theorem. Let $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ be a continuous function. Further we assume that $W \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$ is an element in the universal enveloping algebra of $\mathfrak{m}$ such that*

$$\pi_s^\infty(W) = \psi(\zeta_s^\infty)\cdot\mathrm{Id}$$

is a scalar operator for every $s \in S$ and

$$\rho^\infty(W) \neq \psi(\eta^\infty)\cdot\mathrm{Id}$$

on the subspace of ρ-smooth vectors. Then for appropriate $f \in \mathcal{C}_0^\infty(M)$, the functions $h = W * f$ and f satisfy the conditions of Theorem 5.1. Here $W * f$ denotes the action of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ on $\mathcal{C}_0^\infty(M)$ obtained by differentiating the left regular representation of M in $L^2(M)$.

Proof. The Gårding space, i.e., the subspace generated by vectors of the form $\rho(f)\varphi$ with $f \in \mathcal{C}_0^\infty(M)$ and $\varphi \in \mathfrak{H}_\rho$, is dense in $\mathfrak{H}_\rho$. In [10] Dixmier and Malliavin have shown that the Gårding space is equal to $\mathfrak{H}_\rho^\infty$.

The transpose $W \mapsto W^t$ of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ is the unique $\mathbb{C}$-linear anti-automorphism of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ extending the automorphism $A^t = -A$ of $\mathfrak{m}$. The formal transpose of the unbounded operator $\rho^\infty(W)$ is defined on $\mathfrak{H}_\rho^\infty$ and given by $\rho^\infty(W)^t = \rho^\infty(W^t)$.

Now it is easy to see that the assumption $\rho^\infty(W) \neq \psi(\eta^\infty)\cdot\mathrm{Id}$ implies the existence of a function $f \in \mathcal{C}_0^\infty(M)$ such that

$$\rho(W * f) = \rho^\infty(W)\rho(f) \neq \psi(\eta^\infty)\rho(f)\ .$$

Further we have

$$\pi_s(W * f) = \pi_s^\infty(W)\pi_s(f) = \psi(\zeta_s^\infty)\pi_s(f)\ .$$

This finishes the proof of our lemma. □

Remark 5.5. Let M, Z, π_s, ρ, W and ψ be given as in Theorem 5.1 and Lemma 5.4. We briefly discuss the possibility of relaxing the assumptions on ψ. Define

$$\Lambda = \{\eta^\infty\} \cup \{\zeta_s^\infty : s \in S\} \subset \mathfrak{z}^*\ .$$

In order to make the proof of our theorem work we have to find $f, g \in L^1(M)$, not necessarily continuous, such that $\rho^\infty(W)\rho(f) \neq \psi(\eta^\infty)\rho(f)$ and such that 5.3 holds but only for $x \in \mathbb{R}^k$ outside a set N of measure zero and $\xi \in \Lambda$.

We can assume that η^∞ is contained in the closure of $\{\zeta_s^\infty : s \in S\}$ in $\mathfrak{z}^*$. Otherwise Relation 5.2 holds trivially.

The following question arises: Is it possible to find such f and g under weaker assumptions on ψ? At least in the following situation the answer is negative. If ψ has a singularity in η^∞ or if the restriction of ψ to Λ is not continuous in η^∞, then the validity of Equation 5.3 implies $\widehat{f}(x, -\eta^\infty) = 0$ for $x \notin N$, because for these x the functions $\xi \mapsto \widehat{g}(x, \xi)$ and $\xi \mapsto \widehat{f}(x, \xi)$ are continuous and bounded. So we obtain the contradiction $\rho(f) = 0$.

Remark. Often $\rho^\infty(W)$ is also a scalar operator. Nevertheless, W is not necessarily central in the whole algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$.

The following corollary turns out to be useful in our applications. Typically ψ_0 is composed of functions of the form $\xi \mapsto \xi\log(\xi)$ for $\xi > 0$ and $\xi \mapsto 0$ for $\xi \leq 0$, which become differentiable in $\xi = 0$ by taking powers.

Corollary 5.6. *Let M, Z, π_s and ρ be given as usual. Assume that there exists a $W_1 \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$ such that $\rho^\infty(W_1)$ is a non-zero scalar operator, and a continuous function $\psi_1 : \mathfrak{z}^* \longrightarrow \mathbb{C}$ such that $\psi_1(\eta^\infty) = 0$ and*

$$\pi_s^\infty(W_1) = \psi_1(\zeta_s^\infty)\cdot\mathrm{Id}\ .$$

Assume further that for some $j \geq 1$ all derivatives of ψ_1^j up to order $l+1$ exist, are continuous, and have polynomial growth. Then we have

$$\bigcap_{s\in S} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho\ .$$

Proof. We can apply Theorem 5.1 and Lemma 5.4 to $\psi = \psi_1^j$ and $W = W_1^j$. □

Remark. Let $\mathcal{Q}$ denote the space of all smooth functions $f : M \longrightarrow \mathbb{C}$ whose support is contained in a strip $K \cdot Z$ for some compact subset K of M and which are rapidly decreasing in the sense that

$$(x,z) \mapsto (1+|z|)^r \, D_x^\alpha D_z^\beta f(x,z)$$

is bounded for all $\alpha \in \mathbb{N}^k$, $\beta \in \mathbb{N}^l$ and $r \geq 0$. Note that $\mathcal{Q}$ contains $\mathcal{C}_0^\infty(M)$.

The definition of $\mathcal{Q}$ does not depend on the choice of the coexponential basis $\mathcal{B} = \{b_1, \ldots, b_k\}$ for $\mathfrak{z}$ in $\mathfrak{m}$. However, the Schwartz functions $\mathcal{S}(\mathbb{R}^k \times \mathfrak{z})$ do not yield a function space on M independent of the choice of the coexponential basis $\mathcal{B}$.

The Lie algebra $\mathfrak{z}$ acts on $\mathcal{Q}$ from the right by

$$\begin{aligned}(A * f)(x,z) &= \frac{d}{dt}_{|t=0} \, f\,(\,(x,z)\cdot\exp(tA)\,) \\ &= \langle \partial f(x,z), A\rangle\end{aligned} \tag{5.7}$$

for $A \in \mathfrak{z}$ and $f \in \mathcal{Q}$. Here $\partial f : M \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathfrak{z}, \mathbb{C})$ is the derivative of f with respect to the variable z. As usual, we extend this action to the universal enveloping algebra $\mathcal{U}(\mathfrak{z}_{\mathbb{C}})$. From 5.7 we deduce

$$(-iA * f)\widehat{\ }(x,\xi) \;=\; \langle \xi, A\rangle \; \widehat{f}(x,\xi) \tag{5.8}$$

for all $A \in \mathfrak{z}$.

Let $\bar{\mathcal{P}}(\mathfrak{z}^*)$ denote the vector space of all smooth functions $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ such that all derivatives of ψ have polynomial growth. Note that $\bar{\mathcal{P}}(\mathfrak{z}^*)$ is an associative algebra under pointwise multiplication, containing the algebra $\mathcal{P}(\mathfrak{z}^*)$ of all complex-valued polynomial functions, which is generated by the constant 1 and the linear functions $\xi \mapsto \langle \xi, A\rangle$ for $A \in \mathfrak{z}$.

We can define a $\bar{\mathcal{P}}(\mathfrak{z}^*)$-module structure on $\mathcal{Q}$: Lemma 5.20 and Remark 5.23 show that for every $\psi \in \bar{\mathcal{P}}(\mathfrak{z}^*)$ and $f \in \mathcal{Q}$ there exists a unique function $T_\psi f$ in $\mathcal{Q}$ such that

$$(T_\psi \, f)\widehat{\ }(x,\xi) = \psi(\xi) \; \widehat{f}(x,\xi) \; . \tag{5.9}$$

If we regard $T_\psi f$ as a function on M, then its definition does not depend on the choice of $\mathcal{B}$.

In view of Equation 5.8, we observe that the linear map $\mathfrak{z}_{\mathbb{C}} \longrightarrow \mathfrak{z}_{\mathbb{C}}$, $A \mapsto -iA$ induces an isomorphism of associative algebras from $\mathcal{S}(\mathfrak{z}_{\mathbb{C}})$ onto $\mathcal{U}(\mathfrak{z}_{\mathbb{C}})$ because $\mathfrak{z}$ is commutative. Further, there is a natural isomorphism between $\mathcal{S}(\mathfrak{z}_{\mathbb{C}})$ and $\mathcal{P}(\mathfrak{z}^*)$ which is uniquely determined by the property that it assigns the linear function $\xi \mapsto <\xi, A>$ on $\mathfrak{z}^*$ to $A \in \mathfrak{z}_{\mathbb{C}}$. So we obtain an isomorphism between $\mathcal{U}(\mathfrak{z}_{\mathbb{C}})$ and $\mathcal{P}(\mathfrak{z}^*)$.

Via this isomorphism, let $W \in \mathcal{U}(\mathfrak{z}_{\mathbb{C}})$ correspond to its symbol $p \in \mathcal{P}(\mathfrak{z}^*)$. Then it follows from Equation 5.8 and the uniqueness of the solution of 5.9 that $W * f$ in the sense of 5.7 is equal to $T_p \, f$ in the sense of 5.9. Altogether, we have extended the features of $\mathcal{U}(\mathfrak{z}_{\mathbb{C}}) \subset \mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ from polynomial functions to functions of polynomial growth.

The next remark explains the heading of this section.

Remark. Let $A \in \mathfrak{z}$ be a central element. We know that $(-iA) * -$ acts as a differential operator in $L^1(M)$. We want to declare the notion of functions of this operator.

Let $f \in \mathcal{Q}$ and $\psi_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be a function such that all its derivatives up to order $l+1$ exist, are continuous, and have polynomial growth so that $\psi(\xi) = \psi_0(\,\langle \xi, A\rangle\,)$ is in $\mathcal{P}(\mathfrak{z}^*)$. It follows from Equation 5.8 that the operator $(-iA) * -$ is diagonalized by partial Fourier transformation. It is a basic idea of any definition of $\psi_0(-iA)$ that

$$(\,\psi_0(-iA)f\,)\widehat{\ }(x,\xi) = \psi_0(\,\langle \xi, A\rangle\,)\,\widehat{f}(x,\xi)$$

should hold. But it follows from Lemma 5.20 that there exists a function $T_\psi f$ in $L^1(M)$ such that this equality is satisfied so that the definition $\psi_0(-iA)f := T_\psi\, f$ appears to be reasonable. Thus in particular we have established a functional calculus for central elements.

Definition 5.10. Let W be in $\mathcal{U}(\mathfrak{m}_\mathbb{C})$, $p \in \mathcal{P}(\mathfrak{m}^*)$ be a complex valued polynomial function, and $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ a continuous function. Recall $l = \dim \mathfrak{z}$. Let $\{f_s : s \in S\}$ be a subset of $\mathfrak{m}^*$ and $g \in \mathfrak{m}^*$. We set $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$. Then we say that the triple (W, p, ψ) separates ρ from $\{\pi_s : s \in S\}$ if the following conditions are satisfied:

- the derivatives of ψ up to order $l+1$ exist, are continuous, and have polynomial growth
- $\pi_s^\infty(W) = p(f_s)$ for all s and $\rho(W) = p(g)$
- $p(f_s) = \psi(f_s\,|\,\mathfrak{z})$ for all s and $p(g) \neq \psi(g\,|\,\mathfrak{z})$

Remark 5.11. We point out that $g\,|\,\mathfrak{z}$ might be contained in the closure of the set $\{f_s\,|\,\mathfrak{z} : s \in S\}$. Then the last condition of the preceding definition states that p and ψ diverge in the limit. In particular $f_s\,|\,\mathfrak{z} \longrightarrow g\,|\,\mathfrak{z}$ does not imply $p(f_s) \longrightarrow p(g)$.

If (W, p, ψ) separates ρ from $\{\pi_s : s \in S\}$, then we can apply Theorem 5.1. It follows that the point $\ker_{L^1(M)} \rho$ is not contained in the closure of the subset $\{\ker_{L^1(M)} \pi_s : s \in S\}$ in $\mathrm{Prim}_* \, L^1(M)$, which carries the Jacobson topology.

Now we assume in addition that $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{m}$. Let f_s' resp. g' denote the restriction of f_s resp. g to $\mathfrak{n}$. If g' is not contained in the closure of the subset $\{f_s' : s \in S\}$ in $\mathfrak{n}^*$, then we see that

$$\bigcap_{s \in S} \ker_{C^*(N)} \pi_s \not\subset \ker_{C^*(N)} \rho$$

because the Kirillov map $\mathcal{K}$ is a homeomorphism. Since N is $*$-regular as a connected nilpotent Lie group, we obtain

$$\bigcap_{s \in S} \ker_{L^1(N)} \pi_s \not\subset \ker_{L^1(N)} \rho\,.$$

Of course, the same relation holds in $L^1(M)$. Thus we can assume $g' \in \overline{\{f_s' : s \in S\}}$ to avoid trivialities. Finally, we observe that p cannot be contained in the subspace of $\mathcal{P}(\mathfrak{m}^*)$ corresponding to $\mathcal{S}(\mathfrak{n}_\mathbb{C})$ if (W, p, ψ) separates ρ from $\{\pi_s : s \in S\}$.

The rest of this section is devoted to the proof of Theorem 5.17 and Theorem 5.18, which are nothing but a variant of Theorem 5.1. The proof of Theorem 5.17 relies on the fact that the multiplier problem

$$\widehat{g}(\xi) = \xi \log(\xi)\ \widehat{f}(\xi)$$

for $\xi > 0$ has a solution g in $L^1(\mathbb{R})$ for any given Schwartz function f. The problem is that the function $\xi \longrightarrow \xi \log(\xi)$ is not differentiable in $\xi = 0$.

Remark 5.12. First we recall a result on the Fourier transform of certain tempered distributions, which can be found in Gelfand and Shilov, [13].

(i) Let $r > 0$ be real and $s \geq 0$ be an integer. Let us define the continuous function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\psi(\xi) = \xi^r \log^s(\xi)$$

for $\xi > 0$ and $\psi(\xi) = 0$ for $\xi \leq 0$. Since ψ has polynomial growth, this function defines a tempered distribution on $\mathbb{R}$. It is a general result that there exists a tempered distribution u on $\mathbb{R}$ such that $\widehat{u} = \psi$. In [13], Chapter II, Section 2.4, this distribution u has been computed explicitly by means of Cauchy's theorem and analytic continuation, see pp. 172-175 of [13]. The result is that u is essentially given by a smooth function k on $\mathbb{R}\backslash\{0\}$ which has an algebraic singularity in 0 so that there exists an integer $q > 0$ such that $z \mapsto |z|^q\, k(z)$ is continuous. Further k vanishes at infinity and there exist constants $C > 0$ and $c > 1$ such that $|k(z)| \leq C\,|z|^{-c}$ for all $|z| \geq 1$. The distribution u is given by regularization of the divergent integral

$$\langle u, \varphi\rangle \ = \int_{-\infty}^{+\infty} k(y)\varphi(y)\ dy$$

for Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R})$.

(ii) Let $r > 0$ be real and $s \geq 0$ be an integer. Let us consider the continuous, spherically symmetric function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\psi(\xi) = |\xi|^r\ \log^s |\xi|$$

for $\xi \in \mathbb{R}^n$ where $|\xi|$ denotes the Euclidean norm. There exists a tempered distribution u on $\mathbb{R}^n$ such that $\widehat{u} = \psi$. It is shown in [13], see pp. 194, that u is essentially given by a smooth function k on $\mathbb{R}^n\backslash\{0\}$ which has an algebraic singularity in 0 so that there exists an integer $q > 0$ such that $z \mapsto |z|^q\, k(z)$ is continuous. Further there exist $C > 0$ and $c > n$ such that $|k(z)| \leq C\,|z|^{-c}$ for all $|z| \geq 1$. Again the distribution u is given by regularization of k.

Let us explain the procedure of regularization, see also [13], Chapter I, Section 1.7. Let $I = [-1, 1]^n$. If $k : \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{C}$ is smooth with a singularity in 0 as above, then

$$\varphi \mapsto \int_I k(y)\left(\varphi(y) - P_q^{\varphi}(y)\right)\ dy + \int_{\mathbb{R}^n\backslash I} k(y)\varphi(y)$$

is a well-defined tempered distribution which coincides with $\int_{-\infty}^{+\infty} k(y)\,\varphi(y)\;dy$ for Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subset \mathbb{R}^n\backslash\{0\}$. Here P_q^φ denotes the Taylor polynomial of φ of order q, i.e.,

$$P_q^\varphi(y) = \sum_{|\nu|\leq q} \frac{1}{\nu!}\,(\partial^\nu \varphi)(0)\,y^\nu$$

so that the remainder term is given by

$$\varphi(y) - P_q^\varphi(y) = \sum_{|\nu|=q+1} \frac{1}{\nu!}\,(\partial^\nu \varphi)(\vartheta y)\,y^\nu$$

for some $0 \leq \vartheta \leq 1$ depending on y. Such a regularization of k is not unique. The difference of any two such regularizations is a tempered distribution concentrated on 0 and thus a linear combination of Dirac's delta distribution and its derivatives, see e.g. [12], p. 290. Thus we obtain

$$\langle u, \varphi\rangle \;=\; \int\limits_I k(y)\,\big(\,\varphi(y) - P_q^\varphi(y)\,\big)\;dy + \int\limits_{\mathbb{R}^n\backslash I} k(y)\varphi(y)\;dy + \sum_{\nu=0}^{q} a_\nu(\partial^\nu\varphi)(0)$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$ if the $a_\nu \in \mathbb{C}$ are chosen appropriately for $|\nu| \leq q$.

The next lemma is the key to the proof of Theorem 5.17 and 5.18.

Lemma 5.13. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a smooth function such that for any multi-index $\nu \in \mathbb{N}^n$ there exists a real number $b > n$ such that the function $z \mapsto |z|^b\;(\partial^\nu f)(z)$ is bounded. In particular $\partial^\nu f \in L^1(\mathbb{R}^n)$.*

Let $k : \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{C}$ be a smooth function having an algebraic singularity in 0 so that there exists an integer $q > 0$ such that $z \mapsto |z|^q\,k(z)$ is continuous. Let $C > 0$ and $c > n$ be such that $|k(z)| \leq C\,|z|^{-c}$ for all $|z| \geq 1$. Let u denote a tempered distribution which is given by regularization of k. Suppose further that $\psi = \widehat{u}$ is a continuous function of polynomial growth.

Then there exists a smooth L^1-function $g : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$\widehat{g}(\xi) = \psi(\xi)\;\widehat{f}(\xi)$$

for all $\xi \in \mathbb{R}^n$. For any multi-index $\nu \in \mathbb{N}^n$ there exists a constant $b > n$ such that $z \mapsto |z|^b\;(\partial^\nu g)(z)$ is bounded.

Proof. Clearly $g = u * f$ is the (unique) solution of our multiplier problem because

$$\widehat{g} = (u * f)\widehat{\;\;} = \widehat{f}\,\widehat{u} = \psi\,\widehat{f}\;.$$

It is known that $u * f$ is a smooth and slowly increasing function, see e.g. [12], Chapter 9.2. Its derivatives are given by $\partial^\nu(u * f) = u * (\partial^\nu f)$. Thus it suffices to prove that $u * f$ is in $L^1(\mathbb{R}^n)$ for any function f satisfying the conditions of this lemma.

In order to prove that $u * f \in L^1(\mathbb{R}^n)$, we verify that there exists a real number $b > 1$ such that $z \mapsto |z|^b\,(u * f)(z)$ is bounded. Recall that $(u * f)\,(z) = \langle u, \tau_z \tilde{f}\rangle$ with translation $(\tau_z f)\,(y) = f(y - z)$ and reflection $\tilde{f}(y) = f(-y)$. Note that

$$(5.14)\quad \langle u, \tau_z \tilde{f}\rangle \;=\; \int\limits_I k(y) \left((\tau_z \tilde{f})(y) - P_q^{\tau_z \tilde{f}}(y) \right) \, dy$$

$$+ \int\limits_{\mathbb{R}^n \backslash I} k(y)\, f(z-y)\, dy + \sum_{\nu=0}^{q} a_\nu (\partial^\nu f)(z) \, .$$

Let us choose $n < b < c$ such that the functions $z \mapsto |z|^b\,(\partial^\nu f)(z)$ are bounded for all $|\nu| \leq q+1$. Obviously, the third summand of 5.14 is under control. Now we estimate the first integral of 5.14. Keeping z fixed, we apply Taylor's formula to the function $\tau_z \tilde{f}$. Since $\partial^\nu(\tau_z \tilde{f})(y) = (-1)^{|\nu|}\,(\partial^\nu f)(z-y)$, we obtain

$$|z|^b \left| \int\limits_I k(y) \left((\tau_z \tilde{f})(y) - P_q^{\tau_z \tilde{f}}(y) \right) dy \right|$$

$$\leq \sum_{|\nu|=q+1} |z|^b \int\limits_I |k(y)|\, \frac{1}{\nu!}\, |(\partial^\nu f)(z - \theta y)|\, |y^\nu|\, dy$$

$$\leq \sum_{|\nu|=q+1} \frac{1}{\nu!}\, |z|^b \sup_{|y|\leq 1} |(\partial^\nu f)(z-y)| \int\limits_I |y|^{q+1}\, |k(y)| dy \, .$$

The integral on the right hand side is finite and the whole expression on the right is bounded in z. In order to complete the proof, we must estimate the second summand of 5.14, i.e., the expression

$$(5.15)\qquad |z|^b \int\limits_{\mathbb{R}^n \backslash I} |k(y)\, f(z-y)|\, dy \, .$$

Let $z \neq 0$ be arbitrary and fixed. We consider the measurable subsets

$$A_1 = \{ y \in \mathbb{R}^n \backslash I : 2|y| > |z| \}$$

and $A_2 = \mathbb{R}^n \backslash (I \cup A_1)$ of $\mathbb{R}^n$ which form a partition of $\mathbb{R}^n \backslash I$. Let $D > 0$ such that $|y|^b\,|k(y)| \leq D$ for all $y \in \mathbb{R}^n \backslash I$ and $|y|^b\,|f(y)| \leq D$ for all $y \in \mathbb{R}^n$. If $y \notin I$ and $|z|/|y| < 2$, then

$$|z|^b\,|k(y) f(z-y)| \leq D\,|z|^b\,|y|^{-b}\,|f(z-y)| \leq 2^b\,D\,|f(z-y)| \, .$$

This implies

$$|z|^b \int\limits_{A_1} |k(y) f(z-y)|\, dy \leq 2^b\, D \int\limits_{A_1} |f(z-y)|\, dy \leq 2^b\, D \int\limits_{\mathbb{R}^n} |f(y)|\, dy \, .$$

On the other hand, if $y \notin I$ and $|y|/|z| \leq 1/2$, then $|z|/|z-y| \leq 2$. Since

$$|z|^b\,|k(y) f(z-y)| \leq D \left(\frac{|z|}{|z-y|} \right)^b |k(y)| \leq D\, 2^b\, |k(y)| \, ,$$

it follows

$$|z|^b \int\limits_{A_2} |k(y)f(z-y)|\, dy \le D\, 2^b \int\limits_{A_2} |k(y)|\, dy\ .$$

Thus we see that 5.15 is bounded in z and the proof is complete. $\square$

A further inspection of the proof of Lemma 5.13 shows that moreover the following is true.

Lemma 5.16. *Let $w : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a continuous function. Let $f : \mathbb{R}^d_x \times \mathbb{R}^n_z \longrightarrow \mathbb{C}$ be a continuous function such that all derivatives in z-direction exist and are continuous. We assume that for every multi-index $\nu \in \mathbb{N}^n$ there exists a constant $b > n$ such that the function $(x,z) \mapsto w(x)\,|z|^b\,(\partial_z^\nu f)(x,z)$ is bounded.*

Let k, u, and ψ be given as in Lemma 5.13. Then there exists a continuous function $g : \mathbb{R}^d_x \times \mathbb{R}^n_z \longrightarrow \mathbb{C}$ which is smooth in z-direction and satisfies

$$\widehat{g}(x,\xi) = \psi(\xi)\widehat{f}(x,\xi)$$

for all x, ξ. Here $\widehat{f}$ denotes partial Fourier transformation with respect to z. For any $\nu \in \mathbb{N}^n$ there exists some $b > n$ such that $(x,z) \mapsto w(x)\,|z|^b\,(\partial_z^\nu g)(x,z)$ is bounded.

The preceding lemma immediately implies the next theorem which is of the greatest importance for our further investigations.

Theorem 5.17. *Let M be an exponential solvable Lie group. Let Z be a closed, connected, l-dimensional subgroup of the center of M. Let $\{\pi_s : s \in S\}$ be a family of unitary representations of M such that the restriction of π_s to Z is a character ζ_s, and ρ a unitary representation of M whose restriction to Z equals the character η. Let us fix a basis $e_1, \dots, e_l$ of $\mathfrak{z}$ and let $e_1^*, \dots, e_l^*$ denote the dual basis of $\mathfrak{z}^*$. We assume that $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ is a continuous function which is a sum of functions of the form*

$$\xi \mapsto a\ \xi_1^{r_1}\cdot\ldots\cdot\xi_l^{r_l}\cdot\log^{s_1}(\xi_1)\cdot\ldots\cdot\log^{s_l}(\xi_l)$$

if $\xi_\nu > 0$ for all $1 \le \nu \le l$ and $\xi \mapsto 0$ else, in the coordinates of the basis $e_1^*, \dots, e_l^*$. Here $a \in \mathbb{C}$, $r_1, \dots, r_l > 0$ are real numbers, and $s_1, \dots, s_l \ge 0$ are integers. Further let $h, f \in L^1(M)$ be smooth functions such that

$$\pi_s(h) = \psi(\zeta_s^\infty)\pi_s(f)$$

for all $s \in S$ and

$$\rho(h) \ne \psi(\eta^\infty)\rho(f)\ .$$

More exactly, we suppose that for every multi-index $\nu \in \mathbb{N}^l$ there exist real numbers $a_1, \dots, a_l > 1$ such that the for the derivatives in z-direction the functions

$$(x,z) \mapsto |x|^{k+1}\cdot|z_1|^{a_1}\cdot\ldots\cdot|z_l|^{a_l}\cdot(\partial_z^\nu f)(x,z)$$

are bounded. Here $k = \dim M/Z$. This growth condition is satisfied e.g. for $f \in \mathcal{C}_0^\infty(M)$ or for Schwartz functions f in the coordinates given by Φ. Then the relation

$$\bigcap_{s\in S} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of these representations.

Proof. As in the proof of Theorem 5.1 it suffices to show that there exists a smooth function $g \in L^1(M)$ satisfying $\widehat{g}(x,-\xi) = \psi(\xi)\widehat{f}(x,-\xi)$ for all $x \in \mathbb{R}^k$ and $\xi \in \mathfrak{z}^*$. We can omit the minus signs and thus have to solve the multiplier problem

$$\widehat{g}(x,\xi) = \psi(\xi)\widehat{f}(x,\xi)$$

in $L^1(M)$. Without loss of generality we can assume

$$\psi(\xi) = \, a\, \xi_1^{r_1} \cdot \ldots \cdot \xi_l^{r_l} \cdot \log^{s_1}(\xi_1) \cdot \ldots \cdot \log^{s_l}(\xi_l)\ .$$

In view of Remark 5.12 all we have to do is to apply Lemma 5.16 with multiplier $\xi_\nu \mapsto \xi_\nu^{r_\nu} \log^{s_\nu}(\xi_\nu)$ in direction z_ν for $1 \le \nu \le l$. □

The last theorem of this subsection is just a slight modification of the preceding one. In our applications we will have $\dim \mathfrak{z}_\nu = 1$ or $\dim \mathfrak{z}_\nu = 2$. This theorem becomes important in the presence of complex weights.

Theorem 5.18. *Let M, Z, π_s, and ρ be given as in Theorem 5.17. Let us fix a direct sum decomposition $\mathfrak{z} = \mathfrak{z}_0 \oplus \ldots \oplus \mathfrak{z}_l$ of the Lie algebra $\mathfrak{z}$ of Z. This induces a decomposition of $\mathfrak{z}^*$. Assume that $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ is a continuous function which is a sum of functions of the form*

$$\xi \mapsto a\,|\xi_1|^{r_1} \cdot \ldots \cdot |\xi_l|^{r_l} \cdot \log^{s_1}|\xi_1| \cdot \ldots \cdot \log^{s_l}|\xi_l|$$

where $\xi_\nu \in \mathfrak{z}_\nu^*$ and $|\xi_\nu|$ denotes the Euclidean norm with respect to some basis of $\mathfrak{z}_\nu^*$. Here $a \in \mathbb{C}$, $r_1, \ldots, r_l > 0$ are real numbers, and $s_1, \ldots, s_l$ are integers. Further let $h, f \in L^1(M)$ be smooth functions such that

$$\pi_s(h) = \psi(\zeta_s^\infty)\pi_s(f)$$

for all s and

$$\rho(h) \neq \psi(\eta^\infty)\rho(f)\ .$$

More exactly, we suppose that for every multi-index ν there exist real numbers $a_1, \ldots, a_l$ such that $a_\nu > \dim \mathfrak{z}_\nu$ and such that the for the derivatives in z-direction the functions

$$(x,z) \mapsto |x|^{k+1} \cdot |z_1|^{a_1} \cdot \ldots \cdot |z_l|^{a_l} \cdot (\partial_z^\nu f)(x,z)$$

are bounded. Here $k = \dim M/Z$. Then the relation

$$\bigcap_{s \in S} \ker_{L^1(M)} \pi_s \;\not\subset\; \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of these representations.

Proof. Again we have to solve the multiplier problem

$$\widehat{g}(x,\xi) = \psi(\xi)\widehat{f}(x,\xi)$$

in $L^1(M)$. Here we can assume

$$\xi \mapsto a\,|\xi_1|^{r_1} \cdot \ldots \cdot |\xi_l|^{r_l} \cdot \log^{s_1}|\xi_1| \cdot \ldots \cdot \log^{s_l}|\xi_l|\ .$$

In view of Remark 5.12 we apply Lemma 5.16 with multiplier $\xi_\nu \mapsto |\xi_\nu|^{r_\nu} \log^{s_\nu}|\xi_\nu|$ in direction z_ν for $1 \le \nu \le l$. □

5.2 Completion of the proof of Theorem 5.1

The purpose of this section is to prove the technical Lemma 5.20 which completes the proof of Theorem 5.1. To begin with, we recall some elementary facts of Euclidean Fourier analysis (without proof).

For $f \in L^1(\mathbb{R}^l_z)$ we define the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^l} f(z) e^{-i<\xi,z>}\, dz$$

and for functions $b \in L^1(\mathbb{R}^l_\xi)$ we define the inverse Fourier transform

$$b^{\#}(z) = (2\pi)^{-l} \int_{\mathbb{R}^l} b(\xi) e^{i<\xi,z>}\, dz\ .$$

On $\mathbb{R}^l$ we consider the differential operators

$$D_j = (-i)\, \partial_j$$

and for every multi-index $\alpha \in \mathbb{N}^l$

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_l^{\alpha_l} = (-i)^{|\alpha|}\, \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_l^{\alpha_l}\ .$$

Roughly speaking, the following formulas are valid:

$$(D_z^\alpha f)^\frown = \xi^\alpha \widehat{f} \qquad\qquad D_\xi^\alpha \widehat{f} = (-1)^{|\alpha|}\, (z^\alpha f)^\frown$$

and

$$(D_\xi^\alpha b)^{\#} = (-1)^{|\alpha|}\, z^\alpha b^{\#} \qquad\qquad D_z^\alpha b^{\#} = (\xi^\alpha b)^{\#}$$

For a sample of precise statements see the following lemmata.

Remark. Let $f : \mathbb{R}^l \longrightarrow \mathbb{C}$ be a continuous function and $r > 0$ be an integer.

1. Then $|z|^r f$ is bounded on $\mathbb{R}^l$ iff $z^\alpha f$ is bounded for every multi-index $\alpha \in \mathbb{N}^l$ such that $r = |\alpha| = \sum_{j=1}^l \alpha_j$.
2. The function $(1 + |z|)^r f$ is bounded iff $z^\alpha f$ is bounded for all $|\alpha| \leq r$.

The same statements hold if 'bounded' is replaced by 'L^1-integrable'. All these properties of f are independent of the choice of the norm on $\mathbb{R}^l$.

Lemma. *Let $f \in \mathcal{C}^r(\mathbb{R}^l_z)$ such that $D_z^\alpha f \in L^1(\mathbb{R}^l_z)$ for all $|\alpha| \leq r$. Then the formula*

$$(D_z^\alpha f)^\frown = \xi^\alpha \widehat{f}$$

holds and $\xi^\alpha \widehat{f}$ is a continuous, bounded function for all $|\alpha| \leq r$. In particular $(1+|\xi|)^r \widehat{f}$ is bounded.

Proof. This is a consequence of partial integration. $\square$

Lemma. *Let $f : \mathbb{R}^l_z \longrightarrow \mathbb{C}$ be a continuous function such that $(1+|z|)^r f \in L^1(\mathbb{R}^l_z)$. Then $D^\alpha_\xi \widehat{f}$ exists and is continuous for all $|\alpha| \leq r$ and the formula*

$$D^\alpha_\xi \widehat{f} = (-1)^{|\alpha|} \, (z^\alpha f)^\frown$$

holds.

Proof. This is a consequence of Lebesgue's theorem on dominated convergence. □

Needless to say, analog lemmata can be stated for the inverse Fourier transformation.

The following standard result is essential for our proof of Lemma 5.20.

Lemma 5.19 (Fourier inversion)**.** *Let $b \in L^1(\mathbb{R}^l_\xi)$ be a continuous function such that $b^\# \in L^1(\mathbb{R}^l_z)$. Then we have $b = (b^\#)^\frown$.*

The next lemma is extremely useful.

Lemma. *Let $f : \mathbb{R}^l \longrightarrow \mathbb{C}$ be a continuous function such that $(1+|z|)^{(l+1)} f$ is bounded. Then $f \in L^1(\mathbb{R}^l)$.*

Definition. Let $\mu \geq 0$ be an integer. A family $\{f_i : i \in I\}$ of functions on $\mathbb{R}^l$ has polynomial growth of order less or equal μ, if there exists a constant $C > 0$ such that

$$|f_i(z)| \leq C(1+|z|)^\mu$$

for all $z \in \mathbb{R}^l$ and $i \in I$.

Lemma 5.20. *Let $\psi : \mathbb{R}^l_\xi \longrightarrow \mathbb{C}$ be a continuous function such that $D^\gamma_\xi \psi$ exists and is continuous for all multi-indices $\gamma \in \mathbb{N}$ with $|\gamma| \leq l+1$ and such that $\{D^\gamma_\xi \psi : \|\gamma| \leq l+1\}$ has polynomial growth of order less or equal μ.*

Let $f : \mathbb{R}^k_x \times \mathbb{R}^l_\xi \longrightarrow \mathbb{C}$ be a continuous function such that $D^\alpha_x D^\beta_\xi f$ exists and is continuous for all $|\alpha| \leq r$ and $|\beta| \leq \mu + 2l + 2 + s$. Assume further that these derivatives are rapidly decreasing in the sense that

$$(x,z) \mapsto D^\alpha_x D^\beta_z f(x,z) x^\gamma z^\delta$$

is bounded for $|\alpha| \leq r$, $|\beta| \leq \mu + 2l + 2 + s$, $|\gamma| \leq k+1$ and $|\delta| \leq s$.

Then there exists a continuous function $g \in L^1(\mathbb{R}^k \times \mathbb{R}^l)$ such that

$$\widehat{g}(x,\xi) = \psi(\xi) \widehat{f}(x,\xi) \tag{5.21}$$

for all $x \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^l$. The derivatives $D^\alpha_x D^\beta_\xi g$ exist and are continuous for all $|\alpha| \leq r$ and $|\beta| \leq s$. In this situation g is uniquely determined and given by

$$g(x,z) = (2\pi)^{-l} \int_{\mathbb{R}^l} \psi(\xi) \widehat{f}(x,\xi) e^{i<\xi,z>} \, d\xi \, . \tag{5.22}$$

Remark. The substitution of $\psi(-\xi)$ by $\psi(\xi)$ is harmless, of course.

If $f \in \mathcal{C}_0^\infty(\mathbb{R}^k \times \mathbb{R}^l)$ or if $f \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^l)$, then all the assumptions of the lemma are satisfied. In this case g is a smooth L^1-function. If f has compact support, there exists a compact subset K of $\mathbb{R}^l$ such that $x \notin K$ implies $\widehat{g}(x,\xi) = 0$ for all ξ.

Proof. The crucial idea is to verify Formula 5.22 and apply the Fourier inversion Lemma 5.19. The proof is carried out in four steps.

First we prove that $D_x^\alpha D_\xi^\beta \widehat{f}$ exists and is continuous for all $|\alpha| \leq r$ and $|\beta| \leq \mu + 2l + 2 + s$.

Since

$$(x,z) \mapsto D_x^\alpha f(x,z) z^\beta$$

is bounded, there exists a constant $M > 0$ such that $|D_x^\alpha f(x,z) z^\beta| \leq M$ for all (x,z) and all $|\beta| \leq \mu + 2l + 2 + s$. Hence there exists a $C > 0$ such that

$$|D_x^\alpha f(x,z) z^\beta| \leq C(1 + |z|)^{-(l+1)}$$

for all (x,z) and all $|\beta| \leq \mu + l + 1 + s$. This justifies the application of Lebesgue's theorem on dominated convergence. By induction we see that the derivatives of $\widehat{f}$ exist and can be computed by taking limits under the integral sign. We obtain

$$D_x^\alpha D_\xi^\beta \widehat{f}(x,\xi) = (-1)^{|\beta|} \int\limits_{\mathbb{R}^l} (D_x^\alpha f)(x,z) z^\beta e^{-i<\xi,z>} \, dz$$

for all $|\alpha| \leq r$ and $|\beta| \leq \mu + l + 1 + s$.

A similar argument shows that there exists an $M > 0$ such that

$$\int\limits_{\mathbb{R}^l} |D_x^\alpha D_z^\beta f(x,z) x^\gamma z^\delta| \, dz \leq M$$

for all x, all $|\alpha| \leq r$, $|\beta| \leq \mu + l + 1 + s$, $|\gamma| \leq k+1$ and $|\delta| \leq l+1$. This intermediate result is very important for the rest of this proof.

The second step is to verify that g is well-defined as the partial inverse Fourier transform of

$$b(x,\xi) = \psi(\xi) \widehat{f}(x,\xi) \; .$$

But since

$$\int\limits_{\mathbb{R}^l} |D_z^\beta f(x,z)| \, dz \leq M \; ,$$

we get

$$|\widehat{f}(x,\xi) \xi^\beta| = |(D_z^\beta f)\widehat{\;}(x,\xi)| \leq M$$

for all (x,ξ) and $|\beta| \leq \mu + l + 1$. Hence there exists a constant $M_1 > 0$ such that

$$|\widehat{f}(x,\xi)|(1 + |\xi|)^{\mu+l+1} \leq M_1 \; .$$

Since ψ has polynomial growth of order $\leq \mu$, we obtain

$$|b(x,\xi)| = |\psi(\xi)\widehat{f}(x,\xi)| \leq M_2(1+|\xi|)^{-(l+1)} .$$

This shows that $\xi \mapsto b(x,\xi)$ is L^1-integrable and that

$$g(x,z) = (2\pi)^{-l} \int\limits_{\mathbb{R}^l} \psi(\xi)\widehat{f}(x,\xi)e^{i<\xi,z>}\, d\xi$$

is well-defined for all x.

In the third part we verify that $D_x^\alpha D_z^\beta g$ exists and is continuous for all $|\alpha| \leq r$ and $|\beta| \leq s$.

We have

$$\int\limits_{\mathbb{R}^l} |D_x^\alpha D_z^\beta f(x,z)|\, dz \leq M .$$

By applying the Fourier transformation we see

$$|D_x^\alpha \widehat{f}(x,\xi)\xi^\beta| \leq M$$

for all $|\beta| \leq \mu + l + 1 + s$ and thus

$$|\psi(\xi) D_x^\alpha \widehat{f}(x,\xi)\xi^\beta| \leq M_1(1+|\xi|)^{-(l+1)}$$

for all (x,ξ), $|\alpha| \leq r$ and $|\beta| \leq s$. Again we can apply Lebesgue's theorem to see that

$$D_x^\alpha D_z^\beta g(x,z) = (2\pi)^{-l} \int\limits_{\mathbb{R}^l} \psi(\xi) D_x^\alpha \widehat{f}(x,\xi)\xi^\beta e^{i<\xi,z>}\, d\xi$$

exists and is continuous.

The last step is to prove that $g \in L^1(\mathbb{R}^k \times \mathbb{R}^l)$.

We know

$$\int\limits_{\mathbb{R}^l} |D_z^\beta f(x,z) x^\gamma z^\delta|\, dz \leq M$$

and hence it follows by Fourier transformation that the function

$$(x,\xi) \mapsto D_\xi^\delta \widehat{f}(x,\xi) x^\gamma \xi^\beta$$

is bounded for $|\beta| \leq \mu + l + 1$, $|\gamma| \leq k+1$ and $|\delta| \leq l+1$. Recall that

$$D_\xi^\nu b = \sum_{\lambda+\delta=\nu} c_{\lambda,\delta}\, (D_\xi^\lambda \psi)(D_\xi^\delta \widehat{f}\,)$$

and that $\{D_\xi^\lambda \psi : |\lambda| \leq l+1\}$ has polynomial growth of order $\leq \mu$. Thus we see that

$$(x,\xi) \mapsto D_\xi^\nu b(x,\xi) x^\gamma \xi^\beta$$

is bounded for $|\beta| \leq l+1$ which implies

$$|D_\xi^\nu b(x,\xi)x^\gamma| \leq M_1(1+|z|)^{-(l+1)}$$

for $|\gamma| \leq k+1$ and $|\nu| \leq l+1$. We apply the inverse Fourier transformation and obtain

$$|g(x,z)x^\gamma z^\nu| \leq M_2 \; .$$

So the function

$$(x,z) \mapsto g(x,z)(1+|x|)^{k+1}(1+|z|)^{l+1}$$

is bounded and g is L^1-integrable.

The following conclusion finishes the proof of this lemma. Keeping x fixed we see that $\xi \mapsto b(x,\xi)$ and $z \mapsto b^{\#}(x,z) = g(x,z)$ are continuous L^1-functions. We can apply the Fourier inversion Lemma 5.19 and obtain the desired equality

$$\widehat{g}(x,\xi) = \psi(\xi)\widehat{f}(x,\xi) \; .$$

□

Remark 5.23. Let $\mathcal{Q}$ denote the subspace of all smooth functions $f : \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{C}$ such that $f(x,z) = 0$ for all $z \in \mathbb{R}^l$ and all x outside a compact subset of $\mathbb{R}^k$, and such that

$$(x,z) \mapsto (1+|z|)^r \, D_x^\alpha D_z^\beta f(x,z)$$

is bounded for all $\alpha \in \mathbb{N}^k$, $\beta \in \mathbb{N}^l$ and $r \geq 0$. The definition of $\mathcal{Q}$ does not depend on the choice of the norm on $\mathbb{R}^l$.

Suppose that $\psi : \mathbb{R}^l \longrightarrow \mathbb{C}$ is a smooth function such that all its derivatives have polynomial growth. If $f \in \mathcal{Q}$, then it follows from Lemma 5.20 that there exists a smooth L^1-function $g \in \mathcal{Q}$ such that Formula 5.21 holds. Arguments similar to those in the proof of Lemma 5.20 show that g is again in $\mathcal{Q}$. This assertion remains true, if we replace $\mathcal{Q}$ by the space of Schwartz functions $\mathcal{S}(\mathbb{R}^k \times \mathbb{R}^l)$.

6 Computation of infinitesimal representations

Let M be an exponential solvable Lie group with Lie algebra $\mathfrak{m}$, and $\pi \in \widehat{M}$ an irreducible representation. In this section we describe two particular situations in which it is easy to compute the infinitesimal representation $d\pi$ explicitly.

The assumptions on which these computations are carried out appear to be very special, but we will see in the examples that they cover many relevant cases, in particular for $\dim \mathfrak{m} \leq 6$.

Recall that, in order to apply Theorem 5.1 and Lemma 5.4, it is necessary to find a triple (W, p, ψ) which separates ρ from the set $\{\pi_s : s \in S\}$. The results of this section turn out to be useful in that context.

We begin with some preliminary remarks. Since the Kirillov map

$$\mathcal{K} : \mathfrak{m}/\operatorname{Ad}^*(M) \longrightarrow \widehat{M}$$

is a bijection, there exists an $f \in \mathfrak{m}^*$ such that $\pi = \mathcal{K}(f)$. Let $\mathfrak{p}$ be a Pukanszky polarization at f. By the definition of $\mathcal{K}$, we obtain

$$\pi = \operatorname{ind}_P^M \chi_f \ ,$$

where P denotes the connected subgroup of M with Lie algebra $\mathfrak{p}$, and χ_f the character of P given by

$$\chi_f(\exp(X)) = e^{if(X)} \ .$$

There exists a coexponential basis for $\mathfrak{p}$ in $\mathfrak{m}$ because M is connected, simply connected, and solvable. Thus there is a smooth section

$$s : M/P \hookrightarrow M$$

for the quotient map $q : M \twoheadrightarrow M/P$. Further, there exists a relatively invariant measure $d\mu(\xi)$ on M/P. This shows that we can realize the unitary representation π in the Hilbert space $L^2(M/P, d\mu)$ such that $\pi(m)$ is given by

$$\pi(m)\varphi\,(\xi) = \Delta_{M,P}(m)^{-1/2}\, \chi_f\left(s(\xi)^{-1}\, m\, s(m^{-1}\cdot\xi)\right)\, \varphi(m^{-1}\cdot\xi) \tag{6.1}$$

for $\varphi \in \mathcal{C}_0(M/P)$ and $\xi \in M/P$. Compare pp. 26-27 of [23].

In order to get concrete expressions for the unitary operators $\pi(m)$, we have to compute the section s, the modular function $\Delta_{M,P}$, and the action of M on M/P explicitly. Our goal is to calculate

$$d\pi(X)\varphi \ = \ \frac{d}{dt}_{|t=0}\, \pi(\exp(tX))\varphi \tag{6.2}$$

for $\varphi \in L^2(M/P)^\infty$, the subspace of π-smooth vectors in $L^2(M/P)$.

6.1 Representations in general position

Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{m}$ such that $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{n}$, and $f \in \mathfrak{m}^*$ in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$.

Since f vanishes on the ideal $[\mathfrak{m},\mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f,\mathfrak{z}\mathfrak{n}]$, we obtain $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$. Note that $\mathfrak{z}\mathfrak{m}$ is contained in any polarization at f.

Further, we assume that there exists a Pukanszky polarization $\mathfrak{p}$ at f such that

- $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{n}$
- there exists a commutative subalgebra $\mathfrak{c}$ of $\mathfrak{n}$ such that $\mathfrak{m} = \mathfrak{c} \oplus \mathfrak{p}$, and such that the map
$$\mathfrak{c} \times P \longrightarrow M,\ (C,p) \mapsto \exp(C)p$$
is a global diffeomorphism.

Clearly, the last assumption is equivalent to $\mathfrak{c} \longrightarrow M/P,\ C \mapsto q(\exp(C))$ being a diffeomorphism, so that we can identify M/P and $\mathfrak{c}$. We obtain the smooth section

$$s : M/P = \mathfrak{c} \hookrightarrow M,\ s(\xi) = \exp(\xi)$$

for the quotient map $q : M \twoheadrightarrow M/P = \mathfrak{c}$, which is given by $q(\exp(C)p) = C$.

Lemma 6.3.

(i) The action of M on $M/P = \mathfrak{c}$ is given by $\exp(C)^{-1}\cdot\xi = \xi - C$ for $C \in \mathfrak{c}$ and

$$p^{-1}\cdot\xi = \mathrm{Pr}_{\mathfrak{c}}\left(\mathrm{Ad}(p)^{-1}\xi\right)$$

for $p \in P$, where $\mathrm{Pr}_{\mathfrak{c}}$ denotes the linear projection onto the first summand of the direct sum $\mathfrak{n} = \mathfrak{c} \oplus (\mathfrak{p} \cap \mathfrak{n})$.

(ii) The Lebesgue measure $d\mu(\xi)$ of the vector space $\mathfrak{c}$ is relatively invariant for the action of M. Its modular function $\Delta_{M,P}$ is trivial on N, and given by

$$\Delta_{M,P} = \det\left(\mathrm{Ad}_{\mathfrak{m}/\mathfrak{p}}(p)\right)$$

for $p \in P$.

Proof. First, we prove $q \circ \exp = \mathrm{Pr}_{\mathfrak{c}}$ as maps from $\mathfrak{n}$ to $\mathfrak{c}$. Let $C \in \mathfrak{c}$ and $X \in \mathfrak{p} \cap \mathfrak{n}$. The Campbell-Baker-Hausdorff-formula yields

$$\begin{aligned}\exp(C+X) &= \exp(C)\exp(-C)\exp(C+X)\\ &= \exp(C)\exp\left(H(-C,C+X)\right)\\ &= \exp(C)\exp\left(X + H'(-C,C+X)\right),\end{aligned}$$

where H denotes the Hausdorff-series, and H' the series obtained from H by omitting the terms of first order. Since $\mathfrak{n}$ is nilpotent, H and H' are polynomials. We obtain

$$H'(-C,C+X) \in \mathfrak{p} \cap \mathfrak{n}$$

because $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{n}$. This proves $q(\exp(C+X)) = C$.

Now we obtain

$$\begin{aligned} \exp(C)^{-1}\cdot\xi &= q\left(\exp(C)^{-1}\exp(\xi)\right) = q(\exp(\xi - C)) \\ &= \xi - C \end{aligned}$$

for $C \in \mathfrak{c}$ and

$$\begin{aligned} p^{-1}\cdot\xi &= q\left(p^{-1}\exp(\xi)\right) = q\left(p^{-1}\exp(\xi)p\right) \\ &= \mathrm{Pr}_{\mathfrak{c}}\left(\mathrm{Ad}(p)^{-1}\xi\right) \end{aligned}$$

for $p \in P$. This finishes the proof of *(i)*. Since $\Delta_{M,P}(p) = \Delta_P(p)\Delta_M(p)^{-1}$ for $p \in P$ and $\Delta_M(m) = \det\left(\mathrm{Ad}_{\mathfrak{m}}(m)\right)^{-1}$ holds for the modular function of any connected Lie group, part *(ii)* becomes obvious. □

An application of Lemma 6.3 yields the following explicit formulas for 6.1 and 6.2.

(i) Compute $d\pi(X)$ for $X \in \mathfrak{p}$ such that $[X, \mathfrak{c}] \subset \mathfrak{c}$.

At first, we have

$$\Delta_{M,P}(\exp(X)) = e^{\mathrm{tr}(\mathrm{ad}_{\mathfrak{m}/\mathfrak{p}}(X))}$$

for all $X \in \mathfrak{p}$. Since $\mathfrak{c}$ is $\mathrm{ad}(X)$-invariant, we get

$$\exp(X)^{-1}\cdot\xi = \mathrm{Ad}\left(\exp(X)\right)^{-1}\xi\ .$$

From this it follows

$$\pi\left(\exp(X)\right)\varphi\ (\xi) \ = \ e^{-\frac{1}{2}\mathrm{tr}(\mathrm{ad}_{\mathfrak{m}/\mathfrak{p}}(X))}\ e^{if(X)}\ \varphi\left(\mathrm{Ad}(\exp(X))^{-1}\xi\right)$$

and hence

$$\boldsymbol{d\pi(X)\varphi\ (\xi) \ = \ - < \nabla\varphi(\xi) \mid [X,\xi] > \ - \left(\frac{1}{2}\mathbf{tr}(\mathbf{ad}_{\mathfrak{m}/\mathfrak{p}}(X)) - if(X)\right)\varphi(\xi)}$$

for all $X \in \mathfrak{p}$ such that $[X, \mathfrak{c}] \subset \mathfrak{c}$.

(ii) Compute $d\pi(C)$ for $C \in \mathfrak{c}$.

We have

$$\pi\left(\exp(C)\right)\varphi\ (\xi) \ = \ \varphi(\xi - C)$$

and thus

$$\boldsymbol{d\pi(C)\varphi\ (\xi) \ = \ - < \nabla\varphi(\xi) \mid C >}$$

for all $C \in \mathfrak{c}$.

(iii) Compute $d\pi(Y)$ for $Y \in \mathfrak{p} \cap \mathfrak{n}$.

First, $\Delta_{M,P}(\exp(Y)) = 1$ for $Y \in \mathfrak{p} \cap \mathfrak{n}$. Since $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{n}$, we get

$$\exp(Y)^{-1} \cdot \xi = \mathrm{Pr}_{\mathfrak{c}}\left(\mathrm{Ad}(\exp(Y))\xi \right) = \xi \,.$$

From this it follows

$$\pi\left(\exp(Y)\right)\varphi\left(\xi\right) \;=\; e^{if\left(\mathrm{Ad}(\exp(\xi))^{-1}Y \right)}\,\varphi(\xi)$$

and hence

$$\boldsymbol{d\pi(Y)\varphi\;(\xi) \;=\; if\left(\mathbf{Ad}(\mathbf{exp}(\xi))^{-1}Y \right)\,\varphi(\xi)}$$

for all $Y \in \mathfrak{p} \cap \mathfrak{n}$.

(iv) Compute $d\pi(Z)$ for $Z \in \mathfrak{z}\mathfrak{m}$.

The center of M acts trivially on M/P. This observation shows

$$\pi\left(\exp(Z)\right)\varphi\left(\xi\right) \;=\; e^{if(Z)}\varphi(\xi)$$

and thus

$$\boldsymbol{d\pi(Z)\varphi\;(\xi) \;=\; if(Z)\varphi(\xi)}$$

for all $Z \in \mathfrak{z}\mathfrak{n}$.

6.2 Representations of semi-direct products

Let us assume that $\mathfrak{m}$ is a semi-direct sum of a subalgebra $\mathfrak{c}$ and an ideal $\mathfrak{q}$. Then the exponential solvable Lie group $M = C \ltimes Q$ is a semi-direct product. Therefore we can identify M/Q and C. The Haar measure of C is invariant with respect to the action of M on $M/Q = C$.

Further, let $g \in \mathfrak{m}^*$ such that $\mathfrak{q}$ is a Pukanszky polarization at g. It is a basic result that the induced representation

$$\rho = \mathrm{ind}_Q^M \chi_g$$

can be realized in $L^2(C)$ such that

$$\rho(r)\varphi\ (\xi) = \varphi(r^{-1}\xi)$$

for $r \in C$ and

$$\rho(x)\varphi\ (\xi) = \chi_g(\xi^{-1}x\xi)\ \varphi(\xi)$$

for $x \in Q$. If $\mathfrak{c}$ is commutative, then we obtain the following explicit formulas.

(i) Compute $d\rho(C)$ for $C \in \mathfrak{c}$.

We have
$$\rho(\exp C)\varphi\ (\xi) \ = \ \varphi(\xi - C)$$
and hence
$$\boldsymbol{d\rho(C)\varphi\ (\xi) \ = \ - < \nabla\varphi(\xi) \mid C >}$$
for all $C \in \mathfrak{c}$.

(ii) Compute $d\rho(X)$ for $X \in \mathfrak{q}$.

We obtain
$$\rho(\exp X)\varphi\ (\xi) \ = \ e^{ig\left(\mathrm{Ad}(\exp(\xi))^{-1}X\right)}\ \varphi(\xi)$$
and thus
$$\boldsymbol{d\rho(X)\varphi\ (\xi) \ = \ ig\left(\mathbf{Ad}(\mathbf{exp}(\xi))^{-1}X\right)\ \varphi(\xi)}$$
for all $X \in \mathfrak{q}$.

If $\mathfrak{c}$ is not commutative, then we choose a Malcev basis $b_1, \ldots, b_k$ of $\mathfrak{c}$ defining the diffeomorphism

$$\Phi : \mathbb{R}^k \longrightarrow C,\ \Phi(x) = \exp(x_1 b_1) \ldots \exp(x_k b_k)\ .$$

Using this identification, we realize the representation ρ in the Hilbert space $L^2(\mathbb{R}^k)$ and obtain
$$d\rho(X)\varphi\ (\xi) \ = \ ig\left(\mathrm{Ad}(\Phi(\xi))^{-1}X\right)\ \varphi(\xi)$$

for $X \in \mathfrak{q}$.

7 Restriction to subquotients

This section is divided into three subsections. In the first one we investigate the possibility of restricting to subquotients for certain locally compact groups. The next one contains some notation and conventions for the exponential case. In the third part we treat certain orbits of characters of the abelian Lie group $K = \mathbb{R}^{m+1}$. The motivation for the last part is that 'many' problems for exponential Lie groups can be reduced to the commutative case by applying the results of the first subsection.

7.1 The Heisenberg group as a normal subgroup

In this section we present another tool for the purpose of proving

$$\bigcap_{s\in S} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$$

for certain unitary representations of a locally compact group M. The method of restricting to subquotients, which we will apply here, relies on the results of Poguntke in [26]. Under certain conditions it suffices to prove

$$\bigcap_{s\in S} \ker_{L^1(K,w)} \kappa_s \not\subset \ker_{L^1(K,w)} \lambda$$

where K is proper subgroup of M and w is a continuous weight function on K. The representations κ_s and λ of the Beurling algebra $L^1(K,w)$ are determined by π_s and λ. They have a considerably simpler form than the original ones. The Beurling algebra $L^1(K,w)$ is related to subquotients of the group algebra $L^1(M)$. The purpose of this subsection is to illustrate and prove Theorem 7.10.

First we describe the structure of the groups under investigation. Let M be a locally compact group which contains the 3-dimensional, connected, and simply connected Heisenberg group B as a normal subgroup. Assume that the center Z of B is central in M. Furthermore, let us suppose that there exists a two-dimensional, commutative, and connected subgroup A of B such that $Z \subset A$ and such that A is normal in M.

Let $\mathfrak{z} \subset \mathfrak{a} \subset \mathfrak{b}$ denote the corresponding Lie algebras. Let us choose a basis b_1, b_2, b_3 of $\mathfrak{b}$ such that $[b_1, b_2] = b_3$ and such that $\mathfrak{a} = \langle\, b_2, b_3\,\rangle$. As usual we work with coordinates of the second kind given by

$$(x, y, z) = \exp(xb_1)\exp(yb_2 + zb_3)\ .$$

Here we suppress the diffeomorphism from $\mathbb{R}^3$ onto B. Then the group multiplication of B is given by

$$(x, y, z)\cdot(\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x},\ y + \bar{y},\ z + \bar{z} - \bar{x}y)\ .$$

The group M acts on the normal subgroup B by conjugation. Since $I_m : B \longrightarrow B$,

$$I_m(x, y, z) = m\cdot(x, y, z)\cdot m^{-1}$$

is a continuous group homomorphism, it follows from Theorem 4.2 on p. 84 of [16] that I_m is smooth for every $m \in M$. It is easy to see that $\mathrm{Ad} : M \longrightarrow \mathrm{Aut}(\mathfrak{b})$, $\mathrm{Ad}(m) = dI_m$ is continuous. We obtain

$$\mathrm{Ad}(m) = \begin{pmatrix} \delta(m) & 0 & 0 \\ -\tau(m) & \delta(m)^{-1} & 0 \\ -\sigma(m) & \omega(m) & 1 \end{pmatrix}$$

with respect to the basis b_1, b_2, b_3 from above, where $\delta : M \longrightarrow \mathbb{R}_{>0}$ and $\sigma, \tau, \omega : M \longrightarrow \mathbb{R}$ are continuous functions. Using the Campbell-Baker-Hausdorff formula for the two-step nilpotent Lie group B, we obtain

$$I_m(x, y, z) = (\delta(m)x,\ \delta(m)^{-1}y - \tau(m)x,\ z - \sigma(m)x + \omega(m)y + \frac{1}{2}\delta(m)\tau(m)x^2)\ .$$

Since $\mathrm{Ad} : M \longrightarrow \mathrm{Aut}(\mathfrak{b})$ is a group homomorphism, we get the following formulas for δ, σ, τ, and ω:

$$\begin{aligned} \delta(m\bar{m}) &= \delta(m)\delta(\bar{m}) \\ \sigma(m\bar{m}) &= \sigma(m)\delta(\bar{m}) + \omega(m)\tau(\bar{m}) + \sigma(\bar{m}) \\ \tau(m\bar{m}) &= \tau(m)\delta(\bar{m}) + \delta(m)^{-1}\tau(\bar{m}) \\ \omega(m\bar{m}) &= \omega(m)\delta(\bar{m})^{-1} + \omega(\bar{m}) \end{aligned}$$

For $(x, y, z) \in B$ we have $\delta(x, y, z) = 1$, $\sigma(x, y, z) = y$, $\tau(x, y, z) = 0$ and $\omega(x, y, z) = x$.

Now we describe the relevant unitary representations. Let us consider the fixed character $\chi \in \widehat{A}$ given by $\chi(0, y, z) = e^{iz}$. The closed subgroup $H = \{\omega = 0\}$ is the stabilizer of χ in M, because

$$(m\cdot\chi)\,(0, y, z) = \chi\left(m^{-1}(0, y, z)m \right) = e^{i(z+\omega(m)y)}\ .$$

In this section we study unitary representations κ of the subgroup H in some Hilbert space $\mathfrak{K}$ such that

$$\kappa(0, y, z) = \chi(0, y, z)\cdot\mathrm{Id}\ .$$

The following simple observation will be useful.

Lemma. *Let P be a closed subgroup of H such that $A \subset P$. Further let σ be a unitary representation of P such that $\sigma(a) = \chi(a)\cdot\mathrm{Id}$ for all $a \in A$. Then the induced representation $\kappa = \mathrm{ind}_P^H\,\sigma$ also satisfies $\kappa(a) = \chi(a)\cdot\mathrm{Id}$.*

Now we compute the induced representation $\pi = \mathrm{ind}_H^M\,\kappa$ in terms of δ, σ and τ. We can identify the homogeneous space M/H with $\mathbb{R}$ such that the quotient map is given by $q(m) = \delta(m)\omega(m)$. Then $s(\xi) = (\xi, 0, 0) = \exp(\xi b_1)$ defines a continuous cross-section for q. The action of M on $\mathbb{R}$ is given by

$$m^{-1}\cdot\xi = \delta(m)^{-1}\xi - \omega(m)\ .$$

Obviously, the Lebesgue measure is relatively invariant for this action with modular function $\Delta_{M,H} = \delta$. These considerations show how to realize the induced representation π in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}, \mathfrak{K})$, compare also Section 6. Here we obtain

$$\pi\left((x, 0, 0)\cdot h\right)\varphi\,(\xi) = \delta(h)^{-1/2}\, e^{i\mu(x-\xi,h)}\, \kappa(h)\, \varphi\left(\delta(h)^{-1}(\xi - x)\right)$$

for $h \in H$, where

$$\mu(x-\xi,h) = \delta(h)^{-1}\sigma(h)(x-\xi) + \frac{1}{2}\delta(h)^{-1}\tau(h)(x-\xi)^2 \ .$$

In particular we get

$$\pi(x,y,z)\varphi\ (\xi) = e^{iz}e^{i(x-\xi)y}\ \varphi(\xi - x)$$

for $(x,y,z) \in B$. Note that $\pi|Z = \chi{\cdot}\mathrm{Id}$ and $\pi|B = \mathrm{ind}_A^B(\chi{\cdot}\mathrm{Id})$.

The following lemma is a standard result, see for example pp. 57-62 in [5]. The proof of Proposition 7.1 is also motivated by [5].

Lemma. *Let π be defined as above. Let $\mathfrak{H}^\infty$ denote the dense subspace of π-smooth vectors in $\mathfrak{H} = L^2(\mathbb{R},\mathfrak{K})$. Then every $\varphi \in \mathfrak{H}^\infty$ is continuous.*

Proof. By definition $\pi(x,0,0) = \tau(x)$ is a translation operator in $\mathfrak{H}$. If $\varphi \in \mathfrak{H}^\infty$, then the limit

$$\lim_{h \longrightarrow 0} \frac{1}{h}(\tau(-h)\varphi - \varphi)$$

exists in L^2. Hence φ is weakly differentiable in L^2. Since $\mathfrak{K}$ is a Hilbert space, the Fourier transformation

$$\widehat{\varphi}(\eta) = \int_{-\infty}^{\infty} e^{-i\xi\eta}\ \varphi(\xi)\, d\xi$$

defines an isometric isomorphism from $L^2(\mathbb{R},\mathfrak{K})$ onto itself. This observation allows us to prove a Sobolev lemma for $\mathfrak{K}$-valued functions. The claim of the lemma is now obvious. □

The conclusion of the following proposition is actually a consequence of Mackey's theory.

Proposition 7.1. *The induced representation $\pi = \mathrm{ind}_H^M\, \kappa$ is irreducible if and only if κ is irreducible.*

Proof. If κ is reducible, then π is also reducible because the inducing procedure respects the formation of direct sums.

For the opposite implication, assume that κ is irreducible. By Schur's lemma it suffices to prove that the commutant $\pi(M)'$ in $\mathcal{B}\left(L^2(R,\mathfrak{K})\right)$ is one-dimensional. Let $A \in \pi(M)'$. Since $\pi(0,y,0)$ is equal to the multiplication operator M_{χ_y} with $\chi_y(\xi) = e^{-iy\xi}$, we see that A commutes with M_{χ_y} for all $y \in \mathbb{R}$. But the subspace generated by the characters $\{\chi_y : y \in \mathbb{R}\}$ is dense in $L^\infty(\mathbb{R})$ with respect to the $\sigma(L^\infty, L^1)$-topology. Thus it follows that A commutes with M_ψ for all $\psi \in L^\infty(\mathbb{R})$.

The next step is to verify that for all $\xi \in \mathbb{R}$ there exists a bounded operator $A_\xi : \mathfrak{K} \longrightarrow \mathfrak{K}$ such that $A\varphi\ (\xi) = A_\xi\ \varphi(\xi)$. By the preceding lemma the subspace $\mathfrak{K}^\infty = \{\varphi(\xi) : \varphi \in \mathfrak{H}^\infty\}$ of $\mathfrak{K}$ is well-defined. The definition of $\mathfrak{K}^\infty$ is independent of ξ because $\mathfrak{H}^\infty$ is invariant under translations. Obviously, $\mathfrak{K}^\infty$ is dense in $\mathfrak{K}$. Let $\xi \in \mathbb{R}$ and $\varphi \in \mathfrak{H}^\infty$. It suffices to prove that

$$|A\varphi\ (\xi)| \leq |A|\, |\varphi(\xi)|\ .$$

Let $\epsilon > 0$ be arbitrary. Since φ and $A\varphi$ are continuous, there exists a $\delta > 0$ such that $|\varphi(\eta)| \leq |\varphi(\xi)| + \epsilon$ and $|A\varphi\ (\eta)| \geq |A\varphi\ (\xi)| - \epsilon$ for all $\eta \in U = (\xi - \delta,\ \xi + \delta)$. Let ψ denote the characteristic function of the interval U. Finally, the inequality

$$\int_U |A\varphi\ (\eta)|^2 d\eta = |(M_\psi A)\ \varphi|_2^2 = |(AM_\psi)\ \varphi|_2^2 \leq |A|^2 |M_\psi\ \varphi|_2^2 = |A|^2 \int_U |\varphi(\eta)|^2 d\eta$$

implies

$$|A\varphi\ (\xi)| - \epsilon\ \leq |A|\ (|\varphi(\xi)| + \epsilon)\ .$$

Let $\epsilon \longrightarrow 0$. This finishes the proof of the existence of the A_ξ. We have shown that A is the direct integral of the bounded operators A_ξ.

Since A commutes with the translation operators $\pi(x, 0, 0)$ for $x \in \mathbb{R}$, it follows that $A_0 = A_\xi$ for all $\xi \in \mathbb{R}$. We know

$$A\pi(h)\varphi\ (\xi) = \delta(h)^{-1/2}\ e^{i\mu(-\xi,h)}\ A_0\ \kappa(h)\ \varphi(\delta(h)^{-1}\xi)$$

and

$$\pi(h)A\varphi\ (\xi) = \delta(h)^{-1/2}\ e^{i\mu(-\xi,h)}\ \kappa(h)A_0\ \varphi(\delta(h)^{-1}\xi)$$

for all $\xi \in \mathbb{R}$ and $h \in H$. This implies $A_0\ \kappa(h) = \kappa(h)A_0$ for all $h \in H$. Since κ is irreducible, we obtain $A_0 = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. This proves $A = \lambda\,\mathrm{Id}$. $\square$

Next we introduce a certain quotient of $L^1(M)$. Let $\mathcal{C}_0(M)_\chi$ denote the space of continuous functions $f : M \longrightarrow \mathbb{C}$ such that $f(mz) = \overline{\chi(z)} f(m)$ for all $m \in M$ and $z \in Z$, and such that the support of $|f|$ is compact modulo Z. Let $L^1(M)_\chi$ denote the completion of $\mathcal{C}_0(M)_\chi$ with respect to the norm

$$|f|_1 = \int_{M/Z} |f(m)| dm\ .$$

We provide $L^1(M)_\chi$ with the structure of a Banach $*$-algebra:

$$(f * g)\ (m) = \int_{M/Z} f(mn)g(n^{-1})\ dn \quad \text{and} \quad f^*\ (m) = \Delta_{M/Z}(m)^{-1}\ \overline{f(m^{-1})}\ .$$

The map $T_\chi : L^1(M) \longrightarrow L^1(M)_\chi$ given by

$$T_\chi\ f\ (m) = \int_Z f(mz)\chi(z)\ dz$$

is a homomorphism of Banach $*$-algebras. It is even a quotient map of Banach spaces. This is easy to see, if there exists a continuous cross-section s for the quotient map $q : M \longrightarrow M/Z$: Let $v \in \mathcal{C}_0(Z)$ such that $v \geq 0$ and $|v|_1\ = 1$. Then

$$S_\chi\ f\ (m) = f(m)\ v\left(s(q(m))^{-1}m\right)$$

defines a linear, isometric cross-section for T_χ.

Further, a non-degenerate, unitary representation π of M factors over T_χ if and only if $\pi|Z = \chi\cdot\mathrm{Id}$.

Similarly, we define the algebra $L^1(B)_\chi$ as a quotient of $L^1(B)$. Note that $L^1(B)_\chi$ is contained in the adjoint algebra $L^1(M)^b_\chi$ of $L^1(M)_\chi$ by convolution from the left:

$$(a * f)\,(m) = \int\limits_{B/Z} a(b) f(b^{-1}m)\; db$$

For the definition of the adjoint algebra see §3 of [19]. There exists an approximate identity for $L^1(M)_\chi$ in $L^1(B)_\chi$, i.e., there exists a net (u_λ) in $L^1(B)_\chi$ such that

$$|\, u_\lambda\, * f - f|_1 \longrightarrow 0$$

for all $f \in L^1(M)_\chi$. It is well-known that $L^1(B)_\chi$ is isomorphic to the covariance algebra

$$L^1\left(\mathbb{R}, L^1(\mathbb{R})\right) \;\cong\; L^1\left(\mathbb{R}, \mathcal{A}(\mathbb{R})\right)$$

where $\mathcal{A}(\mathbb{R})$ denotes the Fourier algebra. Here the action of $\mathbb{R}$ on $L^1(\mathbb{R})$ is given by $a^t(x) = e^{-itx}a(x)$. The action of $\mathbb{R}$ on $\mathcal{A}(\mathbb{R})$ is given by translation: $a^t(x) = a(t+x)$.

There are many results on covariance algebras of the form $L^1(R, \mathcal{A})$ where R is a locally compact group and $\mathcal{A} \subset \mathcal{C}_\infty(R)$ is a translation invariant subalgebra satisfying certain additional assumptions, see e.g. [21], [22], and [18]. The following theorem has been obtained by Leptin in 1975. The conclusion of the theorem is true under weaker assumptions on R and $\mathcal{A}$. A proof can be found in §2 of [21] or in §3, Theorem 4 of [22]. The idea for the proof of the density of $\mathcal{E}$ goes back to [18].

Theorem 7.2. *Let R be a locally compact abelian group. In this case Δ_R is trivial. Let $\mathcal{A}$ be an involutive subalgebra of $\mathcal{C}_\infty(R)$ such that the following conditions are satisfied:*

(i) $\mathcal{A}$ is a Banach algebra under a norm $|\cdot|$ such that $|a|_\infty \le |a|$ for all $a \in \mathcal{A}$. The inclusion $\mathcal{A} \longrightarrow \mathcal{C}_\infty(R)$ is continuous with respect to the norm topologies.

(ii) $\mathcal{A}$ is invariant under translations $a^t(x) = a(tx)$ by elements of R. The action of R on $\mathcal{A}$ is isometric and strongly continuous with respect to the norm $|\cdot|$

(iii) $\mathcal{A}$ is a regular function algebra. It is dense in $\mathcal{C}_\infty(R)$ in the ∞-norm by the Stone-Weierstraß theorem

(iv) $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{C}_0(R)$ is dense in $\mathcal{A}$ with respect to the norm $|\cdot|$

If $a, b \in \mathcal{A}$, then $(a \circ b)(x) = a^x\,\bar{b}$ defines a continuous function $R \longrightarrow \mathcal{A}$. If $a, b \in \mathcal{A}_0$, then the support of $a \circ b$ is compact: $\mathrm{supp}(a \circ b) \subset \mathrm{supp}(a)\,\mathrm{supp}(b)^{-1}$.

The linear subspace $\mathcal{E}$ generated by the subset $\{a \circ b \,:\, a, b \in \mathcal{A}_0\}$ is dense in the covariance algebra $\mathcal{B} = L^1(R, \mathcal{A})$ in the L^1-norm.

The algebra $\mathcal{B} = L^1(R, \mathcal{A})$ is (topologically) simple: If $I \lhd \mathcal{B}$ is a closed, two-sided ideal of $\mathcal{B}$, then $I = 0$ or $I = \mathcal{B}$.

Remark 7.3. Let $a, b, c, d \in \mathcal{A}$ and $f, g \in \mathcal{B}$.

(i) The product in $\mathcal{B}$ is given by the convolution

$$f * g\,(x) = \int\limits_R f(xy)^y\, g(y^{-1})\, dy\ .$$

(ii) If $a \circ b,\ c \circ d \in \mathcal{B}$, then $(a \circ b) * (c \circ d)$ is in $\mathcal{B}$ and equal to $< c, b > (a \circ d)$. It holds $(a \circ b)^* = b \circ a$.

(iii) Let $a \in \mathcal{A} \cap L^1(R)$ such that $|a|_2 = 1$ and such that $(a \circ a) \in \mathcal{B}$. Then $p = a \circ a$ is a minimal hermitian idempotent in $\mathcal{B}$, i.e., $p^2 = p = p^*$ and $p * \mathcal{B} * p \cong \mathbb{C}$.

(iv) Let $a, b \in \mathcal{A} \cap L^1(R)$ such that $|a|_2 = |b|_2 = 1$ and $(a \circ a), (b \circ b) \in \mathcal{B}$. Then the equalities $a \circ b = (a \circ a) * (a \circ b)$ and $(a \circ b) * (b \circ a) = a \circ a$ are valid.

Remark 7.4. Let us recall two useful density statements. Let $p = a \circ a$ be a non-zero idempotent in $\mathcal{B} = L^1(R, \mathcal{A})$ as above. By Theorem 7.2 we know that $\mathcal{B}$ is simple. Thus the two-sided ideal $\mathcal{B} * p * \mathcal{B}$ is dense in $\mathcal{B}$. Additionally, let us assume that $\mathcal{A}$ has an approximate identity. From §1 of [20] it follows that the covariance algebra $\mathcal{B} = L^1(R, \mathcal{A})$ has an approximate identity, too. In particular $\mathcal{B} * \mathcal{B} * \mathcal{B}$ is dense in $\mathcal{B}$. Now Theorem 7.2 implies that

$$\sum_{a, b \in \mathcal{A}_0} (a \circ a) * \mathcal{B} * (b \circ b)$$

is dense in $\mathcal{B}$.

We confine ourselves to the case $R = \mathbb{R}$ again.

Remark 7.5. Let u be the unique Gauß function in $\mathcal{S}(\mathbb{R})$ such that $\hat{u}(\xi) = Ce^{-\xi^2}$, where we choose $C > 0$ such that $|\hat{u}|_2 = 1$. Then

$$p(x, y, z) = e^{-iz}\, (u^x * u)(y)$$

defines a minimal hermitian idempotent in $L^1(B)_\chi$. An explicit evaluation of the convolution $u^x * u$ shows that p is L^1-integrable modulo Z. If we consider p as an element of $L^1\left(\mathbb{R}, L^1(\mathbb{R})\right)$, then we have $p(x) = u^x * u$. In $L^1\left(\mathbb{R}, \mathcal{A}(\mathbb{R})\right)$ we have $p = \hat{u} \circ \hat{u}$.

It is well-known that $\mathcal{A}(\mathbb{R})$ is a Wiener algebra, see Chapter 1 of [30]. The isomorphism $L^1(\mathbb{R}) \longrightarrow \mathcal{A}(\mathbb{R})$ given by Fourier transformation provides $\mathcal{A}(\mathbb{R})$ with a complete norm. The subspace $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{C}_0(\mathbb{R})$ is dense in this norm. Clearly, the functions in $\mathcal{A}$ separate points from closed sets. Now Theorem 7.2 and the preceding remarks yield the following

Lemma 7.6. *There exists a set $\mathcal{Q} \subset L^1(B)_\chi$ of minimal hermitian idempotents with the following properties:*

(i) $q^2 = q = q^*$ *and* $q * L^1(B)_\chi * q \cong \mathbb{C}$ *for all* $q \in \mathcal{Q}$.

(ii) *the two-sided ideal* $L^1(B)_\chi * q * L^1(B)_\chi$ *is dense in* $L^1(B)_\chi$ *for* $q \in \mathcal{Q}$.

(iii) *For every pair* $(q_1, q_2) \in \mathcal{Q} \times \mathcal{Q}$ *there exists an element* v_{q_1, q_2} *of* $L^1(B)_\chi$ *such that* $v^*_{q_1, q_2} = v_{q_2, q_1}$, $q_1 * v_{q_1, q_2} = v_{q_1, q_2}$, *and* $v_{q_1, q_2} * v_{q_2, q_1} = q_1$ *for all* $q_1, q_2 \in \mathcal{Q}$.

(iv) The subspace $\sum_{q_1,\, q_2 \in \mathcal{Q}} q_1 * L^1(B)_\chi * q_2$ *is dense in* $L^1(B)_\chi$ *.*

(v) The projector $p(x,y,z) = e^{-iz}\,(u^x * u)(y)$ *is contained in* $\mathcal{Q}$ *.*

The existence of such a 'rich' set of idempotents has been exploited in [25] in order to prove the symmetry of certain covariance algebras.

Let us define the closed subgroup $K = \{\omega = \sigma = 0\}$ of H. Since $Y = \exp(\mathbb{R}b_2)$ is a normal subgroup of H, we see that H is the semi-direct product of K and Y. We consider the symmetric, continuous weight function $w : K \longrightarrow \mathbb{R}_{>0}$ given by

$$w(h) = \Big(\, 4\,\big(\,\delta(h) + \delta(h)^{-1}\big)^2 + \tau(h)^2\Big)^{1/4}$$

and form the Beurling algebra $L^1(K,w)$ as well as the quotient $L^1(K,w)_\chi$. Note that $(x,h) \mapsto (x,0,0)\,h$ defines a homeomorphism from $\mathbb{R}\times H$ onto M. The Haar measure of M is given by

$$\int_M f(m)\,dm = \int_{-\infty}^{+\infty}\int_H \delta(h)^{-1}\, f\,(\,(x,0,0)\,h\,)\;dh\,dx\ .$$

In [26] Poguntke obtained the following remarkable result.

Theorem 7.7. *For any minimal hermitian idempotent* $q \in \mathcal{Q}$ *there exists an isomorphism* $S_q = S_{\chi,q}$ *of Banach* $*$*-algebras from the quotient* $L^1(K,w)_\chi$ *of the Beurling algebra* $L^1(K,w)$ *onto the subquotient* $q * L^1(M)_\chi * q$ *of* $L^1(M)$ *such that the following holds:*

Let κ *be an arbitrary unitary representation of the subgroup* H *of* M *in the Hilbert space* $\mathfrak{K}$ *such that* $\kappa(0,y,z) = e^{iz}\cdot\mathrm{Id}$ *for all* $(0,y,z) \in A$*. Let* $\pi = \mathrm{ind}_H^M\,\kappa$ *denote the induced representation in* $\mathfrak{H}$*. Then there exists an isomorphism of Hilbert spaces* $V_q : \mathfrak{K} \longrightarrow \pi(q)\mathfrak{H}$ *such that*

$$\pi(S_q\, f)\varphi = \big(V_q\,\kappa(f)\,V_q^{-1}\big)\,\varphi$$

holds for all $f \in L^1(K,w)_\chi$ *and* $\varphi \in \pi(q)\mathfrak{H}$*.*

Proof. Let $p(x,y,z) = e^{-iz}\,(u^x * u)(y)$ be the projector from above. Then

$$(V_p\,\eta)\;(\xi) = \hat{u}(\xi)\eta$$

gives a unitary isomorphism from $\mathfrak{K}$ onto $\pi(p)\mathfrak{H}$. For $f \in L^1(K,w)_\chi$ we define

$$(S_p\,f)\;(\,(x,0,0)\,h\,) = f(\dot{h})\;\Phi(x,h)$$

where $h \mapsto \dot{h}$ denotes the quotient map $H \longrightarrow H/Y = K$ and the continuous function $\Phi : \mathbb{R}\times H \longrightarrow \mathbb{C}$ is given by

$$\Phi(x,h) = \frac{1}{2\pi}\;\delta(h)^{1/2}\int_{-\infty}^{+\infty} \hat{u}(-s)\hat{u}(x-\delta(h)s)\;e^{-\frac{1}{2}i\delta(h)\tau(h)s^2}\;e^{-i\sigma(h)s}\;ds\ .$$

In Section (F), Theorem 1 of [26] Poguntke has shown that

$$S_p : L^1(K,w)_\chi \longrightarrow p * L^1(M)_\chi * p$$

is an isometric isomorphism of Banach $*$-algebras which transforms π into κ in the sense of this theorem. The proof of the isometry of S_p depends on the following fact:

$$w(h) \;=\; \delta(h)^{-1} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} |\Phi\,(x,\; h(0,y,0))\;|\; dy\, dx \;.$$

An explicit computation of this integral is merely possible because $\hat{u}$ is a Gauß function.

Now it is easy to prove the existence of S_q for arbitrary $q \in \mathcal{Q}$: Let us consider the map

$$\psi = \psi_{q,p} \;:\; p * L^1(M)_\chi * p \longrightarrow q * L^1(M)_\chi * q$$

given by $\psi(f) = v_{q,p} * f * v_{p,q}$. It is obvious that ψ is linear and bounded. Since $v_{p,q} * v_{q,p} = p$, it holds $\psi(f*g) = \psi(f)*\psi(g)$. Since $v_{p,q}^* = v_{q,p}$, we have $\psi(f^*) = \psi(f)^*$. Of course, the inverse of ψ is given by $\psi^{-1}(f) = v_{p,q} * f * v_{q,p}$. Hence ψ is an isomorphism of Banach $*$-algebras.

It is clear that $S_q = \psi_{q,p} \circ S_p$ and $V_q = \pi(v_{q,p}) \circ V_p$ are isomorphisms which satisfy

$$\pi(S_q\, f)\varphi = \left(V_q\, \kappa(f)\, V_q^{-1}\right)\varphi \;.$$

$\square$

Remark 7.8. One verifies easily that the two-sided ideal $L^1(M)_\chi * \; q \; * L^1(M)_\chi$ is dense in $L^1(M)_\chi$. Similarly, it follows that the subspace $\sum_{q_1,\, q_2 \in \mathcal{Q}} \; q_1 \; * L^1(M)_\chi \; * q_2$ is dense in $L^1(M)_\chi$: Since $L^1(B)_\chi$ contains an approximate identity for $L^1(M)_\chi$, the subspace $L^1(B)_\chi \; * L^1(M)_\chi \; * L^1(B)_\chi$ is dense in $L^1(M)_\chi$. On the other hand, this subspace is contained in the closure of $\sum_{q_1,\, q_2 \in \mathcal{Q}} \; q_1 \; * L^1(M)_\chi * q_2$ by property *(iv)* of Lemma 7.6.

Remark 7.9. Let π be an irreducible representation of $L^1(M)_\chi$ in a Hilbert space $\mathfrak{H}$. Let U be a non-zero subspace of $\mathfrak{H}$. Then the subspace $\sum_{q \in \mathcal{Q}} \; \pi(q)U$ is dense in $\mathfrak{H}$. This follows immediately because this subspace is non-zero and its closure is $\pi(M)$-invariant. Here we use the fact that $\sum_{q_1,\, q_2 \in \mathcal{Q}} \; q_1 \; * L^1(M)_\chi * q_2$ is dense in $L^1(M)_\chi$.

Theorem 7.10. *Let $\{\kappa_s : s \in S\}$ and λ be irreducible, unitary representations of the subgroup H of M such that $\kappa_s(0,y,z) = \lambda(0,y,z) = e^{iz}\cdot\mathrm{Id}$ for all $(0,y,z) \in A$. Let $\pi_s = \mathrm{ind}_H^M\, \kappa_s$ and $\rho = \mathrm{ind}_H^M\, \lambda$ denote the corresponding induced representations. Then*

$$\bigcap_{s \in S}\; \ker_{L^1(M)}\; \pi_s \;\subset\; \ker_{L^1(M)}\; \rho$$

if and only if

$$\bigcap_{s \in S}\; \ker_{L^1(K,w)}\; \kappa_s \;\subset\; \ker_{L^1(K,w)}\, \lambda \;.$$

Proof. It is needless to say that $\bigcap \ker_{L^1(M)} \pi_s \subset \ker_{L^1(M)} \rho$ is equivalent to the corresponding inclusion in $L^1(M)_\chi$. The analogous statement holds for the kernels of the representations κ_s and λ in $L^1(K,w)$ and $L^1(K,w)_\chi$. Let us fix a $q \in \mathcal{Q}$.

Let $f \in L^1(K,w)_\chi$ such that $\kappa_s(f) = 0$ for all s. We apply Theorem 7.7 and obtain

$$\pi_s(S_q\, f)\varphi = \left(V_q\, \kappa_s(f)\, V_q^{-1}\right)\varphi = 0$$

for all s and $\varphi \in \pi(q)\mathfrak{H}$. This implies $\pi_s(S_q\; f)\varphi = 0$ for all $\varphi \in \mathfrak{H}$ because $S_q\, f \in q * L^1(M)_\chi * q$. Now our assumption implies $S_q\, f \in \ker_{L^1(M)_\chi} \rho$ and hence $f \in \ker_{L^1(K,w)_\chi} \lambda$.

For the opposite implication, let $f \in L^1(M)_\chi$ such that $\pi_s(f) = 0$ for all s. It follows $\pi_s(q * f * q) = 0$. By Theorem 7.7 there exists a function $h \in L^1(K,w)_\chi$ such that $S_q\, h = q * f * q$. It holds $\kappa_s(h) = 0$ for all s. Now our assumption implies $\lambda(h) = 0$ and thus $\rho(q * f * q) = 0$. Since $\pi_s(f) = 0$ implies $\pi_s(g_1 * f * g_2) = 0$, the same argument shows that we even get $\rho(q * g_1 * f * g_2 * q) = 0$ for all $g_1, g_2 \in L^1(M)_\chi$. Since the ideal $L^1(M)_\chi * q * L^1(M)_\chi$ is dense in $L^1(M)_\chi$, we obtain $\rho(f) = 0$. □

The preceding theorem states that $\kappa \mapsto \pi = \mathrm{ind}_H^M \kappa$ gives a homeomorphism from $\mathrm{Prim}_* L^1(K,w)_\chi$ onto a subset of $\mathrm{Prim}_* L^1(M)_\chi$ where both of these sets of primitive ideals carry the Jacobson topology.

Remark 7.11. We will apply Theorem 7.10 in order to prove

$$\bigcap_{s\in S} \ker_{L^1(M)}\ \pi_s \ \not\subset\ \ker_{L^1(M)}\ \rho\,.$$

The representations κ_s and λ have a considerably simpler form than the induced representations π_s and ρ. It may be fairly easy to verify

$$\bigcap_{s\in S} \ker_{L^1(K,w)}\ \kappa_s \ \not\subset\ \ker_{L^1(K,w)} \lambda\,.$$

To establish this relation, one has to find an $f \in L^1(K)$ which is integrable against the weight function w and satisfies $\kappa_s(f) = 0$ for all s and $\lambda(f) \neq 0$.

The method of restricting to subquotients is a useful tool. The following proposition allows us to iterate this procedure.

Proposition 7.12. *Let $q \in \mathcal{Q}$ be a minimal hermitian idempotent and w the weight function on K defined above. Let w_0 be a (continuous, symmetric) weight function on M which is constant on $B-$cosets. Then S_q maps $L^1(K, ww_0)_\chi$ onto $q*L^1(M,w_0)_\chi*q$.*

Proof. Let us begin with a preliminary remark. We write

$$|f|_{1,w_0} \;=\; \int_{M/Z} |f(m)|\; w_0(m)\; dm$$

for $f \in L^1(M,w_0)_\chi$. Since w_0 is constant on cosets of the normal subgroup B, we obtain the estimation

$$|a * f|_{1,w_0} \;\leq |a|_1\; |f|_{1,w_0}$$

for $a \in L^1(B)_\chi$ and $f \in L^1(M, w_0)_\chi$. The analogous estimation holds for $f * a$. This proves that the subalgebra $L^1(M, w_0)_\chi$ is invariant under convolutions by elements of $L^1(B)_\chi$ from both sides. In particular

$$q * L^1(M, w_0)_\chi * q \;=\; q * L^1(M)_\chi * q \,\cap\, L^1(M, w_0)\;.$$

First we consider $p(x, y, z) = e^{-iz}(u^x * u)(y)$. We modify the proof of the isometry of S_p in [26]. Since w_0 is constant on B-cosets, we obtain

$$\begin{aligned}
|S_p\, f|_{1,w_0} &= \int_{-\infty}^{+\infty}\int_{K/Z}\int_{-\infty}^{+\infty} \delta(h)^{-1}|f(h)|\, |\Phi\,(x,\ h(0,y,0))\,|\, w_0\,((x,0,0)h(0,y,0))\;dy\,dh\,dx \\
&= \int_{K/Z} |f(h)|\, w_0(h)\,\delta(h)^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\Phi\,(x,\ h(0,y,0))\,|\;dy\,dx\,dh \\
&= \int_{K/Z} |f(h)|\, w(h) w_0(h)\;dh \\
&= |f|_{1,ww_0}
\end{aligned}$$

This shows that S_p yields an isometric isomorphism from $L^1(K, ww_0)_\chi$ into the subquotient $p * L^1(M, w_0)_\chi * p$. For the surjectivity of S_p we consider

$$(T_p\, F)(h) \;=\; \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi\,(x,\ h(0,y,0))\;\; F\,(\,(x,0,0)h(0,y,0)\,)\;\,dy\,dx$$

for $F \in L^1(M)_\chi$ and $h \in K$, where $\Psi : \mathbb{R} \times H \longrightarrow \mathbb{C}$ is given by

$$\Psi(x, h) = \delta(h)^{-1/2} \int_{-\infty}^{+\infty} \hat{u}(-s)\;\hat{u}(x - \delta(h)s)\; e^{\frac{1}{2}i\delta(h)\tau(h)s^2}\; e^{i\sigma(h)s}\;ds\;.$$

In [26] it has been shown that T_p is a continuous, linear map from $L^1(M)_\chi$ onto $L^1(K, w)_\chi$ such that

$$p * F * p = S_p(T_p\, F)\;.$$

To prove the continuity of T_p with respect to the norms $|\cdot|_{1,w}$ and $|\cdot|_1$, one uses the fact that

$$\delta(h)\,|\Psi\,(\,x,\ h(0,y,0)\,)\ |\ w(h)\ \le 2$$

for all $x, y \in \mathbb{R}$ and all $h \in K$. Similarly, we obtain that T_p is continuous with respect to the norms $|\cdot|_{1,w_0}$ on $L^1(M, w_0)_\chi$ and $|\cdot|_{1,ww_0}$ on $L^1(K, ww_0)_\chi$. This proves that S_p maps $L^1(K, ww_0)_\chi$ onto $p * L^1(M, w_0)_\chi * p$. Since the $\psi_{q,p}$ leave $L^1(M, w_0)_\chi$ invariant, we see that the conclusion of the lemma holds for arbitrary $q \in \mathcal{Q}$. □

On the Heisenberg group B we define the symmetric, polynomial weight function

$$w_1(x, y, z) = (1 + |x|)(1 + |y|)\;.$$

Note that w_1 is constant on the center Z of B and hence constant on Z-cosets. Let us consider the quotient $L^1(B, w_1)_\chi$ of the Beurling algebra $L^1(B, w_1)$. It is easy to see that the projector $p(x,y,z) = e^{-iz}\,(u^x * u)(y)$ is in $L^1(B,w_1)_\chi$. If we recall the arguments for the proof of Lemma 7.6, then we see that there exists a subset $\mathcal{Q}'$ of $L^1(B,w_1)_\chi$ such that the conditions *(i)* to *(v)* of Lemma 7.6 are satisfied with $L^1(B,w_1)_\chi$ instead of $L^1(B)_\chi$.

In analogy to Proposition 7.12 the following proposition treats a situation when the weight function w_0 is not constant on B-cosets.

Proposition 7.13. *Let $q \in \mathcal{Q}'$ be arbitrary and w the weight function defined above. Let w_0 be a symmetric weight function on M and $D_0 > 0$ such that*

$$w_0(b) \le D_0\, w_1(b)$$

for all $b \in B$. Then S_q maps $L^1(K, w^9 w_0)$ into $q * L^1(M, w_0) * q$.

Proof. Let w_0' denote the restriction of w_0 to B. Our assumption on w_0 implies $L^1(B,w_1)_\chi \subset L^1(B,w_0')_\chi$. The inequality

$$|\, a * f \,|_{1,w_0} \;\le\; |\, a \,|_{1,w_0'}\; |\, f \,|_{1,w_0}$$

shows that $L^1(M,w_0)_\chi$ is invariant under convolution by elements of $L^1(B,w_0')_\chi$. In particular the $\psi_{q,p}$ leave $L^1(M,w_0)_\chi$ invariant. Thus it suffices to prove the assertion of the lemma for the projector $p(x,y,z) = e^{-iz}\,(u^x * u)(y)$. We have to prove the existence of a constant $D > 0$ such that

$$|\, S_p\, f \,|_{1,w_0} \;\le\; D\, |\, f \,|_{1,w^9 w_0}$$

for all $f \in L^1(K, w^9 w_0)_\chi$. Let $D = D_0^2/2$. It will be convenient to define

$$\alpha(h) = \Big(\, (1+\delta(h)^2)^2 + \frac{1}{4}\delta(h)^2\tau(h)^2 \Big)^{1/2}$$

for $h \in H$. Then we have $\alpha(h) \ge 1 + \delta(h)^2$. Since $w(h)^2 = 2\delta(h)^{-1}\alpha(h)$, we obtain $\delta(h) \le \frac{1}{2} w(h)^2$. A first estimation is

$$\begin{aligned}
|\, S_p f \,|_{1,w_0} &= \int_{-\infty}^{+\infty}\int_K \int_{-\infty}^{+\infty} \delta(h)^{-1} |(S_p f)\,(\,(x,0,0)h(0,y,0)\,)\,|\; w_0\,(\,(x,0,0)h(0,y,0)\,)\; dy dh dx \\
&\le D_0^2 \int_K |f(h)| w_0(h) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(h)^{-1} |\Phi\,(x,\; h(0,y,0)\,)\,|\;(1+|x|)(1+|y|) dy dx dh
\end{aligned}$$

We have to control the inner double integral. By definition Φ is essentially the Fourier transform of a Gauss function with certain parameters. We compute the absolute value of Φ explicitly and obtain

$$\begin{aligned}
|\Phi\,(x,\; h(0,y,0)\,)\,| &= \frac{1}{\pi\sqrt{2}}\; \delta(h)^{1/2}\alpha(h)^{-1/2}\; e^{-\left(y^2 + (\delta(h)y + \delta(h)\tau(h)x)^2 + 4(1+\delta(h)^2)x^2\right)/4\alpha(h)^2} \\
&\le \frac{1}{\pi\sqrt{2}}\; \delta(h)^{1/2}\alpha(h)^{-1/2}\; e^{-y^2/4\alpha(h)^2}\; e^{-(1+\delta(h)^2)x^2/\alpha(h)^2}\;.
\end{aligned}$$

Now we use the fact that

$$\int\limits_{-\infty}^{+\infty} (1+|y|)\, e^{-ay^2}\, dy \leq \frac{2\sqrt{\pi}}{a}$$

for $0 < a \leq 1$. This observation yields

$$\begin{aligned}&\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \delta(h)^{-1}\, |\Phi\,(x,\ h(0,y,0)\,)\ |(1+|x|)(1+|y|)\, dy\, dx \\ &\leq \delta(h)\,\big(\,2\delta(h)^{-1}\alpha(h)\,\big)^{7/2}\ \leq\ w(h)^9/2\ .\end{aligned}$$

Finally we see $|\, S_p\, f\,|_{1,w_0}\ \leq\ D\,|\, f\,|_{1,w^9 w_0}$. This finishes the proof of our lemma. $\square$

7.2 The case of exponential Lie groups

In the preceding subsection we studied certain representations of a locally compact group M containing a 3-dimensional Heisenberg group B as a normal subgroup such that M acts exponentially on its Lie algebra $\mathfrak{b}$ by adjoint representation. In this section we consider the case of M itself being an exponential Lie group. The mere purpose of this part is to relate the results of Subsection 7.1 to the Kirillov picture of M.

Let M be an exponential Lie group and $\mathfrak{m}$ its Lie algebra. Assume that the 3-dimensional Heisenberg group B is a normal subgroup of M as in Subsection 7.1. Let $\mathfrak{h}$ denote the Lie algebra of the closed, connected subgroup $H = \{\omega = 0\}$, and $\mathfrak{k}$ the Lie algebra of $K = \{\omega = \sigma = 0\}$. Obviously $\mathfrak{z}\mathfrak{m} \subset \mathfrak{k}$. Note that $\mathfrak{m} = \mathbb{R}b_1 + \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{k} + \mathbb{R}b_2$. We omit the proof of the following simple

Lemma. *Let $X \in \mathfrak{m}$. Then $X \in \mathfrak{h}$ if and only if $[X, b_2] \in \mathbb{R}b_2$. Further $X \in \mathfrak{k}$ if and only if $[X, b_2] \in \mathbb{R}b_2$ and $[X, b_1] \in \mathbb{R}b_1 + \mathbb{R}b_2$.*

Let $f \in \mathfrak{m}^*$ such that $f(b_3) = 1$ and $f(b_2) = 0$. Let $\bar{f} = f|\mathfrak{h}$. There exists a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{h}$ at $\bar{f} \in \mathfrak{h}^*$ such that $\mathfrak{a} \subset \mathfrak{p}$. Since $[\mathfrak{h}, \mathfrak{a}] \subset \ker f$ and

$$\mathrm{Ad}^*(\exp(yb_2))f\,(b_1) = f(b_1) + y\ ,$$

it follows that $\mathfrak{p}$ is a Pukanszky polarization at $f \in \mathfrak{m}^*$. Note that $\kappa = \mathrm{ind}_P^H \chi_f$ satisfies $\kappa(0, y, z) = e^{iz}$. By definition of the Kirillov map and by induction in stages we obtain

$$\pi = \mathcal{K}(f) = \mathrm{ind}_H^M \kappa\ .$$

If we regard κ as a representation of the quotient $K = H/Y$, then $\kappa = \mathrm{ind}_{K\cap P}^K \chi_f$.

Remark 7.14. Let $g \in \mathfrak{m}^*$ and $\{f_s : s \in S\} \subset \mathfrak{m}^*$ such that $f_s(b_3) = g(b_3) = 1$ and $f_s(b_2) = g(b_2) = 0$. Assume that there exists a common Pukanszky polarization $\mathfrak{p}$ at f_s and g such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{h}$. Let us define $\pi_s = \mathrm{ind}_P^M \chi_{f_s}$, $\rho = \mathrm{ind}_P^M \chi_g$, $\kappa_s = \mathrm{ind}_{K\cap P}^K \chi_{f_s}$ and $\lambda = \mathrm{ind}_{K\cap P}^K \chi_g$. It follows from Theorem 7.10 that the relation

$$\bigcap_{s\in S} \ker_{L^1(M)}\ \pi_s\ \not\subset\ \ker_{L^1(M)}\ \rho$$

is equivalent to

$$\bigcap_{s\in S} \ker_{L^1(K,w)}\ \kappa_s\ \not\subset\ \ker_{L^1(K,w)} \lambda\ .$$

Remark 7.15. Finally we describe an advantageous choice of coordinates for M: Let $e_1, \ldots, e_k$ be in $\mathfrak{k}$ such that the canonical images of these vectors in $\mathfrak{k}/\mathfrak{z}\mathfrak{m}$ form a Malcev basis. If we define $\Phi_1 : \mathbb{R}^k \longrightarrow M$,

$$\Phi_1(w) = \exp(w_1 e_1) \cdot \ldots \cdot \exp(w_k e_k)\ ,$$

then

$$\Phi(x, w, y, Z) = \exp(xb_1)\Phi_1(w)\exp(yb_2 + Z)$$

gives a global diffeomorphism form $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times \mathfrak{z}\mathfrak{m}$ onto M. Note that the restriction of Φ to $\mathbb{R}^k \times \mathfrak{z}\mathfrak{m}$ gives a coordinate diffeomorphism for K.

7.3 Orbits in the dual of commutative Beurling algebras

Let us describe the aim of this subsection: Let K be a vector group, i.e., the additive group of a finite-dimensional real vector space. We will prove the relation

$$\bigcap_{s\in S} \ker_{L^1(K,w)} \chi_s \not\subset \ker_{L^1(K,w)} \zeta$$

for certain subsets (orbits) $\Omega = \{\chi_s : s \in S\} \subset \widehat{K}$ and certain characters $\zeta \in \widehat{K} \setminus \overline{\Omega}$. Here S is an arbitrary index set and $L^1(K,w)$ denotes the Beurling algebra with respect to the exponential weight function w on K. One can easily guess the way in which these results enter in our investigation: If M is an exponential Lie group, then the problem of verifying

$$\bigcap_{s\in S} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$$

can sometimes be reduced to the commutative situation described above by applying the method of restricting to subquotients. Especially Theorem 7.17 will turn out to be helpful.

Let $\mathfrak{k} = \mathbb{R}e_0 \oplus \mathfrak{z}$ and $\mathfrak{z} = \mathfrak{z}_1 \oplus \ldots \oplus \mathfrak{z}_m$ be a direct sum decomposition of the Lie algebra $\mathfrak{k}$ of K, which induces a decomposition of $\mathfrak{k}^*$. Here $e_0 \in \mathfrak{k}$ and the $\mathfrak{z}_\nu$ are subspaces of $\mathfrak{k}$, typically of dimension one or two. Let us fix a Euclidean norm on $\mathfrak{z}_\nu^*$ for each ν. This gives a Euclidean norm on $\mathfrak{z}^*$: $|\,\xi\,|^2 = \sum_{\nu=1}^m |\,\xi_\nu\,|^2$. As usual we identify $\widehat{K}$ and $\mathfrak{k}^*$ by $\chi(X) = e^{if(X)}$ so that the Fourier transformation is given by $\widehat{h}(\chi) = \widehat{h}(f)$ and

$$\widehat{h}(\tau,\xi) = \int_{-\infty}^{+\infty} \int_{\mathfrak{z}} h(t,Z)\, e^{-it\tau}\, e^{-i\langle Z,\xi\rangle}\, dt\, dZ$$

for $h \in L^1(K)$. Here $\langle Z,\xi\rangle$ denotes the standard duality between $\mathfrak{z}$ and $\mathfrak{z}^*$.

For $c > 0$ we consider the exponential weight function $w(t,Z) = e^{c|t|}$ on K. Then $L^1(K,w)$ becomes an involutive Banach algebra because w is continuous and symmetric. Since $\widehat{h}$ can be extended to a holomorphic function in the strip $-c < \mathrm{Im}(t) < c$ of the complex plane, we see that for $h \in L^1(K,w)$ the Fourier transform $\widehat{h}$ is real analytic in t. This is a severe constraint on $\widehat{h}$.

Now we sketch the leading idea for the proof of the following theorems: Let us assume that the subset $\Omega = \{f_s : s \in S\} \subset \mathfrak{k}^*$ is a graph in the sense that $f_s(e_0) = \gamma(\,f_s\,|\,\mathfrak{z}\,)$ for a suitable function γ on $\mathfrak{z}^*$. If we define

$$\widehat{h}(\tau,\xi) = \left(e^{\tau} - e^{\gamma(\xi)}\right)\, e^{-\tau^2-|\xi|^2}\,,$$

then $\widehat{h}(f_s) = 0$ for all s. The essential step is to verify that the inverse Fourier transform h of $\widehat{h}$ is well-defined and in $L^1(K,w)$. Let $d = \dim \mathfrak{z}$. If γ is such that $v(\xi) = e^{\gamma(\xi)}$ is continuously differentiable up to order $d+1$ and such that $|v(\xi)| \leq e^{a|\xi|}$ holds for some $a > 0$ and all $\xi \in \mathfrak{z}^*$, then indeed $h \in L^1(K,w)$. In the sequel we will consider the case that γ has the form $\gamma(\xi) = \gamma_0(\,|\xi_1|, \ldots, |\xi_m|\,)$ where γ_0 is a slowly increasing, smooth function on the open subset $V = \{r : r_j > 0 \text{ for all } j\}$ of $\mathbb{R}^m$. If r tends to the boundary of V in $\mathbb{R}^m$, then we allow singularities of the form $\gamma_0(r) \longrightarrow -\infty$. Typically γ_0 is a polynomial in $\log r_j$. We provide the details and begin with the following simple

Lemma 7.16. *Let $g \in \mathfrak{k}^*$ and $\{f_s : s \in S\}$ be a subset of $\mathfrak{k}^*$ such that $f_s(e_0) = f_0$ for all $s \in S$ and $g(e_0) \neq f_0$. Then there exists a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$.*

Proof. If we define

$$\widehat{h}(\tau, \xi) = \left(e^{\tau - f_0} - 1 \right) \, e^{-\tau^2 - |\xi|^2} \, ,$$

then $\widehat{h}(f_s) = 0$ and $\widehat{h}(g) \neq 0$. The explicit computation of the inverse Fourier transform h of $\widehat{h}$ shows that h is also a sum of two Gauß functions and thus in $L^1(K, w)$. □

A similar argument yields

Theorem 7.17. *Let $g \in \mathfrak{k}^*$ and $\{f_s : s \in \mathbb{R}^m\} \subset \mathfrak{k}^*$ such that $|\, f_s \,|\, \mathfrak{z}_\nu \,| = e^{-s_\nu}$ for $1 \leq \nu \leq m$ and*

$$f_s(e_0) = f_0 + \sum_{\nu=1}^{m} \alpha_\nu s_\nu \, .$$

Here s_ν denotes the ν-th component of the vector $s \in \mathbb{R}^m$ and $\alpha_1, \ldots, \alpha_m$ are fixed coefficients. We define $I = \{1 \leq \nu \leq m : \alpha_\nu \neq 0\}$ and $I_0 = \{\nu \in I : g \,|\, \mathfrak{z}_\nu = 0\}$. We assume that one of the following conditions is satisfied:

(i) $I_0 = \emptyset$ and $g(e_0) \neq f_0 - \sum_{\nu=1}^{m} \alpha_\nu \log |\, g \,|\, \mathfrak{z}_\nu \,|$,

(ii) $I_0 \neq \emptyset$ and either $\{\alpha_\nu : \nu \in I_0\} \subset (-\infty, 0)$ or $\{\alpha_\nu : \nu \in I_0\} \subset (0, +\infty)$.

Then there exists a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all $s \in \mathbb{R}^m$ and $\widehat{h}(g) \neq 0$.

Remark. The preceding theorem includes the case $I = \emptyset$ in which $f_s(e_0) = f_0$. Let $\Omega = \{f_s : s \in S\}$. If $I_0 \neq \emptyset$, then the assumption concerning the signs of the coefficients $\{\alpha_\nu : \nu \in I_0\}$ is necessary. If this condition is not satisfied, then $g \in \overline{\Omega}$ is possible and this means that g cannot be separated from Ω by a continuous function $\widehat{h}$.

Proof of Theorem 7.17. Let $d = \max\{\dim \mathfrak{z}_\nu : 1 \leq \nu \leq m\}$. In any case the assumptions of our theorem imply that there exists an $\alpha \in \mathbb{R}^\times$ such that $\alpha\alpha_\nu > 2d + 3$ for all $\nu \in I_0$. Further there is an $\epsilon > 0$ such that $|\, g \,|\, \mathfrak{z}_\nu \,| > \epsilon$ for all $\nu \in I \setminus I_0$. Let us choose a function $u_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that $u_0 = 0$ on $[-\epsilon/2, \epsilon/2]$ and $u_0 = 1$ on $\mathbb{R} \setminus (-\epsilon, \epsilon)$. Let

$$u(\xi) = \prod_{\nu \in I \setminus I_0} u_0(\,|\, \xi_\nu \,|\,)$$

for $\xi \in \mathbb{R}^m$. Further we set

$$v(\xi) = \prod_{\nu \in I} |\, \xi_\nu \,|^{\alpha\alpha_\nu}$$

Note that the function uv is continuously differentiable up to order $d+1$. Now we can define a function $\widehat{h}$ on $\mathfrak{k}^* = \mathbb{R} \oplus \mathfrak{z}^*$ by

$$\widehat{h}(\tau, \xi) = u(\xi) \left(e^{-\alpha(\tau - f_0)} - v(\xi) \right) e^{-\tau^2 - |\xi|^2} \, .$$

By definition $\widehat{h}(f_s) = 0$ for all $s \in \mathbb{R}^m$. If $I_0 = \emptyset$ and $g_0 \neq f_0 - \sum_{\nu=1}^{m} \alpha_\nu \log|\, g \,|\, \mathfrak{z}_\nu \,|$, then

$$\widehat{h}(g) = \left(e^{-\alpha(g_0 - f_0)} - e^{\sum_{\nu=1}^{m} \alpha\alpha_\nu \, \log|\, g \,|\, \mathfrak{z}_\nu \,|} \right) \, e^{-\sum_{\nu=0}^{m} |\, g \,|\, \mathfrak{z}_\nu \,|^2} \neq 0 \, .$$

If $I_0 \neq \emptyset$, then

$$\widehat{h}(g) = e^{-\alpha(g(e_0) - f_0)} \, e^{-\sum_{\nu=0}^{m} |\, g \,|\, \mathfrak{z}_\nu \,|^2} \neq 0 \, .$$

Thus we see that $\widehat{h}(g) \neq 0$. Clearly $\widehat{h}$ has the form $\widehat{h}(\tau, \xi) = \widehat{h}_1(\tau, \xi) - \widehat{h}_2(\tau, \xi)$ where

$$\widehat{h}_j(\tau, \xi) = \widehat{h}_{j,0}(\tau) \widehat{h}_{j,1}(\xi_1) \cdot \ldots \cdot \widehat{h}_{j,m}(\xi_m)$$

(for $j = 1, 2$) is a tensor product of functions $\widehat{h}_{j,\nu}$ on $\mathfrak{z}_\nu^*$ which are continuously differentiable up to order $d+1$ such that these derivatives are also in $L^1(\mathfrak{z}_\nu^*)$. The Fourier inversion theorem implies that there exist functions $h_{j,\nu}$ in $L^1(\mathfrak{z}_\nu^*)$ whose Fourier transform equals $\widehat{h}_{j,\nu}$.

Note that the functions $\widehat{h}_{j,0}$ are Gauß functions. An explicit computation of the inverse Fourier transform shows that the $h_{j,0}$ are also Gauß functions.

If we define $h_j = h_{j,0} \otimes \ldots \otimes h_{j,m}$ for $j = 1, 2$ and $h = h_1 - h_2$, then it is easy to see that $h \in L^1(K, w)$ and that $\widehat{h}$ is indeed the Fourier transform of h. This finishes the proof of our theorem. □

Proposition 7.18. *Let $m = 1$. Let $g \in \mathfrak{k}^*$ and $\{f_s : s \in \mathbb{R}\}$ a subset of $\mathfrak{k}^*$ such that $|\, f_s \,|\, \mathfrak{z} \,| = e^{-s}$ and*

$$f_s(e_0) = f_0 + Q(s)$$

where Q is a polynomial function in one real variable such that $Q(0) = 0$. Assume that one of the following conditions holds:

(i) $Q = 0$ *and* $g(e_0) \neq f_0$ *,*

(ii) $Q \neq 0$*,* $g \,|\, \mathfrak{z} \neq 0$*, and* $g(e_0) \neq f_0 + Q(\, -\log|\, g \,|\, \mathfrak{z} \,|\,)$ *,*

(iii) $Q \neq 0$ *and* $g \,|\, \mathfrak{z} = 0$ *.*

Then there is a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all $s \in \mathbb{R}$ and $\widehat{h}(g) \neq 0$.

Proof. If $Q = 0$, then we define $v(\xi) = 1$ for all ξ. If $Q \neq 0$, then we choose an $\alpha \neq 0$ such that the leading coefficient of the polynomial αQ is greater than $2d + 3$. Now we define

$$v(\xi) = e^{-\alpha Q(-\log|\, \xi \,|\,)}$$

for $\xi \in \mathfrak{z}^*$. The derivatives of first order of v are given by

$$(\partial_j v)(\xi) = \alpha \xi_j \, Q'(-\log|\, \xi \,|\,) \, e^{-Q_1(-\log|\, \xi \,|\,)}$$

for $1 \leq j \leq d$ where $Q_1(\eta) = \alpha Q(\eta) - 2\eta$ for $\eta \in \mathbb{R}$. By induction it follows that

$$(\partial^\nu v)(\xi) = R_\nu(\xi) \, S_\nu(-\log|\, \xi \,|\,) \, e^{-Q_\nu(-\log|\, \xi \,|\,)}$$

for any multi-index $\nu \in \mathbb{N}^d$ such that $|\, \nu \,| \leq d$. Here R_ν is a polynomial function on $\mathfrak{z}^*$, S_ν and Q_ν are polynomial functions on $\mathbb{R}$ such that the leading coefficient of Q_ν

is greater than $2(d - |\nu|) + 1$. This shows that v is continuously differentiable up to order $d+1$. Clearly, the functions $(\partial^\nu v)(\xi)\, e^{-|\xi|^2}$ are in $L^1(\mathfrak{z}^*)$ for all $|\nu| \leq d+1$. Now we can define the function $\widehat{h}$ on $\mathfrak{k}^*$ by

$$\widehat{h}(\tau, \xi) = \left(e^{-\alpha(\tau - f_0)} - v(\xi) \right) \, e^{-\tau^2 - \xi^2} \, .$$

Obviously $\widehat{h}(f_s) = 0$ for all $s \in \mathbb{R}$ and $\widehat{h}(g) \neq 0$. Furthermore we have $\widehat{h} = \widehat{h}_1 - \widehat{h}_2$ where $\widehat{h}_j = \widehat{h}_{j,1} \otimes \widehat{h}_{j,2}$ for $j = 1, 2$ is a tensor product of a Gauss function $\widehat{h}_{j,0}$ and a function $\widehat{h}_{j,1}$ which is differentiable up to order d+1 such that these derivatives are in $L^1(\mathfrak{z}^*)$. By the Fourier inversion theorem we find a function $h = h_1 - h_2$ in $L^1(K, w)$ whose Fourier transform equals $\widehat{h}$. This completes our proof. □

The following theorem is a slight generalization of the preceding assertions.

Theorem 7.19. *Let $\{f_s : s \in \mathbb{R}^m\} \subset \mathfrak{k}^*$ such that $|\, f_s \,|\, \mathfrak{z}_\nu \,| = e^{-s_\nu}$ for $1 \leq \nu \leq m$ and*

$$f_s(e_0) = f_0 + \sum_{\nu=1}^{m} Q_\nu(s_\nu)$$

where Q_ν is a polynomial function in one real variable such that $Q_\nu(0) = 0$. We define $I = \{1 \leq \nu \leq m : \deg Q_\nu \geq 1\}$ and $I_0 = \{\nu \in I : g \,|\, \mathfrak{z}_\nu = 0\}$. For $\nu \in I$ let α_ν denote the leading coefficient of the polynomial Q_ν. Let $g \in \mathfrak{k}^*$ such that one of the following conditions is satisfied:

(i) $I_0 = \emptyset$ *and* $g(e_0) \neq f_0 + \sum_{\nu=1}^{m} Q(-\log|\, g \,|\, \mathfrak{z}_\nu \,|)$,

(ii) $I_0 \neq \emptyset$ *and either* $\{\alpha_\nu : \nu \in I_0\} \subset (-\infty, 0)$ *or* $\{\alpha_\nu : \nu \in I_0\} \subset (-\infty, 0)$.

Then there exists a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all $s \in \mathbb{R}^m$ and $\widehat{h}(g) \neq 0$.

Proof. Let $\alpha \neq 0$ such that $\alpha\alpha_\nu > 2d + 3$ for all $\nu \in I_0$, where d is defined as above. Let $\epsilon > 0$ be such that $|\, g \,|\, \mathfrak{z}_\nu \,| > 0$ for all $\nu \in I \setminus I_0$. Define u_0 and u as in the proof of Theorem 7.17. Let

$$v(\xi) = \prod_{\nu \in I} e^{-\alpha Q_\nu(-\log|\, \xi_\nu \,|)}$$

if $\xi_\nu > 0$ for all $\nu \in I$ and $v(\xi) = 0$ else. Then we can define

$$\widehat{h}(\tau, \xi) = u(\xi) \, \left(e^{-\alpha(\tau - f_0)} - v(\xi) \right) \, e^{-\tau^2 - |\xi|^2}$$

and proceed as in the proof of Theorem 7.17. □

Remark 7.20. We insist that the point in the proof of Theorem 7.17 and 7.19 is the solution of a Fourier multiplier problem: The crucial step is to find a solution $g \in L^1(\mathbb{R}^m)$ of the equation

$$\widehat{g}(\xi) = u(\xi) v(\xi) \, e^{-|\xi|^2}$$

where the Gauss function is given and the multiplier $\psi(\xi) = u(\xi)v(\xi)$ is sufficiently often continuously differentiable. Note that $\xi \mapsto \psi(\xi) e^{-|\xi|^2}$ is decaying rapidly.

8 Exponential modules

In this section we provide some results of linear algebra concerning weight space decompositions of modules over nilpotent Lie algebras, compare Chapter 11 of [32]. In doing so we pay attention to the particularities of exponential actions. This exposition should serve as background knowledge for the classification of nilpotent Lie algebras $\mathfrak{s}$ acting on stabilizers $\mathfrak{m}$ containing a given nilpotent Lie algebra $\mathfrak{n}$, that will be performed in the next sections.

Let $\mathfrak{s}$ be a real Lie algebra. Let V be a finite-dimensional, real vector space and $V_{\mathbb{C}}$ its complexification. We assume that **$\mathfrak{s}$ acts on V as a nilpotent Lie algebra** of linear endomorphisms. If $\varphi : \mathfrak{s} \longrightarrow \mathrm{End}(V)$ denotes this representation, then $\varphi(\mathfrak{s}) \subset \mathrm{End}(V)$ is a nilpotent subalgebra.

It is well-known that there exists a weight space decomposition of V in this situation, see Chapter 11, Section 3 of [32]. We regard $\mathfrak{s}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathbb{C})$ as a complex vector space. Let $\Delta \subset \mathfrak{s}^*$ denote the set of all weights of the $\mathfrak{s}$-module V. If $\gamma \in \Delta$, then $(V_{\mathbb{C}})_\gamma$ denotes the non-trivial weight space of weight γ. We obtain the following direct sum decomposition:

$$V_{\mathbb{C}} = \bigoplus_{\gamma \in \Delta} (V_{\mathbb{C}})_\gamma \, .$$

If γ is real, then $(V_{\mathbb{C}})_\gamma$ is invariant under complex conjugation. In this case we define $V_\gamma = (V_{\mathbb{C}})_\gamma \cap V$. If γ is not real, then $\bar{\gamma} \in \Delta$ because the action of $\mathfrak{s}$ on $V_{\mathbb{C}}$ commutes with complex conjugation. The direct sum $(V_{\mathbb{C}})_\gamma \oplus (V_{\mathbb{C}})_{\bar{\gamma}}$ is invariant under complex conjugation and we define $V_\gamma = (\,(V_{\mathbb{C}})_\gamma \oplus (V_{\mathbb{C}})_{\bar{\gamma}}\,) \cap V$. Let us choose a subset Δ_0 of Δ which contains the real weights γ in Δ and a representative γ for any pair $(\gamma, \bar{\gamma})$ in Δ such that $\gamma \neq \bar{\gamma}$. From these definitions it follows that

$$V = \bigoplus_{\gamma \in \Delta_0} V_\gamma \, .$$

Without proof we note the following fact.

Lemma 8.1. *Assume that the Lie algebra $\mathfrak{s}$ acts as a (nilpotent) Lie algebra of derivations on the Lie algebra V. Then the commutator relations*

$$[V_\gamma, V_\delta] \subset V_{\gamma+\delta} + V_{\gamma+\bar{\delta}}$$

hold for the weight spaces of the $\mathfrak{s}$-module V.

Additionally we assume that **$\mathfrak{s}$ acts exponentially on the vector space V** in the sense that $\gamma(d) \notin i\mathbb{R}^\times$ for all $d \in \mathfrak{s}$ and $\gamma \in \Delta$. Equivalent conditions are that $\ker\gamma = \ker\mathrm{Re}\,\gamma \subset \ker\mathrm{Im}\,\gamma$, or that there exists a $\lambda \in \mathbb{R}$ such that $\gamma = (1+i\lambda)\,\mathrm{Re}\,\gamma$, for any $\gamma \in \Delta$. In a slightly more general setting we obtain

Lemma 8.2. *Let $\mathfrak{s}$ be a real vector space and $\gamma_1, \ldots, \gamma_m \in \mathfrak{s}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathbb{C})$ such that $\gamma_\nu(d) \notin i\mathbb{R}^\times$ for all $d \in \mathfrak{s}$ and $1 \leq \nu \leq m$.*

(i) Then $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \bigcap_{k=1}^{m} \ker\gamma_k = \dim_{\mathbb{C}} \langle \gamma_1, \ldots, \gamma_m \rangle$. This means that the $\mathbb{C}$-linear span of $\gamma_1, \ldots, \gamma_m$ is equal to $(V / \bigcap_{k=1}^{m} \ker\gamma_k)^*$.

(ii) If $\gamma_1, \ldots, \gamma_m$ are $\mathbb{C}$-linearly independent, then there exist elements $d_1, \ldots, d_m$ in $\mathfrak{s}$ such that $\operatorname{Re}\gamma_\mu(d_\nu) = \delta_{\mu\nu}$ for all $1 \le \mu, \nu \le m$.

Proof. If $m = 1$, the claim of *(i)* is obvious. Assume that the assertion of *(i)* holds for some $m \ge 1$. Set $U_m = \cap_{k=1}^m \ker \gamma_k$. Then it follows

$$\begin{aligned}\dim_{\mathbb{R}} U_{m+1} &= \dim_{\mathbb{R}} U_m - \dim_{\mathbb{R}} U_m/U_{m+1} \\ &= \dim_{\mathbb{R}} V - \dim_{\mathbb{C}}\langle\, \gamma_1, \ldots, \gamma_m \,\rangle - \dim_{\mathbb{R}} U_m/U_{m+1} \\ &= \dim_{\mathbb{R}} V - \dim_{\mathbb{C}}\langle\, \gamma_1, \ldots, \gamma_{m+1} \,\rangle\end{aligned}$$

by induction. Here we use the fact that $\gamma_{m+1} = 0$ on U_m if and only if $\gamma_{m+1} \in \langle\, \gamma_1, \ldots, \gamma_m \,\rangle$. The proof of *(ii)* is again by induction and uses *(i)*. □

Let $\mathfrak{s}$ be a Lie algebra acting as nilpotent Lie algebra and exponentially on the real vector space V. The ideal $\mathfrak{s}_0 = \cap_{\gamma\in\Delta} \ker \gamma$ consists of all elements of $\mathfrak{s}$ acting nilpotently on V. Let $\{\gamma_1, \ldots, \gamma_m\}$ be a maximal set of linearly independent weights in Δ. By the way, note that the set $\{\gamma, \bar{\gamma}\}$ is linearly dependent. The preceding lemma implies $m = \dim_{\mathbb{R}} \mathfrak{s}/\mathfrak{s}_0$. Further we can choose a basis $d_1, \ldots, d_m$ of $\mathfrak{s}$ modulo $\mathfrak{s}_0$ such that

$$\operatorname{Re}\gamma_\mu(d_\nu) = \delta_{\mu\nu}$$

for all $1 \le \mu, \nu \le m$. If γ_μ is not real, then there exists a $\lambda_\mu \in \mathbb{R}$ such that

$$\gamma_\mu(d_\nu) = (1 + i\lambda_\mu)\delta_{\mu\nu}$$

for all $1 \le \nu \le m$. The following general result is useful.

Remark. Let V be a real vector space. Any $A \in \operatorname{End}(V)$ extends to linear endomorphism A of the complexification $V_{\mathbb{C}}$. The transpose A^t of A is given by

$$(A^t f)(v) = f(Av)$$

for $f \in V^* = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $v \in V$.

Let γ be an eigenvalue of A. If $\gamma \in \mathbb{R}$, then there exists an eigenvector $v \in V$ for γ which spans the one-dimensional A-invariant subspace $U = \mathbb{R}v$. In coordinates of this basis of U, A is multiplication by γ on $\mathbb{R}$. Further A^t is multiplication by γ in coordinates of the dual basis of U^*.

If $\gamma \in \mathbb{C}\backslash\mathbb{R}$, then there exist an eigenvector $v \in V_{\mathbb{C}}$ for γ. Since A commutes with complex conjugation, $\bar{v}$ is an eigenvector for $\bar{\gamma}$. The vectors $v_1 = \operatorname{Im} v$ and $v_2 = \operatorname{Re} v$ span a two-dimensional A-invariant subspace $U = \mathbb{R}v_1 + \mathbb{R}v_2$ of V. Note that $A \operatorname{Re} v = \operatorname{Re} Av$ and $A \operatorname{Im} v = \operatorname{Im} Av$. In coordinates of this basis of U, A is (complex) multiplication by γ on $\mathbb{R}^2 = \mathbb{C}$. Further A^t is multiplication by $\bar{\gamma}$ in coordinates of the dual basis of U^*.

Remark 8.3. Let S be a Lie group with Lie algebra $\mathfrak{s}$. Let Φ be a representation of the group S in a finite-dimensional, real vector space V. Then $\mathfrak{s}$ acts on V via the associated infinitesimal representation φ. Assume that $\mathfrak{s}$ acts as a nilpotent Lie algebra and exponentially on V so that there exists a weight space decomposition.

Then we can choose $\gamma_1, \ldots, \gamma_m$ and $d_1, \ldots, d_m$ as above. Further we define a smooth map $E : \mathbb{R}^m \longrightarrow S$ by

$$E(s) = \exp(s_1 d_1) \cdot \ldots \cdot \exp(s_m d_m)$$

using the exponential function $\exp : \mathfrak{s} \longrightarrow S$.

Let $1 \leq \mu \leq m$ be arbitrary. If γ_μ is real, then we define U_μ to be the one-dimensional subspace generated by some eigenvector $v \in V$ of weight γ_μ. This basis defines coordinates and a Euclidean norm on U_μ. If γ_μ is not real, then we define U_μ to be the two-dimensional subspace spanned by $\operatorname{Im} v$ and $\operatorname{Re} v$ for some eigenvector $v \in V_{\mathbb{C}}$ of weight γ_μ, compare the preceding remark. Again this defines a coordinate system and a Euclidean norm on U_μ.

Let $f \in V^*$ be such that $f_\mu = f \,|\, U_\mu$ is non-zero for all $1 \leq \mu \leq m$. We can scale the basis vectors of U_μ such that $|f_\mu| = 1$. Here the norm is the Euclidean one with respect to the dual basis of U_μ^*. Recall that the dual representations of Φ and φ are given by

$$\Phi(a) = \Phi(a^{-1})^t \quad \text{and} \quad \varphi^*(d) = -\varphi(d)^t$$

for $a \in S$ and $d \in \mathfrak{s}$. Now let us define

$$f_s = \Phi^*(E(s))f$$

for $s \in \mathbb{R}^m$. Since $\Phi^*(\exp d) = \exp(\varphi^*(d))$, we know the dual action of S on U_μ^* by the preceding remark. It is easy to see that $f_s \,|\, U_\mu = e^{-s_\mu} f_\mu$ if γ_μ is real, and $f_s \,|\, U_\mu = e^{-s_\mu(1-i\lambda_\mu)} f_\mu$ if γ_μ is not real. In any case we see that we can retrieve the variable s by

$$s_\mu = -\log |\, f_s | U_\mu \,|$$

from the dual action of the group S.

We conclude this section with the following observation: The Euclidean norm on $\mathfrak{z}_\nu^*$ defines the polynomial function $c_\mu(h) = |h|^2$ on V^* so that $c_\mu(f_s) = e^{-2s_\mu}$. If γ_μ is real, then $c_\mu(f) = v^2(f) = f(v)^2$ where $v \in V$ is an eigenvector of weight γ_μ. If γ_μ is not real, $c_\mu(f) = (v_1^2 + v_2^2)(f) = f(v_1)^2 + f(v_2)^2$ where $v_1 = \operatorname{Im} v$ and $v_2 = \operatorname{Re} v$ for some eigenvector $v \in V_{\mathbb{C}}$.

Lemma 8.4. *Let $\mathfrak{s}$, V, $\gamma_1, \ldots, \gamma_m$, and $d_1, \ldots, d_m$ be as above. For $1 \leq \mu \leq m$ there exists an ($\mathfrak{s}$-irreducible) subspace U_μ of V and a Euclidean norm on U_μ^* such that*

$$s_\mu = -\log |\, f_s \,|\, U_\mu \,| \,.$$

These Euclidean norms define the polynomial functions $c_\mu(h) = |\, h \,|\, U_\mu \,|^2$ on V^*.

9 Nilradical is a filiform algebra

More precisely, in this section we study the representation theory of an exponential solvable Lie group G whose Lie algebra $\mathfrak{g}$ contains a coabelian, nilpotent ideal $\mathfrak{n}$ which is a trivial extension of a filiform algebra of arbitrary dimension. This section is divided into three subsections. In the first subsection we describe the algebraic structure of $\mathfrak{g}$. The next two subsections are devoted to the investigation of the unitary representations of G, first in the central case and then in the non-central case.

9.1 The structure of $\mathfrak{g}$

Let $\mathfrak{g}$ be an exponential solvable Lie algebra which contains a nilpotent ideal $\mathfrak{n}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Let $k \geq 2$. Assume that $\mathfrak{n}$ is $(k+1)$-step nilpotent and that

$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} \mathfrak{b} \underset{d-1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \ldots \underset{1}{\supset} C^k\mathfrak{n} \underset{1}{\supset} \{0\} \tag{9.1}$$

is a descending series of characteristic ideals of $\mathfrak{n}$ where $\mathfrak{b} = C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$ and $C^1\mathfrak{n}$ is commutative. The centralizer $\mathfrak{c}$ of $C^1\mathfrak{n}$ in $\mathfrak{n}$ is also a commutative ideal. The center $\mathfrak{z}\mathfrak{n}$ is d-dimensional. The nilpotent ideal $\mathfrak{n}$ is $(k+2+d)$-dimensional and for $1 \leq \nu \leq k$ the commutator ideal $C^\nu\mathfrak{n}$ has dimension $k+1-\nu$. Note that our assumptions include the case of the $(k+1)$-step nilpotent filiform algebra if $d = \dim \mathfrak{z}\mathfrak{n} = 1$.

Further let $\mathfrak{m}$ be a non-nilpotent ideal of $\mathfrak{g}$ with $\mathfrak{n} \subset \mathfrak{m}$ and such that there exists an $f \in \mathfrak{m}^*$ with $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Assume that f is in general position in the following sense: If $\mathfrak{a} \subset \mathfrak{m}$ is a non-trivial ideal of $\mathfrak{g}$, then $f(\mathfrak{a}) \neq 0$. Since f vanishes on the ideal $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$ of $\mathfrak{g}$, it follows $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

If $\mathfrak{g} = \mathfrak{m}$, then in particular $\mathfrak{g} = \mathfrak{g}_{f'} + \mathfrak{n}$ so that the orbit $\mathrm{Ad}^*(G)f' = \mathrm{Ad}^*(N)f'$ is closed. Consequently there are no functionals $g \in \mathfrak{m}^*$ which are critical for the orbit $\mathrm{Ad}^*(G)f$, compare Theorem 3.23. Thus we can assume $\mathfrak{m} \neq \mathfrak{g}$.

It is well-known that $\mathfrak{g}$ possesses Cartan subalgebras, i.e., there exist nilpotent subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ which coincide with their normalizer $N_\mathfrak{g}(\mathfrak{h})$ in $\mathfrak{g}$. Since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$, any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfies the condition $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$. Furthermore $\mathfrak{h}$ is a maximal nilpotent subalgebra.

Let us fix a nilpotent subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$. Here it will be advantageous to choose $\mathfrak{s}$ as small as possible. In particular, if $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ is a semi-direct sum of a commutative subalgebra $\mathfrak{s}$ and the ideal $\mathfrak{n}$, then we choose $\mathfrak{s}$ as indicated. Let us define $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$. We regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules and benefit from the existence of weight space decompositions of these modules.

There exist two weights $\alpha, \beta \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{c}$ and $\mathfrak{c} = (\mathfrak{n}_\beta \cap \mathfrak{c}) + \mathfrak{b}$, and a weight $\gamma_0 \in \mathfrak{s}^*$ such that $C^k\mathfrak{n} \subset \mathfrak{n}_{\gamma_0}$ and $\tilde{\gamma}_0 = 0$. Here $\tilde{\gamma}_0$ denotes the restriction of $\gamma_0 \in \mathfrak{s}^*$ to $\mathfrak{t}$. These definitions imply $[\mathfrak{n}_\alpha, \mathfrak{n}_\alpha] \subset C^2\mathfrak{n}$: If $[\mathfrak{n}_\alpha, \mathfrak{n}_\alpha] \not\subset C^2\mathfrak{n}$, then $C^1\mathfrak{n} = (\mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}$ because $C^1\mathfrak{n}/C^2\mathfrak{n}$ is one-dimensional. Since $\mathfrak{c}$ is commutative, it follows by induction that

$$C^\nu\mathfrak{n} = \big(\mathfrak{n}_{(\nu+1)\alpha} \cap C^\nu\mathfrak{n}\big) + C^{\nu+1}\mathfrak{n}$$

for all $1 \le \nu \le k$. Now $\mathfrak{n}_{(k+1)\alpha} \cap C^k\mathfrak{n} = C^k\mathfrak{n} \subset \mathfrak{n}_{\gamma_0}$ implies $\tilde{\alpha} = \tilde{\gamma}_0 = 0$. Furthermore $\mathfrak{n}_\beta \cap \mathfrak{c} \not\subset \mathfrak{z}\mathfrak{n}$ yields

$$0 \neq [\mathfrak{n}, \mathfrak{n}_\beta \cap \mathfrak{c}] = [\mathfrak{n}_\alpha, \mathfrak{n}_\beta \cap \mathfrak{c}] \subset \mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}$$

so that $\alpha + \beta = \nu\alpha$ for some $2 \le \nu \le k+1$. In particular $\tilde{\beta} = 0$. Thus we have shown that $[\mathfrak{n}_\alpha, \mathfrak{n}_\alpha] \not\subset C^2\mathfrak{n}$ implies that all weights δ of the $\mathfrak{s}$-module $\mathfrak{n}$ satisfy $\tilde{\delta} = 0$ in contradiction to the assumption $\mathfrak{m}$ not nilpotent. This proves $[\mathfrak{n}_\alpha, \mathfrak{n}_\alpha] \subset C^2\mathfrak{n}$. Consequently

$$C^1\mathfrak{n} = (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}.$$

By induction we obtain

$$C^\nu\mathfrak{n} = (\mathfrak{n}_{\nu\alpha+\beta} \cap C^\nu\mathfrak{n}) + C^{\nu+1}\mathfrak{n}$$

for $1 \le \nu \le k$. Finally, the equality $C^k\mathfrak{n} = \mathfrak{n}_{k\alpha+\beta} \cap C^k\mathfrak{n}$ implies $\gamma_0 = k\alpha + \beta$. Since $\tilde{\gamma}_0 = 0$ and since $\mathfrak{m}$ is not nilpotent, we see that $\tilde{\alpha} \neq 0$. This shows that the weights α, $\gamma_0 - k\alpha, \ldots, \gamma_0 - \alpha$, and γ_0 are pairwise distinct. Note that $\mathfrak{n}_0 = \mathfrak{m}_0 \cap \mathfrak{n}$ is not necessarily trivial and that $\mathfrak{t} \cap \mathfrak{n} \subset \mathfrak{n}_0 \subset \mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$. Here $\mathfrak{m}_0$ resp. $\mathfrak{n}_0$ denotes the weight space of the $\mathfrak{s}$-module $\mathfrak{m}$ resp. $\mathfrak{n}$ of weight 0.

We sum up our results: If $\gamma_0 \neq 0$, then we obtain the decomposition

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{\gamma_0 - k\alpha} \oplus \ldots \oplus \mathfrak{n}_{\gamma_0 - \alpha} \oplus \mathfrak{n}_{\gamma_0} \oplus \mathfrak{v}\ .$$

Here $\mathfrak{v} \subset \mathfrak{z}\mathfrak{n}$ is a direct sum of weight spaces corresponding to weights $\gamma \neq 0$ such that $\tilde{\gamma} = 0$. The other weight spaces are one-dimensional. If $\gamma_0 = 0$, then we have the decomposition

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{-k\alpha} \oplus \ldots \oplus \mathfrak{n}_{-\alpha} \oplus \mathfrak{v}$$

with $\mathfrak{v}$ as above and $C^k\mathfrak{n} \subset \mathfrak{m}_0$. Again the other weight spaces are one-dimensional.

Remark. If $k = 1$ so that $\mathfrak{n}$ is a central extension of the three-dimensional Heisenberg algebra, then there does not exist a one-codimensional characteristic ideal $\mathfrak{c}$. But a simple argument yields the weight space decomposition $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{\gamma_0 - \alpha} \oplus \mathfrak{n}_{\gamma_0} \oplus \mathfrak{v}$ if $\gamma_0 \neq 0$ and $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{-\alpha} \oplus \mathfrak{v}$ if $\gamma_0 = 0$ in this case, too. From this it follows that all results of the rest of this section are valid if $k = 1$.

Since $\mathfrak{n} + \mathfrak{z}\mathfrak{m}$ is also a trivial extension of a filiform algebra, we can assume without loss of generality that $\mathfrak{z}\mathfrak{m}$ is contained in $\mathfrak{n}$ so that $\mathfrak{z}\mathfrak{n} = \mathfrak{z}\mathfrak{m}$. There exist vectors $b_0, \ldots, b_l$ in $\mathfrak{t}$ such that $\alpha(b_0) = -1$ and $\alpha(b_\nu) = 0$ for $1 \le \nu \le l$ and such that their canonical images in $\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{n})$ form a basis of $\mathfrak{t}/(\mathfrak{t} \cap \mathfrak{n})$. Further we can find non-zero vectors $e_1 \in \mathfrak{n}_\alpha$ and $e_\nu \in \mathfrak{n}_{\gamma_0 - (k+2-\nu)\alpha}$ for $2 \le \nu \le k+2$ such that

$$[e_1, e_\nu] = e_{\nu+1} \tag{9.2}$$

for $2 \le \nu \le k+1$,

$$[b_0, e_1] = -e_1\ , \quad \text{and} \quad [b_0, e_\nu] = (k+2-\nu)e_\nu \tag{9.3}$$

for $2 \le \nu \le k+2$. We observe that the weight space decomposition of the $\mathfrak{s}$-module $\mathfrak{n}$ shows that the ideal $\ker\tilde{\alpha}$ of $\mathfrak{t}$ satisfies $[\ker\tilde{\alpha}, \mathfrak{n}] = 0$ because $\tilde{\gamma}_0 = 0$ and the weight spaces $\mathfrak{n}_\alpha$ and $\mathfrak{n}_{\gamma_0 - \nu\alpha}$ are one-dimensional. Obviously, the vectors

$b_0, \dots, b_l, e_1, \dots, e_{k+1}$ form a basis of $\mathfrak{m}$ modulo the center $\mathfrak{z}\mathfrak{m}$.

We have $f(e_{k+2}) \neq 0$ because f is in general position. The equations

$$\mathrm{Ad}^*(\exp ve_1)f\,(e_{k+1}) = f(e_{k+1}) - vf(e_{k+2})$$

and

$$\mathrm{Ad}^*(\exp xe_{k+1})f\,(e_1) = f(e_1) + xf(e_{k+2})$$

show that we can achieve $f(e_1) = 0$ and $f(e_{k+1}) = 0$ by choosing f appropriately on the orbit $\mathrm{Ad}^*(N)f$. Now let $X \in \mathfrak{m}_f$ be arbitrary and write $X = \sum_{\nu=0}^{l} \alpha_\nu b_\nu + ve_1 + Y$ with $Y \in \mathfrak{b} = C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. Since f vanishes on $[\mathfrak{m}_f, \mathfrak{m}]$, we obtain

$$0 = f([X, e_{k+1}]) = vf(e_{k+2})$$

and thus $v = 0$. Let $2 \leq \mu \leq k+1$ be arbitrary. Then

$$0 = f([X, e_\mu]) = (k+2-\mu)\,\alpha_0\, f(e_\mu)\ .$$

If $f(e_\mu)$ were non-zero, then it would follow $\alpha_0 = 0$ and this would be a contradiction to our assumption $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Thus we have shown $f(e_\mu) = 0$ for all $1 \leq \mu \leq k+1$. In addition we obtain the following characterization of the stabilizer $\mathfrak{m}_f$.

Lemma 9.4. *Let $X = \sum_{\nu=0}^{l} \alpha_\nu b_\nu + \sum_{\nu=1}^{k+1} x_\nu e_\nu + Z$ with $Z \in \mathfrak{z}\mathfrak{m}$ be an arbitrary element of $\mathfrak{m}$. Then $X \in \mathfrak{m}_f$ if and only if $x_1 = x_{k+1} = 0$ and $\sum_{\nu=0}^{l} \alpha_\nu\, f([b_\nu, b_\mu]) = 0$ for all $0 \leq \mu \leq l$.*

Proof. First we compute $[X, e_\nu] = \alpha_0(k+2-\nu)e_\nu + x_1 e_{\nu+1}$ for $2 \leq \nu \leq k+1$, $[X, e_1] = -\alpha_0 e_1 - \sum_{\nu=0}^{k+1} x_\nu e_{\nu+1}$, and $[X, b_\mu] = \sum_{\nu=0}^{l} \alpha_\nu\,[b_\nu, b_\mu]$ for $0 \leq \mu \leq l$. The assertion of this lemma follows at once if we apply f to these equations and take into account that $f(e_\nu) = 0$ for $1 \leq \nu \leq k+1$. $\square$

Let $\mathfrak{d}$ be the subspace generated by $b_0, \dots, b_l$ so that $\mathfrak{m} = \mathfrak{d} \oplus \mathfrak{n}$. The characterization of $\mathfrak{m}_f$ given by the preceding lemma shows us that

$$\mathfrak{m}_f = (\mathfrak{m}_f \cap \mathfrak{d}) \oplus (\mathfrak{m}_f \cap \mathfrak{n})$$

and thus $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n} = (\mathfrak{m}_f \cap \mathfrak{d}) \oplus \mathfrak{n}$. This proves $\mathfrak{d} = \mathfrak{m}_f \cap \mathfrak{d} \subset \mathfrak{m}_f$. Since $\mathfrak{t} \cap \mathfrak{n} \subset \mathfrak{n}_0 \subset \mathfrak{z}\mathfrak{m}$ and $\mathfrak{t} = \mathfrak{d} + (\mathfrak{t} \cap \mathfrak{n})$, we obtain $\mathfrak{t} \subset \mathfrak{m}_f$.

Furthermore $[\,\ker\tilde{\alpha}, \mathfrak{n}\,] = 0$ implies $[\,\mathfrak{g}, \ker\tilde{\alpha}\,] = [\,\mathfrak{s}, \ker\tilde{\alpha}\,] \subset \mathfrak{s} \cap \mathfrak{n} \subset \mathfrak{n}_0 \subset \mathfrak{z}\mathfrak{m}$ and

$$[\,\mathfrak{g}, [\ker\tilde{\alpha}, \mathfrak{m}]\,] \subset [\,\ker\tilde{\alpha}, [\mathfrak{m}, \mathfrak{g}]\,] + [\,\mathfrak{m}, [\mathfrak{g}, \ker\tilde{\alpha}]\,] = 0\ .$$

This shows us that $[\,\ker\tilde{\alpha}, \mathfrak{m}\,] \subset \mathfrak{z}\mathfrak{g}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{t} \subset \mathfrak{m}_f$, we obtain $[\,\ker\tilde{\alpha}, \mathfrak{m}\,] \subset \ker f$. Thus $[\,\ker\tilde{\alpha}, \mathfrak{m}\,] = 0$ because f is in general position.

We sum up our results: We have shown $\ker\tilde{\alpha} = \mathfrak{t} \cap \mathfrak{z}\mathfrak{m} = \mathfrak{t} \cap \mathfrak{n}$ so that $\dim \mathfrak{m}/\mathfrak{n} = \dim \mathfrak{t}/(\mathfrak{t} \cap \mathfrak{n}) = 1$ and $l = 0$. In particular $\mathfrak{n}$ is the nilradical (the maximal nilpotent ideal) of $\mathfrak{m}$.

By the way, we observe that a Lie algebra $\mathfrak{m}$ is determined uniquely up to isomorphism by the following conditions: $\mathfrak{m}$ is not nilpotent, the nilradical $\mathfrak{n}$ of $\mathfrak{m}$ is a trivial extension of a filiform algebra, and $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

Let us write $e_0 = b_0$. Consequently there exists a basis $e_0, \ldots, e_{k+1}$ of $\mathfrak{m}/\mathfrak{z}\mathfrak{m}$ and a vector $e_{k+2} \in C^k\mathfrak{n}$ such that $[e_1, e_\nu] = e_{\nu+1}$ for $2 \le \nu \le k+1$, $[e_0, e_1] = -e_1$, and $[e_0, e_\nu] = (k+2-\nu)e_\nu$ for $2 \le \nu \le k+2$.

Next we compute the coadjoint action of M on $\mathfrak{m}^*$. For arbitrary $h \in \mathfrak{m}^*$ and $A, B \in \mathfrak{m}$ we apply the general formula

$$\mathrm{Ad}^*(\exp A)h\ (B) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\ h\left(\mathrm{ad}(A)^j\, B\right)\ .$$

For $0 \le \nu \le k+2$ we abbreviate $h_\nu = h(e_\nu)$. Then for $2 \le \mu, \nu \le k+1$ we obtain

$$\begin{aligned}
\mathrm{Ad}^*(\exp x e_\mu)h\ (e_0) &= h_0 + (k+2-\mu)x h_\mu \\
(e_1) &= h_1 + x h_{\mu+1} \\
(e_\nu) &= h_\nu
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}^*(\exp v e_1)h\ (e_0) &= h_0 - v h_1 \\
(e_1) &= h_1 \\
(e_\nu) &= h_\nu + \sum_{j=1}^{k+2-\nu} \frac{(-v)^j}{j!}\ h_{\nu+j} \\
\mathrm{Ad}^*(\exp t e_0)h\ (e_0) &= h_0 \\
(e_1) &= e^t\, h_1 \\
(e_\nu) &= e^{-(k+2-\nu)t}\, h_\nu
\end{aligned}$$

We work with coordinates of the second kind: Since the canonical images of $e_0, \ldots, e_{k+1}$ in $\mathfrak{m}/\mathfrak{z}\mathfrak{m}$ form a Malcev basis and since $\mathfrak{c}$ is commutative, the map $\Phi : \mathbb{R}^{k+2} \times \mathfrak{z}\mathfrak{m} \longrightarrow M$ given by

$$\Phi(t, v, x, Z) = \exp(t e_0)\exp(v e_1)\exp\left(\sum_{\nu=2}^{k+1} x_\nu e_\nu + Z\right)$$

is a global diffeomorphism. Now we compute

$$\begin{aligned}
\mathbf{Ad}^*(\boldsymbol{\Phi}(\boldsymbol{t}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{Z}))\boldsymbol{f}\ (\boldsymbol{e_0}) &= \boldsymbol{f_0} - \boldsymbol{v x_{k+1}}\ \boldsymbol{f_{k+2}} \\
(\boldsymbol{e_1}) &= \boldsymbol{e^t x_{k+1}}\ \boldsymbol{f_{k+2}} \\
(\boldsymbol{e_\nu}) &= \boldsymbol{e^{-(k+2-\nu)t}}\ \frac{(-\boldsymbol{v})^{\boldsymbol{k+2-\nu}}}{(\boldsymbol{k+2-\nu})!}\ \boldsymbol{f_{k+2}} \\
(\boldsymbol{Y}) &= \boldsymbol{f}(\boldsymbol{Y})
\end{aligned} \tag{9.5}$$

where $2 \le \nu \le k+1$ and $Y \in \mathfrak{z}\mathfrak{m}$.

Our next aim is to find a polynomial function p_0 on $\mathfrak{m}^*$ which is constant on the orbit $\mathrm{Ad}^*(M)f$ for all $f \in \mathfrak{m}^*$ such that $f(e_\nu) = 0$ for $1 \le \nu \le k+1$. This polynomial p_0 will be defined as a linear combination of products of the basis vectors $\{e_\nu : 0 \le \nu \le k+2\}$. Here e_ν means the linear function $f \mapsto f(e_\nu)$ on $\mathfrak{m}^*$. Products are taken in the commutative algebra $\mathcal{P}(\mathfrak{m}^*)$ of complex-valued polynomials on $\mathfrak{m}^*$. The definition of p_0 will not depend on f.

Recall that M acts on $\mathcal{P}(\mathfrak{m}^*)$ by

$$\mathrm{Ad}(m)p\,(f) = p\left(\mathrm{Ad}^*(m)^{-1}f\right)\ .$$

If we embed $\mathfrak{m}$ in $\mathcal{P}(\mathfrak{m}^*)$ as described above, then this action extends the adjoint representation of M in $\mathfrak{m}$. Similarly, $\mathfrak{m}$ acts as a Lie algebra of derivations in $\mathcal{P}(\mathfrak{m}^*)$ extending the adjoint representation of $\mathfrak{m}$ in $\mathfrak{m}$.

It will turn out that, in general, p_0 is **not** $\mathrm{Ad}(M)$-invariant. Recall that $\mathrm{Ad}^*(M)$-invariance is equivalent to $\mathrm{ad}^*(\mathfrak{m})$-invariance. In fact, p_0 will not be constant on $\mathrm{Ad}^*(M)f$ for all $f \in \mathfrak{m}^*$ in general position.

Before we come to the general case, we investigate the three-step nilpotent filiform algebra (k=2).

Remark. Assume that $\mathfrak{m}$ is generated by the basis vectors $\{e_\nu : 0 \le \nu \le 4\}$ which satisfy the commutator relations $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = 2e_2$, and $[e_0, e_3] = e_3$. Let $f \in \mathfrak{m}^*$ such that $f_\nu = 0$ for $1 \le \nu \le 3$ and $\lambda = f_4 \ne 0$. Set $m = \Phi(t, v, x, y, z)$. Then

$$\begin{aligned}
\mathrm{Ad}^*(m)f\,(e_0) &= f_0 - vy\lambda \\
(e_1) &= e^t y\lambda \\
(e_2) &= \frac{1}{2}e^{-2t}v^2\lambda \\
(e_3) &= -e^{-t}v\lambda \\
(e_4) &= \lambda\ .
\end{aligned}$$

If we define

$$p_0 = e_0e_0e_4 - 2e_0e_1e_3 + 2e_1e_1e_2\ ,$$

then we obtain

$$p_0\left(\,\mathrm{Ad}^*(m)f\,\right) = (f_0 - vy\lambda)^2\lambda + 2(f_0 - vy\lambda)vy\lambda^2 + v^2y^2\lambda^3 = f_0^2\lambda\ .$$

Thus the function p_0 is constant on $\mathrm{Ad}^*(M)f$. It is easy to see that $\mathrm{ad}^*(e_3)p_0 \ne 0$. Consequently p_0 is **not** $\mathrm{ad}^*(\mathfrak{m})$-invariant.

Of course, there are some other polynomial functions which are constant on $\mathrm{Ad}^*(M)f$: The disadvantage of $p = e_0e_0e_4 - 2e_1e_1e_2 - 2f_0e_1e_3$ is that its definition depends on f. The polynomials $p = e_0e_4 - e_1e_3$ and $p = e_4$ are less profitable for our purposes. The decisive role of the summand $e_1e_1e_2$ of p_0 will become apparent soon.

Next we show that the definition of p_0 can be generalized to arbitrary dimensions. Let $m = \Phi(t, v, x, Z)$ for $t, v \in \mathbb{R}$, $x \in \mathbb{R}^k$ and $Z \in \mathfrak{z}\mathfrak{m}$. Set $\lambda = f_{k+2} \neq 0$. For $0 \le \nu \le k$ we define the polynomial function

$$q_\nu = (k-\nu)!\,(-1)^{k-\nu}\, e_0^\nu\, e_1^{k-\nu}\, e_{\nu+2}$$

and compute

$$q_\nu\,(\,\mathrm{Ad}^*(m)f\,) = (f_0 - v x_{k+1}\lambda)^\nu \;\; \lambda\;\, (v x_{k+1}\lambda)^{k-\nu} \;\; .$$

Now we see that

$$\boldsymbol{p_0 = \sum_{\nu=0}^{k} \binom{k}{\nu}\; q_\nu}$$

satisfies

$$p_0\,(\,\mathrm{Ad}^*(m)f\,) = \sum_{\nu=0}^{k} \binom{k}{\nu}\; (f_0 - v x_{k+1}\lambda)^\nu \;\; \lambda\;\, (v x_{k+1}\lambda)^{k-\nu} = f_0^k\lambda \; .$$

Thus the function p_0 is constant on $\mathrm{Ad}^*(M)f$. This definition of p_0 generalizes the definition for the special case $k = 2$. Observe that the coefficient of $e_1^k e_2$ in p_0 is non-zero: it is given by $(-1)^k\; k!$.

There is a natural isomorphism of associative algebras between the symmetric algebra $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ and $\mathcal{P}(\mathfrak{m}^*)$. Further the modified symmetrization map which is defined by

$$X_1\cdot\;\ldots\;\cdot X_r \mapsto \frac{1}{r!}\sum_{\sigma\in S_r}(-iX_{\sigma(1)})\cdot\;\ldots\;\cdot(-iX_{\sigma(r)})$$

gives a linear isomorphism between $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$, compare Chapter 3.3 of [5]. Composing these two isomorphisms, we obtain a linear, $\mathrm{Ad}(M)$-equivariant isomorphism from $\mathcal{P}(\mathfrak{m}^*)$ onto $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$, which maps the subspace of invariant polynomials onto the center of $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$.

Under this identification the polynomial function p_0 corresponds to an element W_0 in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$. Note that W_0 is (in general) **not** in the center of this algebra.

The polynomial function p_0 and the element W_0 of the enveloping algebra will play a very important role in the subsequent investigation.

Since $\mathfrak{s}$ acts nilpotently on $\mathfrak{t} \subset \mathfrak{m}_0$, there exists a minimal natural number $q \ge 0$ such that $\mathrm{ad}(\mathfrak{s})^{q+1}\,\cdot\,\mathfrak{t} = 0$. Recall that $e_0 \in \mathfrak{t}$. Thus for $d \in \mathfrak{s}$ and arbitrary $h \in \mathfrak{m}^*$ we obtain

$$\mathrm{Ad}^*(\exp(sd))h\;(e_0) = h_0 + \sum_{j=1}^{q}\frac{(-s)^j}{j!}\;h\left(\mathrm{ad}(d)^j\,\cdot\,e_0\right)\;\; . \tag{9.6}$$

Finally, we multiply the vectors $\{e_\nu : 2 \le \nu \le k+2\}$ by $1/f_{k+2}$ so that $f_{k+2} = 1$. From now on, we shall keep this normalization.

We have to distinguish whether $C^k\mathfrak{n}$ is contained in the center $\mathfrak{z}\mathfrak{g}$ of $\mathfrak{g}$ or not. These two cases are essentially different.

9.2 The central case: $C^k\mathfrak{n}$ is contained in $\mathfrak{z}\mathfrak{g}$

In this case $\gamma_0 = 0$. Let $\mathfrak{s}/\mathfrak{t} \cong \mathfrak{g}/\mathfrak{m}$ have dimension m. We fix a maximal set α, $\gamma_1, \ldots, \gamma_{m'}$ of $\mathbb{C}$-linearly independent weights in $\mathfrak{s}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathbb{C})$ of the $\mathfrak{s}$-module $\mathfrak{m}$. Recalling the results of Section 8 we conclude that there exist vectors $d_1, \ldots, d_m$ in $\mathfrak{s}$

- $\mathbb{R}$-linearly independent modulo $\mathfrak{t}$,
- such that $\alpha(d_\nu) = 0$ for $1 \le \nu \le m$,
- such that $\mathrm{Re}\,\gamma_\mu(d_\nu) = \delta_{\mu,\nu}$ for $1 \le \mu, \nu \le m'$,
- and such that $\mathrm{ad}(d_\nu)$ is nilpotent for $m' + 1 \le \nu \le m$.

The Malcev basis $d_1, \ldots, d_m, e_0, \ldots, e_{k+1}$ of $\mathfrak{g}$ modulo $\mathfrak{z}\mathfrak{m}$ defines a smooth map

$$E(s) = \exp(s_1 d_1) \ldots \exp(s_m d_m)$$

and coordinates of the second kind

$$\mathbb{R}^m \times \mathbb{R}^{k+2} \times \mathfrak{z}\mathfrak{m} \longrightarrow G,\ (s, t, v, x, Z) \mapsto E(s)\Phi(t, v, x, Z)\ .$$

Combining 9.5 and 9.6, we obtain the important formulas

$$\begin{aligned}
\mathbf{Ad}^*\,(\,\boldsymbol{E(s)\Phi(0,v,x,Z)}\,)\,\boldsymbol{f}\ \boldsymbol{(e_0)} &= \boldsymbol{f_0 - vx_{k+1} + Q(s)}\\
\boldsymbol{(e_1)} &= \boldsymbol{x_{k+1}}\\
\boldsymbol{(e_\nu)} &= \boldsymbol{\frac{(-v)^{k+2-\nu}}{(k+2-\nu)!}}\\
\boldsymbol{(e_{k+2})} &= \mathbf{1}\\
\boldsymbol{(Y)} &= \boldsymbol{f}\left(\mathbf{Ad}(\boldsymbol{E(s)})^{-1}\boldsymbol{Y}\right)
\end{aligned}$$

where $Y \in \mathfrak{z}\mathfrak{m}$. In the first equation one finds the polynomial function

$$Q(s) = \sum_{1 \le j_1 + \ldots + j_m \le q} c_{j_1, \ldots, j_m}\, s_1^{j_1} \ldots s_m^{j_m}$$

in m variables whose coefficients are given by

$$c_{j_1, \ldots, j_m} = \frac{(-1)^{j_1 + \ldots + j_m}}{j_1! \ldots j_m!}\, f\left(\mathrm{ad}(d_m)^{j_m} \ldots \mathrm{ad}(d_1)^{j_1} \cdot e_0\right) \quad .$$

It is immediate from the definition of these coefficients that Q does not depend on the variable s_ν if $[d_\nu, \mathfrak{t}] \subset \ker f$.

For $1 \le \nu \le m'$ we we fix $\mathfrak{s}$-invariant, $\mathfrak{s}$-irreducible subspaces $\mathfrak{z}_\nu$ of $\mathfrak{z}_{\gamma_\nu} \subset \mathfrak{z}\mathfrak{m}$, where $\dim \mathfrak{z}_\nu = 1$ if γ_ν is real and $\dim \mathfrak{z}_\nu = 2$ else. We choose basis vectors of the $\mathfrak{z}_\nu$ according to Remark 8.3 so that $|\, f|_{\mathfrak{z}_\nu}| = 1$ in the Euclidean norm of $\mathfrak{z}_\nu^*$.

Let $g \in \mathfrak{m}^*$ be a critical functional with respect to the orbit $\mathrm{Ad}^*(G)f$. Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, we have $g_{k+2} = 1$. Without loss of generality we can assume $g_\nu = 0$ for $1 \le \nu \le k+1$.

Let us define $f_s = \mathrm{Ad}^*(E(s))f$ so that

$$\mathrm{Ad}^*(G)f = \bigcup_{s\in\mathbb{R}^m} \mathrm{Ad}^*(M)f_s \; .$$

Further we define $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$. Note that $\mathfrak{p} = \langle e_0, e_2, \ldots, e_{k+1}\rangle + \mathfrak{zm}$ is a common Pukanszky polarization at g and f_s for all s.

The following simple lemma is useful for detecting critical functionals $g \in \mathfrak{m}^*$ for the orbit $\mathrm{Ad}^*(G)f$.

Lemma 9.7. *Let us define the subalgebra $\mathfrak{k} = \mathbb{R}e_0 + \mathfrak{zm}$ of $\mathfrak{m}$. Then*

(i) $\mathrm{Ad}^*(G)g$ is contained in the closure of $\mathrm{Ad}^*(G)f$ if and only if $g\,|\,\mathfrak{k}$ is in the closure of $\{f_s\,|\,\mathfrak{k} : s \in S\}$ in the topology of $\mathfrak{k}^*$

(ii) $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$ if and only if $g\,|\,\mathfrak{zm}$ is in the closure of $\{f_s\,|\,\mathfrak{zm} : s \in S\}$ in the topology of $\mathfrak{zm}^*$

We apply the method of restricting to subquotients developed in Section 7 in order to prove that

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho \; . \tag{9.8}$$

As a first step in this direction we consider the subgroup M_0 of M whose Lie algebra is given by $\mathfrak{m}_0 = \langle e_0, e_1, e_{k+1}\rangle + \mathfrak{zm}$. This is a central extension of $\mathfrak{g}_{4,9}(0)$. We observe that the restrictions $\pi_{0,s} = \pi_s|M_0$ and $\rho_0 = \rho|M_0$ are irreducible representations of M_0 in $L^2(\mathbb{R})$ because

$$\pi_s(\exp ve_1)\varphi\;(\xi) = \varphi(\xi - v)$$

and

$$\pi_s(\exp ye_{k+1})\varphi\;(\xi) = e^{-iy\xi}\varphi(\xi) \; .$$

In order to prove 9.8 it suffices to verify that

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M_0)} \pi_{0,s} \not\subset \ker_{L^1(M_0)} \rho_0 \; . \tag{9.9}$$

The Heisenberg algebra $\mathfrak{b} = \langle e_1, e_{k+1}, e_{k+2}\rangle$ is an ideal of $\mathfrak{m}_0$. Coordinates of B are given by $(v, y, z) = \exp(ve_1)\exp(ye_{k+1} + ze_{k+2})$. Further $\mathfrak{a} = \langle e_{k+1}, e_{k+2}\rangle$ is a commutative ideal of $\mathfrak{m}_0$ and $\mathfrak{z} = \langle e_{k+2}\rangle$ is contained in $\mathfrak{zm}_0$. Note that $\mathfrak{h} = \langle e_0, e_{k+1}\rangle + \mathfrak{zm}_0$ is the Lie algebra of the stabilizer H of the character $\chi(0, y, z) = e^{iz}$ of A. The irreducible representations $\pi_{0,s}$ and ρ_0 are given by $\pi_{0,s} = \mathrm{ind}_H^{M_0} \chi_{f_s}$ and $\rho_0 = \mathrm{ind}_H^{M_0} \chi_g$. It holds

$$\chi_{f_s}(0, y, z) = \chi_g(0, y, z) = e^{iz} \; .$$

Let K be the closed, connected, commutative subgroup of M_0 corresponding to the Lie algebra $\mathfrak{k} = \langle e_0\rangle + \mathfrak{zm}_0$. Now we are exactly in the situation of Section 7.1. By Theorem 7.10 it follows that Relation 9.9 is equivalent to

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(K,w)} \chi_{f_s} \not\subset \ker_{L^1(K,w)} \chi_g \; .$$

Since $\delta(t,Z) = e^{-t}$ and $\tau(t,Z) = 0$, the weight function w on K is given by

$$w(t,Z) = \Big(4\left(\delta(t,Z) + \delta(t,Z)^{-1}\right)^2 + \tau(t,Z) \Big)^{1/4} = 2\cosh^{1/2}(t)\ .$$

Obviously w is dominated by an exponential weight of the form $(t,Z) \mapsto be^{c|t|}$ for suitable constants $b, c > 0$. We sum up our results in the following

Lemma 9.10. *Let $c > 0$ be as above and $w(t,Z) = e^{c|t|}$ the exponential weight function on K. Then for*

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M)}\ \pi_s\ \not\subset\ \ker_{L^1(M)}\ \rho$$

it is sufficient that there exists a function $h \in L^1(K,w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$.

Definition. Let $\Delta \subset \mathfrak{s}^*$ be the set of non-zero weights of the $\mathfrak{s}$-module $\mathfrak{z}\mathfrak{n} = \mathfrak{z}\mathfrak{m}$. We say that Δ is almost independent if every $\delta \in \Delta$ has the form $\delta = c\,\gamma_\nu$ for some $c \in \mathbb{C}^\times$ and $1 \le \nu \le m'$.

The preceding lemma and the results of Section 7.3 yield

Theorem 9.11. *Let $\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g},\mathfrak{g}]$ be as in Subsection 9.1. In particular $\mathfrak{n}$ is a trivial extension of a $(k+1)$-step nilpotent filiform algebra. Further we assume $C^k\mathfrak{n} \subset \mathfrak{z}\mathfrak{g}$ and that one of the following conditions holds:*

(i) $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ *is a semi-direct sum,*

(ii) $\dim \mathfrak{g}/\mathfrak{n} \le 2$*,*

(iii) $\mathfrak{g}/\mathfrak{n}$ *acts semi-simply on* $\mathfrak{z}\mathfrak{n}$ *such that* Δ *is almost independent,*

(iv) $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$ *and* $\dim \mathfrak{z}\mathfrak{n} \le 3$.

Let $\mathfrak{m}$ be a proper, non-nilpotent ideal of $\mathfrak{g}$ with $\mathfrak{n} \subset \mathfrak{m} \subset \mathfrak{g}$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ be critical with respect to the orbit $\mathrm{Ad}^*(G)f$. Then it follows that

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M)}\ \pi_s\ \not\subset\ \ker_{L^1(M)}\ \rho \tag{9.12}$$

holds for the irreducible representations $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$ of M.

Proof. We begin with a preliminary remark. If $m' = 0$, then $\mathrm{ad}(d_\nu)$ is nilpotent for $1 \le \nu \le m$. Consequently the orbit

$$\mathrm{Ad}^*(G)f = \mathrm{Ad}^*\left(\, E(\mathbb{R}^m)\,\right)\mathrm{Ad}^*(N)f$$

is closed in $\mathfrak{m}^*$ because $\mathrm{Ad}^*(G)f$ is the orbit of a connected group acting unipotently on the real vector space $\mathfrak{m}^*$, compare Theorem 3.1.4 (Chevalley-Rosenlicht) on p. 82 of [5]. In this case there are no linear functionals $g \in \mathfrak{m}^*$ which are critical with respect to the orbit $\mathrm{Ad}^*(G)f$. Hence we can always suppose $m' \ge 1$.

First we assume $[\mathfrak{s},\mathfrak{s}] = 0$ so that $Q = 0$. Since g is critical for the orbit

$\mathrm{Ad}^*(G)f$, it follows $g(e_0) \neq f_0$. Now Lemma 9.10 and Theorem 7.17.*(i)* yield our claim. The case $\dim \mathfrak{g}/\mathfrak{n} \leq 1$ being trivial, next we assume that $\dim \mathfrak{g}/\mathfrak{n} = 2$. Let $d \in \mathfrak{s}$ such that $\alpha(d) = 0$ and $\mathfrak{g} = \mathbb{R}d + \mathfrak{m}$. We know

$$f_s(e_0) = f_0 + Q(s)$$

for $s \in \mathbb{R}$. If $\mathrm{ad}(d)\,|\,\mathfrak{zn}$ has a real eigenvalue, then there exists an $\mathrm{ad}(d)$-eigenvector $e_{k+3} \in \mathfrak{zn}$ such that $[d, e_{k+3}] = e_{k+3}$ holds after renormalization of d and such that $f(e_{k+3}) = 1$ if e_{k+3} is chosen appropriately. If $\mathrm{ad}(d)\,|\,\mathfrak{zn}$ has only non-real complex eigenvalues, then there exists a $\lambda \in \mathbb{R}$ and a two-dimensional subspace $\mathfrak{z}_1$ of $\mathfrak{zn}$ such that $\mathrm{ad}(d)$ is multiplication by $1 + i\lambda$ on $\mathfrak{z}_1$ in the coordinates of a suitable basis as in Remark 8.3. In any case

$$s = -\log|\, f_s\,|\,\mathfrak{z}_1\,| \ .$$

If $Q = 0$, then the assumption $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ implies $g(e_0) \neq f_0$ so that Lemma 9.10 and Proposition 7.18.*(i)* apply. Now we assume $Q \neq 0$ and $g\,|\,\mathfrak{z}_1 \neq 0$. Since g is critical for the orbit $\mathrm{Ad}^*(G)f$, we see that

$$g(e_0) \neq f_0 + Q(\,-\log|\, g\,|\,\mathfrak{z}_1\,|\,) \ .$$

Hence Lemma 9.10 and Proposition 7.18.*(ii)* imply the assertion of this theorem. Note that $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$ in this case. If $Q \neq 0$ and $g\,|\,\mathfrak{z}_1 = 0$, then the validity of Relation 9.12 follows by Lemma 9.10 and Proposition 7.18.*(iii)*. This proves our theorem if $\mathfrak{g}/\mathfrak{n}$ is two-dimensional.

Now we assume that $\mathfrak{g}/\mathfrak{n}$ acts semi-simply on $\mathfrak{zn}$ such that Δ is almost independent. At first we recall that for $1 \leq \nu \leq m$ there exists a subspace $\mathfrak{z}_\nu$ of $\mathfrak{zn}$ and a Euclidean norm on $\mathfrak{z}_\nu^*$ such that $s_\nu = -\log|\, f_s\,|\,\mathfrak{z}_\nu\,|$, see Lemma 8.4. Since $\mathfrak{g}/\mathfrak{n}$ acts semi-simple, it follows that $\mathfrak{s}$ acts trivial on the weight space $\mathfrak{n}_0$ of the $\mathfrak{s}$-module $\mathfrak{n}$ of weight 0 in this case. We see that Q has the form

$$Q(s) = \sum_{\nu=1}^{m} \alpha_\nu s_\nu \ .$$

Note that $\mathrm{ad}(d_\nu) = 0$ on $\mathfrak{n}$ for $m' + 1 \leq \nu \leq m$ because of semi-simplicity. If α_ν were non-zero for some $m' + 1 \leq \nu \leq m$, then it would follow $\mathrm{Ad}^*(G)f = \mathrm{Ad}^*(G)f + \mathfrak{n}^\perp$ and thus $g \in (\mathrm{Ad}^*(G)f)^{-}$. This contradicts the fact that g is critical with respect to $\mathrm{Ad}^*(G)f$. Thus we can assume $\alpha_\nu = 0$ for all $m' + 1 \leq \nu \leq m$.

Let us define $I = \{1 \leq \nu \leq m' : \alpha_\nu \neq 0\}$ and $I_0 = \{\nu \in I : g\,|\,\mathfrak{z}_\nu = 0\}$. First we assume $I_0 = \emptyset$. Then the assumption $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ implies

$$g(e_0) \neq f_0 - \sum_{\nu=1}^{m} \alpha_\nu \log|\, g\,|\,\mathfrak{z}_\nu\,|$$

so that Lemma 9.10 and Theorem 7.17.*(i)* yield the assertion of this theorem. Now we assume that $I_0 \neq \emptyset$, and that there exist $\nu_1, \nu_2 \in I_0$ such that $\alpha_{\nu_1} > 0$ and $\alpha_{\nu_2} < 0$. But now it follows that $g \in (\mathrm{Ad}^*(G)f)^{-}$: Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, there exist sequences $s_n \in \mathbb{R}^m$ and $X_n \in \mathfrak{n}$ such that $f_n' \longrightarrow g'$ where

$$f_n = \mathrm{Ad}^*(E(s_n)\Phi(X_n))f \ .$$

Here f_n' and g' denote the restrictions to $\mathfrak{n}$. Then $s_{n,\nu_j} \longrightarrow +\infty$ for $j = 1, 2$ because $g \,|\, \mathfrak{z}_\nu = 0$. Since Δ is almost independent, it is possible to modify the components s_{n,ν_1} and s_{n,ν_2} of the sequence (s_n) without affecting the convergence $f_n' \longrightarrow g'$: We choose the growth of the sequence $s_{n,\nu_1} \longrightarrow +\infty$ such that

$$u_n = \sum_{\substack{1 \le \nu \le m' \\ \nu \ne \nu_2}} \alpha_\nu s_{n,\nu} \longrightarrow +\infty \ .$$

Further we define

$$s_{n,\nu_2} = \alpha_{\nu_2}^{-1} \ (\, g(e_0) - f_0 - u_n \,) \ .$$

Then in particular $s_{n,\nu_2} \longrightarrow +\infty$. Since $[d_\nu, \mathfrak{n}_{\gamma_\mu}] = 0$ for $\mu \ne \nu$ by semi-simplicity, it is easy to see that $f_n \longrightarrow g$. This contradiction shows that we can suppose that the set of coefficients $\{\alpha_\nu : \nu \in I_0\}$ is either contained in $(0, +\infty)$ or in $(-\infty, 0)$. Hence all assumptions of Theorem 7.17.*(ii)* are satisfied and the assertion of this theorem follows in the semi-simple case.

Finally let us assume that $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and that $\dim \mathfrak{z}\mathfrak{n} \le 3$. In particular we have $m' = m = \dim \mathfrak{g}/\mathfrak{m}$. Let us write $d = \dim \mathfrak{z}\mathfrak{n}$. If $d = 1$, then $\mathfrak{m} = \mathfrak{g}$. If $d = 2$, then either $m' = 0$ or $m' = 1$ and $\mathfrak{g}/\mathfrak{n}$ is two-dimensional. If $d = 3$, then $m' = 0$, $m' = 1$ or $\mathfrak{s}$ acts semi-simply on $\mathfrak{z}\mathfrak{n}$ such that Δ is almost independent. This finishes the proof of our theorem. □

Remark. One may ask whether the conclusion of Theorem 9.11 holds if $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $d = \dim \mathfrak{z}\mathfrak{n} = 4$. In this case we know that $0 \le m' \le 3$.

First we exclude some trivial cases: If $m' = 0$, then $\mathfrak{g} = \mathfrak{m}$. If $m' = 1$, then $\mathfrak{g}/\mathfrak{m}$ is one-dimensional and Theorem 9.11.*(i)* applies. If $m' = 3$, then $\mathfrak{g}/\mathfrak{n}$ acts semi-simple on $\mathfrak{z}\mathfrak{n}$ such that Δ is almost independent. In this case Theorem 9.11.*(ii)* applies. Left over is the following situation: There are two linearly independent weights γ_1, γ_2 such that the action of $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{z}\mathfrak{n}$ is not semi-simple or such that Δ is not almost independent. More exactly, one of the following conditions is satisfied:

(i) The $\mathfrak{s}$-module $\mathfrak{z}\mathfrak{n}$ admits the decomposition

$$\mathfrak{z}\mathfrak{n} = (\mathfrak{z}\mathfrak{n})_0 \oplus (\mathfrak{z}\mathfrak{n})_{\gamma_1} \oplus (\mathfrak{z}\mathfrak{n})_{\gamma_2} \ .$$

The action of $\mathfrak{s}$ on $(\mathfrak{z}\mathfrak{n})_{\gamma_2}$ is not semi-simple. In particular $\dim(\mathfrak{z}\mathfrak{n})_{\gamma_2} = 2$. The other weight spaces are one-dimensional.

(ii) The decomposition of the $\mathfrak{s}$-module $\mathfrak{z}\mathfrak{n}$ is the same as that in *(i)*. Here $\mathfrak{g}/\mathfrak{n}$ is not semi-simple on $(\mathfrak{z}\mathfrak{n})_0$.

(iii) The $\mathfrak{s}$-module $\mathfrak{z}\mathfrak{n}$ admits the decomposition

$$\mathfrak{z}\mathfrak{n} = (\mathfrak{z}\mathfrak{n})_{\gamma_1} \oplus (\mathfrak{z}\mathfrak{n})_{\gamma_2} \oplus (\mathfrak{z}\mathfrak{n})_{\gamma_3}$$

with one-dimensional weight spaces. It holds $\gamma_3 = a_1\gamma_1 + a_2\gamma_2$ with $a_1 \ne 0$ and $a_2 \ne 0$.

By the way, we note that the assumption $\dim \mathfrak{z}\mathfrak{n} = 4$ implies $\dim \mathfrak{g} \geq 9$. As above we define the commutative subalgebra $\mathfrak{k} = \mathbb{R}e_0 + \mathfrak{z}\mathfrak{m}$. Recall that $\mathfrak{z}\mathfrak{n} = \mathfrak{z}\mathfrak{m}$. Let us describe each of the preceding cases more extensively.

We begin with *(i)*. In this case there exists a basis $e_0, b_1, \ldots, b_4$ of $\mathfrak{k}$ such that the action of $\mathfrak{s}$ on $\mathfrak{k}$ is given by $[d_1, e_0] = -\alpha_1 b_1$, $[d_1, b_2] = b_2$, $[d_1, b_3] = -a_3 b_4$, $[d_2, e_0] = -\alpha_2 b_1$, $[d_2, b_3] = b_3 - a_2 b_4$, and $[d_2, b_4] = b_4$. Let $f \in \mathfrak{k}^*$ such that $f(b_1) = f(b_2) = f(b_4) = 1$. Let $a_1 = f(b_3)$ and $f_0 = f(e_0)$. Then we get

$$\begin{aligned}
f_s(e_0) &= f_0 + \alpha_1 s_1 + \alpha_2 s_2 \\
f_s(b_1) &= 1 \\
f_s(b_2) &= e^{-s_1} \\
f_s(b_3) &= a_1 e^{-s_2} + a_2 s_2 e^{-s_2} + a_3 s_1 e^{-s_2} \\
f_s(b_4) &= e^{-s_2}
\end{aligned}$$

Let us define $\Omega = \{f_s : s \in \mathbb{R}^2\} \subset \mathfrak{k}^*$ and $\Omega_0 = \Omega \,|\, \mathfrak{z}\mathfrak{m}$. Let $g \in \mathfrak{k}^*$ and $g_0 = g \,|\, \mathfrak{z}\mathfrak{m}$. The critical case is $g \notin \overline{\Omega}$ and $g_0 \in (\Omega_0)^{-}$.

Now the problem is to find a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$. If $g(b_2) \neq 0$ or $g(b_4) \neq 0$, then we can apply Theorem 7.17. Thus we assume $g(b_2) = g(b_4) = 0$. If $\alpha_1\alpha_2 \geq 0$, then we can also apply Theorem 7.17. Thus we assume $\alpha_1\alpha_2 < 0$. If $a_3 = 0$, then we obtain the contradiction $g \in \overline{\Omega}$. If $g_3 = 0$, then $g \in \overline{\Omega}$, too. If $a_3 g_3 < 0$, then $g_0 \notin (\Omega_0)^{-}$. Thus we assume $a_3 g_3 > 0$. Is there a function $h \in L^1(K, w)$ as above in this particular case?

Now we describe *(ii)*. In this case there exists a basis $e_0, b_1, \ldots, b_4$ of $\mathfrak{k}$ such that the action of $\mathfrak{s}$ on $\mathfrak{k}$ is given by $[d_1, e_0] = -\alpha_3 b_1 - \alpha_4 b_2$, $[d_1, b_1] = -\alpha_1 b_2$, $[d_1, b_3] = b_3$, $[d_2, e_0] = -\alpha_5 b_1 - \alpha_6 b_2$, $[d_2, b_1] = -\alpha_2 b_2$, and $[d_2, b_4] = b_4$. The Jacobi identity implies $\alpha_1\alpha_5 = \alpha_2\alpha_3$. Let $f \in \mathfrak{k}^*$ such that $f(b_2) = f(b_3) = f(b_4) = 1$. Let $\alpha_0 = f(b_1)$ and $f_0 = f(e_0)$. Then we see

$$\begin{aligned}
f_s(e_0) &= f_0 + (\alpha_4 + \alpha_0\alpha_3)s_1 + (\alpha_6 + \alpha_0\alpha_5)s_2 \\
&\qquad\qquad + \frac{1}{2}\alpha_1\alpha_3 s_1^2 + \alpha_2\alpha_3 s_1 s_2 + \frac{1}{2}\alpha_2\alpha_5 s_2^2 \\
f_s(b_1) &= \alpha_0 + \alpha_1 s_1 + \alpha_2 s_2 \\
f_s(b_2) &= 1 \\
f_s(b_3) &= e^{-s_1} \\
f_s(b_4) &= e^{-s_2}
\end{aligned}$$

As usual we define $\Omega = \{f_s : s \in \mathbb{R}^2\}$ and $\Omega_0 = \Omega \,|\, \mathfrak{z}\mathfrak{m}$. Let $g \in \mathfrak{k}^*$ such that $g \notin \overline{\Omega}$ and $g_0 \in (\Omega_0)^{-}$. If $\alpha_1\alpha_2 > 0$, then the assumption $g_0 \in (\Omega_0)^{-}$ implies $g(e_3) \neq 0$ and $g(e_4) \neq 0$. In this situation we can apply (a variant of) Theorem 7.19*(i)*. Thus we can assume $\alpha_1\alpha_2 \leq 0$. If $\alpha_1 = \alpha_2 = 0$, the we can apply Theorem 7.17. If ($\alpha_1 = 0$ and $\alpha_2 \neq 0$) or ($\alpha_1 \neq 0$ and $\alpha_2 = 0$), then we can apply Theorem 7.19. Thus we can assume $\alpha_1\alpha_2 < 0$. Is there a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$ in this situation?

Finally we come to *(iii)*. In this case there exists a basis $e_0, b_1, \ldots, b_4$ of $\mathfrak{k}$ such that the action of $\mathfrak{s}$ on $\mathfrak{k}$ is given by $[d_1, e_0] = -\alpha_1 b_1$, $[d_1, b_2] = b_2$, $[d_1, b_4] = a_1 b_4$, $[d_2, e_0] = -\alpha_2 b_1$, $[d_2, b_3] = b_3$, and $[d_2, b_4] = a_2 b_4$. Let $f \in \mathfrak{k}^*$ such that $f(b_\nu) = 1$ for $1 \le \nu \le 4$. Then we obtain

$$\begin{aligned} f_s(e_0) &= f_0 + \alpha_1 s_1 + \alpha_2 s_2 \\ f_s(b_1) &= 1 \\ f_s(b_2) &= e^{-s_1} \\ f_s(b_3) &= e^{-s_2} \\ f_s(b_4) &= e^{-a_1 s_1 - a_2 s_2} \end{aligned}$$

Let us define Ω and Ω_0 as above. Let $g \in \mathfrak{k}^*$ such that $g \notin \overline{\Omega}$ and $g_0 \in (\Omega_0)^{—}$. If $g(b_2) \neq 0$ or $g(b_3) \neq 0$, then Theorem 7.17 applies. Thus we can assume $g(b_2) = g(b_3) = 0$. If $a_1 = 0$ or $a_2 = 0$, then Theorem 7.17 applies as well.

If $a_1 > 0$ and $a_2 > 0$, then the assumption $g \in (\Omega_0)^{—}$ yields the contradiction $g(b_2) \neq 0$ and $g(b_3) \neq 0$. If $a_1 < 0$ and $a_2 < 0$, then Theorem 7.17 furnishes a solution of our problem. Thus we can assume $a_1 a_2 < 0$.

If $\alpha_1 \alpha_2 \ge 0$, then again Theorem 7.17 applies. Thus we can assume $\alpha_1 \alpha_2 < 0$. If $g(e_4) \neq 0$, then we can define a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$: There exist unique real numbers $\beta, \gamma \in \mathbb{R}$ such that $\alpha_1 s_1 + \alpha_2 s_2 = \beta s_1 + \gamma(a_1 s_1 + a_2 s_2)$ for all s because the vectors $(1, 0)$ and (a_1, a_2) are linearly independent. If $\beta = 0$, then our assumption $g \notin \overline{\Omega}$ implies $g(e_0) \neq f_0 - \gamma \log g(e_4)$. So we obtain the solution

$$h(\tau, \xi_1, \ldots, \xi_4) = u_0(\xi_4) \left(e^{-(\tau - f_0)} - \xi_4^\gamma \right) \, e^{-\tau^2 - |\xi|^2} \,,$$

compare the proof of Theorem 7.17. If $\beta \neq 0$, then we choose a $\beta' \in \mathbb{R}$ such that $\beta\beta' > 2$ and define the solution

$$h(\tau, \xi_1, \ldots, \xi_4) = u_0(\xi_4) \left(e^{-\beta'(\tau - f_0)} - \xi_2^{\beta\beta'} \xi_4^\gamma \right) \, e^{-\tau^2 - |\xi|^2} \,.$$

These considerations show that we can assume $g(e_4) = 0$. If $a_1 \alpha_2 > a_2 \alpha_1$, then we see $g \in \overline{\Omega}$. Thus we assume $a_1 \alpha_2 \le a_2 \alpha_1$. Does there exist a function $h \in L^1(K, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) = 0$ in this particular case?

9.3 The non-central case: $C^k\mathfrak{n}$ is not contained in $\mathfrak{z}\mathfrak{g}$

In this case we have $\gamma_0 \neq 0$. Let $\mathfrak{s}/\mathfrak{t} \cong \mathfrak{g}/\mathfrak{m}$ have dimension $m+1$. We fix a maximal set $\alpha,\ \gamma_0, \ldots, \gamma_{m'}$ of $\mathbb{C}$-linearly independent weights in $\mathfrak{s}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathbb{C})$ of the $\mathfrak{s}$-module $\mathfrak{m}$. Arguments similar to those following Lemma 8.2 show that there exist vectors $d_0, \ldots, d_m$ in $\mathfrak{s}$

- $\mathbb{R}$-linearly independent modulo $\mathfrak{t}$,
- such that $\alpha(d_\nu) = 0$ for $0 \leq \nu \leq m$,
- such that $\mathrm{Re}\,\gamma_\mu(d_\nu) = \delta_{\mu,\nu}$ for $0 \leq \mu, \nu \leq m'$,
- and such that $\mathrm{ad}(d_\nu)$ is nilpotent for $m'+1 \leq \nu \leq m$.

Note that $d_0, \ldots, d_m, e_0, \ldots, e_{k+1}$ is a Malcev basis of $\mathfrak{g}$ modulo $\mathfrak{z}\mathfrak{m}$. We define

$$E(r,s) = \exp(rd_0)\exp(s_1 d_1)\ldots\exp(s_m d_m)$$

and work with coordinates of the second kind given by the diffeomorphism

$$\mathbb{R}^{m+1} \times \mathbb{R}^{k+2} \times \mathfrak{z}\mathfrak{m} \longrightarrow G,\ (r,s,t,v,x,Z) \mapsto E(r,s)\Phi(t,v,x,Z)\ .$$

Combining 9.5 and 9.6, we obtain the important formulas

$$\begin{aligned}
\mathbf{Ad}^*\left(\boldsymbol{E(r,s)\Phi(0,v,x,Z)}\right)\boldsymbol{f}\ (\boldsymbol{e_0}) &= \boldsymbol{f_0 - vx_{k+1} + Q(r,s)}\\
(\boldsymbol{e_1}) &= \boldsymbol{x_{k+1}}\\
(\boldsymbol{e_\nu}) &= \boldsymbol{e^{-r}}\ \frac{\boldsymbol{(-v)^{k+2-\nu}}}{\boldsymbol{(k+2-\nu)!}}\\
(\boldsymbol{Y}) &= \boldsymbol{f}\left(\mathbf{Ad}(\boldsymbol{E(r,s)})^{-1}\boldsymbol{Y}\right)\\
(\boldsymbol{e_{k+2}}) &= \boldsymbol{e^{-r}}
\end{aligned}$$

where $2 \leq \nu \leq k+2$ and $Y \in \mathfrak{z}\mathfrak{m}$. The last equation is a special case of the preceding one. In the first equation one finds the polynomial function

$$Q(r,s) = \sum_{1 \leq j+j_1+\ldots+j_m \leq q} c_{j,j_1,\,\ldots\,,j_m}\, r^j s_1^{j_1}\ \ldots\ s_m^{j_m}$$

in $m+1$ variables whose coefficients are given by

$$c_{j,j_1,\,\ldots\,,j_m} = \frac{(-1)^{j+j_1+\ldots+j_m}}{j!\ j_1!\ \ldots\ j_m!}\ f\left(\mathrm{ad}(d_m)^{j_m}\ \ldots\ \mathrm{ad}(d_1)^{j_1}\ \mathrm{ad}(d_0)^j \cdot e_0\right) \quad .$$

It is immediate from the definition of these coefficients that Q does not depend on the variable s_ν if $[d_\nu, \mathfrak{t}] \subset \ker f$.

Let $\mathfrak{s}_c = \ker\gamma_0$ be the centralizer of $C^k\mathfrak{n}$ in $\mathfrak{s}$. Note that $[\mathfrak{s}_c, \mathfrak{t}] \subset \ker f$ is equivalent to $[\mathfrak{s}_c, \mathfrak{t}] = 0$ because f is in general position. If this is true, then the function Q depends only on the variable r. If even $[\mathfrak{s}, \mathfrak{t}] = 0$, then $Q = 0$.

Let $\mathfrak{s}_n$ denote the nilpotent part of $\mathfrak{s}$, i.e., the set of all $d \in \mathfrak{s}$ such that $\mathrm{ad}(d)$ is nilpotent. If $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then $\mathfrak{s}_n \subset \mathfrak{z}\mathfrak{n}$. In this case $m = m'$.

Definition 9.13. A triple Γ consisting of a nilpotent Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, a maximal set $\alpha, \gamma_0, \dots, \gamma_{m'}$ of linearly independent weights as above, and a coexponential basis $d_0, \dots, d_m$ for $\mathfrak{t}$ in $\mathfrak{s}$ as above is called structure data for $\Delta = (\mathfrak{g}, \mathfrak{m}, \mathfrak{n}, f)$.

Our next aim is to describe the closure of the orbit $\mathrm{Ad}^*(G)f$. We want to detect linear functionals $g \in \mathfrak{m}^*$ which satisfy the conditions $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ and $\mathrm{Ad}^*(G)g' \subset (\mathrm{Ad}^*(G)f')^{-}$. Further we require $\mathrm{Ad}^*(G)g' \neq \mathrm{Ad}^*(G)f'$. These linear functionals $g \in \mathfrak{m}^*$ are called critical for the orbit $\mathrm{Ad}^*(G)f$.

There is no loss of generality in assuming $\mathfrak{n}$ to be the nilradical of $\mathfrak{m}$, which is equivalent to $\mathfrak{z}\mathfrak{m} = \mathfrak{z}\mathfrak{n}$. It is only natural to choose $\mathfrak{n}$ as large as possible in order to keep the set of critical functionals for $\mathrm{Ad}^*(G)f$ small.

We assert that an arbitrary linear functional $g \in \mathfrak{m}^*$ satisfies

$$\mathrm{Ad}^*(G)g' \;\subset\; (\,\mathrm{Ad}^*(G)f'\,)^{-}$$

if and only if there exist sequences $r_n \in \mathbb{R}$, $s_n \in \mathbb{R}^m$, $v_n \in \mathbb{R}$, and $x_n \in \mathbb{R}^k$ such that

$$\begin{aligned} x_{n,k+1} &\longrightarrow g_1 \\ e^{-r_n}\,\frac{(-v_n)^{k+2-\nu}}{(k+2-\nu)!} &\longrightarrow g_\nu \\ \text{and } f\left(\mathrm{Ad}(E(r_n,s_n))^{-1}Y\right) &\longrightarrow g(Y) \end{aligned}$$

for $2 \le \nu \le k+2$ and every $Y \in \mathfrak{z}\mathfrak{n}$. If so, then in particular $e^{-r_n} \longrightarrow g_{k+2}$. Of course, we always take the limit for $n \longrightarrow \infty$. There is a certain degree of freedom in the choice of these sequences. The stronger condition

$$\mathrm{Ad}^*(G)g \;\subset\; (\,\mathrm{Ad}^*(G)f\,)^{-}$$

is fulfilled if and only if these sequences can be chosen such that in addition

$$f_0 - v_n x_{n,k+1} + Q(r_n, s_n) \longrightarrow g_0\ .$$

We introduce some notation that will be useful in the sequel. Let $1 \le \nu \le l$ be arbitrary. According to Remark 8.3 we choose an $\mathfrak{s}$-invariant, $\mathfrak{s}$-irreducible subspace $\mathfrak{z}_\nu$ of $\mathfrak{z}_{\gamma_\nu}$ which has dimension one if γ_ν is real and dimension two else. As in Remark 8.3 we fix a basis of $\mathfrak{z}_\nu$ and its dual basis of $\mathfrak{z}_\nu^*$ defining a coordinate system and a Euclidean norm such that $|\, f\,|\,\mathfrak{z}_\nu\,| = 1$. If sequences r_n, s_n, v_n, and x_n are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*(\,E(r_n,s_n)\Phi(0,v_n,x_n,0)\,)f\ . \tag{9.14}$$

Lemma 9.15. *Let $g \in \mathfrak{m}^*$ such that the conditions $\mathrm{Ad}^*(G)g' \subset (\mathrm{Ad}^*(G)f')^{-}$ and $g_{k+2} = 0$ are satisfied. Then $g_\nu = 0$ for all $3 \le \nu \le k+2$.*

Proof. Assume that $g_\nu \neq 0$ for some $3 \le \nu \le k+1$. Suppose that $f_n' \longrightarrow g'$ for suitably chosen sequences r_n, s_n, and x_n. Since $g_{k+2} = 0$, it follows $r_n \longrightarrow +\infty$. We consider the quotient $f_n(e_{\nu-1})/f_n(e_\nu)$ and obtain

$$\frac{-v_n}{k+3-\nu} \longrightarrow \frac{g_{\nu-1}}{g_\nu}$$

for $n \longrightarrow +\infty$. But this is a contradiction to

$$e^{-r_n}\,\frac{(-v_n)^{k+2-\nu}}{(k+2-\nu)!} \longrightarrow g_\nu \neq 0$$

and the proof of the lemma is complete. □

In order to obtain more concrete results, we require some additional assumptions.

Assumption 9.16 For all $m'+1 \leq \nu \leq m$ the polynomial function Q defined by f and Γ does **not** depend on the variable s_ν.

Permuting the weights $\gamma_1, \dots, \gamma_{m'}$ and the vectors $d_1, \dots, d_{m'}$, we can even suppose that there exists a $0 \leq l \leq m'$ such that Q depends on s_ν if and only if $1 \leq \nu \leq l$. Here we use the fact that $\mathfrak{s}$ acts as a commutative algebra on $\mathfrak{m}$. This follows from the obvious relation $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{z}\mathfrak{n}$. We call l the **critical index**.

Note that $[\mathfrak{s}_n, \mathfrak{t}] \subset \ker f$ is equivalent to $[\mathfrak{s}_n, \mathfrak{t}] = 0$. In this case Assumption 9.16 is satisfied. If $[\mathfrak{s}_n, e_0] \not\subset \ker f$, then 9.16 is violated.

Assumption 9.17 The condition $g\,|\,\mathfrak{z}_\nu \neq 0$ is satisfied for all $1 \leq \nu \leq l$. In this case we say that g is **admissible** with respect to f and Γ.

Remark. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then there exist sequences r_n, s_n, v_n, x_n such that $f_n' \longrightarrow g'$ where f_n is defined by 9.14. For $1 \leq \nu \leq l$ let $\mathfrak{z}_\nu$ be the $\mathfrak{s}$-irreducible subspace defined above. Since f is in general position, we have $f\,|\,\mathfrak{z}_\nu \neq 0$. From

$$|\, f_n \,|\, \mathfrak{z}_\nu \,| = e^{-s_{n,\nu}}\; |\, f \,|\, \mathfrak{z}_\nu \,| \longrightarrow |\, g \,|\, \mathfrak{z}_\nu \,| \neq 0$$

it follows that the sequence $s_{n,\nu}$ is convergent and hence remains bounded. Clearly, the condition $g\,|\,\mathfrak{z}_\nu \neq 0$ does not depend on the choice of $\mathfrak{z}_\nu$. We summarize:

The postulates 9.16 and 9.17 enforce that r is the only among the first $l+1$ variables (the arguments of Q) which may tend to infinity.

Remark 9.18. Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)f' \subset (\,\mathrm{Ad}^*(G)f'\,)^{-}$. Suppose that $g_{k+2} \neq 0$ and $g\,|\,\mathfrak{z}_\nu \neq 0$ for all $1 \leq \nu \leq m'$ and all $\mathfrak{s}$-irreducible subspaces $\mathfrak{z}_\nu$ of $\mathfrak{z}_{\gamma_\nu}$. Then it follows $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$, compare Remark 12.15. In particular this $g \in \mathfrak{m}^*$ is not critical for the orbit $\mathrm{Ad}^*(G)f$.

Let us recall the definition of the polynomial function p_0 on $\mathfrak{m}^*$ introduced in Subsection 9.1 which is constant on the orbits $\mathrm{Ad}^*(M)f_{r,s}$ in general position. We know that

$$p_0\,(\,\mathrm{Ad}^*(\,E(r,s)\Phi(t,v,x,Z)\,)f\,) \;=\; e^{-r}\;(\,f_0 + Q(r,s)\,)^k\;.$$

Observe that p_0 is also constant on orbits $\mathrm{Ad}^*(M)g$ for $g \in \mathfrak{m}^*$ such that $g_\nu = 0$ for all $3 \leq \nu \leq k+2$. Further we recall that

$$p_1 = e_0 e_{k+2} - e_1 e_{k+1}$$

is an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$ and hence constant on all $\mathrm{Ad}^*(M)$-orbits. Profiting by the existence of these polynomials p_0 and p_1, we now obtain the following description of (the admissible part of) the closure of the orbit $\mathrm{Ad}^*(G)f$.

Lemma 9.19. *Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Suppose that there exists some structure data Γ for $\Delta = (\mathfrak{g}, \mathfrak{m}, \mathfrak{n}, f)$ such that Assumption 9.16 holds. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ in the sense of 9.17 such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then*

$$\mathrm{Ad}^*(G)g \subset (\,\mathrm{Ad}^*(G)f\,)^{-}$$

if and only if one of the following conditions holds:

(i) $g_{k+2} \neq 0$ *and* $p_1(g) = g_{k+2}\,(\,f_0 + Q(-\log g_{k+2}, -\log|\,g\,|\,\mathfrak{z}_1\,|, \ldots, -\log|\,g\,|\,\mathfrak{z}_l\,|\,)\,)$,

(ii) $g_{k+2} = 0$ *and* $p_0(g) = 0$.

Proof. First we prove *(i)*. Let $g_{k+2} \neq 0$. Without loss of generality we can choose a representative g on the orbit $\mathrm{Ad}^*(M)g$ such that $g_{k+1} = 0$. If sequences r_n, s_n, v_n, and x_n are chosen such that $f_n' \longrightarrow g$ with f_n as in 9.14, then the relations

$$f_n(e_{k+2}) = e^{-r_n} \longrightarrow g_{k+2} \neq 0 \quad \text{and} \quad f_n(e_{k+1}) = -e^{-r_n}\, v_n \longrightarrow g_{k+1} = 0$$

imply $v_n \longrightarrow 0$ and thus $g_\nu = 0$ for all $2 \leq \nu \leq k+1$. Since $g_{k+2} \neq 0$ and since g is admissible for the orbit $\mathrm{Ad}^*(G)f$, we obtain

$$r_n \longrightarrow -\log g_{k+2} \quad \text{and} \quad s_{n,\mu} \longrightarrow -\log|\,g\,|\,\mathfrak{z}_\mu\,|$$

for $1 \leq \mu \leq l$. Now it is easy to see that $\mathrm{Ad}^*(G)g$ is contained in the closure of $\mathrm{Ad}^*(G)f$ if and only if

$$f_n(e_0) = f_0 + Q(r_n, s_n) \longrightarrow g_0 = f_0 + Q\,(\,-\log g_{k+2}, -\log|\,g\,|\,\mathfrak{z}_1\,|, \ldots, -\log|\,g|\mathfrak{z}_l\,|\,)$$

which is equivalent to

$$p_1(g) = g_{k+2}\, g_0 = g_{k+2}\ \ (\,f_0 + Q(-\log g_{k+2}, -\log|\,g\,|\,\mathfrak{z}_1\,|, \ldots, -\log|\,g\,|\,\mathfrak{z}_l\,|)\,)\ .$$

Now we prove *(ii)*. Let $g_{k+2} = 0$. Then Lemma 9.15 already implies $g_\nu = 0$ for all $3 \leq \nu \leq k+2$. We observe that the condition $p_0(g) = k!\,(-1)^k\, g_1^k\, g_2 = 0$ is equivalent to ($g_1 = 0$ or $g_2 = 0$). So let us assume $p_0(g) = 0$.

Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of the orbit $\mathrm{Ad}^*(G)f'$, there exist sequences r_n and s_n such that

$$f\left(\mathrm{Ad}(E(r_n, s_n)^{-1}Y\right) \longrightarrow g(Y)$$

for all $Y \in \mathfrak{z}\mathfrak{n}$. In particular $e^{-r_n} \longrightarrow 0$ and thus $r_n \longrightarrow +\infty$. Since g is admissible for the orbit $\mathrm{Ad}^*(M)f$, it follows that the sequences $s_{n,\mu}$ converge for $1 \leq \mu \leq l$.

We distinguish three cases: If $g_1 \neq 0$ and $g_2 = 0$, then we define

$$v_n = \frac{1}{g_1}\,(f_0 + Q(r_n, s_n) - g_0) \quad \text{and} \quad x_{n,k+1} = g_1\ .$$

Then $f_0 - v_n x_{n,k+1} + Q(r_n, s_n) = g_0$. The boundedness of the sequences $s_{n,\mu}$ for $1 \le \mu \le l$ implies

$$e^{-r_n} \frac{(-v_n)^{k+2-\nu}}{(k+2-\nu)!} \longrightarrow 0$$

for all $2 \le \nu \le k+2$. This proves the inclusion $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$.

Next we consider the case $g_1 = 0$ and $g_2 \ne 0$. First we assume $g_2 > 0$. If we define

$$v_n = -(k!\, g_2)^{1/k}\, e^{r_n/k} \quad \text{and} \quad x_{n,k+1} = -(k!\, g_2)^{-1/k}\, e^{-r_n/k}\, (f_0 + Q(r_n, s_n) - g_0)\ ,$$

then we obtain

$$e^{-r_n} \frac{(-v_n)^k}{k!} = g_2 \quad \text{and} \quad e^{-r_n} \frac{(-v_n)^{k+2-\nu}}{(k+2-\nu)!} \longrightarrow 0$$

for all $3 \le \nu \le k+2$. Further we see $f_0 - v_n x_{n,k+1} + Q(r_n, s_n) = g_0$ and $x_{n,k+1} \longrightarrow 0$ because the sequence $s_{n,\mu}$ is bounded for $1 \le \mu \le l$. If $g_2 < 0$, then k is odd. In this case we replace $-(k!\, g_2)^{\frac{1}{k}}$ by $(k!\, |g_2|)^{\frac{1}{k}}$. This shows $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$.

Finally we treat the case $g_1 = 0$ and $g_2 = 0$. If we define

$$v_n = e^{r_n/(2k)} \quad \text{and} \quad x_{n,k+1} = e^{-r_n/(2k)}\ (\, f_0 + Q(r_n, s_n) - g_0\,)\ ,$$

then the assertion $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ follows.

In order to prove the opposite implication of *(ii)*, let us suppose that $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ and $g_1 g_2 \ne 0$. Then there exist sequences r_n, s_n, v_n, and x_n such that $f_n \longrightarrow g$. First we see that $r_n \longrightarrow +\infty$ because $f_n(e_{k+2}) \longrightarrow 0$. Note that $e^{-r_n/k} Q(r_n, s_n) \longrightarrow 0$ because the sequence $s_{n,\mu}$ is bounded for all $1 \le \mu \le l$. From this and $e^{-r_n/k}\, f_n(e_0) \longrightarrow 0$ we conclude that

$$e^{-r_n}\, v_n^k\, x_{n,k+1}^k \longrightarrow 0\ .$$

On the other hand, we know that

$$f_n(e_1)^k\, f_n(e_2) = e^{-r_n} \frac{(-v_n)^k}{k!}\, x_{n,k+1}^k \longrightarrow g_1^k\, g_2 \ne 0$$

which is a contradiction. This completes the proof of our lemma. □

Now we turn to the investigation of the relevant (unitary) representation theory of the group M. As a starting point we choose the observation that the subset $\mathrm{Ad}^*(G)f$ of $\mathfrak{m}^*$ decomposes into $\mathrm{Ad}^*(M)$-orbits. More precisely, since M is a normal subgroup of G, we obtain

$$\mathrm{Ad}^*(G)f = \bigcup_{(r,s)\in\mathbb{R}\times\mathbb{R}^m} \mathrm{Ad}^*(M) f_{r,s}$$

where

$$f_{r,s} = \mathrm{Ad}^*\,(\,E(r,s)\,)\,f\ .$$

The link to representation theory is given by the Kirillov map $\mathcal{K} : \mathfrak{m}^*/\,\mathrm{Ad}^*(M) \longrightarrow \widehat{M}$. It is a well-known fact that $\mathcal{K}$ is a G-equivariant homeomorphism, if $\mathfrak{m}^*/\,\mathrm{Ad}^*(M)$

carries the quotient topology, and $\widehat{M} = \operatorname{Prim} C^*(M)$ the Jacobson topology, see [23].

The Kirillov map $\mathcal{K}$ maps $\mathrm{Ad}^*(G)f$ bijectively onto the subset $\{\pi_{r,s} : (r,s) \in \mathbb{R}^{m+1}\}$ of $\widehat{M}$ where

$$\pi_{r,s} = \mathcal{K}(\mathrm{Ad}^*(M)f_{r,s}) = \mathcal{K}(f_{r,s}) \; .$$

Our intention is to compute the infinitesimal operators of these representations.

First, note that $f_{r,s}(e_{k+2}) = e^{-r}$ and $f_{r,s}(e_\nu) = 0$ for all $1 \leq \nu \leq k+2$. From this it becomes obvious that the subalgebra

$$\mathfrak{p} = \langle e_0, e_2, \ldots, e_{k+1} \rangle + \mathfrak{z}\mathfrak{n}$$

is a polarization at $f_{r,s}$ for all r and s. This reflects the $\mathfrak{s}$-invariance of $\mathfrak{p}$. The equations

$$\begin{aligned} \mathrm{Ad}^*(\exp x e_{k+1}) f_{r,s}\ (e_1) &= e^{-r} x \\ (Y) &= f_{r,s}(Y) \end{aligned}$$

for all $Y \in \mathfrak{p}$ show that the Pukanszky condition

$$\mathrm{Ad}^*(P) f_{r,s} = f_{r,s} + \mathfrak{p}^\perp$$

is also satisfied for all r and s. We notice that $\mathfrak{c} = \mathbb{R}e_1$ is a commutative subalgebra of $\mathfrak{m}$, which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. Further $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{m}$. Hence the results of Section 6 for representations in general position apply. We see that the infinitesimal operators of

$$\pi_{r,s} = \operatorname{ind}_P^M \chi_{f_{r,s}}$$

are given by

$$\begin{aligned} d\pi_{r,s}(e_0) &= \left(\frac{1}{2} + i f_0 + i Q(r,s) \right) + \xi \partial_\xi \\ d\pi_{r,s}(e_1) &= -\partial_\xi \\ d\pi_{r,s}(e_\nu) &= i e^{-r} \, \frac{(-\xi)^{k+2-\nu}}{(k+2-\nu)!} \\ d\pi_{r,s}(Y) &= i f \left(E(r,s)^{-1} Y \right) \; . \end{aligned} \tag{9.20}$$

where $Y \in \mathfrak{z}\mathfrak{n}$. Now let $g \in \mathfrak{m}^*$ be a singular functional in the sense that $g_\nu = 0$ for all $3 \leq \nu \leq k+2$ and ($g_1 \neq 0$ or $g_2 \neq 0$). The inclusion $[\mathfrak{n}, \mathfrak{n}] \subset \ker g$ shows that $\mathfrak{n}$ is a polarization at $g \in \mathfrak{m}^*$. Further we see that the Pukanszky condition

$$\mathrm{Ad}^*(N) g = g + \mathfrak{n}^\perp$$

is also satisfied because

$$\begin{aligned}\mathrm{Ad}^*(\exp ve_1)g\ (e_0) &= g_0 + vg_1\\ \mathrm{Ad}^*(\exp ve_1)g\ (Y) &= g(Y)\\ \mathrm{Ad}^*(\exp xe_2)g\ (e_0) &= g_0 - kxg_2\\ \mathrm{Ad}^*(\exp xe_2)g\ (Y) &= g(Y)\end{aligned}$$

for $Y \in \mathfrak{n}$. Finally $\mathfrak{c} = \mathbb{R}e_0$ is a commutative subalgebra of $\mathfrak{m}$ which is coexponential for $\mathfrak{n}$ in $\mathfrak{m}$. Consequently the results of Subsection 6.2 for representations of semi-direct products show that the infinitesimal operators of

$$\rho = \mathrm{ind}_N^M\ \chi_g$$

are given by

$$\begin{aligned}&\boldsymbol{d\rho(e_0) = -\partial_\xi}\\ &\boldsymbol{d\rho(e_1) = ie^{\xi}\, g_1}\\ &\boldsymbol{d\rho(e_2) = ie^{-k\xi}\, g_2}\\ &\boldsymbol{d\rho(e_\nu) = 0}\\ &\boldsymbol{d\rho(Y) = ig(Y)}\end{aligned} \tag{9.21}$$

for all $3 \leq \nu \leq k+2$ and $Y \in \mathfrak{z}\mathfrak{n}$.

The purpose of the rest of this subsection is to prove

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

for irreducible representations $\rho = \mathcal{K}(g)$ which correspond to critical functionals $g \in \mathfrak{m}^*$. The results of Section 5.1 turn out to be valuable in this context.

First we consider the $\mathrm{ad}^*(\mathfrak{m})$-invariant polynomial function $p_1 = e_0 e_{k+2} - e_1 e_{k+1}$ on $\mathfrak{m}^*$. Let W_1 be the image of p_1 in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map, i.e.,

$$W_1 = \frac{1}{2}\,(\,e_1 e_{k+1} - e_{k+1} e_1\,) - e_0 e_{k+2}\ .$$

Now it is easy to verify the crucial equation

$$d\pi_{r,s}(W_1) = p_1(f_{r,s})\cdot \mathrm{Id}$$

where $p_1(f_{r,s}) = e^{-r}\ (\,f_0 + Q(r,s)\,)$. Note that $f_{r,s}(e_{k+2}) = e^{-r}$. As above we consider the $\mathfrak{s}$-invariant, $\mathfrak{s}$-irreducible subspaces $\mathfrak{z}_\nu$ of $\mathfrak{z}\mathfrak{m}$ and Euclidean norms on $\mathfrak{z}_\nu^*$ such that $|\, f_{r,s}\,|\,\mathfrak{z}_\nu\,| = e^{-s_\nu}$ for $1 \leq \nu \leq l$, compare Remark 8.3 and Lemma 8.4. Further we form the direct sum $\mathfrak{z} = \mathbb{R}e_{k+2} \oplus \mathfrak{z}_1 \oplus \ldots \oplus \mathfrak{z}_l$ of these spaces. Now we can prove

Lemma 9.22. *Let f and Γ be as above. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ. Assume that $g_{k+2} \neq 0$ and that $\mathrm{Ad}^*(G)g \not\subset (\,\mathrm{Ad}^*(G)f\,)^-$. Then*

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.

Proof. It follows from Lemma 9.19 that

$$p_1(g) \neq g_{k+2}\,(\,f_0 + Q(-\log g_{k+2}, -\log|\,g|_{\mathfrak{z}_1}\,|, \ldots, -\log|\,g|_{\mathfrak{z}_l}\,|\,)\,)\ .$$

The following modification of the element $W_1 \in \mathcal{U}(\mathfrak{m})_{\mathbb{C}}$ is necessary: According to Remark 8.3 we consider the polynomial functions $c_\nu \in \mathcal{P}(\mathfrak{z}_\nu^*)$ given by $c_\nu(h) = |h|^2$ for $h \in \mathfrak{z}_\nu^*$ so that in particular $c_\nu(f_{r,s}\,|\,\mathfrak{z}_\nu) = e^{-2s_\nu}$. Let C_ν denote the image of c_ν in $\mathcal{U}(\mathfrak{z}_\nu)_{\mathbb{C}}$ under the modified symmetrization map. Then we define

$$W = W_1 \cdot C_1 \cdot \ldots \cdot C_l\ .$$

Further we consider the continuous function $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ given by

$$\psi(\xi) = \xi_0 \cdot |\,\xi_1\,|^2 \cdot \ldots \cdot |\,\xi_l\,|^2 \cdot (\,f_0 + Q(-\log \xi_0, -\log|\,\xi_1\,| \ldots, -\log|\,\xi_l\,|)\,)$$

if $\xi_0 > 0$ and $\psi(\xi) = 0$ else. Then we obtain

$$d\pi_{r,s}(W) = e^{-2(s_1 + \ldots + s_l)}\, p_1(f_{r,s}) \cdot \mathrm{Id} = \psi(f_{r,s}\,|\,\mathfrak{z}) \cdot \mathrm{Id}$$

and

$$d\rho(W) = |\,g|_{\mathfrak{z}_1}\,|^2 \cdot \ldots \cdot |\,g|_{\mathfrak{z}_l}\,|^2\, p_1(g) \cdot \mathrm{Id} \neq \psi(g\,|\,\mathfrak{z}) \cdot \mathrm{Id}\ .$$

Now we can apply Lemma 5.4 and a variant of Theorem 5.18 to $\pi_{r,s}$, ρ, W, and ψ. The claim of this lemma follows. □

Next we consider the polynomial function p_0 on $\mathfrak{m}^*$ defined in Subsection 9.1 which is constant on orbits $\mathrm{Ad}^*(M)f$ in general position for $f \in \mathfrak{m}^*$ such that $f(e_\nu) = 0$ for all $1 \leq \nu \leq k+1$, and on singular orbits $\mathrm{Ad}^*(M)g$ for $g \in \mathfrak{m}^*$ such that $g(e_\nu) = 0$ for all $3 \leq \nu \leq k+2$. One could expect that the following is true: If W_0 denotes the image of p_0 in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map, then it holds $d\pi_{r,s}(W_0) = p_0(f_{r,s}) \cdot \mathrm{Id}$ and $d\rho(W_0) = p_o(g) \cdot \mathrm{Id}$ for the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$. We discuss this question for the four-dimensional filiform algebra (k=2).

Remark. Assume that there exists basis $e_0, \ldots, e_4$ of $\mathfrak{m}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = 2e_2$, and $[e_0, e_3] = e_3$. Here the polynomial p_0 is given by

$$p_0 = e_0 e_0 e_4 - 2 e_0 e_1 e_3 + 2 e_1 e_1 e_2\ .$$

The image W_0 of p_0 under the modified symmetrization map equals

$$W_0 = i\left[\,e_0 e_0 e_4 - 2 e_0 e_3 e_1 + 2 e_2 e_1 e_1 - e_0 e_4 + 2 e_3 e_1 + \frac{1}{3} e_4\right]\ .$$

The infinitesimal operators of $\pi_{r,s} = \mathcal{K}(f_{r,s})$ are given by

$$\begin{aligned} d\pi_{r,s}(e_0) &= \frac{1}{2} + if_0 + iQ(r,s) + \xi\partial\xi \\ d\pi_{r,s}(e_1) &= -\partial_\xi \\ d\pi_{r,s}(e_2) &= \frac{1}{2} ie^{-r}\xi^2 \\ d\pi_{r,s}(e_3) &= -ie^{-r}\xi \\ d\pi_{r,s}(e_4) &= ie^{-r} \end{aligned}$$

From this it follows that

$$d\pi_{r,s}(W_0) = -\frac{1}{12}\, e^{-r} + p_0(f_{r,s})$$

where $p_0(f_{r,s}) = e^{-r}\,(f_0 + Q(r,s))^2$. This means that $d\pi_{r,s}(W_0)$ is essentially given by $p_0(f_{r,s})$, but it is not equal to $p_0(f_{r,s})\cdot\mathrm{Id}$. On the other hand, the equations

$$\begin{aligned} d\rho(e_0) &= -\partial_\xi \\ d\rho(e_1) &= ie^{\xi}\, g_1 \\ d\rho(e_2) &= ie^{-2\xi}\, g_2 \\ d\rho(e_3) &= d\rho(e_4) = 0 \end{aligned}$$

imply $d\rho(W_0) = 2g_1^2 g_2\cdot\mathrm{Id} = p_0(g)$. Unfortunately we could not compute $d\pi_{r,s}(W_0)$ if k is arbitrary and W_0 is the image of p_0 under the symmetrization map. However, in general we obtain the following result.

Lemma 9.23. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ be given as in Subsection 9.1. In particular $\mathfrak{n}$ is a central extension of a $(k+1)$-step nilpotent filiform algebra. Let $f \in \mathfrak{m}^*$ be in general position such that $f(e_\nu) = 0$ for all $1 \le \nu \le k+1$. Let $g \in \mathfrak{m}^*$ be singular in the sense that $g(e_\nu) = 0$ for $3 \le \nu \le k+2$. We define the complex polynomial function*

$$\alpha(r,s) = f_0 + Q(r,s) - i/2\,.$$

Then there exists an element W_0 in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ such that

$$d\pi_{r,s}(W_0) = e^{-r}\,\alpha(r,s)^k \quad \textit{and} \quad d\rho(W_0) = (-1)^k\, k!\, g_1^k\, g_2$$

*holds for the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$. In general W_0 is **not** in the center of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$, and W_0 is **not** equal to the image of p_0 under the symmetrization map.*

Proof. Let us define $A = -ie_0$, $C = -ie_1$, and $B_\nu = (-i)\,(-1)^\nu\,\nu!\,e_{k+2-\nu}$. These are elements of the complexification of $\mathfrak{m}$. One verifies

$$\begin{aligned} d\pi_{r,s}(A) &= \alpha(r,s) + \xi D_\xi \\ d\pi_{r,s}(B_\nu) &= e^{-r}\,\xi^\nu \\ d\pi_{r,s}(C) &= -D_\xi \end{aligned}$$

$$\begin{aligned} d\rho(A) &= -D_\xi \\ d\rho(B_\nu) &= 0 \\ d\rho(B_k) &= (-1)^k\,k!\,e^{-k\xi}\,g_2 \\ d\rho(C) &= e^\xi\,g_1 \end{aligned}$$

where $1 \le \nu \le k-1$. The following assertion is easily proved by induction: Let $\lambda \ge 1$ be arbitrary. Then for $1 \le \mu \le \lambda$ there exist positive numbers $a_{\lambda,\mu}$ such that

$$(\xi\partial_\xi)^\lambda = \sum_{\mu=1}^{\lambda} a_{\lambda,\mu}\,\xi^\mu \partial_\xi^\mu\ .$$

The proof of this equality shows $a_{\lambda+1,\mu} = \mu a_{\lambda,\mu} + a_{\lambda,\mu-1}$ for $\lambda \ge 1$ and $1 \le \mu \le \lambda+1$. Here one reads $a_{\lambda,0} = 0$ and $a_{\lambda,\lambda+1} = 0$ for $\lambda \ge 1$. Obviously $a_{\lambda,\lambda} = a_{\lambda,1} = 1$ for $\lambda \ge 1$. Let us define $a_{0,0} = 1$ and $a_{0,\mu} = 0$ for $1 \le \mu \le \lambda$. From these definitions it follows

$$(\xi\partial_\xi)^\lambda = \sum_{\mu=0}^{\lambda} a_{\lambda,\mu}\,\xi^\mu \partial_\xi^\mu\ .$$

for all $\lambda \ge 0$. Using this identity and applying the binomial theorem to the commuting operators $\alpha(r,s) + \xi D_\xi$ and $-\xi D_\xi$, we obtain

$$\begin{aligned} e^{-r}\,\alpha(r,s)^k &= e^{-r}\ \left(\alpha(r,s) + \xi D_\xi - \xi D_\xi\right)^k \\ &= e^{-r}\ \sum_{\nu=0}^{k} \binom{k}{\nu}\ (\alpha(r,s)+\xi D_\xi)^\nu\ \ (-\xi D_\xi)^{k-\nu} \\ &= \sum_{\nu=0}^{k}\ \sum_{\mu=0}^{k-\nu}\ i^{k-\nu-\mu}\ \binom{k}{\nu}\ a_{k-\nu,\mu}\ (\alpha(r,s)+\xi D_\xi)^\nu\ \ (e^{-r}\xi^\mu)\ (-D_\xi)^\mu\ . \end{aligned}$$

These computations motivate the definition

$$W_0 = \sum_{\nu=0}^{k}\sum_{\mu=0}^{k-\nu} i^{k-\nu-\mu}\ \binom{k}{\nu}\ a_{k-\nu,\mu}\ A^\nu\ B_\mu\ C^\mu$$

of the element W_0 in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$. This definition does not depend on r, s, f, or g. Immediately we get

$$d\pi_{r,s}(W_0) = e^{-r}\ \alpha(r,s)^k\ .$$

Since $d\rho(B_\nu) = 0$ for $1 \le \nu \le k-1$, we obtain

$$d\rho(W_0) = (-1)^k\,k!\,g_1^k\,g_2\ .$$

This completes the proof of our lemma. □

Remark 9.24. Let us suppose that there exists an $X \in \mathfrak{z}\mathfrak{m}$ such that $f_{r,s}(X) = 1$ for all r and s. Then $d\pi_{r,s}(X) = i$. Clearly, this condition is satisfied if $Q \neq 0$. Let f and g be as in the preceding lemma. If p_0 is the polynomial function on $\mathfrak{m}^*$ defined in Subsection 9.1, which is constant on the orbits $\mathrm{Ad}^*(M)f$ and $\mathrm{Ad}^*(M)g$, then

$$p_0(f_{r,s}) = (\, f_0 + Q(r,s)\,)^k \quad \text{and} \quad p_0(g) = (-1)^k\, k!\, g_1^k\, g_2\ .$$

Let us replace A by $\tilde{A} = A + \frac{1}{2}X$ in the proof of Lemma 9.23. Then

$$d\pi_{r,s}(\tilde{A}) = (f_0 + Q(r,s)) + \xi D_\xi$$

and the proof of Lemma 9.23 with $\tilde{A}$ instead of A shows that the following conclusion holds: There exists an element W_0 in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ such that

$$d\pi_{r,s}(W_0) = p_0(f_{r,s}) \quad \text{and} \quad d\rho(W_0) = p_0(g)\ .$$

By now we have done most of the work which is necessary to prove the following lemma.

Lemma 9.25. *Let f and Γ be as above. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ. Assume that $g_{k+2} = 0$ and that g is critical for the orbit $\mathrm{Ad}^*(G)f$. Then*

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of the irreducible representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.

Proof. Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f$, Lemma 9.15 implies that $g_\nu = 0$ for all $3 \leq \nu \leq k+2$. Since $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$, it follows from Lemma 9.19 that

$$p_0(g) = (-1)^k\ k!\ g_1^k\, g_2 \neq 0\ .$$

As before we consider $\mathfrak{z} = \mathbb{R}e_{k+2} \oplus \mathfrak{z}_1 \oplus \ldots \oplus \mathfrak{z}_l$ and appropriate Euclidean norms on $\mathfrak{z}_\nu^*$. We fix an element $W_0 \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$ whose existence is guaranteed by Lemma 9.23. The following modification of W_0 is necessary: According to Remark 8.3 we define the polynomial functions $c_\nu \in \mathcal{P}(\mathfrak{z}_\nu^*)$ by $c_\nu(h) = |h|^2$ for $h \in \mathfrak{z}_\nu^*$ so that in particular $c_\nu(f_{r,s}|\mathfrak{z}_\nu) = e^{-2s_\nu}$. Let C_ν denote the image of c_ν in $\mathcal{U}(\mathfrak{z}_\nu)_\mathbb{C}$ under the modified symmetrization map. Then we define

$$W = W_0 \cdot C_1 \cdot \ldots \cdot C_l\ .$$

Further we consider the continuous function $\psi : \mathfrak{z}^* \longrightarrow \mathbb{C}$ given by

$$\psi(\xi) = \xi_0\, |\, \xi_1\, |^2 \cdot \ldots \cdot |\, \xi_l\, |^2 \cdot \alpha(-\log \xi_0, -\log|\, \xi_1\, |, \ldots, -\log|\, \xi_l\, |)^k$$

if $\xi_0 > 0$ and $\psi(\xi) = 0$ else. Here the complex-valued polynomial function α is defined as in Lemma 9.23. Then we obtain

$$d\pi_{r,s}(W) = e^{-2(s_1+\ldots+s_l)}\ e^{-r}\ \alpha(r,s)^k \ = \ \psi(f_{r,s}\,|\,\mathfrak{z})$$

and

$$d\rho(W) = (-1)^k\ k!\ g_1^k\, g_2\, |\, g|\mathfrak{z}_1\, |^2\ \ldots\ |\, g|\mathfrak{z}_l\, |^2 \ \neq\ \psi(g\,|\,\mathfrak{z}) = 0\ .$$

Here we use the fact that g is admissible with respect to f so that $g\,|\,\mathfrak{z}_\nu \neq 0$ for $1 \leq \nu \leq l$. An application of Lemma 5.4 and a slight modification of Theorem 5.18 to $\pi_{r,s}$, ρ, W, and ψ finishes this proof. ☐

Remark 9.26. Once again we point out that the modification (multiplication by C_ν for $1 \leq \nu \leq l$) of the elements W_0 and W_1 of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ is absolutely necessary in order to avoid singularities of ψ. Such singularities make the application of Theorem 5.1 or Theorem 5.17 impossible, as we have already noticed in Remark 5.5.

There is no doubt about the significance of the Assumptions 9.16 and 9.17 for our treatise. Concerning the orbit space of the coadjoint action, these assumptions are indispensable for a concrete characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$ in $\mathfrak{m}^*$, see Lemma 9.19.

From the representation theoretical point of view, these postulates guarantee that (W, p, ψ) separates ρ from $\{\pi_{r,s} : (r,s) \in \mathbb{R}^{m+1}\}$, see Definition 5.10 and Lemmata 9.22 and 9.25.

In the next theorem we sum up some of the results obtained in this subsection.

Theorem 9.27. *Let $\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ be as in Subsection 9.1. In particular $\mathfrak{n}$ is a central extension of a $(k+1)$-step nilpotent filiform algebra. Here we assume $C^k\mathfrak{n} \not\subset \mathfrak{z}\mathfrak{g}$. Let $\mathfrak{m}$ be a proper, non-nilpotent ideal of $\mathfrak{g}$ with $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$. Let us assume that there exists a nilpotent subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ and $[\mathfrak{s}_c, \mathfrak{t}] = 0$ where $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$ and $\mathfrak{s}_c$ is the centralizer of $C^k\mathfrak{n}$ in $\mathfrak{s}$.*

Further let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows that

$$\bigcap_{(r,s)\in\mathbb{R}^{m+1}} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$ of M.

Proof. Let us choose a basis of the weights of the $\mathfrak{s}$-module $\mathfrak{m}$, and a coexponential basis for $\mathfrak{t}$ in $\mathfrak{s}$ as in the beginning of this subsection. Now we observe that the additional Assumptions 9.16 and 9.17 are always satisfied: First, the polynomial function Q depends only on the variable r because $[\mathfrak{s}_c, \mathfrak{t}] = 0$. Hence for every $f \in \mathfrak{m}^*$ in general position, Assumption 9.16 holds with $m' = l = 0$. Any $g \in \mathfrak{m}^*$ is admissible with respect to f and Γ. Now an application of Lemma 9.22 or 9.25 completes this proof. □

Though including semi-direct sums $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$, the condition $[\mathfrak{s}_c, \mathfrak{t}] = 0$ does not reach far beyond the case $\dim \mathfrak{g}/\mathfrak{m} = 1$, i.e., a one-parameter group $\mathrm{Ad}(\exp rd_0)$ acting on the stabilizer $\mathfrak{m}$, non-trivially on the central ideal $C^k\mathfrak{n}$.

If $\dim \mathfrak{g}/\mathfrak{m} > 1$ and $g \in \mathfrak{m}^*$ is critical for $\mathrm{Ad}^*(G)f$, but not admissible with respect to f and Γ, then we must admit that the situation remains somewhat mysterious.

Open Problem 9.28 The following 8-dimensional example gives a first impression of the phenomena which may occur if $\dim \mathfrak{g}/\mathfrak{m} > 1$. Let us assume that there exists a basis $d_0, d_1, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that the commutator relations are given by

$$[e_1, e_2] = e_3, \quad [e_0, e_1] = -e_1, \quad [e_0, e_2] = e_2$$

and

$$[d_0, e_0] = -ae_5, \quad [d_0, e_3] = e_3, \quad [d_1, e_0] = -be_5, \quad [d_1, e_4] = e_4 \; .$$

The stabilizer $\mathfrak{m}$ is the ideal spanned by $e_0, \dots, e_5$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. We can assume $f(e_\nu) = 0$ for $\nu = 1, 2$ and $f(e_\nu) = 1$ for $\nu = 3, 4, 5$. Now we obtain

$$\begin{aligned} \mathrm{Ad}^*\,(\,E(r,s)\,\Phi(0,v,x,z)\,)\,f\;(e_0) &= f_0 - vx + ar + bs \\ (e_1) &= x \\ (e_2) &= -e^{-r}v \\ (e_3) &= e^{-r} \\ (e_4) &= e^{-s} \\ (e_5) &= 1 \end{aligned}$$

where $v, x \in \mathbb{R}$ and $z \in \mathbb{R}^3$.

Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_5 = 1$ and $g_3 \geq 0$. We restrict ourselves to non-admissible g. Such linear functionals g satisfy $g_4 = 0$ and exist if and only if $b \neq 0$, what we shall assume henceforth.

The following lemma contains a description of the non-admissible part of the closure of $\mathrm{Ad}^*(G)f$ in this example.

Lemma 9.29. *Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Assume that $g \in \mathfrak{m}^*$ is not admissible with respect to f and that that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ if and only if one of the following conditions is satisfied:*

(i) $g_3 = 0$ *and* $b\,g_1\,g_2 < 0$,

(ii) $g_3 = 0$ *and* $g_1\,g_2 = 0$.

Proof. First we suppose that $g_3 \neq 0$ and $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$. Then there exist sequences r_n, s_n, v_n, and x_n such that $f_n \longrightarrow g$. In particular $f_n(e_1) = x_n \longrightarrow g_1$ and $f_n(e_3) = e^{-r_n} \longrightarrow g_3 \neq 0$ so that $r_n \longrightarrow -\log g_3$ converges. Further $f_n(e_4) = e^{-s_n} \longrightarrow g_4 = 0$ implies $s_n \longrightarrow +\infty$. From $f_n(e_4) = -e^{-r_n}v_n \longrightarrow g_2$ it follows that $v_n \longrightarrow g_2/g_3$ converges, too. But this is a contradiction to $f_n(e_0) = f_0 - v_n x_n + ar_n + bs_n \longrightarrow g_0$ because $b \neq 0$ and $s_n \longrightarrow +\infty$. Thus we have shown that $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ if $g_3 \neq 0$.

Now we assume that $g_3 = 0$ and $bg_1g_2 < 0$. Then we define $r_n = n$, $v_n = -e^{r_n}g_2$, $x_n = g_1$, and

$$s_n = -\frac{1}{b}(f_0 - v_n x_n + ar_n - g_0) \; .$$

These settings imply $f_n(e_0) = g_0$ and $s_n \longrightarrow +\infty$ because $bg_1g_2 > 0$. This proves $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ in this case.

For the opposite implication we assume $g_3 = 0$ and $b\, g_1\, g_2 > 0$. Let r_n, s_n, v_n, and x_n be arbitrary sequences such that $f_n' \longrightarrow g'$. Since

$$\frac{1}{b}\, f_n(e_1)\, f_n(e_2) = -\frac{1}{b}\, e^{-r_n}\, v_n x_n \longrightarrow \frac{g_1\, g_2}{b} > 0\,,$$

there exists a $c > 0$ such that $-\frac{1}{b} v_n\, x_n \geq c e^{r_n}$ for almost all n. But this implies

$$\frac{1}{b}\, f_n(e_0) = \frac{f_0}{b} - \frac{v_n\, x_n}{b} + \frac{a r_n}{b} + s_n \longrightarrow +\infty$$

which is a contradiction. It follows $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ in this case.

In order to verify *(ii)*, we have to distinguish three cases: Assume $g_3 = 0$. If $g_2 = 0$ and $g_1 \neq 0$, then we define $r_n = s_n = n$, $x_n = g_1$, and

$$v_n = \frac{1}{g_1}\, (f_0 + a r_n + b s_n - g_0)\;.$$

If $g_2 \neq 0$ and $g_1 = 0$, then we set $r_n = s_n = n$, $v_n = -e^{r_n} g_2$, and

$$x_n = -\frac{1}{g_2}\, e^{-r_n}\, (f_0 + a r_n + b s_n - g_0)\;.$$

If $g_2 = 0$ and $g_1 = 0$, then we define $r_n = s_n = n$, $v_n = e^{r_n/2}$, and

$$x_n = e^{-r_n/2}\, (f_0 + a r_n + b s_n - g_0)\;.$$

In any case it follows $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$. $\square$

We conclude this section with the following open question: Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Assume that $g \in \mathfrak{m}^*$ is non-admissible and critical for the orbit $\mathrm{Ad}^*(G)f$. This means that in addition to $g_5 = 1$ and $g_4 = 0$ one of the following conditions is satisfied: $g_3 \neq 0$ or ($g_3 = 0$ and $b g_1 g_2 > 0$). Does the relation

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \;\not\subset\; \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of the irreducible representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$ in this 8-dimensional example? Once again we stress that the results of Section 5.1 do not apply here. A solution of this problem lies beyond the scope of our investigation in this work.

10 Nilradical is a Heisenberg algebra

In this section we study the representation theory of an exponential solvable Lie group G such that, roughly speaking, the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{g}$ is a Heisenberg algebra. In the first subsection we describe the algebraic structure of $\mathfrak{g}$ in case of a Heisenberg algebra $\mathfrak{n}$ of arbitrary dimension. In particular we see that the stabilizer $\mathfrak{m} = \mathfrak{g}_{f'} + \mathfrak{n}$ of a linear functional $f' \in \mathfrak{n}^*$ in general position is equal to the centralizer of the one-dimensional ideal $C^1\mathfrak{n} = \mathfrak{z}\mathfrak{n}$ in $\mathfrak{g}$. In the second subsection we treat the case of a three-dimensional Heisenberg algebra $\mathfrak{n}$. In the third subsection we give a list of all possible exponential solvable Lie algebras $\mathfrak{g}$ containing the five-dimensional Heisenberg algebra $\mathfrak{n}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$ and such that the centralizer $\mathfrak{m}$ of $C^1\mathfrak{n}$ in $\mathfrak{g}$ is not nilpotent. In the subsequent subsections we study these algebras case by case.

10.1 The structure of $\mathfrak{g}$

Assume that $\mathfrak{g}$ is an exponential solvable Lie algebra containing a $(2k+1)$-dimensional Heisenberg algebra $\mathfrak{n}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. In this case

(10.1) $$\mathfrak{n} \underset{2k}{\supset} C^1\mathfrak{n} \underset{1}{\supset} \{0\}$$

is the descending central series of $\mathfrak{n}$ and $\mathfrak{z}\mathfrak{n} = C^1\mathfrak{n}$. Let $\mathfrak{m}$ denote the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ be a linear functional in general position so that $f \neq 0$ on $C^1\mathfrak{n}$. As usual we denote by f' its restriction to $\mathfrak{n}$. From

(10.2) $$\mathrm{Ad}^*(N)f' = f' + (\mathfrak{z}\mathfrak{n})^\perp$$

it follows that $\mathfrak{m} = \mathfrak{g}_{f'} + \mathfrak{n}$. If $\mathfrak{m} = \mathfrak{g}$, then the orbit $\mathrm{Ad}^*(G)f' = \mathrm{Ad}^*(N)f'$ is closed. Consequently Theorem 3.23 implies that there are no functionals $g \in \mathfrak{m}^*$ which are critical for the orbit $\mathrm{Ad}^*(G)f$. Thus we can assume $\mathfrak{m} \neq \mathfrak{g}$ so that $\dim \mathfrak{g}/\mathfrak{m} = 1$.

The structure of the algebra $\mathrm{Der}(\mathfrak{n})$ of derivations of the Heisenberg algebra is well-known. What we need here is

Lemma. *The inner derivations of the Heisenberg algebra $\mathfrak{n}$ are given by*

$$\mathrm{Inn}(\mathfrak{n}) = \{\, D \in \mathrm{End}(\mathfrak{n}) : D\cdot\mathfrak{n} \subset \mathfrak{z}\mathfrak{n} \text{ and } D = 0 \text{ on } \mathfrak{z}\mathfrak{n} \,\}\,.$$

Proof. Let D be as above. Let $e_1, \ldots, e_{2k}, Z$ be a basis of $\mathfrak{n}$ such that $Z \in \mathfrak{z}\mathfrak{n}$ and $[e_i, e_{n+j}] = \delta_{ij} Z$ for $1 \le i, j \le n$. Then there exist $\lambda_i \in \mathbb{R}$ such that $D\cdot e_i = \lambda_i Z$. Now it is easy to see that

$$D = \sum_{i=1}^{n} \lambda_i \,\mathrm{ad}(e_{n+i}) - \sum_{i=1}^{n} \lambda_{n+i} \,\mathrm{ad}(e_i)$$

is in $\mathrm{Inn}(\mathfrak{n})$. The opposite inclusion is obvious. □

Lemma 10.3. *Let $\mathfrak{g}$, $\mathfrak{n}$, and f be as above and consider the subspace $\mathfrak{v} = \ker f'$ of $\mathfrak{n}$. Then*

$$\mathfrak{s} = \{X \in \mathfrak{g} : [X, \mathfrak{v}] \subset \mathfrak{v}\}$$

is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ and $\mathfrak{g}_{f'} \subset \mathfrak{s}$. Further $\mathfrak{s}$ acts as a commutative algebra of derivations on $\mathfrak{m}$ so that there exists a weight space decomposition of the $\mathfrak{s}$-module $\mathfrak{m}$.

Proof. Clearly $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}_{f'} \subset \mathfrak{s}$ and $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{s} \cap \mathfrak{n} \subset \mathfrak{z}\mathfrak{n}$ so that $[\mathfrak{s},\mathfrak{s}]$ acts trivially on $\mathfrak{m}$. Since $\mathfrak{m} \neq \mathfrak{g}$, there exists an element $D \in \mathfrak{g}$ such that $\mathrm{ad}(D) \neq 0$ on $\mathfrak{z}\mathfrak{n}$. The preceding lemma implies that there exists an $A \in \mathfrak{n}$ such that $D_0 = D + A$ satisfies $[D_0, \mathfrak{v}] \subset \mathfrak{v}$ and $[D_0, \mathfrak{z}\mathfrak{n}] \neq 0$ which means $D_0 \in \mathfrak{s}$. This proves $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ because $D_0 \notin \mathfrak{m}$. □

In the rest of this section we regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules and work with the weight space decompositions of these modules. Let us define $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m} = \mathfrak{g}_{f'}$. We observe that there exists a real weight $\gamma \in \mathfrak{s}^*$ such that $C^1\mathfrak{n} \subset \mathfrak{n}_\gamma$ and $\gamma \neq 0$. Clearly $\ker\gamma = \mathfrak{t}$ so that $\tilde{\gamma} = 0$. As usual the tilde indicates restriction to $\mathfrak{t}$.

Note that $B(X,Y) = f([X,Y])$ defines a skew, bilinear form on $\mathfrak{n}$ which is $\mathfrak{m}$-equivariant, i.e., it satisfies

$$B([T,X], Y) = -B(X, [T,Y]) \tag{10.4}$$

for all $T \in \mathfrak{m}$ and $X, Y \in \mathfrak{n}$. In the following lemma we resume a few basic properties of the weight space decomposition of the $\mathfrak{s}$-module $\mathfrak{n}$ which are caused by this symplectic structure.

Lemma 10.5. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as above.*

(i) If $\alpha \in \mathfrak{s}^* = \mathrm{Hom}(\mathfrak{s}, \mathbb{C})$ is a weight of the $\mathfrak{s}$-module $\mathfrak{n}$ such that $\mathfrak{n}_\alpha \not\subset \mathfrak{z}\mathfrak{n}$, then it follows that $\gamma - \alpha$ is a weight, too. Possibly α is not real.

(ii) There exists at least one weight $\alpha \in \mathfrak{s}^*$ such that $\tilde{\alpha} \neq 0$. If α is a non-real weight of the $\mathfrak{s}$-module $\mathfrak{n}$, then $\tilde{\alpha} = 0$.

(iii) The weights $\alpha_1, \dots, \alpha_r, \gamma$ are $\mathbb{C}$-linearly independent if and only if $\tilde{\alpha}_1, \dots \tilde{\alpha}_r$ are $\mathbb{C}$-linearly independent.

(iv) The weight space $\mathfrak{n}_\gamma$ is one-dimensional if and only if 0 is not a weight of the $\mathfrak{s}$-module $\mathfrak{n}$. If $\dim \mathfrak{n}_\gamma = 1$, then $\mathfrak{g}$ is the semi-direct sum of a commutative algebra $\mathfrak{s}_0$ and the ideal $\mathfrak{n}$. In this case $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$.

Proof. We begin with *(i)* and use the fact that $\mathfrak{n}$ decomposes into a direct sum of weight spaces $\mathfrak{n}_\alpha$. Recall that if α is not real, then

$$\mathfrak{n}_\alpha = ((\mathfrak{n}_\mathbb{C})_\alpha \oplus (\mathfrak{n}_\mathbb{C})_{\bar{\alpha}}) \cap \mathfrak{n} = \mathfrak{n}_{\bar{\alpha}} .$$

Let α be an arbitrary weight of the $\mathfrak{s}$-module $\mathfrak{n}$ such that $\mathfrak{n}_\alpha \not\subset \mathfrak{z}\mathfrak{n}$ so that there exists a weight β such that

$$\{0\} \neq [\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset C^1\mathfrak{n} \cap (\mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\bar{\alpha}+\beta}) .$$

This implies $\gamma = \alpha + \beta$ or $\gamma = \bar{\alpha} + \beta$ because $C^1\mathfrak{n} \subset \mathfrak{n}_\gamma$. Since both β and $\bar{\beta}$ are weights and γ is real, we see that $\gamma - \alpha$ is also a weight of the $\mathfrak{s}$-module $\mathfrak{n}$. Note that the case $\alpha = \beta = \gamma/2$ is possible. This proves *(i)*.

The first claim of *(ii)* is obvious because $\mathfrak{m}$ is not nilpotent. Let us suppose that there exists a non-real weight α such that $\tilde{\alpha} \neq 0$. We conclude that there exists a $\sigma \in \mathbb{R}$, $\sigma \neq 0$ such that $\mathrm{Im}\,\alpha = \sigma\,\mathrm{Re}\,\alpha$ because $\mathfrak{g}$ is exponential. Further it follows from *(i)* that $\beta = \gamma - \alpha$ is a non-real weight, too. Thus there exists a $\tau \in \mathbb{R}$, $\tau \neq 0$ such that $\operatorname{Im}\beta = \tau \operatorname{Re}\beta$. If we restrict to $\mathfrak{t}$, then it follows $\operatorname{Re}\tilde{\alpha} + \operatorname{Re}\tilde{\beta} = 0$ and $\sigma \operatorname{Re}\tilde{\alpha} + \tau \operatorname{Re}\tilde{\beta} = 0$ because $\tilde{\gamma} = 0$. Since $\operatorname{Re}\tilde{\alpha} \neq 0$, we see $\sigma = \tau$. Now we obtain the contradiction

$$\gamma = \operatorname{Re}\alpha + \operatorname{Re}\beta = \frac{1}{\sigma}(\operatorname{Im}\alpha + \operatorname{Im}\beta) = 0\,.$$

This proves *(ii)*. In order to prove *(iii)*, let us assume that the weights $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ are $\mathbb{C}$-linearly dependent, i.e., there exist $\sigma_1, \ldots, \sigma_r \in \mathbb{C}$ not all zero such that $\sum_{\nu=1}^r \sigma_\nu \tilde{\alpha}_\nu = 0$. It follows that there exists a $\sigma \in \mathbb{C}$ such that $\sigma\gamma = \sum_{\nu=1}^r \sigma_\nu \alpha_\nu$ because $\mathfrak{s}/\mathfrak{t}$ is one-dimensional, $\gamma \neq 0$ and $\tilde{\gamma} = 0$. This proves the first direction. The opposite direction is trivial.

Finally we come to the proof of *(iv)*. The first claim is a consequence of *(i)*. Assume that $\mathfrak{n}_\gamma$ is one-dimensional. Let $\mathfrak{t}_0$ be the weight space of weight 0 of the $\mathfrak{s}$-module $\mathfrak{t} = \mathfrak{g}_{f'}$. Clearly $\mathfrak{m} = \mathfrak{t}_0 + \mathfrak{n}$ and $[\mathfrak{s}, \mathfrak{t}_0] \subset \mathfrak{t}_0 \cap \mathfrak{n} = 0$. If we choose $D \in \mathfrak{s}$ such that $\operatorname{ad}(D) \neq 0$ on $\mathfrak{zn}$, then $\mathfrak{s}_0 = \mathbb{R}D + \mathfrak{t}_0$ is a commutative subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s}_0 + \mathfrak{n}$ and $\mathfrak{s}_0 \cap \mathfrak{n} = 0$. Furthermore $[\mathfrak{t}_0, \mathfrak{g}] = [\mathfrak{t}_0, \mathfrak{n}] \subset \ker f$ implies $\mathfrak{t}_0 \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. This completes the proof of our lemma. □

The subspace $\mathfrak{v} = \ker f'$ of $\mathfrak{n}$ is a symplectic vector space with respect to the skew, bilinear form $B = B_f$ and $\mathfrak{n} = \mathfrak{v} \oplus C^1\mathfrak{n}$. Furthermore there exists a direct sum decomposition $\mathfrak{v} = \bigoplus_\alpha \mathfrak{w}_\alpha$ into $\mathfrak{s}$-invariant, symplectic subspaces $\mathfrak{w}_\alpha = \mathfrak{v}_\alpha + \mathfrak{v}_{\gamma-\alpha}$ where $\mathfrak{v}_\alpha$ denotes the weight spaces of the $\mathfrak{s}$-module $\mathfrak{v}$. This sum is taken over all representatives α for pairs $(\alpha, \gamma-\alpha)$ of real weights, and for 4-tuples $(\alpha, \gamma-\alpha, \bar{\alpha}, \gamma-\bar{\alpha})$ of non-real weights. This sum is orthogonal with respect to the bilinear form $B = B_f$ which is non-degenerate on $\mathfrak{w}_\alpha$ for all α. It holds $\mathfrak{n}_\gamma = \mathfrak{v}_\gamma \oplus C^1\mathfrak{n}$ and $\mathfrak{n}_\alpha = \mathfrak{v}_\alpha$ for $\alpha \neq \gamma$. Further we note that $\mathfrak{v}_\alpha$ and $\mathfrak{v}_{\gamma-\alpha}$ are $\mathfrak{s}$-invariant, Lagrangian subspaces of $\mathfrak{w}_\alpha$ for $\operatorname{Re}\alpha \neq \gamma/2$. If $\operatorname{Re}\alpha = \gamma/2$, then there is no such decomposition for $\mathfrak{w}_\alpha$.

These assertions are also part of the following lemma in which we describe the action of $\mathfrak{t}$ on $\mathfrak{v}_\alpha$.

Lemma 10.6. *The skew, bilinear form $B = B_f$ is non-degenerate on $\mathfrak{v}$ and induces a $\mathfrak{t}$-equivariant isomorphism from $\mathfrak{v}$ onto $\mathfrak{v}^*$. More exactly, the restriction of B to each summand $\mathfrak{w}_\alpha$ of the above orthogonal sum is non-degenerate. If $\operatorname{Re}\alpha \neq \gamma/2$, then we obtain a $\mathfrak{t}$-equivariant isomorphism $\mathfrak{v}_{\gamma-\alpha} \longrightarrow \mathfrak{v}_\alpha^*$.*

Proof. First we prove that B is non-degenerate on $\mathfrak{v}$: Let $X \in \mathfrak{v}$ such that $B(X,Y) = 0$ for all $Y \in \mathfrak{v}$. Since $f \neq 0$ on $C^1\mathfrak{n}$, it follows $[X,Y] = 0$ for all $Y \in \mathfrak{v}$, even for all $Y \in \mathfrak{n}$. This means $X \in \mathfrak{zn}$ and thus $X = 0$. We have shown that $\varphi(X)(Y) = B(X,Y)$ defines a linear isomorphism $\varphi : \mathfrak{v} \longrightarrow \mathfrak{v}^*$. It follows from 10.4 that φ is $\mathfrak{t}$-equivariant:

$$\varphi(\operatorname{ad}(T)X)\ (Y) = B(\,[T,X]\,,\,Y\,) = -B(\,X\,,\,[T,Y]\,) = \operatorname{ad}^*(T)\varphi(X)\ (Y)\,.$$

Let X be in $\mathfrak{v}_\alpha$. Since $X \notin \mathfrak{zn}$, there exists some weight β and an element $Y \in \mathfrak{v}_\beta$ such that $B_f(X,Y) = f([X,Y]) \neq 0$. Now it follows $\beta = \gamma - \alpha$ because $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset \mathfrak{n}_{\alpha+\beta}$. This shows us that the restriction of B to any summand $\mathfrak{w}_\alpha$ of the above orthogonal sum is non-degenerate. Assume $\operatorname{Re}\alpha \neq \gamma/2$. Since $\mathfrak{v}_\alpha$ and $\mathfrak{v}_{\gamma-\alpha}$ are Lagrangian subspaces (isotropic subspaces of maximal dimension), we get a $\mathfrak{t}$-equivariant isomorphism $\mathfrak{v}_{\gamma-\alpha}$ onto $\mathfrak{v}_\alpha^*$. This finishes the proof of our lemma. □

Remark 10.7. Let us explain the relation between the action of $\mathfrak{s}$ on $\mathfrak{v}_\alpha$ and on $\mathfrak{v}_{\gamma-\alpha}$. Assume that $\alpha \neq \gamma/2$. Then the preceding lemma implies $\dim \mathfrak{v}_\alpha = \dim \mathfrak{v}_{\gamma-\alpha}$. Let $B = B_f$ be as above. If $e_1, \ldots, e_r$ is a basis of $\mathfrak{v}_\alpha$, then there exists a basis $e_1^*, \ldots, e_r^*$ of $\mathfrak{v}_{\gamma-\alpha}$ such that

$$B(e_\nu, e_\mu^*) = \delta_{\mu,\nu}$$

for all $1 \leq \mu, \nu \leq r$. This is just the dual basis under the identification of $\mathfrak{v}_{\gamma-\alpha}$ and $\mathfrak{v}_\alpha^*$ given by $\varphi = \varphi_B$. Let $S \in \mathfrak{s}$ be arbitrary. We have

$$B(\, X, [S, Y]\,) = \gamma(S)\, B(X, Y) - B(\, [S, X], Y\,)$$

for $X, Y \in \mathfrak{n}$. From this equality we conclude that if the action of S on $\mathfrak{v}_\alpha$ is given by a matrix $S_{\mu,\nu}$ with respect to the basis $e_1, \ldots, e_r$, then the action of S on $\mathfrak{v}_{\gamma-\alpha}$ is given by the matrix

$$S^*_{\mu,\nu} = \gamma(S)\, \delta_{\mu,\nu} - S_{\nu,\mu}$$

with respect to the basis $e_1^*, \ldots, e_r^*$. In particular we see that S acts semi-simple on $\mathfrak{v}_\alpha$ if and only if S is semi-simple on $\mathfrak{v}_{\gamma-\alpha}$. Note that if $T \in \mathfrak{t}$ acts by $T_{\mu,\nu}$ on $\mathfrak{v}_\alpha$, then T acts by the inverse transpose $T^*_{\mu,\nu} = -T_{\nu\,\mu}$ on $\mathfrak{v}_{\gamma-\alpha}$.

Remark 10.8. We emphasize the importance of the moment map μ in the subsequent investigation of stabilizers $\mathfrak{m}$ of a Heisenberg algebra $\mathfrak{n}$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Note that $\mathfrak{v} = \ker f'$ is an $\mathrm{ad}(\mathfrak{m}_{f'})$-invariant subspace of $\mathfrak{n}$ and that B_f defines a symplectic form on $\mathfrak{v}$. Further the stabilizer $M_{f'}$ of $f' = f \,|\, \mathfrak{n}$ in M acts as a group of symplectic isomorphisms on $\mathfrak{v}$. We define the moment map $\mu : \mathfrak{v} \longrightarrow \mathfrak{m}^*_{f'}$ as the $M_{f'}$ - equivariant, polynomial map

$$\mu(X, T) = f\,(\,[X, [X, T]]\,)\ .$$

On the one hand, the moment map partly determines the orbit $\mathrm{Ad}^*(M)f$ as it follows from

$$\mathrm{Ad}^*(\exp X)f\,(T) = f(T) + \frac{1}{2}\,\mu(X, T)$$

for $T \in \mathfrak{m}_{f'}$ and $X \in \mathfrak{n}$. One the other hand, the moment map μ defines invariant polynomial functions on $\mathfrak{m}^*$ as we will explain next:

Since the symplectic form B_f on $\mathfrak{v}$ induces a canonical isomorphism from $\mathfrak{v}$ onto $\mathfrak{v}^*$, we can define a linear map Φ from $\mathfrak{m}^*$ onto $\mathfrak{v}$: If $g \in \mathfrak{m}^*$, then there exists a unique element $X = \Phi(g)$ in $\mathfrak{v}$ such that $B_f(X, Y) = g(Y)$ for all $Y \in \mathfrak{v}$. Since $\mathfrak{v}$ is an $\mathrm{Ad}(M_{f'})$-invariant subspace, it follows

$$\mathrm{Ad}^*(a)g\,(Y) = B_f(X, \mathrm{Ad}(a)^{-1}Y) = B_f(\mathrm{Ad}(a)X, Y)$$

for all $Y \in \mathfrak{v}$, $g \in \mathfrak{m}^*$, and $a \in M_{f'}$ so that

$$\Phi(\mathrm{Ad}^*(a)g) = \mathrm{Ad}(a)\, \Phi(g)\ .$$

This means that Φ is $M_{f'}$ - equivariant. Similarly the computation

$$\begin{aligned}\mu\,(\mathrm{Ad}(a)X, T) &= B_f\,(\,\mathrm{Ad}(a)X, [\mathrm{Ad}(a)X, T]\,) = B_f\,\big(\,X, [X, \mathrm{Ad}(a)^{-1}T]\,\big)\\ &= \mu\,\big(X, \mathrm{Ad}(a)^{-1}T\big)\end{aligned}$$

shows that $\mu : \mathfrak{v} \longrightarrow \mathfrak{m}^*_{f'}$ as well as the composition $\Psi = \mu \circ \Phi$ are $M_{f'}$ - equivariant. Let us fix an element $T \in \mathfrak{m}_{f'}$ and write

$$\Psi_T(g) = \Psi(g, T) = \mu(\Phi(g), T) \ .$$

Clearly Ψ_T is a real-valued, polynomial function on $\mathfrak{m}^*$ such that $\mathrm{Ad}(a)\Psi_T = \Psi_{\mathrm{Ad}(a)T}$ for all $a \in M_{f'}$. Since $\mathrm{Ad}(a)T = T$ modulo $\mathfrak{n}$, it follows from the definition of μ that $\mathrm{Ad}(a)\Psi_T = \Psi_T$ for all $a \in M_{f'}$, i.e., Ψ_T is $\mathrm{Ad}(M_{f'})$-invariant.

Furthermore we note that if $M_{f'}$ is a commutative group, then the symmetric algebra $\mathcal{S}(\mathfrak{m}_{f'})$ yields an algebra of $\mathrm{Ad}(M_{f'})$-invariant, polynomial functions on $\mathfrak{m}^*$.

Finally we close with the following crucial observation: It will turn out that in the subsequent examples there exist $\mathrm{Ad}(M)$-invariant, polynomial functions p on $\mathfrak{m}^*$ such that $p = Q + R$ is a sum of polynomials $Q \in \mathcal{S}(\mathfrak{m}_{f'})$ and $R = \Psi_T \in \mathcal{S}(\mathfrak{n})$ which are both defined by some $T \in \mathfrak{m}_{f'}$.

10.2 The three-dimensional Heisenberg algebra

In the beginning we point out that the considerations of this subsection are nothing but a special case of the results of Subsection 9.3. However, it seems to be useful to treat the three-dimensional Heisenberg algebra before we come to the five-dimensional case.

Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ be as in Subsection 10.1 such that $\dim \mathfrak{n} = 3$. Assume that $\mathfrak{n}$ is equal to the nilradical of $\mathfrak{g}$. Clearly the $\mathfrak{s}$-module $\mathfrak{n}$ admits the weights α, $\gamma - \alpha$, γ where α, γ are linearly independent. In particular $\dim \mathfrak{g}/\mathfrak{n} = 2$. It follows that there exists a basis $d, e_0, \ldots, e_3$ of $\mathfrak{g}$ such that the commutator relations $[e_1, e_2] = e_3$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[d, e_2] = e_2$, and $[d, e_3] = e_3$ hold. Let $f \in \mathfrak{m}^*$ be in general position so that $f_3 \neq 0$. Without loss of generality we can assume $f_3 = 1$. It follows from 10.2 that $f_1 = f_2 = 0$ for a suitable representative f of the orbit $\mathrm{Ad}^*(M)f$. It is easy to see that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. As usual we work with coordinates of the second kind for M with respect to the above Malcev basis of $\mathfrak{m}$. A diffeomorphism $\Phi : \mathbb{R}^4 \longrightarrow M$ is given by

$$\Phi(t, x, y, z) = \exp(te_0)\exp(xe_1)\exp(ye_2 + ze_3) \ .$$

For the coadjoint action of G in $\mathfrak{m}^*$ we obtain

$$\begin{aligned} \mathrm{Ad}^*\left(\exp(rd)\Phi(0, x, y, z)\right) f \ (e_0) &= f_0 - xy \\ (e_1) &= y \\ (e_2) &= -e^{-r}x \\ (e_3) &= e^{-r} \end{aligned} \tag{10.9}$$

If sequences r_n, x_n, and y_n are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*\left(\exp(r_n d)\,\Phi(0, x_n, y_n, 0)\right) \ .$$

It is easy to see that $p = e_0\,e_3 - e_1\,e_2$ defines an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$ so that p is constant on all $\mathrm{Ad}^*(M)$-orbits. Note that $p(f_n) = f_0\,e^{-r_n}$. Profiting by the existence of p we achieve the following characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$.

Lemma 10.10. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that* $\mathrm{Ad}^*(G)g'$ *is contained in the closure of* $\mathrm{Ad}^*(G)f'$*. In particular $g_3 \geq 0$. Then* $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ *if and only if $p(g) = f_0\, g_3$. If $g_3 \neq 0$, then we have* $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.

Proof. First we assume $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ so that there exist sequences r_n, x_n, y_n such that $f_n \longrightarrow g$. Since $f_n(e_3) = e^{-r_n} \longrightarrow g_3$ and p is continuous, it follows $p(f_n) = f_0\, e^{-r_n} \longrightarrow p(g) = f_0\, g_3$.

Now we verify the opposite implication. Suppose $g_3 \neq 0$ so that without loss of generality we can assume $g_1 = g_2 = 0$. If we define $r_n = -\log g_3$ and $x_n = y_n = 0$, then $p(g) = g_0\, g_3 = f_0\, g_3$ implies $f_0 = g_0$ and $\mathrm{Ad}^*(G)g = \mathrm{Ad}^*(G)f$.

Next we suppose $g_3 = 0$. We must distinguish three cases. Note that $p(g) = -g_1\, g_2$. If $g_1 \neq 0$ and $g_2 = 0$, then we define $r_n = n$, $y_n = g_1$, and

$$x_n = \frac{1}{g_1}\,(f_0 - g_0)\,.$$

This implies $f_n \longrightarrow g$ and thus $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ in this case. If $g_1 = 0$ and $g_2 \neq 0$, then we define $r_n = n$, $x_n = -g_2 e^n$, and

$$y_n = -\frac{1}{g_2}\, e^{-n}\,(f_0 - g_0)\,.$$

If $g_1 = g_2 = 0$, then we set $r_n = n$, $x_n = e^{n/2}$, and $y_n = e^{-n/2}\,(f_0 - g_0)$. This proves $f_n \longrightarrow g$ and thus $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ in all three cases. The proof of this lemma is complete. ☐

Now we investigate the unitary representation theory of G. Let $f_r = \mathrm{Ad}^*(\exp(rd))f$ so that $\mathrm{Ad}^*(G)f = \cup_{r\in\mathbb{R}}\, \mathrm{Ad}^*(M)f_r$. First we describe the irreducible representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_1) = f_r(e_2) = 0$ and $f_r(e_3) = e^{-r}$, it is clear that $\mathfrak{p} = \langle\, e_0, e_2, e_3\,\rangle$ is a common Pukanszky polarization at f_r for all $r \in \mathbb{R}$. Note that $\mathfrak{p}$ is $\mathfrak{s}$-invariant. Further $\mathfrak{c} = \langle\, e_1\,\rangle$ is a commutative subalgebra which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. From Section 6.1 we learn that the infinitesimal operators of $\pi_r = \mathrm{ind}_P^M\, \chi_{f_r}$ are given by

$$\begin{aligned} d\pi_r(e_0) &= \frac{1}{2} + if_0 + \xi\partial_\xi \\ d\pi_r(e_1) &= -\partial_\xi \\ d\pi_r(e_2) &= -ie^{-r}\xi \\ d\pi_r(e_3) &= ie^{-r} \end{aligned}$$

Now let $g \in \mathfrak{m}^*$ such that $g_3 = 0$ and ($g_1 \neq 0$ or $g_2 \neq 0$). Then we see that the nilradical $\mathfrak{n} = \langle\, e_1, e_2, e_3\,\rangle$ is a Pukanszky polarization at g, and that $\mathfrak{c} = \langle\, e_0\,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply that the infinitesimal operators of $\rho = \mathrm{ind}_N^M\, \chi_g$ are given by

$$\begin{aligned} d\rho(e_0) &= -\partial_\xi \\ d\rho(e_1) &= ie^{\xi}\, g_1 \\ d\rho(e_2) &= ie^{-\xi}\, g_2 \\ d\rho(e_3) &= 0 \end{aligned}$$

Let W denote the image of p in $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ under the modified symmetrization map so that
$$W = \frac{1}{2}(e_1\, e_2 + e_2\, e_1) - e_0\, e_3\ .$$
One verifies easily $d\pi_r(W) = p(f_r)\cdot\mathrm{Id}$ for all $r \in \mathbb{R}$ and $d\rho(W) = p(g)\cdot\mathrm{Id}$. By now we have done most of the work which is necessary to prove the following

Theorem 10.11. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra such that the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is a three-dimensional Heisenberg algebra. Let $\mathfrak{m}$ denote the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{g}$. Let $f \in \mathfrak{m}^*$ be in general position and $g \in \mathfrak{m}^*$ critical for the orbit $\mathrm{Ad}^*(G)f$. Then the relation*
$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)}\, \pi_r \not\subset \ker_{L^1(M)}\, \rho$$
holds for the L^1-kernels of the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

Proof. Let us fix a function $h \in \mathcal{C}_0^\infty(M)$ such that $\rho(h) \neq 0$. Further we define the element $\tilde{W} = W + i f_0\, e_3$ in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$. It is easy to see that
$$\pi_r(\tilde{W} * h) = \big(\, p(f_r) - f_0\, e^{-r}\,\big)\ \pi_r(h) = 0$$
for all $r \in \mathbb{R}$. Note that $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^{-}$ because g is critical for the orbit $\mathrm{Ad}^*(G)f$. Consequently Lemma 10.10 implies $p(g) \neq f_0\, g_3$ and thus
$$\rho(\tilde{W} * h) = (\, p(g) - f_0\, g_3\,)\ \rho(h) \neq 0\ .$$
These considerations prove our theorem. □

10.3 The five-dimensional Heisenberg algebra

Our aim is to classify those exponential solvable Lie algebras $\mathfrak{g}$ whose nilradical is a five-dimensional Heisenberg algebra by means of properties of the weight space decomposition of the $\mathfrak{s}$-module $\mathfrak{n}$. Afterwards these algebras are studied case by case.

Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 10.1 such that $\dim\mathfrak{n} = 5$. Assume that $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. First we exclude the possibility of non-real weights: If α is a non-real weight of the $\mathfrak{s}$-module $\mathfrak{n}$, then we have the weight space decomposition $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{\gamma-\alpha} \oplus \mathfrak{n}_\gamma$ where $\mathfrak{n}_\gamma = C^1\mathfrak{n}$ is one-dimensional and $\mathfrak{n}_\alpha$, $\mathfrak{n}_{\gamma-\alpha}$ are two-dimensional, $\mathfrak{s}$-irreducible subspaces. From Lemma 10.5.*(ii)* it follows $\tilde{\alpha} = 0$. This means that $\mathfrak{m}$ is nilpotent, a contradiction. Hence we can assume that all weights are real if $\dim\mathfrak{n} = 5$.

With the aid of Lemma 10.5 we see that the following cases occur:

- There are five distinct weights α, $\gamma-\alpha$, β, $\gamma-\beta$, γ where α, β, γ are linearly independent. All weight spaces are one-dimensional.
- There are five distinct weights α, $\gamma-\alpha$, β, $\gamma-\beta$, γ where α, γ are linearly independent and $\beta \in \langle\,\alpha, \gamma\,\rangle$. Again all weight spaces are one-dimensional.
- There are four distinct weights α, $\gamma-\alpha$, $\gamma/2$, γ where $\tilde{\alpha} \neq 0$. This implies that α, γ are linearly independent. In this case $\mathfrak{n}_{\gamma/2}$ has dimension two and the other weight spaces are one-dimensional.

- There are four distinct weights α, $\gamma-\alpha$, 0, γ where $\tilde{\alpha} \neq 0$. Here $\mathfrak{n}_\gamma$ is two-dimensional and the other weight spaces are one-dimensional.
- There are three distinct weights α, $\gamma-\alpha$, γ such that $\tilde{\alpha} \neq 0$. Here $\mathfrak{n}_\alpha$, $\mathfrak{n}_{\gamma-\alpha}$ are two-dimensional and $\mathfrak{n}_\gamma$ is one-dimensional.

In the next subsections these Lie algebras $\mathfrak{g}$ will be investigated case by case. In doing so we will see the validity of

Theorem 10.12. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra such that the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is a five-dimensional Heisenberg algebra. Let $\mathfrak{m}$ denote the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{g}$. Let $f \in \mathfrak{m}^*$ be a linear functional in general position and $g \in \mathfrak{m}^*$ critical for the orbit $\mathrm{Ad}^*(G)f$. Then the relation*

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

Proof. This theorem is a consequence of the results of the Subsections 10.3.1 to 10.3.5. □

10.3.1 Five distinct weights α, $\gamma-\alpha$, β, $\gamma-\beta$, γ where α, β, γ are linearly independent

In this case $\dim \mathfrak{g}/\mathfrak{n} = 3$ and thus $\dim \mathfrak{g} = 8$. There exists a basis $d, e_0, \ldots, e_6$ of $\mathfrak{g}$ such that the commutator relations $[e_2, e_3] = e_6$, $[e_4, e_5] = e_6$, $[e_0, e_2] = -e_2$, $[e_0, e_3] = e_3$, $[e_1, e_4] = -e_4$, $[e_1, e_5] = e_5$, $[d, e_3] = e_3$, $[d, e_5] = e_5$, and $[d, e_6] = e_6$ hold.

Let $f \in \mathfrak{m}^*$ be in general position so that $f_6 \neq 0$. Without loss of generality we can assume $f_6 = 1$ and $f_\nu = 0$ for $2 \leq \nu \leq 5$. Obviously $\mathfrak{t} \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ and $\mathrm{Ad}^*(M)f = \mathrm{Ad}^*(N)f$. As usual we work with coordinates of the second kind for M given by

$$\Phi(t, u, x_1, y_1, x_2, y_2, z) = \exp(te_0 + ue_1)\, \exp(x_1e_2 + x_2e_4)\, \exp(y_1e_3 + y_2e_5 + ze_6)\ .$$

The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned} \mathrm{Ad}^*\left(\exp(rd)\, \Phi(0,0,x_1,y_1,x_2,y_2,z)\right) f\ (e_0) &= f_0 - x_1y_1 \\ (e_1) &= f_1 - x_2y_2 \\ (e_2) &= y_1 \\ (e_3) &= -e^{-r}x_1 \\ (e_4) &= y_2 \\ (e_5) &= -e^{-r}x_2 \\ (e_6) &= e^{-r} \end{aligned} \tag{10.13}$$

It is easy to see that $p_1 = e_0\, e_6 - e_2\, e_3$ and $p_2 = e_1\, e_6 - e_4\, e_5$ are $\mathrm{ad}(\mathfrak{m})$-invariant polynomial functions on $\mathfrak{m}^*$ which are constant on all $\mathrm{Ad}^*(M)$-orbits. Profiting by the existence of p_1 and p_2 we achieve the following characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$.

Lemma 10.14. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_6 \geq 0$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p_1(g) = f_0\, g_6$ and $p_2(g) = f_1\, g_6$. If $g_6 \neq 0$, then we have $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.*

Proof. The proof is very similar to that of Lemma 10.10. We omit the details. □

Let $f_r = \mathrm{Ad}^*(\exp(rd))f$. Next we describe the unitary representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_\nu) = 0$ for $2 \leq \nu \leq 5$ and $f_r(e_6) = e^{-r}$, it follows that $\mathfrak{p} = \langle\, e_0, e_1, e_3, e_5, e_6 \,\rangle$ is a common Pukanszky polarization at f_r for all $r \in \mathbb{R}$. Note that $\mathfrak{p}$ is $\mathfrak{s}$-invariant. Further $\mathfrak{c} = \langle\, e_2, e_4 \,\rangle$ is a commutative subalgebra of $\mathfrak{n}$ which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. The results of Section 6.1 yield:

$$\begin{aligned}
d\pi_r(e_0) &= \frac{1}{2} + i f_0 + \xi_1 \partial_{\xi_1} \\
d\pi_r(e_1) &= \frac{1}{2} + i f_1 + \xi_2 \partial_{\xi_2} \\
d\pi_r(e_2) &= -\partial_{\xi_1} \\
d\pi_r(e_3) &= -i e^{-r} \xi_1 \\
d\pi_r(e_4) &= -\partial_{\xi_2} \\
d\pi_r(e_5) &= -i e^{-r} \xi_2 \\
d\pi_r(e_6) &= i e^{-r}
\end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_6 = 0$ and ($g_2 \neq 0$ or $g_3 \neq 0$) and ($g_4 \neq 0$ or $g_5 \neq 0$). Then we see that the nilradical $\mathfrak{n} = \langle\, e_2, \ldots, e_6 \,\rangle$ is a Pukanszky polarization at g. Further $\mathfrak{c} = \langle\, e_0, e_1 \,\rangle$ is a commutative, coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$.
The results of Section 6.2 imply

$$\begin{aligned}
d\rho(e_0) &= -\partial_{\xi_1} \\
d\rho(e_1) &= -\partial_{\xi_2} \\
d\rho(e_2) &= i e^{\xi_1}\, g_2 \\
d\rho(e_3) &= i e^{-\xi_1}\, g_3 \\
d\rho(e_4) &= i e^{\xi_2}\, g_4 \\
d\rho(e_5) &= i e^{-\xi_2}\, g_5 \\
d\rho(e_6) &= 0
\end{aligned}$$

For $\nu = 1, 2$ let W_ν denote the image of p_ν in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map so that

$$W_1 = \frac{1}{2}(e_2\, e_3 + e_3\, e_2) - e_0 e_6 \quad \text{and} \quad W_2 = \frac{1}{2}(e_4\, e_5 + e_5\, e_4) - e_1 e_6\ .$$

One verifies easily $d\pi_r(W_\nu) = p_\nu(f_r)\cdot\mathrm{Id}$ for all r and $d\rho(W_\nu) = p_\nu(g)\cdot\mathrm{Id}$. Now we can prove the assertion of Theorem 10.12.

Lemma 10.15. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra such that the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is a five-dimensional Heisenberg algebra. Let $\mathfrak{s}$ be a nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$. Let us assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α,$\gamma - \alpha$, β, $\gamma - \beta$, γ where α, β, γ are linearly independent.*

Let $\mathfrak{m}$ denote the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{g}$. Let $f \in \mathfrak{m}^*$ be in general position and $g \in \mathfrak{m}^*$ critical for the orbit $\mathrm{Ad}^*(G)f$. Then the relation

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

Proof. Let us fix a function $h \in \mathcal{C}_0^\infty(M)$ such that $\rho(h) \neq 0$. Further we define the elements $\tilde{W}_1 = W_1 + if_0\, e_6$ and $\tilde{W}_2 = W_2 + if_1\, e_6$ in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$. It is easy to see that $\pi_r(\tilde{W}_1 * h) = 0$ and $\pi_r(\tilde{W}_2 * h) = 0$ for all $r \in \mathbb{R}$. Note that $\mathrm{Ad}^*(G)g \not\subset (\mathrm{Ad}^*(G)f)^-$ because g is critical for the orbit $\mathrm{Ad}^*(G)f$. Consequently Lemma 10.14 implies $p_1(g) \neq f_0\, g_6$ or $p_2(g) \neq f_1\, g_6$ so that $\rho(\tilde{W}_1 * h) \neq 0$ or $\rho(\tilde{W}_2 * h) \neq 0$. This proves our lemma. □

10.3.2 Five distinct weights α, $\gamma-\alpha$, β, $\gamma-\beta$, γ such that α, γ are linearly independent and $\beta \in \langle\, \alpha, \gamma\,\rangle$

In this case $\dim \mathfrak{g}/\mathfrak{n} = 2$ so that $\dim \mathfrak{g} = 7$. There exist $\sigma, \tau \in \mathbb{R}$ such that $\beta = \sigma\alpha + \tau\beta$. We conclude that there exists a basis $\langle\, d, e_0, \ldots, e_5\,\rangle$ of $\mathfrak{g}$ such that the commutator relations $[e_1, e_2] = e_5$, $[e_3, e_4] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = -\sigma e_3$, $[e_0, e_4] = \sigma e_4$, $[d, e_2] = e_2$, $[d, e_3] = \tau e_3$, $[d, e_4] = (1-\tau)e_4$, and $[d, e_5] = e_5$ hold.

Let $f \in \mathfrak{m}^*$ be a linear functional in general position so that $f_5 \neq 0$. Without loss of generality we can assume $f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 4$. Obviously $\mathfrak{t} \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ and $\mathrm{Ad}^*(M)f = \mathrm{Ad}^*(N)f$. As usual we work with coordinates of the second kind for M given by

$$\Phi(t, x_1, y_1, x_2, y_2, z) = \exp(te_0)\, \exp(x_1 e_1 + x_2 e_3)\, \exp(y_1 e_2 + y_2 e_4 + z e_5)\ .$$

The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned}
\mathrm{Ad}^*\left(\exp(rd)\, \Phi(0, x_1, y_1, x_2, y_2, 0)\right) f\ (e_0) &= f_0 - x_1 y_1 - \sigma x_2 y_2 \\
(e_1) &= y_1 \\
(e_2) &= -e^{-r} x_1 \\
(e_3) &= e^{-\tau r} y_2 \\
(e_4) &= -e^{-(1-\tau) r} x_2 \\
(e_5) &= e^{-r}
\end{aligned}$$

If sequences r_n , x_{1n} , y_{1n} , x_{2n} , and y_{2n} are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*\left(\exp(r_n d)\, \Phi(0, x_{1n}\, , y_{1n}\, , x_{2n}\, , y_{2n}\, , 0)\right) f\ .$$

One verifies easily that $p = e_0\, e_5 - e_1\, e_2 - \sigma e_3\, e_4$ defines an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$ so that p is constant on all $\mathrm{Ad}^*(M)$-orbits. Profiting by the existence of p we obtain the following characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$.

Lemma 10.16. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_5 \geq 0$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p(g) = f_0\, g_5$. If $g_5 \neq 0$, then we have $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.*

Proof. At first we assume $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ so that there exist sequences r_n, x_{1n}, y_{1n}, x_{2n}, y_{2n} such that $f_n \longrightarrow g$. Since $f_n(e_5) = e^{-r_n} \longrightarrow g_5$, it follows $p(f_n) = f_0\, e^{-r_n} \longrightarrow p(g) = f_0\, g_5$.

Now we prove the opposite implication. First we suppose $g_5 \neq 0$. We can establish $g_\nu = 0$ for $1 \leq \nu \leq 4$. If we define $r_n = -\log g_5$ and $x_{1n} = y_{1n} = x_{2n} = y_{2n} = 0$, then $p(g) = g_0\, g_5 = f_0\, g_5$ implies $g_0 = f_0$ and $\mathrm{Ad}^*(G)g = \mathrm{Ad}^*(G)f$.

Now we suppose $g_5 = 0$. In this case $p(g) = -g_1 g_2 - \sigma g_3 g_4 = 0$. Let $r_n = n$, $x_{2n} = -e^{(1-\tau)n}\, g_4$, and $y_{2n} = e^{\tau n}\, g_3$. Here we must distinguish three cases: If $g_1 \neq 0$, then we define $y_{1n} = g_1$ and

$$x_{1n} = -e^n\, g_2 + \frac{1}{g_1}\,(f_0 - g_0)\,.$$

This implies $f_n(e_0) = g_0$ so that $f_n \longrightarrow g$ and thus $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ in this case. If $g_2 \neq 0$, then we define $x_{1n} = -e^n\, g_2$ and

$$y_{1n} = g_1 - \frac{1}{g_2}\, e^{-n}\,(f_0 - g_0)\,.$$

Again we have $f_n(e_0) = g_0$. Thus $f_n \longrightarrow g$ and $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ if $g_2 \neq 0$. If $g_1 = g_2 = 0$, then we set $x_{1n} = e^{n/2}$ and $y_{1n} = e^{-n/2}\,(f_0 - g_0)$. This proves $f_n \longrightarrow g$ in the third and last case. The proof of this lemma is complete. $\square$

Let $f_r = \mathrm{Ad}^*(\exp(rd))f$. Next we describe the unitary representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_\nu) = 0$ for $1 \leq \nu \leq 4$ and $f_r(e_5) = e^{-r}$, it is clear that $\mathfrak{p} = \langle\, e_0, e_2, e_4, e_5 \,\rangle$ is a Pukanszky polarization at f_r for all r. Note that $\mathfrak{p}$ is $\mathfrak{s}$-invariant. Further $\mathfrak{c} = \langle\, e_1, e_3 \,\rangle$ is a commutative subalgebra of $\mathfrak{n}$ which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. From Section 6.1 we learn that the infinitesimal operators of $\pi_r = \mathrm{ind}_P^M\, \chi_{f_r}$ are given by

$$\begin{aligned}
d\pi_r(e_0) &= \frac{1+\sigma}{2} + if_0 + \xi_1\partial_{\xi_1} + \sigma\xi_2\partial_{\xi_2}\\
d\pi_r(e_1) &= -\partial_{\xi_1}\\
d\pi_r(e_2) &= -ie^{-r}\xi_1\\
d\pi_r(e_3) &= -\partial_{\xi_2}\\
d\pi_r(e_4) &= -ie^{-r}\xi_2\\
d\pi_r(e_5) &= ie^{-r}
\end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_5 = 0$ and $g_\nu \neq 0$ for some $1 \leq \nu \leq 4$. Then we see that the nilradical $\mathfrak{n} = \langle\, e_1, \ldots, e_5 \,\rangle$ is a Pukanszky polarization at g. Further $\mathfrak{c} = \langle\, e_0 \,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply

$$\begin{aligned}
d\rho(e_0) &= -\partial_\xi\\
d\rho(e_1) &= ie^{\xi}\, g_1\\
d\rho(e_2) &= ie^{-\xi}\, g_2\\
d\rho(e_3) &= ie^{\sigma\xi}\, g_3\\
d\rho(e_4) &= ie^{-\sigma\xi}\, g_4\\
d\rho(e_5) &= 0
\end{aligned}$$

Let W denote the image of p in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map so that

$$W = \frac{1}{2}(e_1\, e_2 + e_2\, e_1) + \frac{\sigma}{2}(e_3\, e_4 + e_4\, e_3) - e_0 e_5 \ .$$

One verifies easily that $d\pi_r(W) = p(f_r)\cdot\mathrm{Id}$ for all r and $d\rho(W) = p(g)\cdot\mathrm{Id}$. Now it is easy to see that the preceding considerations imply the validity of Theorem 10.12 in the case that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α,$\gamma-\alpha$, β, $\gamma-\beta$, γ where α, γ are linearly independent and $\beta \in \langle\, \alpha, \gamma\,\rangle$. The proof is similar to that of Theorem 10.11 and Lemma 10.15.

10.3.3 Four distinct weights α, $\gamma-\alpha$, $\gamma/2$, γ where α, γ are linear independent

In this case $\dim \mathfrak{g} = 7$. There exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that the commutator relations $[e_1, e_2] = e_5$, $[e_3, e_4] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = be_4$, $[d, e_2] = e_2$, $[d, e_3] = \frac{1}{2}e_3 + ae_4$, $[d, e_4] = \frac{1}{2}e_4$, and $[d, e_5] = e_5$ hold where $a, b \in \mathbb{R}$ are arbitrary constants. By scaling the basis vectors appropriately we could establish $b \in \{0, 1\}$, but this is not necessary for the following considerations.

Let $f \in \mathfrak{m}^*$ be in general position so that $f_5 \neq 0$. We can assume $f_5 = 1$ and $f_\nu = 0$ for all $1 \leq \nu \leq 4$. Then $\mathfrak{t} \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ and $\mathrm{Ad}^*(M)f = \mathrm{Ad}^*(N)f$. As usual we work with coordinates of the second kind given by the diffeomorphism

$$\Phi(t, x_1, y_1, x_2, y_2, z) = \exp(te_0)\, \exp(x_1 e_1 + x_2 e_3)\, \exp(y_1 e_2 + y_2 e_4 + z e_5)\ .$$

For the coadjoint action of G in $\mathfrak{m}^*$ we obtain

$$\begin{aligned}
\mathrm{Ad}^*\,(\,\exp(rd)\,\Phi(t, x_1, y_1, x_2, y_2, z)\,)\, f\ (e_0) &= f_0 - x_1 y_1 - \frac{b}{2}\, x_2^2 \\
(e_1) &= e^t\, y_1 \\
(e_2) &= -e^{-r}\, e^{-t}\, x_1 \\
(e_3) &= e^{-r/2}\,(y_2 + arx_2 + btx_2) \\
(e_4) &= -e^{-r/2}\, x_2 \\
(e_5) &= e^{-r}
\end{aligned}$$

If sequences r_n , x_{1n} , y_{1n} , x_{2n} , and y_{2n} are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*\,(\,\exp(r_n d)\,\Phi(0,\, x_{1n}\,,\, y_{1n}\,,\, x_{2n}\,,\, y_{2n}\,, 0)\,)\, f\ .$$

Let us define the polynomial function

$$p = e_0\, e_5 - e_1\, e_2 + \frac{b}{2}\, e_4\, e_4$$

on $\mathfrak{m}^*$. We point out that p is $\mathrm{ad}(\mathfrak{m})$-invariant and thus constant on all $\mathrm{Ad}^*(M)$-orbits. We have $p\,(\,\mathrm{Ad}^*(\exp(rd)\Phi(t, \ldots, z))\,) = f_0\, e^{-r}$. Profiting by the existence of p we obtain the following characterization of the closure of $\mathrm{Ad}^*(G)f$.

Lemma 10.17. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_5 \geq 0$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p(g) = f_0\, g_5$. If $g_5 \neq 0$, then we have $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.*

Proof. Let $g \in \mathfrak{m}^*$ such that $g_5 = 0$ and $p(g) = -g_1 g_2 + \frac{b}{2} g_4^2 = 0$. We set $r_n = n$, $x_{2n} = -e^{n/2}\, g_4$, and $y_{2n} = e^{n/2}\,(g_3 - ane^{n/2}\, g_4)$. We must distinguish three cases: If $g_1 \neq 0$, then we define $y_{1n} = g_1$ and

$$x_{1n} = -e^n\, g_2 + \frac{1}{g_1}\,(f_0 - g_0)\ .$$

If $g_2 \neq 0$, then we define $x_{1n} = -e^n\, g_2$ and

$$y_{1n} = g_1 - \frac{1}{g_2}\, e^{-n}\,(f_0 - g_0)\ .$$

If $g_1 = g_2 = 0$, then we set $x_{1n} = e^{n/2}$ and $y_{1n} = e^{-n/2}\,(f_0 - g_0)$. In particular $f_n(e_0) = g_0$ because $p(g) = 0$. It follows immediately that $f_n \longrightarrow g$ and hence $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-\!-}$ in all three cases. The proof for the remaining assertions of this lemma can be copied from the proof of Lemma 10.16. □

Let $f_r = \mathrm{Ad}^*(\exp(rd))f$. Next we describe the unitary representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_\nu) = 0$ for $1 \leq \nu \leq 4$ and $f_r(e_5) = e^{-r}$, it is clear that $\mathfrak{p} = \langle\, e_0, e_2, e_4, e_5 \,\rangle$ is a Pukanszky polarization at f_r for all r. Furthermore $\mathfrak{c} = \langle\, e_1, e_3 \,\rangle$ is a commutative subalgebra of $\mathfrak{n}$ which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. Note that $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{n}$. But the subalgebra $\mathfrak{c}$ is not $\mathrm{ad}(e_0)$-invariant in this case so that we cannot apply the results of Section 6.1 in order to compute $\pi_r(\exp(te_0))$. However, an elementary computation using the Campbell-Baker-Hausdorff formula

$$\exp(A)\, \exp(B) = \exp(\, A + B + \frac{1}{2}[A, B]\,)$$

for elements A and B of the two-step nilpotent algebra $\mathfrak{n}$ yields

$$\pi_r(\exp(te_0))\varphi\ (\xi_1, \xi_2) = e^{t/2}\, e^{if_0 t}\, e^{-\frac{1}{2} ibe^{-r} t \xi_2^2}\ \varphi(e^t \xi_1, \xi_2)\ .$$

Now it follows that the infinitesimal operators of $\pi_r = \mathrm{ind}_P^M\, \chi_{f_r}$ are given by

$$\begin{aligned}
d\pi_r(e_0) &= \frac{1}{2} + if_0 + \xi_1 \partial_{\xi_1} - \frac{1}{2}\, ibe^{-r} \xi_2^2 \\
d\pi_r(e_1) &= -\partial_{\xi_1} \\
d\pi_r(e_2) &= -ie^{-r} \xi_1 \\
d\pi_r(e_3) &= -\partial_{\xi_2} \\
d\pi_r(e_4) &= -ie^{-r} \xi_2 \\
d\pi_r(e_5) &= ie^{-r}
\end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_5 = 0$ and ($g_1 \neq 0$ or $g_2 \neq 0$). Then $\mathfrak{n} = \langle\, e_1, \ldots, e_5 \,\rangle$ is a Pukanszky polarization at g. Further $\mathfrak{c} = \langle\, e_0 \,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply

$$\begin{aligned}
d\rho(e_0) &= -\partial_\xi \\
d\rho(e_1) &= ie^\xi\, g_1 \\
d\rho(e_2) &= ie^{-\xi}\, g_2 \\
d\rho(e_3) &= i(g_3 - b\xi g_4) \\
d\rho(e_4) &= ig_4 \\
d\rho(e_5) &= 0
\end{aligned}$$

Let W denote the image of p in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map so that

$$W = \frac{1}{2}(e_1\, e_2 + e_2\, e_1) - e_0\, e_5 - \frac{b}{2}\, e_4\, e_4\ .$$

One verifies easily that $d\pi_r(W) = p(f_r)\cdot\mathrm{Id}$ for all r and $d\rho(W) = p(g)\cdot\mathrm{Id}$. Again the preceding considerations imply the validity of Theorem 10.12 in the case of four distinct weights α,$\gamma - \alpha$, $\gamma/2$, γ where α, γ are linearly independent.

10.3.4 Four distinct weights α, $\gamma - \alpha$, 0, γ where α, γ are linearly independent

In this case $\dim \mathfrak{g} = 7$. There exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ with commutator relations $[e_1, e_2] = e_5$, $[e_3, e_4] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[d, e_0] = ae_3$, $[d, e_2] = e_2$, $[d, e_4] = e_4$, and $[d, e_5] = e_5$ where $a \in \mathbb{R}$ is an arbitrary constant. Since $\mathrm{ad}(d)$ is a derivation, it follows $a = 0$.

Let $f \in \mathfrak{m}^*$ be in general position so that $f_5 \neq 0$. We can assume $f_5 = 1$ and $f_\nu = 0$ for all $1 \le \nu \le 4$. Then $\mathfrak{t} \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ and $\mathrm{Ad}^*(M)f = \mathrm{Ad}^*(N)f$. As usual we work with coordinates of the second kind given by the diffeomorphism

$$\Phi(t, x_1, y_1, x_2, y_2, z) = \exp(te_0)\, \exp(x_1 e_1 + x_2 e_3)\, \exp(y_1 e_2 + y_2 e_4 + z e_5)\ .$$

For the coadjoint action of G in $\mathfrak{m}^*$ we obtain

$$\begin{aligned} \mathrm{Ad}^*\left(\exp(rd)\, \Phi(t, x_1, y_1, x_2, y_2, z)\right) f\ (e_0) &= f_0 - x_1 y_1 \\ (e_1) &= e^t\, y_1 \\ (e_2) &= -e^{-r}\, e^{-t}\, x_1 \\ (e_3) &= y_2 \\ (e_4) &= -e^{-r}\, x_2 \\ (e_5) &= e^{-r} \end{aligned}$$

It is easy to see that $p = e_0\, e_5 - e_1\, e_2$ defines an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$ so that p is constant on all $\mathrm{Ad}^*(M)$-orbits. With the aid of p we obtain the following characterization of the closure of $\mathrm{Ad}^*(G)f$.

Lemma 10.18. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_5 \geq 0$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p(g) = f_0\, g_5$. If $g_5 \neq 0$, then we have $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.*

Proof. The proof is similar to that of Lemma 10.10. We omit the details. □

Let $f_r = \mathrm{Ad}^*(\exp(rd))f$. Next we describe the unitary representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_\nu) = 0$ for $1 \le \nu \le 4$ and $f_r(e_5) = e^{-r}$, it is clear that $\mathfrak{p} = \langle\, e_0, e_2, e_4, e_5\, \rangle$ is an $\mathfrak{s}$-invariant Pukanszky polarization at f_r. Furthermore $\mathfrak{c} = \langle\, e_1, e_3\, \rangle$ is a commutative subalgebra of $\mathfrak{n}$ which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. The results of Section 6.1 yield

$$\begin{aligned}
d\pi_r(e_0) &= \frac{1}{2} + if_0 + \xi_1\partial_{\xi_1} \\
d\pi_r(e_1) &= -\partial_{\xi_1} \\
d\pi_r(e_2) &= -ie^{-r}\xi_1 \\
d\pi_r(e_3) &= -\partial_{\xi_2} \\
d\pi_r(e_4) &= -ie^{-r}\xi_2 \\
d\pi_r(e_5) &= ie^{-r}
\end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_5 = 0$ and ($g_1 \neq 0$ or $g_2 \neq 0$). Then the nilradical $\mathfrak{n}$ is a Pukanszky polarization at g and $\mathfrak{c} = \langle\, e_0 \,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply

$$\begin{aligned}
d\rho(e_0) &= -\partial_\xi \\
d\rho(e_1) &= ie^{\xi}\, g_1 \\
d\rho(e_2) &= ie^{-\xi}\, g_2 \\
d\rho(e_3) &= ig_3 \\
d\rho(e_4) &= ig_4 \\
d\rho(e_5) &= 0
\end{aligned}$$

Let W denote the image of p in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map so that

$$W = \frac{1}{2}(e_1\, e_2 + e_2\, e_1) - e_0\, e_5 \;.$$

One verifies easily that $d\pi_r(W) = p(f_r)\cdot\mathrm{Id}$ for all r and $d\rho(W) = p(g)\cdot\mathrm{Id}$. Again the standard argument shows the validity of Theorem 10.12 in the case of four distinct weights α,$\gamma-\alpha$, 0, γ where α, γ are linearly independent.

10.3.5 Three distinct weights α, $\gamma - \alpha$, γ where α, γ are linearly independent

In this case $\dim \mathfrak{g} = 7$. It follows from Remark 10.7 that there exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that the commutator relations $[e_1, e_3] = e_5$, $[e_2, e_4] = e_5$, $[e_0, e_1] = -e_1 - be_2$, $[e_0, e_2] = -e_2$, $[e_0, e_3] = e_3$, $[e_0, e_4] = be_3 + e_4$, $[d, e_1] = -ae_2$, $[d, e_3] = e_3$, $[d, e_4] = ae_3 + e_4$, and $[d, e_5] = e_5$ hold.

Let $f \in \mathfrak{m}^*$ be in general position so that $f_5 \neq 0$. We can assume $f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 4$. Clearly $\mathfrak{t} \subset \mathfrak{g}_f$ so that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ and $\mathrm{Ad}^*(M)f = \mathrm{Ad}^*(N)f$. As usual we work with coordinates of the second kind given by the diffeomorphism

$$\Phi(t, x_1, x_2, y_1, y_2, z) = \exp(te_0)\, \exp(x_1e_1 + x_2e_2)\, \exp(y_1e_3 + y_2e_4 + ze_5)\;.$$

For the coadjoint action of G in $\mathfrak{m}^*$ we obtain

$$\begin{aligned}\mathrm{Ad}^*\left(\exp(rd)\,\Phi(t,x_1,x_2,y_1,y_2,z)\right)f\ (e_0) &= f_0 - x_1y_1 - bx_1y_2 - x_2y_2\\ (e_1) &= e^t\,(y_1 + asy_2 + bty_2)\\ (e_2) &= e^t\,y_2\\ (e_3) &= -e^{-s}\,e^{-t}\,x_1\\ (e_4) &= -e^{-s}\,e^{-t}\,(x_2 - asx_1 - btx_1)\\ (e_5) &= e^{-r}\end{aligned}$$

If sequences r_n , x_{1n} , x_{2n} , y_{1n} , and y_{2n} are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*\left(\exp(r_nd)\,\Phi(0,\,x_{1n}\,,\,x_{2n}\,,\,y_{1n}\,,\,y_{2n}\,,0)\right)f\ .$$

Then we have

$$\begin{aligned}f_n(e_0) &= f_0 - x_{1n}y_{1n} - bx_{1n}y_{2n} - x_{2n}y_{2n}\\ (e_1) &= y_{1n} + ar_ny_{2n}\\ (e_2) &= y_{2n}\\ (e_3) &= -e^{-r_n}\,x_{1n}\\ (e_4) &= -e^{-r_n}\,(x_{2n} - ar_nx_{1n})\\ (e_5) &= e^{-r_n}\end{aligned}$$

It is easy to see that $p = e_0\,e_5 - e_1\,e_3 - be_2\,e_3 - e_2\,e_4$ defines an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$ so that p is constant on all $\mathrm{Ad}^*(M)$-orbits. Note that $p(f_n) = f_0\,e^{-r_n}$. Profiting by the existence of p we obtain the following characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$.

Lemma 10.19. *Let $f \in \mathfrak{m}^*$ be a linear functional in general position. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_5 \geq 0$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ if and only if $p(g) = f_0\,g_5$. If $g_5 \neq 0$, then we have $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.*

Proof. Let $g \in \mathfrak{m}^*$ such that $g_5 = 0$ and $p(g) = -g_1g_3 - bg_2g_3 - g_2g_4 = 0$. Let $r_n = n$. Here we must distinguish four cases. If $g_2 \neq 0$, then we define $y_{1n} = g_1 - ar_ny_{2n}$, $y_{2n} = g_2$, $x_{1n} = -e^{r_n}\,g_3$, and

$$x_{2n} = -e^{r_n}\,g_4 + ar_nx_{1n} + \frac{1}{g_2}\,(f_0 - g_0)\ .$$

Now let $x_{2n} = -e^{r_n}\,g_4 + ar_nx_{1n}$. If $g_2 = 0$ and $g_3 \neq 0$, then we define $x_{1n} = -e^{r_n}\,g_3$, $y_{2n} = 0$ and

$$y_{1n} = g_1 - \frac{1}{g_3}\,e^{-r_n}\,(f_0 - g_0)\ .$$

If $g_2 = g_3 = 0$ and $g_1 \neq 0$, then we define $y_{2n} = 0$, $y_{1n} = g_1$, and

$$x_{1n} = \frac{1}{g_1}\,(f_0 - g_0)\ .$$

If $g_1 = g_2 = g_3 = 0$, then we define $y_{2n} = 0$, $y_{1n} = e^{-r_n/2}\,(f_0 - g_0)$, and $x_{1n} = e^{r_n/2}$. In all four cases we obtain $f_n(e_0) = g_0$ because $p(g) = 0$. Now it follows $f_n \longrightarrow g$ and thus $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$. The proof for the remaining assertions of this lemma can be copied from the proof of Lemma 10.16. □

Let $f_r = \mathrm{Ad}^*(\exp(rd))f$. Next we describe the unitary representations $\pi_r = \mathcal{K}(f_r)$ in general position. Since $f_r(e_\nu) = 0$ for $1 \le \nu \le 4$ and $f_r(e_5) = e^{-r}$, it is clear that $\mathfrak{p} = \langle\, e_0, e_3, e_4, e_5 \,\rangle$ is a Pukanszky polarization at f_r. Furthermore $\mathfrak{c} = \langle\, e_1, e_2 \,\rangle$ is a commutative subalgebra of $\mathfrak{n}$ which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. The results of Section 6.1 yield

$$\begin{aligned}
d\pi_r(e_0) &= 1 + if_0 + \xi_1\partial_{\xi_1} + (\xi_2 + b\xi_1)\partial_{\xi_2}\\
d\pi_r(e_1) &= -\partial_{\xi_1}\\
d\pi_r(e_2) &= -\partial_{\xi_2}\\
d\pi_r(e_3) &= -ie^{-r}\,\xi_1\\
d\pi_r(e_4) &= -ie^{-r}\,\xi_2\\
d\pi_r(e_5) &= ie^{-r}
\end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_5 = 0$ and ($g_2 \neq 0$ or $g_3 \neq 0$). Then the nilradical $\mathfrak{n}$ is a Pukanszky polarization at g and $\mathfrak{c} = \langle\, e_0 \,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply

$$\begin{aligned}
d\rho(e_0) &= -\partial_\xi\\
d\rho(e_1) &= ie^{\xi}\,(g_1 + b\xi g_2)\\
d\rho(e_2) &= ie^{\xi}\,g_2\\
d\rho(e_3) &= ie^{-\xi}\,g_3\\
d\rho(e_4) &= ie^{-\xi}\,(g_4 - b\xi g_3)\\
d\rho(e_5) &= 0
\end{aligned}$$

Let W denote the image of p in $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ under the modified symmetrization map so that

$$W = \frac{1}{2}(e_1\,e_3 + e_3\,e_1) + be_2\,e_3 + \frac{1}{2}(e_2\,e_4 + e_4\,e_2) - e_0\,e_5\ .$$

One verifies easily that $d\pi_r(W) = p(f_r)\cdot\mathrm{Id}$ for all r and $d\rho(W) = p(g)\cdot\mathrm{Id}$. These considerations prove the validity of Theorem 10.12 in the case of three distinct weights α,$\gamma - \alpha$, γ where α, γ are linearly independent.

11 Nilradical is the algebra $\mathfrak{g}_{5,2}$

In this section we study the unitary representation theory of an exponential solvable Lie group G such that the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{g}$ is the five-dimensional, 2-step nilpotent Lie algebra $\mathfrak{g}_{5,2}$. This section is divided into two subsections.

In the first one we describe the algebraic structure of these Lie algebras $\mathfrak{g}$. In some sense we give a classification of them. Let $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ denote the stabilizer of a linear functional $f \in \mathfrak{g}^*$ in general position. We will define certain polynomial functions on the linear dual $\mathfrak{m}^*$ which are constant on the $\mathrm{Ad}^*(M)$-orbits contained in the closure of $\mathrm{Ad}^*(G)f$. Finally we will compute the infinitesimal operators of the relevant unitary representations of M.

The considerations of the first subsection imply that there exist four algebras $\mathfrak{g}$ of this kind. In the second subsection these algebras $\mathfrak{g}$ are studied one by one. In each case we describe the closure in $\mathfrak{m}^*$ of the orbit $\mathrm{Ad}^*(G)f$ in general position. If $g \in \mathfrak{m}^*$ is critical for $\mathrm{Ad}^*(G)f$, then we show how to separate $\rho = \mathcal{K}(g)$ from the set $\{\pi_s = \mathcal{K}(f_s) : s \in S\}$ by L^1-functions.

11.1 The structure of $\mathfrak{g}$

Let $\mathfrak{g}$ be an exponential solvable Lie algebra and let $\mathfrak{n}$ denote the nilradical of $\mathfrak{g}$. In particular $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. Assume that $\mathfrak{n}$ is 2-step nilpotent such that

$$\mathfrak{n} \underset{3}{\supset} C^1\mathfrak{n} \underset{2}{\supset} \{0\} \tag{11.1}$$

is the descending series of $\mathfrak{n}$ where $\mathfrak{z}\mathfrak{n} = C^1\mathfrak{n}$. Let $\mathfrak{m}$ be a non-nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{m}$. Assume that there exists an $f \in \mathfrak{m}^*$ in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Since f vanishes on $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$, this ideal of $\mathfrak{g}$ must be zero. This proves $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$. As usual we assume $\mathfrak{m} \neq \mathfrak{g}$.

On page 72 of [23] it is shown that there exist nilpotent subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$. We fix such an $\mathfrak{s}$ and define $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$. We regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules and benefit from the existence of weight space decompositions of these modules.

To begin with let us assume that there exists a single weight $\alpha \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + C^1\mathfrak{n}$. Then $C^1\mathfrak{n} = [\mathfrak{n}_\alpha, \mathfrak{n}_\alpha] = \mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n}$ implies $\tilde{\alpha} = 0$. This means that $\mathfrak{m}$ is nilpotent in contradiction to our assumption. We conclude that there exist a least two distinct weights $\alpha, \beta \in \mathfrak{s}^*$ such that $\mathfrak{n}_\alpha \not\subset C^1\mathfrak{n}$ and $\mathfrak{n}_\beta \not\subset C^1\mathfrak{n}$.

Now we assume $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{n}_\beta + C^1\mathfrak{n}$ for distinct weights $\alpha, \beta \in \mathfrak{s}^*$ such that $\tilde{\alpha} \neq 0$. Then it follows

$$C^1\mathfrak{n} = [\mathfrak{n}_\alpha + \mathfrak{n}_\beta, \mathfrak{n}_\alpha + \mathfrak{n}_\beta] = (\mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n}) + (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + (\mathfrak{n}_{2\beta} \cap C^1\mathfrak{n}) \ .$$

Since $\tilde{\alpha} \neq 0$, we see that $\mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n} = 0$. If $\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}$ were zero, then we would obtain the contradiction $\mathfrak{n}_\alpha \subset \mathfrak{z}\mathfrak{n}$. Thus $\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n} \neq 0$ so that $\tilde{\beta} = -\tilde{\alpha} \neq 0$. But this implies

$\mathfrak{n}_{2\beta} \cap C^1\mathfrak{n} = 0$, too. Now let us define $\gamma = \alpha + \beta$. Then we see that in this case $\mathfrak{n}$ admits the weight space decomposition

$$\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{\gamma-\alpha} \oplus \mathfrak{n}_\gamma$$

where, without loss of generality, $\mathfrak{n}_\alpha$ is one-dimensional, and $\mathfrak{n}_{\gamma-\alpha}$ and $\mathfrak{n}_\gamma = C^1\mathfrak{n}$ are two-dimensional. Note that these three weight functions are real. Here $\dim \mathfrak{g} = 7$. In this situation we find

Lemma 11.2. *Let $0 \neq e_1 \in \mathfrak{n}_\alpha$ be arbitrary. Then $D = \mathrm{ad}(e_1) : \mathfrak{n}_{\gamma-\alpha} \longrightarrow \mathfrak{n}_\gamma$ is a linear isomorphism which is* $(\ker \alpha)$*-equivariant.*

Proof. Since $C^1\mathfrak{n} = [e_1, \mathfrak{n}_{\gamma-\alpha}] = \mathfrak{n}_\gamma$, it follows that D is surjective and hence an isomorphism. Let $S \in \mathfrak{s}$ such that $\alpha(S) = 0$. Then the equality

$$[\,e_1, [S, X]\,] = [\,[e_1, S], X\,] + [\,S, [e_1, X]\,] = [\,S, [e_1, X]\,]$$

shows that D is $(\ker \alpha)$-equivariant. □

Next we discuss the possibility of three distinct weights $\alpha, \beta_1, \beta_2 \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{n}_{\beta_1} + \mathfrak{n}_{\beta_2} + C^1\mathfrak{n}$ where $\tilde{\alpha} \neq 0$ and all weight spaces are one-dimensional. Then

$$C^1\mathfrak{n} = (\mathfrak{n}_{\alpha+\beta_1} \cap C^1\mathfrak{n}) + (\mathfrak{n}_{\alpha+\beta_2} \cap C^1\mathfrak{n}) + (\mathfrak{n}_{\beta_1+\beta_2} \cap C^1\mathfrak{n})$$

holds for the three distinct weights $\alpha + \beta_1$, $\alpha + \beta_2$, and $\beta_1 + \beta_2$. It follows that one and only one of these weight spaces is zero. Without loss of generality we can assume $\mathfrak{n}_{\beta_1+\beta_2} \cap C^1\mathfrak{n} = 0$. Let us define $\gamma = \alpha + \beta_1$ and $\delta = \alpha + \beta_2$. Then we see that $\mathfrak{n}$ admits the weight space decomposition

$$\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{\gamma-\alpha} \oplus \mathfrak{n}_{\delta-\alpha} \oplus \mathfrak{n}_\gamma \oplus \mathfrak{n}_\delta\ .$$

At first we treat the case of real weight functions. If α, γ, δ are $\mathbb{R}$-linearly independent, then $\dim \mathfrak{g} = 8$. Now let us suppose that α, γ are $\mathbb{R}$-linearly independent and $\delta \in \langle\, \alpha, \gamma\,\rangle$. Then in particular $\dim \mathfrak{g} = 7$. There exist constants $a, b \in \mathbb{R}$ such that $\delta = a\gamma + b\alpha$. Since $\tilde{\alpha} \neq 0$, the equality $0 = \tilde{\delta} = b\tilde{\alpha}$ implies $b = 0$ so that $\delta = a\gamma$ for some $a \neq 1$. If $\delta = 0$, then $\dim \mathfrak{m}_0 = 2$ and the action of $\mathfrak{s}$ on $\mathfrak{m}_0$ may be non-trivial.

Now we take the possibility of non-real weight functions into account. Since $\mathfrak{g}$ is an exponential, solvable Lie algebra, it follows that α is real and that $\overline{\gamma} = \delta$. Moreover we conclude that there exists a $\lambda \in \mathbb{R}\backslash\{0\}$ such that $\mathrm{Im}\,\gamma = \lambda\,\mathrm{Re}\,\gamma$. Clearly α, γ are $\mathbb{C}$-linearly independent and $\delta \in \mathbb{C}\gamma$.

Altogether we have shown that the following cases occur:

- The $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct real weights α, $\gamma - \alpha$, $\delta - \alpha$, γ, δ where α, γ, δ are linearly independent. All weight spaces of the $\mathfrak{s}$-module $\mathfrak{m}$ are one-dimensional.

- There are five distinct real weights α, $\gamma - \alpha$, $\delta - \alpha$, γ, δ where α, γ are linearly independent and $\delta = a\gamma$ for $a \in \mathbb{R}\backslash\{0, 1\}$. Again all weight spaces are one-dimensional.

- There are five distinct real weights α, $\gamma-\alpha$, $-\alpha$, γ, 0 where α, γ are linearly independent. Here $\mathfrak{m}_0$ is two-dimensional and the action of $\mathfrak{s}$ on $\mathfrak{m}_0$ may be non-trivial.

- There exist five distinct weights α, $\gamma-\alpha$, $\delta-\alpha$, γ, δ such that α is real and $\overline{\gamma}=\delta$ is non-real.

- There are three distinct weights α, $\gamma-\alpha$, γ where α, γ are linearly independent. Here $\mathfrak{n}_{\gamma-\alpha}$ and $\mathfrak{n}_\gamma$ are two-dimensional and the action of $\mathfrak{s}$ is possibly not semi-simple on these subspaces.

In any case there exists a basis $e_0,\ldots,e_5$ of $\mathfrak{m}$ such that the commutator relations $[e_1,e_2]=e_4$, $[e_1,e_3]=e_5$, $[e_0,e_1]=-e_1$, $[e_0,e_2]=e_2+be_3$, and $[e_0,e_3]=e_3$ hold. Furthermore the Jacobi identity for e_0, e_1, e_2 implies $b=0$. As usual we work with coordinates of the second kind given by the diffeomorphism

$$\Phi(t,v,w,x,y,z)=\exp(te_0)\,\exp(ve_1)\,\exp(we_2+xe_3+ye_4+ze_5)\;.$$

Now we can prove

Lemma 11.3. *Let $f\in\mathfrak{m}^*$ be in general position such that $\mathfrak{m}=\mathfrak{m}_f+\mathfrak{n}$. Then there exists a representative f on the orbit $\mathrm{Ad}^*(M)f=\mathrm{Ad}^*(N)f$ such that $f_1=f_2=f_3=0$.*

Proof. Since f is in general position, we have $f_4\neq 0$ or $f_5\neq 0$. The equations

$$\begin{aligned}\mathrm{Ad}^*(\exp(xe_3))f\;(e_1)&=f_1+xf_5\\ \mathrm{Ad}^*(\exp(xe_3))f\;(e_\nu)&=f_\nu\\ \mathrm{Ad}^*(\exp(we_2))f\;(e_1)&=f_1+wf_4\\ \mathrm{Ad}^*(\exp(we_2))f\;(e_\nu)&=f_\nu\end{aligned}$$

for $2\le\nu\le 5$ show that we can establish $f_1=0$. Since $\mathfrak{m}=\mathfrak{m}_f+\mathfrak{n}$, there exists an element $X=te_0+ve_1+we_2+xe_3+ye_4+ze_5$ of $\mathfrak{m}_f$ such that $t\neq 0$. Now

$$\begin{aligned}[X,e_1]&=-te_1-we_4-xe_5\\ [X,e_2]&=te_2+ve_4\\ [X,e_3]&=te_3+ve_5\end{aligned}$$

and $X\in\mathfrak{m}_f$ implies

$$\begin{aligned}0&=-wf_4-xf_5\\ 0&=tf_2+vf_4\\ 0&=tf_3+vf_5\end{aligned}$$

If $f_5\neq 0$, then the explicit formula for $\mathrm{Ad}^*(\exp(ue_1))f$ shows that we can establish $f_3=0$. From the preceding three equations it follows $v=0$. If f_2 were non-zero, then we would obtain the contradiction $t=0$. Thus $f_2=0$ in this case, too. The case $f_5=0$ and $f_4\neq 0$ can be treated similarly. $\square$

Until the end of this section let $f \in \mathfrak{m}^*$ such that ($f_4 \neq 0$ or $f_5 \neq 0$) and $f_\nu = 0$ for $1 \leq \nu \leq 3$. The coadjoint representation of M in $\mathfrak{m}^*$ is given by

$$\begin{aligned} \operatorname{Ad}^*(\Phi(t,v,w,x,y,z))f\ (e_0) &= f_0 - vwf_4 - vxf_5 \\ (e_1) &= e^t\,(wf_4 + xf_5) \\ (e_2) &= -e^{-t}\,vf_4 \\ (e_3) &= -e^{-t}\,vf_5 \\ (e_4) &= f_4 \\ (e_5) &= f_5 \end{aligned} \tag{11.4}$$

These formulas motivate the definition of the polynomial function $p_1 = e_0\,e_5 - e_1\,e_3$ on $\mathfrak{m}^*$ which is constant on the orbit $\operatorname{Ad}^*(M)f$. But p_1 is not $\operatorname{ad}(\mathfrak{m})$-invariant and thus not constant on all $\operatorname{Ad}^*(M)$-orbits. The polynomial function $p_2 = e_0\,e_4 - e_1\,e_2$ has the same properties. We will see that these two polynomial functions determine whether $g \in \mathfrak{m}^*$ lies in the closure of $\operatorname{Ad}^*(G)f$ or not. Furthermore one can decide with the aid of $p_3 = e_2\,e_5 - e_3\,e_4$ if g' is in the closure of $\operatorname{Ad}^*(G)f'$.

Next we describe the relevant unitary representations. Let $\pi = \mathcal{K}(f)$. It is easy to see that $\mathfrak{p} = \langle\, e_0, e_2, e_3, e_4, e_5 \,\rangle$ is a Pukanszky polarization at f and that $\mathfrak{c} = \langle\, e_1 \,\rangle$ is a coexponential subalgebra for $\mathfrak{p}$ in $\mathfrak{m}$. The results of Section 6.1 yield

$$\begin{aligned} d\pi_r(e_0) &= \frac{1}{2} + if_0 + \xi\partial_\xi \\ d\pi_r(e_1) &= -\partial_\xi \\ d\pi_r(e_2) &= -if_4\,\xi \\ d\pi_r(e_3) &= -if_5\,\xi \\ d\pi_r(e_4) &= if_4 \\ d\pi_r(e_5) &= if_5 \end{aligned}$$

Now let $g \in \mathfrak{m}^*$ be such that $g_5 = g_4 = 0$ and ($g_1 \neq 0$ or $g_2 \neq 0$ or $g_3 \neq 0$). Then $\mathfrak{n} = \langle\, e_1, \ldots, e_5 \,\rangle$ is a Pukanszky polarization at g. Further $\mathfrak{c} = \langle\, e_0 \,\rangle$ is a coexponential subalgebra for $\mathfrak{n}$ in $\mathfrak{m}$. The results of Section 6.2 imply

$$\begin{aligned} d\rho(e_0) &= -\partial_\xi \\ d\rho(e_1) &= ie^{\xi}\,g_1 \\ d\rho(e_2) &= ie^{-\xi}\,g_2 \\ d\rho(e_3) &= ie^{-\xi}\,g_3 \\ d\rho(e_4) &= 0 \\ d\rho(e_5) &= 0 \end{aligned}$$

The images of p_1 and p_1 under the modified symmetrization map in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ are given by

$$W_1 = \frac{1}{2}\,(e_1\,e_3 + e_3\,e_1) - e_0\,e_5$$

and

$$W_2 = \frac{1}{2}\,(e_1\,e_2 + e_2\,e_1) - e_0\,e_4$$

respectively. Clearly we have $d\pi(W_\nu) = p_\nu(f)\cdot\mathrm{Id}$ and $d\rho(W_\nu) = p_\nu(g)\cdot\mathrm{Id}$ for $\nu \in \{1, 2\}$.

11.2 Representation theory of G

Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ be as in Subsection 11.1. Here we study those four Lie algebras $\mathfrak{g}$ which result from the classification of the preceding subsection. Recall the definition of the polynomial functions p_1 and p_2 on $\mathfrak{m}^*$. In each case we will describe the closure in $\mathfrak{m}^*$ of the orbit $\mathrm{Ad}^*(G)f$ in general position, i.e., we will prove a statement quite similar to the following

Lemma. *Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$.*

Remark. If $g \in \mathfrak{m}^*$ such that $g_4\, g_5 \neq 0$ and such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, then $p_3(g) = 0$ so that $g_2\, g_5 = g_3\, g_4$. From this equation it follows that the assertions $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$ are equivalent.

If $\dim \mathfrak{g}/\mathfrak{m} = 1$, then we choose $d \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}d + \mathfrak{m}$ and define $f_s = \mathrm{Ad}^*(\exp(sd))f$. If $\dim \mathfrak{g}/\mathfrak{m} = 2$, then we fix $d_0, d_1 \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}d_0 + \mathbb{R}d_1 + \mathfrak{m}$ and set $f_s = \mathrm{Ad}^*(\exp(s_0 d_0)\exp(s_1 d_1))f$. The rest of this section is devoted to the proof of

Theorem 11.5. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra whose nilradical $\mathfrak{n}$ is the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,2}$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows*

$$\bigcap_s \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$.

11.2.1 Five distinct weights α, $\gamma - \alpha$, $\delta - \alpha$, γ, δ such that α, γ, δ are linearly independent

In this case $\dim \mathfrak{g} = 8$. There exists a basis $d_0, d_1, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = e_3$, $[d_0, e_2] = e_2$, $[d_0, e_4] = e_4$, $[d_1, e_3] = e_3$, and $[d_1, e_5] = e_5$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Then $f_4 \neq 0$ and $f_5 \neq 0$. Without loss of generality we can assume $f_4 = f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 3$. The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned}
\mathrm{Ad}^*(\exp(rd_0)\, \exp(sd_1)\, \Phi(0, v, w, x, 0, 0))f\ (e_0) &= f_0 - v(w + x) \\
(e_1) &= w + x \\
(e_2) &= -e^{-r}\, v \\
(e_3) &= -e^{-s}\, v \\
(e_4) &= e^{-r} \\
(e_5) &= e^{-s}
\end{aligned}$$

Lemma 11.6. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights such that α, γ, δ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ if and only if $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$.*

Proof. First we assume $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ so that there exist sequences r_n, s_n, v_n, w_n, and x_n such that $f_n \longrightarrow g$ where

$$f_n = \mathrm{Ad}^*(\exp(r_n d_0)\exp(s_n d_1)\Phi(0, v_n, w_n, x_n, 0, 0))f \ .$$

Since $f_n(e_5) = e^{-s_n} \longrightarrow g_5$ and $f_n(e_4) = e^{-r_n} \longrightarrow g_4$, it is obvious that $p_1(f_n) = f_0\, e^{-s_n} \longrightarrow p_1(g) = f_0\, g_5$ and $p_2(f_n) = f_0\, e^{-r_n} \longrightarrow p_2(g) = f_0\, g_4$.

In order to prove the opposite implication, we assume $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$. At first we obtain $p_3(g) = 0$, i.e., $g_2\, g_5 = g_3\, g_4$ because $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Now we must distinguish several cases: In any case we set $x_n = 0$. If $g_4 \neq 0$ and $g_5 \neq 0$, then we define $r_n = -\log g_4$, $s_n = -\log g_5$, $v_n = -e^{s_n} g_3$, and $w_n = g_1$. Since $p_3(g) = 0$, it follows $f_n(e_2) = g_2$ and $f_n(e_3) = g_3$. Since $p_1(g) = g_0 g_5 - g_1 g_3 = f_0 g_5$, we obtain $f_n(e_0) = g_0$. This shows $f_n = g$. Next we assume $g_4 \neq 0$ and $g_5 = 0$. Since $p_3(g) = 0$, we see $g_3 = 0$. We define $r_n = -\log g_4$, $s_n = n$, $v_n = -e^{s_n}\, g_3$, and $w_n = g_1$. Now $f_n(e_0) = g_0$ because $p_2(g) = f_0 g_4$. This proves $f_n \longrightarrow g$. The case $g_4 = 0$ and $g_5 \neq 0$ can be treated similarly. Now we assume $g_4 = g_5 = 0$ and $g_1 \neq 0$. It follows $g_2 = g_3 = 0$ because $p_1(g) = p_2(g) = 0$. If we put $r_n = s_n = n$, $w_n = g_1$, and

$$v_n = \frac{1}{g_1}\,(f_0 - g_0)\ ,$$

then $f_n \longrightarrow g$. The next case is $g_1 = g_4 = g_5 = 0$ and ($g_2 \neq 0$ or $g_3 \neq 0$). Note that there exist sequences r_n, s_n, v_n such that $f_n(e_\nu) \longrightarrow g_\nu$ for $2 \leq \nu \leq 5$. In particular $|v_n| \longrightarrow +\infty$. If we define

$$w_n = \frac{1}{v_n}\,(f_0 - g_0)\ ,$$

then $f_n \longrightarrow g$. Finally we suppose $g_\nu = 0$ for $1 \leq \nu \leq 5$. Here we define $r_n = s_n = n$, $v_n = e^{r_n/2}$, and $w_n = e^{-r_n/2}\,(f_0 - g_0)$. Again $f_n \longrightarrow g$. This finishes the proof of our lemma. ☐

The characterization of the closure of $\mathrm{Ad}^*(G)f$ given by the preceding lemma yields

Lemma 11.7. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $\delta-\alpha$, γ, δ such that α, γ, δ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows*

$$\bigcap_{(r,s)\in\mathbb{R}^2} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.

Proof. Let us set $\mathfrak{z} = \langle\, e_4, e_5 \,\rangle$ and define continuous functions $\psi_\nu : \mathfrak{z}^* \longrightarrow \mathbb{R}$ by

$$\psi_\nu(\xi_1 e_4^* + \xi_2 e_5^*) = f_0\, \xi_{3-\nu}$$

for $\nu \in \{1,2\}$. Note that $p_\nu(f_{r,s}) = \psi_\nu(\, f_{r,s}\,|\,\mathfrak{z}\,)$ for all r,s. Since g is critical for $\mathrm{Ad}^*(G)f$, it follows from Lemma 11.6 that there exists a $\nu \in \{1,2\}$ such that $p_\nu(\,g\,) \neq \psi_\nu(\,g\,|\,\mathfrak{z}\,)$. Now it suffices to apply Theorem 5.1 and Lemma 5.4 to π_s, ρ, W_ν, p_ν, and ψ_ν. The conclusion of this lemma follows at once. ☐

11.2.2 Five distinct weights α, $\gamma-\alpha$, $\delta-\alpha$, γ, δ such that α, γ are linearly independent and $\delta = a\gamma \neq 0$

In this case $\dim \mathfrak{g} = 7$. There exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = e_3$, $[d, e_2] = e_2$, $[d, e_3] = ae_3$, $[d, e_4] = e_4$, and $[d, e_5] = ae_5$. Here $a \neq 0$. Replacing d by $\frac{1}{a}d$ we can establish $|a| \geq 1$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Then $f_4 \neq 0$ and $f_5 \neq 0$. Without loss of generality we can assume $f_4 = f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 3$. The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned} \mathrm{Ad}^*(\exp(rd)\,\Phi(0, v, w, x, 0, 0))f\ (e_0) &= f_0 - v(w + x) \\ (e_1) &= w + x \\ (e_2) &= -e^{-r}\,v \\ (e_3) &= -e^{-ar}\,v \\ (e_4) &= e^{-r} \\ (e_5) &= e^{-ar} \end{aligned}$$

Lemma 11.8. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights such that α, γ are linearly independent and $\delta = a\gamma \neq 0$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ if and only if $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$.*

Proof. First we assume $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ so that there exist sequences r_n, v_n, w_n, and x_n such that $f_n \longrightarrow g$. Since $f_n(e_5) = e^{-ar_n} \longrightarrow g_5$, it is obvious that $p_1(f_n) = f_0\, e^{-ar_n} \longrightarrow p_1(g) = f_0\, g_5$. The same argument works for p_2.

In order to prove the opposite implication, we assume $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$. At first we suppose $g_4 \neq 0$ and $g_5 \neq 0$. Since $\mathrm{Ad}^*(G)g' \subset (\mathrm{Ad}^*(G)f')^{-}$, there exists a sequence r_n such that $e^{-r_n} \longrightarrow g_4$ and $e^{-ar_n} \longrightarrow g_5$. This implies $a \log g_4 = \log g_5$. If we define $r_n = -\log g_4$, $v_n = -e^{ar_n} g_3$, $w_n = g_1$, and $x_n = 0$, then $f_n = g$.

Next we assume $g_4 = 0$ or $g_5 = 0$. Then we conclude $g_4 = g_5 = 0$ and $a > 0$. Without loss of generality we can assume $a \geq 1$. Now we must distinguish several subcases. In any case we set $x_n = 0$. First we assume $g_1 \neq 0$. Since $p_1(g) = p_2(g) = 0$, it follows $g_2 = g_3 = 0$. We define $r_n = n$, $w_n = g_1$, and

$$v_n = \frac{1}{g_1}\,(f_0 - g_0)$$

so that $f_n \longrightarrow g$. Next we assume $g_1 = 0$ and ($g_2 \neq 0$ or $g_3 \neq 0$). In this case we choose sequences r_n and v_n such that $f_n(e_\nu) \longrightarrow g_\nu$ for $2 \leq \nu \leq 5$. In particular $r_n \longrightarrow +\infty$ and $|v_n| \longrightarrow +\infty$. Further we set

$$w_n = \frac{1}{v_n}\,(f_0 - g_0)\,.$$

Then we obtain $f_n \longrightarrow g$. Finally we assume $g_\nu = 0$ for $1 \leq \nu \leq 5$. We define $r_n = n$, $v_n = e^{r_n/2}$ and $w_n = e^{-r_n/2}\,(f_0 - g_0)$. These definitions imply $f_n \longrightarrow g$. This completes the proof of our lemma. □

In the same way as in the preceding subsection one can prove

Lemma 11.9. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $\delta-\alpha$, γ, δ such that α, γ are linearly independent and $\delta = a\gamma \neq 0$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows*

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

11.2.3 Five distinct weights α, $\gamma-\alpha$, $-\alpha$, γ, 0 such that α, γ are linearly independent

In this case $\dim \mathfrak{g} = 7$. There exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = e_3$, $[d, e_0] = -ae_5$, $[d, e_2] = e_2$, and $[d, e_4] = e_4$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Then $f_4 \neq 0$ and $f_5 \neq 0$. Without loss of generality we can assume $f_4 = f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 3$. The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned}
\mathrm{Ad}^*(\exp(rd)\,\Phi(0,v,w,x,0,0))f\ (e_0) &= f_0 - v(w+x) + ar\\
(e_1) &= w + x\\
(e_2) &= -e^{-r}\,v\\
(e_3) &= -v\\
(e_4) &= e^{-r}\\
(e_5) &= 1
\end{aligned}$$

Lemma 11.10. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $-\alpha$, γ, 0 such that α, γ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. In particular $g_4 \geq 0$, $g_5 = 1$ and $g_2 = g_3\, g_4$. If $a = 0$, then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ holds if and only if $p_1(g) = f_0$ and $p_2(g) = f_0\, g_4$. If $a \neq 0$, then this inclusion is valid if and only if $g_4 \neq 0$ and $p_1(g) = f_0 - a \log g_4$.*

Proof. First we assume $a = 0$. The only-if-part of our claim is obvious. In order to prove the opposite implication, we assume $p_1(g) = f_0$. There exist sequences r_n, v_n, w_n, and x_n such that $f'_n \longrightarrow g'$. In particular $v_n \longrightarrow -g_3$ and $w_n + x_n \longrightarrow g_1$. Since $p_1(g) = f_0$, it follows $f_0 - v_n(w_n + x_n) \longrightarrow f_0 + g_1 g_3 = g_0$ and thus $f_n \longrightarrow g$.

Next we treat the case $a \neq 0$. If $f_n \longrightarrow g$, then $p_1(f_n) = f_0 + ar_n \longrightarrow p(g)$ and the sequence r_n is convergent. Consequently $e^{-r_n} \longrightarrow g_4 \neq 0$ and $f_0 + ar_n \longrightarrow f_0 - a \log g_4$. In order to prove the opposite implication, we assume $g_4 \neq 0$ and $p_1(g) = f_0 - a \log g_4$. There exist sequences r_n, v_n, w_n, and x_n such that $f'_n \longrightarrow g'$. In particular $r_n \longrightarrow -\log g_4$, $v_n \longrightarrow -g_3$, and $w_n + x_n \longrightarrow g_1$. Since $p_1(g) = g_0 - g_1 g_3 = f_0 - a \log g_4$, it follows $f_0 - v_n(w_n + x_n) + ar_n \longrightarrow g_0$. Thus $f_n \longrightarrow g$ and the proof is complete. $\square$

Lemma 11.11. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $-\alpha$, γ, 0 such that α, γ are linearly* *independent. Let* $f \in \mathfrak{m}^*$ *be in general position such that* $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. *Let* $g \in \mathfrak{m}^*$ *be critical for the orbit* $\mathrm{Ad}^*(G)f$. *Then it follows*

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_r = \mathcal{K}(f_r)$ *and* $\rho = \mathcal{K}(g)$.

Proof. First we assume $a = 0$. Since g is critical for $\mathrm{Ad}^*(G)f$, Lemma 11.10 implies $p_1(g) \neq f_0$. Let $\mathfrak{z} = \mathbb{R}e_5$ and $\psi_1 : \mathfrak{z}^* \longrightarrow \mathbb{R}$, $\psi_1(\xi e_5^*) = f_0$ so that $p_1(f_r) = \psi_1(f_r \,|\, \mathfrak{z})$ and $p_1(g) \neq \psi_1(g \,|\, \mathfrak{z})$. If we apply Theorem 5.1 and Lemma 5.4 to π_r, ρ, p_1, W_1, and ψ_1, then the assertion of this lemma follows.

The case $a \neq 0$ is more delicate. First of all we can establish $g_3 = 0$ which implies $g_2 = 0$, too. Let M_0 denote the connected subgroup of M whose Lie algebra is the one-codimensional ideal $\mathfrak{m}_0 = \langle\, e_0, e_1, e_3, e_4, e_5 \,\rangle$ of $\mathfrak{m}$. We consider the restrictions $\pi_{0r} = \pi_r \,|\, M_0$ and $\rho_0 = \rho \,|\, M_0$ of the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$. Note that $\mathfrak{p} = \langle\, e_0, e_2, e_3, e_4, e_5 \,\rangle$ is a Pukanszky polarization at f_r and g. If we define $f_{0r} = f_r \,|\, \mathfrak{m}_0$ and $g_0 = g \,|\, \mathfrak{m}_0$, then it follows $\pi_{0r} = \mathcal{K}(f_{0r})$ and $\rho_0 = \mathcal{K}(g_0)$ because $\mathfrak{m}_{f_s} \not\subset \mathfrak{m}_0$ and $\mathfrak{m}_g \not\subset \mathfrak{m}_0$ so that $\mathfrak{h} = \mathfrak{p} \cap \mathfrak{m}_0$ is a Pukanszky polarization at f_{0r} and g_0. For a proof of this assertion see also the second part of the proof of Proposition 3.4. We observe that in order to prove

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

it suffices to verify the relation

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M_0)} \pi_{0r} \not\subset \ker_{L^1(M_0)} \rho_0 \;. \tag{11.12}$$

Now we explain how to apply the method of restricting to subquotients developed in Section 7 in this case. We consider the chain $\mathfrak{b} \supset \mathfrak{a} \supset \mathfrak{z}$ of ideals of $\mathfrak{m}_0$ where $\mathfrak{b} = \langle\, e_1, e_3, e_5 \,\rangle$ is a three-dimensional Heisenberg algebra, $\mathfrak{a} = \langle\, e_3, e_5 \,\rangle$ is commutative, and $\mathfrak{z} = \mathbb{R}e_5$ is contained in the center of $\mathfrak{m}_0$. We observe that $\pi_{0r} = \mathrm{ind}_H^{M_0} \chi_{f_{0r}}$ and $\rho_0 = \mathrm{ind}_H^{M_0} \chi_{g_0}$. Furthermore the condition

$$\chi_{f_{0r}}(\exp(xe_3 + ze_5)) = \chi_{g_0}(\exp(xe_3 + ze_5)) = e^{iz}$$

is satisfied. Hence we are exactly in the situation of Theorem 7.10. Let K denote the connected subgroup of M_0 whose Lie algebra is the commutative subalgebra $\mathfrak{k} = \langle\, e_0, e_4, e_5 \,\rangle$ of $\mathfrak{m}_0$. Theorem 7.10 shows us that Relation 11.12 is equivalent to

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(K,w)} \chi_{f_{0r}} \not\subset \ker_{L^1(K,w)} \chi_{g_0}$$

where the weight function w on K is given by

$$w(t, y, z) = 2\cosh^{1/2}(t) \;.$$

Note that $f_{0r}(e_0) = f_0 + ar$, $f_{0r}(e_4) = e^{-r}$, and $f_{0r}(e_5) = g_0(e_5) = 1$. We see that the assertion of this lemma follows if we can find a function $h \in L^1(K, w)$ such that

$$\widehat{h}(f_0 + ar, e^{-r}, 1) = 0 \quad \text{for all } r \text{ and } \quad \widehat{h}(g_0, g_4, 1) \neq 0 \;.$$

If $g_4 \neq 0$, then Lemma 11.10 implies $g_0 \neq f_0 - a\log g_4$ because g is critical for the orbit $\mathrm{Ad}^*(G)f$. In this case the existence of h follows from Theorem 7.17.*(i)*. If $g_4 = 0$, then we can apply Theorem 7.17.*(ii)*. The proof is complete. $\square$

11.2.4 Five distinct weights α, $\gamma-\alpha$, $\overline{\gamma}-\alpha$, γ, $\overline{\gamma}$ where α is real and γ is non-real such that α, γ are $\mathbb{C}$-linearly independent

In this case $\dim\mathfrak{g} = 7$. Since $\mathfrak{g}$ is exponential, there exists a $\lambda \in \mathbb{R}\backslash\{0\}$ and a $\delta \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s},\mathbb{R})$ such that $\gamma = (1+i\lambda)\delta$. Let us choose $d, e_0 \in \mathfrak{s}$ such that $\alpha(d) = 0$, $\alpha(e_0) = -1$, $\delta(d) = 1$, and $\delta(e_0) = 0$. Now we see that there exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[e_0, e_3] = e_3$, $[d, e_2] = e_2 + \lambda e_3$, $[d, e_3] = -\lambda e_2 + e_3$, $[d, e_4] = e_4 + \lambda e_5$, and $[d, e_5] = -\lambda e_4 + e_5$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Then $f_4^2 + f_5^2 \neq 0$. Without loss of generality we can assume $f_\nu = 0$ for $1 \le \nu \le 3$. The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned}
\mathrm{Ad}^*(\exp(rd)\,\Phi(0,v,w,x,0,0))f\ (e_0) &= f_0 - v\,(wf_4 + xf_5)\\
(e_1) &= wf_4 + xf_5\\
(e_2) &= -e^{-r}\,(\,\cos(\lambda r)f_4 + \sin(\lambda r)f_5\,)\,v\\
(e_3) &= -e^{-r}\,(\,-\sin(\lambda r)f_4 + \cos(\lambda r)f_5\,)\,v\\
(e_4) &= e^{-r}\,(\,\cos(\lambda r)f_4 + \sin(\lambda r)f_5\,)\\
(e_5) &= e^{-r}\,(\,-\sin(\lambda r)f_4 + \cos(\lambda r)f_5\,)
\end{aligned}$$

Lemma 11.13. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $\overline{\gamma}-\alpha$, γ, $\overline{\gamma}$ where α is real and γ is non-real such that α, γ are $\mathbb{C}$-linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ holds if and only if $p_1(g) = f_0\,g_5$ and $p_2(g) = f_0\,g_4$.*

Proof. The only-if-part is obvious. In order to prove the opposite implication, let us assume $p_1(g) = f_0\,g_5$ and $p_2(g) = f_0\,g_4$. At first we suppose $g_5 \neq 0$. Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, there exist sequences r_n, v_n, w_n, x_n such that $f_n' \longrightarrow g'$. Now $f_n(e_5) = e^{-r_n}\,(\,-\sin(\lambda r_n)f_4 + \cos(\lambda r_n)f_5\,) \longrightarrow g_5 \neq 0$ implies that the sequence r_n remains bounded. From $p_1(g) = g_0g_5 - g_1g_3 = f_0g_5$ and $f_n' \longrightarrow g'$ we conclude that

$$e^{-r_n}\,(\,-\sin(\lambda r_n)f_4 + \cos(\lambda r_n)f_5\,)\,(\,f_0 - v_n\,(w_nf_4 + x_nf_5)\,) \longrightarrow f_0g_5 + g_1g_3 = g_0g_5\ .$$

This shows $f_n \longrightarrow g$. If $g_4 \neq 0$, then the last convergence follows similarly with the aid of the polynomial p_2.

Now we treat the case $g_4 = g_5 = 0$. We must distinguish several subcases. First we assume $g_1 \neq 0$. Since $p_1(g) = p_2(g) = 0$, it follows $g_2 = g_3 = 0$. We choose sequences w_n, x_n such that $w_nf_4 + x_nf_5 = g_1$. Further we define $r_n = n$ and

$$v_n = \frac{1}{g_1}\,(f_0 - g_0)\ .$$

These definitions imply $f_n \longrightarrow g$. Next we assume $g_1 = 0$ and ($g_2 \neq 0$ or $g_3 \neq 0$). Since $\mathrm{Ad}^*(G)g' \subset (\mathrm{Ad}^*(G)f')^{-}$, there exist sequences r_n, v_n such that $f_n(e_\nu) \longrightarrow g_\nu$ for $2 \le \nu \le 5$. Now

$$f_n(e_4)^2 + f_n(e_5)^2 = e^{-2r_n}\,(f_4^2 + f_5^2) \longrightarrow 0$$

implies $r_n \longrightarrow +\infty$. Further

$$f_n(e_2)^2 + f_n(e_3)^2 = e^{-2r_n}\, v_n^2\,(f_4^2 + f_5^2) \longrightarrow g_2^2 + g_3^2 \neq 0$$

implies $|v_n| \longrightarrow +\infty$. Further we choose sequences w_n, x_n such that

$$w_n f_4 + x_n f_5 = \frac{1}{v_n}\,(f_0 - g_0)\,.$$

These considerations yield $f_n \longrightarrow g$. If $g_\nu = 0$ for $1 \leq \nu \leq 5$, then we define $r_n = n$ and $v_n = e^{r_n/2}$. Further we choose w_n, x_n such that

$$w_n f_4 + x_n f_5 = e^{-r_n/2}\,(f_0 - g_0)\,.$$

Again we see $f_n \longrightarrow g$. This completes the proof of our lemma. □

In the same way as in Subsection 11.2.1 we obtain

Lemma 11.14. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits five distinct weights α, $\gamma-\alpha$, $\overline{\gamma}-\alpha$, γ, $\overline{\gamma}$ such that α, γ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows*

$$\bigcap_{r\in\mathbb{R}} \ker_{L^1(M)}\, \pi_r \not\subset \ker_{L^1(M)}\, \rho$$

for the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

11.2.5 Three distinct weights α, $\gamma-\alpha$, γ such that α, γ are linearly independent

In this case $\dim\mathfrak{g} = 7$. There exists a basis $d, e_0, \ldots, e_5$ of $\mathfrak{g}$ such that $[e_1,e_2] = e_4$, $[e_1,e_3] = e_5$, $[e_0,e_1] = -e_1$, $[e_0,e_2] = e_2$, $[e_0,e_3] = e_3$, $[d,e_2] = e_2 + ae_3$, $[d,e_3] = e_3$, $[d,e_4] = e_4 + ae_5$, and $[d,e_5] = e_5$. We could establish $a \in \{0,1\}$, but this is not necessary for the following considerations. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. In particular $f_5 \neq 0$. Without loss of generality we can assume $f_5 = 1$ and $f_\nu = 0$ for $1 \leq \nu \leq 3$. The coadjoint representation of G in $\mathfrak{m}^*$ is given by

$$\begin{aligned}
\mathrm{Ad}^*(\exp(rd)\,\Phi(0,v,w,x,0,0))f\ (e_0) &= f_0 - v(wf_4 + x)\\
(e_1) &= wf_4 + x\\
(e_2) &= -e^{-r}\,v\,(f_4 - ar)\\
(e_3) &= -e^{-r}\,v\\
(e_4) &= e^{-r}\,(f_4 - ar)\\
(e_5) &= e^{-r}
\end{aligned}$$

Lemma 11.15. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits three distinct weights α, $\gamma-\alpha$, γ such that α, γ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^-$ holds if and only if $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$.*

Proof. First we assume $\mathrm{Ad}^*(G)g \subset (\mathrm{Ad}^*(G)f)^{-}$ so that there exist sequences r_n, v_n, w_n, x_n such that $f_n \longrightarrow g$. Since $f_n(e_5) = e^{-r_n} \longrightarrow g_5$ and $f_n(e_4) = e^{-r_n}(f_0 - ar_n) \longrightarrow g_4$, it follows $p_1(f_n) = f_0\, e^{-r_n} \longrightarrow p(g) = f_0\, g_5$ and $p_2(f_n) = f_0\, e^{-r_n}(f_0 - ar_n) \longrightarrow p(g) = f_0\, g_4$.

In order to prove the opposite implication, we assume $p_1(g) = f_0\, g_5$ and $p_2(g) = f_0\, g_4$. At first we suppose $g_5 \neq 0$. There exist sequences r_n, v_n, w_n, x_n such that $f_n(e_\nu) \longrightarrow g_\nu$ for $1 \le \nu \le 5$. In particular $e^{-r_n} \longrightarrow g_5$ so that $r_n \longrightarrow -\log g_5$. This implies $g_4 = (f_4 + a \log g_5)\, g_5$. Finally we see

$$e^{-r_n}\left(f_0 - v_n(w_n f_4 + x_n)\right) \longrightarrow f_0\, g_5 + g_1\, g_3 = g_0\, g_5$$

because $p_1(g) = f_0\, g_5$. This proves $f_n \longrightarrow g$.

Next we assume $g_5 = 0$. In this case $r_n \longrightarrow +\infty$ so that $g_4 = 0$, too. Now we must distinguish several subcases. In any case we set $w_n = 0$. First we assume $g_1 \neq 0$. Since $p_1(g) = p_2(g) = 0$, it follows $g_2 = g_3 = 0$. We define $r_n = n$, $x_n = g_1$, and

$$v_n = \frac{1}{g_1}\,(f_0 - g_0)$$

so that $f_n \longrightarrow g$. Next we assume $g_1 = 0$ and $g_3 \neq 0$. Then it follows $g_2 = g_3\, f_4$ and $a = 0$. Here we define $r_n = n$, $v_n = -e^{r_n}\, g_3$, and

$$x_n = -\frac{1}{g_3}\, e^{-r_n}\,(f_0 - g_0)\,.$$

Then we obtain $f_n \longrightarrow g$. The next case is $g_1 = g_3 = 0$ and $g_2 \neq 0$. Here we have $a \neq 0$. We define $r_n = n$,

$$v_n = -g_2\, e^{r_n}\, \frac{1}{f_4 - ar_n} \quad \text{and} \quad x_n = \frac{1}{v_n}\,(f_0 - g_0)\,.$$

This proves $f_n \longrightarrow g$. Finally we assume $g_1 = g_2 = g_3 = 0$. In this situation we define $r_n = n$, $v_n = e^{r_n/2}$, and $x_n = e^{-r_n/2}(f_0 - g_0)$. These definitions imply $f_n \longrightarrow g$. This finishes the proof of our lemma. □

In the same way as in Subsection 11.2.1 we obtain

Lemma 11.16. *Let $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$ and $\mathfrak{s}$ be as in Subsection 11.1. Assume that the $\mathfrak{s}$-module $\mathfrak{n}$ admits three distinct weights α, $\gamma - \alpha$, γ such that α, γ are linearly independent. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $g \in \mathfrak{m}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. Then it follows*

$$\bigcap_{r \in \mathbb{R}} \ker_{L^1(M)} \pi_r \not\subset \ker_{L^1(M)} \rho$$

for the unitary representations $\pi_r = \mathcal{K}(f_r)$ and $\rho = \mathcal{K}(g)$.

12 Nilradical is the algebra $\mathfrak{g}_{5,3}$

More precisely, we study the unitary representation theory of an exponential solvable Lie group G such that its Lie algebra $\mathfrak{g}$ contains a coabelian, nilpotent ideal $\mathfrak{n}$ which is a trivial extension of the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,3}$. This section is divided into four subsections. In the first one we describe the algebraic structure of $\mathfrak{g}$. Let $\mathfrak{m} \subset \mathfrak{g}$ be the stabilizer of a linear functional $f \in \mathfrak{g}^*$ in general position. We prove the existence of an $\mathrm{ad}(\mathfrak{m})$-invariant polynomial function on $\mathfrak{m}^*$. The next two sections are devoted to the investigation of the unitary representations of G, first in the central case and then in the non-central case. In part four of this section we prove two multiplier theorems in order to complete the proof of Proposition 12.26 and Theorem 12.34.

12.1 The structure of $\mathfrak{g}$

Let $\mathfrak{g}$ be an exponential solvable Lie algebra which contains a nilpotent ideal $\mathfrak{n}$ such that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{n}$. Assume that $\mathfrak{n}$ is 3-step nilpotent and that

$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{2}{\supset} C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n} \underset{d-1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} \{0\} \tag{12.1}$$

is a descending series of characteristic ideals of $\mathfrak{n}$. Here $\mathfrak{c}$ denotes the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$ and satisfies

$$\{0\} \neq [\mathfrak{c},\mathfrak{c}] \subset C^2\mathfrak{n}\ . \tag{12.2}$$

The center $\mathfrak{z}\mathfrak{n}$ is d-dimensional. In particular for $d = 1$ our assumptions include the case of the nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g}_{5,3}$.

Let $\mathfrak{m}$ be a non-nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{m}$. Assume that there exists an $f \in \mathfrak{m}^*$ in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Since f vanishes on $[\mathfrak{m},\mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f,\mathfrak{z}\mathfrak{n}]$, this ideal of $\mathfrak{g}$ must be zero. This proves $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

If $\mathfrak{m} = \mathfrak{g}$, then the orbit $\mathrm{Ad}^*(G)f' = \mathrm{Ad}^*(N)f'$ is closed. Consequently there are no functionals $g \in \mathfrak{m}^*$ which are critical for $\mathrm{Ad}^*(G)f$, compare Theorem 3.23. Thus we can assume $\mathfrak{m} \neq \mathfrak{g}$.

Let us fix a nilpotent subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$. A proof for the existence of such subalgebras $\mathfrak{s}$ can be found on p. 72 of [23]. Another proof of this fact uses the existence of Cartan subalgebras of $\mathfrak{g}$. Let us define $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$. We regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules and benefit from the existence of weight space decompositions of these modules.

There exists a weight $\alpha \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{c}$, and a weight $\gamma_0 \in \mathfrak{s}^*$ such that $C^2\mathfrak{n} \subset \mathfrak{n}_{\gamma_0}$ and $\tilde{\gamma}_0 = 0$. Here $\tilde{\gamma}_0$ is the restriction of γ_0 to $\mathfrak{t}$. Suppose $\mathfrak{c} = (\mathfrak{n}_{\tilde{\beta}} \cap \mathfrak{c}) + C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$ where $\tilde{\beta} \in \mathfrak{t}^*$ and $\mathfrak{n}_{\tilde{\beta}}$ denotes the weight space for $\tilde{\beta}$ of the $\mathfrak{t}$-module $\mathfrak{n}$. Then

$$C^1\mathfrak{n} = [\mathfrak{n},\mathfrak{c}] = \Big[\mathfrak{n}_{\tilde{\alpha}} + \mathfrak{c}\ ,\ (\mathfrak{n}_{\tilde{\beta}} \cap \mathfrak{c}) + C^1\mathfrak{n}\Big] = (\mathfrak{n}_{\tilde{\alpha}+\tilde{\beta}} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}$$

and

$$C^2\mathfrak{n} = [\mathfrak{n}, C^1\mathfrak{n}] = \Big[\mathfrak{n}_{\tilde{\alpha}} + \mathfrak{c}\ ,\ (\mathfrak{n}_{\tilde{\alpha}+\tilde{\beta}} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}\Big] = \mathfrak{n}_{2\tilde{\alpha}+\tilde{\beta}} \cap C^2\mathfrak{n}\ .$$

On the other hand we have

$$C^2\mathfrak{n} = [\mathfrak{c},\mathfrak{c}] = \Big[(\mathfrak{n}_{\tilde{\beta}} \cap \mathfrak{c}) + C^1\mathfrak{n}\ ,\ (\mathfrak{n}_{\tilde{\beta}} \cap \mathfrak{c}) + C^1\mathfrak{n}\Big] = \mathfrak{n}_{2\tilde{\beta}} \cap C^2\mathfrak{n}\ .$$

Since $C^2\mathfrak{n} \subset \mathfrak{z}\mathfrak{n}$, we obtain $2\tilde{\alpha} + \tilde{\beta} = 0$ and $2\tilde{\beta} = 0$. Thus $\tilde{\alpha} = \tilde{\beta} = 0$. This contradicts the assumption that $\mathfrak{m}$ is not nilpotent.

Hence there exist distinct weights $\beta, \delta \in \mathfrak{s}^*$ such that $\mathfrak{n}_\beta \cap \mathfrak{c} \not\subset C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$ and $\mathfrak{n}_\delta \cap \mathfrak{c} \not\subset C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. Without loss of generality we can assume $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta \cap \mathfrak{c}] \subset C^2\mathfrak{n}$ and $[\mathfrak{n}_\alpha, \mathfrak{n}_\delta \cap \mathfrak{c}] \not\subset C^2\mathfrak{n}$. Then it follows

$$C^1\mathfrak{n} = \big[\mathfrak{n}_\alpha + \mathfrak{c}\ ,\ (\mathfrak{n}_\beta \cap \mathfrak{c}) + (\mathfrak{n}_\delta \cap \mathfrak{c}) + C^1\mathfrak{n}\big] = (\mathfrak{n}_{\alpha+\delta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}$$

and

$$C^2\mathfrak{n} = \big[\mathfrak{n}_\alpha + \mathfrak{c}\ ,\ (\mathfrak{n}_{\alpha+\delta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}\big] = \mathfrak{n}_{2\alpha+\delta} \cap C^2\mathfrak{n}\ .$$

Similarly, we have

$$[\mathfrak{c},\mathfrak{c}] = \mathfrak{n}_{\beta+\delta} \cap C^2\mathfrak{n}\ .$$

This implies $2\alpha + \delta = \gamma_0$ and $\beta + \delta = \gamma_0$. Since $\tilde{\gamma}_0 = 0$ and since $\mathfrak{m}$ is not nilpotent, we have $\tilde{\alpha} \neq 0$. This shows that the weights α, 2α, $\gamma_0 - 2\alpha$, $\gamma_0 - \alpha$ and γ_0 are pairwise distinct. Note that $\mathfrak{n}_0 = \mathfrak{m}_0 \cap \mathfrak{n}$ is not necessarily trivial and that $\mathfrak{t} \cap \mathfrak{n} \subset \mathfrak{n}_0 \subset \mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$. Here $\mathfrak{m}_0$ resp. $\mathfrak{n}_0$ denotes the weight space of the $\mathfrak{s}$-module $\mathfrak{m}$ resp. $\mathfrak{n}$ of weight 0.

If $\gamma_0 \neq 0$, then we obtain the decomposition

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha} \oplus \mathfrak{n}_{\gamma_0 - 2\alpha} \oplus \mathfrak{n}_{\gamma_0 - \alpha} \oplus \mathfrak{n}_{\gamma_0} \oplus \mathfrak{v}\ .$$

Here $\mathfrak{v} \subset \mathfrak{z}\mathfrak{n}$ is a direct sum of weight spaces corresponding to weights $\gamma \neq 0$ such that $\tilde{\gamma} = 0$. The other weight spaces are one-dimensional. If $\gamma_0 = 0$, then we have the decomposition

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha} \oplus \mathfrak{n}_{-2\alpha} \oplus \mathfrak{n}_{-\alpha} \oplus \mathfrak{v}$$

with $\mathfrak{v}$ as above and $C^2\mathfrak{n} \subset \mathfrak{m}_0$.

Since $\mathfrak{n} + \mathfrak{z}\mathfrak{m}$ is also a trivial extension of $\mathfrak{g}_{5,3}$, we can assume without loss of generality that $\mathfrak{z}\mathfrak{m}$ is contained in $\mathfrak{n}$ so that $\mathfrak{z}\mathfrak{n} = \mathfrak{z}\mathfrak{m}$. As in Subsection 9.1 one can prove that the assumptions $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and f in general position imply that $\dim \mathfrak{m}/\mathfrak{n} = 1$ so that $\mathfrak{n}$ is the nilradical of $\mathfrak{m}$.

We choose $e_0 \in \mathfrak{t}$ such that $\alpha(e_0) = -1$. It is easy to see that we can find vectors $e_1, \ldots, e_5$ in the weight spaces $\mathfrak{n}_\alpha, \ldots, \mathfrak{n}_{\gamma_0}$ such that

$$(12.3) \qquad [e_1, e_3] = e_4\ , \quad [e_1, e_4] = e_5\ , \quad [e_2, e_3] = e_5\ ,$$

and

$$(12.4) \qquad [e_0, e_1] = -e_1\ , \quad [e_0, e_2] = -2e_2\ , \quad [e_0, e_3] = 2e_3\ , \quad [e_0, e_4] = e_4\ .$$

Obviously, the vectors $e_0, \ldots, e_4$ form a basis of $\mathfrak{m}$ modulo the center $\mathfrak{z}\mathfrak{m}$.

By the way, we see that $\mathfrak{m}$ is uniquely determined up to isomorphism by the following conditions: $\mathfrak{m}$ is not nilpotent, the nilradical $\mathfrak{n}$ of $\mathfrak{m}$ is a trivial extension of the nilpotent Lie algebra $\mathfrak{g}_{5,3}$, and $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

Next we compute the coadjoint action of M on $\mathfrak{m}^*$. For arbitrary $f \in \mathfrak{m}^*$ and $X, Y \in \mathfrak{m}$ the general formula

$$\mathrm{Ad}^*(\exp X)f\ (Y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\ f\left(\mathrm{ad}(X)^j(Y)\right)$$

yields $\mathrm{Ad}^*(\exp Z)f = f$ and $\mathrm{Ad}^*(\exp X)f\ (Z) = f(Z)$ for all $X \in \mathfrak{m}$ and $Z \in \mathfrak{z}\mathfrak{m}$. Further we obtain

$$\begin{aligned}
\mathrm{Ad}^*(\exp ye_4)f(e_0) &= f_0 + yf_4\\
(e_1) &= f_1 + yf_5\\
(e_\nu) &= f_\nu \text{ for } 2 \le \nu \le 4
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}^*(\exp xe_3)f(e_0) &= f_0 + 2xf_3\\
(e_1) &= f_1 + xf_4\\
(e_2) &= f_2 + xf_5\\
(e_\nu) &= f_\nu \text{ for } 3 \le \nu \le 4
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}^*(\exp we_2)f(e_0) &= f_0 - 2wf_2\\
(e_\nu) &= f_\nu \text{ for } 1 \le \nu \le 2\\
(e_3) &= f_3 - wf_5\\
(e_4) &= f_4
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}^*(\exp ve_1)f(e_0) &= f_0 - vf_1\\
(e_\nu) &= f_\nu \text{ for } 1 \le \nu \le 2\\
(e_3) &= f_3 - vf_4 + \frac{1}{2}v^2 f_5\\
(e_4) &= f_4 - vf_5
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ad}^*(\exp te_0)f(e_0) &= f_0\\
(e_1) &= e^t f_1\\
(e_2) &= e^{2t} f_2\\
(e_3) &= e^{-2t} f_3\\
(e_4) &= e^{-t} f_4\ .
\end{aligned}$$

We work with coordinates of the second kind. Since the canonical images of $e_0, \ldots, e_4$ in $\mathfrak{m}/\mathfrak{z}\mathfrak{m}$ form a Malcev basis, the map $\Phi : \mathbb{R}^5 \times \mathfrak{z}\mathfrak{m} \longrightarrow M$ given by

$$\Phi(t, v, w, x, y, Z) = \exp(te_0) \exp(ve_1) \exp(we_2) \exp(xe_3) \exp(ye_4) \exp(Z)$$

is a global diffeomorphism.

Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. The last condition is equivalent to $\mathfrak{m} = \mathfrak{m}_{f'} + \mathfrak{n}$ because $\mathfrak{m}/\mathfrak{n}$ is one-dimensional. In particular we have $f_5 \neq 0$. The equations

$$\mathrm{Ad}^*(\exp ve_1)f\ (e_4) = f_4 - vf_5$$

$$\begin{aligned} \mathrm{Ad}^*(\exp we_2)f\ (e_3) &= f_3 - wf_5 \\ (e_4) &= f_4 \end{aligned}$$

$$\begin{aligned} \mathrm{Ad}^*(\exp xe_3)f\ (e_2) &= f_2 + xf_5 \\ (e_\nu) &= f_\nu \text{ for } 3 \leq \nu \leq 4 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ad}^*(\exp ye_4)f\ (e_1) &= f_1 + yf_5 \\ (e_\nu) &= f_\nu \text{ for } 2 \leq \nu \leq 4 \end{aligned}$$

show that we can achieve $f_\nu = 0$ for all $1 \leq \nu \leq 4$ by choosing f appropriately on the orbit $\mathrm{Ad}^*(N)f$.

Let $Z \in \mathfrak{z}\mathfrak{m}$ and $X = te_0 + ve_1 + we_2 + xe_3 + ye_4 + Z \in \mathfrak{m}$. Then we obtain

$$\begin{aligned} \mathrm{Ad}^*(\Phi(X))f\ (e_0) &= f_0 - vyf_5 - 2wxf_5 \\ (e_1) &= e^t yf_5 \\ (e_2) &= e^{2t} xf_5 \\ (e_3) &= -e^{-2t} wf_5 + \frac{1}{2} e^{-2t} v^2 f_5 \\ (e_4) &= -e^{-t} vf_5 \\ (Y) &= f(Y) \text{ for } Y \in \mathfrak{z}\mathfrak{m}\ . \end{aligned} \tag{12.5}$$

These formulas motivate the definition of the polynomial function

$$\boldsymbol{p_0 = e_0 e_5 e_5 - e_1 e_4 e_5 - 2 e_2 e_3 e_5 + e_2 e_4 e_4}$$

on $\mathfrak{m}^*$ which is constant on the orbit $\mathrm{Ad}^*(M)f$. Here e_ν means the linear function $f \mapsto f(e_\nu)$ on $\mathfrak{m}^*$ and the products are taken in the commutative algebra $\mathcal{P}(\mathfrak{m}^*)$ of complex-valued polynomials on $\mathfrak{m}^*$.

Recall that M acts on $\mathcal{P}(\mathfrak{m}^*)$ by

$$\mathrm{Ad}(m)p\ (f) = p\left(\mathrm{Ad}^*(m)^{-1} f\right)\ .$$

If we embed $\mathfrak{m}$ in $\mathcal{P}(\mathfrak{m}^*)$ as described above, then this action extends the adjoint representation of M in $\mathfrak{m}$. Similarly, $\mathfrak{m}$ acts as a Lie algebra of derivations in $\mathcal{P}(\mathfrak{m}^*)$ extending the adjoint representation of $\mathfrak{m}$ in $\mathfrak{m}$.

It turns out that p_0 is actually an $\mathrm{Ad}(M)$-invariant polynomial and hence constant on any $\mathrm{Ad}^*(M)$-orbit in $\mathfrak{m}^*$. This is easily verified by checking the $\mathrm{ad}(\mathfrak{m})$-invariance of p_0.

There is a natural isomorphism of associative algebras between the symmetric algebra $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ and $\mathcal{P}(\mathfrak{m}^*)$ which maps $Y \in \mathfrak{m} \subset \mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ onto the linear function $f \mapsto f(Y)$. Further there is a linear isomorphism between $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$ which is uniquely determined by linearity and the property

$$X_1 \cdot \ldots \cdot X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-iX_{\sigma(1)}) \cdot \ldots \cdot (-iX_{\sigma(r)}) \; .$$

Here S_r denotes the permutation group on r elements. The product on the left-hand side is in $\mathcal{S}(\mathfrak{m}_{\mathbb{C}})$ and on the right-hand side is in the non-commutative algebra $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$. This is a slight modification of the so-called symmetrization map, compare chapter 3.3 of [5].

Composing these two isomorphisms, we obtain a linear, $\mathrm{Ad}(M)$-equivariant isomorphism from $\mathcal{P}(\mathfrak{m}^*)$ onto $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$, which maps the subspace of invariant polynomials onto the center of $\mathcal{U}(\mathfrak{m}_{\mathbb{C}})$.

Under this identification the $\mathrm{Ad}(M)$-invariant polynomial p_0 corresponds to the central element

$$\boldsymbol{W_0 = i\left[e_0e_5e_5 - \frac{1}{2}(e_1e_4e_5 + e_4e_1e_5) - (e_2e_3e_5 + e_3e_2e_5) + e_2e_4e_4\right]} \; .$$

The invariant polynomial p_0 and the central element W_0 play a very important role in the subsequent investigation.

Since $\mathfrak{s}$ acts nilpotently on $\mathfrak{t} \subset \mathfrak{m}_0$, there exists a minimal $q \geq 0$ such that $\mathrm{ad}(\mathfrak{s})^{q+1} \cdot \mathfrak{t} = 0$. Recall that $e_0 \in \mathfrak{t}$. Thus for $d \in \mathfrak{s}$ and arbitrary $h \in \mathfrak{m}^*$ we obtain

$$\mathrm{Ad}^*(\exp(sd))h\,(e_0) = h_0 + \sum_{j=1}^{q} \frac{(-s)^j}{j!}\, h\left(\mathrm{ad}(d)^j \cdot e_0\right) \; . \tag{12.6}$$

Finally, we scale the vectors e_1, e_2, and e_5 (multiply them by $1/f_5$) so that $f_5 = 1$. From now on, we shall keep this normalization.

We have to distinguish whether $C^2\mathfrak{n}$ is contained in the center $\mathfrak{z}\mathfrak{g}$ of $\mathfrak{g}$ or not. These two cases are essentially different.

12.2 The central case: $C^2\mathfrak{n}$ is contained in $\mathfrak{zg}$

In this case $\gamma_0 = 0$. Let $\mathfrak{s}/\mathfrak{t} \cong \mathfrak{g}/\mathfrak{m}$ have dimension m. We fix a maximal set α, $\gamma_1, \ldots, \gamma_{m'}$ of linearly independent weights of the $\mathfrak{s}$-module $\mathfrak{m}$. If we recall the results of Section 8, then we find that there exist vectors $d_1, \ldots, d_m$ in $\mathfrak{s}$

- linearly independent modulo $\mathfrak{t}$,
- such that $\alpha(d_\nu) = 0$ for $1 \leq \nu \leq m$,
- such that $\gamma_\mu(d_\nu) = \delta_{\mu,\nu}$ for $1 \leq \mu, \nu \leq m'$,
- and such that $\mathrm{ad}(d_\nu)$ is nilpotent for $m' + 1 \leq \nu \leq m$.

Remark. For simplicity we assume that all weights of the $\mathfrak{s}$-module $\mathfrak{m}$ are **real** here. However, all results of this subsection (in particular Theorem 12.10) are valid, if there exist complex weights. In Section 9.2 we treat this situation explicitly.

The Malcev basis $d_1, \ldots, d_m, e_0, \ldots, e_4$ of $\mathfrak{g}$ modulo $\mathfrak{zm}$ defines a smooth map

$$E(s) = \exp(s_1 d_1) \ldots \exp(s_m d_m)$$

and coordinates of the second kind

$$\mathbb{R}^m \times \mathbb{R}^5 \times \mathfrak{zm} \longrightarrow G,\ (s, t, v, w, x, y, Z) \mapsto E(s)\Phi(t, v, w, x, y, Z)\ .$$

Combining 12.5 and 12.6, we obtain the important formulas

$$\begin{aligned}
\mathbf{Ad}^* \left(\boldsymbol{E(s)\Phi(0, v, w, x, y, Z)}\right) \boldsymbol{f}\ (\boldsymbol{e_0}) &= \boldsymbol{f_0 - vy - 2wx + Q(s)} \\
(\boldsymbol{e_1}) &= \boldsymbol{y} \\
(\boldsymbol{e_2}) &= \boldsymbol{x} \\
(\boldsymbol{e_3}) &= \boldsymbol{-w + \frac{1}{2}v^2} \\
(\boldsymbol{e_4}) &= \boldsymbol{-v} \\
(\boldsymbol{e_5}) &= \mathbf{1} \\
(\boldsymbol{Y}) &= \boldsymbol{f}\left(\mathbf{Ad}(\boldsymbol{E(s)})^{-1}\boldsymbol{Y}\right)
\end{aligned}$$

where $Y \in \mathfrak{zm}$. In the first equation one finds the polynomial function

$$Q(s) = \sum_{1 \leq j_1 + \ldots + j_m \leq q} c_{j_1, \ldots, j_m}\ s_1^{j_1}\ \ldots\ s_m^{j_m}$$

in m variables whose coefficients are given by

$$c_{j_1, \ldots, j_m} = \frac{(-1)^{j_1 + \ldots + j_m}}{j_1!\ \ldots\ j_m!}\ f\left(\mathrm{ad}(d_m)^{j_m}\ \ldots\ \mathrm{ad}(d_1)^{j_1}\ \cdot e_0\right)\quad .$$

It is immediate from the definition of these coefficients that Q does not depend on the variable s_ν if $[d_\nu, \mathfrak{t}] \subset \ker f$.

For $1 \leq \nu \leq m'$ we fix $\mathfrak{s}$-eigenvectors $e_{5+\nu} \in \mathfrak{zn}$ of weight γ_ν such that $f(e_{5+\nu}) = 1$.

Let $g \in \mathfrak{m}^*$ be a critical functional with respect to the orbit $\mathrm{Ad}^*(G)f$. Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, we have $g_5 = 1$. Without loss of generality we can assume $g_\nu = 0$ for $1 \le \nu \le 4$.

Let us define $f_s = \mathrm{Ad}^*(E(s))f$ so that

$$\mathrm{Ad}^*(G)f = \bigcup_{s\in\mathbb{R}^m} \mathrm{Ad}^*(M)f_s\ .$$

Further we set $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$. Note that $\mathfrak{p} = \langle e_0, e_3, e_4\rangle + \mathfrak{z}\mathfrak{m}$ is a common Pukanszky polarization at g and f_s for all s.

We apply the method of restricting to subquotients developed in Section 7 in order to prove that

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M)}\ \pi_s\ \not\subset\ \ker_{L^1(M)}\ \rho\ . \tag{12.7}$$

The Heisenberg algebra $\mathfrak{b}_0 = \langle e_1, e_4, e_5\rangle$ is an ideal of $\mathfrak{m}$. Coordinates of B_0 are given by $(v, y, z) = \exp(ve_1)\exp(ye_4 + ze_5)$. Further $\mathfrak{a}_0 = \langle e_4, e_5\rangle$ is an ideal of $\mathfrak{m}$ and $\mathfrak{z}_0 = \langle e_5\rangle$ is contained in $\mathfrak{z}\mathfrak{m}$. Let $\mathfrak{h}_0 = \langle e_0, e_2, e_3, e_4\rangle + \mathfrak{z}\mathfrak{m}$ denote the Lie algebra of the stabilizer H_0 in M of the character $\chi(0, y, z) = e^{iz}$ of A_0. If we define $\kappa_s = \mathrm{ind}_P^{H_0}\chi_{f_s}$ and $\lambda = \mathrm{ind}_P^{H_0}\chi_g$, then

$$\kappa_s(0, y, z) = \lambda(0, y, z) = e^{iz}\ .$$

By induction in stages we see that $\pi_s = \mathrm{ind}_{H_0}^M\kappa_s$ and $\rho = \mathrm{ind}_{H_0}^M\lambda$. Let K_0 be the closed, connected subgroup of M corresponding to the Lie algebra

$$\mathfrak{k}_0 = \langle e_0, e_2, e_3\rangle + \mathfrak{z}\mathfrak{m}\ .$$

Note that we are exactly in the situation of Section 7.1. By Theorem 7.10 it follows that Relation 12.7 is equivalent to

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(K_0,w_0)}\ \kappa_s\ \not\subset\ \ker_{L^1(K_0,w_0)}\ \lambda \tag{12.8}$$

where the weight function w_0 on K_0 is given by

$$w_0(h) = \big(\ 4(\delta_0(h) + \delta_0(h)^{-1})^2 + \tau_0(h)^2\ \big)^{1/4}\ .$$

The Lie group K_0 is a central extension of $G_{4,9}(0)$. Once more we restrict to a suitable subquotient: Note that $\mathfrak{b}_1 = \langle e_2, e_3, e_5\rangle$ is a Heisenberg algebra and an ideal of $\mathfrak{k}_0$. Coordinates of B_1 are $(w, x, z) = \exp(we_2)\exp(xe_3 + ze_5)$. Let $\mathfrak{a}_1 = \langle e_3, e_5\rangle$ and $\mathfrak{z}_1 = \langle e_5\rangle$. Further let $\mathfrak{h}_1 = \langle e_0, e_3\rangle + \mathfrak{z}\mathfrak{m}$ denote the Lie algebra of the stabilizer H_1 in K_0 of the character $\chi(0, x, z) = e^{iz}$ of A_1. We have $\mathfrak{h}_1 = \mathfrak{p}\cap\mathfrak{k}_0$. Passing over to the quotient $K_0 = H_0/\exp(\mathbb{R}e_4)$, we see that $\kappa_s = \mathrm{ind}_{H_1}^{K_0}\chi_{f_s}$ and $\lambda = \mathrm{ind}_{H_1}^{K_0}\chi_g$. We observe that the restriction of the weight w_0 to the Heisenberg group B_1 is given by

$$w_0(w, x, z) = (16 + x^2)^{1/4}$$

because $\delta_0(w, x, z) = 1$ and $\tau_0(w, x, z) = 0$. Hence the restriction of w_0 to B_1 is dominated by a polynomial weight so that the assumptions of Proposition 7.13 are satisfied. Let K_1 denote the subgroup of K_0 corresponding to the Lie algebra $\mathfrak{k}_1 = \langle e_0 \rangle + \mathfrak{z}\mathfrak{m}$. By Proposition 7.13, the validity of

$$\bigcap_{s \in \mathbb{R}^m} \ker_{L^1(K_1, w_0 w_1^9)} \chi_{f_s} \not\subset \ker_{L^1(K_1, w_0 w_1^9)} \chi_g$$

is sufficient for Relation 12.8 to be true. Note that K_1 is a commutative Lie group. Since $w_0(t, Z) = 2\cosh(t)^{1/2}$ and $w_1(t, Z) = 2\cosh(2t)^{1/2}$, the weight function $w_0 w_1^9$ on K_1 is dominated by an exponential weight of the form $(t, Z) \mapsto b e^{c|t|}$ for suitable constants $b, c > 0$. This proves

Lemma 12.9. *Let $c > 0$ be as above and $w(t, Z) = e^{c|t|}$ the exponential weight function on K_1. Then for*

$$\bigcap_{s \in \mathbb{R}^m} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$$

it is sufficient that there exists a function $h \in L^1(K_1, w)$ such that $\widehat{h}(f_s) = 0$ for all s and $\widehat{h}(g) \neq 0$.

Definition. Let $\Delta \subset \mathfrak{s}^*$ be the set of non-zero weights of the $\mathfrak{s}$-module $\mathfrak{z}\mathfrak{n} = \mathfrak{z}\mathfrak{m}$. We say that Δ is almost independent if every $\delta \in \Delta$ has the form $\delta = c\,\gamma_\nu$ for some $c \in \mathbb{R}^\times$ and $1 \leq \nu \leq m'$.

The preceding lemma and the results of Section 7.3 yield

Theorem 12.10. *Let $\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ be as in Section 12.1 such that $C^2\mathfrak{n} \subset \mathfrak{z}\mathfrak{g}$. In particular $\mathfrak{n}$ is a trivial extension of the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,3}$. Further we assume that one of the following conditions holds:*

(i) $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ *is a semi-direct sum,*

(ii) $\dim \mathfrak{g}/\mathfrak{n} \leq 2$,

(iii) $\mathfrak{g}/\mathfrak{n}$ *acts semi-simply on* $\mathfrak{z}\mathfrak{n}$ *such that* Δ *is almost independent,*

(iv) $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$ *and* $\dim \mathfrak{z}\mathfrak{n} \leq 3$.

Let $\mathfrak{m}$ be a proper, non-nilpotent ideal of $\mathfrak{g}$ with $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$. Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ be critical with respect to the orbit $\mathrm{Ad}^*(G)f$. Then it follows that

$$\bigcap_{s \in \mathbb{R}^m} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho \tag{12.11}$$

holds for the unitary representations $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$ of M.

Proof. We begin with a preliminary remark. If $m' = 0$, then $\mathrm{ad}(d_\nu)$ is nilpotent for $1 \leq \nu \leq m$. Consequently the orbit $\mathrm{Ad}^*(G)f = \mathrm{Ad}^*\left(E(\mathbb{R}^m) \right) \mathrm{Ad}^*(N)f$ is closed in $\mathfrak{m}^*$ because $\mathrm{Ad}^*(G)f$ is the orbit of a connected group acting unipotently on the real vector space $\mathfrak{m}^*$, compare Theorem 3.1.4 (Chevalley-Rosenlicht) on p. 82 of [5]. In this case there are no linear functionals $g \in \mathfrak{m}^*$ which are critical with respect to the orbit $\mathrm{Ad}^*(G)f$. Hence we can always suppose $m' \geq 1$.

First we assume $[\mathfrak{s}, \mathfrak{s}] = 0$ so that $Q = 0$. Since g is critical for $\mathrm{Ad}^*(G)f$, it follows $g(e_0) \neq f_0$. Now Lemma 12.9 and Theorem 7.17 yield our claim. The case $\dim \mathfrak{g}/\mathfrak{n} \leq 1$ being trivial, next we assume $\dim \mathfrak{g}/\mathfrak{n} = 2$. Then we find a $d \in \mathfrak{s}$ such that $\alpha(d) = 0$ and $\mathfrak{g} = \mathbb{R}d + \mathfrak{m}$, and an $\mathrm{ad}(d)$-eigenvector $e_6 \in \mathfrak{z}\mathfrak{n}$ such that $[d, e_6] = e_6$ and $f(e_6) = 1$. If $g(e_6) \neq 0$, then it follows $\mathrm{Ad}^*(G)f' = \mathrm{Ad}^*(G)g'$. Thus we can assume $g(e_6) = 0$. We know

$$f_s(e_0) = f_0 + Q(s)$$

for $s \in \mathbb{R}$. If $Q = 0$, then $g(e_0) \neq f_0$. In this case relation 12.11 follows by Lemma 12.9 and Lemma 7.16. Now let $Q \neq 0$. Then we see the validity of 12.11 by Lemma 12.9 and Proposition 7.18. This proves our theorem in the case that $\mathfrak{g}/\mathfrak{n}$ is two-dimensional.

Now we assume that $\mathfrak{g}/\mathfrak{n}$ acts semi-simply on $\mathfrak{z}\mathfrak{n}$. In this case $\mathfrak{s}$ acts trivial on the weight space $\mathfrak{n}_0$ of the $\mathfrak{s}$-module $\mathfrak{n}$ of weight 0. We see that Q has the form

$$Q(s) = \sum_{\nu=1}^{m} \alpha_\nu s_\nu \ .$$

Note that $\mathrm{ad}(d_\nu) = 0$ on $\mathfrak{n}$ for $m' + 1 \leq \nu \leq m$ because of semi-simplicity. If α_ν were non-zero for some $m' + 1 \leq \nu \leq m$, then it would follow $\mathrm{Ad}^*(G)f = \mathrm{Ad}^*(G)f + \mathfrak{n}^\perp$ and thus $g \in (\mathrm{Ad}^*(G)f)^{-}$. This contradicts the fact that g is critical with respect to $\mathrm{Ad}^*(G)f$. Thus we can assume $\alpha_\nu = 0$ for all $m' + 1 \leq \nu \leq m$.

Let us define $I = \{1 \leq \nu \leq m' : \alpha_\nu \neq 0\}$ and $I_0 = \{\nu \in I : g(e_\nu) = 0\}$. First we assume that there exist $\nu_1, \nu_2 \in I_0$ such that $\alpha_{\nu_1} > 0$ and $\alpha_{\nu_2} < 0$. But in this case it will follow that $g \in (\mathrm{Ad}^*(G)f)^{-}$: Since $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, there exist sequences $s_n \in \mathbb{R}^m$ and $X_n \in \mathfrak{n}$ such that $f_n' \longrightarrow g'$ where

$$f_n = \mathrm{Ad}^*(E(s_n)\Phi(X_n))f \ .$$

Here f_n' and g' denote the restrictions to $\mathfrak{n}$. Then $s_{n,\nu_j} \longrightarrow +\infty$ because $g(e_{5+\nu_j}) = 0$ for $j = 1, 2$. Since Δ is almost independent, it is possible to modify the components s_{n,ν_1} and s_{n,ν_2} of the sequence (s_n) without affecting the convergence $f_n' \longrightarrow g'$: We choose the growth of the sequence $s_{n,\nu_1} \longrightarrow +\infty$ such that

$$u_n = \sum_{\substack{1 \leq \nu \leq m' \\ \nu \neq \nu_2}} \alpha_\nu s_{n,\nu} \ \longrightarrow +\infty \ .$$

Further we define

$$s_{n,\nu_2} = \alpha_{\nu_2}^{-1} \ (\ g(e_0) - f_0 - u_n\ ) \ .$$

Then in particular $s_{n,\nu_2} \longrightarrow +\infty$. Since $[d_\nu, \mathfrak{n}_{\gamma_\mu}] = 0$ for $\mu \neq \nu$ by semi-simplicity, it is easy to see that $f_n \longrightarrow g$. This contradiction shows that we can suppose that the set of coefficients $\{\alpha_\nu : \nu \in I_0\}$ is either contained in $(0, +\infty)$ or in $(-\infty, 0)$. Hence all assumptions of Theorem 7.17 are satisfied. By Theorem 12.9 and Theorem 7.17 the assertion of this theorem follows in the semi-simple case.

Finally let us assume that $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and that $\dim \mathfrak{z}\mathfrak{n} \leq 3$. In particular we have $m' = m = \dim \mathfrak{g}/\mathfrak{m}$. Let us write $d = \dim \mathfrak{z}\mathfrak{n}$. If $d = 1$, then $\mathfrak{m} = \mathfrak{g}$. If $d = 2$, then $m' = 1$ and $\mathfrak{g}/\mathfrak{n}$ is two-dimensional. If $d = 3$, then $m' = 1$ or $\mathfrak{s}$ acts semi-simply on $\mathfrak{z}\mathfrak{n}$ and Δ is almost independent. $\square$

12.3 The non-central case: $C^2\mathfrak{n}$ is not contained in $\mathfrak{z}\mathfrak{g}$

In this case we have $\gamma_0 \neq 0$. Let $\mathfrak{s}/\mathfrak{t} \cong \mathfrak{g}/\mathfrak{m}$ have dimension $m+1$. We fix a maximal set α, $\gamma_0, \ldots, \gamma_{m'}$ of linearly independent weights of the $\mathfrak{s}$-module $\mathfrak{m}$. The remarks following Lemma 8.2 show that we can choose vectors $d_0, \ldots, d_m$ in $\mathfrak{s}$

- linearly independent modulo $\mathfrak{t}$,
- such that $\alpha(d_\nu) = 0$ for $0 \leq \nu \leq m$,
- such that $\gamma_\mu(d_\nu) = \delta_{\mu,\nu}$ for $0 \leq \mu, \nu \leq m'$,
- and such that $\mathrm{ad}(d_\nu)$ is nilpotent for $m'+1 \leq \nu \leq m$.

Remark. For simplicity we assume that all weights of the $\mathfrak{s}$-module $\mathfrak{m}$ are **real** in this subsection. Note that α and γ_0 are always real. However, all results of this subsection (in particular Lemma 12.23, Lemma 12.24, Proposition 12.26 and Theorem 12.34) are still valid in the presence of complex weights. In the course of Section 9.3 we take the possibility of complex weights into account.

Note that $d_0, \ldots, d_m, e_0, \ldots, e_4$ is a Malcev basis of $\mathfrak{g}$ modulo $\mathfrak{z}\mathfrak{m}$. We define

$$E(r,s) = \exp(rd_0)\exp(s_1 d_1)\ldots\exp(s_m d_m)$$

and work with coordinates of the second kind given by the diffeomorphism

$$\mathbb{R}^{m+1} \times \mathfrak{m} \longrightarrow G,\ (r,s,X) \mapsto E(r,s)\Phi(X)\ .$$

Combining 12.5 and 12.6, we obtain the important formulas

$$\begin{aligned}
\mathbf{Ad}^*\left(\boldsymbol{E(r,s)\Phi(X)}\right)\boldsymbol{f}\ \boldsymbol{(e_0)} &= \boldsymbol{f_0 - vy - 2wx + Q(r,s)}\\
\boldsymbol{(e_1)} &= \boldsymbol{y}\\
\boldsymbol{(e_2)} &= \boldsymbol{x}\\
\boldsymbol{(e_3)} &= \boldsymbol{-e^{-r}w + \frac{1}{2}e^{-r}v^2}\\
\boldsymbol{(e_4)} &= \boldsymbol{-e^{-r}v}\\
\boldsymbol{(Y)} &= \boldsymbol{f}\left(\mathbf{Ad}\boldsymbol{(E(r,s))^{-1}Y}\right)\\
\boldsymbol{(e_5)} &= \boldsymbol{e^{-r}}
\end{aligned}$$

where $X \in \mathfrak{n}$ and $Y \in \mathfrak{z}\mathfrak{m}$. The last equation is a special case of the preceding one. In the first equation one finds the polynomial function

$$Q(r,s) = \sum_{1 \leq j+j_1+\ldots+j_m \leq q} c_{j,j_1,\,\ldots\,,j_m}\, r^j s_1^{j_1}\ \ldots\ s_m^{j_m}$$

in $m+1$ variables whose coefficients are given by

$$c_{j,j_1,\,\ldots\,,j_m} = \frac{(-1)^{j+j_1+\ldots+j_m}}{j!\ j_1!\ \ldots\ j_m!}\ f\left(\mathrm{ad}(d_m)^{j_m}\ \ldots\ \mathrm{ad}(d_1)^{j_1}\ \mathrm{ad}(d_0)^j \cdot e_0\right) \quad .$$

It is immediate from the definition of these coefficients that Q does not depend on the variable s_ν if $[d_\nu, \mathfrak{t}] \subset \ker f$.

Let $\mathfrak{s}_c = \ker \gamma_0$ be the centralizer of $C^2\mathfrak{n}$ in $\mathfrak{s}$. Note that $[\mathfrak{s}_c, \mathfrak{t}] \subset \ker f$ is equivalent to $[\mathfrak{s}_c, \mathfrak{t}] = 0$ because f is in general position. If this is true, then the function Q depends only on the variable r. If even $[\mathfrak{s}, \mathfrak{t}] = 0$, then $Q = 0$.

Let $\mathfrak{s}_n$ denote the nilpotent part of $\mathfrak{s}$, i.e., the set of all $d \in \mathfrak{s}$ such that $\mathrm{ad}(d)$ is nilpotent. If $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then $\mathfrak{s}_n \subset \mathfrak{z}\mathfrak{n}$. In this case $m = m'$.

Definition 12.12. A triple Γ consisting of a nilpotent Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, a maximal set $\alpha, \gamma_0, \ldots, \gamma_{m'}$ of linearly independent weights as above, and a coexponential basis $d_0, \ldots, d_m$ for $\mathfrak{t}$ in $\mathfrak{s}$ as above is called structure data for $\Delta = (\mathfrak{g}, \mathfrak{m}, \mathfrak{n}, f)$.

Our next aim is to describe the closure of the orbit $\mathrm{Ad}^*(G)f$. In particular we want to determine all critical functionals $g \in \mathfrak{m}^*$ for this orbit.

There is no loss of generality in assuming $\mathfrak{n}$ to be the nilradical of $\mathfrak{m}$, which is equivalent to $\mathfrak{z}\mathfrak{m} = \mathfrak{z}\mathfrak{n}$. It is only natural to choose $\mathfrak{n}$ as large as possible in order to keep the set of critical functionals for $\mathrm{Ad}^*(G)f$ small.

We assert that an arbitrary linear functional $g \in \mathfrak{m}^*$ satisfies

$$\mathrm{Ad}^*(G)g' \subset \left(\mathrm{Ad}^*(G)f' \right)^{-}$$

if and only if there exist sequences r_n, s_n, and $X_n = v_n e_1 + w_n e_2 + x_n e_3 + y_n e_4 \in \mathfrak{n}$ such that

$$\begin{aligned} y_n &\longrightarrow g_1 \\ x_n &\longrightarrow g_2 \\ -e^{-r_n} w_n + \frac{1}{2} e^{-r_n} v_n^2 &\longrightarrow g_3 \\ -e^{-r_n} v_n &\longrightarrow g_4 \\ \text{and } f\left(\mathrm{Ad}(E(r_n, s_n))^{-1} Y\right) &\longrightarrow g(Y) \end{aligned}$$

for every $Y \in \mathfrak{z}\mathfrak{n}$. If so, then in particular $e^{-r_n} \longrightarrow g_5$. Of course, we always take the limit for $n \longrightarrow \infty$. There is a certain degree of freedom in the choice of these sequences. The stronger condition

$$\mathrm{Ad}^*(G)g \subset \left(\mathrm{Ad}^*(G)f \right)^{-}$$

is fulfilled if and only if these sequences can be chosen such that in addition

$$f_0 - v_n y_n - 2 w_n x_n + Q(r_n, s_n) \longrightarrow g_0 \,.$$

In order to obtain more concrete results, we require some additional assumptions.

Assumption 12.13 For all $m' + 1 \leq \nu \leq m$ the polynomial function Q defined by f and Γ does **not** depend on the variable s_ν.

Permuting the weights $\gamma_1, \ldots, \gamma_{m'}$ and the vectors $d_1, \ldots, d_{m'}$, we can even suppose that there exists a $0 \le l \le m'$ such that Q depends on s_ν if and only if $1 \le \nu \le l$. Here we use the fact that $\mathfrak{s}$ acts as a commutative algebra on $\mathfrak{m}$. This follows from the obvious relation $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{z}\mathfrak{n}$. We call l the **critical index**.

Note that $[\mathfrak{s}_n, \mathfrak{t}] \subset \ker f$ is equivalent to $[\mathfrak{s}_n, \mathfrak{t}] = 0$. In this case Assumption 12.13 is satisfied. If $[\mathfrak{s}_n, e_0] \not\subset \ker f$, then 12.13 is violated.

Assumption 12.14 It holds $g(Y) \neq 0$ for all $1 \le \nu \le l$ and all $\mathfrak{s}$-eigenvectors $Y \in \mathfrak{z}\mathfrak{n}$ of weight γ_ν. We say that g is **admissible** with respect to f and Γ.

Remark. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ such that

$$\mathrm{Ad}^*(G)g' \ \subset\ (\,\mathrm{Ad}^*(G)f'\,)^{-} \,.$$

Then there exist sequences r_n, s_n, X_n such that

$$\mathrm{Ad}^*\left(E(r_n, s_n)\Phi(X_n)\right) f' \longrightarrow g' \,.$$

Let $1 \le \nu \le l$ and $Y \in \mathfrak{z}\mathfrak{n}$ be an $\mathfrak{s}$-eigenvector of weight γ_ν. Since f is in general position, we have $f(Y) \neq 0$. From

$$\mathrm{Ad}^*\left(E(r_n, s_n)\Phi(X_n)\right) f'\ (Y) = e^{-s_{n,\nu}}\ f(Y) \longrightarrow g(Y)$$

it follows that the sequence $s_{n,\nu}$ is convergent and hence remains bounded. Clearly, $g(Y) \neq 0$ for one such Y implies this inequality for all such Y.

The postulates 12.13 and 12.14 enforce that r is the only among the first $l+1$ variables (the arguments of Q) which may tend to infinity.

Remark 12.15. Let $g \in \mathfrak{m}^*$ such that $\mathrm{Ad}^*(G)f' \subset (\,\mathrm{Ad}^*(G)f'\,)^{-}$, $g_5 \neq 0$ and $g(Y) \neq 0$ for all $1 \le \nu \le m'$ and all $\mathfrak{s}$-eigenvectors $Y \in \mathfrak{z}\mathfrak{n}$ of weight γ_ν. There exist sequences r_n, s_n and X_n such that

$$\mathrm{Ad}^*\left(E(r_n, s_n)\Phi(X_n)\right) f' \longrightarrow g'$$

for $n \longrightarrow \infty$. Let us write s' for the first $m'+1$ coordinates of s, and s'' for the last ones. As in the preceding remark it follows that $r_n \longrightarrow r_0$ and $s'_n \longrightarrow s'_0$. Applying $\mathrm{Ad}^*\left(E(r_n, s'_n, 0)\right)^{-1}$ to both sides of the above relation, we obtain

$$\mathrm{Ad}^*\left(E(0, 0, s''_n)\Phi(X_n)\right) f' \longrightarrow \mathrm{Ad}^*\left(E(r_0, s'_0, 0)\right)^{-1} g' \,.$$

Since $\mathfrak{n}_0 = \mathfrak{s}_0 + \mathfrak{n}$ is a nilpotent ideal of $\mathfrak{g}$, the connected subgroup N_0 of G acts unipotently on $\mathfrak{n}^*$. It is a standard result, see Theorem 3.1.4 of [5], that the orbit $\mathrm{Ad}^*(N_0)f'$ is closed in $\mathfrak{n}^*$. So $\mathrm{Ad}^*\left(E(r_0, s'_0, 0)\right)^{-1} g' \in \mathrm{Ad}^*(N_0)f'$ and thus $\mathrm{Ad}^*(G)g' = \mathrm{Ad}^*(G)f'$.

We conclude that $g \in \mathfrak{m}^*$ is not critical for the orbit $\mathrm{Ad}^*(G)f$ if $g(Y) \neq 0$ for all $1 \le \nu \le m'$ and all eigenvectors Y of weight γ_ν.

We introduce some notation that will be used in the following lemma. For $1 \leq \nu \leq l$ we fix $\mathfrak{s}$-eigenvectors $e_{5+\nu} \in \mathfrak{z}\mathfrak{n}$ of weight γ_ν such that $f(e_{5+\nu}) = 1$. We write $g_\nu = g(e_\nu)$ for all $0 \leq \nu \leq 5+l$ and any $g \in \mathfrak{m}^*$.

If sequences r_n, s_n, and X_n are chosen, then we abbreviate

$$f_n = \mathrm{Ad}^*(\, E(r_n, s_n)\Phi(X_n)\,) f \ . \tag{12.16}$$

Let us recall the definition of the $\mathrm{Ad}(M)$-invariant polynomial p_0 introduced in Subsection 12.1. We observe that

$$p_0\,(\,\mathrm{Ad}^*(\,E(r,s)\Phi(X)\,)f\,) \;=\; e^{-2r}\;\left(\,f_0 + Q(r,s')\,\right)\ .$$

If $Q = 0$, then p_0 is semi-invariant for the action of $\mathfrak{s}$ on $\mathfrak{m}$.

Profiting by the existence of the polynomial p_0, we obtain the following description of (the admissible part of) the closure of the orbit $\mathrm{Ad}^*(G)f$ in $\mathfrak{m}^*$.

Lemma 12.17. *Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Suppose that there exists some structure data Γ such that Assumption 12.13 holds. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ in the sense of 12.14, and such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\,\mathrm{Ad}^*(G)f\,)^{-}$ if and only if one of the following conditions holds:*

(i) $g_5 \neq 0$ *and* $p_0(g) = g_5 g_5\,(\,f_0 + Q(-\log g_5, \ldots, -\log g_{5+l})\,)$,

(ii) $g_5 = 0$, $g_4 \neq 0$, *and* $g_2 = 0$,

(iii) $g_5 = g_4 = 0$, $g_2 \neq 0$, *and* $g_3 \geq 0$,

(iv) $g_5 = g_4 = g_2 = 0$.

Proof.

(i) Let $g_5 \neq 0$ and $\mathrm{Ad}^*(G)g \subset (\,\mathrm{Ad}^*(G)f\,)^{-}$. Hence there exist sequences r_n, s_n, X_n such that the sequence f_n defined by 12.16 converges to g. It follows

$$p_0(f_n) = e^{-2r_n}(\,f_0{+}Q(r_n, s_n')\,) \longrightarrow p_0(g) = g_5 g_5\,(\,f_0 + Q(-\log g_5, \ldots, -\log g_{5+l})\,)\ .$$

To prove the opposite direction, let us assume

$$p_0(g) = g_5 g_5(\,f_0 + Q(-\log g_5, \ldots, -\log g_{5+l})\,)\ .$$

Since $g_5 \neq 0$, we can further establish $g_1 = g_2 = g_3 = g_4 = 0$. From the definition of p_0 and again from $g_5 \neq 0$, it follows $g_0 = f_0 + Q(-\log g_5, \ldots, -\log g_{5+l})$. Now our claim $g \in (\mathrm{Ad}^*(G)f\,)^{-}$ is obvious.

(ii) Let $g_5 = 0$ and $g_4 \neq 0$. The equations

$$\begin{aligned} \mathrm{Ad}^*(\exp xe_3)g\,(e_1) &= g_1 + xg_4 \\ (e_\nu) &= g_\nu \text{ for } 2 \leq \nu \leq 5 \end{aligned}$$

and

$$\begin{aligned}\mathrm{Ad}^*(\exp ve_1)g\,(e_3) &= g_3 - vg_4\\ (e_\nu) &= g_\nu \text{ for } 1 \le \nu \le 5, \nu \ne 3\end{aligned}$$

show that we can also assume $g_1 = g_3 = 0$. If $g \in (\,\mathrm{Ad}^*(G)f\,)^{-}$, then there exist sequences r_n, s_n, X_n such that

$$\mathrm{Ad}^*\left(E(r_n, s_n)\right) f\,(Y) \longrightarrow g(Y) \tag{12.18}$$

for all $Y \in \mathfrak{z}\mathfrak{n}$. In particular $r_n \longrightarrow +\infty$ because $g_5 = 0$. Hence

$$p_0(f_n) = e^{-2r_n}\left(f_0 + Q(r_n, s_n')\right) \longrightarrow 0 = p_0(g) = g_2g_4g_4$$

because the sequence s_n' is bounded. This proves $g_2 = 0$ because $g_4 \ne 0$.

For the converse assume $g_2 = 0$. Choose sequences r_n, s_n such that 12.18 holds and define $y_n = 0$, $v_n = -e^{r_n}g_4$,

$$\begin{aligned} w_n &= \frac{1}{2}\,e^{2r_n}g_4^2\,,\\ \text{and}\quad x_n &= \frac{1}{g_4^2}\,e^{-2r_n}\,\left(f_0 - g_0 + Q(r_n, s_n')\right)\,.\end{aligned}$$

Now it is clear that $f_n \longrightarrow g$.

(iii) Let $g_5 = g_4 = 0$ and $g_2 \ne 0$. If $g \in (\,\mathrm{Ad}^*(G)f\,)^{-}$, then there exist sequences r_n, s_n, X_n such that $f_n \longrightarrow g$. In particular

$$e^{-r_n}(\,f_0 - v_ny_n - 2w_nx_n + Q(r_n, s_n')\,) \longrightarrow 0\,.$$

Since s_n' is bounded, $r_n \longrightarrow +\infty$, and $e^{-r_n}v_n \longrightarrow 0$, we obtain $2e^{-r_n}w_nx_n \longrightarrow 0$. From $x_n \longrightarrow g_2 \ne 0$ we even get $e^{-r_n}w_n \longrightarrow 0$. Hence we see

$$-e^{-r_n}w_n + \frac{1}{2}e^{-r_n}v_n^2 \longrightarrow g_3 \ge 0\,.$$

For the converse assume $g_5 = g_4 = 0$, $g_2 \ne 0$, and $g_3 \ge 0$. Let r_n, s_n satisfy 12.18 and set $x_n = g_2$, $y_n = g_1$,

$$\begin{aligned} v_n &= (\,2\,e^{r_n}\,g_3\,)^{1/2}\\ \text{and}\quad w_n &= \frac{1}{2g_2}\left(f_0 - g_0 - g_1(2\,e^{r_n}\,g_3)^{1/2} + Q(r_n, s_n')\right)\,.\end{aligned}$$

Note that $e^{-r_n}w_n \longrightarrow 0$ because s_n' is bounded. Again we see $f_n \longrightarrow g$.

(iv) Finally, assume $g_5 = g_4 = g_2 = 0$. If $g_3 \neq 0$, then we choose $v_n = 0$, $w_n = -e^{r_n} g_3$, $y_n = g_1$, and

$$x_n = -\frac{1}{2g_3}\, e^{-r_n} \, \left(f_0 - g_0 + Q(r_n, s_n') \right) \, .$$

If $g_3 = 0$, then we define $v_n = 0$, $w_n = e^{r_n/2}$, $y_n = g_1$, and

$$x_n = \frac{1}{2}\, e^{-r_n/2} \, \left(f_0 - g_0 + Q(r_n, s_n') \right) \, .$$

It is easy to see that $g \in (\operatorname{Ad}^*(G)f\,)^{-}$ holds in either case.

□

Now we turn to the investigation of the relevant (unitary) representation theory of the group M. As a starting point we choose the observation that the subset $\operatorname{Ad}^*(G)f$ of $\mathfrak{m}^*$ decomposes into $\operatorname{Ad}^*(M)$-orbits. More precisely, since M is a normal subgroup of G, we obtain

$$\operatorname{Ad}^*(G)f = \bigcup_{(r,s)\in\mathbb{R}\times\mathbb{R}^m} \operatorname{Ad}^*(M)f_{r,s}$$

where

$$f_{r,s} = \operatorname{Ad}^* \left(E(r,s) \right) f \, .$$

The link to representation theory is given by the Kirillov map $\mathcal{K} : \mathfrak{m}^*/\operatorname{Ad}^*(M) \longrightarrow \widehat{M}$. It is a well-known fact that $\mathcal{K}$ is a G-equivariant homeomorphism, if $\mathfrak{m}^*/\operatorname{Ad}^*(M)$ carries the quotient topology, and $\widehat{M} = \operatorname{Prim} C^*(M)$ the Jacobson topology, see [23].

The Kirillov map $\mathcal{K}$ maps $\operatorname{Ad}^*(G)f$ bijectively onto the subset $\{\pi_{r,s} : (r,s) \in \mathbb{R}^{m+1}\}$ of $\widehat{M}$ where

$$\pi_{r,s} = \mathcal{K}(\operatorname{Ad}^*(M)f_{r,s}) = \mathcal{K}(f_{r,s}) \, .$$

Our intention is to compute the infinitesimal operators of these representations.

First, note that $f_{r,s}(e_5) = e^{-r}$ and $f_{r,s}(e_\nu) = 0$ for all $1 \leq \nu \leq 4$. From this it becomes obvious that the subalgebra

$$\mathfrak{p} = \langle e_0, e_3, e_4 \rangle + \mathfrak{z}\mathfrak{n}$$

is a polarization at $f_{r,s}$ for all r and s. This reflects the $\mathfrak{s}$-invariance of $\mathfrak{p}$. The equations

$$\begin{aligned} \operatorname{Ad}^* \left(\exp(xe_3) \exp(ye_4) \right) f_{r,s} \, (e_1) &= e^{-r} y \\ (e_2) &= e^{-r} x \\ (Y) &= f_{r,s}(Y) \text{ for all } Y \in \mathfrak{p} \end{aligned}$$

show that the Pukanszky condition

$$\operatorname{Ad}^*(P) f_{r,s} = f_{r,s} + \mathfrak{p}^\perp$$

is also satisfied for all r and s. We notice that $\mathfrak{c} = \langle e_1, e_2 \rangle$ is a commutative subalgebra of $\mathfrak{m}$, which is coexponential for $\mathfrak{p}$ in $\mathfrak{m}$. Further $\mathfrak{p} \cap \mathfrak{n}$ is an ideal of $\mathfrak{m}$. Hence the results of Section 6 for representations in general position apply. We see that the infinitesimal operators of

$$\pi_{r,s} = \operatorname{ind}_P^M \chi_{f_{r,s}}$$

are given by

$$
\begin{aligned}
\boldsymbol{d\pi_{r,s}(e_0)} &= \boldsymbol{\left(\frac{3}{2} + if_0 + iQ(r,s')\right) + \xi_1\partial_{\xi_1} + 2\xi_2\partial_{\xi_2}} \\
\boldsymbol{d\pi_{r,s}(e_1)} &= \boldsymbol{-\partial_{\xi_1}} \\
\boldsymbol{d\pi_{r,s}(e_2)} &= \boldsymbol{-\partial_{\xi_2}} \\
\boldsymbol{d\pi_{r,s}(e_3)} &= \boldsymbol{-ie^{-r}\xi_2 + \frac{1}{2}ie^{-r}\xi_1^2} \\
\boldsymbol{d\pi_{r,s}(e_4)} &= \boldsymbol{-ie^{-r}\xi_1} \\
\boldsymbol{d\pi_{r,s}(Y)} &= \boldsymbol{if\left(E(r,s)^{-1}Y\right)} \quad \textbf{for } \boldsymbol{Y \in \mathfrak{zn}}\ .
\end{aligned}
\tag{12.19}
$$

The purpose of the rest of this subsection is to prove

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho \tag{12.20}$$

for irreducible representations $\rho = \mathcal{K}(g)$ which correspond to critical functionals $g \in \mathfrak{m}^*$. The results of Section 5.1 turn out to be valuable in this context.

To begin with, we recall the definition of the central element $W_0 \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$. By means of 12.19, it is then easy to verify the crucial equation

$$\boldsymbol{d\pi_{r,s}(W_0) = p_0(f_{r,s})\cdot \mathrm{Id}}\ .$$

Let $\mathfrak{z}$ denote the central subalgebra of $\mathfrak{m}$ generated by the eigenvectors $e_5, \dots, e_{5+l}$. Note that $f_{r,s}(e_5) = e^{-r}$ and $f_{r,s}(e_{5+\nu}) = e^{-s_\nu}$ for $1 \le \nu \le l$. We work in coordinates with respect to this basis of $\mathfrak{z}$. We define a real-valued function ψ on $\mathfrak{z}^*$ by

$$\psi(\eta) = \eta_0^2\eta_1 \dots \eta_l\ \left(f_0 + Q(-\log\eta_0, \dots, -\log\eta_l)\right)$$

if $\eta_\nu > 0$ for all $0 \le \nu \le l$, and $\psi(\eta) = 0$ else. For ψ to be a continuous function, the factor $\eta_0^2\eta_1 \dots \eta_l$ is necessary. Further, for $j \ge 1$ large enough, the derivatives of ψ^j up to order $l+1$ exist, are continuous, and have polynomial growth. We modify the central element W_0 by

$$W = W_0\,(-ie_6)\dots(-ie_{5+l}), \tag{12.21}$$

so that the equation

$$d\pi_{r,s}(W) = e^{-(s_1+\dots+s_l)}\ p_0(f_{r,s})\cdot\mathrm{Id} = \psi(\,f_{r,s}\,|\,\mathfrak{z}\,)\cdot\mathrm{Id} \tag{12.22}$$

becomes true. Up to this point, we have done most of the work which is necessary to prove the following two lemmata.

Lemma 12.23. *Let f and Γ be as above. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ. Assume that $g_5 \neq 0$ and that $\mathrm{Ad}^*(G)g \not\subset (\,\mathrm{Ad}^*(G)f\,)^-$. Then Relation 12.20 holds for the L^1-kernels of the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.*

Proof. By Lemma 12.17 we have

$$p_0(g) \neq g_5^2\ \left(\, f_0 + Q(-\log g_5, \ldots, -\log g_{5+l})\,\right)$$

and thus we obtain

$$d\rho(W) = g_6 \ldots g_{5+l}\, p_0(g) \cdot \mathrm{Id} \neq \psi(\, g \,|\, \mathfrak{z}\,) \cdot \mathrm{Id}$$

because $g_\nu \neq 0$ for all $5 \leq \nu \leq 5+l$. Now we can apply Lemma 5.4 and Theorem 5.17 to $\pi_{r,s}$, ρ, W, and ψ. The assertion of the lemma follows. □

Lemma 12.24. *Let f and Γ be as above. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ. Assume $g_5 = 0$, $g_4 \neq 0$, and that $\mathrm{Ad}^*(G)g$ is not contained in the closure of $\mathrm{Ad}^*(G)f$. Then*

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \ \not\subset\ \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.

Proof. Lemma 12.17 implies $g_2 \neq 0$ in this case. In the proof of this lemma we have already seen that we can assume $g_1 = g_3 = 0$. It is easy to verify that

$$\mathfrak{q} = \langle e_2, e_3, e_4 \rangle + \mathfrak{z}\mathfrak{n}$$

is a Pukanszky polarization at $g \in \mathfrak{m}^*$ simply because

$$\begin{aligned} \mathrm{Ad}^*(\exp y e_4)g\ (e_0) &= g_0 + y g_4 \\ (e_1) &= g_1 \\ (Y) &= f(Y) \text{ for } Y \in \mathfrak{q} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ad}^*(\exp x e_3)g\ (e_0) &= g_0 \\ (e_1) &= g_1 + x g_4 \\ (Y) &= f(Y) \text{ for } Y \in \mathfrak{q}\ . \end{aligned}$$

Further, $\mathfrak{m}$ is a semi-direct sum of the non-commutative algebra $\mathfrak{c} = \langle e_0, e_1 \rangle$ and the polarization $\mathfrak{q}$. We can apply the results of Section 6 for representations of semi-direct products to

$$\rho = \mathrm{ind}_Q^M \chi_g$$

and obtain

$$\boldsymbol{d\rho(e_0) = -\partial_{\xi_1}}$$

$$\boldsymbol{d\rho(e_1) = -e^{\xi_1}\partial_{\xi_2}}$$

$$\boldsymbol{d\rho(e_2) = ie^{2\xi_1}\, g_2}$$

$$\boldsymbol{d\rho(e_3) = -ie^{-2\xi_1}\xi_2\, g_2}$$

$$\boldsymbol{d\rho(e_4) = ie^{-\xi_1}\, g_4}$$

$$\boldsymbol{d\rho(Y) = ig(Y) \quad \textbf{for } Y \in \mathfrak{z}\mathfrak{n}\ .}$$

From $p_0(g) = g_2 g_4 g_4 \neq 0$, $g_\nu \neq 0$ for $6 \leq \nu \leq 5 + l$, and $d\rho(W_0) = p_0(g) \cdot \mathrm{Id}$, it follows that $d\rho(W)$ is a non-zero scalar operator. Hence all assumptions of Lemma 5.4 and Theorem 5.17 are satisfied. This finishes the proof of our lemma. □

Remark 12.25. There is no doubt about the significance of the Assumptions 12.13 and 12.14 for our treatise. Concerning the orbit space of the coadjoint action, these assumptions are indispensable for a concrete characterization of the closure of the orbit $\mathrm{Ad}^*(G)f$ in $\mathfrak{m}^*$, see Lemma 12.17.

From the representation theoretical point of view, these postulates guarantee that (W, p, ψ) separates ρ from $\{\pi_{r,s} : (r, s) \in \mathbb{R}^{m+1}\}$, compare Lemma 12.24.

In this context we mention that the modification of the central element W given by Equation 12.21 is absolutely necessary in order to avoid singularities of ψ. Such singularities make the application of Theorem 5.1 impossible, as we have already noticed in Remark 5.5.

We point out that Theorem 5.1 does not apply in the situation of the following proposition. In this case $d\rho(W_0) = p_0(g) \cdot \mathrm{Id} = 0$. Thus L^1-functions of the form $h = W * H$ with $H \in \mathcal{C}_0^\infty(M)$ are inadequate and the triple (W, p, ψ) does not separate the quite singular representation $\rho = \mathcal{K}(g)$ from the subset $\{\pi_{r,s} : (r, s) \in \mathbb{R}^{m+1}\}$ of representations in general position in the sense of Definition 5.10.

Our goal is to prove

Proposition 12.26. *Let f and Γ be as above. Let $g \in \mathfrak{m}^*$ be admissible with respect to f and Γ. Assume $g_5 = g_4 = 0$, $g_2 \neq 0$, and $g_3 < 0$. Then the relation*

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

holds for the L^1-kernels of $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$.

First of all, we can assume $g_3 = -1$. Before we come to the proof of Proposition 12.26, we have to make several preparations.

However, it will turn out that the underlying ideas of the proof of Theorem 5.1 are still applicable. Let H be a suitable smooth function satisfying the conditions of Remark 12.28. As in Subsection 5.1 we will verify the following two assertions:

I. There exists a smooth function $b \in L^1(M)$ such that

$$\pi_{r,s}(b) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,)\ \pi_{r,s}(H) \quad \text{and} \quad \rho(b) \neq 0\ .$$

II. There exists a smooth function $c \in L^1(M)$ such that

$$\pi_{r,s}(c) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,)\ \pi_{r,s}(H) \quad \text{and} \quad \rho(c) = 0\ .$$

We anticipate that the function b will be defined with the aid of W in a way generalizing the definition $b = W * H$ for $H \in L^1(M)$. The proof of the existence of b requires a thorough investigation of the differential operator W in $L^1(M)$. Ultimately we are interested in the L^1-functions b and c. However, it will be necessary to leave the framework of $L^1(M)$ temporarily. In fact, H will not be in $L^1(M)$. The proof of the existence of b and c will be reduced to certain multiplier problems of Euclidean Fourier analysis. A proof of these multiplier theorems can be found in Section 12.4.

In order to describe the relevant unitary representations, we introduce partial Fourier transformation with respect to the variables (x, y, z): Let us fix an arbitrary complementary subspace $\mathfrak{v} \subset \mathfrak{zm}$ such that $\mathfrak{zm} = \mathbb{R}e_5 \oplus \mathfrak{v}$. Any element of $\mathfrak{zm}$ can be written uniquely as $ze_5 + Z$ with $z \in \mathbb{R}$ and $Z \in \mathfrak{v}$. The partial Fourier transform of a function $h \in L^1(M)$ is given by

$$\widehat{h}(t, v, w, \xi_3, \xi_4, \lambda, Z) = \int_{\mathbb{R}^3} h(t, v, w, x, y, z, Z)\ e^{-i\xi_3 x}\ e^{-i\xi_4 y}\ e^{-i\lambda z}\ dx\, dy\, dz\ .$$

For the one-parameter subgroups generated by the basis vectors $e_0, \ldots, e_5$ we compute the unitary operators of the representations $\pi_{r,s}$ and ρ. The results of Section 6 yield

$$\pi_{r,s}(\exp(te_0))\varphi\ (\vartheta_1, \vartheta_2) = e^{3t/2}\ e^{itf_{r,s}(e_0)}\ \varphi(e^t\vartheta_1, e^{2t}\vartheta_2)$$

$$\pi_{r,s}(\exp(ve_1))\varphi\ (\vartheta_1, \vartheta_2) = \varphi(\vartheta_1 - v, \vartheta_2)$$

$$\pi_{r,s}(\exp(we_2))\varphi\ (\vartheta_1, \vartheta_2) = \varphi(\vartheta_1, \vartheta_2 - w)$$

$$\pi_{r,s}(\exp(xe_3))\varphi\ (\vartheta_1, \vartheta_2) = e^{ixe^{-r}(\vartheta_1^2/2 - \vartheta_2)}\ \varphi(\vartheta_1, \vartheta_2)$$

$$\pi_{r,s}(\exp(ye_4))\varphi\ (\vartheta_1, \vartheta_2) = e^{-iye^{-r}\vartheta_1}\ \varphi(\vartheta_1, \vartheta_2)$$

$$\pi_{r,s}(\exp(Z))\varphi\ (\vartheta_1, \vartheta_2) = e^{if_{r,s}(Z)}\ \varphi(\vartheta_1, \vartheta_2)$$

and

$$\rho(\exp(te_0))\varphi\ (\vartheta) = \varphi(\vartheta - t)$$

$$\rho(\exp(ve_1))\varphi\ (\vartheta) = e^{ive^{\vartheta}g_1}\ \varphi(\vartheta)$$

$$\rho(\exp(we_2))\varphi\ (\vartheta) = e^{iwe^{2\vartheta}g_2}\ \varphi(\vartheta)$$

$$\rho(\exp(xe_3))\varphi\ (\vartheta) = e^{-ixe^{-2\vartheta}}\ \varphi(\vartheta)$$

$$\rho(\exp(ye_4))\varphi\ (\vartheta) = \varphi(\vartheta)$$

$$\rho(\exp(Z)\varphi\ (\vartheta) = \varphi(\vartheta)\ .$$

Next a simple computation shows

$$\pi_{r,s}(t,\dots,Z)\varphi\ (\vartheta_1,\vartheta_2) = e^{3t/2}\, e^{itf_{r,s}(e_0)}\, e^{ixe^{-r}(\,(e^t\vartheta_1-v)^2/2-(e^{2t}\vartheta_2-w)\,)}$$
$$e^{-iye^{-r}(e^t\vartheta_1-v)}\, e^{ize^{-r}}\, e^{if_{r,s}(Z)}\ \varphi(e^t\vartheta_1 - v,\ e^{2t}\vartheta_2 - w)\ .$$

We consider $\pi_{r,s}$ as a representation of the group algebra $L^1(M)$ and obtain

$$\pi_{r,s}(h)\varphi\ (\vartheta_1,\vartheta_2)$$
$$= \int\limits_{\mathbb{R}^3\times\mathfrak{v}} \widehat{h}\big(\,t,v,w,e^{-r}(e^{2t}\vartheta_2 - w) - e^{-r}(e^t\vartheta_1 - v)^2/2, e^{-r}(e^t\vartheta_1 - v), -e^{-r}, Z\,\big)$$
$$e^{3t/2}\, e^{itf_{r,s}(e_0)}\, e^{if_{r,s}(Z)}\ \varphi(e^t\vartheta_1 - v,\ e^{2t}\vartheta_2 - w)\ dt\,dv\,dw\,dZ\ .$$

Similarly, we see

$$\rho(t,\dots,Z)\varphi\ (\vartheta) = e^{ive^{\vartheta-t}g_1}\, e^{iwe^{2(\vartheta-t)}g_2}\, e^{-ixe^{-2(\vartheta-t)}}\, e^{ig(Z)}\ \varphi(\vartheta - t)$$

and

$$\rho(h)\varphi\ (\vartheta) = \int\limits_{\mathbb{R}^3\times\mathfrak{v}} \widehat{h}(\,t,v,w,e^{-2(\vartheta-t)},0,0,Z\,)\ e^{ive^{\vartheta-t}g_1}\ e^{iwe^{2(\vartheta-t)}g_2}$$
$$e^{ig(Z)}\ \varphi(\vartheta - t)\ dt\,dv\,dw\,dZ\ .$$

In the following lemma we give a characterization of $\ker_{L^1(M)}\rho$.

Lemma 12.27. *Let $\tilde{h}$ denote the partial Fourier transformation with respect to the variables (v,w,x,y,z,Z) of a function $h \in L^1(M)$. Then $h \in \ker_{L^1(M)}\rho$ if and only if*

$$\tilde{h}(\,t,-e^{\vartheta}g_1,-e^{2\vartheta}g_2,e^{-2\vartheta},0,0,g\,|\,\mathfrak{v}\,) = 0$$

for all $\vartheta \in \mathbb{R}$ and almost all $t \in \mathbb{R}$.

Proof. We have $h \in \ker_{L^1(M)} \rho$ if and only if

$$0 = \langle \rho(h)\varphi \, , \psi \rangle = \int_{-\infty}^{+\infty} \rho(h)\varphi \,(\vartheta) \, \overline{\psi(\vartheta)} \, d\vartheta$$

for all $\varphi, \psi \in \mathcal{C}_0(M)$. Since

$$\rho(h)\varphi \,(\vartheta) = \int_{-\infty}^{+\infty} \tilde{h}(\, t, -e^{\vartheta - t} g_1, -e^{2(\vartheta - t)} g_2, e^{-2(\vartheta - t)}, 0, 0, g \,|\, \mathfrak{v} \,) \; \varphi(\vartheta - t) \; dt \; ,$$

the claim of the lemma follows by Fubini's theorem and the substitution $\vartheta \longrightarrow \vartheta + t$. $\square$

Remark 12.28. Now we explain how to deal with functions which are not necessarily elements of $L^1(M)$. The above formula for $\pi_{r,s}(h)$ suggests to define the bounded operator

$$\pi_{r,s}(H)\varphi \,(\vartheta_1, \vartheta_2) = \int_{\mathbb{R}^3 \times \mathfrak{v}} H(\ldots) \; e^{3t/2} \; e^{itf_{r,s}(e_0)} \; e^{if_{r,s}(Z)} \; \varphi(e^t \vartheta_1 - v, e^{2t} \vartheta_2 - w) \; dt \, dv \, dw \, dZ$$

for continuous functions $H = H(t, v, w, \xi_3, \xi_4, \lambda)$ which satisfy

$$|\, H \,|'_\lambda = \int_{\mathbb{R}^3 \times \mathfrak{v}} e^{3|t|/2} \sup_{\xi_3, \xi_4} |H(t, v, w, \xi_3, \xi_4, \lambda)| \; dt \, dv \, dw \, dZ \; < \; \infty$$

for all λ or even

$$|\, H \,|' = \int_{\mathbb{R}^3 \times \mathfrak{v}} e^{3|t|/2} \sup_{\xi_3, \xi_4, \lambda} |H(t, v, w, \xi_3, \xi_4, \lambda)| \; dt \, dv \, dw \, dZ \; < \; \infty \; .$$

Here and in the sequel, three dots represent the argument

$$\big(t, v, w, e^{-r}(e^{2t}\vartheta_2 - w) - e^{-r}(e^t \vartheta_1 - v)^2/2, e^{-r}(e^t \vartheta_1 - v), -e^{-r}, Z\,\big)$$

of H. A sufficient condition for $|H|' < \infty$ is that

$$(1 + t^2)(1 + v^2)(1 + w^2)(1 + |Z|)^{d+1} \; e^{3|t|/2} \; |H(t, v, w, \xi_3, \xi_4, \lambda, Z)|$$

is bounded in all seven variables. Here $d = \dim \mathfrak{v}$.

Note that there is an obvious way of defining a convolution and an involution for functions H such that $|H|' < \infty$. Altogether, we have defined a family of continuous $*$-representations $\pi_{r,s}$ of a normed $*$-algebra, which is, roughly speaking, larger than the group algebra $L^1(M)$. This extension is comparable to the completion $C^*(A) = \mathcal{C}_\infty(\,\widehat{A}\,)$ of $L^1(A)$ for locally compact abelian groups A in the following sense: Let R denote the connected subgroup of M with Lie algebra $\mathfrak{r} = \langle e_0, e_1, e_2 \rangle + \mathfrak{v}$ and A the connected normal subgroup of M whose Lie algebra is the commutative ideal $\mathfrak{a} = \langle e_3, e_4, e_5 \rangle$ so that M can be considered as a semi-direct product $M = R \ltimes A$. Using the technique of covariance algebras, we have, again roughly speaking, extended the group algebra $L^1(M) = L^1(R, L^1(A))$ to the larger algebra $L^1(R, \mathcal{C}_\infty(\,\widehat{A}\,))$.

The following observation is very important: For $\pi_{r,s}$ to be well-defined for all r and s, it is sufficient that H is only declared for $\lambda < 0$ and satisfies $|H|'_\lambda < \infty$ for all $\lambda < 0$. In this way functions which are only defined on one half of the real axis enter our considerations naturally. We even allow H to have a singularity for $\lambda \longrightarrow 0$.

Now let us consider the infinitesimal left regular representation $d\Lambda$ of M in $L^2(M)$ which is given by

$$d\Lambda(X)f\ (y) = \frac{d}{d\tau}_{|\tau=0}\, f(\,\exp(-\tau X)y\,)$$

for $X \in \mathfrak{m}$ and $f \in \mathcal{C}_0^\infty(M)$. Sometimes we write $X * f = d\Lambda(X)f$. Working with the differential operator $D_x = -i\partial_x$ instead of ∂_x, we obtain

$$d\Lambda(ie_0) = D_t$$

$$d\Lambda(ie_1) = e^t\, D_v$$

$$d\Lambda(ie_2) = e^{2t}\, D_w$$

$$d\Lambda(ie_3) = e^{-2t}\,(\,D_x - vD_y - wD_z + \frac{1}{2}v^2 D_z\,)$$

$$d\Lambda(ie_4) = e^{-t}\,(\,D_y - vD_z\,)$$

$$d\Lambda(ie_5) = D_z$$

If $\partial_Z\, f : M \longrightarrow \mathfrak{z}\mathfrak{m}^*$ denotes the derivative of f with respect to the central variable Z of our coordinates, then $d\Lambda(Z)f = -\,\langle \partial_Z\, f, Z\rangle$ for all $Z \in \mathfrak{z}\mathfrak{m}$. Since

$$W_0 = ie_0e_5e_5 - \frac{1}{2}i(e_1e_4e_5 + e_4e_1e_5) - i(e_2e_3e_5 + e_3e_2e_5) + ie_2e_4e_4$$

as an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$, it follows that $W_0 = d\Lambda(W_0)$ is given by

$$W_0 = -D_tD_z^2 + \frac{3}{2}iD_z^2 + D_vD_yD_z - vD_vD_z^2 + 2D_wD_xD_z - D_wD_y^2 - 2wD_wD_z^2$$

as a differential operator in $L^1(M)$. It is easy to see that this differential operator factors over partial Fourier transformation to

$$\boldsymbol{\widehat{W_0} = \lambda^2\, W_2 + \xi_4\lambda\, D_v + (2\xi_3\lambda - \xi_4^2)\, D_w}$$

where

$$W_2 = -D_t + \frac{3}{2}i - v\,D_v - 2w\,D_w\ .$$

Recall that the differential operator $\widehat{W_0}$ is linked to the function

$$\psi_0(\lambda,\eta) = \lambda^2\,(\,f_0 + Q(-\log\lambda, -\log\eta_1,\ldots,-\log\eta_l)\,)$$

on $\mathfrak{z}^*$. This assertion is confirmed by the following lemma.

Lemma 12.29. *For all $\lambda < 0$ let $H = H(t, v, w, \xi_3, \xi_4, \lambda, Z)$ be a function such that all partial derivatives of first order with respect to (t, v, w, ξ_3, ξ_4) exist and are continuous. Keeping $\lambda < 0$ fixed, assume that these derivatives multiplied by the factor*

$$e^{4|t|}\,(1+|v|)^4\,(1+|w|)^4\,(1+|\xi_3|)^2\,(1+|\xi_4|)^2$$

are bounded. Then

$$\pi_{r,s}(\widehat{W_0} \,*\, H) = \psi_0\,(\,f_{r,s}\,|\,\mathfrak{z}\,)\;\pi_{r,s}(H)\;.$$

Proof. The assumptions of this lemma imply that $\pi_{r,s}(\widehat{W_0} \,*\, H)$ is well-defined and that we can apply partial integration below. From Remark 12.28 we recall the term which is represented by the three dots in the argument of H. First we compute

$$\begin{aligned}
D_t\,[\,H(\ldots)\,] &= (D_tH)(\ldots) + e^{-r}\left(2e^{2t}\vartheta_2 - (e^t\vartheta_1 - v)e^t\vartheta_1\right)(D_{\xi_3}H)(\ldots)\\
&\quad + e^{-r}e^t\vartheta_1(D_{\xi_4}H)(\ldots)\\
D_v\,[\,H(\ldots)\,] &= (D_vH)(\ldots) + e^{-r}(e^t\vartheta_1 - v)(D_{\xi_3}H)(\ldots) - e^{-r}(D_{\xi_4}H)(\ldots)\\
D_w\,[\,H(\ldots)\,] &= (D_wH)(\ldots) - e^{-r}(D_{\xi_3}H)(\ldots)
\end{aligned}$$

Then we see

$$\begin{aligned}
&(\widehat{W_0} \,*\, H)\,(\ldots)\\
&= e^{-2r}\left(-(D_tH)(\ldots) + \frac{3}{2}iH(\ldots) - e^t\vartheta_1(D_vH)(\ldots) - 2e^{2t}\vartheta_2(D_wH)(\ldots)\right)\\
&= e^{-2r}\left(-D_t\,[H(\ldots)] + \frac{3}{2}iH(\ldots) - e^t\vartheta_1 D_v\,[H(\ldots)] - 2e^{2t}\vartheta_2 D_w\,[H(\ldots)]\right)
\end{aligned}$$

If we apply partial integration to the integral

$$\int\limits_{\mathbb{R}^3\times\mathfrak{v}} D_t\,[H(\ldots)]\;e^{3t/2}\,e^{itf_{r,s}(e_0)}\,e^{if_{r,s}(Z)}\;\varphi(e^t\vartheta_1 - v, e^{2t}\vartheta_2 - w)\;dt\,dv\,dw\,dZ$$

and to the integrals over $D_v\,[H(\ldots)]$ and $D_w\,[H(\ldots)]$ as well, then the claim of this lemma becomes obvious. □

By the way, note that the variable λ can be treated like a constant in most of these computations. Another important observation is that the differential operator $\widehat{W_0}$ respects tensor products: If

$$H(t, v, w, \xi_3, \xi_4, \lambda, Z) = H_1(t, v, w, Z)\;H_2(\xi_3, \xi_4, \lambda)\;,$$

then

$$\widehat{W_0} \,*\, H = (W_2 \,*\, H_1) \otimes (\lambda^2 H_2) + (D_vH_1) \otimes (\xi_4\lambda H_2) + (D_wH_1) \otimes (2\xi_3\lambda - \xi_4^2)H_2\;.$$

Still we keep in mind that $\mathfrak{m}$ is a semi-direct sum of the subalgebra $\mathfrak{r} = \langle e_0, e_1, e_2\rangle + \mathfrak{v}$ and the commutative ideal $\mathfrak{a} = \langle e_3, e_4, e_5\rangle$.

Our aim is to solve the following problem: Let $H = H(t, v, w, \xi_3, \xi_4, \lambda, Z)$ be a smooth function defined for $\lambda < 0$ and having a singularity of first order for $\lambda \longrightarrow 0$. Is there a function $h \in L^1(M)$ such that $\widehat{h} = \widehat{W_0} \,*\, H$ for all $\lambda < 0$?

If H is a tensor product as above, then this question reduces to a problem in the three variables (ξ_3, ξ_4, λ). Each summand of $\widehat{W_0} * H$ can be treated separately. The third summand is the most critical one. Is there a function $h_2 \in L^1(\mathbb{R}^3)$ such that

$$\widehat{h}_2(\xi_3, \xi_4, \lambda) = (2\xi_3\lambda - \xi_4^2)\; H_2(\xi_3, \xi_4, \lambda)$$

for all ξ_3, ξ_4 and all $\lambda < 0$? The existence of h_2 imposes the following restriction on the singular behavior of H_2 in $\lambda = 0$: Since $h_2 \in L^1(\mathbb{R}^3)$, it follows that $\widehat{h}_2$ is continuous. Division by $(2\xi_3\lambda - \xi_4^2)$ shows that

$$H_2(\xi_3, \xi_4, \lambda) \longrightarrow -\frac{1}{\xi_4^2}\, \widehat{h}_2(\xi_3, \xi_4, 0)$$

for $\lambda \longrightarrow 0$ if $\xi_4 \neq 0$. Further H_2 must vanish at infinity rapidly enough.

On the other hand, we want to establish $\rho(h) \neq 0$ which is guaranteed by the condition $\widehat{h}_2(1, 0, 0) \neq 0$, compare Lemma 12.27. This requires a singularity of H_2 for $\lambda \longrightarrow 0$. Clearly we can restrict ourselves to functions satisfying $H_2(\xi_3, \xi_4, \lambda) = 0$ if $|\xi_3 - 1| \geq 1/2$. Note that $2\xi_3 - \xi_4^2/\lambda > 1$ for $|\xi_3 - 1| < 1/2$ and $\lambda < 0$.

The preceding considerations make the following approach plausible: Let

$$H_2(\xi_3, \xi_4, \lambda) = \frac{1}{2\xi_3\lambda - \xi_4^2}\;\; K(\xi_3, \xi_4, \lambda)$$

where K is a Schwartz function (defined for all λ) such that

$$K(1, 0, 0) \neq 0$$

and

$$K(\xi_3, \xi_4, \lambda) = 0 \quad \text{if} \quad |\xi_3 - 1| \geq \frac{1}{2}\,.$$

Note that H is well-defined for all $\lambda < 0$. Now the difficulty is to treat the first and the second summand of $\widehat{W_0} * H$. The original question of the existence of h_2 splits into two multiplier problems as we will see in the proof of

Lemma 12.30. *Let $K \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz function such that $K(\xi_3, \xi_4, \lambda) = 0$ for $|\xi_3 - 1| \geq 1/2$ so that*

$$H_2(\xi_3, \xi_4, \lambda) = \frac{1}{2\xi_3\lambda - \xi_4^2}\, K(\xi_3, \xi_4, \lambda)$$

is well-defined for all $\lambda < 0$. Let $H_1 \in \mathcal{S}(\mathbb{R}^3 \times \mathfrak{v})$ denote the Gauss function

$$H_1(t, v, w, Z) = e^{-(t^2+v^2+w^2+|Z|^2)/2}$$

If $H = H_1 \otimes H_2$, then there exists a smooth function $h \in L^1(M)$ such that $\widehat{h} = \widehat{W_0} * H$. In particular $\pi_{r,s}(h) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,)\; \pi_{r,s}(H)$ and $\rho(h) \neq 0$.

Proof. Clearly there exists a Schwartz function $h_{2,3} \in \mathcal{S}(\mathbb{R}^3)$ such that

$$\widehat{h}_{2,3}(\xi_3, \xi_4, \lambda) = (2\xi_3\lambda - \xi_4^2)\; H_2(\xi_3, \xi_4, \lambda) = K(\xi_3, \xi_4, \lambda)$$

for all λ. Further it follows from Theorem 12.49 and 12.50 that there exist smooth functions $h_{2,1}$ and $h_{2,2}$ in $L^1(\mathbb{R}^3)$ such that

$$\widehat{h}_{2,1}(\xi_3, \xi_4, \lambda) = \frac{-\lambda^2}{\xi_4^2 - 2\xi_3\lambda}\; K(\xi_3, \xi_4, \lambda) \tag{12.31}$$

and

$$\widehat{h}_{2,2}(\xi_3, \xi_4, \lambda) = \frac{-\xi_4\lambda}{\xi_4^2 - 2\xi_3\lambda}\; K(\xi_3, \xi_4, \lambda) \tag{12.32}$$

for all $\lambda < 0$. If we think of K as $K = \widehat{k}$, then the functions $h_{2,\nu}$ for $1 \leq \nu \leq 3$ turn out to be solutions of Fourier multiplier problems in $L^1(\mathbb{R}^3)$. Let us define the smooth function

$$h = (W_2 \,*\, H_1) \otimes h_{2,1} + (D_v H_1) \otimes h_{2,2} + (D_w H_1) \otimes h_{2,3}$$

in $L^1(M)$ so that $\widehat{h} = \widehat{W_0} \,*\, H$ holds for all $\lambda < 0$. In particular Lemma 12.29 implies $\pi_{r,s}(h) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,)\pi_{r,s}(H)$. Furthermore it holds $\rho(h) \neq 0$ as we will now show: With regard to Lemma 12.27 let $\tilde{h}$ denote the partial Fourier transform of a function h in $L^1(M)$ with respect to the variables (v, w, x, y, z, Z). If $h = h_1 \otimes h_2$, then $\tilde{h} = \widehat{h}_1 \otimes \widehat{h}_2$ where $\widehat{h}_1$ is the Fourier transform of h_1 with respect to (v, w, Z). Here we obtain

$$\tilde{h} = (W_2 \,*\, H_1)\widehat{\ } \otimes (\lambda^2 H_2) + (\xi_1 \widehat{H}_1) \otimes (\xi_4 \lambda H_2) + (\xi_2 \widehat{H}_1) \otimes K\ .$$

If we set $(\xi_3, \xi_4) = (1, 0)$ and take the limit for $\lambda \longrightarrow 0$, then we see that

$$\tilde{h}(t, \xi_1, \xi_2, 1, 0, 0, \zeta) = \xi_2\; \widehat{H}_1(t, \xi_1, \xi_2, \zeta)\; K(1, 0, 0)\; \neq 0$$

for $\xi_2 \neq 0$ and thus $\rho(h) \neq 0$ by Lemma 12.27. Here we have used the fact that the function $\lambda \mapsto \tilde{h}(t, \xi_1, \xi_2, 1, 0, \lambda, \zeta)$ is continuous for almost all t and all ξ_1, ξ_2, ζ. This finishes the proof of our lemma. □

Henceforth we change our notation slightly.

Proof of Proposition 12.26. Let $\mathfrak{z}_0$ be the subspace of $\mathfrak{z}\mathfrak{m}$ generated by the vectors $e_6, \ldots, e_{5+l}$ and let $\mathfrak{z} = \mathbb{R}e_5 \oplus \mathfrak{z}_0$. Further we fix a complementary subspace $\mathfrak{v}$ of $\mathfrak{z}\mathfrak{m}$ such that $\mathfrak{z}\mathfrak{m} = \mathfrak{z} \oplus \mathfrak{v}$. If $h \in L^1(M)$, then $\widehat{h}$ denotes the partial Fourier transform of h with respect to the commutative ideal $\mathfrak{a} = \langle e_3, e_4 \rangle \oplus \mathfrak{z}$. Recall the definition of the element $W = W_0\,(-ie_6)\,\ldots\,(-ie_{5+l})$ of the center of $\mathcal{U}(\mathfrak{m})$ and the differential operator $\widehat{W} = \eta_1\,\ldots\,\eta_l\,\widehat{W_0}$. Now Lemma 12.29 implies that the continuous function

$$\psi(\lambda, \eta) = \lambda^2\, \eta_1\, \ldots\, \eta_l\;\; (\, f_0 + Q(-\log\lambda, -\log\eta_1, \ldots, -\log\eta_l)\,)$$

on $\mathfrak{z}^*$ corresponds to $\widehat{W}$ in the sense that

$$\pi_{r,s}(\widehat{W} \,*\, H) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,)\; \pi_{r,s}(H)$$

for continuous functions H satisfying the differentiability and growth conditions of Lemma 12.29. As before we fix a Gauss function H_1 in the variables (t, v, w, Z) and a Schwartz function $K \in \mathcal{S}(\mathbb{R}^3)$ such that $K(1,0,0) \neq 0$ and $K(\xi_3, \xi_4, \lambda) = 0$ if $|\xi_3 - 1| \geq \frac{1}{2}$. If we define

$$H_2(\xi_3, \xi_4, \lambda, \eta) = \frac{1}{2\xi_3\lambda - \xi_4^2} \, K(\xi_3, \xi_4, \lambda) \, e^{-|\eta|^2/2}$$

and $H = H_1 \otimes H_2$ for $\lambda < 0$, then Lemma 12.30 implies that there exists a smooth function $b_0 \in L^1(M)$ such that $\widehat{b_0} = \widehat{W_0} * H$ for $\lambda < 0$. Clearly $b = D_{\eta_1} \ldots D_{\eta_l} b_0$ satisfies $\widehat{b} = \widehat{W} * H$ so that

$$\pi_{r,s}(b) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,) \, \pi_{r,s}(H) \, .$$

As in the proof of Proposition 12.26 one can show that $\rho(b) \neq 0$. Here one uses the fact that $g(e_{5+\nu}) \neq 0$ for $1 \leq \nu \leq l$ because g is admissible with respect to f. This proves assertion I.

It remains to be shown that there exists a smooth function $c \in L^1(M)$ such that

$$\pi_{r,s}(c) = \psi(\, f_{r,s} \,|\, \mathfrak{z}\,) \, \pi_{r,s}(H) \quad \text{and} \quad \rho(c) = 0 \, .$$

If c is in $L^1(M)$ such that $\widehat{c}$ furnishes a solution of the multiplier problem

$$\widehat{c}(t, v, w, Z, \xi_3, \xi_4, \lambda, \eta) = \psi(-\lambda, -\eta) \, H(t, v, w, Z, \xi_3, \xi_4, \lambda, \eta)$$

for all η and $\lambda < 0$, then c has the desired property. Since $H = H_1 \otimes H_2$ is a tensor product, we set $c = H_1 \otimes C$ and look for L^1-functions C on $\mathfrak{a}$ such that

$$\widehat{C}(\xi_3, \xi_4, \lambda, \eta) = \frac{\psi(-\lambda, -\eta)}{\xi_4^2 - 2\xi_3\lambda} \, K(\xi_3, \xi_4, \lambda) \, e^{-|\eta|^2/2}$$

for all η and $\lambda < 0$. The definition of ψ by the polynomial Q shows us that without loss of generality we can assume $\psi(\lambda, \eta) = \lambda^2 \log^j(\lambda) \, \psi_2(\eta)$ for $\lambda > 0$ where $j \geq 0$ is an integer and ψ_2 is defined on $\mathfrak{z}_0^*$ by

$$\psi_2(\eta) = \sum_{0 \leq j_1 + \ldots + j_l \leq q} a_{j_1, \ldots, j_l} \quad \eta_1 \, \ldots \, \eta_l \, \log^{j_1}(\eta_1) \, \ldots \, \log^{j_l}(\eta_l)$$

if $\eta_\nu > 0$ for all $1 \leq \nu \leq l$ and $\psi_2(\eta) = 0$ else. Since ψ_2 meets the assumptions of Lemma 5.13, it follows as in Theorem 5.17 that there exists a smooth L^1-function C_2 on $\mathfrak{z}_0$ such that

$$\widehat{C}_2(\eta) = \psi_2(-\eta) \, e^{-|\eta|^2/2} \, .$$

Furthermore it follows from Theorem 12.49 that there is a smooth L^1-function C_1 on the commutative ideal $\langle e_3, e_4, e_5 \rangle$ such that

$$\widehat{C}_1(\xi_3, \xi_4, \lambda) = \frac{\lambda^2 \log^j(-\lambda)}{\xi_4^2 - 2\xi_3\lambda} \, K(\xi_3, \xi_4, \lambda) \tag{12.33}$$

for all $\lambda < 0$. Now it is easy to see that $c = H_1 \otimes C_1 \otimes C_2$ solves our problem. This proves assertion II and finishes the proof of our proposition. □

Theorem 12.34. *Let $\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ be as in Section 12.1 such that $C^2\mathfrak{n} \not\subset \mathfrak{z}\mathfrak{g}$. Here $\mathfrak{n}$ is a trivial extension of the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,3}$. Further let $\mathfrak{m}$ be a proper, non-nilpotent ideal of $\mathfrak{g}$ with $\mathfrak{g} \supset \mathfrak{m} \supset \mathfrak{n}$. Let us assume that there exists a nilpotent subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ and $[\mathfrak{s}_c, \mathfrak{t}] = 0$ where $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$ and $\mathfrak{s}_c$ is the centralizer of $C^2\mathfrak{n}$ in $\mathfrak{s}$.*

Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$ and $g \in \mathfrak{m}^*$ be critical with respect to the orbit $\mathrm{Ad}^*(G)f$. Then it follows that

$$\bigcap_{(r,s)\in\mathbb{R}^{m+1}} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

holds for the unitary representations $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$ of M.

Proof. Let us choose a basis of the weights of the $\mathfrak{s}$-module $\mathfrak{m}$, and a coexponential basis for $\mathfrak{t}$ in $\mathfrak{s}$ as in the beginning of this subsection. Now we observe that the additional Assumptions 12.13 and 12.14 are always satisfied: First, the polynomial function Q depends only on the variable r. Hence for every $f \in \mathfrak{m}^*$ in general position, Assumption 12.13 holds with $m' = l = 0$. Any $g \in \mathfrak{m}^*$ is admissible with respect to f and Γ. Now an application of Lemma 12.23, Lemma 12.24 or Proposition 12.26 completes this proof. □

Though including semi-direct sums $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$, the condition $[\mathfrak{s}_c, \mathfrak{t}] = 0$ does not reach far beyond the case $\dim \mathfrak{g}/\mathfrak{m} = 1$, i.e., a one-parameter group $\mathrm{Ad}(\exp rd_0)$ acting on the stabilizer $\mathfrak{m}$, non-trivially on the central ideal $C^2\mathfrak{n}$.

If $\dim \mathfrak{g}/\mathfrak{m} > 1$ and $g \in \mathfrak{m}^*$ is critical for $\mathrm{Ad}^*(G)f$, but not admissible with respect to f and Γ, then we must admit that the situation remains somewhat mysterious.

Open Problem 12.35 The following 10-dimensional example gives a first impression of the phenomena which may occur in the case $\dim \mathfrak{g}/\mathfrak{m} > 1$.

We assume that $\mathfrak{g}$ is a 10-dimensional exponential solvable Lie algebra whose nilradical $\mathfrak{n}$ is 7-dimensional and satisfies 12.1 and 12.2. In particular $d = \dim \mathfrak{z}\mathfrak{n} = 3$. Let $d_0, d_1, e_0, \ldots, e_7$ be a basis of $\mathfrak{g}$ such that, in addition to the commutator relations given by 12.3 and 12.4, we have

$$[d_0, e_0] = -ae_7\ , \quad [d_0, e_3] = e_3\ , \quad [d_0, e_4] = e_4 \quad [d_0, e_5] = e_5\ ,$$

and

$$[d_1, e_0] = -be_7\ , \quad [d_1, e_6] = e_6\ .$$

There is a unique (and obvious) choice of a set of structure data Γ in this example.

Let $\mathfrak{m}$ be the ideal spanned by $e_0, \ldots, e_7$. If $f \in \mathfrak{m}^*$ is in general position, then we can assume $f_\nu = 0$ for $1 \le \nu \le 4$ and $f_\nu = 1$ for $5 \le \nu \le 7$. Note that f defines the polynomial function

$$Q(r, s) = ar + bs\ .$$

Let $g \in \mathfrak{m}^*$ be such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$, so that $g_5 \geq 0$ and $g_7 = 1$. We restrict ourselves to non-admissible g. Such g satisfy $g_6 = 0$ and exist only if $b \neq 0$, what we shall assume henceforth.

The following lemma contains a description of the non-admissible part of the closure of $\mathrm{Ad}^*(G)f$ in this example.

Lemma 12.36. *Let $f \in \mathfrak{m}^*$ be in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Assume that $g \in \mathfrak{m}^*$ is not admissible with respect to f and Γ, and such that $\mathrm{Ad}^*(G)g'$ is contained in the closure of $\mathrm{Ad}^*(G)f'$. Then $\mathrm{Ad}^*(G)g \subset (\,\mathrm{Ad}^*(G)f\,)^-$ if and only if one of the following conditions holds:*

(i) $g_5 = 0,\ g_4 \neq 0,$ *and* $g_2 = 0,$

(ii) $g_5 = 0,\ g_4 \neq 0,$ *and* $bg_2 > 0,$

(iii) $g_5 = g_4 = 0,$ *and* $bg_2 > 0,$

(iv) $g_5 = g_4 = g_2 = 0.$

Proof. We begin with a preliminary remark. It turns out to be useful to write

$$q_n = \frac{1}{b}\,(f_0 - v_n y_n - 2w_n x_n + a r_n)$$

if sequences $r_n \in \mathbb{R}$ and $X_n \in \mathfrak{n}$ are chosen. Then $\mathrm{Ad}^*(G)g \subset (\,\mathrm{Ad}^*(G)f\,)^-$ if and only if there exist sequences r_n and X_n such that $q_n \longrightarrow -\infty$ and

$$\begin{aligned} y_n &\longrightarrow g_1 \\ x_n &\longrightarrow g_2 \\ -e^{-r_n} w_n + \frac{1}{2} e^{-r_n} v_n^2 &\longrightarrow g_3 \\ -e^{-r_n} v_n &\longrightarrow g_4 \\ e^{-r_n} &\longrightarrow g_5\ . \end{aligned} \tag{12.37}$$

If $g \in (\,\mathrm{Ad}^*(G)f\,)^-$, then $b(q_n + s_n) \longrightarrow g_0$ and $s_n \longrightarrow +\infty$ because $g_6 = 0$. This shows $q_n \longrightarrow -\infty$. To prove the converse, it suffices to define $s_n = \frac{g_0}{b} - q_n$. Then $s_n \longrightarrow +\infty$ and $b(q_n + s_n) = g_0$.

Suppose $g_5 \neq 0$ and $g \in (\,\mathrm{Ad}^*(G)f\,)^-$. Then there exist sequences r_n, s_n, X_n such that 12.37 holds. But from $g_5 \neq 0$, it follows that the sequences r_n, v_n, w_n, x_n, and y_n are convergent. This contradicts $q_n \longrightarrow -\infty$. It results $\mathrm{Ad}^*(G)g \not\subset (\,\mathrm{Ad}^*(G)f\,)^-$.

Now we come to the proof of *(i)*. If $g_5 = g_2 = 0$ and $g_4 \neq 0$, then we can also assume $g_1 = g_3 = 0$. We define $v_n = -e^{r_n} g_4$, $y_n = 0$, $w_n = \frac{1}{2} e^{2r_n} g_4^2$, and

$$x_n = \frac{b}{g_4^2}\, e^{-r_n/2}\ .$$

Then we see that $e^{-\frac{3}{2} r_n}\, q_n \longrightarrow -1$. This proves $q_n \longrightarrow -\infty$ and thus $\mathrm{Ad}^*(G)g$ is contained in the closure of $\mathrm{Ad}^*(G)f$.

Next we prove *(ii)*. Let r_n, X_n be sequences such that 12.37 holds. The third and the fourth convergence of 12.37 yield

$$e^{-2r_n}\, w_n \longrightarrow \frac{1}{2} g_4^2$$

and hence

$$e^{-2r_n}\, q_n \longrightarrow -\frac{g_2 g_4^2}{b}\ .$$

This shows $q_n \longrightarrow -\infty$ if $bg_2 > 0$, and $q_n \longrightarrow +\infty$ if $bg_2 < 0$. The proof of *(ii)* is complete.

In order to prove *(iii)*, we assume $g_5 = g_4 = 0$. If r_n, X_n are chosen such that 12.37 holds, then $r_n \longrightarrow +\infty$ and $-e^{-r_n} v_n \longrightarrow 0$. This implies

$$\lim_{n \longrightarrow \infty} e^{-r_n}\, q_n = \lim_{n \longrightarrow \infty} -\frac{2}{b}\, e^{-r_n}\, w_n x_n = \lim_{n \longrightarrow \infty} \frac{2}{b} \left(g_3 - \frac{1}{2}\, e^{-r_n}\, v_n^2 \right) x_n$$

in the following sense: If one of these limits (possibly $\pm\infty$) exists, then all limits exist and are equal.

If $bg_2 < 0$, then we obtain

$$\lim_{n \longrightarrow \infty} \frac{2}{b} \left(g_3 - \frac{1}{2}\, e^{-r_n}\, v_n^2 \right) x_n \geq \frac{g_2 g_3}{b}\ .$$

Here we pass to a convergent subsequence if necessary. But this contradicts $q_n \longrightarrow -\infty$.

If $bg_2 > 0$, then we define $y_n = g_1$, $x_n = g_2$, $v_n = e^{\frac{2}{3} r_n}$, and

$$w_n = \frac{1}{2}\, e^{\frac{4}{3}\, r_n} - e^{r_n}\, g_3\ .$$

Obviously, 12.37 holds and $q_n \longrightarrow -\infty$. This finishes the proof of *(iii)*.

The last step is to prove *(iv)*. If $g_5 = g_4 = g_2 = 0$, then we define $y_n = g_1$, $x_n = \frac{b}{n}$, and v_n, w_n as in part *(iii)*, $bg_2 < 0$. Then it is easy to see $q_n \longrightarrow -\infty$. This finishes the proof of the lemma. □

We conclude this section with the following open question concerning this 10-dimensional example: Assume that $g \in \mathfrak{m}^*$ is not admissible with respect to f, and critical for the orbit $\mathrm{Ad}^*(G)f$, i.e., g does not satisfy any of the conditions of Lemma 12.36. In this case, is it true that

$$\bigcap_{r,s} \ker_{L^1(M)} \pi_{r,s} \not\subset \ker_{L^1(M)} \rho$$

holds for $\pi_{r,s} = \mathcal{K}(f_{r,s})$ and $\rho = \mathcal{K}(g)$? Once again, we stress that the results of Section 5.1 do not apply here.

12.4 Two multiplier theorems

The aim of this subsection is to complete the proof of Proposition 12.26 in Section 12.3. To this end we must find smooth L^1-functions h which solve the following multiplier problems: Let $k \in \mathcal{S}(\mathbb{R}^3)$ be a given Schwartz function such that $\widehat{k}(1,0,0) \neq 0$ and $\widehat{k}(\xi_3, \xi_4, \lambda) = 0$ if $|\xi_3 - 1| \geq 1/2$. Assume that the multiplier $\widehat{m}$ is defined for $\xi_3 > 0$ and $\lambda < 0$ by one of the following expressions:

$$\widehat{m}(\xi_3, \xi_4, \lambda) = \frac{-\lambda^2}{\xi_4^2 - 2\xi_3\lambda}$$
$$\widehat{m}(\xi_3, \xi_4, \lambda) = \frac{-\xi_4\lambda}{\xi_4^2 - 2\xi_3\lambda}$$
$$\widehat{m}(\xi_3, \xi_4, \lambda) = \frac{\lambda^2 \log^j(-\lambda)}{\xi_4^2 - 2\xi_3\lambda}$$

where $j \geq 0$ is an integer. Does there exist a smooth function $h \in L^1(\mathbb{R}^3)$ such that

$$\widehat{h}(\xi_3, \xi_4, \lambda) = \widehat{m}(\xi_3, \xi_4, \lambda)\, \widehat{k}(\xi_3, \xi_4, \lambda)\ .$$

for all ξ_3, ξ_4 and all $\lambda < 0$? The essential step in proving the existence of h is to solve the simplified multiplier problem given below. Starting from the original one, we assume that $\xi_3 = 1$ is constant, replace λ by $-\lambda/2$, and ignore multiplicative constants in the equality defining $\widehat{m}$. Then we are in the following situation: Let $g \in \mathcal{S}(\mathbb{R}^2)$ be a Schwartz function such that $\widehat{g}(0,0) \neq 0$. Assume that the multiplier $\widehat{m}$ is defined for $\lambda > 0$ by

$$\widehat{m}(\xi, \lambda) = \frac{\lambda^2 \log^j(\lambda)}{\xi^2 + \lambda} \quad \text{or} \quad \widehat{m}(\xi, \lambda) = \frac{\xi\lambda}{\xi^2 + \lambda}\ .$$

Does there exist a smooth function $f \in L^1(\mathbb{R}^2)$ such that $\widehat{f}(\xi, \lambda) = \widehat{m}(\xi, \lambda)\ \widehat{g}(\xi, \lambda)$ for all ξ and $\lambda > 0$? In Proposition 12.46 and 12.48 we answer this question in the affirmative. The proof of these propositions relies on

Proposition 12.38 (Hausdorff-Young trick)**.** *Let $1 < p \leq 2$ and S be a finite subset of $\mathbb{R}$. Assume that f in $\mathcal{C}_\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is continuously differentiable in $\mathbb{R}\backslash S$ such that its derivative $\partial_y f$ is in $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Then it follows $\widehat{f} \in L^1(\mathbb{R})$. Further there exists $C > 0$ such that*

$$|\, \widehat{f}\, |_1 \ \leq\ C\, (\, |\, f\, |_1 + |\, \partial_y f\, |_p\,)$$

for all these f.

Proof. We have to estimate the integral $\int_{-\infty}^{+\infty} |\widehat{f}(\xi)|\, d\xi$ for the Fourier transform $\widehat{f}$ which is a continuous and bounded function. Since

$$\int_{-1}^{1} |\widehat{f}(\xi)|\, d\xi \leq 2\, |\, \widehat{f}\, |_\infty \leq 2\, |\, f\, |_1\ ,$$

it suffices to estimate

$$\int_{1}^{+\infty} |\widehat{f}(\xi)|\, d\xi = \int_{1}^{+\infty} \frac{1}{\xi}\, |\xi \widehat{f}(\xi)|\, d\xi\ .$$

Partial integration yields

$$\xi \widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(y)\, \xi e^{-iy\xi}\, dy = -i \int_{-\infty}^{+\infty} (\partial_y f)(y)\, e^{-iy\xi}\, dy = (D_y f)\widehat{\ }(\xi)\ .$$

In this computation one must break up the integral into parts because of the non-differentiability of f in S. Note that the boundary values cancel out because f is continuous and vanishes at infinity. Now Hölder's inequality and the Hausdorff-Young theorem imply

$$\int_1^{\infty} |\widehat{f}(\xi)|\, d\xi \le K \left(\int_1^{+\infty} |(\partial_y f)\widehat{\ }(\xi)|^q\, d\xi \right)^{1/q} \le K\, |\, (\partial_y f)\widehat{\ }\,|_q \ \le\ K\, |\, \partial_y f\, |_p$$

where $K = \left(\int_1^{+\infty} \xi^{-p}\, d\xi \right)^{1/p} = (p-1)^{-1/p}$. The integral $\int_{-\infty}^{1} |\widehat{f}(\xi)|\, d\xi$ can be treated similarly. If we define $C := \max\{2, 2K\}$, then the assertion of this proposition becomes evident. □

We stress that this proposition treats only of functions in one real variable. Although one would expect this result to be contained in any textbook on Fourier analysis including the Hausdorff-Young theorem, I could not find a reference for it. Poguntke pointed out that trick to me. This proposition furnishes a practical sufficient criterion for functions f to be in the Fourier algebra $\mathcal{A}(\mathbb{R})$: If f is piecewise continuously differentiable such that the singularities of $\partial_\lambda f$ are L^p-integrable for some $1 < p \le 2$, then it follows $\widehat{f} \in L^1(\mathbb{R})$. Now the Fourier inversion theorem yields $f \in \mathcal{A}(\mathbb{R})$.

The next lemma is of a technical nature. Its purpose will become apparent in the proof of Proposition 12.41. The point of the proof of this lemma is to control the behavior of the integrand ω for $\lambda \longrightarrow 0$ and $|y| \longrightarrow +\infty$.

Lemma 12.39. *Assume that $r, s, k \ge 0$ are real such that $1 + 2r > k$. Let us consider the function*

$$\omega(y, \lambda) = \lambda^r\, |\log \lambda|^s\, |y|^k\, e^{-\lambda^{1/2}|y|}\, (1+\lambda)^{-(r+2)} \tag{12.40}$$

on $\mathbb{R} \times (0, +\infty)$. Then there exists an index $1 < p_0 \le 2$ depending only on r and k such that the mixed L^1-L^p-norm

$$|\, \omega\, |_{1,p} = \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} \omega(y, \lambda)^p\, d\lambda \right)^{1/p} dy$$

of ω is finite for all $1 \le p \le p_0$. If $1 + 2r = k$, then $|\, \omega\, |_{1,1}$ is not finite.

Proof. At first we observe that Fubini's theorem and the substitution $y \longrightarrow y/\lambda^{1/2}$ show that

$$\begin{aligned} |\, \omega\, |_{1,1} &= \int_{-\infty}^{+\infty} \int_0^{+\infty} \lambda^r\, |\log \lambda|^s\, |y|^k\, e^{-\lambda^{1/2}|y|}\, (1+\lambda)^{-(r+2)}\, d\lambda\, dy \\ &= \left(\int_0^{+\infty} \lambda^{r-k/2-1/2}\, |\log \lambda|^s\, (1+\lambda)^{-(r+2)}\, d\lambda \right) \left(\int_{-\infty}^{+\infty} |y|^k\, e^{-|y|}\, dy \right) \end{aligned}$$

is not finite if $1+2r=k$ because in this case $r-k/2-1/2=-1$ and the singularity of $1/\lambda$ in $\lambda=0$ is not integrable. Clearly $1+2r>k$ implies $|\,\omega\,|_{1,1}<+\infty$.

Now we explain how to choose the index p_0: Since $1+2r>k$ is equivalent to $k-2r+1<2$, we can choose $1<p_0\le 2$ such that $k-2r+1<2/p_0$. Furthermore we choose $0<\epsilon<1$ such that $k-2r+1<2(1-\epsilon)/p_0$. Now let $1\le p\le p_0$ be arbitrary. In order to prove this lemma, it suffices to verify that the integrals

$$I_1=\int\limits_{|y|\le 1}\left(\int\limits_0^{+\infty}\omega(y,\lambda)^p\,d\lambda\right)^{1/p}dy$$

and

$$I_2=\int\limits_{|y|\ge 1}\left(\int\limits_0^{+\infty}\omega(y,\lambda)^p\,d\lambda\right)^{1/p}dy$$

are both finite. For $|y|\le 1$ we get

$$\left(\int\limits_0^{+\infty}\omega(y,\lambda)^p\,d\lambda\right)^{1/p}\le\left(\int\limits_0^{+\infty}\frac{\lambda^{rp}\,|\log\lambda|^{sp}}{(1+\lambda)^{(r+2)p}}\,d\lambda\right)^{1/p}$$

where the integral on the right hand side does not depend on y. This integral is finite because the singularity of $|\log\lambda|^{sp}$ in $\lambda=0$ is integrable. If $r>0$, then the integrand is even a continuous function for all $\lambda\ge 0$. This shows us that the integral I_1 is finite.

Next we assume $|y|\ge 1$. We have to estimate the inner integral

$$\int\limits_0^{+\infty}\omega(y,\lambda)^p\,d\lambda=\int\limits_0^{+\infty}|y|^{kp}\;\frac{\lambda^{rp}\,|\log\lambda|^{sp}}{(1+\lambda)^{(r+2)p}}\;e^{-p\lambda^{1/2}|y|}\,d\lambda\,.$$

To this end we substitute λ by $\lambda/|y|^2$. If we choose $A>0$ such that

$$\frac{\lambda^{\epsilon/p}\,|\log\lambda|^s}{(1+\lambda)^{r+2}}\le A$$

for all $\lambda\ge 0$, then we can estimate the new integrand

$$\frac{1}{|y|^2}\,|y|^{kp}\;\frac{(\lambda/|y|^2)^{rp}\,|\log(\lambda/|y|^2)\,|^{sp}}{(1+\lambda/|y|^2)^{(r+2)p}}\;e^{-p\lambda^{1/2}}\;\le\;A^p\,|y|^{kp-2rp-2+2\epsilon}\,\lambda^{rp-\epsilon}\,e^{-p\lambda^{1/2}}$$

so that

$$\left(\int\limits_0^{+\infty}\omega(y,\lambda)^p\,d\lambda\right)^{1/p}\le A\,|y|^{k-2r-2(1-\epsilon)/p}\left(\int\limits_0^{+\infty}\lambda^{rp-\epsilon}\,e^{-p\lambda^{1/2}}\,d\lambda\right)^{1/p}.$$

Note that the integral over λ on the right hand side is finite because $rp-\epsilon>-1$. Integration of both sides over $y\in[1,+\infty)$ shows us that the integral I_2 is finite because $k-2r-2(1-\epsilon)/p<-1$. This finishes the proof of our lemma. □

The next proposition will be used many times in the rest of this subsection. The boundaries of the integral defining Φ will often be equal to $a(y) = -\infty$ and $b(y) = +\infty$.

Proposition 12.41. *Let $\vartheta \in L^1(\mathbb{R})$ and $a, b : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be continuous functions. Assume that φ is a continuous function on $\mathbb{R}^2 \times (0, +\infty)$ such that*

$$|\,\varphi(y,t,\lambda)\,| \leq B\,\omega(y,\lambda)\,|\vartheta(t)| \tag{12.42}$$

holds for all $(y,t,\lambda) \in \mathbb{R}^2 \times (0,+\infty)$. Here $B > 0$ and ω is given by Equation 12.40 for $r, s, k \geq 0$ such that $1 + 2r > k$ as in Lemma 12.39. Then

$$\Phi(y,\lambda) = \int_{a(y)}^{b(y)} \varphi(y-t,t,\lambda)\,dt$$

is a well-defined continuous function on $\mathbb{R} \times (0,+\infty)$ such that the mixed L^1-L^p-norm

$$|\,\Phi\,|_{1,p} \;=\; \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} |\,\Phi(y,\lambda)\,|^p\,d\lambda \right)^{1/p} dy \;\leq\; B\,|\,\omega\,|_{1,p}\,|\,\vartheta\,|_1 \tag{12.43}$$

is finite for all $1 \leq p \leq p_0$ if we choose $1 < p_0 \leq 2$ as in Lemma 12.39.

Proof. If $\lambda > 0$ is fixed, then there exists some $C(\lambda) > 0$ such that $\omega(y,\lambda) \leq C(\lambda)$. Equation 12.42 implies $|\,\varphi(y,t,\lambda)\,| \leq B\,C(\lambda)\,|\vartheta(t)|$ so that the integral defining Φ exists. Further Lebesgue's theorem on dominated convergence shows us that Φ is continuous. We observe that

$$|\Phi(y,\lambda)| \leq \int_{-\infty}^{+\infty} |\varphi(y-t,t,\lambda)|\,dt\,.$$

Applying the Minkowski inequality for integrals (see e.g. p. 194 of [12]), we obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left(\int_0^{+\infty} |\Phi(y,\lambda)|^p\,d\lambda \right)^{1/p} dy &\leq \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} |\varphi(y-t,t,\lambda)|\,dt \right)^p d\lambda \right)^{1/p} dy \\
&\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} |\,\varphi(y-t,t,\lambda)\,|^p\,d\lambda \right)^{1/p} dt\,dy \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} |\,\varphi(y,t,\lambda)\,|^p\,d\lambda \right)^{1/p} dy\,dt
\end{aligned}$$

The last equality is a consequence of Fubini's theorem and the simple substitution $y \longrightarrow y + t$. Now Equation 12.42 yields

$$\left(\int_0^{+\infty} |\,\varphi(y,t,\lambda)\,|^p\,d\lambda \right)^{1/p} \;\leq\; B\,\left(\int_0^{+\infty} \omega(y,\lambda)^p\,d\lambda \right)^{1/p} |\vartheta(t)|$$

and our claim becomes obvious by integration over y and t. $\square$

Remark 12.44. Let $r, s, k \geq 0$ be real. If $\tilde{g} \in \mathcal{S}(\mathbb{R}^2)$ is a Schwartz function, then there exists some $B > 0$ such that

$$|\,\tilde{g}(t,\lambda)\,| \;\leq\; B\,(1+|\lambda|)^{-(r+2)}\,(1+|t|^2)^{-1}$$

for all $(t,\lambda) \in \mathbb{R}^2$. This implies that the function

$$\varphi(y,t,\lambda) = \lambda^r\,\log^s(\lambda)\,|y|^k\,e^{-\lambda^{1/2}\,|y|}\,\tilde{g}(t,\lambda)$$

for $\lambda > 0$ satisfies the assumptions of Proposition 12.41.

Remark 12.45. It is well-known that for any $a > 0$ the Fourier transform of the L^1-function $m(y) = 1/(2a)\,e^{-a|y|}$ can be computed explicitly: $\widehat{m}(\xi) = 1/(\xi^2 + a^2)$.

Now we come to the solution of the first simplified multiplier problem.

Proposition 12.46. *Let $j \geq 0$ be an integer and $g \in \mathcal{S}(\mathbb{R}^2)$ an arbitrary Schwartz function. Then there exists a smooth function $f \in L^1(\mathbb{R}^2)$ such that*

$$\widehat{f}(\xi,\lambda) \;=\; \frac{\lambda^2\,\log^j(\lambda)}{\xi^2+\lambda}\quad\widehat{g}(\xi,\lambda) \tag{12.47}$$

for all ξ and $\lambda > 0$. Furthermore all derivatives $\partial_y^\alpha \partial_z^\beta f$ are in $L^1(\mathbb{R}^2)$.

Proof. First of all we observe that $|\widehat{m}(\xi,\lambda)| \leq \lambda\,|\log\lambda|^j$ for $\lambda > 0$ so that in particular $\widehat{m}(\xi,\lambda) \longrightarrow 0$ for $(\xi,\lambda) \longrightarrow (\xi_0, 0)$. Thus the definition $\widehat{m}(\xi,\lambda) = 0$ for $\lambda \leq 0$ extends $\widehat{m}$ to a continuous function on $\mathbb{R}^2$. Our aim is to show that the inverse Fourier transform f of the function $\widehat{f} = \widehat{m}\,\widehat{g}$, given by Equation 12.47 for $\lambda > 0$, is in $L^1(\mathbb{R}^2)$.

As a first step, let $\tilde{f}$, $\tilde{m}$, and $\tilde{g}$ denote the inverse Fourier transform w. r. t. the variable ξ of the functions $\widehat{f}$, $\widehat{m}$, and $\widehat{g}$ respectively. Note that

$$\tilde{g}(y,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{g}(\xi,\lambda)\,e^{i\xi y}\,d\xi$$

is a Schwartz function. For $\lambda > 0$ fixed, the function $\xi \mapsto \widehat{m}(\xi,\lambda)$ is in $L^1(\mathbb{R})$ and Remark 12.45 implies

$$\tilde{m}(y,\lambda) = \frac{1}{2}\,\lambda^{3/2}\,\log^j(\lambda)\,e^{-\lambda^{1/2}\,|y|}\,.$$

Further we know that $\tilde{f}(-,\lambda)$ is the convolution product of $\tilde{m}(-,\lambda)$ and $\tilde{g}(-,\lambda)$ which means $\tilde{f}(y,\lambda) = 0$ for $\lambda < 0$ and

$$\tilde{f}(y,\lambda) = \int_{-\infty}^{+\infty} \frac{1}{2}\,\lambda^{3/2}\,\log^j(\lambda)\;e^{-\lambda^{1/2}\,|y-t|}\,\tilde{g}(t,\lambda)\,dt\,.$$

for $\lambda > 0$. Clearly $\tilde{f}$ is continuous on $\mathbb{R}^2$ and $\tilde{f}(y,-)$ is in $\mathcal{C}_\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ for all y. Finally we define f as the inverse Fourier transform of $\tilde{f}$ w. r. t. the variable λ. Since f

is also the Fourier transform of the L^1-function $\widehat{f}$, it is clear that f is continuous and bounded. In order to show $f \in L^1(\mathbb{R}^2)$ we must prove that the integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y,z)|\; dz\; dy$$

is finite. Let us fix the variable y for the moment. Let $1 < p \leq 2$ be arbitrary. The value of p will be specified later. It is easy to see that $(\partial_\lambda f)(y,-)$ is in $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ because

$$(\partial_\lambda f)(y,\lambda) = \int_{-\infty}^{+\infty} c(y-t,\lambda)\; e^{-\lambda^{1/2}\,|y-t|}\; \tilde{g}(t,\lambda)\; dt$$

$$+ \int_{-\infty}^{+\infty} \frac{1}{2}\; \lambda^{3/2}\; \log^j(\lambda)\; e^{-\lambda^{1/2}\,|y-t|}\; (\partial_\lambda \tilde{g})(t,\lambda)\; dt$$

for $\lambda > 0$ with

$$c(y,\lambda) = \frac{3}{4}\;\lambda^{1/2}\;\log^j(\lambda) + \frac{j}{2}\;\lambda^{1/2}\;\log^{j-1}(\lambda) - \frac{1}{4}\;\lambda\;\log^j(\lambda)\;|y|\;.$$

The assumptions of Proposition 12.38 being satisfied, we conclude

$$\int_{-\infty}^{+\infty} |f(y,z)|\; dz \leq C \int_{0}^{+\infty} |\tilde{f}(y,\lambda)|\; d\lambda + C\left(\int_{0}^{+\infty} |(\partial_\lambda \tilde{f})(y,\lambda)|^p\; d\lambda\right)^{1/p}$$

and thus

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y,z)|\; dz\; dy \;\leq\; C|\,\tilde{f}\,|_{1,1} + C|\,\partial_\lambda \tilde{f}\,|_{1,p}$$

by integration over y. We will prove that the L^1-norm and the mixed L^1-L^p-norm on the right hand side are finite. By the triangle inequality of the L^1-L^p-norm we can treat each summand of $\partial_\lambda \tilde{f}$ separately. If we expand the formula for $\partial_\lambda \tilde{f}$ by linearity of the integral, then up to a multiplicative constant the function $\tilde{f}$ and the summands of $\partial_\lambda \tilde{f}$ have the form

$$\Phi(y,\lambda) = \int_{-\infty}^{+\infty} \varphi(y-t,t,\lambda)\; dt$$

with

$$\varphi(y,t,\lambda) = \lambda^r\;\log^s(\lambda)\;|y|^k\; e^{-\lambda^{1/2}\,|y|}\;(\partial_\lambda^\gamma \tilde{g})(t,\lambda)$$

where $k,\gamma,s \geq 0$ are integers and $r \geq 0$ is real such that $r \geq k/2$. In particular $2r+1 > k$. Furthermore we observe that $r > 0$ whenever $s > 0$. From Remark 12.44 we deduce that φ satisfies all assumptions of Proposition 12.41 so that there exists an index $1 < p_0 \leq 2$ such that $|\,\Phi\,|_{1,p} < +\infty$ for all $1 \leq p \leq p_0$. If we choose p to be the minimum of all the p_0's that we have chosen for $\Phi = \tilde{f}$ and for the summands Φ of $\partial_\lambda \tilde{f}$, then we see that the integrals

$$\int_{-\infty}^{+\infty} |\tilde{f}(y,\lambda)|\; d\lambda\; dy \quad \text{and} \quad \int_{-\infty}^{+\infty}\left(\int_{0}^{+\infty} |(\partial_\lambda \tilde{f})(y,\lambda)|^p\; d\lambda\right)^{1/p} dy$$

are both finite. This proves $f \in L^1(\mathbb{R}^2)$.

So far we have established the first part of our proposition: If $g \in \mathcal{S}(\mathbb{R}^2)$ is a given Schwartz function, then there exists a function $f \in L^1(\mathbb{R}^2)$ such that $\widehat{f}(\xi,\lambda) = \widehat{m}(\xi,\lambda)\widehat{g}(\xi,\lambda)$ for all λ,ξ. If $\alpha,\beta \geq 0$ are integers, then the preceding considerations imply that there exist solutions $f_{\alpha,\beta} \in L^1(\mathbb{R}^2)$ of the Fourier multiplier problems

$$\widehat{f}_{\alpha,\beta}(\xi,\lambda) = \widehat{m}(\xi,\lambda)\,\xi^\alpha\lambda^\beta\,\widehat{g}(\xi,\lambda)$$

for the Schwartz functions $\xi^\alpha\lambda^\beta\,\widehat{g}$. Since the $\widehat{f}_{\alpha,\beta}$ are in $L^1(\mathbb{R}^2)$, it follows inductively by Fourier inversion that the functions $f_{\alpha,\beta}$ are the derivatives $D_y^\alpha D_z^\beta f$ of f. This completes the proof of our proposition. □

The solution of the second simplified multiplier problem is very similar to that of the first one.

Proposition 12.48. *Let $g \in \mathcal{S}(\mathbb{R}^2)$ be a given Schwartz function. Then there exists a smooth function $f \in L^1(\mathbb{R}^2)$ such that*

$$\widehat{f}(\xi,\lambda) \;=\; \frac{\lambda\,\xi}{\xi^2+\lambda} \quad \widehat{g}(\xi,\lambda)$$

for all ξ and $\lambda > 0$, and such that all derivatives $\partial_y^\alpha\partial_z^\beta f$ are in $L^1(\mathbb{R}^2)$.

Proof. First of all we observe that $|\widehat{m}(\xi,\lambda)| \leq |\xi|$ for $\lambda > 0$ so that in particular $\widehat{m}(\xi,\lambda) \longrightarrow 0$ for $(\xi,\lambda) \longrightarrow (\xi_0,0)$. Thus the definition $\widehat{m}(\xi,\lambda) = 0$ for $\lambda \leq 0$ extends $\widehat{m}$ to a continuous function on $\mathbb{R}^2$. Our aim is to show that the inverse Fourier transform f of the function $\widehat{f}$ defined by the equality $\widehat{f} = \widehat{m}\,\widehat{g}$ is in $L^1(\mathbb{R}^2)$. If we define

$$\widehat{m}_0(\xi,\lambda) = \frac{\lambda}{\xi^2+\lambda}$$

for $\lambda > 0$ and $\widehat{m}_0(\xi,\lambda) = 0$ for $\lambda \leq 0$, then $\widehat{m}_0$ is a continuous function on $\mathbb{R}^2\backslash\{(0,0)\}$ such that $\widehat{m}(\xi,\lambda) = \widehat{m}_0(\xi,\lambda)\cdot\xi$. Let $\tilde{f}$, $\tilde{m}$, and $\tilde{g}$ denote the inverse Fourier transform w. r. t. the variable ξ of the functions $\widehat{f}$, $\widehat{m}$, and $\widehat{g}$ respectively. Note that $\tilde{g}$ is again a Schwartz function. Remark 12.45 implies

$$\tilde{m}_0(y,\lambda) = \frac{1}{2}\,\lambda^{1/2}\,e^{-\lambda^{1/2}\,|y|}\,.$$

Further we know that $\tilde{f}(-,\lambda)$ is the convolution product of $\tilde{m}_0(-,\lambda)$ and $(D_t\tilde{g})(-,\lambda)$ which means

$$\begin{aligned}
\tilde{f}(y,\lambda) &= \int_{-\infty}^{+\infty} \frac{1}{2}\,\lambda^{1/2}\,e^{-\lambda^{1/2}\,|y-t|}\,(D_t\tilde{g})(t,\lambda)\,dt \\
&= \int_{y}^{+\infty} \frac{1}{2}\,\lambda\,e^{-\lambda^{1/2}\,|y-t|}\,\tilde{g}(t,\lambda)\,dt - \int_{-\infty}^{y} \frac{1}{2}\,\lambda\,e^{-\lambda^{1/2}\,|y-t|}\,\tilde{g}(t,\lambda)\,dt
\end{aligned}$$

The last equality is a consequence of partial integration. There are no boundary values because $t \mapsto 1/2\ \lambda^{1/2}\ e^{-\lambda^{1/2}|y-t|}$ is a continuous function vanishing at infinity. The different signs of these integrals correspond to the fact that the derivative of $t \mapsto |t|$ is $t \mapsto \operatorname{sgn}(t) = t/|t|$ in the sense of distributions. Differentiation under the integral sign yields

$$(\partial_\lambda \tilde{f})(y,\lambda) = \int_y^{+\infty} c(y-t,\lambda)\, e^{-\lambda^{1/2}\,|y-t|}\, \tilde{g}(t,\lambda)\, dt + \int_y^{+\infty} \frac{1}{2}\, e^{-\lambda^{1/2}\,|y-t|}\, (\partial_\lambda \tilde{g})(t,\lambda)\, dt$$
$$- \int_{-\infty}^{y} c(y-t,\lambda)\, e^{-\lambda^{1/2}\,|y-t|}\, \tilde{g}(t,\lambda)\, dt - \int_{-\infty}^{y} \frac{1}{2}\, e^{-\lambda^{1/2}\,|y-t|}\, (\partial_\lambda \tilde{g})(t,\lambda)\, dt$$

where

$$c(y,\lambda) = \frac{1}{2} - \frac{1}{4}\lambda^{1/2}\,|y|\ .$$

In order to show that the inverse Fourier transform f of $\tilde{f}$ w. r. t. the variable λ is in $L^1(\mathbb{R}^2)$ we must prove that the double integral $\int\int |f(y,z)|\ dy\, dz$ is finite. Let $1 < p \leq 2$ be arbitrary. Keeping y fixed, we observe that $\tilde{f}(y,-)$ is in $\mathcal{C}_\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $(\partial_\lambda \tilde{f})(y,-)$ is in $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. The assumptions of Proposition 12.38 being satisfied, we conclude

$$\int_{-\infty}^{+\infty} |f(y,z)|\ dz \leq C \int_0^{+\infty} |\tilde{f}(y,\lambda)|\ d\lambda + C \left(\int_0^{+\infty} |(\partial_\lambda \tilde{f})(y,\lambda)|^p\ d\lambda \right)^{1/p}$$

and thus

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(y,z)|\ dz\ dy\ \leq\ C|\,\tilde{f}\,|_{1,1} + C|\,\partial_\lambda \tilde{f}\,|_{1,p}\ .$$

It remains to be shown that these two norms are finite. To this end we can treat $\tilde{f}$ and each summand of $\partial_\lambda \tilde{f}$ separately. Up to multiplicative constants these summands have the form

$$\Phi(y,\lambda) = \int_y^{+\infty} \varphi(y-t,t,\lambda)\ dt$$

or

$$\Phi(y,\lambda) = \int_{-\infty}^{y} \varphi(y-t,t,\lambda)\ dt$$

with

$$\varphi(y,t,\lambda) = \lambda^r\ |y|^k\ e^{-\lambda^{1/2}\,|y|}\ (\partial_\lambda^\gamma \tilde{g})(t,\lambda)$$

where $k, \gamma \geq 0$ are integers and $r \geq 0$ is real such that $r \geq k/2$. In particular $1+2r > k$. Clearly φ satisfies the assumptions of Proposition 12.41. This time either $a(y) = y$ and $b(y) = +\infty$ or $a(y) = -\infty$ and $b(y) = y$. If we choose $1 < p \leq 2$ to be the minimum of all the p_0's that we get for $\Phi = \tilde{f}$ and for the summands Φ of $\partial_\lambda \tilde{f}$, then it follows $|\,\tilde{f}\,|_{1,1} < +\infty$ and $|\,\partial_\lambda \tilde{f}\,|_{1,p} < +\infty$ which implies $f \in L^1(\mathbb{R}^2)$. As in the proof of Proposition 12.46 it follows that f is smooth and all its derivatives are in $L^1(\mathbb{R}^2)$. □

After these preparations it is now easy to solve the original multiplier problems in three variables (ξ_3, ξ_4, λ). A reformulation of the first and the third one is given by

Theorem 12.49. *Let $j \geq 0$ be an integer and $g \in \mathcal{S}(\mathbb{R}^3)$ a Schwartz function such that $\widehat{g}(\xi_3, \xi_4, \lambda) = 0$ whenever $|\xi_3 - 2| \geq 1$. Then there exists a smooth function $f \in L^1(\mathbb{R}^3)$ such that*

$$\widehat{f}(\xi_3, \xi_4, \lambda) \;=\; \frac{\lambda^2 \log^j(\lambda)}{\xi_4^2 + \xi_3 \lambda} \quad \widehat{g}(\xi_3, \xi_4, \lambda)$$

for all $(\xi_3, \xi_4) \in \mathbb{R}^2$ and $\lambda > 0$, and such that all derivatives of f are in $L^1(\mathbb{R}^3)$.

Proof. Note that $\widehat{m}(\xi_3, \xi_4, \lambda) = \lambda^2 \log^j(\lambda) \,/\, (\xi_4^2 + \xi_3 \lambda)$ for $\lambda > 0$ and $\widehat{m}(\xi_3, \xi_4, \lambda) = 0$ for $\lambda \leq 0$ defines a continuous multiplier $\widehat{m}$ on $(0, +\infty) \times \mathbb{R}^2$. If $\tilde{f}$ denotes the inverse Fourier transform of $\widehat{f} = \widehat{m}\,\widehat{g}$ w. r. t. the variable ξ_4, then

$$\tilde{f}(\xi_3, y, \lambda) = \int_{-\infty}^{+\infty} \frac{1}{2}\, \xi_3^{-1/2}\, \lambda^{3/2}\, \log^j(\lambda)\, e^{-(\xi_3 \lambda)^{1/2}\,|y-t|}\, \tilde{g}(\xi_3, t, \lambda)\, dt$$

for $\lambda > 0$. In particular $\tilde{f}(\xi_3, \xi_4, \lambda) = 0$ if $|\xi_3 - 2| \geq 1$. Furthermore let $\bar{f}$ denote the inverse Fourier transform of $\tilde{f}$ w. r. t. the variable λ, and f the inverse Fourier transform of $\bar{f}$ w. r. t. the variable ξ_3. Clearly f is a continuous and bounded function. Our aim is to show that the integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x, y, z)|\, dz\, dy\, dx$$

is finite. To this end, it suffices to prove that there exists a constant $D > 0$ such that

$$(1 + x^2) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x, y, z)|\, dz\, dy \;\leq\; D$$

for all $x \in \mathbb{R}$. In other words, it suffices to prove that for for $0 \leq \alpha \leq 2$ the function $x \mapsto \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x^\alpha f(x, y, z)|\, dz\, dy$ is bounded. Since $x^\alpha f(-, y, z)$ is the inverse Fourier transform of $D_{\xi_3}^{\alpha} \bar{f}(-, y, z)$, it follows

$$|\, x^\alpha\, f(x, y, z)\,| \;\leq\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \bar{f})(\xi_3, y, z)\,|\, d\xi_3$$

so that

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\, x^\alpha f(x, y, z)\,|\, dz\, dy \;&\leq\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \bar{f})(\xi_3, y, z)\,|\, dy\, dz \right) d\xi_3 \\
&\leq\; \frac{1}{\pi} \sup_{1 \leq \xi_3 \leq 3} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \bar{f})(\xi_3, y, z)\,|\, dy\, dz
\end{aligned}$$

because $\bar{f}(\xi_3, y, z) = 0$ for all y, z whenever $|\xi_3 - 2| \geq 1$. Let $1 < p \leq 2$ be arbitrary. Since differentiation in direction of ξ_3 commutes with Fourier transformation w. r. t. to the variable ξ_4, it follows that $\partial_{\xi_3}^\alpha \bar{f}$ is the inverse Fourier transform of $\partial_{\xi_3}^\alpha \tilde{f}$. Hence Proposition 12.38 yields

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\,(\partial_{\xi_3}^\alpha \bar{f})(\xi_3,y,z)\,|\,dz\,dy \;\le\; C\int_{-\infty}^{+\infty}\int_{0}^{+\infty} |\,(\partial_{\xi_3}^\alpha \tilde{f})(\xi_3,y,\lambda)\,|\,d\lambda\,dy$$
$$+\,C\int_{-\infty}^{+\infty}\left(\int_{0}^{+\infty} |\,(\partial_{\xi_3}^\alpha \partial_\lambda \tilde{f})(\xi_3,y,\lambda)\,|^p\,d\lambda\right)^{1/p} dy\;.$$

It remains to be shown that the L^1-norm $|\,(\partial_{\xi_3}^\alpha \tilde{f})(\xi_3,-,-)\,|_{1,1}$ and the mixed L^1-L^p-norm $|\,(\partial_{\xi_3}^\alpha \partial_\lambda \tilde{f})(\xi_3,-,-)\,|_{1,p}$ are bounded as functions of ξ_3. Again it suffices to do this for each summand because $\partial_{\xi_3}^\alpha \tilde{f}$ and $\partial_{\xi_3}^\alpha \partial_\lambda \tilde{f}$ are linear combinations of functions Φ of the form

$$\Phi(\xi_3,y,\lambda) = \int_{-\infty}^{+\infty} \varphi(\xi_3,y-t,t,\lambda)\,dt$$

with

$$\varphi(\xi_3,y,t,\lambda) = \xi_3^q\;\lambda^r\;\log^s(\lambda)\;|y|^k\;e^{-(\xi_3\lambda)^{1/2}\,|y|}\;(\partial_{\xi_3}^\beta \partial_\lambda^\gamma \tilde{g})(\xi_3,t,\lambda)$$

where $s,k,\beta,\gamma \ge 0$ are integers, $q \in \mathbb{R}$, and $r \ge 0$ is real such that the crucial inequality $r \ge k/2$ is satisfied. Note that differentiation with respect to ξ_3 respects this condition whereas r and k do not satisfy this inequality for derivatives of φ with respect to λ of order ≥ 2. The point is that differentiating $e^{-(\xi_3\lambda)^{1/2}\,|y|}$ with respect to ξ_3 gives a factor $\lambda^{1/2}|y|$.

Since $\partial_{\xi_3}^\beta \partial_\lambda^\gamma \tilde{g}$ is a Schwartz function, there exists some $B_0 > 0$ such that

$$|\,(\partial_{\xi_3}^\beta \partial_\lambda^\gamma \tilde{g})(\xi_3,t,\lambda)\,| \;\le\; B_0\;(1+|\lambda|)^{-(r+2)}\;(1+|t|^2)^{-1}$$

for all $(\xi_3,t,\lambda) \in \mathbb{R}^3$. Furthermore $\varphi(\xi_3,y,t,\lambda) = 0$ for $|\xi_3 - 2| \ge 1$ and

$$e^{-(\xi_3\lambda)^{1/2}\,|y|} \le e^{-\lambda^{1/2}\,|y|}$$

for $1 \le \xi_3 \le 3$ implies

$$|\,\varphi(\xi_3,y,t,\lambda)\,| \;\le\; 3^{|q|}\,B_0\;w(y,t)\;\vartheta(t)$$

for all $(\xi_3,y,t,\lambda) \in \mathbb{R}^3 \times (0,+\infty)$ where ω is defined by Equation 12.40 depending on $r,s,k \ge 0$ and $\vartheta(t) = (1+|t|^2)^{-1}$. We observe that $B = 3^{|q|}\,B_0$, ω, and ϑ do not depend on ξ_3. If we apply Proposition 12.41 to the function $\varphi(\xi_3,-,-,-)$, then we obtain an index $1 < p_0 \le 2$ depending only on r and k such that

$$|\,\Phi(\xi_3,-,-)\,|_{1,p} \;\le\; B\,|\,\omega\,|_{1,p}\,|\,\vartheta\,|_1$$

holds for all $1 \le p \le p_0$. We emphasize that the right hand side is finite and does not depend on ξ_3. If we choose p to be the minimum of all the p_0's that we obtain for the summands of $\partial_{\xi_3}^\alpha \tilde{f}$ and $\partial_{\xi_3}^\alpha \partial_\lambda \tilde{f}$, then we see that their mixed L^1-L^p-norms are bounded as functions of ξ_3. Consequently $f \in L^1(\mathbb{R}^3)$ which proves our theorem. $\square$

In the last theorem of this section we consider the second original multiplier problem in the three variables (ξ_3, ξ_4, λ). More precisely, we treat the multiplier problem which we obtain from the original one by the change of variables $\lambda \longrightarrow -\lambda$ and $\xi_3 \longrightarrow 2\xi_3$.

Theorem 12.50. *Let $g \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz function such that $\widehat{g}(\xi_3, \xi_4, \lambda) = 0$ whenever $|\xi_3 - 2| \geq 1$. Then there exists a smooth function $f \in L^1(\mathbb{R}^3)$ such that*

$$\widehat{f}(\xi_3, \xi_4, \lambda) \;=\; \frac{\lambda \xi_4}{\xi_4^2 + \xi_3 \lambda} \quad \widehat{g}(\xi_3, \xi_4, \lambda)$$

for all $(\xi_3, \xi_4) \in \mathbb{R}^2$ and $\lambda > 0$, and such that all derivatives of f are in $L^1(\mathbb{R}^3)$.

Proof. Just like the proof of Theorem 12.49, the proof of this theorem consists of the following steps: First we compute the function $\tilde{f}$ and its derivatives $\partial_{\xi_3}^{\beta} \partial_\lambda \tilde{f}$ as in the proof of Proposition 12.48. Then we prove the existence of an upper bound for the right hand side of the inequality

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \bar{f})(\xi_3, y, z)\,|\; dz\; dy \;\leq\; & C \int_{-\infty}^{+\infty}\int_{0}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \tilde{f})(\xi_3, y, \lambda)\,|\; d\lambda\; dy \\ & + C \int_{-\infty}^{+\infty} \left(\int_{0}^{+\infty} |\,(\partial_{\xi_3}^{\alpha} \partial_\lambda \tilde{f})(\xi_3, y, \lambda)\,|^p\; d\lambda \right)^{1/p} dy\;.\end{aligned}$$

considered as a function of ξ_3. From this we conclude $f \in L^1(\mathbb{R}^3)$. We omit the details of this proof. $\square$

13 Nilradical is the algebra $\mathfrak{g}_{5,4}$

In this section we study the representation theory of an exponential solvable Lie group G such that the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{g}$ is a central extension of the five-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,4}$. It turns out that the primitive ideals $\ker_{C^*(G)} \pi$ for locally faithful, irreducible representations $\pi = \mathcal{K}(f)$ of G are induced from the nilpotent normal stabilizer subgroup M of G whose Lie algebra is given by $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. Recall that a representation π is called locally faithful if π is non-trivial on any non-trivial, connected, normal subgroup. Clearly $\pi = \mathcal{K}(f)$ is locally faithful if and only if f is in general position.

Let $\mathfrak{g}$ be an exponential solvable Lie algebra which contains a nilpotent ideal $\mathfrak{n}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Assume that

$$\mathfrak{n} \underset{2}{\supset} C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n} \underset{d-1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{2}{\supset} \{0\} \tag{13.1}$$

is a descending series of characteristic ideals of $\mathfrak{n}$ such that the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$ is equal to $C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. In particular $C^1\mathfrak{n}$ is commutative. Note that our assumptions include the case of the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,4}$ if $d = 1$.

Let $f \in \mathfrak{g}^*$ be in general position. We define the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. Since f vanishes on $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$, this ideal of $\mathfrak{g}$ must be zero. This proves $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

Again we fix a nilpotent subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$. Let us define $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$. We regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules and benefit from the existence of weight space decompositions of these modules. First we assume that there exists one weight $\alpha \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + C^1\mathfrak{n}$. Then we obtain the following commutator relations:

$$\begin{aligned} C^1\mathfrak{n} &= [\mathfrak{n}_\alpha + C^1\mathfrak{n}\,,\, \mathfrak{n}_\alpha + C^1\mathfrak{n}] = (\mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}\,, \\ C^2\mathfrak{n} &= [\mathfrak{n}_\alpha + C^1\mathfrak{n}\,,\, (\mathfrak{n}_{2\alpha} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}] = \mathfrak{n}_{3\alpha} \cap C^2\mathfrak{n}\,. \end{aligned}$$

Since $\mathfrak{n}_{3\alpha} \cap C^2\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$, it follows $\tilde{\alpha} = 0$. Here the tilde indicates restriction to $\mathfrak{t}$. Consequently $\mathfrak{m}$ is nilpotent in this case.

Now we assume that there exist two distinct weights $\alpha, \beta \in \mathfrak{s}^*$ such that $\mathfrak{n}_\alpha \not\subset C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$ and $\mathfrak{n}_\beta \not\subset C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. In particular $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{n}_\beta + C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. Here we obtain:

$$\begin{aligned} C^1\mathfrak{n} &= [\mathfrak{n}_\alpha + \mathfrak{n}_\beta + C^1\mathfrak{n}\,,\, \mathfrak{n}_\alpha + \mathfrak{n}_\beta + C^1\mathfrak{n}] = (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}\,, \\ C^2\mathfrak{n} &= [\mathfrak{n}_\alpha + \mathfrak{n}_\beta + C^1\mathfrak{n}\,,\, (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}] = (\mathfrak{n}_{2\alpha+\beta} \cap C^2\mathfrak{n}) + (\mathfrak{n}_{\alpha+2\beta} \cap C^2\mathfrak{n})\,. \end{aligned}$$

Since $C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$ is equal to the centralizer of $C^1\mathfrak{n}$ in $\mathfrak{n}$, it follows that $\mathfrak{n}_{2\alpha+\beta} \cap C^2\mathfrak{n} \neq \{0\}$ and $\mathfrak{n}_{\alpha+2\beta} \cap C^2\mathfrak{n} \neq \{0\}$. Both of these subspaces are contained in $\mathfrak{z}\mathfrak{m}$. This implies $2\tilde{\alpha} + \tilde{\beta} = 0$ and $\tilde{\alpha} + 2\tilde{\beta} = 0$, i.e., $\tilde{\alpha} = \tilde{\beta} = 0$. Thus $\mathfrak{m}$ is nilpotent in this case, too. We have shown

Proposition 13.2. *Let G be an exponential Lie group whose Lie algebra $\mathfrak{g}$ contains a nilpotent ideal $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ which is a central extension of the five-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,4}$. If π is a locally faithful, irreducible representation of G, then the primitive ideal $\ker_{C^*(G)} \pi$ is induced from a nilpotent normal subgroup M.*

14 Nilradical is the algebra $\mathfrak{g}_{5,6}$

In this section we study the representation theory of an exponential solvable Lie group G such that the nilradical $\mathfrak{n}$ of its Lie algebra $\mathfrak{g}$ is a central extension of the five-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,6}$. It turns out that the primitive ideals $\ker_{C^*(G)} \pi$ for locally faithful, irreducible representations $\pi = \mathcal{K}(f)$ are induced from the nilpotent normal stabilizer subgroup M of G whose Lie algebra is given by $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$.

Let $\mathfrak{g}$ be an exponential solvable Lie algebra which contains a nilpotent ideal $\mathfrak{n}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Assume that

$$\mathfrak{n} \underset{1}{\supset} \mathfrak{c} \underset{1}{\supset} C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n} \underset{d-1}{\supset} C^1\mathfrak{n} \underset{1}{\supset} C^2\mathfrak{n} \underset{1}{\supset} C^3\mathfrak{n} \underset{1}{\supset} \{0\} \tag{14.1}$$

is a descending series of characteristic ideals of $\mathfrak{n}$ where $\mathfrak{c}$ is the centralizer of $C^2\mathfrak{n}$ in $\mathfrak{n}$ satisfying $[\mathfrak{c}, \mathfrak{c}] = C^3\mathfrak{n}$. In particular $C^1\mathfrak{n}$ is commutative. Note that our assumptions include the case of the five-dimensional, nilpotent Lie algebra $\mathfrak{g}_{5,6}$ if $d = 1$.

Let $f \in \mathfrak{g}^*$ be in general position. We define the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. Since f vanishes on $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$, this ideal of $\mathfrak{g}$ must be zero. This proves $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

Let $\mathfrak{s}$ be a nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ and let $\mathfrak{t} = \mathfrak{s} \cap \mathfrak{m}$. As usual we regard $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathfrak{s}$-modules. There exist two weights $\alpha, \beta \in \mathfrak{s}^*$ such that $\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{c}$ and $\mathfrak{c} = (\mathfrak{n}_\beta \cap \mathfrak{c}) + C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}$. These definitions yield the following commutator relations:

$$\begin{aligned}
C^1\mathfrak{n} &= [\mathfrak{n}_\alpha + \mathfrak{c}, (\mathfrak{n}_\beta \cap \mathfrak{c}) + C^1\mathfrak{n} + \mathfrak{z}\mathfrak{n}] = (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}\,, \\
C^2\mathfrak{n} &= [\mathfrak{n}_\alpha + \mathfrak{c}, (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}] = (\mathfrak{n}_{2\alpha+\beta} \cap C^2\mathfrak{n}) + C^3\mathfrak{n}\,, \\
C^3\mathfrak{n} &= [\mathfrak{n}_\alpha + \mathfrak{c}, (\mathfrak{n}_{2\alpha+\beta} \cap C^2\mathfrak{n}) + C^3\mathfrak{n}] = \mathfrak{n}_{3\alpha+\beta} \cap C^3\mathfrak{n}\,.
\end{aligned}$$

On the other hand we obtain

$$[\mathfrak{c}, \mathfrak{c}] = [\mathfrak{n}_\beta + C^1\mathfrak{n}, (\mathfrak{n}_{\alpha+\beta} \cap C^1\mathfrak{n}) + C^2\mathfrak{n}] = \mathfrak{n}_{\alpha+2\beta} \cap C^3\mathfrak{n}$$

and consequently $3\alpha + \beta = \alpha + 2\beta$, i.e., $\beta = 2\alpha$. Since $\mathfrak{n}_{5\alpha} \cap C^3\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$, it follows $\tilde{\alpha} = \tilde{\beta} = 0$. Here the tilde indicates restriction to $\mathfrak{t}$. This proves that $\mathfrak{m}$ is a nilpotent ideal of $\mathfrak{g}$. We have shown

Proposition 14.2. *Let G be an exponential Lie group whose Lie algebra $\mathfrak{g}$ contains a nilpotent ideal $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ which is a central extension of the five-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,6}$. If π is a locally faithful, irreducible representations of G, then the ideal $\ker_{C^*(G)} \pi$ is induced from a nilpotent normal subgroup M.*

15 Exponential solvable Lie groups in low dimensions

In this section we will prove the main result of this work: All exponential solvable Lie groups of dimension less or equal seven are primitive $*$-regular. This assertion is a consequence of the well-known classification of all nilpotent Lie algebras of dimension less or equal five (see Section 4) and the next four propositions, in which we prove certain families of exponential solvable Lie groups in arbitrary dimensions to be primitive $*$-regular. To a great extent the proof of these propositions relies on the results of Sections 9 to 14 for representations in general position. The rest of the proof is merely a simple induction.

We note that all Lie algebras which are verified to be primitive $*$-regular in this section are in some sense close to being a semi-direct sum of a commutative subalgebra $\mathfrak{s}$ and a nilpotent ideal $\mathfrak{n}$. Furthermore we will see that the results obtained so far do not allow us to tone down the quite restrictive assumptions *(i)* to *(iii)* of Propositions 15.2 to 15.5. On the other hand, the results of Sections 9 to 14 are not yet exhausted: Although these results might not suffice to prove the primitive $*$-regularity of G, it is still possible to separate certain admissible critical representations ρ from certain representations π in general position in the sense of Relation 15.3 without one of the conditions *(i)* to *(iii)* being satisfied.

Definition 15.1. We say that a Lie algebra $\mathfrak{n}$ is a trivial extension of $\dot{\mathfrak{n}}$ if it is a split central extension, i.e., a direct sum of $\dot{\mathfrak{n}}$ and a commutative algebra.

First we treat algebras related to filiform algebras.

Proposition 15.2. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Let $k \geq 1$. Suppose that $\mathfrak{n}$ is either a commutative algebra or a trivial extension of the $(k+1)$-step nilpotent filiform algebra. Further we assume that one of the following conditions is satisfied:*

(i) there is a commutative subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ is a semi-direct sum,

(ii) $\dim \mathfrak{g}/\mathfrak{n} \leq 2$,

(iii) $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$ *and* $\dim \mathfrak{z}\mathfrak{n} \leq 2$.

Then G is primitive $*$-regular.

Proof. If $\mathfrak{n}$ is commutative ($k = 0$), then $\mathfrak{g}$ is metabelian and hence $*$-regular by Theorem 3.5 of [3]. Now let $k \geq 1$ be arbitrary. Assume that either $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ is a semi-direct sum or $\dim \mathfrak{g}/\mathfrak{n} \leq 2$. We carry out the basic strategy developed in Remark 3.30. First we consider proper quotients of $\mathfrak{g}$: If $\mathfrak{a}$ is a non-trivial minimal ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{z}\mathfrak{n}$, then we can pass to the quotient $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$. Clearly either $\dot{\mathfrak{g}}$ is a semi-direct sum of $\mathfrak{s}$ and $\dot{\mathfrak{n}} = \mathfrak{n}/\mathfrak{a}$ or $\dim \dot{\mathfrak{g}}/\dot{\mathfrak{n}} \leq 2$. Note that the quotient $\dot{\mathfrak{n}}$ is either commutative or a trivial extension of a filiform algebra. Consequently $\dot{\mathfrak{g}}$ is known to be primitive $*$-regular by induction.

Now let $f \in \mathfrak{g}^*$ be in general position, i.e., $f \neq 0$ on any non-trivial ideal $\mathfrak{a}$ as above. By Theorem 3.13 and 3.23 we can assume that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$. Let $g \in \mathfrak{g}^*$ be critical for the orbit $\mathrm{Ad}^*(G)f$. If $C^k\mathfrak{n} \subset \mathfrak{z}\mathfrak{g}$, then Theorem 9.11.*(i)* or *(ii)* implies

$$\bigcap_{s\in\mathbb{R}^m} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho \tag{15.3}$$

for the unitary representations $\pi_s = \mathcal{K}(f_s)$ and $\rho = \mathcal{K}(g)$. If $C^k\mathfrak{n} \not\subset \mathfrak{z}\mathfrak{g}$, then Relation 15.3 follows from Theorem 9.27. In any case we see that the ideal $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined.

Finally the weight space decomposition of the $\mathfrak{s}$-module $\mathfrak{n}$ that we obtained in Section 9.1 shows that the validity of condition *(iii)* of this proposition implies that either *(i)* or *(ii)* is true. These considerations prove the primitive $*$-regularity of G in any case. □

We point out that the tools 'Functional calculus for central elements' and 'Restriction to subquotients' developed in Section 5 and 7 allow no result for the 8-dimensional exponential Lie group G presented in Remark 9.28. The nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ of this group is a trivial extension of the 3-dimensional Heisenberg algebra such that $\dim \mathfrak{g}/\mathfrak{n} = 3$ and $\dim \mathfrak{z}\mathfrak{n} = 3$. Clearly none of the conditions *(i)* to *(iii)* is satisfied. The investigation of this 8-dimensional Lie group is beyond the scope of this work. In this sense the assumptions *(i)* to *(iii)* are necessary.

Proposition 15.4. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. Suppose that $\mathfrak{n}$ is a trivial extension of the five-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,3}$. Further we assume that one of the following conditions is satisfied:*

(i) $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ *is a semi-direct sum,*

(ii) $\dim \mathfrak{g}/\mathfrak{n} \leq 2$,

(iii) $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$ *and* $\dim \mathfrak{z}\mathfrak{n} \leq 2$.

Then G is primitive $*$-regular.

Proof. This proof is very similar to that of Proposition 15.2. Again we proceed as in Remark 3.30. The first step is to consider proper quotients of $\mathfrak{g}$: If $\mathfrak{a}$ is a non-trivial minimal ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{z}\mathfrak{n}$ and $C^2\mathfrak{n} \cap \mathfrak{a} = 0$, then the quotient $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ has the same form as $\mathfrak{g}$. Thus $\dot{\mathfrak{g}}$ is known to be primitive $*$-regular by induction. If $\mathfrak{a} = C^2\mathfrak{n}$, then $\dot{\mathfrak{n}} = \mathfrak{n}/\mathfrak{a}$ is a trivial extension of the 4-dimensional filiform algebra so that $\dot{\mathfrak{g}}$ is primitive $*$-regular by Proposition 15.2.

Now let $f \in \mathfrak{g}$ be in general position such that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$. As in the proof of the preceding proposition it follows from Theorem 12.10 in case of $C^2\mathfrak{n} \subset \mathfrak{z}\mathfrak{g}$ and from Theorem 12.34 in case of $C^2\mathfrak{n} \not\subset \mathfrak{z}\mathfrak{g}$ that the ideal $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined. This finishes our proof. □

Again we have no result for the 10-dimensional Lie group given in Remark 12.35.

Proposition 15.5. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra and $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{g}$ such that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{n}$ is a trivial extension of the five-dimensional nilpotent Lie algebras $\mathfrak{g}_{5,4}$ or $\mathfrak{g}_{5,6}$ and if one of the conditions (i) to (iii) as in the preceding propositions is satisfied, then G is primitive $*$-regular.*

Proof. Let $\mathfrak{n}$ be a trivial extension of $\mathfrak{g}_{5,4}$ or $\mathfrak{g}_{5,6}$. If $\mathfrak{a}$ is a non-trivial minimal ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{z}\mathfrak{n}$, then either the quotient $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ has the same form as $\mathfrak{g}$ or $\dot{\mathfrak{g}}$ satisfies the assumptions of Proposition 15.2. In any case it follows that $\dot{\mathfrak{g}}$ is primitive $*$-regular. If $f \in \mathfrak{g}^*$ is in general position and $\pi = \mathcal{K}(f)$, then Proposition 13.2 and Proposition 14.2 imply that $\mathfrak{m}$ is nilpotent so that $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined by Theorem 3.13. $\square$

Proposition 15.6. *Let $\mathfrak{g}$ be an exponential solvable Lie algebra such that the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is a 5-dimensional Heisenberg algebra or the 2-step nilpotent Lie algebra $\mathfrak{g}_{5,2}$. Then G is primitive $*$-regular.*

Proof. First we assume that $\mathfrak{n}$ is a 3- or 5-dimensional Heisenberg algebra. The quotient $\dot{\mathfrak{g}} = \mathfrak{g}/C^1\mathfrak{n}$ is metabelian and hence $*$-regular by Theorem 3.5 of [3]. If $f \in \mathfrak{g}^*$ such that $f \neq 0$ on $C^1\mathfrak{n}$ and such that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$, then Theorem 10.11 or 10.12 implies that $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined.

Finally we assume $\mathfrak{n} = \mathfrak{g}_{5,2}$. Again the quotient $\dot{\mathfrak{g}} = \mathfrak{g}/C^1\mathfrak{n}$ is metabelian and hence $*$-regular. If $\mathfrak{a}$ is a one-dimensional ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subset C^1\mathfrak{n}$, then $\dot{\mathfrak{n}} = \mathfrak{n}/\mathfrak{a}$ is a trivial extension of the 3-dimensional Heisenberg algebra such that $\dim \mathfrak{z}\dot{\mathfrak{n}} = 2$. Thus $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ is primitive $*$-regular by Proposition 15.2. If $f \in \mathfrak{g}^*$ is in general position such that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of $\mathfrak{g}$, then $\ker_{C^*(G)} \pi$ is $L^1(G)$-determined by Theorem 11.5. These considerations prove our proposition. $\square$

Finally we come to the main result of this paper. The efforts of Sections 9 to 14 culminate in

Theorem 15.7. *All exponential solvable Lie groups G of dimension ≤ 7 are primitive $*$-regular.*

Proof. Let $\mathfrak{n}$ denote the nilradical of $\mathfrak{g}$. It follows from Lemma 3.27 that we can assume $\dim \mathfrak{g}/\mathfrak{n} \geq 2$. This implies $\dim \mathfrak{n} \leq 5$ so that $\mathfrak{n}$ is either commutative or one of the algebras of the list in Section 4. In particular, if $\mathfrak{n}$ is a trivial extension of a filiform algebra, then $\dim \mathfrak{z}\mathfrak{n} \leq 2$. Now it is easy to see that the preceding propositions imply the primitive $*$-regularity of G. $\square$

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