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Numerical analysis of the method of freezing traveling waves

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Contents

Introduction

Traveling wave solutions of parabolic equations occur, besides other examples of pattern formation, in different areas of biology, chemistry and physics. They describe transport phenomena such as spread of populations [39], nerve conduction [24], [19] as well as oscillatory modes in models of superconductivity [44].

This thesis deals with the numerical computation and stability of traveling wave solutions (and more generally relative equilibria) of parabolic partial differential equations (PDEs) on the real line

$$u_t = Au_{xx} + f(u, u_x), \quad u : \mathbb{R} \times [0, \infty) \to \mathbb{R}^n, A \in \mathbb{R}^{n,n}. \tag{1}$$

Traveling waves are solutions which can be written in the form $u(x, t) = \bar{u}(x - \bar{\lambda}t)$, where $\bar{u} : \mathbb{R} \to \mathbb{R}^m$ denotes the waveform and $\bar{\lambda}$ the velocity.

In a comoving frame $v(\xi, t) = u(\xi + \bar{\lambda}t, t)$ equation (1) is transformed into

$$v_t = Av_{\xi\xi} + \bar{\lambda}v_\xi + f(v, v_\xi), \quad \xi \in \mathbb{R}, \quad t \geq 0. \tag{2}$$

For this equation $\bar{u}(\xi)$ is a stationary solution, i.e. $(\bar{u}, \bar{\lambda})$ solves the second order ODE:

$$0 = Av'' + \lambda v' + f(v, v'). \tag{3}$$

It is of particular interest to examine stability with asymptotic phase of this stationary solution for the dynamic equation (2), i.e. solutions of (2) with initial values close to $\bar{u}$ that converge in a suitable norm to a shifted version of the profile $\bar{u}$.

For strongly parabolic systems on the real line, which we consider in this thesis, there exist well known results [23], [49] which relate nonlinear and spectral stability. More precisely, consider the linearization of the right hand side of (2) at the wave form $\bar{u}$, given by

$$\Lambda u = Au_{xx} + (\bar{\lambda}I + D_2 f(\bar{u}, \bar{u}_x))u_x + D_1 f(\bar{u}, \bar{u}_x)u.$$

Then "asymptotic stability with asymptotic phase" of the traveling wave is related to the location of the spectrum of Λ. Thus, in order to gain information about the stability of a traveling wave, one has to study properties of the spectrum of the generally unbounded linear operator Λ in appropriate function spaces with appropriate norms. These are determined by the type of perturbation w.r.t. which stability is considered. Note that these operators may not only have discrete eigenvalues but continuous spectrum as well. Investigations of the spectrum of Λ have been conducted for many systems. To detect isolated eigenvalues of finite multiplicity one often uses the so called Evans function [2], [57], [49], which is an analytic function that measures the angle between subspaces of modes that decay for $x \to -\infty$ and $x \to +\infty$. The location of the rest of the spectrum, the so called essential spectrum is determined by the constant coefficient operators $\Lambda^{\pm}$ that are obtained by letting x tend to $\pm\infty$ in the coefficients of Λ. The spectrum of $\Lambda^{\pm}$ can be calculated from the so called dispersion relation [49], [23], [60]. In many applications neither the traveling wave nor its spectrum and the Evans function are known analytically. Therefore one has to resort to numerical methods to approximate not only the wave but the spectrum (or at least isolated eigenvalues of finite multiplicity) as well.

Suppose, the system (1) has a traveling wave with nonzero velocity. One is interested in solving the Cauchy problem (1) for initial data $u(\cdot,0) = v^0$ that are close to the wave $\bar{u}$ or at least converge to it after sufficiently long time. One simulates the PDE (1) directly by restricting it to a finite interval $J = [x_-, x_+]$ and using finite boundary conditions. Then one employs some method of discretization for the corresponding initial bounday value problem. It may then happen, that the solution leaves the interval before it reaches the traveling wave form, or it reaches the traveling wave form which then dies out when reaching the finite boundary. Therefore one would like to work in a comoving frame, i.e. solve equation (2) numerically. However, the velocity $\bar{\lambda}$ is generally unknown. This leads to the idea of freezing the traveling wave as in [7] by introducing the unknown (time dependent) velocity of the frame as an additional independent variable and by employing a so called phase condition in order to deal with the additional degree of freedom. The original PDE (1) is now transformed via $u(x,t) = v(x-\gamma(t),t)$ into a partial differential-algebraic equation (PDAE)

$$\begin{aligned} v_t &= Av_{xx} + f(v, v_x) + \lambda v_x, \quad \gamma_t = \lambda, \quad v(\cdot,0) = v^0,\ \gamma(0) = 0 \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle. \end{aligned} \tag{4}$$

Here $\hat{v}$ is an appropriate reference function, for example $\hat{v} = v^0$. The last equation in (4) constitutes an additional algebraic constraint.

The purpose of this thesis is to investigate the asymptotic behavior $t \to \infty$ of such systems for two cases: the continuous case on the whole line, which is dealt with in Chapter 1, and the spatially discrete case, which arises from a simple spatial discretization with finite differences on a finite interval. Here the discrete analog of (4) reads

$$v_n' = A(\delta_+\delta_- v)_n + \lambda(\delta_0 v)_n + f(v_n, \delta_0 v_n), \quad n \in J = [n_-, n_+],\ t > 0 \tag{5}$$

$$\eta = P_- v_{n_-} + Q_-(\delta_0 v)_{n_-} + P_+ v_{n_+} + Q_+(\delta_0 v_{n_+}) \tag{6}$$

$$0 = h \sum_{n=n_-}^{n_+} (\delta_0 \hat{v})_n^T (v_n - \hat{v}_n) =: \Psi(v). \tag{7}$$

where $\delta_-, \delta_+, \delta_0$ denote forward, backward and central finite differences respectively, the integers $n_\pm$ determine the finite interval J and $P_\pm, Q_\pm \in \mathbb{R}^{2m,m}$ are suitable matrices (cf. Section 2.2). The approximation properties of stationary solutions of (5)–(7) are examined in Chapter 2.

Note that (1) and (4) are equivalent, whereas on finite intervals J, the DAE formulation (5)–(7) is no longer equivalent to the direct discretization of the PDE (1) on J given by

$$\begin{aligned} u_n' &= A(\delta_+\delta_- u)_n + f(u_n, \delta_0 u_n), \quad n \in J = [n_-, n_+],\ t > 0 \\ \eta &= P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u_{n_+}). \end{aligned} \tag{8}$$

This can be seen clearly in numerical computations: In the PDE case (8) (if the velocity $\bar{\lambda}$ is not zero) a pulse or front will eventually leave the compuational domain, whereas in the PDAE case the wave form will stabilize in the interval (if the initial conditions are reasonable), see [7] for numerical experiments.

Now the question arises, if the traveling wave solution is a stable solution of the PDAE (4) under the same conditions which ensure its stability with asymptotic phase as a solution of the PDE (1) (using the appropriate notion of stability in each case). The main result Theorem 1.13 in Chapter 1 is a positive answer for the PDAE. In Chapter 4 we show an analogous result in Theorem 4.2 for the differential algebraic equation (5)–(7) provided the boundary matrices $P_{\pm}, Q_{\pm}$ satisfy an appropriate regularity assumption.

In both cases the method of proof is quite similar; as in the stability proofs for the PDE (see [60], [23], [36], [63]) we will use semigroup methods to define a solution of the nonlinear system via a variation of constants formula. Then we use the properties of the spectrum of the corresponding linear operator as well as the fact that the phase condition removes the eigenvalue zero.

In Chapter 2 we prove that the discretized stationary equations

$$\begin{aligned} 0 &= A(\delta_+\delta_- v)_n + \lambda(\delta_0 v)_n + f(v_n, \delta_0 v_n), \quad n \in J = [n_-, n_+],\ t > 0 \\ \eta &= P_- v_{n_-} + Q_-(\delta_0 v)_{n_-} + P_+ v_{n_+} + Q_+(\delta_0 v_{n_+}) \\ 0 &= \Psi(v) \end{aligned} \tag{9}$$

have a solution $(\tilde{u}, \tilde{\lambda})$, that approximates the traveling wave $(\bar{u}, \bar{\lambda})$. The dependence of the error estimate on the grid size h and the size of the interval J is quantified. The corresponding approximation results for discrete eigenvalues as well as resolvent estimates for the discrete operators are proven in Chapter 3.

The numerical approach of approximating the derivatives by finite differences is widely used [30], [13] besides other (global) methods such as Galerkin or (pseudo-)spectral methods [59], collocation [38] or finite elements [37]. Therefore the results concerning the approximation in dependence on h and T are interesting from a numerical analysis point of view. We expect our results to hold in an analogous manner for these other discretization methods.

In the thesis we need these results on the approximation of the wave as well as on the spectral properties of the discretized system in order to prove resolvent estimates. These are used in Chapter 4 for obtaining precise estimates of the discrete solution operator of the linear equation.

The methods used in Chapters 2 and 3 are mainly dynamical systems tools, namely exponential dichotomies for finite difference equations. These allow to decompose the space of initial values into subspaces which give rise to solutions that decay exponentially either in forward or backward x-direction.

Such methods have been used for discrete dynamical systems in [26], [4] and in [64] to study connecting orbits of discrete systems on $\mathbb{Z}$ (i.e. without boundary conditions). The numerical approximation of (3) gives rise to such a discrete dynamical system in space which inherits many properties of the continuous system. Combining this with the methods, used in [26] in order to deal with boundary conditions, we can prove approximation results for the traveling wave as well as for simple, isolated eigenvalues and for the resolvent equation.

The influence of the boundary conditions on the approximation of the wave in the continuous case (i.e. without discretization) has been dealt with in [3], [60], and on the spectrum in [6] and [51],[50]. The two latter papers also analyze the dependence of the essential spectrum on boundary conditions. At the end of Chapter 3 we comment on similar spectral behavior that is observed for spatially discretized systems in Chapter 5. If periodic boundary conditions are used the eigenvalues of the system on the finite grid cluster near the essential spectrum of the discrete operator on the infinite grid.

The freezing approach in [7] is not restricted to traveling waves. It is possible to deal with general relative equilibria [9], [53] such as rotating waves on the real line and even spiral waves in two space dimensions. We outline this more general approach here in order to indicate, how the results of this thesis may be extended to more general equations.

Consider an evolution equation in a Banach space X of the form

$$u_t = F(u) \tag{10}$$

with an equivariant right hand side F, i.e. $a(\gamma)F(u) = F(a(\gamma)u)$ where $a : G \to GL(X), \gamma \to a(\gamma)$ denotes the action of a Lie group G on X. The equation (10) can be transformed via the ansatz $v(t) = a(\gamma(t))u(t)$ into the equivalent system

$$v_t = F(v) - a(\gamma)^{-1} a_\gamma(\gamma) v\lambda, \quad \lambda = \gamma_t. \tag{11}$$

A traveling wave is a special type of a relative equilibrium of equivariant evolution equations, where the action is given by translation, $[a(\gamma)u](x) = u(x - \gamma),\ \gamma \in \mathbb{R}$. Most of our results concerning convergence and stability can be generalized to equivariant parabolic equations on $\mathbb{R}$. We will indicate the necessary modifications in the proofs for this case at the end of the corresponding chapters.

As indicated before, the theory becomes more difficult in higher space dimensions, although the freezing approach works in this case as well. The main difficulty is the lack of a spectral gap, since in this case the essential spectrum touches the imaginary axis [49]. Moreover, the use of dynamical systems tools such as exponential dichotomies, relies on the fact that the space is one dimensional.

In Chapter 5 we demonstrate the convergence properties of the solution of the boundary value problem (9) (different intervals and grid sizes) as well as the behavior of the spectrum under discretization for two different numerical examples. The first example is the scalar Nagumo equation for which an exact traveling front solution is known. The second more general example is the quintic complex Ginzburg Landau equation (QCGL), which is equivariant w.r.t. the action of the group $G = S^1 \times \mathbb{R}$ on $\mathbb{R}^2$. The action is given by translation in the domain and rotation in the image, i.e.

$$[a(\gamma)u](x) = R_{-\gamma_r} u(x - \gamma_t), \quad \gamma = (\gamma_r, \gamma_t) \in G,\ x \in \mathbb{R},\ u(x) \in \mathbb{R}^2,\ R_\gamma = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix}.$$

In both cases, the numerical convergence behavior confirms the theoretical predictions from Chapter 2.

For the convergence of the eigenvalues of the discrete system near zero similar computations are performed. The error of the eigenvalue and of the eigenfunction and the corresponding invariant subspace (in the QCGL case) is computed for various values of the grid size h and of the interval length.

Furthermore the stability properties which are discussed in Chapter 4 are examined numerically. We compare the spectral data with the rate of exponential convergence of the solution of the time dependent system (5)–(7) towards the solution of the boundary value problem (9).

At last, we show the spectrum (i.e. all eigenvalues) of the discrete operator on J for different boundary conditions. These results have led to the conjectures concerning the approximation of the essential spectrum in Chapter 3.

In Appendix A we summarize functional analytic tools as well as some well known facts for exponential dichotomies. Several symbols that are used frequently in the text, especially function spaces are listed in Appendix B.

In summary, this thesis gives a detailed study of the existence and stability of traveling waves for a newly developed equation and a widely used numerical discretization. The extension of our results to much more general patterns seems possible and provides new questions.

Chapter 1

Stability of traveling waves as PDAE solutions

In this chapter we deal with the stability of traveling wave solutions of parabolic systems in one space dimension. After stating well known results about stability with asymptotic phase, we prove stability for the PDAE formulation (cf. (4)). This PDAE contains an additional phase condition that singles out a unique solution from the continuum of shifted traveling waves.

We first introduce the appropriate notion of stability for traveling waves and state sufficient conditions on the spectrum of the elliptic operator which ensure nonlinear stability. Note that the existence of such a solution will always be assumed, existence proofs can be found for example in [63], [11].

We employ semigroup theory for the solution of the PDE and generalize some of the results to the special PDAE under consideration. As general references for the theory of analytic semigroups and sectorial operators we use the monographs [23], [36], [46], [43]. For more general theory on abstract PDAEs see [17].

XXX

In Chapter 4 a similar approach will be used to show the stability of a traveling wave for a discretization of the PDAE (4) with finite differences and appropriate boundary conditions.

Consider the following strongly parabolic PDE

$$u_t = Au_{xx} + f(u, u_x),\ u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^m,\ A \in \mathbb{R}^{m,m}, \tag{1.1}$$

where $A > 0$, i.e. $\langle v, Av\rangle > 0\ \forall v \in \mathbb{R}^m \setminus \{0\}$. Assume that equation (1.1) has a traveling wave solution u, i.e. u can be written as

$$u(x,t) = \bar{u}(\xi), \quad \xi = x - \bar{\lambda}t,\ \bar{u} \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}^m), \tag{1.2}$$

where the waveform $\bar{u} \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}^m)$ possesses bounded derivatives up to order 2 and has the properties

$$\lim_{\xi\to\pm\infty} \bar{u}(\xi) = u_\pm, \quad \lim_{\xi\to\pm\infty} \bar{u}'(\xi) = 0. \tag{1.3}$$

In a comoving frame, i.e. for $v(\xi,t) = u(\xi+\bar{\lambda}t, t)$ equation (1.1) reads

$$v_t = Av_{\xi\xi} + \bar{\lambda}v_\xi + f(v, v_\xi), \quad \xi \in \mathbb{R}, \quad t \geq 0 \tag{1.4}$$

and $\bar{u}$ is a stationary solution of this equation.

1.1 Stability with asymptotic phase

For such solutions the correct notion of stability is the so called "asymptotic stability with asymptotic phase" which will be given in Definition 1.1 (see [63],[23],[49]). The term "asymptotic phase" refers to the fact, that solutions starting close to the wave do not necessarily converge to the wave itself but to some suitably shifted profile. This is reasonable since with $\bar{u}$, each shifted function $\bar{u}(\cdot+\gamma)$ is also a solution of (1.1). A numerical procedure for computing the traveling wave has to single out one unique solution of this family. This is done by employing a so called phase condition as discussed in later sections.

Definition 1.1 *The wave $(\bar{u}, \bar{\lambda})$ is called "asymptotically stable with asymptotic phase" with respect to a norm $\|\cdot\|$ in a Banach space X, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each solution v of (1.4) with $v(\cdot, 0) \in X$ and*

$$\|v(\cdot\,, 0) - \bar{u}\| \leq \delta$$

there exists a phase shift $\gamma \in \mathbb{R}$ such that

$$\begin{aligned} &\|v(\cdot\,,t) - \bar{u}(\cdot\,{+}\gamma)\| \leq \epsilon, \quad \forall t \geq 0 \\ &\|v(\cdot\,,t) - \bar{u}(\cdot\,{+}\gamma)\| \to 0, \quad \text{as } t \to \infty. \end{aligned}$$

The Banach space X will we specified later, for the moment we just assume that X satisfies $C_0^\infty(\mathbb{R}, \mathbb{R}^m) \subset X \subset \mathcal{L}_2(\mathbb{R}, \mathbb{R}^m)$. Note also that the solution $\bar{u}$ itself need not be an element of X, rather Def. 1.1 assumes that $v(\cdot, t) - \bar{u}(\cdot - \gamma)$ is in X for each $\gamma \in \mathbb{R}$ and $t \geq 0$. As has been shown in [63], [60], [49],[18] asymptotic stability is determined by the linearization of the right hand side of (1.1) about the traveling wave profile $(\bar{u}, \bar{\lambda})$ which is given by

$$\Lambda u = Au'' + Bu' + Cu. \tag{1.5}$$

Here $B : \mathbb{R} \to \mathbb{R}^{m,m}$, $C : \mathbb{R} \to \mathbb{R}^{m,m}$ are defined as follows

$$B(x) = \bar{\lambda}I + D_2 f(\bar{u}(x), \bar{u}'(x)), \qquad C(x) = D_1 f(\bar{u}(x), \bar{u}'(x)).$$

Note that B and C converge as $x \to \pm\infty$ to

$$\lim_{x\to\pm\infty} B(x) = \bar{\lambda}I + D_2 f(u_\pm, 0) =: B_\pm, \quad \lim_{x\to\pm\infty} C(x) = D_1 f(u_\pm, 0) =: C_\pm.$$

Sufficient conditions for asymptotic stability of $(\bar{u}, \bar{\lambda})$ with asymptotic phase are (see Theorem 1.8 below)

Spectral condition (SC):

There exist $\sigma > 0$, $\beta > 0$, such that for s with $\mathrm{Re}\, s \geq -\beta$ the solutions λ of the quadratic eigenvalue problems

$$\det(\lambda^2 A + \lambda B_\pm + C_\pm - sI) = 0 \tag{1.6}$$

satisfy: $|\mathrm{Re}\, \lambda| \geq \sigma$.

Eigenvalue condition (EC):

Considered as an operator in X the differential operator Λ has a simple eigenvalue 0 and there exists $\beta > 0$ such that there are no other isolated eigenvalues s of finite multiplicity with $\mathrm{Re}\, s \geq -\beta$.

Before we proceed to the main stability result for the PDAE, we note two important consequences of these conditions that are used in the proof of the stability theorem 1.8 below. We recall the definitions for resolvent and (essential) spectrum in definition A.1 in the appendix. The spectral condition (SC) implies that the essential spectrum $\sigma_{\mathrm{ess}}(\Lambda)$ is contained in the left half plane as the following Theorem shows.

Theorem 1.2 *Let B,$C : \mathbb{R} \to \mathbb{R}^{m,m}$ be bounded, continuous matrix functions with*

$$\lim_{x\to\pm\infty} B(x) =: B_\pm, \qquad \lim_{x\to\pm\infty} C(x) =: C_\pm$$

and let $A \in \mathbb{R}^{m,m}$ satisfy $A > 0$.
Consider the operator

$$\Lambda u = Au'' + B(\cdot)u' + C(\cdot)u. \tag{1.7}$$

in $\mathcal{L}_p(\mathbb{R}, \mathbb{R}^m)$, $1 \leq p \leq \infty$, define the set

$$S_\pm = \{s \in \mathbb{C} : \det(-\kappa^2 A + i\kappa B_\pm + C_\pm - sI) = 0, \textit{for some } \kappa \in \mathbb{R}\}.$$

and let M be the complement of the connected component of $\mathbb{C} \setminus \{S_+ \cup S_-\}$ that contains the right half plane.
Then the essential spectrum $\sigma_{\mathrm{ess}}(\Lambda)$ satisfies

$$S_- \cup S_+ \subset \sigma_{\mathrm{ess}}(\Lambda) \subset M.$$

Note that the set $S_\pm$ is a variety which is symmetric w.r.t. the real line. Theorem 1.2 as stated above is a slight generalization of [23], Chapter 5, Thm. A.2 to non-symmetric A.

The eigenvalue condition (EC) ensures that the rest of the spectrum, i.e. all isolated eigenvalues of finite multiplicity, except for the eigenvalue 0, have real part $\leq -\beta < 0$. Due to translational invariance the eigenvalue 0 is always present. This can be seen by differentiating the equation for the phase shifted solutions

$$0 = A\bar{u}''(x+\lambda) + \bar{\lambda}\bar{u}'(x+\lambda) + f(\bar{u}(x+\lambda), \bar{u}'(x+\lambda)), \quad x \in \mathbb{R}$$

with respect to the parameter λ at $\lambda = 0$. One obtains $\Lambda\bar{u}' = 0$, thus the eigenfunction corresponding to 0 is $\bar{u}'$ if $\bar{u}' \in X$.
The following estimate for the resolvent of Λ follows from (EC) ,(SC) :

Lemma 1.3 *Let $A \in \mathbb{R}^{m,m}$ with $A > 0$ and let Λ be given by (1.5). Then there exists an open sector*

$$\mathcal{S}_{\omega,\zeta} = \{s \in \mathbb{C} : |\arg(s+\omega)| < \zeta,\ s \neq -\omega\},\ \text{where } \zeta > \frac{\pi}{2}$$

and a constant $K > 0$, such that for each $s \in \mathcal{S}_{\omega,\zeta}$ and each $u \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ the following estimate holds

$$|s|^2\|u\|^2_{\mathcal{L}_2} + |s|\|u'\|^2_{\mathcal{L}_2} \leq K\|f\|^2_{\mathcal{L}_2}$$

for

$$f = (sI - \Lambda)u.$$

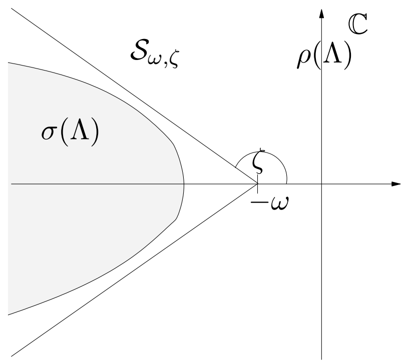


Figure 1.1: The sector $\mathcal{S}_{\omega,\zeta}$ contained in the resolvent set

This has been shown in [60] for the symmetric case and the extension to the nonsymmetric case is immediate. Together with the conditions (SC) and (EC) this shows that Λ is a sectorial operator in $\mathcal{L}_2$.

Definition 1.4 *Let X be a Banach space and let $\Lambda : \mathcal{D}(\Lambda) \to X$ be a linear operator on X. Λ is called sectorial if*

1. *Λ is closed and densely defined*
2. *there exist $\zeta \in (\frac{\pi}{2}, \pi)$, $M \geq 1$, $\omega \in \mathbb{R}$, defining the closed sector*

   $$\bar{\mathcal{S}}_{\omega,\zeta} = \{s \in \mathbb{C} : |\arg(s+\omega)| \leq \zeta,\ s \neq -\omega\},$$

   such that the resolvent set $\rho(\Lambda)$ contains $\bar{\mathcal{S}}_{\omega,\zeta}$ and obeys the following estimate

   $$\|(sI - \Lambda)^{-1}\| \leq \frac{M}{|s+\omega|}, \quad \forall s \in \bar{\mathcal{S}}_{\omega,\zeta}. \tag{1.8}$$

We recall the definition of the solution of a semilinear evolution equation with sectorial operator Λ as given in [36], [60]. Note, that this is a modified version of the solution definition in [23], which is necessary in order to guarantee the uniqueness of solutions (cf. [35], [36])

Definition 1.5 *Let Λ be a sectorial operator in $\mathcal{L}_2$ with $\mathcal{D}(\Lambda) = \mathcal{H}^2$ and $g : \mathcal{H}^1 \to \mathcal{L}_2$. A function $u : [0, \tau) \to \mathcal{H}^1$ is called a solution of the autonomous equation*

$$u' = \Lambda u + g(u), \quad u(0) = u^0 \in \mathcal{H}^1$$

in the interval $(0, \tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$ if

1. *$g(u(\cdot)) : [0, \tau) \to \mathcal{L}_2$ is continuous*
2. *$u : [0, \tau) \to \mathcal{H}^1$ is continuous, $u(t) \in \mathcal{H}^2$ for $t \in (0, \tau)$ and $u(0) = u^0$*
3. *$u'(t) \in \mathcal{L}_2$ exists and $u'(t) = \Lambda u(t) + g(u(t))$ for $t \in (0, \tau)$*

For the nonlinear stability problem we have to deal with solutions of (1.4) of the form $\bar{u} + v$ only, where v is supposed to lie in the correct function space. If u solves (1.4) then $v = u - \bar{u}$ solves

$$v_t = Av_{xx} + \bar{\lambda} v_x + g(v, v_x) \tag{1.9}$$

where

$$g(v, w) = f(\bar{u} + v, \bar{u}' + w) - f(\bar{u}, \bar{u}')$$

Then we can define a solution of equation (1.1) as follows.

Definition 1.6 *A function u is called a solution of equation (1.1) if $v = u - \bar{u}$ solves (1.9) in the sense of Definition 1.5.*

Remark 1.7 Note that here we require only the difference to the traveling wave solution to lie in $\mathcal{L}_2$, e.g. traveling fronts do not lie in $\mathcal{L}_2$. For $f \in \mathcal{C}^1$ we obtain even $\bar{u}'$ in $\mathcal{H}^2$: From the PDE (1.4) we obtain that $\bar{u}'$ solves the variational equation $\Lambda u = 0$ which implies $\bar{u}' \in \mathcal{C}^2$. With $z = (u, u')$ this equation is transformed to the first order equation $Lz = 0$ of which $\bar{z} = (\bar{u}', \bar{u}'')$ is a bounded solution. Since L has exponential dichotomies on $\mathbb{R}^-, \mathbb{R}^+$ (see Section A.3) this implies that $(\bar{u}', \bar{u}'')$ is actually exponentially decaying for $x \to \pm\infty$, i.e.

$$\|\bar{u}(x) - u_\pm\| \le K\mathrm{e}^{\mp\varrho x} \quad \text{as well as} \quad \|\bar{u}^{(k)}(x)\| \le K\mathrm{e}^{-\varrho|x|},\ k = 1, 2$$

for some $\rho > 0$. Thus $\bar{u}', \bar{u}''$ are in $\mathcal{L}_2$. With $\Lambda\bar{u}' = 0$ we obtain using the definition of Λ in (1.5) that $\bar{u}''' \in \mathcal{L}_2$ as well, which implies $\bar{u}' \in \mathcal{H}^2$.

From the resolvent estimate (1.8) and the two properties of the spectrum (EC) and (SC) the nonlinear stability of the traveling wave solution follows. This has been shown in [63],[60] for the special case, where f depends on u only, and is summarized in the following theorem. Note that (EC) can be verified in certain situations (see [63], [12]).

Theorem 1.8 (Asymptotic stability of traveling waves) *Let $\bar{u}$ be a traveling wave solution of (1.1) and assume that the conditions (SC) and (EC) hold. Assume further that the map $g : u \mapsto f(\bar{u} - u) - f(\bar{u})$ is in $\mathcal{C}^1(\mathcal{H}^1, \mathcal{L}_2)$.*

Then the traveling wave solution $\bar{u}$ is asymptotically stable with asymptotic phase w.r.t. $\|\cdot\|_{\mathcal{H}^1}$. More precisely, there exist $\epsilon > 0$, $M > 0$ such that the equation (1.4) possesses for each initial value $u^0 = \bar{u} + v^0$, with $v^0 \in \mathcal{H}^1$ and $\|v^0\|_{\mathcal{H}^1} \le \epsilon$ a unique solution $u = \bar{u} + v$ with $v(t) \in \mathcal{H}^2$ for $t > 0$, and there exists a $\gamma \in \mathbb{R}$ such that the exponential estimate

$$\|u(\cdot\,, t) - \bar{u}(\cdot + \gamma)\| \le M\mathrm{e}^{-\beta t}\|v^0\|$$

holds for $t \ge 0$.

We consider the more general situation where $f = f(u, u_x)$ under the following main nonlinearity assumption.

Hypothesis 1.9 *The function $f \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ is of the form*

$$f(u, u_x) = f_1(u)u_x + f_2(u), \quad f_1 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^{m,m}), f_2 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$$

where f_1, f_2, f_1', f_2' are globally Lipschitz.

Remark 1.10 Hypothesis 1.9 implies that f_1', f_2' are globally bounded and with

$$D_1 f(u, w) = f_1'(u)(w, \cdot) + f_2'(u), \quad D_2 f(u, w) = f_1(u),$$

we obtain for $u, w, \delta_u, \delta_w \in \mathbb{R}^m$

$$\begin{aligned} \|D_1 f(u + \delta_u, w + \delta_w) - D_1 f(u, w)\| &\leq L(\|\delta_u\| + \|\delta_w\|), \\ \|D_2 f(u + \delta_u, w + \delta_w) - D_2 f(u, w)\| &\leq L\|\delta_u\|. \end{aligned} \tag{1.10}$$

Note that the above condition includes the nonlinearity $f(u, u_x) = -uu_x$ of Burger's equation. Moreover, one can show that it implies the composition operator $g : u \mapsto f(u, u_x)$ to lie in $\mathcal{C}^1(\mathcal{H}^1, \mathcal{L}_2)$.

1.2 The PDAE formulation

If we transform equation (1.1) to a co-moving frame with unknown position $\gamma(t)$, i.e. insert $v(x, t) = u(x + \gamma(t), t)$, we get

$$v_t = Av_{xx} + f(v, v_x) + \lambda v_x, \tag{1.11}$$

where $\lambda = \gamma_t$. In order to compensate for this additional parameter we have to introduce an additional phase condition $\Psi(v) = 0$ which together with (1.11) forms a PDAE [7]. The actual position γ can then be calculated by integration from the ODE

$$\gamma_t = \lambda, \quad \gamma(0) = 0.$$

We use a phase condition which requires that the distance to a reference function $\hat{u}$,

$$\delta(\gamma) = \|v(\cdot + \gamma) - \hat{u}\|_{\mathcal{L}_2}$$

attains its minimum at $\gamma = 0$. This leads to the condition

$$0 = \Psi_{\mathrm{fix}}(v) = \langle \hat{u}', v - \hat{u} \rangle = \int_{\mathbb{R}} \hat{u}'(x)^T (v(x) - \hat{u}(x))\, dx. \tag{1.12}$$

This is the same phase condition that was proposed in [15] for the computation of the traveling wave by solving the following boundary value problem for (u, λ)

$$\begin{aligned} 0 &= Au'' + f(u, u') + \lambda u', \\ 0 &= \langle \hat{u}', u - \hat{u} \rangle. \end{aligned} \tag{1.13}$$

Similar to the proof of Theorem 1.8 we will prove the asymptotic stability of $(\bar{u}, \bar{\lambda})$ as a stationary solution of the PDAE

$$\begin{aligned} v_t &= Av_{xx} + f(v, v_x) + \lambda v_x, \quad v(\cdot, 0) = u^0 \\ 0 &= \langle \hat{u}', v - \hat{u} \rangle \end{aligned} \tag{1.14}$$

under the same conditions which ensure asymptotic stability (with asymptotic phase) of the family $\bar{u}(\gamma)$.

Before we give a precise definition of solution for the PDAE (1.14) we show how on a formal level one can recover a solution u of (1.1) from a solution (v, λ) of (1.14).

Let (v, λ) be a solution of (1.14) and define $\gamma(t)$ by $\gamma_t(t) = \lambda(t)$, $\gamma(0) = 0$ and $u(x,t) = v(x - \gamma(t), t)$. Inserting this into the first equation of (1.14) we obtain that u solves (1.1).

The proper generalization of the notion of a solution for a semilinear PDAE is given in the following definition.

Definition 1.11 *Let Λ be a sectorial operator in $\mathcal{L}_2$ with $\mathcal{D}(\Lambda) = \mathcal{H}^2$, $\psi \in \mathcal{H}^1$ and $g : \mathcal{H}^1 \times \mathbb{R} \to \mathcal{L}_2$. A function $(v, \mu) : [0, \tau) \to \mathcal{H}^1 \times \mathbb{R}$ is called a solution of*

$$\begin{aligned} v' &= \Lambda v + g(v, \mu), \quad v(0) = v^0 \in \mathcal{H}^1 \\ 0 &= \langle \psi, v \rangle \end{aligned}$$

in $(0, \tau)$, $\tau \in (0, \infty]$ if the following conditions hold

1. *$g(v(\cdot), \mu(\cdot)) : [0, \tau) \to \mathcal{L}_2$ is continuous*
2. *$v : [0, \tau) \to \mathcal{H}^1$ is continuous, $v(t) \in \mathcal{H}^2$ for $t \in (0, \tau)$ and $v(0) = v^0$*
3. *μ is continuous in $[0, \tau)$*
4. *$v'(t) \in \mathcal{L}_2$ exists and $v'(t) = \Lambda v(t) + g(v(t), \mu(t))$ for $t \in (0, \tau)$*
5. *$\langle \psi, v(t) \rangle = 0$ $\forall t \in [0, \tau)$.*

Using the ansatz $v = u - \bar{u}$, $\mu = \lambda - \bar{\lambda}$ and defining $\phi = \bar{u}'$ and $\psi = \hat{u}'$, we get the equivalent formulation of (1.14), namely

$$\begin{aligned} v_t &= \Lambda v + g(v, \mu), \\ 0 &= \langle \psi, v \rangle. \end{aligned} \tag{1.15}$$

Here Λ is the linearization of (1.4) about $(\bar{u}, \bar{\lambda})$, which has been defined in (1.5) and

$$g(v, \mu) = \phi\mu + \omega(v) + v_x \mu,$$

where $\omega : \mathcal{H}^1 \to \mathcal{L}_2$ denotes the composition operator given by

$$\omega(v) = f(\bar{u} + v, \bar{u}' + v_x) - f(\bar{u}, \bar{u}') - D_1 f(\bar{u}, \bar{u}')v - D_2 f(\bar{u}, \bar{u}')v_x. \tag{1.16}$$

Using this ansatz we define a solution of (1.14) via the transformed equation (1.15) and Definition 1.11.

Definition 1.12 *We call (u, λ) a solution of (1.14) if the difference $(u - \bar{u}, \lambda - \bar{\lambda})$ is a solution of (1.15) in the sense of Definition 1.11.*

1.2.1 Stability of the PDAE solution

The main result of this section is the following stability theorem for the PDAE (1.14).

Theorem 1.13 *Let $A \in \mathbb{R}^{m,m}$ be given with $A > 0$ and assume that the function $f \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ satisfies Hypothesis 1.9.*

Let $(\bar{u}, \bar{\lambda})$, $\bar{u} \in \mathcal{C}_b^2$ be a stationary solution of the PDAE (1.14), i.e.

$$\begin{aligned} 0 &= A\bar{u}'' + \bar{\lambda}\bar{u}' + f(\bar{u}, \bar{u}') \\ 0 &= \langle \hat{u}', \bar{u} - \hat{u} \rangle \end{aligned}$$

where $\hat{u} \neq 0$ is a given reference function with $\hat{u} - \bar{u} \in \mathcal{H}^2$ and $\langle \hat{u}', \bar{u}' \rangle \neq 0$. Furthermore, assume that (EC) and (SC) hold.

Then $(\bar{u}, \bar{\lambda})$ is asymptotically stable, i.e. there exists $\delta > 0$ such that for each u^0 with $u^0 - \bar{u} \in \mathcal{H}^1$, $\langle \hat{u}', u^0 - \hat{u} \rangle = 0$ and $\|u^0 - \bar{u}\|_{\mathcal{H}^1} < \delta$ there exists a unique solution $(u(t), \lambda(t))$ of (1.14) on $[0, \infty)$ and the following exponential estimate holds for some $K > 0, \alpha > 0$

$$\|u(t) - \bar{u}\|_{\mathcal{H}^1} + |\lambda(t) - \bar{\lambda}| \leq K\mathrm{e}^{-\alpha t}\|u^0 - \bar{u}\|_{\mathcal{H}^1} \quad \forall t \geq 0. \tag{1.17}$$

Thus in order to prove the stability of $(\bar{u}, \bar{\lambda})$ as a solution of (1.14) it is sufficient to consider the stability of the zero solution $(\bar{u}, \bar{\lambda}) = 0$ of (1.15). In the next paragraph we will solve this problem by directly analyzing the linearizations of the PDAE 1.15

Before following this path of proof we outline an alternative of proving stability which solely uses well known results of stability of traveling waves ([23], [63], [49]). Let (v, λ) be a solution of (1.14) then substituting $v(x, t)$ by $u(x + \gamma(t), t)$ in the second equation of (1.14) and differentiating w.r.t. t we obtain

$$\begin{aligned} 0 &= \langle \hat{u}', u_x(\cdot + \gamma(t), t)\gamma_t(t) + u_t(\cdot + \gamma(t), t) \rangle \\ &= \langle \hat{u}', u_x(\cdot + \gamma(t), t)\ \gamma_t(t) \rangle + \langle \hat{u}', Au_{xx}(\cdot + \gamma(t), t) + f(u(\cdot + \gamma(t), t), u_x(\cdot + \gamma(t), t)) \rangle. \end{aligned}$$

This implies that (u, γ) solves

$$\gamma_t = g(u, \gamma), \quad \gamma(0) = 0, \tag{1.18}$$

where

$$g(u, \gamma) = -\frac{\langle \hat{u}'(\cdot + \gamma), Au_{xx} + f(u, u_x) \rangle}{\langle \hat{u}'(\cdot + \gamma), u_x \rangle}.$$

On the other hand, let u solve (1.1) and define $\gamma(t)$ by solving (1.18). Then (v, λ), given by $v(\cdot, t) = u(\cdot + \gamma(t), t)$, $\lambda(t) = \gamma_t(t)$, solves (1.14).

Therefore the stability of an equilibrium $(\bar{v}, \bar{\lambda})$ of (1.14) can be concluded from the stability of a family of traveling wave solutions $\bar{u}(\cdot - \gamma)$ of (1.1). However, this works only if the spatial domain is the whole real line. Since our ultimate goal is to prove in Chapter 4 stability of a traveling wave solution for the discretized system on a finite interval, we prove the stability of (v, λ) directly. The methods developed here can then be transferred to the discretized equations.

1.3 The semilinear equation

For proving the stability of the zero solution of (1.15) we will reduce this PDAE to a corresponding PDE by eliminating the parameter μ via the hidden constraint, which one gets by differentiating the algebraic condition w.r.t. t.

This is analogous to the treatment of higher index DAEs. Using the definition of differential index for PDAEs which is given in [33] the PDAE (1.15) is of index 2. This index definition is completely analogous to the DAE case (see [22]). In order to be able to solve (1.15) we need consistent initial values $(v(0), \mu(0))$ which solve the algebraic condition as well as an extra consistency equation obtained by differentiating the algebraic constraint w.r.t. time. The solution of this projected equation can then be found using the well known arguments [23], [36] in the context of analytic semigroups and sectorial operators.

We define a weighted norm for $(v, \mu) \in \mathcal{H}^1 \times \mathbb{R}$ by

$$\|(v,\mu)\|_{w,\mathcal{H}^1} = w\|v\|_{\mathcal{H}^1} + |\mu|$$

and denote the ball of radius δ in this norm around (v, μ) by

$$B_{\delta,w}(v,\mu) = \{(u,\lambda) \in \mathcal{H}^1 \times \mathbb{R} : \ \|(v-u, \mu-\lambda)\|_{w,\mathcal{H}^1} \leq \delta\}.$$

Consider a general semilinear equation

$$\begin{aligned} v_t &= \Lambda v + \mu\phi + \varphi(v,\mu), \quad v(0) = v^0 \\ 0 &= \langle \psi, v \rangle, \end{aligned} \tag{1.19}$$

where the right hand side φ satisfies the following hypothesis:

Hypothesis 1.14 *Assume that $\varphi : \mathcal{H}^1 \times \mathbb{R} \to \mathcal{L}_2$ satisfies $\varphi(0,0) = 0$ and there exist $\varrho_0, K, C_L > 0$ such that for all $\varrho < \varrho_0$ and $(v,\mu), (u,\lambda) \in B_{\varrho,1}(0)$ the following inequalities hold:*

$$\|\varphi(v,\mu) - \varphi(u,\lambda)\|_{\mathcal{L}_2} \leq C_L(\|v-u\|_{\mathcal{H}^1} + \max\{\|v\|_{\mathcal{H}^1}, \|u\|_{\mathcal{H}^1}\}|\mu - \lambda|) \tag{1.20}$$

$$\|\varphi(v,\mu)\|_{\mathcal{L}_2} \leq K\varrho(\|v\|_{\mathcal{H}^1} + |\mu|). \tag{1.21}$$

Now we can formulate the main stability theorem.

Theorem 1.15 *Let Λ be the operator defined in (1.7) and assume that (EC) and (SC) hold. Assume that φ satisfies Hypothesis 1.14 and that $\mathcal{N}(\Lambda) := \operatorname{span}\{\phi\}$, $\psi \in \mathcal{H}^1$ and $\langle \psi, \phi \rangle \neq 0$.*
Then zero is a stable stationary solution of the PDAE (1.19). More precisely, there exists $\rho > 0$ such that for each v^0 with $\|v^0\|_{\mathcal{H}^1} < \rho$ there exists a unique solution $(v(t), \mu(t))$ on $(0, \infty)$ of (1.19) which satisfies the exponential estimate

$$\|v(t)\|_{\mathcal{H}^1} + |\mu(t)| \leq C\mathrm{e}^{-\nu t}\|v^0\|_{\mathcal{H}^1}, \ \forall t \geq 0, \tag{1.22}$$

for some $\nu, C > 0$.

Here we consider the kernel of Λ in $\mathcal{H}^2$. Since $\phi = \bar{u}' \in \mathcal{H}^2$ by Remark 1.7, it makes no difference if we consider it in $\mathcal{H}^2$ or $\mathcal{H}^1$.

Before we proceed with the proof of this stability theorem we show how the proof of Theorem 1.13 follows from an application of Theorem 1.15 to the PDAE (1.15).

Proof of Theorem 1.13:
We show that $\varphi(v,\mu) = \omega(v) + \mu v_x$ (see (1.16)) satisfies (1.20). Clearly, $\varphi(0,0) = 0$ by construction and using Hypothesis 1.9 together with the Sobolev imbedding $\mathcal{H}^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$, from which we obtain $\|v\|_\infty \le C\|v\|_{\mathcal{H}^1}$ we have for all $(v,\mu),(u,\lambda) \in B_{\varrho,1}(0)$ the following estimates (we suppress the argument x in order to improve readability and denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^m$):

$$\begin{aligned}
\|\omega(v) - \omega(u)\|^2_{\mathcal{L}_2} = \int_{\mathbb{R}} &\|f(\bar{u}+v, \bar{u}'+v_x) - f(\bar{u}+u, \bar{u}'+u_x) \\
&\qquad - D_1 f(\bar{u},\bar{u}')(v-u) - D_2 f(\bar{u},\bar{u}')(v_x - u_x)\|^2\, dx \\
= \int_{\mathbb{R}} &\|f_1(\bar{u}+v)(\bar{u}'+v_x) - f_1(\bar{u}+u)(\bar{u}'+u_x) - f_1'(\bar{u})(\bar{u}', v-u) - f_1(\bar{u})(v_x-u_x) \\
&+ f_2(\bar{u}+v) - f_2(\bar{u}+u) - f_2'(\bar{u})(v-u)\|^2\, dx \\
\le c \int_{\mathbb{R}} &\|(f_1(\bar{u}+v) - f_1(\bar{u}+u))\bar{u}'\|^2 + \|(f_1(\bar{u}+v) - f_1(\bar{u}+u))v_x\|^2 \\
&+ \|(f_1(\bar{u}+u) - f_1(\bar{u}))(v_x-u_x)\|^2 + \|f_1'(\bar{u})(\bar{u}', v-u)\|^2 \\
&\quad + \|f_2(\bar{u}+v) - f_2(\bar{u}+u)\|^2 + \|f_2'(\bar{u})(v-u)\|^2\, dx \\
\le cc_1 \int_{\mathbb{R}} &\|v-u\|^2 + \|v-u\|^2\|v_x\|^2 + \|u\|^2\|v_x-u_x\|^2\, dx \\
\le cc_1 (&\|v-u\|^2_{\mathcal{L}_2} + \|v-u\|^2_{\mathcal{H}^1}\|v\|^2_{\mathcal{H}^1} + \|u\|^2_{\mathcal{H}^1}\|v-u\|^2_{\mathcal{H}^1}) \\
\le c\|&v-u\|^2_{\mathcal{H}^1}
\end{aligned}$$

and

$$\begin{aligned}
\|\mu v_x - \lambda u_x\|_{\mathcal{L}_2} &\le \|v_x\|_{\mathcal{L}_2}|\mu-\lambda| + |\lambda|\|v_x - u_x\|_{\mathcal{L}_2} \\
&\le \|v\|_{\mathcal{H}^1}|\mu-\lambda| + |\lambda|\|v-u\|_{\mathcal{H}^1} \le \varrho\|(v-u, \mu-\lambda)\|_{1,\mathcal{H}^1}.
\end{aligned}$$

The consequence (1.10) of Hypothesis 1.9 leads for $\|v\|_{\mathcal{H}^1} + |\mu| \le \varrho$ to

$$\begin{aligned}
\|\omega(v)\|_{\mathcal{L}_2}^2 &\le \int_{\mathbb{R}} \|f(\bar{u}+v, \bar{u}'+v_x) - f(\bar{u}, \bar{u}') \\
&\qquad - D_1 f(\bar{u}, \bar{u}')v - D_2 f(\bar{u}, \bar{u}')v_x\|^2 \, dx \\
&\le 2 \int_{\mathbb{R}} \int_0^1 \|[D_1 f(\bar{u}+tv, \bar{u}'+tv_x) - D_1 f(\bar{u}, \bar{u}')]v\|^2 \, dt \\
&\qquad + \int_0^1 \|[D_2 f(\bar{u}+tv, \bar{u}'+tv_x) - D_2 f(\bar{u}, \bar{u}')]v_x\|^2 \, dt \, dx \\
&\le 4L^2 \int_{\mathbb{R}} \int_0^1 t^2(\|v\| + \|v_x\|)^2 \|v\|^2 \, dt + \int_0^1 t^2 \|v\|^2 \|v_x\|^2 \, dt \, dx \\
&\le \frac{4}{3} L^2 \int_{\mathbb{R}} ((\|v\| + \|v_x\|)^2 + \|v_x\|^2) \|v\|^2 \, dx \\
&\le 4L^2 \|v\|_\infty^2 \int_{\mathbb{R}} (\|v\| + \|v_x\|)^2 \, dx \le 4(Lc)^2 \|v\|_{\mathcal{H}^1}^2 \|v\|_{\mathcal{H}^1}^2 \\
&\le (2Lc\varrho \|v\|_{\mathcal{H}^1})^2.
\end{aligned}$$

□

Remark 1.16 Note that most of the proofs below are valid as well, if the following weaker variant of the eigenvalue condition (EC) is satisfied.

Weak eigenvalue condition (ECw):
Considered as an operator in X the differential operator Λ has a simple isolated eigenvalue 0.

This includes the case of unstable traveling waves, where the whole construction of a solution via semigroups works in the same way. Clearly, the stability result does not hold, since the estimates for the solution operator of the linear equation are not exponentially decaying in time in that case. In order to streamline the presentation we restrict ourselves to the stable case Re $(\sigma(\Lambda) \setminus \{0\}) < 0$ and indicate the changes in the proofs that are necessary for the unstable case.

In the following we always assume without further notice that for the operator Λ defined in (1.7) the conditions (EC) and (SC) hold.

1.3.1 The linear inhomogenous equation

A first step will be the proof of a "variation of constants" formula for the linear inhomogeneous equation which will then lead to an integral representation of the solution of (1.19).

We consider an inhomogenous linear equation of the type

$$v_t = \Lambda v + \phi\mu + r, \quad v(0) = v^0 \tag{1.23}$$

$$0 = \langle \psi, v \rangle \tag{1.24}$$

where $r : (0, \tau) \to \mathcal{L}_2$. Assume that the initial value $v^0 \in \mathcal{H}^1$ is consistent,i.e. $\langle \psi, v^0 \rangle = 0$.

The solution of the PDAE (1.23), (1.24) can be reduced to the solution of a corresponding projected PDE as follows. We define the bilinear form $a : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{R}$ via

$$a(u,v) = \int_{\mathbb{R}} -u_x(x)^T A v_x(x) + u(x)^T (B(x)v_x(x) + C(x)v(x))\, dx$$

where $A, B(\cdot), C(\cdot)$ are the bounded matrix functions defined in (1.5). For $\psi \in \mathcal{H}^1$ we get via integration by parts

$$a(\psi, v) = \langle \psi, \Lambda v \rangle \quad \text{for} \quad v \in \mathcal{H}^2 \tag{1.25}$$

and

$$|a(\psi, v)| \leq C_\psi \|v\|_{\mathcal{H}^1}. \tag{1.26}$$

Furthermore the condition $\langle \psi, \phi \rangle \neq 0$ implies

$$|\langle \psi, \phi \rangle^{-1}| \leq C_{\psi,\phi} \tag{1.27}$$

and we define the projector P onto $\psi^\perp$ along ϕ by

$$Pv = v - \phi \langle \psi, \phi \rangle^{-1} \langle \psi, v \rangle. \tag{1.28}$$

Under the assumptions (1.27) the boundedness of P follows for $\diamond \in \{\mathcal{L}_2, \mathcal{H}^1\}$ from

$$\|Pv\|_\diamond \leq \|v\|_\diamond + \|\phi\|_\diamond |\langle \psi, \phi \rangle^{-1}||\langle \psi, v \rangle| \leq (1 + C_{\psi,\phi} \|\phi\|_\diamond \|\psi\|_{\mathcal{L}_2}) \|v\|_\diamond.$$

Note that (1.25) implies for $v \in \mathcal{H}^2$

$$P\Lambda v = \Lambda v - \phi \langle \psi, \phi \rangle^{-1} a(\psi, v). \tag{1.29}$$

With these definitions we have the following lemma:

Lemma 1.17 *Let $r \in \mathcal{C}([0,\tau), \mathcal{L}_2)$ and let the estimate (1.27) hold. If the pair (v, μ) is a solution of (1.23), (1.24) on the interval $(0,\tau)$ with consistent initial conditions*

$$v^0 \in \mathcal{H}^1, \langle \psi, v^0 \rangle = 0$$

then v is a solution on $(0,\tau)$ of the PDE

$$v_t = P(\Lambda v + r), \qquad v(0) = v^0 \in \mathcal{H}^1 \cap \mathcal{R}(P) \tag{1.30}$$

and μ satisfies on $[0,\tau)$

$$\mu(t) = -\langle \psi, \phi \rangle^{-1} (a(\psi, v(t)) + \langle \psi, r(t) \rangle). \tag{1.31}$$

Proof: Differentiating the algebraic condition (1.24) with respect to $t \in (0,\tau)$ we get (1.31). Inserting this expression for μ into (1.23) one arrives at (1.30).

From the continuity of $a(\psi,\cdot)$, $v \in \mathcal{C}([0,\tau), \mathcal{H}^1)$ and $r \in \mathcal{C}([0,\tau), \mathcal{L}_2)$ follows $\mu \in C([0,\tau), \mathbb{R})$.

Conversely, from v being a solution of (1.30) equation (1.24) follows. And with (1.31) and (1.29) we obtain from (1.30)

$$\begin{aligned} v_t &= P(\Lambda v + r) = \Lambda v - \phi \langle \psi, \phi \rangle^{-1} a(\psi, v) + Pr \\ &= \Lambda v + \phi(\mu + \langle \psi, \phi \rangle^{-1} \langle \psi, r \rangle) + r - \phi \langle \psi, \phi \rangle^{-1} \langle \psi, r \rangle = \Lambda v + \phi\mu + r. \end{aligned}$$

□

We consider the operator $P\Lambda$ in the subspace $\mathcal{R}(P) \cap \mathcal{L}_2$. If we can show, that $\Lambda_P := P\Lambda_{|\mathcal{R}(P)}$ is sectorial then we can solve the linear inhomogenous equation (1.30) via

$$v(t) = \mathrm{e}^{\Lambda_P t} v^0 + \int_0^t \mathrm{e}^{\Lambda_P (t-s)}\, Pr(s)\, ds$$

where the solution operator $\mathrm{e}^{\Lambda_P t}$ is defined using the resolvent $R_s(\Lambda_P) := (sI - \Lambda_P)^{-1}$ as the Dunford integral (see [36], [23])

$$\mathrm{e}^{\Lambda_P t} = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{st} R_s(\Lambda_P)\, ds \tag{1.32}$$

and the curve Γ has to be defined appropriately.

Using this projected system we can now construct the solution of the PDAE (1.23),(1.24) via a "variation of constants" formula (compare [23], Thm. 3.2.2 and [36], Thm. 6.2.3 for the PDE case).

Lemma 1.18 *Let $r : [0,\tau) \to \mathcal{L}_2$ be bounded and Lipschitz continuous and assume $\psi \in \mathcal{H}^1$.*

Then there exists $\tau > 0$ such that a unique solution (v, μ) of

$$\begin{aligned} v_t &= \Lambda v + \mu\phi + r, \\ 0 &= \langle \psi, v \rangle \end{aligned}$$

on $(0,\tau)$ exists for initial values $v(0) = v^0 \in \mathcal{H}^1 \cap \mathcal{R}(P)$, namely

$$\begin{aligned} v(t) &= \mathrm{e}^{\Lambda_P t} v^0 + \int_0^t \mathrm{e}^{\Lambda_P (t-s)} P\, r(s)\, ds, \\ \mu(t) &= -\langle \psi, \phi \rangle^{-1} (a(\psi, v(t)) + \langle \psi, r(t) \rangle), \quad t \in [0,\tau). \end{aligned}$$

In order to prove this lemma we need resolvent estimates which justify the integral representation in (1.32) and lead to estimates of $\mathrm{e}^{\Lambda_P t}$ which are exponentially decaying in t.

1.3.2 Resolvent estimates

We will discuss the resolvent estimates in the following three regions in $\mathbb{C}$:

$$\Omega_\epsilon \; : \; |s| < \epsilon, \; \mathrm{Re}\; s \geq -\beta$$

$$\Omega_{C_0} \; : \; \epsilon \leq |s| \leq K, \; \mathrm{Re}\; s \geq -\beta$$

$$\Omega_\infty \; : \; |s| > K, |\arg(s)| < \zeta \in (\tfrac{\pi}{2}, \pi)$$

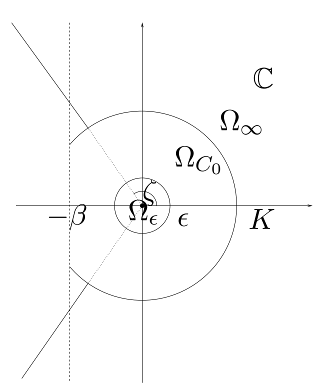


Figure 1.2: Regions for resolvent estimates

As has been noted before, resolvent estimates have been shown for Λ in a sector in (1.3). More precisely, for $s \in \varrho(\Lambda)$ and large $|s|$ an estimate depending on $|s|$, and for s in a compact set a uniform estimate has been shown in [6], [60]. This is summarized in the following lemma.

Lemma 1.19 *There exists a sector $\mathcal{S}_{\omega,\zeta} \subset \rho(\Lambda)$ such that if we define $v = R_s(\Lambda)r$, then there exists $K > 0$ such that for each $s \in \mathcal{S}_{\omega,\zeta}$ with $|s| > K$*

$$|s|^2\|v\|^2_{\mathcal{L}_2} + |s|\|v\|^2_{\mathcal{H}^1} \leq C\|r\|^2_{\mathcal{L}_2}. \tag{1.33}$$

For s in a compact set $S_C \subset \rho(L)$ we have a uniform estimate

$$\|v\|_{\mathcal{H}^2} \leq C\|r\|_{\mathcal{L}_2}. \tag{1.34}$$

Note, that from (EC) and (SC) we conclude that there exists $K > 0$ such that the estimate (1.34) holds in Ω_{C_0} and (1.33) in Ω_∞.

These results will be used to show corresponding estimates for the projected system, which then lead to estimates for solutions of a bordered system by introducing an appropriate parameter μ:

Lemma 1.20 *Let $r \in \mathcal{L}_2$, then $v \in \mathcal{H}^2$ solves the resolvent equation*

$$(sI - P\Lambda)v = Pr \tag{1.35}$$

and μ satisfies

$$\mu = -\langle\psi, \phi\rangle^{-1} a(\psi, v) \tag{1.36}$$

if and only if the pair $(v, \mu) \in \mathcal{H}^2 \times \mathbb{R}$ is a solution of the bordered system

$$(sI - \Lambda)v - \phi\mu = Pr \tag{1.37}$$

$$\langle\psi, v\rangle = 0. \tag{1.38}$$

Proof: Let (v, μ) be a solution of (1.35),(1.36), then $v \in \mathcal{R}(P)$, i.e. $\langle\psi, v\rangle = 0$ and using (1.29) we get

$$Pr = (sI - P\Lambda)v = (sI - \Lambda)v + \phi\langle\psi, \phi\rangle^{-1} a(\psi, v) = (sI - \Lambda)v + \phi\mu.$$

Conversely, left multiplication of (1.37) with ψ gives

$$0 = \langle\psi, (sI - \Lambda)v\rangle + \langle\psi, \phi\rangle\mu = s\langle\psi, v\rangle - a(\psi, v) + \langle\psi, \phi\rangle\mu.$$

This implies with (1.38) equation (1.36). Inserting this expression into (1.37) one arrives at (1.35). □

The projection P has the effect, that zero is removed from the spectrum of Λ_P. Note that in the proof of Thm. 2.18 in [60] and Ex. 6 in [23] which deal with the stability of relative equilibria, a special projection with ψ being the left zero eigenfunction of Λ has been used to achieve the same effect. We emphasize that this assumption is not made here. In numerical approximations we cannot assume to know even approximately the left eigenfunction (see Chapter 4).

In the following we will prove estimates of the solutions of (1.37), (1.38) in the regions $\Omega_\epsilon, \Omega_{C_0}, \Omega_\infty$ which will ensure the existence of the integrals in (1.32).

Lemma 1.21 *Let Λ be the operator defined in (1.5) and assume that (EC) and (SC) hold. Let $\mathcal{N}(L) = \mathrm{span}\{\phi\}$ and assume that $\psi \in \mathcal{H}^1$ obeys condition (1.27).*

Then there exist constants $C_R, K > 0$ such that for each $s \in \Omega_{C_0} \cup \Omega_\infty$ there exists a solution (v, μ) of (1.37),(1.38) for which the following estimates hold

$$\|v\|_{\mathcal{H}^1} + |\mu| \leq C_R \|r\|_{\mathcal{L}_2}, \qquad \text{as} \quad s \in \Omega_{C_0} \tag{1.39}$$

and

$$|s|^2 \|v\|^2_{\mathcal{L}_2} + |s| \|v\|^2_{\mathcal{H}^1} + |\mu|^2 \leq C_R \|r\|^2_{\mathcal{L}_2}, \qquad \text{as} \quad s \in \Omega_\infty. \tag{1.40}$$

Proof: By Lemma 1.19 there exists $K > 0$ such that the resolvent estimate (1.34) holds in the bounded set Ω_{C_0}, and (1.33) holds in Ω_∞. For $s \in \rho(\Lambda)$ we can solve equation (1.37) by taking $\phi\mu$ to the right hand side and get

$$v = R_s(\Lambda)(Pr + \phi\mu).$$

By inserting v into (1.38) we obtain

$$\mu = -\langle \psi, R_s(\Lambda)\phi \rangle^{-1} \langle \psi, R_s(\Lambda) Pr \rangle$$

which leads to

$$v = Q R_s(\Lambda) Pr$$

where the projector Q is defined by

$$Qw = w - R_s(\Lambda)\phi \ \langle \psi, R_s(\Lambda)\phi \rangle^{-1} \langle \psi, w \rangle.$$

In order to estimate μ and Q we need a lower bound of $|\langle \psi, R_s(\Lambda)\phi \rangle|$. Use

$$\phi = R_s(\Lambda)\Lambda\phi - s R_s(\Lambda)\phi = -s R_s(\Lambda)\phi \tag{1.41}$$

and multiply with ψ from the left. This gives

$$\langle \psi, \phi \rangle = -s \langle \psi, R_s(\Lambda)\phi \rangle$$

which implies

$$|\langle \psi, R_s(\Lambda)\phi \rangle|^{-1} = |s| |\langle \psi, \phi \rangle|^{-1} \leq |s| C_{\psi,\phi}. \tag{1.42}$$

Together with $|s| \leq C$ we can estimate Q by

$$\|Qw\|_{\mathcal{H}^1} \leq \|w\|_{\mathcal{H}^1} + \|R_s(\Lambda)\phi\|_{\mathcal{H}^1} \ |\langle \psi, R_s(\Lambda)\phi \rangle|^{-1} |\langle \psi, w \rangle| \leq C_Q \|w\|_{\mathcal{H}^1}.$$

Using the uniform estimate $\|R_s(\Lambda) Pr\|_{\mathcal{H}^1} \leq C_K \|r\|_{\mathcal{H}^1}$ from (1.34) we obtain

$$\|v\|_{\mathcal{H}^1} \leq \|Q R_s(\Lambda) Pr\|_{\mathcal{H}^1} \leq C_Q C_R \|Pr\|_{\mathcal{L}_2} \leq C \|r\|_{\mathcal{L}_2}.$$

It remains to estimate μ:

$$|\mu| \le |\langle\psi, R_s(\Lambda)\phi\rangle^{-1}||\langle\psi, R_s(\Lambda)Pr\rangle| \le |s|C_{\psi,\phi}\|\psi\|_{\mathcal{L}_2}C_R C_P\|r\|_{\mathcal{L}_2} \le C\|r\|_{\mathcal{L}_2}.$$

For $s \in \Omega_\infty$ equation (1.33) states

$$\|R_s(\Lambda)r\|_{\mathcal{L}_2} \le \frac{C_R}{|s|}\|r\|_{\mathcal{L}_2}, \quad \text{and} \quad \|R_s(\Lambda)r\|_{\mathcal{H}^1} \le \frac{C_R}{\sqrt{|s|}}\|r\|_{\mathcal{L}_2}.$$

From this follows with (1.42)

$$\|Qw\|_{\mathcal{L}_2} \le \|w\|_{\mathcal{L}_2} + \frac{C_R}{|s|}\|\phi\|_{\mathcal{L}_2}|s|C_{\psi,\phi}|\langle\psi, w\rangle| \le C_Q\|w\|_{\mathcal{L}_2},$$

as well as

$$\|Qw\|_{\mathcal{H}^1} \le \|w\|_{\mathcal{H}^1} + \frac{C_R}{\sqrt{|s|}}\|\phi\|_{\mathcal{L}_2}\sqrt{|s|}C_{\psi,\phi}|\langle\psi, w\rangle| \le C_Q\|w\|_{\mathcal{H}^1}.$$

Thus we obtain

$$\|v\|_{\mathcal{L}_2} \le C_Q\frac{C_R}{|s|}C_P\|r\|_{\mathcal{L}_2} \le \frac{C}{|s|}\|r\|_{\mathcal{L}_2}$$

and similarly

$$\|v\|_{\mathcal{H}^1} \le \frac{C}{\sqrt{|s|}}\|r\|_{\mathcal{L}_2}.$$

□

Note that the above result is still true, if we use in (1.41) for ϵ small the weaker condition $\|\Lambda\phi\|_{\mathcal{L}_2} < \epsilon$ instead of $\Lambda\phi = 0$.

It remains to prove a resolvent estimate in Ω_ϵ for a sufficiently small ϵ, i.e. to find a solution of (1.37),(1.38). This will be constructed in a similar fashion as in the proof of Theorem 3.7 in [64]. Therefore we need some results concerning exponential dichotomies for ODEs, which are summarized in the Appendix.

Lemma 1.22 *Under the same assumptions as in Lemma 1.21, there exists $\epsilon > 0$ such that (1.37),(1.38) possesses a unique solution (v, μ) for $s \in B_\epsilon(0)$ which satisfies the following uniform estimate in s*

$$\|v\|_{\mathcal{H}^1} + |\mu| \le K\|r\|_{\mathcal{L}_2}. \tag{1.43}$$

Proof: Using $z = (v, v')$ we can transform (1.37),(1.38) into the first order system

$$L(s)z = R - \Phi\mu, \tag{1.44}$$
$$\langle\Psi, z\rangle = 0 \tag{1.45}$$

where

$$L(s)z = z' - M(\cdot, s)z, \text{ with } M(x,s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C(x)) & -A^{-1}B(x) \end{pmatrix}, \tag{1.46}$$

$$R = \begin{pmatrix} 0 \\ -A^{-1}Pr \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 \\ -A^{-1}\phi \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}.$$

Here $A, B(\cdot), C(\cdot)$ are the matrices defined in (1.5). It has been shown in Lemma 3.30, [60] and [28] that (SC) implies that the matrices $M^{\pm}(s) = \lim_{x\to\pm\infty} M(x,s)$ are hyperbolic for all $s \in \mathbb{C}$ with Re $s > -\beta$. Thus for these s the operator $L(s)$ has exponential dichotomies on both half-axes $\mathbb{R}^{\pm}$ with data $(K_{\pm}, \alpha_{\pm}, \pi_{\pm})$. This implies that the operators $L(s)$ are Fredholm operators of index 0 ([49], Remark 3.3). Thus it is sufficient to show the solvability of (1.44), (1.45) for $s = 0$. Then a regular perturbation argument can be used to conclude the solvability for $s \in B_\epsilon(0)$, where $\epsilon > 0$ has to be small enough.

As in the proof of Theorem 3.7 in [64] we construct solutions $z^{\pm}$ of (1.44) for $s = 0$ on each half line using the ansatz

$$z^{\pm} = S(\cdot, 0)z_0^{\pm} + \bar{s}^{\pm}(R - \Phi\mu)$$

where S denotes the solution operator of the linear equation (A.11) and $\bar{s}^{\pm}(r)$ is the corresponding solution of the linear inhomogeneous equation on $\mathbb{R}^{\pm}$ as given in (A.17) in the appendix. The function $z(x) = \begin{cases} z^+(x), & x \geq 0 \\ z^-(x), & x < 0 \end{cases}$ is a solution, if $z^-(0) = z^+(0) \in \mathcal{N}(P_-(0)) \cap \mathcal{R}(P_+(0))$ and if z solves the phase condition (1.45).

This is equivalent to (cf. Proof of Theorem 3.7 in [64])

$$T(z_0^-, z_0^+, \mu) = \begin{pmatrix} \rho \\ \delta \end{pmatrix} \tag{1.47}$$

where $T : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{2m} \times \mathbb{R}$ is given by

$$T = \begin{pmatrix} I & -I & \Omega \\ \Theta & \Lambda & \Xi \end{pmatrix}$$

with

$$\begin{aligned}
\Omega &= [\bar{s}^+(\Phi)](0) - [\bar{s}^-(\Phi)](0), \\
\Theta &= \int_{-\infty}^0 \Psi(x)^T S(x,0)\, dx, \quad \Lambda = \int_0^\infty \Psi(x)^T S(x,0)\, dx, \\
\Xi &= -\int_{-\infty}^0 \Psi(x)^T [\bar{s}^-(\Phi)](x)\, dx - \int_0^\infty \Psi(x)^T [\bar{s}^+(\Phi)](x)\, dx
\end{aligned}$$

and

$$\begin{aligned}
\rho &= [\bar{s}^+(R)](0) - [\bar{s}^-(R)](0) \\
\delta &= -\int_{-\infty}^0 \Psi(x)^T [\bar{s}^-(R)](x)\, dx - \int_0^\infty \Psi(x)^T [\bar{s}^+(R)](x)\, dx
\end{aligned}$$

The injectivity of T can be shown in the same way as in the proof of Theorem 3.7 in [64]. In the following we indicate only the main steps. From the eigenvalue condition (EC) we have $\mathcal{N}(L(0)) = \text{span}\{\phi\}$. For the transformed system this yields the nondegeneracy condition

$$z' - M(0)z = \Phi\mu \implies \mu = 0, \text{ and } z = c\begin{pmatrix} \phi \\ \phi' \end{pmatrix}, \ c \in \mathbb{R}. \tag{1.48}$$

This implies the injectivity of T, since for any solution of $T(z_0^-, z_0^+, \mu) = 0$ we can construct

$$v(x) = S(x,0)z_0^{\pm} + [\bar{s}^{\pm}(\Phi)](x), \text{for} \pm x \geq 0$$

which would then yield a bounded solution of

$$z' - M(0)z = \Phi\mu.$$

From the nondegeneracy condition (1.48) follows $\mu = 0$ and $v = c\phi$ and from $\langle \phi, \psi \rangle \neq 0$ we obtain $c = 0$.

Since T is a map between finite dimensional spaces, it follows that T is invertible. Thus there exists a solution of (1.47) which can be estimated for $R \in \mathcal{L}_2$ as follows:

$$\|z_0^-\| + \|z_0^+\| + |\mu| \leq C(\|\delta\| + \|\rho\|) \leq C\|R\|_{\mathcal{L}_2},$$

since we have from (A.18)

$$\|\delta\| \leq C\|\Psi\|_{\mathcal{L}_2}\|\bar{s}^{\pm}(R)\|_{\mathcal{L}_2} \leq C\|\Psi\|_{\mathcal{L}_2}\|R\|_{\mathcal{L}_2}$$

and

$$\|\rho\| \leq \|[\bar{s}^+(R)](0)\| + \|[\bar{s}^-(R)](0)\| \leq C\|R\|_{\mathcal{L}_2}.$$

From this, and by using the dichotomy estimates, we obtain

$$\|z\|_{\mathcal{L}_2} + |\mu| \leq C\|R\|_{\mathcal{L}_2}.$$

Finally, using the definition of z and R we obtain for v and μ the desired estimate (1.43). □

A particular consequence of the uniform estimate in $\Omega_\epsilon \cup \Omega_{C_0}$ and the s dependent estimate in Ω_∞ are the following sectorial estimates:

Corollary 1.23 *There exist $C > 0$ and a sector $\bar{\mathcal{S}}_{a,\theta} \subset \rho(\Lambda_P)$ with $a > 0$, $\theta \in (\frac{\pi}{2}, \pi)$ such that for all $s \in \bar{\mathcal{S}}_{a,\theta}$ for*

$$v = (sI - P\Lambda)^{-1}Pr$$

the estimates

$$\|v\|_{\mathcal{L}_2} \leq \frac{C}{|s+a|}\|r\|_{\mathcal{L}_2}, \qquad \|v\|_{\mathcal{H}^1} \leq \frac{C}{\sqrt{|s+a|}}\|r\|_{\mathcal{L}_2} \tag{1.49}$$

hold.

Proof: We summarize the estimates (1.43),(1.39), (1.40) in

$$\|v\|_{\mathcal{H}^1} + |\mu| \leq C\|r\|_{\mathcal{L}_2}, \quad \text{for } s \in \Omega_\epsilon \cup \Omega_{C_0} \tag{1.50}$$

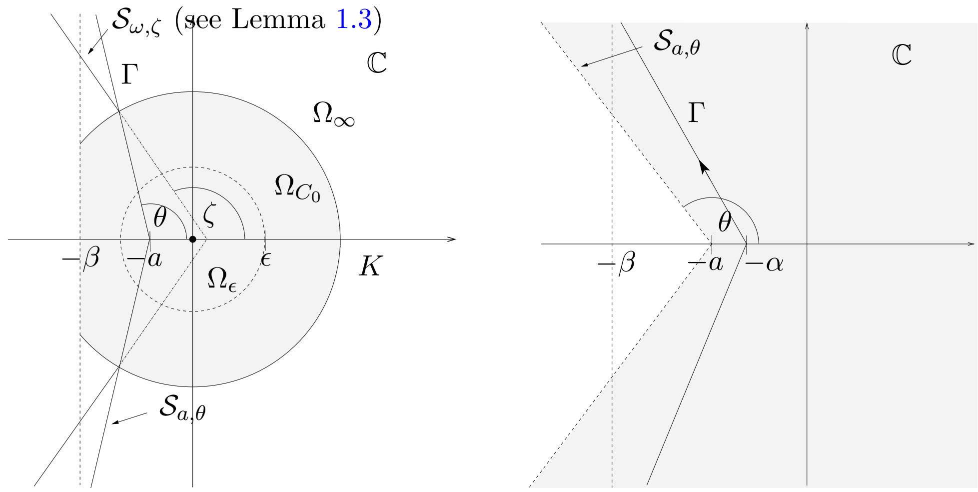
and

$$|s|^2\|v\|^2_{\mathcal{L}_2} + |s|\|v\|^2_{\mathcal{H}^1} + |\mu|^2 \leq C\|r\|^2_{\mathcal{L}_2}, \quad \text{for } s \in \Omega_\infty, \tag{1.51}$$

where $C > 0$ does not depend on r and s.

Thus we can construct a sector as depicted in Figure 1.3(a) such that the estimates (1.49) hold for some $a \in (0, \beta)$. □

Similar estimates with $a \in \mathbb{R}$ can be shown in the case where unstable eigenvalues exist.



(a) Overview of regions for estimates

(b) Path of integration

Figure 1.3: Path of integration for definition (1.32)

1.3.3 Estimates of the solution operator

The estimates (1.50), (1.51) show that Λ_P is sectorial. Therefore an application of [23], Theorem 1.3.4 and Theorem 1.4.3 or of [36], Theorem 4.5.10, Corollary 4.5.11 and Theorem 4.5.14 implies that the solution operator $e^{\Lambda_P t}$ is well defined via (1.32) and satisfies the corresponding estimates for sectorial operators. We summarize this fact in the following lemma.

Lemma 1.24 *Assume that the sectorial operator Λ_P satisfies (1.49). Assume further that for ψ, which occurs in the definiton of P in (1.28), condition (1.26) holds.*

Then $e^{\Lambda_P t}$ is well defined via (1.32) and for $r \in \mathcal{L}_2 \cap \mathcal{R}(P)$ the exponential estimates

$$\|e^{\Lambda_P t} r\|_{\mathcal{L}_2} \le K e^{-\alpha t} \|r\|_{\mathcal{L}_2}, \quad \|e^{\Lambda_P t} r\|_{\mathcal{H}^1} \le K e^{-\alpha t} t^{-\frac{1}{2}} \|r\|_{\mathcal{L}_2} \tag{1.52}$$

hold for some $K > 0$. For $t > 0$ the derivative w.r.t. t exists and

$$\frac{d}{dt} e^{\Lambda_P t} = \Lambda_P e^{\Lambda_P t}.$$

Note that $\Lambda_P^{\frac{1}{2}} e^{\Lambda_P t} = e^{\Lambda_P t} \Lambda_P^{\frac{1}{2}}$ implies with (1.52) $\|e^{\Lambda_P t} r\|_{\mathcal{H}^1} \le K e^{-\alpha t} \|r\|_{\mathcal{H}^1}$ for $r \in \mathcal{H}^1$.

Since $\alpha > 0$, the above estimates are exponentially decaying for $t \to \infty$. This will be used in the proof of the stability theorem 1.15.

The definition (1.32) is valid for an unstable equilibrium as well. Then the above estimates are not decaying anymore. Nevertheless $e^{\Lambda_P t}$ allows an estimate by $e^{\alpha t}$. The path $\Gamma \subset \mathcal{S}_{a,\theta}$ in (1.32) can be chosen as follows (see Figure 1.3(b))

$$\Gamma = \{\gamma(t),\ t \in \mathbb{R}\}, \text{ where } \gamma(t) = \begin{cases} \gamma_-(t) = -\alpha + t e^{-i\theta}, & t \le 0 \\ \gamma_+(t) = -\alpha + t e^{i\theta}, & t > 0 \end{cases}$$

where $\alpha \in (0, a)$ and $\theta \in (\frac{\pi}{2}, \pi)$.

In order to ensure regularity of the solution of the inhomogenous equation in Lemma 1.18 we use Theorem 6.2.1 and Theorem 6.2.3 in [36] (cf. Lemma 3.2.1 in [23]). Below we state a version which has been adapted to the function spaces used here. For the definition and main properties of the Bochner integral see [36], Section 4.2.

Lemma 1.25 *Let $r : [0,T] \to \mathcal{L}_2$ be Bochner integrable and bounded. For $t \in [0,T]$ define*

$$R(t) = \int_0^t e^{\Lambda_P(t-s)}\, Pr(s)\, ds$$

Then $R \in \mathcal{C}([0,T], \mathcal{H}^1) \cap \mathcal{C}^1((0,T), \mathcal{L}_2)$ with $R(t) \in \mathcal{D}(\Lambda_P)$ for $t \in (0,T)$, $R(0) = 0$ and

$$\frac{d}{dt} R(t) = \Lambda_P R(t) + Pr(t), \quad \text{for } t \in (0,T).$$

Now we can prove Lemma 1.18 using the above Lemma and the estimates (1.52).
Proof of Lemma 1.18: Using Lemma 1.17 we reduce the solution of (1.23),(1.24) to the solution of (1.30). Note that by definition $\mathcal{R}(P) = \{v \in \mathcal{L}_2 : \ \langle \psi, v \rangle = 0\}$ and by Lemma 1.25 we get that v defined by (1.30) satisfies conditions 2. and 4. of Definition 1.11. From the continuity of $a(\psi, \cdot)$ and the properties of r follows $\mu \in C([0,\tau), \mathbb{R})$ and therefore $\varphi(v, \mu)$ is continuous from $[0, \tau)$ into $\mathcal{L}_2$ as well. □

Using the result for the inhomogeneous equation we can prove now a "variation of constants" formula along the lines of Theorem 3.2.2 and Lemma 3.3.2 in [23] taking into account the modified definition of solution due to [35] as in Thm. 6.4.3 in [36].

Lemma 1.26 *Let $\tau \in (0, \infty]$ be given such that $\varphi : \mathcal{H}^1 \times \mathbb{R} \to \mathcal{L}_2$ is locally Lipschitz, i.e. there exists $\rho > 0$ such that for $(u, \lambda), (v, \mu) \in B_{\rho,1}(0)$*

$$\|\varphi(u, \lambda) - \varphi(v, \mu)\|_{\mathcal{L}_2} \le K_L(\|u - v\|_{\mathcal{H}^1} + |\lambda - \mu|).$$

Then any solution (v, μ) of

$$\begin{aligned} v_t &= \Lambda v + \mu\phi + \varphi(v, \mu), \\ 0 &= \langle \psi, v \rangle \end{aligned} \tag{1.53}$$

on $(0, \tau)$ with consistent initial value $v(0) = v^0 \in \mathcal{H}^1 \cap \mathcal{R}(P)$ satisfies

$$\begin{aligned} v(t) &= e^{\Lambda_P t} v^0 + \int_0^t e^{\Lambda_P(t-s)} P\ \varphi(v(s), \mu(s))\, ds \\ \mu(t) &= -\langle \psi, \phi \rangle^{-1} (a(\psi, v(t)) + \langle \psi, \varphi(v(t), \mu(t)) \rangle),\ t \in [0, \tau) \end{aligned} \tag{1.54}$$

where P is the projector defined in (1.28).

Conversely, if $v : [0, \tau) \to \mathcal{H}^1$ is continuous, $v(0) \in \mathcal{H}^1 \cap \mathcal{R}(P)$ and if (1.54) holds, then (v, μ) is a solution of (1.53) on $(0, \tau)$.

Proof: The first part follows from Lemma 1.18 applied to $r(s) = \varphi(v(s), \mu(s))$ and the definition of solution 1.11. that $\varphi(v(\cdot), \mu(\cdot))$ is locally Lipschitz

$$\|\varphi(v(s), \mu(s)) - \varphi(v(t), \mu(t))\|_{\mathcal{L}_2} \le C|t - s|$$

and $\varphi(v(\cdot), \mu(\cdot))$ is Bochner integrable (cf. Thm. 6.4.3 in [36]).

Conversely, if (v, μ) is a solution of the integral equation (1.54) then the regularity estimate in Lemma 1.25 implies that v is continuous from $[0, \tau)$ to $\mathcal{H}^1$. Using the representation of μ in (1.54) this implies the continuity of μ in $[0, \tau)$. □

1.3.4 Local existence and uniqueness

Lemma 1.26 will be used to establish the local existence of a solution of the PDAE (1.53). We can now formulate a local existence result similar to Theorem 3.3.3 in [23].

Lemma 1.27 *Let P the projection defined in (1.28) and $\varphi : U \to \mathcal{L}_2$, $U \subset \mathcal{H}^1 \times \mathbb{R}$ be given with $\varphi(0, 0) = 0$ and assume that (1.20) holds for all $(v, \mu), (u, \lambda) \in B_{\hat{\rho},1}(0)$ for some $\hat{\rho} > 0$.*

Then there exist $\delta > 0$ and a weight $w > 1$ such that for any consistent initial condition $v^0 \in \mathcal{H}^1 \cap \mathcal{R}(P)$ with $\|v^0\|_{\mathcal{H}^1} \leq \delta$ the following holds.

There exists a solution μ^0 of the consistency condition

$$\mu^0 = -\langle \psi, \phi \rangle^{-1} (a(\psi, v^0) + \langle \psi, \varphi(v^0, \mu^0) \rangle) \tag{1.55}$$

and there exists $\tau = \tau(v_0) > 0$ such that (1.53) has a solution (v, μ) with

$$\|(v(t), \mu(t))\|_{w,\mathcal{H}^1} < \hat{\rho} \quad \forall t \in (0, \tau). \tag{1.56}$$

Proof: For $\rho \in (0, \min\{C_L, \hat{\rho}\}]$ we obtain from (1.20) for each $w > 1$ and all $(v, \mu), (u, \lambda) \in B_{\rho,w}(0)$ the inequality

$$\|\varphi(v, \mu) - \varphi(u, \lambda)\|_{\mathcal{L}_2} \leq C_L(\|v - u\|_{\mathcal{H}^1} + \frac{1}{w}|\mu - \lambda|). \tag{1.57}$$

Choose $w > \max(4C_{\psi,\phi}(C_\psi + \|\psi\|_{\mathcal{L}_2} C_L), 1)$, $\delta \in (0, \frac{\rho}{4w})$ and define $S_\rho = \{\mu : |\mu| \leq \rho\}$. In order to show the solvability of the consistency equation (1.55) for v^0 with $\|v^0\|_{\mathcal{H}^1} \leq \delta$ we prove that $g : S_{\frac{\rho}{4}} \to S_{\frac{\rho}{4}}$ given by

$$g(\mu) = -\langle \psi, \phi \rangle^{-1} (a(\psi, v^0) + \langle \psi, \varphi(v^0, \mu) \rangle)$$

maps $S_{\frac{\rho}{4}}$ into itself and is contracting. For $\mu \in S_{\frac{\rho}{4}}$ we have with (1.26),(1.27) and (1.57)

$$|g(\mu)| \leq |\langle \psi, \phi \rangle^{-1}||a(\psi, v^0) + \langle \psi, \varphi(v^0, \mu) \rangle| \leq C_{\psi,\phi}(C_\psi \delta + \|\psi\|_{\mathcal{L}_2} C_L(\delta + \frac{1}{w}|\mu|)) < \frac{1}{8}\rho.$$

Similarly (1.27) and (1.57) imply

$$|g(\mu) - g(\lambda)| \leq |\langle \psi, \phi \rangle^{-1}||\langle \psi, \varphi(v^0, \mu) - \varphi(v^0, \lambda) \rangle| \leq C_{\psi,\phi}\|\psi\|_{\mathcal{L}_2} \frac{C_L}{w}|\mu - \lambda| \leq \frac{1}{4}|\mu - \lambda|.$$

Thus the fixed point μ^0 of g exists and lies in $S_{\frac{\rho}{4}}$, and with $w\|v^0\| + |\mu^0| \leq \frac{\rho}{4} + \frac{\rho}{4}$ follows $(v^0, \mu^0) \in B_{\frac{\rho}{2},w}(0)$. Choose $\tau > 0$ such that

$$\|(\mathrm{e}^{\Lambda_P t} - I)v^0\|_{\mathcal{H}^1} < \frac{\rho}{8w}, \quad \forall t \in (0, \tau) \tag{1.58}$$

$$KC_L \int_0^\tau \frac{\mathrm{e}^{-\alpha s}}{\sqrt{s}}\, ds < \frac{w}{4(w+1)} \tag{1.59}$$

where K, α are the constants from (1.52). Using Lemma 1.26, it is sufficient to find a solution of the integral equation (1.54). For $(v,\mu) \in \mathcal{C}([0,\tau],\mathcal{H}^1\times\mathbb{R})$ we define the norm

$$\|(v,\mu)\|^{\tau}_{w,\mathcal{H}^1} = \sup_{t\in[0,\tau]} \|(v(t),\mu(t))\|_{w,\mathcal{H}^1}$$

and denote the set of functions which stay for $t \in [0,\tau]$ in a weighted $\frac{\rho}{2}$-ball around (v^0,μ^0) by S, i.e.

$$S = \{\ (v,\mu) \in \mathcal{C}([0,\tau],\mathcal{H}^1\times\mathbb{R}):\ \|(v-v^0,\mu-\mu^0)\|^{\tau}_{w,\mathcal{H}^1} \le \frac{\rho}{2}\ \}.$$

Then condition (1.57) holds for all $(v,\mu)\in S$.

For $(v,\mu)\in S$ we define $G(v,\mu): [0,\tau]\to\mathcal{L}_2\times\mathbb{R}$ by

$$G(v,\mu)(t) = \begin{pmatrix} \mathrm{e}^{\Lambda_P t}v^0 + \int_0^t \mathrm{e}^{\Lambda_P(t-s)}\varphi(v(s),\mu(s))\,ds \\ -\langle\psi,\phi\rangle^{-1}\left(a(\psi,v(t)) + \langle\psi,\varphi(v(t),\mu(t))\rangle\right)\end{pmatrix}$$

and show that G maps S into itself and is strictly contracting.

From (1.59) follows

$$\left(C_L\frac{\rho}{2} + w\|\varphi(v^0,\mu^0)\|_{\mathcal{L}_2}\right)K\int_0^{\tau}\frac{\mathrm{e}^{-\alpha s}}{\sqrt{s}}\,ds < \frac{\rho}{8}$$

and for $t\in[0,\tau]$ we have with (1.58)

$$\begin{aligned}
\|G(v,\mu)(t)-(v^0,\mu^0)\|_{\mathcal{H}^1,w} &\le w\|(\mathrm{e}^{\Lambda_P t}-I)v^0\|_{\mathcal{H}^1} + w\int_0^t \|\mathrm{e}^{\Lambda_P(t-s)}\varphi(v(s),\mu(s))\|_{\mathcal{H}^1}\,ds \\
&\quad + |-\langle\psi,\phi\rangle^{-1}(a(\psi,v(t))+\langle\psi,\varphi(v(t),\mu(t))\rangle) - \mu^0| \\
&< \frac{\rho}{8} + w\int_0^t K\mathrm{e}^{-\alpha(t-s)}\frac{1}{\sqrt{t-s}}\|\varphi(v(s),\mu(s))\|_{\mathcal{L}_2}\,ds \\
&\quad + |\langle\psi,\phi\rangle^{-1}(a(\psi,v^0)+\langle\psi,\varphi(v^0,\mu^0)\rangle)+\mu^0| \\
&\quad + |\langle\psi,\phi\rangle^{-1}|(|a(\psi,v(t)-v^0)| + |\langle\psi,\varphi(v(t),\mu(t))-\varphi(v^0,\mu^0)\rangle|) \\
&\le \frac{\rho}{8} + (C_L\frac{\rho}{2} + w\|\varphi(v^0,\mu^0)\|_{\mathcal{L}_2})K\int_0^{\tau}\frac{\mathrm{e}^{-\alpha s}}{\sqrt{s}}\,ds + C_{\psi,\phi}(C_\psi + \|\psi\|_{\mathcal{L}_2}C_L)\frac{\rho}{w} \\
&< \frac{\rho}{8} + \frac{\rho}{8} + \frac{\rho}{4} = \frac{\rho}{2}.
\end{aligned}$$

G is contracting for $(u,\lambda),(v,\mu)\in S$ since we have for $t\in[0,\tau]$ by (1.57) and (1.59)

$$\begin{aligned}
\|G(u,\lambda)(t)-G(v,\mu)(t)\|_{w,\mathcal{H}^1} &\le w\int_0^t \|\mathrm{e}^{\Lambda_P(t-s)}(\varphi(u(s),\lambda(s))-\varphi(v(s),\mu(s)))\|_{\mathcal{H}^1}d\ s \\
&\quad + |\langle\psi,\phi\rangle^{-1}|\ (|a(\psi,u(t)-v(t))| + |\langle\psi,\varphi(u(t),\lambda(t))-\varphi(v(t),\mu(t))\rangle|) \\
&\le w\int_0^t K\mathrm{e}^{-\alpha(t-s)}\frac{1}{\sqrt{t-s}}\|\varphi(u(s),\lambda(s))-\varphi(v(s),\mu(s))\|_{\mathcal{L}_2}\,ds \\
&\quad + C_{\psi,\phi}(C_\psi\|u(t)-v(t)\|_{\mathcal{H}^1} + \|\psi\|_{\mathcal{L}_2}C_L\frac{1}{w}\|(u,\lambda)-(v,\mu)\|^{\tau}_{w,\mathcal{H}^1}) \\
&\le KC_L\int_0^{\tau}\frac{\mathrm{e}^{-\alpha s}}{\sqrt{s}}\,ds\ \|(u,\lambda)-(v,\mu)\|^{\tau}_{w,\mathcal{H}^1} \\
&\quad + \frac{1}{w}C_{\psi,\phi}(C_\psi + \|\psi\|_{\mathcal{L}_2}C_L)\|(u,\lambda)-(v,\mu)\|^{\tau}_{w,\mathcal{H}^1} \\
&< \frac{1}{2}\|(u,\lambda)-(v,\mu)\|^{\tau}_{w,\mathcal{H}^1}.
\end{aligned}$$

Taking the supremum over t gives $\|G(u,\lambda)-G(v,\mu)\|^{\tau}_{w,\mathcal{H}^1} \le \frac{1}{2}\|(u,\lambda)-(v,\mu)\|^{\tau}_{w,\mathcal{H}^1}$. Using the contraction mapping theorem we get a fixed point $(v,\mu)\in S$ which is a solution of the integral equation (1.54) and thus a solution of (1.53) which satisfies the estimate (1.56). □

1.3.5 Proof of the stability theorem

Now we can give the proof of Theorem 1.15 which is similar to the proof of Theorem 5.5.1 in [23].

Proof: Choose the weight w as in Lemma 1.27 and choose $\nu\in(0,\alpha)$ and $\sigma>0$ so small that

$$wK\sigma\int_0^\infty \frac{e^{-(\alpha-\nu)s}}{\sqrt{s}}\,ds \le \frac{1}{4} \quad \text{and} \quad \sigma C_{\psi,\phi}\|\psi\|_{\mathcal{L}_2} \le \frac{1}{4}.$$

Choose $\rho\le w\sigma$, then (1.21) implies

$$\|\varphi(v,\mu)\|_{\mathcal{L}_2} \le \sigma\|(v,\mu)\|_{w,\mathcal{H}^1} \quad \text{for } \|(v,\mu)\|_{w,\mathcal{H}^1}\le\rho.$$

If $\|v^0\|_{\mathcal{H}^1}\le\delta=\frac{\rho}{4wK}$, $v^0\in\mathcal{R}(P)$ then from Lemma 1.27 follows that there exists $\tau>0$ such that a solution (v,μ) of exists on $(0,\tau)$ with $\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}\le\rho$.

Then we have with the estimates (1.52) for some $C\ge 1$

$$\begin{aligned}
\|(v(t),\mu(t))\|_{w,\mathcal{H}^1} &\le w\|e^{\Lambda_P t}v^0\|_{\mathcal{H}^1} + w\int_0^t \|e^{\Lambda_P(t-s)}P\varphi(v(s),\mu(s))\|_{\mathcal{H}^1}\,ds \\
&\quad + |\langle\psi,\phi\rangle^{-1}(a(\psi,v(t))+\langle\psi,\varphi(v(t),\mu(t))\rangle)| \\
&\le wCe^{-\alpha t}\|v^0\|_{\mathcal{H}^1} + wC\int_0^t \frac{1}{\sqrt{t-s}}e^{-\alpha(t-s)}\|\varphi(v(s),\mu(s))\|_{\mathcal{L}_2}\,ds \\
&\quad + C_{\psi,\phi}(C_\psi\|v(t)\|_{\mathcal{H}^1}+\|\psi\|_{\mathcal{L}_2}\sigma\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}) \\
&\le \frac{\rho}{4} + C\sigma\int_0^\infty \frac{1}{\sqrt{s}}e^{-\alpha s}\,ds\|(v,\mu)\|^{\tau}_{w,\mathcal{H}^1} + C_{\psi,\phi}(\frac{C_\psi}{w}\rho+\|\psi\|_{\mathcal{L}_2}\sigma\rho) \\
&\le \frac{3}{4}\rho.
\end{aligned}$$

Since the PDAE (1.14) is autonomous, this leads to $\tau=\infty$ using the usual arguments: If $(0,\tau_*)$ is the maximal interval of existence of a solution (v,μ) of (1.53) with $\|v(t),\mu(t)\|_{w,\mathcal{H}^1}\le\rho$, then by the above estimate we have $\|v(t),\mu(t)\|_{w,\mathcal{H}^1}\le\frac{3}{4}\rho$. Thus we can solve (1.14) at $\tau_0=\tau_*-\frac{\tau}{2}$, where τ is given by Lemma 1.27 and therewith continue the solution to $\tilde\tau>\tau_*$, which contradicts the maximality of τ_*. From this the existence of (v,μ) in $(0,\infty)$ follows with $\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}<\rho$ for all $t\in[0,\infty)$.

It remains to prove the exponential estimate. Define

$$n(t)=\sup_{s\in[0,t]}\{e^{\nu s}\|(v(s),\mu(s))\|_{w,\mathcal{H}^1}\}.$$

Then we obtain

$$
\begin{aligned}
\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}\mathrm{e}^{\nu t} &\le wK\mathrm{e}^{(\nu-\alpha)t}\|v^0\|_{\mathcal{H}^1} \\
&\quad + wK\sigma\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{\nu t}\|(v(s),\mu(s))\|_{w,\mathcal{H}^1}\,ds \\
&\quad + C_{\psi,\phi}(C_\psi \mathrm{e}^{\nu t}\|v(t)\|_{\mathcal{H}^1} + \|\psi\|_{\mathcal{L}_2}\sigma \mathrm{e}^{\nu t}\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}) \\
&\le wK\|v^0\|_{\mathcal{H}^1} + wK\sigma\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{(\nu-\alpha)(t-s)}\mathrm{e}^{\nu s}\|(v(s),\mu(s))\|_{w,\mathcal{H}^1}\,ds \\
&\quad + C_{\psi,\phi}(C_\psi \mathrm{e}^{\nu t}\|v(t)\|_{\mathcal{H}^1} + \|\psi\|_{\mathcal{L}_2}\sigma \mathrm{e}^{\nu t}\|(v(t),\mu(t))\|_{w,\mathcal{H}^1}) \\
&\le wK\|v^0\|_{\mathcal{H}^1} + \frac{1}{4}n(t) + C_{\psi,\phi}(C_\psi\frac{1}{w} + \|\psi\|_{\mathcal{L}_2}\sigma)n(t) \\
&< wK\|v^0\|_{\mathcal{H}^1} + \frac{3}{4}n(t).
\end{aligned}
$$

Taking the supremum on both sides gives $n(t) < 4wK\|v^0\|_{\mathcal{H}^1} < \rho$ for $t \ge 0$, and choosing $C = 4wK$ the estimate (1.22) follows. □

Remark 1.28 There is an alternative way of proving the above stability result which uses the linearity of g in μ, i.e. one assumes, that $\varphi(v,\mu)$ is of the following form

$$\varphi(v,\mu) = \tilde{\varphi}(v) + Sv\mu,$$

where $S : \mathcal{H}^1 \to \mathcal{L}_2$ is the linear operator $Sv = v_x$. One can eliminate μ from (1.19) directly using

$$\mu(t) = -\langle\psi, \phi - Sv\rangle^{-1}(a(\psi, v(t)) + \langle\psi, \tilde{\varphi}(v(t))\rangle).$$

Setting $g(t,v) = P(\tilde{\varphi}(v) + Sv\mu(t)) = P(\tilde{\varphi}(v) + Sv\varphi(v(t)))$, it remains to consider the nonautonomous system

$$v_t = P\Lambda v + g(t,v), \quad v(0) = v^0.$$

This method is similar to the stability proof in [23], Ex. 6, [60], Thm. 2.17. where a special projection with the left eigenfunction has been used in order to remove the zero eigenvalue. For this choice resolvent estimates for the projected system are not necessary since the operator $P\Lambda$ equals the restriction of Λ to $\mathcal{R}(\Lambda)$.

Remark 1.29 To complete the stability discussion, one needs an instability result similar to Thm. 5.1.3 in [23] which states that if Re $(\sigma(L)) > 0$, then the solution $(\bar{u},\bar{\lambda})$ of (1.14) is unstable. More precisely, there exist $\epsilon_0 > 0$ and a sequence of initial data $\{(u_n,\lambda_n)$ with $\|(u_n,\lambda_n)\|_{1,\mathcal{H}^1} \to 0$ as $n \to \infty$ but $\sup_{t\ge 0}\|u(t) - \bar{u}\| \ge \epsilon_0$, where u denotes the solution of (1.14) with $u(0) = u_n$. With the tools at hand, it seems possible to show such a result in a similar fashion as in [23], but we have not pursued the details of the proof.

1.4 Stability of relative equilibria

1.4.1 Abstract framework

A natural extension of the question of stability of traveling waves is the stability of relative equilibria of equivariant evolution equations in Banach spaces. We explain the abstract concept (based on [9], [47], [7]) without going into details of the numerical implementation. Consider a general evolution equation

$$\begin{aligned} u_t = F(u), \quad u(0) = u^0, \\ F : Y \subset X \longmapsto X \end{aligned} \tag{1.60}$$

where Y is a dense subspace of the Banach space X.

Assume that F is equivariant w.r.t. a (noncompact) Lie group G acting on X via a homomorphism

$$a : G \to GL(X),\ \gamma \mapsto a(\gamma)$$

where

$$a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2), \quad a(\mathbb{1}) = I, \quad \mathbb{1} = \text{unit element in } G.$$

Equivariance means that the following relation holds

$$\begin{aligned} F(a(\gamma)u) = a(\gamma)F(u) \quad \forall u \in Y, \gamma \in G \\ a(\gamma)(Y) \subset Y \quad \forall \gamma \in G. \end{aligned}$$

We assume that for any $v \in X$ the map

$$a(\cdot)v : G \to X,\ \gamma \mapsto a(\gamma)v$$

is continuous and it is continuously differentiable for any $v \in Y$ with derivative denoted by

$$a_\gamma(\gamma)v : T_\gamma G \to X,\ \lambda \mapsto [a_\gamma(\gamma)v]\ \lambda.$$

Here we use $T_\gamma G$ to denote the tangent space of G at γ. Note that in general we can neither expect the action a to be differentiable from G into $GL(X)$ nor the map $\gamma \mapsto a(\gamma)u$ to be differentiable for any fixed $u \in X$.

Such systems have been widely studied in the context of bifurcation theory for equivariant dynamical systems (see the monograph [9]). In a series of papers [18],[52],[53] a center manifold reduction theory has been developed for (1.60) especially for the case where differentiability is an issue.

In contrast to this reduction ansatz, it is more convenient for numerical purposes to extend the equation. This has been done for traveling waves in (1.14) by adding an additional parameter and a phase condition (see [7], [47]). In that case the Lie group is $G = \mathbb{R}$ and the action is given by $[a(\gamma)u](x) = u(x - \gamma)$.

Example 1.30 *In the numerical applications in Chapter 5 we will consider a more general example, where* $\gamma = (\gamma_r, \gamma_t) \in G = S^1 \times \mathbb{R}$ *with* $(\gamma_r, \gamma_t) \circ (\tilde{\gamma}_r, \tilde{\gamma}_t) = (\gamma_r + \tilde{\gamma}_r, \gamma_t + \tilde{\gamma}_t)$. *The action is given by*

$$[a(\gamma)u](x) = R_{-\gamma_r} u(x - \gamma_t),$$

where

$$R_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \tag{1.61}$$

denotes the rotation about the angle φ.

Then using the transformation $v(t) = a(\gamma)u(t)$ one obtains an equivalent formulation of (1.60) (see [7]), namely

$$v_t = F(v) - a(\gamma^{-1})[a_\gamma(\gamma)v]\gamma_t. \tag{1.62}$$

The evolution of $\gamma(t)$ then describes the motion on the group.

Introducing Lagrange parameters $\lambda(t) = \gamma_t(t) \in T_\gamma G$ we consider

$$v_t = F(v) - a(\gamma^{-1})[a_\gamma(\gamma)v]\lambda \tag{1.63}$$
$$\gamma_t = \lambda \tag{1.64}$$
$$0 = \pi(v, \lambda) \tag{1.65}$$

with a phase condition $\pi : Y \times T_\gamma G \to \mathbb{R}^p$, $p = \dim G$ which has to satisfy some regularity conditions.

We denote the derivative of the left multiplication with γ by $d\gamma_l$

$$\gamma_l : G \to G, \quad g \mapsto \gamma \circ g, \qquad d\gamma_l(g) : T_g G \to T_{\gamma\circ g} G, \quad \mu \mapsto D\gamma_l(g)\mu$$

and the derivative of the right multiplication with γ by $d\gamma_r$

$$\gamma_r : G \to G, \quad g \mapsto g \circ \gamma, \qquad d\gamma_r(g) : T_g G \to T_{g\circ\gamma} G, \quad \mu \mapsto D\gamma_r(g)\mu.$$

Note that $d\gamma_\diamond(\mathbb{1})$ is a linear homeomorphism between the Lie algebra $T_\mathbb{1} G$ and $T_\gamma G$ for $\diamond \in \{l, r\}$. Differentiating the relation

$$a(\gamma)(a(g)v) = a(\gamma \circ g)v$$

with respect to g at $g = \mathbb{1}$, one obtains for $\mu \in T_\mathbb{1} G,\ v \in Y$

$$a(\gamma)[a_\gamma(\mathbb{1})v]\mu = [a_\gamma(\gamma)v](d\gamma_l(\mathbb{1})\mu), \tag{1.66}$$

and similarly

$$[a_\gamma(\mathbb{1})a(\gamma)v]\mu = [a_\gamma(\gamma)v](d\gamma_r(\mathbb{1})\mu). \tag{1.67}$$

Using (1.66), defining $\mu \in T_\mathbb{1} G$ via $\lambda = d\gamma_l(\mathbb{1})\mu$ and setting $\psi(v, \mu) = \pi(v, d\gamma_l(\mathbb{1})\mu)$, equation (1.62) is transformed into

$$v_t = F(v) - [a_\gamma(\mathbb{1})v]\mu \tag{1.68}$$
$$\gamma_t = d\gamma_l(\mathbb{1})\mu \tag{1.69}$$
$$0 = \psi(v, \mu). \tag{1.70}$$

Note that the first equation does not depend explicitly on γ any more, thus it suffices to consider the first and the last equation as a PDAE and address equation (1.69) in a postprocessing step.

The fixed phase condition (1.12), generalizes in this setting to

$$0 = \psi_{\text{fix}}(v) = \langle [a_\gamma(\mathbb{1})\hat{u}]\mu, v - \hat{u}\rangle \quad \forall \mu \in T_\mathbb{1} G$$

where $\hat{u} \neq 0$ is a given reference function with $\hat{u} - \bar{v} \in \mathcal{H}^2$.

Definition 1.31 *We define* $(\bar v, \bar\mu)$ *to be a relative equilibrium of the PDE* (1.60) *if* $(\bar v, \bar\mu)$ *is an equilibrium of* (1.68)*, i.e.*

$$0 = F(\bar v) - [a_\gamma(\mathbb{1})\bar v]\bar\mu. \tag{1.71}$$

Note that in [9] the whole group orbit $\mathcal{O}(\bar v) = \{a(\gamma)\bar v,\ \gamma \in G\}$ is called a relative equilibrium.

If $(\bar v, \bar\mu)$ is a relative equilibrium of (1.60) and $\gamma(t)$ solves

$$\gamma_t = d\gamma_l(\mathbb{1})\bar\mu,$$

then $(\bar v, \bar\lambda(t) = d\gamma_l(t)(\mathbb{1})\bar\mu)$ satisfies (cf. (1.63))

$$0 = F(\bar v) - a(\gamma(t)^{-1})[a_\gamma(\gamma(t))\bar v]\bar\lambda(t),$$

and $\bar u(t) = a(\gamma(t))\bar v$ solves (1.60).

Spectral problem

The corresponding spectral problem, which gives information about stability of the PDAE solution $(\bar v, \bar\mu)$ can be derived as follows. With $(\bar v, \bar\mu)$ all functions in the family $\{(a(\gamma)\bar v,\ z(\gamma)\bar\mu)\}_{\gamma\in G}$, where

$$z(\cdot)\bar\mu : G \to T_{\mathbb{1}}G,\ \gamma \mapsto d\gamma_r(\mathbb{1})^{-1}d\gamma_l(\mathbb{1})\bar\mu,$$

are solutions of (1.71), since we obtain with (1.66) and (1.67) for $\tilde v = a(\gamma)\bar v$

$$\begin{aligned}
0 &= F(\bar v) - [a_\gamma(\mathbb{1})\bar v]\bar\mu \\
&= F(a(\gamma)^{-1}\tilde v) - [a_\gamma(\mathbb{1})a(\gamma)^{-1}\tilde v]\bar\mu \\
&= a(\gamma)^{-1}F(\tilde v) - a(\gamma)^{-1}[a_\gamma(\gamma)a(\gamma)^{-1}\tilde v](d\gamma_l(\mathbb{1})\bar\mu) \\
&= a(\gamma)^{-1}\left(F(\tilde v) - [a_\gamma(\mathbb{1})\tilde v](d\gamma_r(\mathbb{1})^{-1}d\gamma_l(\mathbb{1})\bar\mu)\right).
\end{aligned}$$

For $\tilde\mu = z(\gamma)\bar\mu \in T_{\mathbb{1}}G$ this is equivalent to

$$0 = F(\tilde v) - [a_\gamma(\mathbb{1})\tilde v]\tilde\mu.$$

Differentiating the equation

$$0 = F(a(\gamma)\bar v) - [a_\gamma(\mathbb{1})a(\gamma)\bar v](z(\gamma)\bar\mu)$$

with respect to γ at $\gamma = \mathbb{1}$ and denoting the corresponding derivative of $z(\cdot)\bar\mu$ by $z_\gamma(\mathbb{1})\mu$, we obtain for $\mu \in T_{\mathbb{1}}G$

$$\begin{aligned}
0 &= DF(a(\mathbb{1})\bar v)[a_\gamma(\mathbb{1})\bar v]\mu - [a_\gamma(\mathbb{1})[a_\gamma(\mathbb{1})\bar v]\mu](z(\mathbb{1})\bar\mu) - [a_\gamma(\mathbb{1})a(\mathbb{1})\bar v](z_\gamma(\mathbb{1})\bar\mu)\mu \\
&= DF(\bar v)[a_\gamma(\mathbb{1})\bar v]\mu - [a_\gamma(\mathbb{1})[a_\gamma(\mathbb{1})\bar v]\mu]\bar\mu - [a_\gamma(\mathbb{1})\bar v](z_\gamma(\mathbb{1})\bar\mu)\mu.
\end{aligned}$$

Note that if the group G is Abelian, then $z(\gamma) = d\gamma_r(\mathbb{1})^{-1}d\gamma_l(\mathbb{1})$ is the identity in $T_{\mathbb{1}}G$, and we have

$$0 = DF(\bar v)[a_\gamma(\mathbb{1})\bar v]\mu - [a_\gamma(\mathbb{1})[a_\gamma(\mathbb{1})\bar v]\mu]\bar\mu.$$

Thus all functions $\bar w = [a_\gamma(\mathbb{1})\bar v]\mu, \mu \in T_{\mathbb{1}}G$ are eigenfunctions of the linear operator

$$\Lambda w = DF(\bar u)w - [a_\gamma(\mathbb{1})w]\bar\mu \tag{1.72}$$

corresponding to the eigenvalue 0. Of course, the operator has to be defined in an appropriate function space.

In the numerical examples in Chapter 5 the group G will always be Abelian.

We expect that the spectrum of the operator Λ gives information about stability in this general case as well. At the end of this Chapter we give the expected stability result for the parabolic case.

1.4.2 Realization

For numerical computations we choose a basis $\{e^1, \dots, e^p\}$ in the Lie algebra $T_{\mathbb{1}}G$, where p is the dimension of G. Writing $\mu = \sum_{i=1}^p \mu_i e^i$ and defining $S_i v = -a_\gamma(\mathbb{1}) v e^i$, the generalization of equation (1.14) now reads

$$\begin{aligned} v_t &= F(v) + \sum_{i=1}^p \mu_i S_i v, \\ 0 &= \Psi(v, \vec{\mu}). \end{aligned} \tag{1.73}$$

Here $\Psi(v, \vec{\mu}) = \psi(v, \sum_{i=1}^p \mu_i e^i)$ is a map from $Y \times \mathbb{R}^p$ to $\mathbb{R}^p$ and $\vec{\mu}$ denotes the vector $(\mu_1, \dots, \mu_p) \in \mathbb{R}^p$. An example of such a phase condition is given by the following generalization of the fixed phase condition in (1.13).

$$0 = (\Psi_{\text{fix}}(v))_i = \langle S_i v^0, u - v^0 \rangle, \quad i = 1, \dots, p.$$

Another possibility mentioned in [7] is the orthogonality of v_t and the group orbit $\{a(\gamma)v : \gamma \in G\}$ at $\gamma = \mathbb{1}$:

$$\langle a_\gamma(\mathbb{1}) v \mu, v_t \rangle = 0 \quad \forall \mu \in T_{\mathbb{1}}G.$$

Using the differential equation (1.68) we rewrite this as

$$\psi_{\text{orth}}(v, \mu) = \langle a_\gamma(\mathbb{1}) v \eta, F(v) - a_\gamma(\mathbb{1}) v \mu \rangle = 0 \quad \forall \eta \in T_{\mathbb{1}}G.$$

Setting $\Psi_{\text{orth}}(v, \vec{\mu}) = \psi_{\text{orth}}(v, \sum_{i=1}^p \mu_i \mathrm{e}^i)$ this leads to the condition

$$0 = (\Psi_{\text{orth}}(v, \vec{\mu}))_i = \langle S_i v, v_t \rangle = \langle S_i v, F(v) - \sum_{j=1}^p \mu_j S_j v \rangle, \quad i = 1, \dots, p.$$

Using this phase condition, the resulting PDAE is of differentiation index 1 (generalizing the notion for DAEs [22] to PDAEs; for a different definition which focuses on consistent initialisation by Cauchy data, see [33]), whereas it is of index 2 for Ψ_{fix} (as mentioned before). After discretization this leads to a DAE of differentiation index 2 and 1 respectively. We will not discuss the phase condition Ψ_{orth} any further in this thesis.

The operator Λ in (1.72) is given by

$$\Lambda v = DF(\bar{u}) v + \sum_{j=1}^p \bar{\mu}_j S_j v$$

and the functions $\bar{w} = S_i \bar{u}$, $i = 1, \dots, p$ are eigenfunctions of Λ, corresponding to the eigenvalue 0.

To simplify notation, we drop the arrow which distinguishes between $\mu \in T_{\mathbb{1}}G$ and $\vec{\mu} \in \mathbb{R}^p$ in the following, if no confusion is possible.

Realization for the parabolic PDE

Now we consider these stability problems for the parabolic PDE (1.1). Assume that the operators S_i in (1.73) are linear differential operators of order ≤ 1. and the generalization (1.73) of the PDAE (1.14) reads

$$\begin{aligned} v_t &= Av_{xx} + \sum_{i=1}^{p} \mu_i (S_i^0 v + S_i^1 v_x) + f(v, v_x) \\ 0 &= \Psi(v, \mu) \end{aligned} \tag{1.74}$$

where $\mu \in \mathbb{R}^p$, $S_i^0, S_i^1 \in \mathbb{R}^{m,m}$, $i \in \{1, \ldots, p\}$. The linear operator Λ is given by

$$\Lambda u = Au'' + Bu' + Cu \tag{1.75}$$

where

$$B(x) = D_2 f(\bar{u}(x), \bar{u}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_i^1, \quad C(x) = D_1 f(\bar{u}(x), \bar{u}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_i^0.$$

Example 1.32
For Example 1.30 we have $[a_\gamma(\mathbb{1})v]e^1 = v_x$, $[a_\gamma(\mathbb{1})v]e^2 = R_{\frac{\pi}{2}} v$ *i.e.* $S_1^1 = I$, $S_2^0 = R_{\frac{\pi}{2}}, S_1^0 = S_2^1 = 0$ *and* $\mu_r, \mu_t \in \mathbb{R}$. *Thus the equation (1.74) reads*

$$\begin{aligned} v_t &= Av_{xx} + \mu_t v_x + \mu_r R_{\frac{\pi}{2}} v + f(v, v_x) \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle, \quad 0 = \langle R_{\frac{\pi}{2}} v, v - \hat{v} \rangle \end{aligned}$$

and the operator Λ *in (1.75) is given by*

$$\Lambda v = Av'' + (\mu_t I + D_2 f(\bar{v}, \bar{v}'))v' + (\mu_r R_{\frac{\pi}{2}} + D_1 f(\bar{v}, \bar{v}'))v.$$

The general stability problem

The stability theory in this chapter as well as the approximation results in the following chapters can be generalized to this case.

In this situation we can formulate the generalization of the stability Theorem 1.13 using the following generalized eigenvalue condition:

Eigenvalue condition (EC') :
Assume that the differential operator Λ in (1.75) has an eigenvalue 0 of multiplicity p and there exists $\beta > 0$ such that there are no other isolated eigenvalues s of finite multiplicity with $\text{Re}\, s \geq -\beta$.

We suspect the following generalization of Theorem 1.13 to be true.

Theorem 1.33 *Let* $A \in \mathbb{R}^{m,m}$ *be given with* $A > 0$ *and assume that the PDE (1.1) is equivariant w.r.t. the action* $a(\gamma)$ *of a group* G *of dimension* $\dim G = p$. *Assume further that the function* $f \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ *satisfies Hypothesis 1.9.*

Let $(\bar{u}, \bar{\mu}) \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}^p$ *be a stationary solution of the PDAE* (1.74)*, i.e.*

$$\begin{aligned} 0 &= A\bar{u}'' + \sum_{i=1}^{p} \bar{\mu}_i (S_i^0 \bar{u} + S_i^1 \bar{u}') + f(\bar{u}, \bar{u}') \\ 0 &= \langle S_j \hat{u}, \bar{u} - \hat{u} \rangle = 0, \;\; j = 1, \ldots, p \end{aligned}$$

where $\hat{u} \neq 0$ *is a given reference function with* $S_j \hat{u} \in \mathcal{H}^1, \; j = 1, \ldots, p, \; \hat{u} - \bar{u} \in \mathcal{H}^1$ *and for which the* $p \times p$ *matrix*

$$\langle [S_1(\hat{u}), \ldots, S_p(\hat{u})], [S_1(\bar{u}), \ldots, S_p(\bar{u})] \rangle = (\langle S_i(\hat{u}), S_j(\bar{u}) \rangle)_{i,j=1,\ldots,p}$$

is nonsingular. Furthermore, assume that (EC') and (SC) hold.

Then $(\bar{u}, \bar{\mu})$ *is asymptotically stable, i.e. there exists* $\delta > 0$ *such that for each* u^0 *with* $u^0 - \bar{u} \in \mathcal{H}^1$*,* $\langle S_j \hat{u}, u^0 - \hat{u} \rangle = 0, \;\; j = 1, .., p$ *and* $\|u^0 - \bar{u}\|_{\mathcal{H}^1} < \delta$ *there exists a unique solution* $(u(t), \mu(t))$ *of* (1.14) *on* $[0, \infty)$ *and the following exponential estimate holds for some* $\nu, K > 0$

$$\|u(t) - a(\gamma)\bar{u}\|_{\mathcal{H}^1} + \|\mu(t) - \bar{\mu}\| \leq K \mathrm{e}^{-\nu t} \|u^0 - \bar{u}\|_{\mathcal{H}^1} \;\; \forall t \geq 0 \tag{1.76}$$

where γ *is the solution of*

$$\gamma_t = d\gamma_l(\mathbb{1})\bar{\mu}, \quad \gamma(0) = \mathbb{1}.$$

In order to prove this theorem, one has to adapt the proof of Theorem 1.13 to the general case. For a similar adaptation see e.g. the generalization of the proof of Theorem 2.18 which deals with the asymptotic stability of a family of equilibria, as indicated in [23] at the end of Exercise 6 in Chapter 5.

Chapter 2

Approximation via difference equations

In this chapter we will prove convergence results for the numerical approximation of traveling wave solutions of (1.1) with finite differences on an equidistant grid. Furthermore we consider the approximation of isolated eigenvalues of finite multiplicity and we derive resolvent estimates for the discretized system.

We apply the linear results of the preceding section to prove several approximation results

- approximation of the traveling wave solution and it's velocity
- approximation of simple eigenvalues
- resolvent estimates in compact sets which do not contain eigenvalues
- resolvent estimates for large absolute values of the resolvent parameter.

A general principle for proving the invertibility of the occuring linear operators, is to show the invertibility of a nearby operator which is linked via its h-flow to a continuous system that has well known properties.

2.1 Auxiliary results

We define a discrete interval in $\mathbb{Z} \cup \{\pm\infty\}$

$$J = [n_-, n_+] = \{n \in \mathbb{Z} : \; n_- \le n \le n_+, \text{ where } n_\pm \in \mathbb{Z} \cup \{\pm\infty\} \; \}$$

as well as extended intervals

$$J_r = [n_-, n_+ + 1], \; J_l = [n_- - 1, n_+], \; J_e = [n_- - 1, n_+ + 1]$$

and a corresponding equidistant grid with grid size $h > 0$ and shift $x_0 \in \mathbb{R}$

$$\mathbb{G}_{J,h,x_0} = \{x_n : \; x_n = x_0 + nh, \; n \in J\}.$$

We denote sequences in a Banach space X which are indexed by J by

$$X^J = \{(z_n)_{n\in J},\ z_n \in X\}.$$

It is well known that X^J provided with the supremum norm

$$\|z\|_\infty = \sup_{n\in J} \|z_n\|$$

is a Banach space which we denote by $S_J(X)$. If X is clear from the context, we drop the dependency on X.

We consider the spatial discretization of the stationary equation

$$0 = Au'' + \lambda u' + f(u,u'), \quad x \in \mathbb{R},\ u(x) \in \mathbb{R}^m \tag{2.1}$$

on the grid $\mathbb{G}_{J,h,x_0}$ which uses second order finite difference operators for the approximation of the derivatives of u at x_n

$$u_n' \approx (\delta_0 u)_n, \qquad u_n'' \approx (\delta_+\delta_- u)_n,$$

where $u_n = u(x_n)$ and $\delta_0 : S_{J_e} \to S_J,\ \delta_+ : S_{J_r} \to S_J,\ \delta_- : S_{J_l} \to S_J$ are defined as usual by

$$(\delta_0 u)_n = \frac{1}{2h}(u_{n+1} - u_{n-1}), \qquad (\delta_+ u)_n = \frac{1}{h}(u_{n+1} - u_n), \qquad (\delta_- u)_n = \frac{1}{h}(u_n - u_{n-1}).$$

We obtain the following difference equation on J

$$A(\delta_+\delta_- u)_n + \lambda(\delta_0 u)_n + f(u_n, (\delta_0 u)_n) = 0, \quad n \in J. \tag{2.2}$$

Remark 2.1 The error estimates for $u \in C^4(\mathbb{R},\mathbb{R}^m)$ are given by:

$$\|(\delta_0 u)_n - u'(x_n)\| \le Ch^2\psi_n, \quad \text{with} \quad \psi_n = \max_{\xi\in[x_{n-1},x_{n+1}]} \|u^{(3)}(\xi)\|.$$

and

$$\|(\delta_+\delta_- u)_n - u''(x_n)\| \le Ch^2\phi_n, \quad \text{with} \quad \phi_n = \max_{\xi\in[x_{n-1},x_{n+1}]} \|u^{(4)}(\xi)\|.$$

Note that from $\|\bar{u}^{(k)}(x)\| \le C\mathrm{e}^{-\rho|x|},\ k = 1,\dots,4$ follows:

$$\|(\delta_0\bar{u})_n - \bar{u}'(x_n)\| \le Ch^2\mathrm{e}^{-\varrho h|n|}, \qquad \|(\delta_+\delta_-\bar{u})_n - \bar{u}''(x_n)\| \le Ch^2\mathrm{e}^{-\varrho h|n|}.$$

For sequences $u, v \in S_J(\mathbb{R}^m)$, $J = [n_-, n_+]$ we define

$$\langle u,v\rangle_{r,s} = \sum_{n=r}^{s} hu^T v, \qquad \langle u,v\rangle_h = \langle u,v\rangle_{n_-,n_+}$$

and introduce norms which include the approximations of higher derivatives by

$$\|z\|_{1,\infty} = \|z\|_\infty + \|\delta_+ z\|_\infty, \qquad \|z\|_{2,\infty} = \|z\|_{1,\infty} + \|\delta_+\delta_- z\|_\infty.$$

One has to keep in mind that the supremum is taken in different intervals for the different difference operators.

In particular, the space of bounded biinfinite sequences is denoted as

$$l_\infty = S_{\mathbb{Z}}(X) = \{z \in X^{\mathbb{Z}} : \ \|z\|_\infty = \sup_{n\in\mathbb{Z}} \|z_n\| < \infty\}.$$

If necessary, we can embed each $z \in S_J(X)$ in $S_{\mathbb{Z}}(X)$ by setting $z_n = 0$ for $n \in \mathbb{Z} \setminus J$. This will be done without any further notice.

We further introduce suitably scaled discrete approximations to the $\mathcal{L}_2$-norm, the $\mathcal{H}^1$-norm and the $\mathcal{H}^2$-norm by

$$\|z\|_{\mathcal{L}_{2,h}} = (\sum_{n=n_-}^{n_+} h\|z_n\|^2)^{\frac{1}{2}}, \qquad \|z\|_{\mathcal{H}_h^1} = (\|z\|^2_{\mathcal{L}_{2,h}} + \|\delta_+ z\|^2_{\mathcal{L}_{2,h}})^{\frac{1}{2}},$$
$$\|z\|_{\mathcal{H}_h^2} = (\|z\|^2_{\mathcal{H}_h^1} + \|\delta_+\delta_- z\|^2_{\mathcal{L}_{2,h}})^{\frac{1}{2}}$$

and denote X^J employed with these norms by

$$\mathcal{L}_{2,h}(J,X), \quad \mathcal{H}_h^1(J,X) \quad \text{and} \quad \mathcal{H}_h^2(J,X).$$

If no confusion is possible, we drop the dependency on X (in the following we will always use $X = \mathbb{R}^m$ or $X = \mathbb{C}^m$) as well as on J if $J = \mathbb{Z}$.

In order to simplify notation we will often use the following abbreviations

$$\|z\|_{1,\mathcal{L}_{2,h}} = \|z\|_{\mathcal{H}_h^1}, \qquad \|z\|_{2,\mathcal{L}_{2,h}} = \|z\|_{\mathcal{H}_h^2}. \tag{2.3}$$

The general method for all approximation results will be the following: we transform the discrete system (2.2) via $z_n = (u_n, (\delta_- u)_n)$ into a difference equation of the form

$$N_n z_{n+1} - K_n z_n = r_n \tag{2.4}$$

and use the corresponding first order transformation of the continuous system in order to prove important properties of (2.4). The estimates for (2.4) will be transformed back to the original system by using the following facts about the norms: If $z = (u, \delta_- u) \in S_{J_r}(\mathbb{R}^{2m})$ then

$$\|u\|_{2,\infty} \le C\|z\|_{1,\infty}, \qquad \|u\|_{\mathcal{H}_h^2} \le C\|z\|_{\mathcal{H}_h^1} \tag{2.5}$$

and for $r = (0, hg) \in S_J(\mathbb{R}^{2m})$

$$\|r\|_\infty = h\|g\|_\infty, \qquad \|r\|_{\mathcal{L}_{2,h}} = h\|g\|_{\mathcal{L}_{2,h}} \tag{2.6}$$

hold.

We will prove a stability inequality for the transformed system and conclude a stability inequality for the original system. The convergence of the solution of the original system is then proved using consistency and stability.

The main tool for constructing solutions of the discrete equations are “exponential dichotomies”. For a definition of exponential dichotomy in the continuous case see A.5. The definition of an exponential dichotomy for difference equations is given below.

In order to obtain exponential dichotomies for the finite difference approximation (2.2) of (2.1) use the fact that both are linked via the time-h map of the flow of the continuous system. This link has been used in [64] for proving the existence and approximation property of connecting orbits for the discrete system on the whole line.
Consider the linear difference equation

$$z_{n+1} = M_n z_n, \quad n \in J \tag{2.7}$$

with $M_n \in \mathbb{R}^{k,k}$ for all $n \in J$. If the matrices M_n are invertible for all $n \in J$ then the map $\Phi : J^2 \to \mathbb{R}^{k,k}$ given by

$$\Phi(n,m) = \begin{cases} M_{n-1} \cdot \cdots \cdot M_m, & \text{for } n > m \\ I, & \text{for n=m} \\ M_n^{-1} \cdot \cdots \cdot M_{m-1}^{-1}, & \text{for } n < m \end{cases}$$

is a solution operator for (2.7), which has the cocycle property

$$\Phi(n,l)\Phi(l,m) = \Phi(n,m) \quad \forall\, l,m,n \in J.$$

Definition 2.2 (Exponential dichotomy)
The linear difference equation (2.7) has an exponential dichotomy with data (K, α, P) on $J \subset \mathbb{Z}$ if M_n is invertible for all $n \in J$ and there exist a bound $K > 0$, a rate $\alpha > 0$ and projectors P_n such that the following holds

$$\Phi(n,m)P_m = P_n \Phi(n,m) \tag{2.8}$$

and the Green's function

$$G(n,m) = \begin{cases} \Phi(n,m)P_m, & \text{for } n \geq m \\ -\Phi(n,m)(I - P_m), & \text{for } n < m \end{cases} \tag{2.9}$$

satisfies an exponential estimate

$$\|G(n,m)\| \leq K \mathrm{e}^{-\alpha|n-m|}, \quad n,m \in J. \tag{2.10}$$

The connection between the two definitions via the time h-map follows directly from the definition: If we define $x_n = x_0 + hn$ for fixed x_0 and $\Phi(n,m) = S(x_n, x_m)$ we obtain the following lemma.

Lemma 2.3 *Let the linear differential operator L from (A.11) given by*

$$Lz = z' - Mz, \quad x \in J \subset \mathbb{R},\ M : J \to \mathbb{R}^{m,m}$$

have an exponential dichotomy with data (K_J, α_J, π_J) on $J = \mathbb{R}^{\pm}, \mathbb{R}$.

Then the difference operator

$$\hat{L}z = (z_{n+1} - \Phi(n+1,n)z_n)_{n \in \hat{J}},$$

has an exponential dichotomy on $\hat{J} = \mathbb{Z}^{\pm}, \mathbb{Z}$ with data $(K_J, \alpha_J h, P^J)$ where $P_n^J = \pi_J(x_n)$.

Furthermore, the discrete Green's function defined in (2.9) is given by

$$G(n,m) = \begin{cases} S(x_n,x_m)\pi_J(x_m), & \text{for } n \geq m \\ -S(x_n,x_m)(I - \pi_J(x_m)), & \text{for } n < m. \end{cases} \tag{2.11}$$

Note that for $\hat{L} : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$

$$\begin{aligned}\mathcal{N}(\hat{L}) &= \{(\Phi(n,0)z)_{n\in\mathbb{Z}}, \qquad z \in \mathcal{N}(P_0^-) \cap \mathcal{R}(P_0^+)\} \\ &= \{(S(x_n,x_0)z)_{n\in\mathbb{Z}}, \quad z \in \mathcal{N}(\pi^-(x_0)) \cap \mathcal{R}(\pi^+(x_0))\}\end{aligned}$$

and if $\mathcal{N}(L) = \text{span}\{\phi^1,\dots,\phi^p\}$ then $\phi^i(x) = S(x,\xi)\phi^i(\xi)$.

A main tool will be a "roughness theorem" (see [40]) which allows to transfer an exponential dichotomy of the constant coefficient operators $L^\infty = z' - M^\infty z$, $M^\infty = \lim_{x\to\pm\infty} M(x)$ to the variable coefficient operator L.

2.1.1 The linear difference equation

The existence of exponential dichotomies ensures that certain boundary value problems can be solved that arise later in the construction of solutions of more general equations.

We use a slightly adapted version of Lemma 1.1.6 in [26] or Lemma 2.7 in [42].

Lemma 2.4 *Let the linear difference operator*

$$L : S_{J_r} \to S_J, z \mapsto (z_{n+1} - M_n z_n)_{n\in J}$$

have an exponential dichotomy with data (K,β,P) on $J = [n_-,n_+] \subset \mathbb{Z}$ where $n_\pm = \pm\infty$ is allowed.

For each $r \in S_J$ there exists a unique solution $\tilde{z} \in S_{J_r}$ of the inhomogenous equation

$$(Lz)_n = r_n, \quad n \in J \tag{2.12}$$

$$P_{n_-}z_{n_-} = \rho_- \in \mathcal{R}(P_{n_-}) \qquad \text{if } n_- \in \mathbb{Z} \tag{2.13}$$

$$(I - P_{n_+})z_{n_+} = \rho_+ \in \mathcal{R}(I - P_{n_+}) \quad \text{if } n_+ \in \mathbb{Z}. \tag{2.14}$$

It is given by

$$\begin{aligned}\tilde{z}_n &= R_n^-(\rho_-) + R_n^+(\rho_+) + \hat{s}_n(r), \quad n \in J, \\ \tilde{z}_{n_++1} &= M_{n_+}\tilde{z}_{n_+} + r_{n_+}\end{aligned}$$

where $\hat{s}$ is defined with G from (2.9) as follows:

$$\hat{s}_n(r) = \sum_{m=n_-}^{n_+-1} G(n,m+1)\; r_m, \quad n \in J \tag{2.15}$$

and

$$R_n^\pm(\rho) = \begin{cases}\Phi(n,n_\pm)\rho, & \textit{in case } \pm n_\pm < \infty \\ 0, & \textit{otherwise}\end{cases}.$$

Furthermore, the following estimate holds for $n \in J$

$$\|\hat{s}_n(r)\| \le K\; C_\beta \|r\|_\infty, \quad \textit{where} \quad C_\beta = \frac{1+\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}}. \tag{2.16}$$

In addition, if $r \in \mathcal{L}_{2,h}(J)$ then

$$\|\hat{s}_n(r)\| \le K\sqrt{\frac{C_{2\beta}}{h}}\; \|r\|_{\mathcal{L}_{2,h}} \quad \forall n \in J \tag{2.17}$$

and

$$\|\hat{s}(r)\|_{\mathcal{L}_{2,h}} \le KC_\beta \ \|r\|_{\mathcal{L}_{2,h}}. \tag{2.18}$$

In case $\pm n_\pm < \infty$ we obtain for the boundary terms the estimates

$$\|R_n^\pm(\rho_\pm)\| \le K\mathrm{e}^{-\beta|n-n_\pm|}\|\rho_\pm\| \tag{2.19}$$

as well as

$$\|R^\pm(\rho_\pm)\|_{\mathcal{L}_{2,h}} \le K\sqrt{hC_{2\beta}}\|\rho_\pm\|. \tag{2.20}$$

Proof: For $r \in S_J$ we get from the exponential dichotomy

$$\|\hat{s}_n(r)\| \le K \sum_{m=n_-}^{n_+-1} \mathrm{e}^{-\beta|n-m-1|}\|r_m\| \le K\|r\|_\infty \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\beta|n-m|} \le KC_\beta\|r\|_\infty.$$

The $\mathcal{L}_{2,h}$ estimate is completely analogous to the continuous case: For $r \in \mathcal{L}_{2,h}$ we have

$$\begin{aligned}
\|\hat{s}_n(r)\|^2 &\le K^2 \left(\sum_{m=n_-+1}^{n_+} \mathrm{e}^{-\frac{\beta}{2}|n-m|}\left(\mathrm{e}^{-\frac{\beta}{2}|n-m|}\|r_{m-1}\|\right)\right)^2 \\
&\le K^2 \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\beta|n-m|} \sum_{m=n_-+1}^{n_+} \mathrm{e}^{-\beta|n-m|}\|r_{m-1}\|^2 \\
&\le K^2 \ C_\beta \sum_{m=n_-}^{n_+-1} \mathrm{e}^{-\beta|n-m-1|}\|r_m\|^2.
\end{aligned}$$

Summing over all $n \in J$ gives

$$\begin{aligned}
\|\hat{s}(r)\|^2_{\mathcal{L}_{2,h}} = h \sum_{n=n_-}^{n_+-1} \|\hat{s}_n(r)\|^2 &\le K^2C_\beta \ h \sum_{n=n_-}^{n_+-1} \sum_{m=n_-+1}^{n_+} \mathrm{e}^{-\beta|n-m|}\|r_{m-1}\|^2 \\
&= K^2C_\beta \ h \sum_{m=n_-+1}^{n_+} \|r_{m-1}\|^2 \sum_{n=n_-}^{n_+-1} \mathrm{e}^{-\beta|n-m|} \\
&\le K^2C_\beta^2 \ h \sum_{m=n_-}^{n_+-1} \|r_m\|^2 \le (KC_\beta)^2\|r\|^2_{\mathcal{L}_{2,h}}.
\end{aligned}$$

For $r \in \mathcal{L}_{2,h}(J)$ one obtains

$$\begin{aligned}
\|\hat{s}_n(r)\|^2 &\le K^2 \left(\sum_{m=n_-+1}^{n_+} \mathrm{e}^{-\beta|n-m|}\|r_{m-1}\|\right)^2 \\
&\le K^2 \sum_{m=-\infty}^{\infty} \mathrm{e}^{-2\beta|n-m|} \sum_{m=n_-}^{n_+-1} |r_m|^2 \le \frac{1}{h}K^2 \ C_{2\beta}\|r\|^2_{\mathcal{L}_{2,h}}
\end{aligned}$$

It remains to estimate the boundary terms. From the dichotomy estimates we obtain directly (2.19) which imply

$$\|R^\pm(\rho_\pm)\|^2_{\mathcal{L}_{2,h}} \le \sum_{n=n_-}^{n_+} h\|\Phi(n,n_\pm)\rho_\pm\|^2 \le K^2\|\rho_\pm\|^2 h \sum_{n=n_-}^{n_+} \mathrm{e}^{-2\beta|n-n_\pm|} \le K^2\|\rho_\pm\|^2 hC_{2\beta}.$$

□

Note that the C_β does not depend on the interval J but only on the dichotomy data. Note further that $C_{h\alpha}$ is of order $\mathcal{O}(\frac{1}{h})$ for small h.

We can now use the above lemma to construct solutions on half intervals $J^\pm$ of $\mathbb{Z}$ which match in a special way at 0. This is similar to Lemma 1.1.6 in [26] or Lemma 2.7 in [42].

Lemma 2.5 *Let the linear difference operator*

$$L : S_\mathbb{Z} \to S_\mathbb{Z},\ z \mapsto (z_{n+1} - M_n z_n)_{n\in\mathbb{Z}}$$

have exponential dichotomies on $\mathbb{Z}^-$ and $\mathbb{Z}^+$ with data (K_-, β_-, P^-) and (K_+, β_+, P^+).

Consider the boundary value problems

$$\begin{aligned} Lz_n &= r_n, \quad n \in J^- = [n_-, -1] \\ P^-_{n_-} z_{n_-} &= \rho_- \in \mathcal{R}(P^-_{n_-}), \\ (I - P^-_0) z_0 &= \eta_- \in \mathcal{N}(P^-_0) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} Lz_n &= r_n, \quad n \in J^+ = [0, n_+] \\ P^+_0 z_0 &= \eta_+ \in \mathcal{R}(P^+_0), \\ (I - P^+_{n_+}) z_{n_+} &= \rho_+ \in \mathcal{N}(P^+_{n_+}). \end{aligned} \tag{2.22}$$

Then for each $r \in S_\mathbb{Z}$ there exists $N > 0$ such that for $\pm n_\pm > N$ exist unique solutions $\tilde{z}^\pm(r) \in S_{J_r^\pm}$ on $J_r^- = [n_-, 0]$ and $J_r^+ = [0, n_+ + 1]$ which are given by

$$\tilde{z}^-_n(r) = \Phi(n, n_-)\rho_- + \Phi(n, 0)\eta_- + \hat{s}^-_n(r), \quad n \in [n_-, 0] \tag{2.23}$$

$$\tilde{z}^+_n(r) = \Phi(n, 0)\eta_+ + \Phi(n, n_+)\rho_+ + \hat{s}^+_n(r), \quad n \in [0, n_+], \quad \tilde{z}^+_{n_++1} = M_{n_+}\tilde{z}_{n_+} + r_{n_+} \tag{2.24}$$

Here $\hat{s}^-(r) \in S_{J^-}$ and $\hat{s}^+(r) \in S_{J^+}$ are the special solutions of $Lz = r$ on J^- and J^+ defined in (2.15), which read

$$\hat{s}^-_n(r) = \sum_{m=n_-}^{n-1} \Phi(n, m+1) P^-_{m+1}\, r_m - \sum_{m=n}^{-1} \Phi(n, m+1)(I - P^-_{m+1})\, r_m, \quad n \in [n_-, 0],$$

$$\hat{s}^+_n(r) = \sum_{m=0}^{n-1} \Phi(n, m+1) P^+_{m+1}\, r_m - \sum_{m=n}^{n_+-1} \Phi(n, m+1)(I - P^+_{m+1})\, r_m, \quad n \in [0, n_+].$$

For $\beta = \alpha h$ the we can estimate the solutions $\tilde{z}^\pm$ as follows:

Corollary 2.6 *If $\beta = \alpha h$ then the partial solutions $\tilde{z}^\pm$ defined in Lemma 2.5 obey the estimate*

$$\begin{aligned} \|\tilde{z}^-\|_\infty &\le C(\frac{1}{h}\|r\|_\infty + \|\rho_-\| + \|\eta_-\|) \\ \|\tilde{z}^+\|_\infty &\le C(\frac{1}{h}\|r\|_\infty + \|\eta_+\| + \|\rho_+\|) \end{aligned} \tag{2.25}$$

and for $r \in \mathcal{L}_{2,h}$ *additionally*

$$\begin{aligned} \|\tilde{z}^-\|_{\mathcal{L}_{2,h}} &\le C(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\rho_-\| + \|\eta_-\|) \\ \|\tilde{z}^+\|_{\mathcal{L}_{2,h}} &\le C(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\eta_+\| + \|\rho_+\|) \end{aligned} \tag{2.26}$$

Proof: Applying the estimate (2.16) we get with $\beta = h\alpha$

$$\|\hat{s}^{\pm}(r)\|_\infty \le \frac{K}{h}\|r\|_\infty$$

and for $r \in \mathcal{L}_{2,h}$ with (2.18)

$$\|\hat{s}^{\pm}(r)\|_{\mathcal{L}_{2,h}} \le \frac{K}{h}\|r\|_{\mathcal{L}_{2,h}}$$

It remains to estimate the boundary terms. By application of (2.19) to $J = [0, n_+]$ we obtain for $n \in [0, n_+]$

$$\|\Phi(n,0)\eta_+\| \le K_- \mathrm{e}^{-h\alpha_+|n_+-n|}\|\eta_+\| \le C\|\eta_+\|$$

and with $C_{2\alpha h} \le \frac{C}{h}$ and (2.20)

$$\|\Phi(\cdot,0)\eta_+\|_{\mathcal{L}_{2,h}} \le K_+\sqrt{hC_{2\alpha_+ h}} \le C\|\eta_+\|.$$

In a similar fashion one gets

$$\|\Phi(\cdot,n_+)\rho_+\|_\diamond \le C\|\rho_+\|, \quad \text{for } \diamond \in \{\infty, \mathcal{L}_{2,h}\}.$$

The estimate for the boundary terms of $\tilde{z}^-$ is analogous. Thus the estimates (2.25) and (2.26) hold. □

In the following we transfer the proof in [26] and [60] to the discrete case along the lines of the method used in [64] and [65].

We define a class of functions for which all derivatives decay exponentially and give some convergence results for it.

Definition 2.7 *We define a function* $g : I \to \mathbb{R}^m$, $I \subset \mathbb{R}$ *to be in* $\mathcal{E}_\rho(I, \mathbb{R}^m)$ *if there exists* $K > 0$ *such that*

$$\|g(x)\| \le K\mathrm{e}^{-\varrho|x|} \quad \textit{and} \quad \|g'(x)\| \le K\mathrm{e}^{-\varrho|x|}.$$

Note that $\bar{u}'$ is in this class (see Remark 1.7). Similar to [64] we have the following Lemma.

Lemma 2.8 *Let* $g \in \mathcal{E}_\rho(\mathbb{R}^+, \mathbb{R}^m)$ *and* $\tilde{g} \in S_\mathbb{Z}$ *be given*

$$\|g(x_m) - \tilde{g}_m\| \le Ch\mathrm{e}^{-\varrho x_m}, \quad x_m = x_0 + mh$$

Then the estimates

$$\|\int_0^\infty g(x)\,dx - h\sum_{m=0}^{n_+-1}\tilde{g}_m\| \le c(x_0 + h^2 + \mathrm{e}^{-\varrho x_{n_+}}) \tag{2.27}$$

and

$$\|\int_{x_0}^\infty g(x)\,dx - h\sum_{m=0}^{n_+-1}\tilde{g}_m\| \le c(h^2 + \mathrm{e}^{-\rho x_{n_+}}) \tag{2.28}$$

hold.

Proof:

$$\|\int_0^\infty g(x)\,dx - h\sum_{m=0}^{n_+-1}\tilde{g}_m\| \le \|\int_0^{x_{n_+}} g(x)\,dx - h\sum_{m=0}^{n_+-1} g(x_m)\| + h\sum_{m=0}^{n_+}\|g(x_m)-\tilde{g}_m\| + \int_{x_{n_+}}^\infty \|g(x)\|\,dx.$$

The last term can be estimated by

$$\int_{x_{n_+}}^\infty \|g(x)\|\,dx \le \frac{K}{\varrho}\mathrm{e}^{-\varrho x_{n_+}}.$$

Choose x_*, h_0 small enough such that for all $0 \le x_0 \le x_*, h < h_0$. The estimate for the first term is

$$\begin{aligned}
&\|\int_0^{x_{n_+}} g(x)\,dx - h\sum_{m=0}^{n_+} g(x_m)\| \\
&\quad\le \|\int_0^{x_0} g(x)\,dx\| + \sum_{m=0}^{n_+-1}\int_{x_m}^{x_{m+1}} \|g(x)-g(x_m)\|\,dx \\
&\quad\le x_0\|g\|_\infty + C_1\sum_{m=0}^{n_+-1} h \sup_{\xi\in[x_m,x_{m+1}]}\|g'(\xi)\| \\
&\quad\le x_0\|g\|_\infty + hn_+C_2\sum_{m=0}^{\infty}\mathrm{e}^{-\varrho mh} \le x_0\|g\|_\infty + C_2h^2\frac{1}{1-\mathrm{e}^{-\varrho h}} \\
&\quad\le C_3(x_0+h^2).
\end{aligned} \tag{2.29}$$

Decrease h_0 further, such that we have for the second term

$$h\sum_{m=0}^{n_+}\|g(x_m)-\tilde{g}_m\| \le Ch^2\sum_{m=0}^{n_+}\mathrm{e}^{-\varrho x_m} \le Ch^2.$$

If we start the integration in (2.29) at x_0 instead of 0 we see directly that the first error term in (2.29) vanishes and we arrive at (2.28). □

Note that the same can be done with a function $g : \mathbb{R}^- \to \mathbb{R}$, and for a general $g : \mathbb{R} \to \mathbb{R}$ the estimate (2.27) follows by combining the estimates for $\mathbb{R}^+$ and $\mathbb{R}^-$.

With the help of Lemma 2.8 we can prove the convergence of the solutions to the discrete linear boundary value problem (2.12)-(2.14) to corresponding continuous expressions. Consider the solutions $s^\pm$ of (A.16) for $J = \mathbb{R}^\pm$ defined in (A.17) by

$$[s^-(\bar{r})](0) = \int_{-\infty}^0 S(0,x)\pi^-(x)\,\bar{r}(x)dx \quad \text{and} \quad [s^+(\bar{r})](0) = -\int_0^\infty S(0,x)(I-\pi^+(x))\,\bar{r}(x)dx$$

or more generally

$$s^\pm(\bar{r})(0) = \int_{\mathbb{R}^\pm} G_c(0,x)\bar{r}(x)dx.$$

The operators $s^\pm$ are approximated by the solution operators of the corresponding discrete system (2.12) given in (2.15) as the following Lemma shows.

Lemma 2.9 *Let* $\bar r : \mathbb{R} \to \mathbb{R}^m$ *and* $\hat r \in S_J(\mathbb{R}^m)$ *be given with* $\|\bar r\|_\infty < \infty$, $\|\bar r'\|_\infty < \infty$ *and*

$$\|\hat r_n - h\bar r(x_n)\| \le Ch^2 \quad \forall n \in J$$

Then for each $\epsilon > 0$ *there exist* $h_0, T > 0$*, such that for* $x_0 < h < h_0$*,* $hn_+ > T$ *the estimate*

$$\|s^\pm(\bar r)(0) - \hat s_0^\pm(\hat r)\| \le \epsilon$$

holds.

Proof: For $x > 0$ we set $g(x) = G_c(0,x)\bar r(x)$ and $\tilde g_m = \frac{1}{h}G(0,m)\hat r_m = \frac{1}{h}G_c(x_0,x_m)\hat r_m$ and obtain the following estimates:

$$\begin{aligned}
\|g(x)\| &\le \|G_c(0,x)\|\|\bar r(x)\| \le K\mathrm{e}^{-\alpha x}\|\bar r\|_\infty \\
\|g'(x)\| &\le \|\frac{d}{dx}G_c(0,x)\|\|\bar r(x)\| + \|G_c(0,x)\|\|\bar r'(x)\| \\
&\le \|G_c(0,x)\|(\|M(x)\|\|\bar r(x)\| + \|\bar r'(x)\|) \le K\mathrm{e}^{-\alpha x}(\|M\|_\infty\|\bar r\|_\infty + \|\bar r'\|_\infty)
\end{aligned}$$

as well as

$$\begin{aligned}
\|\tilde g_m - g(x_m)\| &\le \|G_c(x_0,x_m)\frac{1}{h}\hat r_m - G_c(0,x_m)\bar r(x_m)\| \\
&\le \|(G_c(x_0,0) - I)G_c(0,x_m)\|\|\bar r(x_m)\| + \|G_c(x_0,x_m)\|\|\frac{1}{h}\hat r_m - \bar r(x_m)\| \\
&\le C_0x_0\mathrm{e}^{-\alpha x_m} + C_1h\mathrm{e}^{-\alpha x_m} \le Ch\mathrm{e}^{-\alpha hm}.
\end{aligned}$$

Thus we can apply Lemma 2.8 from which the statements of Lemma 2.9 follow. □

The main linear result in this section deals with the existence of solutions $(z,\lambda) \in S_{J_r}(\mathbb{R}^k)\times\mathbb{R}^p$ of the following linear inhomogenous boundary value problem

$$z_{n+1} - \hat M_n z_n - \hat V_n\lambda = r_n, \quad n \in J = [n_-, n_+] \tag{2.30}$$

$$B_-z_{n_-} + B_+z_{n_+} = \eta \quad \in \mathbb{R}^k, \tag{2.31}$$

$$\hat\Pi(z) = \omega \quad \in \mathbb{R}^p \tag{2.32}$$

where

$$\hat M_n = \Phi(n+1,n) = S(x_{n+1},x_n),\ x_n = x_0 + hn. \tag{2.33}$$

Here $S(x,\xi)$ denotes the solution operator of the linear nonautonomous equation $Lz = z' - \mathcal{M}(\cdot)z$.

Hypothesis 2.10 *L has exponential dichotomies on* $\mathbb{R}^\pm$ *with data* $(K^\pm, \alpha^\pm, \pi^\pm)$ *and* $\mathcal{N}(L) = \mathrm{span}\{\phi^1,\dots,\phi^p\}$.

Hypothesis 2.11 *The matrix*

$$\begin{pmatrix} B_-\hat X_-^s & B_+\hat X_+^u \end{pmatrix} \in \mathbb{R}^{k,k} \tag{2.34}$$

is nonsingular, where the columns of $\hat X_-^s$ *span the stable subspace* X_-^s *of* M_- *and the columns of* X_+^u *span the unstable subspace of* M_+ *and* $\mathcal{M}_\pm = \lim_{x\to\pm\infty} M(x)$.

The phase condition $\hat{\Pi} : S_J(\mathbb{R}^k) \to \mathbb{R}^p$ is the discrete approximation of the linear integral condition $\langle \psi, v\rangle = 0$ in (1.19) and is given by

$$\hat{\Pi}(z) = h \sum_{n=n_-}^{n_+} \hat{\psi}(x_n)^T z_n, \tag{2.35}$$

where $\hat{\psi} : \mathbb{R} \to \mathbb{R}^{k,p}$ is a given reference function which satisfies the following hypothesis.

Hypothesis 2.12 *Assume that $\hat{\psi} \in \mathcal{E}_\rho(\mathbb{R}, \mathbb{R}^{k,p})$ (see Definition 2.7) and that the $p \times p$ matrix defined using ϕ^i from Hypothesis 2.10 by*

$$F = \int_{\mathbb{R}} \hat{\psi}(x)^T [\phi^1(x), \ldots, \phi^p(x)]\, dx. \tag{2.36}$$

is nonsingular.

Hypothesis 2.13 *The matrices $\hat{V}_n$ are of the form*

$$\hat{V}_n = hV(x_n) + \mathcal{O}(h^2) \in \mathbb{R}^{k,p} \tag{2.37}$$

for some continuous function $V \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{k,p})$ for which the following nondegeneracy condition holds. The matrix E given by

$$E = \int_{\mathbb{R}} [\psi_1, \ldots, \psi_p](x)^T V(x)\, dx \in \mathbb{R}^{p,p}$$

is nonsingular, where $\mathcal{N}(L^*) = \operatorname{span}\{\psi^1, \ldots, \psi^p\}$. (for the definition of the adjoint operator L^* see (A.14))

Now we can formulate the main linear existence result from which we obtain the existence of solutions of (2.30)–(2.32) as well as corresponding estimates. This lemma will be used in all of our approximation results which follow in the next sections.

Lemma 2.14 *Consider (2.30)–(2.32) and let Hypotheses 2.10–2.13 be satisfied.*

There exist $h_0 > 0$, $T > 0$ such that for $h < h_0$ and $\pm h n_\pm > T$ the equation (2.30) - (2.32) has a unique solution $(\tilde{z}, \tilde{\lambda}) \in S_{J_r}(\mathbb{R}^k) \times \mathbb{R}^p$ for any $r \in S_J(\mathbb{R}^k)$, $\eta \in \mathbb{R}^k$, $\omega \in \mathbb{R}^p$.

Furthermore the following estimate holds for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$:

$$\|\tilde{z}\|_{1,\diamond} + \|\tilde{\lambda}\| \le c(\frac{1}{h}\|r\|_\diamond + \|\eta\| + \|\omega\|) \tag{2.38}$$

Remark 2.15 Note that in the traveling wave case we have $p = 1$. But in order to be able to deal with more general symmetries (compare 1.4) we prove the result for general $p \ge 1$. This allows to prove approximation results for the general case (see 2.3.1) where the dimension p of the group G is larger than one.

Proof: From Hypothesis 2.10 and Lemma 2.3 one obtains that the operator $\hat{L}z : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ defined by

$$\hat{L}z = (z_{n+1} - \hat{M}_n z_n)_{n\in\mathbb{Z}}$$

possesses exponential dichotomies on $\mathbb{Z}^{\pm}$ with data $(K_{\pm}, \alpha_{\pm} h, P^{\pm})$ and

$$\mathcal{N}(\hat{L}) = \operatorname{span}\{\hat{q}^1, \dots, \hat{q}^p\}, \quad \text{where} \quad \hat{q}^i = \phi^i_{|J}, \; i = 1, \dots, p.$$

We use Lemma 2.5 to define partial solutions on $J^- = [n_-, 0]$ and $J^+ = [0, n_+]$ and construct for each $r \in S_J$

$$\begin{aligned}
\tilde{z}^-_n &= \hat{s}^-_n(r + \hat{V}\lambda) + \Phi(n,0) z^-_0 + \Phi(n, n_-)\rho_-, \quad n \in [n_-, 0],\\
\tilde{z}^+_n &= \hat{s}^+_n(r + \hat{V}\lambda) + \Phi(n,0) z^+_0 + \Phi(n, n_+)\rho_+, \quad n \in [0, n_+],\\
\tilde{z}^+_{n_++1} &= \hat{M}_{n_+} \tilde{z}^+_{n_+} + r_{n_+}
\end{aligned}$$

with

$$z^-_0 \in \mathcal{N}(P^-_0), \; z^+_0 \in \mathcal{R}(P^+_0), \; \rho_- \in \mathcal{R}(P^-_{n_-}), \; \rho_+ \in \mathcal{N}(P^+_{n_+}).$$

We define $\tilde{z} \in S_{J_r}$ by

$$\tilde{z}_n = \begin{cases} \tilde{z}^-_n, & \text{for } n \in [n_-, -1] \\ \tilde{z}^+_n, & \text{for } n \in [0, n_+ + 1] \end{cases} \tag{2.39}$$

which is a solution of (2.30)-(2.32) if the following system is solved

$$\tilde{z}^-_0 = \tilde{z}^+_0 \qquad \in \mathbb{R}^k \tag{2.40}$$

$$B_- \tilde{z}_{n_-} + B_+ \tilde{z}_{n_+} = \eta \qquad \in \mathbb{R}^k \tag{2.41}$$

$$\hat{\Pi}(\tilde{z}) = \omega \qquad \in \mathbb{R}^p. \tag{2.42}$$

Note that the parameter $\lambda \in \mathbb{R}^p$ is hidden in the definition of $\tilde{z}^{\pm}_n$ and is yet to be determined.

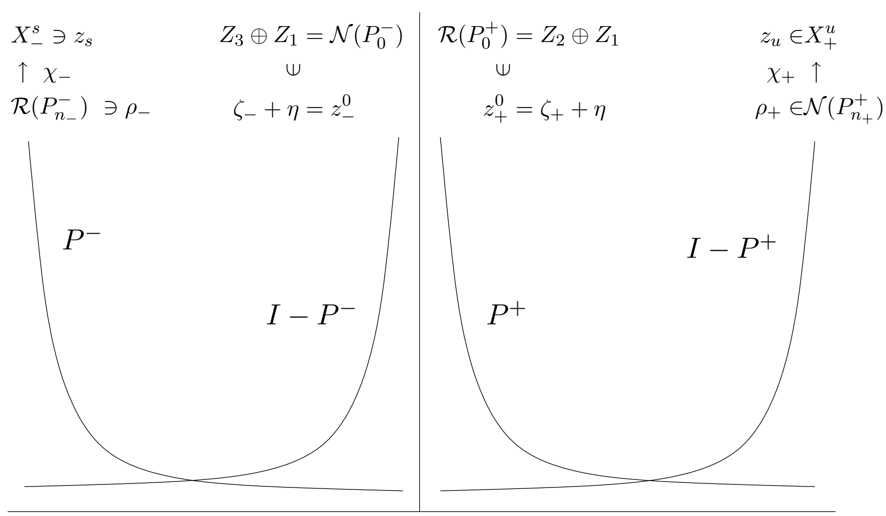


Figure 2.1: Overview over dichotomy estimates

We decompose $\mathbb{R}^k$ as follows: Let $Z_1 = \mathcal{R}(P^+_0) \cap \mathcal{N}(P^-_0)$. According to Hypothesis 2.10 we have $\dim(Z_1) = p$, we complement Z_1 by subspaces Z_2 and Z_3 such that

$$\mathcal{R}(P^+_0) = Z_1 \oplus Z_2, \quad \mathcal{N}(P^-_0) = Z_1 \oplus Z_3$$

Since $\dim(Z_1 \oplus Z_2 \oplus Z_3) = k_s^+ + k_u^- - p = k - p$ there exists a subspace Z_4 with $\dim Z_4 = p$ such that $Z_1 \oplus Z_2 \oplus Z_3 \oplus Z_4 = \mathbb{R}^k$ is a complete decomposition of $\mathbb{R}^k$.

We can change the projectors $P_0^\pm$ in such a way that (see [42], Prop. 2.3)

$$\mathcal{N}(P_0^+) = Z_3 \oplus Z_4, \quad \mathcal{R}(P_0^-) = Z_2 \oplus Z_4,$$

without changing the other dichotomy data.

From the choice of z_0^-, z_0^+ follows

$$\begin{aligned} (I - P_0^-)\tilde{z}_0^- &= z_0^- \quad \in \mathcal{N}(P_0^-) = Z_3 \oplus Z_1 \\ P_0^+ \tilde{z}_0^+ &= z_0^+ \quad \in \mathcal{R}(P_0^+) = Z_2 \oplus Z_1. \end{aligned}$$

We use the ansatz $z_0^- = \zeta_- + \eta_-$, $z_0^+ = \zeta_+ + \eta_+$, where $\zeta_- \in Z_3$, $\zeta_+ \in Z_2$, $\eta_\pm \in Z_1$ and from (2.40) we obtain $\eta_+ = \eta_- =: \eta$. Equation (2.40) now reads

$$\zeta_- - \zeta_+ + \Phi(0, n_-)\rho_- - \Phi(0, n_+)\rho_+ + (\hat{s}_0^-(\hat{V}) - \hat{s}_0^+(\hat{V}))\lambda = \hat{s}_0^+(r) - \hat{s}_0^-(r).$$

The left hand side of this equation has no component in Z_1. We transform the boundary values ρ_-, ρ_+ to coordinates (z_s, z_u) which are independent of J as follows: Denote by E_-^s the projector onto X_-^s along X_-^u and by E_+^u the projector onto X_+^u along X_+^s, where $X_\pm^{s,u}$ are defined in Hypothesis 2.11. We define the transformations

$$\chi_- : \mathcal{R}(P_{n_-}^-) \to X_-^s,\ \rho_- \mapsto z_s, \qquad \chi_+ : \mathcal{N}(P_{n_+}^+) \to X_+^u,\ \rho_+ \mapsto z_u$$

by

$$\chi_- = I + E_-^s - P_{n_-}^-, \qquad \chi_+ = I - E_+^s + P_{n_+}^+.$$

From the roughness theorem A.6 we have $\lim\limits_{x\to\pm\infty} \pi^\pm(x) = E_\pm^s$ and with $P_n^\pm = \pi^\pm(x_0 + hn)$ the invertibiliy of χ_- and χ_+ follows for $\pm hn_\pm > T$, T large and $h, x_0 \to 0$ as well as the estimates

$$\|\chi_\pm^{-1}\| \le \frac{1}{1 - \|P_{n_\pm}^\pm - E_\pm^s\|} \le 2. \tag{2.43}$$

Furthermore, for all $(z_s, z_u) \in X_-^s \times X_+^u$ we have

$$\|(I - E_-^s)\chi_-^{-1} z_s\| \le \frac{\|P_{n_-}^- - E_-^s\| \|z_s\|}{1 - \|P_{n_-}^- - E_-^s\|}, \qquad \|E_+^s \chi_+^{-1} z_u\| \le \frac{\|P_{n_+}^+ - E_+^s\| \|z_u\|}{1 - \|P_{n_+}^+ - E_+^s\|}. \tag{2.44}$$

Defining the maps $c : Z_3 \times Z_2 \times \mathbb{R}^p \to S_\mathbb{Z}(\mathbb{R}^k)$ and $d : X_-^s \times X_+^u \to S_\mathbb{Z}(\mathbb{R}^k)$ by

$$c_n(\zeta_-, \zeta_+, \lambda) = \begin{cases} \Phi(n,0)\zeta_- + \hat{s}_n^-(\hat{V})\lambda, & n < 0 \\ \Phi(n,0)\zeta_+ + \hat{s}_n^+(\hat{V})\lambda, & n \ge 0 \end{cases}, \tag{2.45}$$

$$d_n(z_s, z_u) = \begin{cases} d_n^-(z_s), & n < 0 \\ d_n^+(z_u), & n \ge 0, \end{cases} \tag{2.46}$$

$$\text{where } d_n^\pm(\eta) = \Phi(n, n_\pm)\chi_\pm^{-1}\eta,$$

we can rewrite $\tilde{z}$ defined in (2.39) as follows

$$\begin{aligned} \tilde{z}_n &= c_n(\zeta_-, \zeta_+, \lambda) + d_n(z_s, z_u) + \Phi(n,0)\eta + \begin{cases} \hat{s}_n^-(r), & n \in [n_-, -1] \\ \hat{s}_n^+(r), & n \in [0, n_+], \end{cases} \\ \tilde{z}_{n_++1} &= \hat{M}_{n_+}\tilde{z}_{n_+} + r_{n_+}. \end{aligned}$$

Using (2.46) equation (2.40) now reads

$$\zeta_- - \zeta_+ + (\hat{s}_0^-(\hat{V}) - \hat{s}_0^+(\hat{V}))\lambda + d_0^-(z_s) - d_0^+(z_u) = \hat{s}_0^+(r) - \hat{s}_0^-(r). \tag{2.47}$$

Define $\hat{Q} \in S_{\mathbb{Z}}(\mathbb{R}^{m,p})$ by

$$\hat{Q}_n = \begin{cases} \Phi(n,0)(I - P_0^-)[\phi^1(x_0), \dots, \phi^p(x_0)], & n < 0, \\ \Phi(n,0)P_0^+[\phi^1(x_0), \dots, \phi^p(x_0)], & n \geq 0. \end{cases} \tag{2.48}$$

Since the columns of $\hat{Q}_0$ span a basis of $\mathcal{N}(P_0^-) \cap \mathcal{R}(P_0^+)$ we can write $\eta \in Z_1$ as $\eta = \hat{Q}_0\kappa$ for some $\kappa \in \mathbb{R}^p$. We obtain for the boundary conditions (2.41)

$$\begin{aligned} &B_- c_{n_-}(\zeta_-, \zeta_+, \lambda) + B_+ c_{n_+}(\zeta_-, \zeta_+, \lambda) + B_- d_{n_-}^-(z_s) + B_+ d_{n_+}^+(z_u) \\ &\quad + (B_-\Phi(n_-, 0) + B_+\Phi(n_+, 0))\, \hat{Q}_0\kappa \\ &= \eta - (B_+(\hat{s}_{n_+}^+(r) - B_-\hat{s}_{n_-}^-(r)). \end{aligned}$$

and the phase condition (2.42) reads

$$\hat{\Pi}(c(\zeta_-, \zeta_+, \lambda)) + \hat{\Pi}(d(z_s, z_u)) + \hat{\Pi}(\hat{Q}\kappa) = \omega - \hat{\Pi}(\hat{s}(r)).$$

We summarize the equations in

$$T \begin{pmatrix} (\zeta_-, \zeta_+, \lambda) \\ (z_s, z_u) \\ \kappa \end{pmatrix} = \begin{pmatrix} \hat{s}_0^+(r) - \hat{s}_0^-(r) \\ \eta - \left(B_+(\hat{s}_{n_+}^+(r) - B_-\hat{s}_{n_-}^-(r)\right) \\ \omega - \hat{\Pi}(\hat{s}(r)) \end{pmatrix} \tag{2.49}$$

where $T : (Z_2 \times Z_3 \times \mathbb{R}^p) \times (X_-^s \times X_+^u) \times \mathbb{R}^p \to (Z_2 \oplus Z_3 \oplus Z_4) \times \mathbb{R}^k \times \mathbb{R}^p$ has the following structure

$$T = \begin{pmatrix} X & \sigma & 0 \\ \Delta & Y & \varrho \\ \Theta & \Lambda & Z \end{pmatrix}$$

where

$$\begin{aligned} X(\zeta_-, \zeta_+, \lambda) &= \zeta_- - \zeta_+ + (\hat{s}_0^-(\hat{V}) - \hat{s}_0^+(\hat{V}))\lambda \\ \sigma(z_s, z_u) &= \Phi(0, n_-)\chi_-^{-1} z_s - \Phi(0, n_+)\chi_+^{-1} z_u \\ \Delta(\zeta_-, \zeta_+, \lambda) &= B_- c_{n_-}(\zeta_-, \zeta_+, \lambda) + B_+ c_{n_+}(\zeta_-, \zeta_+, \lambda) \\ Y(z_s, z_u) &= B_-\chi_-^{-1} z_s + B_+\chi_+^{-1} z_u, \\ \Theta(\zeta_-, \zeta_+, \lambda) &= h \sum_{n=n_-}^{n_+} \hat{\psi}(x_n)^T c_n(\zeta_-, \zeta_+, \lambda) \\ \Lambda(z_s, z_u) &= \hat{\Pi}(d(z_s, z_u)) = h \sum_{n=n_-}^{n_+} \hat{\psi}(x_n)^T d_n(z_s, z_u) \\ \rho(\kappa) &= (B_-\Phi(n_-, 0) + B_+\Phi(n_+, 0))\hat{Q}_0\kappa \\ Z(\kappa) &= \hat{\Pi}(\hat{Q}\kappa) = h \sum_{n=n_-}^{n_+} \hat{\psi}(x_n)^T \hat{Q}_n\kappa. \end{aligned}$$

We have to show the invertibility of T as well as an estimate of the inverse of T. The terms σ, ρ can be estimated using the exponential dichotomy of $\hat{L}$ by

$$\|\sigma\| \leq K\mathrm{e}^{-\alpha h \min(-n_-, n_+)} \quad \to 0 \quad \text{as } h \min\{-n_-, n_+\} \to \infty$$

and using $\mathcal{R}(\hat{Q}_0) = \mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)$ we get

$$\begin{aligned} \|\varrho\| &\leq \|B_-\| \|\Phi(n_-, 0)\hat{Q}_0\| + \|B_+\| \|\Phi(n_+, 0)\hat{Q}_0\| \\ &= \|B_-\| \|\Phi(n_-, 0)(I - P_0^-)\hat{Q}_0\| + \|B_+\| \|\Phi(n_+, 0)P_0^+\hat{Q}_0\| \\ &\leq \left(K_- \mathrm{e}^{\alpha_- h n_-} + K_+ \mathrm{e}^{-\alpha_+ h n_+}\right) \|\hat{Q}_0\| \\ &\to 0 \quad \text{as } h \min\{-n_-, n_+\} \to \infty \end{aligned}$$

The boundedness of the operators Δ, Λ, Θ will be shown as follows: The term $\|c_n(\zeta_-, \zeta_+, \lambda)\|$ can be estimated for all $n \in J$ using Lemma 2.6 and the estimate $\|\hat{V}\|_\infty \leq Ch\|V\|_\infty$ which follows from (2.37) by

$$\begin{aligned} \|c_n(\zeta_-, \zeta_+, \lambda)\| &\leq \begin{cases} \|\Phi(n, 0)(I - P_0^-)\zeta_-\| + \|\tilde{s}_n^-(\hat{V})\| \|\lambda\|, & \text{for } n < 0 \\ \|\Phi(n, 0)P_0^+\zeta_+\| + \|\tilde{s}_n^+(\hat{V})\| \|\lambda\|, & \text{for } n \geq 0 \end{cases} \\ &\leq \begin{cases} K_- \mathrm{e}^{\alpha_- h n} \|\zeta_-\| + C\|V\|_\infty \|\lambda\|, & \text{for } n < 0 \\ K_+ \mathrm{e}^{-\alpha_+ h n} \|\zeta_+\| + C\|V\|_\infty \|\lambda\|, & \text{for } n \geq 0 \end{cases} \\ &\leq K(\|\zeta_-\| + \|\zeta_+\| + \|\lambda\|). \end{aligned}$$

Therefore we get for Δ

$$\begin{aligned} \|\Delta(\zeta_-, \zeta_+, \lambda)\| &\leq \|B_-\| \|c_{n_-}(\zeta_-, \zeta_+, \lambda)\| + \|B_+\| \|c_{n_+}(\zeta_-, \zeta_+, \lambda)\| \\ &\leq K(\|\zeta_-\| + \|\zeta_+\| + \|\lambda\|). \end{aligned}$$

The properties of ζ in Hypothesis 2.12 ensure that the map $\hat{\Pi} : S_J(\mathbb{R}^k) \to \mathbb{R}^p$ is uniformly bounded in J. Using the dichotomy estimates again we obtain

$$\|\Theta(\zeta_-, \zeta_+, \lambda)\| \leq K\|c(\zeta_-, \zeta_+, \lambda)\|_\infty \leq K\|\hat{\Pi}\|(\|\zeta_-\| + \|\zeta_+\| + \|\lambda\|)$$

and finally

$$\|\Lambda(z_s, z_u)\| = \|\hat{\Pi}(d(z_s, z_u))\| \leq K\|(z_s, z_u)\|.$$

From (2.44) and Hypothesis 2.11 follows that Y has a uniformly bounded inverse, therefore it remains to show the invertibility of the remaining operators on the diagonal X and Z.

Application of Lemma 2.9 shows that X and Z converge for $x_0, h \to 0$ and $\pm h n_\pm \to \infty$ to $\bar{X}$ and $\bar{Z}$ given by

$$\bar{X}(\zeta_-, \zeta_+, \lambda) = \zeta_- - \zeta_+ + (s^-(V)(0) - s^+(V)(0))\lambda,$$

where

$$\begin{aligned} s^-(V)(x) &= \int_{-\infty}^{x} S(x, \xi)\pi^-(\xi)V(\xi)\, d\xi - \int_x^0 S(x, \xi)(I - \pi^-(\xi))V(\xi)\, d\xi, \quad \text{for } x \leq 0 \\ s^+(V)(x) &= \int_0^x S(x, \xi)\pi^+(\xi)V(\xi)\, d\xi - \int_x^\infty S(x, \xi)(I - \pi^+(\xi))V(\xi)\, d\xi, \quad \text{for } x \geq 0 \end{aligned}$$

and

$$\bar{Z}(\kappa) = \int_{\mathbb{R}} \hat{\psi}(x)^T [\phi^1(x), \ldots, \phi^p(x)]\, dx\ \kappa.$$

The invertiblity of the operator $\bar{Z}$ is ensured by Hypothesis 2.12 and the invertibility of $\bar{X}$ follows from the nondegeneracy condition Hypothesis 2.13 similar to [3], [60] by multiplying the equation

$$0 = \zeta_- - \zeta_+ + (s^-(V)(0) - s^+(V)(0))\lambda$$

from the left by $[\psi^1, \ldots, \psi^k]$. Then we obtain that X and Z are invertible for x_0, h small enough and $\mp hn_\pm$ large enough with a uniform bound for the inverse.

Summing up the estimates for the right hand side in (2.49) we get for $\pm hn_\pm > T$

$$\begin{aligned}
&\|\zeta_-\| + \|\zeta_+\| + \|\lambda\| + \|z_s\| + \|z_u\| + \|\kappa\| \\
&\quad \le C\Big(\|\hat{s}_0^+(r)\| + \|\hat{s}_0^-(r)\|) + \|\eta\| + \|B_+\hat{s}_{n_+}^+(r)\| + \|B_-\hat{s}_{n_-}^-(r)\| + \|\omega\| + \|\hat{\Pi}(\hat{s}(r))\| \Big)
\end{aligned}$$

With the estimate (2.16) for $\hat{s}^\pm$ in Lemma 2.5 and the properties of $\hat{\Pi}$ one obtains

$$\|\zeta_-\| + \|\zeta_+\| + \|\lambda\| + \|z_s\| + \|z_u\| + \|\kappa\| \le C(\frac{1}{h}\|r\|_\infty + \|\eta\| + \|\omega\|) \tag{2.50}$$

and additionally for $r \in \mathcal{L}_{2,h}$ using (2.17) with $\beta = \alpha h$

$$\|\zeta_-\| + \|\zeta_+\| + \|\lambda\| + \|z_s\| + \|z_u\| + \|\kappa\| \le C(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\eta\| + \|\omega\|). \tag{2.51}$$

From Corollary 2.6 we get estimates of the partial solution $\tilde{z}^- \in S_{J^-}$ for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$

$$\|\tilde{z}^-\|_\diamond \le C(\frac{1}{h}\|r + \hat{V}\lambda\|_\diamond + \|z_0^-\| + \|\rho_-\|) \le C(\frac{1}{h}\|r\|_\diamond + \|\lambda\| + \|z_0^-\| + \|\rho_-\|)$$

using $\|\hat{V}\|_\infty \le h\|V\|_\infty$ as well as $\|\hat{V}\|_{\mathcal{L}_{2,h}} \le Ch\|V\|_{\mathcal{L}_2}$.

Now (2.43), (2.50) and for $r \in \mathcal{L}_{2,h}$ (2.51) yield for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$

$$\|z_0^-\| \le \|\zeta_-\| + \|\hat{Q}_0\|\|\kappa\| \le C(\frac{1}{h}\|r\|_\diamond + \|\eta\| + \|\omega\|)$$

and

$$\|\rho_-\| = \|\chi_-^{-1} z_s\| \le 2\|z_s\| \le C(\frac{1}{h}\|r\|_\diamond + \|\eta\| + \|\omega\|)$$

giving the desired estimate of $\tilde{z}^-$. Similar estimates hold for $\tilde{z}^+$, which leads for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ to

$$\|\tilde{z}_{|J}\|_\diamond \le C(\frac{1}{h}\|r\|_\diamond + \|\eta\| + \|\omega\|).$$

It remains to consider the contribution at $n_+ + 1$. We have

$$\|\tilde{z}_{n_+ + 1}\| \le \|M_{n_+}\|\|\tilde{z}_{n_+}^+\| + \|r_{n_+}\| \le C(\frac{1}{h}\|r\|_\infty + \|\eta\| + \|\omega\|)$$

for $h < 1$. This implies for $r \in \mathcal{L}_{2,h}$ with $\|r\|_\infty \le \frac{1}{\sqrt{h}}\|r\|_{\mathcal{L}_{2,h}}$

$$\|\tilde{z}\|_{\mathcal{L}_{2,h}} \le \|\tilde{z}_{|J}\|_{\mathcal{L}_{2,h}} + \sqrt{h}\|\tilde{z}_{n_+ + 1}\| \le C(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\eta\| + \|\omega\|).$$

Thus we can estimate $(\tilde{z}, \tilde{\lambda})$ for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ by

$$\|\tilde{z}\|_\diamond + \|\tilde{\lambda}\| \leq C(\frac{1}{h}\|r\|_\diamond + \|\eta\| + \|\omega\|) = C\|(r, \eta, \omega)\|_\diamond^*.$$

Using the difference equation (2.30) and

$$\hat{V}_n = hV(x_n) + \mathcal{O}(h^2), \quad \hat{M}_n = I + \mathcal{O}(h)$$

which hold by (2.37) and (2.33), this can be improved for h small enough to the $\|\cdot\|_{1,\infty}$ resp. $\|\cdot\|_{\mathcal{H}_h^1}$ estimates (2.38). Since for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ we obtain, again using $\|\hat{V}\|_{\mathcal{L}_{2,h}} \leq Kh\|V\|_{\mathcal{L}_2}$,

$$\begin{aligned}
\|\delta_+\tilde{z}\|_\diamond &= \frac{1}{h}\|(\tilde{z}_{n+1} - \tilde{z}_n)_{n\in J}\|_\diamond \leq \frac{1}{h}\|((\hat{M}_n - I)\tilde{z}_n + \hat{V}_n\lambda + r_n)_{n\in J}\|_\diamond \\
&\leq \frac{1}{h}(\sup_{n\in J}(\|\hat{M}_n - I\|)\|z\|_\diamond + h\|V\|_\diamond\|\lambda\| + \|r\|_\diamond) \\
&\leq \frac{1}{h}(Ch\|(r, \eta, \omega)\|_\diamond^* + \|r\|_\diamond) \leq \tilde{C}\|(r, \eta, \omega)\|_\diamond^*.
\end{aligned}$$

□

Remark 2.16 If the operator L possesses an exponential dichotomy on the whole line $\mathbb{R}$ then Lemma 2.14 holds with $p = 0$, i.e. the phase condition (2.32) and the parameter λ do not occur. The estimate (2.38) simplifies to

$$\|\tilde{z}\|_{1,\diamond} \leq C(\frac{1}{h}\|r\|_\diamond + \|\eta\|), \quad \diamond \in \{\infty, \mathcal{L}_{2,h}\}$$

A solution of a small perturbation of (2.30) can be estimated as well.

Corollary 2.17 *Let $(z^\Delta, \lambda^\Delta)$ be a solution of the perturbed equation*

$$z_{n+1} - (\hat{M}_n + \Delta M_n)z_n - (\hat{V}_n + \Delta V_n)\lambda = r_n, \quad n \in J = [n_-, n_+] \tag{2.52}$$

$$B_- z_{n_-} + B_+ z_{n_+} = \eta \quad \in \mathbb{R}^k, \tag{2.53}$$

$$\hat{\Pi}(z) = \omega \quad \in \mathbb{R}^p \tag{2.54}$$

where $\hat{M}, \hat{V}, \hat{\Pi}$ and $B_\pm$ are defined in Lemma 2.14 and the error terms can be estimated by

$$\|\Delta M\|_\infty \leq \sigma(h, T)h, \quad \|\Delta V\|_{\mathcal{L}_{2,h}} \leq \sigma(h, T)h,$$

where $\lim_{h\to 0,\ T\to\infty} \sigma(h, T) = 0$.

Then $(z^\Delta, \lambda^\Delta)$ can be estimated by

$$\|z^\Delta\|_{\mathcal{H}^1} + |\lambda^\Delta| \leq \frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\eta\| + \|\omega\|. \tag{2.55}$$

Proof: By (2.52) we obtain

$$z_{n+1} - \hat{M}_n z_n - \hat{V}_n\lambda = r_n + \Delta M_n z_n + \Delta V_n\lambda.$$

Applying the estimate (2.38) results in

$$\begin{aligned}\|z\|_{\mathcal{H}_h^1}+\|\lambda\| &\le \frac{1}{h}(\|r\|_{\mathcal{L}_{2,h}}+\|(\Delta M_n z_n)_{n\in J}\|_{\mathcal{L}_{2,h}}+\|\Delta V\|_{\mathcal{L}_{2,h}}\lambda)+\|\eta\|+\|\omega\| \\ &\le \frac{1}{h}(\|r\|_{\mathcal{L}_{2,h}}+h\sigma(h,T)\|z\|_{\mathcal{L}_{2,h}}+h\sigma(h,T)\|\lambda\|)+\|\eta\|+\|\omega\| \\ &\le \sigma(h,T)(\|z\|_{\mathcal{L}_{2,h}}+\|\lambda\|)+\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}}+\|\eta\|+\|\omega\| \\ &\le \frac{1}{2}(\|z\|_{\mathcal{L}_{2,h}}+\|\lambda\|)+\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}}+\|\eta\|+\|\omega\|\end{aligned}$$

for $h<h_0,\ T>T_0$. This implies

$$\|z\|_{\mathcal{H}_h^1}+\|\lambda\| \le 2(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}}+\|\eta\|+\|\omega\|).$$

□

2.2 Approximation of the traveling wave

As motivated above, the discretized equation will be transformed to a first order system. Using the results for the linear difference equation the following main approximation result for the traveling wave solution can be proved by using the fixed point Theorem A.3 which is stated in the appendix. Let $(\bar{u},\bar{\lambda})\in\mathcal{C}_b^4(\mathbb{R},\mathbb{R}^m)\times\mathbb{R}$ be a solution of (2.1), i.e.

$$A\bar{u}''+\bar{\lambda}\bar{u}'+f(\bar{u},\bar{u}')=0$$

with $\lim_{x\to\pm\infty}\bar{u}(x)=u_\pm$.

Consider the corresponding discrete boundary value problem (2.2) with affine-linear boundary conditions and a phase condition, given by

$$A(\delta_+\delta_-u)_n+\lambda(\delta_0u)_n+f(u_n,(\delta_0u)_n)=0,\quad n\in J \tag{2.56}$$

$$P_-u_{n_-}+Q_-(\delta_0u)_{n_-}+P_+u_{n_+}+Q_+(\delta_0u)_{n_+}=\eta \tag{2.57}$$

$$\tilde{\Psi}(u)=0. \tag{2.58}$$

The phase condition $\hat{\Psi}:S_J(\mathbb{R}^m)\to\mathbb{R}$ is the discrete approximation of the integral condition (1.12) and is given by

$$\tilde{\Psi}(u)=\langle\delta_0\hat{u},u-\hat{u}\rangle_h=h\sum_{n=n_-}^{n_+}(\delta_0\hat{u})_n^T(u_n-\hat{u}_n)=0, \tag{2.59}$$

where $\hat{u}:\mathbb{R}\to\mathbb{R}^m$ is a given reference function which satisfies the following hypothesis.

Hypothesis 2.18 *Assume* $\hat{u}-\bar{u}\in\mathcal{H}^1(\mathbb{R},\mathbb{R}^m),\ \hat{u}'\in\mathcal{E}_\alpha(\mathbb{R},\mathbb{R}^m)$, $\langle\hat{u}',\bar{u}'\rangle\neq 0$ *and* $\langle\hat{u}',\bar{u}-\hat{u}\rangle=0$.

The phase condition $\tilde{\Psi}$ is a discrete approximation of the integral condition (1.12).

Hypothesis 2.19 *The boundary condition* (2.57) *is satisfied at the stationary points* $u_\pm$*, i.e.*

$$\eta = P_- u_- + P_+ u_+.$$

Assume further, that the following regularity condition holds

$$\det\left(\begin{pmatrix}P_- & Q_-\end{pmatrix}\begin{pmatrix}Y_-^s \\ Y_-^s\Lambda_-^s\end{pmatrix} \quad \begin{pmatrix}P_+ & Q_+\end{pmatrix}\begin{pmatrix}Y_+^u \\ Y_+^u\Lambda_+^u\end{pmatrix}\right) \neq 0. \tag{2.60}$$

where Y_-^s, Y_+^u *and* Λ_-^s, Λ_+^u *are defined in A.8.*

In the following we list the general assumptions on the operator Λ and the nonlinearity f which will be used throughout the thesis.

Hypothesis 2.20 *The operator* Λ *satisfies the assumptions (SC) and (ECw) (see Remark 1.16) and the nonlinearity* f *satisfies Hypothesis 1.9.*

Then the following theorem holds.

Theorem 2.21 *Assume that Hypotheses 2.20, 2.19 and are 2.18 are satisfied.*

Then there exist $\varrho > 0$, $T > 0$, $h_0 > 0$ *such that for* $h < h_0$ *and* $\pm hn_\pm > T$ *the boundary value problem* (2.56)-(2.58) *has a unique solution* $(\tilde{u}, \tilde{\lambda})$ *in a neighborhood* $B_\varrho(\bar{u}, \bar{\lambda}) = \{(u, \lambda) \in S_{J_e} \times \mathbb{R} : \|\bar{u} - u\|_{2,\infty} + |\bar{\lambda} - \lambda| < \varrho\}$ *which obeys the following estimate for* $C > 0, \alpha > 0$

$$\|\bar{u}_{|J} - \tilde{u}\|_{2,\infty} + |\bar{\lambda} - \tilde{\lambda}| \leq C(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}). \tag{2.61}$$

Proof: A solution of (2.56)–(2.58) is a zero of the operator $F : S_{J_e}(\mathbb{R}^m) \times \mathbb{R} \to S_J(\mathbb{R}^m) \times \mathbb{R}^{2m} \times \mathbb{R}$ where

$$F(u, \lambda) = \begin{pmatrix} (A(\delta_+\delta_- u)_n + \lambda(\delta_0 u)_n + f(u_n, \delta_0 u_n))_{n\in J} \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} - \eta \\ \tilde{\Psi}(u) \end{pmatrix}.$$

The derivative at the exact traveling wave $(\bar{u}_{|J}, \bar{\lambda})$ then reads

$$DF(\bar{u}, \bar{\lambda})(u, \lambda) = \begin{pmatrix} (A(\delta_+\delta_- u)_n + B_n(\bar{u}_{|J}, \bar{\lambda})(\delta_0 u)_n + C_n(\bar{u}_{|J}) u_n + D_n(\bar{u}_{|J})\lambda)_{n\in J} \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} \\ \tilde{\Pi}(u) \end{pmatrix} \tag{2.62}$$

where, setting $w_n = (\delta_0 u)_n = \frac{1}{2h}(u_{n+1} - u_{n-1})$,

$$B_n(u, \lambda) = \lambda I + D_2 f(u_n, w_n), \quad C_n(u) = D_1 f(u_n, w_n), \quad D_n(u) = (\delta_0 u)_n,$$
$$\tilde{\Pi}(u) = h \sum_{n=n_-}^{n_+} (\delta_0 \hat{u})_n^T u_n.$$

We want to apply the fixed point Theorem A.3 to F with $Y = S_{J_e}(\mathbb{R}^m) \times \mathbb{R}$, $Z = S_J(\mathbb{R}^m) \times \mathbb{R}^{2m} \times \mathbb{R}$, with norms

$$\|(u, \lambda)\|_Y = \|u\|_{2,\infty} + |\lambda|, \qquad \|(r, \eta, \omega)\|_Z = \|r\|_\infty + \|\eta\| + |\omega|$$

at the approximative zero $\bar{y} = (\bar{u}, \bar{\lambda})$.

Using Lemma 2.8 we obtain from Hypothesis 2.18 the estimate

$$\|\tilde{\Psi}(\bar{u}_{|_J})\| \le C(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}).$$

Together with the approximation properties of the difference operators and the exponential convergence of $\bar{u}$ towards the stationary points this implies consistency

$$\begin{aligned}
\|F(\bar{u}_{|_J}, \bar{\lambda})\|_Z &\le \sup_{n\in J} \|A(\delta_+\delta_-\bar{u})_n + \bar{\lambda}(\delta_0\bar{u})_n + f(\bar{u}_n, (\delta_0\bar{u})_n)\| \\
&\quad + \|P_-\bar{u}_{n_-} + Q_-(\delta_0\bar{u})_{n_-} + P_+\bar{u}_{n_+} + Q_+(\delta_0\bar{u})_{n_+}\| + \|\tilde{\Psi}(\bar{u}_{|_J})\| \\
&\le \sup_{n\in J} \|A\bar{u}''_n + \bar{\lambda}\bar{u}'_n + f(\bar{u}_n, \bar{u}'_n)\| + \mathcal{O}(h^2)\mathrm{e}^{-\varrho h n} \\
&\quad + \|P_-u_- + P_+u_+ - \eta\| + \|P_-(\bar{u}_{n_-} - u_-)\| + \|P_+(\bar{u}_{n_+} - u_+)\| \\
&\quad\ + \|Q_-\bar{u}'_{n_-}\| + \mathcal{O}(h^2)\|\bar{u}''\|_\infty + \|Q_+\bar{u}'_{n_+}\| + \mathcal{O}(h^2)\|\bar{u}''\|_\infty \\
&\quad\quad + \|\tilde{\Psi}(\bar{u}_{|_J})\| \\
&\le C(h^2 + \|\bar{u}_{n_-} - u_-\| + \|\bar{u}_{n_+} - u_+\| + \|\bar{u}'_{n_-}\| + \|\bar{u}'_{n_+}\| + h^2\|\bar{u}''\|_\infty) \\
&\quad + \|\tilde{\Psi}(\bar{u}_{|_J})\| \\
&\le C(h^2 + \mathrm{e}^{-\tilde{\alpha} T}).
\end{aligned} \tag{2.63}$$

From Hypothesis 1.9 we obtain the estimates

$$\begin{aligned}
\|B_n(\tilde{u}, \tilde{\lambda}) - B_n(\bar{u}_{|_J}, \bar{\lambda})\| &\le |\tilde{\lambda} - \bar{\lambda}| + \|D_2 f(\tilde{u}_n, \tilde{w}_n) - D_2 f(\bar{u}_n, \bar{w}_n)\| \\
&\le C(|\tilde{\lambda} - \bar{\lambda}| + \|\tilde{u}_n - \bar{u}_n\| + \|\delta_0(\tilde{u}_n - \bar{u}_n)\| \\
\|C_n(\tilde{u}, \tilde{\lambda}) - C_n(\bar{u}_{|_J}, \bar{\lambda})\| &= \|D_1 f(\tilde{u}_n, \tilde{w}_n) - D_1 f(\bar{u}_n, \bar{w}_n)\| \\
&\le C(\|\tilde{u}_n - \bar{u}_n\| + \|\delta_0(\tilde{u}_n - \bar{u}_n)\|) \\
\|D_n(\tilde{u}) - D_n(\bar{u}_{|_J})\| &\le C\|\delta_0(\tilde{u}_n - \bar{u}_n)\|
\end{aligned}$$

from which follows

$$\begin{aligned}
\|(DF(\tilde{u}, \tilde{\lambda}) - DF(\bar{u}_{|_J}, \bar{\lambda}))(u, \lambda)\|_\infty &\le \sup_{n\in J} \|B_n(\tilde{u}, \tilde{\lambda}) - B_n(\bar{u}_{|_J}, \bar{\lambda})\| \|\delta_0 u\|_\infty \\
&\quad + \sup_{n\in J} \|C_n(\tilde{u}, \tilde{\lambda}) - C_n(\bar{u}_{|_J}, \bar{\lambda})\| \|u\|_\infty + \sup_{n\in J} \|D_n(\tilde{u}) - D_n(\bar{u}_{|_J})\| |\lambda| \\
&\le C(\|\tilde{u} - \bar{u}\|_{1,\infty} + |\tilde{\lambda} - \bar{\lambda}|)(\|u\|_{1,\infty} + |\lambda|).
\end{aligned} \tag{2.64}$$

In order to show the invertibility of $DF(\bar{u}_{|_J}, \bar{\lambda})$ we transform the variational equation

$$DF(\bar{u}, \bar{\lambda})(u, \lambda) = (g, \eta, \omega) \tag{2.65}$$

to first order using $z_n = (u_n, v_n),\ v_n = (\delta_- u)_n$ and obtain for (z, λ) the equivalent equation

$$\tilde{\Lambda}(z, \lambda) = (r, \eta, \omega), \tag{2.66}$$

where $w_n = (\delta_0 u)_n = \frac{1}{2}(v_{n+1} + v_n)$ and

$$\tilde{\Lambda}(z, \lambda) = \begin{pmatrix} \Gamma(z, \lambda) \\ P_-u_{n_-} + Q_-w_{n_-} + P_+u_{n_+} + Q_+w_{n_+} \\ \hat{\Pi}(z) \end{pmatrix},$$

$$r_n = \begin{pmatrix} 0 \\ hg_n \end{pmatrix}, \quad \hat{\Pi}(z) = h \sum_{n=n_-}^{n_+} \begin{pmatrix} (\delta_0\hat{u})_n \\ 0 \end{pmatrix}^T z_n.$$

The operator $\Gamma : S_{J_r}(\mathbb{R}^{2m}) \times \mathbb{R} \to S_J(\mathbb{R}^{2m})$ is given by

$$(\Gamma(z,\lambda))_n = N_n z_{n+1} - K_n z_n - W_n \lambda$$

where

$$N_n = \begin{pmatrix} I & -hI \\ 0 & E_n^+ \end{pmatrix}, \quad K_n = \begin{pmatrix} I & 0 \\ -hC_n & E_n^- \end{pmatrix}, \quad W_n = -\begin{pmatrix} 0 \\ h(\delta_0 \bar{u}_{|_J})_n \end{pmatrix}$$

and

$$B_n = B_n(\bar{u}_{|_J}, \bar{\lambda}), \quad C_n = C_n(\bar{u}_{|_J}, \bar{\lambda}), \quad E_n^{\pm} = A \pm \frac{h}{2} B_n. \tag{2.67}$$

We consider $\tilde{\Lambda}$ as an operator from $S_{J_r}(\mathbb{R}^{2m}) \times \mathbb{R}, \|(z,\lambda)\|_{1,\infty} = \|z\|_{1,\infty} + |\lambda|$ into $S_J(\mathbb{R}^{2m}) \times \mathbb{R}^{2m} \times \mathbb{R}, \|(r,\eta,\omega)\|_\infty^* = \frac{1}{h}\|r\|_\infty + \|\eta\| + |\omega|$.

In order to relate (2.66) with a corresponding continuous system we consider a perturbation of $\tilde{\Lambda}$ which is given by

$$\Lambda_i(z,\lambda) = \begin{pmatrix} (\hat{N} z_{n+1} - \hat{K}_n z_n - \hat{W}_n \lambda)_{n \in J} \\ (P_- \ Q_-) z_{n_-} + (P_+ \ Q_+) z_{n_+} \\ \hat{\Pi}(z) \end{pmatrix} \tag{2.68}$$

where

$$\hat{K}_n = \begin{pmatrix} I & hI \\ -h\hat{C}_n & A - h\hat{B}_n \end{pmatrix}, \quad \hat{C}_n = D_1 f(\bar{u}_n, \bar{u}_n'), \ \hat{B}_n = \bar{\lambda} I + D_2 f(\bar{u}_n, \bar{u}_n')$$
$$\hat{N} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \quad \hat{W}_n = -\begin{pmatrix} 0 \\ h\bar{u}_n' \end{pmatrix}.$$

Using for $\bar{w}_n = (\delta_0 \bar{u})_n$ the estimate $\|\bar{w}_n - \bar{u}_n'\| \le Ch^2$ as well as Hypothesis 1.9, we obtain

$$\|B_n - \hat{B}_n\| = \|D_2 f(\bar{u}_n, \bar{w}_n) - D_2 f(\bar{u}_n, \bar{u}_n')\| \le Ch^2$$
$$\|C_n - \hat{C}_n\| = \|D_1 f(\bar{u}_n, \bar{w}_n) - D_1 f(\bar{u}_n, \bar{u}_n')\| \le Ch^2.$$

Thus we have the estimates

$$\|N_n - \hat{N}\| = \left\| \begin{pmatrix} 0 & -hI \\ 0 & \frac{h}{2} B_n \end{pmatrix} \right\| \le Ch$$

and

$$\|\hat{K}_n - K_n + N_n - \hat{N}\| \le h(\|C_n - \hat{C}_n\| + \|B_n - \hat{B}_n\|) \le Ch^3$$

as well as

$$\|W_n - \hat{W}_n\| = \|h(\bar{v}_n - \bar{w}_n)\| = \frac{h^2}{2} \|(\delta_+ \delta_- \bar{u})_n\| \le Ch^2.$$

In the last inequality we have used the relation

$$w_n - v_n = (\delta_0 u)_n - (\delta_- u)_n = \frac{h}{2} (\delta_+ \delta_- u)_n = \frac{h}{2} (\delta_+ v)_n.$$

The above estimates imply that Λ_i is an order h perturbation of $\tilde{\Lambda}$:

$$\begin{aligned}\|(\tilde{\Lambda}-\Lambda_i)(z,\lambda)\|_\infty^* \leq & \frac{1}{h}\sup_{n\in J}\Big(\|(N_n-\hat{N})z_{n+1}-(K_n-\hat{K}_n)z_n\| \\ & +\|(W_n-\hat{W}_n)\lambda\|\Big)+\|Q_-(w_{n_-}-v_{n_-})\|+\|Q_+(w_{n_+}-v_{n_+})\| \\ \leq & \sup_{n\in J}(\|N_n-\hat{N}\|\|\delta_+ z_n\|)+\frac{1}{h}\sup_{n\in J}\Big(\|\hat{K}_n-K_n+N_n-\hat{N}\|\|z_n\|\Big) \\ & +\frac{1}{h}\sup_{n\in J}\|W_n-\hat{W}_n\||\lambda|+\frac{h}{2}(\|Q_-\|\|(\delta_+ v)_{n_-}\|+\|Q_+\|\|(\delta_+ v)_{n_+}\|) \\ \leq & Ch(\|\delta_+ z\|_\infty+\|z\|_\infty+|\lambda|).\end{aligned}$$

Define $\hat{M}_n = S(x_{n+1}, x_n)$, where S denotes the solution operator of the linear equation

$$Lz = \bar{r}, \quad \text{where } Lz = z' - M(\cdot)z \tag{2.69}$$

where

$$M(x) = \begin{pmatrix} 0 & I \\ -A^{-1}C(x) & -A^{-1}B(x)\end{pmatrix} = \hat{N}^{-1}\begin{pmatrix} 0 & I \\ -C(x) & -B(x)\end{pmatrix} \tag{2.70}$$

and A, B, C define the operator Λ in (1.5).

Then the operator

$$\hat{\Lambda}_i(z,\lambda) = \begin{pmatrix} (\hat{N}z_{n+1}-\hat{N}\hat{M}_n z_n - \hat{W}_n\lambda)_{n\in J} \\ (P_-\ Q_-)z_{n_-}+(P_+\ Q_+)z_{n_+} \\ \hat{\Pi}(z)\end{pmatrix}$$

is a order h perturbation of Λ_i, since

$$\hat{M}_n = S(x_{n+1},x_n) = I + hM(x_n) + h^2 E_n \tag{2.71}$$

and the equality (cf. (2.68))

$$\hat{K}_n = \hat{N}(I + hM(x_n)),$$

lead to

$$\|\Lambda_i - \hat{\Lambda}_i\|_\infty^* \leq \frac{1}{h}\sup_{n\in J}\|\hat{K}_n - \hat{N}\hat{M}_n\|\|z\|_\infty \leq Ch\|E\|_\infty\|z\|_\infty. \tag{2.72}$$

Setting $\hat{V}_n = \hat{N}^{-1}\hat{W}_n$, the equation $\hat{\Lambda}_i(z,\lambda) = (r,\eta,\omega)$ can be equivalently written as

$$\begin{aligned} z_{n+1} - \hat{M}_n z_n - \hat{V}_n\lambda &= \hat{N}^{-1}r_n \\ (P_-\ Q_-)z_{n_-} + (P_+\ Q_+)z_{n_+} &= \eta \\ \hat{\Pi}(z) &= \omega. \end{aligned} \tag{2.73}$$

In order to apply the linear Lemma 2.14 to (2.73) we show that Hypotheses 2.11,2.12 and 2.13 are satisfied.

The spectral condition (SC) and the eigenvalue condition (EC) ensure that equation (2.69) possesses exponential dichotomies on $\mathbb{R}^\pm$ with data $(K^\pm, \alpha^\pm, \pi^\pm)$ and we have $\mathcal{N}(L) = \text{span}\{\bar{z}'\}$.

From the solvability condition (2.60) follows that Hypothesis 2.11 is satisfied for $B_\pm = (P_\pm\ Q_\pm)$, since the invariant subspaces of $M^\pm = \lim_{x\to\pm\infty} M(x)$ are given by $X_-^s = W_-^s(0)$ and $X_-^u = W_+^u(0)$ which are defined in Definition A.8 (compare Lemma 3.29 in [60]).

The definition of $\hat{V}_n$ in (2.68) and the definition of $\hat{N}$ shows that Hypothesis 2.13 is satisfied with

$$V(x) = \begin{pmatrix} 0 \\ -A^{-1}\bar{u}'(x) \end{pmatrix} \tag{2.74}$$

and

$$\|V(x)\| \leq C\|\bar{u}'(x)\| \leq C\mathrm{e}^{-\alpha|x|}.$$

Application of Lemma 2.8 shows that $\tilde{\Pi}(\bar{u}'_{|J})$ converges for $h \to 0, \pm hn_\pm \to \infty$ to $\langle \hat{u}', \bar{u}' \rangle$, i.e.

$$h \sum_{n=n_-}^{n_+} (\delta_0 \hat{u})_n^T \bar{u}'_n \to \int_{\mathbb{R}} \hat{u}(x)^T \bar{u}'(x)\, dx.$$

Thus it follows from Hypothesis 2.18 that Hypothesis 2.12 is satisfied.

The nondegeneracy condition in Hypothesis 2.13 follows from the fact that $\mathcal{N}(\Lambda) = \operatorname{span}\{\bar{u}'\}$. In this case

$$Au'' + Bu' + Cu + \lambda\bar{u}' = 0$$

implies $u = a\bar{u}',\ a \in \mathbb{R}$ and $\lambda = 0$. Since A is nonsingular this is equivalent to

$$z' - M(\cdot)z - V(\cdot)\lambda = 0 \implies z = a\bar{z},\ a \in \mathbb{R} \text{ and } \lambda = 0.$$

As has been shown in [3], Proposition 2.1, this is equivalent to Hypothesis 2.13.

By applying Lemma 2.14 to (2.73) and multiplying with the bounded matrix $\hat{N}$ we obtain the invertibility of $\hat{\Lambda}_i$ as well as the uniform bound for the inverse

$$\|\hat{\Lambda}_i^{-1}(r, \eta, \omega)\|_{1,\infty} \leq c(\frac{1}{h}\|r\|_\infty + \|\eta\| + |\omega|) = c\|(r, \eta, \omega)\|_\infty^*.$$

Using the perturbation estimates (2.72) as well as (2.72) the invertibility of $\tilde{\Lambda}$ follows with the same bound for a probably different constant c.

Note that, if $z = (u, \delta_- u)$ then $\Gamma(z, \lambda)$ has the following structure

$$(\Gamma(z, \lambda))_n = \begin{pmatrix} (hv_{n+1} - (u_{n+1} - u_n))_{n \in J} \\ h(A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + C_n u_n + D_n \lambda) \end{pmatrix}.$$

This implies for any z of the form $z = (u, \delta_- u)$ (i.e. the first m rows $\Gamma(z, \lambda)$ are zero) for $\diamond \in \{\mathcal{H}_h^1,\ (1, \infty)\}$

$$\begin{aligned}
\|\hat{\Lambda}(z, \lambda)\|_\diamond^* &= \frac{1}{h}\|\Gamma(z, \lambda)\|_\diamond + \|\hat{\Pi}(z)\| \\
&\quad + \|P_- u_{n_-} + Q_- \frac{1}{2}((\delta_- u)_{n_-+1} + (\delta_- u)_{n_-}) + P_+ u_{n_+} + Q_- \frac{1}{2}((\delta_- u)_{n_++1} + (\delta_- u)_{n_+})\| \\
&= \|A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + C_n u_n + D_n\lambda\|_\diamond + \|\tilde{\Pi}u\| \\
&\quad + \|P_- u_{n_-} + Q_- \delta_0 u_{n_-} + P_+ u_{n_+} + Q_- \delta_0 u_{n_+}\| \\
&= \|DF(\bar{u}, \bar{\lambda})(u, \lambda)\|_\diamond.
\end{aligned}$$

Together with

$$\|z\|_{1,\infty} = \|z\|_\infty + \|\delta_+ z\|_\infty \leq c(\|u\|_\infty + \|\delta_0 u\|_\infty + \|\delta_+\delta_- u\|_\infty) = c\|u\|_{2,\infty}$$

and $\|z\|_{\mathcal{H}_h^1} \le c\|u\|_{\mathcal{H}_h^2}$ it follows that $DF(\bar{u}_{|_J}, \bar{\lambda}))$ is also invertible with

$$\|DF(\bar{u}_{|_J}, \bar{\lambda})(r, \eta, \omega)\|_{2,\infty} \le c(\|g\|_\infty + \|\eta\| + |\omega|).$$

Therefore we can find $\sigma > 0$ with

$$\|DF(\bar{u}_{|_J}, \bar{\lambda})^{-1}\|^*_{2,\infty} \le \frac{1}{\sigma},$$

i.e. for $h < h_0$ and $\pm hn_\pm > K$ condition (A.6) is satisfied. Using (2.64) we obtain $\rho > 0$ such that for $(\tilde{u}, \tilde{\lambda}) \in U_\rho((\bar{u}_{|_J}, \bar{\lambda})) = \{(u, \lambda) : \|u - \bar{u}\|_{1,\infty} + |\lambda| \le \rho\}$

$$\|DF(\tilde{u}, \tilde{\lambda}) - DF(\bar{u}_{|_J}, \bar{\lambda})\|_\infty \le \frac{\sigma}{2},$$

which implies (A.5) with $\kappa = \frac{\sigma}{2}$.

Application of the nonlinear perturbation Theorem A.3 now gives the existence of a zero $(\tilde{u}, \tilde{\lambda})$ of F and the desired stability inequality

$$\|u_1 - u_2\|_{2,\infty} + |\lambda_1 - \lambda_2| \le C\|F(u_1, \lambda_1) - F(u_2, \lambda_2)\|_\infty$$

for $(u_1, \lambda_1), (u_2, \lambda_2) \in U_\delta((\bar{u}_{|_J}, \bar{\lambda}))$. Together with the consistency estimate (2.63) this leads to estimate (2.61). □

Besides the $\|\cdot\|_\infty$ estimate (2.61) we obtain a $\|\cdot\|_{\mathcal{L}_{2,h}}$ estimate in the following corollary.

Corollary 2.22 *The constants ϱ, T, h_0 in Theorem 2.21 can be choosen such that the solution $(\tilde{u}, \tilde{\lambda})$ obeys the following estimate for $C > 0, \alpha > 0$*

$$\|\bar{u}_{|_J} - \tilde{u}\|_{\mathcal{H}_h^2} + |\bar{\lambda} - \tilde{\lambda}| \le C(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}). \tag{2.75}$$

Proof: In order to estimate the difference $(u_\Delta, \lambda_\Delta) = (\bar{u}_{|_J} - \tilde{u}, \bar{\lambda} - \tilde{\lambda})$ in the $\|\cdot\|^*_{\mathcal{H}_h^2}$ norm we show that $(u_\Delta, \lambda_\Delta)$ solves a linear equation to which Corollary 2.17 can be applied. For $(\bar{u}, \bar{\lambda})$ and $(\tilde{u}, \tilde{\lambda})$ we have

$$\begin{aligned} A(\delta_+\delta_-\tilde{u})_n + \tilde{\lambda}(\delta_0\tilde{u})_n + f(\tilde{u}_n, \delta_0\tilde{u}_n) &= 0, \; n \in J \\ A\bar{u}_n'' + \bar{\lambda}\bar{u}_n' + f(\bar{u}_n, \bar{u}_n') &= 0. \end{aligned}$$

Thus $(u_\Delta, \lambda_\Delta)$ solves

$$\begin{aligned} A(\delta_+\delta_- u)_n + B_n^\Delta(\delta_0 u)_n + C_n^\Delta u_n + D_n^\Delta \lambda &= g_n \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} &= \eta \\ \langle \delta_0 \hat{u}, u\rangle_h &= \omega \end{aligned} \tag{2.76}$$

where $w_n(t) = \tilde{u}_n + t(\bar{u}_n - \tilde{u}_n)$ and

$$\begin{aligned} B_n^\Delta &= \frac{1}{2}(\bar{\lambda} + \tilde{\lambda}) + \int_0^1 D_2 f(w_n(t), \delta_0 w_n(t))\, dt, \\ C_n^\Delta &= \int_0^1 D_1 f(w_n(t), \delta_0 w_n(t))\, dt, \quad D_n^\Delta = \frac{1}{2}\delta_0(\tilde{u} + \bar{u}_{|_J})_n \\ g_n &= A(\bar{u}_n'' - (\delta_+\delta_-\bar{u})_n) + \bar{\lambda}(\bar{u}_n' - (\delta_0\bar{u})_n) + f(\bar{u}_n, \bar{u}_n') - f(\bar{u}_n, \delta_0\bar{u}_n) \\ \eta &= P_-(\bar{u}_{n_-} - \bar{u}_-) + Q_-(\delta_0\bar{u}_{|_J})_{n_-} + P_+(\bar{u}_{n_+} - \bar{u}_+) + Q_+(\delta_0\bar{u}_{|_J})_{n_+} \\ \omega &= \langle \delta_0\hat{u}, \bar{u} - \hat{u}\rangle_h. \end{aligned}$$

Furthermore, we have the estimates

$$\begin{aligned}
\|B_n^\Delta - B_n\| &\le \frac{1}{2}|\tilde\lambda - \bar\lambda| + \int_0^1 \|D_2 f(w_n(t), \delta_0 w_n(t)) - D_2 f(\bar u_n, \delta_0 \bar u_n)\|\, dt \\
\|C_n^\Delta - C_n\| &\le \int_0^1 \|D_1 f(w_n(t), \delta_0 w_n(t)) - D_1 f(\bar u_n, \delta_0 \bar u_n)\|\, dt \\
\|D_n^\Delta - D_n\| &\le \frac{1}{2}\|\delta_0(\tilde u - \bar u_{|J})_n\| \le K(h^2 + \mathrm{e}^{-\alpha T})
\end{aligned}$$

as well as

$$\sup_{t\in(0,1)} \|D_1 f(\bar u_n + t(\tilde u_n - \bar u_n), \delta_0(\bar u_n + t(\tilde u_n - \bar u_n))) - D_1 f(\bar u_n, \delta_0 \bar u_n)\|$$
$$\le c(\|\tilde u_n - \bar u_n\| + \|\delta_0(\tilde u - \bar u)_n\|) \le c\|\tilde u - \bar u\|_{1,\infty}$$

and

$$\sup_{t\in(0,1)} \|D_2 f(\bar u_n + t(\tilde u_n - \bar u_n), \delta_0(\bar u_n + t(\tilde u_n - \bar u_n))) - D_2 f(\bar u_n, \delta_0 \bar u_n)\|$$
$$\le c\|(\tilde u_n - \bar u_n)\| \le c\|\tilde u - \bar u\|_\infty.$$

which follow from (1.10).

Equation (2.76) is transformed via $z = (u, \delta_- u)$ into the system

$$\begin{aligned}
z_{n+1} - M_n^\Delta z_n - V_n^\Delta \lambda &= r_n \\
P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} &= \eta \\
\langle \delta_0 \hat u, u\rangle_h &= \omega
\end{aligned} \tag{2.77}$$

where

$$M_n^\Delta = \begin{pmatrix} I - h^2(E_n^{+\Delta})^{-1}C_n^\Delta & h(E_n^{+\Delta})^{-1}E_n^{-\Delta} \\ -h(E_n^{+\Delta})^{-1}C_n^\Delta & (E_n^{+\Delta})^{-1}E_n^{-\Delta} \end{pmatrix}, \quad E_n^{\pm\Delta} = A \pm \frac{h}{2}B_n^\Delta,$$
$$V_n^\Delta = -h\begin{pmatrix} hI \\ I \end{pmatrix}(E_n^{+\Delta})^{-1}D_n^\Delta, \quad r_n = h\begin{pmatrix} hI \\ I \end{pmatrix}(E_n^{+\Delta})^{-1}g_n.$$

Then (2.71) and (2.37) imply

$$\hat M_n - M_n^\Delta = \begin{pmatrix} h^2(E_n^+)^{-1}C_n - (E_n^{+\Delta})^{-1}C_n^\Delta) & h(I - (E_n^{+\Delta})^{-1}E_n^{-\Delta}) \\ -h((E_n^+)^{-1}C_n - (E_n^{+\Delta})^{-1}C_n^\Delta) & (E_n^+)^{-1}E_n^- - (E_n^{+\Delta})^{-1}E_n^{-\Delta} \end{pmatrix}.$$

Using the above estimates, as well as

$$(A+hB)^{-1} = A^{-1} - hA^{-1}BA^{-1} + \mathcal{O}(h^2), \quad (A+hB)^{-1}(A-hB) = I - 2hA^{-1}B + \mathcal{O}(h^2)$$

we obtain

$$\|\hat M_n - M_n^\Delta\| = \begin{pmatrix} \mathcal{O}(h^2) & \mathcal{O}(h^2) \\ -h(\mathrm{err}(h,T) + \mathcal{O}(h)) & h(\mathrm{err}(h,T) + \mathcal{O}(h)) \end{pmatrix}$$

and

$$\|\hat V_n - V_n^\Delta\| = h\|\begin{pmatrix} hI \\ I \end{pmatrix}(E_n^+)^{-1}D_n - (E_n^{+\Delta})^{-1}D_n^\Delta\| = h(\mathrm{err}(h,T) + \mathcal{O}(h)),$$

where $\mathrm{err}(h,T)$ denotes the discretization error and $\mathrm{err}(h,T) \le c(h^2 + \mathrm{e}^{-\alpha T})$.

By applying Corollary 2.17 to (2.77) we obtain the following $\mathcal{H}_h^1$ estimate of $(\bar{z}_{|_J} - \tilde{z}, \bar{\lambda} - \tilde{\lambda})$ for each $r \in \mathcal{L}_{2,h}$

$$\|z^\Delta\|_{\mathcal{H}_h^1} + |\lambda^\Delta| \le c(\frac{1}{h}\|r\|_{\mathcal{L}_{2,h}} + \|\eta\| + |\omega|).$$

This yields the $\mathcal{H}_h^2$ estimate for $(u_\Delta, \lambda_\Delta)$, since via the estimates in Remark 2.1 for the difference operators and (1.10) we get

$$\begin{aligned}
\|u_\Delta\|_{\mathcal{H}_h^2} + |\lambda_\Delta| &\le c(\|g\|_{\mathcal{L}_{2,h}} + \|\eta\| + |\omega|) \\
&\le c(\|(A(\bar{u}_n'' - (\delta_+\delta_-\bar{u})_n) + \bar{\lambda}(\bar{u}_n' - (\delta_0\bar{u})_n) + f(\bar{u}_n, \bar{u}_n') - f(\bar{u}_n, \delta_0\bar{u}_n))_{n\in J}\|_{\mathcal{L}_{2,h}} \\
&\quad + \|P_-(\bar{u}_{n_-} - \bar{u}_-) + Q_-(\delta_0\bar{u})_{n_-} + P_+(\bar{u}_{n_+} - \bar{u}_+) + Q_+(\delta_0\bar{u})_{n_+}\| \\
&\quad + |\langle \delta_0\hat{u}_{|_J}, (\bar{u} - \hat{u})_{|_J}\rangle_h|) \\
&\le c(h^2 + \mathrm{e}^{-\alpha h \min(-n_-, n_+)}).
\end{aligned}$$

□

2.3 Extensions

The above results can be extended in different directions. First it is possible to consider more general symmetries as has been indicated in section 1.4 already. Second, one can generalize the above results to prove theorems about the discretization of "connecting orbits" on finite intervals, extending the results in [64].

2.3.1 Generalization to higher symmetries

In this section we indicate how the proofs in section 2.2 have to be modified for more general symmetries, i.e. in order to prove approximation of relative equilibria as described in section 1.4. In the simplest case the generalization (1.73) of the PDE, which describes stationary solutions of (1.14), reads

$$\begin{aligned}
A(\delta_+\delta_-u)_n + \sum_{i=1}^{p} \mu_i(S_i^0 u + S_i^1(\delta_0 u)_n) + f(u_n, (\delta_0 u)_n) &= 0, \quad n \in J \\
P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} &= \eta \\
h \sum_{n=n_-}^{n_+} (\hat{w}_n^i)^T (u_n - \hat{u}_n) &= 0, \quad i = 1, \ldots, p
\end{aligned} \tag{2.78}$$

where $\hat{w}_n^i = S_i^0 \hat{u}_n + S_i^1(\delta_0\hat{u})_n \in \mathbb{R}^{m,p}$.

In this setting the discretization of the operator Λ given in (1.75) which decides upon stability is given by

$$\hat{L}u = A(\delta_+\delta_-u)_n + B_n(\delta_0 u)_n + C_n u_n,$$

where with $\bar{w}_n = (\delta_0 \bar{u})_n$

$$B_n = D_2 f(\bar{u}_n, \bar{w}_n) + \sum_{i=1}^{p} \bar{\mu}_i S_i^1, \quad C_n = D_1 f(\bar{u}_n, \bar{w}_n) + \sum_{i=1}^{p} \bar{\mu}_i S_i^0,$$
$$D_n = D_1 f(\bar{u}_n, \bar{w}_n) + \sum_{i=1}^{p} \bar{\mu}_i S_i^0.$$

The equation (2.62) can then be written as

$$DF(\bar{u}, \bar{\lambda})(u, \lambda) = \begin{pmatrix} (A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + C_n u_n + D_n \lambda)_{n \in J} \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} \\ \tilde{\Pi}(u) \end{pmatrix}$$

In order to generalize the approximation result Theorem 2.21 we have to check the approximation properties of

$$\|\hat{B}_n - B_n\|, \ \|\hat{C}_n - C_n\|, \ \|\hat{S}_n^i - S_n^i(\bar{u}_{|_J})\|,$$

where

$$\begin{aligned} \hat{B}_n &= D_2 f(\bar{u}_n, \bar{u}'_n) + \sum_{i=1}^{p} \mu_i S_i^1, \qquad \hat{C}_n = D_1 f(\bar{u}_n, \bar{u}'_n) + \sum_{i=1}^{p} \mu_i S_i^0, \\ \hat{S}_n^i &= (S_i^0 \bar{u}_{|_J})_n + (S_i^1 \bar{u}'_{|_J})_n. \end{aligned} \tag{2.79}$$

It remains to check Hypothesis 2.12, which requires that the matrices

$$\int_{\mathbb{R}} [S^1(\hat{u})(x), \ldots, [S^p(\hat{u})](x)]^T [S^1(\bar{u})(x), \ldots, S^p(\bar{u})(x)] \, dx \in \mathbb{R}^{p,p}$$

are nonsingular. This follows, since $\mathcal{N}(L) = \operatorname{span}\{S_i \bar{u}, \ i = 1, \ldots, s\} = \mathcal{R}(S(\bar{u}))$ (see section 1.4), where $S_i \bar{u} = -a_\gamma(\mathbb{1}) \bar{u} e^i$.

Example 2.23 *For the example 1.30 where*

$$u(x) \in \mathbb{R}^2, x \in \mathbb{R}, [a(\gamma)u](x) = R_{-\gamma_r} u(x - \gamma_t), \ \gamma = (\gamma_r, \gamma_t) \in G = \mathbb{R} \times S^1$$

equation (1.74) reads

$$u_t = A u_{xx} + f(u, u_x) + \lambda_t u_x + \lambda_r R_{\frac{\pi}{2}} u$$

and we obtain for the discretization $S_1^1 = I, S_2^0 = R_{\frac{\pi}{2}}, \ S_1^0 = S_2^1 = 0$. Therefore the discrete system (2.78) is given by

$$\begin{aligned} A(\delta_+\delta_- u)_n + \lambda_t(\delta_0 u)_n + \lambda_r R_{\frac{\pi}{2}} u_n + f(u_n, (\delta_0 u)_n) = 0, \quad n \in J \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} = \eta \\ h \sum_{n=n_-}^{n_+} \begin{pmatrix} (\delta_0 \hat{u})_n^T \\ (R_{\frac{\pi}{2}} \hat{u}_n)^T \end{pmatrix} (u_n - \hat{u}_n) = 0 \in \mathbb{R}^2. \end{aligned}$$

With $\mathcal{N}(L) = \operatorname{span}\{\bar{u}', \ R_{\frac{\pi}{2}} \bar{u}\}$ we have that

$$\int_{\mathbb{R}} \begin{pmatrix} \bar{u}'(x)^T \\ R_{\frac{\pi}{2}} u(x)^T \end{pmatrix} \begin{pmatrix} \phi_1(x) & \phi_2(x) \end{pmatrix} \, dx \in \mathbb{R}^{2,2}$$

is invertible for all linearly independent $\phi_1, \phi_2 \in \mathcal{N}(L)$. Thus Hypothesis 2.12 is satisfied. Hypothesis 2.13 is satisfied as well, since in this case

$$V(x) = \begin{pmatrix} 0 \\ -A^{-1}[S_1^0\bar{u} + S_1^1\bar{v}, \ldots, S_p^0\bar{u} + S_p^1\bar{v}] \end{pmatrix} = \begin{pmatrix} 0 \\ -A^{-1}[\bar{u}', R_{\frac{\pi}{2}}\bar{u}] \end{pmatrix}$$

and the matrix

$$\int_{\mathbb{R}} [\bar{u}'(x), R_{\frac{\pi}{2}}\bar{u}(x)]^T A^{-1}[\bar{u}'(x), R_{\frac{\pi}{2}}\bar{u}(x)]\, dx = \langle [\bar{u}', R_{\frac{\pi}{2}}\bar{u}], A^{-1}[\bar{u}', R_{\frac{\pi}{2}}\bar{u}]\rangle$$

is invertible.

2.3.2 Discretization of connecting orbits

The method we used above for the proof of the approximation theorems for the traveling wave can be used to extend the approximation result for connecting orbits as discussed in [64], to boundary value problems. In the following we outline the results and indicate the line of proof without giving details.

Consider a parameter dependent ODE

$$z' = G(z, \lambda), \quad x \in \mathbb{R},\ \lambda \in \mathbb{R}^p,\ z(x) \in \mathbb{R}^k \tag{2.80}$$

which will be compared later with a difference equation arising from a one step method. Special solutions of (2.80) are given by the following definition

Definition 2.24 (connecting orbit) *A solution $\bar{z} \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^k)$ of (2.80) at parameter $\lambda = \bar{\lambda}$ with*

$$\lim_{x \to \pm\infty} \bar{z}(x) = \bar{z}_\pm$$

*is called a **connecting orbit** between the limiting values $\bar{z}_-$ and $\bar{z}_+$. The pair $(\bar{z}, \bar{\lambda})$ is a **connecting orbit pair (COP)** between $\bar{z}_-$ and $\bar{z}_+$.*

COPs aris as are intersections of stable and unstable manifolds and they are robust w.r.t. perturbations if these intersection is transversal. This leads to the following nondegeneracy condition for COPs.

Definition 2.25 (nondegenerate) *A COP $(\bar{z}, \bar{\lambda})$ of the system (2.80) is called **nondegenerate** if the matrices $M_\pm = \lim_{x\to\pm\infty} G_z(\bar{z}(x), \bar{\lambda})$ are hyperbolic (i.e. there are no purely imaginary eigenvalues), and for the number $k_s^\pm$ of stable and $k_u^\pm$ of unstable eigenvalues of $M_\pm$ we have $p = k + 1 - k_u^- - k_s^+ = k_s^- - k_s^+$, and for any solution (z_0, λ_0) of the variational equation*

$$z' - G_z(\bar{z}, \bar{\lambda})z - G_\lambda(\bar{z}, \bar{\lambda})\lambda = 0,$$

we have

$$\lambda_0 = 0 \quad \textit{and} \quad z_0 = \alpha\bar{z}',\ \alpha \in \mathbb{R}.$$

The following lemma [3], Prop. 3.1, gives an equivalent condition:

Lemma 2.26 *A COP* $(\bar{z},\bar{\lambda})$ *of* (2.87) *is nondegenerate if and only if we have for the linear operator*

$$L : \mathcal{C}_b^1 \to \mathcal{C}_b,\ z \mapsto z' - G_z(\bar{z},\bar{\lambda})z$$

- $\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*) = 1$
- *if* $\operatorname{span}\{\psi\} = \mathcal{N}(L^*)$ *then*

$$\langle \psi, G_\lambda(\bar{z},\bar{\lambda})\rangle := \int_{\mathbb{R}} \psi^T G_\lambda(\bar{z}(x),\bar{\lambda})\ dx\ \neq 0. \tag{2.81}$$

In this setting the linear operator L is the linearization of a G operator about the equilibrium $(\bar{z},\bar{\lambda})$, i.e.

$$Lz = z' - M(x)z, \quad \text{where} \quad M(x) = G_z(\bar{z}(x),\bar{\lambda})$$

As before, we denote the corresponding solution operator by S.

In this situation Lemma 2.3 can be slightly extended, since $\phi_z(\bar{z}(x),\bar{\lambda},h)$ solves the equation $Lz = 0$, see Lemma 3.3 in [64].

Lemma 2.27 *Consider the nonlinear parameter dependent equation*

$$z' = G(z,\lambda) \tag{2.82}$$

with flow $\phi(z,\lambda,x)$*. If the operator* L *has an exponential dichotomy with data* (K,α,π) *on* J *then the corresponding difference equation defined via the* h*-flow of* (2.82)

$$z_{n+1} = \phi_z(\bar{z}(x_n),\bar{\lambda},h)z_n = S(x_{n+1},x_n)z_n, \quad n\in\mathbb{Z} \tag{2.83}$$

has an exponential dichotomy on J *with data* $(K,\alpha h,P)$ *where* $P_n = \pi(x_n)$*.*

Note that if $(\bar{z},\bar{\lambda})$ is a COP of equation (2.80) and we define $\bar{z}_n = (\bar{u}(x_n),\bar{u}'(x_n)),\ n\in\mathbb{Z}$, then $(\bar{z}_{|_J},\bar{\lambda})$ is a connecting orbit for the discrete system

$$0 = A(\delta_+\delta_- u)_n + \lambda(\delta_0 u)_n + f(u_n), \quad n\in\mathbb{Z}. \tag{2.84}$$

Now we consider a one-step method for (2.80) given by

$$z_{n+1} = \psi_h(z_n,\lambda)$$

with order p, i.e.

$$\|\phi(z,\lambda,h) - \psi_h(z,\lambda)\| \le Ch^p.$$

For the explicit Euler method one can show for (z,λ) in compact sets the estimate

$$\|\phi(z,\lambda,h) - \psi_h(z,\lambda)\| \le Ch^2.$$

We define the operator

$$\hat{L} : S_{\mathbb{Z}} \times \mathbb{R}^l \to S_{\mathbb{Z}}\ ; \quad (z,\lambda) \mapsto (z_{n+1} - \psi_h(z_n,\lambda))_{n\in\mathbb{Z}}$$

and consider the restriction of Γ_h to a finite interval $J = [n_-, n_+]$ with boundary condition

$$b(z_{n_-}, z_{n_+}) = 0,$$

and a phase condition $\Psi(z) = 0$, where Ψ is defined by

$$\Psi(z) = \langle \delta_0 \hat{z}, z - \hat{z} \rangle_h = \sum_{n \in \mathbb{Z}} (\delta_0 \hat{z})_n^T (z_n - \hat{z}_n) = 0,$$

where $\hat{z} : \mathbb{R} \to \mathbb{R}^m$ is an appropriate reference function. We summarize the equations in the operator $F_{J,h} : S_J \times \mathbb{R}^l \to S_J \times \mathbb{R}^k \times \mathbb{R}$, defined by

$$F_{J,h}(z, \lambda) = \begin{pmatrix} \hat{L}(z, \lambda) \\ b(z_{n_-}, z_{n_+}) \\ \Psi(z) \end{pmatrix}.$$

The derivative $DF_{J,h}(\bar{z}, \bar{\lambda}) : S_J \times \mathbb{R}^l \to S_J \times \mathbb{R}^k \times \mathbb{R}$ of $F_{J,h}$ at $(\bar{z}, \bar{\lambda})$ is then given by

$$DF_{J,h}(\bar{z}, \bar{\lambda})(z, \lambda) = \begin{pmatrix} (z_{n+1} - \hat{M}_n z_n - \hat{V}_n \lambda)_{n \in J} \\ D_1 b(\bar{z}_{n_-}, \bar{z}_{n_+}) z_{n_-} + D_2(\bar{z}_{n_-}, \bar{z}_{n_+}) z_{n_+} \\ \langle \hat{\psi}_{|J}, z \rangle_h \end{pmatrix}, \tag{2.85}$$

where

$$\hat{M}_n = S(x_{n+1}, x_n) = \phi_z(\bar{z}(x_n), \bar{\lambda}, h), \quad \hat{V}_n = \phi_\lambda(\bar{z}(x_n), \bar{\lambda}, h), \quad \hat{\psi}(x) = \delta_0 \hat{z}(x).$$

The conditions on b and Ψ are chosen in such a way, that Hypotheses 2.11 and 2.12 hold, and (a variant of) Lemma 2.14 can be applied.

Hypothesis 2.28 *The $k \times k$ matrix*

$$\begin{pmatrix} D_1 b(\bar{z}_{n_-}, \bar{z}_{n_+}) X_s^- & D_2(\bar{z}_{n_-}, \bar{z}_{n_+}) X_u^+ \end{pmatrix}$$

is nonsingular, where the columns of $\hat{X}_-^s$ span the stable subspace X_-^s of M_- and the columns of X_+^u span the unstable subspace of M_+. Furthermore, we assume that the boundary condition is satisfied at the stationary points, i.e.

$$b(z_-, z_+) = 0.$$

For the phase condition we assume the same Hypothesis 2.12 as in Lemma 2.14. The simplest possibility is to take $\hat{z} = \bar{z}$. However, this is not useful for the numerical compuations, since $\bar{z}$ is the unknown solution we are looking for.

Then for the approximation of the nondegenerate COP by a discrete boundary value problem the following theorem holds.

Theorem 2.29 *Let $(\bar{z}, \bar{\lambda})$ be a nondegenerate COP of equation (2.80) and let Hypotheses 2.28 and 2.12 be satisfied.*

There exist $h_0, T > 0$, $K > 0$, $\rho > 0$, such that for $h < h_0$, $\pm n_\pm > T$, the discrete boundary value problem $F_{J,h}(z, \lambda) = 0$ has a unique solution $(\tilde{z}, \tilde{\lambda}) \in S_J \times \mathbb{R}^l$ in the neighborhood $B_\rho(\bar{z}_{|J}, \bar{\lambda}) = \{(z, \lambda) \in S_J \times \mathbb{R}^l : \ \|z - \bar{z}_{|J}\|_\infty + |\lambda - \bar{\lambda}| \leq \rho\}$ which obeys the following estimate

$$\|\bar{z}_{|J} - \tilde{z}\|_{\mathcal{H}_h^1} + \|\bar{\lambda} - \tilde{\lambda}\| \leq K(h^p + \|b(\bar{z}_{n_-}, \bar{z}_{n_+})\| + |\Psi(\bar{z}_{|J})|). \tag{2.86}$$

In the following we indicate the main steps of the proof which is similar to the proof of Theorem 2.21.

We apply the fixed point Theorem A.3 to the following situation: $Y = S_J \times \mathbb{R}^l$, $Z = S_J \times \mathbb{R}^{k+l}$, with norm $\|(z,\eta,\omega)\|_Z = \frac{1}{h}\|z\| + \|\eta\| + \|\omega\|$ $\bar{y} = (\bar{z}, \bar{\lambda})$ and $F : S_J \times \mathbb{R}^p \to S_J \times \mathbb{R}^{k+l}$ is given by $\hat{L}(z,\lambda)$.

The inhomogenous equation $DF(\bar{z},\bar{\lambda}) = (r,\eta,\omega)$ in J has the form (2.30)–(2.32) where $\hat{M}_n$ and $\hat{V}_n$ are defined in (2.85). Lemma 2.14 implies the existence of a unique zero $(\tilde{z},\tilde{\lambda}) \in S_J \times \mathbb{R}^l$ which obeys the estimate (2.38).

Thus we can use Theorem A.3 and obtain for each large enough J and small enough h a unique solution $(\tilde{z},\tilde{\lambda}) \in S_J(\mathbb{R}^k) \times \mathbb{R}^l$.

It remains to check Hypotheses 2.10–2.13:

Since the COP $(\bar{z},\bar{\lambda})$ is assumed to be nondegenerate, we obtain that the matrices $M^{\pm}$ are hyperbolic and $\mathcal{N}(L) = \operatorname{span}\{\bar{z}'\}$. The solvability condition of the boundary condition Hypothesis 2.11 is equivalent to Hypothesis 2.28. As has been proven in [64], we have $\hat{V}_n = hG_\lambda(\bar{z},\bar{\lambda}) + \mathcal{O}(h^2)$, thus Hypothesis 2.13 holds. We have $\Psi(\bar{z}_{|_J}) = 0$ and

$$\Psi(\bar{z}'_{|_J}) = h \sum_{n=n_-}^{n_+} (\bar{z}'_n)^T \bar{z}'_n = \|\bar{z}'_{|_J}\|^2_{\mathcal{L}_{2,h}} \geq \delta > 0,$$

which implies with $\mathcal{N}(L) = \operatorname{span}\{\bar{z}'\}$, that the matrix F defined in (2.36), which in this case is just a number, is nonsingular.

Application of a variant of Lemma 2.14 shows the existence of a unique solution $(\tilde{z},\tilde{\lambda})$ of the boundary value problem (1.13) in B_ρ which can be estimated by (2.86).

Note that for the approximation of the traveling wave we could apply the previous lemma directly. Since $(\bar{u},\bar{\lambda})$ solve (2.1) we obtain that $(\bar{z},\bar{\lambda})$ with $\bar{z} = (\bar{u},\bar{u}')$ is a solution of the first order equation

$$z' = G(z,\lambda), \quad \text{where} \quad G(z,\lambda) = \begin{pmatrix} v \\ -A^{-1}(\lambda v + f(u,v)) \end{pmatrix}, \tag{2.87}$$

which one obtains from (2.1) via the transformation $z = (u,u') = (u,v)$. The partial derivatives of G are given by

$$G_z(\bar{z},\bar{\lambda}) = \begin{pmatrix} 0 & I \\ -A^{-1}D_1 f(\bar{u},\bar{u}') & -A^{-1}(\bar{\lambda} I + D_2 f(\bar{u},\bar{u}')) \end{pmatrix} = M(\cdot)$$

and

$$G_\lambda(\bar{z},\bar{\lambda}) = \begin{pmatrix} 0 \\ -A^{-1}\bar{u}' \end{pmatrix} = V(\cdot),$$

where $M(\cdot)$ and $V(\cdot)$ are defined in (2.70) and (2.74). The nondegeneracy condition (2.81) corresponds to Hypothesis 2.13 and $(\bar{z},\bar{\lambda})$ is a nondegenerate COP of (2.87) with $\lim_{x\to\pm\infty} \bar{z}(x) = (u_\pm, 0)$ (see [60], [3]).

Chapter 3

Resolvent estimates and approximation of eigenvalues

In this chapter we prove resolvent and eigenvalue estimates for the discretized system on a finite interval. At the end of the chapter we present a result on the essential spectrum for the discretized operator on the whole line, which is the discrete analog of Theorem 1.2, as well as some conjectures concerning the influence of the boundary conditions on the essential spectrum for the discrete operator.

3.1 Resolvent estimates

In this section we construct solutions for the resolvent equation in the discrete setting using a similar method as in [6].

In the following we consider the discrete resolvent equation for $u \in S_{J_e}(\mathbb{C}^m)$ on the grid $\mathbb{G}_{J,h,x_0}$ with right hand side $\hat{g} \in S_J(\mathbb{C}^m)$

$$A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + (C_n - sI)u_n = \hat{g}_n, \quad n \in J \tag{3.1}$$

where

$$B_n = \tilde{\lambda} I + D_2 f(\tilde{u}_n, (\delta_0 \tilde{u})_n), \quad C_n = D_1 f(\tilde{u}_n, (\delta_0 \tilde{u})_n)$$

with boundary conditions

$$P_- u_{n_-} + Q_- \delta_0 u_{n_-} + P_+ u_{n_+} + Q_+ \delta_0 u_{n_+} = \eta, \quad P_\pm, Q_\pm \in \mathbb{R}^{2m,m} \tag{3.2}$$

for s in different regions of $\mathbb{C}$.

Here $(\tilde{u}, \tilde{\lambda}) \in S_{J_e}(\mathbb{R}^m) \times \mathbb{R}$ denotes the solution of (2.56)–(2.58) which approximates the exact traveling wave solution as estimated in (2.61).

We have to discuss the invertibility of the linear operators $F_s : S_{J_e}(\mathbb{C}^m) \to S_J(\mathbb{C}^m) \times \mathbb{C}^{2m}$ defined by

$$F_s u = \begin{pmatrix} (A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + (C_n - sI)u_n)_{n \in J} \\ P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u)_{n_+} \end{pmatrix}.$$

Our standing assumption in this chapter is the following: The operator Λ defined in (1.5) satisfies the conditions of the stability Theorem 1.13, i.e. (SC) and (EC) hold, with $\mathcal{N}(\Lambda) = \text{span}\{\phi\}$, and Hypothesis 1.9 holds.

Similar to the continuous case we consider the resolvent in several different regions of $\mathbb{C}$ (cf. Figure 3.1). The quantities ϵ, C_0 will be determined later while $\delta > 0$ will be chosen such that $|\arg \mu| \le \frac{\pi}{2} - \delta$ for all eigenvalues μ of A. For s in a compact set which does not contain zero, a similar method as in the proof of the approximation Theorem 2.21 can be used. Although in Chapter 2 we have formulated Lemma 2.14 for $\hat{g} \in \mathbb{R}^m$ only, the same holds for $\hat{g} \in \mathbb{C}^m$ as well. For large $|s|$ a different approach is necessary, since the analogy between the discrete and the continuous system is no longer valid.

$$\Omega_\epsilon : \ |s| < \epsilon, \ \text{Re } s \ge -\beta$$
$$\Omega_{C_0} : \ |s| \in [\epsilon, C_0], \ \text{Re } s \ge -\beta$$
$$\Omega^h_{C_0} : \ |s| \in (C_0, \frac{C_0}{h^2}], \quad |\arg(s)| \le \frac{\pi}{2} + \frac{2}{3}\delta$$
$$\Omega^h_\infty : \ |s| > \frac{C_0}{h^2}$$

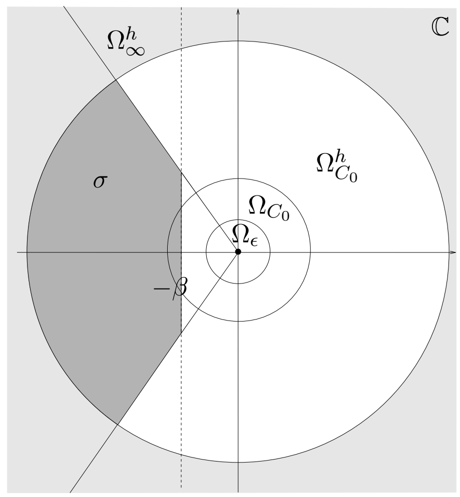


Figure 3.1: Regions for resolvent estimates

3.1.1 Compact subsets

We estimate the resolvent for s in the compact set

$$\Omega_{C_0} = \{s \in \mathbb{C} : \ \text{Re } s \ge -\beta, \ \text{and } |s| \in [\epsilon, C_0]\}$$

where $\epsilon > 0$ using the same approach as for the traveling wave in Section 2.2. These estimates will hold for any given pair of positive constants ϵ, C_0. The following condition is similar to (2.60).

Hypothesis 3.1 *Assume that the following regularity condition holds*

$$\det \left(\begin{pmatrix} P_- & Q_- \end{pmatrix} \begin{pmatrix} Y^s_-(s) \\ Y^s_-(s)\Lambda^s_-(s) \end{pmatrix} \quad \begin{pmatrix} P_+ & Q_+ \end{pmatrix} \begin{pmatrix} Y^u_+(s) \\ Y^u_+(s)\Lambda^u_+(s) \end{pmatrix} \right) \ne 0 \ \forall s \in \Omega_{C_0}, \tag{3.3}$$

where $Y^s_-(s), Y^u_+(s)$ and $\Lambda^s_-(s), \Lambda^u_+(s)$ are defined in Definition A.8.

Theorem 3.2 *Consider the boundary value problem (3.1)-(3.2) and let Hypothesis 3.1 be satisfied.*

Then there exist $C > 0$, $T > 0$, $h_0 > 0$ such that for $h < h_0$ and $\pm h n_\pm > T$ the resolvent equation (3.1)-(3.2) possesses for each $s \in \Omega_{C_0}$ and every $\hat{g} \in S_J$ a unique solution $\tilde{u} \in S_{J_e}$ wich obeys for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ the following estimate

$$\|\tilde{u}\|_{2,\diamond} \le C(\|\hat{g}\|_\diamond + \|\eta\|). \tag{3.4}$$

Proof: We transform equation (3.1), (3.2) using $z_n = (u_n, \delta_- u_n) = (u_n, v_n)$ to the equivalent equation (cf. (2.66))

$$\tilde{\Lambda}(s)z = (\hat{r}, \eta)$$

where with $w_n = \frac{1}{2}(v_n + v_{n+1})$,

$$\tilde{\Lambda}(s)(z,\lambda) = \begin{pmatrix} \Gamma(s)z \\ P_- u_{n_-} + Q_- w_{n_-} + P_+ u_{n_+} + Q_+ w_{n_+} \end{pmatrix}, \quad \hat{r}_n = \begin{pmatrix} 0 \\ h\hat{g}_n \end{pmatrix}$$

and

$$(\Gamma_r(s)z)_n = N_n z_{n+1} - K_n(s) z_n, \quad n \in J$$

with

$$N_n = \begin{pmatrix} I & -hI \\ 0 & E_n^+ \end{pmatrix}, \; K_n(s) = \begin{pmatrix} I & 0 \\ h(sI - C_n) & E_n^- \end{pmatrix}, \; E_n^{\pm} = A \pm \frac{h}{2} B_n. \tag{3.5}$$

As before we show that $\tilde{\Lambda}(s)$ is a perturbation of

$$\Lambda_i(s)z = \begin{pmatrix} (\hat{N} z_{n+1} - \hat{K}_n(s) z_n)_{n\in J} \\ (P_- \; Q_-) z_{n_-} + (P_+ \; Q_+) z_{n_+} \end{pmatrix}$$

where

$$\hat{K}_n(s) = \begin{pmatrix} I & hI \\ h(sI - \hat{C}_n) & A - h\hat{B}_n \end{pmatrix} \tag{3.6}$$

and $\hat{N}, \; \hat{C}_n, \hat{B}_n$ are defined in (2.68).

Similar to section 2.2 the estimate $\|N_n - \hat{N}\| \le ch$ holds, and using (2.61) we get with

$$\|C_n - \hat{C}_n\| \le c(h^2 + \mathrm{e}^{-\alpha T}), \quad \|B_n - \hat{B}_n\| \le c(h^2 + \mathrm{e}^{-\alpha T})$$

the uniform estimate

$$\|\hat{K}_n(s) - K_n(s) + N_n - \hat{N}\| \le Ch\,(h^2 + \mathrm{e}^{-\alpha T}).$$

This leads to

$$\begin{aligned} \|(\tilde{\Lambda}(s) - \Lambda_i(s))z\|_\infty^* &\le \frac{1}{h} \sup_{n\in J} \|(N_n - \hat{N}) z_{n+1} - (K_n(s) - \hat{K}_n(s)) z_n\| \\ &\quad + \|Q_-(w_{n_-} - v_{n_-})\| + \|Q_+(w_{n_+} - v_{n_+})\| \\ &\le \sup_{n\in J}(\|N_n - \hat{N}\|\,\|\delta_+ z_n\|) + \frac{1}{h}\sup_{n\in J}(\|\hat{K}_n(s) - K_n(s) + N_n - \hat{N}\|\,\|z_n\|) \\ &\quad + \|Q_-\|\,\|v_{n_-+1} - v_{n_-}\| + \|Q_+\|\,\|v_{n_++1} - v_{n_+}\| \\ &\le \sup_{n\in J} Ch\|\delta_+ z_n\| + \frac{1}{h}\sup_{n\in J} Ch\,(h^2 + \mathrm{e}^{-\alpha T})\|z_n\| + Ch(\|\delta_+ z_{n_-}\| + \|\delta_+ z_{n_+}\|) \\ &\le C(h\|\delta_+ z\|_\infty + (h^2 + \mathrm{e}^{-\alpha T})\|z\|_\infty) \le \sigma(h,T)\|z\|_{1,\infty} \end{aligned}$$

where $\lim_{h\to 0, T\to\infty} \sigma(h,T) = 0$ uniformly for all $s \in \mathbb{C}$.

The operators $\Lambda_i(s)$ are perturbations of $\hat{\Lambda}_i(s)$ defined by

$$\hat{\Lambda}_i(s)z = \begin{pmatrix} (\hat{N} z_{n+1} - \hat{N}\hat{M}_n(s) z_n)_{n\in J} \\ (P_- \; Q_-) z_{n_-} + (P_+ \; Q_+) z_{n_+} \end{pmatrix}$$

with $\hat{M}_n(s) = S(x_{n+1}, x_n, s)$, where $S(\cdot,\cdot,s)$ denotes the solution operator corresponding to the differential operator $L(s)$

$$L(s) = z' - M(\cdot, s)z, \quad \text{where}$$
$$M(x,s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C(x)) & -A^{-1}B(x) \end{pmatrix}.$$

In fact, the expansion (cf. (2.50))

$$S(x_{n+1}, x_n, s) = I + hM(x_n, s) + h^2 E_n(s)$$

and the definition of $\hat{K}_n$ (cf. (3.6))

$$\hat{K}_n(s) = \hat{N}(I + hM(x_n, s)),$$

lead to

$$\|\Lambda_i(s) - \hat{\Lambda}_i(s)\|_\infty^* \leq \frac{1}{h} \sup_{n\in J} \|\hat{K}_n(s) - \hat{N}\hat{M}_n(s)\| \|z\|_\infty \leq Ch\|E(s)\|_\infty \|z\|_\infty.$$

For $s \in \Omega_{C_0}$ the error term $E(s)$ is uniformly bounded in s. Note that for arbitrary large $|s|$ this does not hold any more, therefore this case is dealt with separately in subsection 3.1.2.

The operators $L(s)$ have exponential dichotomies on $\mathbb{R}$ with data $(K, \alpha, \pi(s))$ if $s \in \Omega_{C_0}$ lies in the resolvent of $L(0)$, i.e. $s \in \rho(L(0)) \cap \Omega_{C_0}$ and the dichotomy constants K, α do not depend on s (see [6]).

Thus we can apply the linear Lemma 2.14 with $k = 2m,\ p = 0$ to the explicit version of $\hat{\Lambda}_i(s)(z, \lambda) = (\hat{r}, \eta)$ which reads

$$\begin{aligned} z_{n+1} - \hat{M}_n(s) z_n &= \hat{N}^{-1}\hat{r}_n \\ (P_-\ Q_-)z_{n_-} + (P_+\ Q_+)z_{n_+} &= \eta. \end{aligned} \tag{3.7}$$

Hypothesis 3.1 ensures that Hypothesis 2.11 holds and the other Hypotheses are void in the case $p = 0$. We obtain that (3.7) is solvable for each $r \in S_J$ for $h < h_0, \pm n_\pm h > T$, $s \in \Omega_{C_0}$.

Applying Lemma 2.14 to (3.7) we obtain using that $\hat{N}$ is independent of s and h, that the operators $\hat{\Lambda}_i(s)$ considered as operators from $S_{J_r}(\mathbb{C}^{2m}), \|\cdot\|_{1,\diamond}^*$ to $S_J(\mathbb{C}^{2m}) \times \mathbb{C}^{2m}, \|\cdot\|_\diamond^*$, where $\|(r,\eta)\|_\diamond^* = \frac{1}{h}\|r\|_\diamond + \|\eta\|$, $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ are invertible for any $s \in \Omega_{C_0}$ with a uniform bound, i.e.

$$\|\hat{\Lambda}_i(s)^{-1}(r,\eta)\|_\diamond \leq C\|(r,\eta)\|_\diamond^* \quad \forall s \in \Omega_{C_0}.$$

Transforming these estimates back using (2.5),(2.6), we obtain the existence of a solution of (3.1),(3.2) as well as (3.4). □

3.1.2 $|s|$ large

In the case of $|s|$ large we cannot relate the discrete resolvent equation (3.1), (3.2) to corresponding continuous systems uniformly in s. Instead we prove its solvability directly by modifying some of the techniques for the continuous case in [6].

From $A > 0$ we find some $\delta > 0$ such that $|\arg(\mu)| < \frac{\pi}{2} - \delta \quad \forall \mu \in \sigma(A^{-1})$. Let $\sqrt{z}$ be the principal branch of the square root defined for $z = re^{i\phi}$, $\phi \in (-\pi, \pi), r > 0$ by $\sqrt{z} = \sqrt{r}e^{i\frac{\phi}{2}}$. Let $B^{\frac{1}{2}}$ be the corresponding matrix square root defined for $B \in \mathbb{C}^{m,m}$ with $\sigma(B) \subset \mathbb{C} \setminus \mathbb{R}^-$. For $z \in \mathbb{C}$ with $|\arg(z)| \leq \frac{\pi}{4} + \frac{\delta}{3}$ and $\mu \in \sigma(A^{-1})$ we obtain $|\arg(z^2\mu + 1)| < |\arg(z^2\mu)| \leq 2(\frac{\pi}{4} + \frac{\delta}{3}) + \frac{\pi}{2} - \delta = \pi - \frac{\delta}{3}$. Therefore the following matrix function is well defined

$$\Delta(z) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}}(I + z^2A^{-1})^{\frac{1}{2}}A^{-\frac{1}{2}}, \quad |\arg(z)| \leq \frac{\pi}{4} + \frac{\delta}{3}. \tag{3.8}$$

For $|z|$ large we have Re $(\sigma(\frac{1}{z^2}I + A^{-1})) > 0$ and we define for some $C > 0$

$$\Delta(z) = \frac{z}{(1+|z|^2)^{\frac{1}{2}}}(\frac{1}{z^2}I + A^{-1})^{\frac{1}{2}}A^{-\frac{1}{2}}, \quad |z| > C. \tag{3.9}$$

Note that for $|z|$ large and $|\arg(z)| < \frac{\pi}{4} + \frac{\delta}{3}$ both definitions coincide, since then we have $|\arg(z^2)| < \pi$, $\arg(\sigma(\frac{1}{z^2}I + A^{-1})) < \pi$ and $|\arg(\sigma(I + z^2A^{-1}))| < \pi$ and hence the functional equation $(z^2)^{\frac{1}{2}}(\frac{1}{z^2}I + A^{-1})^{\frac{1}{2}} = (I + z^2A^{-1})^{\frac{1}{2}}$ holds.

As in Chapter 3 we assume that the matrices $P_\pm, Q_\pm$ in the boundary conditions are divided into a Neumann and a Dirichlet part as follows:

Hypothesis 3.3 *The matrix $(Q_- Q_+)$ is of rank $r \in [0, 2m]$ and we assume that the boundary conditions are partitioned into a Dirichlet and Neumann part, i.e. the matrices $(P_\pm, Q_\pm) \in \mathbb{R}^{2m,2m}$ have the following structure*

$$(P_\pm, Q_\pm) = \begin{pmatrix} P_\pm^N & Q_\pm^N \\ P_\pm^D & 0 \end{pmatrix}, \qquad P_\pm^N, Q_\pm^N \in \mathbb{R}^{r,m},\ P_\pm^D \in \mathbb{R}^{2m-r,m} \tag{3.10}$$

Assume that there exists $C > 0$ such that the matrices

$$\Gamma(z) = \begin{pmatrix} Q_-^N\Delta(z) & -Q_+^N\Delta(z) \\ P_-^D & P_+^D \end{pmatrix} \tag{3.11}$$

have uniformly bounded inverses for

$$z \in \mathbb{C} : \ \arg(z) \leq \frac{\pi}{4} + \frac{\delta}{3}, \quad \text{or} \quad |z| \geq C. \tag{3.12}$$

Discussion of Hypothesis 3.3

Remark 3.4 Note that the following statements are equivalent

1. $\Gamma(z)$ has a uniformly bounded inverse for all $|\arg(z)| \leq \frac{\pi}{4} + \frac{\delta}{3}$ and for $|z| \geq C$.

2. The matrices $\Gamma_0 = \begin{pmatrix} Q_-^N A^{-\frac{1}{2}} & -Q_+^N A^{-\frac{1}{2}} \\ P_-^D & P_+^D \end{pmatrix}$ and $\Gamma_\infty = \begin{pmatrix} Q_-^N A^{-1} & -Q_+^N A^{-1} \\ P_-^D & P_+^D \end{pmatrix}$ are nonsingular and $\Gamma(z)$ is nonsingular for $|\arg(z)| \leq \frac{\pi}{4} + \frac{\delta}{3},\ z \neq 0$.

This equivalence follows from $\Gamma_0 = \Gamma(0)$ and $\Gamma(z) \sim \begin{pmatrix} \frac{z}{(1+|z|^2)^{\frac{1}{2}}} I & 0 \\ 0 & I \end{pmatrix} \Gamma_\infty$ as $|z| \to \infty$. The nonsingularity of Γ_0 corresponds to the corresponding condition (see Theorem 2.1 in [6]) which is necessary for resolvent estimates on finite intervals for large $|s|$ in the continuous case. The nonsingularity of Γ_∞ will be used in Chapter 4.

For $A = I$ the matrix $\Delta(z)$ has the form $\Delta(z) = \alpha I$ for some $\alpha \in \mathbb{C}$. Therefore it remains to check the invertibility of

$$\begin{pmatrix} Q_-^N & -Q_+^N \\ P_-^D & P_+^D \end{pmatrix}.$$

For the boundary conditions which are used in the numerical computations in Chapter 5 we obtain:

Neumann b.c. $\delta_0 u_{n_-} = \delta_0 u_{n_+} = 0,\ r = 2m$**:**

$$Q_- = \begin{pmatrix} I \\ 0 \end{pmatrix},\ Q_+ = \begin{pmatrix} 0 \\ I \end{pmatrix},\ P_- = P_+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} :\ \Gamma(z) = \begin{pmatrix} \Delta(z) & 0 \\ 0 & \Delta(z) \end{pmatrix}.$$

Then Hypothesis 3.3 requires the invertiblity of $\Delta(z)$ in the domains 3.12, which is always satisfied.

periodic b.c. $u_{n_-} = u_{n_+}, \delta_0 u_{n_-} = \delta_0 u_{n_+}$**,** $r = m$**:**

$$P_- = \begin{pmatrix} 0 \\ I \end{pmatrix},\ P_+ = \begin{pmatrix} 0 \\ -I \end{pmatrix},\ Q_- = \begin{pmatrix} I \\ 0 \end{pmatrix},\ Q_+ = \begin{pmatrix} -I \\ 0 \end{pmatrix} :\ \Gamma(z) = \begin{pmatrix} \Delta(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix},$$

and again Hypothesis 3.3 holds true.

Dirichlet b.c. $u_{n_-} = u_{n_+} = 0, r = 0$**:**

$$P_- = \begin{pmatrix} I \\ 0 \end{pmatrix},\ P_+ = \begin{pmatrix} 0 \\ I \end{pmatrix},\ Q_- = Q_+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} :\ \Gamma(z) = I.$$

Here Hypothesis 3.3 is automatically satisfied.

We consider $s \in \mathbb{C}$ in the following two regions $\Omega_{C_0}^h$, Ω_∞^h (cf. Figure 3.1)

$$\Omega_{C_0}^h = \left\{ s \in \mathbb{C} :\ |s| \in (C_0, \frac{C_0}{h^2}], \quad |\arg(s)| \le \frac{\pi}{2} + \frac{2\delta}{3} \right\} \tag{3.13}$$

$$\Omega_\infty^h = \left\{ s \in \mathbb{C} :\ |s| > \frac{C_0}{h^2} \right\} \tag{3.14}$$

where the constant C_0 will be chosen later.

In order to simplify the presentation we will restrict ourselves to diagonalizable A. The main result of this section is the following resolvent estimate, which will be used together with the estimates in Theorem 3.2 in Chapter 4.

Theorem 3.5 *Consider the resolvent equation* (3.1)-(3.2) *with diagonalizable* $A > 0$ *and assume that Hypothesis* (3.3) *holds.*

Then C_0 *can be chosen such that there exist* $c > 0$, $T > 0$, $h_0 > 0$ *such that for* $h < h_0$ *and* $\pm h n_\pm > T$ *and* s *restricted by* (3.13) *or* (3.14) *the following holds. The resolvent* *equation* (3.1) *with boundary conditions* (3.2) *possesses for each* $\hat{g} \in S_J(\mathbb{C}^m)$ *and each* $\eta = (\eta^N, \eta^D)^T$, $\eta^N \in \mathbb{C}^r, \eta^D \in \mathbb{C}^{2m-r}$ *a unique solution* $u \in S_{J_e}(\mathbb{C}^m)$. *Furthermore,* u *can be estimated for* $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ *by*

$$|s|^2\|u\|_\diamond^2 + |s|\|\delta_+ u\|_\diamond^2 \le c(\|\hat{g}\|_\diamond^2 + |s|\|\eta^N\|^2 + |s|^2\|\eta^D\|^2), \quad \text{for } s \in \Omega_{C_0}^h \tag{3.15}$$

$$|s|^2\|u_{|_J}\|_\diamond^2 + |s|\|\delta_+(u_{|_J})\|_\diamond^2 \le c(\|\hat{g}\|_\diamond^2 + |s|\|\eta^N\|^2 + |s|^2\|\eta^D\|^2), \quad \text{for } s \in \Omega_\infty^h. \tag{3.16}$$

Note that similar estimates have been obtained directly using energy estimates in [60], Lemma 4.9 for Dirichlet and periodic boundary conditions.

Before we start with a series of Lemmas which are needed for the proof of Theorem 3.5, we give a short outline:

The equation (3.1), (3.2) is transformed to first order via the scaled transformation $(u_n, \frac{1}{\rho}\delta_- u_n) = (u_n, v_n)$. The transformed system is approximated by constant coefficient operators $\hat{L}(s,\rho)z_n = z_{n+1} - \hat{M}(s,\rho)z_n$, for small h and large ρ. The matrices $\hat{M}(s,\rho)$ are hyperbolic for $s \in \Omega_{C_0}^h \cup \Omega_\infty^h$. This will imply that $\hat{L}(s,\rho)$ has exponential dichotomies on $\mathbb{Z}$. In order to obtain estimates for the solution of the corresponding boundary value problem for large ρh we need to take into account the structure of the right hand side of the transformed system. Therefore we cannot apply the linear theory in Chapter 2 directly. Nevertheless the proofs follow the lines in Section 2.1.1.

Using the assumption that A is diagonalizable, we can pretransform (3.1),(3.2) as follows Let $U \in \mathbb{C}^{m,m}$ be given such that $UAU^{-1} = \tilde{A} = \operatorname{diag}(\mu_1, \dots, \mu_m)$ and define $\tilde{B}_n = UB_nU^{-1}$, $\tilde{C}_n = UC_nU^{-1}$ for $n \in J$ as well as $\tilde{P}_\pm = P_\pm U^{-1}$, $\tilde{Q}_\pm = Q_\pm U^{-1}$. Then $u \in S_J$ solves (3.1),(3.2) if and only if $w = Uu$ solves

$$\begin{aligned} \tilde{A}(\delta_+\delta_- w)_n + \tilde{B}_n(\delta_0 w)_n + (\tilde{C}_n - sI)w_n &= U\hat{g}_n, \\ \tilde{P}_- w_{n_-} + \tilde{Q}_-\delta_0 w_{n_-} + \tilde{P}_+ w_{n_+} + \tilde{Q}_+\delta_0 w_{n_+} &= \eta. \end{aligned}$$

The relation $\Delta(z) = U^{-1}\tilde{\Delta}(z)U$, where $\tilde{\Delta}(z)$ is defined by (3.8),(3.9) with $\tilde{A}$ instead of A, leads to

$$\Gamma(z) = \begin{pmatrix} \tilde{Q}_-^N\tilde{\Delta}(z)U & -\tilde{Q}_+^N\tilde{\Delta}(z)U \\ \tilde{P}_-^D U & \tilde{P}_+^D U \end{pmatrix} = \begin{pmatrix} \tilde{Q}_-^N\tilde{\Delta}(z) & -\tilde{Q}_+^N\tilde{\Delta}(z) \\ \tilde{P}_-^D & \tilde{P}_+^D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

Thus Hypothesis 3.3 is invariant under diagonalization. In the following we drop the tildes and assume w.l.o.g. that A is diagonal. Transformation to first order via $z_n = (u_n, \frac{1}{\rho}\delta_- u_n) = (u_n, v_n)$, $n = n_-, \dots, n_+ + 1$, for some $\rho > 0$ leads to the equation

$$N_n(\rho)z_{n+1} - K_n(s,\rho)z_n = \hat{r}_n, \quad n \in J = [n_-, n_+] \tag{3.17}$$

$$R(\rho)z = \hat{\eta} \tag{3.18}$$

where

$$N_n(\rho) = \begin{pmatrix} I & -h\rho I \\ 0 & E_n^+ \end{pmatrix}, \quad K_n(s,\rho) = \begin{pmatrix} I & 0 \\ \frac{h}{\rho}(sI - C_n) & E_n^- \end{pmatrix}, \quad E_n^\pm = A \pm \frac{h}{2}B_n,$$

$$R(\rho)z = B_-(\rho)z_{n_-} + \hat{B}_- z_{n_-+1} + B_+(\rho)z_{n_+} + \hat{B}_+ z_{n_++1} \tag{3.19}$$

and

$$\hat{r}_n = \begin{pmatrix} 0 \\ \frac{h}{\rho}\hat{g}_n \end{pmatrix}, \quad B_\pm(\rho) = \begin{pmatrix} \frac{1}{\rho}P_\pm^N & \frac{1}{2}Q_\pm^N \\ P_\pm^D & 0 \end{pmatrix}, \quad \hat{B}_\pm = \begin{pmatrix} 0 & \frac{1}{2}Q_\pm^N \\ 0 & 0 \end{pmatrix}, \quad \hat{\eta} = \begin{pmatrix} \frac{1}{\rho}\eta^N \\ \eta^D \end{pmatrix}.$$

We consider the explicit formulation of (3.17) which is given by

$$(\tilde{L}(s,\rho)z)_n = \frac{h}{\rho}\begin{pmatrix} h\rho I \\ I \end{pmatrix} E_n^{+-1}\hat{g}_n, \quad n \in J \tag{3.20}$$

where

$$(\tilde{L}(s,\rho)z)_n = z_{n+1} - M_n(s,\rho)z_n, \tag{3.21}$$

$$M_n(s,\rho) = N_n(\rho)^{-1}K_n(s,\rho) = \begin{pmatrix} I + h^2 E_n^{+-1}(sI - C_n) & h\rho E_n^{+-1}E_n^- \\ \frac{h}{\rho}E_n^{+-1}(sI - C_n) & E_n^{+-1}E_n^- \end{pmatrix}. \tag{3.22}$$

In order to obtain solutions of (3.20), (3.18) we will use the following constant coefficient difference equation, given by

$$(\hat{L}(s,\rho)z)_n = \frac{h}{\rho}\begin{pmatrix} h\rho I \\ I \end{pmatrix}\hat{g}_n, \quad n \in J \tag{3.23}$$

where

$$(\hat{L}(s,\rho)z)_n = z_{n+1} - \hat{M}(s,\rho)z_n, \tag{3.24}$$

$$\hat{M}(s,\rho) = \hat{N}(\rho)^{-1}\hat{K}(s,\rho) = I + h\rho\begin{pmatrix} \frac{h}{s}\rho A^{-1} & I \\ \frac{s}{\rho^2}A^{-1} & 0 \end{pmatrix} \tag{3.25}$$

and

$$\hat{N}(\rho) = \begin{pmatrix} I & -h\rho I \\ 0 & A \end{pmatrix}, \quad \hat{K}(s,\rho) = \begin{pmatrix} I & 0 \\ \frac{h}{\rho}sI & A \end{pmatrix}.$$

As we will show later, $\hat{L}(s,\rho)$ is a small perturbation of $\tilde{L}(s,\rho)$ for $|s|$ large. If we set

$$s = \rho^2 \mathrm{e}^{2i\theta}, \quad \rho = \sqrt{|s|}$$

then we obtain

$$\hat{M}(s,\rho) = I + h\rho\begin{pmatrix} h\rho\mathrm{e}^{2i\theta}A^{-1} & I \\ \mathrm{e}^{2i\theta}A^{-1} & 0 \end{pmatrix}.$$

We will prove in the next lemma that the matrices $\hat{M}(s,\rho)$ are hyperbolic for $s \in \Omega_{C_0}^h$ and $s \in \Omega_\infty^h$. Then $\hat{L}(s,\rho)$ possesses an exponential dichotomy on $\mathbb{Z}$, which will be used to construct a solution of (3.23), (3.18).

The following lemma deals with the eigenvalues of matrices which have the same structure as $\hat{M}(s,\rho)$.

Lemma 3.6 *Consider*

$$M = I + \kappa N(\kappa), \ \text{where } N(\kappa) = \begin{pmatrix} \kappa S & I \\ S & 0 \end{pmatrix}$$

with $\kappa > 0$*, and* $S \in \mathbb{C}^{m,m}$ *a nonsingular diagonal matrix. Then there exist* $\delta, C_0 > 0$ *such that the following holds: If either (*$\kappa \le C_0$ *and* $\arg(\sigma(S)) \le \pi - \delta$*) or* $\kappa > C_0$ *then* M *is a hyperbolic matrix with* m *stable eigenvalues* $\nu_{s,i}$ *and* m *unstable eigenvalues* $\nu_{u,i}$*,* $i = 1, \ldots, m$*.*

Moreover, there exist $\alpha, a > 0$*,* $\epsilon \in (0, C_0]$ *such that for* $i = 1, \ldots, m$*, the following estimates hold:*

$$a\kappa^2 \ge |\nu_{u,i}| \ge \alpha\kappa^2, \qquad \frac{\alpha}{\kappa^2} \le |\nu_{s,i}| \le \frac{a}{\kappa^2} \qquad \text{for } \kappa > C_0 \tag{3.26}$$

$$|\nu_{u,i}| \ge 1 + \alpha, \qquad |\nu_{s,i}| \le \frac{1}{1+\alpha} \qquad \text{for } \kappa \in [\epsilon, C_0],\ \arg(\sigma(S)) \le \pi - \delta \tag{3.27}$$

$$|\nu_{u,i}| \ge 1 + \alpha\kappa, \qquad |\nu_{s,i}| \le \frac{1}{1+\alpha\kappa} \qquad \text{for } \kappa \in (0, \epsilon),\ \arg(\sigma(S)) \le \pi - \delta \tag{3.28}$$

Proof: Let $\mu \in \mathbb{C}$ be an eigenvalue of S with eigenvector u. Then λ is an eigenvalue of $N(\kappa)$ with eigenvector v if and only if λ is a solution of

$$\lambda^2 - \lambda\kappa\mu - \mu = 0 \tag{3.29}$$

and $v = \begin{pmatrix} \lambda S^{-1} u \\ u \end{pmatrix}$. The solutions of (3.29) are given by

$$\lambda_\pm = \begin{cases} \frac{1}{2}\big(\kappa\mu \pm \sqrt{\kappa^2\mu^2 + 4\mu}\big), & \text{if } \kappa > 0,\ |\arg(\mu)| \le \pi - \delta, \\ \frac{\kappa\mu}{2}\big(1 \pm \sqrt{1 + \frac{4}{\mu\kappa^2}}\big), & \text{if } \kappa > C_0. \end{cases} \tag{3.30}$$

Note that both definitions coincide on the common domain of definition, and that

$$\lambda_+ - \lambda_- = \begin{cases} \sqrt{\kappa^2\mu^2 + 4\mu}\big), & \text{if } \kappa > 0,\ |\arg(\mu)| \le \pi - \delta, \\ \frac{\kappa\mu}{2}\sqrt{1 + \frac{4}{\mu\kappa^2}}, & \text{if } \kappa > C_0 \end{cases}$$

implies a lower estimate

$$|\lambda_+ - \lambda_-| \ge c\max(\kappa, 1), \qquad \text{for some } c > 0. \tag{3.31}$$

The eigenvalues $\nu_\pm$ of M are given by $\nu_\pm = 1 + \kappa\lambda_\pm$. From $\lambda_-\lambda_+ = -\mu$,$\lambda_- + \lambda_+ = \kappa\mu$ and (3.29) we obtain $1 + \kappa\lambda_- = (1 + \kappa\lambda_+)^{-1}$.

We consider $\nu_\pm$ for κ in three different regions:

1. Large κ:

Use the expansion $\sqrt{1+z} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$ to obtain

$$|1 + \kappa\lambda_+| = |1 + \frac{\mu\kappa^2}{2}(1 + \sqrt{1 + \frac{4}{\mu\kappa^2}})| \ge \alpha\kappa^2 \text{ if } \kappa > C_0.$$

This implies $|\nu_{u,i}| \ge \alpha\kappa^2$, as well as $|\nu_{s,i}| < \frac{1}{\alpha\kappa^2}$ for $\kappa > C_0$, $i = 1, \ldots, m$.

2. Small κ, $|\arg(\mu)| \le \pi - \delta$
For small κ and $|\arg(\mu)| \le \pi - \delta$ we have the expansion

$$1+\kappa\lambda_+ = 1+\frac{\kappa^2\mu}{2}+\kappa\sqrt{\mu}\sqrt{1+\frac{\kappa^2\mu}{4}} = 1+\frac{\kappa^2\mu}{2}+\kappa\sqrt{\mu}(1+\frac{\kappa^2\mu}{8}+\mathcal{O}(\kappa^4)) = 1+\kappa\sqrt{\mu}+\mathcal{O}(\kappa^2).$$

From $|\arg(\mu)| \le \pi - \delta$ we obtain Re $\sqrt{\mu} > 0$ and hence $|\nu_{u,i}| \ge 1 + \alpha\kappa$, $|\nu_{s,i}| \le \frac{1}{1+\alpha\kappa}$ for some $\alpha > 0$ and $\kappa \in (0, \epsilon)$.

3. κ in the compact set $\kappa \in [\epsilon, C_0]$, $|\arg(\mu)| \le \pi - \delta$
Let $\kappa > 0$, $|\arg(\mu)| \le \pi - \delta$. In particular Re $\mu > 0$. Then Re $\sqrt{\kappa^2\mu^2 + 4\mu} \ge 0$ by definition. Hence Re $\lambda_+ =$ Re $\frac{\kappa\mu}{2}$ + Re $\sqrt{\kappa^2\mu^2+4\mu} \ge$ Re $\frac{\kappa\mu}{2} \ge c\kappa$ for some $c > 0$. Therefore Re $(1+\kappa\lambda_+) \ge 1 + c\kappa^2$ and $|1+\kappa\lambda_+| > 1$. Since κ varies in a compact interval this proves the assertion (3.27).

□

By application of the previous Lemma with $S = \mathrm{e}^{2i\theta}A^{-1}$ and $\kappa = \rho h$ we obtain that the constant coefficient operators $\hat{L}(s,\rho)$ possess an exponential dichotomy on $\mathbb{Z}$ if $s \in \Omega^h_{C_0} \cup \Omega^h_\infty$ as the following corollary shows.

Corollary 3.7 *Assume that $A > 0$ is diagonal. Then there exist $C_0, \epsilon, \delta > 0$ such that the operators $\hat{L}(s,\rho)$ possess exponential dichotomies on $\mathbb{Z}$ if $s = \rho^2\mathrm{e}^{2i\theta}$ is restricted by (3.13) or (3.14). The dichotomy data are (K, β, P), where K is independent of ρ and h, and for some $\alpha > 0$*

$$\beta = \ln(\alpha(\rho h)^2) \qquad \textit{for } \rho > \frac{C_0}{h}, \tag{3.32}$$

$$\beta = \ln(1+\alpha) \qquad \textit{for } \rho \in [\frac{\epsilon}{h}, \frac{C_0}{h}],\ |\theta| \le \frac{\pi}{4}+\frac{\delta}{3}, \tag{3.33}$$

$$\beta = \ln(1+\alpha\rho h) \qquad \textit{for } \rho \in [C_0, \frac{\epsilon}{h}],\ |\theta| \le \frac{\pi}{4}+\frac{\delta}{3} \tag{3.34}$$

and the projector P is given by

$$P = \begin{pmatrix} (\Lambda_s - \Lambda_u)^{-1}\Lambda_s & -(\Lambda_s-\Lambda_u)^{-1} \\ -\Lambda_u(\Lambda_s-\Lambda_u)^{-1}\Lambda_s & \Lambda_s(\Lambda_s-\Lambda_u)^{-1} \end{pmatrix}. \tag{3.35}$$

Here Λ_s and Λ_u are defined by

$$\Lambda_s = \mathrm{diag}(\lambda_{-,i})_{i=1,\dots,m}, \quad \Lambda_u = \mathrm{diag}(\lambda_{+,i})_{i=1,\dots,m} \tag{3.36}$$

where $\lambda_{\pm,i}$ are defined for each $i = 1,\dots,m$ by (3.30) with $\mu = \mu_i \in \sigma(\mathrm{e}^{2i\theta}A^{-1})$.

Proof: Denote the eigenvalues of A^{-1} by $r\mathrm{e}^{-2i\phi}$, then the eigenvalues of $\mathrm{e}^{2i\theta}A^{-1}$ are given by $r\mathrm{e}^{2i(\theta-\phi)}$ and for $|\theta| < \frac{\pi}{4}+\frac{\delta}{3}$ and $|2\phi| \le \frac{\pi}{2} - \delta$ we obtain $2|\theta-\phi| < \pi - \frac{\delta}{3}$. Then application of Lemma 3.6 with $S = \mathrm{e}^{2i\theta}A^{-1}$ implies that the matrix $\hat{M}(s,\rho)$ given by

$$I + h\rho \begin{pmatrix} h\rho\,\mathrm{e}^{2i\theta}A^{-1} & I \\ \mathrm{e}^{2i\theta}A^{-1} & 0 \end{pmatrix} \tag{3.37}$$

is hyperbolic for $|\theta| < \frac{\pi}{4} + \frac{\delta}{3}$. Furthermore, the m stable eigenvalues $\nu_{s,i} = 1 + h\rho\lambda_{s,i}$ and the m unstable eigenvalues $\nu_{u,i} = \nu_{s,i}^{-1}$, $i = 1, \dots, m$ can be estimated using (3.26)–(3.28) by

$$|\nu_{u,i}| \geq \alpha(\rho h)^2, \qquad |\nu_{s,i}| \leq \frac{\alpha}{(\rho h)^2}, \qquad \text{for } \rho > \frac{C_0}{h} \tag{3.38}$$

$$|\nu_{u,i}| \geq 1 + \alpha, \qquad |\nu_{s,i}| \leq \frac{1}{1+\alpha}, \qquad \text{for } \rho \in [\frac{\epsilon}{h}, \frac{C_0}{h}] \tag{3.39}$$

$$|\nu_{u,i}| \geq 1 + \alpha\rho h, \qquad |\nu_{s,i}| \leq \frac{1}{1+\alpha\rho h}, \qquad \text{for } \rho \in [C_0, \frac{\epsilon}{h}]. \tag{3.40}$$

The matrices $\hat{M}(s,\rho)$ can be transformed to diagonal form via $TD = \hat{M}(s,\rho)T$ with

$$D = \begin{pmatrix} D_s & 0 \\ 0 & D_s^{-1} \end{pmatrix}, \qquad D_s = I + \kappa\Lambda_s, \quad D_u = I + \kappa\Lambda_u, \quad \kappa = \rho h \tag{3.41}$$

and

$$T = \begin{pmatrix} -I & -I \\ \Lambda_u & \Lambda_s \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} (\Lambda_s - \Lambda_u)^{-1} & 0 \\ 0 & (\Lambda_s - \Lambda_u)^{-1} \end{pmatrix} \begin{pmatrix} -\Lambda_s & -I \\ \Lambda_u & I \end{pmatrix}. \tag{3.42}$$

Note the relations

$$\begin{gathered} \Lambda_u\Lambda_s = \Lambda_s\Lambda_u = -S, \quad \Lambda_s + \Lambda_u = \kappa S, \quad D_u = D_s^{-1}, \\ \Lambda_u D_s = -\Lambda_s, \quad \Lambda_s = \frac{1}{\kappa}(D_s - I). \end{gathered} \tag{3.43}$$

From this the existence of an exponential dichotomy on $\mathbb{Z}$ for the constant coefficient operators $\hat{L}(s,\rho)$ follows by Remark 2.5 in [42] with data (K, β, P) with $\beta = -\ln \nu_s$ where $|\nu_{s,i}| < \nu_s < 1$, $i = 1, \dots, m$ and P is defined in (3.35). □

Using the exponential dichotomy, the Green's function is given by (2.9) where in this case the dichotomy projector P and the matrix $\hat{M}$ are constant. The following Lemma is an adaptation of Lemma 2.4 to the current situation.

Lemma 3.8 *Let s be restricted by (3.13) or (3.14). Then there exist $h_0, T > 0$ such that for $h < h_0, \pm n_\pm h > T$ and for each $\hat{g} \in S_J(\mathbb{C}^m)$ there exists a unique solution $\tilde{z} \in S_{J_e}(\mathbb{C}^{2m})$ of the boundary value problem*

$$(\hat{L}(s,\rho)z)_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \hat{g}_n, \quad n \in J \tag{3.44}$$

$$P z_{n_-} = \rho_- \in \mathcal{R}(P) \tag{3.45}$$

$$(I - P) z_{n_+} = \rho_+ \in \mathcal{R}(I - P) \tag{3.46}$$

where P is the dichotomy projector defined in (3.35). The solution has the form

$$\tilde{z}_n = z_n^{\text{hom}} + \hat{z}_n(\hat{g}), \quad n \in J, \qquad \tilde{z}_{n_++1} = \hat{M}\tilde{z}_{n_+} + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \hat{g}_{n_+} \tag{3.47}$$

where

$$z_n^{\text{hom}} = \Phi(n, n_-)\rho_- + \Phi(n, n_+)\rho_+, \tag{3.48}$$

and

$$\hat{z}_n(\hat{g}) = \frac{h}{\rho} \sum_{n=n_-}^{n_+} G(n, m+1) P \begin{pmatrix} h\rho I \\ I \end{pmatrix} \hat{g}_n \tag{3.49}$$
$$= \frac{h}{\rho} \Big(\sum_{m=n_-}^{n-1} \Phi(n, m+1) P \begin{pmatrix} h\rho I \\ I \end{pmatrix} \hat{g}_m - \sum_{m=n}^{n_+-1} \Phi(n, m+1)(I-P) \begin{pmatrix} h\rho I \\ I \end{pmatrix} \hat{g}_m \Big).$$

In order to obtain the necessary estimates of $\hat{z}$, especially for the case $h\rho > C_0$, we have to take into account the special structure of the right hand side. Therefore we diagonalize equation (3.44) using the transformation T given in (3.42). For $w_n = T^{-1} z_n$ equation (3.23) reads

$$w_{n+1} - \begin{pmatrix} D_s & 0 \\ 0 & D_s^{-1} \end{pmatrix} w_n = \frac{h}{\rho} T^{-1} \begin{pmatrix} h\rho I \\ I \end{pmatrix} \hat{g}_n, \quad n \in J = [n_-, n_+].$$

In order to be able to distinguish estimates in the different components we introduce the following vector norm notation. For $z = (u, v) \in \mathbb{R}^m \times \mathbb{R}^m$, $\|z\|_{\text{vec}} = \begin{pmatrix} n_u \\ n_v \end{pmatrix}$ means $\|u\| = n_u$, $\|v\| = n_v$ and $\|z\|_{\text{vec}} \le \begin{pmatrix} c_u \\ c_v \end{pmatrix}$ means the componentwise estimates $\|u\| \le c_u$ and $\|v\| \le c_v$. With this notation we obtain the following estimates for the Green's function.

Lemma 3.9 *Let $|\sigma(D_s)| < \nu_s < 1$. Then the following holds.*

$$\left\| \Phi(n, m+1) P \begin{pmatrix} h\rho I \\ I \end{pmatrix} \right\|_{\text{vec}} \le \frac{c}{\max(\rho h, 1)} \begin{pmatrix} \nu_s \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n-m-1}, \quad n \ge m \tag{3.50}$$
$$\left\| \Phi(n, m+1)(I-P) \begin{pmatrix} h\rho I \\ I \end{pmatrix} \right\|_{\text{vec}} \le \frac{c}{\max(\rho h, 1)} \begin{pmatrix} 1 \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{m-n}, \quad n < m \tag{3.51}$$

and

$$\begin{aligned} \|\Phi(n, n_-) T_-\|_{\text{vec}} &\le \begin{pmatrix} \nu_s \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n-n_--1}, \\ \|\Phi(n, n_+) T_+\|_{\text{vec}} &\le \begin{pmatrix} 1 \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n_+-n}, \end{aligned} \tag{3.52}$$

where $T = (T_-,\ T_+)$ with T defined by (3.42).

Proof: With

$$\Phi(n, m) = T D^{n-m} T^{-1}, \quad P = T E^s T^{-1}, \quad E^s = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.53}$$

we obtain using $D_s = I + h\rho\Lambda_s$

$$\Phi(n, m+1) P \begin{pmatrix} h\rho I \\ I \end{pmatrix} = T \begin{pmatrix} D_s^{n-m-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \begin{pmatrix} h\rho I \\ I \end{pmatrix}$$
$$= - \begin{pmatrix} -I & -I \\ \Lambda_u & \Lambda_s \end{pmatrix} \begin{pmatrix} D_s^{n-m}(\Lambda_s - \Lambda_u)^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} I \\ -\Lambda_u \end{pmatrix} D_s^{n-m} (\Lambda_s - \Lambda_u)^{-1}$$
$$= \begin{pmatrix} D_s \\ \frac{1}{(\rho h)}(D_s - I) \end{pmatrix} D_s^{n-m-1} (\Lambda_s - \Lambda_u)^{-1}$$

as well as

$$\begin{aligned}\Phi(n,m+1)(I-P)\begin{pmatrix}h\rho I\\ I\end{pmatrix} &= T\begin{pmatrix}0 & 0\\ 0 & D_s^{m-n+1}\end{pmatrix}T^{-1}\begin{pmatrix}h\rho I\\ I\end{pmatrix}\\ &= \begin{pmatrix}-I\\ \Lambda_s\end{pmatrix}D_s^{m-n}(\Lambda_s-\Lambda_u)^{-1}\\ &= \begin{pmatrix}-I\\ \frac{1}{(\rho h)}(D_s-I)\end{pmatrix}D_s^{m-n}(\Lambda_s-\Lambda_u)^{-1}.\end{aligned}$$

This implies the estimates (3.50), (3.51). Similarly with (3.31)

$$\Phi(n,n_-)T_- = T\hat{M}^{n-n_-}T^{-1}T_- = \begin{pmatrix}-I\\ \Lambda_u\end{pmatrix}D_s^{n-n_-} = \begin{pmatrix}-D_s\\ \frac{1}{\rho h}(D_s-I)\end{pmatrix}D_s^{n-n_--1}$$

and

$$\Phi(n,n_+)T_+ = T\hat{M}^{n-n_+}T^{-1}T_+ = \begin{pmatrix}-I\\ \Lambda_s\end{pmatrix}D_s^{n_+-n} = \begin{pmatrix}-I\\ \frac{1}{(\rho h)}(D_s-I)\end{pmatrix}D_s^{n_+-n}$$

lead to (3.52). □

The special solution is estimated in the following Lemma.

Lemma 3.10 *Let s be restricted by (3.13) or (3.14). Then there exist $c,h_0,T>0$ such that for $h<h_0,\pm n_\pm h>T$ for each $\hat{g}\in S_J(\mathbb{C}^m)$ the special solution $\hat{z}(\hat{g})\in S_J(\mathbb{C}^{2m})$ given by (3.49) can be estimated for $\diamond\in\{\mathcal{L}_{2,h},\infty\}$ by*

$$\|\hat{z}(\hat{g})\|_\diamond \le \frac{c}{\rho^2}\|\hat{g}\|_\diamond. \tag{3.54}$$

Moreover, we obtain

$$\|\hat{M}\hat{z}_{n_+}(\hat{g})\|_{\text{vec}} \le c\begin{pmatrix}h^2+\frac{h}{\rho}+\frac{1}{\rho^2}\\ \frac{h}{\rho}+\frac{1}{\rho^2}\end{pmatrix}\|\hat{g}\|_\infty. \tag{3.55}$$

Proof: Using the estimates (3.50), (3.51) we obtain for $n\in J$ for $\hat{z}(\hat{g})=(\hat{u},\hat{v})$ with $\nu_s<1$

$$\|\hat{u}_n\| \le \frac{c}{\max(\rho h,1)}\frac{h}{\rho}\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\| \le c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s}\|\hat{g}\|_\infty \tag{3.56}$$

where $c_u(h,\rho)=\frac{ch}{\rho\max(\rho h,1)}$. Then we obtain

$$\|\hat{u}_n\| \le \frac{c}{\rho^2}\|\hat{g}\|_\infty, \quad \forall n\in J \tag{3.57}$$

provided we can show

$$c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s} \le \frac{c}{\rho^2}. \tag{3.58}$$

For $\rho h>C_0$ this holds, since by (3.38)

$$c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s} \le \frac{c}{\rho^2}\frac{\alpha(\rho h)^2+1}{\alpha(\rho h)^2-1} \le \frac{c}{\rho^2}.$$

For $\rho h < \epsilon$ we obtain (3.58) using (3.40)

$$c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s} \le c\frac{h}{\rho}\ \frac{2+\alpha\rho h}{\alpha\rho h} \le \frac{c}{\rho^2},$$

and (3.39) implies for $\rho h \in [\epsilon, C_0]$

$$c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s} \le \frac{c}{\rho^2}.$$

The estimate of the second coordinate is even easier. From the second coordinate of (3.50), (3.51) and $\frac{c}{\rho^2 \max(\rho h,1)} \le \frac{c}{\rho^2}$ we obtain

$$\begin{aligned}
\|\hat{v}_n\| &\le \frac{c}{\rho^2}\frac{1-\nu_s}{\max(\rho h,1)}\Big(\sum_{m=n_-}^{n-1}\nu_s^{n-m-1}\|\hat{g}_m\| + \sum_{m=n}^{n_+-1}\nu_s^{m-n}\|\hat{g}_m\|\Big)\\
&\le \frac{c}{\rho^2}\|\hat{g}\|_\infty(1-\nu_s)\Big(\frac{1-\nu_s^{n-n_-}}{1-\nu_s} + \frac{1-\nu_s^{n_+-n}}{1-\nu_s}\Big)\\
&\le \frac{c}{\rho^2}\|\hat{g}\|_\infty(2-(\nu_s^{n-n_-}+\nu_s^{n_+-n}))\\
&\le \frac{c}{\rho^2}\|\hat{g}\|_\infty.
\end{aligned} \tag{3.59}$$

The estimates (3.57), (3.59) imply (3.54) with $\diamond = \infty$. The $\mathcal{L}_{2,h}$ estimate is similar to the estimate in Lemma 2.4. From (3.56) we find

$$\begin{aligned}
\|\hat{u}_n\|^2 &\le c_u(h,\rho)^2\Big(\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\|\Big)^2 \le c_u(h,\rho)^2\sum_{m=-\infty}^{\infty}\nu_s^{-|n-m|}\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\|^2\\
&\le c_u(h,\rho)^2\frac{1+\nu_s}{1-\nu_s}\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\|^2 \ \le c_u(h,\rho)\frac{c}{\rho^2}\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\|^2
\end{aligned}$$

which implies by summation over all $n \in J$ with (3.58)

$$\begin{aligned}
\|\hat{u}\|^2_{\mathcal{L}_{2,h}} &= \sum_{n=n_-}^{n_+} h\|\hat{u}_n\|^2 \le \frac{ch}{\rho^2}c_u(h,\rho)\sum_{n=n_-}^{n_+}\sum_{m=n_-}^{n_+-1}\nu_s^{-|n-m|}\|\hat{g}_m\|^2\\
&\le \frac{ch}{\rho^2}c_u(h,\rho)\sum_{m=n_-}^{n_+-1}\|\hat{g}_m\|^2\sum_{n=n_-}^{n_+}\nu_s^{-|n-m|}\\
&\le \frac{ch}{\rho^2}c_u(h,\rho)\frac{1+\nu_s}{1-\nu_s}\sum_{m=n_-}^{n_+-1}\|\hat{g}_m\|^2 \le \Big(\frac{c}{\rho^2}\Big)^2 h\sum_{m=n_-}^{n_+-1}\|\hat{g}_m\|^2 = \Big(\frac{c}{\rho^2}\Big)^2\|\hat{g}_m\|^2_{\mathcal{L}_{2,h}}.
\end{aligned}$$

Similarly, (3.59) implies with $c_v(h,\rho) = (\rho^2 \max(\rho h, 1))^{-1}$

$$
\begin{aligned}
\|\hat{v}_n\|^2 &\le c c_v(h,\rho)^2 (1-\nu_s)^2 \Big[\Big(\sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|\hat{g}_m\| \Big)^2 + \Big(\sum_{m=n}^{n_+-1} \nu_s^{m-n} \|\hat{g}_m\| \Big)^2 \Big] \\
&\le c c_v(h,\rho)^2 (1-\nu_s)^2 \Big[\sum_{m=-\infty}^{n-1} \nu_s^{n-m-1} \sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|\hat{g}_m\|^2 + \sum_{m=n}^{\infty} \nu_s^{m-n} \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|\hat{g}_m\|^2 \Big] \\
&\le c c_v(h,\rho)^2 (1-\nu_s)^2 \Big[\frac{1}{1-\nu_s} \sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|\hat{g}_m\|^2 + \frac{1}{1-\nu_s} \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|\hat{g}_m\|^2 \Big] \\
&\le c c_v(h,\rho)^2 (1-\nu_s) \Big[\sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|\hat{g}_m\|^2 + \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|\hat{g}_m\|^2 \Big]
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|\hat{v}\|_{\mathcal{L}_{2,h}}^2 = \sum_{n=n_-}^{n_+} h \|\hat{v}_n\|^2 &\le c c_v(h,\rho)^2 (1-\nu_s) h \sum_{n=n_-}^{n_+} \Big[\sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|\hat{g}_m\|^2 + \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|\hat{g}_m\|^2 \Big] \\
&\le c c_v(h,\rho)^2 (1-\nu_s) h \sum_{m=n_-}^{n_+-1} \|\hat{g}_m\|^2 \Big[\sum_{n=m+1}^{n_+} \nu_s^{n-m-1} + \sum_{m=n_-}^{m} \nu_s^{m-n} \Big] \\
&\le c c_v(h,\rho)^2 h \sum_{m=n_-}^{n_+-1} \|\hat{g}_m\|^2 = \frac{c}{\rho^4} \|\hat{g}\|_{\mathcal{L}_{2,h}}^2.
\end{aligned}
$$

Finally the estimate (3.55) follows from the definition of $\hat{M}$ in (3.37)

$$
\|\hat{M}\hat{z}_{n+}(\hat{g})\|_{\text{vec}} \le c \begin{pmatrix} (1+(\rho h)^2)\|\hat{u}_{n_+}\| + \rho h \|\hat{v}_{n_+}\| \\ \rho h \|\hat{u}_{n_+}\| + \|\hat{v}_{n_+}\| \end{pmatrix} \le c \begin{pmatrix} h^2 + \frac{h}{\rho} + \frac{1}{\rho^2} \\ \frac{h}{\rho} + \frac{1}{\rho^2} \end{pmatrix} \|\hat{g}\|_\infty.
$$

□

Remark 3.11 Note that for $\rho h < C_0$ no special structure of the right hand side is needed for the estimate of the special solution $\hat{z}(\hat{g})$. In this case we can use the dichotomy constants for $\hat{L}$ given in Corollary 3.7 directly to obtain with Lemma 2.4 the estimate

$$
\|\hat{z}(\hat{g})\|_\diamond \le C_\beta (h^2 + \frac{h}{\rho}) \|\hat{g}\|_\diamond, \quad \diamond \in \{\infty, \mathcal{L}_{2,h}\},
$$

where C_β is the constant defined in (2.16) via the dichotomy exponent β wich is defined in (3.32)–(3.34). Using

$$
\begin{aligned}
C_\beta &= \frac{2+\alpha}{\alpha} \le c, && \text{for } \rho \in [\frac{\epsilon}{h}, \frac{C_0}{h}] \\
C_\beta &= \frac{2+\alpha\rho h}{\alpha \rho h} \le \frac{c}{\rho h}, && \text{for } \rho \in (C_0, \frac{\epsilon}{h})
\end{aligned}
$$

we obtain for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$

$$
\begin{aligned}
\|\hat{z}(\hat{g})\|_\diamond &\le c(h^2 + \frac{h}{\rho}) \|\hat{g}\|_\diamond \le \frac{cC_0}{\rho^2} \|\hat{g}\|_\diamond && \text{for } \rho \in [\frac{\epsilon}{h}, \frac{C_0}{h}] \\
\|\hat{z}(\hat{g})\|_\diamond &\le \frac{C}{\rho h}(h^2 + \frac{h}{\rho}) \|\hat{g}\|_\diamond \le C(\epsilon+1)\frac{1}{\rho^2} \|\hat{g}\|_\diamond && \text{for } \rho \in (C_0, \frac{\epsilon}{h}).
\end{aligned}
$$

However for $\rho h > C_0$ we have $C_\beta < c$ wich leads only to

$$\|\hat{z}(\hat{g})\|_\diamond \leq c(h^2 + \frac{h}{\rho})\|\hat{g}\|_\diamond.$$

Inserting the ansatz for $\tilde{z}$ in (3.8) into the boundary conditions we obtain the following lemma.

Lemma 3.12 *Let s be restricted by (3.13) or (3.14) and assume Hypothesis 3.3. Then there exists $h_0, T > 0$ such that the following holds. If $h < h_0$ and $\pm h n_\pm > T$ then for each $\hat{g} \in S_J(\mathbb{C}^m)$ there exists a unique solution $\tilde{z} \in S_{J_r}(\mathbb{C}^{2m})$ of (3.23) which satisfies the boundary conditions (3.18), i.e.*

$$R(\rho)z = \hat{\eta} = \begin{pmatrix} \frac{1}{\rho}\eta^N \\ \eta^D \end{pmatrix}. \tag{3.60}$$

The solution $\tilde{z} \in S_{J_r}(\mathbb{C}^{2m})$ can be estimated for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ as follows

$$\|\tilde{z}\|_\diamond \leq c(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond), \quad \textit{for } s \in \Omega^h_{C_0}, \tag{3.61}$$

$$\|\tilde{z}_{|_{\hat{J}}}\|_\diamond \leq c(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond), \quad \textit{for } s \in \Omega^h_\infty, \quad \hat{J} = [n_- + 1, \dots, n_+]. \tag{3.62}$$

Proof: Inserting the ansatz (3.47) into the boundary condition (3.60) one obtains

$$\begin{aligned} B_-(\rho)(\rho_- + \Phi(n_-, n_+)\rho_+) + \hat{B}_-(\Phi(n_- + 1, n_-)\rho_- + \Phi(n_- + 1, n_+)\rho_+) \\ + B_+(\rho)(\Phi(n_+, n_-)\rho_- + \rho_+) + \hat{B}_+\hat{M}(\Phi(n_+, n_-)\rho_- + \rho_+) \\ = \hat{\eta} - \left(B_-(\rho)\hat{z}_{n_-}(\hat{g}) + \hat{B}_-\hat{z}_{n_-+1}(\hat{g}) + B_+(\rho)\hat{z}_{n_+}(\hat{g}) + \hat{B}_+\left[\hat{M}\hat{z}_{n_+}(\hat{g}) + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \hat{g}_{n_+}\right]\right). \end{aligned}$$

This equation has to be solved for ρ_- and ρ_+. We can write $\rho\pm = T_\pm \xi_\pm,\ \xi_\pm \in \mathbb{C}^m$ where $T = (T_-\ T_+)$. After rearranging terms we obtain from the previous equation

$$R_\rho(\xi_-, \xi_+) + \Delta R_\rho(\xi_-, \xi_+) = \hat{\eta} - F_\rho(\hat{g}) \tag{3.63}$$

where

$$\begin{aligned} R_\rho(\xi_-, \xi_+) &= B_-(\rho)T_-\xi_- + \hat{B}_-\Phi(n_- + 1, n_-)T_-\xi_- + B_+(\rho)T_+\xi_+ + \hat{B}_+\hat{M}T_+\xi_+ \\ \Delta R_\rho(\xi_-, \xi_+) &= \big(B_-(\rho)\Phi(n_-, n_+) + \hat{B}_-\Phi(n_- + 1, n_+)\big)T_+\xi_+ \\ &\quad + (B_+(\rho) + \hat{B}_+\hat{M})\Phi(n_+, n_-)T_-\xi_- \\ F_\rho(\hat{g}) &= \left(B_-(\rho)\hat{z}_{n_-}(\hat{g}) + \hat{B}_-\hat{z}_{n_-+1}(\hat{g}) + B_+(\rho)\hat{z}_{n_+}(\hat{g}) + \hat{B}_+\left[\hat{M}\hat{z}_{n_+}(\hat{g}) + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \hat{g}_{n_+}\right]\right) \end{aligned}$$

With (3.53) and $\hat{M} = TDT^{-1}$ as well as $T^{-1}T_- = \binom{I}{0}$, $T^{-1}T_+ = \binom{0}{I}$ and

$$TD = \begin{pmatrix} -I & -I \\ \Lambda_u & \Lambda_s \end{pmatrix} \begin{pmatrix} D_s & 0 \\ 0 & D_s^{-1} \end{pmatrix} = \begin{pmatrix} -D_s & -D_s^{-1} \\ \Lambda_u D_s & \Lambda_s D_s^{-1} \end{pmatrix}$$

these terms can be calculated as follows:

$$\begin{aligned}
R_\rho(\xi_-,\xi_+) &= \begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} T_-\xi_- + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} TD \begin{pmatrix} I \\ 0 \end{pmatrix} \xi_- \\
&\quad + \begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} T_+\xi_+ + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} TD \begin{pmatrix} 0 \\ I \end{pmatrix} \xi_+ \\
&= \left(\begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_u \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_u \end{pmatrix} D_s \right) \xi_- \\
&\quad + \left(\begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_s \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_s \end{pmatrix} D_s^{-1} \right) \xi_+ \\
&= \mathcal{B} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B} &= \begin{pmatrix} -\frac{1}{\rho}P_-^N + \frac{1}{2}Q_-^N\Lambda_u(I+D_s) & -\frac{1}{\rho}P_+^N + \frac{1}{2}Q_+^N\Lambda_s(I+D_s^{-1}) \\ -P_-^D & -P_+^D \end{pmatrix} \\
&= -\begin{pmatrix} \frac{1}{\rho}P_-^N - \frac{1}{2}Q_-^N(\Lambda_u - \Lambda_s) & \frac{1}{\rho}P_+^N + \frac{1}{2}Q_+^N(\Lambda_u - \Lambda_s) \\ P_-^D & P_+^D \end{pmatrix}.
\end{aligned}$$

The last equation follows from $\Lambda_u(I + D_s) = \Lambda_u - \Lambda_s$ wich is implied by (3.43). From (3.30) we get with $z = \frac{1}{2}\rho h e^{i\theta}$, $\delta(\theta, z) = 2e^{i\theta}(1 + |z|^2)^{\frac{1}{2}}$ and the definition of $\Delta(z)$ in (3.8),(3.9)

$$\begin{aligned}
\Lambda_u - \Lambda_s &= \begin{cases} ((\rho h e^{2i\theta})A^{-1} + 4I)^{\frac{1}{2}} e^{i\theta} A^{-\frac{1}{2}}, & \text{if } \rho h > 0,\ |\theta| \le \frac{\pi}{4} + \frac{\delta}{3}, \\ \rho h e^{2i\theta} A^{-1}(1 + \frac{4}{(\rho h)^2} e^{-2i\theta} A)^{\frac{1}{2}}, & \text{if } \rho h > C_0 \end{cases} \\
&= \delta(\theta, z)\Delta(z).
\end{aligned}$$

With these notations the matrix $\mathcal{B}$ reads $\mathcal{B} = \mathcal{S}\mathcal{B}_s$ where

$$\mathcal{S} = \begin{pmatrix} -\delta(\theta, z) I_r & 0 \\ 0 & -I_{2m-r} \end{pmatrix}, \tag{3.64}$$

and

$$\mathcal{B}_s = \begin{pmatrix} \frac{2}{\rho\delta(\theta,z)}P_-^N + Q_-^N\Delta(z) & \frac{2}{\rho\delta(\theta,z)}P_+^N - Q_+^N\Delta(z) \\ P_-^D & P_+^D \end{pmatrix}.$$

From Hypothesis 3.3 and (3.13), (3.14) we obtain that

$$\hat{\mathcal{B}}_s = \begin{pmatrix} Q_-^N\Delta(z) & -Q_+^N\Delta(z) \\ P_-^D & P_+^D \end{pmatrix}$$

has a uniformly bounded inverse. From $c_1 \max(1, |z|) \le |\delta(\theta, z)| \le c_2 \max(1, |z|)$ we find

$$\frac{1}{|\delta(\theta, z)|} \le c \min\left(1, \frac{1}{\rho h}\right) \le c. \tag{3.65}$$

Therefore the difference $\|\mathcal{B}_s - \hat{\mathcal{B}}_s\|$ can be estimated by

$$\|\mathcal{B}_s - \hat{\mathcal{B}}_s\| \le \frac{2}{\rho|\delta(\theta, z)|}\left(\|P_-^N\| + \|P_+^N\|\right) \le \frac{c}{\rho},$$

which tends to zero as $\rho \to \infty$. Choosing C_0 in (3.13) large enough, we obtain $\|\mathcal{B}^{-1}\| \le C$ for some $C > 0$.

For the error term ΔR_ρ we get

$$\begin{aligned}
\Delta R_\rho(\xi_-, \xi_+) &= (B_-(\rho)\Phi(n_-, n_+) + \hat{B}_-\Phi(n_- + 1, n_+))T_+\xi_+ \\
&\quad + (B_+(\rho) + \hat{B}_+\hat{M})\Phi(n_+, n_-)T_-\xi_- \\
&= \left(\begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} TD^{(n_- - n_+)} + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} TD^{(n_- - n_+ + 1)}\right) \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix} \\
&\quad + \left(\begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} TD^{(n_+ - n_-)} + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} TD^{(n_+ - n_- + 1)}\right) \begin{pmatrix} \xi_- \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_s \end{pmatrix} D_s^{(n_+ - n_-)}\xi_+ + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_s \end{pmatrix} D_s^{(n_+ - n_- - 1)}\xi_+ \\
&\quad + \begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_u \end{pmatrix} D_s^{(n_+ - n_-)}\xi_- + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I \\ \Lambda_u \end{pmatrix} D_s^{(n_+ - n_- + 1)}\xi_- \\
&= \Delta\mathcal{B} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}
\end{aligned}$$

where

$$\Delta\mathcal{B} = \mathcal{B}\begin{pmatrix} 0 & D_s^{(n_+ - n_-)} \\ D_s^{(n_+ - n_-)} & 0 \end{pmatrix} = \mathcal{S}\mathcal{B}_s \begin{pmatrix} 0 & D_s^{(n_+ - n_-)} \\ D_s^{(n_+ - n_-)} & 0 \end{pmatrix}.$$

Here $\mathcal{S}$ denotes the scaling matrix defined in (3.64). Furthermore $\nu_s^{(n_+ - n_-)}$ vanishes as $n_+ - n_- \to \infty$ and

$$\|\mathcal{B}_s\| \le c(\frac{1}{\rho|\delta(\theta, z)|} + \|\Delta(z)\|) \le c(\frac{1}{\rho} + C) \le c$$

implies that $\Delta\mathcal{B}_s = \mathcal{S}^{-1}\Delta\mathcal{B}$ vanishes as $n_+ - n_- \to \infty$.

The right hand side of (3.63) can be rewritten as follows:

$$\begin{aligned}
F_\rho(\hat{g}) &= \begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} \hat{z}_{n_-}(\hat{g}) + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} \hat{z}_{n_- + 1}(\hat{g}) + \begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} \hat{z}_{n_+}(\hat{g}) \\
&\quad + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} (\hat{M}\hat{z}_{n_+}(\hat{g}) + \begin{pmatrix} h^2 I \\ \frac{h}{\rho}I \end{pmatrix} \hat{g}_{n_+}) \\
&= \begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_{n_-} \\ \hat{v}_{n_-} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_{n_- + 1} \\ \hat{v}_{n_- + 1} \end{pmatrix} + \begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_{n_+} \\ \hat{v}_{n_+} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} \gamma_u \\ \gamma_v \end{pmatrix} + \begin{pmatrix} h^2 I \\ \frac{h}{\rho}I \end{pmatrix} \hat{g}_{n_+}\right] \\
&= \begin{pmatrix} \frac{1}{\rho}P_-^N\hat{u}_{n_-} + \frac{1}{2}Q_-^N(\hat{v}_{n_-} + \hat{v}_{n_- + 1}) + \frac{1}{2}Q_+^N(\gamma_v + \frac{h}{\rho}\hat{g}_{n_+}) + \frac{1}{\rho}P_+^N\hat{u}_{n_+} \\ P_-^D\hat{u}_{n_-} + P_+^D\hat{u}_{n_+} \end{pmatrix}
\end{aligned}$$

where we used the notation $\hat{M}\hat{z}_{n_+}(\hat{g}) = (\gamma_u, \gamma_v)^T$. Using (3.56), (3.59), (3.55) we obtain

$$\|F_\rho(\hat{g})\|_{\text{vec}} \le c \begin{pmatrix} \frac{1}{\rho^2} + \frac{h}{\rho} \\ \frac{1}{\rho^2} \end{pmatrix} \|\hat{g}\|_\infty.$$

Then the scaled version of $F_\rho(\hat{g})$ can be estimated by

$$\left\| \begin{pmatrix} \frac{1}{\delta(\theta,z)} I_r & 0 \\ 0 & I_{2m-r} \end{pmatrix} F_\rho(\hat{g}) \right\| \le c\Big(\min(1, \frac{1}{\rho h})(\frac{1}{\rho^2} + \frac{h}{\rho}) + \frac{1}{\rho^2}\Big)\|\hat{g}\|_\infty \le \frac{c}{\rho^2}\|\hat{g}\|_\infty.$$

Equation (3.63) is equivalent to

$$(\mathcal{B}_s + \Delta\mathcal{B}_s)\begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho\delta(\theta,z)}\eta^N \\ \eta^D \end{pmatrix} + \begin{pmatrix} \frac{1}{\delta(\theta,z)} I_r & 0 \\ 0 & I_{2m-r} \end{pmatrix} F_\rho(\hat{g}),$$

thus we can estimate the solution (ξ_-, ξ_+) using (3.65) by

$$\|(\xi_-, \xi_+)\| \le c\big(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty\big). \tag{3.66}$$

The homogenous solution $z^{\mathrm{hom}} = (u^{\mathrm{hom}}, v^{\mathrm{hom}})$ can be estimated using (3.52) as follows: The estimates

$$\begin{aligned} \|\Phi(n,n_-)\rho_-\|_{\mathrm{vec}} = \|\Phi(n,n_-)T_-\xi_-\|_{\mathrm{vec}} &\le \begin{pmatrix} \nu_s \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n-n_--1}\|\xi_-\|, \\ \|\Phi(n,n_+)\rho_+\|_{\mathrm{vec}} = \|\Phi(n,n_+)T_+\xi_+\|_{\mathrm{vec}} &\le \begin{pmatrix} 1 \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n_+-n}\|\xi_+\| \end{aligned} \tag{3.67}$$

imply for all $n \in J$

$$\|u_n^{\mathrm{hom}}\| \le c(\nu_s^{n-n_-}\|\xi_-\| + \nu_s^{n_+-n}\|\xi_-\|) \le c(\|\xi_+\| + \|\xi_+\|) \tag{3.68}$$

and for $n \in \hat{J} = [n_- + 1, n_+]$

$$\|v_n^{\mathrm{hom}}\| \le c\frac{1-\nu_s}{\rho h}\big(\nu_s^{n-n_--1}\|\xi_-\| + \nu_s^{n_+-n}\|\xi_+\|\big) \le c(\|\xi_-\| + \|\xi_+\|). \tag{3.69}$$

From (3.38)–(3.40) and (3.26) we obtain

$$\|v_{n_-}^{\mathrm{hom}}\| \le c\frac{1-\nu_s}{\rho h}\big(\nu_s^{-1}\|\xi_-\| + \nu_s^{n_+-n_-}\|\xi_+\|\big) \le c(\max(1,\rho h)\|\xi_-\| + \|\xi_+\|). \tag{3.70}$$

The estimates (3.68) and (3.54) lead for $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$ defined in (3.47) for all $n \in J$ to

$$\begin{aligned} \|\tilde{u}_n\| \le \|u_n^{\mathrm{hom}}\| + \|\hat{z}\|_\infty &\le c\big(\|\xi_-\| + \|\xi_+\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty\big) \\ &\le c\big(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty\big) \end{aligned}$$

and for $n \in \hat{J} = [n_- + 1, n_+]$

$$\begin{aligned} \|\tilde{v}_n\| \le \|v_n^{\mathrm{hom}}\| + \|\hat{z}\|_\infty &\le c(\|\xi_-\| + \|\xi_+\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty) \\ &\le c\big(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty\big). \end{aligned}$$

Finally $\frac{1}{\rho h}(\nu_s^{-1} - 1) \le c\max(1, \rho h)$ implies with (3.70)

$$\|\tilde{v}_{n_-}\| \le \|v_{n_-}^{\mathrm{hom}}\| + \|\hat{z}\|_\infty \le c\max(1,\rho h)\big(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\infty\big)$$

and from

$$\left\|\hat{M} z_{n+}^{\text{hom}}\right\|_{\text{vec}} \leq c\Big(\begin{pmatrix}(\rho h)^2\nu_s^{n_+-n_-}\\(1-\nu_s)\nu_s^{n_+-n_--1}\end{pmatrix}\|\xi_-\| + \begin{pmatrix}(\rho h)^2\nu_s^{n_+-n_-}\\(1-\nu_s)\nu_s^{n_+-n_-}\end{pmatrix}\|\xi_+\|\Big)$$

and with $n_+ - n_- > 1$ end up with

$$\|\hat{M} z_{n+}^{\text{hom}}\| \leq c(\|\xi_-\| + \|\xi_+\|). \tag{3.71}$$

Together with (3.55) we obtain (3.75) for $\diamond = \infty$.

By (3.39),(3.40) we obtain for $\rho \in (C_0, \frac{C_0}{h}]$ the estimate $\frac{h}{1-\nu_s^2} < c$ as well as $\frac{h}{1-\nu_s^2} < h$ for $\rho h > C_0$ by (3.40). This leads to

$$\begin{aligned}\|u^{\text{hom}}\|^2_{\mathcal{L}_{2,h}} &\leq c\big(\sum_{n=n_-}^{n_+} h\nu_s^{2(n-n_-)}\|\xi_-\|^2 + \sum_{n=n_-}^{n_+} h\nu_s^{2(n_+-n)}\|\xi_+\|^2\big)\\ &\leq c\frac{h}{1-\nu_s^2}(\|\xi_-\|^2 + \|\xi_+\|^2) \leq c(\|\xi_-\|^2 + \|\xi_+\|^2).\end{aligned} \tag{3.72}$$

In the restricted intervall $\hat{J} = [n_- + 1, n_+]$ we obtain in the same way

$$\begin{aligned}\|v^{\text{hom}}_{|_{\hat{J}}}\|^2_{\mathcal{L}_{2,h}} &\leq c\big(\sum_{n=n_-+1}^{n_+} h\frac{(1-\nu_s)^2}{(\rho h)^2}\nu^{2(n-n_--1)}\|\xi_-\|^2 + \sum_{n=n_-+1}^{n_+} h\nu^{2(n_+-n)}\|\xi_+\|^2\big),\\ &\leq c\big(\|\xi_-\|^2 + \|\xi_+\|^2\big)\end{aligned} \tag{3.73}$$

and with (3.26) we arrive at

$$\begin{aligned}\|v^{\text{hom}}\|^2_{\mathcal{L}_{2,h}} &\leq ch\big(\frac{1-\nu_s}{(\rho h)^2\nu_s^2(1+\nu_s)}\|\xi_-\|^2 + \frac{1}{1-\nu_s^2}\|\xi_+\|^2\big)\\ &\leq c\big(\max(1,(\rho h)^2)\|\xi_-\|^2 + \|\xi_+\|^2\big).\end{aligned} \tag{3.74}$$

Using (3.54),(3.55), (3.72),(3.74) and (3.66) we obtain (3.61) with $\rho h < C_0$

$$\begin{aligned}\|\tilde{z}\|_{\mathcal{L}_{2,h}} &\leq \|\hat{z}\|_{\mathcal{L}_{2,h}} + \|z^{\text{hom}}\|_{\mathcal{L}_{2,h}} + \sqrt{h}(\|\hat{M} z_{n_+}^{\text{hom}}\| + \|\hat{M}\hat{z}_{n_+}\|)\\ &\leq c(\frac{1}{\rho^2}\|\hat{g}\| + \max(1,\rho h)\|\xi_-\| + \|\xi_+\| + (h^2 + \frac{h}{\rho} + \frac{1}{\rho^2})\|\hat{g}\|_{\mathcal{L}_{2,h}})\\ &\leq c\big(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_{\mathcal{L}_{2,h}}\big).\end{aligned}$$

In the same way (3.54),(3.72),(3.73) and (3.66) lead to (3.62). □

Remark 3.13 The restriction to $\hat{J}$ in (3.62) is necessary, since from (3.55),(3.70) and (3.71) we obtain for $s \in \Omega_\infty^h$ only

$$\|\tilde{z}\|_\diamond \leq c\max(1,(\rho h)^2)(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond). \tag{3.75}$$

From the above estimates the invertibility of (3.20),(3.18) now follows from a regular perturbation argument.

Lemma 3.14 *Let $A > 0$ be diagonalizable and assume Hypothesis 3.3 Then there exist $\epsilon, C_0, h_0, T > 0$, such that for s restricted by (3.13) or (3.14) and $h < h_0$, $\pm n_\pm h > T$ the following holds. For each $\hat{g} \in S_J(\mathbb{C}^m)$, there exists a unique solution $z \in S_{J_r}(\mathbb{C}^m)$ of (3.20), (3.18) which can be estimated for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ in the following way*

$$\|z\|_\diamond \leq c(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond), \quad \textit{for } s \in \Omega^h_{C_0} \tag{3.76}$$

$$\|z_{|\hat{J}}\|_\diamond \leq c(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond), \quad \textit{for } s \in \Omega^h_\infty, \ \hat{J} = [n_- + 1, n_+]. \tag{3.77}$$

Proof: Write (3.20) as

$$z_{n+1} - \hat{M}(s,\rho)z_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} E_n^{+^{-1}} \hat{g}_n + (M_n(s,\rho) - \hat{M}(s,\rho))z_n, \qquad n \in J$$

and define the space

$$S = \left\{(\hat{r},\hat{\eta}) \in S_{J_r}(\mathbb{C}^{2m}) \times \mathbb{R}^{2m} : \ \hat{r}_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \hat{g}_n, \ n \in J_r, \ \hat{g} \in S_{J_r}(\mathbb{C}^m)\right\}$$

equipped with the norm

$$\|(\hat{r},\hat{\eta})\|^*_\diamond = \frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|\hat{g}\|_\diamond, \quad \hat{\eta} = \begin{pmatrix} \frac{1}{\rho}\eta^N \\ \eta^D \end{pmatrix}, \eta^N \in \mathbb{R}^m, \eta^D \in \mathbb{R}^{2m-r}.$$

Then Lemma 3.12 implies that the operator $\hat{\Lambda}(\rho) : S_{J_r} \to S$ defined by

$$\hat{\Lambda} = \begin{pmatrix} \hat{L}(s,\rho) \\ R(\rho) \end{pmatrix}$$

where $\hat{L}(s,\rho)$, $R(\rho)$ are defined in (3.24), (3.18), is nonsingular for $s \in \Omega^h_{C_0} \cup \Omega^h_\infty$ with a uniform bound for the inverse for $s \in \Omega^h_{C_0}$. Using (3.22), (3.25) we obtain for $z = (u, v)$

$$(M_n(s,\rho) - \hat{M}(s,\rho))z_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \Big[(s(E_n^{+^{-1}} - A^{-1}) - C_n)u_n + (\frac{\rho}{h}(E_n^{+^{-1}}E_n^- - I))v_n\Big].$$

Combinining this with the error estimate

$$\frac{1}{\rho^2}\|(s(E_n^{+^{-1}} - A^{-1}) - C_n)u_n + (\frac{\rho}{h}(E_n^{+^{-1}}E_n^- - I))v_n\| \leq c(h + \frac{1}{\rho^2} + \frac{1}{\rho})\|z_n\|$$

implies for $\rho > C_0$

$$\left\|\begin{pmatrix} \tilde{L}(s,\rho) - \hat{L}(s,\rho) \\ 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\eta} \end{pmatrix}\right\|^*_\diamond \leq c(h + \frac{1}{\rho})\|\hat{g}\|_\diamond.$$

Taking h small and ρ large and using $\|E_n^{+^{-1}}\| \leq c$ we find that the system (3.20), (3.18) has a unique solution for $s \in \Omega^h_{C_0}$ which can be estimated by (3.76). In a similar way we obtain the existence of a unique solution of (3.20),(3.18) for $s \in \Omega^h_\infty$ which satisfies the estimate (3.77). □

The estimate (3.15),(3.16) now follows for $\diamond \in \{\mathcal{L}_{2,h}, \infty\}$ directly with $\|\delta_- u\|_\diamond = \|\delta_+ u\|_\diamond$ which implies

$$\|u\|^2_\diamond + \frac{1}{\rho^2}\|\delta_+ u\|^2_\diamond \leq c(\|u\|^2_\diamond + \|v\|^2_\diamond).$$

3.1.3 Eigenvalues of finite multiplicity

The aim of this section is an approximation theorem for simple, isolated eigenvalues.

Let $(\bar{u}, \bar{\lambda})$ be the solution of (2.1) and $\phi \in \mathcal{H}^2(\mathbb{R}, \mathbb{C}^m)$ an eigenfunction of Λ which corresponds to the simple eigenvalue σ, i.e. $(u, s) = (\phi, \sigma)$ solves

$$Au'' + B(\cdot)u' + (C(\cdot) - sI)u = 0, \quad x \in \mathbb{R}.$$

The corresponding discrete boundary value problem on the grid $\mathbb{G}_{J,h,x_0}$ reads

$$0 = A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + (C_n - sI)u_n, \quad n \in J \tag{3.78}$$

with homogenous boundary conditions

$$0 = P_- u_{n_-} + Q_- \delta_0 u_{n_-} + P_+ u_{n_+} + Q_+ \delta_0 u_{n_+} \tag{3.79}$$

and a linear phase condition

$$1 = h \sum_{n=n_-}^{n_+} \hat{u}_n^H u_n = \langle \hat{u}_{|_J}, u \rangle_h \tag{3.80}$$

where $\hat{u} \in \mathcal{E}_\rho(\mathbb{R} \to \mathbb{C}^m)$ is a given normalizing function which satisfies $|\langle \hat{u}, \phi \rangle| > 0$ as well as $\langle \hat{u}, \phi \rangle = 1$.

Here we can drop the eigenvalue condition (EC) and consider unstable eigenvalues as well.

Theorem 3.15 *Consider the boundary value problem (3.78), (3.79) and assume, that for $P_\pm$, $Q_\pm$ the solvability condition (3.3) holds with $s = \sigma$.*

Then there exist $K > 0$, $\rho > 0$, $T > 0$, $h_0 > 0$, such that for $h < h_0$ and $\pm h n_\pm > T$ there exists a unique solution $(\tilde{v}, \tilde{s})$ of the boundary value problem (3.78)-(3.80) in a neigborhood $B_\rho(\phi, \sigma) := \{(v, s) \in S_\mathbb{Z}(\mathbb{C}^m) \times \mathbb{C} : \|\phi_{|_J} - v\|_\infty + |\sigma - s| < \rho\}$, which satifies for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ the following estimate

$$\|\phi_{|_J} - \tilde{v}\|_{2,\diamond} + |\sigma - \tilde{s}| \le K(h^2 + \mathrm{e}^{-\alpha T}). \tag{3.81}$$

Proof: Similar to the proof of Theorem 2.21 we apply the fixed point Theorem A.3 to the operator $F : S_{J_e}(\mathbb{C}^m) \times \mathbb{C} \to S_J(\mathbb{C}^m) \times \mathbb{C}^{2m} \times \mathbb{C}$

$$F(u, s) = \begin{pmatrix} (A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + (C_n - sI)u_n)_{n \in J} \\ P_- u_{n_-} + Q_- \delta_0 u_{n_-} + P_+ u_{n_+} + Q_+ \delta_0 u_{n_+} \\ \langle \hat{u}, u \rangle_h - 1 \end{pmatrix}.$$

Therefore we have to discuss for given $(\hat{g}, \eta, \omega) \in S_J(\mathbb{C}^m) \times \mathbb{C}^{2m} \times \mathbb{C}$ solutions of the equation

$$DF(\phi_{|_J}, \sigma)(u, \lambda) = (\hat{g}, \eta, \omega), \tag{3.82}$$

where the derivative of F at $(\phi_{|_J}, \sigma) \in S_J(\mathbb{C}^m) \times \mathbb{C}$ reads

$$DF(\phi_{|_J}, \sigma)(u, \lambda) = \begin{pmatrix} (A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + (C_n - \sigma I)u_n - \phi_n \lambda)_{n \in J} \\ P_- u_{n_-} + Q_- \delta_0 u_{n_-} + P_+ u_{n_+} + Q_+ \delta_0 u_{n_+} \\ \langle \hat{u}, u \rangle_h \end{pmatrix}.$$

By transformation of (3.82) to first order using $z_n = (u_n, \delta_- u_n) = (u_n, v_n)$ we obtain the equivalent equation for the operator $\tilde{\Lambda} : S_{J_r}(\mathbb{C}^{2m}) \times \mathbb{C} \to S_J(\mathbb{C}^{2m}) \times \mathbb{C}^{2m} \times \mathbb{C}$

$$\tilde{\Lambda}(z,\lambda) = (\hat{r}, \eta, \omega), \tag{3.83}$$

where $\bar{z} = (\phi_{|J}, \delta_- \phi_{|J})$ and

$$\tilde{\Lambda}(z,\lambda) = \begin{pmatrix} \tilde{L}(\bar{z},\sigma)(z,\lambda) \\ P_- u_{n_-} + Q_- w_{n_-} + P_+ u_{n_+} + Q_+ w_{n_+} \\ \hat{\Pi}(z) \end{pmatrix}$$

with

$$w_n = (\delta_0 u)_n = \frac{1}{2}(v_{n+1} + v_n), \qquad \hat{r}_n = \begin{pmatrix} 0 \\ h\hat{g}_n \end{pmatrix}, \qquad \hat{\Pi}(z) = \langle \hat{u}, u \rangle_h$$

and

$$\tilde{L}(\bar{z},\sigma)(z,\lambda) = (N_n z_{n+1} - K_n(\sigma) z_n - V_n(\bar{z})\lambda)_{n \in \hat{J}} \tag{3.84}$$

where

$$N_n = \begin{pmatrix} I & -hI \\ 0 & E_n^+ \end{pmatrix}, \quad K_n(s) = \begin{pmatrix} I & 0 \\ h(sI - C_n) & E_n^- \end{pmatrix}, \quad W_n(z) = \begin{pmatrix} 0 \\ hu_n \end{pmatrix}$$

and $E_n^\pm$ are defined in (3.5).

As before we compare this to a corresponding system

$$\hat{\Lambda}_i(z,\lambda) = \begin{pmatrix} (\hat{N} z_{n+1} - \hat{K}_n z_n - \hat{W}_n \lambda)_{n \in J} \\ (P_- \; Q_-) z_{n_-} + (P_+ \; Q_+) z_{n_+} \\ \hat{\Pi}(z) \end{pmatrix}$$

where

$$\hat{K}_n = \begin{pmatrix} I & hI \\ h(\sigma I - \hat{C}_n) & A - h\hat{B}_n \end{pmatrix}, \quad \hat{W}_n = \begin{pmatrix} 0 \\ h\phi_n \end{pmatrix}$$

and $\hat{N}$, $\hat{C}_n, \hat{B}_n$ are defined in (2.68). As in the previous section the estimates

$$\|N_n - \hat{N}\| \le Ch$$

and

$$\|\hat{K}_n - K_n + N_n - \hat{N}\| \le Ch\,(h^2 + \mathrm{e}^{-\alpha T})$$

hold. With the equality $\hat{W}_n = W_n(\bar{z})$ this leads to

$$\|(\tilde{\Lambda} - \Lambda_i)(z,\lambda)\|_\infty^* \le \varrho(h,T)(\|z\|_{1,\infty} + |\lambda|)$$

where $\lim_{h\to 0, T\to\infty} \varrho(h,T) = 0$. The equation (3.83) is equivalent to

$$\begin{aligned} z_{n+1} - \hat{M}_n z_n - \hat{N}^{-1}\hat{W}_n \lambda &= \hat{N}^{-1}\hat{r}_n, \quad n \in J \\ (P_- \; Q_-) z_{n_-} + (P_+ \; Q_+) z_{n_+} &= \eta \\ \hat{\Pi}(z) &= \omega \end{aligned}$$

where $\hat{M}_n = S(x_{n+1}, x_n)$ and S is the solution operator corresponding to the linear differential operator L_σ given by (cf. (A.20)).

$$L_\sigma z = z' - M(\cdot,\sigma) z, \quad \text{with} \quad M(x,\sigma) = \begin{pmatrix} 0 & I \\ A^{-1}(\sigma I - C(x)) & -A^{-1}B(x) \end{pmatrix}$$

The spectral condition (SC) implies that these operators have exponential dichotomies on $\mathbb{R}^{\pm}$. From the simplicity of the eigenvalue σ follows $\mathcal{N}(\Lambda - \sigma I) = \operatorname{span}\{\phi\}$. As in the proof of Theorem 2.21, this implies the nondegeneracy Hypothesis 2.13. By the definition of $\hat{\Pi}$ (cf. (3.80)) and $|\langle \hat{u}, \phi\rangle| > 0$ we obtain directly that Hypothesis 2.12 is satisfied.

Now Lemma 2.14 yields the existence of a solution (v, s) of (3.82) which can be estimated by (2.38). As in the proof of Theorem 2.21 this implies that $DF(\phi_{|_J}, \sigma))$ is invertible as well with

$$\|DF(\phi_{|_J}, \sigma)(r, \eta, \omega)\|_{2,\infty} \le c(\|g\|_\infty + \|\eta\| + |\omega|).$$

Using the same arguments as in the proof of Theorem 3.15 we arrive at (3.81). □

3.2 Essential spectrum

In this section we state some results about the approximation of the essential spectrum for the continuous and the discrete case.

In the continuous case, the essential spectrum can be controlled by the constant coefficient operators $L_\pm$. For the corresponding problem on a finite inverval, the choice of boundary conditions determines which sort of spectrum is approximated for increasing intervals $J \to \mathbb{R}$. As has been shown in [50] (see [51] for a short overview over the results), for periodic boundary conditions the essential spectrum is approximated whereas separated boundary conditions lead to the approximaton of the so called absolute spectrum.

3.2.1 Influence of discretization

Similar to (1.2) in the continuous case, the essential spectrum is determined from the constant coefficient operators obtained by letting $n \to \pm\infty$ in the coefficient. This is a general result for discrete operators which has been shown in [10] for the scalar case. We state a version of Theorem 4.3 in [10] where we use the result of Corollary 4.10 in [10] for $s_1, s_2 > 0$.

Theorem 3.16 *Let $\Lambda^h : S_{\mathbb{Z}}(\mathbb{R}) \to S_{\mathbb{Z}}(\mathbb{R})$ be given by*

$$(\Lambda^h u)_n = \sum_{k=-s_1}^{s_2} \alpha_n^k u_n,$$

where $\alpha_n^k \in \mathbb{R}$ for $k = 0, 1, ..., s_1 + s_2$, $s_1, s_2 > 0$, $n \in \mathbb{Z}$ and

$$\lim_{n\to\pm\infty} \alpha_n^k = \alpha_\pm^k.$$

Define the curves

$$\Sigma_\pm = \left\{ s \in \mathbb{C} \,:\, s = \sum_{k=-s_1}^{s_2} \mathrm{e}^{-ik\omega} \alpha_\pm^k, \quad \omega \in \mathbb{R} \right\}$$

and denote by $I_\pm$ the interior of $\Sigma_\pm$ (i.e. $\mathbb{C} \setminus I$ is the open connected component of $\mathbb{C} \setminus \Sigma_- \cup \Sigma_+$ which is unbounded).

Then the essential spectrum $\sigma_{\text{ess}}(\Lambda^h)$ *satisfies*

$$\Sigma_- \cup \Sigma_+ \subset \sigma_{\text{ess}}(\Lambda^h) \subset \Sigma_- \cup \Sigma_+ \cup I_- \cup I_+.$$

Note, that the proof proceeds along the same lines as the proof of Theorem 1.2 (Thm. A.2 in [23]). Consider the discrete operators $\Lambda^h : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ given by

$$(\Lambda^h u)_n = a(\delta_-\delta_+ u)_n + b_n(\delta_0 u)_n + c_n u_n \tag{3.85}$$

where $a \in \mathbb{R}, b, c \in S_{\mathbb{Z}}(\mathbb{R})$. Applying Theorem 3.16 to Λ^h we obtain with $s_1 = s_2 = 1$ and

$$\alpha_n^{\pm 1} = \frac{1}{h^2}a \pm \frac{1}{h}b_n, \quad \alpha_n^0 = -\frac{2}{h^2}a + b_n,$$

the following Corollary.

Corollary 3.17 *Consider the operators* Λ^h *in* $\mathcal{L}_{2,h}$ *or* $S_{\mathbb{Z}}$ *given by* (3.85) *and define*

$$\Sigma_\pm = \left\{ s \in \mathbb{C} : \ s = \tfrac{2}{h^2}(\cos(\omega) - 1)a + \tfrac{i}{h}\sin(\omega)b_\pm + c_\pm, \quad \omega \in \mathbb{R} \right\}$$

Then the essential spectrum of Λ^h *satisfies*

$$\Sigma_- \cup \Sigma_+ \subset \sigma_{\text{ess}}(\Lambda^h) \subset \Sigma_- \cup \Sigma_+ \cup \{\textit{interior of } \Sigma_-\} \cup \{\textit{interior of } \Sigma_+\}$$

.

Similar to Theorem 1.2 this result can be adapted to the matrix case.

Lemma 3.18 *Consider the operators* Λ^h *in* $\mathcal{L}_{2,h}$ *or* $S_{\mathbb{Z}}$ *defined by*

$$(\Lambda^h u)_n = A(\delta_-\delta_+ u)_n + B_n(\delta_0 u)_n + C_n u_n, \quad n \in \mathbb{Z}, \tag{3.86}$$

y where B_n*,* C_n *are given in* (2.67) *and define*

$$\Sigma_\pm = \left\{ s \in \mathbb{C} : \ \det\left(\tfrac{2}{h^2}(\cos(\omega) - 1)A + \tfrac{i}{h}\sin(\omega)B_\pm + C_\pm - sI \right) = 0, \quad \omega \in \mathbb{R} \right\}.$$

Denote the interior of $\mathbb{C} \setminus \Sigma_- \cup \Sigma_+$ *by* I *(i.e.* $\mathbb{C} \setminus I$ *is the open connected component of* $\mathbb{C} \setminus \Sigma_- \cup \Sigma_+$ *which is unbounded). Then the essential spectrum of* Λ^h *satisfies*

$$\Sigma_- \cup \Sigma_+ \subset \sigma_{\text{ess}}(\Lambda^h) \subset I.$$

3.2.2 Influence of boundary conditions in the continuous case

The numerical computations in Chapter 5 suggest that the eigenvalues of the restriction ${\Lambda^h}_{|J}$ of the discrete operator Λ^h on the whole lattice $\mathbb{Z}$ approximate in a certain sense the essential spectrum $\sigma_{\text{ess}}(\Lambda^h)$ as the interval size tends to infinity. This is observed for periodic boundary conditions whereas for Dirichlet or Neumann boundary conditions the eigenvalues of ${\Lambda^h}_{|J}$ change dramatically.

A first step to understand this phenomenon is to recall the results concerning the influence of the boundary conditions on the spectrum of the operator in the continuous case. These have been given in [50], where it has been clarified in which way the choice of boundary conditions influences the essential spectrum. For periodic boundary conditions the essential spectrum is approximated, whereas for separated boundary conditions, such as Dirichlet or Neumann conditions, the so called absolute spectrum is approximated.

Consider the restriction of the operator Λ which has been defined in (1.7) to an interval $J = [-T, T]$ given by

$$\Lambda_J v = \begin{pmatrix} Av'' + B(\cdot)v' + C(\cdot)v, x \in J \\ (P_- Q_-) \begin{pmatrix} v(x_-) \\ v'(x_-) \end{pmatrix} + (P_+ Q_+) \begin{pmatrix} v(x_+) \\ v'(x_+) \end{pmatrix} \end{pmatrix}. \tag{3.87}$$

In order to state the corresponding theorems of [50], we need some more definitions:

Definition 3.19 (Absolute spectrum) *Denote the $2m$ solutions of the quadratic eigenvalue problems (1.6) at $s = 0$ by $\nu_i^{\pm}$, $i = 1, \ldots, 2m$, i.e. solutions of*

$$\det(\lambda^2 A + \lambda B_{\pm} + C_{\pm} - sI) = 0$$

Sort them by real part: $\text{Re}\,(\nu_1^{\pm}) \leq \ldots \leq \text{Re}\,(\nu_{2m}^{\pm})$*. Then each $s \in \mathbb{C}$ where* $\text{Re}\,(\nu_m^{\pm}) = \text{Re}\,(\nu_{m+1}^{\pm})$ *belongs to the absolute spectrum σ_{abs} of L.*

The absolute spectrum plays a role in the case of separated boundary conditions.

Definition 3.20 (Separated boundary conditions) *Boundary conditions of the form*

$$B_- z_{n_-} + B_+ z_{n_+} = \eta$$

are called separated if

$$B_{\pm} = \begin{pmatrix} B_{\pm}^{I} \\ B_{\pm}^{II} \end{pmatrix} \in \mathbb{R}^{2m,2m}, \text{ and } B_-^{I} = B_+^{II} = 0 \in \mathbb{R}^{m,m}.$$

Neumann and Dirichlet boundary conditions are separated boundary conditions.

The definition of essential spectrum used in [51], [50] differs slightly from our definition: Instead of considering the spectrum of Λ directly, they use the so called B-spectrum of the family of corresponding first order operators

$$L(s)z = z' + M(\cdot, s)z, \quad x \in \mathbb{R},\ M : \mathbb{R} \times \mathbb{C} \to \mathbb{C}^{2m,2m} \tag{3.88}$$

with boundary conditions

$$B_- z(x_-) + B_+ z(x_+) = \eta, \quad B_{\pm} = (P_{\pm}\ Q_{\pm}) \in \mathbb{R}^{2m,2m}$$

which is given by (see Definition 3.2 in [50]):

Definition 3.21 *The spectrum Σ of the family of operators $\{L(s)\}_{s \in \mathbb{C}}$ consists of those points $s \in \mathbb{C}$ where $L(s) : \mathcal{H}^2 \to \mathcal{L}_2$ is not invertible.*

The point spectrum Σ_{pt} consists of those $s \in \Sigma$ for which $L(s)$ is a Fredholm operator of index zero.

The essential spectrum is defined as $\Sigma_{\text{ess}} = \Sigma \setminus \Sigma_{\text{pt}}$.

One can show that $s \notin \Sigma_{\text{ess}}$ if, and only if the operator $L(s)$ has exponential dichtomies on $\mathbb{R}^{\pm}$. As has been discussed in [51], [50], the spectrum Σ of the family $L(s)$ coincides with the spectrum σ of Λ. Note, that the definition of Σ_{pt} is slightly different from σ_{pt}: By definition Σ_{pt} consists of all points where $L(s)$ is Fredholm of index 0 whereas σ_{pt} consists of all isolated eigenvalues of Λ of finite multiplicity. Therefore Σ_{ess} and σ_{ess} are different as well (cf. Definition A.1). This difference is removed by Hypothesis 3 in [50] which requires that all eigenvalues in $\mathbb{C} \setminus \Sigma_{\text{ess}}$ are isolated eigenvalues of finite multiplicity. This will be our standing hypothesis for the rest of the section. In order to formulate the convergence results one more definition is necessary for the family of first order operators corresponding to Λ_J given by

$$L_J(s)z = z' + M(\cdot, s)z, \quad x \in J,\ M : J \times \mathbb{C} \to \mathbb{C}^{2m,2m}. \tag{3.89}$$

The extrapolated essential spectral set is introduced in [50], Definition 5.7 as follows.

Definition 3.22 (Extrapolated essential spectral set) *The extrapolated essential spectral set of a family of operators $\{L_J(s)\}$ is defined as the complement of all points $s \in \mathbb{C}$ for which exist a neighbourhood $U(s)$, a minimal interval length $T_0 > 0$ and a maximal order $l \in \mathbb{N}$ such that $\{L_J(s)\}$ has eigenvalues at most of order l in $U(s)$ for $T \geq T_0$, or in short notation:*
$\Sigma_{\text{ext}} = \mathbb{C} \setminus \{s :\ \exists U(s) \subset \mathbb{C},\ l \in \mathbb{N},\ T_0 > 0$ *such that* $\{L_J(s)\}$ *has eigenvalues at most of order* l *in* $U(s)$ *for* $T \geq T_0.\}$

The definition in [50] uses the Evans function [2], but as shown in Lemma 4.2 in [50] the zeros of the Evans function correspond to the eigenvalues of $L_J(s)$. A more heuristic description of Σ_{ess} is the following:
The extrapolated essential spectral set Σ_{ext} consists of those points in $\mathbb{C}$ where infinitely many eigenvalues of L_J accumulate as the interval size tends to infinity.

The main theorems in [50] now state the following under some additional hypotheses, which are satisfied for Λ:

The eigenvalues of the restriction of Λ to the finite interval J with periodic boundary conditions accumulate at the essential spectrum of Λ as T tends to infinity (Proposition 4 in [50]), i.e.$\Sigma_{\text{ext}}^{\text{per}} \subset \Sigma_{\text{ess}}$.

If one additional reducibility condition (Hypothesis 6 in [50]) is satisfied, then equality holds, i.e. $\Sigma_{\text{ext}}^{\text{per}} = \Sigma_{\text{ess}}$.

An analogous result, Proposition 5 in [50], holds for separated boundary conditions. The eigenvalues of the restriction of Λ on the finite interval J with separated boundary conditions accumulate at the absolute spectrum of Λ as T tends to infinity, i.e. $\Sigma_{\text{ext}}^{\text{sep}} \subset \Sigma_{\text{abs}}$.

If again a reducibility condition (Hypothesis 8 in [50]) holds, then Theorem 5 in [50] states $\Sigma_{\text{ext}}^{\text{sep}} = \Sigma_{\text{abs}}$.

These results, which clarify the influence of the boundary conditions can be observed in the numerical computations. In the following we give an example how the essential and the absolute spectrum can be calculated for a given PDE and discuss later how the above theorems should be transferred to the discrete case. This discussion will be mostly heuristic, but helps to understand some of the spectral pictures in Chapter 5.

Example 3.23 *The essential spectrum of a scalar linear operator*

$$\Lambda u = u'' + \bar{\lambda} u' + f'(\bar{u})u, \quad f : \mathbb{R} \to \mathbb{R}$$

is bounded by a parabola in the left half plane: Theorem 1.2 implies that that the essential spectrum is located by the following curves parametrized by $\omega \in \mathbb{R}$

$$s_{\pm}(\omega) = -\omega^2 + i\bar{\lambda}\omega + f'(u_{\pm}). \tag{3.90}$$

which are parabolas over the imaginary axis. The essential spectrum σ_{ess} *of* Λ *lies to the left of the rightmost of the curves* s_-, s_+. *Hence the spectral gap between zero and the essential spectrum is at least* $\min(f'(u_-), f'(u_+))$.

For the Nagumo equation, where $f(u) = u(1-u)(u-\mu), \mu \in (0, \frac{1}{2})$ *(see (5.15) in Chapter 5) we obtain with‘* $f'(u_-) = -\mu$ *and* $f'(u_+) = \mu - 1$ *that* σ_{ess} *is bounded by* s_- *and the spectral gap in this case is* $\frac{1}{4}$ *for* $\mu = \frac{1}{4}$. *This is the parameter value used in the numerical computations.*

The linearized operators of first order $L(s)$ *read*

$$L(s)z = z' - M(\cdot, s)z, \quad \text{where } M(x,s) = \begin{pmatrix} 0 & 1 \\ s - f'(\bar{u}(x)) & -\bar{\lambda} \end{pmatrix}$$

and the eigenvalues of $M^{\pm}(s) = \lim_{x\to\infty} M(\cdot, s)$ *are given by*

$$\nu_{\pm}^{+}(s) = -\frac{\bar{\lambda}}{2} \pm \sqrt{\frac{\bar{\lambda}^2}{4} + s - f'(u_+)}, \quad \nu_{\pm}^{-}(s) = -\frac{\bar{\lambda}}{2} \pm \sqrt{\frac{\bar{\lambda}^2}{4} + s - f'(u_-)}.$$

The absolute spectrum which has been defined in Definition 3.19, consists of points $s \in \mathbb{C}$ *where* $\operatorname{Re} \nu_-^+ = \operatorname{Re} \nu_+^+$ *and* $\operatorname{Re} \nu_-^- = \operatorname{Re} \nu_+^-$, *i.e. were*

$$\frac{\bar{\lambda}^2}{4} + s - f'(u_{\pm}) < 0.$$

In the Nagumo case, this gives with $\bar{\lambda} = -\frac{\sqrt{2}}{4}$

$$\sigma_{abs} = (-\infty, \max(f'(u_-), f'(u_+)) - \frac{\bar{\lambda}^2}{4}].$$

Thus from $\bar{\lambda} \neq 0$ *the essential and the absolute spectrum differ. For the Nagumo system with* $\mu = \frac{1}{4}$ *this reads* $\sigma_{abs} = (-\infty, -\frac{9}{32}]$.

Corollary 3.17 yields that the essential spectrum of the discrete operator on the whole line is enclosed by shifted ellipses with semi-major axis of size $\frac{2}{h^2}$ *and semi-minor axis of size* $\frac{\bar{\lambda}}{h}$. *These are parametrized by*

$$\sigma_-(\omega) = \frac{2}{h^2}(\cos(\omega) - 1) + \bar{\lambda}\frac{i}{h}\sin(\omega) - \mu$$
$$\sigma_+(\omega) = \frac{2}{h^2}(\cos(\omega) - 1) + \bar{\lambda}\frac{i}{h}\sin(\omega) + \mu - 1.$$

The observations in the previous section lead to the following assumption (which we will not formulate as a Theorem, since we have no proof)

The eigenvalues of the linear discrete operators $\Lambda_J^h : S_{J_e}(\mathbb{R}^m) \to S_J(\mathbb{R})$ given by

$$\Lambda_J^h u = (A(\delta_+\delta_- u)_n + B_n(\delta_0 u)_n + C_n u_n)_{n\in J}$$

subject to periodic boundary conditions accumulate at the essential spectrum of the operator Λ^h defined on the whole lattice $\mathbb{Z}$ (see (3.86)).

Chapter 4

Stability of the discretized system

In this Chapter we analyze stability of traveling wave solutions for a discretized version of the frozen system from Chapter 1 (cf. equation (4.2)). In particular, we show asymptotic stability of the steady state that has been shown to exist in Chapter 2. This is an overall justification of the freezing method and is in accordance with the numerical results in Chapter 5.

Here we have to take into account the additional boundary conditions which constitute additional algebraic conditions besides the phase condition. Thus we cannot follow the lines of Chapter 1 directly, rather the Dirichlet part of the boundary conditions and the phase condition are both used to reduce the DAE to an ODE. We transform the system with equilibrium $(\tilde{u}, \tilde{\lambda})$, to a semilinear DAE with equilibrium $u \equiv 0, \lambda = 0$. The solution of this equation can be estimated (uniformly in h and J) using the solution of a reduced ODE. As in Chapter 1 we obtain exponential estimates for the solution operator of the corresponding linear equation using its integral representation and resolvent estimates which follow from the resolvent estimates of the previous chapter.

4.1 The nonlinear time dependent system

Consider the spatial discretization of the time dependent PDE

$$u_t = Au_{xx} + \lambda u_x + f(u, u_x) \tag{4.1}$$

on the grid $\mathbb{G}_{J,h,x_0}$ with finite differences, given by

$$u_n' = A(\delta_+\delta_- u)_n + \lambda(\delta_0 u)_n + f(u_n, \delta_0 u_n), \quad n \in J,\ t > 0 \tag{4.2}$$

$$\eta = Ru \tag{4.3}$$

$$0 = \langle \delta_0 \hat{u}, u_{|_J} - \hat{u}_{|_J} \rangle_h. \tag{4.4}$$

Here $\hat{u} \in \mathcal{E}_\alpha(\mathbb{R}, \mathbb{R}^m)$ is a given reference function which satisfies Hypothesis 2.18. As in Chapter 2, the boundary conditions are assumed to be linear, i.e. $R : S_{J_e}(\mathbb{R}^m) \to \mathbb{R}^{2m}$ reads

$$Ru = P_- u_{n_-} + Q_-(\delta_0 u)_{n_-} + P_+ u_{n_+} + Q_+(\delta_0 u_{n_+}), \quad P_\pm, Q_\pm \in \mathbb{R}^{2m,m}. \tag{4.5}$$

We also need Hypothesis 3.3 for the boundary conditions, since we use the resolvent estimates of Chapter 3.

Consider a general DAE of the form

$$\begin{aligned} (\pi u)' &= f_{\text{diff}}(u,\lambda), \quad u(0)=u^0, \lambda(0)=\lambda^0 \\ 0 &= f_{\text{alg}}(u,\lambda) \end{aligned} \tag{4.6}$$

where $f_{\text{diff}} : S_{J_e}(\mathbb{R}^m) \times \mathbb{R} \to S_J(\mathbb{R}^m)$, $f_{\text{alg}} : S_{J_e}(\mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}^{2m+1}$, and $\pi : S_{J_e}(\mathbb{R}^m) \to S_J(\mathbb{R}^m)$ denotes the restriction to J defined by

$$\pi : \ (u_{n_- -1}, \dots, u_{n_+ +1}) \mapsto (u_{n_-}, \dots, u_{n_+}). \tag{4.7}$$

The proper notion of a solution of (4.6) is the following (cf. Definition 1.11).

Definition 4.1 *A function $(u,\lambda) : [0,\tau) \to S_{J_e}(\mathbb{R}^m) \times \mathbb{R}$ is called a solution of (4.6) in $(0,\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$ if*

1. *$f_{\text{diff}}(u(\cdot),\lambda(\cdot)) : [0,\tau) \to S_J$ is continuous*
2. *$(u,\lambda) : [0,\tau) \to S_{J_e}(\mathbb{R}^m) \times \mathbb{R}$ is continuous*
3. *$(\pi u)'(t)$ exists, $(\pi u)'(t) = f_{\text{diff}}(u(t),\lambda(t)) \in S_J(\mathbb{R}^m)$ for $t \in (0,\tau)$, and $(u(0),\lambda(0)) = (u^0,\lambda^0)$*
4. *$f_{\text{alg}}(u(t),\lambda(t)) = 0 \ \forall t \in [0,\tau)$.*

Assume that $(\tilde{u},\tilde{\lambda})$ is an equilibrium of (4.6), i.e. $0 = f_{\text{diff}}(\tilde{u},\tilde{\lambda})$,$0 = f_{\text{alg}}(\tilde{u},\tilde{\lambda})$. As in Section 1.2 we are interested in the stability of $(\tilde{u},\tilde{\lambda})$ as a stationary solution of (4.2)–(4.4). This system is a differential algebraic equation (DAE) of differentiation index 2 [22] (cf. Section 5.1). In particular, initial values have to satisfy additional consistency conditions, which are defined as follows: Denoting the boundary conditions which constitute the Neumann and the Dirichlet part by

$$\begin{aligned} R^N u &= P_-^N u_{n_-} + Q_-^N \delta_0 u_{n_-} + P_+^N u_{n_+} + Q_+^N \delta_0 u_{n_+}, \\ R^D u &= P_-^D u_{n_-} + P_+^D u_{n_+}. \end{aligned} \tag{4.8}$$

equation (4.3) is split into one part that does not depend on the external variables $u_{n_- -1}, u_{n_+ +1}$ and one part depending on $u_{|J}$. Then (4.3) reads

$$R^N u = \eta^N, \tag{4.9}$$

$$R^D \pi u = \eta^D. \tag{4.10}$$

The initial values u^0, λ^0 are then called consistent if they solve the algebraic constraints (4.3),(4.4) as well as the equations which are obtained by differentiating (4.10),(4.4) w.r.t. time t and inserting (4.2):

$$R^D(A\delta_+\delta_- u + \lambda\,\delta_0 u + f(u,\delta_0 u)) = 0 \tag{4.11}$$

$$\langle \tilde{\psi}, A\delta_+\delta_- u + \lambda\,\delta_0 u + f(u,\delta_0 u)\rangle_h = 0. \tag{4.12}$$

The main result of this chapter is the following stability theorem, which is the discrete analog of Theorem 1.13. Recall that the system (4.2)–(4.4) has a stationary solution $(\tilde{u},\tilde{\lambda})$ close to the original wave by Theorem 2.21.

Theorem 4.2 *Assume that the linear operator Λ defined in (1.7) and the nonlinearity f satisfy the conditions of Theorem 1.13. Assume further that Hypotheses 3.3, 3.1, 2.18 hold.*

Then there exist $h_0 > 0, T > 0$ such that for $h < h_0$, $\mp hn_\pm > T$ the stationary solution $(\tilde{u}, \tilde{\lambda}) \in S_J(\mathbb{R}^m) \times \mathbb{R}$ of (4.2)–(4.4) is asymptotically stable.

More precisely, there exist $K, \nu, \rho, h_0, T > 0$ such that for $h < h_0$, $\mp hn_\pm > T$ the following statements hold for $\diamond = \infty$ and also for $\diamond = \mathcal{L}_{2,h}$ if additonally $\mathrm{e}^{-\alpha T} < c\sqrt{h}$ for some $c > 0$, where α denotes the constant in Hypothesis 2.18:

For each consistent $(u^0, \lambda^0) \in S_{J_e}(\mathbb{R}^m) \times \mathbb{R}$ (i.e. (4.3), (4.4), (4.11), (4.12) are satisfied) with $\|u^0 - \tilde{u}\|_{1,\diamond} + |\lambda^0 - \tilde{\lambda}| \le \rho$, there exists a unique solution (u, λ) of (4.2)–(4.4) with initial condition $(u(0), \lambda(0)) = (u^0, \lambda^0)$ which obeys the estimate

$$\|u(t) - \tilde{u}\|_{1,\diamond} + |\lambda(t) - \tilde{\lambda}| \le K\mathrm{e}^{-\nu t}(\|u^0 - \tilde{u}\|_{1,\diamond} + |\lambda^0 - \tilde{\lambda}|). \tag{4.13}$$

Remark 4.3 Combining the estimate (4.13) with the approximation result (2.61), we obtain for $h > h_0, \pm n_\pm > T$ and a sufficiently large $\tau_0 > 0$:

$$\|u(t) - \bar{u}\|_{1,\infty} + |\lambda(t) - \bar{\lambda}| \le K(\mathrm{e}^{-\nu t} + h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}) \quad \forall t > \tau_0.$$

Remark 4.4 We will show later that if one prescribes the initial value u^0 on the grid J and the so called essential conditions (4.10),(4.4) are satisfied, then the external points $u^0_{n_- -1}, u^0_{n_+ +1}$ of u^0 and the initial parameter λ^0 can be chosen in such a way, that (u^0, λ^0) solves (4.3), (4.4), (4.11), (4.12).

The system (4.2)–(4.4) has the special structure of an inital boundary value problem with an additional constraint. Therefore we will reduce the algebraic constraints directly and try to match the semigroup approach developed in Chapter 1 as far as possible, using the resolvent estimates which have been proven in Chapter 3. The proof of Theorem 4.2 will proceed along the same lines as the proof of Theorem 1.13.

Note that the standard (P)DAE methods [21], [22] either rely on the transformation of the DAE into Weierstrass form or deal with general PDAEs [17], [34]. In the latter case it is difficult to check the abstract conditions for the special system considered here, whereas the transformation into Weierstrass form needs detailed information about the spectrum. But we have detailed information about the resolvent only. For a rather up-to-date account on DAE theory see [45].

As in Chapter 1 we transform the system (4.2)–(4.4) into a semilinear equation which has 0 as a stationary solution and prove a stability result for this system.

4.2 The semilinear equation

Let $(\tilde{u}, \tilde{\lambda})$ be a solution of the boundary value problem (4.2)-(4.4). Inserting the ansatz $u = \tilde{u} + v$, $\lambda = \tilde{\lambda} + \mu$ into (4.2) we obtain

$$\begin{aligned} v_n' &= A(\delta_+\delta_- v)_n + B_n(\delta_0 v)_n + C_n v_n + (\delta_0\tilde{u})_n\mu + \hat{\varphi}_n(v, \mu), \quad n \in J \\ &= (\Lambda^h v)_n + \delta_0\tilde{u}_n\ \mu + \hat{\varphi}_n(v, \mu) \end{aligned} \tag{4.14}$$

where B_n, C_n are given by (cf. (3.1)):

$$B_n = \tilde{\lambda} I + D_2 f(\tilde{u}_n, (\delta_0 \tilde{u})_n), \quad C_n = D_1 f(\tilde{u}_n, (\delta_0 \tilde{u})_n),$$

and $\hat{\varphi} : S_{J_e}(\mathbb{R}^m) \times \mathbb{R} \to S_J(\mathbb{R}^m)$ is defined by

$$\hat{\varphi}_n(v, \mu) = \hat{\omega}_n(v) + \delta_0 v_n \, \mu, \quad n \in J \tag{4.15}$$

where

$$\hat{\omega}_n(v) = f(\tilde{u}_n + v_n, \delta_0 \tilde{u}_n + \delta_0 v_n) - f(\tilde{u}_n, \delta_0 \tilde{u}_n) - D_1 f(\tilde{u}_n, \delta_0 \tilde{u}_n) v_n - D_2 f(\tilde{u}_n, \delta_0 \tilde{u}_n) \delta_0 v_n.$$

The boundary condition (4.3) is transformed into

$$0 = Rv, \tag{4.16}$$

and the phase condition (4.4) reads

$$0 = \langle \delta_0 \hat{u}, v_{|J} \rangle_h. \tag{4.17}$$

Then $(0,0)$ is a stationary solution of (4.14), (4.16), (4.17) and the stability of $(\tilde{u}, \tilde{\lambda})$ is now equivalent to the stability of $(0,0)$. Using the notations $\tilde{\psi} = \delta_0 \hat{u}$, $\tilde{\phi} = \delta_0 \tilde{u}$ we have to prove the stability of the zero solution of a semilinear equation of the form

$$\pi v' = \Lambda^h v + \tilde{\phi} \mu + \hat{\varphi}(v, \mu), \tag{4.18}$$

$$0 = R^N v, \tag{4.19}$$

$$0 = R^D \pi v, \tag{4.20}$$

$$0 = \langle \tilde{\psi}, \pi v \rangle_h. \tag{4.21}$$

where $\hat{\varphi}_n : S_{J_e} \times \mathbb{R} \to S_J$, $\tilde{\phi}, \tilde{\psi} \in S_J$ and $v \in S_{J_e}$. For $(v, \mu) \in S_J \times \mathbb{R}$ we use the notation

$$B_\rho^{1,\diamond}((v, \mu)) = \{(u, \lambda) \in S_{J_e} \times \mathbb{R} : \ \|v - u\|_{1,\diamond} + |\mu - \lambda| \leq \rho\}.$$

where $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$. Recall the definition of $\|\cdot\|_{1,\mathcal{L}_{2,h}} = \|\cdot\|_{\mathcal{H}_h^1}$ in (2.3).

As in Chapter 1 the main assumptions on $\hat{\varphi}$ are summarized in the following hypothesis.

Hypothesis 4.5 *Assume that $\hat{\varphi} : S_{J_e} \times \mathbb{R} \to S_J$ satisfies $\hat{\varphi}(0,0) = 0$ and that there exist $\rho_0, C_L > 0$ such that the following holds: There exist $h_0, T > 0$ such that for $h < h_0$, $\pm n_\pm h > T$ for all $(v, \mu), (u, \lambda) \in B_\rho^{1,\diamond}(0)$, $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ with $\rho < \rho_0$, the uniform estimates*

$$\|\hat{\varphi}(v, \mu) - \hat{\varphi}(u, \lambda)\|_\diamond \leq C_L(\|v - u\|_{1,\diamond} + \max(\|v\|_{1,\diamond}, \|u\|_{1,\diamond})|\mu - \lambda|) \tag{4.22}$$

$$\|\hat{\varphi}(v, \mu)\|_\diamond \leq K\rho(\|v\|_{1,\diamond} + |\mu|) \tag{4.23}$$

hold, where C_L, K are independent of h, $J = [n_-, n_+]$.

For the semilinear equation (4.18)–(4.21), the consistency conditions (4.11),(4.12) read

$$0 = R^D(\Lambda^h v + \tilde{\phi} \mu + \hat{\varphi}(v, \mu)), \tag{4.24}$$

$$0 = \langle \tilde{\psi}, \Lambda^h v + \tilde{\phi} \mu + \hat{\varphi}(v, \mu) \rangle_h. \tag{4.25}$$

The main result of this chapter is the following stability theorem for the zero solution of the DAE (4.18)–(4.21), which is the discrete analog of Theorem 1.15.

Theorem 4.6 *Let Λ satisfy the same conditions as in Theorem 1.15 and let $\hat{\varphi}$ satisfy Hypothesis 4.5. Assume further that $\tilde{\psi} = \delta_0 \hat{u}$, where $\hat{u}$ satisfies Hypothesis 2.18 and that the boundary conditions satisfy Hypotheses 3.1,3.3.*

Then there exist $h_0 > 0, T > 0$, $C > 0$ such that for $h < h_0$, $\mp h n_\pm > T$ the stationary solution $0 \in S_{J_e} \times \mathbb{R}$ of (4.14),(4.16),(4.17) is asymptotically stable.

More precisely, there exist $K, \nu, \rho, h_0, T > 0$ such that for $h < h_0$, $\mp h n_\pm > T$ the following statements hold for $\diamond = \infty$ and also for $\diamond = \mathcal{L}_{2,h}$ if additionally $\mathrm{e}^{-\alpha T} < c\sqrt{h}$ for some $c > 0$, where α denotes the constant in Hypothesis 2.18:

For each consistent initial value $(v^0, \mu^0) \in S_{J_e} \times \mathbb{R}$ (i.e. (4.3), (4.4), (4.11), (4.12) are satisfied) with $\|v^0\|_{1,\diamond} + |\mu^0| < \rho$ there exists a unique solution (v, μ) of (4.14),(4.16),(4.17) which obeys the estimate

$$\|v(t)\|_{1,\diamond} + |\mu(t)| \leq K\mathrm{e}^{-\nu t}(\|v^0\|_{1,\diamond} + |\mu^0|) \quad \forall t \geq 0. \tag{4.26}$$

We will prove the above Theorem by reducing the DAE (4.18),(4.19)–(4.21) to an ODE an a subspace of S_J where the so called essential algebraic conditions (4.20),(4.21) are satisfied. We define this space as follows:

$$S_J^{\text{ess}} = \{u \in S_J(\mathbb{R}^m) : \; R^D u = 0, \; \langle \tilde{\psi}, u \rangle_h = 0\}. \tag{4.27}$$

Remark 4.7 We will show in Lemma 4.18, that there exists $\delta > 0$ such that for each $u^0 \in S_J^{\text{ess}}$ with $\|u^0\| \leq \delta$, there exists a unique extension $(v^0, \mu^0) \in S_{J_e} \times \mathbb{R}$, which satisfies $\pi v^0 = u^0$ and solves (4.3), (4.4), (4.11), (4.12).

Let us first show that Theorem 4.6 implies the stability result Theorem 4.2. The proof is similar to the proof of Theorem 1.13 in Chapter 1.

Proof of Theorem 4.2:
For $\hat{\varphi}(v, \mu) = \hat{\omega}(v) + \mu\delta_0 v$, (see (4.15)) we show that Hypothesis 4.5 is satisfied. We obtain for $v, u \in B_\rho^{1,\infty}(0)$

$$\begin{aligned}
\|\hat{\omega}_n(v) - \hat{\omega}_n(u)\| &= \|f(\tilde{u}_n + v_n, \tilde{\phi}_n + \delta_0 v_n) - f(\tilde{u}_n + u_n, \tilde{\phi}_n + \delta_0 u_n) \\
&\qquad - D_1 f(\tilde{u}_n, \tilde{\phi}_n)(v_n - u_n) - D_2 f(\tilde{u}_n, \tilde{\phi}_n)(\delta_0 v_n - \delta_0 u_n)\| \\
&= \|f_1(\tilde{u}_n + v_n)(\tilde{\phi}_n + \delta_0 v_n) - f_1(\tilde{u}_n + u_n)(\tilde{\phi}_n + \delta_0 u_n) - f_1'(\tilde{u}_n)(\tilde{\phi}_n, v_n - u_n) \\
&\qquad - f_1(\tilde{u}_n)(\delta_0 v_n - \delta_0 u_n) + f_2(\tilde{u}_n + v_n) - f_2(\tilde{u}_n + u_n) - f_2'(\tilde{u}_n)(v_n - u_n)\| \\
&\leq c\|(f_1(\tilde{u}_n + v_n) - f_1(\tilde{u}_n + u_n))\tilde{\phi}_n\| + \|(f_1(\tilde{u}_n + v_n) - f_1(\tilde{u}_n + u_n))\delta_0 v_n\| \\
&\qquad + \|(f_1(\tilde{u}_n + u_n) - f_1(\tilde{u}_n))(\delta_0 v_n - \delta_0 u_n)\| + \|f_1'(\tilde{u}_n)(\tilde{\phi}_n, v_n - u_n)\| \\
&\qquad + \|f_2(\tilde{u}_n + v_n) - f_2(\tilde{u}_n + u_n)\| + \|f_2'(\tilde{u}_n)(v_n - u_n)\| \\
&\leq c(\|v_n - u_n\| + \|v_n - u_n\|\|\delta_0 v_n\| + \|u_n\|\|\delta_0(v - u)_n\|) \qquad (4.28) \\
&\leq c\|u - v\|_{1,\infty}.
\end{aligned}$$

This implies for all $(v, \mu), (u, \lambda) \in B_\rho^{\mathcal{H}_h^1}(0)$ using (4.28), Hypothesis 1.9 as well as the

Sobolev imbedding $\|v\|_\infty \le C\|v\|_{\mathcal{H}_h^1}$ (see A.4) for a generic constant $c > 0$:

$$
\begin{aligned}
\|\hat{\omega}(v) - \hat{\omega}(u)\|^2_{\mathcal{L}_{2,h}} &= \sum_{n=n_-}^{n_+} h\|\hat{\omega}_n(v) - \hat{\omega}_n(u)\|^2 \\
&\le ch \sum_{n=n_-}^{n_+} \|v_n - u_n\|^2 + \|v_n - u_n\|^2\|\delta_0 v_n\|^2 + \|u_n\|^2\|\delta_0(v-u)_n\|^2 \\
&\le c\Big(\sum_{n=n_-}^{n_+} h\|v_n - u_n\|^2 + \|\delta_0 v\|^2_\infty \sum_{n=n_-}^{n_+} h\|v_n - u_n\|^2 + \|u\|^2_\infty \sum_{n=n_-}^{n_+} h\|\delta_0(v-u)_n\|^2\Big) \\
&\le c(\|v-u\|^2_{\mathcal{L}_{2,h}} + \|v-u\|^2_{\mathcal{H}_h^1}\|v\|^2_{\mathcal{H}_h^1} + \|u\|^2_{\mathcal{H}_h^1}\|v-u\|^2_{\mathcal{H}_h^1}) \\
&\le c\|v-u\|^2_{\mathcal{H}_h^1}.
\end{aligned}
$$

Furthermore, (1.20) leads for $\|v\|_{1,\infty} \le \rho$ to

$$
\begin{aligned}
\|\hat{w}_n(v)\| &\le \|f(\tilde{u}_n + v_n, \tilde{\phi}_n + \delta_0 v_n) - f(\tilde{u}_n, \tilde{\phi}_n) \\
&\qquad - D_1 f(\tilde{u}_n, \tilde{\phi}_n)v_n - D_2 f(\tilde{u}_n, \tilde{\phi}_n)\delta_0 v_n\| \\
&\le \int_0^1 \|[D_1 f(\tilde{u}_n + tv_n, \tilde{\phi}_n + t\delta_0 v_n) - D_1 f(\tilde{u}_n, \tilde{\phi}_n)]v_n\|\, dt \\
&\quad + \int_0^1 \|[D_2 f(\tilde{u}_n + tv_n, \tilde{\phi}_n + t\delta_0 v_n) - D_2 f(\tilde{u}_n, \tilde{\phi}_n)]\delta_0 v_n\|\, dt \\
&\le c\int_0^1 t(\|v_n\| + \|\delta_0 v_n\|)\|v_n\|\, dt + \int_0^1 t\|v_n\|\|\delta_0 v_n\|\, dt \\
&\le c(\|v_n\| + \|\delta_0 v_n\|)\|v_n\| \qquad (4.29)\\
&\le c\|v\|_{1,\infty}\|v\|_\infty \le c\rho\|v\|_{1,\infty}.
\end{aligned}
$$

Equation (4.29) implies for $\|v\|_{\mathcal{H}_h^1} \le \rho$

$$
\begin{aligned}
\|\hat{w}(v)\|^2_{\mathcal{L}_{2,h}} &\le \sum_{n=n_-}^{n_+} h\|\hat{\omega}_n\|^2 \le c\sum_{n=n_-}^{n_+} h(\|v_n\| + \|\delta_0 v_n\|)^2\|v_n\|^2 \\
&\le c\|v\|^2_\infty h\sum_{n=n_-}^{n_+} (\|v_n\| + \|\delta_0 v_n\|)^2 \le c\|v\|^2_{\mathcal{H}_h^1}\|v\|^2_{\mathcal{H}_h^1} \\
&\le c\rho^2\|v\|^2_{\mathcal{H}_h^1}.
\end{aligned}
$$

These estimates show together with

$$
\begin{aligned}
\|\mu\delta_0 v - \lambda\delta_0 u\|_\infty &\le \|\delta_0 v\|_\infty|\mu - \lambda| + |\lambda|\|\delta_0(v-u)\|_\infty \\
&\le \|v\|_{1,\infty}|\mu - \lambda| + |\lambda|\|v-u\|_{1,\infty} \le \rho(\|v-u\|_{1,\infty} + |\mu - \lambda|)
\end{aligned}
$$

as well as

$$
\|\mu\delta_0 v - \lambda\delta_0 u\|_{\mathcal{L}_{2,h}} \le \|v\|_{\mathcal{H}_h^1}|\mu - \lambda| + |\lambda|\|v-u\|_{\mathcal{H}_h^1} \le \rho(\|v-u\|_{\mathcal{H}_h^1} + |\mu - \lambda|)
$$

and $\hat{\varphi}(0,0) = 0$ that Hypothesis 4.5 holds.

Finally, (v^0, μ^0) satisfies (4.16), (4.17) and (4.24), (4.25) if and only if (u^0, λ^0) satisfies (4.3),(4.4) and (4.11),(4.12). □

In the following we will use equations (4.19), (4.24),(4.25) to reduce the system (4.18)–(4.21) to an ODE in the space S_J^{ess} where the essential initial conditions (4.20), (4.21) are satisfied.

The proof of Theorem 4.6 needs several preparations which are done in the sections 4.2.1–4.2.4.

4.2.1 The linear inhomogeneous equation

In the following we discuss the solution of the linear inhomogenous equation

$$\pi v' = \Lambda^h v + \mu\tilde{\phi} + r \tag{4.30}$$

together with the constraints (4.19)–(4.21) for $r \in \mathcal{C}(\mathbb{R}^+, S_J)$ with initial conditions $(v(0), \mu(0)) = (v^0, \mu^0) \in S_{J_e} \times \mathbb{R}$.

The conditions (4.24), (4.25) are in this situation given by:

$$0 = R^D(\Lambda^h v + \tilde{\phi}\mu + r), \tag{4.31}$$

$$0 = \langle\tilde{\psi}, \Lambda^h v + \tilde{\phi}\mu + r\rangle_h. \tag{4.32}$$

Reduction to an ODE

The following lemma states conditions under which a consistent $(v, \mu) \in S_{J_e} \times \mathbb{R}$ can be uniquely determined from a given $u \in S_J^{\text{ess}}$ with $\pi v = u$. Here only the limiting case $|z| \to \infty$ of Hypothesis 3.3 is needed.

Hypothesis 4.8 *Assume that the matrices $P_\pm, Q_\pm$ are partitioned into a Neumann and a Dirichlet part as in (3.10) with* $\operatorname{rank}(Q_- Q_+) = r \in [0, 2m]$ *and assume that the matrix*

$$\begin{pmatrix} Q_-^N A^{-1} & -Q_+^N A^{-1} \\ P_-^D & P_+^D \end{pmatrix} \tag{4.33}$$

is nonsingular.

Lemma 4.9 *For each $u \in S_J^{\text{ess}}$ and each $r \in S_J$ there exists a unique extension $(v, \mu) \in S_{J_e} \times \mathbb{R}$ such that $\pi v = u$ and (4.19), (4.31), (4.32) hold. The map $(u, r) \mapsto (v, \mu)$ is linear in u and r. Moreover, for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ the following estimates hold*

$$\|v\|_{2,\diamond} + |\mu| \le c(\frac{1}{h^2}\|u\|_\diamond + \|r\|_\diamond). \tag{4.34}$$

Proof: Let $u \in S_J^{\text{ess}}$ be given and set $v = (v_{n_- -1}, u_{n_-}, \ldots, u_{n_+}, v_{n_+ +1})$. It remains to compute the external points $v_{n_- -1}, v_{n_+ +1}$ and μ from the equations (4.19), (4.31), (4.32) which read

$$\begin{aligned} 0 &= P_-^N v_{n_-} + Q_-^N \delta_0 v_{n_-} + P_+^N v_{n_+} + Q_+^N \delta_0 v_{n_+} \\ 0 &= P_-^D(\Lambda^h v_{n_-} + \tilde{\phi}_{n_-}\mu + r_{n_-}) + P_+^D(\Lambda^h v_{n_+} + \tilde{\phi}_{n_+}\mu + r_{n_+}) \\ 0 &= \langle\tilde{\psi}, \Lambda^h v\rangle_h + \langle\tilde{\psi}, \tilde{\phi}\rangle_h \mu + \langle\tilde{\psi}, r\rangle_h. \end{aligned} \tag{4.35}$$

We use the relation

$$\delta_+\delta_- v_n = \frac{2}{h}(\delta_0 v_n + \delta_- v_n) = \frac{2}{h}(-\delta_0 v_n + \delta_+ v_n) \tag{4.36}$$

as well as the definition of Λ^h in (4.14) to obtain the equivalent system for $w = (w_-, w_+) = (\delta_0 v_{n_-}, \delta_0 v_{n_+})$ and μ

$$\mathcal{M}\begin{pmatrix} w \\ \mu \end{pmatrix} = \mathcal{R}^u u + \mathcal{R}^r r \tag{4.37}$$

where

$$\mathcal{M} = \begin{pmatrix} Q_-^N & Q_+^N & 0 \\ -P_-^D(A - \frac{h}{2}B_{n_-}) & P_+^D(A + \frac{h}{2}B_{n_+}) & \frac{h}{2}(P_-^D\tilde{\phi}_{n_-} + P_+^D\tilde{\phi}_{n_+}) \\ -\tilde{\psi}_{n_-}^T(A - \frac{h}{2}B_{n_-}) & \tilde{\psi}_{n_+}^T(A + \frac{h}{2}B_{n_+}) & \frac{1}{2}\langle\tilde{\psi}, \tilde{\phi}\rangle_h \end{pmatrix},$$

$$\mathcal{R}^u u = \begin{pmatrix} -P_-^N u_{n_-} - P_+^N u_{n_+} \\ -P_-^D A\delta_+ u_{n_-} - P_+^D A\delta_- u_{n_+} - \frac{h}{2}(P_-^D C_{n_-} u_{n_-} + P_+^D C_{n_+} u_{n_+}) \\ -\tilde{\psi}_{n_-}^T(A\delta_+ u_{n_-} + \frac{h}{2}C_{n_-}u_{n_-}) - \tilde{\psi}_{n_+}^T(A\delta_- u_{n_+} + \frac{h}{2}C_{n_+}u_{n_+}) - \frac{h}{2}\sum_{n=n_-+1}^{n_+-1}\tilde{\psi}_n^T\Lambda^h u_n \end{pmatrix},$$

$$\mathcal{R}^r r = -\frac{1}{2}\begin{pmatrix} 0 \\ h(P_-^D r_{n_-} + P_+^D r_{n_+}) \\ \langle\tilde{\psi}, r\rangle_h \end{pmatrix}.$$

For $h \to 0$, $-hn_-, hn_+ \to \infty$ the matrix $\mathcal{M}$ converges to

$$\hat{\mathcal{M}} = \begin{pmatrix} Q_-^N & Q_+^N & 0 \\ -P_-^D A & P_+^D A & 0 \\ -\tilde{\psi}_{n_-}^T A & \tilde{\psi}_{n_+}^T A & \frac{1}{2}\langle\hat{u}', \bar{u}'\rangle \end{pmatrix} \tag{4.38}$$

which is invertible due to condition (4.33) and $\langle\hat{u}', \bar{u}'\rangle \neq 0$. Therefore the solution $(\hat{w}, \hat{\mu})$ of $\hat{\mathcal{M}}(w, \mu)^T = \mathcal{R}^u u + \mathcal{R}^r r$ can be estimated by

$$\|\hat{w}\| \leq c(\frac{1}{h}\|u\|_\infty + h\|r\|_\infty), \tag{4.39}$$

and we obtain the same estimate for $w = (w_-, w_+)$ with a different c. This implies, together with the relations

$$v_{n_--1} = -2hw_- + u_{n_-+1} = -2hw_-, \qquad v_{n_++1} = 2hw_+ + u_{n_+-1} = 2hw_+, \tag{4.40}$$

the estimate

$$\|v_{n_--1}\| + \|v_{n_++1}\| \leq ch\|w\| \leq c(\|u\|_\infty + h^2\|r\|_\infty). \tag{4.41}$$

Furthermore, the relation

$$\delta_+ v_{n_+} = 2\delta_0 v_{n_+} - \delta_+ u_{n_+-1} = 2w_+, \quad \delta_+ v_{n_--1} = \delta_- v_{n_-} = 2w_- \tag{4.42}$$

leads with (4.39) to

$$\|\delta_+ v\|_\infty \leq c(\frac{1}{h}\|u\|_\infty + h\|r\|_\infty). \tag{4.43}$$

Similarly, by (4.36) we find

$$\delta_+\delta_- v_{n_-} = \frac{2}{h}(-w_- + \delta_+ u_{n_-}) = -\frac{2}{h}w_-, \quad \delta_+\delta_- v_{n_+} = \frac{2}{h}(w_+ - \delta_+ u_{n_+-1}) = \frac{2}{h}w_+,$$

which implies with (4.39)

$$\|\delta_+\delta_- v\|_\infty \leq c(\frac{1}{h^2}\|u\|_\infty + \|r\|_\infty). \tag{4.44}$$

Finally we obtain using (4.35)

$$|\mu| \leq |\langle\tilde{\psi},\tilde{\phi}\rangle_h|^{-1}(|\langle\tilde{\psi},\Lambda^h v\rangle_h| + |\langle\tilde{\psi},r\rangle_h|) \leq c(\frac{1}{h^2}\|v\|_\infty + \|r\|_\infty).$$

Together with (4.41),(4.43),(4.44) this leads to (4.34) for $\diamond = \infty$.

In a similar way the estimate for $\diamond = \mathcal{L}_{2,h}$ follows. □

Define the space of consistent initial conditions by

$$\mathcal{S}^{co} = \{(v,\mu) \in S_{J_e} \times \mathbb{R} : \ (v,\mu) \text{ satisfies } (4.19), (4.20), (4.21), (4.31), (4.32)\}.$$

Then Lemma 4.9 implies that the map $\mathcal{S}^{co} \to S_J^{\text{ess}}, (v,\mu) \mapsto \pi v$ is invertible, with a uniform bound for the inverse. Moreover, we can write (v,μ) as

$$v = M_v u + R_v r, \qquad \mu = M_\mu u + R_\mu r, \tag{4.45}$$

where $M_v, R_v : S_J \to S_{J_e}$, $M_\mu, R_\mu : S_J \to \mathbb{R}$, are linear. Thus for any $(v^0,\mu^0) \in \mathcal{S}^{co}$ the solution of the DAE (4.30), (4.19)–(4.21) with inital values $(v^0,\mu^0) \in \mathcal{S}^{co}$ is obtained from the solution of the reduced ODE

$$\begin{aligned} u' &= (\Lambda^h M_v + \tilde{\phi} M_\mu)u + (\Lambda^h R_v + \tilde{\phi} R_\mu + I)r, \\ &=: \Lambda_P^h u + \Pi r \end{aligned} \tag{4.46}$$

with inital value $u^0 = \pi v^0$ by

$$v(t) = M_v u(t) + R_v r(t), \quad \mu(t) = M_\mu u(t) + R_\mu r(t). \tag{4.47}$$

Note that by construction $\Pi r \in S_J^{\text{ess}}$. Therefore it is sufficient to consider (4.46) in S_J^{ess}. Thus we have reduced the bordered system (4.30) to an ODE in a similar fashion as in Lemma 1.17. The inhomogenous ODE (4.46) in S_J^{ess} is then solved as usual via

$$u(t) = \hat{S}_P(t)u^0 + \int_0^t \hat{S}_P(t-s)\, r(s)\, ds,$$

where the operator $\hat{S}_P(t)$ is defined via the Dunford integral

$$\hat{S}_P(t) = \frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{st}(sI - \Lambda_P^h)^{-1}\, ds \tag{4.48}$$

and Γ is a closed curve which encloses the spectrum of Λ_P^h. In the following section we give estimates similar to (1.52) for the resolvent of the operator $\Lambda_P^h : S_J^{\text{ess}} \to S_J^{\text{ess}}$, which lead to estimates of $\hat{S}_P(t)$.

Resolvent estimates

We use the technique of Chapter 1 of proving resolvent estimates for the discretized system in different regions $\Omega_\epsilon, \Omega_{C_0}, \Omega_{C_0}^h, \Omega_\infty^h$ (cf. 1.2). These estimates will be used to estimate the solution operator $\hat{S}_P$ for the reduced system (4.46). The following Lemma shows that the resolvent equations in Chapter 3 are equivalent to the resolvent equations for the operator Λ_P^h. To this end we transform the resolvent equation for the projected operator Λ_P^h back into a bordered equation. This is accomplished by reintroducing the algebraic variables. A direct application of Lemma 4.9 leads to the following lemma.

Lemma 4.10 *Let $r \in S_J$, then $u \in S_J^{\text{ess}}$ solves*

$$(sI - \Lambda_P^h)u = \Pi r \tag{4.49}$$

and

$$v = M_v u + R_v r, \quad \mu = M_\mu u + R_\mu r \tag{4.50}$$

if and only if the pair $(v, \mu) \in \mathcal{S}^{co}$ is a solution of the bordered system

$$\begin{aligned} (s\pi - \Lambda^h)v - \tilde{\phi}\mu &= r \\ Rv &= 0 \\ \langle \tilde{\psi}, \pi v \rangle_h &= 0. \end{aligned} \tag{4.51}$$

Using this equivalence, we obtain that the resolvent estimates in Chapter 3 imply a uniform estimate in a compact set and an estimate for large $|s|$ for equation (4.49).

Lemma 4.11 *There exist $C_0 > 0$ and $h_0, T > 0$ such that for each $h < h_0$, $\pm n_\pm > T$ there exists for each $s \in \Omega_{C_0} \cup \Omega_{C_0}^h \cup \Omega_\infty^h$ and each $\hat{g} \in S_J$ a solution (v, μ) of (4.51) which can for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ be estimated by*

$$\|v\|_{1,\diamond} + |\mu| \le C\|\hat{g}\|_\diamond, \quad \text{as } s \in \Omega_{C_0} \tag{4.52}$$

$$|s|\|v\|_\diamond + \sqrt{|s|}\|v\|_{1,\diamond} \le C\|\hat{g}\|_\diamond, \quad \text{as } s \in \Omega_{C_0}^h \tag{4.53}$$

where $C > 0$ does not depend on r, s, h and T.

The construction of a solution of (4.51) for $s \in \Omega_\epsilon$ together with a resolvent estimate will proceed along the same lines as Lemma 1.22.

Lemma 4.12 *Under the same assumptions as in the previous lemma, there exist $C, \epsilon > 0$ and $h_0, T > 0$ such that for each $h < h_0$, $\pm n_\pm > T$ the following holds. For each for $s \in \Omega_\epsilon$ and $g \in S_J$ the resolvent equation (4.51) possesses a unique solution $(v, \mu) \in S_{J_e} \times \mathbb{R}$ which satisfies the following uniform estimate in s*

$$\|v\|_{2,\diamond} + |\mu| \le C\|g\|_\diamond, \quad \diamond \in \{\infty, \mathcal{L}_{2,h}\}. \tag{4.54}$$

Proof: We transform equation (4.51) to first order using $z = (u, v)$, $v = \delta_- u$. With the same abbreviations as in Chapter 3 we obtain the equivalent equation

$$\tilde{\Lambda}(z, \mu) = (r, \eta, \omega) \tag{4.55}$$

with $w_n = \delta_0 v_n$ and

$$\tilde{\Lambda}(z,\mu) = \begin{pmatrix} \tilde{L}(s)(z,\mu) \\ P_- u_{n_-} + Q_- w_{n_-} + P_+ u_{n_+} + Q_+ w_{n_+} \\ h \sum_{n=n_-}^{n^+} \tilde{\psi}_n^T u_n \end{pmatrix}, \quad r_n = \begin{pmatrix} 0 \\ h g_n \end{pmatrix}.$$

Here $\tilde{L}(s)$ is given by

$$(\tilde{L}(s)(z,\mu))_n = N_n z_{n+1} - K_n(s) z_n - W_n \mu$$

where

$$N_n = \begin{pmatrix} I & -hI \\ 0 & E_n^+ \end{pmatrix}, \; K_n(s) = \begin{pmatrix} I & 0 \\ h(sI - C_n) & E_n^- \end{pmatrix}, \; W_n = \begin{pmatrix} 0 \\ h\tilde{\phi}_n \end{pmatrix}$$

and $E_n^{\pm}$ is defined in (3.5). Notice the similarity to the operator $\tilde{L}(\tilde{\phi}, s)$, which has been defined in (3.84).

As in the previous chapters we show invertibility for a perturbation of $\tilde{\Lambda}$ which is given by

$$\Lambda_i z = \begin{pmatrix} (\hat{N} z_{n+1} - \hat{K}_n(s) z_n - \hat{W}_n \mu)_{n \in J} \\ (P_- \; Q_-) z_{n_-} + (P_+ \; Q_+) z_{n_+} \\ \sum_{n=n_-}^{n^+} h \tilde{\psi}_n^T u_n \end{pmatrix}$$

where

$$\hat{K}_n = \begin{pmatrix} I & hI \\ h(sI - \hat{C}_n) & A - h\hat{B}_n \end{pmatrix}, \quad \hat{W}_n = \begin{pmatrix} 0 \\ h\bar{u}_n' \end{pmatrix}$$

and $\hat{N}, \; \hat{B}_n, \hat{C}_n$ are defined in (2.68).

Using (2.61) we obtain for $h < h_0$, $\pm h n_{\pm} > T$ the estimates

$$\|\hat{K}_n(s) - K_n(s) + N_n - \hat{N}\| \le ch(h^2 + \mathrm{e}^{-\alpha T})$$

as well as

$$\|W_n - \hat{W}_m\| = h\|\delta_0 \tilde{u}_n - \bar{u}_n'\| \le c(h^2 + \mathrm{e}^{-\alpha T}).$$

Together with

$$\|N_n - \hat{N}\| \le ch$$

this leads to

$$\|(\tilde{\Lambda} - \Lambda_i)(z,\mu)\|_\infty^* \le \rho(h,T)(\|z\|_{1,\infty} + |\mu|)$$

where $\lim_{h\to 0, T\to\infty} \rho(h,T) = 0$. (see the proof of Theorem 2.21).

In the same way as in Chapter 3, we use the fact that the spectral condition (SC) implies that $L(s)$ has exponential dichotomies on $\mathbb{R}^{\pm}$ for all s with $\operatorname{Re} s > -\beta$. Thus, for these s the operator

$$\tilde{L}(s) z = (z_{n+1} - \hat{N}^{-1} \hat{K}_n(s) z_n)_{n \in \mathbb{Z}}$$

possesses an exponential dichotomy on $\mathbb{Z}^{\pm}$ with data $(K^{\pm}, \alpha_{\pm} h, P^{\pm})$ by Lemma 2.3. Moreover, the Hypothesis 2.11 follows from condition (2.60) for $P_{\pm}, Q_{\pm}$ and (1.27) implies Hypothesis 2.12. The definition of $\hat{W}$ together with Lemma 2.8 implies that Hypothesis 2.13 is satisfied as well. Application of Lemma 2.14 implies the existence of solutions of

$$\Lambda_i(z,\mu) = (r, \eta, \omega)$$

which can be estimated by (2.38) from which we obtain (4.54). □

Remark 4.13 Note that from (4.34) we obtain

$$\|\Lambda_P^h u\|_\diamond \leq \|(\Lambda^h M_v + \tilde{\phi} M_\mu)u\|_\diamond \leq \frac{c}{h^2}\|u\|_\diamond, \quad \diamond \in \{\infty, \mathcal{L}_{2,h}\}$$

which leads for the spectrum of Λ_P^h to the bound $|\sigma(\Lambda_P^h)| < C_1 h^{-2}$ for some $C_1 > 0$. Thus, using the estimates (4.52),(4.53) and (4.54) we obtain, similar to Lemma 1.21, resolvent estimates for Λ_P^h in a sector and in an annular region (cf. Figure 4.1).

Corollary 4.14 *There exist $\alpha > 0$, $\phi \in (\frac{\pi}{2}, \pi), C_1 > 0$ such that $s \in \rho(\Lambda^h)$ if $s \in \bar{S}_{\alpha,\phi}$ or $|s| \geq C_1 h^{-2}$. Furthermore, for any $C_0 > C_1$ there exist $K > 0$ such that, defining the annulus $A^h_{C_1,C_0} = \{s \in \mathbb{C} : \ |s| \in [\frac{C_1}{h^2}, \frac{C_0}{h^2}]\ \}$ the following estimates hold for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ for $s \in \bar{S}_{\alpha,\phi} \cup A^h_{C_1,C_0}$:*

$$\|v\|_\diamond \leq \frac{K}{|s+\alpha|}\|r\|_\diamond, \qquad \|v\|_{1,\diamond} \leq \frac{K}{\sqrt{|s+\alpha|}}\|r\|_\diamond. \tag{4.56}$$

4.2.2 Estimates of the solution operator

From (4.56) an estimate of $\hat{S}_P(t)$ which is uniform in h and T follows. Under the same assumptions on Λ as in Lemma 1.21 we obtain:

Lemma 4.15 *Let Λ satisfy the same assumptions as in Lemma 1.21 and assume that for $\tilde{\psi}$ Hypothesis 2.18 holds.*

Then there exist $h_0, T > 0$ such that for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ all $h < h_0$ and $\pm n_\pm h > T$ the solution operator $\hat{S}_P(t)$ can be estimated by

$$\|\hat{S}_P(t)r\|_\diamond \leq K\mathrm{e}^{-\alpha t}\|r\|_\diamond, \qquad \|\hat{S}_P(t)r\|_{1,\diamond} \leq K\mathrm{e}^{-\alpha t}\frac{1}{\sqrt{t}}\|r\|_\diamond. \tag{4.57}$$

The proof is similar to the proof of Lemma 1.24. Note, that in this case it is suffient to estimate the integral

$$\frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{st}(sI - \Lambda_P^h)^{-1}\, ds$$

along a closed curve, which encloses the spectrum of Λ_P^h.

We take a path Γ around the eigenvalues of Λ_P^h and can assume Re $s < 0$ $\forall s \in \Gamma$ (see Figure 4.1).

We introduce the following notation for a function $g : \Gamma \to \mathbb{R}$, where $\Gamma = \{\gamma(\xi) : \xi \in [0, l]\}$ is a closed curve

$$\oint_\Gamma g(z)|dz| := \int_0^l g(\gamma(\xi))|\gamma'(\xi)|\, d\xi.$$

We denote the resolvent by $G(s) = (sI - \Lambda_P^h)^{-1}$ and obtain for $r \in S_J^{\text{ess}}$ with (4.56) for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$ for $t > 0$ the following:

$$\begin{aligned}
\|\hat{S}_P(t)\, r\|_\diamond &= \left\|\frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{st}G(s)r\, ds\right\|_\diamond = \left\|\frac{1}{2\pi i}\oint_{\Gamma-\alpha} \mathrm{e}^{st}G(s)r\, ds\right\|_\diamond \\
&= \left\|\frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{(s-\alpha)t}G(s-\alpha)r\, ds\right\|_\diamond \leq \frac{1}{2\pi}\mathrm{e}^{-\alpha t}\oint_\Gamma |\mathrm{e}^{st}|\|G(s-\alpha)r\|_\diamond |ds| \\
&\leq \frac{1}{2\pi}\mathrm{e}^{-\alpha t}\oint_\Gamma \left|\frac{\mathrm{e}^\lambda}{t}\right| \left\|G(\frac{\lambda}{t}-\alpha)r\right\|_\diamond |d\lambda| \leq K\mathrm{e}^{-\alpha t}\|r\|_\diamond \oint_\Gamma \frac{|\mathrm{e}^\lambda|}{|\lambda|}|d\lambda| \\
&\leq C\mathrm{e}^{-\alpha t}\|r\|_\diamond.
\end{aligned}$$

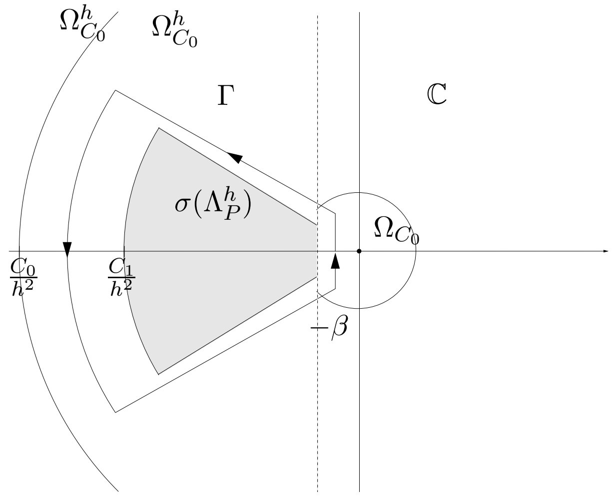


Figure 4.1: Path of integration

Here we have used the fact that we can move the curve Γ to the left up to $\Gamma - \alpha$ for $\alpha < \beta$ small enough without changing the integral. Along the rays this is the standard estimate for sectorial operators (see [36],[23]). Along the arc $\gamma(\xi) = R\mathrm{e}^{i\xi},\ \xi \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ we obtain

$$\int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} R|\mathrm{e}^{tR\mathrm{e}^{i\xi}}| \|G(R\mathrm{e}^{i\xi})r\|_\diamond \, d\xi \leq \|r\|_\diamond \int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} R\mathrm{e}^{tR\cos(\xi)} \frac{1}{R} \, d\xi < \frac{\pi}{2}\|r\|_\diamond.$$

In a similar way we obtain

$$\|\hat{S}_P(t)\, r\|_{1,\diamond} \leq C\mathrm{e}^{-\alpha t} \frac{1}{\sqrt{t}} \|r\|_\diamond.$$

Using this representation, the solution of the original inhomogenous equation can be obtained, as the following Lemma shows.

Lemma 4.16 *Let $r \in \mathcal{C}([0,\tau), S_J)$ and assume that Hypothesis 2.12 holds.*

If the pair $(v,\mu) \in \mathcal{C}(\mathbb{R}^+, \mathcal{S}^{co})$ is a solution of (4.30) on the interval $(0,\tau)$ with consistent initial values $(v^0,\mu^0) \in \mathcal{S}^{co}$ then $u = \pi v$ is a solution on $(0,\tau)$ of (4.46). Furthermore, (v,μ) is given on $[0,\tau)$ by

$$v(t) = M_v u(t) + R_v r(t), \quad \mu(t) = M_\mu u(t) + R_\mu r(t). \tag{4.58}$$

Remark 4.17 Setting $y = (\mathrm{vec}(v_{n_-}, \ldots, v_{n_+}, v_{n_++1}, v_{n_--1}), \mu)$, $M = n_+ - n_- + 1$ and $r = \mathrm{vec}(\hat{g})$ we can write (4.30),(4.19)–(4.21) in matrix notation as

$$\tilde{\mathcal{B}}y' = \tilde{\mathcal{A}}y + \begin{pmatrix} r \\ 0 \end{pmatrix} \in \mathbb{R}^{m(M+3)+1} \tag{4.59}$$

$$\tilde{\mathcal{A}} = \begin{pmatrix} \mathcal{A} & \Phi \\ h\Psi^T & 0 \end{pmatrix}, \quad \tilde{\mathcal{B}} = \begin{pmatrix} I_{Mm} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m(M+3)+1, m(M+3)+1}$$

The definition of $\mathcal{A}, \mathcal{B}, \Phi, \Psi$ will be given in section 5.1. This system can be dealt with using standard DAE methods [21],[56]. We can define the solution operator of the linear homogenous equation $\tilde{\mathcal{B}}y' = \tilde{\mathcal{A}}$ via the generalized resolvent of the matrix pencil $\tilde{\mathcal{A}} - s\tilde{\mathcal{B}}$ by (see [56])

$$\mathcal{F}(t) = \frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{st}(\tilde{\mathcal{A}} - s\tilde{\mathcal{B}})^{-1}\,ds.$$

Note that the equation

$$(\tilde{\mathcal{A}} - s\tilde{\mathcal{B}})y = \begin{pmatrix} r \\ 0 \end{pmatrix}, \quad y \in \mathbb{C}^{(n_+-n_-+4)m+1},\ r \in \mathbb{C}^{(n_+-n_-+2)m}$$

is equivalent to the resolvent equation (3.1), (3.2). Then transforming (4.59) into Kronecker canonical form, [21], [22], [56] one can obtain a solution of the inhomogenous equation via a generalized "variation of constants" formula.

We did not pursue this ansatz, since it requires more knowledge on the Jordan structure of the matrix pencil $\tilde{\mathcal{A}} - s\tilde{\mathcal{B}}$. Instead we have eliminated the boundary conditions and the phase condition directly similar to Section 1.3.1. In our case this is feasible, since the algebraic conditions are linear and do not depend on time.

4.2.3 The nonlinear system

In order to reduce the semilinear DAE (4.18)–(4.21) to an ODE we need a nonlinear version of Lemma 4.9 which guarantees the solvability of the equations (4.19), (4.24), (4.25) which define the transformation $S_J^{\mathrm{ess}} \ni u \to (v, \mu) \in \mathcal{S}^{co}$. This corresponds to the first part of the proof of Lemma 1.27.

Lemma 4.18 *Assume the same as in Theorem 4.6. Then there exist $h_0, T > 0$ such that for all $h < h_0, \pm hn_\pm > T$ the following statements hold for $\diamond = \infty$ and for $\diamond = \mathcal{L}_{2,h}$, if additionally $\mathrm{e}^{-\alpha T} > c\sqrt{h}$.*

For each $u \in S_J^{\mathrm{ess}}$ there exists a unique extension $S_{J_e} \times \mathbb{R} \ni (v, \mu) = (T_v(u), T_\mu(u))$ such that $\pi v = u$, $T_v(0) = 0$,$T_\mu(0) = 0$ and (4.19), (4.24), (4.25) hold.

Moreover, for the map $\tilde{\varphi} : S_J^{\mathrm{ess}} \to S_J^{\mathrm{ess}}$ defined by

$$\tilde{\varphi}(u) = \Lambda^h(T_v(u) - M_v u) + \tilde{\phi}(T_\mu(u) - M_\mu u) + \hat{\varphi}(T_v(u), T_\mu(u)), \tag{4.60}$$

where M_v, M_μ are the linear operators defined in Lemma (4.9), the following holds:

$$\|\tilde{\varphi}(u) - \tilde{\varphi}(v)\|_\diamond \le C_L \|u - v\|_{1,\diamond}, \tag{4.61}$$

and for each $\sigma > 0$ there exists $\rho > 0$ such that

$$\|\tilde{\varphi}(u)\|_\diamond \le \sigma \|u\|_{1,\diamond}, \quad \text{as } \|u\|_{1,\diamond} \le \rho. \tag{4.62}$$

Proof: Let $u \in S_J$ be given and set $v = (v_{n_--1}, u_{n_-}, \dots, u_{n_+}, v_{n_++1})$. It remains to compute the external points v_{n_--1}, v_{n_++1} and μ from the equations (4.19), (4.24), (4.25) which read

$$\begin{aligned}
0 &= P_-^N v_{n_-} + Q_-^N \delta_0 v_{n_-} + P_+^N v_{n_+} + Q_+^N \delta_0 v_{n_+} \\
0 &= P_-^D(\Lambda^h v_{n_-} + \tilde{\phi}_{n_-}\mu + \hat{\varphi}_{n_-}(v,\mu)) + P_+^D(\Lambda^h v_{n_+} + \tilde{\phi}_{n_+}\mu + \hat{\varphi}_{n_+}(v,\mu)) \\
0 &= \langle \tilde{\psi}, \Lambda^h v + \tilde{\phi}\mu + \hat{\varphi}(v,\mu)\rangle_h
\end{aligned} \tag{4.63}$$

Define the map $\chi : S_J \times \mathbb{R}^{2m} \to S_{J_e}, (u, w) \mapsto v,\ w = (w_-, w_+)$ by

$$v_n = u_n,\ n = n_-, \dots, n_+, \quad v_{n_- -1} = -2hw_- + u_{n_-+1}, \quad v_{n_++1} = 2hw_+ + u_{n_+-1}.$$

Then $\delta_0 v_{n_\pm} = w_\pm$ and we obtain

$$\|\chi(u, w) - \chi(u, z)\|_{\mathcal{L}_{2,h}} \le ch\sqrt{h}\|w - z\|. \tag{4.64}$$

The relation (4.42) leads to

$$\|\chi(u, w) - \chi(u, z)\|_{\mathcal{H}^1_h} \le c\sqrt{h}\|w - z\|, \tag{4.65}$$

and also to

$$\|\chi(u, w)\|_{\mathcal{H}^1_h} \le c(\|u\|_{\mathcal{H}^1_h} + h\|w\|), \quad \|\chi(u, w)\|_{1,\infty} \le c(\|u\|_\infty + \|w\|). \tag{4.66}$$

In the same way as in the proof of Lemma 4.9 we obtain with (4.36) the following system which is equivalent to (4.63).

$$\mathcal{M}\begin{pmatrix} w \\ \mu \end{pmatrix} = \mathcal{R}^u u + g(u, w, \mu), \tag{4.67}$$

where $\mathcal{M},\ \mathcal{R}^u$ are given by (4.37) and (cf. $\mathcal{R}^r$ in (4.37))

$$g(u, w, \mu) = -\frac{1}{2}\begin{pmatrix} 0 \\ h(P^D_- \hat{\varphi}_{n_-}(\chi(u, w), \mu) + P^D_+ \hat{\varphi}_{n_+}(\chi(u, w), \mu)) \\ \langle \tilde{\psi}, \hat{\varphi}(\chi(u, w), \mu)\rangle_h \end{pmatrix}.$$

For $h < h_0$ $\pm hn_\pm > T$ the matrix $\mathcal{M}$ is nonsingular and we can define $G : S_J \times \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m} \times \mathbb{R}$ by

$$G(u, w, \mu) = \mathcal{M}^{-1}(\mathcal{R}^u u + g(u, w, \mu)),$$

the fixed point of which is a solution of (4.67). To apply the parametrized contraction mapping theorem A.2 we have to verify (A.1),(A.2). From (4.23),(4.66) we obtain

$$\|\hat{\varphi}(\chi(u, 0), 0)\|_\diamond \le K\rho\|\chi(u, 0)\|_{1,\diamond} \le c\rho\|u\|_{1,\diamond} \tag{4.68}$$

which implies

$$\sqrt{h}\|\hat{\varphi}(\chi(u, 0), 0)\|_\infty \le \|\hat{\varphi}(\chi(u, 0), 0)\|_{\mathcal{L}_{2,h}} \le c\rho\|u\|_{\mathcal{H}^1_h} \tag{4.69}$$

as well as with Cauchy-Schwarz, Hypothesis 2.18 and (4.66)

$$|\langle \tilde{\psi}, \hat{\varphi}(\chi(u, 0), 0)\rangle_h| \le c\|\chi(u, 0)\|_\diamond \le c\rho\|u\|_{1,\diamond}. \tag{4.70}$$

Using (4.22) we obtain with (4.65) and (4.66)

$$\begin{aligned} \|\hat{\varphi}(\chi(u, w), \mu) - \hat{\varphi}(\chi(u, z), \lambda)\|_{\mathcal{L}_{2,h}} &\le C_L(\|\chi(u, w) - \chi(u, z)\|_{\mathcal{H}^1_h} \\ &\quad + \max(\|\chi(u, w)\|_{\mathcal{H}^1_h}, \|\chi(u, z)\|_{\mathcal{H}^1_h})|\mu - \lambda|) \\ &\le c(\sqrt{h}\|w - z\| + (\|u\|_{\mathcal{H}^1_h} + h\max(\|w\|, \|z\|))|\mu - \lambda|). \end{aligned} \tag{4.71}$$

Equation (4.71) leads for $\|u\|_{\mathcal{H}^1_h} < \rho$ to

$$\|\hat{\varphi}(\chi(u, w), \mu) - \hat{\varphi}(\chi(u, z), \lambda)\|_{\mathcal{L}_{2,h}} \le c(\sqrt{h} + \rho + h\delta)(\|w - z\| + |\mu - \lambda|)$$

as well as for $\|u\|_{\mathcal{H}_h^1} \le \sqrt{h}\|u\|_{1,\infty} < \sqrt{h}\rho$ to

$$\|\hat{\varphi}(\chi(u,w),\mu) - \hat{\varphi}(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}} \le c(\sqrt{h}(1+\rho+\delta)(\|w-z\| + |\mu-\lambda|).$$

Thus (4.68), (4.69), (4.70) imply for $\|u\|_{\mathcal{H}_h^1} \le \rho$

$$\begin{aligned}\|g(u,0,0)\| &\le h(\|\hat{\varphi}_{n_-}(\chi(u,0),0)\| + \|\hat{\varphi}_{n_+}(\chi(u,0),0)\| + |\langle\tilde{\psi}, \hat{\varphi}(\chi(u,0),0)\rangle_h|) \\ &\le c\rho\|u\|_{\mathcal{H}_h^1}\end{aligned} \tag{4.72}$$

as well as for $\|u\|_{1,\infty} \le \rho$

$$\|g(u,0,0)\| \le c\rho\|u\|_{1,\infty}. \tag{4.73}$$

Similarly, with (4.71) we find

$$\begin{aligned}\|g(u,w,\mu) - g(u,z,\lambda)\| &\le c(h\|\hat{\varphi}(\chi(u,w),\mu) - \hat{\varphi}(\chi(u,z),\lambda)\|_\infty \\ &\qquad + |\langle\tilde{\psi}, \hat{\varphi}(\chi(u,w),\mu) - \hat{\varphi}(\chi(u,z),\lambda)\rangle_h|) \\ &\le c\|\hat{\varphi}(\chi(u,w),\mu) - \hat{\varphi}(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}}.\end{aligned} \tag{4.74}$$

It remains to estimate $\|\mathcal{R}^u u\|$: As in Chapter 1, the summation by parts formula (A.10)

$$\langle\tilde{\psi}, A\delta_-\delta_+u\rangle_{n_-+1,n_+-1} = -\langle\delta_+\tilde{\psi}, A\delta_+u\rangle_{n_-,n_+-2} + \tilde{\psi}_{n_-}^T A(\delta_+u)_{n_-} - \tilde{\psi}_{n_+-1}^T A(\delta_+u)_{n_+-1}$$

leads for $\hat{J} = [n_-+1, n_+-1]$ with

$$\langle\tilde{\psi}_{|\hat{J}}, \Lambda^h u\rangle_h = \langle\tilde{\psi}_{|\hat{J}}, A\delta_-\delta_+u\rangle_h + \langle\tilde{\psi}_{|\hat{J}}, B\delta_0 u\rangle_h + \langle\tilde{\psi}_{|\hat{J}}, Cu\rangle_h$$

to

$$|\langle\tilde{\psi}_{|\hat{J}}, \Lambda^h u\rangle_h| \le c\|u\|_{1,\infty}. \tag{4.75}$$

Using Hypothesis 2.18 for $\pm hn_\pm > T$ we find

$$|\langle\tilde{\psi}_{|\hat{J}}, \Lambda^h u\rangle_h| \le c(\|u\|_{\mathcal{H}_h^1} + h^{-\frac{1}{2}}\mathrm{e}^{-\alpha T}\|\delta_+u\|_{\mathcal{L}_{2,h}}) \le c(1 + h^{-\frac{1}{2}}\mathrm{e}^{-\alpha T})\|u\|_{\mathcal{H}_h^1}.$$

This implies with the definition of $\mathcal{R}^u$ in (4.37) and (4.75)

$$\|\mathcal{R}^u u\| \le c(\|u\|_{1,\infty} + |\langle\tilde{\psi}_{|\hat{J}}, \Lambda^h u\rangle_h|) \le c\|u\|_{1,\infty}$$

as well as

$$\|\mathcal{R}^u u\| \le c(h^{-\frac{1}{2}}\mathrm{e}^{-\alpha T}\|\delta_+u\|_{\mathcal{L}_{2,h}} + \sqrt{h}\|u\|_{\mathcal{L}_{2,h}} + |\langle\tilde{\psi}_{|\hat{J}}, \Lambda^h u\rangle_h|) \le c(1 + h^{-\frac{1}{2}}\mathrm{e}^{-\alpha T})\|u\|_{\mathcal{H}_h^1}.$$

For $\|u\|_{1,\infty} \le \rho$ we obtain with (4.73)

$$\|G(u,0,0)\| \le c(\|u\|_{1,\infty} + \|g(u,0,0)\|) \le c(1+\rho)\|u\|_{1,\infty} \le c_0\rho$$

and similarly, if $h^{-\frac{1}{2}}\mathrm{e}^{-\alpha T} < c_2$ for $\|u\|_{\mathcal{H}_h^1} \le \rho$ with (4.72)

$$\|G(u,0,0)\| \le c(\|u\|_{\mathcal{H}_h^1} + \|g(u,0,0)\|) \le c(1+\rho)\|u\|_{\mathcal{H}_h^1} \le c_0\rho$$

For $(w,\mu),(z,\lambda) \in B_\delta(0) \subset \mathbb{R}^{2m+1}$ equation (4.74) leads for $\|u\|_{1,\infty} \le \rho$ or $\|u\|_{\mathcal{H}_h^1} \le \rho$ to

$$\|G(u,w,\mu) - G(u,z,\lambda)\| \le c_1(\sqrt{h} + \rho + h\delta)(|\mu-\lambda| + \|w-z\|).$$

Choosing $h, \delta < 1$ so small that $\sqrt{h} + (\frac{1}{2c_0} + h)\delta < \frac{1}{c_1}$ and $\rho < \min(1, \frac{\delta}{2c_0})$ we can apply Theorem A.2 with $q = \frac{1}{2}$. This yields a unique solution $(\bar{w}, \bar{\mu}) \in B_\delta(0)$ of (4.67). Equation (A.4) implies with the continuity of G

$$\|T_v(u_1) - T_v(u_2)\|_\diamond + |T_\mu(u_1) - T_\mu(u_2)| \le c\|u_1 - u_2\|_{1,\diamond}, \tag{4.76}$$

which implies with $T_v(0) = 0, T_\mu(0) = 0$

$$\|T_v(u)\|_\diamond + |T_\mu(u)| \le c\|u\|_{1,\diamond}. \tag{4.77}$$

It remains to prove the Lipschitz estimates for $\tilde{\varphi}$. Using the definition of $T_v(\cdot), T_\mu(\cdot)$ and M_v, M_μ and subtracting (4.31), (4.32) from (4.24), (4.25) we obtain that $v^\Delta = T_v(u) - M_v u$, $\mu^\Delta = T_\mu(u) - M_\mu u$ solves $\pi v^\Delta = 0$ and

$$\begin{aligned}
0 &= R^N v^\Delta \\
0 &= R^D(\Lambda^h v^\Delta + \tilde{\phi}\mu^\Delta + \hat{\varphi}(T_v(u), T_\mu(u))), \\
0 &= \langle \tilde{\psi}, \Lambda^h v^\Delta + \tilde{\phi}\mu^\Delta + \hat{\varphi}(T_v(u), T_\mu(u)) \rangle_h.
\end{aligned}$$

Application of estimate (4.34) in Lemma 4.9 to (v^Δ, μ^Δ) leads to

$$\|T_v(u) - M_v u\|_{2,\diamond} + |T_\mu(u) - M_\mu u| \le c\|\hat{\varphi}(T_v(u), T_\mu(u))\|_\diamond, \quad \diamond \in \{\infty, \mathcal{L}_{2,h}\}.$$

Thus we have for $\tilde{\varphi}$ defined in (4.60) by (4.77) and (4.23)

$$\begin{aligned}
\|\tilde{\varphi}(u)\|_\diamond &\le \|\Lambda^h(T_v(u) - M_v u)\|_\diamond + \|\tilde{\phi}(T_\mu(u) - M_\mu u)\|_\diamond + \|\hat{\varphi}(T_v(u), T_\mu(u))\|_\diamond \\
&\le c\|\hat{\varphi}(T_v(u), T_\mu(u))\|_\diamond \le K\rho(\|T_v(u)\|_\diamond + |T_\mu(u)|)
\end{aligned}$$

which leads to

$$\|\tilde{\varphi}(u)\|_\infty \le c\rho\|u\|_{1,\infty},$$

as well as for $h^{-\frac{1}{2}} \mathrm{e}^{-\alpha T} < c_2$ to

$$\|\tilde{\varphi}(u)\|_{\mathcal{L}_{2,h}} \le c\rho\|u\|_{\mathcal{H}_h^1}.$$

In the same way we obtain for $u_1, u_2 \in S_J^{\mathrm{ess}}$ that $v^\Delta = T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2)$, $\mu^\Delta = T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2)$ solves $\pi v^\Delta = 0$ and

$$\begin{aligned}
0 &= R^N v^\Delta \\
0 &= R^D(\Lambda^h v^\Delta + \tilde{\phi}\mu^\Delta + \hat{\varphi}(T_v(u_1), T_\mu(u_1)) - \hat{\varphi}(T_v(u_2), T_\mu(u_2))), \\
0 &= \langle \tilde{\psi}, \Lambda^h v^\Delta + \tilde{\phi}\mu^\Delta + \hat{\varphi}(T_v(u_1), T_\mu(u_1)) - \hat{\varphi}(T_v(u_2), T_\mu(u_2)) \rangle_h.
\end{aligned}$$

Again, application of estimate (4.34) in Lemma 4.9 to (v^Δ, μ^Δ) implies for $\diamond \in \{\infty, \mathcal{L}_{2,h}\}$

$$\begin{aligned}
&\|T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2)\|_{2,\diamond} + |T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2)| \\
&\quad \le c\|\hat{\varphi}(T_v(u_1), T_\mu(u_1)) - \hat{\varphi}(T_v(u_2), T_\mu(u_2))\|_\diamond.
\end{aligned}$$

Thus we obtain with (4.76) and (4.22)

$$\begin{aligned}
\|\tilde{\varphi}(u_1) - \tilde{\varphi}(u_2)\|_\diamond &\le \|\Lambda^h(T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2))\|_\diamond \\
&\qquad + \|\tilde{\phi}(T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2))\|_\diamond \\
&\qquad\quad + \|\hat{\varphi}(T_v(u_1), T_\mu(u_1)) - \hat{\varphi}(T_v(u_2), T_\mu(u_2))\|_\diamond \\
&\le c\|\hat{\varphi}(T_v(u_1), T_\mu(u_1)) - \hat{\varphi}(T_v(u_2), T_\mu(u_2))\|_\diamond \\
&\le c\|u_1 - u_2\|_{1,\diamond}.
\end{aligned}$$

□

As in the linear case we use transformations T_v, T_μ to reduce the semilinear DAE (4.18)–(4.21) to an ODE in S_J^{ess} as follows. Let $(v(t), \mu(t))$ a solution of (4.18)–(4.21) for consistent initial values $(v^0, \mu^0) \in \mathcal{S}^{co}$ on $(0, \tau)$. Then differentiating (4.20), (4.21) w.r.t. time we obtain by (4.18) that $(v(t), \mu(t))$ solves (4.24), (4.25),i.e. $(v(t), \mu(t)) \in \mathcal{S}^{co}$ for $t \in (0, \tau)$. For $u = \pi v$ we can insert $v = T_v(u),\ \mu = T_\mu(u)$ into (4.18) to obtain

$$
\begin{aligned}
u' = \pi v' &= \Lambda^h v + \tilde{\phi}\mu + \hat{\varphi}(v, \mu) \\
&= \Lambda^h T_v(u) + \tilde{\phi} T_\mu(u) + \hat{\varphi}(T_v(u), T_\mu(u)) \\
&= (\Lambda^h M_v + \tilde{\phi} M_\mu)u + \Lambda^h(T_v(u) - M_v u) + \tilde{\phi}(T_\mu(u) - M_\mu u) + \hat{\varphi}(T_v(u), T_\mu(u)) \\
&= \Lambda_P^h u + \tilde{\varphi}(u).
\end{aligned}
$$

Conversely, if u solves the reduced ODE

$$
u' = \Lambda_P^h u + \tilde{\varphi}(u), \quad u(0) = u^0 \in S_J^{\text{ess}} \cap B_\delta^{1,\diamond}(0) \tag{4.78}
$$

then Lemma 4.18 implies that $v(t) = T_v(u(t)), \mu(t) = T_\mu(u(t))$ is a solution of (4.18)–(4.21) in $B_\rho^{1,\diamond}(0) \subset \mathcal{S}^{co}$ for some $\rho > 0$ in the sense of in the sense of (4.1). The above arguments lead to the following lemma:

Lemma 4.19 *Assume the same as in Theorem 4.6. Then there exist $h_0, T > 0$ such that for $h < h_0$, $\pm n_\pm h > T$ we have the following equivalence.*

For each $\rho > 0$ there exists $\delta > 0$ such that, if $u \in \mathcal{C}([0, \tau), S_J^{\text{ess}} \cap B_\delta^{1,\diamond}(0))$ is a solution on $(0, \tau)$ of (4.78) with $u(0) = u^0$, then $(v(t), \mu(t)) = (T_v(u(t)), T_\mu(u(t))) \in \mathcal{C}([0, t), \mathcal{S}^{co})$ is a solution of (4.18)–(4.21) on $(0, \tau)$ with $v(0) = T_v(u^0), \mu(0) = T_\mu(u^0)$ and $\|v(t)\|_{1,\diamond} + |\mu(t)| \leq \rho$.

Conversely, there exists $\rho > 0$ such that if $(v(t), \mu(t)) \in \mathcal{C}([0, t), \mathcal{S}^{co})$ is a solution of (4.18)–(4.21) on $(0, \tau)$ with $v(0) = v^0, \mu(0) = \mu^0$ with $\|v(t)\|_{1,\diamond} + |\mu(t)| \leq \rho$, then $u = \pi v$ is a solution of (4.78) with $\|u(t)\|_{1,\diamond} < \rho$.

4.2.4 The semilinear reduced system

Local existence and uniqueness

In this section we prove the solvability of the integral equation together with some estimates. Note that the existence of a solution of (4.78) follows from standard ODE theory.

Lemma 4.20 *Assume the same as in Lemma 4.19. There exists $h_0, T > 0$ such that for $h < h_0$, $\pm h n_\pm > T$ the following statements hold:*

For each $\rho > 0$ there exist $\delta > 0$ such that for each $u^0 \in S_J^{\text{ess}}$ with $\|u^0\|_\diamond < \delta$ there exists $\tau(h, J) > 0$ such that a unique solution of (4.78) on $(0, \tau(h, J))$ such that $\|u(t)\|_{1,\diamond} \leq \rho$ for $t \in [0, \tau(h, J))$.

Proof: For each fixed $h, J = [n_-, n_+]$ we use the fact that all norms are equivalent, i.e. we have

$$C_1(h,J)\|u\| \leq \|u\|_\diamond \leq C_2(h,J)\|u\|.$$

Moreover, by Lemma 4.18 there exists $\rho > 0$ such that for $\|u\|_{1,\diamond} < \rho$ the map $\tilde{\varphi}$ is Lipschitz. Thus we can apply the standard Picard-Lindelöf theorem in $\mathbb{R}^{n_+ - n_- + 1}$ to obtain the existence of a solution of (4.78) for $[0, \tau(h,J))$. We can further achieve that $\|u\| \leq C_2(h,J)^{-1}\rho$ in $\tau(h,J)$ such that $\|u\|_\diamond \leq \rho$ for all $t \in [0, \tau(h,J))$. □

Stability for the reduced system

The stability of 0 as a solution of the reduced system is the usual Lyapunov type estimate. We repeat it here, since we are interested not only in the stability of the solution of a single DAE but we aim at a uniform stability estimate for a whole family of solutions of DAEs corresponding to discretizations with different stepsizes and intervals. Therefore we have to mimic the method of the continuous case as far as possible.

Lemma 4.21 *Let $\tilde{\varphi} : S_J^{\mathrm{ess}}(\mathbb{R}^m) \to S_J^{\mathrm{ess}}(\mathbb{R}^m)$ be given which satisfies (4.61),(4.62) in $B_\delta^{1,\diamond}(0)$ and assume that (4.57) holds for the solution operator of the linear system.*

Then there exist $\rho, h_0, T > 0$ such that for any $h < h_0$, $\pm n_\pm h > T$ and any consistent initial condition $u^0 \in S_J^{\mathrm{ess}}$ with $\|u^0\|_{1,\diamond} \leq \rho$ the following holds: There exists a unique solution u of (4.78) which can be estimated by

$$\|u(t)\|_{1,\diamond} \leq C\mathrm{e}^{-\nu t}, \quad \forall t \geq 0. \tag{4.79}$$

where $\nu, C > 0$ are independent of h, J.

Proof: We choose $\nu \in (0, \alpha)$ and $\sigma > 0$ so small that

$$K\sigma \int_0^\infty \frac{\mathrm{e}^{-(\alpha-\nu)s}}{\sqrt{s}}\, ds \leq \frac{3}{4}$$

and $\delta > 0$ so small (using (4.62)) that

$$\|\tilde{\varphi}(u)\|_\diamond \leq \sigma\|u\|_{1,\diamond} \quad \text{for } \|u\|_{1,\diamond} \leq \delta.$$

Then for each h, J we find by Lemma 4.20 some $\rho > 0$ such that for $u^0 \in S_J^{\mathrm{ess}}$ with $\|u^0\|_{1,\diamond} \leq \rho$ a solution u of (4.78) exists on $(0, \tau(h,J))$ with $\|u(t)\|_{1,\diamond} \leq \delta$ for $t \in [0, \tau(h,J)$. This solution is given by the "variation of constants" formula

$$u(t) = \hat{S}_P(t)u^0 + \int_0^t \hat{S}_P(t-s)\, \tilde{\varphi}(u(s))\, ds$$

and the estimates (4.57) lead for $C \geq 1$ to

$$\begin{aligned}
\|u(t)\|_{1,\diamond} &\leq \|\hat{S}_P(t)u^0\|_{1,\diamond} + \int_0^t \|\hat{S}_P(t-s)\tilde{\varphi}(u(s))\|_{1,\diamond}\, ds \\
&\leq C\mathrm{e}^{-\alpha t}\|u^0\|_{1,\diamond} + C\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{-\alpha(t-s)}\|\tilde{\varphi}(u(s))\|_\diamond\, ds \\
&\leq \frac{\delta}{4} + C\sigma \int_0^\infty \frac{1}{\sqrt{s}}\mathrm{e}^{-\alpha s}\, ds\ \|u\|_{1,\diamond}^\tau \\
&\leq \frac{3}{4}\delta.
\end{aligned}$$

Since the ODE (4.78) is autonomous, this leads to $\tau(h,J)=\infty$ using the usual arguments: If $(0,\tau_*)$ is the maximal interval of existence of a solution u with $\|u(t)\|_{1,\diamond}\le\delta$, then by the above estimate we have $\|u(t)\|_{1,\diamond}\le\frac{3}{4}\delta$ on $(0,\tau_*)$. Thus we can solve (4.78) for each h,J with initial condition at $t_0=\tau_*-\frac{\tau}{2}$, where $\tau=\tau(h,J)$ is given by Lemma 4.20. In this way we continue the solution to $\tilde\tau>\tau_*$, which contradicts the maximality of τ_*. From this the existence of u in $(0,\infty)$ follows with $\|u(t)\|_{1,\diamond}<\delta$ for all $t\in[0,\infty)$ and small enough h and large enough J.

It remains to prove the exponential estimate. Define

$$n(t)=\sup_{s\in[0,t]}\{\mathrm{e}^{\nu s}\|u(s)\|_{1,\diamond}\}$$

then

$$\begin{aligned}\|u(t)\|_{1,\diamond}\mathrm{e}^{\nu t}&\le K\mathrm{e}^{(\nu-\alpha)t}\|u^0\|_{1,\diamond}+K\sigma\int_0^t\frac{1}{\sqrt{t-s}}\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{\nu t}\|u(s)\|_{1,\diamond}\,ds\\&\le K\|u^0\|_{1,\diamond}+K\sigma\int_0^t\frac{1}{\sqrt{t-s}}\mathrm{e}^{(\nu-\alpha)(t-s)}\mathrm{e}^{\nu s}\|u(s)\|_{1,\diamond}\,ds\\&<K\|u^0\|_{1,\diamond}+\frac{1}{4}n(t).\end{aligned}$$

Taking the supremum on both sides gives $n(t)<4K\|u^0\|_{1,\diamond}<\delta$ for $t\ge0$ and we obtain the estimate (4.79). □

4.2.5 Proof of the stability theorem

Now the proof of the stability Theorem 4.6 is effortless: For any $(v^0,\mu^0)\in\mathcal{S}^{co}\cap B_\rho^{1,\diamond}(0)$ we have $u^0=\pi v^0\in\mathcal{S}^{co}\cap B_\rho^{1,\diamond}(0)$ and using Lemma 4.21 we obtain a solution u of (4.78) on $(0,\infty)$ which satisfies (4.79). By Lemma 4.19 we find that

$$v(t)=T_v(u(t)),\ \mu(t)=T_\mu(u(t))$$

solves (4.18)–(4.21) with $v(0)=T_v(u^0)=v^0$, $\mu(0)=T_\mu(u^0)=\mu^0$. Moreover, the estimates (4.77),(4.79) imply that (v,μ) can be estimated by (4.26).

Chapter 5

Numerical results

In this chapter we test the approximation results of Chapter 2 and 3 on two different examples of reaction-diffusion equations for which exact solutions are known. We compare the order of approximation for different grid sizes h and interval sizes J with the expected behavior from Theorems 2.21 and 3.15. The essential spectrum for the continuous and the discrete operator on the whole line is compared to the eigenvalues of the discrete operator with periodic boundary conditions.

First we describe the implementation of the solution of the DAE resulting from the freezing ansatz which results in the boundary value problem (2.56)–(2.58) for the wave. Then the solution procedure for the spectral problem (3.78)–(3.80) is described.

Then we deal with the Nagumo equation which is a scalar example. The quintic Ginzburg-Landau equation is a $2D$ example which has besides the translational symmetry an additional rotational symmetry.

5.1 Implementation

For a given interval $J = [x_-, x_+]$ and number of grid points $M + 1$ we discretize the PDAE (4.1) using finite differences for the spatial derivatives. We use the notation $x_j = x_- + hj$, $u_j = u(x_j) \in \mathbb{R}^m$, $j = 0, \dots, M$, and sort differential and algebraic variables in $y = (\mathrm{vec}(u_0), \dots, \mathrm{vec}(u_M))$ and $z = (\mathrm{vec}(u_{M+1}), \mathrm{vec}(u_{-1}), \mu)$ where $\mu \in \mathbb{R}^p$, $y \in \mathbb{R}^{m(M+1)}$ and $y \in \mathbb{R}^{2m+p}$ (compare section 1.4.2). Here we use the vec notation for $u \in S_J(\mathbb{R}^m)$: $\mathrm{vec}(u) = ((u_0)_1, \dots, (u_0)_m, \dots, (u_M)_1, \dots, (u_M)_m)$. Then we obtain a DAE of the form

$$\begin{aligned} y' &= f(y, z) \in \mathbb{R}^{m(M+1)} \\ 0 &= g(y, z) \in \mathbb{R}^{2m+p} \end{aligned} \tag{5.1}$$

where

$$f(y, z) = \Big(A(\delta_+\delta_- u)_j + f(u_j, (\delta_0 u)_j) + \sum_{k=1}^{p} \mu_k (S_k(u))_j \Big)_{j=0,\dots,M},$$

$$g(y, z) = \begin{pmatrix} P_- u_0 + Q_-(\delta_0 u)_0 + P_+ u_M + Q_+(\delta_0 u)_M - \gamma \\ \Big(\sum_{j=0}^{M} h S_k(\hat{u})_j^T (u_j - \hat{u}_j) \Big)_{k=1,\dots,p} \end{pmatrix}.$$

and $S(u) = (S_1(u), ..., S_p(u)),\ \ S_k(u) \in \mathbb{R}^{m,M+1},\ \ k = 1, \dots, p$. Since

$$g_z(y^0, z^0) = \frac{1}{2h}\begin{pmatrix} Q_+ & -Q_- & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2m+p,2m+p}$$

cis singular, this is a DAE of differentiation index 2 (see [22]) and we have to choose initial values (y^0, z^0) which solve the consistency conditions

$$\begin{aligned} 0 &= g(y, z) \\ 0 &= Dg(y, z)f(y, z). \end{aligned} \tag{5.2}$$

In order to illustrate the stability of the solution as has been proven in Chapter 4 we show the time evolution of the wave and the parameter μ starting from some initial condition.

The resulting stationary solution (u, μ) can then be used as an initial value for the Newton's method for solving the boundary value problem (2.56)–(2.58). It's solution is then compared to the exact solution for different grid sizes h and intervals J.

We solve (5.1) using a Matlab implementation of the Radau IIA method of order 5 [16] of Hairer / Wanner [22] and compare the results with a θ method with fixed time steps.
For the following detailed description of the implementation we differentiate not between differential and algebraic variables but between the wave u and the parameter μ and denote the differential resp. the algebraic part of the right hand side of (5.1) by f_{diff} and f_{alg} respectively. For $v = (y, z) \in \mathbb{R}^{m(M+3)+p}$ we have to solve the DAE

$$\mathcal{B}v' = F(v) \in \mathbb{R}^{m(M+3)+p}, \quad v(0) = v^0 \tag{5.3}$$

where $\mathcal{B} = \begin{pmatrix} I_{m(M+1)} & 0 \\ 0 & 0 \end{pmatrix}$ and

$$F(v) = \begin{pmatrix} f_{\text{diff}}(u,\mu) \\ \hline f_{\text{alg}}(u) \end{pmatrix} = \left(\begin{array}{c} \vdots \\ A(\delta_+\delta_- u)_j + f(u_j, (\delta_0 u)_j) + (\sum_{k=1}^p S_k(u)\mu_k)_j, \quad j = 1, \dots, M \\ \vdots \\ \hline P_- u_0 + Q_-(\delta_0 u)_0 + P_+ u_M + Q_+(\delta_0 u)_M - \gamma \\ \left(\sum_{j=0}^M h S_k(\hat{u})_j^T (u_j - \hat{u}_j)\right)_{k=1,\dots,p} \end{array}\right).$$

The Jacobian at v is given by

$$\begin{aligned} DF(v) &= \begin{pmatrix} D_u f_{\text{diff}}(u,\mu) & D_\mu f_{\text{diff}}(u,\mu) \\ \hline D_u f_{\text{alg}}(u) & 0 \end{pmatrix} \\ &= \left(\begin{array}{cccccccc} Y_1 & Z_1 & & & & & X_1 & \phi_1 \\ X_2 & Y_2 & Z_2 & & & & & \vdots \\ & \ddots & \ddots & \ddots & & & & \vdots \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & X_M & Y_M & Z_M & & \phi_M \\ \hline P_- & \frac{1}{2h}Q_- & \dots & -\frac{1}{2h}Q_+ & P_+ & \frac{1}{2h}Q_+ & -\frac{1}{2h}Q_- & \\ \psi_1^T & \dots & \dots & \dots & \psi_M^T & & & \end{array}\right) \end{aligned} \tag{5.4}$$

with

$$X_j = \frac{1}{h^2}A - \frac{1}{2h}B_j, \quad Y_j = -\frac{2}{h^2}A + C_j, \quad Z_j = \frac{1}{h^2}A + \frac{1}{2h}B_j,$$

where B_j, C_j are given as in (2.79) by

$$B_j = D_2 f(u_j, \delta_0 u_j) + \sum_{k=1}^{p} \mu_k S_k^1 \qquad C_j = D_1 f(u_j, \delta_0 u_j) + \sum_{k=1}^{p} \mu_k S_k^0,$$
$$\psi_i = (S_1(\hat{u})_i, \ldots, S_p(\hat{u})_i), \quad \phi_i = (S_1(u)_i, \ldots, S_p(u)_i) \in \mathbb{R}^{m,p},\ i = 1, \ldots, M.$$

The consistency condition (5.2) for the initial values $v^0 = (u^0, \mu^0)$ yields a system of the form

$$G(v^0) = \begin{pmatrix} f_{\text{alg}}(u^0) \\ D_u f_{\text{alg}}(u^0)\ f_{\text{diff}}(u^0, \mu^0) \end{pmatrix} = 0.$$

This underdetermined system is solved by using a Gauss-Newton method starting from suitable initial conditions. A better procedure, which allows to prescribe the initial values at the inner points $i = 1, \ldots, M$, has been introduced in Chapter 4. Assume that η and the matrices $P_\pm, Q_\pm$ are blocked into a Neumann and a Dirichlet part as in (3.10). For any given $(u_0, \ldots, u_M)$ which satisfies the essential conditions (4.10),(4.4)

$$\begin{aligned} \eta^D &= P_-^D u_0 + P_+^D u_M, \\ 0 &= \sum_{j=0}^{M} h S_k(\hat{u})_j^T (u_j - \hat{u}_j), \quad k = 1, \ldots, p \end{aligned}$$

the remaining values $(\text{vec}(u_{-1}), \text{vec}(u_{M+1}), \mu_1, \ldots, \mu_p)$ can be computed from (4.9), (4.11),(4.12), here given by

$$\begin{aligned} \eta^N &= P_-^N u_0 + Q_-^N \delta_0 u_0 + P_+^N u_M + Q_+^N \delta_0 u_M, \\ 0 &= P_-^D (A\delta_+\delta_- u_0 + f(u_0, \delta_0 u_0) + \sum_{k=1}^{p} \mu_k S_k(u)_0) \\ &\quad + P_+^D (A\delta_+\delta_- u_M + f(u_M, \delta_0 u_M) + \sum_{k=1}^{p} \mu_k S_k(u)_M) \\ 0 &= \sum_{j=0}^{M} h S_k(\hat{u})_j^T (A\delta_+\delta_- u_j + f(u_j, \delta_0 u_j) + \sum_{k=1}^{p} \mu_k S_k(u)_j), \quad k = 1, \ldots, p. \end{aligned}$$

However, we did not implement this procedure for determining consistent initial values.

The linear system (3.78)–(3.80) for the computation of discrete eigenvalues of the operator Λ defined in (1.5), which has been discussed in section 3.1.3 reads for $p = 1$

$$(\mathcal{A} - s\mathcal{B})u = 0 \tag{5.5}$$
$$\langle \hat{u}, u \rangle = \omega \tag{5.6}$$

where

$$\mathcal{A} = \begin{pmatrix} D_u f_{\text{diff}}(\tilde{u}, \tilde{\mu}) \\ R \end{pmatrix} = \left(\begin{array}{ccccccc} Y_1 & Z_1 & & & & & X_1 \\ X_2 & Y_2 & Z_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & X_M & Y_M & Z_M & \\ \hline P_- & \frac{1}{2h} Q_- & \ldots & -\frac{1}{2h} Q_+ & P_+ & \frac{1}{2h} Q_+ & -\frac{1}{2h} Q_- \end{array}\right)$$

with

$$Ru = P_- u_1 + Q_- \frac{1}{2h}(u_2 - u_0) + P_+ u_M + Q_+ \frac{1}{2h}(u_{M+1} - u_{M-1}).$$

Here $(\tilde{u}, \tilde{\mu})$ denotes the numerical solution of (5.3). The equation (5.5) is the approximation of the spectral problem for the operator Λ which determines the stability of the traveling wave and (5.6) is an appropriate normalizing condition which is needed to regularize the eigenvalue problem.

Note that the boundary conditions and therewith the matrices $P_\pm, Q_\pm$ need not be the same as in equation (5.4) which defines approximation of the traveling wave. As has been discussed in Section 3.2 already, the choice of boundary conditions has a great influence on the eigenvalues. For the numerical tests we solve (5.5),(5.6) with Newton's method, starting from the exact eigenfunctions and eigenvalues which are known in the test cases. Clearly, this is not an option when no initial values are known. In that case, one has to use a general eigenvalue solver.

A natural generalization of (5.5) to more general symmetries, i.e. the case described in Section 1.4.2 where $p = \dim \mathcal{N}(D_u f_{\text{diff}}(\tilde{u}, \tilde{\mu})) > 1$ is given by

$$\mathcal{A}V - \mathcal{B}VD = 0 \tag{5.7}$$

$$\hat{V}(V - \hat{V}) = 0 \tag{5.8}$$

where $V = [v^1, \ldots, v^p] \in \mathbb{R}^{m(M+3),p}$ and $D \in \mathbb{R}^{p,p}$. Here we compute a p-dimensional invariant subspace which belongs to the p eigenvalues near zero. In this case in each Newton step one has to solve linear equations of the form

$$\begin{aligned} \mathcal{A}V_\delta - \mathcal{B}(V\Lambda_\delta + V_\delta D) &= \mathcal{A}V - \mathcal{B}VD \\ \hat{V}^T V_\delta &= \hat{V}^T(V - \hat{V}) \end{aligned}$$

for $(V_\delta, \Lambda_\delta) \in \mathbb{R}^{m(M+3)+p,p} \times \mathbb{R}^{p,p}$. This is accomplished by using a Bartels-Stewart algorithm which is described in [27], [5]. The error in the invariant subspace is measured via the angle between the two subspaces V and $\hat{V}$. The cosines $\cos(\theta_1), .., \cos(\theta_p)$ of the principal angles $0 \le \theta_1 \le \ldots \le \theta_p \le \frac{\pi}{2}$ are given by the singular values of $V^T\hat{V}$, provided the two matrices are orthogonal. Then we define

$$\text{dist}(V, \hat{V}) = \sin(\theta_p) = \sqrt{1 - \sigma_{\min}^2}, \tag{5.9}$$

where $\sigma_{\min}$ is the minimal singular value of $V^T\hat{V}$ (see [20], [27]).

For the computation of the whole spectrum we solve the generalized eigenvalue problem (5.7) for $V, \Lambda \in \mathbb{R}^{m(M+3)+p, m(M+3)+p}$ with the standard Matlab eigenvalue solver which is an implementation of the QZ algorithm. Here the additional condition (5.8) needed in the application of Newton's method is not necessary.

To examine the influence of the bordering of Λ^h which has been introduced in Chapter 4 in order to remove the eigenvalue near zero from the spectrum, we consider the following bordered generalized eigenvalue problem

$$(\tilde{\mathcal{A}} - s\tilde{\mathcal{B}})v = 0, \quad v \in \mathbb{R}^{m(M+3)+p} \tag{5.10}$$

where

$$\tilde{\mathcal{A}} = \begin{pmatrix} \mathcal{A} & \Phi \\ \Psi^T & 0 \end{pmatrix}, \qquad \tilde{\mathcal{B}} = \begin{pmatrix} I_{m(M+1)} & 0 \\ 0 & 0 \end{pmatrix}$$

and $\Psi, \Phi \in \mathbb{R}^{m(M+3),p}$ are defined as in Section 4.2 by

$$\Psi = \begin{pmatrix} \mathrm{vec}(S_1(\tilde{u}))^T & \dots & \mathrm{vec}(S_p(\tilde{u}))^T \\ 0 & \dots & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \mathrm{vec}(S_1(\hat{u}))^T & \dots & \mathrm{vec}(S_p(\hat{u}))^T \\ 0 & \dots & 0 \end{pmatrix}.$$

The number of infinite eigenvalues of (5.5),(5.10) can be computed as follows. The multiplicity of $s = \infty$ corresponds to the multiplicity of $\lambda = 0$ of the problem

$$(\lambda\tilde{\mathcal{A}} - \tilde{\mathcal{B}})v = 0, \quad v \in \mathbb{C}^{m(M+3)+p}.$$

Since the span of the generalized eigenvectors corresponding to $\lambda = 0$ is given by $\mathcal{N}(\tilde{\mathcal{B}})$, which has dimension $2m + p$, a principal vector w is defined by $\tilde{\mathcal{B}}w = \tilde{\mathcal{A}}v$ where $v \in \mathcal{N}(\tilde{\mathcal{B}}) = \mathrm{span}\{e^{m(M+1)+1}, \dots, e^{m(M+3)+p}\}$. With the notation $v = \mathrm{vec}(v_0, \dots, v_{M+1}, v_{-1}, \mu), v_n \in \mathbb{R}^m, \mu \in \mathbb{R}^p$ and $w = \mathrm{vec}(w_0, \dots, w_{M+1}, w_{-1}, \lambda), w_n \in \mathbb{R}^m, \lambda \in \mathbb{R}^p$ we obtain that w is defined by $w_0 = X_1 v_{-1} + (\Phi\lambda)_0$, $w_M = Z_M v_{M+1} + (\Phi\lambda)_M$, $w_i = (\Phi\lambda)_i$, $i = 1, \dots, M-1$ if v satisfies

$$0 = P_-^N v_0 + Q_-^N \delta_0 v_0 + P_+^N v_M + Q_+^N \delta_0 v_M, \tag{5.11}$$

$$0 = P_-^D v_0 + P_+^D v_M \tag{5.12}$$

$$0 = \sum_{j=0}^{M} h S_k(\hat{u})_j^T v_j, \quad k = 1, \dots, p. \tag{5.13}$$

The equations (5.12),(5.13) are satisfied automatically and (5.11) reduces to r conditions

$$0 = -Q_-^N v_{-1} + Q_+^N v_{M+1}. \tag{5.14}$$

There exist no further principal generalized eigenvectors, since $Rw = 0$ and (5.14) imply $v = 0$. Thus the number of infinite eigenvalues is given by $2m+p+(2m+p-r) = 4m+2p-r$.

	m=1			m=2		
b.c.	r	p=0	p=1	r	p=0	p=2
Dirichlet	0	4	6	0	8	12
periodic	1	3	5	2	6	10
Neumann	2	2	4	4	4	8

Table 5.1: Number of ∞-eigenvalues

5.2 The Nagumo equation

The first example is the well known scalar Nagumo equation [25]

$$u_t = u_{xx} + u(1-u)(u-\lambda), \quad u(x,t) \in \mathbb{R}, \ x \in \mathbb{R}, \ t > 0, \tag{5.15}$$

where $\lambda \in (0, \frac{1}{2})$.

This equation is often used for testing algorithms since a traveling wave solution $u(x,t) = \bar{u}(x - \bar{\mu}t)$ connecting the stationary points $u_- = 0$, $u_+ = 1$ of this equation is explicitly known

$$\bar{u}(x) = \left(1 + \mathrm{e}^{\frac{-x}{\sqrt{2}}}\right)^{-1}, \quad \bar{\mu} = -\sqrt{2}\,(\tfrac{1}{2} - \lambda) \tag{5.16}$$

besides other explicit solutions, such as pulses, sources and sinks [8], [1], which we do not deal with here.

For the following computations we choose $\lambda = 0.25$ which leads to $\bar{\mu} \approx -0.3536$. The exact profile on the interval $[-40, 40]$ is displayed in Figure 5.1.

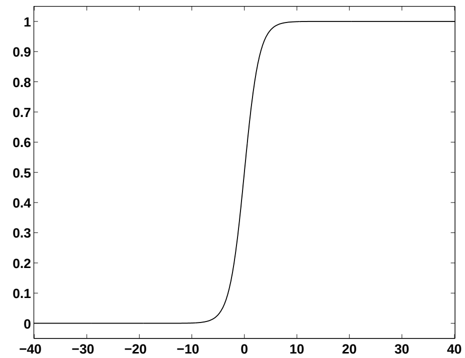
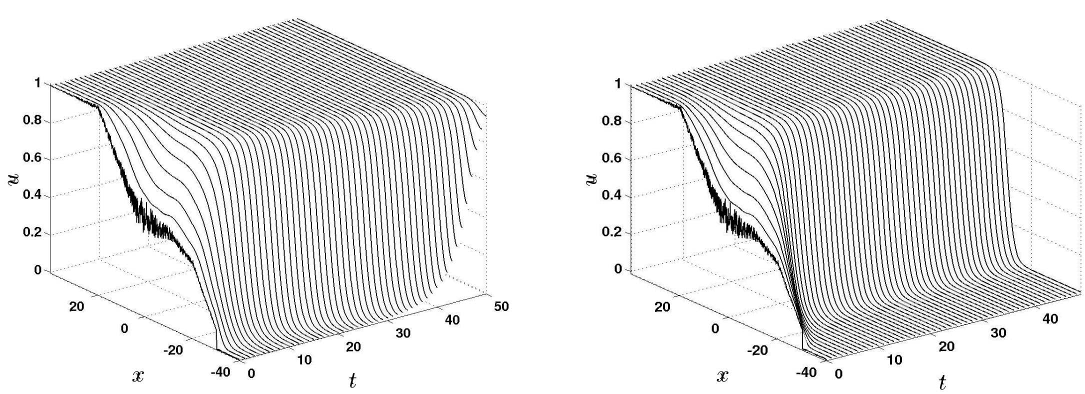


Figure 5.1: Nagumo, traveling front

The time dependent system

The time evolution of the frozen wave u starting from a randomly perturbed step-like initial profile, is shown in Figure 5.2 and compared to the corresponding traveling wave. In Figure 5.3 the development of the parameter μ is displayed. We use the Radau IIA method for the solution of the DAE which arises from a discretization with $h = 0.1$ on an interval $J = [-40, 40]$ with Neumann boundary conditions. We employ the fixed phase condition (2.59) and use exact solution $\bar{u}$ given in (5.16) for the reference function $\hat{u}$.



(a) traveling (b) frozen

Figure 5.2: Nagumo wave, evolution of $u(t)$

After a transient phase, the frozen wave is stabilized and the parameter μ converges to the exact velocity $\bar{\mu}$. In contrast, the traveling wave shown in 5.2(a) travels to the left and leaves the computational domain at $t \approx 50$.

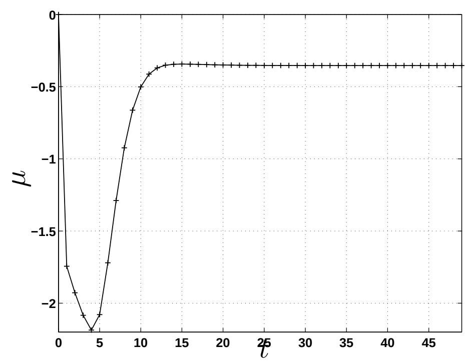


Figure 5.3: Frozen wave, evolution of the parameter $\mu(t)$

In Figure 5.4 we show the time evolution of the difference $|\mu(t) - \bar{\mu}|$ and $\|u(t) - \bar{u}_{|J}\|_\infty$ to the exact solution $(\bar{u}, \bar{\mu})$ defined in (5.16) of the boundary value problem (1.13). One clearly observes exponential convergence in time until the error reaches machine precision at $t \approx 80$. The exponential rate of convergence in this region is about $\alpha \approx -0.29$. This behavior matches the prediction from the stability result 4.6, and rate of convergence is in good agreement with the spectral information, since in this case $\max(\mathrm{Re}\,(\sigma(L) \setminus \{0\})) \approx 0.283$ (cf. Figure 5.8).

Note that, although it is not covered by the theory in this thesis, we get similar results using the orthogonality phase condition $0 = \Psi_{\text{orth}}$. Since here the resulting DAE is of index 1 only, we were able to perform the computations not only with the Radau and the θ-method, but as well with the standard Matlab DAE solvers for index 1 DAEs ode15s and ode23t [55],[54]. As expected, the results in this cases are similar to the ones reported above.

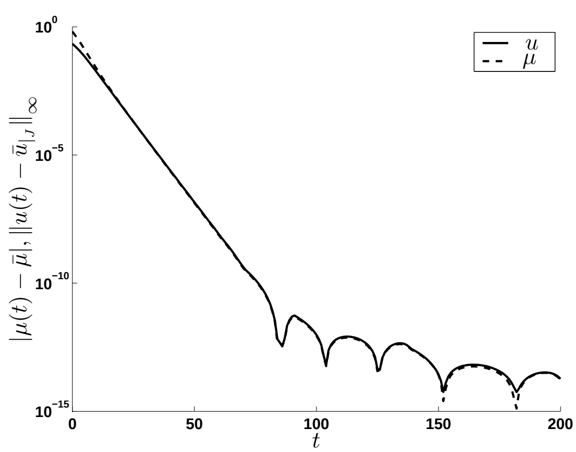
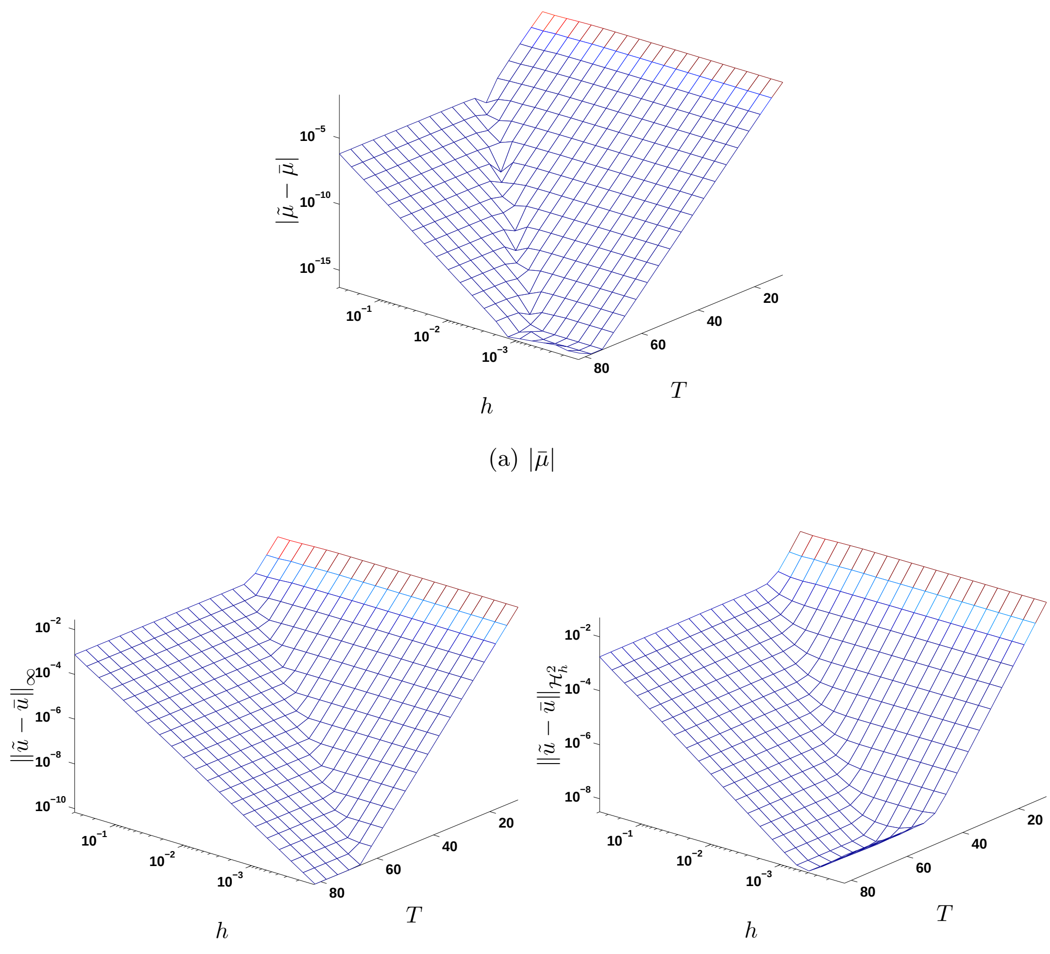


Figure 5.4: Nagumo, time evolution of $|\mu(t) - \bar{\mu}|$ and $\|u(t) - \bar{u}_{|J}\|_\infty$

Approximation of the traveling wave

Figures 5.5 and 5.6 show the approximation error of the traveling wave for periodic and Neumann boundary conditions. The grid size h has been varied exponentially in



(a) $|\bar{\mu}|$

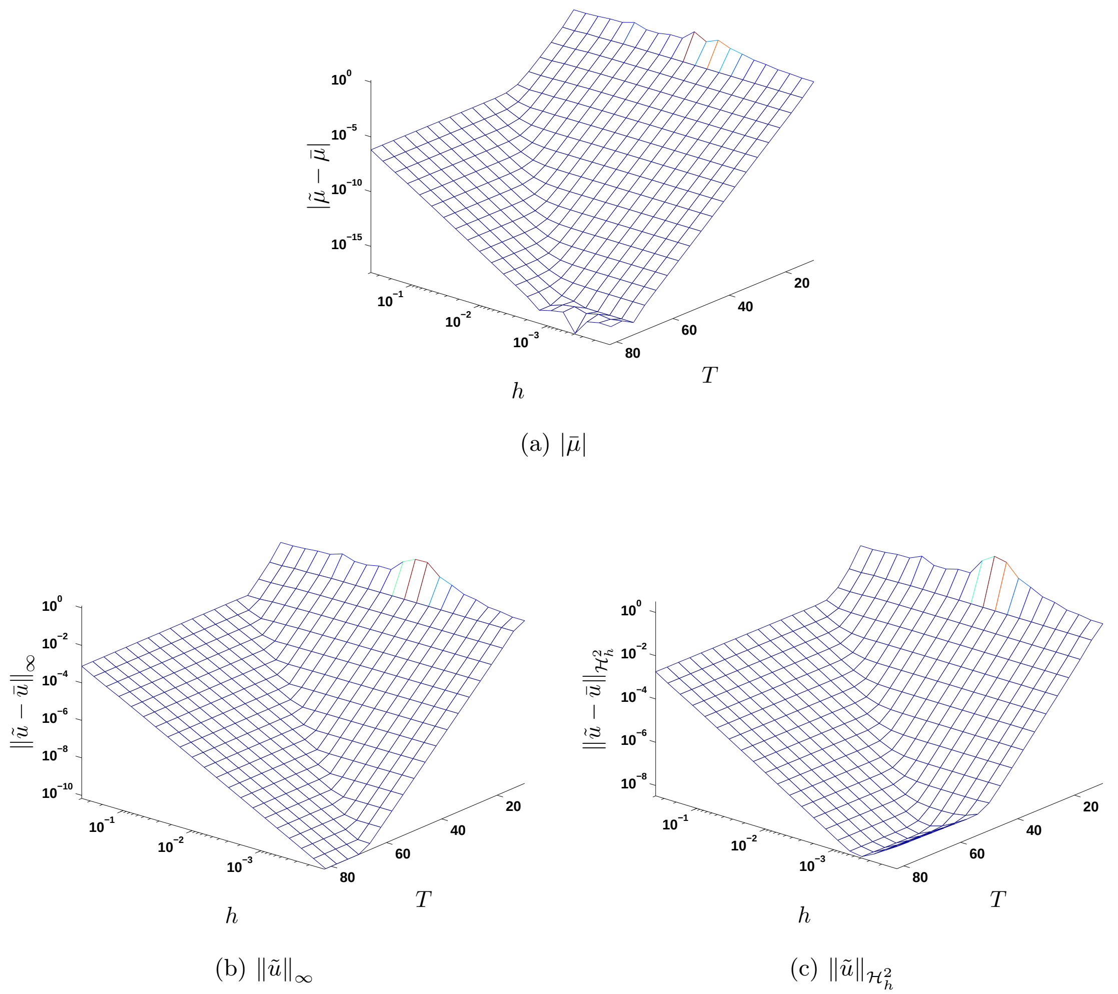
(b) $\|\tilde{u}\|_\infty$

(c) $\|\tilde{u}\|_{\mathcal{H}_h^2}$

Figure 5.5: Approximation error, Dirichlet b.c.

$[10^{-4}, 10^{-1}]$ and the size of the symmetric interval linearly in $[20, 80]$. We observe, that the convergence of the wave form $\tilde{u}$ to the exact solution $\bar{u}$ is linear in T and quadratic in h. This is in good agreement with the prediction of the approximation Theorem 2.21. The exponential factor in T is about $\alpha \approx 0.35$, i.e. $\|\bar{u} - \tilde{u}\|_\diamond \le K(h^2 + \mathrm{e}^{-\alpha T})$, $\diamond \in \{\infty, \mathcal{H}_h^2\}$. The parameter μ converges twice as fast in h to the exact velocity $\bar{\mu}$; here we observe $|\bar{\mu} - \tilde{\mu}| \le K(h^4 + \mathrm{e}^{-\alpha T})$, where $\alpha \approx 0.5$. This is a superconvergence phenomenon, which has been studied in [48]. For very small grid sizes $h < 10^{-3.5}$, the $\|\bar{u} - \tilde{u}\|_{\mathcal{H}_h^2}$ increases slightly. This is due to the fact, that the equations which have to be solved in the Newton Iteration become very ill conditioned. In later examples, this effect will become even more prominent.

Comparison of 5.5 and 5.6 shows, that the behavior is similar for Neumann and Dirichlet conditions. For small interval sizes, Neumann conditions perform worse than Dirichlet conditions. It is well known [3],[48], that the order of convergence can be improved using projection boundary conditions, but these have not been considered in this thesis.



(a) $|\bar{\mu}|$

(b) $\|\tilde{u}\|_\infty$ (c) $\|\tilde{u}\|_{\mathcal{H}_h^2}$

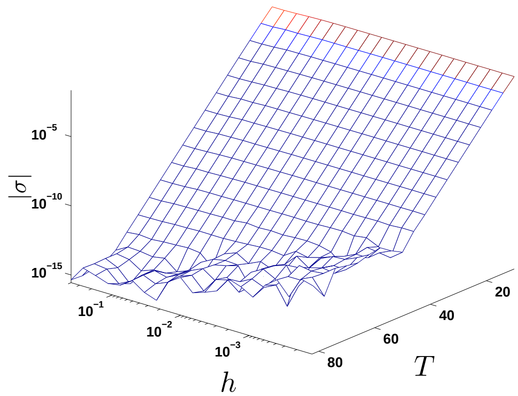
Figure 5.6: Approximation error, Neumann b.c.

Approximation of the eigenvalue 0

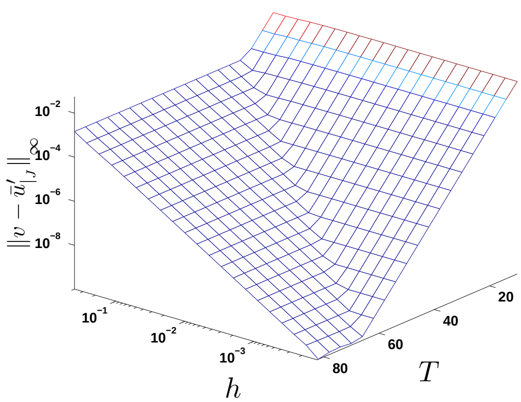
For the Nagumo equation the eigenvalue 0 which corresponds to the translational eigenmode $\bar{u}'$ (which is always present, since the equation (1.1) is equivariant w.r.t. translations) is simple. Thus the approximation Theorem 3.15 can be applied directly.

In order to document the dependence of convergence on the grid size h and the interval length T, we perform similar computations as above for the boundary value problem (3.78),(3.79),(3.80) near the eigenvalue $\sigma = 0$. We use the numerical solution $(\tilde{u}, \tilde{\mu})$ from above as linearization point and the exact eigenfunction $\bar{u}'$ restricted to the grid as reference function $\hat{u}$. As initial values we chose $(v^0, \mu^0) = (\bar{u}'_{|J}, 0)$. As before, we vary h in $[10^{-4}, 10^{-1}]$ and T in $[10, 80]$ and use homogenous Dirichlet boundary conditions.

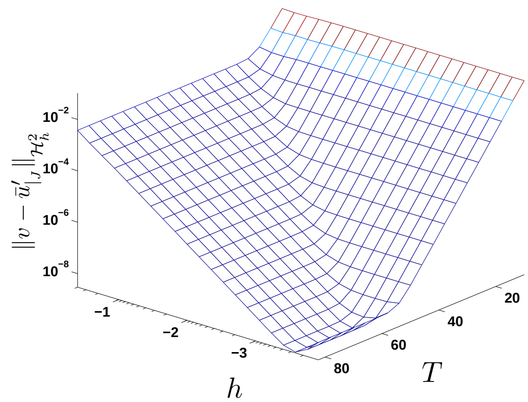
The error in v which is displayed in Figure 5.7 decreases linearly in T and quadratically in h. However, the error in the eigenvalue σ_h is constant for decreasing h in Figure 5.7(a). The bad conditioning of the matrices for small h and large T is clearly visible. But since the error in v is always larger, the overall behavior is still in accordance with the statement of theorem 3.15.



(a) $|\sigma|$



(b) $\|v - \bar{u}'_{|J}\|_\infty$



(c) $\|v - \bar{u}'_{|J}\|_{\mathcal{H}_h^2}$

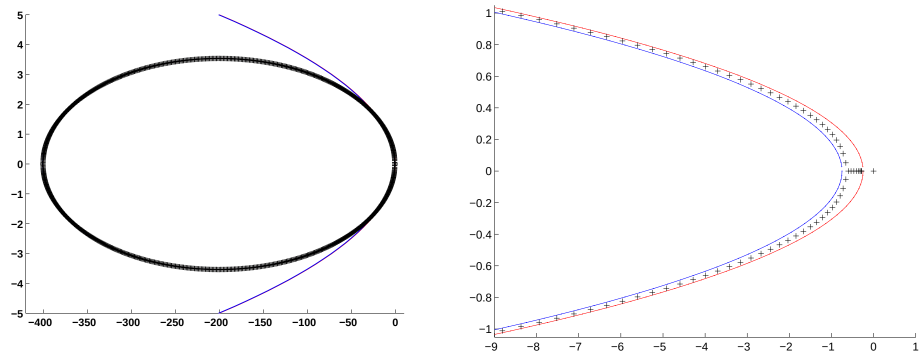
Figure 5.7: Eigenvalue near 0, approximation error, Dirichlet b.c.

The essential spectrum

In Figure 5.8 all eigenvalues of the generalized eigenvalue problem (5.5) with periodic boundary conditions are displayed with black crosses. The solid parabolas are the curves $s_\pm$ defined in (3.90) in Example 3.23. As has been discussed already there, the essential spectrum of Λ lies in the part of the left half plane which is bounded by s_-. Most eigenvalues lie on an ellipsis, which encloses the essential spectrum of the discrete operator Λ^h on the whole lattice $\mathbb{Z}$ defined by

$$(\Lambda^h u)_n = (\delta_+\delta_- u)_n + \bar{\lambda}(\delta_0 u)_n + f'(\bar{u}_n)u_n, \quad n \in \mathbb{Z}.$$

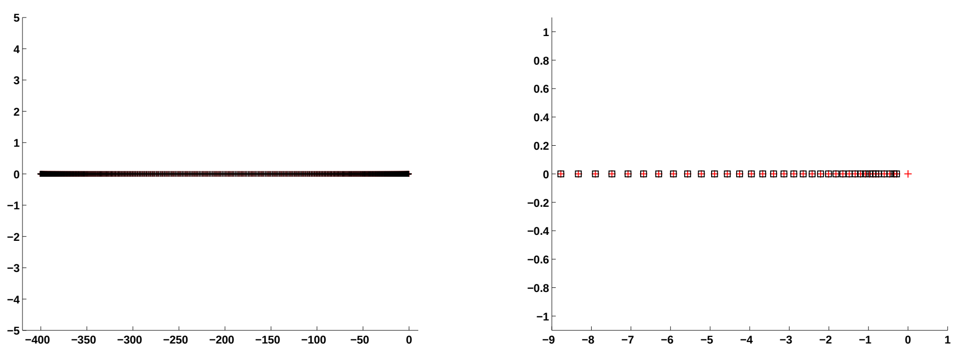
This has been discussed in Example 3.23. The zoom into the region near 0 in Figure 5.8(b) shows a simple eigenvalue near zero, which is separated by a gap from the rest of the spectrum. This gap is bounded by the parabola s_-. Using Dirichlet boundary conditions, one obtains the approximation of the absolute spectrum, which is given (cf. Example 3.23) by $\sigma_{abs} = (-\infty, -0.28]$ This is shown in Figure 5.9(a). The zoom in Figure 5.9(b) displays the eigenvalues of the of the bordered generalized eigenvalue problem (5.10)



(a) overview (b) zoom in

Figure 5.8: Nagumo, spectra

as well. Here the bordering is given by $\Psi = \mathrm{vec}(\delta_0 \tilde{u})^T$ and $\Phi = \mathrm{vec}(\delta_0 \hat{u})^T$. The spectrum is the same, exept for the zero eigenvalue which is removed from the spectrum as expected. The same is true for periodic boundary conditions. Note that the number of ∞-eigenvalues depends on the choice of boundary conditions, we obtain the predicted quantity $4m+2p-r$ (see Table 5.1).



(a) overview (b) zoom in, with border

Figure 5.9: Nagumo, spectra

5.3 The quintic complex Ginzburg Landau equation

The second example is the cubic quintic Ginzburg Landau equation [58], [62], [14]

$$u_t = au_{xx} + \delta u + g(u), \quad g(u) = \beta |u|^2 u + \gamma |u|^4 u, \quad u(x,t) \in \mathbb{C}\,, \delta \in \mathbb{R},\ a, \beta, \gamma \in \mathbb{C}\,. \tag{5.17}$$

This equation is an amplitude equation which describes the slow modulation in space and time of the envelope of the finite wavelength pattern for traveling wave systems just above the onset of a finite-wavelength instability. It shows a variety of coherent structures, like stable pulse solutions, fronts, sources, sinks, etc. Moreover, the equation has regimes where the behavior is intrinsically chaotic.

The equation is equivariant w.r.t. the group $G = S^1 \times \mathbb{R}$ with action

$$a(\gamma)u(x) = e^{-i\gamma_r} u(x - \gamma_t) \quad \text{for } \gamma = (\gamma_r, \gamma_t) \in G$$

and thus the functions $\bar{u}'$ and $i\bar{u}$ are eigenfunctions of

$$\Lambda u = au_{xx} + \delta u + \bar{\mu}_t u_x + i\bar{\mu}_r u + Dg(\bar{u})u$$

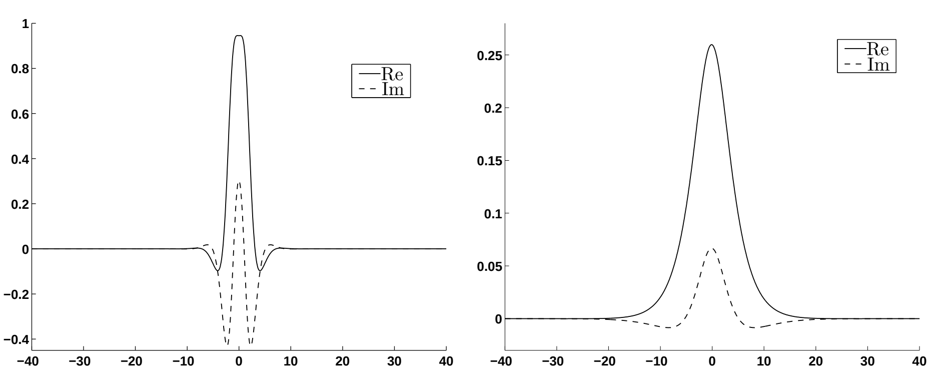
corresponding to zero. Thus here the condition of zero being a simple eigenvalue of Λ is not satisfied and the approximation results Theorem 2.21 and Theorem 3.15 do not apply directly. For numerical computations we write (5.17) in real variables. Introducing the two parameters μ_t and μ_r the operators arising from the symmetries (translation and rotation) are then given by

$$S_1 u = R_{\frac{\pi}{2}} u,\ S_2 u = u_x,$$

where R_φ denotes the rotation defined in (1.61). For certain parameter values, this equation possesses stable rotating pulses (the so called Thual-Fauve pulse [58]) and unstable pulses, as well as rotating and traveling fronts. All these solutions can be written (in complex notation) in the form

$$u(x,t) = e^{-i\bar{\mu}_r t} \bar{u}(x - \bar{\mu}_t t),$$

where for the rotating pulses, we have $\bar{\mu}_t = 0$. It depends on the choice of initial conditions which type of solution is selected.



(a) stable pulse (b) unstable pulse

Figure 5.10: QCGL, stable and unstable pulse

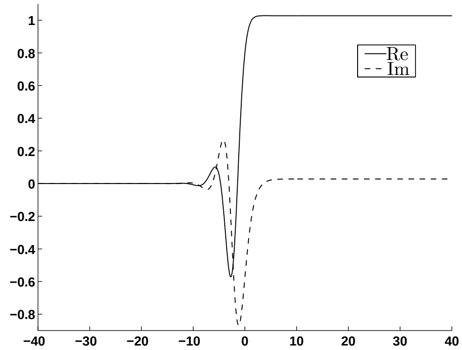


Figure 5.11: QCGL, front

For the parameter set $a = 1,\ \delta = -0.1,\ \beta = 3 + i,\ \gamma = -2.75 + i$, which has been used in [58], we found numerically a stable pulse with rotational velocity $\mu_r \approx -1.30$ as well as a rotating front. Here we used a grid size $h = 0.1$ and Dirichlet boundary conditions for the pulse and Neumann boundary conditions for the front on the interval $[-40, 40]$. These solutions are depicted on Figure 5.10(a) and 5.11.

Using Painlevé methods, some exact solutions have been constructed explicitly in [32]. With $\xi = x - \bar{\mu}_t t$ the explicit expression for an unstable pulse reads

$$u(x,t) = u_0 \mathrm{e}^{i(a_0\theta_0\xi - \bar{\mu}_r t)}(\cosh(k\xi) - \cosh(\rho))^{ia_0}\sqrt{\frac{k\sinh(\rho)}{\cosh(k\xi) + \cosh(\rho)}} \tag{5.18}$$

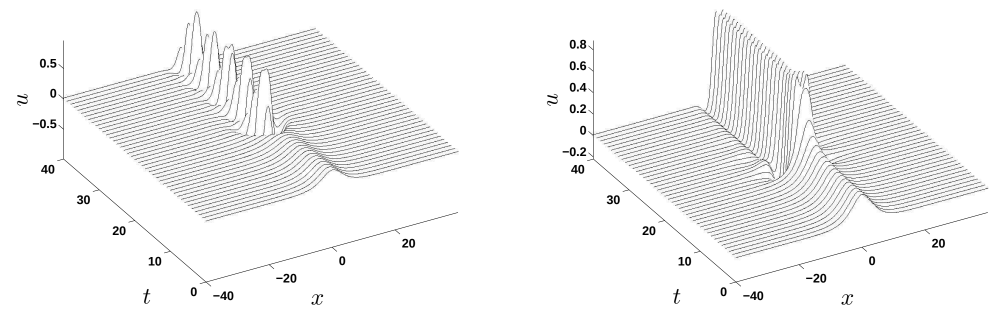
where $u_0, a_0, \theta_0, \rho, \bar{\mu}_r, \bar{\mu}_t, k$ are parameters that can be computed explicitly from a, δ, β and γ using quite complicated formulae which are given in [32] and which we do not want to restate here. For the used parameter set, we have $\bar{\mu}_t = 0,\ \bar{\mu}_r \approx 0.0573$ and all other parameters are real. Starting a Newton iteration with this explicit solution we found an unstable pulse with $\tilde{\mu}_r \approx 0.0573$ for the discretized equation on $J = [-40, 40]$ as well. This solution is shown in Figure 5.10(b).

The time dependent system

The time evolution of the real part of the stable pulse is compared for the frozen and the rotating system in Figure 5.12 on the interval $J = [-40, 40]$ with grid size $h = 0.1$. We start with the exact unstable pulse solution given in (5.18) and use Neumann boundary conditions. After a transient phase until $t \approx 15$, the rotating pulse rotates with a fixed rotational velocity $\bar{\mu}_r$. In contrast, the frozen pulse is stabilized. As is shown in Figure 5.14(a) the parameter μ_r converges to a fixed velocity $\bar{\mu}_r$ whereas the translational speed μ_t stays at zero.

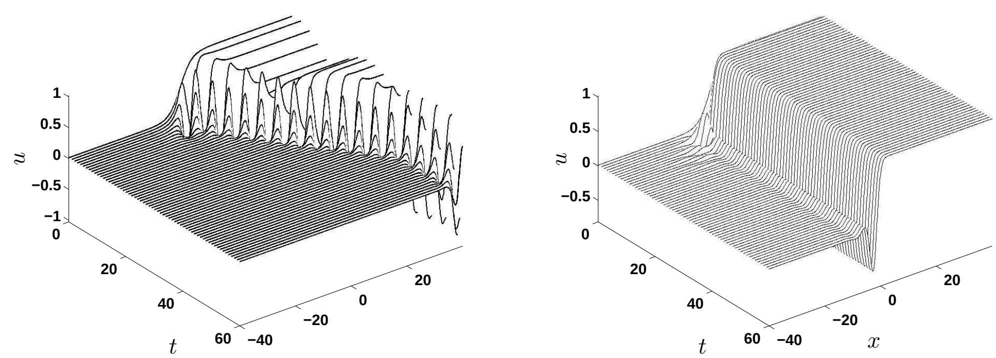
The comparison of the rotating and traveling with the frozen front in Figure 5.13 shows a similar situation. The frozen wave stabilizes quickly, whereas the non-frozen front continues to rotate and travels out of the computational domain at $t \approx 60$.

The parameters μ_t and μ_r converge to the same translational speed and rotational velocity that are observed in the non-frozen system. This is displayed in Figure 5.14(b).



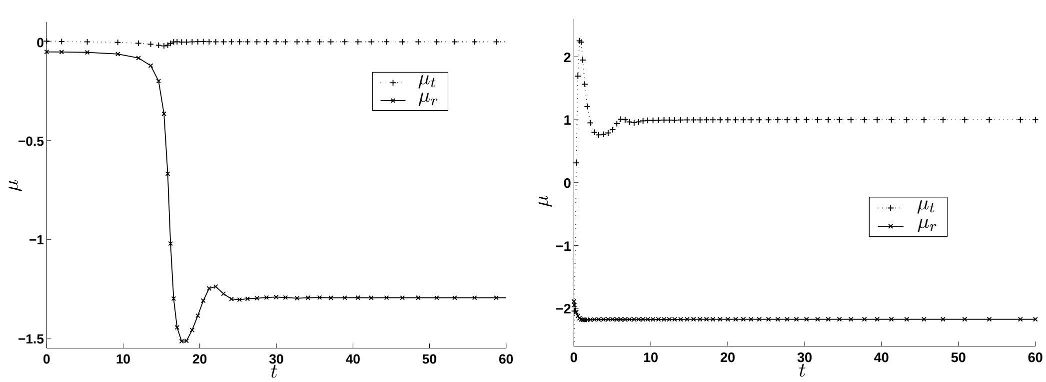
(a) rotating (b) frozen

Figure 5.12: QCGL, rotating vs. frozen pulse



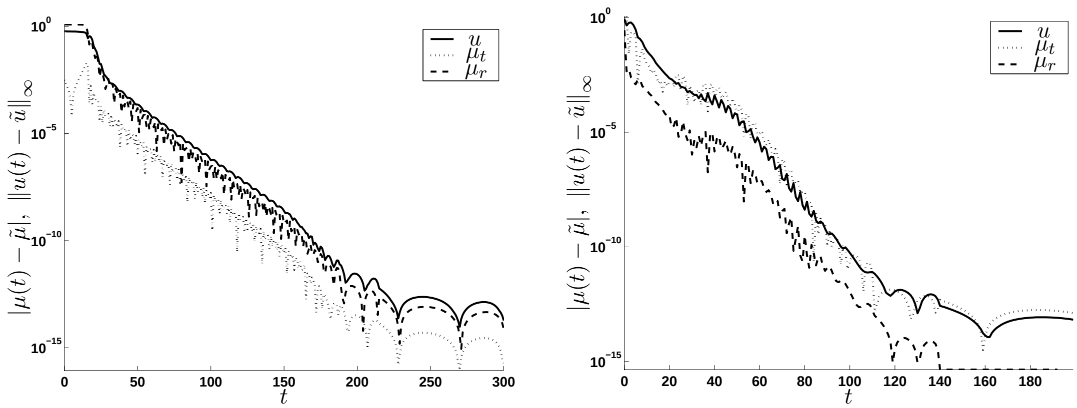
(a) rotating (b) frozen

Figure 5.13: QCGL, rotating vs. frozen front



(a) pulse (b) front

Figure 5.14: QCGL, time evolution of μ_r, μ_t



(a) pulse (b) front

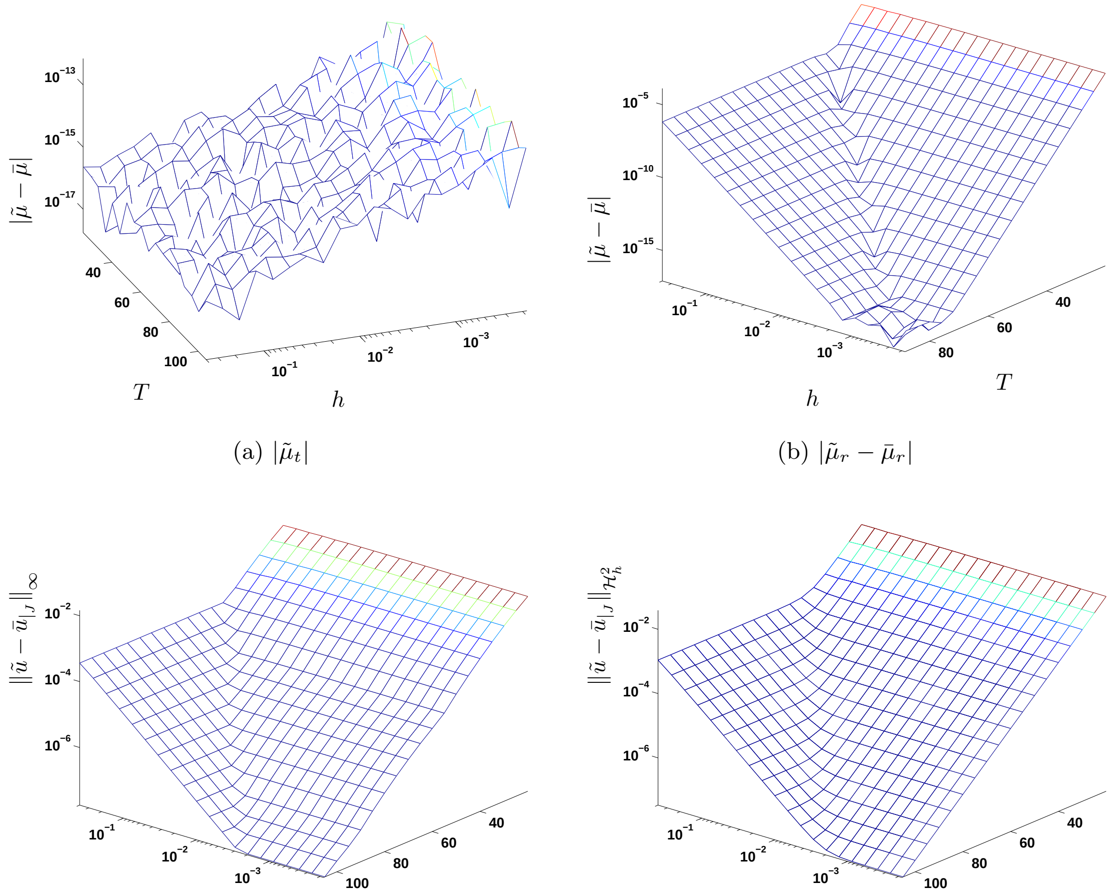
Figure 5.15: QCGL, time evolution of $|\mu(t) - \tilde{\mu}|$ and $\|u(t) - \tilde{u}\|_\infty$

The rate of this convergence is discussed in Figure 5.15, where the time evolution of the difference to the solution of the boundary value problem (2.56)–(2.58) is shown.

The error $|\mu(t) - \tilde{\mu}|$ in the parameters μ_t, μ_r is displayed as well as the error in the waveform $\|u(t) - \tilde{u}\|_\infty$. As in the previous example, the exponential convergence in time matches the prediction from the stability result Theorem 4.6. Here the convergence rate of $\alpha \approx 0.12$ for the pulse and $\alpha \approx 0.2$ for the front is again in good agreement with the spectral information (see Figure 5.18(a) and 5.20).

Approximation of the unstable pulse

As in the previous example we compare the approximation error of the solution of the boundary value problem (2.56), (2.57),(2.58) with the estimates in Theorem 2.21. For the unstable pulse the exact solution is explicitly given by (5.18). Figure 5.16 shows the approximation error of the pulse for Dirichlet boundary conditions. The grid size h is varied exponentially in $[10^{-4}, 10^{-1}]$ and the size of the symmetric interval J linearly in $[20, 80]$. As shown in Figure 5.16 the parameters μ_t, μ_r converge much faster than the wave form $\tilde{u}$ to the exact values. The rate of convergence of μ_r to $\bar{\mu}_r$ is of order 4 in h and the exponential rate in T is $\alpha \approx 0.5$. In contrast, μ_t reaches quickly the range of machine precision where rounding errors dominate and the bad conditioning of the equations in the Newton iteration becomes prominent. The wave $\hat{u}$ itself converges as predicted with quadratic order in h and with $\alpha \approx 0.16$ in T. This can be observed in $\|\cdot\|_{\mathcal{H}_h^2}$ as well as in $\|\cdot\|_\infty$ (see Figures 5.16(c), 5.16(d)). In all cases the overall behavior matches the predictions made in Theorem 2.21.



(a) $|\tilde{\mu}_t|$

(b) $|\tilde{\mu}_r - \bar{\mu}_r|$

(c) $\|\tilde{u} - \bar{u}_{|J}\|_{\infty}$

(d) $\|\tilde{u} - \bar{u}_{|J}\|_{\mathcal{H}_h^2}$

Figure 5.16: QCGL, approximation error for the unstable pulse

Approximation of discrete eigenvalues

The corresponding linearization of the transformed equation at the exact solution $\bar{u} = (\bar{v}, \bar{w})$ here reads

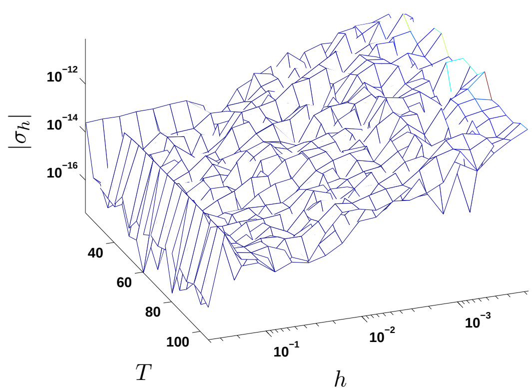
$$\Lambda u = A u_{xx} + \bar{\mu}_t u_x + (\delta I + B M_1 + G M_2 + \bar{\mu}_r R_{\frac{\pi}{2}}) u$$

where

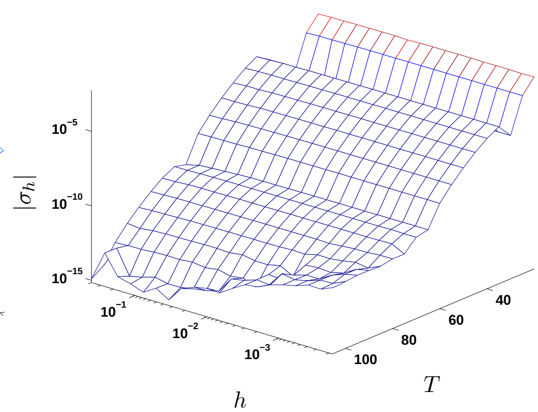
$$A = M_{\mathrm{Re}}\,(\alpha),\; B = M_{\mathrm{Re}}\,(\beta),\; G = M_{\mathrm{Re}}\,(\gamma) \quad \text{with } M_{\mathrm{Re}}\,(z) = \begin{pmatrix} \mathrm{Re}\, z & -\mathrm{Im}\, z \\ \mathrm{Im}\, z & \mathrm{Re}\, z \end{pmatrix}$$

and

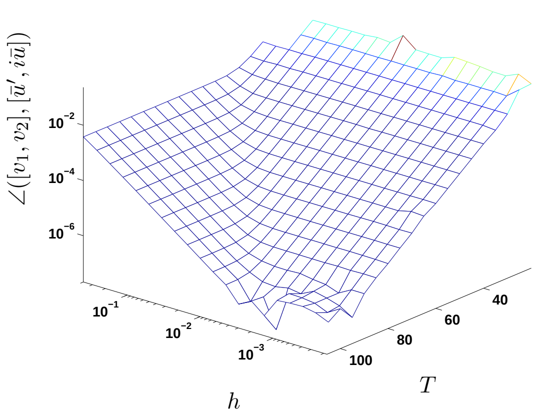
$$M_1 = \begin{pmatrix} 3\bar{v}^2 + \bar{w}^2 & 2\bar{v}\bar{w} \\ 2\bar{v}\bar{w} & \bar{v}^2 + 3\bar{w}^2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 5\bar{v}^4 + 6\bar{v}^2\bar{w}^2 + \bar{w}^4 & 4(\bar{v}^3\bar{w} + \bar{v}\bar{w}^3) \\ 4(\bar{v}^3\bar{w} + \bar{v}\bar{w}^3) & \bar{v}^4 + 6\bar{v}^2\bar{w}^2 + 5\bar{w}^4 \end{pmatrix}$$



(a) $|\sigma_t|$



(b) $|\sigma_r|$



(c) $\angle([v_1, v_2], [\bar{u}', i\bar{u}])$

Figure 5.17: QCGL, approximation error for the double zero eigenvalue

In order to solve the eigenvalue problem (5.7),(5.8) we use a Newton method, starting from $\hat{V} = [\hat{u}'_{|J}, i\hat{u}_{|J}]$. Alternatively we use the function eigs of the Matlab implementation of Arpack [31] compute the two eigenvalues of smallest magnitude of the generalized eigenvalue problem (5.7) iteratively.

The errors in the subspaces as defined in (5.9) and the absolute values of the two eigenvalues near 0 are shown in Figure 5.17 for the unstable pulse. Here σ_t denotes the eigenvalue which belongs to the approximation of the translational eigenfunction $\bar{u}'$ and σ_r is the eigenvalue which belongs to the approximation of the rotational eigenfunction $i\bar{u}$. It can be seen that the translations eigenvalue σ_t is in the range of machine precision, thus the errors increase for decreasing h, since the condition of the eigenvalue problem gets worse. The error in the rotational eigenvalue σ_r is nearly constant for different h, but decreases for increasing T, as expected. For very small h and large T the increase in error due to the conditioning becomes visible as well. The angle between the invariant subspace which belongs to σ_t and σ_r and the span of restriction of the exact eigenfunctions $\bar{u}'_{|J}$ and $i\bar{u}_{|J}$ to the grid shows the expected behavior. It decreases quadratically in h and linearly in T with a rate of ca. -0.32 until the range of machine precision is reached.

Note that the choice of boundary conditions decides about the multiplicity of zero. For example, zero is a double eigenvalue for the pulse with periodic boundary conditions, whereas it is a simple eigenvalue for the front, since $i\bar{u}$ is not periodic. For the continuous operator the same is true: $i\bar{u}$ is not in $\mathcal{L}_2$ if $\bar{u}$ is a front.

The essential spectrum

The dispersion relation (1.6) is given by

$$\det(-\kappa^2 I + i\kappa\bar{\mu}_t I + \bar{\mu}_r R_{\frac{\pi}{2}} + \delta I + BM_1^{\pm} + GM_2^{\pm} - sI) = 0 \tag{5.19}$$

where $M_1^{\pm}$, $M_2^{\pm}$ are given by inserting the stationary points $(v_{\pm}, w_{\pm})$ in M_1, M_2.

Similarly, the essential spectrum of the operator on the whole line is determined by (see Lemma 3.18) the solutions $s \in \mathbb{C}$ of

$$\det\left(\frac{2}{h^2}(\cos(\kappa) - 1)I + \frac{i}{h}\sin(\kappa)\bar{\mu}_t I + \bar{\mu}_r R_{\frac{\pi}{2}} + \delta I + BM_1^{\pm} + GM_2^{\pm} - sI\right) = 0,\ \kappa \in \mathbb{R}. \tag{5.20}$$

Inserting the data $\bar{\mu}_t = 0$, $\bar{u}_{\pm} = 0$ of the (stable or unstable) pulse , we obtain $M_i^{\pm} = 0$ and (5.19) simplifies to

$$\det\begin{pmatrix} -\kappa^2 + \delta - s & -\bar{\mu}_r \\ \bar{\mu}_r & -\kappa^2 + \delta - s \end{pmatrix} = 0, \quad \kappa \in \mathbb{R}.$$

Thus the essential spectrum of the linearization of the operator Λ on the whole line at an pulse, consists of the two half lines which are given by

$$s_{\pm}(\kappa) = -\kappa^2 + \delta \pm i\bar{\mu}_r$$

which is $\pm i\bar{\mu} + [-\infty, \delta]$.

Similarly, the solution of (5.20) simplifies to

$$\det\begin{pmatrix} \frac{2}{h^2}(\cos(\kappa) - 1) + \delta I - sI & -\bar{\mu}_r \\ \bar{\mu}_r & \frac{2}{h^2}(\cos(\kappa) - 1) + \delta I - sI \end{pmatrix} = 0,\ \kappa \in \mathbb{R}$$

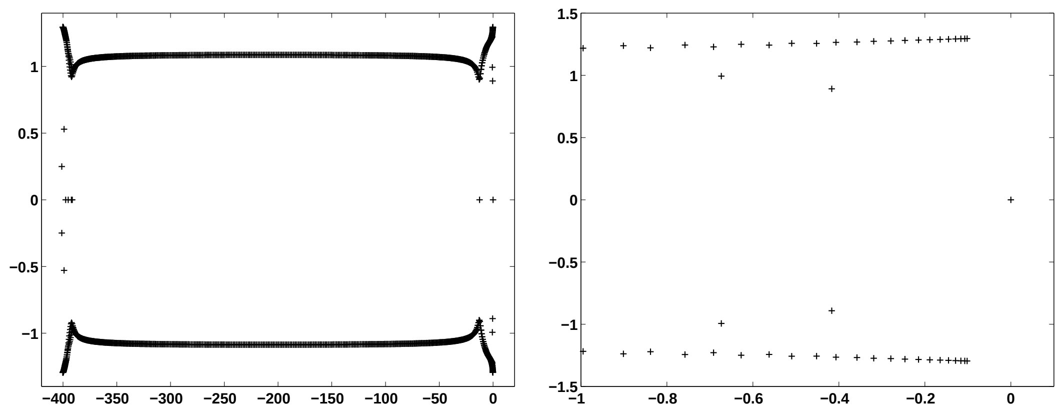
which is solved by

$$\Sigma_{\pm}^{h}(\kappa) = \frac{2}{h^2}(\cos(\kappa) - 1) + \delta \pm i\bar{\mu}_r.$$

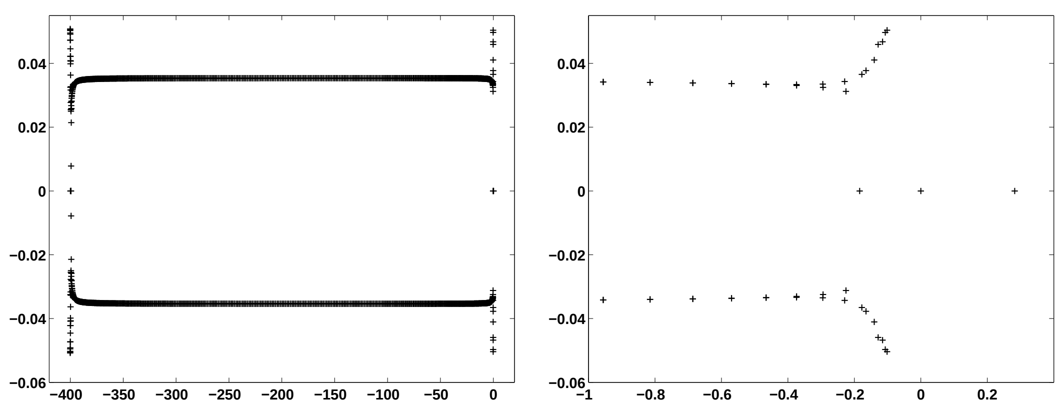
This are line segments, given by $\pm i\bar{\mu}_r + [-\frac{4}{h^2}, \delta]$.

In Figure 5.18 the solutions of the generalized eigenvalue problem (5.5) for the stable and the unstable pulse are compared. The zoom near zero shows that for the stable pulse only the (double) zero eigenvalue is present, whereas for the unstable pulse an eigenvalue with real part > 0 exists as well. In order to approximate the essential spectrum, we have used periodic boundary conditions (compare section 3.2), but the approximation of the lines mentioned above is still rather coarse.

Note that one has to be careful interpreting the numerical for the whole spectrum. For small grid sizes h and large T the condition of the eigenvalues of (5.5) becomes quite bad.



(a) stable pulse



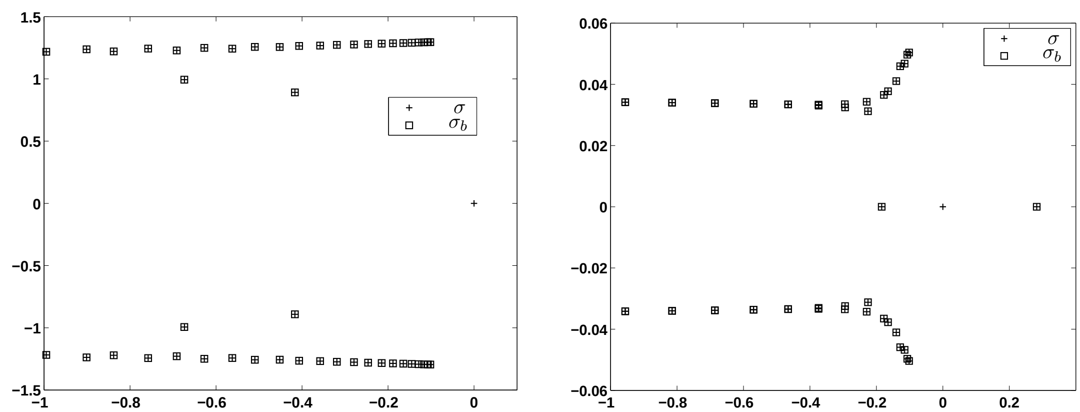
(b) unstable pulse

Figure 5.18: QCGL, spectra

In Figure 5.19 we compare solutions σ of the system (5.5) with the eigenvalues σ_b of the bordered system (5.10). Here the bordering is given by $\Psi = \begin{pmatrix}\delta_0\hat{u} & i\hat{u}\end{pmatrix}, \quad \Phi = \begin{pmatrix}\delta_0\tilde{u} & i\tilde{u}\end{pmatrix}$. It can be clearly seen, that the zero eigenvalue is removed from the spectrum σ_b of the bordered operator and that this procedure works for the unstable situation as well (although there one cannot make use of it).

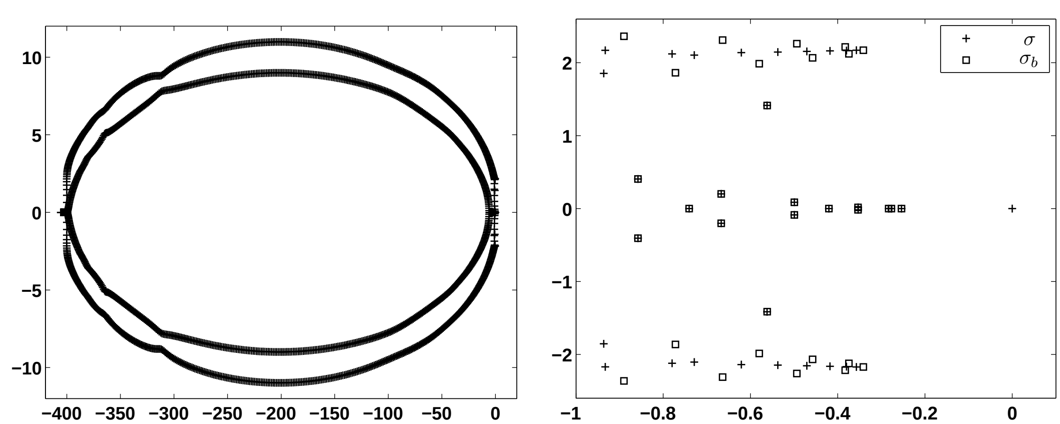
The same is shown in Figure 5.20 for the stable front. Here it becomes visible that the bordering does not only remove zero from the spectrum, but has an effect on the other eigenvalues as well. Nevertheless no additional eigenvalues are created on the right of the spectral gap at ca. -0.22, as expected by the resolvent estimates (3.4), (3.15),(3.16).

For the number of infinite eigenvalues of the generalized eigenvalue problem we obtain the predicted quantity $4m + 2p - r$ (see Table 5.1).



(a) stable pulse (b) unstable pulse

Figure 5.19: QCGL, bordered system, zoom in, spectra



(a) spectrum (b) zoom-in with bordering

Figure 5.20: QCGL, front, spectrum, zoom in with border

Appendix A

Auxiliary results

A.1 Functional analytic notions

First recall that $s \in \mathbb{C}$ is in the resolvent set of the operator $L : \mathcal{D}(L) \subset X \to X$, if $L - sI$ has a bounded inverse on X and that the essential spectrum $\sigma_{\text{ess}}(L)$ contains all $s \in \mathbb{C}$ that are neither in the resolvent set nor eigenvalues of finite multiplicity.

Definition A.1 *Let X be a Banach space and $L : X \supseteq \mathcal{D}(L) \to X$ a linear operator.*

1. *The operator $R_s(\Lambda) = (sI - L)^{-1}$ with domain $\mathcal{D}(R_s(\Lambda))$ is called the resolvent of L in s.*
2. *The resolvent set $\rho(L)$ contains all $\lambda \in \mathbb{C}$ for which*
   - *R_λ exists*
   - *R_λ is bounded*
   - *$\mathcal{D}(R_s(\Lambda)) = R(sI - L)$ is dense in X.*
3. *The complement of the resolvent set $\sigma(L) = \mathbb{C} \setminus \rho(L)$ is called spectral set. It can be divided into two subsets $\sigma(L) = \sigma_{\text{ess}}(L) \cup \sigma_{\text{pt}}(L)$, where the point spectrum $\sigma_{\text{pt}}(L)$ contains all isolated eigenvalues of finite multiplicity and $\sigma_{\text{ess}}(L) = \sigma(L) \setminus \sigma_{\text{pt}}(L)$ is called the essential spectrum.*

A.2 Fixed point theorems

Parameter dependent contraction Lemma

Theorem A.2 *Let X, Y be Banach spaces and $F : (X \times Y) \supset B_\varrho(0) \times B_\delta(0) \to Y$ be a continuous mapping, which satisfies the following estimates for $q \in [0, 1)$:*

$$\|F(x, y_1) - F(x, y_2)\| \leq q\|y_1 - y_2\| \quad \forall x \in B_\rho(0),\ y_1, y_2 \in B_\delta(0) \tag{A.1}$$

$$\|F(x, 0)\| \leq \delta(1 - q) \quad \forall x \in B_\rho(0) \tag{A.2}$$

Then for each $x \in B_\rho(0)$ there exists a unique fixed point $\bar{y} = g(x)$ of $F(x,\cdot)$, i.e. $F(x,g(x)) = g(x)$ and the following estimate holds

$$\|y_1 - y_2\| \le \frac{1}{1-q}\|y_1 - F(x,y_1) - (y_2 - F(x,y_2))\| \quad \forall x \in B_\rho(0), y_1, y_2 \in B_\delta(0). \tag{A.3}$$

Note that (A.3) implies the continuity of g in $B_\rho(0)$, since

$$\begin{aligned}\|g(x_1) - g(x_2)\| &\le \frac{1}{1-q}\|g(x_1) - F(x_1,g(x_1)) - (g(x_2) - F(x_1,g(x_2)))\| \\ &= \frac{1}{1-q}\|F(x_2,g(x_2)) - F(x_1,g(x_2))\|.\end{aligned} \tag{A.4}$$

Nonlinear perturbation theorem

(see [61], HS 50 or [3], Lemma 3.1)

Theorem A.3 *Let $F : Y \supset B_\varrho(\bar{y}) \to Z$ be a C^1 mapping between two Banach spaces Y and Z and let $(DF(\bar{y}))^{-1} \in L[Z,Y]$ exist. Assume the following estimates for $\kappa, \sigma > 0$*

$$\|DF(y) - DF(\bar{y})\|_{Y\to Z} \le \kappa < \sigma \le \frac{1}{\|DF(\bar{y})^{-1}\|_{Z\to Y}} \quad \forall y \in B_\varrho(\bar{y}), \tag{A.5}$$

$$\|F(\bar{y})\|_Z \le (\sigma - \kappa)\varrho. \tag{A.6}$$

Then F has a unique zero y_0 in $B_\varrho(\bar{y}) = \{y : \ \|y - \bar{y}\|_Y \le \varrho\}$ and the following estimates hold

$$\|y_0 - \bar{y}\|_Y \le \frac{1}{(\sigma - \kappa)}\|F(\bar{y})\|_Z \tag{A.7}$$

$$\|y_1 - y_2\|_Y \le \frac{1}{(\sigma - \kappa)}\|F(y_1) - F(y_2)\|_Z \quad \forall y_1, y_2 \in B_\varrho(\bar{y}). \tag{A.8}$$

Discrete Sobolev embedding

We need a discrete version of the Sobolev embedding $\mathcal{H}^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$ which has been proven for intervals of length one in the appendix of [29]. Here we prove a modified version which takes into account the variability of the interval length.

Lemma A.4 *Let $u \in S_J(\mathbb{C}^m)$, then for any $C > 1$ there exists $T > 0$ such that for any $h > 0$ and $\pm hn_\pm > T$ the following discrete Sobolev inequality holds*

$$\|u\|_\infty \le C\|u\|_{\mathcal{H}^1_h}. \tag{A.9}$$

Proof: With the notation

$$\langle u, v\rangle_{r,s} = h\sum_{n=r}^{s} u_n^H v_n, \qquad \|u\|_{r,s}^2 = \langle u, u\rangle_{r,s}$$

we have the following version of the summation by parts formula

$$\langle u, \delta_+ v\rangle_{r,s} = -\langle \delta_- u, v\rangle_{r+1,s+1} + u_{s+1}^H v_{s+1} - u_r^H v_r. \tag{A.10}$$

Let

$$\|u_k\| = \min_{n\in J} \|u_n\|, \qquad \|u_l\| = \max_{n\in J} \|u_n\| = \|u\|_\infty$$

and assume w.l.o.g. $l > k$. With (A.10) we obtain

$$\begin{aligned}\|u_l\|^2 - \|u_k\|^2 &= \langle u, \delta_- u\rangle_{k,l-1} + \langle \delta_+ u, u\rangle_{k+1,l} \\ &\le 2\|u\|_{\mathcal{L}_{2,h}} \|\delta_+ u\|_{\mathcal{L}_{2,h}} \le \|u\|^2_{\mathcal{L}_{2,h}} + \|\delta_+ u\|^2_{\mathcal{L}_{2,h}}.\end{aligned}$$

Since

$$\|u_k\|^2 \le \frac{1}{h(n_+ - n_- + 1)} \sum_{n=n_-}^{n_+} h\|u_n\|^2 \le \frac{1}{2T}\|u\|^2_{\mathcal{L}_{2,h}}$$

we obtain for $T \ge \frac{1}{2(C^2-1)}$

$$\begin{aligned}\|u\|^2_\infty = \|u_k\|^2 + \|u_l\|^2 - \|u_k\|^2 &\le \frac{1}{2T}\|u\|^2_{\mathcal{L}_{2,h}} + \|u\|^2_{\mathcal{L}_{2,h}} + \|\delta_+ u\|^2_{\mathcal{L}_{2,h}} \\ &\le (1 + \frac{1}{2T})\|u\|^2_{\mathcal{H}^1_h} \le C^2\|u\|^2_{\mathcal{H}^1_h}.\end{aligned}$$

□

A.3 Exponential dichotomies for ordinary differential equations

In this section we repeat some well known results about exponential dichotomies for ordinary differential equations which can be found in [42], [60], and some facts about the operator L defined in (1.46).

Definition A.5 (Exponential dichotomy)
The linear differential operator

$$Lz = z' - M(\cdot)z, \quad x \in J \subset \mathbb{R},\ M(\cdot) : J \to \mathbb{R}^{m,m} \tag{A.11}$$

with solution operator $S(x,\xi)$ has an **exponential dichotomy (ED)** *in the interval $J = [x_-, x_+]$, $x_\pm \in \mathbb{R} \cup \{\pm\infty\}$ with data (K, α, π) if there exist a bound $K > 0$, a rate $\alpha > 0$ and a function $\pi : J \ni x \mapsto \pi(x)$, $\pi(x)$ a projector, such that the following holds*

$$S(x,\xi)\pi(\xi) = \pi(x)S(x,\xi) \tag{A.12}$$

and the Green's function

$$G_c(x,\xi) = \begin{cases} S(x,\xi)\pi(\xi), & x \ge \xi \\ -S(x,\xi)(I - \pi(\xi)), & x < \xi, \end{cases}$$

satisfies an exponential estimate

$$\|G_c(x,\xi)\| \le K\mathrm{e}^{-\alpha|x-\xi|}, \quad x, \xi \in J. \tag{A.13}$$

If $J = (-\infty, 0]$ then the kernel of the projector $\pi(0)$ is given by

$$\mathcal{N}(\pi(0)) = \{z_0 \in \mathbb{R}^m : \sup_{x \le 0} \|S(x,0)z_0\| < \infty\}$$

and for $J = [0, \infty)$ the image of $\pi(0)$ is given by

$$\mathcal{R}(\pi(0)) = \{z_0 \in \mathbb{R}^m : \sup_{x \ge 0} \|S(x,0)z_0\| < \infty\},$$

(see [41], Section 2). If L has an exponential dichotomy on $(-\infty, 0]$ and $[0, \infty)$ with data $(K_\pm, \alpha_\pm, \pi_\pm)$, then the kernel of L is given by

$$\mathcal{N}(L) = \{S(\cdot, 0)z_0 : \ z_0 \in \mathcal{N}(\pi_-(0)) \cap \mathcal{R}(\pi_+(0))\}.$$

Note that, if L has exponential dichotomies on $\mathbb{R}^\pm$ with data $(K_\pm, \alpha_\pm, \pi_\pm)$ then the adjoint operator

$$L^* : \mathcal{L}_2 \to \mathcal{H}^2, \ z \mapsto z' + M^T(\cdot)z \tag{A.14}$$

also has exponential dichotomies on $\mathbb{R}^\pm$ with projectors $I - \pi_\pm^T$ and

$$\mathcal{N}(L) = \{S(x,0)z_0 : \ z_0 \in \mathcal{R}(\pi^+) \cap \mathcal{N}(\pi^-)\}.$$

Thus for $\phi \in \mathcal{N}(L)$ we obtain the exponential estimate

$$\|\phi(x)\| \le K\mathrm{e}^{-\alpha|x|}, \ x \in \mathbb{R}. \tag{A.15}$$

Note that, G_c being a Green's function means that the solution of the linear inhomogeneous equation

$$Lz = \bar{r}, \quad x \in J \tag{A.16}$$

is given by $z(x) = \int_J G(x,\xi)r(\xi)\, d\xi$.

Thus if the operator L has exponential dichotomies on $\mathbb{R}^\pm$ with data $(K_\pm, \alpha_\pm, \pi_\pm)$, then solutions of (A.16) on $J = \mathbb{R}^\pm$ are given by

$$\begin{aligned}
[\bar{s}^-(\bar{r})](x) &= \int_{-\infty}^0 G_c^-(x,\xi)\bar{r}(\xi)\, d\xi \\
&= \int_{-\infty}^x S(x,\xi)P^-(\xi)\bar{r}(\xi)\, d\xi - \int_x^0 S(x,\xi)(I - P^-(\xi))\bar{r}(\xi)\, d\xi \\
[\bar{s}^+(\bar{r})](x) &= \int_0^\infty G_c^+(x,\xi)\bar{r}(\xi)\, d\xi \\
&= \int_0^x S(x,\xi)P^+(\xi)\bar{r}(\xi)\, d\xi - \int_x^\infty S(x,\xi)(I - P^+(\xi))\bar{r}(\xi)\, d\xi.
\end{aligned} \tag{A.17}$$

Using the dichotomy estimates, these solutions can be estimated for $\bar{r} \in \mathcal{L}_2$ by (cf. Lemma 3.21 in [60])

$$\|\bar{s}^\pm(\bar{r})\|_{\mathcal{L}_2} + \|[\bar{s}^\pm(\bar{r})](0)\| \le C\|\bar{r}\|_{\mathcal{L}_2}. \tag{A.18}$$

In order to infer the existence of exponential dichotomies on $\mathbb{R}^\pm$ for the operator L defined in (1.46) from the existence of exponential dichotomies for the constant coefficient operators $L^\pm = \lim_{x\to\infty} \frac{d}{dx} - M(x)$, the following well known "Roughness Theorem" ([41], [3]) is used. It describes the persistence of exponential dichotomies under perturbations which decay for $x \to \infty$ to zero.

Lemma A.6 *Assume that the operator $Lz = z' - M(\cdot)z$ possesses an exponential dichotomy on $J = [x_0, \infty), x_0 \in \mathbb{R}$ with data (K, α, π). Consider the perturbed operator*

$$\tilde{L}z = z' - (M(\cdot) + \Delta(\cdot))z$$

with $\Delta \in \mathcal{C}(J, \mathbb{R}^{m,m})$ and $\|\Delta(x)\| \to 0$ as $x \to \infty$. Then $\tilde{L}$ has an exponential dichotomy auf $[x_0, \infty)$ with data $(\tilde{K}, \tilde{\alpha}, \tilde{\pi})$, and

$$\|\tilde{\pi}(x) - \pi(x)\| \to 0 \quad as \quad x \to \infty.$$

It has been shown in [3], Lemma 2.1, the existence of exponential dichotomies for L on $\mathbb{R}^{\pm}$ follows from the hyperbolicity of the matrices $M^{\pm} = \lim_{x\to\pm\infty} M(x)$.

Corollary A.7 *Assume that the matrix $M \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{m,m})$ has limits*

$$M_{\pm} = \lim_{x\to\pm\infty} M(x),$$

which are hyperbolic. Let $X^s_{\pm}$ and $X^u_{\pm}$ be the stable and unstable invariant subspace of $M_{\pm}$, respectively.

Then L has an exponential dichotomy on $\mathbb{R}^- = (-\infty, 0]$ and $\mathbb{R}^+ = [0, \infty)$ with data $(K_{\pm}, \alpha_{\pm}, \pi_{\pm})$. The projectors π_- and π_+ satisfy

$$\lim_{x\to-\infty} (I - \pi_-(x)) = E^u_-, \qquad \lim_{x\to+\infty} \pi_+(x) = E^s_+,$$

where E^u_- denotes the projector onto X^u_- and E^s_+ the projector onto X^s_+.

If the number of stable and unstable eigenvalues of $M_{\pm}$ is m, then we have

$$\dim \mathcal{N}(\pi_-(0)) = \dim \mathcal{R}(E^u_-) = m, \qquad \dim \mathcal{R}(\pi_+(0)) = \dim \mathcal{R}(E^s_+) = m.$$

Moreover, the operator L is a Fredholm operator of index $k^+_s - k^-_s = k^-_u - k^+_u$ where $k^{\pm}_u$ resp. $k^{\pm}_s$ denotes the number of unstable resp. stable eigenvalues of $M_{\pm}$.

Instead of a single matrix function $M(\cdot)$ we often consider families of matrix functions $M(\cdot, s)$, in general the hyperbolicity of the matrices $M^{\pm}(s) = \lim_{x\to\infty} M(x, s)$ is related to the characteristic equations (1.6) as follows. A solution $(Y, \Lambda) \in \mathbb{R}^{m,p} \times \mathbb{R}^{p,p}$ of the quadratic eigenvalue problem

$$AY\Lambda^2 + BY\Lambda + (C - sI)Y = 0, \qquad A, B, C \in \mathbb{R}^{m,m}.$$

is related via linearization to the eigenvalue problem

$$M(s)W - W\Lambda$$

for the matrix

$$M(s) = \begin{pmatrix} 0 & I \\ -A^{-1}(C - sI) & -A^{-1}B \end{pmatrix} \in \mathbb{R}^{2m,2m} \tag{A.19}$$

via

$$W = \begin{pmatrix} Y \\ Y\Lambda \end{pmatrix}.$$

Thus the spectral condition (SC) implies that the matrices $M^{\pm}(s)$ are hyperbolic for all s with Re $(s) > -\beta$ with m stable and m unstable eigenvalues (cf. Lemma 3.30 in [60]).

It has been shown in [3], [60] that this implies that the operators

$$L(s)z = z' - M(\cdot, s)z, \quad \text{with} \qquad (A.20)$$
$$M(x,s) = \begin{pmatrix} 0 & I \\ A^{-1}(sI - C(x)) & -A^{-1}B(x) \end{pmatrix}$$

possesses exponential dichotomies on both half lines $\mathbb{R}^{\pm}$ if Re $(s) > -\beta$.

Note that $\Lambda - sI$ and $L(s)$ are strongly related. As has been proven in [50], the Jordan-block structures of $\Lambda - sI$ and $L(s)$ are the same, as well as the Fredholm properties.

In the following we fix the notation for the corresponding invariant subspaces and its projectors.

Definition A.8 *We denote the (orthogonal) projector onto the stable subspace of $M^-(s)$ by $E_-^s(s)$, i.e. $\mathcal{R}(E_-^s(s)) = \mathcal{R}(W_-^s(s))$, for*

$$W_-^s(s) = \begin{pmatrix} Y_-^s(s) \\ Y_-^s(s)\Lambda_-^s(s) \end{pmatrix} \in \mathbb{R}^{2m,m}$$

where $Y_-^s(s), \Lambda_-^s(s)$ solve the quadratic eigenvalue problem

$$AY\Lambda^2 + B_- Y\Lambda + (C_- - sI)Y = 0$$

and Re $\sigma(\Lambda_-^s(s)) < 0$.
Similarly, we denote the projector onto the unstable subspace of $M^+(s)$ by $E_+^u(s)$, i.e. $\mathcal{R}(E_+^u(s)) = \mathcal{R}(W_+^u(s))$, for

$$W_+^u(s) = \begin{pmatrix} Y_+^u(s) \\ Y_+^u(s)\Lambda_+^u(s) \end{pmatrix}$$

where $Y_+^u(s), \Lambda_+^u(s)$ solve the quadratic eigenvalue problem

$$AY\Lambda^2 + B_+ Y\Lambda + (C_+ - sI)Y = 0$$

and Re $\sigma(\Lambda_+^u(s)) > 0$.

In case $s = 0$, we omit the s dependency, e.g. write just Y_+^u,Λ_-^s.

Appendix B

Notation

$\mathcal{D}(P)$	domain of definition of the operator P.
$\mathcal{N}(P)$	nullspace or kernel of P.
$\mathcal{R}(P)$	image or range of P.
$\|P\|_{X\to Y}$	norm of a bounded operator $P: X \to Y$: $\|P\| = \sup\limits_{\substack{x\in\mathcal{D}(P)\\ x\neq 0}} \left\{\frac{\|Px\|_Y}{\|x\|_X}\right\}$.
$\sigma(L), \varrho(L)$	spectrum and resolvent of an operator L
$\mathcal{C}(X,Y)$	bounded continuous operators from X to Y with sup norm.
$\mathcal{C}^k(X,Y)$	k-times continuous differentiable operators from X to Y.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$\mathcal{C}^k(\mathbb{R},\mathbb{K}^n)$	k-times continuous differentiable functions from $\mathbb{R}$ to $\mathbb{K}^n$.
$\mathcal{C}_b^k(\mathbb{R},\mathbb{K}^n)$	functions, which possess continuous, bounded derivatives $f^{(j)} = \frac{d^j}{dx^j} f$ up to order k equipped with the norm $$\|f\|_{k,\infty} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty = \sum_{j=0}^{k} \sup_{x\in\mathbb{R}} \|f^{(j)}(x)\|.$$
$\mathcal{L}_p(\mathbb{R},\mathbb{K}^n)$	Lebesgue integrable functions from $\mathbb{R}$ to $\mathbb{K}^n$, with norm $$\|f\|_{\mathcal{L}_p} := \left(\int_{\mathbb{R}} \|f(x)\|^p \, dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$
$\langle u, v\rangle$	$\mathcal{L}_2$ inner product, $\langle u, v\rangle = \int_{\mathbb{R}} u(x)^H v(x)\, dx$
$\mathcal{H}^k(\mathbb{R},\mathbb{K}^n)$	Sobolev space of functions $f \in \mathcal{L}_2(\mathbb{R},\mathbb{K}^n)$, which possess $\mathcal{L}_2$-integrable derivatives up to order k with norm $$\|f\|_{\mathcal{H}^k} := \left(\sum_{j=0}^{k} \|f^{(j)}\|_{\mathcal{L}_2}^2\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} \sum_{j=0}^{k} \|f^{(j)}(x)\|^2 \, dx\right)^{\frac{1}{2}}.$$

u'	derivative of a function $u(x)$
u_x, u_t	partial derivatives of a function $u(x,t)$.
$B_\rho^\diamond(x)$	closed ball of radius ρ around $x \in X$: $B_\rho(x) = \{y \in X : \|x-y\|_\diamond \le \rho\}$
$J,\ J_e,$	discrete intervals: $J = [n_-, n_+],\ J_e = [n_- - 1, n_+ + 1]$,
$J_r,\ J_l$	$J_r = [n_-, n_+ + 1],\ J_l = [n_- - 1, n_+]$
$\mathbb{G}_{J,h,x_0}$	equidistant finite grid $\mathbb{G}_{J,h,x_0} = \{x_n :\ x_n = x_0 + nh,\ n \in J\}$.
$S_J(\mathbb{K}^m)$	Banach space of sequences $\{z_n\}_{n\in J}$, $z_n \in \mathbb{K}^m$ with $\|z\|_\infty = \sup_{n\in J} \|z_n\|$
$\delta_+, \delta_-, \delta_0$	finite difference operators: $\delta_+ : S_{J_r} \to S_J,\ \delta_- : S_{J_l} \to S_J,\ \delta_0 : S_{J_e} \to S_J$ $(\delta_+ u)_n = \frac{1}{h}(u_{n+1} - u_n), \quad (\delta_- u)_n = \frac{1}{h}(u_n - u_{n-1}),$ $(\delta_0 u)_n = \frac{1}{2h}(u_{n+1} - u_{n-1})$
For $z \in S_J$:	
$\|\cdot\|_{1,\infty}$	$\|z\|_{1,\infty} = \|z\|_\infty + \|\delta_+ z\|_\infty$
$\|\cdot\|_{2,\infty}$	$\|z\|_{2,\infty} = \|z\|_{1,\infty} + \|\delta_+\delta_- z\|_\infty$
$\|\cdot\|_{\mathcal{L}_{2,h}}$	discrete $\mathcal{L}_2$-norm for $z \in S_J$: $\|z\|_{\mathcal{L}_{2,h}} = (\sum_{n=n_-}^{n_+} h\|z_n\|^2)^{\frac{1}{2}}$
$\|\cdot\|_{\mathcal{H}_h^1}, \|\cdot\|_{1,\mathcal{L}_{2,h}}$	discrete $\mathcal{H}^1$-norm $\|z\|_{1,\mathcal{L}_{2,h}} = \|z\|_{\mathcal{H}_h^1} = (\|z\|_{\mathcal{L}_{2,h}}^2 + \|\delta_+ z\|_{\mathcal{L}_{2,h}}^2)^{\frac{1}{2}}$,
$\|\cdot\|_{\mathcal{H}_h^2}, \|\cdot\|_{2,\mathcal{L}_{2,h}}$	discrete $\mathcal{H}^2$-norm $\|z\|_{2,\mathcal{L}_{2,h}} = \|z\|_{\mathcal{H}_h^2} = (\|z\|_{\mathcal{H}_h^1}^2 + \|\delta_+\delta_- z\|_{\mathcal{L}_{2,h}}^2)^{\frac{1}{2}}$
$\langle u,v\rangle_{r,s}$	$\langle u,v\rangle_{r,s} = \sum_{n\in r}^{s} h\, u_n^H v_n$
$\langle u,v\rangle_h$	$\mathcal{L}_{2,h}$ inner product in S_J, $J = [n_-, n_+]$: $\langle u,v\rangle_h = \langle u,v\rangle_{n_-,n_+}$
$\mathcal{E}_\rho$	functions which decay with its derivative, i.e. $g \in \mathcal{E}_\rho(J, \mathbb{R}^m)$ if $\|g(x)^{(k)}\| \le K\mathrm{e}^{-\varrho\|x\|}$, $k = 0,1$ for some $K > 0$
$\mathrm{vec}(u)$	$u \in S_J(\mathbb{R}^m) : \mathrm{vec}(u) = (u_{n_-}^T, \ldots, u_{n_+}^T) \in \mathbb{R}^{m(n_+ - n_- + 1)}$
$\mathcal{O}, o$	Landau symbols

List of Figures

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