Games with Fuzzy Coalitions: Concepts Based on the Choquet Extension

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Contents

Introduction

Economic models in cooperative game theory reflect the potential of groups or 'coalitions' to acquire bundles of goods, utility-payments, or simply political power. Such models usually assume that a group/coalition signs a contract, the realisation of which requires the involved individuals (players) to cooperate with full commitment in order to reach the desired objectives by a joint, possibly coordinated action.

There are some approaches which assume tacitly that players are able to divide or split their activities. For example, the concept of a balanced system of coalitions (which plays an important role for the proof of the existence of core-elements) allows for an interpretation that players can partition the intensity of their cooperation to different coalitions of the system. This is a quite realistic assumption since, in general, contracts are not explicitly formed for the cooperation in only one group. The simultaneous cooperation of an individual within several groups with different goals (possibly contradicting each other) seems to be the norm.

More general, there is a whole set of game theoretic approaches which are interested in coalitions with percentage varying activity. However, this term has to be specified first. Since a coalition is in general represented as a subset of a 'set of players', we have to turn our attention to 'fuzzy sets'.

When considering a 'crisp' set there are exactly two possibilities: an object is either a member of this set or it is not. However, for many questions this sharp concept represents a restriction which is too strong. There are problems in which the objects under consideration may not only have one or zero as their degree of membership but may adopt principally every degree between this two extremes.

Examples are the 'set of all natural numbers that a particular person considers to be much greater than 1' or the 'set of humans that a particular person considers to be large' (for these and other examples see Zimmermann [32]). In the first case, there are a lot of numbers for which one cannot definitely say whether they are an element of the described set or not. Of course, 30 is greater than 1 from the mathematical point of view. However, the term 'much greater' is not sharply defined. Now, it is possible to assign a value of $0,8$ to the number 30 to express that 30 belongs with a high degree to this class. For the treatment of questions dealing with such 'diffuse' classes, Zadeh [31] establishes the concept of a fuzzy set: For a given space of points/objects, a fuzzy set is a mapping on this space to the unit interval, i. e. a function which assigns to each point in this space a real number between zero and one. The value of this mapping at a certain point is interpreted as the degree of membership of this 'player'.

Butnariu and Klement [6] consider cooperative games with fuzzy coalitions. Their understanding of fuzzy coalitions is in line with Zadeh's concept of fuzzy sets, i. e. in a fuzzy coalition the players have a certain degree of membership which lies between zero and one. As Butnariu and Klement restrict themselves to cooperative games with transferable utility, the fuzzy coalitional function is a mapping from the set of (feasible) fuzzy coalitions to the real numbers. Hence, they define a fuzzy game to be a tripel consisting of a set of players, a fuzzy coalitional function, and a set of feasible fuzzy coalitions. A possible application for such a game can be seen in the field of cost sharing. Butnariu and Klement discuss in [6, Section 28] the problem of getting a fair rate for services in bulk. Here, they mainly think about the determination of the fixing of prices for utilities like energy or water. A similar topic is treated in the paper of Billera et al. [5], where the authors deal with the problem of finding telephone billing rates. Another example for fuzzy games is provided by Dhingra and Rao [9]. They consider a multiple optimisation problem with a certain degree of indefiniteness which leads them to consider a cooperative game with fuzzy coalitions and non-transferable utility.

The definition and motivation of fuzzy games is the main topic of Chapter 1. Especially, our aim is to answer the question "How can a crisp game with transferable utility be extended to a game with fuzzy coalitions"? In principal, there are infinitely many possibilities. However, we restrict ourselves to the extension which is given by the Choquet integral [7]. This extension is a very intuitive one what can be seen especially for the important unanimous games, where the Choquet extension is nothing else but the infimum function.

As for crisp games, one tries to establish solution concepts for games with fuzzy coalitions like the core or the Shapley value. The main problem turns out to be the fact that, even in the case of finitely many players, there is a continuum of fuzzy coalitions. As shown in [29], the core of a Choquet game coincides with the underlying crisp game. For a certain class of 'general fuzzy games', there exists a nice formula to calculate the Shapley value (cf. [6, Theorem 18.4]). However, if we consider a problem with at least three players, the extension given by the Choquet integral is not an element of this class. Mertens [16], [17] and Weiß [29] consider some possibilities to repair this gap.

In Chapter 2, we make an attempt to establish the Shapley value even for countably many players. A reason for dealing with this large number of actors is given by Rosenmüller [24, page 469]: "The set of players is assumed to be the set of natural numbers. This should not necessarily be interpreted as a belief that a countable set of players reflects a real life situation. Rather we think that the behaviour of solution concepts can be studied when there is an ocean of small players and a few influential or important players have dominant influence but nevertheless do not rule the game on their own. Thus, it is predominantly a statistically motivated consideration which leads to modeling games on a countable player set."

First of all, one should mention that it is not possible to obtain a value in the classical sense for all games. One very important counter-example is the unanimous game, where symmetry and additivity are mutually exclusive. However, for the countable case there are several frameworks to determine a Shapley value for some classes of crisp games ([26], [1], [20]). We will show how far these approaches can be extended to the fuzzy case. Furthermore, we will investigate whether the smoothing procedure of Weiß [29] can be applied. It will turn out that, for games belonging to the closure of the linear hull of the games with finite carrier, this framework provides the same value as the 'fuzzified' approach of Artstein [1]. Finally, we show how a quasi-value for a class of Choquet games can be constructed which contains all infimum functions, i. e. all Choquet integrals of the unanimous games. This is done by a limiting argument with respect to weak convergence similar to that used by Rosenmüller in [22].

In our last chapter, we leave the special class of games with TU character and consider games with non-transferable utility (NTU games). We answer the question of how NTU games with fuzziness can be understood and be defined. Again, our special interest will be in the Choquet extension of such games. As there is no canonical way to do this, our aim is to preserve as many properties of the original Choquet integral as possible. We will see that comonotonic additivity together with continuity cannot be satisfied by all games. Hence, we will outline a possibility for a comonotonic additive as well as for a continuous Choquet extension. At the end, our focus is on the core and we show that, under the assumption of monotonicity, the core of a crisp NTU game coincides with both that of the comonotonic additive and that of the continuous Choquet extension.

Chapter 1

Fuzzy Games

This thesis is mainly concerned with fuzzy games and solution concepts on such games. In the first section of this chapter, we provide a definition and motivation for games with fuzzy coalitions. In particular, we will discuss their meaning and possible fields of application. Afterwards, we show how some concepts known from crisp games like monotonicity and convexity can be imbedded in a fuzzy context.

Obviously, the restriction of a fuzzy game to the crisp coalitions is a crisp game. For the converse direction there are a lot of possibilities to extend a crisp game to a game with fuzziness. In Section 3, we present the Choquet integral, which constitutes an extension, which gives a nice result for the unanimity games, and which satisfies a lot of desirable properties.

The final section of this chapter is devoted to the value for fuzzy games. A short overview of the different approaches concerning the value is given.

1.1 Definition and Motivation

Zadeh was the first who used in [31] the expression "fuzzy set". In the case of a crisp set, there are two possibilities: Either an element is a member of this set or it is not. In other words, we have two possible degrees of membership. In contrast to this, a fuzzy set is a class with a continuum of grades. This extended concept of a set provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership.

Let Ω be a set of points. Then a fuzzy set A in Ω is characterized by its membership function $f_A : \Omega \longrightarrow [0,1]$. The value of $f_A(\omega)$ at $\omega \in \Omega$ represents the degree of membership of ω in A. To get an impression of a fuzzy set, Zadeh [31] gives the following example: Let Ω be the real line $\mathbb{R}$ and let A be a fuzzy set of numbers which are much greater than 1. Then a (quite subjective) characterization of A is given by f_A where some representative values are $f_A(1) = 0; f_A(5) = 0{,}01; f_A(10) = 0{,}2; f_A(100) = 0{,}95; f_A(500) = 1$. Other examples of fuzzy sets include terms like "tall men" or "beautiful women" (cf. Zimmermann [32]).

Most models in cooperative game theory assume that a coalition of players signs a contract the realization of which requires the participants to cooperate with full commitment. However, there are approaches which are interested in coalitions with varying degree of activity. Hence, we have to deal with the question of how a game with fuzziness can be defined.

In the context of cooperative game theory, a fuzzy coalition is, from the mathematical point of view, nothing else but a fuzzy set, i. e. it is a coalition to which the players can belong with different degrees of membership. Again, the characteristic function of a fuzzy coalition is a mapping defined on the set of players Ω to the unit interval $[0,1]$. The relationship between a fuzzy coalition and its membership function is the same as the relationship between a crisp coalition and its corresponding indicator function. For $S \subseteq \Omega$, the indicator function $1_S : \Omega \longrightarrow \{0,1\}$ is defined by

$$1_S(i) = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S. \end{cases}$$

In the following, we will not distinguish between a fuzzy coalition and its characteristic function, and will represent both by a small latin letter. Whenever we consider a crisp coalition, we will use a capital letter to underline the fact that we do not have any fuzziness at that moment.

Let us now define a fuzzy game as the major tool of the following chapters. A (crisp) cooperative game with transferable utility is a tripel $(\Omega, \mathcal{P}, v)$ (cf. [23, Chapter 3, Definition 1.1]) where Ω is the (not necessarily finite) set of players, $\mathcal{P}$ is the set of feasible coalitions (in the case of finitely many players we generally take the power set), and $v : \mathcal{P} \to \mathbb{R}$, $v(\emptyset) = 0$, is the coalitional function.

As we will see later on (cf. Example 1.2 and Example 1.3), there are games where the use of crisp coalitions seems to be insufficient. Hence, we will deal with fuzzy coalitions and start with the definition of a fuzzy game:

Definition 1.1 *A **cooperative fuzzy game with transferable utility** is a tripel* $(\Omega, \overline{\mathcal{P}}, v^F)$*, where*

- Ω *denotes the set of players,*
- $\overline{\mathcal{P}} \subseteq [0,1]^\Omega$ *is the set of feasible fuzzy coalitions (in the following we will often use* $\overline{\mathcal{P}} = [0,1]^\Omega$*),*
- $v^F : \overline{\mathcal{P}} \to \mathbb{R}$*,* $v^F(\emptyset) = 0$*, is a fuzzy coalitional function.*

Fuzzy coalitions have been used in game theory yet. For example, Mertens has used fuzzy coalitions in [16] and [17] as technical tools to extend the Shapley value on *pNA* (a space that we will define later on) to larger spaces of games. Aubin stated in [2]: "In many instances, the core remains too large. Since we would like to have solution concepts yielding a set of solutions as small as possible, the question arises as to whether it is possible to shrink the core again by enlarging the set of coalitions. This is done by embedding the set of coalitions into the set of fuzzy coalitions." Both authors use fuzzy coalitions only for technical reasons. The following examples show that one can think of cases where fuzzy coalitions are interesting for themselves, i. e. they deal with real world problems which contain fuzziness.

Example 1.2 *LP-games with fuzziness*

A class of games where one can introduce fuzziness in a quite natural way are the LP-games (cf. Rosenmüller [24, p. 362]). In this type of games the set of players $\Omega = \{1, \ldots, n\}$, a vector $c \in \mathbb{R}^m_+$, n vectors $b^j \in \mathbb{R}^l_+$ and a positive $l \times m$ matrix $A = (a_{jk})_{j,k}$ are given. We have $j = 1, \ldots, m$ production processes and $k = 1, \ldots, l$ factors (resources). The decision to produce quantity x_k of commodity k in process k requires the use of a_{jk} units of factor j per unit of k. Each player $i \in \Omega$ has got an initial endowment b^i of resources. The function $b : \underline{\underline{P}}(\Omega) \to \mathbb{R}^l$ is now given via $b(S) = \sum_{i \in S} b^i$. Then, $(\Omega, \underline{\underline{P}}(\Omega), v)$ with

$$v(S) = \max\{cx | \, x \in \mathbb{R}^m_+, Ax \leq b(S)\}$$

defines the L.P.-game.

Now one can think of the case where the players do not want to give all of their endowments to the production process. Hence, we can interpret a fuzzy coalition f in such a way that if player j is a member of f she is willing to give the rate $f(j)$ of her resources to the processes. Therefore, $\sum_{j=1}^n f(j) b^j_k$ is the amount of factor k available to f, and this term presents the limit of resource k for the production processes. The corresponding fuzzy game is now defined by $(\Omega, [0,1]^\Omega, v^F)$ with

$$v^F(f) = \max\left\{ cx \middle| \, x \in \mathbb{R}^m_+, Ax \leq \sum_{j=1}^n f(j) b^j \right\}.$$

Example 1.3 *Multiple Objective Design Optimization*

Here, we want to give another possible use of fuzzy games. First of all, we will define multiple objective design optimization (MODO). Hwang and Masud have stated in [13]: "Decision making is the process of selecting a possible course of action from all the available alternatives. In almost all such problems the multiplicity of criteria for judging the alternatives is pervasive. That is, for many such problems, the decision maker wants to attain more than one objective or goal in selecting the course of action while satisfying the constraints dictated by environment, processes, and resources." Mathematically, a multiobjective problem is based on n decision variables $x \in \mathbb{R}^n$, k objectives, $y_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, \ldots, k$, and m constraints, $z_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$. The problem can now be written as

$$\max \left\{ \left(y_1(x), y_2(x), \ldots, y_k(x) \right) \middle| \, z_i(x) \leq 0, \, i = 1, \ldots, m \right\}. \tag{1.1}$$

In order to use (1.1), we have to know the goals and constraints in a very precise way. However, these are often imprecise in nature, i. e. we have to deal with vagueness. Here, we can use fuzzy set theories to alleviate such modelling difficulties (cf. Dhingra and Rao [9] for further thoughts). A fuzzy goal as a fuzzy constraint is nothing else but a function $f : \mathbb{R}^n \longrightarrow [0, 1]$.

As an example for fuzziness in MODO one can think of the following (cf. Zimmermann [32, Example 12.5]): "A board of directors is trying to find the "optimal" dividend to be paid to the shareholders. For financial reasons it ought to be attractive and for reasons of wage negotiations it should be modest." Here we have a fuzzy objective "attractive dividend" and a fuzzy constraint "modest dividend".

Dhingra and Rao [9] assert that the following system is the correct way to involve fuzziness in (1.1): We have again k crisp objectives y_j and m crisp constraints z_i. Moreover, there are fuzzy goals $f_i : \mathbb{R}^n \longrightarrow [0, 1]$, $i = 1 \ldots, l$, and o fuzzy constraints $g_j : \mathbb{R}^n \longrightarrow [0, 1]$. "To capture the essence of the bargaining model by permitting a tradeoff between conflicting fuzzy goals", Dhingra and Rao use $\mu_{obj} : \prod_{i=1}^{l} [0, 1]^{\mathbb{R}^n} \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\mu_{obj}(f_1, \ldots, f_l, x) := \left[\sum_{i=1}^{l} f_i(x) \right]^{\frac{1}{l}}$$

as aggregation operator of the fuzzy objectives. The operator $\mu_{con} : \prod_{j=1}^{o} [0, 1]^{\mathbb{R}^n} \times \mathbb{R}^n \to \mathbb{R}$ for the fuzzy constraints is defined as

$$\mu_{con}(g_1, \ldots, g_o, x) := \min_{j=1,\ldots,o} g_j(x).$$

Now the problem is given by

$$\max \left\{ \left(y_1(x), y_2(x), \ldots, y_k(x), \mu^f(x) \bigwedge \mu^g(x) \right) \middle| \, z_i(x) \leq 0, \, i = 1, \ldots, m \right\}. \tag{1.2}$$

We have a problem similar to (1.1). However, in this case we want to satisfy the fuzzy goals/constraints as good as possible. To come to a solution we will use some concepts known from cooperative game theory. We define

$$U := \{(y_1(x), \ldots, y_k(x), \lambda) \Big| z_i(x) \leq 0, i = 1, \ldots, m, \lambda \leq \min\{\mu^f(x), \mu^g(x)\}\}.$$

If there exists, for each $i = 1, \ldots, k$, a point $r_i \in \mathbb{R}$ s.t. r_i denotes the worst value of objective function y_i that the decision maker is willing to accept, we can bring the problem to a NTU context. As far as $r := (r_1, \ldots, r_k, 0)$ is an element of U, we can define U as the set of feasible payoffs and r as the threat point. Now we are able to look for Pareto optimal solutions for (1.2). Dhingra and Rao [9] suggest the Nash solution [18]. For further thoughts about fuzzy NTU games, see Chapter 3.

One can think of further examples: In a market game (see [23, Chapter 5, Definition 1.1 and Definition 3.1]), each player has got an initial allocation of commodities. The initial endowment of a crisp coalition S is reallocated in such a way that the sum over the utilities of the players in this coalition S is maximized. If the players are allowed to give fractions of their initial endowments to different coalitions, we have to deal with fuzziness again.

Another possible application of fuzzy games lies in the field of rate problems for services in bulk. Butnariu and Klement consider in [6, Section 28] the problem of getting fair rates for some utilities like electricity or water. In order to obtain a solution, they construct a fuzzy game and calculate the corresponding Shapley value.

1.2 Extension of some Crisp Concepts

We have to think about the question of how we can bring some well known concepts for crisp games to the fuzzy context. In the following, we consider operations on fuzzy coalitions. These operations are defined pointwise. For example, $\Omega - f, f \in [0, 1]^\Omega$, denotes nothing else but the fuzzy coalition where each player

i has $1 - f(i)$ as her degree of membership. The next definition is due to Zadeh [31].

Definition 1.4 *Let Ω be a set of players and let f and g be fuzzy coalitions of Ω.*

- *We say that the fuzzy coalition f is* **contained** *in g if $f(i) \leq g(i)$ is valid for every player $i \in \Omega$. Hence, for this case we use the notation $\boldsymbol{f} \leq \boldsymbol{g}$.*
- *The* **complement $\boldsymbol{f^C}$ of a fuzzy coalition** *f is defined by $f^C = \Omega - f$.*

There are different possibilities to define the intersection of fuzzy coalitions (cf. Butnariu and Klement [6, Section 1] or Zimmermann [32, pp.31 and 32]) all of which coincide with the notion of intersection in the case of crisp coalitions. We mainly deal with two alternatives:

The possibility Butnariu and Klement prefer is given by means of the formula

$$f \cap_{BK} g = \max(\emptyset,\ f + g - \Omega). \tag{1.3}$$

Here, the degree of membership of each player in the intersection of f and g is given as the sum of her two single values minus one — a very surprising interpretation of an intersection. As we require $f(i) + g(i) = (f \cup_{BK} g)(i) + (f \cap_{BK} g)(i)$, we get

$$f \cup_{BK} g = \min(\Omega, f + g)$$

as union. This means that, according to Butnariu and Klement, the degree of membership is additive in a certain way. Though one can find some nice mathematical applications, in the context of fuzzy coalitions additivity does not seem to be the intuitive correct representation of the union. As an example consider the following case: Player i has the degrees of membership $f(i) = 0,5$ and $g(i) = 0,6$. Hence, her degree in the intersected coalition is $(f \cap_{BK} g)(i) = 0,1$, which is hard to justify. Another point is that a fuzzy coalition f which is contained in another coalition g should satisfy $f \cap_{BK} g = f$, but this equality is not valid any longer. On the other hand, we have $f \cap_{BK} f^C = \emptyset$ and $f \cup_{BK} f^C = \Omega$.

Another definition of an intersection is given by Zadeh [31]. He uses the minimum to represent the intersection, i. e.

$$f \cap g = f \wedge g (= \min(f, g)). \tag{1.4}$$

The corresponding union is nothing else but the maximum. This equation is much easier to explain as the one of Butnariu and Klement. A player who is a member of f and g takes in the intersection the minimum of the degrees that she has assigned to the two coalitions. For $f \leq g$, we obviously get $f \cap g = f$. Zadeh [31] has shown that the minimum of two fuzzy coalitions f and g is the largest fuzzy coalition which is contained in both f and g. Furthermore, Yager [30] states that the minimum is the only intersection operator that satisfies the following conditions:

1. Commutativity: $f \cap g = g \cap f$
2. Associativity: $[f \cap g] \cap h = f \cap [g \cap h]$
3. Two valued equivalence to ordinary logic: $\Omega \cap \Omega = \Omega, \Omega \cap \emptyset = \emptyset \cap \Omega = \emptyset, \emptyset \cap \emptyset = \emptyset$
4. Monotonicity: $f_1 \geq f_2$ and $g_1 \geq g_2$ implies $f_1 \cap g_1 \geq f_2 \cap g_2$
5. Continuity: $x_n \to x$ implies $(f \cap g)(x_n) \to (f \cap g)(x)$
6. Idempotency: $f \cap f = f$

Obviously, idempotency is not satisfied by the intersection concept of Butnariu and Klement. For the reasons given above, we will take the minimum as intersection, i. e.

Definition 1.5 *Let Ω be a set of players and $f, g \in [0,1]^\Omega$. The* **intersection** *of f and g is defined by $f \cap g := \min(f, g)$.*

As in some later definitions the set of players may be an arbitrary set, we should discuss briefly our understanding of a set of feasible fuzzy coalitions. At least we require such a set to be a **fuzzy tribe** in the sense of Butnariu and Klement [6, Definition 2.5], i. e. a subset of $[0,1]^\Omega$ containing $\emptyset$ and being closed under countable BK-intersection ($f_i \in \overline{\mathcal{P}} \Rightarrow \bigcap_{BK} f_i \in \overline{\mathcal{P}}$) and under complement. One should remark that a fuzzy tribe is closed under countable intersection ($f_i \in \overline{\mathcal{P}} \Rightarrow \bigwedge f_i \in \overline{\mathcal{P}}$)[6, Proposition 2.7(ii)]. For a finite or countable set of players, we are using $\overline{\mathcal{P}} = [0,1]^\Omega$.

Now we will define some expressions for fuzzy coalitional functions all of which can be found in the book of Butnariu and Klement [6].

Definition 1.6 *1. As in the case of crisp games a fuzzy coalition function v^F is said to be* ***monotone****, if for $f, g \in \overline{\mathcal{P}}$ with $f \geq g$ the inequality $v^F(f) \geq v^F(g)$ is valid.*

2. A fuzzy coalitional function v^F on $\overline{\mathcal{P}}$ is said to be of ***bounded variation****, if v^F can be written as the difference of two monotone fuzzy coalitional functions, i e.*

$$v^F = u^F - w^F, \qquad u^F, w^F \text{ monotone.}$$

The family of fuzzy functions with bounded variation is denoted **FBV**.

3. The ***variation norm*** *is defined on FBV by*

$$\|v^F\| = \inf\{u^F(\Omega) + w^F(\Omega) |\, u^F, w^F \text{ monotone}, v^F = u^F - w^F\}. \qquad (1.5)$$

The following theorem provides another possibility to state the variation norm:

Theorem 1.7 *[6, Lemma 15.5] $v^F \in FBV$, if and only if*

$$\sup\left\{\sum_{i=1}^{k} |v^F(f_i) - v^F(f_{i-1})| \,\middle|\, f_i \in \overline{\mathcal{P}}, i = 1, \ldots, k, f_0 \leq f_1 \leq \ldots \leq f_k\right\} \qquad (1.6)$$

is finite. For $v^F \in FBV$ the expression (1.6) equals $\|v^F\|$.

Definition 1.8 *A fuzzy coalitional function v^F is called* ***fuzzy measure****, if for $f_1, f_2, \ldots \in \overline{\mathcal{P}}$ with $\sum_{i\in\mathbb{N}} f_i \leq \Omega$ always*

$$v^F\left(\sum_{i\in\mathbb{N}} f_i\right) = \sum_{i\in\mathbb{N}} v^F(f_i)$$

is true.

As defined on page 12, the intersection of two fuzzy coalitions is given by the minimum. However, in the definition of a fuzzy measure we are using the intersection in the sense of Butnariu and Klement [6]. To be more precise, $\sum_{i\in\mathbb{N}} f_i \leq \Omega$ is tantamount to $\bigcap_{BK} f_i = \max(\emptyset, \sum_{i\in\mathbb{N}} f_i - \Omega) = \emptyset$. Hence, v^F is a fuzzy measure if for $f_1, f_2, \ldots \in \overline{\mathcal{P}}$ with $\bigcap_{BK} f_i = \emptyset$ always $v^F(\bigcup_{BK} f_i) = \sum v^F(f_i)$ is true. One should remark that in particular $\bigcap_{BK} f_i = \emptyset$ implies $f_j \cap_{BK} f_k = \emptyset$ for all $j \neq k$.

We will use (finitely additive) fuzzy measures mainly in the context of a solution concept, and there additivity (as given by the intersection concept of Butnariu and Klement) is meaningful. For fuzzy coalitions f, g with $f + g \leq \Omega$ we require a solution concept Ψ to satisfy $\Psi(v^F)(f) + \Psi(v^F)(g) = \Psi(v^F)(f+g)$. Of course, one can define a **weak fuzzy measure** by

$$v^F\left(\bigvee_{i\in\mathbb{N}} f_i\right) = \sum_{i\in\mathbb{N}} v^F(f_i) \tag{1.7}$$

for $f_j \wedge f_k = \emptyset$, $j \neq k$, i.e. for fuzzy coalitions with pairwise disjoint carriers. Obviously such a collection (f_i) satisfies $\sum_{i\in\mathbb{N}} f_i \leq \Omega$, i.e. each fuzzy measure is a weak fuzzy measure. As we want to use the concept of additivity for as many fuzzy coalitions as possible, we prefer the concept of a fuzzy measure to a weak fuzzy measure not only for solution concepts but also for fuzzy games.

A fuzzy set function v^F is called **homogeneous** if for each $f \in \overline{\mathcal{P}}$ and for each $\lambda \in [0,1]$ s.t. $\lambda f \in \overline{\mathcal{P}}$ we have that

$$v^F(\lambda f) = \lambda v^F(f).$$

For reasons of simplicity, we assume that we always have a set of feasible fuzzy coalitions which contains with a fuzzy coalition f always λf for all $\lambda \in [0,1]$.

Proposition 1.9 *In the case of countably many players a weak fuzzy measure m with bounded variation is a fuzzy measure if and only if m is homogeneous.*

Proof Let a fuzzy coalition f be given. In the case of finitely many players, $\Omega = \{1, \ldots, n\}$, f can be extended to $[0,1]^{\mathbb{N}}$ by simply setting $f(j) = 0$ for $j > n$. Now $f(j)j$, $j \in \mathbb{N}$, can be interpreted as that fuzzy-coalition where player j has the degree of membership $f(j)$ and all other players in $\mathbb{N}\setminus\{j\}$ do not participate at all. Hence, f can be written as $f = \sum_{j\in\mathbb{N}} f(j)j$.

Let m be a homogeneous weak fuzzy measure and $\sum_{i\in\mathbb{N}} f_i \leq \mathbb{N}$. Then

$$\begin{aligned}
m\left(\sum_{i\in\mathbb{N}} f_i\right) &= m\left(\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} f_i(j)j\right) \\
&= m\left(\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} f_i(j)j\right) \\
&= m\left(\bigvee_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} f_i(j)j\right) \\
&= \sum_{j\in\mathbb{N}} m\left(\sum_{i\in\mathbb{N}} f_i(j)j\right) \\
&= \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} f_i(j)m(j) \\
&= \sum_{i\in\mathbb{N}} m\left(\bigvee_{j\in\mathbb{N}} f_i(j)j\right) \\
&= \sum_{i\in\mathbb{N}} m(f_i)
\end{aligned}$$

Note that the bounded variation of m was necessary to use the double series theorem of Cauchy.

Now, let m be a (signed) fuzzy measure. Then m can be written as $m = m^+ - m^-$ where m^+ and m^- are non-negative fuzzy measures i. e. we have a Jordan decomposition (cf. [6, Theorem 10.2]). Lemma 2.1.3 in [29] states that each finitely additive non-negative fuzzy set function is homogeneous. This is especially true for m^+ and m^-. Thus, m itself has this property. **q.e.d.**

An example of a weak fuzzy measure that is no fuzzy measure is given by $m(f) = (f(1))^2$.

1.3 The Choquet Extension

If one restricts a fuzzy set function to $\underline{\underline{P}}(\Omega)$, one obviously obtains a crisp coalitional function. Vice versa, it is a very interesting question how to extend a given coalition function to $[0,1]^\Omega$. This may be of some interest if one has a crisp problem (game) and is of the opinion that some "fuzzy decision" should be allowed for a better representation of the reality. For this extension, there are various possibilities. An example for the case of finitely many players is the rather popular multilinear extension of Owen [19], that is, for $\Omega = \{1, \dots, n\}$, given by

$$v^O(f) = \sum_{S \in \underline{\underline{P}}(\Omega)} \left\{ \prod_{i \in S} f(i) \prod_{i \notin S} (1 - f(i)) \right\} v(S). \tag{1.8}$$

However, in the following we will concentrate on the extension which is based on the Choquet integral [7].

Definition 1.10 *The* ***Choquet extension*** v^C *of a monotone outcome function* v *is given by*

$$v^C(f) = \int_0^1 v(\{j \mid f(j) > t\})\, dt. \tag{1.9}$$

For finitely many players, i. e. $\Omega = \{1, 2, \dots, n\}$, one can find a permutation π of the set of players such that $f(\pi^{-1}(1)) \geq f(\pi^{-1}(2)) \geq \dots \geq f(\pi^{-1}(n))$. The Choquet integral can be written as

$$v^C(f) = \sum_{i=1}^{n} f(i)[v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1})], \tag{1.10}$$

where $S^\pi_i := \{j \mid \pi(j) \leq i\}$ for $i \in \{1, \dots, n\}$ and $S^\pi_0 := \emptyset$.

The general definition of the Choquet extension only allows for monotone v, since the term $v(\{j \mid f(j) > t\})$ has to be Lebesgue-integrable as a function in t. In the finite case we do not have such problems, and therefore we will drop this requirement here.

Why are we interested in Choquet games? An answer to this question can be found when considering the unanimous game e^T, which is defined by

$$e^T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

Unanimous games are quite important as in the case of finitely many players they build a basis for $\mathcal{V} := \{v \,|\, v : \underline{\underline{P}}(\Omega) \to \mathbb{R}, v(\emptyset) = 0\}$. The Choquet integral w. r. t. e^T provides

$$(e^T)^C(f) = \inf_{j \in T} f(j) \tag{1.11}$$

for all $f \in [0,1]^\Omega$.

This seems to be the intuitively correct extension. In the extension of the unanimous game e^T only that player in T with the lowest degree of membership fixes the outcome — just as in the crisp game. All players in T have to agree, and the maximal degree where an agreement should be possible is $\inf_{j \in T} f(j)$. For a better understanding, one can think of the following story: A decision maker asks for every level in the unit interval whether the players in T do agree or not. He starts at 0 and increases continuously the level until the first player disagrees. Obviously, this procedure yields the predicted outcome.

Another possibility to explain formula (1.11) is given by the "intensity of participation". If $f(i) = \frac{1}{2}$, then player $i \in T$ works with 50 per cent of his potential in the fuzzy coalition f. Hence, the over-all intensity of activity in f is given by $\min_{i \in T} f(i)$.

One should remark that the extension of Owen given by formula (1.8) provides for an unanimous game

$$(e^T)^O(f) = \prod_{i \in T} f(i),$$

a result that is much harder to justify than the one given by the Choquet integral.

Another nice property of the Choquet extension is homogeneity (cf. [8, Proposition 5.1]), i. e. $v^C(\lambda f) = \lambda v^C(f)$ for all $\lambda \in [0,1]$. If all players play with half the intensity used before, the outcome is precisely half as high as before.

Another aspect that stresses the importance of the Choquet integral in game theory is given by Strassen in [27]. He was the first who has given an extension of the Shapley value to a continuum of players. This extension was done by a continuity argument with the help of the Choquet integral.

On pages 13 and 14 we discussed weak versus strong additivity and decided to use the strong variant. Since the Choquet extension is homogeneous, Proposition 1.9 states that there is no difference between the two kinds of additivity for the Choquet integral provided there are at most countably many players.

1.4 The Value for Fuzzy Games

Butnariu and Klement give in [6, Chapter 18] a formula for a fuzzy value by carrying the approach of Aumann and Shapley [3] to *pFNA*, which is a certain class of fuzzy games and will be defined soon. First of all we repeat this formula. Thereafter we discuss possibilities of extending the value to larger sets as *pFNA* which can be found in literature.

Again, let Ω denote the (arbitrary) set of players, and let $\overline{\mathcal{P}}$ be the set of feasible fuzzy coalitions.

Definition 1.11 *1. A fuzzy measure m is called **non-atomic**, if for each $f \in \overline{\mathcal{P}}$ with $m(f) \neq 0$ there exists a fuzzy coalition $g \in \overline{\mathcal{P}}$, $g \leq f$, s. t. $m(g) \notin \{0, m(f)\}$.*

2. *The family of all finite, non-atomic fuzzy measures on $\overline{\mathcal{P}}$ is denoted as* **FNA**. **pFNA** *is the closed linear hull of $\{m^k | \, m \in FNA^+, k \in \mathbb{N}\}$, where FNA^+ denotes the space of monotone functions on FNA.*

3. *Let* **FBVA** *denote the set of fuzzy coalitional functions with bounded variation, which are finitely additive. Here, a mapping $p : \overline{\mathcal{P}} \to \mathbb{R}$ is called finitely additive if for any $f_1, f_2 \in \overline{\mathcal{P}}$ with $f_1 + f_2 \leq \Omega$ always $p(f_1 + f_2) = p(f_1) + p(f_2)$ is valid.*

Let $\pi : \Omega \to \Omega$ be a one-to-one mapping. We call π a permutation of $(\Omega, \overline{\mathcal{P}})$ if πf and $\pi^{-1} f$ are contained in $\overline{\mathcal{P}}$ whenever $f \in \overline{\mathcal{P}}$. Here, πf is the fuzzy coalition defined by $(\pi f)(i) = f(\pi^{-1}(i))$. A permutation π can easily be interpreted as an operator on *FBV* by the formula

$$\pi v^F(f) = v^F(\pi f).$$

A linear subspace Q of *FBV* is called **symmetric**, if $\pi Q \subseteq Q$ is valid for all permutations π of $(\Omega, \overline{\mathcal{P}})$. It is easily seen that, for example, *pFNA* is a symmetric subspace of *FBV*.

An operator $\varphi : Q \longrightarrow FBV$ is called **positive**, if it preserves monotonicity, i. e. if φv^F is monotone whenever v^F is monotone.

Now, we are well prepared to give the definition of a Shapley value on fuzzy games [6, Definition 18.2]:

Definition 1.12 *Let Q be a symmetric subspace of FBV. An* **(Aumann-Shapley-) value** *on Q is a positive linear operator $\varphi : Q \to FBVA$ which satisfies*

- *Symmetry: If π is a permutation of Ω, then*

$$\varphi(\pi v^F) = \pi(\varphi v^F).$$

- *Efficiency: For all $v^F \in Q$*

$$\varphi v^F(\Omega) = v^F(\Omega).$$

This definition states some properties the value should have which are quite similar to those of the axiomatization of the Shapley value for crisp games (cf. Rosenmüller [23, Chapter 3, Definition 7.6] or Theorem A.5 in the appendix). The problem is that we do not have a formula for the fuzzy value yet. The following theorem [6, Theorem 18.3] provides the so called diagonal formula:

Theorem 1.13 *There exists a value on pFNA which is norm continuous s. t. for each vector m of non-atomic fuzzy measures, $m = (m_1, \ldots, m_n), n \in \mathbb{N}$, and* *for each continuously differentiable function* $p : \mathcal{R}(m) \to \mathbb{R}$ *with* $p(0) = 0$ *the following formula holds:*

$$\varphi(p \circ m)(f) = \int_0^1 p_{m(f)}(t \cdot m(\Omega))\, dt, \tag{1.12}$$

where $\mathcal{R}(m) = \{m(f) | f \in [0,1]^\Omega\}$ *is the range of* m *and* $p_{m(f)}$ *denotes the derivative of* p *in the direction* $m(f)$*.*

The formula (1.12) is called **diagonal formula** and the operator φ is said to be the **diagonal value**. One should remark that under the assumptioms of the last theorem $\mathcal{R}(m)$ is compact and convex, and that $v := p \circ m$ is really an element of *pFNA* (cf. Butnariu and Klement [6, Theorem 14.2 respectively Proposition 17.4]).

Owen shows in [19] that his extension of a game can be written in such a way that formula (1.12) can be applied. Now, we will present some possibilities to determine a value for the Choquet-games for finite Ω , $\Omega = \{1, \ldots, n\}$, $\overline{\mathcal{P}} = [0,1]^\Omega$. For these games the problem occurs that it is not possible to decompose them as required in Theorem 1.13. Of course, one can define a non-atomic fuzzy vector measure m by $m_i(f) = f(i)$ for all $f \in [0,1]^\Omega$ and a function $p^C : \mathcal{R}(m) \to \mathbb{R}$ with $p^C(0) = 0$ by

$$p^C(q) = \sum_{i=1}^n q_i \left[v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1}) \right], \tag{1.13}$$

where we have assumed $q_{\pi^{-1}(1)} \geq \ldots \geq q_{\pi^{-1}(n)}$. As one can immediately see, p^C is not differentiable except for an additive v. As an example for the non-differentiability one can consider the unanimous game e^T where $p^C_{e^T}$ is given by $p^C_{e^T}(q) = \min_{j \in T} q_j$. This deficiency of the Choquet integral is repaired in the next two subsections. The further subsections deal with two other ways to obtain a value for the minimum function.

Butnariu and Klement show in [6, Chapter V] some possibilities to extend the diagonal value beyond *pFNA*. However, they show that the largest space to which they can extend the value does not contain the glove game (i. e. the Choquet extension of a unanimity game) if one has more than two players [6, Example 25.5].

1.4.1 The Smoothing Procedure

Let us consider the game $(\{1,\ldots,n\},[0,1]^{\{1,\ldots,n\}},v^C)$. As stated above, p_v^C as given by formula (1.13) is not differentiable at the diagonal except for an additive v. The main idea of the smoothing procedure (see Section 3.3 in [29]) is now to "smooth" the sharp bend (i. e. the non-differentiability) of the Choquet extension and to obtain a value by a limit argument.

To be more precise, one is looking for a sequence (p^i) of continuously differentiable functions $p^i:[0,1]^\Omega \longrightarrow \mathbb{R}$ with $p^i(0)=0$ such that p^i converges uniformly to p^C. With these p^i one gets a sequence of fuzzy coalitional functions $v^i := p^i \circ m$, and these functions satisfy all assumptions of Theorem 1.12. Therefore, we have a sequence of diagonal formulas:

$$\varphi v^i(f) = \sum_{j=1}^{n} f(i) \int_0^1 \frac{\partial p^i}{\partial x_j}(te)dt. \tag{1.14}$$

Here, $e=(1,\ldots,1)\in\mathbb{R}^n$ denotes the unit vector. Because of the convergence of p^i to p^C we are interested in the limiting behaviour of this expression, i. e. we hope to obtain a value for v^C by

$$\lim_{i\to\infty} \varphi v^i(f),$$

which has to be independent of the chosen sequence (p^i). This value could then be interpreted as the diagonal value $\varphi_{SP}v^C$.

One should remark that this procedure weakens one part of Theorem 1.12, namely that one, saying, that the value is continuous w. r. t. the variation norm. It is a direct consequence of formula (1.6) that convergence w. r. t. the variation norm implies uniform convergence. Thus, we use a form of convergence which is not stronger as the one we used before. Moreover, uniform convergence is strictly weaker as convergence w. r. t. the variation norm. One immediately proves that $p^i:[0,1]\longrightarrow\mathbb{R}, i\in\mathbb{N}$, defined by

$$p^i(q) = \begin{cases} \frac{1}{i} - 2q, & q\in[0,\frac{1}{i}] \\ -\frac{1}{i} + 2(q-\frac{1}{i}), & q\in]\frac{1}{i},\frac{2}{i}] \\ \vdots & \vdots \\ (-1)^{i-1}\frac{1}{i} + (-1)^i 2(q-\frac{i-1}{i}), & q\in]\frac{i-1}{i},1] \end{cases}$$

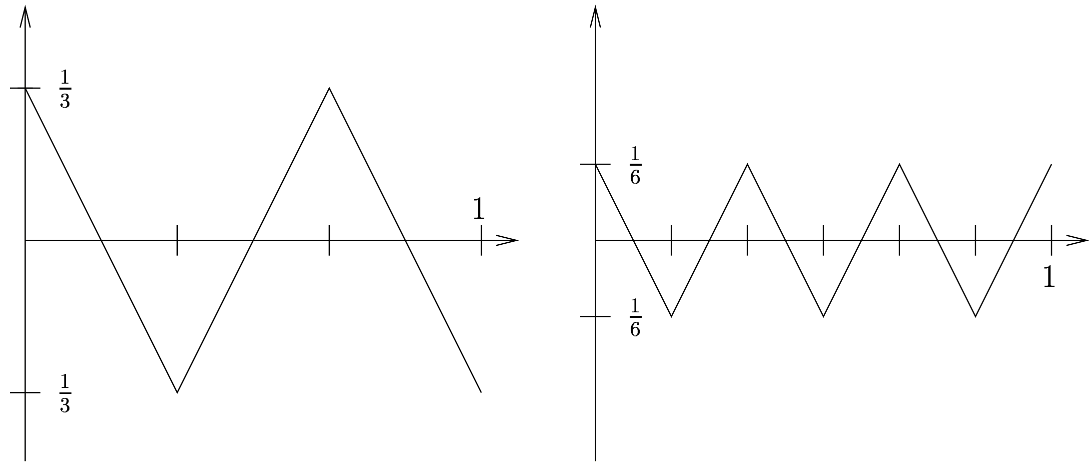

Figure 1.1: The graph of p^3 and p^6

converges uniformly to the null function. The graphs of p^3 and p^6 can be seen in Figure 1.1.

On the other hand we have

$$\|p^i\| = \sum_{j=1}^{i} \left| p^i(\frac{j}{i}) - p^i(\frac{j-1}{i}) \right| = \sum_{j=1}^{i} 2\frac{1}{i} = 2$$

for all $i \in \mathbb{N}$, i. e. there is no convergence of (p^i) to the null function w. r. t. the variation norm $\| \bullet \|$.

Using Choquet games, we have to be content with uniform convergence, as for $(e^T)^C$ with $|T| \geq 2$ the convergence of p^i to p^C w. r. t. $\| \bullet \|$ would imply $(e^T)^C \in$ *pFNA*. This is a consequence of Proposition 17.1 in [6] which states that *pFNA* is a closed subspace of *FBV*. However, Butnariu and Klement have shown in [6, p. 160], that the glove game is not an element of *pFNA*.

There are several examples showing that the requirement of a sequence of continuously differentiable functions p^i which converges uniformly to p^C without making further assumptions is not sufficient to gain a (unique) value with $\lim_{i\to\infty} \varphi v^i$.

We will consider again the Choquet extension of an unanimous game $(e^T)^C$ with $|T| \geq 2$. Let a sequence (p^i) be given as required s. t. $\lim_{i\to\infty} \varphi v^i$ exists. Now we define another sequence (p^i_π) by the following settings: Let π be a permutation of Ω s. t. $\pi(j) \in T$ for all $j \in T$. For a given $q \in [0,1]^n$ we define $q^\pi \in [0,1]^n$ by

$$q^\pi_{\pi^{-1}(l)} = q_l, \quad l \in \Omega, \tag{1.15}$$

and, furthermore,

$$p^i_\pi(q) = p^i(q^\pi). \tag{1.16}$$

Obviously, p^i_π converges uniformly to p^C since

$$|p^i_\pi(q) - p^C(q)| = |p^i(q^\pi) - p^C(q)| = |p^i(q^\pi) - p^C(q^\pi)|$$

is valid. It is also not difficult to verify continuous differentiability, since the following is true:

$$\frac{\partial p^i}{\partial x_l}(q) = \frac{\partial p^i_\pi}{\partial x_{\pi^{-1}(l)}}(q^\pi).$$

Consequently, all in all we have

$$\lim_{i\to\infty} \varphi v^i_\pi(f) = \lim_{i\to\infty} \varphi v^i(f^\pi),$$

where $v^i_\pi := p^i_\pi \circ m$ and $f^\pi(i) := f(\pi^{-1}(i))$, i. e. the limit of φv^i exists here, too. Under the assumption that the diagonal value for $(e^T)^C$ is unique, $\lim_{i\to\infty} \varphi v^i(f)$ has to equal $\lim_{i\to\infty} \varphi v^i_\pi(f)$ for all fuzzy coalitions, i. e., in particular,

$$\lim_{i\to\infty} \int_0^1 \frac{\partial p^i}{\partial x_j}(te)dt = \lim_{i\to\infty} \int_0^1 \frac{\partial p^i_\pi}{\partial x_j}(te)dt = \lim_{i\to\infty} \int_0^1 \frac{\partial p^i}{\partial x_{\pi(j)}}(te)dt. \tag{1.17}$$

This implies that all players in T will receive the same amount. In [29] the sequence $(p^i_{e^T})$ is required to be symmetric, i. e. $p^i_{e^T}(q) = p^i_{e^T}(q^\pi)$ has to be true for all $q \in [0,1]^n$ and all permutations π of Ω with $\pi(T) = T$.

A very reasonable additional assumption on the smoothness sequence is that of preserving of carriers, i. e. $p^i(q + \lambda e_l) = p^i(q)$ shall be valid for $l \notin T$, for all $q \in [0,1]^n$, and all $\lambda \in \mathbb{R}$. Concerning this, we have to say that p^i has to respect null players in the limit because of the uniform convergence:

$$\begin{aligned} \left|p^i(q) - p^i(q|_T)\right| &\leq \left|p^i(q) - p^C(q|_T)\right| + \left|p^C(q|_T) - p^i(q|_T)\right| \\ &= \left|p^i(q) - p^C(q)\right| + \left|p^C(q|_T) - p^i(q|_T)\right| \\ &\to 0. \end{aligned}$$

Therefore, we assume respecting null players, and this implies $\frac{\partial p^i}{\partial x_l}(q) = 0$, and hence

$$\lim_{i\to\infty}\int_0^1 \frac{\partial p^i}{\partial x_l}(te)dt = 0 \tag{1.18}$$

for $l \notin T$. All in all, one wants to achieve a Shapley value, i. e. especially efficiency:

$$\lim_{i\to\infty}\sum_{j=1}^{n}\int_0^1 \frac{\partial p^i}{\partial x_j}(te)dt = (e^T)^C(\Omega) = 1. \tag{1.19}$$

Together with (1.17) and (1.18) this means nothing else but

$$\lim_{i\to\infty}\int_0^1 \frac{\partial p^i}{\partial x_j}(te)dt = \frac{1}{|T|} \text{ for } j \in T,$$

and this leaves

$$\varphi_{SP}(e^T)^C(f) = \frac{1}{|T|}\sum_{j=1}^{n} f(j) \tag{1.20}$$

as the only reasonable value for $(e^T)^C$. Requiring linearity for this new value, one has found a value for all games, namely

$$\varphi_{SP}v^C(f) = \sum_{j=1}^{n} f(j)\Phi_j(v), \tag{1.21}$$

where Φ is the Shapley value for crisp games. Hence, the fuzzy value is nothing else but an evaluation of the well known Shapley value with the help of the degrees of membership of the respective players.

As was shown, a nice value can be achieved using some rather weak and reasonable assumptions. However, it remains to show that a suitable sequence (p^i) exists. The proof of the existence is quite long. It can be found in [29] in detail and shall be omitted here.

1.4.2 Averaging over Small Perturbations

Mertens extends in [16] and [17] the diagonal formula (1.12) (which is defined on *pFNA*) to a much wider class of games. In the following, Ω again denotes the set of players, and $\mathcal{C}$ is a $\sigma-$field of subsets of the set Ω. The main idea of the approach in [17] is "taking the derivative not on the diagonal, but at some small perturbation of it - say $t\Omega + \varepsilon g$ instead of $t\Omega$ - and by averaging the result for some probability distribution over perturbations". Mertens considers in detail a certain closed subspace $\mathcal{Q}$ of *FBV* (which shall not be decribed here in detail) with $pFNA \subsetneq \mathcal{Q}$ and an invariant cylinder measure $\mathcal{P}$ on $B(\Omega, \mathcal{C})$, where $B(\Omega, \mathcal{C})$ denotes the space of bounded measurable functions on $(\Omega, \mathcal{C})$. (If we take the space $B_1^+(\Omega, \mathcal{C}) = \{f | \, f \in B(\Omega, \mathcal{C}), 0 \leq f \leq 1\}$ of "ideal sets", we again have the concept of fuzzy coalitions.) As stated in [17, Theorem 2], the two sided derivative $D_g^{\widetilde{v}}(f)$ of a given $\widetilde{v} \in \mathcal{Q}$ at g in the direction of f exists for every f, for $\mathcal{P}-$ almost every g:

$$D_g^{\widetilde{v}}(f) := \lim_{\tau \to 0} \frac{\widetilde{v}(g + \tau f) - \widetilde{v}(g - \tau f)}{2\tau}.$$

Furthermore, the mapping $\varphi_{av} : \mathcal{Q} \longrightarrow FBVA$ defined by

$$\varphi_{av}(\widetilde{v})(f) = \int D_g^{\widetilde{v}}(f) \, d\mathcal{P}(g) \tag{1.22}$$

is a value which is independent of the particular invariant $\mathcal{P}$ chosen. Let λ be the Lebesgue measure. Then formula 1.22 can be rewritten in the form

$$\varphi_{av}(\widetilde{v})(f) = \lim_{\tau \to 0} \int_0^1 \frac{\widetilde{v}(t\Omega + \tau f) - \widetilde{v}(t\Omega - \tau f)}{2\tau} \, d\lambda(t).$$

It is shown in [17] that the n-handed glove game $\widehat{v}$, $\widehat{v}(f) = \min_{\{i=1\ldots,n\}} \mu_i(f)$ where $\mu_1, \ldots, \mu_n$ are non-atomic fuzzy measures, is an element of $\mathcal{Q}$. Let us now consider the special case $\widehat{v} = (e^T)^C$ for a finite set T of players. Then we obtain for a fixed fuzzy coalition f and for each $g \in [0,1]^\Omega$ s. t. there exists a $j \in T$ with $g(j) < g(k)$ for all $k \in T \setminus j$:

$$\begin{aligned} D_g^{\widehat{v}}(f) &= \lim_{\tau \to 0} \frac{\min_{i \in T} [g(i) + \tau f(i)] - \min_{i \in T} [g(i) - \tau f(i)]}{2\tau} \\ &= f(j). \end{aligned}$$

Since we get

$$\mathcal{P}(g | \, \text{There exists a } j \in T \text{ s. t. } g(j) < g(k) \, \forall k \in T \setminus j) = \frac{1}{|T|}$$

for all invariant $\mathcal{P}$, formula (1.22) provides the value

$$\varphi_{av}(\widehat{v})(f) = \frac{1}{|T|} \sum_{i \in T} f(i).$$

Hence, in the case of a finite set of players, we get exactly the same result as with the smoothing procedure though the two approaches are different in many aspects.

One should remark that it is not possible to use Mertens' framework for $(e^T)^C$ for a countable T.

An example where formula (1.22) can be used for fuzzy games with countably many players is given by weighted majority games [17, Remark 2] (for the definition of these games, of pivoting, and of the measure $\mathcal{P}$, the reader is referred to Section 2.2). For these games, Mertens' formula provides

$$\varphi_{av}(v)(f) = \sum_{i \in \mathbb{N}} f(i)\mathcal{P}(i \text{ pivots}).$$

1.4.3 The Axiomatic Approach

Tauman deals with glove games in [25]. He defines $\mathcal{Q}^n, n \in \mathbb{N}$, to be the linear space generated by all games of the form

$$v = \min(\mu_1, \ldots, \mu_n)$$

where $(\mu_1, \ldots, \mu_n)$ is a vector of n measures defined on a measurable space $(\mathcal{I}, \mathcal{C})$ with the property that each μ_i is a non-atomic probability measure and if $i \neq j$ then μ_i and μ_j are mutually singular, i. e., for each pair (i, j), $i \neq j$, there exists a set $B_{ij} \in \mathcal{C}$ s. t. $\mu_i(B_{ij}) = 0 = \mu_j(B_{ij}^C)$. Tauman states in [25, Theorem 1] that there is one and only one value φ on $\mathcal{Q}^n$, i. e. an operator that satisfies the properties of Definition 1.12. This φ satisfies

$$\varphi(\min(\mu_1, \ldots, \mu_n)) = \frac{\mu_1 + \ldots + \mu_n}{n}.$$

1.4.4 A Further Axiomatization

The three approaches described previously are all based on the definition of Aumann and Shapley [3]. The framework in this subsection differs from everything we have presented so far inasmuch here the value has to satisfy different properties. We mention the paper of Tsurumi, Tanino, and Inuiguchi [28] only for reasons of completeness and will use later on Definition 1.12 whenever thinking of a fuzzy value.

Let $\Omega = \{1, \ldots, n\}, n \in \mathbb{N}$, and a fuzzy game $(\Omega, [0,1]^\Omega, v^F)$ be given. Tsurumi, Tanino, and Inuiguchi [28] call a fuzzy coalition g a fuzzy carrier in $f \in [0,1]^\Omega$ for v^F if it satisfies $g \leq f$ and

$$v^F(g \wedge h) = v^F(h) \text{ for all } h \leq f.$$

The set of all fuzzy carriers in f for v^F is denoted by $FC(f, v^F)$.

Let $f, g \in [0,1]^\Omega$ with $g \leq f$ and a pair (i, j) with $i, j \in \Omega, i \neq j$ be given. The fuzzy coalition $g^f_{ij} \leq f$ is defined by

$$g^f_{ij}(k) = \begin{cases} g(i) \wedge f(j) & \text{if } k = i \\ g(j) \wedge f(i) & \text{if } k = j \\ g(k) & \text{otherwise.} \end{cases}$$

For any $f \in [0,1]^\Omega$, the fuzzy coalition $h_{ij}[f]$ is given by

$$h_{ij}[f](k) = \begin{cases} f(j) & \text{if } k = i \\ f(i) & \text{if } k = j \\ f(k) & \text{otherwise.} \end{cases}$$

Now we are well prepared to state the properties which have to be satisfied by a fuzzy value in the eyes of Tsurumi, Tanino, and Inuiguchi.

Definition 1.14 *[28, Definition 10]*

Let $\mathcal{F}(\Omega)$ be a set of fuzzy games, $v^F \in \mathcal{F}(\Omega)$, and $f \in [0,1]^\Omega$. A function

$$\varphi : \mathcal{F}(\Omega) \to (\mathbb{R}^n_+)^{[0,1]^\Omega}$$

is said to be a Shapley value on $\mathcal{F}(\Omega)$ if it satisfies the following four axioms:

1. $\sum_{i \in \Omega} \varphi_i(v^F)(f) = v^F(f)$ *and* $\varphi_i(v^F)(f) = 0$ *if* $f(i) = 0$.

2. *For* $g \in FC(f, v^F)$*, we have* $\varphi_i(v^F)(f) = \varphi_i(v^F)(g)$ *for all* $i \in \Omega$*.*

3. *If* $f_{ij}^f \in FC(f, v^F)$ *and* $v^F(g) = v^F(h_{i,j}[g])$ *for any* $g \le f_{ij}^f$*, then* $\varphi_i(v^F)(f) = \varphi_j(v^F)(f)$*.*

4. *Let* $v_1^F, v_2^F \in \mathcal{F}(\Omega)$ *be given s. t.* $v_1^F + v_2^F \in \mathcal{F}(\Omega)$*. Then* $\varphi_i(v_1^F + v_2^F) = \varphi_i(v_1^F) + \varphi_i(v_2^F)$ *for any* $i \in \Omega$*.*

One should remark that while φ is additive, $\varphi(v^F)$ does not have this property, i. e. for $f, g \in [0,1]^\Omega$, $f + g \le \Omega$, the equation $\varphi(v^F)(f) + \varphi(v^F)(g) = \varphi(v^F)(f+g)$ does not hold in general.

The following theorem provides a value for all Choquet games with finitely many players.

Theorem 1.15 *[28, Theorem 7]*

Let the set $\mathcal{V}$ *be defined as* $\mathcal{V} := \{v \mid v : \underline{\underline{P}}(\Omega) \to \mathbb{R}, v(\emptyset) = 0\}$ *and let the function* $\overline{\varphi} : \mathcal{V} \to (\mathbb{R}_+^n)^{\underline{\underline{P}}(\Omega)}$ *be given by*

$$\overline{\varphi}_i(v)(S) = \sum_{T \subseteq S} \frac{(t-1)!(s-t)!}{s!}[v(T) - v(T-i)], i \in \Omega.$$

W. l. o. g. we may consider a fuzzy coalition f *with* $f(1) \ge \ldots \ge f(n)$*. Then the function* $\varphi : \mathcal{V}^C \to (\mathbb{R}_+^n)^{[0,1]^\Omega}$ *defined by*

$$\varphi_i(v^C)(f) = \sum_{i \in \Omega} \overline{\varphi}_i(v)(S_i)[f(i) - f(i+1)]$$

is a Shapley value on $\mathcal{V}^C$*.*

The foregoing theorem provides exactly one value. However, it is not clear whether there are other functions satisfying the four axioms of Definition 1.14.

Example 1.16 *The unanimous game*

Let us consider the case $\Omega = \{1,2,3\}, v = e^{\{1,2,3\}}$ and $f \in [0,1]^\Omega$ with $f(1) = \frac{1}{2}, f(2) = \frac{1}{4}, f(3) = \frac{1}{8}$. Then, the value of Tsurumi, Tanino, and Inuiguchi given by Theorem 1.15 provides

$$\varphi_i(e^\Omega)^C(f) = \frac{1}{24} \text{ for all } i \in \Omega.$$

Chapter 2

A Fuzzy Value for Countably Many Players

In the following, we will have a look at games with $\Omega = \mathbb{N}$, i. e. we consider the set of players to be countable. A reason for doing this is that in the countable case we have some large players set against a bulk of very small players which are more and more unimportant but can never be fully neglected. Thus, the countable model reflects some typical situations with large and small players (cf. Rosenmüller [24, Chapter 7]).

In this chapter, we will discuss in how far it is possible to get a fuzzy value for Choquet games in this case. First of all, one has to mention that it is not possible to receive a value for all Choquet extensions with bounded variation. To demonstrate this statement, one can consider the important example $(e^{\mathbb{N}})^C$. It is easy to prove that there is no $\sigma-$additive value for this game since for any permutation π we have $\pi(e^{\mathbb{N}})^C = (e^{\mathbb{N}})^C$. However, there is no $\sigma-$additive fuzzy measure m satisfying both $m(\mathbb{N}) = 1$ and $\pi m = m$ for all π.

Unfortunately, it is not even possible to find a finitely additive value. As mentioned above, $(e^{\mathbb{N}})^C$ is invariant under all automorphisms of $\mathbb{N}$ and thus for the sets

$$S_1 := \{1, 4, 7, 10, \ldots\}$$

$$S_2 := \{2, 5, 8, 11, \ldots\}$$
$$S_3 := \{3, 6, 9, 12, \ldots\}$$

$\varphi((e^{\mathbb{N}})^C)(S_1) = \varphi((e^{\mathbb{N}})^C)(S_2) = \varphi((e^{\mathbb{N}})^C)(S_3)$ has to be valid. Since it is easy to find a permutation π which maps S_1 in $S_2 + S_3$, we have that $\varphi((e^{\mathbb{N}})^C)(S_1) = \varphi((e^{\mathbb{N}})^C)(S_2 + S_3) = 2\varphi((e^{\mathbb{N}})^C)(S_1)$. This last equation implies $\varphi((e^{\mathbb{N}})^C)(S_i) = 0$ for all i, i. e $\varphi((e^{\mathbb{N}})^C)(\mathbb{N}) = \sum_{j=1}^{3} \varphi((e^{\mathbb{N}})^C)(S_j) = 0$. However, this contradicts efficiency. Hence, we have to restrict ourselves to certain classes of games when looking for a value.

This is done in the first three sections in which we extend some crisp approaches (Artstein [1], Shapley [26], Pallaschke and Rosenmüller [20]) to the fuzzy case. We show how we can get most of their results with only some slight changes.

In Section 2.4, we try to apply the smoothing procedure to the countable case. We show that a value for all games in the closure of the linear hull of all games with finite carrier does exist. However, as the smoothing procedure is based strongly on efficiency and invariance under permutations, it is not even possible to get something like a quasi-value for $(e^{\mathbb{N}})^C$.

In the last section of this chapter, we examine an approach of Rosenmüller [22] who constructs a quasi-value for a continuum of players. One can use the main part of this framework for the countable case to get a result for a class of games which contains the unanimous games. Restricting ourselves to $\mathbb{N}$ as the player set, we can avoid some measurability problems which occurred in Rosenmüller's framework. Without these problems we can get some quite reasonable results.

2.1 The Value on $\mathbf{spanC_0}$

In this section, we consider Choquet games on countably many players and allow all coalitions to be feasible, i. e. we consider the coalitional functions to be defined on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$. We define $\boldsymbol{C_0}$ as the set of Choquet functions with a finite carrier, i. e. for each $v^C \in C_0$ there exists a coalition $K \in \underline{\underline{P}}(\mathbb{N})$ with $|K| < \infty$ s. t. $v^C|_K = v^C$. Considering such a finite coalition to be the set of players, Mertens

[17] and Weiß [29] have shown how a fuzzy value could look like. Now, the assumption of the null player property implies a canonical way to obtain a fuzzy value on C_0.

As the diagonal value of Butnariu and Klement [6, Theorem 18.4] is norm-continuous, and as we hope that our value preserves as many properties as possible, one can think of extending the value found in [29] to the set $spanC_0$, where $spanC_0$ denotes the closure of the linear hull of C_0. The examination of this space is exactly our task in this chapter, and we will show that it is possible to establish a value on this space. To do this, we will use the approach of Artstein [1], who made similar thoughts for games with crisp coalitions. In this section, we will mainly follow Rosenmüller [24, Chapter 7.5], who gives a quite detailed description of Artstein's ideas.

Let $\boldsymbol{BV}$ denote the set of crisp coalitional functions with bounded variation. As can be easily verified, each element of BV has got a Choquet extension. The set $\boldsymbol{CBV}$ ($\subsetneq FBV$) may consist of all Choquet extensions of the crisp games with bounded variation. Since we will deal with closed subspaces of CBV, it is an interesting and important question whether CBV is complete w. r. t. the variation norm. Proposition 2.2 will give an answer to this topological problem. In the proof of this proposition, we will need a specific decomposition of the elements of FBV, which will be described in the following lemma (cf. [6, Corollary 15.8]):

Lemma 2.1 *If v^F has bounded variation, then there exist monotone fuzzy coalitional functions $\boldsymbol{v^{+F}}$ and $\boldsymbol{v^{-F}}$ (the* ***upper*** *and* ***lower variation of v*** *) which satisfy*

$$v^F = v^{+F} - v^{-F} \tag{2.1}$$

and

$$\|v^F\| = v^{+F}(\mathbb{N}) + v^{-F}(\mathbb{N}) \tag{2.2}$$

The previously mentioned fuzzy coalitional functions are defined by

$$\begin{aligned} v^{+F}(f) &= \sup\left\{ \sum_{i=1}^{k} \left(v^F(f_i) - v^F(f_{i-1})\right)^+ \,\middle|\, f_0 \leq \ldots \leq f_k \leq f \right\} & (2.3)\\ v^{-F}(f) &= v^{+F}(f) - v^F(f). & (2.4) \end{aligned}$$

The variation norm for crisp games is defined analogously to the one for fuzzy games. Since in the following it will be clear whether we want to calculate a crisp or a fuzzy variation norm, we will not distinguish between the two norms.

Proposition 2.2 *CBV is a Banach space with the total variation as its norm.*

Proof $\| \bullet \|$ is obviously a norm. Butnariu and Klement show in [6, Proposition 15.9] that $(FBV, \| \bullet \|)$ is a Banach space. Hence every Cauchy sequence $(v_n^C)_{n\in\mathbb{N}}$ in $CBV (\subseteq FBV)$ converges to an element $v^F \in FBV$. It remains to show that this limit v^F is also an element of CBV.

For this reason, we define a coalitional function v on $\underline{\underline{P}}(\mathbb{N})$ as the restriction of v^F to the crisp coalitions, i. e. v is defined on $\underline{\underline{P}}(\mathbb{N})$ by

$$v(S) := v^F(S)$$

for all $S \in \underline{\underline{P}}(\mathbb{N})$. As v^F has bounded variation, v has this property, too. We denote by v^C the Choquet extension of this v. To complete our proof, it is sufficient to show that $v^C = v^F$. To show this, we define the sequence (v_n) on $\underline{\underline{P}}(\mathbb{N})$ by

$$v_n(S) := v_n^C(S).$$

Because of the convergence of $(v_n^C)_n$ to v^F w. r. t. the variation norm, we obviously have $\|v_n - v\| \to 0$ $(n \to \infty)$. As $v_n^C \in CBV$, every v_n can be represented as the difference of two monotone setfunctions defined on $\underline{\underline{P}}(\mathbb{N})$. In particular, we can choose the upper and lower variation of v_n, $v_n = v_n^+ - v_n^-$.

Claim: $\lim_{n\to\infty} v_n^+(S) = v^+(S)$ is valid for all $S \subseteq \mathbb{N}$, and, therefore, also $\lim_{n\to\infty} v_n^-(S) = v^-(S)$.

In order to verify this, one should observe that the following is true for all $n \in \mathbb{N}$

$$\begin{aligned} v_n^+(S) &= \sup\left\{ \sum_{i=1}^{k} [v_n(S_i) - v_n(S_{i-1})]^+ \,\middle|\, S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq S \right\} \\ &\leq \sup\left\{ \sum_{i=1}^{k} [(v_n - v)(S_i) - (v_n - v)(S_{i-1})]^+ \,\middle|\, S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq S \right\} \end{aligned}$$

$$
\begin{aligned}
&+\sup\left\{\sum_{i=1}^{k}[v(S_i)-v(S_{i-1})]^+ \middle| S_0\subseteq S_1\subseteq\ldots\subseteq S_k\subseteq S\right\}\\
= \quad &(v_n-v)^+(S)+v^+(S), \qquad (2.5)
\end{aligned}
$$

and analogously

$$v^+(S)\le(v-v_n)^+(S)+v_n^+(S). \qquad (2.6)$$

Formula (2.2) states that

$$\|v_n-v\|=(v_n-v)^+(\mathbb{N})+(v_n-v)^-(\mathbb{N}).$$

Since $\|v_n-v\|\to 0$, we have that $(v_n-v)^+(\mathbb{N})\to 0$ $(n\to\infty)$, and, because of the monotonicity of the upper variation, also $(v_n-v)^+(S)\to 0$ is valid for all $S\in\underline{\underline{P}}(\mathbb{N})$. Together with (2.5), we have that

$$\lim_{n\to\infty} v_n^+(S)\le v^+(S)$$

follows for all $S\in\underline{\underline{P}}(\mathbb{N})$. Using (2.6), one gets in an analogous way

$$\lim_{n\to\infty} v_n^+(S)\ge v^+(S).$$

This completely proves the Claim.

Now, a sequence $(\widetilde{v}_j)_{j\in\mathbb{N}}$ is defined on $\underline{\underline{P}}(\mathbb{N})$ by

$$\widetilde{v}_j=\inf_{i\ge j}v_i^+-\sup_{i\ge j}v_i^-.$$

One can easily verify the following properties:

1. $\widetilde{v}_j:\underline{\underline{P}}(\mathbb{N})\to\mathbb{R}$ for all $j\in\mathbb{N}$: $\lim_{i\to\infty}v_i^+=v^+$ and $\lim_{i\to\infty}v_i^-=v^-$ are again setfunctions. Hence, both $\inf_{i\ge j}v_i^+(S)<\infty$ and $\sup_{i\ge j}v_i^-(S)<\infty$ is valid for all $S\in\underline{\underline{P}}(\mathbb{N})$ and every $j\in\mathbb{N}$.

2. $\widetilde{v}_j\in BV$ for all $j\in\mathbb{N}$: If we take the infimum or the supremum over monotone functions, we get a monotone function. ($S\subseteq T\Rightarrow v_i^+(S)\le v_i^+(T)$ for all $i\Rightarrow \inf_{i\ge j}v_i^+(S)\le\inf_{i\ge j}v_i^+(T)$ for all j)

3. $\widetilde{v}_j\le v_j$ for all $j\in\mathbb{N}$ as $\inf_{i\ge j}v_i^+\le v_j^+$ and $\sup_{i\ge j}v_i^-\ge v_j^-$

4. $\widetilde{v}_j \leq \widetilde{v}_{j+1}$ since $\inf_{i\geq j} v_i^+ \leq \inf_{i\geq j+1} v_i^+$ and also $\sup_{i\geq j} v_i^- \geq \sup_{i\geq j+1} v_i^-$.

5. $\lim_{i\to\infty} \widetilde{v}_i(S) = v(S)$ for all $S \in \underline{\underline{P}}(\mathbb{N})$:

   For every $S \subseteq \mathbb{N}$ and every $\varepsilon > 0$ there exists an $\overline{n} \in \mathbb{N}$ s.t. $|v_n^+(S) - v^+(S)| < \varepsilon$ for all $n \geq \overline{n}$. Hence, $|\inf_{i\geq n} v_i^+(S) - v^+(S)| < \varepsilon$ is valid for all $n \geq \bar{n}$, and, consequently, $\lim_{i\to\infty} \inf_{j\geq i} v_j^+(S) = v^+(S)$. The same can be shown for $v^-(S)$. All in all, we have that

$$\begin{aligned}\lim_{i\to\infty} \widetilde{v}_i(S) &= \lim_{i\to\infty}[\inf_{j\geq i} v_j^+(S) - \sup_{j\geq i} v_j^-(S)] \\ &= v^+(S) - v^-(S) = v(S).\end{aligned}$$

Because of property 5., we have, in particular, that $\lim_{i\to\infty} \widetilde{v}_i(f > t) = v(f > t)$ is valid for all $f \in [0,1]^{\mathbb{N}}$, and, because of 4., $(\widetilde{v}_n(f > t))_n$ is monotone increasing in n. Therefore, applying the monotone convergence theorem given by Denneberg [8, Proposition 5.2(iv)], we have that

$$\begin{aligned}\lim_{i\to\infty} \widetilde{v}_i^C(f) &= \lim_{i\to\infty} \int_0^1 \widetilde{v}_i(f > t)dt \\ &= \int_0^1 \lim_{i\to\infty} \widetilde{v}_i(f > t)dt \\ &= \int_0^1 v(f > t)dt \\ &= v^C(f).\end{aligned}$$

Proposition 5.2(iii) in [8] states that $v \leq u$ implies $v^C \leq u^C$. Because of property 3., we have that $\widetilde{v}_i^C \leq v_i^C$, and, consequently,

$$v^C = \lim_{i\to\infty} \widetilde{v}_i^C \leq \lim_{i\to\infty} v_i^C = v^F.$$

By defining $\overline{v}_n := \sup_{j\geq n} v_j^+ - \inf_{j\geq n} v_j^-$, one can show that $v^C = \lim_{i\to\infty} \overline{v}_i^C \geq v^F$. As the final result one obtains $v^C = v^F$, and this had to be shown. **q.e.d.**

Later on, we will need another possibility of representing $spanC_0$. Let $\boldsymbol{A}_+^{\boldsymbol{\sigma}}$ be the set of $\sigma-$additive, non-negative functions on $\mathbb{N}$, and the space $\boldsymbol{AC}$ be given by

$$AC := span\left(\{p \circ m \mid m \in A_+^{\sigma}, p : [0, m(\mathbb{N})] \to \mathbb{R}, p \text{ absolutely continuous}\}\right).$$

Lemma 2.3 $AC^C = spanC_0$.

Proof We define V_0 as the space of crisp setfunctions with finite carrier. Rosenmüller shows in [24, Theorem 7.4.13] that $AC = spanV_0$ is valid. Since it is obvious that V_0^C equals C_0, it remains to show that $(spanV_0)^C = spanV_0^C$. This, however, follows easily from the linearity of the Choquet integral (cf. Denneberg [8, Proposition 5.2(i) und (ii)]):

$$\begin{aligned} v^C \in spanV_0^C \iff & \text{ for } \varepsilon > 0 \text{ there exist } c_k \in \mathbb{R} \text{ and } v_k^C \in V_0^C, \\ & k = \{1, \ldots, K\}, K \in \mathbb{N}, \text{ s.t.} \\ & \widetilde{v}^C = \sum_{k=1}^{K} c_k v_k^C = \left(\sum_{k=1}^{K} c_k v_k\right)^C \\ & \text{satisfies } ||\widetilde{v}^C - v^C|| < \varepsilon \\ \iff & v^C \in (spanV_0)^C. \end{aligned}$$

q.e.d.

For every Choquet function v^C on $[0,1]^{\mathbb{N}}$, we define a sequence $v^{(n)C} : [0,1]^{\mathbb{N}} \to \mathbb{R}$ $(n \in \mathbb{N})$ by

$$v^{(n)C}(f) = v^C(\widetilde{f}^n),$$

where $\widetilde{f}^n \in [0,1]^{\mathbb{N}}$ is defined by

$$\widetilde{f}^n(j) = \begin{cases} f(j), & \text{if } j \leq n, \\ f(n), & \text{if } j > n. \end{cases}$$

Later on, we will often use $v^{0(n)C}$ which denotes the restriction of $v^{(n)C}$ to $[0,1]^{\{1,\ldots,n\}}$, i. e.

$$v^{0(n)C}(f^n) := v^{(n)C}(f \cap \{1, \ldots, n\}),$$

$f^n \in [0,1]^{\{1,\dots,n\}}$. Since the linear combination of Choquet functions with finite carrier is nothing else but a Choquet function with a finite carrier, we know that for every $v^C \in spanC_0$ there exists a sequence (v_n^C) converging to v^C w. r. t. $\|\bullet\|$ s. t. every v_n^C has got a finite carrier. The next proposition shows that $(v^{(n)C})_n$ does this job.

Proposition 2.4 *For every $v^C \in spanC_0$, we have that $v^{(n)C} \longrightarrow_{n\to\infty} v^C$ w. r. t. the variation norm.*

To prove this statement, we have to make some further thoughts first (among other things, we will define another sequence of Choquet functions and state two lemmas).

Definition 2.5 1. ***CM*** *is the set of monotone Choquet functions, i. e. for $v^C \in CM$ and $f \geq g$, $f, g \in [0,1]^{\mathbb{N}}$, we have that $v^C(f) \geq v^C(g)$.*

2. *$v^C \in CM$ is said to be **upper $\sigma-$continuous**, if for all $f, f_n \in [0,1]^{\mathbb{N}}$ such that $f_n \uparrow f (n \to \infty)$ we have that $v^C(f_n) \uparrow v^C(f)(n \to \infty)$.*

3. *$v^C \in CBV$ is said to be **convex**, if*

$$v^C(f) + v^C(g) \leq v^C(f \vee g) + v^C(f \wedge g)$$

is valid for all $f, g \in [0,1]^{\mathbb{N}}$.

As far as upper $\sigma-$continuity and convexity are considered, there is a strong relationship between a Choquet function and its underlying crisp set function concerning, as the following lemma states:

Lemma 2.6 1. *v is upper $\sigma-$continuous $\Longleftrightarrow v^C$ is upper $\sigma-$continuous.*

2. *v is convex $\Longleftrightarrow v^C$ is convex.*

Proof "$\Longleftarrow$" is obvious for both cases since it is sufficient to restrict v^C to $\underline{\underline{P}}(\mathbb{N})$.

1. "$\Longrightarrow$" $f_n \uparrow f$ implies $v(f_n > t) \uparrow v(f > t)$, and hence it is possible to apply the monotone convergence theorem to get

$$v^C(f_n) = \int v(f_n > t)dt \to \int v(f > t)dt = v^C(f).$$

2. "$\Longrightarrow$" It is easy to verify that the equalities $\{(f \wedge g) > t\} = \{(f > t) \cap (g > t)\}$ and $\{(f \vee g) > t\} = \{(f > t) \cup (g > t)\}$ are valid. Therefore, it follows that

$$\begin{aligned} v^C(f \wedge g) + v^C(f \vee g) &= \int v((f \wedge g) > t)dt + \int v((f \vee g) > t)dt \\ &= \int [v((f > t) \cap (g > t)) + v((f > t) \cup (g > t))]dt \\ &\geq \int [v(f > t) + v(g > t)]dt = v^C(f) + v^C(g). \end{aligned}$$

q.e.d.

Remark 2.7 *If $v^C \in CBV$ is convex and non-negative ($v^C \in \boldsymbol{CBV_+^C}$), then v^C is monotone.*

Proof Let S and T be crisp coalitions ($S, T \subseteq \mathbb{N}$) and $S \subseteq T$. We have $v^C(S) + v^C(T \setminus S) \leq v^C(\emptyset) + v^C(T)$, and hence $v^C(S) \leq v^C(T)$ because of the non-negativity. But this inequality states nothing else than $v(S) \leq v(T)$, i. e. the underlying v is monotone. Since the Choquet extension preserves monotonicity [8, Proposition 5.1(iv)], we have proven the remark. **q.e.d.**

We do not know whether the statement of remark 2.7 holds true for every fuzzy coalitional function. In the case of crisp coalitions one can use the well known fact that convexity implies superadditivity and deduce monotonicity from this fact. Superadditivity for fuzzy coalitions is defined as

$$v^F(f \vee g) \geq v^F(f) + v^F(g)$$

for $f \wedge g = \emptyset$. Again, convexity is a special case of superadditivity. However, now it is not possible to conclude monotonicity. If one defines superadditivity in the way Butnariu and Klement [6] do, i. e. $v^F(f+g) \geq v^F(f) + v^F(g)$ for $f + g \leq \Omega$, one would get monotonicity. The problem is that this kind of superadditivity is not implied by our form of convexity. To show this, one can look at a convex and non-negative v with $v(\Omega) > 0$ and define a fuzzy coalitional function v^E by

$$v^E(f) = v(T_f) \text{ where } T_f := \{\omega | f(\omega) \geq \frac{1}{2}\}, f \in [0,1]^\Omega.$$

v^E is convex as

$$v^E(f) + v^E(g) = v(T_f) + v(T_g) \leq v(T_f \cap T_g) + v(T_f \cup T_g) = v^E(f \wedge g) + v^E(f \vee g)$$

is valid. However, v^E is not superadditive in the sense of Butnariu and Klement. To see this, consider for example $f = g = \frac{1}{2}\Omega$. In this case we have that

$$v^E(f) + v^E(g) = v(\Omega) + v(\Omega) > v(\Omega) = v^E(f+g).$$

After these general thoughts concerning fuzzy coalitional functions we continue with some considerations of convergence. For this reason, we define $\boldsymbol{C_+^{\sigma,C}}$ to be the space of all $v^C \in CBV$ which are convex, non-negative, and upper $\sigma-$continuous.

Lemma 2.8 *[24, Theorem 7.4.9] Let $v^C \in C_+^{\sigma,C}$ be given. Then, $v^{nC} := v^C|_{\{1,\dots,n\}}$ satisfies $v^{nC} \to v^C$ (w. r. t. $\| \bullet \|$).*

Proof Let $f, g \in [0,1]^{\mathbb{N}}$ and $f \leq g$. $f|_{\{1,\dots,n\}}$ is defined by

$$f|_{\{1,\dots,n\}}(i) = \begin{cases} f(i), & \text{if } i \leq n \\ 0, & \text{else.} \end{cases}$$

We have

$$\begin{aligned} v^C(f) + v^C(g|_{\{1,\dots,n\}}) &\leq v^C(f|_{\{1,\dots,n\}}) + v^C(g|_{\{1,\dots,n\}} \vee f) \\ &\leq v^C(f|_{\{1,\dots,n\}}) + v^C(g). \end{aligned}$$

Thus, $v^C - v^{nC}$ is monotone, and hence

$$||v^C - v^{nC}|| = v^C(\mathbb{N}) - v^{nC}(\mathbb{N}) = v(\mathbb{N}) - v(\{1,\ldots,n\}) \to 0\ (n \to \infty).$$

q.e.d.

Lemma 2.9 *[24, Theorem 7.4.12]* $v^C \in C_+^{\sigma,C}$ *satisfies* $v^{(n)C} \to_{n\to\infty} v^C$ *(w. r. t.* $\|\bullet\|$*).*

Proof First of all, we show the monotonicity of $v^{(n)C} - v^{nC}$. To do this, we take $f, g \in [0,1]^{\mathbb{N}}$ s. t. $f \leq g$. Because of convexity and monotonicity of v^C, the following is valid:

$$\begin{aligned}
& v^C(f|_{\{1,\ldots,n\}}) + v^C\left(g|_{\{1,\ldots,n\}} + \sum_{i=n+1}^{\infty} g(n)i\right) \\
\geq\ & v^C(g|_{\{1,\ldots,n\}}) + v^C\left(f|_{\{1,\ldots,n\}} + \sum_{i=n+1}^{\infty} g(n)i\right) \\
\geq\ & v^C(g|_{\{1,\ldots,n\}}) + v^{(n)C}\left(f|_{\{1,\ldots,n\}} + \sum_{i=n+1}^{\infty} f(n)i\right).
\end{aligned}$$

One obtains nothing else but

$$v^{(n)C}(g) - v^{nC}(g) \geq v^{(n)C}(f) - v^{nC}(f),$$

and this had to be shown. Therefore,

$$\begin{aligned}
\|v^{(n)C} - v^{nC}\| &= v^{(n)C}(\mathbb{N}) - v^{nC}(\mathbb{N}) \\
&= v(\mathbb{N}) - v(\{1,\ldots,n\}) \to 0 (n \to \infty).
\end{aligned}$$

All in all, together with Lemma 2.8, we have that

$$\|v^{(n)C} - v^C\| \leq \|v^{(n)C} - v^{nC}\| + \|v^{nC} - v^C\| \to 0\ \ (n \to \infty).$$

q.e.d.

Lemma 2.8 and 2.9 are also valid for general convex, non-negative and upper σ-continuous fuzzy coalitional functions v^F, provided that v^F preserves monotonicity. After all these preparing thoughts, we are finally able to present the still outstanding proof:

Proof (of Proposition 2.4) By Lemma 2.3 we know that $spanC_0 = AC^C$ is true. Consider $v^C \in AC^C$ and $\varepsilon > 0$. There can be found a $\overline{v}^C = (p \circ m)^C$, where p and m are as in the definition of AC s. t. $\|v^C - \overline{v}^C\| < \varepsilon$. Because of Lemma 7.4.7 in [24], there exists a $\widetilde{v}^C = (\widetilde{p} \circ m)^C$ for $\overline{v}^C$ with piecewise linear $\widetilde{p}$ s. t. $\|\overline{v}^C - \widetilde{v}^C\| < \varepsilon$. This $\widetilde{p}$ is a linear combination of some $p^\alpha_\beta : [0, m(\mathbb{N})] \to \mathbb{R}$ where $p^\alpha_\beta(x) = \beta(x-\alpha)^+$ (cf. Rosenmüller [24, Lemma 7.4.11]). One can easily check that $(p^\alpha_\beta \circ m)^C$ is an element of $C^{\sigma,C}_+$, and hence the same statement is valid for $\widetilde{v}^C$. For this reason, one gets $\|\widetilde{v}^C - \widetilde{v}^{(n)C}\| < \varepsilon$ for n large enough (Lemma 2.9). To sum up, we have until now that

$$\|v^C - \widetilde{v}^{(n)C}\| \leq \|v^C - \overline{v}^C\| + \|\overline{v}^C - \widetilde{v}^C\| + \|\widetilde{v}^C - \widetilde{v}^{(n)C}\| < 3\varepsilon.$$

Since $\|v^C - w^C\| < \varepsilon$ naturally implies $\|v^{(n)C} - w^{(n)C}\| < \varepsilon$, all in all, the following inequality is valid for sufficiently large n:

$$\|v^{(n)C} - v^C\| \leq \|v^{(n)C} - \overline{v}^{(n)C}\| + \|\overline{v}^{(n)C} - \widetilde{v}^{(n)C}\| + \|\widetilde{v}^{(n)C} - v^C\| < 5\varepsilon.$$

q.e.d.

The next statements are concerned with the value and its properties.

Definition 2.10 *We define the operator $\varphi^{(n)} : CBV \longrightarrow FBVA$ for every $n \in \mathbb{N}$ by*

$$\varphi^{(n)}(v^C)(f) = \varphi^n_{SP}(v^{0(n)C})(f^{\{1,\dots,n\}}),\ f \in [0,1]^{\mathbb{N}}, \tag{2.7}$$

where $\varphi^n_{SP} = \varphi^n_{av}$ is the fuzzy value on $\{1,\dots,n\}$ given by the smoothing procedure, and $f^{\{1,\dots,n\}} \in [0,1]^{\{1,\dots,n\}}$ is defined by $f^{\{1,\dots,n\}}(j) = f(j)$ for $j \leq n$.

Proposition 2.11 *The set $B^3 := \{v^C \in CBV|$ For each $i \in \mathbb{N}$ and each $f \in [0,1]^{\mathbb{N}}$ the expression $\lim_{n\to\infty} \varphi^{(n)}(v^C)(f(i)i)$ exists$\}$ is a closed linear subspace of CBV.*

This theorem can be proven in a similar way as Theorem 7.5.4 in [24].

Lemma 2.12 *For $v^C \in B^3$ and $f \in [0,1]^{\mathbb{N}}$, we define*

$$\varphi(v^C)(f(i)i) := \lim_{n\to\infty} \varphi^{(n)}(v^C)(f(i)i). \tag{2.8}$$

With this definition, the inequality

$$\sum_{i\in\mathbb{N}} |\varphi(v^C)(f(i)i)| \leq \|v^C\| \tag{2.9}$$

is valid.

Proof $v^C \in B^3$ implies that v^C can be written as the difference of two monotone Choquet functions, and that $\lim_{n\to\infty} \varphi_i^{(n)}(v)$ exists where $\varphi_i^{(n)}(v) := \varphi^{(n)}(v^C)(i)$.

$$\sum_{i\in\mathbb{N}} |\varphi_i(v)| \leq \|v\| \leq \|v^C\| \tag{2.10}$$

is true, where the first inequality has been shown by Artstein [1, Proposition 5.2]. The second inequality is obviously true: With the definition of the variation norm, we have that

$$\|v^C\| = \sup\left\{\sum_{k=1}^{l} |v^C(f_k) - v^C(f_{k-1})| \,\Big|\, f_0 \leq \ldots \leq f_l \leq \mathbb{N}\right\}$$

and

$$\|v\| = \sup\left\{\sum_{k=1}^{l} |v(S_k) - v(S_{k-1})| \,\Big|\, S_0 \subseteq \ldots \subseteq S_l \subseteq \mathbb{N}\right\}.$$

The feasible set in the second equation is contained in the feasible set of the first "supremum", i. e. $\|v^C\| \geq \|v\|$. Together with formula (1.21), one obtains

$$|\varphi^{(n)}(v^C)(f(i)i)| = f(i)|\varphi_i^{(n)}(v)| \leq |\varphi_i^{(n)}(v)|,$$

and, therefore,

$$|\varphi(v^C)(f(i)i)| \leq |\varphi_i(v)|$$

in the limit. Thus,

$$\sum_{i\in\mathbb{N}} |\varphi(v^C)(f(i)i)| \leq \sum_{i\in\mathbb{N}} |\varphi_i(v)|$$

is naturally given, and, together with (2.10), the statement of the lemma follows.
q.e.d.

Because of inequality (2.9), the following definition is possible

Definition 2.13 *$\varphi : B^3 \longrightarrow FBVA$ is given by*

$$\varphi(v^C)(f) \;:=\; \sum_{i=1}^{\infty} \varphi(v^C)(f(i)i), \tag{2.11}$$

where $\varphi(v^C)(f(i)i)$ is given by formula (2.8).

We have that

1. $\varphi(v^C)(f(i)i) = \lim_{n\to\infty} \varphi^{(n)}(v^C)(f(i)i) = \lim_{n\to\infty} f(i)\varphi_i^{(n)}(v) = f(i)\varphi_i(v)$, where again formula (1.21) has been used.

2. $\varphi(v^C)$ is $\sigma-$additive:

   Consider a sequence $(f_j)_{j\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ with $\sum_{j=1}^{\infty} f_j \leq \mathbb{N}$. Then $f := \sum_{j=1}^{\infty} f_j \in [0,1]^{\mathbb{N}}$ is a fuzzy coalition, and one obtains

$$\begin{aligned}
\sum_{j=1}^{\infty} \varphi(v^C)(f_j) &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \varphi(v^C)(f_j(i)i) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f_j(i)\varphi_i(v) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_j(i)\varphi_i(v) \\
&= \sum_{i=1}^{\infty} f(i)\varphi_i(v) \\
&= \varphi(v^C)(f) \\
&= \varphi(v^C)\left(\sum_{j=1}^{\infty} f_j\right)
\end{aligned}$$

Here, we have used the double series theorem of Cauchy: Since

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}|f_j(i)\varphi_i(v)| \leq \sum_{i\in\mathbb{N}}|\varphi_i(v)| \leq \|v\| < \infty,$$

the order of the two summation signs of the term on the left side can be switched.

Proposition 2.14 *$\|\varphi(v^C))\| \leq \|v^C\|$ is valid, i. e. $\varphi : B^3 \to A^\sigma$ is a norm-continuous, linear operator.*

Proof

$$\begin{aligned}
\|\varphi(v^C)\| &= \sup\left\{\sum_{k=1}^{l}|\varphi(v^C)(f_k) - \varphi(v^C)(f_{k-1})| \;\middle|\; f_0 \leq \ldots \leq f_l \leq \mathbb{N}\right\}\\
&= \sup\left\{\sum_{k=1}^{l}|\varphi(v^C)(f_k - f_{k-1})| \;\middle|\; f_0 \leq \ldots \leq f_l \leq \mathbb{N}\right\}\\
&= \sup\left\{\sum_{k=1}^{l}|\varphi(v^C)(g_k)| \;\middle|\; \sum_{k=1}^{l} g_k \leq \mathbb{N}\right\}\\
&= \sup\left\{\sum_{k=1}^{l}\left|\sum_{i\in\mathbb{N}}\varphi(v^C)(g_k(i)i)\right| \;\middle|\; \sum_{k=1}^{l} g_k \leq \mathbb{N}\right\}\\
&\leq \sup\left\{\sum_{k=1}^{l}\sum_{i\in\mathbb{N}} g_k(i)|\varphi_i(v)| \;\middle|\; \sum_{k=1}^{l} g_k \leq \mathbb{N}\right\}\\
&= \sup\left\{\sum_{i\in\mathbb{N}}\sum_{k=1}^{l} g_k(i)|\varphi_i(v)| \;\middle|\; \sum_{k=1}^{l} g_k \leq \mathbb{N}\right\}\\
&= \sum_{i\in\mathbb{N}}|\varphi_i(v)| = \|\varphi(v)\|\\
&\leq \|v\| \leq \|v^C\|.
\end{aligned}$$

Here, we have again used inequality (2.10). **q.e.d.**

The proof shows in addition, that $\|\varphi(v^C)\|$ equals $\|\varphi(v)\|$. To be precise, only $\|\varphi(v^C)\| \leq \|\varphi(v)\|$ has been proven. However, "$\geq$" is obviously true.

Corollary 2.15 $B^2 := \{v^C \in B^3 | \; \varphi(v^C)(\mathbb{N}) = v(\mathbb{N})\}$ *is a closed, linear subspace of CBV.*

Proof Linearity is obvious. To prove that B^2 is closed, we have to show that for each $v^C \in B^3$ and each sequence $(v_k^C)_k$ of Choquet functions in B^2 with $v_k^C \to v^C (k \to \infty)$ we always have that $v^C \in B^2$. Because of

$$\|v_k^C - v^C\| \geq |(v_k^C - v^C)(\mathbb{N})|,$$

one gets

$$v_k^C(\mathbb{N}) \to v^C(\mathbb{N}) \;\; (k \to \infty).$$

$\|\varphi(v^C)\| \leq \|v^C\|$ implies $\varphi(v_k^C)(\mathbb{N}) \to \varphi(v^C)(\mathbb{N})$, and, thus, one obtains

$$\varphi(v^C)(\mathbb{N}) = \lim_{k\to\infty} \varphi(v_k^C)(\mathbb{N}) = \lim_{k\to\infty} v_k^C(\mathbb{N}) = v^C(\mathbb{N}) = v(\mathbb{N}).$$

q.e.d.

The proof of the next corollary is just as easy and shall therefore be omitted

Corollary 2.16 *Let us define*

$$B^1 := \{v^C \in B^2 | \; \pi v^C \in B^2, \varphi(\pi v^C) = \pi\varphi(v^C) \;\; (\pi \in \Pi)\},$$

where Π *denotes the set of permutations on* $\mathbb{N}$. *Then,* B^1 *is a closed, linear subspace of CBV and is invariant under permutations.*

Theorem 2.17 *The operator* φ *defined on the space* $spanC_0$ *is a norm-continuous value that preserves carriers.*

Proof First of all we will prove that $spanC_0$ is invariant under permutations: Consider a $v^C \in spanC_0$ and $v_k^C \in C_0$ s. t. $v_k^C \to v^C$ w. r. t. $\| \bullet \|$. One gets

$$\begin{aligned} \|\pi v_k^C - \pi v^C\| &= \|v_k^C - v^C\| \to 0 \\ &\Rightarrow \pi v_k^C \to \pi v^C \text{ w. r. t. } \| \bullet \| \\ &\Rightarrow \pi v^C \in spanC_0. \end{aligned}$$

Now we will show that φ has the desired properties: $C_0 = V_0^C \subseteq B^1$ can be easily seen. Since B^1 is a closed subset of CBV, one obtains $AC^C = spanC_0 \subseteq B^1 \subseteq B^2 \subseteq B^3$. Thus, φ is invariant under permutations and Pareto-efficient on AC^C.

Additivity and the preservation of carriers can be shown immediately, where the latter is valid as every null player of v^C is a null player of $v^{(n)C}$ for every n. Norm-continuity has already been shown in Theorem 2.14. **q.e.d.**

We have proven that φ defined by

$$\varphi(v^C)(f) = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \varphi^n(v^{0(n)C})(f(i)i) \tag{2.12}$$

is a very intuitive operator on $spanC_0$. The aim of this section is to obtain a value with the help of a continuity argument. In other words, we hope to get a value $\boldsymbol{\varphi_{Art}}$ by the formula

$$\varphi_{Art}(v^C)(f) = \lim_{n\to\infty} \varphi^{(n)}(v^C)(f) = \lim_{n\to\infty} \sum_{i\in\mathbb{N}} \varphi^n(v^{0(n)C})(f(i)i).$$

In the following theorem, we will show that the ordering of the limit and the summation can be exchanged, i. e. that φ given by formula (2.12) is nothing else but our desired value.

Theorem 2.18 *The operator φ (see Definition 2.13) satisfies*

$$\varphi v^C := \lim_{n\to\infty} \varphi^n v^{0(n)C}. \tag{2.13}$$

Proof Let v^C be monotone for the beginning, and let $n \in \mathbb{N}$ be given. Defining $\varphi_j^n(v^{0(n)}) = 0$ for $j > n$, the formula

$$\begin{aligned} v^C(\mathbb{N}) &= v^{(n)C}(\mathbb{N}) = v^{0(n)C}(\{1,\dots,n\}) \\ &= \sum_{j=1}^{\infty} \varphi_j^n(v^{0(n)}) \geq \sum_{j=1}^{\infty} f(j)\varphi_j^n(v^{0(n)}). \end{aligned}$$

is valid for every $n \in \mathbb{N}$. Hence, for fixed $f \in [0,1]^{\mathbb{N}}$, the sequence $(a_{n,f})_n := (\sum_{j=1}^{\infty} f(j)\varphi_j^n(v^{0(n)}))_n$ is bounded by 0 and $v^C(\mathbb{N})$, i. e. the sequence has got a convergent subsequence $(a_{n_k,f})_k$. Since $a_{n_k,f}+a_{n_k,f^C} = v^{0(n_k)}(\{1,\ldots,n_k\})$ is true, also $(a_{n_k,f^C})_k$ is convergent. For every $K \in \mathbb{N}$, the inequality

$$\begin{aligned}
\sum_{j=1}^{K} f(j) \lim_{k\to\infty} \varphi_j^{n_k}(v^{0(n_k)}) &= \lim_{k\to\infty} \sum_{j=1}^{K} f(j)\varphi_j^{n_k}(v^{0(n_k)}) \\
&\leq \lim_{k\to\infty} \sum_{j=1}^{\infty} f(j)\varphi_j^{n_k}(v^{0(n_k)}) \qquad (2.14)
\end{aligned}$$

is obviously true. We know by inequality (2.9) that $\sum_{j=1}^{\infty} \lim_{n\to\infty} f(j)\varphi_j^n(v^{0(n)})$ exits, and, by observing (2.14), that $\sum_{j=1}^{\infty} \lim_{n\to\infty} f(j)\varphi_j^n(v^{0(n)}) \leq \lim_{k\to\infty} a_{n_k,f}$.

Let us assume now that $\varphi(v^C)(f) < \lim_{k\to\infty} a_{n_k,f}$. However, this immediately leads to a contradiction since

$$\begin{aligned}
v^C(\mathbb{N}) &= \varphi(v^C)(\mathbb{N}) = \varphi(v^C)(f) + \varphi(v^C)(f^C) \\
&< \lim_{k\to\infty} \sum_{j=1}^{\infty} f(j)\varphi_j^{n_k}(v^{0(n_k)}) + \lim_{k\to\infty} \sum_{j=1}^{\infty} f^C(j)\varphi_j^{n_k}(v^{0(n_k)}) \\
&= \lim_{k\to\infty} \sum_{j=1}^{\infty} \varphi_j^{n_k}(v^{0(n_k)}) \\
&= \lim_{k\to\infty} v^{0(n_k)C}(\{1,\ldots,n_k\}) \\
&= v^C(\mathbb{N}).
\end{aligned}$$

Knowing that $\lim_k a_{n_k,f} = \varphi(v^C)(f)$, it is easy to show that $(a_{n,f})$ itself is a convergent sequence. Assuming the contrary is tantamount to the existence of another accumulation point b_f of $(a_{n,f})$, $b_f \neq \varphi(v^C)(f)$. However, this would result in the existence of a subsequence $(a_{m_l,f})$ with $\lim_l a_{m_l,f} = b_f > \varphi(v^C)(f)$, and this can be disproven as easy as before. All in all, we have shown that

$$\sum_{j=1}^{\infty} \lim_{n\to\infty} f(j)\varphi_j^n(v^{0(n)}) = \lim_{n\to\infty} \sum_{j=1}^{\infty} f(j)\varphi_j^n(v^{0(n)}),$$

and, thus, the statement of the theorem.

At the beginning of the proof we restricted our considerations to monotone Choquet functions. However, this is no real restriction as every $v^C \in spanC_0$ can be written as the difference of two monotone Choquet functions. Hence, linearity of the value shows the theorem even for the general case. **q.e.d.**

Because of the last theorem, we call the operator φ_{Art} the value on $spanC_0$. As φ_{Art} is norm-continuous, we know that our value is independent of the sequence used to approximate $v^C \in spanC_0$ (we have taken $v^{(n)C}$).

It should be mentioned that φ_{Art} has norm 1, i. e.

$$\|\varphi_{Art}\| := \sup_{v^C \in spanC_0} \frac{\|\varphi_{Art}(v^C)\|}{\|v^C\|} = 1.$$

$\|\varphi_{Art}\| \leq 1$ is shown by Theorem 2.14. Equality follows for each monotone $v^C \in C_0$. This is another property which φ_{Art} preserves from the diagonal value.

Lemma 2.19 *Let* Ψ *be a value on a symmetric subspace* $\mathbb{Q}$ *of* FBV. *Then* $\|\Psi\| \leq 1$. *In particular,* Ψ *is norm-continuous.*

Proof Consider a $v^F \in FBV$ and the corresponding upper and lower variation v^{F+} and v^{F-}. One obtains

$$\begin{aligned} \|\Psi(v^F)\| &= \|\Psi(v^{F+} - v^{F-})\| = \|\Psi(v^{F+}) - \Psi(v^{F-})\| \\ &\leq \|\Psi(v^{F+})\| + \|\Psi(v^{F-})\| = v^{F+}(\mathbb{N}) + v^{F-}(\mathbb{N}) \\ &= \|v^F\|. \end{aligned}$$

q.e.d.

If $\mathbb{Q}$ contains at least one monotone v^F, we have that $\|\Psi(v^F)\| = v^F(\mathbb{N})$. Hence, in this case $\|\Psi\| = 1$ is valid.

The operator φ_{Art} is quite nice since it satisfies a lot of reasonable properties. The question is whether or not there can be found a similar value on $spanC_0$. The answer is no:

Theorem 2.20 *There is one and only one value on $spanC_0$ that preserves carriers (and this value is our φ_{Art}).*

Proof The statement that φ_{Art} satisfies all required properties is proven in Lemma 2.17.

Therefore, only uniqueness remains to be shown. In the case of ordinary games with finite carrier the Shapley value is defined uniquely by additivity, efficiency, symmetry, and the null player property (see for example Rosenmüller [23, Chapter 3, Theorem 7.7]). We have to show a similar result for fuzzy games.

Let $\Psi : AC^C \longrightarrow FBVA$ be a function that satisfies the desired properties. First, we consider $\Psi(v^C)$ for $v^C \in C_O$. For $T \in \underline{\underline{P}}(\mathbb{N})$ with $|T| < \infty$, one obtains

1. $\Psi((e^T)^C)(\mathbb{N}) = (e^T)^C(\mathbb{N}) = 1$ (efficiency)
2. $\Psi((e^T)^C)(f(i)i) = 0$ for $i \notin T$ (null player property)
3. $\Psi((e^T)^C)(i) = \Psi((e^T)^C)(j)$ for $i, j \in T$ (symmetry).

These properties imply that $\Psi((e^T)^C)(i) = \frac{1}{|T|}$ is valid for every $i \in T$. In particular, $\Psi((e^T)^C)$ is non-negative. Hence, we can apply Lemma 2.1.3 in [29] which states that a finitely additive, non-negative fuzzy function is homogeneous. All in all, we obtain that

$$\begin{aligned}
\Psi((e^T)^C)(f) &= \Psi((e^T)^C)(f|_T) + \Psi((e^T)^C)(f|_{T^C}) \\
&= \sum_{i\in T} \Psi((e^T)^C)(f(i)i) \\
&= \sum_{i\in T} f(i)\Psi((e^T)^C)(i) \\
&= \frac{1}{|T|}\sum_{i\in T} f(i) \\
&= \varphi_{Art}((e^T)^C)(f)
\end{aligned}$$

is valid (here we have used additivity and homogeneity). In a similar way it can be shown that $\Psi(\lambda(e^T)^C) = \varphi_{Art}(\lambda(e^T)^C)$ for every $\lambda \in \mathbb{N}$. Thus, for a Choquet game $(\mathbb{N}, [0,1]^{\mathbb{N}}, v^C)$ with $C(v^C) = \Omega$, $|\Omega| < \infty$, the following is true:

$$\begin{aligned}\Psi(v^C) &= \Psi(\textstyle\sum_{S\in\underline{\underline{P}}(\Omega)}\alpha_S(v)(e^S)^C) = \sum_{S\in\underline{\underline{P}}(\Omega)}\Psi(\alpha_S(v)(e^S)^C)\\ &= \textstyle\sum_{S\in\underline{\underline{P}}(\Omega)}\varphi_{Art}(\alpha_S(v)(e^S)^C)\\ &= \varphi_{Art}(v^C).\end{aligned}$$

Hence, the value on C_0 is unique. Lemma 2.19 states that a value is norm-continuous. Therefore, φ_{Art} is the unique value on $spanC_0$. **q.e.d.**

An analysis of the proof of the uniqueness of the value on $spanC_0$ yields that we have not used positivity directly. We used norm-continuity which is implied by the other properties of a value (cf. Lemma 2.19).

Requiring norm-continuity instead of positivity would lead to the same result as the one given by Theorem 2.20. Since we obtain a unique value that is non-negative for monotone v^C, positivity is implied by the other properties. Hence, we can say that an operator is a value on $spanC_0$ if it satisfies additivity, efficiency, symmetry, norm-continuity, and if it preserves carriers. These are exactly the properties Rosenmüller uses in [24, Chapter 7, Definition 5.1] for the definition of a value in the case of countably many players.

In the next two sections, we will consider two further possibilities to extend the diagonal value to a larger class of games. The two processes can be extended much easier to the Choquet games than the last one since we have to use only the linearity of the Choquet integral.

2.2 The Fuzzy Value for Weighted Majority Games

First, let us consider the finite case, i. e. a tupel $(\Omega, [0,1]^\Omega, v^C)$ with $\Omega = \{1, 2, \dots, n\}$. Under consideration of the diagonal formula (1.21), one obtains

$$\begin{aligned}\varphi_{SP}(v^C)(f) &= \sum_{i\in\Omega} f(i)\Phi_i(v) \\ &= \sum_{i\in\Omega} f(i)\frac{1}{n!}\sum_{\pi}[v(S^{\pi}_{\pi(i)}) - v(S^{\pi}_{\pi(i)-1})],\end{aligned}$$

where the summation is carried out over all permutations π on Ω, and Φ is the "crisp" value. A special case is the fuzzy coalition $f(i)i$:

$$\varphi_{SP}(v^C)(f(i)i) = f(i)\frac{1}{n!}\sum_{\pi}[v(S^{\pi}_{\pi(i)}) - v(S^{\pi}_{\pi(i)-1})].$$

One defines the space $\Pi := \{\pi |\ \pi : \Omega \to \Omega \text{ permutation}\}$ and the probability distribution p on $\underline{\underline{P}}(\Pi)$ by means of the formula $p(\pi) = \frac{1}{n!}$. The random variable

$$g^{v^C}_{f(i)} : \underline{\underline{P}}(\Pi) \to \mathbb{R}$$

given by

$$g^{v^C}_{f(i)}(\pi) = f(i)[v(S^{\pi}_{\pi(i)}) - v(S^{\pi}_{\pi(i)-1})]$$

has

$$E_p g^{v^C}_{f(i)} = \varphi_{SP}(v^C)(f(i)i)$$

as its expectation w. r. t. p.

Shapley considers in [26] the case of countably many players and shows a way to obtain a value in a similar way as described above. For this reason, he defines a measure P on $(\Sigma, \underline{\underline{\Sigma}})$, where $\Sigma := \{\sigma |\ \sigma \text{ is an ordering of } \mathbb{N}\}$ and $\underline{\underline{\Sigma}}$ is an "appropriate" $\sigma-$algebra. An ordering σ of $\mathbb{N}$ is a binary relation $\mathbb{N}\times\mathbb{N}$ that is reflexive, transitive, complete, and anti-symmetric. For the explicit definition of the measure P and its properties, we refer to Rosenmüller [24, Chapter 7, Section 1].

Before we are able to present the value which is constructed by means of the probability measure P, we need some more definitions. $\boldsymbol{A_1^{\sigma+}}$ is the set of all non-negative, $\sigma-$additive functions on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$ with normalization to total mass 1. A coalitional function v constitutes a **weighted majority game** $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}), v)$, if there is an $m \in A_1^{\sigma+}$ and a $0 < \beta < 1$ s. t. $v = 1_{[\beta,1]} \circ m$. In this section, we will restrict ourselves to such v.

Definition 2.21 *Let* $m \in A_1^{\sigma+}, \beta \in (0,1)$. *Then* $i \in \mathbb{N}$ *pivots* $\sigma \in \Sigma$ *w. r. t.* (m, β), *if*

$$m(\{j | j \prec_\sigma i\}) < \beta \leq m(\{j | j \preceq_\sigma i\}),$$

where the set $\{j|\, j \preceq_\sigma i\}$ *contains those* j *which satisfy* $(j,i) \in \sigma$, *and* $\{j|\, j \prec_\sigma i\}$ *consists of those* j *with* $(j,i) \in \sigma,\ j \neq i$.

Now, we are ready to define the value on $\mathbb{N}$ for the Choquet extension of the weighted majority games. In the following, $\Phi^{WM}(v)$ denotes the Shapley value of v as it is defined in [24, Definition 7.2.4].

Definition 2.22 *Let* $m \in A_1^{\sigma+}$ *and* $\beta \in (0,1)$, *s. t. the crisp coalitional function* v *can be written as* $v = 1_{[\beta,1]} \circ m$. *Then, the Shapley value of* v^C *is the* $\sigma-$*additive, non-negative fuzzy-function* $\varphi^{WM}(v^C)$, *which is given by means of the formula*

$$\begin{aligned}
\varphi^{WM}(v^C)(f(i)i) &= \int_\Sigma f(i)[v(S^{\sigma,i}) - v(S^{\sigma,i} - i)]dP(\sigma) \\
&= f(i)P(\{\sigma | i \ piv\ \sigma\ w.\,r.\,t.\ (m,\beta)\}) \\
&= f(i)\Phi_i^{WM}(v), \\
\varphi^{WM}(v^C)(f) &= \sum_{i\in\mathbb{N}} \Phi^{WM}(v^C)(f(i)i),
\end{aligned}$$

where $S^{\sigma,i} := \{j \in \mathbb{N} | j \preceq_\sigma i\}$.

This is a very simple extension of Shapley' s approach. We just have to use the homogeneity of the integral.

As in this chapter only weighted majority games will be considered, one could ask how the Choquet extension of such a game looks like. For an answer, take any $f \in [0,1]^{\mathbb{N}}$ and define the ordering σ_f as follows:

$$i \prec_{\sigma_f} j, \text{ if } \begin{cases} f(i) > f(j) \text{ or} \\ f(i) = f(j) \text{ and } i < j. \end{cases}$$

Verbally, those players who have a high degree of membership in the fuzzy coalition f are placed "to the front" w. r. t. the ordering σ_f ; if two players have the same value in f, they are placed according to the lexicographic ordering. We have that $P(\{\sigma_f | \exists i \text{ s. t. } i \text{ piv } \sigma_f \text{ w. r. t. } (m, \beta)\}) = 1$ [24, Theorem 7.2.10]. Hence, there exists $P-$ almost sure an i^* s. t. $m(S^{\sigma_f, i^*}) \geq \beta > m(S^{\sigma_f, i^*} - i^*)$. For this reason,

$$v(f > t) = \begin{cases} 1, & \text{if } S^{\sigma_f, i^*} \subseteq \{f > t\} \\ 0, & \text{otherwise} \end{cases}$$

is true for $t \in [0, 1]$. However, $S^{\sigma_f, i^*} \subseteq \{f > t\}$ is tantamount to $f(i^*) > t$ (this follows immediately from the construction of σ_f). Thus, one obtains

$$v^C(f) = \int_0^1 v(f > t)dt = f(i^*).$$

To conclude, we show that Φ is indeed a value:

Theorem 2.23 *Let* $(\mathbb{N}, v)$ *be a weighted majority game. Then, we have that*

1. $C(\varphi^{WM}(v^C)) \subseteq C(v^C)$
2. $\pi\varphi^{WM}(v^C) = \varphi^{WM}(\pi v^C)$ *for all* $\pi \in \Pi$
3. $\varphi^{WM}(v^C)(\mathbb{N}) = 1$

Proof

1. The equations $C(\varphi^{WM}(v^C)) = C(\Phi^{WM}(v))$ and $C(v^C) = C(v)$ are easily verified, and $C(\Phi^{WM}(v)) \subseteq C(v)$ follows from Theorem 7.2.5 in [24].

2.
$$\begin{aligned} \varphi^{WM}(\pi v^C)(f) &= \sum_{i \in \mathbb{N}} f(i)\Phi_i^{WM}(\pi v) = \sum_{i \in \mathbb{N}} f(i)\Phi_{\pi^{-1}(i)}^{WM}(v) \\ &= \sum_{k \in \mathbb{N}} f(\pi(k))\Phi_k^{WM}(v) = \varphi^{WM}(v^C)(\pi^{-1}f) \\ &= \pi\varphi^{WM}(v^C)(f) \end{aligned}$$

Here, we use that $\pi\Phi^{WM}(v) = \Phi^{WM}(\pi v)$ [24, Chapter 7, Theorem 2.6]

3. $\varphi^{WM}(v^C)(\mathbb{N}) = \sum_{k\in\mathbb{N}} \varphi^{WM}(v^C)(k) = \sum_{k\in\mathbb{N}} \Phi_k^{WM}(v) = \Phi^{WM}(v)(\mathbb{N}) = 1$,

   where the last equality follows from the Theorem of Berbee [24, Theorem 7.2.10].

q.e.d.

2.3 A Value Using the Structure of the Symmetric Group of $\mathbb{N}$

Here, we demonstrate an approach to obtain a value that is similar in at least some aspects to the one presented in the last section. Again, we consider

$$E_p g_{f(i)}^{v^C} = \varphi_{SP}(v^C)(f(i)i)$$

for a set of finitely many players, where p and $g_{f(i)}^{v^C}$ are defined as in the previous section. In the countable case, a $\sigma-$additive value is constructed with the help of a measure μ. However, different from the last section, this measure μ is defined on the set Π of permutations of $\mathbb{N}$. As can be shown very easily, Π is a proper subset of Σ. Using the measure P of Shapley and Shapiro, we even have $P(\Pi) = 0$. For the exact definition of μ, the reader is referred to Pallaschke and Rosenmüller [20].

Definition 2.24 *1. For $i \in \mathbb{N}$, one defines*

$$\begin{aligned}
\varphi^{PR}(v^C)(f(i)i) &= \int_\Pi f(i)[v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1})]d\mu(\pi) \\
&= f(i)\int_\Pi [v(S^\pi_{\pi(i)}) - v(S^\pi_{\pi(i)-1})]d\mu(\pi) \\
&= f(i)\Phi_i^{PR}(v),
\end{aligned}$$

where $\Phi^{PR}(v)$ is the value given by Pallaschke and Rosenmüller in [20].

2. *If, for each* $f \in [0,1]^{\mathbb{N}}$, $(\varphi^{PR}(v^C)(f(i)i))_i$ *is an absolutely convergent sequence, the Shapley value is defined by means of the formula*

$$\varphi^{PR}(v^C)(f) = \sum_{i\in\mathbb{N}} \varphi^{PR}(v^C)(f(i)i).$$

Though we call $\varphi^{PR}(v^C)$ a value, we have to be aware of the fact that not necessarily all properties of Definition 1.1 are satisfied. In particular, efficiency cannot be guaranteed for all games. The following equivalence relations are valid:

$$\begin{aligned} \Phi^{PR}(v) \text{ exists} &\iff (\Phi_i^{PR}(v))_i \text{ is an absolutely convergent series} \\ &\iff (\varphi^{PR}(v^C)(f(i)i))_i \text{ is an absolutely convergent series for all} \\ &\qquad f \in [0,1]^{\mathbb{N}} \text{ (since } |\Phi_i^{PR}(v)| \geq |f(i)\Phi_i^{PR}(v)|) \qquad (2.15) \\ &\iff \varphi^{PR}(v^C) \text{ exists.} \end{aligned}$$

Proposition 3.5 in [20] states that Φ^{PR} is linear on BV and satisfies $\|\Phi^{PR}(v)\| \leq \|v\|$ ($\| \bullet \|$ is here again the variation norm). As CBV is the largest space on which we have defined the Choquet integral, it follows under the consideration of (2.15), that $\varphi^{PR}(v^C)$ exists for all v^C.

Let Π_n be the set of permutations of $\{1,\ldots,n\}$ and $\Pi^* := \bigcup_{n\in\mathbb{N}} \Pi_n$. For the Shapley value as it is constructed in this section, the following holds:

Theorem 2.25 1. *For all* $v^C \in CBV$, *the value* $\varphi^{PR}(v^C)$ *exists.*

2. *For each* $\pi \in \Pi^*$, *we have that* $\pi\varphi^{PR}(v^C) = \varphi^{PR}(\pi v^C)$.

3. *Let* v^C *be monotone, which implies, in particular, that* v *is monotone. Then,* $\varphi^{PR}(v^C)(f(i)i) \geq 0$, *and* $\sum_{i\in\mathbb{N}} \varphi^{PR}(v^C)(f(i)i) \leq v^C(\mathbb{N})$ *is valid for all* $i \in \mathbb{N}$ *and each* $f \in [0,1]^{\mathbb{N}}$.

The proofs of these statements can be found in [20] (Proposition 3.5, Lemma 3.2, and Lemma 3.4, respectively).

The following two theorems compare the value defined in this chapter with that defined by Artstein and that by Shapley and Shapiro, respectively.

Theorem 2.26 *[20, Theorem 3.6] φ^{PR} equals Artstein's value φ_{Art} on AC^C.*

Theorem 2.27 *[20, Theorem 4.5] φ^{PR} coincides with Shapley's and Shapiro's value φ^{WM} on the set of weighted majority games.*

2.4 The Smoothing Procedure for Countably Many Players

At the beginning of this section, we will check how the diagonal formula (1.12) looks like for countably many players. Of course, even in this case, every fuzzy coalitional function v^F can be decomposed into $v^F = p^F \circ m$, where $m = (m_1, m_2, \ldots)$, each m_j is a fuzzy measure on $[0,1]^{\mathbb{N}}$ with $m_j(f) = f(j)$, and $p : \mathcal{R}(m) \to \mathbb{R}$ with $p(0) = 0$ (cf. Section 1.2). We aim at giving an answer to the question whether or not there exists a diagonal value for those v^F with continuously differentiable p^F.

As a second step, we want to examine the case of Choquet extensions. Here, p^C is usually not differentiable. However, we would like to ask whether it is possible to get a sequence $(p_n^C)_n$ which converges uniformly to p^C and where each p_n^C is continuously differentiable. Does there exist a limit for $\varphi(p_n^C \circ m)$, is it independent of the selected sequence, and does the limit provide a value for v^C?

First, we we should mention some rules of calculus in $\mathbb{R}^{\mathbb{N}}$. Some well known results from the "finite" analysis remain valid in the countable case. However, other results for functions on $\mathbb{R}^n$ to $\mathbb{R}^m$, $n, m \in \mathbb{N}$, may not remain valid in the countable case.

In the following, "continuous differentiability of the function $p^F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$" is used to express that each partial derivative $\frac{\partial p^F}{\partial x_j}$ exits and is continuous; furthermore, we assume absolute differentiability, i. e. for each $q \in \mathbb{N}$ there shall be a linear operator $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ s. t.

$$p^F(q + \xi) = p^F(q) + A\xi + \psi(\xi) \tag{2.16}$$

is valid in a neighbourhood of q, where ψ is a function defined on a neighbourhood of 0 with

$$\lim_{\xi \to 0} \frac{\psi(\xi)}{\|\xi\|_{\sup}} = 0.$$

Here, $\| \bullet \|_{\sup}$ denotes the supremum norm on $\mathbb{R}^{\mathbb{N}}$. In the finite case, it is a well known fact that continuity of all partial derivatives implies absolute differentiability. However, it is not known whether this statement remains valid for functions on $\mathbb{R}^{\mathbb{N}}$. Therefore, we explicitly assume the existence of a linear operator A as above.

Besides the fact that there may occur some problems when considering functions on $\mathbb{R}^{\mathbb{N}}$, one can show that most of the theorems known for finite analysis remain valid for the countable case. Recapitulating the proofs of the respective theorems known from "finite analysis", one can see, for example, that the operator A in formula (2.16) is nothing else but the $\mathbb{N}$-vector which consists of the partial derivatives of p^F evaluated at q. Furthermore, the chain rule remains valid, and the theorem which states that the directional derivative of a function equals the product of the direction and the vector of gradients holds true as well.

In the following theorem, we will show how we can extend the diagonal value from finitely many to countably many players. The proof uses various spaces for different sets of players. To make clear which set of players is actually considered, the respective space is followed by a subset of $\mathbb{N}$. For example, $FNA^+(\{1, \ldots, n\})$ denotes the space of all non-negative, finite, non-atomic measures on $[0,1]^{\{1,\ldots,n\}}$.

Theorem 2.28 *Consider a $v^E \in C_+^{\sigma,C}$ and assume that this can be written as $v^E = p^E \circ m$, where $m = (m_1, m_1, \ldots)$ is the vector of countably many fuzzy measures $m_j, j \in \mathbb{N}$, given by $m_j(f) = f(j)$, and $p^E : \mathcal{R}(m) \to \mathbb{R}$ is continuously differentiable with $p^E(0) = 0$. Then, the following holds: $v^E \in pFNA(\mathbb{N})$, and, under the assumption of the null player property, the diagonal value can be written as*

$$\varphi v^E(f) = \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p^E}{\partial x_j}(te)dt. \tag{2.17}$$

Proof First of all, we will prove that v^E is an element of $pFNA(\mathbb{N})$. The assumptions of the theorem allow the use of Lemma 2.9 which states that $v^{(n)E} \to v^E$

w. r. t. the variation norm. The corresponding set functions $v^{0(n)E}$ can be decomposed into $v^{0(n)E} = p^{0(n)E} \circ \overline{m}^n$, $\overline{m}^n = (m_1, \ldots, m_n)$. Butnariu and Klement show in [6, Proposition 17.4] that such functions are an element of $pFNA(\{1, \ldots, n\})$. Thus, for each $\varepsilon > 0$ there always exists a $J \in \mathbb{N}$ s. t. $\|v^{0(n)E} - \sum_{j=1}^{J} c_j \overline{r}_j^{k_j}\| < \varepsilon$ is true for some $c_j \in \mathbb{R}$, $k_j \in \mathbb{N}$, and $\overline{r}_j \in FNA^+(\{1, \ldots, n\})$, $j = 1, \ldots, J$. We define $r_j : [0,1]^{\mathbb{N}} \to \mathbb{R}$ by $r_j(f^n) = \overline{r}_j(f)$, where $f^n \in [0,1]^{\mathbb{N}}$ is given by $f^n(j) = f(j)$ for $j \leq n$ and $f(j) = 0$ for $j > n$. One can see immediately that $r_j \in FNA^+(\mathbb{N})$ and $\|v^{(n)E} - \sum_{j=1}^{J} c_j r_j^{k_j}\| < \varepsilon$, i. e. the $v^{(n)E}$ are elements of the space $pFNA(\mathbb{N})$. Since this latter space is closed (see [6, Theorem 17.1]), we are done.

Because of the null player property, $\varphi v^{(n)E}(f) = \varphi v^{(n)E}(f|_{\{1,\ldots,n\}})$ is valid for all $f \in [0,1]^{\mathbb{N}}$. Hence, we have that

$$\varphi v^{(n)E}(f) = \varphi v^{0(n)E}(f^{\{1,\ldots,n\}}) = \int_0^1 \sum_{j=1}^{n} f(j) \frac{\partial p^{0(n)E}}{\partial x_j}(te^{\{1,\ldots,n\}})dt,$$

where $f^{\{1,\ldots,n\}} \in [0,1]^{\{1,\ldots,n\}}$ is given by $f^{\{1,\ldots,n\}}(j) = f(j)$ and $e^{\{1,\ldots,n\}} = (1, \ldots, n)$ denotes the unit vector in $\mathbb{R}^n$. Since the diagonal formula is norm-continuous, it remains to show that

$$\int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p^E}{\partial x_j}(te)dt = \lim_{n\to\infty} \int_0^1 \sum_{j=1}^{n} f(j) \frac{\partial p^{0(n)E}}{\partial x_j}(te^{\{1,\ldots,n\}})dt$$

is true. We now calculate the partial derivatives of $p^{0(n)E}$:

$$\frac{\partial p^{0(n)E}}{\partial x_j}(te^{\{1,\ldots,n\}}) = \begin{cases} \frac{\partial p^E}{\partial x_j}(te), & \text{if } j < n \\ \sum_{j=n}^{\infty} \frac{\partial p^E}{\partial x_j}(te), & \text{if } j = n. \end{cases}$$

The first equality is obviously true, the second follows from

$$\begin{aligned} \frac{\partial p^{0(n)E}}{\partial x_n}(te^{\{1,\ldots,n\}}) &= \lim_{\xi\to 0} \frac{p^{0(n)E}(te^{\{1,\ldots,n\}} + \xi e_n^{\{1,\ldots,n\}}) - p^{0(n)E}(te^{\{1,\ldots,n\}})}{\xi} \\ &= \lim_{\xi\to 0} \frac{p^E(te + \sum_{j=n}^{\infty} \xi e_j) - p^E(te)}{\xi} \\ &= \sum_{j=n}^{\infty} \frac{\partial p^E}{\partial x_j}(te). \end{aligned}$$

The last equation uses the fact that even in the countable case the directional derivative is nothing else but the product of the vector of gradients and the given direction. Now, $\varphi v^{(n)E}$ can be written as

$$\varphi v^{(n)E}(f) = \int_0^1 \sum_{j=1}^{n} f(j)\frac{\partial p^E}{\partial x_j}(te)dt + \int_0^1 f(n) \sum_{j=n+1}^{\infty} \frac{\partial p^E}{\partial x_j}(te)dt.$$

Because of monotonicity of v^E we get

$$\begin{aligned} \left|\varphi v^{(n)E}(f) - \int_0^1 \sum_{j=1}^{\infty} f(j)\frac{\partial p^E}{\partial x_j}(te)dt\right| &= \left|\int_0^1 \sum_{j=n+1}^{\infty} (f(n) - f(j))\frac{\partial p^E}{\partial x_j}(te)dt\right| \\ &\leq \int_0^1 \sum_{j=n+1}^{\infty} \frac{\partial p^E}{\partial x_j}(te)dt. \qquad (2.18) \end{aligned}$$

We know that the sequence $v^{nE} := v^E|_{\{1,\dots,n\}}$ is another possibility to approximate v^E when using the variation norm (cf. Lemma 2.8). For these functions, the partial derivatives are given by

$$\frac{\partial p^{nE}}{\partial x_j}(te) = \begin{cases} \frac{\partial p^E}{\partial x_j}(te), & \text{if } j \leq n \\ 0, & \text{if } j > n. \end{cases}$$

Thus, we have that

$$\varphi v^{nE}(\mathbb{N}) = \int_0^1 \sum_{j=1}^{n} \frac{\partial p^E}{\partial x_j}(te)dt = v^{nE}(\mathbb{N}) \to v^E(\mathbb{N}).$$

Because of the efficiency of $\varphi v^{(n)E}$, we finally obtain

$$v^E(\mathbb{N}) \leftarrow v^{(n)E}(\mathbb{N}) = \varphi v^{(n)E}(\mathbb{N}) = v^{nE}(\mathbb{N}) + \int_0^1 \sum_{j=n+1}^{\infty} \frac{\partial p^E}{\partial x_j}(te)dt,$$

i.e.

$$\int_0^1 \sum_{j=n+1}^{\infty} \frac{\partial p^E}{\partial x_j}(te)dt \to 0.$$

Because of inequality (2.18), this concludes the proof. **q.e.d.**

As an example for a Choquet extension that satisfies all assumptions of Theorem 2.28, we consider a $\sigma-$additive v:

Proposition 2.29 *If $v \in BV$ is $\sigma-$additive, then v^C is $\sigma-$additive, too.*

Proof Because of our assumption, v is, in particular, additive. Following Denneberg [8, Corollary 6.5], the same is true for the Choquet extension. Furthermore, one can easily show that, because of the $\sigma-$additivity of v and the definition of the upper and lower variation (v^+ and v^-, respectively), v^+ und v^- are also $\sigma-$additive. One gets, for example, $v^+(S) = \sum_{j \in S}(v(j))^+$.

As $\sigma-$additivity always implies upper $\sigma-$continuity, we obtain with the help of the monotone convergence theorem, that both v^{+C} and v^{-C} are upper $\sigma-$continuous (cf. Lemma 2.6). Because of the equality $v^C = v^{+C} - v^{-C}$, the Choquet extension of v is upper $\sigma-$continuous. And now we know from measure theory that this fact together with the additivity implies the statement of the proposition. **q.e.d.**

As one checks immediately, the Choquet extension of a $\sigma-$additive v is a function which is feasible for formula (2.17). We have that

$$p^C(q) = \sum_{j \in \mathbb{N}} q_j v(j), \quad \text{i.e.} \quad \frac{\partial p^C}{\partial x_j}(q) = v(j),$$

and, thus,

$$\varphi v^C(f) = \sum_{j \in \mathbb{N}} f(j) v(j)$$

as result for the diagonal value.

Now, we want to examine general fuzzy functions $v^E \in FBV$ which are not necessarily an element of $C_+^{\sigma,C}$, and which can be written as $v^E = p^E \circ m$, where m and p^E are defined as in Theorem 2.28. It is not obvious whether such fuzzy functions are in general elements of the space *pFNA*. The proof of Aumann and Shapley [3, Proposition 7.1] for such functions in the finite case seems to be based strongly on the finiteness assumption. Nevertheless, we will examine the diagonal formula for this case. For this reason, we define ***PM*** to be the space which contains all these fuzzy functions $v^E = p^E \circ m$ with bounded variation, i. e.

$$\begin{aligned} PM \ := \ & \Big\{ v^E \Big| \, v^E : [0,1]^{\mathbb{N}} \to \mathbb{R}, v^E(\emptyset) = 0, \|v^E\| < \infty, v^E = p^E \circ m, \\ & p^E : [0,1]^{\mathbb{N}} \to \mathbb{R} \text{ continuously differentiable}, m \text{ a vector of} \\ & \text{countably many fuzzy measures } m_j \text{ with } m_j(f) = f(j). \Big\} \end{aligned}$$

The operator φ, $\varphi : PM \to FBVA$, is given by means of the formula

$$\varphi(p^E \circ m)(f) = \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p^E}{\partial x_j}(te) dt. \tag{2.19}$$

This expression satisfies all properties of a value:

- φ is obviously positive and linear.
- For each permutation π of $\mathbb{N}$, we have that

$$\begin{aligned} \varphi(p^E \circ m)(\pi f) &= \int_0^1 \sum_{j=1}^{\infty} f(\pi^{-1}(j)) \frac{\partial p^E}{\partial x_j}(te) dt \\ \varphi(\pi(p^E \circ m))(f) &= \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p^E}{\partial x_{\pi(j)}}(te) dt. \end{aligned}$$

  To see the second equation, define a mapping $\overline{p}^E$ by $\overline{p}^E(q) := p^E(q^{\pi^{-1}})$ with $q_l^{\pi^{-1}} = q_{\pi^{-1}(l)}$. For this function, one sees immediately that $(\overline{p}^E \circ m)(f) = (p^E \circ m)(\pi f)$ and that

$$\frac{\partial \overline{p}^E}{\partial x_j}(q) = \frac{\partial p^E}{\partial x_{\pi(j)}}(q^{\pi}).$$

  For fixed $k \in \mathbb{N}$ and $j := \pi^{-1}(k)$, the equation

$$f(j) \frac{\partial p^E}{\partial x_{\pi(j)}}(te) = f(\pi^{-1}(k)) \frac{\partial p^E}{\partial x_k}(te) \tag{2.20}$$

  is valid. Next, we consider the mapping $\widetilde{p}^E : [0,1]^{\mathbb{N}} \to \mathbb{R}$ given by $\widetilde{p}^E(0) = 0$ and

$$\frac{\partial \widetilde{p}^E}{\partial x_j}(q) = \left| \frac{\partial p^E}{\partial x_j}(q) \right| \ \text{ for all } q \in [0,1]^{\mathbb{N}}, \ \text{ for each } j \in \mathbb{N}.$$

  Since, in particular, we have that $f(j)\frac{\partial \widetilde{p}^E}{\partial x_j}(te) \geq f(j)\frac{\partial p^E}{\partial x_j}(te)$ is valid for every $j \in \mathbb{N}$ and each $f \in [0,1]^{\mathbb{N}}$, we have a majorant for the series

$\sum_{j=1}^{\infty} f(\pi^{-1}(j))\frac{\partial p^E}{\partial x_j}(te)$. Hence, a permutation of the terms of the series does not change the value. Having in mind formula (2.20), we have established the claim.

- To prove efficiency, define a function $g : [0,1] \to \mathbb{R}$ by $g(t) = p^E(te)$. Now, we have that

$$\begin{aligned} v^E(\mathbb{N}) &= p^E(e) - p^E(0) = g(1) - g(0) \\ &= \int_0^1 g'(t)dt = \int_0^1 \sum_{j=1}^{\infty} \frac{\partial p^E}{\partial x_j}(te)dt. \end{aligned}$$

Because of our latter thoughts, the operator φ defined as in (2.19) is called the **diagonal value on *PM***.

Now, we want to come back to the Choquet extension, which, in general, is not differentiable at the diagonal. However, if there exists a sequence (v_n^C), $v_n^C = p_n^C \circ m$, converging uniformly to v^C s. t. each p_n^C is continuously differentiable, we can use the smoothing procedure as in the finite case. If $\lim_{n\to\infty} \varphi(p_n^C \circ m)$ exists and is independent of the sequence (v_n^C), then we call this limit the value of v^C and denote it with $\varphi_{SP}(v^C)$. We will show later on that it is not possible to find such a sequence for all $v^C \in CBV$. However, on $spanC_0$ our idea works:

Proposition 2.30 *Each $v^C \in spanC_0$ can be written as $v^C = p^C \circ m$ with $m = (m_1, m_2, \ldots)$, $m_j(f) = f(j)$, and $p^C : [0,1]^{\mathbb{N}} \to \mathbb{R}$, $p^C(0) = 0$. Furthermore, there exists a sequence (p_n^C) with the following properties:*

1. *(p_n^C) converges uniformly to p^C.*
2. *Each p_n^C is continuously differentiable.*
3. *$\lim_{n\to\infty} \varphi(p_n^C \circ m)$ exists and provides a value that preserves carriers.*

For each sequence (p_n^C) satisfying these three properties, $\lim \varphi(p_n^C \circ m)$ equals Artstein's value, i. e. we have that $\varphi_{SP} = \varphi_{Art}$.

Proof It is quite easy to show that each $v^C \in spanC_0$ can be decomposed as required. The sequence $(v^{(n)C})_n$ defined as in Lemma 2.8 converges to v^C w. r. t. the variation norm. As each $p^{(n)C}$ has a finite carrier, we can construct, with the help of the (finite) smoothing procedure, sequences $(p_k^{(n)C})_k$ s. t. each $p_k^{(n)C}$ is continuously differentiable and the sequence $(p_k^{(n)C})$ converges uniformly to $p^{(n)C}$ (for fixed n). The value of $v^{(n)C}$ provides for player j, $1 \leq j \leq n$,

$$\begin{aligned}\varphi_{SP}(v^{(n)C})(j) &= \lim_{k\to\infty} \int_0^1 \frac{\partial p_k^{(n)C}}{\partial x_j}(te)\, dt \\ &= \varphi_{SP}^n(v^{0(n)C})(j).\end{aligned}$$

Obviously, there exists a diagonal sequence $(p_{k_n}^{(n)C})_n$ which converges uniformly to p^C s. t.

$$\left| \int_0^1 \frac{\partial p_{k_n}^{(n)C}}{\partial x_j}(te)\, dt - \varphi_{SP}^n(v^{0(n)C})(j) \right| < \frac{1}{n^2}$$

is valid for all n and all $j \leq n$. Hence, we have that

$$\begin{aligned}&\left|\varphi_{Art}(v^C)(f) - \varphi_{SP}(v^C)(f)\right| \\ = &\left| \lim_{n\to\infty} \left(\sum_{j=1}^n f(j) \left[\varphi_{SP}^n(v^{0(n)C})(j) - \int_0^1 \frac{\partial p_{k_n}^{(n)C}}{\partial x_j}(te)\, dt \right] \right) \right| \\ \leq &\lim_{n\to\infty} \sum_{j=1}^n f(j) \left| \varphi_{SP}^n(v^{0(n)C})(j) - \int_0^1 \frac{\partial p_{k_n}^{(n)C}}{\partial x_j}(te) dt \right| \\ \leq &\lim_{n\to\infty} \sum_{j=1}^n f(j) \frac{1}{n^2} \\ \leq &\lim_{n\to\infty} \frac{n}{n^2} \\ = &\ 0.\end{aligned}$$

Theorem 2.20 implies that the smoothing procedure is independent from the actual sequence chosen. **q.e.d.**

For finitely many players, the unanimous games are the most important ones since they build a basis for all coalitional functions. Now, we would like to examine whether the smoothing procedure can provide a result for the game $(e^{\mathbb{N}})^C$. We know that there exists no real value for this game since optimality and symmetry are mutually exclusive. Nevertheless, we will try to find something like a quasi-value for $(e^{\mathbb{N}})^C$, i. e. we will have to drop at least one of the usual properties of a value.

We would like to analyse which consequences a sequence $(p_n^C)_n$ entails that converges uniformly to $p_{e^{\mathbb{N}}}^C$ where each p_n^C is continuously differentiable. Assuming that we have already found such a sequence, we can define another one by $p_{n,\pi}^C(q) = p_n^C(q^\pi)$, where π is a permutation of $\mathbb{N}$ and $q^\pi \in [0,1]^{\mathbb{N}}$ is given by $q_j^\pi = q_{\pi(j)}$. As one can easily verify, also $(p_{n,\pi}^C)_n$ converges uniformly to $p_{e^{\mathbb{N}}}^C$. We would like to obtain a value by a limiting procedure and this value shall be independent of the sequence, i. e. we have that

$$\begin{aligned}
\varphi_{SP}((e^{\mathbb{N}})^C)(f) &= \lim_{n\to\infty} \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p_n^C}{\partial x_j}(te)\, dt \\
&= \lim_{n\to\infty} \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p_{n,\pi}^C}{\partial x_j}(te)\, dt \\
&= \lim_{n\to\infty} \int_0^1 \sum_{j=1}^{\infty} f(j) \frac{\partial p_n^C}{\partial x_{\pi(j)}}(te)\, dt \\
&= \lim_{n\to\infty} \int_0^1 \sum_{j=1}^{\infty} f(\pi^{-1}(j)) \frac{\partial p_n^C}{\partial x_j}(te)\, dt \\
&= \varphi_{SP}((e^{\mathbb{N}})^C)(\pi f).
\end{aligned}$$

This means that independence of the sequence implies symmetry of the value. Furthermore,

$$\begin{aligned}
\varphi_{SP}((e^{\mathbb{N}})^C)(\mathbb{N}) &= \lim_{n\to\infty} \varphi(v_n^C)(\mathbb{N}) \\
&= \lim_{n\to\infty} v_n^C(\mathbb{N}) \\
&= v^C(\mathbb{N})
\end{aligned}$$

is true. However, as we have shown before, this is a contradiction. Thus, the smoothing procedure does not work for $(e^{\mathbb{N}})^C$. It is not even possible to get a quasi value in the sense that we drop the requirement of efficiency or symmetry.

2.5 A Value for the Unanimous Game

In this section, we will give a quasi value for a class of Choquet games $\overline{\mathcal{V}} \subseteq CBV$ which contains all unanimous games. Our quasi value does not satisfy the axioms of definition 1.12 since it is not symmetric. However, we receive a quite reasonable result for $(e^{\mathbb{N}})^C$.

In the first part, we will follow the framework presented by Rosenmüller [22] which leads us to a fuzzy value for continuously many players. However, this quasi value for $(e^{\mathbb{N}})^C$ allocates, for example, a positive weight to fuzzy coalitions with finite carrier. This can be avoided when considering the countable case.

2.5.1 A Value for a Continuum of Players

Let us consider one possible representation of the Shapley value Φ for crisp coalitions in the case of finitely many players $N = \{1, \ldots, n\}$ given by

$$\Phi(v) = \sum_{S \in \underline{\underline{P}}(N)} c_S(v) g^S, \tag{2.21}$$

where $c_S(v)$ is defined as

$$c_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T) \tag{2.22}$$

and $g^S : \underline{\underline{P}}(\mathbb{N}) \to \mathbb{R}$ is the uniform distribution on S, i. e.

$$g^S(T) = \frac{|T \cap S|}{|S|}.$$

As the Choquet integral is linear (cf. Denneberg [8, Proposition 5.2, (i) and (ii)]), we have for each $v \in BV$ that

$$v^C = \sum_{S \in \underline{\underline{P}}(N)} c_S(v)(e^S)^C.$$

Since we require the value of a fuzzy game also to be linear, the corresponding formula to (2.21) for the Choquet extension is

$$\varphi(v^C) = \sum_{S \in \underline{\underline{P}}(N)} c_S(v) \widetilde{g}^S, \tag{2.23}$$

where for each $S \in \underline{\underline{P}}(N)$ the function $\widetilde{g}^S : [0,1]^N \longrightarrow \mathbb{R}$ is the fuzzy value of $(e^S)^C$, i. e.

$$\widetilde{g}^S(f) = \frac{1}{|S|}\sum_{i=1}^{n} f(i).$$

Rosenmüller extends in [22] formula (2.21) for some games to the player set $\Omega = [0,1]$. To be more precise, he looks at weak convergence:

Definition 2.31 *Let $\Omega = [0,1]$ be the set of players and $\mathcal{B}$ be the $\sigma-$field of Borel sets in Ω. Furthermore, a coalitional function $v : \mathcal{B} \to \mathbb{R}$ shall be given. We say that a sequence (v^n)* ***converges weakly*** *to v if*

$$\int f\, dv^n \to \int f\, dv$$

is valid for all elements f of the cone of non-negative continuous functions on Ω.

Let for the rest of this subsection the set of players and the set of feasible coalitions be given as in the previous definition.

Proposition 2.32 *For each $v \in BV$ there exists a sequence $(v^n) \in BV$ s. t. each v^n has a finite carrier and $v^n \to v$ weakly.*

Proof Most parts of this proof can be found in the proof of [22, Theorem 3.1] which states that for each monotone v that is both continuous from above and continuous from below such an appropriate sequence (v^n) exists. However, it can be shown that the statement is valid even for our more general case.

For now, let us consider a monotone v. Furthermore, let a dense increasing sequence of partitions $(S_j^n)_n$ $(j \leq n)$ of the set of players be given, i. e. $\Omega = S_1^n + \ldots + S_n^n$. Let, for every $n \in \mathbb{N}$, a point $\omega_i^n \in S_i^n$ be given $(i = 1, \ldots, n)$, and define $v^n : \mathcal{B} \longrightarrow \mathbb{R}_+$ by

$$v^n(F) = v\left(\sum_{i \mid \omega_i^n \in F} S_i^n\right). \tag{2.24}$$

Then, v^n has $(\{\omega_1^n, \ldots, \omega_n^n\})$ as its carrier. We will show next that (v^n) converges weakly to v.

Let an $\varepsilon > 0$ and a non-negative continuous function f be given. As f is uniformly continuous on $[0,1]$, there exists an $\overline{n} \in \mathbb{N}$ s.t.

$$|f(\omega) - f(\eta)| < \frac{\varepsilon}{2}$$

is true for all $n \geq \overline{n}$, and for all $\omega, \eta \in S_i^n, i \leq n$. We have that

$$\begin{aligned} v^n(\{f > t\}) &= v\left(\sum_{i|\, f(\omega_i^n)>t} S_i^n\right) \\ &\leq v\left(\sum_{i|\, f>t-\varepsilon \text{ on } S_i^n} S_i^n\right) \\ &\leq v(\{f > t - \varepsilon\}) \end{aligned}$$

and

$$\begin{aligned} v^n(\{f > t\}) &\geq v\left(\sum_{i|\, f>t+\varepsilon \text{ on } S_i^n} S_i^n\right) \\ &\geq v(\{f > t + \varepsilon\}) \end{aligned}$$

for each $t \in [0,1]$ and all $n \geq N$.

Let $\|f\|_{\sup}$ denote the sup norm of f. Then the following is true:

$$\begin{aligned} \left| \int f\, dv - \int f\, dv^n \right| &\leq \int_0^{\|f\|_{\sup}} |v(\{f > t\}) - v^n(\{f > t\})|\, dt \\ &\leq \int_0^{\|f\|_{\sup}} v(\{f > t - \varepsilon\}) - v(\{f > t + \varepsilon\})\, dt \\ &= \int_{-\varepsilon}^{\|f\|_{\sup}-\varepsilon} v(\{f > t\})\, dt - \int_{\varepsilon}^{\|f\|_{\sup}+\varepsilon} v(\{f > t\})\, dt \\ &\leq \int_{-\varepsilon}^{\varepsilon} v(\{f > t\})\, dt \\ &\leq 2\varepsilon v(\Omega). \end{aligned}$$

Now, we consider the general case, i. e. we consider a $v \in BV$. Since a v with bounded variation can be written as the difference of two monotone coalitional functions u and w, we have that

$$\begin{aligned}\int f\,dv &= \int f\,du - \int f\,dw \\ &= \lim_{n\to\infty}\int f\,du^n - \lim_{n\to\infty}\int f\,dw^n \\ &= \lim_{n\to\infty}\int f\,d(u^n - w^n),\end{aligned}$$

i.e. $(u^n - w^n)_n$ converges weakly to v. **q.e.d.**

Remark 2.33 *If (v^n) converges to v w.r.t. the variation norm, then $v^n \to v$ weakly.*

Proof The proof can be found indirectly in the proof of Proposition 2.2. There we have shown that

$$\lim_{n\to\infty}\int v^n(f > t)\,dt = \int v(f > t)\,dt$$

is valid for all fuzzy coalitions f if (v^n) converges to v w.r.t. $\|\bullet\|$. It is an easy task to generalize this statement to the space of all non-negative continuous functions. **q.e.d.**

To proceed with the extension of the Shapley value, we define

$$\overline{\Omega} := \{K \mid K \subseteq \Omega,\ K\text{compact}\}.$$

Consider now an arbitrary metric d on Ω.

Lemma 2.34 *For $A_1, A_2 \in \overline{\Omega}$, we define*

$$\overline{d}(A_1, A_2) := \inf\{\varepsilon > 0 \mid A_1 \subset B_\varepsilon(A_2) \text{ and } A_2 \subset B_\varepsilon(A_1)\},$$

where $B_\varepsilon(A) = \{x \in \Omega \mid \inf\{d(x,a);\ a \in A\} < \varepsilon\}$. $\overline{d}$ is a metric on $\overline{\Omega}$ (called the Hausdorff metric), and we have that

$$\overline{d}(A, B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a,b),\ \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

It is a well known fact that $\overline{\Omega}$ together with the Hausdorff metric is a compact metrizable space. Let $\overline{\mathcal{B}}$ denote the Borelian $\sigma-$ algebra on $\overline{\Omega}$. Then, for a v with finite carrier $\{\omega_1, \ldots, \omega_n\}$, a discrete signed measure $\mu = \mu_v$ is given on $(\overline{\Omega}, \overline{\mathcal{B}})$ by

$$\mu(K) := \begin{cases} \sum_{S \subseteq K} (-1)^{|K|-|S|} v(S), & \text{if } K \subseteq \{\omega_1, \ldots, \omega_n\} \\ 0, & \text{otherwise.} \end{cases} \tag{2.25}$$

μ as defined above has mass only on those points of $\overline{\Omega}$ which are subsets of the carrier of v.

In the following proposition, we will show how we can obtain a unique measure μ_v for each $v \in BV$. The proof requires a lemma which was first proven by Strassen [27, Lemma 4.1]. Here $C(\Omega)$ and $C(\overline{\Omega})$ denote the cone of non-negative continuous functions on Ω and $\overline{\Omega}$, respectively . For each $f \in C(\Omega)$, we define $\overline{\bigwedge} f \in C(\overline{\Omega})$ by

$$\overline{\bigwedge} f(\overline{S}) = \min_{s \in \overline{S}} f(s), \qquad \overline{S} \in \overline{\mathcal{B}}.$$

Lemma 2.35 *The linear hull of $\overline{\bigwedge} C(\Omega)$ is dense in $C(\overline{\Omega})$.*

For an arbitrary $v \in BV$, consider a sequence (v^n) with finite carrier and $v^n \to v$ weakly. Then, we obtain a sequence (μ_{v^n}) of discrete signed measures (cf. formula(2.25)) which admits a Hahn decomposition $\mu_{v^n} = \mu^+_{v^n} - \mu^-_{v^n}$.

Proposition 2.36 *If $\mu^+_{v^n}(\overline{\Omega})$ is bounded, then $(\mu^n) := (\mu_{v^n})$ has a unique weak accumulation point μ_v.*

If this μ_v exists, it replaces the $c_\bullet(v)$ in formula (2.23) and is exactly the S-measure of Rosenmüller [22, Definition 3.3]. Rosenmüller does not prove the existence part of the latter Proposition.

Proof First of all, we will prove the existence of μ: Let $(\overline{T}_i^m)_m, 1 \leq i \leq m$, be a dense increasing partition of $\overline{\Omega}$. For an arbitrarily chosen continuous $\overline{f}$ on $\overline{\Omega}$ and for each $\varepsilon > 0$, there exists an $M = M(\varepsilon, \overline{f}) \in \mathbb{N}$ s. t. $|\overline{f}(\overline{\omega}) - \overline{f}(\overline{\eta})| < \frac{\varepsilon}{2}$ for all $\overline{\omega}, \overline{\eta} \in \overline{T}_i^M, 1 \leq i \leq M$. As $\mu^{n+}(\overline{\Omega})$ is bounded, there exists a subsequence

$(\mu^{n_k})_k$ and a measure μ on $(\overline{\Omega}, \overline{\mathcal{B}})$ s. t. $\mu^{n_k}(\overline{T}_i^M) \to \mu(\overline{T}_i^M)$ is true for each $1 \le i \le M$. In particular, we have that $|\mu^{n_k}(\overline{T}_i^M) - \mu(\overline{T}_i^M)| < \frac{\varepsilon}{M}$ for all $1 \le i \le M$ and k sufficiently large. W. l. o. g., we assume that (μ^n) itself satisfies the latter inequality. We have that

$$\begin{aligned} \mu^n(\{\overline{f} > t\}) &\le \mu^n\left(\sum_{i|\overline{f}>t-\varepsilon \text{ on } \overline{T}_i^M} \overline{T}_i^M\right) \\ &< \mu\left(\sum_{i|\overline{f}>t-\varepsilon \text{ on } \overline{T}_i^M} \overline{T}_i^M\right) + \varepsilon \\ &\le \mu(\{\overline{f} > t - \varepsilon\}) + \varepsilon. \end{aligned}$$

Similarly, it can be shown that

$$\mu^n(\overline{f} > t) > \mu(\overline{f} > t + \varepsilon) - \varepsilon$$

All in all, the following is true:

$$\begin{aligned} \left| \int \overline{f}\, d\mu^n - \int \overline{f}\, d\mu \right| &\le \int_0^1 \mu(\overline{f} > t - \varepsilon) + \varepsilon - (\mu(\overline{f} > t + \varepsilon) - \varepsilon)\, dt \\ &\le 2\varepsilon\mu(\overline{\Omega}) + 2\varepsilon. \end{aligned}$$

To prove the uniqueness part, we will follow the steps in Rosenmüller [22, pp. 97 and 98]. Let $(\omega_1^n, \ldots, \omega_n^n)$ be the carrier of v^n. For each compact $E \subseteq \overline{\Omega}$, an induction argument over $l(E) := \#(E \cap (\omega_1^n, \ldots, \omega_n^n))$ shows that $\mu^n(\overline{T}|\overline{T} \subseteq E) = v^n(E)$ is true. Hence, we have that

$$\begin{aligned} \int f\, dv^n &= \int_0^1 v^n(\{f > t\})\, dt \\ &= \int_0^1 \mu^n\left(\left\{\overline{\bigwedge} f > t\right\}\right) dt \\ &= \int \overline{\bigwedge} f\, d\mu^n. \end{aligned}$$

Let (μ^{n_k}) again be a subsequence with $\mu^{n_k} \to \mu$ weakly for $k \to \infty$, where μ is a signed measure on $(\overline{\Omega}, \overline{\mathcal{B}})$. Then

$$\int f\, dv = \lim_{k\to\infty} \int f\, dv^{n_k} = \lim_{k\to\infty} \int \overline{\bigwedge} f\, d\mu^{n_k} = \int \overline{\bigwedge} f\, d\mu. \tag{2.26}$$

It is a well known fact that, given two measures on a metric space $(S, \mathcal{S})$, $\mathcal{S}$ the Borel-σ-algebra of S, whenever the integrals w. r. t. the measures coincide for all continuous bounded functions on S, then also the two measures coincide. Knowing this statement and having in mind Lemma 2.35, formula (2.26) determines μ uniquely. **q.e.d.**

The next step is now the construction of a space of games for which the measure μ exists.

Definition 2.37 *A fuzzy coalitional function v^F is **totally monotone** if it is non-negative and , for every $n \geq 2$ and $f_1, \ldots, f_n \in [0,1]^{\mathbb{N}}$,*

$$v^F\left(\bigvee_{i=1}^{n} f_i\right) \geq \sum_{I \mid \emptyset \neq I \subset \{1,\ldots,n\}} (-1)^{|I|+1} v^F\left(\bigwedge_{i \in I} f_i\right).$$

One should remark that v is totally monotone if and only if v^C is totally monotone. This can be shown as easy as the analoguous statement for convexity (cf. Lemma 2.6).

Theorem 2.38 *Consider a $v \in BV$ that can be written as the difference of two totally monotone functions u and w, i. e. $v = u - w$. Then, our method provides a measure μ_v.*

Proof Let us define u^n and w^n, $n \in \mathbb{N}$, as the v^n in the proof of Proposition 2.32, i. e. let us consider some points $\omega_i^n \in S_i^n$ for a dense increasing sequence of partitions $(S_j^n)_n, j \leq n$, of Ω. Then u^n (and correspondingly w^n) is given by

$$u^n(F) = u\left(\sum_{i \mid \omega_i^n \in F} S_i^n\right).$$

Obviously, each u^n and each w^n is totally monotone. We know that the sequences $(u^n)_n$ and $(w^n)_n$ are converging weakly to u and w, respectively, and, hence, one obtains weak convergence of $(v^n)_n$ to v for $v^n := u^n - w^n$. Furthermore, we have that for each $E \subseteq \{\omega_1^n, \ldots, \omega_n^n\}$

$$\begin{aligned} 0 &\leq u^n(E) - \sum_{I \mid \emptyset \neq I \subseteq \{1,\ldots,m\}} (-1)^{|I|+1} u^n \left(\bigcap_{i \in I} T_i \right) \\ &= \sum_{T \subseteq E} (-1)^{|E|-|T|} u^n(T) \\ &= \mu_{u^n}(E), \end{aligned}$$

where $E := \{e_1, \ldots, e_m\}$ and $T_i := E \setminus \{e_i\}$. Following the same lines, one can show non-negativity of μ_{w^n}. The corresponding measures μ_{v^n} can now be written as

$$\begin{aligned} \mu_{v^n}(E) &= \sum_{T \subseteq E} (-1)^{|E|-|T|} v^n(T) \\ &= \sum_{T \subseteq E} (-1)^{|E|-|T|} [u^n(T) - w^n(T)] \\ &= \mu_{u^n}(E) - \mu_{w^n}(E) \end{aligned}$$

for $E \subseteq \{\omega_1^n, \ldots, \omega_n^n\}$, i.e. we get with the measures μ_{u^n} and μ_{w^n} a Hahn decomposition of μ_{v^n}. Since it is easy to verify that $\mu_{u^n}(\overline{\Omega}) = u^n(\Omega)$ is true for all $n \in \mathbb{N}$, we have that

$$\mu_{u^n}(\overline{\Omega}) \to u(\Omega).$$

In particular, we know that $\mu_{u^n}(\overline{\Omega})$ is bounded. Now we can again show that $(\mu_{v^n})_n$ has a weak accumulation point μ_v and that μ_v is the unique signed measure satisfying

$$\int \overline{\bigwedge} f \, d\mu_v = \int f \, dv$$

for all continuous f on Ω. **q.e.d.**

The question remains how to obtain a generalization of the uniform measure g^S for the extended value. For this reason, we define the space $\mathcal{F}$ to be the set of all continuous functions $f : [0,1] \to [0,1]$. $\mathcal{F}$ is a fuzzy tribe (cf. [6, Definition 2.5]). Furthermore, we will use a quasi-kernel $\mathcal{P}$ on $\mathcal{F} \times \overline{\Omega}$ with the properties

1. $\mathcal{P}(f, \cdot)$ is measurable for every f,
2. $\mathcal{P}(\cdot, \overline{S})$ is a (finitely additive) measure on $\mathcal{F}$,
3. $\mathcal{P}(\Omega, \overline{S}) = 1$ for every $\overline{S} \in \overline{\mathcal{B}}$.

The notation quasi-kernel is used to stress that we do not insist on σ-additivity. In our definition of the kernel we follow Rosenmüller's definition in the case of crisp coalitions [22, page 99]:

1. If $\lambda(\overline{S}) > 0$:
$$\mathcal{P}(\cdot, \overline{S}) = \frac{1}{\lambda(\overline{S})} \int_{\overline{S}} \cdot \, d\lambda.$$
2. If $\overline{S}$ is finite, $\overline{S} = \{\omega_1, \ldots, \omega_n\}$:
$$\mathcal{P}(f, \overline{S}) = \frac{1}{n} \sum_{i=1}^{n} f(\omega_i).$$
3. For all remaining $\overline{S}$:
$$\mathcal{P}(f, \overline{S}) = f(k), \; k = \min_K .$$

Measurability can be proven as in [22, Theorem 3.4]. For the first two cases, the kernel $\mathcal{P}$ seems to be the correct generalization of the uniform distribution. The case of a compact and infinite set $\overline{S}$ with Lebesgue measure 0 looks a little bit strange (all is given to the first player in the natural ordering). Rosenmüller states in [22]: "… it might be preferable to take an additive measure that is diffuse in the sense that it assigns 0 to all finite coalitions in $\overline{S}$. It is not clear if such a measure can be found without violating the measurability conditions of the kernel $\mathcal{P}$."

Definition 2.39 *[22, Definition 3.4] Let $v^C \in CBV$, and assume that $\mu = \mu_v$ exists. Then, $\varphi_{Ros} : \mathcal{F} \to \mathbb{R}$ given by*
$$\varphi_{Ros}(\cdot) := \int \mathcal{P}(\cdot, \overline{\omega}) \, \mu(d\overline{\omega})$$
*is called the **value** of v^C.*

One can ask why we restrict the value to continuous fuzzy coalitions. From the mathematical point of view, there is no problem in defining the quasi-kernel $\mathcal{P}$ on all measurable functions $f : [0,1] \to [0,1]$. However, for an arbitrary v it is not possible to obtain a sequence (v^n) with finite carrier and $\lim_{n\to\infty}(v^n)^C(f) \to v^C(f)$ for all measurable f. This can be shown, for example, by considering the unanimous game $(e^\Omega)^C$.

2.5.2 A Value for $(e^{\mathbb{N}})^C$

Now, we would like to consider again the case of countably many players. In the following, we will define a class of games $\widetilde{V} \subseteq CBV$ s. t. for all $v^C \in \widetilde{V}$ the corresponding measure μ_v exists and for each $v^C \in \widetilde{V}$ a sequence (v^{nC}) with finite carrier can be found s. t. $v^{nC}(f) \to v^C(f)$ is true for all $f \in [0,1]^{\mathbb{N}}$. For these games we would like to establish a kernel on $[0,1]^{\mathbb{N}} \times \mathcal{D}$ as a generalization of the uniform measure. Here, $\mathcal{D}$ denotes the space of non-empty subsets of $\mathbb{N}$.

Of course, there is no problem in restricting the player set $[0,1]$ to countably many agents. For example, one can take the intersection between the rational numbers and the unit interval, i. e. $\Omega' := [0,1] \cap \mathbb{Q}$, as the player set. Then, we can follow the same calculations as in the continuous case: Let $\widetilde{\mathcal{F}}$ be the space of all continuous functions $f : \Omega' \to [0,1]$ and let $\widetilde{\Omega}$ be the space of all compact subsets of Ω'. As we do not have any coalitions with Lebesgue-measure greater than zero, the quasi-kernel $\widetilde{\mathcal{P}}$ on $\widetilde{\mathcal{F}} \times \widetilde{\Omega}$ is given by

$$\widetilde{\mathcal{P}}(f,S) = \begin{cases} \frac{1}{|S|}\sum_{i\in S} f(i), & |S| < \infty \\ f(s) & |S| = \infty, s = \min_S . \end{cases} \tag{2.27}$$

However, we would like to allow all coalitions to be feasible. This is not the case for $\widetilde{\Omega}$. For example, Ω' itself is not an element of $\widetilde{\Omega}$ since the closure of Ω' is nothing else but the unit interval. Another argument against $[0,1] \cap \mathbb{Q}$ as the player set is given by the fact that all participants of the game seem to have the same importance. However, in the case of countably many players we would like to reflect the situation of some big players and many small players. In this sense, the natural numbers are a much more intuitive space. Another reason is that we would like to obtain a more reasonable kernel as the one given by formula (2.27).

Therefore, we would like to consider the set of games defined on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$. However, for some mathematical reasons, we cannot match the framework that we have used in the continuous case one to one to the case of countably many players. First of all, we have to deal with the problem that, in the proof of Proposition 2.32, we explicitly need the compactness of the player set Ω. This property is obviously not shared by $\mathbb{N}$. However, the following lemma presents a possibility to obtain a converging sequence even for $\mathbb{N}$ as the set of players. By $\mathcal{V}^{\searrow}$ $[\mathcal{V}^{\nearrow}]$we denote the games in BV which are upper σ-continuous [lower σ-continuous].

Lemma 2.40 *For each $v \in linhull\{\mathcal{V}^{\searrow} \cup \mathcal{V}^{\nearrow}\}$, there exists a sequence $(v^{nC})_n$ s. t. $v^{nC} \to v^C$ pointwise and each v^{nC} has $\{1, \ldots, n\}$ as its carrier.*

Proof For $v \in \mathcal{V}^{\searrow}$, we define v^{nC} by $v^{nC}(f) := v^C(f^n)$ where

$$f^n(j) = \begin{cases} f(j), & \text{if } j \leq n \\ 1, & \text{if } j > n. \end{cases}$$

Obviously, $f^n \searrow f$ is true. Now, one can show as in Lemma 2.6(1) that $v^{nC} \to v^C$ pointwise.

For $v \in \mathcal{V}^{\nearrow}$, one constructs a sequence $(v^{nC})_n$ by $v^{nC}(f) := v^C(f|_{\{1,\ldots,n\}})$. As an immediate consequence of the continuity from below, we have that $v^{nC}(f) \to v^C(f)$ for all f.

For an arbitrary $v \in linhull\{\mathcal{V}^{\searrow} \cup \mathcal{V}^{\nearrow}\}$, $v = \sum_{i=1}^{I} \lambda_i v_i$ with $I \in \mathbb{N}, \lambda_i \in \mathbb{R}$ and $v_i \in \mathcal{V}^{\searrow} \cup \mathcal{V}^{\nearrow}$, we can use the argumentation as above for each single v_i and finally obtain a sequence for v^C as required. **q.e.d.**

Moreover, we have to use some compactness arguments to show uniqueness of μ. Thus, we use the following detour:

For each v on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$, a $\overline{v}$ on $(\overline{\mathbb{N}}, \underline{\underline{P}}(\overline{\mathbb{N}}))$ is given by $\overline{v}(S) := v(S \setminus \{\infty\}), S \subseteq \overline{\mathbb{N}}$, where $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Obviously, the extra player ∞ is a null player for $(\overline{v}, \overline{\mathbb{N}})$. We now define $\mathcal{D}$ as the set of non-empty coalitions in $\mathbb{N}$, and, correspondingly,

$\overline{\mathcal{D}}$ as the set of non-empty subsets of $\overline{\mathbb{N}}$. For $\overline{v}$, one can show again, as in the case of continuously many players, that there exists a unique signed measure $\mu_{\overline{v}}$ on $(\overline{\mathcal{D}}, \underline{\underline{P}}(\overline{\mathcal{D}}))$:

First of all, a metric d on $\overline{\mathbb{N}}$ is given by

$$d(x,y) := \begin{cases} \frac{|x-y|}{xy} & \text{if } x, y \in \mathbb{N} \\ \frac{1}{x} & \text{if } x < y = \infty \\ \frac{1}{y} & \text{if } y < x = \infty \\ 0 & \text{if } x = y = \infty. \end{cases} \tag{2.28}$$

One can easily verify that d as defined above satisfies the three properties of a metric. Especially, this metric satisfies $d(x,y) \leq 1$ and $\lim_{y\to\infty} d(x,y) = \frac{1}{x}$ for $x < \infty$. Furthermore, one can immediately see that for each given $n \in \mathbb{N}$ and every $m \neq n$

$$d(n,m) \geq d(n, n+1) = \frac{1}{n(n+1)}$$

is valid. To get an impression of the corresponding Hausdorff metric $\overline{d}$ (cf. Lemma 2.34), the next remark might be quite helpful:

Remark 2.41 *Let $S, T \in \overline{\mathcal{D}}$ and $n \in \mathbb{N}$ be given.*

1. *If $\overline{d}(S,T) < \frac{1}{n(n+1)}$, then $S|_{\{1,\ldots,n\}} = T|_{\{1,\ldots,n\}}$ is valid.*
2. *If $S|_{\{1,\ldots,n\}} = T|_{\{1,\ldots,n\}}$ and $S|_{\{n+1,\ldots\}} \neq \emptyset \iff T|_{\{n+1,\ldots\}} \neq \emptyset$, then $\overline{d}(S,T) \leq \frac{1}{n}$.*

Proof

1. The assumption $S|_{\{1,\ldots,n\}} \neq T|_{\{1,\ldots,n\}}$ implies that there exists an $a \in \{1,\ldots,n\}$ s. t. $a \in S \cup T$ and $a \notin S \cap T$. One can easily check that the distance between S and T has to be at least $\frac{1}{a(a+1)}$, i. e. we have that $\overline{d}(S,T) \geq \frac{1}{n(n+1)}$.

2. If both S and T contain no element larger than n, we have that $\overline{d}(S,T) = 0$.

   Consider now the second case, i. e. assume that there exists an $\widetilde{s} := min\{j \in S, j > n\}$ and an analogously defined $\widetilde{t}$. We can assume w. l. o. g. that $\widetilde{s} \leq \widetilde{t}$. With these definitions we have that, for $s \in S$,

$$\inf_{t \in T} d(s,t) \begin{cases} = 0 & \text{if } s < \widetilde{s} \\ \leq d(\widetilde{s}, \widetilde{t}) \leq \frac{1}{n} & \text{if } \widetilde{s} \leq s \leq \widetilde{t} \\ \leq d(s, \widetilde{t}) \leq \frac{1}{\widetilde{t}} \leq \frac{1}{n} & \text{if } s > \widetilde{t}. \end{cases}$$

   In the same way, one can show that $\inf_{s \in S} d(s,t) \leq \frac{1}{n}$ for each $t \in T$. All in all, we have proven that $\overline{d}(S,T) \leq \frac{1}{n}$. **q.e.d.**

As is easy to verify, $\overline{\mathcal{D}}$ together with $\overline{d}$ is a compact metrizable space. Hence, one can show, as in Subsection 2.5.1, the existence and uniqueness of $\mu_{\overline{v}}$ as a weak accumulation point of $\mu_{\overline{v}^n}$, where $\overline{v}^n$ and $\mu_{\overline{v}^n}$ are defined according to formula (2.24) and (2.25), respectively.

To obtain a measure μ_v on $(\mathcal{D}, \underline{\underline{P}}(\mathcal{D}))$, a correspondence $p : \mathcal{D} \Rightarrow \overline{\mathcal{D}}$ is defined by

$$p(S) := \{S, S + \{\infty\}\},$$

i. e. each non-empty coalition S in $\mathcal{D}$ is assigned to the set S itself and the set containing all players of S and $\{\infty\}$. The measure μ_v is defined by

$$\mu_v(S) = \begin{cases} \mu_{\overline{v}}(p(S)), & \text{if } \emptyset \neq S \subseteq \mathcal{D}, \\ 0, & \text{if } S = \emptyset. \end{cases}$$

One easily checks that μ_v satisfies $\sigma-$additivity, and that

$$\mu_v(\mathcal{D}) = \mu_{\overline{v}}(\overline{\mathcal{D}}) = \overline{v}(\overline{\mathbb{N}}) = v(\mathbb{N}).$$

One should remark that, if v^n has $K_n := \{\omega_1, \ldots, \omega_n\}$ as its carrier, the same is valid for $\overline{v}^n$. We know that $\mu_{\overline{v}^n}$ has only mass on those points of $\overline{\mathcal{D}}$ which are subsets of K_n. Since we also have that $\mu_{v^n}(S) = \mu_{\overline{v}^n}(p(S)) = \mu_{\overline{v}^n}(S)$ and $\overline{v}^n(S) = v^n(S)$ for $S \subseteq K_n$, our μ_{v^n} is exactly the measure given by formula

(2.25). μ_v defined in this way is unique as $\mu_{\overline{v}}$ is. Hence, we declare μ_v to be the extension of $c_\bullet(v)$.

Let the space $\overline{\mathcal{V}} \subset \mathcal{V}$ be given by the equation

$$\begin{aligned}\overline{\mathcal{V}} \ := \ & \{v \in \mathcal{V} |\, v = u - w, \text{ where } u \text{ and } w \text{ are totally monotone} \\ & \text{and } u, w \in linhull\{\mathcal{V}^{\searrow} \cup \mathcal{V}^{\nearrow}\}\}.\end{aligned}$$

One should remark that $\overline{\mathcal{V}}^C$ is a symmetric subspace of CBV, i.e. for each $v^C \in \overline{\mathcal{V}}^C$ and every permutation π of $\mathbb{N}$, we have that $\pi v^C \in \overline{\mathcal{V}}^C$. The following corollary is a direct consequence of Theorem 2.38 and Lemma 2.40.

Corollary 2.42 *For each $v \in \overline{\mathcal{V}}$ there exists a measure μ_v and a sequence $(v_n)_n$ s. t. $v_n^C(f) \to v^C(f)$ for all $f \in [0,1]^{\mathbb{N}}$, where each v_n has $\{1, \dots, n\}$ as its carrier.*

For the definition of the value, we need a (quasi-) kernel on $[0,1]^{\mathbb{N}} \times \mathcal{D}$. As stated before, the kernel given by Rosenmüller [22] has some disadvantages which we would like to avoid in the countable case. We will make some proposals for a more intuitive kernel later on. For the moment, the minimal properties of a kernel $\mathcal{P}$ (besides the three properties given on page 71) shall be

$$\mathcal{P}(f, S) = \frac{1}{|S|} \sum_{i \in S} f(i) \tag{2.29}$$

for every fuzzy coalition f and each finite $S \subset \mathbb{N}$, and

$$\mathcal{P}(S, S) = 1 \tag{2.30}$$

for all $S \subseteq \mathbb{N}$. We have to insist on property (2.29) as our value shall be the usual Shapley value for finite coalitions S. Condition (2.30) is necessary to express that $\mathcal{P}(\bullet, S)$ should be something like the uniform distribution on S.

Definition 2.43 *Let $v \in \overline{\mathcal{V}}$ and the corresponding μ_v be given. Consider $\mathcal{P}$ to be any kernel on $[0,1]^{\mathbb{N}} \times \mathcal{D}$ that satisfies the properties (2.29) and (2.30). Then, $\varphi_{Ros}^{\mathcal{P}} : [0,1]^{\mathbb{N}} \to \mathbb{R}$ given by*

$$\varphi_{Ros}^{\mathcal{P}}(\bullet) = \int \mathcal{P}(\bullet, \overline{\omega}) \mu(d\overline{\omega}) \tag{2.31}$$

is called the ***value on $\overline{\mathcal{V}}^C$ with respect to $\mathcal{P}$***.

Example 2.44 *The value for the unanimous game*

Since each unanimous game e^K, $K \subseteq \mathbb{N}$, is continuous from below and totally monotone, these games are some quite important elements of $\overline{\mathcal{V}}$. Rosenmüller shows in [22, Theorem 3.5] that, for each e^K, the corresponding measure μ_{e^K} is nothing else but the dirac measure on K. Thus, the value takes the form

$$\varphi^{\mathcal{P}}_{Ros}(e^K)^C = \int \mathcal{P}(\bullet, \omega)\, d\mu_{e^K}(\omega) = \mathcal{P}(\bullet, K).$$

This implies that the value of $(e^K)^C$ is nothing else but an evaluation of the given fuzzy coalition on K with the kernel $\mathcal{P}$. For a finite K, this formula coincides with our result obtained by the smoothing procedure.

We know that the value given by Definition 2.43 is not a value in a sharp sense since, as we have shown before, there cannot exist a measure m on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$ that satisfies both $m(\mathbb{N}) = 1$ and is invariant under permutations (a value for $e^{\mathbb{N}}$ had to satisfy these two properties). In the following, we would like to analyse which properties are still valid:

Proposition 2.45 *Let $\mathcal{P}$ be a kernel on $[0,1]^{\mathbb{N}} \times \mathcal{D}$ that satisfies the properties (2.29) and (2.30). Then, the value $\varphi^{\mathcal{P}}_{Ros}$ is a linear and efficient operator. If v is totally monotone, $\varphi^{\mathcal{P}}_{Ros}(v)$ is non-negative.*

Proof Linearity follows directly from the fact that

$$\mu_{\alpha v + \beta w} = \alpha \mu_v + \beta \mu_w$$

for all $\alpha, \beta \in \mathbb{R}, v, w \in \overline{\mathcal{V}}$. Efficiency can also be shown straightforward:

$$\varphi^{\mathcal{P}}_{Ros}(v^C)(\mathbb{N}) = \int \mathcal{P}(\mathbb{N}, \omega)\, d\mu_v = \int d\mu_v = \mu_v(\mathcal{D}) = v(\mathbb{N}).$$

If $v \in \overline{\mathcal{V}}$ is totally monotone, then, obviously, μ_v is non-negative. Thus, the same is true for $\varphi^{\mathcal{P}}_{Ros}(v^C)$. **q.e.d.**

In particular, our procedure leads to a value for all $(e^K)^C$, $K \in \underline{\underline{P}}(\mathbb{N})$:

$$\varphi^{\mathcal{P}}_{Ros}(e^K)^C(\bullet) = \mathcal{P}(\bullet, K).$$

The result that the value of an unanimous game is a kernel is very reasonable. However, the question remains how this kernel should look like for infinite coalitions K. One possibility is given by formula (2.27), but, as mentioned before, this P has some great disadvantages. Hence, we would like to think about a new and more intuitive kernel.

The kernel shall present something like a uniform distribution. One possibility to approximate this distribution is given by

$$\mathcal{P}_\infty(f, K) = \sum_{j \in \mathbb{N}} \frac{1}{2^j} f(k_j),$$

where $K = \{k_1, k_2, \ldots\}$. The smaller a player is in the natural ordering, the more he gets (a similar result can be received with every absolutely convergent sequence with total mass 1). However, the distribution would be more intuitive if it assigns 0 to all fuzzy coalitions with finite carrier.

One possibility to solve this problem shall be presented now. First of all, we look at the case of crisp coalitions. After this, we try to extend our results to fuzzy games. Let a $K \subseteq \mathbb{N}$ with infinitely many players be given. A filter $\mathcal{F}_K \subseteq \underline{\underline{P}}(K)$ is given by

- $A \in \mathcal{F}_K,\ A \subseteq B \subseteq K \Longrightarrow B \in \mathcal{F}_K$
- $A, B \in \mathcal{F}_K \Longrightarrow A \cap B \in \mathcal{F}_K$
- $\emptyset \notin \mathcal{F}_K$.

A maximal filter is called ultrafilter, and each filter is contained in an ultrafilter. This is a well known application of Zorn' s lemma. If $\mathcal{U}_K$ is an ultrafilter on $\underline{\underline{P}}(K)$ and $A \in \underline{\underline{P}}(K)$ then either A or A^C is a member of $\mathcal{U}_K$.

We define $\widetilde{\mathcal{F}}_K := \{A |\, A \subseteq K \text{ is cofinite}\}$, where a coalition $A \subseteq K$ is called cofinite if $A^C \cap K$ is finite. Obviously, $\widetilde{\mathcal{F}}_K$ is a filter. Let $\widetilde{\mathcal{U}}_K$ be a corresponding ultrafilter. Then, we obtain a finitely additive measure $\mathcal{P}_K$ on $(\mathbb{N}, \underline{\underline{P}}(\mathbb{N}))$ by

$$\mathcal{P}_K(A) = \begin{cases} 1, & \text{if } A \cap K \in \widetilde{\mathcal{U}}_K, \\ 0, & \text{otherwise.} \end{cases}$$

In the crisp case, the measure defined in this way can be seen as our new kernel $\mathcal{P}(\cdot, K)$, i.e. $\mathcal{P}(\cdot, K) = \mathcal{P}_K(\cdot)$. Here, we have found a finitely additive measure that assigns 0 to all finite coalitions in K. As we know that the Choquet extension of an additive set function is itself additive, we can use, for example, the Choquet integral w.r.t. the so defined kernel $\mathcal{P}(\cdot, K)$ for our fuzzy value. Obviously, $\mathcal{P}^C(K, K) = 1$ is valid. Since each additive fuzzy measure is homogeneous (cf. [29, Lemma 2.1.3]), we know that $\mathcal{P}^C(\lambda K, K) = \lambda$ is true for every $\lambda \in [0, 1]$. Furthermore, we have that $\mathcal{P}_K^C(f) = \int_0^1 \mathcal{P}_K(\{j|\, f(j) > t\})\, dt$ and

$$\mathcal{P}_K(\{j|\, f(j) > t\}) = \begin{cases} 1, & \text{if } \{j|\, f(j) > t\} \in \widetilde{\mathcal{U}}_K \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, there exists exactly one $t^* = t^*(f, \widetilde{\mathcal{U}}_K)$ s.t. $\{j|\, f(j) > t\} \in \widetilde{\mathcal{U}}_K$ for all $t < t^*$ and $\{j|\, f(j) > t\} \notin \widetilde{\mathcal{U}}_K$ for all $t > t^*$. Hence, we have that $\mathcal{P}_K^C(f) = t^*$.

Another possibility to get something like a uniform distribution is given in the following. There we think of a filter for fuzzy coalitions on $\mathbb{N}$ and define the corresponding measure as in the crisp case.

Weak Fuzzy Filter

As already mentioned, we define the intersection of two fuzzy coalitions as the minimum. Since we would like to assign only the values zero and one to all fuzzy coalitions, it is not possible to use a strong additive measure here, i.e. a measure P which satisfies $P(f+g) = P(f) + P(g)$ for $f + g \leq \mathbb{N}$. We have the problem that, as shown in [29, Lemma 2.1.3], such measures are always homogeneous, i.e. $P(\lambda\mathbb{N}) = \lambda P(\mathbb{N}) = \lambda$. Hence, we can only take the weak form of additivity: $m(f \vee g) = m(f) + m(g)$ for $f \wedge g = \emptyset$.

Lemma 2.46 *The definition of weak additivity is equivalent to*

$$m(f \vee g) + m(f \wedge g) = m(f) + m(g) \text{ for all } f, g \in [0,1]^{\mathbb{N}}. \tag{2.32}$$

Proof Considering fuzzy coalitions f and g with pairwise disjoint carrier, one can immediately see that equation (2.32) implies weak additivity.

Let us assume now weak additivity. For two fuzzy coalitions f and g, one can define $T := \{i|\, f(i) \geq g(i)\}$. By this setting, the union of f and g can be written as

$$f \vee g = f|_T + g|_{T^C} = f|_T \vee g|_{T^C}.$$

As $f|_T$ and $g|_{T^C}$ have an empty intersection, we obtain the equality

$$m(f \vee g) = m(f|_T \vee g|_{T^C}) = m(f|_T) + m(g|_{T^C}).$$

For the same reasons

$$m(f \wedge g) = m(f|_{T^C} + g|_T) = m(f|_{T^C} \vee g|_T) = m(f|_{T^C}) + m(g|_T)$$

is true. Summarizing, we have that

$$m(f \vee g) + m(f \wedge g) = m(f|_T) + m(g|_{T^C}) + m(f|_{T^C}) + m(g|_T),$$

and this is equation (2.32). **q.e.d.**

We now try to bring the definition of a filter into the context of fuzzy coalitions:

Definition 2.47 *For $K \subseteq \mathbb{N}, |K| = \infty$, a* **(weak) fuzzy filter** $\mathcal{F}_K \subseteq [0,1]^K$ *is given by*

- $f \in \mathcal{F}_K,\ f \leq g \leq K \Longrightarrow g \in \mathcal{F}_K$
- $f, g \in \mathcal{F}_K \Longrightarrow f \wedge g \in \mathcal{F}_K$
- $\emptyset \notin \mathcal{F}_K$

Again, one can show with the help of Zorn' s lemma that every fuzzy filter $\mathcal{F}$ is contained in a fuzzy ultrafilter, which denotes a maximal fuzzy filter:

Let X denote the collection of all fuzzy filters containing $\mathcal{F}$. For all $\mathcal{F}_\alpha$, $\mathcal{F}_\beta$ in X, we say that $\mathcal{F}_\alpha \leq \mathcal{F}_\beta$ if $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. $\leq$ is a partial order on X. Let $\{\mathcal{F}_\alpha | \, \alpha \in A\}$, $A \subseteq \mathbb{R}$, be a chain in X. Then $\bigcup_{\alpha \in A} \mathcal{F}_\alpha \in X$ and $\bigcup_{\alpha \in A} \mathcal{F}_\alpha$ is an upper bound of the chain $\{\mathcal{F}_\alpha | \, \alpha \in A\}$. Consequently, there exists a maximal element $\mathcal{U}$ in X.

Our next steps aim at looking for a weak fuzzy filter on $\underline{\underline{P}}(\mathbb{N})$. For an arbitrary infinite K, one can make similar thoughts. To be more precise, we now try to find a partition of $[0,1]^{\mathbb{N}}$ s. t. the first part of $[0,1]^{\mathbb{N}}$ contains the "large" fuzzy coalitions and each member of this part gets the measure one. All other fuzzy coalitions get zero. To do this, we take a look at

$$\widehat{\mathcal{F}} := \left\{ f \in [0,1]^{\mathbb{N}} \middle| \, f(i) \leq \frac{1}{2} \text{ for at most finitely many } i \right\}.$$

Let $\widehat{\mathcal{U}}$ be a maximal filter containing $\widehat{\mathcal{F}}$. The cofinite crisp coalitions are contained in $\widehat{\mathcal{F}}$, and, hence, especially in $\widehat{\mathcal{U}}$. Since the finite crisp coalitions have an empty intersection with their complements, and since these complements are cofinite, no finite crisp coalition is an element of $\widehat{\mathcal{U}}$.

Lemma 2.48 $f \notin \widehat{\mathcal{U}} \iff \exists g \in \widehat{\mathcal{U}}$ *s. t.* $f \wedge g = \emptyset$.

Proof If there exists a $g \in \widehat{\mathcal{U}}$ s. t. $f \wedge g = \emptyset$, then, obviously, f is not an element of $\widehat{\mathcal{U}}$, since otherwise the property $\emptyset \notin \widehat{\mathcal{U}}$ would be violated.

To prove the other direction, we assume that for a given $f \notin \widehat{\mathcal{U}}$ there is no $g \in \widehat{\mathcal{U}}$ s. t. f and g have an empty intersection. Consider the set $\mathcal{H}$ consisting of all fuzzy coalitions h for which there exists some $g \in \widehat{\mathcal{U}}$ s. t. $h \geq (f \wedge g)$. $\mathcal{H}$ is a weak fuzzy filter. Moreover, $\mathcal{H}$ contains both $\widehat{\mathcal{U}}$ and f. Hence, one knows that $f \in \widehat{\mathcal{U}}$ is valid because of the maximality of $\widehat{\mathcal{U}}$. This is a contradiction. **q.e.d.**

Lemma 2.49 $f \notin \widehat{\mathcal{U}} \Longrightarrow f^C \in \widehat{\mathcal{U}}$

Proof Assume for the contrary that $f^C \notin \widehat{\mathcal{U}}$, i. e. that there exists a $g \in \widehat{\mathcal{U}}$ s. t. $f^C \wedge g = \emptyset$. Since f is not a member of $\widehat{\mathcal{U}}$, there exists a $h \in \widehat{\mathcal{U}}$ with $h \wedge f = \emptyset$.

Summarizing, we have that

$$g(i) > 0 \Longrightarrow f^C(i) = 0 \Longrightarrow f(i) = 1 \Longrightarrow h(i) = 0.$$

In the same way it can be shown $h(i) > 0 \Longrightarrow g(i) = 0$. Therefore, we have that $h \wedge g = \emptyset$, and this provides the contradiction. **q.e.d.**

The equivalence $f \notin \widehat{\mathcal{U}} \Longleftrightarrow f^C \in \widehat{\mathcal{U}}$ is well known for crisp coalitions, but is not valid in the case of fuzzy coalitions. For example, $\lambda\mathbb{N}$ is an element of $\widehat{\mathcal{U}}$ for every $\lambda \in (0,1)$ as the intersection of such a fuzzy coalition with any $\emptyset \neq f \in [0,1]^{\mathbb{N}}$ is non-empty. The complement of $\lambda\mathbb{N}$, $(\lambda\mathbb{N})^C = (1-\lambda)\mathbb{N}$, is obviously in $\widehat{\mathcal{U}}$, too. We will repair this "gap" in the next section.

Nevertheless, we can define a finitely weak additive $\mathcal{P}_w(\bullet, \mathbb{N})$ on $[0,1]^{\mathbb{N}}$ by $\mathcal{P}_w(f, \mathbb{N}) = 1$ for $f \in \widehat{\mathcal{U}}$ and $\mathcal{P}_w(f, \mathbb{N}) = 0$ for $f \notin \widehat{\mathcal{U}}$. To get a measure $\mathcal{P}_w(\bullet, K)$ for infinite K, one has to restrict the fuzzy coalition f to K and follow the steps of the construction of $\mathcal{P}_w(\bullet, \mathbb{N})$.

Strong Fuzzy Filter

As already mentioned, it is not possible to get a strong additive measure by the approach demonstrated above. But as we will show next, it is possible to find a weak additive measure m that satisfies

$$m(f) + m(f^C) = m(\mathbb{N}) = 1 \tag{2.33}$$

for all $f \in [0,1]^{\mathbb{N}}$. One can think about requiring the implication $f \in \mathcal{F} \Rightarrow f^C \notin \mathcal{F}$ as a fourth property of a fuzzy filter. However, the problem is that we have $\frac{1}{2}\mathbb{N} = (\frac{1}{2}\mathbb{N})^C$, i. e. there exists a fuzzy coalition that coincides with its own complement.

Now, we want to present an approach in which a weak fuzzy measure assigns values $0, \frac{1}{2}$, and 1 and satisfies equation (2.33). The first thing to do is to find an "equivalence class" $\Theta \subseteq [0,1]^{\mathbb{N}}$ for $\frac{1}{2}\mathbb{N}$ s. t. both Θ and $\Theta^C (= [0,1]^{\mathbb{N}} \setminus \Theta)$ are closed under taking intersections and complements. To build such an equivalence class, we define

Definition 2.50 *A **fuzzy quasi filter** $\mathcal{G} \in [0,1]^{\mathbb{N}}$ is given by*

- $f \in \mathcal{G} \Longleftrightarrow f^C \in \mathcal{G}$
- $f, g \in \mathcal{G} \Longrightarrow f \wedge g \in \mathcal{G}$
- $\emptyset \notin \mathcal{G}$

Remark 2.51 *1. As easy as for a fuzzy filter one can show that each fuzzy quasi filter is contained in a maximal fuzzy quasi filter.*

2. $f, g \in \mathcal{G} \Longrightarrow f \vee g \in \mathcal{G}$ since $f \vee g = (f^C \wedge g^C)^C$.

3. No crisp coalition T can be an element of a fuzzy quasi filter as $T \wedge T^C = \emptyset$.

4. One can see quite easily that $\frac{1}{2}\mathbb{N}$ is an element of each maximal fuzzy quasi filter.

Obviously $\overline{\mathcal{G}} = \{\frac{1}{2}\mathbb{N}\}$ is a fuzzy quasi filter. Let Θ be a maximal fuzzy quasi filter for $\overline{\mathcal{G}}$. One can verify that Θ is closed under taking complements and intersections. Now, we want to prove the same statement for $\Theta^C = [0,1]^{\mathbb{N}} \setminus \Theta$. To do this, we have to make some thoughts about the appearance of Θ^C.

Lemma 2.52 *$g \in \Theta^C$ iff there exists either $h_1 \in \Theta$ with $h_1 \vee g = \emptyset$ or $h_2 \in \Theta$ with $h_2 \wedge g = \mathbb{N}$.*

Proof If there exists a $h \in \Theta$ with $h \wedge g = \emptyset$ or $h \vee g = \mathbb{N}$, then, obviously, g cannot be an element of Θ.

To show the other direction, we first consider a $g \in \Theta^C \setminus \underline{\underline{P}}(\mathbb{N})$, i. e. a g which is no crisp coalition. Furthermore, we assume for the contrary that there is no $h \in \Theta$ s. t. $h \wedge g = \emptyset$ or $h \vee g = \mathbb{N}$. Then, $\overline{\Theta} := \{\Theta, g, g^C, g \wedge g^C, g \vee g^C, g \vee$

$\Theta, g^C \vee \Theta, (g \wedge g^C) \vee \Theta, (g \vee g^C) \vee \Theta, g \wedge \Theta, g^C \wedge \Theta, (g \wedge g^C) \wedge \Theta, (g \vee g^C) \wedge \Theta\}$ is a fuzzy quasi filter that contains Θ and g, i. e. we have that $g \in \Theta$ because of the maximality of Θ, and this is a contradiction.

We had to exclude the crisp coalitions since $T \wedge T^C = \emptyset$. Now, we want to close this gap: To do this, we first prove the following remark.

Claim 1: If, for a given $g \in [0,1]^{\mathbb{N}}$, there exists an $h \in \Theta$ s. t. $h \wedge g = T$ $[h \vee g = T]$, $T \in \underline{\underline{P}}(\mathbb{N})$, then there exists also an $\overline{h} \in \Theta$ $[\widetilde{h} \in \Theta)$ s. t. $\overline{h} \wedge g = \emptyset$ $[\widetilde{h} \vee g = \mathbb{N}]$.

To be precise, this $\overline{h}$ is given by $\overline{h} := h \wedge h^C$:

$$(h \wedge h^C)(i) = \begin{cases} 0, & \text{if } i \in T \text{ (since } h^C(i) = 0) \\ 0, & \text{if } i \in T^C \text{ and } h(i) = 0 \\ (h \wedge h^C)(i), & \text{if } i \in T^C \text{ and } h(i) \neq 0. \end{cases}$$

For $i \in T^C$ and $h(i) \neq 0$, $g(i)$ always has to be zero, since otherwise the intersection of h and g cannot be T. Therefore, $g \wedge (h \wedge h^C) = \emptyset$.

Of course, there is no problem in showing the analogue statement for $g \vee h = T$, i. e. in showing the existence of a $\widetilde{h} \in \Theta$ with $\widetilde{h} \vee g = \mathbb{N}$. This completes the proof of Claim 1.

Let a crisp coalition T be given. Since $\frac{1}{2}T \wedge \frac{1}{2}T^C$ is nothing else but the empty set, at most one of these fuzzy coalitions can be contained in Θ.

Claim 2: Either $\frac{1}{2}T$ or $\frac{1}{2}T^C$ is an element of Θ. Suppose, for the contrary, that there exist $h_1, h_2 \in \Theta$ s. t. $\frac{1}{2}T \wedge h_1 = \emptyset = \frac{1}{2}T^C \wedge h_2$ (For a crisp set S, the equation $\frac{1}{2}S \vee h = \mathbb{N}$ implies $h = \mathbb{N}$ i. e. especially $h \notin \Theta$). Observing this fact, we have that $h_1(i) = 0$ for $i \in T$ and $h_2(i) = 0$ for $i \in T^C$, i. e. $h_1 \wedge h_2 = \emptyset$. However, this contradicts the fact that both h_1 and h_2 are elements of Θ.

Now, we are able to prove that, for each crisp coalition T, there exists an $h \in \Theta$ s. t. $h \wedge T = \emptyset$ or $h \vee T = \mathbb{N}$. If $\frac{1}{2}T \in \Theta$ we know that $(\frac{1}{2}T)^C \in \Theta$ and $T \vee (\frac{1}{2}T)^C = \mathbb{N}$. For $\frac{1}{2}T^C \in \Theta$ we have that $T \wedge \frac{1}{2}T^C = \emptyset$.

It remains to show that, for a given $g \in \Theta^C$, there is no pair $h_1, h_2 \in \Theta$ s. t. $g \wedge h_1 = \emptyset$ and $g \vee h_2 = \mathbb{N}$. However, this cannot be valid since $h_1 \wedge h_2^C$ would be equal to the empty set. **q.e.d.**

Lemma 2.53 Θ^C *is a closed subset of* $[0,1]^{\mathbb{N}}$ *w. r. t. taking intersections and complements.*

Proof Of course, Θ^C is closed w. r. t. taking complements since Θ is a fuzzy quasi filter.

To show closure w. r. t. intersection, we consider $f, g \in \Theta^C$. Suppose that $f \wedge g \in \Theta$, i. e. that there is no $h \in \Theta$ s. t. $h \wedge (f \wedge g) = \emptyset$. Because of the trivial inequalities $h \wedge f \geq h \wedge (f \wedge g) \leq h \wedge g$, there exist $h_1, h_2 \in \Theta$ s. t. $f \vee h_1 = \mathbb{N}$ and $g \vee h_2 = \mathbb{N}$ referring to Lemma 2.52. Because of these equations, we have that

$$(h_1 \vee h_2) \vee (f \wedge g) = (f \vee h_1 \vee h_2) \wedge (g \vee h_1 \vee h_2) = \mathbb{N} \wedge \mathbb{N} = \mathbb{N}.$$

This is a contradiction, and, hence, our lemma is proven. **q.e.d.**

To define a measure, we first have to look at what happens if we take the intersection or union of an element of Θ with one of Θ^C.

Lemma 2.54 *Let* $f \in \Theta$ *and* $g \in \Theta^C$ *be given. Then the following is valid:*

$$f \wedge g \in \begin{cases} \Theta \\ \Theta^C \end{cases} \iff f \vee g \in \begin{cases} \Theta^C \\ \Theta \end{cases}$$

Proof Since g is an element of Θ^C, there exists either an $h_1 \in \Theta$ with $g \wedge h_1 = \emptyset$ or an $h_2 \in \Theta$ with $g \vee h_2 = \mathbb{N}$ (cf. Lemma 2.52). Let us consider the case $g \wedge h = \emptyset$ for $h \in \Theta$. Of course, $(f \wedge g) \wedge h = \emptyset$ is true, i. e. $(f \wedge g) \in \Theta^C$. As $(f \vee g) \geq f$ is valid, the assumption $(f \vee g) \wedge \widehat{h} = \emptyset$ for a $\widehat{h} \in \Theta$ implies especially $f \wedge \widehat{h} = \emptyset$, i. e. $f \in \Theta^C$. This is a contradiction.

Thus, it remains to prove that there exists no $\widetilde{h} \in \Theta$ with $(f \vee g) \vee \widetilde{h} = \mathbb{N}$ to show $(f \vee g) \in \Theta$. However, Lemma 2.52 states that both $g \vee (f \vee \widetilde{h}) = \mathbb{N}$ and

$g \wedge h = \emptyset$ is not possible. (Here we have used closure of Θ w. r. t. intersection and complement.) Therefore $f \vee g$ is an element of Θ.

Correspondingly, one can show that for any $h \in \Theta$ such that $g \vee h = \mathbb{N}$ we have that $(f \vee g) \in \Theta^C$ and $(f \wedge g) \in \Theta$. **q.e.d.**

Definition 2.55 *A* ***strong fuzzy filter*** $\mathcal{F} \subseteq \Theta^C$ *is given by*

- $f \in \mathcal{F},\ f \leq g \in \Theta^C \Longrightarrow g \in \mathcal{F}$
- $f, g \in \mathcal{F} \Longrightarrow f \wedge g \in \mathcal{F}$
- $\emptyset \notin \mathcal{F}$
- *For each* $f \in \Theta^C$, *the following is true:* $f \in \mathcal{F} \Longleftrightarrow f^C \notin \mathcal{F}$.

Now, we define the subset $\mathcal{U}$ of Θ^C by

$$\mathcal{U} := \{f \in \Theta^C |\, \exists h \in \Theta \text{ s. t. } f \vee h = \mathbb{N}\}.$$

$\mathcal{U}$ is a strong ultra filter on Θ^C, i. e. it satisfies the following properties:

- $f \in \mathcal{U}, f \leq g \in \Theta^C \Longrightarrow g \in \mathcal{U}$

  ($f \vee h = \mathbb{N}$ for a $h \in \Theta \Longrightarrow g \vee h = \mathbb{N}$)
- $f, g \in \mathcal{U} \Longrightarrow f \wedge g \in \mathcal{U}$

  ($f \vee h_1 = \mathbb{N}, g \vee h_2 = \mathbb{N},\ h_1, h_2 \in \Theta \Longrightarrow (f \wedge g) \vee (h_1 \vee h_2) = \mathbb{N}$)
- $\emptyset \notin U$ ($\emptyset \vee h = \mathbb{N} \Longrightarrow h = \mathbb{N} \notin \Theta$)
- $f \in \mathcal{U} \Longrightarrow f^C \notin \mathcal{U}$

  ($\mathbb{N} = f \vee h = (f^C \wedge h^C)^C, h \in \Theta \Longrightarrow (f^C \wedge h^C) = \emptyset \Longrightarrow$ There does not exist an $\overline{h} \in \Theta$ s. t. $f^C \vee \overline{h} = \mathbb{N}$)

- $\mathcal{U}$ is maximal, i.e. for each $f \in \Theta^C$, either f or f^C is an element of $\mathcal{U}$ ($f \notin \mathcal{U} \Longrightarrow f \wedge h = \emptyset$ for a $h \in \Theta \Longrightarrow f^C \vee h^C = \mathbb{N} \Longrightarrow f^C \in \mathcal{U}$)

Now, we are well prepared to analyse the different possibilities for the intersection and the union of fuzzy coalitions again:

- $f, g \in \Theta(\in \mathcal{U}, \in \Theta^C \setminus \mathcal{U}) \Longrightarrow f \wedge g, f \vee g \in \Theta(\in \mathcal{U}, \in \Theta^C \setminus \mathcal{U})$
- $f \in \Theta, g \in \mathcal{U} \Longrightarrow f \wedge g \in \Theta, f \vee g \in \mathcal{U}$
- $f \in \Theta, g \in \Theta^C \setminus \mathcal{U} \Longrightarrow f \wedge g \in \Theta^c \setminus \mathcal{U}, f \vee g \in \Theta$
- $f \in \mathcal{U}, g \in \Theta^C \setminus \mathcal{U} \Longrightarrow f \wedge g \in \Theta^C \setminus \mathcal{U}, f \vee g \in \mathcal{U}$

Now, the measure $\mathcal{P}_s(\bullet, \mathcal{N})$ can be defined on $[0,1]^{\mathbb{N}}$ as follows:

$$\mathcal{P}_s(f, \mathbb{N}) = \begin{cases} 1, & \text{if } f \in \mathcal{U} \\ \frac{1}{2}, & \text{if } f \in \Theta \\ 0, & \text{if } f \in [0,1]^{\mathbb{N}} \setminus (\Theta \cup \mathcal{U}). \end{cases} \tag{2.34}$$

To obtain a kernel on $[0,1]^{\mathbb{N}} \times \mathcal{D}$, we define $\mathcal{P}(f, K), K \subset \mathbb{N}$, in a similar way (cf. the construction of $\mathcal{P}_w$).

2.5.3 Comparison with a Model of Gilboa and Schmeidler

It is an open question whether or not the class of coalitional functions v for which a corresponding measure μ_v exists is a Banach space. If we allow for some changes in our model, an answer can be found in the framework of Gilboa and Schmeidler [11]. They consider all games $v \in \mathcal{V}$ on an arbitrary player space (Ω, Σ), where Σ is an algebra of coalitions of Ω. Σ' is defined by $\Sigma' := \Sigma \setminus \{\emptyset\}$. For each $T \in \Sigma'$, the set $\widetilde{T} \subseteq \Sigma'$ is given by

$$\widetilde{T} := \{S \in \Sigma' |\, S \subseteq T\}$$

and

$$\Theta := \{\widetilde{T} |\, T \in \Sigma'\} \subseteq \underline{\underline{P}}(\Sigma').$$

They use Ψ and $\widetilde{\Psi}$ to denote the algebra and $\sigma-$algebra generated by Θ, respectively.

Theorem 2.56 *[11, Theorem A] For every $v \in \mathcal{V}$ there exists a unique signed finitely additive measure η_v on (Σ', Ψ) s. t.*

$$v = \int_{\Sigma'} e^T \, d\eta_v(T). \tag{2.35}$$

In case that Σ is finite, the composition norm is defined as

$$\|v\|_{comp} := \sum_{T \in \Sigma'} |c_T(v)|,$$

where the reader is referred to formula (A.1) for a definition of $c_T(v)$. For the general case, one defines

$$\|v\|_{comp} := \sup \left\{ \|v|_{\Sigma_0}\|_{comp} \middle| \Sigma_0 \text{ is a finite subalgebra of } \Sigma \right\}.$$

There are two interesting subspaces of $\mathcal{V}$:

$$\begin{aligned} \mathcal{V}^b &:= \left\{ v \in \mathcal{V} \middle| \|v\|_{comp} < \infty \right\} \\ \mathcal{V}^\sigma &:= \left\{ v \in \mathcal{V}^b \middle| \eta_v \text{ is a } \sigma\text{-additive signed measure} \right\}. \end{aligned}$$

Theorem 2.57 *[11, Theorem B] $\mathcal{V}^b$ and $\mathcal{V}^\sigma$ are Banach spaces w. r. t. $\| \bullet \|_{comp}$. Furthermore,*

$$\mathcal{V}^b := \{ v \in \mathcal{V} \big| \, v = u - w, u, w \textit{ are totally monotone} \}.$$

Theorem 2.58 *[11, Theorem D] If Ω is countable, the mapping $v \mapsto \eta_v$ is a bijection from*

$$\{ v \in \mathcal{V} | \, v \textit{ is totally monotone and continuous from above} \}$$

to

$$\{ \eta | \, \eta \textit{ is a measure on } \Psi \}.$$

Theorem 2.59 *[11, Theorem E] Let $v \in \mathcal{V}^\sigma$ and a bounded measurable function $f : \Omega \to \mathbb{R}$ be given. Then,*

$$\int_\Omega f\, dv = \int_{\Sigma'} [\inf_{\omega \in T} f(\omega)]\, d\eta_v(T).$$

By defining $\Omega = \mathbb{N}$ and $\Sigma = \underline{\underline{P}}(\mathbb{N})$, one can get some nice results. In this case, Σ' is nothing else but $\mathcal{D}$, i. e. the space of all non-empty subsets of $\mathbb{N}$. Now, we restrict μ (i. e. the measure defined on page 76) to (Σ', Ψ). First of all, one should remark that each game with finite composition norm is of bounded variation, i. e. $\mathcal{V}^b \subseteq BV$ (cf. [11, page 205]). Let $\mathcal{V}^\mu$ denote the set of coalitional functions v for which μ_v exists. We will show next that $\mathcal{V}^\sigma \subseteq \mathcal{V}^\mu$ and that μ_v coincides with η_v on $\mathcal{V}^\sigma$. Therefore we have found a Banach space w. r. t. the composition norm in which μ_v exists for all games.

Theorem 2.60 *For each $v \in \mathcal{V}^\sigma$, there exists a μ_v, and this measure coincides with η_v.*

Proof Let a v^n with a finite carrier T^n be given. As we have that

$$v^n = \sum_{S \subseteq T^n} e^S \mu_{v^n}(S) = \int_{\Sigma'} e^S\, d\mu_{v^n}(S),$$

we know that $\eta_{v^n} = \mu_{v^n}$ since Theorem 2.56 states that η_{v^n} is uniquely defined by the equation above. In the same way, one can show that $\eta_{\overline{v}^n} = \mu_{\overline{v}^n}$, where $\overline{v}$ is defined on $(\overline{\mathbb{N}}, \underline{\underline{P}}(\overline{\mathbb{N}}))$ by $\overline{v}(S) = v(S \setminus \{\infty\})$ (cf. page 74). Proposition 2.32 provides that, for each $\overline{v} \in \mathcal{V}^\sigma$, there exists a sequence $(\overline{v}^n)_n$ converging weakly to $\overline{v}$, where each $\overline{v}^n$ has a finite carrier. Hence, we have that, for each continuous $\overline{f} : \overline{\mathbb{N}} \to \overline{\mathbb{R}}$,

$$\int \overline{\bigwedge \overline{f}}\, d\mu_{\overline{v}^n} = \int \overline{\bigwedge \overline{f}}\, d\eta_{\overline{v}^n} = \int \overline{f}\, d\overline{v}^n \to \int \overline{f}\, d\overline{v} = \int \overline{\bigwedge \overline{f}}\, d\eta_{\overline{v}}.$$

Now, we can again use Lemma 2.35 to see that $(\mu_{\overline{v}^n})$ converges weakly to $\eta_{\overline{v}}$. Since we have shown the uniqueness of the weak accumulation point $\mu_{\overline{v}}$ of $(\mu_{\overline{v}^n})$ before, we have that $\mu_{\overline{v}} = \eta_{\overline{v}}$.

For a fixed $S \subseteq \mathbb{N}$, we obtain

$$\begin{aligned}
\int e^T(S \setminus \{\infty\})\, d\mu_v &= \mu_v(T \,|\, T \subseteq S \setminus \{\infty\}) \\
&= \mu_{\overline{v}}(\overline{T} \,|\, \overline{T} \subseteq S \cup \{\infty\}) \\
&= \int e^{\overline{T}}(S \cup \{\infty\})\, d\mu_{\overline{v}} \\
&= \overline{v}(S \cup \{\infty\}) = \overline{v}(S \setminus \{\infty\}) \\
&= \int e^{\overline{T}}(S \setminus \{\infty\})\, d\mu_{\overline{v}}.
\end{aligned}$$

Hence, we have shown that

$$\int_{\mathcal{D}} e^T(S \setminus \{\infty\})\, d\mu_v = \int_{\overline{\mathcal{D}}} e^{\overline{T}}(S)\, d\mu_{\overline{v}}$$

is true. This fact implies the following for each $S \subseteq \overline{\mathbb{N}}$:

$$\begin{aligned}
\int_{\mathcal{D}} e^T(S \setminus \{\infty\})\, d\eta_v &= v(S \setminus \{\infty\}) = \overline{v}(S) \\
&= \int_{\overline{\mathcal{D}}} e^{\overline{T}}(S)\, d\eta_{\overline{v}} \\
&= \int_{\overline{\mathcal{D}}} e^{\overline{T}}(S)\, d\mu_{\overline{v}} \\
&= \int_{\mathcal{D}} e^T(S \setminus \{\infty\})\, d\mu_v.
\end{aligned}$$

All in all, we have proven that $\eta_v = \mu_v$ (cf. Theorem 2.56). **q.e.d.**

With the help of this result, we get a nice value on $\mathcal{V}^\sigma$ that is an extension of the Shapley value for games with finitely many players. For each $v \in \mathcal{V}^\sigma \cap$ *linhull*$\{\mathcal{V}^{\searrow} \cup \mathcal{V}^{\nearrow}\}$, there exists a sequence v^n s.t. each v^n has a finite carrier and $\lim_{n\to\infty} v^{nC}(f) = v^C(f)$ is valid for all fuzzy coalitions f. The corresponding sequence (μ^n) has a weak accumulation point μ. This measure μ is unique in a twofold sense: it is the only weak accumulation point of (μ^n), and it is unique in the sense of Theorem 2.56. However, restricting μ to $(\mathcal{D}, \Psi)$ has the big disadvantage that we are not allowed to use all subcoalitions of $\mathbb{N}$, i. e. that we have to care for the measurability of our kernel $\mathcal{P}$ again when considering $\varphi^{\mathcal{P}}_{Ros}$.

One should remark that there are games s. t. the corresponding μ is not $\sigma-$additive. There are also $v \in \mathcal{V}$ s. t. no sequence $(\mu_{v^n})_n$ has a weak accumulation point. Nevertheless, Theorem 2.56 provides a finitely additive η_v s.t. one could think of building a value as in Definition 2.43.

Chapter 3

NTU Games with Fuzzy Coalitions

Throughout the previous chapter, we have considered fuzzy games with transferable utility (TU games). Since games with non-transferable utility (NTU) present a much wider class of games as those with TU character, this chapter is devoted to the question of how one can "fuzzify" NTU games. Mostly, we are interested in obtaining a Choquet extension of an NTU game as this extension seems to be a quite sensible one for the TU case.

We try to preserve as many properties of the Choquet integral as possible. However, there exist NTU games s. t. the corresponding Choquet extension cannot satisfy all properties of the Choquet integral. For example, comonotone additivity and continuity are mutually exclusive for some games in the NTU case.

In Section 2, we will give an intuitive formula for a Choquet game (N, V^C), where the correspondence V^C is not continuous. For this reason, we will present in Section 3 a possibility to define an extension $(N, \overline{V}^C)$ that satisfies even this property. Finally, we will discuss the core, one of the most important solution concepts.

3.1 Fuzzy NTU Games: Definition and Motivation

In this chapter, we would like to examine a game in which no possibility of utility transfer exists, i. e. the case of something like a fuzzy NTU game. Dhingra and Rao give in [9] an example which can be seen as an NTU game with a fuzzy part (cf. Examples 1.3 and 3.4). There, we have k crisp players and one fuzzy player. However, in this chapter we would like to consider the more general class of games where everyone can make fuzzy decisions.

First of all, we will give the definition of a (crisp) NTU game (see for example Rosenmüller [23, Chapter 4, Section 1]). After this, we will make some thoughts about how we can extend this concept to fuzzy coalitions.

Let $n \in \mathbb{N}$ be given, and let S be a subset of $\{1, \ldots, n\}$. Define $\mathbb{R}^{\mathbf{n}}_{\mathbf{S}}$ as the subspace of $\mathbb{R}^n$ spanned by the unit vectors $(e^i)_{i \in S}$. For $x \in \mathbb{R}^n$, let $\mathbf{x^S} \in \mathbb{R}^n_S$ be the vector defined by

$$x_i^S = \begin{cases} x_i, & i \in S \\ 0, & i \in S^C. \end{cases}$$

A set $A \subseteq \mathbb{R}^n$ is called **comprehensive**, if for all $x \in A$ and $y \in \mathbb{R}^n$ the inequality $y \leq x$ implies $y \in A$. $A \subseteq \mathbb{R}^n_S$ is called **S−comprehensive**, if for all $x \in A, y \in \mathbb{R}^n_S, y^S \leq x^S$ implies $y \in A$.

Definition 3.1 *[23, Chapter 4, Definition 1.3] An* **n-person cooperative game without sidepayments (NTU)** *is a pair (N, V), where $N := \{1, \ldots, n\}$ is the set of players, and $V : \underline{\underline{P}}(N) \longrightarrow \underline{\underline{P}}(\mathbb{R}^n)$ is a function that associates with every $S \subseteq N$ a non-empty set $V(S) \subseteq \mathbb{R}^n_S$ such that*

1. *$V(S)$ is S-comprehensive,*

2. *$V(S)$ is closed, and*

3. *for every $x^S \in \mathbb{R}^n_S$, the set $V(S) \cap (x^S + \mathbb{R}^n_{S+})$ is bounded.*

This definition implies, in particular, that $V(\emptyset) = 0$ (here, 0 is the n-dimensional null-vector). The following remark explains how a TU game can be imbedded into the NTU-context:

Remark 3.2 *Let a TU game (N, v) be given, i. e. $v : \underline{\underline{P}}(N) \longrightarrow \mathbb{R}$, $v(\emptyset) = 0$. Then, (N, V_v) given by*

$$V_v(S) := \left\{ x \in \mathbb{R}^n_S \middle| \sum_{i \in S} x_i \leq v(S) \right\}$$

is an NTU game. The latter formula can be found for example in [23, Chapter 4, Example 2.17].

Mareš deals in [15] with NTU games and fuzziness. He suggests the following: A fuzzy NTU game is a pair $(N, \overline{V})$, where $\overline{V} : \underline{\underline{P}}(N) \longrightarrow [0, 1]^{\mathbb{R}^n}$ is a function mapping the set of crisp coalitions into the class of fuzzy subsets of $\mathbb{R}^n$. Here, we have vagueness in the outcome of each coalition, i. e. each crisp coalition is assigned a fuzzy set on $\mathbb{R}^n$. As in the crisp case, $\overline{V}$ is considered to have several properties which we do not mention here in detail. However, we would like to give a short introduction to this kind of a fuzzy game: The function $\overline{V}$ is built in such a way that for every $\emptyset \neq S \in \underline{\underline{P}}(N)$ some payoffs can be taken for granted ($\exists x \in \mathbb{R}^n$ s. t. $\overline{V}(S)(x) = 1$), and some others (more profitable for players) are only possible with various degrees of possibility. Lexicographically larger points are less probable, i. e. $x^S \leq y^S$ implies $\overline{V}(S)(x) \geq \overline{V}(S)(y)$, for all $S \in \underline{\underline{P}}(N)$. Furthermore, there always exists a $y \in \mathbb{R}^n$ with $\overline{V}(S)(y) = 0$.

This approach does not fit to Butnariu's and Klement's definition of a TU game. There, the set function is a mapping on the fuzzy coalitions to the real numbers, i. e. we have different degrees of membership in a coalition and exact outcomes for each fuzzy coalition. Hence, a corresponding fuzzy NTU game should be of the form (N, V^F) where V^F is a mapping on the fuzzy coalitions of N to the power set of $\mathbb{R}^N$. In the following definition, we convert an NTU game to the fuzzy context in a quite intuitive way. Later on, we will show that there are other (in some sense more reasonable) possibilities for the definition of a fuzzy

NTU game. Remember that for a given $f \in [0,1]^N$ the carrier $C(f)$ is the crisp subset of N that consists exactly of those members which have a positive degree of membership in f, i. e. $C(f) := \{i \in N | \, f(i) > 0\}$.

Definition 3.3 *An* **n-person cooperative fuzzy game without sidepayments (NTU)** *is a pair* (N, V^F) *where* $N := \{1, \dots, n\}$ *is again the set of players and* $V^F : [0,1]^N \longrightarrow \underline{\underline{P}}(\mathbb{R}^n)$ *is a function that associates with every fuzzy coalition in* N *a subset of* $\mathbb{R}^n$*.* V^F *satisfies the following properties:*

1. $V^F(f) \subseteq \mathbb{R}^n_{C(f)}$.
2. *For* $f \neq \emptyset$*, the set* $V^F(f)$ *is non-empty and closed.*
3. $V^F(f)$ *is* $C(f)$*-comprehensive.*
4. *For every* $x^{C(f)} \in \mathbb{R}^n_{C(f)}$*, the set* $V^F(f) \cap (x^{C(f)} + \mathbb{R}^n_{C(f)+})$ *is bounded.*

Let us go back to Example 1.3, where Dhingra and Rao [9] describe a multiple objective optimization problem with fuzziness. We would like to have a look at how it is possible to write down this problem in such a way that it fits to our definition of a fuzzy NTU game.

Example 3.4 *Multiple Objective Design Optimization II*

In this example, we have $k+1$ players. The first k players correspond to an objective function and can, for this reason, only make crisp decisions. The last player corresponds to the combined fuzzy objectives and fuzzy constraints. She can have any degree of membership in a fuzzy coalition between zero and one. This means that only a certain subclass of $[0,1]^N$, $N = \{1, \dots, k+1\}$, provides the feasible set of fuzzy coalitions in this example. Obviously, the subclass described above is a fuzzy tribe ([6, Definition 2.5]). Let us assume that U given as in Example 1.3 is bounded from above. Then, $(N, \widetilde{U})$ provides a fuzzy NTU game, where $\widetilde{U} : \{0,1\}^k \times [0,1] \to \underline{\underline{P}}(\mathbb{R}^{k+1})$ is given by

$$\begin{aligned}\widetilde{U}(h) \;:=\; & CCH\Big\{(y_1(x), \dots, y_k(x), \lambda) \,|\, z_i(x) \leq 0, i = 1, \dots, m, \\ & \qquad 0 \leq \lambda \leq \min\{\mu^f(x), \mu^g(x), h(k+1)\}\Big\} \cap \mathbb{R}^{k+1}_{C(h)}.\end{aligned}$$

Here, *CCHA* denotes the convex comprehensive hull of the set A.

Now, we are interested in the question how a fuzzy TU game (N, v^F) can be embedded into an NTU game. This can be done by defining

$$\widehat{V}^F_{v^F}(f) := \left\{ x \in \mathbb{R}^n_{C(f)} \middle| \sum_{i \in C(f)} x_i \leq v^F(f) \right\}. \tag{3.1}$$

This is a similar procedure to the one used for crisp games, and one can show quite easily that we obtain a fuzzy NTU game. All players in the carrier of f can share the total gain $v^F(f)$ equally, i. e. the respective degrees of membership determine the overall outcome but not the share of the single players.

Formula (3.1) is one possibility of embedding a fuzzy TU game into the NTU context. However, it is not the only one. One could take care a little bit more for the various fractions of the players. This can be done by defining

$$\widetilde{V}^F_{v^F}(f) := \left\{ x \in \mathbb{R}^n_{C(f)} \middle| \sum_{i \in C(f)} \frac{1}{f(i)} x_i \leq \frac{1}{f(1)} v^F(f) \right\} \tag{3.2}$$

for $f(1) \geq f(i)$, $i \geq 2$. As in the previous version, each player in the carrier of f takes part in the allocation of the total gain. However, the players are evaluated by their degree of membership. If, for example, $f(2)$ is much smaller than $f(1)$, player 2 is in a weaker position than player 1. Figure 3.1 demonstrates the appearance of this approach for $n = 2$ and $f(1) = 2f(2)$.

An interesting question is whether $\widehat{V}^F_{v^F}$ or $\widetilde{V}^F_{v^F}$ preserve continuity. For this reason, we define continuity for correspondences (cf. Hildenbrand and Kirman [12, Mathematical Appendix III]):

Let a set-valued function $\gamma : S \to T, S \subseteq \mathbb{R}^m, T \subseteq \mathbb{R}^n$ be given. Then, γ is called **upper hemi-continuous (u. h. c.)** at the point $\overline{s} \in S$ if

$$s^l \to \overline{s}, t^l \in \gamma(s^l), t^l \to \overline{t} \quad \Longrightarrow \quad \overline{t} \in \gamma(\overline{s}).$$

The correspondence γ is **lower hemi-continuous (l. h. c.)** at the point $\overline{s} \in S$ if

$$s^l \to \overline{s}, \overline{t} \in \gamma(\overline{s}) \quad \Longrightarrow \quad \text{There exists a sequence } (t^l)_l \text{ in } T \text{ s. t. } t^l \in \gamma(s^l) \text{ and } t^l \to \overline{t}.$$

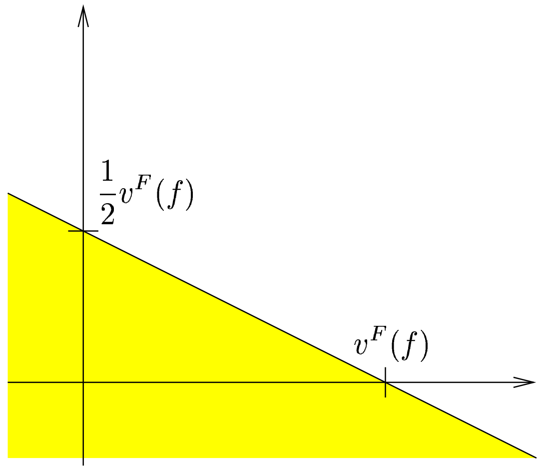

Figure 3.1: $\widetilde{V}^F_{v^F}(f)$ for $f(1) = 2f(2)$

If γ is both u. h. c. and l. h. c., it is called **continuous**.

It is easy to see that neither $\widehat{V}^F_{v^F}$ nor $\widetilde{V}^F_{v^F}$ can be u. h. c.. There are no problems for sequences (f_n) with $C(f_n) \subseteq C(f)$. However, if $C(f) \neq N$, one can consider (f_n) with $C(f_n) = N$ and $f_n \to f$ w. r. t. the maximum norm. As $C(f)$ has no more than $C(f_n) - 1$ players, and since we made the requirements that $V^F(f)$ is $C(f)$-comprehensive and that $V^F(f) \subseteq \mathbb{R}^n_{C(f)}$, we "lose" $(n - C(f))$ dimensions. This means nothing else but that $\widehat{V}^F_{v^F}$ and $\widetilde{V}^F_{v^F}$ cannot be continuous for fuzzy coalitions with non-full carrier.

To repair this gap, one has to modify the definition of a fuzzy NTU game a little bit. Up to now, V^F had to be a subset of $\mathbb{R}^n_{C(f)}$. For the following approach, we will allow negative numbers for players not being in $C(f)$. If a fuzzy coalition comes to a cooperation, and if a player has a degree of membership of 0, the best result he can obtain is 0, but he is free to throw away as much as he wants to. To be precise we replace the first requirement of Definition 3.3, i. e. $V^F(f) \subseteq \mathbb{R}^n_{C(f)}$, with the requirement $V^F(f) \subseteq \mathbb{R}^n_{C(f)} + \mathbb{R}^n_{C(f)^C-} = \{x \in \mathbb{R}^n \,|\, x_i \leq 0 \text{ for } i \notin C(f)\}$. Moreover, $V^F(f)$ is assumed to be comprehensive for every $f \in [0,1]^N$.

The following lemma is concerned about building a $\overline{V}^F_{v^F}$ for two players that preserves continuity. For further thoughts concerning continuity, the reader is referred to Section 3.

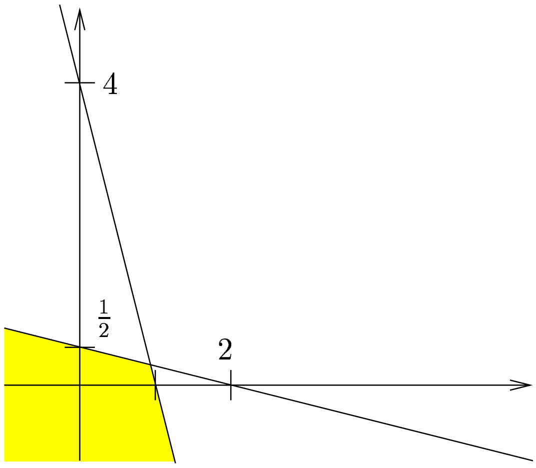

Figure 3.2: The set $\overline{V}^F_{v^F}(f)$ for $f(1) = \frac{1}{2}, f(2) = \frac{1}{4}$

Lemma 3.5 *Let $n = 2$ and v^F be continuous w. r. t. the maximum norm. Then, $\overline{V}^F_{v^F}$ as defined in the follows is also continuous:*

$$\overline{V}^F_{v^F}(f) = \begin{cases} \{x \in \mathbb{R}^2 \,|\, x_i + \frac{1}{f(3-i)}x_{3-i} \leq \frac{1}{f(i)}v^F(f) & \text{if } C(f) = \{1,2\} \text{ and} \\ \quad \text{and } x_i + f(3-i)x_{3-i} \leq v^F(f)\} & \quad f(i) \geq f(3-i) \\ \{x \in \mathbb{R}^2_i \times \mathbb{R}^2_{(3-i)-} \,|\, x_i \leq v^F(f)\} & \text{if } C(f) = \{i\} \\ \mathbb{R}^2_- & \text{if } f = \emptyset \end{cases}$$

In Figure 3.2, one can see how $\overline{V}^F_{v^F}(f)$ looks like for $f(1) = \frac{1}{2}$, $f(2) = \frac{1}{4}$ and $v^F(f) = 1$.

Proof (of the lemma)

First of all, we have to verify that $\overline{V}^F_{v^F}$ is well defined. However, as one can easily see the inequalities for $f(1) \geq f(2)$ coincide with those for $f(2) \geq f(1)$ in the special case where $f(1) = f(2) > 0$.

For a sequence $(f_l)_l$ of fuzzy coalitions converging to f w. r. t. the maximum norm, the case $C(f) \subseteq C(f_l)$ is the only interesting one, since $f(i) > 0$ implies $f_l(i) > 0$ for l sufficiently large.

Now, we are well prepared to show continuity. At the beginning we want to show lower hemi-continuity, i. e.

$$f_l \to f, x \in \overline{V}^F_{v^F}(f) \Longrightarrow \exists (x^l)_l \text{ s. t. } x^l \in \overline{V}^F_{v^F}(f_l), x^l \to x.$$

- $C(f) = \{1, 2\}$

  We can assume w. l. o. g. $f_l(1) \geq f_l(2)$. Then, we define $x_1^l := x_1$ for all l. Since we require x^l to be an element of $\overline{V}^F_{v^F}(f_l)$, we have the following inequality: $x_2^l \leq \min(f_l(2)[\frac{1}{f_l(1)} v^F(f_l) - x_1], \frac{1}{f(2)}[v(f_l) - x_1])$. One can easily see that, for each $l \in \mathbb{N}$, there exists a sequence $(\varepsilon_l)_l$ with $\varepsilon_l > 0$ for all l and $\lim_l \varepsilon_l = 0$ s. t. $x_2^l := x_2 - \varepsilon_l$ satisfies this inequality. Hence, there exists a sequence as required.

- $C(f) = \{1\}$

  If $C(f_l) = \{1\}$, the proof is trivial.

  Consider now the case $C(f_l) = \{1, 2\}$. We have $f_l(1) > f_l(2) \to 0$. For a sequence $(\varepsilon_k)_k, \varepsilon_k > 0$ for all k, $\lim_k \varepsilon_k = 0$, there exists, since v^F is continuous, a sequence $(l_k)_k, l_k = l(\varepsilon_k) \in \mathbb{N}, l_k < l_{k+1}$, s. t. $|v^F(f_{l_k}) - v^F(f)| < \varepsilon_k$. Now, we build a sequence $(x^{l_k})_k, x^{l_k} \in \overline{V}^F_{v^F}(f_{l_k})$. For $x_1^{l_k} := x_1 - \varepsilon_k$ we have to ensure that $x_2^{l_k} \leq \min(f_{l_k}(2)[\frac{1}{f_{l_k}(1)} v^F(f_{l_k}) - x_1 + \varepsilon_k], \frac{1}{f_{l_k}(2)}[v^F(f_{l_k}) - x_1 + \varepsilon_k])$. Since $x \in \overline{V}^F_{v^F}(f)$ is true, we know that $v^F(f_{l_k}) - x_1 + \varepsilon_k$ is non-negative. Hence, we can set $x_2^{l_k} := \min(f_{l_k}(2)[\frac{1}{f_{l_k}(1)} v^F(f_{l_k}) - x_1 + \varepsilon_k], 0)$. This sequence $(x^{l_k})_k$ satisfies the required properties.

- $C(f) = \emptyset$

  The case $C(f_l) \neq \{1, 2\}$ is trivial. If f_l has a full carrier, one can use a similar proof as before.

It remains to show upper hemi-continuity, i. e.

$$f_l \to f, x^l \in \overline{V}^F_{v^F}(f_l), x^l \to x \Longrightarrow x \in \overline{V}^F_{v^F}(f).$$

- $C(f) = \{1, 2\}$

  We may assume that $f(1) \geq f(2)$ and $f_l(1) \geq f_l(2)$ for all l. Because of the convergence of (x^l) to x there exists for each $\varepsilon > 0$ a $L \in \mathbb{N}$ s. t. both

$|\frac{1}{f_l(1)}v^F(f_l) - \frac{1}{f(1)}v^F(f)|$, $|v^F(f_l) - v^F(f)|$, $|x_1 - x_1^l|$, $|f_l(2)x_2^l - f(2)x_2|$, and $|\frac{1}{f_l(2)}x_2^l - \frac{1}{f(2)}x_2|$ are smaller than ε for all $l \geq L$. With these inequalities in mind one sees immediately $x_1 + \frac{1}{f(2)}x_2 \leq \frac{1}{f(1)}v^F(f) + 3\varepsilon$ and $x_1 + f(2)x_2 \leq v^F(f) + 3\varepsilon$.

- $C(f) = \{1\}$

  For $C(f_l) = \{1\}$, the statement is obvious. Therefore, we concentrate on the case $C(f_l) = \{1, 2\}$, i.e. on $f_l(1) > f_l(2) \to 0$. Let us assume that $x_2 > 0$. Then, $x_2^l > 0$ for l large enough, and $\frac{1}{f_l(2)}x_2^l \to \infty$. However, this contradicts $x_1^l + \frac{1}{f_l(2)}x_2^l \leq \frac{1}{f_l(1)}v^F(f_l)$.

  It remains to show that $x_1^l \leq \min(\frac{1}{f_l(1)}v^F(f_l) - \frac{1}{f_l(2)}x_2^l, v^F(f_l) - f_l(2)x_2^l)$ implies $x_1 \leq v^F(f)$. Since the second term in the inequality converges to $v^F(f)$, we are done.

- $C(f) = \emptyset$

  Again, $C(f_l) = \{1, 2\}$ is the only interesting case. Let $f_l(1)$ be at least as great as $f_l(2)$. Then $x_2^l \leq \frac{f_l(2)}{f_l(1)}v^F(f_l) - f_l(2)x_1^l$ implies $x_2 \leq 0$, since $\frac{f_l(2)}{f_l(1)} \leq 1$ and $v^F(f_l) \to 0 \leftarrow f_l(2)$. Together with $x_1^l + f_l(2)x_2^l \leq v^F(f_l)$, we have that $x_1 \leq 0$. This completes the proof. **q.e.d.**

3.2 The Choquet Extension

Now, we would like to examine the same question as in the case of TU games: How can one extend a given crisp NTU game to the "world of fuzziness" in a canonical way. To give an answer, we will concentrate again on the Choquet extension because of several nice properties (see Chapter 1 and Appendix B). First of all, we will give an explicit definition. As we would like to consider only NTU games with finitely many players, we recall the summation formula of the Choquet integral for the TU case, which is given by

$$v^C(f) = \sum_{i=1}^{n} v(S_i)[f(i) - f(i+1)],$$

where $f(1) \geq \ldots \geq f(n)$, $f(n+1) = 0$, and $S_i := \{1, \ldots, i\}$. To give a similar formula for NTU games, we have to make the following settings: For a set $A \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the new set αA is defined as

$$\alpha A := \{x \in \mathbb{R}^n \,|\, \exists a \in A \text{ s.t. } x = \alpha a\}.$$

For $A, B \in \mathbb{R}^n$, we denote by $A + B$ the direct sum of these sets, i. e. $A + B$ is nothing else but the set of vectors which can be written as the sum of a vector from A and a vector from B:

$$A + B := \{x \in \mathbb{R}^n \,|\, \exists a \in A, b \in B \text{ s.t. } x = a + b\}.$$

Now, we make a first attempt to explain the Choquet extension $(N,\, V^C)$ of a given NTU game $(N,\, V)$. This is done for a fuzzy coalition f with $f(1) \geq \ldots \geq f(n)$ by means of the formula

$$V^C(f) = \sum_{i=1}^{n} V(S_i)[f(i) - f(i+1)]. \tag{3.3}$$

We call a NTU game $(N,\, V)$ **monotone** if $V(S) \subset V(T)$ is valid for all $S \subset T$. In particular, we have that $0 \in V(S)$ for all coalitions S. (N, V) is called **convex-valued** if $V(S)$ is convex for every $S \in \underline{\underline{P}}(N)$.

Theorem 3.6 *Let a convex-valued, monotone NTU game (N, V) be given. Then, the corresponding Choquet extension $(N,\, V^C)$ satisfies the following properties for each $f \in [0,1]^N$:*

1. $V^C(f) \subseteq \mathbb{R}^n_{C(f)}$,
2. $V^C(f)$ *is non-empty,*
3. $V^C(f)$ *is $C(f)-$comprehensive,*
4. $V^C(f) \cap (x^{C(f)} + \mathbb{R}^n_{C(f)+})$ *is bounded for every $x \in \mathbb{R}^n$.*

Proof

- $V^C(f) \subseteq \mathbb{R}^n_{C(f)}$:

  Let the carrier of f be given by $C(f) = \{1, \ldots, m\}$, $m \leq n$ (we assume again that f is lexicographically ordered). Then, we have that $V^C(f) = \sum_{i=1}^m V(S_i)[f(i) - f(i+1)]$. As $V(S_i) \subseteq \mathbb{R}^n_{S_i}$ is valid for every i, we are done.

- $V^C(f)$ is non-empty:

  For $f = \emptyset$, there is nothing to prove since $V^C(\emptyset) = 0$. For $f \neq \emptyset$, there exists at least one $i \in \{1, \ldots, n\}$ s. t. $f(i) > f(i+1)$ when setting $f(n+1) = 0$. Hence, we have that $\emptyset \neq V(S_i)[f(i) - f(i+1)]$, and, as a direct consequence, $V^C(f) \neq \emptyset$.

- $V^C(f)$ is $C(f)$-comprehensive:

  Let a $y \in V^C(f)$ be given, and let $C(f) = \{1, \ldots, m\}$. This means that y can be written as $y = \sum_{j=1}^m z^j[f(j) - f(j+1)]$ with $z^j \in V(S_j)$. Consider now an $x \in \mathbb{R}^n$ s. t. $x^{C(f)} \leq y^{C(f)}$, and define $\overline{z}^m = z^m - \frac{1}{f(m)}(y - x^{C(f)})$. Then, $x^{C(f)}$ can be written as $x^{C(f)} = \sum_{j=1}^{m-1} z^j[f(j) - f(j+1)] + f(m)\overline{z}^m$. Since $\overline{z}^m \leq z^m$ is valid, we have that $\overline{z}^m \in V(S_m)$. Thus, the claim is proven.

- $V^C(f) \cap (x^{C(f)} + \mathbb{R}^n_{C(f)+})$ is bounded for every $x \in \mathbb{R}^n$:

  If $x^{C(f)} \notin V^C(f)$, we have that $V^C(f) \cap (x^{C(f)} + \mathbb{R}^n_{C(f)+}) = \emptyset$ because of $C(f)-$comprehensiveness.

  Let us now consider the case $x^{C(f)} \in V^C(f)$. We have that $\lambda V(S) \subseteq V(S)$ for every $\lambda \in [0, 1]$, since $V(S)$ is a convex set containing zero. Hence, it is sufficient to show the following: $[V(S) + V(T)] \cap (x^T + \mathbb{R}^n_{T+})$ is bounded for $S \subset T$, and $x^T \in V(S) + V(T)$. There exist $s \in V(S), t \in V(T)$ s. t. $x = s + t$. Because of monotonicity, we have that $V(S) \subset V(T)$, and, together with convexity, we have that $\frac{1}{2}x = \frac{1}{2}s + \frac{1}{2}t \in V(T)$. Next, we define

  $$\max_{i \in T} \left\{ \overline{t}_i - \frac{1}{2}x_i \middle| \overline{t} \in V(T), \overline{t} \geq \frac{1}{2}x \right\} =: \delta \in \mathbb{R}.$$

  This δ is finite since $V(T) \cap [(\frac{1}{2}x^T) + \mathbb{R}^n_{T+}]$ is bounded. With this definition in

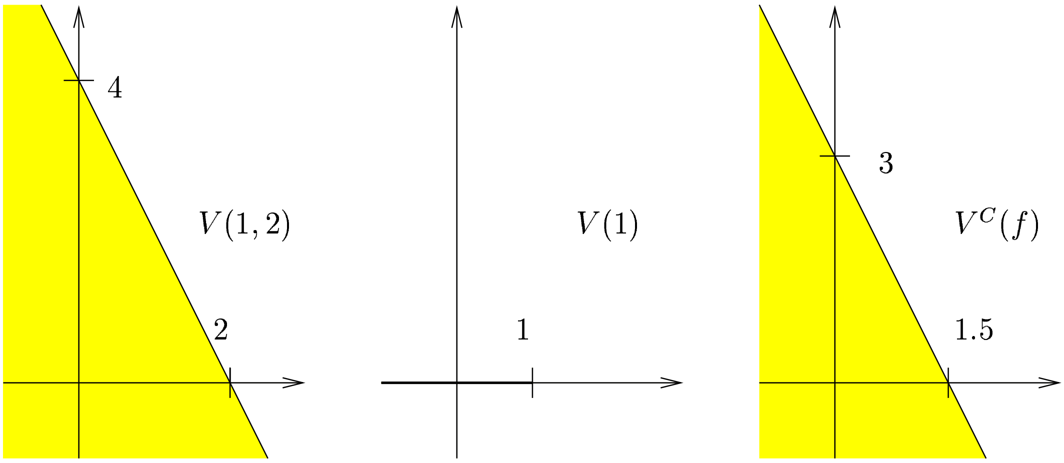

Figure 3.3: Example for a Choquet NTU Game

mind, we have that, for each pair $(\widetilde{s}, \widetilde{t})$, $\widetilde{s} \in V(S), \widetilde{t} \in V(T)$ with $(\widetilde{s}+\widetilde{t}) \geq x$,

$$\|(\widetilde{s}+\widetilde{t}) - x\|_{\sup} = 2\|\frac{1}{2}\widetilde{s} + \frac{1}{2}\widetilde{t} - \frac{1}{2}x\|_{\sup} \leq 2\delta.$$

This completes the proof. **q.e.d.**

Example 3.7

For reasons of simplicity, we consider the two player case. Let a fuzzy coalition f be given by $f(1) = 1, f(2) = \frac{1}{2}$ and $V(1,2) = \{x \in \mathbb{R}^2 \,|\, x_1 + \frac{1}{2}x_2 \leq 2\}$, $V(1) = \{x \in \mathbb{R}^2_1 \,|\, x_1 \leq 1\}$. Then, one can easily calculate the Choquet extension to be $V^C(f) = \{x \in \mathbb{R}^2 \,|\, x_1 + \frac{1}{2}x_2 \leq 1,5\}$ (see Figure 3.3).

Theorem 3.6 states that the Choquet extension given by formula (3.3) satisfies all properties of a fuzzy NTU game except for closure. Later on, we will give conditions for (N, V) that guarantee this property.

In Theorem 3.6, we require the mapping V to be monotone and convex-valued. The following example demonstrates that these properties are necessary to show boundedness of $V^C(f) \cap (x^{C(f)} + \mathbb{R}^n_{C(f)+})$. This means that the Choquet extension in the NTU case is not as general as in the TU case. There, we need the monotonicity of the underlying setfunction v to obtain an integrable function. In particular, for finitely many players no restrictions on v are required.

Example 3.8

1. If we drop monotonicity, the following can happen: Consider the case $n = 3$ and the game $(N,\, V)$ with $V(1,2) := \{x \in \mathbb{R}^n_{\{1,2\}} \,|\, 4x_1 + x_2 \le 0\}$, $V(1,2,3) = \{x \in \mathbb{R}^n_{\{1,2,3\}} \,|\, x_1 + 4x_2 \le 0, x_3 \le 0\}$. We have that $\lambda V(1,2) = V(1,2)$ and $\lambda V(1,2,3) = V(1,2,3)$ for $\lambda > 0$. Hence, we know that for $f(1) = f(2) = 1, f(3) = \frac{1}{2}$, the Choquet extension can be written as $V^C(f) = V(1,2) + V(1,2,3)$. For each $m \in \mathbb{N}$, the vector $(-m, 4m, 0)$ is an element of $V(1,2)$, and $(4m, -m, 0) \in V(1,2,3)$ is also valid. Hence, we have that $(3m, 3m, 0) \in [V(1,2) + V(1,2,3)]$. Obviously, $V^C(f) \cap (0 + \mathbb{R}^n_{C(f)+})$ is not bounded.

2. If we drop the convexity assumption, we can, for three players, define $V(1,2)$ and $V(1,2,3)$ by $V(1,2) = \{x \in \mathbb{R}^n_{\{1,2\}} \,|\, 4x_1 + x_2 \le 0 \text{ or } x_1 + 4x_2 \le 0, \}$, $V(1,2,3) := \{x \in \mathbb{R}^n_{\{1,2,3\}} \,|\, 4x_1 + x_2 \le 0 \text{ or } x_1 + 4x_2 \le 0, x_3 \le 0\}$. Now, one can draw the same conclusions as before.

As we want to get a fuzzy NTU game with the help of the Choquet formula, we still have to find conditions under which $V^C(f)$ is closed for all $f \in [0,1]^N$. For $\lambda \in (0,1]$, one can easily see that $\lambda V(S)$ is closed, since $x \in \partial(\lambda V(S))$ iff $\frac{1}{\lambda}x \in \partial V(S)$. However, the sum of $V(S)$ and $V(T)$, as the sum of two closed sets, does not need to be closed itself. To show this, we can consider a slight modification of an example for three players given by Kern [14, Beispiel 4.16]. A monotone, convex-valued NTU game $(N,\, V)$ is defined by

$$\frac{1}{2}V(1,2) = \left\{x \in \mathbb{R}^3_{\{1,2\}} \,\middle|\, x_1 \le -1, x_2 \le \sqrt{(x_1)^2 - 1}\right\} + (1,0,0),$$

$$\frac{1}{2}V(1,2,3) = \{x \in \mathbb{R}^3 \,|\, x_1 + x_2 \le 1, x_3 \le 0\}$$

and $V(S) = \mathbb{R}^3_{S-}$ for the remaining $S \in \underline{\underline{P}}(N)$. For $f \in [0,1]^N$ given by $f(1) = f(2) = 1$, $f(3) = \frac{1}{2}$, we have that

$$V^C(f) = \frac{1}{2}V(1,2) + \frac{1}{2}V(1,2,3).$$

Let us now consider $V^C \cap \mathbb{R}^n_{\{1,2\}}$. Since the third coordinate of each element of this set is zero, we will concentrate in the following on $proj_{\{1,2\}}(V^C)$, $proj_{\{1,2\}} : \mathbb{R}^3 \to \mathbb{R}^2$, $proj_{\{1,2\}}(x_1, x_2, x_3) = (x_1, x_2)$. Kern states in [14, Korollar 4.4] that the sum of two points of two sets A and B, which have no parallel supporting hyperplanes, is always an interior point of $A + B$. An immediate consequence is that $proj_{\{1,2\}}(V^C(f))$ is an open set, since there exist no parallel hyperplanes of support for $proj_{\{1,2\}}\frac{1}{2}V(1,2)$ and $proj_{\{1,2\}}\frac{1}{2}V(1,2,3)$ at all. One can show that

$$V^C(f) = \{x \in \mathbb{R}^3 \,|\, x_1 + x_2 < 2, x_3 \leq 0\}.$$

Hence, $V^C(f)$ is not closed in general. However, Kern gives in [14, Satz 4.11] conditions under which this is the case. To be precise, he does not consider the Choquet extension but the one of Owen. Furthermore, he deals only with fuzzy coalitions which have full carrier. Since we would like to show the closure of $V^C(f)$ for all $f \in [0,1]^N$, we have to make some slight changes. However, the calculations in his proof can be used without bigger modifications in our case. Thus, we will not prove the next theorem.

For each $x \in \mathbb{R}$, we denote by $CH_S(x^S)$ the **S−comprehensive hull** of x^S, i. e. $CH_S(x^S) := \{y^S |\, y \in \mathbb{R}^n, y^S \leq x^S\}$. A **quasi TU game** (N, V) is an NTU game for which there exists an $a \in \mathbb{R}^n_{++}$ and a $\beta^S \in \mathbb{R}$ for every S s. t. $V(S)$ can be written as $V(S) = \{x \in \mathbb{R}^n_S \,|\, a^S x^S \leq \beta^S\}$.

Theorem 3.9 *[14, Satz 4.11] $V^C(f)$ is a closed set for all $f \in [0,1]^N$ if one of the following conditions is satisfied:*

1. *For each $S \subset N, S \neq N$, there exists an $x^S \in \mathbb{R}^n_S$ s. t. $V(S)$ can be written as $V(S) = CH_S(x^S)$.*
2. *(N, V) is a quasi TU game.*

In the two player case, we have no problems with the closure of $V^C(f)$ since $V(\{i\}) = CH_{\{i\}}(x^{\{i\}})$ for $i \in \{1, 2\}$, i. e. we can use the first part of the last theorem. Another possibility to ensure the closure of V^C is given by the following theorem, which represents an extension of the first condition of the last theorem.

For the following proof, we will use a statement from convex analysis. Let A be a non-empty convex set in $\mathbb{R}^n$ and $0 \neq y \in \mathbb{R}^n$. Then, A **recedes in the direction of y** if and only if $x + \lambda y \in A$ for every $\lambda \geq 0$ and each $x \in A$. The set of all vectors $y \in \mathbb{R}^n$ satisfying this condition, including $y = 0$, will be called the **recession cone** of A and will be denoted by $\mathbf{0^+A}$. Directions in which A recedes will also be referred to as **directions of recession** of A. Finally, the set $(-0^+A) \cap 0^+A$ is called the **lineality space** of A. These definitions can be found, for example, in the book of Rockafellar [21].

Corollary 3.10 *[21, Corollary 9.1.1] Let $A_1, \ldots, A_m$ be non-empty, closed, convex sets in $\mathbb{R}^n$ satisfying the following condition: if $z_1, \ldots, z_m$ are vectors such that $z_i \in 0^+A_i$ for each $i \in \{1, \ldots, n\}$ and $z_1 + \ldots + z_m = 0$, then actually z_i belongs to the lineality space of A_i for $i = 1, \ldots, m$. Then, $A_1 + \ldots + A_m$ is closed.*

Theorem 3.11 *Let (N, V) be a monotone and convex-valued NTU game. For $S \in \underline{P}(N), S \neq N$, $V(S)$ may have the form $V(S) = CH_S K_S$ for a compact set $K_S \subset \mathbb{R}^n_S$. Then, $V^C(f)$ is closed for every $f \in [0,1]^N$.*

Proof One can easily check that, for $\lambda \in (0,1)$ and a compact set $B_T \subseteq \mathbb{R}^n_T$ $(T \subseteq N)$, the equality $CH_T(\lambda B_T) = \lambda CH_T B_T$ is valid. Furthermore, if $B_T \subseteq \mathbb{R}^n_T$ is a convex and compact set, the set λB_T, $\lambda \in (0,1)$, is compact and convex, too.

Let a fuzzy coalition f be given. For $f = \emptyset$, there is nothing to prove. For $f \neq \emptyset$, we can assume w. l. o. g. $f(1) \geq \ldots \geq f(n) \geq f(n+1) := 0$. We define $L(f) = L := \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$, $i_1 < \ldots < i_l$, as the set of players that satisfy

$$f(i_k) > f(i_k + 1), k \in \{1, \ldots, l\} \text{ and}$$
$$f(j) \in \{f(i_k) | \, k \in \{1, \ldots, l\}\} \text{ for every } j \in \{1, \ldots, n\}.$$

Hence, $L(f)$ is the set of all players which have a strictly higher degree of membership in f as the next player in the lexicographic order. If $n \notin L$, $V^C(f)$ can be written as

$$V^C(f) = \sum_{k=1}^{l} CH_{S_{i_k}} \left[f(i_k) - f(i_k + 1) \right] K_{S_{i_k}}.$$

If y is a direction of recession of $CH_{S_{i_k}}\left[f(i_k)-f(i_k+1)\right]K_{S_{i_k}}$, then $y\le 0$ is true. If $z^1,\ldots,z^l$ are vectors s. t. $z^k\in 0^+CH_{S_{i_k}}\left[f(i_k)-f(i_k+1)\right]K_{S_{i_k}}$ and $z^1+\ldots+z^l=0$, each z^k has to equal zero, i. e. each z^k belongs to the lineality space of $CH_{S_{i_k}}\left[f(i_k)-f(i_k+1)\right]K_{S_{i_k}}$. Thus, by Corollary 3.10, $V^C(f)$ is closed.

If $i_l=n$, we take any $z^l\in 0^+\left[f(i_l)V(N)\right]$. If there exists a $j\in N$ with $z_j^l>0$, there has to be a $k\in N$ with $z_k^l<0$ because of boundedness of $V(N)\cap\mathbb{R}_+^n$. Hence, if we consider $z^k\in 0^+CH_{S_{i_k}}\left[f(i_k)-f(i_k+1)\right]K_{S_{i_k}}$, $k\in\{1,\ldots,l-1\}$, with $z^1+\ldots+z^l=0$, we have that $z^k=0$ for all $1\le k\le l$ as in the first case. Now, Corollary 3.10 provides again the stated result. **q.e.d.**

Rosenmüller calls in [23][p. 436] an NTU game (N,V) **compactly generated** if, for each $S\in\underline{\underline{P}}(N)$, there exists a compact set $K_S\subset\mathbb{R}_S^n$ s. t. $V(S)=CH_SK_S$. Since we have made no restrictions to $V(N)$ in the last theorem, the compactly generated, monotone, and convex-valued NTU games are a strict subset of the class of games described above.

The Theorems 3.6, 3.9, and 3.11 provide conditions under which the Choquet extension of a crisp NTU game is a fuzzy NTU game. In particular,

$$V_{v^C}^C(f)=\sum_{i=1}^n V_v(S_i)[f(i)-f(i+1)] \tag{3.4}$$

satisfies all conditions as far as v is monotone, where V_v is given by $V_v(S)=\{x\in\mathbb{R}_S^n|\ \sum_{j\in S}x_j\le v(S)\}$. The question that now arises is: "Does $V_{v^C}^C$ coincide with any proposed $V_{v^C}^F$ of Section 1?" The answer is given by the following proposition:

Proposition 3.12 *Let $(N,\ \underline{\underline{P}}(N),\ v)$ be a monotone TU game. Then, $V_{v^C}^C$ given by formula (3.4) coincides with $\widehat{V}_{v^C}^F$ (see formula (3.1)). Hence, we have that, for a fuzzy coalition f with $f(1)\ge\ldots\ge f(n)$,*

$$\begin{aligned} V_{v^C}^C(f) &= \sum_{i=1}^n V_v(S_i)[f(i)-f(i+1)] \\ &= \left\{x\in\mathbb{R}_{C(f)}^n\Big|\ \sum_{i\in C(f)}x_i\le v^C(f)\right\}. \end{aligned}$$

Proof First of all, we will show that each $y \in \mathbb{R}^n$ which can be written as $y = \sum_{i=1}^n x^i[f(i) - f(i+1)]$ with $\sum_{j \in S_i} x^i_j \leq v(S_i), x^i \in \mathbb{R}^n_{S_i}$ (i.e. $x^i \in V_v(S_i)$) satisfies $y \in \mathbb{R}^n_{C(f)}$ and $\sum_{j \in C(f)} y_j \leq v^C(f)$. The first statement is obviously true, the second can be proven as follows:

$$\begin{aligned} \sum_{j \in C(f)} y_j &= \sum_{i=1}^n \sum_{j \in C(f)} x^i_j[f(i) - f(i+1)] \\ &\leq \sum_{i=1}^n v(S_i)[f(i) - f(i+1)] \\ &= v^C(f). \end{aligned}$$

Now, let a $y \in \mathbb{R}^n_{C(f)}$ with $\sum_{j \in C(f)} y_j \leq v^C(f)$ be given. The question arises whether there exist some vectors $x^i \in \mathbb{R}^n_{S_i}$ with $\sum_{j \in S_i} x^i_j \leq v(S_i)$ s.t. y can be written as $y = \sum_{i=1}^n x^i[f(i) - f(i+1)]$. Let the carrier of f be $C(f) = \{1, \ldots, m\}$. For $f \neq \emptyset$, we can assume w.l.o.g. $f(1) > \ldots > f(m)$. If this is not the case we can build a nonempty subset $T := \{t_1, \ldots, t_k\}$ of $\{1, \ldots, m\}$ with $t_1 < \ldots < t_k$ s.t. $f(t_j) > f(t_j + 1)$ and $f(j) \in \{f(t_j) | \, j \in \{1, \ldots, k\}\}$ for each $j \in \{1, \ldots, m\}$, $f(n+1) := 0$. Now we can make all following calculations with T instead of $C(f)$. Obviously, $\sum_{i=1}^n v(S_i)[f(i) - f(i+1)] = \sum_{j=1}^k v(S_{t_j})[f(t_j) - f(t_{j+1})]$ is valid.

Now, we construct vectors $x^i \in \mathbb{R}^n_{S_i}$, $1 \leq i \leq m$, by the following procedure: For $i > 1$, we define

$$x^i_i := \frac{y_i}{[f(i) - f(i+1)]}, x^i_1 := v(S_i) - \frac{y_i}{[f(i) - f(i+1)]} \text{ and } x^i_j = 0, j \notin \{1, i\}.$$

x^1 is defined by

$$\begin{aligned} x^1_1 &:= \frac{y_1 - \sum_{i=2}^m x^i_1[f(i) - f(i+1)]}{[f(1) - f(2)]}, \\ x^1_j &= 0 \text{ for all } j \geq 2. \end{aligned}$$

With these definitions, we have that $\sum_{i=1}^m x^i[f(i) - f(i+1)] = y$. For $i \geq 2$, the equality $\sum_{j=1}^i x^i_j = v(S_i)$ holds by definition. Moreover, we have that

$$\begin{aligned} x^1_1 &= \frac{y_1 - \sum_{i=2}^m x^i_1[f(i) - f(i+1)]}{[f(1) - f(2)]} \\ &= \frac{y_1 - \sum_{i=2}^m v(S_i)[f(i) - f(i+1)] + \sum_{i=2}^m y_i}{[f(1) - f(2)]} \end{aligned}$$

$$\begin{aligned} &\le \frac{v^C(f) - v^C(f) + v(1)[f(1) - f(2)]}{f(1) - f(2)} \\ &= v(1) = v(S_1). \end{aligned}$$

q.e.d.

We call a NTU game (N, V) **flat**, if V can be written as $V(S) = \sum_{i \in S} V(\{i\})$ for every $S \subseteq N$. Two fuzzy coalitions f and g are called **comonotonic**, if they have the same ordering, i. e. if there is no pair (i, j) with $f(i) > f(j)$ and $g(i) < g(j)$. Next, we will check some properties of the Choquet extension

Theorem 3.13 *Let (N, V) and (N, W) be monotone, convex-valued NTU games s. t. $V^C(f)$ and $W^C(f)$ are closed sets for all $f \in [0,1]^N$. Then, the corresponding Choquet extensions satisfy the following properties:*

1. *$V^C(S) = V(S)$ for all $S \in \underline{\underline{P}}(N)$.*
2. *$V \subseteq W$ (i. e. $V(S) \subseteq W(S)$ for all $S \in \underline{\underline{P}}(N)$) $\Longrightarrow V^C \subseteq W^C$.*
3. *$V^C(\lambda f) = \lambda V^C(f)$ for all $\lambda \in [0,1]$.*
4. *$V^C(f + g) = V^C(f) + V^C(g)$ for f, g comonotonic.*
5. *If V is flat, then $V^C(f) = \sum_{i=1}^n f(i) V(\{i\})$.*
6. *$V^C(f)$ is convex for all $f \in [0,1]^{\mathbb{N}}$, i. e. (N, V^C) is convex-valued.*
7. *V^C is monotone, i. e. $V^C(f) \subseteq V^C(g)$ for $f \le g$.*

Proof 1. to 4. are obviously true.

Let V be flat and f a fuzzy coalition with $f(1) \ge \ldots \ge f(n)$:

$$\begin{aligned} V^C(f) &= \sum_{i \in N} V(S_i)[f(i) - f(i+1)] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{i} V(\{j\})[f(i) - f(i+1)] \\ &= \sum_{i=1}^{n} f(i) V(\{i\}). \end{aligned}$$

Also the convexity of $V^C(f)$ can be shown without any problems. One can easily see that, for a convex set $A \in \mathbb{R}^n$ and a constant $\lambda \in \mathbb{R}_+$, the set λA is convex, too. Furthermore, the sum of two convex sets is a convex set.

It remains to prove monotonicity. We consider, for this reason, $f \leq g$ with $f(1) \geq \ldots \geq f(n)$ and $g(\pi^{-1}(1)) \geq \ldots \geq g(\pi^{-1}(n))$, π a permutation of N. We define now a set $J := \{j_1, \ldots, j_r\} \subseteq N, r \leq n,, j_1 < \ldots < j_r, \pi(j_1) < \ldots < \pi(j_r)$, recursively by the following procedure: We set $j_1 := 1$. Let us assume that we have checked whether the players $\{1, \ldots, l-1\}, 2 \leq l \leq n-1$, belong to J. We consider now the lexicographic maximal element of J in $\{1, \ldots, l-1\}$ which is denoted with j_k for some $1 \leq k \leq l-1$. If $\pi(l) > \pi(j_k)$ is true, we add player l to the set J and define $j_{k+1} := l$. On the other hand, if $\pi(l) < \pi(j_k)$ is valid, l does not join J.

One can easily see that $\pi(j_r) = n$ is true. Assume, for the contrary, that $\pi(j_r) < n$. Then, one possibility is given by $\bar{l} := \pi^{-1}(n) < j_r$. However, this implies $\bar{l} \in J$ and $j_r \notin J$ since $\pi(\bar{l}) > \pi(j_r)$. The other possible case is given by $\bar{l} := \pi^{-1}(n) > j_r$. Obviously $\bar{l}$ is an element of J. However, this means that j_r is not lexicographic maximal in J. Since both cases lead to a contradiction, we have shown the claim.

We now define $l_k := \pi(j_k), k = 1, \ldots, r$. We have, by definition, $l_1 < \ldots < l_r$ and $1 = \pi^{-1}(l_1) < \ldots < \pi^{-1}(l_r)$. Setting $\pi^{-1}(l_{r+1}) := n+1$, one can show that $S_i \subseteq S^\pi_{l_k}$ is valid for $\pi^{-1}(l_k) \leq i < \pi^{-1}(l_{k+1})$. This is done by induction:

First of all, we consider the case $k = 1$, i. e. we have to show that $\pi(i) \leq l_1 = \pi(1)$ for $1 \leq i < \pi^{-1}(l_2)$. For $i = 1$, this inequality is trivial, and for $1 < i < j_2$ the claim follows by the definition of the set J.

Let us assume that everything is shown for $k < \pi^{-1}(l_m)$, $m \in \{1, \ldots, r\}$. For $\pi^{-1}(l_m) \leq i < \pi^{-1}(l_{m+1})$, the statement $S_i \subseteq S^\pi_{l_m}$ is tantamount to $\pi(k) \leq \pi(j_m)$ for all $k \leq i$. However, this follows immediately from the definition of J.

Having this in mind, we obtain

$$V^C(f) \quad = \quad \sum_{i=1}^{n} V(S_i)[f(i) - f(i+1)]$$

$$= \sum_{i=1}^{r} \sum_{j=\pi^{-1}(l_i)}^{\pi^{-1}(l_{i+1})-1} V(S_j)[f(j) - f(j+1)]$$
$$\subseteq \sum_{i=1}^{r} V(S_{l_i}^{\pi})[f(\pi^{-1}(l_i)) - f(\pi^{-1}(l_{i+1}))], \quad (3.5)$$

where $f(n+1) := 0$. Next, we will show that

$$\sum_{i=1}^{r} V(S_{l_i}^{\pi})[f(\pi^{-1}(l_i)) - f(\pi^{-1}(l_{i+1}))] \subseteq \sum_{i=1}^{r} V(S_{l_i}^{\pi})[g(\pi^{-1}(l_i)) - g(\pi^{-1}(l_{i+1}))]. \quad (3.6)$$

Because of $g \geq f$, we have that $V(S_{l_r}^{\pi})f(\pi^{-1}(l_r)) \subseteq V(S_{l_r}^{\pi})g(\pi^{-1}(l_r))$. Now, we assume that

$$\sum_{i=m+1}^{r} V(S_{l_i}^{\pi})[f(\pi^{-1}(l_i)) - f(\pi^{-1}(l_{i+1}))]$$
$$\subseteq \sum_{i=m+1}^{r} V(S_{l_i}^{\pi})[g(\pi^{-1}(l_i)) - g(\pi^{-1}(l_{i+1}))]$$

is valid for some $m < r$. To show the same relation for the sum starting at m, we define $a_k := f(\pi^{-1}(l_k)) - f(\pi^{-1}(l_{k+1}))$ and $b_k := g(\pi^{-1}(l_k)) - g(\pi^{-1}(l_{k+1}))$, $k \in \{m, \ldots, r\}$. If $a_m \leq b_m$ is valid, we are done. Otherwise one can ask whether $a_{m+1} + a_m - b_m \leq b_{m+1}$ is true. If the answer is positive, we have proven the claim, since

$$V(S_{l_m}^{\pi})(a_m - b_m) + V(S_{l_{m+1}}^{\pi})a_{m+1} \subseteq V(S_{l_{m+1}}^{\pi})b_{m+1}.$$

If necessary, we can continue this procedure and have to prove in the last step the inequality $\sum_{i=m}^{r} a_i - \sum_{i=m}^{r-1} b_i \leq b_r$. However, this inequality definitely holds since $\sum_{i=m}^{r} a_i = f(\pi^{-1}(l_m)) \leq g(\pi^{-1}(l_m)) = \sum_{i=m}^{r} b_i$.

Since V is monotone, we have that

$$V(S_{l_k}^{\pi})[g(\pi^{-1}(l_k)) - g(\pi^{-1}(l_{k+1}))] \subseteq \sum_{i=l_k}^{l_{k+1}-1} V(S_i^{\pi})[g(\pi^{-1}(i)) - g(\pi^{-1}(i+1))] \quad (3.7)$$

for each $k \in \{1, \ldots, r\}$. Setting $l_{r+1} = n+1, \pi^{-1}(n+1) := n+1$ and $g(n+1) = 0$, the Choquet extension at g can be written as

$$V^C(g) = \sum_{i=1}^{n} V(S_i^{\pi})[g(\pi^{-1}(i)) - g(\pi^{-1}(i+1))]$$

$$= \sum_{i=1}^{l_1-1} V(S_i^\pi)[g(\pi^{-1}(i)) - g(\pi^{-1}(i+1))]$$
$$+\sum_{k=1}^{r} \sum_{i=l_k}^{l_{k+1}-1} V(S_i^\pi)[g(\pi^{-1}(i)) - g(\pi^{-1}(i+1))].$$

The three formulas (3.5), (3.6), and (3.7) imply now that $V^C(f) \subseteq V^C(g)$. **q.e.d.**

In the TU case, the proof of monotonicity is much easier since we can use the integral representation. However, in the NTU case we only have the summation formula. Within the proof, we had to pay particular attention to the fact that a multiplication of a set $V(S)$ with a negative constant is prohibited since $\lambda V(S) \cap (x^S + \mathbb{R}^n_{S+})$ is not bounded for $\lambda < 0$. For this reason, we cannot use the same proofs as Denneberg [8] for a lot of interesting statements. For example, it is questionable whether or not convexity of V, i. e. $V(S) + V(T) \subseteq V(S \cup T) + V(S \cap T)$ for $S, T \in \underline{\underline{P}}(N)$, implies superadditivity of V^C, i. e. $V^C(f) + V^C(g) \subseteq V^C(f+g)$ for $f+g \leq N$. The proof for this statement in the TU case [8, Corollary 6.4] does definitely not work in our framework.

A property we lose for sure is continuity. Even if we define $V(S)$ to be a subset of $\mathbb{R}^n_S + \mathbb{R}^n_{S^C-}$, we can easily show that, in general, V^C is not continuous. However, one should remark that if we restrict ourselves to the class of compactly generated games, this approach of redefining $V(S)$ implies continuity.

3.3 A Continuous Version of a Choquet Game

In this subsection, we will show how we can obtain a continuous version of a Choquet NTU-game by only some small changes in the original extension. Most properties of the Choquet extension can be preserved, but we have to accept the loss of comonotonic additivity. For the construction of a continuous Choquet game, we will make some thoughts similar to those in Lemma 3.5.

First of all, we have to solve the problem that there is a “dimension gap” when considering a series $(f_k)_k$ of fuzzy coalitions converging to a $f \in [0,1]^N$ with $C(f) \subsetneqq C(f_k)$. Unfortunately, the carrier of a fuzzy coalition f as the set of players i with $f(i) > 0$ itself is a crisp coalition. Independent of how small $f(i)$ is, as long as $f(i)$ is greater than 0, i is a member of $C(f)$, and hence $\mathbb{R}^n_{i-}$ is contained in $V^C(f)$. However, if $f(i)$ equals 0, no vector of $\mathbb{R}^n$ with a non-zero entry in the ith coordinate is feasible for f. One could think that the small step in the degree of membership from a small ε to 0 should not have such strong effects for $V^C(f)$.

To demonstrate the problem, consider, for example, the NTU game $(\{1,2\}, V)$ with $V(\{i\}) = \{x \mid x_i \leq 1\}, i \in \{1,2\}$, and $V(\{1,2\}) = \{x \mid x_1 + x_2 \leq 1\}$. For the sequence $(f_l)_l$ of fuzzy coalitions given by $f_l(1) = 1, f_l(2) = \frac{1}{l}$, we have that

$$\begin{aligned} V^C(f_l) &= (1 - \frac{1}{l})V(1) + \frac{1}{l}V(1,2) \\ &= \{x \mid x_1 \leq 1 - \frac{1}{l}, x_2 \leq 0\} + \{x \mid x_1 + x_2 \leq \frac{1}{l}\} \\ &= V(1,2) \end{aligned}$$

for every $l \in \mathbb{N}$. The degree of membership of the second player is decreasing to zero. However, this has no effect on the possible outcomes at all. In particular, player two can reach a positive outcome all the time.

To avoid this problem, we use the following approach: A player i with $f(i) = 0$ will be considered as an active player in f who has no influence in this fuzzy coalition. He is free to throw away as much money as he likes to. However, the best result he can reach is 0. There is no possibility for him to get a positive outcome. In other words, we assume that we have the free disposal property even for players with a degree of membership of zero.

Since, with this change in the model, $\overline{V}^F(S) \neq V(S)$, one could argue that we do not have a real extension. However, as we explained before, the concept of the carrier is not as sensitive in the case of a fuzzy game as it is in the case of a crisp game. Furthermore, the change is not as dramatic as it may seem on the first view. The players can throw away as much as they want to, but why should they behave in such a way? We just have built a mathematical framework for the players which are not in S, and we still have $\overline{V}^F(S) \cap \mathbb{R}^n_S = V(S)$. Last but not least, note that all solution concepts which satisfy individual rationality do not care for our new possibilities of outcome.

Definition 3.14 *A* **fuzzy NTU game without a dimension gap** *is a pair* $(N, \overline{V}^F)$, *where* $N := \{1, \ldots, n\}$ *is the set of players and* $\overline{V}^F : [0,1]^N \to \underline{\underline{P}}(\mathbb{R}^n)$ *is a correspondence satisfying the following properties:*

1. $\overline{V}^F(f)$ *is a closed and comprehensive set for all* $f \in [0,1]^N$.
2. *The origin is an element of* $\overline{V}^F(f)$ *for all fuzzy coalitions* f.
3. *For each* $x \in \mathbb{R}^n$, *the set* $\overline{V}^F(f) \cap (x + \mathbb{R}^n_+)$ *is bounded.*
4. $\overline{V}^F(f) \subseteq \mathbb{R}^n_{C(f)} + \mathbb{R}^n_{C(f)^C-}$.

With this definition, we have that, for each $f \in [0,1]^N$, $\mathbb{R}^n_- \subseteq \overline{V}^F(f)$, and $x \in \overline{V}^F(f) \Rightarrow x_i \leq 0$ for all $i \in N$, $i \notin C(f)$.

Let (N, V) be a crisp NTU game. A first attempt to obtain a continuous correspondence for the Choquet extension could be based on the idea of adding to each $V(S)$ in formula (3.3) the negative orthant, i. e. to define $\widetilde{V}(S)$ by $\widetilde{V}(S) := V(S) + \mathbb{R}^n_-$ and

$$\widetilde{V}^C(f) := \sum_{i=1}^{n} [f(i) - f(i+1)]\widetilde{V}(S_i)$$

for $f(1) \geq \ldots \geq f(n)$. However, this approach does not provide a continuous correspondence: This becomes clear by recalling the example at the beginning of this subsection, where the sequence $(f_l)_l$ was given by $f_l(1) = 1$ and $f_l(2) = \frac{1}{l}$. Obviously, (f_l) is converging to f, $f(1) = 1, f(2) = 0$. However, we have that $\widetilde{V}^C(f_l) = V(1,2) = \{x |\, x_1 + x_2 \leq 1\}$ for all l and $\widetilde{V}^C(f) = \{x |\, x_1 \leq 1, x_2 \leq 0\}$.

To obtain a positive result, we will slightly modify the $V(S)$ in formula (3.3). We will define for each fuzzy coalition f and each crisp coalition S a set $V_f(S)$ which relates $V(S)$ to $\min_{j \in S} f(j)$. Then, we substitute the $V(S_i)$ by these $V_f(S_i)$ to get a Choquet formula for a continuous extension.

Definition 3.15 *Let (N,V) be a monotone and convex-valued crisp NTU game, and let $f \in [0,1]^N$ be a fuzzy coalition with $f(1) \geq \ldots \geq f(n)$. Then, a* **continuous version of the Choquet extension** *is $(N, \overline{V}^C)$, $\overline{V}^C : [0,1]^N \longrightarrow \underline{\underline{P}}(\mathbb{R}^n)$, given by the formula*

$$\overline{V}^C(f) = \sum_{i=1}^{n}[f(i) - f(i+1)]clCCH\left(\bigcup_{j=1}^{i} V_f(S_j)\right) + \mathbb{R}^n_-, \qquad (3.8)$$

where $S_i := \{1,\ldots,i\}$, $f(n+1) := 0$, and $V_f : \underline{\underline{P}}(N) \to \underline{\underline{P}}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} V_f(S) &:= \bigcup_{j\in S}\left\{(x_1,\ldots,[\min_{l\in S} f(l)]x_j,\ldots,x_n)\right| \qquad (3.9) \\ &\qquad x \in V(S), [\min_{l\in S} f(l)]x_j \geq \max_{k\in S\setminus j} x_k\Big\} + \mathbb{R}^n_- \\ &= \Big\{y \Big| \exists x \in V(S) \text{ with } [\min_{l\in S} f(l)]x_j \geq x_k \text{ for a } j \in S, \text{for all } k \in S\setminus j \\ &\qquad s.\,t.\ y_j \leq [\min_{l\in S} f(l)]x_j \text{ and } y_m = x_m \text{ for all } m \neq j\Big\}. \end{aligned}$$

Here clCCHA denotes the closure of the convex, comprehensive hull of the set A.

In formula (3.8) we have added $\mathbb{R}^n_-$. This is to avoid problems when dealing with the empty coalition. We have defined $V_f(S_i)$ in such a way that $V_f(S_i) = V(S_i) + \mathbb{R}^n_-$ for $f(i) = 1$ and $V_f(S_i) \subseteq V(S_i) + \mathbb{R}^n_-$ for all f, for all i. We will show later on that for a sequence (f_l) with $f_l(i) \to 0$ we have that $V_{f_l}(S_i) \to \mathbb{R}^n_-$: The smaller the minimal degree of membership of the players of S in f, the smaller $V_f(S)$. This becomes quite obvious for $V_f(1,2)$ and a small $f(2)$ (see Example 3.16).

One should remark that

$$\bigcup_{j=1}^{i} V_f(S_j) = \bigcup_{T\subseteq S_i} V_f(T)$$

is valid for every $i \in N$. The direction "$\subseteq$" is obvious. The other inclusion follows from the fact that $V_f(T)$ is a subset of $V_f(S_j)$ for $T \subseteq S_j$ and $j \in T$ since $f(j) = \min_{l\in S_j} f(l)$. Hence, together with the equation

$$\bigcup_{T\subseteq S_i} V_f(T) = \bigcup_{j=1}^{i} \bigcup_{j\in T\subseteq S_j} V_f(T),$$

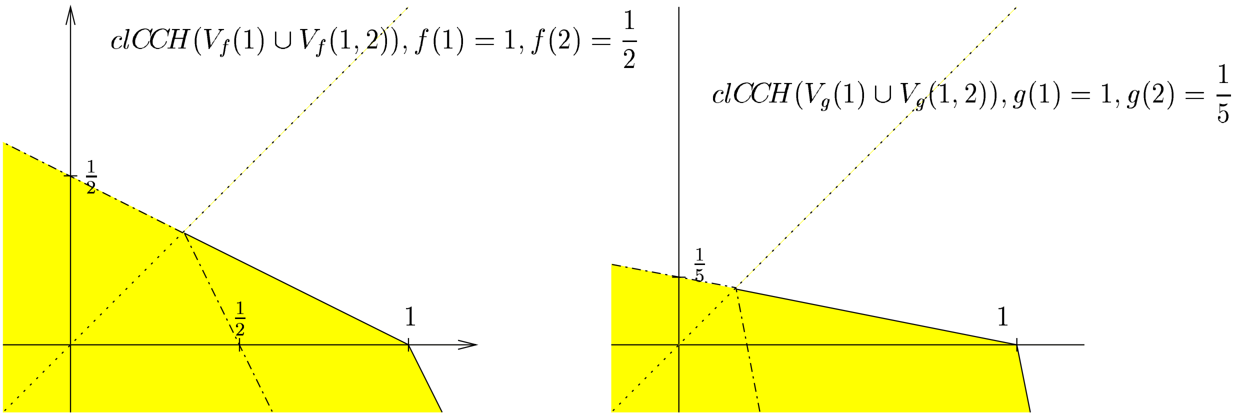

Figure 3.4: Representation of the change of $clCCH(V_\bullet(1) \cup V_\bullet(1,2))$ for $V(1) = \{x|\, x_1 \leq 1\}$, $V(1,2) = \{x|\, x_1 + x_2 \leq 1\}$, and a decreasing degree of membership of the second player

we obtain the stated result. This result implies that formula (3.8) is well defined, i. e. that there are no problems with fuzzy coalitions f that satisfy $f(i) = f(i+1)$ for $i \in \{1, \ldots, n-1\}$. Moreover, we would like to stress that, for a set $A \subseteq \mathbb{R}^n$, there is no difference between the comprehensive convex hull and the convex comprehensive hull of this set. The simple proof is omitted here.

Example 3.16 *Consider the NTU game* $(\{1,2\}, V)$ *with* $V(i) = \{x|\, x_i \leq 1\}, i \in \{1,2\}, V(1,2) = \{x|\, x_1 + x_2 \leq 1\}$. *For a sequence* $(f_l)_l$ *with* $f_l(1) = 1$ *and* $f_l(2) = \frac{1}{l}$, *we have that*

$$\begin{aligned} V_{f_l}(1,2) &= \{x|\, lx_1 + x_2 \leq 1, x_1 \geq x_2\} \cup \{x|\, x_1 + lx_2 \leq 1, x_2 \geq x_1\}, \\ V_{f_l}(1) &= V(1) + \mathbb{R}^n_-, \end{aligned}$$

and, finally,

$$\begin{aligned} \overline{V}^C(f_l) &= (1 - \frac{1}{l})V_{f_l}(1) + \frac{1}{l} clCCH\left(V_{f_l}(1) \cup V_{f_l}(1,2)\right) \\ &= clCCH\left(V_{f_l}(1) \cup V_{f_l}(1,2)\right). \end{aligned}$$

$\overline{V}^C(f_l)$ can be computed to be the comprehensive hull of $(\{x|\, x_1 + lx_2 \leq 1, x_2 \geq 0\} \cup \{x|\, lx_1 + x_2 \leq l, x_2 \leq 0\})$.

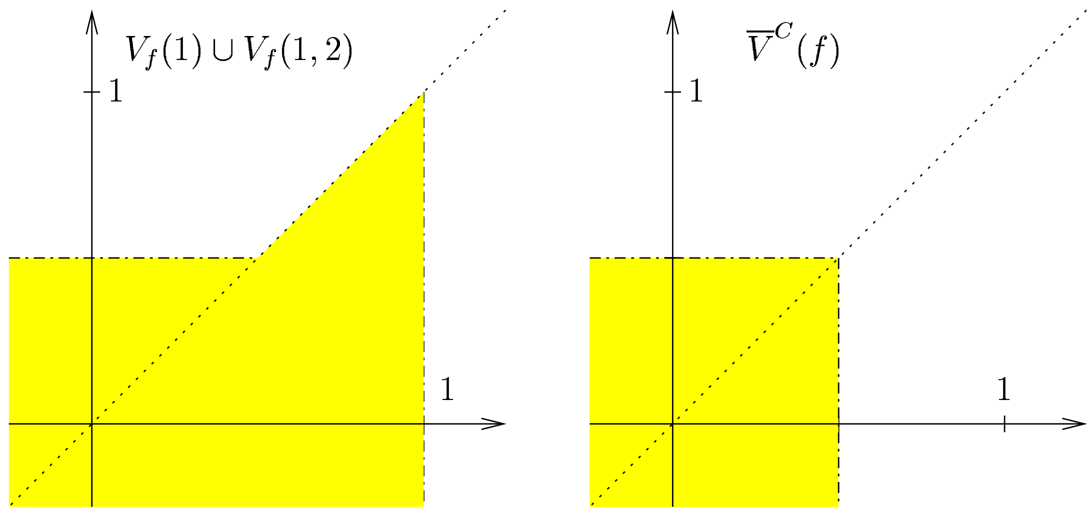

Figure 3.5: Representation of $\overline{V}^C(f)$ for $V(i) = \{x|\, x_i \leq 0\}, i \in \{1,2\}, V(1,2) = \{x|\, x_1 \leq 2, x_2 \leq 1\}$ and $f(1) = f(2) = \frac{1}{2}$

In figure 3.4, the representation of $\overline{V}^C$ is outlined for $f(1) = 1, f(2) = \frac{1}{2}$ and $g(1) = 1, g(2) = \frac{1}{5}$. In this figure, the effect of a decrease in the degree of membership of player 2 becomes particularly clear.

One should remark that $\bigcup_{j=1}^{i} V_f(S_j)$ is not a convex set by itself: In the last example, the point $\overline{a} = (\frac{3}{4}, \frac{1}{8})$ is an element of the convex hull of $[V_f(1) \cup V_f(1,2)]$ for $f(1) = 1, f(2) = \frac{1}{2}$. As one can easily check, $\overline{a}$ is not an element of $[V_f(1) \cup V_f(1,2)]$.

We have to require closure of $CCH \bigcup_{j=1}^{i} V_f(S_j)$, since, otherwise, there may be open sets: $(2,-2) \in \partial CCH[V_f(1) \cup V_f(1,2)]$ but $(2,-2) \notin CCH[V_f(1) \cup V_f(1,2)]$.

Furthermore, one can show that $\bigcup_{j=1}^{i} V_f(S_j)$ is generally not comprehensive:

Example 3.17 *For* $(\{1,2\}, V)$, $f(1) = f(2) = \frac{1}{2}, V(i) = \{x|\, x_i \leq 0\}, i \in \{1,2\}$, *and* $V(1,2) = \{x|\, x_1 \leq 2, x_2 \leq 1\}$, *we have that*

$$V_f(1) \cup V_f(1,2) = \left\{x \middle|\, \frac{1}{2} \geq x_2 \geq x_1\right\} \cup \{x|\, 1 \geq x_1 \geq x_2\},$$

and this set is definitively not comprehensive (figure 3.5)

Theorem 3.18 *Let a monotone and convex-valued NTU game (N, V) and a $f \in [0,1]^N$ be given. Then, $\overline{V}^C$ defined as in formula (3.8) satisfies the following four properties:*

1. *$\overline{V}^C(f)$ is comprehensive,*
2. *$0 \in \overline{V}^C(f)$,*
3. *$\overline{V}^C(f) \cap (x + \mathbb{R}^n_+)$ is bounded for every $x \in \mathbb{R}^n$,*
4. *$x \in \overline{V}^C(f)$ and $i \notin C(f)$ implies $x_i \leq 0$.*

Proof

1. Comprehensiveness of $\overline{V}^C(f)$ is shown by Kern [14]. He states in Lemma 4.8 that, for two comprehensive sets $A, B \subseteq \mathbb{R}^n$ and a constant $\alpha \in \mathbb{R}_+$, the sets αA and $A + B$ are comprehensive.
2. $0 \in \overline{V}^C(f)$ is obviously true since $\mathbb{R}^n_- \subseteq \overline{V}^C(f)$.
3. Boundedness can also be shown without any problems: One can easily verify that $V_f(S_j) \subseteq V(S_i)$ for all $j \leq i$. Since $V(S_i)$ is closed, convex, and comprehensive, we have that $clCCH(\bigcup_{j=1}^{i} V_f(S_j)) \subseteq V(S_i)$, and, hence, $\overline{V}^C(f) \cap \mathbb{R}^n_+ \subseteq V^C(f) \cap \mathbb{R}^n_+$. Since $V^C(f)$ is bounded, we are done.
4. $\overline{V}^C(f) = \overline{V}^C(f)|_{\mathbb{R}^n_{C(f)}} + \mathbb{R}^n_{C(f)^C-}$ can be seen immediately. **q.e.d.**

Theorem 3.18 states that $(N, \overline{V}^C)$ satisfies all properties of a fuzzy NTU game without dimension gap except for closure. As in the non-continuous version, we have the following result:

Theorem 3.19 *Let (N, V) be a monotone and convex-valued NTU game. $\overline{V}^C(f)$ is a closed set for all $f \in [0,1]^{\mathbb{N}}$ if one of the following conditions is satisfied:*

1. *For each $S \subset N, S \neq N$, there exists a compact set $K_S \subseteq \mathbb{R}^n_S$ s. t. $V(S)$ can be written as $V(S) = CH_S(K_S)$.*

2. *(N, V) is a quasi TU game.*

Proof

1. In the proof of boundedness of $\overline{V}^C \cap (x + \mathbb{R}^n_+)$, $x \in \mathbb{R}$, we have shown that $clCCH(\bigcup_{j=1}^i V_f(S_j)) \subseteq V(S_i)$ is valid for every $i \in N$. This implies that every direction of recession of $clCCH(\bigcup_{j=1}^i V_f(S_j))$ is a direction of recession of $V(S_i)$. In the proof of Theorem 3.11 we have checked that we can use Corollary 3.10 to show closure of V^C. As the sets $clCCH(\bigcup_{j=1}^i V_f(S_j)), i \in \{1, \dots, n\}$, are non-empty, closed, and convex, we can use this corollary again. Since $0^+(clCCH(\bigcup_{j=1}^i V_f(S_j))) \subseteq 0^+V(S_i)$ is valid, we are done.

2. Again, we will use Corollary 3.10 to show closure of V^C. Then, the same thoughts as in the first part of this theorem imply the closure of $\overline{V}^C$.

   One can easily check that $0^+V(S) = 0^+(\mu V(S))$ for $\mu > 0$. If we assume that we have an $f \in [0,1]^N$ with $f(1) > \ldots > f(n) > 0$ (what we may do w. l. o. g.), we have to show that for $z^i \in 0^+V(S_i), i = 1, \dots, n$, the equality $z^1 + \ldots + z^n = 0$ implies $z^i \in 0^+V(S_i) \cap 0^+(-V(S_i))$. Let $a \in \mathbb{R}^n_{++}$ and $\beta^S \in \mathbb{R}_{++}, S \in \underline{\underline{P}}(N)$, be the constants determing V. If one has a $y^S \in \mathbb{R}^n_S$ with $a^S y^S = \beta^S$ and a $x^S \in \mathbb{R}^n_S$ with $(y^S + x^S) \in V(S)$, one gets $a^S(y^S + x^S) = \beta^S + a^S x^S \leq \beta^S$, i. e. $a^S x^S \leq 0$. If $a^S x^S \leq 0$ is true, x^S is obviously an element of $0^+V(S)$, and for $x^S \in 0^+V(S)$ we always have that $a^S x^S \leq 0$. Therefore, all in all, $0^+V(S) = \{x^S \in \mathbb{R}^n_S | \, a^S x^S \leq 0\}$ is valid. The lineality space of $V(S)$ is nothing else but $\{x^S | \, a^S x^S = 0\}$. Hence, it remains to prove that $a^{S_i} z^i = a^N z^i = 0$ for all i. However, this can be seen without any problems by recursion:

$$\begin{aligned} 0 &\geq a^N(z^1 + \ldots + z^{n-1}) = a^N(-z^n) \geq 0 \Rightarrow a^N z^n = 0 \\ &\vdots \\ 0 &\geq a^N(z^1 + \ldots + z^{j-1}) = a^N(-z^j) \geq 0 \Rightarrow a^N z^j = 0 \end{aligned}$$

Here, we have used the fact that $(z^1 + \ldots + z^{j-1})$ is an element of $0^+V(N)$ and $(-z^j) \in 0^+(-V(N)), j \in \{2, \ldots, n\}$.

q.e.d.

In the following, we are going to deal with the question of what further properties $\overline{V}^C$ has:

Lemma 3.20 *Let (N, V) and (N, W) be monotone, convex-valued NTU games s. t. $V^C(f)$ and $W^C(f)$ are closed sets for all $f \in [0,1]^N$. Then, the corresponding continuous versions of the Choquet extensions satisfy the following properties:*

1. $\overline{V}^C(S) \cap \mathbb{R}^n_S = V(S)$ *for all* $S \in \underline{\underline{P}}(N)$
2. $V \subseteq W \Longrightarrow \overline{V}^C \subseteq \overline{W}^C$
3. $\overline{V}^C(\lambda f) \subseteq \lambda \overline{V}^C(f)$ *for every* $\lambda \in [0,1]$
4. $\overline{V}^C(f)$ *is convex for all* $f \in [0,1]^N$
5. $\overline{V}^C$ *is monotone*

Proof

1.

$$\begin{aligned}
\overline{V}^C(S) \cap \mathbb{R}^n_S &= clCCH\left(\bigcup_{T \subseteq S} V_S(T)\right) \cap \mathbb{R}^n_S \\
&= clCCH\left(\bigcup_{T \subseteq S} V(T)\right) \\
&= clCCH(V(S)) \\
&= V(S).
\end{aligned}$$

2. $V \subseteq W \Rightarrow V_f \subseteq W_f$ for all $f \in [0,1]^N$. Hence we immediately obtain the stated result.

3. Considering formula (3.9), one can immediately see that $V_{\lambda f}(S) \subseteq V_f(S)$ for all $\lambda \in [0,1]$, all $f \in [0,1]^N$ and every $S \in \underline{\underline{P}}(N)$. Thus, the following is true:

$$\begin{aligned}\overline{V}^C(\lambda f) &= \sum_{i=1}^{n} \lambda[f(i) - f(i+1)] clCCH \left(\bigcup_{j=1}^{i} V_{\lambda f}(S_i) \right) \\ &\subseteq \lambda \sum_{i=1}^{n} [f(i) - f(i+1)] clCCH \left(\bigcup_{j=1}^{i} V_f(S_i) \right) \\ &= \lambda \overline{V}^C(f).\end{aligned}$$

4. Since λA is a convex set whenever A itself is convex, we can use Theorem 3.1 of Rockafellar [21], which states that for two convex sets in $\mathbb{R}^n$ the sum is also convex.

5. Let $f \leq g$, $f, g \in [0,1]^N$ be given. We define a crisp monotone NTU game (N, X) by $X(S) := clCCH(\bigcup_{T \subseteq S} V_g(T)) \cap \mathbb{R}^n_S$ for all $S \in \underline{\underline{P}}(N)$. Since $V_f(T) \subseteq V_g(T)$ is obviously true for every $T \in \underline{\underline{P}}(N)$, we have that $clCCH(\bigcup_{T \subseteq S} V_f(T)) \subseteq clCCH(\bigcup_{T \subseteq S} V_g(T))$. Therefore, we can draw the following conclusions:

$$\begin{aligned}\overline{V}^C(f) &= \sum_{i=1}^{n} [f(i) - f(i+1)] clCCH \left(\bigcup_{j=1}^{i} V_f(S_i) \right) + \mathbb{R}^n_- \\ &\subseteq \sum_{i=1}^{n} [f(i) - f(i+1)] X(S_i) + \mathbb{R}^n_- \\ &= X^C(f) + \mathbb{R}^n_- \\ &\subseteq X^C(g) + \mathbb{R}^n_- \\ &= \overline{V}^C(g)\end{aligned}$$

Here, we have used the monotonicity of X^C (cf. Theorem 3.13).

q.e.d.

The preceding lemma shows that several properties of the Choquet extension are shared by our continuous version. However, there are properties of V^C that are not satisfied by $\overline{V}^C$:

Remark 3.21 *The continuous version of the Choquet extension given by formula (3.8) neither preserves flatness nor is it comonotonic additive.*

Proof Let us consider the case $n = 2$, $f(1) = f(2) = \frac{1}{2}$, and a flat game (N, V) with $V(1) = \{x | \, x_1 \leq 2\}$ and $V(2) = \{x | \, x_2 \leq 1\}$. We have that $\overline{V}^C(f) = \{x | \, x_1 \leq \frac{1}{2}, x_2 \leq \frac{1}{2}\}$, and, on the other hand, $f(1)V(1) + f(2)V(2) = \{x | \, x_1 \leq 1, x_2 \leq \frac{1}{2}\}$.

Let $f \in [0,1]^N$ and (N, V) be given as before. Defining $g := f$, one has two comonotonic fuzzy coalitions. However, we see that $(2,1) \notin \overline{V}^C(f) + \overline{V}^C(g)$ but $(2,1) \in V(1,2) = \overline{V}^C(f+g)$. **q.e.d.**

One might think that comonotonic additivity is a quite important property of the Choquet extension which should be preserved in the continuous version. However, one can show quite easily that there exist NTU games (N, V) s. t. no fuzzy extension $(N, \overline{V}^F)$ can be both continuous and comonotonic additive. To prove this statement, we consider the NTU game (N, V) with $N = \{1, 2\}$ and $V(1,2) = \{x | \, x_1 + x_2 \leq 0\}$, $V(i) = \{x | \, x_i \leq 0\}, i = 1, 2$, and assume that $\overline{V}^F$ is comonotonic additive. For each $l \in \mathbb{N}$, we have that

$$\begin{aligned} \overline{V}^F(N) &= \overline{V}^F\left(\frac{(l-1)}{l}N\right) + \overline{V}^F\left(\frac{1}{l}N\right) \\ &= \ldots = l\overline{V}^F\left(\frac{1}{l}N\right). \end{aligned}$$

As $\overline{V}^F$ is an extension of V, we have that $V(N) = \overline{V}^F(N)$. Furthermore, $V(N) = \frac{1}{l}V(N)$ is true for every $l \in \mathbb{N}$. Hence, we have that $(1, -1) \in \overline{V}^F(N) = \frac{1}{l}V^F(N) = V^F(\frac{1}{l}N)$. As $\frac{1}{l}N$ converges to the empty set for $l \to \infty$, continuity would imply $(1, -1) \in V^F(\emptyset) = \mathbb{R}^2_-$, and this is obviously not correct.

Now, we will quote a technical statement of Rockafellar [21][Theorem 8.3.], which we will use several times in the proof of the following theorem.

Lemma 3.22 *Let A be a non-empty, closed, convex set, and let $0 \neq y \in \mathbb{R}^n$. If there exists an $\overline{x} \in A$ s. t. the half-line $\{\overline{x} + \lambda y |\, \lambda \geq 0\}$ is contained in A, then the same statement is true for every $x \in A$, i. e. one has $y \in 0^+A$.*

Another statement which we need for the next proof is the well known "Separating Hyperplane Theorem":

Lemma 3.23 *Let $A, B \subset \mathbb{R}^n$ be non-empty, closed, convex sets with $A \cap B = \emptyset$. Then, there exist $0 \neq x \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ s. t.*

$$\langle a, x\rangle \leq \beta \text{ for all } a \in A \text{ and } \langle b, x\rangle \geq \beta \text{ for all } b \in B.$$

Let us consider the situation for a convex-valued NTU game (N, V) where we have a set $V(N)$ which satisfies non-emptiness, convexity, and closure. We claim that there exists a pair (x, β) with $x_j > 0$ for all j and $\beta > 0$ s. t. $\langle a, x\rangle \leq \beta$ for all $a \in V(N)$. The non-negativity of x is a direct consequence of comprehensiveness: If $x_k < 0$ was true for an $k \in \{1, \ldots, n\}$, one could consider the sequence $(a^l)_l$, $a^l = \frac{l}{x_k} e^k \in V(N)$ and would receive $\langle x, a^l\rangle = l \to \infty$ for $l \to \infty$. However, this contradicts boundedness.

To complete the proof of the claim, we will consider $[V(N) + e]$ instead of $V(N)$. Because of boundedness, there exists, for each $j \in \{1, \ldots, n\}$, a point $b^j = \gamma_j e^j, \gamma_j \in \mathbb{R}_+$, with $b^j \notin [V(N) + e]$. Now, the separating hyperplane theorem provides for $[V(N) + e]$ and each b^j the existence of a vector $x^j \in \mathbb{R}^n$ and a constant $\beta^j \in \mathbb{R}$ s. t. $\langle a, x^j\rangle \leq \beta^j$ for all $a \in [V(N) + e]$ and $\langle x^j, b^j\rangle \geq \beta^j$. Since $x^j \geq 0$ and $x^j \neq 0$ has to be true, we know that, for each $j \in N$, there is at least one $k \in N$ with $x^j_k > 0$. As e^k is an element of $[V(N) + e]$, we know that $\beta^j \geq x^j_k > 0$ is valid. Moreover, we have that $\langle b^j, x^j\rangle = \gamma_j x^j_j \geq \beta^j > 0$, and this implies $x^j_j > 0$. All in all, we have that

$$\left\langle \frac{1}{n} \sum_{j=1}^{n} x^j, a \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \langle x^j, a\rangle \leq \max_j \beta^j \text{ for all } a \in V(N),$$

i. e. by defining $x := \frac{1}{n}\sum_{j=1}^{n} x^j$ and $\beta := \max_j \beta^j$ we have shown the claim.

Now, we proceed with the most important statement of this section, which guarantees continuity of $\overline{V}^C$:

Theorem 3.24 *Let (N, V) be a monotone, convex-valued NTU game s. t. the correspondence $\overline{V}^C : [0,1]^N \Rightarrow \mathbb{R}^n$ given by formula (3.8) maps all $f \in [0,1]^N$ into closed sets $\overline{V}^C(f)$. Then, $\overline{V}^C$ is continuous.*

Proof Throughout the whole proof, we assume that the fuzzy coalitions f and f_l, $l \in \mathbb{N}$, under consideration are ordered lexicographically, i. e. $f(1) \geq \ldots \geq f(n)$ and $f_l(1) \geq \ldots \geq f_l(n)$ for all $l \in \mathbb{N}$.

First of all, we would like to show lower hemi-continuity of $\overline{V}^C$, i. e. we would like to prove that, for each given $f \in [0,1]^N$, each $x \in \overline{V}^C(f)$, and every sequence $(f_l)_l, f_l \to f$, there exists a sequence of points x^l in $\mathbb{R}^n$ s. t. $x^l \in \overline{V}^C(f_l)$ for every l where (x^l) converges to x.

Let us define $m_i := [\frac{1}{2}i(i+1)]$ for $i \in N$, and recall the definition of $V_f(S_j), j \in N$:

$$V_f(S_j) = \bigcup_{l=1}^{j} \left\{ (x_1, \ldots, f(j)x_l, \ldots, x_n) \middle| x \in V(S_j), f(j)x_l \geq \max_{k \in S_j \setminus l} x_k \right\} + \mathbb{R}^n_-.$$

The point $x \in \overline{V}^C(f)$ can be written as $x = \sum_{i=1}^{n}[f(i) - f(i+1)]y^i$ with $y^i \in clCCH(\bigcup_{j=1}^{i} V_f(S_j))$. Hence, for each $i \in N$, there exist a vector $\overline{y}^i \geq y^i$ s. t., for every given $\varepsilon > 0$, there are $\lambda^{ijk} \in \mathbb{R}, z^{ijk} \in \mathbb{R}^n, 1 \leq j \leq i, 1 \leq k \leq j$, s. t.

$$\left\| \overline{y}^i - \sum_{j=1}^{i}\sum_{k=1}^{j} \lambda^{ijk} z^{ijk} \right\|_{\text{sup}} < \varepsilon$$

is true, where $\lambda^{ijk} \geq 0$, $\sum_{j=1}^{i}\sum_{k=1}^{j} \lambda^{ijk} = 1$, and $z^{ijk} \in V_f(S_j)$. For $f(j) > 0$, the z^{ijk} can be written as

$$z^{ijk} = (r_1^{ijk}, \ldots, f(j)r_k^{ijk}, \ldots, r_n^{ijk})$$

for some $r^{ijk} \in V(S_j)$ with $f(j)r_k^{ijk} \geq r_l^{ijk}$ for all $l \in S_j \setminus k$. We define for every $l \in \mathbb{N}$

$$\delta_l^{ijk} := \begin{cases} [f(j) - f_l(j)]r_k^{ijk}, & \text{if } f_l(j) < f(j) \\ 0, & \text{else.} \end{cases}$$

Then,

$$z^{ijkl} = (r_1^{ijk} - \delta_l^{ijk}, \ldots, f_l(j) r_k^{ijk}, \ldots, r_n^{ijk} - \delta_l^{ijk})$$

defines an element of $V_{f_l}(S_j)$ with $z^{ijkl} \to z^{ijk}$ for $l \to \infty$. For $f(j) = 0$, we have that $z^{ijk} \leq 0$, and we can define $z^{ijkl} := z^{ijk}$. Now,

$$y^{il} := \left[\sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{ijk} z^{ijkl} - (\overline{y}^i - y^i) \right] \to y^i$$

is valid for $l \to \infty$. Since, obviously, $y^{il} \in clCCH(\cup_{j=1}^{i} V_{f_l}(S_j))$ is true, the sequence $x^l = \sum_{i=1}^{n} [f_l(i) - f_l(i+1)] y^{il}$ satisfies the requirements.

It remains to prove upper hemi-continuity, i.e. $f_l \to f, x^l \in \overline{V}^C(f_l), x^l \to x \Rightarrow x \in \overline{V}^C(f)$. We will show this statement in three steps.

Step 1: $V_\bullet(S)$ is u. h. c. for every $S \subseteq N$

There exists a subsequence of $(f_l)_l$ (which we will denote for reasons of simplicity again with $(f_l)_l$) s. t. the space of players N can be partitioned into three subsets N_i:

- $j \in N_1$: $f(j) > 0$,
- $j \in N_2$: $f(j) = 0$, and $f_l(j) = 0$ for all $l \in \mathbb{N}$,
- $j \in N_3$: $f(j) = 0$, and $f_l(j) > 0$ for all $l \in \mathbb{N}$.

Let us now consider a sequence $(x^l)_l$ with $x^l \in V_{f_l}(S)$ and $\lim_l x^l = x$. For $argmin_{j \in S} f(j) \in N_2$, we are done, since $V_{f_l}(S) = V_f(S) = \mathbb{R}^n_-$. For all other cases, we may assume w. l. o. g. that each x^l can be written as

$$x^l = (y_1^l, \ldots, [\min_{j \in S} f_l(j)] y_k^l, \ldots, y_n^l) \text{ for some } k \in S, y^l \in V(S)$$

with $[\min_{j \in S} f_l(j)] y_k^l \geq \max_{r \in S \setminus k} y_r^l$.

For $argmin_{j \in S} f(j) \in N_1$, we have that $\min_{j \in S} f_l(j) > 0$ for l sufficiently large. Therefore, because of closure of $V(S)$, the proof is trivial.

Let us consider the remaining case, $argmin_{j\in S} f(j) \in N_3$. We have to show that $\lim_l \min_{j\in S} f_l(j) y_k^l = x_k \leq 0$. We assume, for the contrary, that $x_k > 0$. For $x_k > \varepsilon > 0$, there exists an $L \in \mathbb{N}$ s. t. $\|x^l - x\|_{\sup} < \varepsilon$ and

$$\frac{x_k - \varepsilon}{\min_{j\in S} f_l(j)} < y_k^l \text{ for all } l \geq L.$$

Now, we define $\overline{y}^l \in \mathbb{R}^n$ by

$$\overline{y}_i^l = \begin{cases} x_i - \varepsilon & \text{for } i \in S \setminus k, \\ \frac{x_k - \varepsilon}{\min_{j\in S} f_l(j)} & \text{for } i = k, \\ 0 & \text{else.} \end{cases}$$

As $V(S)$ is comprehensive and $\overline{y}^l \leq y^l$, we have that $\overline{y}^l \in V(S)$ for all $l \geq L$. However, $\lim_l \overline{y}_k^l \to \infty$ contradicts boundedness. Thus, $x_k \leq 0$ is valid and $V_\bullet(S)$ is upper hemi-continuous.

Step 2: clCCH $\left(\bigcup_{j=1}^{i} V_\bullet(S_i)\right)$ *is u. h. c. for every* $i \in N$

Let a sequence $(y^l)_l$ be given with $y^l \in clCCH\left(\bigcup_{j=1}^{i} V_{f_l}(S_j)\right)$ and $\lim_l y^l = y$. Using the fact that the comprehensive hull of the closure of the set A equals the closure of the comprehensive hull of A, one can see the following: For each $l \in \mathbb{N}$, there exists a vector $\overline{y}^l \geq y^l$ with $\overline{y}^l \to \overline{y}$ s. t., for each $\varepsilon > 0$, there are $\lambda^{jkl} \in \mathbb{R}, z^{jkl} \in \mathbb{R}^n, 1 \leq j \leq i, 1 \leq k \leq j$, with

$$\left\| \overline{y}^l - \sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{jkl} z^{jkl} \right\|_{sup} < \varepsilon. \tag{3.10}$$

The λ^{jkl} are non-negative and satisfy $\sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{jkl} = 1$; provided that $f_l(j) > 0$ (the case $f_l(j) = 0$ is trivial), z^{jkl} is an element of $V_{f_l}(S_j)$, and can be written as $z^{jkl} = (r_1^{jkl}, \dots, f_l(j) r_k^{jkl}, \dots, r_n^{jkl})$ with $r^{jkl} \in V(S_j)$ and $f_l(j) r_k^{jkl} \geq \max_{m\in S\setminus k} r_m^{jkl}$. As mentioned before, the separating hyperplane theorem guarantees the existence of a vector $a \in \mathbb{R}_{++}^n$ and a constant $\beta \in \mathbb{R}_{++}$ s. t. each $x \in V(N)$ satisfies $\langle a, x \rangle \leq \beta$. Observing formula (3.10), we have that, for ε sufficiently small,

$$\left\langle a, \sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{jkl} z^{jkl} - \overline{y}^l \right\rangle \geq -\beta. \tag{3.11}$$

Furthermore, there exists an $m \in \{0, \ldots, n\}$ with $f(m) = 1 > f(m+1)$, where we define $f(0) = 1$ and $f(n+1) = 0$. For $m \geq i$, u. h. c. can be seen without any problems, since

$$clCCH\left(\bigcup_{j=1}^{i} V_{f_l}(S_j)\right) \subseteq V(S_i) = clCCH\left(\bigcup_{j=1}^{i} V_f(S_j)\right).$$

Hence, let us consider the case $m < i$. For $\lambda^{lm} := \sum_{j=1}^{m}\sum_{k=1}^{j}\lambda^{jkl}$ and $z^{lm} := \sum_{j=1}^{m}\sum_{k=1}^{j} z^{jkl}$, we have that $\lambda^{lm} z^{lm} \in V(S_m)$ since, obviously, $z^{jkl} \in V(S_m)$ is true for all $j \leq m$, $k \leq j$.

Next, we will show that $\lambda^{lm} z^{lm}$ and $\lambda^{jkl} z^{jkl}$, $m < j \leq i, 1 \leq k \leq j$, are bounded. If there exists an $o \in N$ s. t. $\lambda^{lm} z_o^{lm} \to \infty$, there exists a $p \in N, p \neq o$, with $\lambda^{lm} z_p^{lm} \to -\infty$ since $V(S_m) \cap (x + \mathbb{R}_+^n)$ is bounded for every $x \in \mathbb{R}^n$. This, however, means that there is at least one pair (j^*, k^*), $i \geq j^* > m$, $j^* \geq k^* \geq 1$ with $\lambda^{j^*k^*l} z_p^{j^*k^*l} \to \infty$ (cf. formula (3.10)). In a similar way, one can show that $\lambda^{jkl} z_o^{jkl} \to -\infty$ for some $i \geq j > m$, $j \geq k \geq 1$ implies that$\lambda^{\overline{j}\overline{k}l} z_p^{\overline{j}\overline{k}l} \to \infty$ for a $p \in N, \overline{j} > m$ and $\overline{j} \geq \overline{k} \geq 1$. Therefore, it is sufficient to prove that $\lambda^{jkl} z_o^{jkl} \to \infty$ is not possible for any j, k, o with $i \geq j > m, j \geq k \geq 1$, and $o \in N$. Assuming the contrary immediately leads to a pair (j^*, k^*), $j^* > m$, with $\lambda^{j^*k^*l} z_{k^*}^{j^*k^*l} \to \infty$ as $z_{k^*}^{j^*k^*l} = f_l(j^*) r_{k^*}^{j^*k^*l} \geq r_o^{j^*k^*l} = z_o^{j^*k^*l}$. With a and β given by the separating hyperplane theorem, we have that

$$\begin{aligned}\langle a, \lambda^{j^*k^*l} z^{j^*k^*l}\rangle &= \langle a, \lambda^{j^*k^*l} r^{j^*k^*l}\rangle - (1 - f_l(j^*)) a_{k^*} \lambda^{j^*k^*l} r_{k^*}^{j^*k^*l} \\ &\leq \beta - (1 - f_l(j^*)) a_{k^*} \lambda^{j^*k^*l} r_{k^*}^{j^*k^*l}.\end{aligned}$$

There exists a constant $M \in \mathbb{N}$ with $-M\beta < \langle a, \overline{y}\rangle$, i. e.

$$-(M+1)\beta < \langle a, \overline{y}^l\rangle$$

for l sufficiently large. For $\lambda^{j^*k^*l} z_{k^*}^{j^*k^*l} > \frac{f_l(j^*)(M+5)\beta}{a_{k^*}(1-f_l(j^*))}$, we have that $\langle a, \lambda^{j^*k^*l} z^{j^*k^*l}\rangle < -(M+4)\beta$, which implies that $\langle a, \sum_{j=1}^{i}\sum_{k=1}^{j}\lambda^{jkl} z^{jkl}\rangle < -(M+3)\beta$. Now, this inequality, together with (3.11), implies that

$$\begin{aligned}-(M+3)\beta &> \left\langle a, \sum_{j=1}^{i}\sum_{k=1}^{j}\lambda^{jkl} z^{jkl}\right\rangle \\ &\geq -\beta + \langle a, \overline{y}^l\rangle \\ &> -(M+2)\beta,\end{aligned}$$

i. e., all in all, we have that $0 > \beta$. This is a contradiction, and,thus, the sequence $(\lambda^{jkl} z^{jkl})_l$ is bounded.

This fact provides the existence of a convergent subseries $(\lambda^{jkl_m} z^{jkl_m})_m$ s. t. also $(\lambda^{jkl_m})_m$ converges. Furthermore, the subsequence $(\lambda^{jkl_m})_m$ can be built in such a way that all (j,k) are an element of one of the following sets $T_i, i \in \{1,2,3\}$:

- $(j,k) \in T_1$, if $\lim_m \lambda^{jkl_m} > 0$,
- $(j,k) \in T_2$, if $\lambda^{jkl_m} = 0$ for all m,
- $(j,k) \in T_3$, if $\lim_m \lambda^{jkl_m} = 0$ and $\lambda^{jkl_m} > 0$ for all m.

For reasons of simplicity we again assume that the subsequences coincide with the corresponding sequences. Let us denote the limit of $(\lambda^{jkl} z^{jkl})$, $(j,k) \in T_3$, with $\overline{z}^{jk}$. We will show next that these $\overline{z}^{jk}$ are directions of recession for $V_f(S_j)$. Since $V_\bullet(S_j)$ is u. h. c., we have that $\overline{z}^{jk} \in V_f(S_j)$. Moreover, there exists a sequence $(\varepsilon_l)_l$ with $\varepsilon_l > 0, \lim_l \varepsilon_l \to 0$, and

$$\frac{1}{\lambda^{jkl}} \overline{z}^{jk} - \varepsilon_l e \leq z^{jkl} \in V_{f_l}(S_j)$$

for all $l \in \mathbb{N}$. For a given $\mu \geq 0$, we have that $\frac{1}{\lambda^{jkl}} > \mu$ for sufficiently large l. As $0 \in [V_{f_l}(S_j) + \varepsilon_l e]$, and since the latter set is convex, we have that $\mu \overline{z}^{jk} \in [V_{f_l}(S_j) + \varepsilon_l e]$. Taking the limit of this expression, and observing upper hemi-continuity of $V_\bullet(S_j)$, provides that $\overline{z}^{jk}$ is a direction of recession.

For each $(j,k) \in T_1$, we have that $\lambda^{jkl} \to \lambda^{jk}, z^{jkl} \to z^{jk}$, and for $(j,k) \in T_3$ we have defined $\overline{z}^{jk} := \lim_l \lambda^{jkl} z^{jkl}$. Let $\varepsilon > 0$ be given and $\mu \geq 1$ sufficiently large such that $\mu \varepsilon e \geq \sum_{(j,k) \in T_1} \lambda^{jk} z^{jk}$ is true. Then, we have the trivial inequality

$$\begin{aligned} \sum_{(j,k) \in T_1} \lambda^{jk} z^{jk} + \sum_{(j,k) \in T_3} \overline{z}^{jk} - \varepsilon e &= \left(1 - \frac{1}{\mu}\right) \sum_{(j,k) \in T_1} \lambda^{jk} z^{jk} + \\ &\quad \frac{1}{\mu} \left(\sum_{(j,k) \in T_1} \lambda^{jk} z^{jk} + \mu \left[\sum_{(j,k) \in T_3} \overline{z}^{jk} - \varepsilon e \right] \right). \end{aligned} \tag{3.12}$$

Since the vectors $\overline{z}^{jk}$, $(j,k) \in T_3$, are directions of recession, one can see that

$$\sum_{(j,k)\in T_1} \lambda^{jk} z^{jk} + \mu \left(\sum_{(j,k)\in T_3} \overline{z}^{jk} - \varepsilon e \right) \leq \mu \sum_{(j,k)\in T_3} \overline{z}^{jk} \in CCH \left(\bigcup_{j=1}^{i} V_f(S_j) \right).$$

Together with (3.12),

$$\sum_{(j,k)\in T_1} \lambda^{jk} z^{jk} + \sum_{(j,k)\in T_3} \overline{z}^{jk} \in clCCH \left(\bigcup_{j=1}^{i} V_f(S_j) \right)$$

is valid. All in all, we have that, for large l,

$$\begin{aligned}
& \left\| \overline{y} - \sum_{(j,k)\in T_1} \lambda^{jk} z^{jk} - \sum_{(j,k)\in T_3} \overline{z}^{jk} \right\|_{\sup} \\
\leq \quad & \left\| \overline{y} - \overline{y}^l \right\|_{\sup} + \left\| \overline{y}^l - \sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{jkl} z^{jkl} \right\|_{\sup} \\
& + \left\| \sum_{j=1}^{i} \sum_{k=1}^{j} \lambda^{jkl} z^{jkl} - \sum_{(j,k)\in T_1} \lambda^{jk} z^{jk} - \sum_{(j,k)\in T_3} \overline{z}^{jk} \right\|_{\sup} \\
< \quad & 3\varepsilon,
\end{aligned}$$

i.e. $\overline{y} \in clCCH \left(\bigcup_{j=1}^{i} V_f(S_j) \right)$. Since $y \leq \overline{y}$, this implies that y is also an element of $clCCH \left(\bigcup_{j=1}^{i} V_f(S_j) \right)$.

Step 3: $\overline{V}^C$ is u. h. c.

Let $x^l = \sum_{i=1}^{n} [f_l(i) - f_l(i+1)] y^{il}$ be given with $y^{il} \in clCCH \left(\bigcup_{j=1}^{i} V_{f_l}(S_j) \right)$ and $\lim_l x^l = x$. As in Step 2, where we have proven boundedness of $(\lambda^{jkl} z^{jkl})_l$, one can show that, for each $i \in N$, there exists a subsequence $([f_{l_m}(i) - f_{l_m}(i+1)] y^{il_m})_m$ which converges. W. l. o. g., we may assume that this subsequence coincides with the sequence. Let us define $I := \{i \in N |\, f(i) = f(i+1)\}$, $f(n+1) := 0$. In Step 2, we have shown that for $(j,k) \in T_3$ the sequence $(\lambda^{jkl} z^{jkl})$ converges to a direction of recession for $V_f(S_j)$. In a similar way, one can prove that for $i \in I$ the limit of $[f_l(i) - f_l(i+1)] y^{il} =: \overline{y}^i$ is a direction of recession for $clCCH \left(\bigcup_{j=1}^{i} V_f(S_j) \right)$. For $i \in N \setminus I$, we define $\lim_l [f_l(i) - f_l(i+1)] y^{il} =: [f(i) - f(i+1)] y^i$.

If $f = \emptyset$, we have that $x = \lim_l x^l = \sum_{i=1}^n \overline{y}^i$. Since each $\overline{y}^i$ is an element of $clCCH\left(\bigcup_{j=1}^i V_f(S_j)\right)$ and $V_f(S_j) = \mathbb{R}_l^n$, we have in this case that $\overline{y}^i \leq 0$. Hence, x cannot be positive, and we are done.

If $f \neq \emptyset$, there exists $1 \leq m \leq n$ with $f(m) > 0$ and $f(m+1) = 0$, where we again define $f(n+1) = 0$. Since, for $k > m$, we always have that $V_f(S_k) = \mathbb{R}_-^n$, one can easily see that $\overline{y}^i \in clCCH\left(\bigcup_{j=1}^m V_f(S_j)\right)$ for all $i \in I$. Hence, we obtain with $\widehat{y}^m := y^m + \sum_{i \in I} \frac{1}{f(m)} \overline{y}^i \in clCCH(\bigcup_{j=1}^m V_f(S_j))$ the following:

$$\begin{aligned} x = \lim_{l \to \infty} x^l &= \lim_{l \to \infty} \sum_{i=1}^n [f_l(i) - f_l(i+1)] y^{i,l} \\ &= \sum_{i \in N \setminus (I \cup m)} [f(i) - f(i+1)] y^i + f(m) y^m + \sum_{i \in I} \overline{y}^i \\ &= \sum_{i \in N \setminus (I \cup m)} [f(i) - f(i+1)] y^i + f(m) \widehat{y}^m \\ &\in \overline{V}^C(f). \end{aligned}$$

q.e.d.

3.4 The Core

In this section, we will answer the questions "How should the core be defined for a fuzzy NTU game?" and "How is the relationship between the core for the Choquet extension and its underlying NTU game?". The second problem is solved by observing that both V^C and $\overline{V}^C$ are monotone.

The following definition is a one-to-one translation of the two definitions for crisp games (cf. Rosenmüller [23][Definition 4.7.1 and Definition 4.7.9]) in the language of fuzzy games.

Definition 3.25 *Let a fuzzy NTU game (N, V^F) be given.*

1. *The* **weak core of V^F** *is given by*

$$\mathcal{C}^W(V^F) := V(N) \cap \bigcap_{f \in [0,1]^N} \mathcal{B}^{V^F}(f),$$

where $\mathcal{B}^{V^F}$ *is defined as*

$$\mathcal{B}^{V^F}(f) := \{x \in \mathbb{R}^n \,|\, \nexists y \in V^F(f) \text{ s. t. } y_i > x_i \text{ for } i \in C(f)\}, f \in [0,1]^N \ .$$

2. *Using*

$$\mathcal{I}^{V^F}(f) := \{x \in \mathbb{R}^n \,|\, \nexists y \in V^F(f) \setminus \{x_{C(f)}\} \text{ s. t. } y_i \geq x_i \text{ for } i \in C(f)\},$$

$f \in [0,1]^N$ *, we define the* **strong core of V^F** *by*

$$\mathcal{C}^S(V^F) := V(N) \cap \bigcap_{f \in [0,1]^N} \mathcal{I}^{V^F}(f).$$

The inclusion Core(N, V^F) $\subseteq$ Core(N, V) is obvious for every extension V^F, especially for the one of Choquet. However, for this case, we can even show equality provided (N, V) is monotone:

Theorem 3.26 *Let (N, V) be a monotone NTU game. Then we have that $\mathcal{C}^W(V^C) = \mathcal{C}^W(V)$ and $\mathcal{C}^S(V^C) = \mathcal{C}^S(V)$.*

Proof Theorem 3.13 provides monotonicity of V^C. In particular, we know that $y \in V^C(f)$ implies $y \in V^C(C(f)) = V(C(f))$. Let us now consider an $S \in \underline{\underline{P}}(N)$, a fuzzy coalition f with $C(f) = S$, and a vector $x \in \mathbb{R}^n$. Monotonicity means that, if there exists no $y \in V(S)$ s. t. $y_i > x_i$ for all $i \in S$, then there is no $y \in V^F(f)$ with $y_i > x_i$ for $i \in C(f)$. Hence, $\mathcal{C}^W(V^C) = \mathcal{C}^W(V)$ is true. Since the same argument can be used for the strong core, we are done. **q.e.d.**

One can see easily that $x \in \mathcal{C}^S(V^C)$ implies $x \geq 0$ (provided (N, V) is monotone). Assuming the contrary easily implies a contradiction since $0 \in V(\{i\})$ is valid for every $i \in N$.

The weak and the strong core of a fuzzy NTU game without dimension gap can be defined equivalently to the corresponding expressions in definition 3.25. The only thing to do is to substitute V^F with $\overline{V}^F$ and to exchange the expression $C(f)$ with N. Because of the monotonicity of $\overline{V}^C$, one can use the same proof as before to show that the crisp and the fuzzy cores are equal.

Other solution concepts for NTU games, like the Nash value or the Kalai-Smorodinsky value, only consider $V(N)$ and $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$ with $\underline{x}_i := \max\{t | \, te^i \in V(\{i\})\}$, i.e. the maximal value each player can get when playing alone. In other words, we only need N and the single coalitions, and do not have to care about the other $S \in \underline{\underline{P}}(N)$. Hence, we can also use these solution concepts, without any changes for the Choquet games.

Appendix A

Crisp Games: Definition, Properties, and Solution Concepts

The definitions, statements, and proofs of this chapter can be found in almost all standard books about cooperative game theory. As an example, we refer to Rosenmüller [23, Chapters 3 and 4].

Let us assume that there is a group of individuals N which is not necessarily finite. These players are participating in a situation calling for decisions. They discuss formation of coalitions and eventually enter a contract. It may happen that not every subset of N is feasible in some sense, either by mathematical reasons or by the structure of the game. Hence, we have to specify a class of admissible coalitions $\mathcal{P} \subseteq \underline{\underline{P}}(N)$. For technical reasons, we require the following: If $S_i, i \in \mathbb{N}$, are elements of $\mathcal{P}$, then $\bigcup_i S_i$ and $\bigcap_i S_i$ are also elements of $\mathcal{P}$, i. e. $\mathcal{P}$ has to be closed under countable union and countable intersection.

A.1 Games with Transferable Utility

If a feasible coalition S chooses to cooperate, it achieves a utility $v(S) \in \mathbb{R}$, i. e. we need a mapping $v : \mathcal{P} \longrightarrow \mathbb{R}$ that describes the incentives to cooperate.

Definition A.1 *A* **cooperative game with transferable utility** *is a tripel* $(N, \mathcal{P}, v)$ *with*

- *the set of players* N*,*
- *the set of feasible coalitions* $\mathcal{P}$ *(in the case of finitely many players, we usually take the power set of* N*), and*
- *the coalitional function* $v : \mathcal{P} \to \mathbb{R}$*,* $v(\emptyset) = 0$*.*

For different reasons, one is interested in set functions v having certain properties. The properties we are using in this dissertation will be stated next.

Definition A.2 *Let* $(N, \mathcal{P}, v)$ *be a TU-game. The set function* v *is called*

- **additive** *if*
$$v(S) + v(T) = v(S \cup T)$$
is valid for $S, T \in \mathcal{P}$ *with* $S \cap T = \emptyset$*.*
- **superadditive** *if*
$$v(S) + v(T) \leq v(S \cup T)$$
is valid for $S, T \in \mathcal{P}$ *with* $S \cap T = \emptyset$*.*
- **convex** *if*
$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$
is true for all $S, T \in \mathcal{P}$*.*
- **monotone** *if*
$$v(S) \leq v(T)$$
is valid for $S, T \in \mathcal{P}$ *with* $S \subseteq T$*.*

- **continuous from below** *if, for every increasing sequence of coalitions* $(S_n)_n, S_n \in \mathcal{P}$, *the following is true:*

$$v\left(\bigcup_{n=1}^{\infty} S_n\right) = \lim_{n\to\infty} v(S_n).$$

- **totally monotone** *if it is non-negative and, for every* $n \geq 2$ *and* $S_1, \ldots, S_n \in \mathcal{P}$,

$$v\left(\bigcup_{i=1}^{n} S_i\right) \geq \sum_{\{I|\emptyset\neq I\subseteq\{1,\ldots,n\}\}} (-1)^{|I|+1} v\left(\bigcap_{i\in I} S_i\right).$$

A solution concept is a mapping from the space of set functions $\mathcal{V} := \{v|\, v : \mathcal{P} \longrightarrow \mathbb{R}, v(\emptyset) = 0\}$ to the power set of $\mathbb{R}^N$, i.e. a solution concept is a proposal to divide the money. We will now present two of the most popular concepts: the core and the Shapley value. Let a game $(N, \mathcal{P}, v)$ be given with a finite set of players, i.e. $N = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, and $\mathcal{P} = \underline{\underline{P}}(N)$. Furthermore, let $\mathcal{A}$ denote the set of additive mappings on $\mathcal{P}$.

The set

$$\mathcal{C}(v) := \{x \in \mathcal{A}|\, x(S) \geq v(S) \text{ for all } S \in \mathcal{P}, x(N) = v(N)\}$$

is called the **core** of the game $(N, \mathcal{P}, v)$.

Theorem A.3 *If* v *is convex, then* $\mathcal{C}(v) \neq \emptyset$.

The **Shapley value** is the mapping $\Phi : \mathcal{V} \longrightarrow \mathcal{A}$ that is given by means of the formula

$$\Phi_i(v) := \frac{1}{n!}\left(\sum_{\pi\in\Pi} \left[v(S^{\pi}_{\pi(i)}) - v(S^{\pi}_{\pi(i)} - i)\right]\right),$$

where Π is the set of permutations of N, i.e. one-to-one functions from N to itself, and $S^{\pi}_k := \{i|\, \pi(i) \leq k\}$.

We define the uniform distribution on $T \in \mathcal{P}$ by $\mu^T \in \mathbb{R}^n$,

$$\mu^T_i = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T, \\ 0, & \text{otherwise.} \end{cases}$$

The unanimity games e^T are given by

$$e^T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

It is a well known fact that these unanimity games form a basis of $\mathcal{V}$. To be more precise, we have that, for each $v \in \mathcal{V}$,

$$v = \sum_{S \in \mathcal{P}, S \neq \emptyset} c_S(v) e^S \text{ with } c_S(v) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{A.1}$$

Theorem A.4 *The Shapley value can be written as*

$$\Phi(v) = \sum_{S \in \mathcal{P}} c_S(v) \mu^S.$$

A third approach for the Shapley value is an axiomatic one. First of all we need some preparatory remarks: A permutation $\pi : N \longrightarrow N$ induces a mapping $\pi : \mathcal{V} \longrightarrow \mathcal{V}$ by $(\pi v)(S) := v(\pi^{-1}(S))$. Let v_T be the restriction of v to T, i. e. $v_T(S) = v(T \cap S)$ for all $S \in \mathcal{P}$. Then, $C(v) = \bigcap_{T, v_T = v} T$ is called the carrier of v. A player $i \notin C(v)$ is called a null player.

Theorem A.5 *The Shapley value Φ is the only mapping $\phi : \mathcal{V} \longrightarrow \mathcal{A}$ that satisfies the following four axioms:*

1. *Additivity: $\phi(v) + \phi(w) = \phi(v + w)$ for all $v, w \in \mathcal{V}$,*
2. *Pareto efficiency: $\phi(v)(N) = v(N)$,*
3. *Symmetry: $\phi(\pi v) = \pi \phi(v)$,*
4. *Respecting of null players: $C(\phi(v)) \subseteq C(v)$.*

A.2 Games with Non-Transferable Utility

In this section, we are only interested in a finite set of players $N = \{1, \ldots, n\}$. Moreover, we declare all coalitions as feasible. As we will see, the games with non-transferable utility are much more general as the TU-games. In the NTU-case, we have a mapping V from the set of the feasible coalitions into the power set of $\mathbb{R}^n$. This mapping V assigns a set $V(S)$ of utility vectors to each coalition S, i. e. S can achieve every vector in $V(S)$ by the coorperation of its members.

Let S be a subset of $\{1, \dots, n\}$. Define $\mathbb{R}^{\boldsymbol{n}}_{\boldsymbol{S}}$ as the subspace of $\mathbb{R}^n$ spanned by the vectors $(e^i)_{i \in S}$ where e^i denotes the ith unit vector. For $x \in \mathbb{R}^n$, let $\boldsymbol{x^S} \in \mathbb{R}^n_S$ be the vector defined by

$$x_i^S = \begin{cases} x_i, & i \in S \\ 0, & i \in S^C. \end{cases}$$

A set $A \subseteq \mathbb{R}^n$ is called **comprehensive**, if for all $x \in A, y \in \mathbb{R}^n, y \leq x$ implies $y \in A$. $A \subseteq \mathbb{R}^n_S$ is called **S−comprehensive**, if for all $x \in A, y \in \mathbb{R}^n_S, y_S \leq x_S$ implies $y \in A$.

Definition A.6 *An* **n-person cooperative game without sidepayments (NTU)** *is a pair* (N, V)*, where* $N := \{1, \dots, n\}$ *is the set of players and* $V : \underline{\underline{P}}(N) \longrightarrow \underline{\underline{P}}(\mathbb{R}^n)$ *is a function that associates with every* $S \subseteq N$ *a nonempty set* $V(S) \subset \mathbb{R}^n_S$ *such that*

1. *$V(S)$ is S-comprehensive,*
2. *$V(S)$ is closed, and*
3. *for every $x^S \in \mathbb{R}^n_S$, the set $V(S) \cap (x^S + \mathbb{R}^n_{S+})$ is bounded.*

An implication of this definition is that $V(\emptyset) = 0$. Here, 0 denotes the n-dimensional null-vector. The following remark explains how a TU-game can be imbedded into the NTU-context:

Remark A.7 *Let a TU-game* $(N, \underline{\underline{P}}(N), v)$ *be given. Then,* (N, V_v) *given by*

$$V_v(S) := \left\{ x \in \mathbb{R}^n_S \middle| \sum_{i \in S} x_i \leq v(S) \right\}$$

is an NTU-game.

Now, we would like to state the definitions of some solution concepts for NTU-games. Each $V(\{i\})$ is a half-open interval, i.e. of the form $V(\{i\}) = (-\infty, \underline{x}_i]$. We denote by $\underline{x} \in \mathbb{R}^n$ the vector $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$. The space $\overline{V}$ may consist of all the functions $V : \underline{\underline{P}}(N) \longrightarrow \underline{\underline{P}}(\mathbb{R}^n)$, where $V(N)$ is convex and $\underline{x} \in V(N)$. For each $V \in \overline{V}$, the function $g^V : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$g^V(x) = \prod_{i \in N} (x_i - \underline{x}_i)$$

has a unique maximizer ν on $U_{\underline{x}} := \{x \mid x \geq \underline{x}\} \cap V(N)$. ν is called the **Nash solution**. For an axiomatization of this concept, the reader is referred to Rosenmüller [23, Chapter 4, Theorem 2.16].

At the end of this chapter, the definition of the weak and the strong core are given: Let a NTU game (N, V) be given. Using

$$\mathcal{B}^V(S) := \{x \in \mathbb{R}^n \mid \nexists y \in V(S) \text{ s. t. } y_i > x_i \text{ for all } i \in S\} \quad (S \in \underline{\underline{P}}(N)),$$

we define the **weak core** of V by

$$\mathcal{C}^W(V) := V(N) \cap \bigcap_{S \in \underline{\underline{P}}(N)} \mathcal{B}^V(S).$$

The **strong core** of V is given by

$$\mathcal{C}^S(V) := V(N) \cap \bigcap_{S \in \underline{\underline{P}}(N)} \mathcal{I}^V(S),$$

where $\mathcal{I}^V$ is defined as

$$\mathcal{I}^V(S) := \{x \in \mathbb{R}^n \mid y \in V(S), y \geq x_S \text{ implies } y = x_S\} \quad (S \in \underline{\underline{P}}(N)).$$

Obviously, $\mathcal{C}^S(V) \subseteq \mathcal{C}^W(V)$ is true. There are games where the strong core is a strict subset of the weak core.

Appendix B

Properties of the Choquet Integral

In this chapter, we will define the Choquet integral [7], and will state some of the most important properties. For a more detailed discussion, the reader is referred to Denneberg [8].

Let N be a set and $\mathcal{B}$ be a $\sigma-$algebra on N. Let $\mu : \mathcal{B} \longrightarrow \mathbb{R}_+$ be a monotone set function. The **outer set function μ^*** and the **inner set function μ_*** are defined on $\underline{\underline{P}}(N)$ by

$$\begin{aligned}\mu^*(A) &= \inf\{\mu(B)|\, A \subseteq B \in \mathcal{B}\},\\ \mu_*(A) &= \sup\{\mu(C)|\, C \in \mathcal{B}, C \subseteq A\}.\end{aligned}$$

It is a well known fact that μ_* and μ^* are the smallest and greatest extension of μ to the power set of N, respectively. A function $f : N \longrightarrow [0, 1]$ is called **upper μ-measurable** if $\mu_*(f > t) = \mu^*(f > t)$ is valid almost everywhere on $[0, 1]$.

Definition B.1 *Let a measurable space $(N, \mathcal{B})$, a coalitional function v, and a fuzzy coalition f be given. If f is upper v-measurable, the* **Choquet integral of f with respect to v** *is defined as*

$$\int f\, dv := \int_0^1 v(f > t)\, dt.$$

The definition given by Denneberg [8] is much more general since he allows f to be a function on N to $\overline{\mathbb{R}}$ and $v(N) = \infty$. Since we use $\mathcal{B} = \underline{\underline{P}}(N)$ in most parts of this work, we do not have to care for measurability.

If v is σ-additive, the Choquet integral coincides with the usual definition of the integral known from measure theory.

For a step function $f = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ with $\alpha_1 \geq \ldots \geq \alpha_n$, the Choquet integral of f w. r. t. v can be written as

$$\begin{aligned} \int f\,dv &= \sum_{i=1}^{n} \alpha_i (v(S_i) - v(S_{i-1})) \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) v(S_i), \end{aligned}$$

where $S_i = A_1 \cup \ldots \cup A_i, i = 1, \ldots, n, S_0 = \emptyset, \alpha_{n+1} = 0$.

Definition B.2 *Two fuzzy coalitions f, g are called* **comonotonic**, *if there is no pair $(i, j) \in N$ s. t. $f(i) < f(j)$ and $g(i) > g(j)$.*

Theorem B.3 *[8, Proposition 5.1, Corollary 6.4, Corollary 6.5]*

Let N be a set, v be a monotone set function on a σ-algebra $\mathcal{B}$ on N, and let f, g be upper v-measurable fuzzy coalitions.

1. *$\int 1_A\,dv = v(A)$ for all $A \in \mathcal{B}$,*
2. *$\int cf\,dv = c \int f\,dv$ for $c \geq 0$,*
3. *$f \leq g$ implies $\int f\,dv \leq \int g\,dv$,*
4. *If f and g are comonotonic, then $\int (f+g)\,dv = \int f\,dv + \int g\,dv$,*
5. *If v is additive, we have that $\int (f+g)\,dv = \int f\,dv + \int g\,dv$,*
6. *If v is convex, we have that $\int (f+g)\,dv \geq \int f\,dv + \int g\,dv$, i. e. the Choquet integral is superadditive in this case.*

A function $f : N \longrightarrow \mathbb{R}$ is called **upper $\mathcal{B}$-measurable**, if it is upper v-measurable for any monotone set function v on $\mathcal{B}$.

Theorem B.4 *[8, Proposition 5.2]*

Let v, w be monotone set functions on the measurable space $(N, \mathcal{B})$, and f be a upper $\mathcal{B}$-measurable fuzzy-coalition. Then:

1. *For $c > 0$, the multiple cv of v is a monotone set function on $\mathcal{B}$ and*

$$\int f\, d(cv) = c \int f\, dv.$$

2. *$v + w$ is a monotone set function on $\mathcal{B}$, and*

$$\int f\, d(v+w) = \int f\, dv + \int f\, dw.$$

3. *$v \leq w$ implies*

$$\int f\, dv \leq \int f\, dw.$$

4. *If (v_n) is a sequence of monotone set functions on $\mathcal{B}$ with $v_n \leq v_{n+1}$ and $\lim_{n\to\infty} v_n(A) = v(A)$ for all $A \in \mathcal{B}$, then*

$$\lim_{n\to\infty} \int f\, dv_n = \int f\, dv.$$

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Index of Notations

e	unit vector
e^i	ith unit vector
$\mathbb{R}^n_S$	subspace of $\mathbb{R}^n$ spanned by $(e^i)_{i\in S}$
x^S	projection of $x \in \mathbb{R}^n$ to $\mathbb{R}^n_S$
$CCHA$	convex, comprehensive hull of the set A
$CH_S(x^S)$	$S-$comprehensive hull of x^S
$intA$	interior of A
0^+A	recession cone of A
$clCCHA$	closure of the convex comprehensive hull of A
$\parallel \bullet \parallel$	variation norm
μ^*	outer set function of μ
μ_*	inner set function of μ
f^C	complement of the fuzzy coalition f
$\bigcap_{BK}$	intersection in the sense of Butnariu and Klement
$\bigcup_{BK}$	union in the sense of Butnariu and Klement
$(\Omega, \mathcal{P}, v)$	crisp TU game
$\underline{\underline{P}}(\Omega)$	power set of Ω
$\mathcal{V}$	space of all coalitional functions
1_S	indicator function of the coalition S
e^T	unanimous game w.r.t. T
$\mathcal{C}(v)$	core of v
$\Phi(v)$	Shapley value of v

$C(v)$	carrier of v
S_i^π	set of players which are ordered by the permutation π in front of i
$(\Omega, \overline{\mathcal{P}}, v^F)$	fuzzy TU game
v^O	Owen's extension of v
v^C	Choquet's extension of v
v^{+F}	upper variation of v
v^{-F}	lower variation of v
$\mathcal{A}$	space of all additive mappings
BV	space of crisp coalitional functions with bounded variation
$\mathcal{A}_1^{\sigma+}$	space of non-negative, σ-additive functions with total mass 1
$V^{\searrow}$	space of $v \in BV$ which are upper σ-continuous
$V^{\nearrow}$	space of $v \in BV$ which are lower σ-continuous
FBV	space of fuzzy functions with bounded variation
FNA	family of all finite non-atomic fuzzy measures
FNA^+	space of all monotone functions in FNA
$pFNA$	closed linear hull of $\{m^k \mid m \in FNA^+, k \in \mathbb{N}\}$
$FBVA$	set of finitely additive fuzzy functions with bounded variation
C_0	space of Choquet functions with finite carrier
$spanC_0$	closure of the linear hull of C_0
CBV	space of the Choquet extensions of the functions in BV
CM	space of monotone Choquet functions
CBV_+^C	space of convex and non-negative Choquet functions in CBV
$C_+^{\sigma,C}$	space of upper σ-continuous Choquet functions in CBV_+^C
φ_{SP}	fuzzy value given by the smoothing procedure
φ_{av}	fuzzy value given by averaging over small perturbations
φ_{Art}	fuzzy value on $spanC_0$ given by Artstein's approach
φ^{WM}	fuzzy value for weighted majority (Shapley's approach)
φ^{PR}	fuzzy value given by the approach of Pallaschke and Rosenmüller
$\varphi_{Ros}^{\mathcal{P}}$	fuzzy value w. r. t. the kernel $\mathcal{P}$

(N, V)	crisp NTU game
(N, V^C)	Choquet NTU game
$(N, \overline{V}^C)$	Choquet NTU game without dimension gap
$\mathcal{C}^W(V)$	weak core of V
$\mathcal{C}^S(V)$	strong core of V

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