Diffusions on Path Spaces over the Real Line with Singular Interaction via Dirichlet Forms

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Contents

1 Introduction

The aim of this work is to examine classical Dirichlet forms on $L^2(C(\mathbb{R},\mathbb{R}),\mu)$ where μ is a Gibbs measure on $C(\mathbb{R},\mathbb{R})$.

Gibbs measures are used to model equilibrium states in statistical mechanics. The first models were on the d-dimensional lattice $\mathbb{Z}^d$. Later it was generalised to other state spaces. In our case we consider Gibbs measures on the space $C(\mathbb{R},\mathbb{R})$ of continuous functions on the real line. This case was treated first by Osada and Spohn in 1999 (see [OS99]). Since there are already several articles on existence and uniqueness we do not cover this area and refer the reader to [Bet03], [Har06] and [BLS05].

The theory of Dirichlet forms was started by Beurling and Deny in the 1950s (cf. [BD58] and [BD59]) with the introduction of the name Dirichlet Form. A connection to stochastics was then developed in the 1970s in works by Fukushima and Silverstein. Early papers in this area are [Fuk71b] and [Fuk71a]. The major books on this subjects are then [Sil74], [Fuk80] and more recently [BH91] and [FOT94].

The theory was extended in two ways: The state space could also be infinite-dimensional, and the forms could be non-symmetric. The first papers on infinite dimensional state spaces appeared in the end of the 1970s, see eg. [AHK77]. Important results on closability can be found in [AR90b]. Results on quasi-regularity are in [RS92] and [RS95]. The development of the non-symmetric case occured during the same time. A book on non-symmetric forms on also infinite-dimensional state spaces is then [MR92]. This was later extended to so-called generalized Dirichlet forms by Stannat in [Sta99].

We are interested whether there exists a stochastic process that is associated to this bilinear form. The process we are interested in, is the Markov-process associated in the sense of [MR92] to the Dirichlet-form $\mathcal{E}(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \, \mathrm{d}\mu$. In this case H is the Sobolev-space $H^{1,2}(\mathbb{R})$ and μ a Gibbs-measure for the specification $(\pi_r^{\mathcal{H}})_{r>0}$.

A result of this kind (but with a slightly different space H) can be found in [HO01]. Hariya and Osada took as H the vector space

$$H := \{h \in C_0(\mathbb{R},\mathbb{R});\ h \text{ is absolutely continuous and } \int \dot{h}(x)^2 \, \mathrm{d}x < \infty\}$$

with the inner product $\langle h_1, h_2\rangle_H = \int \dot{h}_1(x)\dot{h}_2(x)\,\mathrm{d}x$. This space is not a Hilbert space. They also consider the spaces H_r defined as

$$H_r := \{h \in C(\mathbb{R},\mathbb{R});\ h \text{ is absolutely continuous, } h(x) = 0 \text{ if } |x| \geq r, \int \dot{h}(x)^2 \, \mathrm{d}x < \infty\}$$

with the inner product $\langle h_1, h_2\rangle_{H_r} = \int \dot{h}_1(x)\dot{h}_2(x)\,\mathrm{d}x$. They then define forms $\mathcal{E}_r(u,v) := \int \frac{1}{2}\langle \nabla_r u, \nabla_r v\rangle_{H_r} \, \mathrm{d}\mu$ which approximate $\mathcal{E} := \int \frac{1}{2}\langle \nabla u, \nabla v\rangle_H$ in the sense that the resolvents $G_{\alpha,r}$ of $\mathcal{E}_r$ converge strongly in $L^2(C(\mathbb{R},\mathbb{R}),\mu)$. They show then closability for the forms $\mathcal{E}_r$ and conclude that then $\mathcal{E}$ is also closable.

Our aim is to show closability directly for the form $\mathcal{E}(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \,\mathrm{d}\mu$. We examine the quasi-invariant case, i.e. when $\tau_{sh}(\mu) \ll \mu$, where $\tau_h(w) := w + h$ is the shift in direction h, for all $s \in \mathbb{R}$ and enough directions h.

The main result we use to show closability can be found in [AR89]. In comparison with the result in [HO01] we have to use a different space H, namely the Sobolevspace $H^{1,2}(\mathbb{R})$, and we need stronger assumptions on the potential φ.

Now we give an overview how this paper is organized and we give more details on our results.

In chapter 2 we define the kernels of our specification and prove that they are a specification in the sense of [Pre76] and [Geo88]. The specification $(\pi_r^{\mathcal{H}})_{r>0}$, is defined as follows:

Let m_r be a Gaussian measure on $C(\mathbb{R},\mathbb{R})$ that describes a Brownian bridge on the interval $[-r,r]$ and whose support are the functions that are equal to 0 outside the interval $[-r,r]$. The existence of such a measure is shown in the beginning of chapter 2.

Define now

$$H_r(\xi)(t) := \begin{cases} \xi(t), & t \notin [-r,r] \\ \xi(-r) + \frac{t-(-r)}{r-(-r)}(\xi(r) - \xi((-r))), & t \in [-r,r] \end{cases}$$

and $\tau_{r,\xi}(\omega) := \omega + H_r(\xi)$ for $\xi, \omega \in C(\mathbb{R},\mathbb{R})$ and $t, r \in \mathbb{R}$.

Now we can define the kernels π_r und $\pi_r^{\mathcal{H}}$ as follows: $\pi_r(\Lambda,\xi) := m_r \circ \tau_{H_r^{-1}}(\Lambda)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ be measurable, then $\mathcal{H}_r$ and $\pi_r^{\mathcal{H}}$ are defined as:

$$\begin{aligned} \mathcal{H}_r(w) &:= \int_{[-r,r]} \varphi(w(x))\mathrm{d}x + \frac{1}{2}\iint_{|x|,|y|\le r} \psi(x-y, w(x)-w(y))\mathrm{d}x\mathrm{d}y \\ &\quad + \iint_{|x|\le r<|y|} \psi(x-y, w(x)-w(y))\mathrm{d}x\mathrm{d}y \\ \pi_r^{\mathcal{H}}(\Lambda,\xi) &:= \int_\Lambda \frac{1}{\int_{C(\mathbb{R},\mathbb{R})} \exp(-\mathcal{H}_r(v))\,\pi_r(\xi,\mathrm{d}v)} \exp(-\mathcal{H}_r(w))\,\pi_r(\xi,\mathrm{d}w). \end{aligned}$$

The main result in this chapter is Proposition 2.3.1 which states that $(\pi_r^{\mathcal{H}})_{r>0}$ is a specification if $\psi(x,y) = \psi(-x,-y)$ for all $x, y \in \mathbb{R}$. In the end we recall the results on existence and uniquenes from [OS99], [Har06] and [Bet03].

In chapter 3 we define the bilinear form we are interested in and find a criterion for closability and apply it in two examples. More exactly we first define the form $\mathcal{E}$ on $\mathcal{F}C_b^\infty$. Then we show that the measures $\pi_r(\xi,\cdot)$ are k-quasiinvariant for all $k \in$ and calculate the densities $\frac{\mathrm{d}\pi_r(\xi,\cdot)\circ\tau_{sk}^{-1}}{\mathrm{d}\pi_r(\xi,\cdot)}$. This we use then to calculate the densities $\frac{\mathrm{d}\pi_r^{\mathcal{H}}(\xi,\cdot)\circ\tau_{sk}^{-1}}{\mathrm{d}\pi_r^{\mathcal{H}}(\xi,\cdot)}$, for the case that $\pi_r(\xi,\{\mathcal{H}_r = \infty\}) = 0$ for all $\xi \in C(\mathbb{R},\mathbb{R}))$. This assumption is reasonable, because in the case that $\pi_r^{\mathcal{H}}(\xi,\cdot)$ is quasiinvariant, we already have that $\pi_r(\xi, \mathcal{H}_r = \infty) \in \{0,1\}$, and in the case of $\pi_r(\xi,\{\mathcal{H}_r = \infty\}) = 1$ there is no reasonable way to define $\pi_r^{\mathcal{H}}(\xi,\cdot)$. As a last step we the calculate the densities $\frac{\mathrm{d}\mu\circ\tau_{sk}^{-1}}{\mathrm{d}\mu}$ for a Gibbs measure belonging to this specification and get:

$$
\begin{aligned}
a_{sk}(w) &:= \frac{\mathrm{d}\mu \circ \tau_{sk}^{-1}}{\mathrm{d}\mu}(w) = \\
&\exp(-\int sw \,\mathrm{d}(k)' - \frac{1}{2}\int s^2(k'(t))^2 \,\mathrm{d}t) \exp(-\int\limits_{\operatorname{supp} k} \varphi(w(t)-sk(t)) - \varphi(w(t)) \,\mathrm{d}t) \\
&\times \exp(-\int \int\limits_{0\le|r|<|t|\le r_0} \psi(r-t, w(r)-w(t)-s(k(r)-k(t))) - \psi(r-t, w(r)-w(t))) \,\mathrm{d}r \,\mathrm{d}t \\
&\times \exp(-\int \int\limits_{|r|\le r_0<|t|} \psi(r-t, w(r)-w(t)-sk(r)) - \psi(r-t, w(r)-w(t))) \,\mathrm{d}r \,\mathrm{d}t.
\end{aligned}
\tag{1.0.1}
$$

Then we use that the form $\mathcal{E}_k := \frac{1}{2}\int \frac{\partial u}{\partial k}\frac{\partial u}{\partial k}\,\mathrm{d}\mu$ is closable, if the mapping $s \mapsto a_{sk}$ fulfills the Hamza-condition, i.e.:

A $\mathcal{B}(\mathbb{R})$-measurable function $\rho : \mathbb{R} \to \mathbb{R}_+$ fulfills the Hamza-condition if $\rho = 0$ on $\mathbb{R} \setminus R(\rho)$ where

$$
R(\rho) := \left\{ t \in \mathbb{R} \middle| \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho}\,\mathrm{d}s < \infty \text{ for some } \varepsilon > 0 \right\}
$$

Finally we get the following statement: (see 3.5.1)

Suppose that there exists an ONB $(h_n)_{n\in\mathbb{N}} \subset C_0^1(\mathbb{R},\mathbb{R})$ of H such that $t \mapsto a_{th_n}(w)$ fulfills the Hamza-condition (see 3.1.3) for every $n \in \mathbb{N}$, then $(\mathcal{E}, \mathcal{F}C_b^\infty(C(\mathbb{R},\mathbb{R})))$ is closable.

As an application we treat two cases: The first is very similar to the one examined in [HO01]. We have to use stronger assumptions on the potential φ, but then we can show closability directly for the form $(\mathcal{E}, \mathcal{F}C_b^\infty(C(\mathbb{R},\mathbb{R})))$ without having to use approximation techniques. As a second example we show the same under the same conditions (again with the same stronger condition on φ) as in [Bet03].

In chapter 4 we first review some definitions which will be used in this chapter. In the second section we show that the bilinear form $(\mathcal{E}, D(\mathcal{E}))$ which is the closure constructed in chapter 3 is a Dirichletform, in this section we also get some estimates which we use in the next section to show that this form is quasi-regular. In the third section of this chapter we prove that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular and in the last section we apply the methods in [MR92] to construct the process associated to $(\mathcal{E}, D(\mathcal{E}))$. We show that $(\mathcal{E}, D(\mathcal{E}))$ is local and conservative and we conclude that the process is a diffusion. As a final remark we state that this process is a weak solution to a stochastic differential equation of the type $\mathrm{d}X_t = \mathrm{d}W_t + \frac{1}{2}\beta(X_t)\,\mathrm{d}t$, if a β exists such that $k(\beta) = 2\lim_{s\to 0} s^{-1}(1-\sqrt{a_{sk}})$ and this limit exists for all k in a dense linear supspace of E' of well-μ-admissible elements in E.

If we compare our results with the results of [HO01] we had to take a smaller space H, but then we could show closability and quasi-regularity directly for $\mathcal{E}$. In [HO01] they could show closability and quasi-regularity for forms $\mathcal{E}_r$ approximating $\mathcal{E}$ and could then show that $\mathcal{E}$ is closed and densely defined. They could also show that for all $\alpha > 0$

the resolvents $G_{r,\alpha}$ of $\mathcal{E}_r$ converge strongly in $L^2(C(\mathbb{R},\mathbb{R}),\mu)$ to the resolvent G_α of $\mathcal{E}$ as $r\to\infty$. They did not show quasi-regularity for $(\mathcal{E},\mathbb{D}(\mathcal{E}))$.

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2 Defining the Specification for the Considered Gibbs-measures

In this chapter we want to construct families of kernels which are specifications. First we have to construct measures on $C(\mathbb{R},\mathbb{R})$, then describe the kernels in terms of these measures, show that these are indeed kernels and finally check that they form a specification.

We want to consider Gibbs-measures on $C(\mathbb{R},\mathbb{R})$ with the specification $(\pi_r^{\mathcal{H}})_{r>0}$ defined as follows:

$$\pi_r^{\mathcal{H}}(\Lambda,\xi) := \int_\Lambda \frac{1}{\int_{C(\mathbb{R},\mathbb{R})} \exp(-\mathcal{H}_r(v))\,\pi_r(\xi,\mathrm{d}v)} \exp(-\mathcal{H}_r(w))\,\pi_r(\xi,\mathrm{d}w) \tag{2.0.1}$$

where

$$\pi_r(\Lambda,\xi) := \int_{C(\mathbb{R},\mathbb{R})} 1_\Lambda(w)\, m_r \circ \tau^{-1}_{H_r(\xi)}(\mathrm{d}w) \tag{2.0.2}$$

and

$$\begin{aligned}\mathcal{H}_r(w) :=& \int_{[-r,r]} \varphi(w(x))\mathrm{d}x \\ &+ \frac{1}{2}\iint_{|x|,|y|\le r} \psi(x-y,w(x)-w(y))\mathrm{d}x\mathrm{d}y \\ &+ \iint_{|x|\le r<|y|} \psi(x-y,w(x)-w(y))\mathrm{d}x\mathrm{d}y\end{aligned} \tag{2.0.3}$$

where we tacitly assume that these integrals exist for all $w \in C(\mathbb{R},\mathbb{R})$. To do this, we have to define $m_r, r>0$ as probability measures on $C(\mathbb{R},\mathbb{R})$ and the mappings $H_r, r>0$ and $\tau_h, h \in C(\mathbb{R},\mathbb{R})$. Furthermore we have to check that $(\pi_r^{\mathcal{H}})_{r\in\mathbb{R}}$ is a specification. The function φ is usually called the *external potential*, and ψ is called the *interaction potential*. Sometimes one denotes (φ,ψ) as the *potential*.

2.1 Definitions

We want to consider measures on $C(\mathbb{R},\mathbb{R})$ and $\mathbb{R}^{\mathbb{R}}$. The σ-algebras we consider are: $\mathcal{B}(C(\mathbb{R},\mathbb{R}))$ the Borel-σ-algebra on $C(\mathbb{R},\mathbb{R})$ where we consider the topology induced by the following metric:

$$d(f,g) := \sum_{i=1}^{\infty} 2^{-i}(\sup\{|f(x)-g(x)|| -i \le x \le i\} \wedge 1) \tag{2.1.1}$$

and on $\mathbb{R}^{\mathbb{R}}$ we take $\sigma(\{X_t|t \in \mathbb{R}\})$, where $X_t : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}, f \mapsto f(t)$. For simplicity we denote the restriction of X_t to $C(\mathbb{R},\mathbb{R})$ also by X_t.

We define the notion of a specification for the concrete case of $(C(\mathbb{R},\mathbb{R}), \mathcal{B}(C(\mathbb{R},\mathbb{R})))$ with sub-σ-algebras $\sigma(\{X_t|t \in (-r,r)^c\})$.

Definition 2.1.1. [cf. [Pre76, Chapter 2] or [Geo88]]
A family of kernels on $(C(\mathbb{R},\mathbb{R}), \mathcal{B}(C(\mathbb{R},\mathbb{R}))$ is called a specification if it fulfills the following properties:

1. $\pi_r X$ is $\sigma(\{X_t|t \in (-r,r)^c\})$-measurable for every bounded $\mathcal{B}(C(\mathbb{R},\mathbb{R}))$-measurable function X.
2. (Consistency condition) For each $s > r$ and every bounded $\mathcal{B}(C(\mathbb{R},\mathbb{R}))$-measurable function X and for every bounded $\sigma(\{X_t|t \in (-r,r)^c\})$-measurable function Z we have that $\pi_r(Z\pi_s X) = \pi_r(ZX)$.

Definition 2.1.2. A measure μ is called a Gibbs-measure for the specification $(\pi_r)_{r>0}$ if the following equation holds for all $r > 0$:

$$\mu\pi_r = \mu \qquad \text{i.e. } \forall A \in \mathcal{B}(C(\mathbb{R},\mathbb{R})) : \int_{C(\mathbb{R},\mathbb{R})} \pi_r(\xi, A)\, \mu(\mathrm{d}\xi) = \mu(A) \tag{2.1.2}$$

This equation is called the Dobrushin-Lanford-Ruelle-equation, or in short DLR-equation.

2.2 Preliminaries—the measures m_r

We want to construct Gaussian measures m_r on $C(\mathbb{R},\mathbb{R})$ such that they describe a Brownian Bridge on the interval $(-r,r)$ and the paths are outside of $(-r,r)$ equal to zero.

Let us define

$$K_r(s,t) := \begin{cases} (s \wedge t - (-r))(r - s \vee t)/(2r), & s,t \in [-r,r] \\ 0, & \text{otherwise} \end{cases}$$

Lemma 2.2.1. For all $t_1, \dots, t_n \in \mathbb{R}$, $n \in \mathbb{N}$ we have that $(K_r(t_i,t_j))_{i,j}$ is positive semidefinite, i.e. for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $t_1, \dots, t_n \in \mathbb{R}$, $n \in \mathbb{N}$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j K_r(t_i,t_j) \geq 0.$$

Proof. Denote by m the Lebesgue-measure on $[-r,r]$ and consider the following probability space: $([-r,r], \mathcal{B}([-r,r]), \frac{1}{2r}m)$. Then we can consider the following random variables: $Z_t : [-r,r] \to \mathbb{R}, x \mapsto 1_{[-r,x]}$, $t \in [-r,r]$. We then get:

$$\operatorname{cov}(Z_s, Z_t) = \frac{1}{2r}\int_{-r}^{r} 1_{[-r,s]}(x) 1_{[-r,t]}(x)\,\mathrm{d}x - \frac{1}{2r}\int_{-r}^{r} 1_{[-r,s]}(x)\,\mathrm{d}x \frac{1}{2r}\int_{-r}^{r} 1_{[-r,t]}(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{2r}(s \wedge t - (-r)) - \frac{1}{2r}(s - (-r))\frac{1}{2r}(t - (-r)) \\
&= \frac{1}{4r^2}\Big(2r(s \wedge t + r) - (s + r)(t + r)\Big) \\
&= \frac{1}{4r^2}\Big(2r(s \wedge t) + 2r^2 - st - r^2 - rs - rt\Big) \\
&= \frac{1}{4r^2}\Big(r^2 - st + r\big((s \wedge t - s) + (s \wedge t - t)\big)\Big) \\
&= \frac{1}{4r^2}\Big(r^2 - (s \wedge t)(s \vee t) + r(s \wedge t - \vee t)\Big) \\
&= \frac{1}{4r^2}\big((s \wedge t + r)(r - s \vee t)\big) \\
&= \frac{1}{2r}K(s,t)
\end{aligned}
$$

Let now $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{R}$, $n \in \mathbb{N}$. We can assume that $-r \leq \alpha_i \leq r$ for all $1 \leq i \leq n$, because otherwise all terms with t_i and α_i are equal to 0. Then we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j K_r(t_i, t_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j 2r \operatorname{cov}(Z_{t_i}, Z_{t_j}) = 2r \operatorname{Var}(\sum_{i=1}^{n} \alpha_i Z_{t_i}) \geq 0.$$

□

Now we can apply the following corollary from [Sim05, Corollary 2.4]

Corollary 2.2.2. Let $c(t,s)$ be a jointly continuous real-valued function on $K \times K$ where K is a separable topological space. Suppose that for any $t_1, \ldots, t_n \in K$, $c(t_i, t_j)$ is a positive semidefinite matrix. Then, there exits an essentially unique measure space $(X, \mathcal{F}, \mu)$ and a random variable $q(t)$ for each $t \in K$ so that the $q(t)$ are jointly Gaussian with covariance c.

$K(s,t)$ is obviously jointly continuous on $\mathbb{R} \times \mathbb{R}$. We then consider the image measure of μ under q on $\mathbb{R}^{\mathbb{R}}$ and get a Gaussian process $(\mathbb{R}^{\mathbb{R}}, \sigma(\{X_t | t \in \mathbb{R}\}, X_t)$ on $\mathbb{R}^{\mathbb{R}}$ with $\operatorname{cov}(X_s, X_t) = K_r(s,t)$ and $E[X_s] = 0$. We want to show that there exists a continuous version of this process.

Step 1: Restrict this process to $\mathbb{R}^{[-r-1, r+1]}$ in the obvious way.

Step 2: Apply the following theorem by Fernique ([Fer64] and [Fer65, Théorème 1]) which can be found in [Dud67, Theorem 7.1] in English.

Theorem 2.2.3. Suppose $(X_t)_{t \in T}$ is a Gaussian process where T is a bounded subset of $\mathbb{R}^k$. Suppose φ is a nonnegative real-valued function such that

1. $E|X_s - X_t|^2 \leq \varphi(|s-t|)^2$ for all $s, t \in T$.

2. $\varphi(u)$ is monotone-increasing on some interval $0 < u < \alpha$.

3. $\int_M^\infty \varphi(e^{-x^2})\,\mathrm{d}x < \infty$ for some $M < \infty$.

Then X_t is sample continuous, i.e. X_t has a continuous version.

Let us check the conditions:

1. $E[|X_s - X_t|^2] = K_r(s,s) - 2K_r(s,t) + K_r(t,t) \le |s-t| = \sqrt{(s-t)^2}$, since:

$s,t \notin [-r,r]$ In this case we have $E[|X_s - X_t|^2] = 0 \le |s-t|$.

$s < -r \le t \le r$ In this case we have $(s-(-r))(r-s)/(r-(-r)) = (t-(-r))\frac{r-t}{r-(-r)}$ $\le t-(-r) \le |t-s|$ since $s < -r \le t$.

$-r \le s \le r < t$ Here we have $(s-(-r))(r-s)/(r-(-r)) = (r-s)\frac{s-(-r)}{r-(-r)} \le r-s \le |t-s|$ since $s \le r < t$.

$s,t \in [-r,r]$ $\frac{(s-(-r))(r-t)-2(s-(-r))(r-t)+(t-(-r))(r-t)}{2r}$
$= \frac{(s-(-r))((r-s)-(r-t))+(r-t)((t-(-r))-(s-(-r)))}{2r} = \frac{(s-(-r))(t-s)+(r-t)(t-s)}{2r}$
$= (t-s)\frac{(r-(-r))-(t-s)}{2r} \le t-s = |t-s|$

2. $x \mapsto \sqrt{x}$ is monotone increasing

3. $\int_0^\infty \sqrt{\exp(-x^2)}\,\mathrm{d}x = \int_0^\infty \exp(-\frac{x^2}{2})\,\mathrm{d}x < \infty$.

Then there exists a continuous version of this process on $\mathbb{R}^{[-r-1,r+1]}$ and since the paths are almost surely constant outside the interval $[-r,r]$, there exists a continuous version of the original process on $\mathbb{R}^{\mathbb{R}}$. Let us now take the restriction of this process to $C(\mathbb{R},\mathbb{R})$ and denote the measure by m_r.

Let us define the following mappings:

$$H_r(\xi)(t) := \begin{cases} \xi(t), & t \notin [-r,r] \\ \xi(-r) + \frac{t-(-r)}{r-(-r)}(\xi(r) - \xi((-r))), & t \in [-r,r] \end{cases} \tag{2.2.1}$$

$$\tau_{r,\xi}(\omega) := \omega + H_r(\xi) \tag{2.2.2}$$

Now we can define the following families $(\pi_r)_{r\in\mathbb{R}}$ and $(\pi_r^{\mathcal{H}})_{r\in\mathbb{R}}$ of kernels on $C(\mathbb{R},\mathbb{R})$:

$$\pi_r(\xi, A) := m_r \circ \tau_{r,\xi}^{-1}(A) \tag{2.2.3}$$

$$\pi_r(\xi, f) := \int f(w)\pi_r(\xi, \mathrm{d}w) \tag{2.2.4}$$

$$\pi_r^{\mathcal{H}}(\xi, f) := \frac{1}{\pi_r(\xi, \exp(-\mathcal{H}_r))}\pi_r(\xi, e^{-\mathcal{H}_r} f) \tag{2.2.5}$$

$$\pi_r^{\mathcal{H}}(\xi, A) := \pi_r(\xi, 1_A) \tag{2.2.6}$$

We still have to show that these really are kernels, this will happen in the next section.

2.3 The family $(\pi_r^{\mathcal{H}})_{r>0}$ is a specification

In this section we want to prove that the family $(\pi_r^{\mathcal{H}})_{r>0}$ is a specification. We will show the following proposition:

Proposition 2.3.1. Assume that $\psi(x,y) = \psi(-x,-y)$ for all $x, y \in \mathbb{R}$, then the family $(\pi_r^{\mathcal{H}})_{r>0}$ is a specification, i.e. it has the following properties (cf. 2.1.1)

1. $(\pi_r^{\mathcal{H}})_{r>0}$ is a family of probability kernels on $(C(\mathbb{R},\mathbb{R}), \mathcal{B})$.
2. $\pi_r^{\mathcal{H}} f$ is $\sigma(\{X_t \| t| \geq r\})$-measurable for all $f \in \mathcal{B}_b$.
3. $\pi_r^{\mathcal{H}} f \pi_s^{\mathcal{H}} g = \pi_r^{\mathcal{H}} f g$ for all $g \in \mathcal{B}_b$ and $f \in \sigma(\{X_t \| t| \geq s\})_b$.

2.3.1 The family $\pi_r^{\mathcal{H}}$ is a family of kernels

In this section we will show properties 1 and 2 from the proposition above. First we will show it for the kernels π_r and then for $\pi_r^{\mathcal{H}}$.

Lemma 2.3.2. Let $n \in \mathbb{N}$ and $t_1 \leq \cdots \leq t_n$. Define $p : C(\mathbb{R},\mathbb{R}) \to \mathbb{R}^n$ by $p(\omega) = (\omega(t_1), \ldots, \omega(t_n))$. Denote by $\mathfrak{F}$ the Fourier-transform. Then we have

$$\mathfrak{F}\left(\pi_r(\xi, \cdot) \circ p^{-1}\right)(y) = e^{i\langle y, p(H_r(\xi))\rangle} \mathfrak{F}\left(m_r \circ p^{-1}\right)(y). \tag{2.3.1}$$

In particular the mapping $\xi \mapsto \mathfrak{F}\left(\pi_r(\xi, \cdot) \circ p^{-1}\right)(y)$ is $\sigma(\{X_t | t \in (-r,r)^c\})$-measurable for every $y \in \mathbb{R}^n$.

Proof.

$$\begin{aligned}
&\mathfrak{F}(\pi_r(\xi, \cdot) \circ p^{-1})(y) \\
&= \int_{\mathbb{R}^n} e^{i\langle y, x\rangle_{\mathbb{R}^n}} \pi_r(\xi, p^{-1})(\mathrm{d}x)) \\
&= \int_{\mathbb{R}^n} e^{i\langle y, x\rangle_{\mathbb{R}^n}} (m_r \circ \tau_{r,\xi}^{-1} \circ p^{-1})(\mathrm{d}x) \\
&= \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, p(\omega)\rangle_{\mathbb{R}^n}) m_r \circ \tau_{r,\xi}^{-1}(\mathrm{d}\omega) \\
&= \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, p \circ \tau_{r,\xi}(\omega)\rangle_{\mathbb{R}^n}\, m_r(\mathrm{d}\omega) \\
&= \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, (\omega(t_1) + H_r(\xi)(t_1), \ldots, \omega(t_n) + H_r(\xi)(t_n))\rangle_{\mathbb{R}^n} m_r(\mathrm{d}\omega) \\
&= \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, (H_r(\xi)(t_1), \ldots, H_r(\xi)(t_n))\rangle_{\mathbb{R}^n} + i\langle y, (\omega(t_1), \ldots, \omega(t_n))\rangle_{\mathbb{R}^n} m_r(\mathrm{d}\omega) \\
&= \exp(i\langle y, p(H_r(\xi))\rangle_{\mathbb{R}^n} \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, (\omega(t_1), \ldots, \omega(t_n))\rangle_{\mathbb{R}^n} m_r(\mathrm{d}\omega)
\end{aligned}$$

$$\begin{aligned}&= \exp(i\langle y, p(H_r(\xi))\rangle_{\mathbb{R}^n} \int_{C(\mathbb{R},\mathbb{R})} \exp(i\langle y, p(\omega)\rangle_{\mathbb{R}^n} m_r(\mathrm{d}\omega) \\ &= \exp(i\langle y, p(H_r(\xi))\rangle_{\mathbb{R}^n} \mathfrak{F}\left(m_r \circ p^{-1}\right)(y).\end{aligned}$$

Since $\xi \mapsto H_r(\xi)$ is $\sigma(\{X_t | t \in (-r,r)^c\})$-measurable and since the mapping $\eta \mapsto e^{i\langle y, p(\eta)\rangle_{\mathbb{R}^n}} \mathfrak{F}\left(m_{(-r,r)} \circ p^{-1}\right)(y)$ is continuous in η, we have the desired measureability. □

Proposition 2.3.3. For every $\sigma(\{X_t | t \in \mathbb{R}\})$-measurable function f we have that the function $\xi \mapsto \int f \pi_r(\xi, \mathrm{d}\omega)$ is $\sigma(\{X_t | |t| \geq r\})$-measurable. In particular we have that for every $r \in \mathbb{R}$ π_r is a kernel.

Proof. By Lemma 2.3.2 we know this already for functions f of the type $f(\omega) = \exp(i\langle y, (\omega(t_1), \ldots, \omega(t_n))\rangle_H)$ where $y \in \mathbb{R}^n$, $t_1 \leq \cdots \leq t_n$ and $n \in \mathbb{N}$. By linearity we can extend this to functions of the sets $\mathcal{S}_n, \mathcal{C}_n$ and $\mathcal{A}$ defined as follows:

$$\begin{aligned}\mathcal{S}_n &:= \{\sin(\langle x, (\omega(t_1), \ldots, \omega(t_n))\rangle_{\mathbb{R}^n} | x \in \mathbb{R}^n, t_1 \leq \cdots \leq t_n\} \\ \mathcal{C}_n &:= \{\cos(\langle x, (\omega(t_1), \ldots, \omega(t_n))\rangle_{\mathbb{R}^n} | x \in \mathbb{R}^n, t_1 \leq \cdots \leq t_n\} \\ \mathcal{A} &:= \operatorname{span}(\bigcup_{n \in \mathbb{N}} (\mathcal{S}_n \cup \mathcal{C}_n)).\end{aligned}$$

The set $\mathcal{A}$ is an algebra of functions: That $\mathcal{A}$ is a vectorspace is obvious. For the closedness under multiplication note that

$$\begin{aligned}\sin x * \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y) \\ \sin x * \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \\ \cos x * \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\end{aligned}$$

Since we want to apply the monotone class theorem, we have to show that the set

$$\mathcal{H} := \left\{ f \in \sigma(\{X_t | t \in \mathbb{R}\})_b \middle| \xi \mapsto \int f \pi_r(\xi, \mathrm{d}\omega) \text{ is } \sigma(\{X_t | |t| \geq r\})\text{-measurable} \right\}$$

is a monotone vectorspace.

Since $\int 1 \pi_r(\xi, \mathrm{d}\omega) = 1$ for every ξ we have that $1 \in \mathcal{H}$.

Since $\int (\alpha f + \beta g) \pi_r(\xi, \mathrm{d}\omega) = \alpha \int f \pi_r(\xi, \mathrm{d}\omega) + \int g \pi_r(\xi, \mathrm{d}\omega)$, and since measurable functions form a vector space, we have that $\mathcal{H}$ is a vector space.

Finally let $0 \leq f_1 \leq \cdots \leq f_n \leq \ldots$ be an increasing sequence of positive functions in $\mathcal{H}$ such that $f := \sup_n f_n$ is bounded. Then we have $\int f \pi_r(\xi, \mathrm{d}\omega) = \int \sup_n f_n \pi_r(\xi, \mathrm{d}\omega) = \sup_n \int f_n \pi_r(\xi, \mathrm{d}\omega)$ by Beppo Levi. And since $\sup_n g_n$ is measurable for a sequence of measurable functions g_n, we have shown that $\mathcal{H}$ is a monotone vector space.

Now we can apply the monotone class theorem and conclude the proof, since $\sigma(\mathcal{A}) = \sigma(\{X_t | t \in \mathbb{R}\}(= \mathcal{B}(C(\mathbb{R}, \mathbb{R}))$, because $\mathcal{A}$ separates the points of $C(\mathbb{R}, \mathbb{R})$. □

Now we have to show the same results for the family $(\pi_r^{\mathcal{H}})$.

Lemma 2.3.4. For every $r \in \mathbb{R}$ we have that $\pi_r^{\mathcal{H}}$ is a probability kernel from $C((\mathbb{R},\mathbb{R}),\mathcal{B})$ to $C(\mathbb{R},\mathbb{R}), \sigma(\{X_t||t| \geq r\})$.

Proof. We have $\pi_r^{\mathcal{H}}(\xi, A) = \frac{1}{\int \exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)} \int_A \exp(-\mathcal{H}_r(\omega))\pi_r(\xi, \mathrm{d}\omega)$ If we can show that $\omega \mapsto \mathcal{H}_r(\omega)$ is $\sigma(\{X_t|t \in \mathbb{R}\})$-measurable then we can apply 2.3.3 and are finished. Since $(\omega, t) \mapsto \omega(t)$ and $(\omega, s, t) \mapsto (s - t, \omega(s) - \omega(t))$ are continuous, hence measurable and since φ and ψ are measurable, we have finished our proof. $\square$

Using the results of this section we have shown properties 1 and 2 of proposition 2.3.1.

2.3.2 The kernels $\pi_r^{\mathcal{H}}$ form a specification

Lemma 2.3.5. Let $r, r' \in \mathbb{R}_+$ such that $r < r'$, then $H_r(m_{r'}) * m_r = m_{r'}$.

Proof. Since $H_r(m_{r'})$ is a linear transform of the Gaussian measure $m_{r'}$ and since convolutions of Gaussian measures are again Gaussian, both sides of the equations are Gaussian measures. Hence it is sufficient to show that the covariance and the mean value of both sides coincide.

Let $(X_t)_{t\in\mathbb{R}}$ be the canonical process on $(C(\mathbb{R},\mathbb{R}), m_{r'})$ and $(Y_t)_{t\in\mathbb{R}}$ be the canonical process on $(C(\mathbb{R},\mathbb{R}), m_r)$, then $Z_t := H_r(X_t) + Y_t$ is distributed under $H_r(m_{r'}) * m_r$.

As usual we can assume without loss of generality that $s \leq t$. Then we get:

$$\begin{aligned}
\mathrm{cov}(Z_s, Z_t) &= E[Z_s Z_t] \\
&= E[(H_r(X_s) + Y_s)(H_r(X_t) + Y_t)] \\
&= E[H_r(X_s)H_r(X_t)] + E[H_r(X_s)Y_t] + E[Y_s H_r(X_t)] + E[Y_s Y_t] \\
&= E[H_r(X_s)H_r(X_t)] + E[H_r(X_s)]E[Y_t] + E[Y_s]E[H_r(X_t)] + E[Y_s Y_t] \\
&= E[H_r(X_s)H_r(X_t)] + E[Y_s Y_t]
\end{aligned}$$

For further calculation we have to look at several different cases.

$s \notin [-r', r']$ or $t \notin [-r, r']$ In this cases we have $Y_s = 0$ and $H_r(X_s) = X_s = 0$ a.s. or $Y_t = 0$ and $H_r(X_t) = X_t = 0$ a.s., so we get in the end $\mathrm{cov}(Z_s, Z_t) = 0$.

$s, t \in [-r', r'] \setminus [-r, r]$ In this case $Y_s = 0$ and $Y_t = 0$, so $E[Y_s Y_t] = 0$. For the other summand we get

$$\begin{aligned}
&E[H_r(X_s)H_r(X_t)] \\
&\qquad = E[X_s X_t] \\
&\qquad = \frac{(s + r')(t - r')}{2r'}
\end{aligned}$$

$\boldsymbol{-r < s \leq t < r}$ In this case we have to calculate everything.

$$
\begin{aligned}
&E[H_r(X_s)H_r(X_t)] + E[Y_sY_t] \\
&\quad = E\Big[(X_{-r} + \frac{s-(-r)}{r-(-r)}(X_r - X_{-r}))(X_{-r} + \frac{t-(-r)}{r-(-r)}(X_r - X_{-r}))\Big] \\
&\qquad + \frac{(s-(-r))(r-t)}{r-(-r)} \\
&\quad = E[(X_{-r}X_{-r})] + E\Big[\frac{s+r}{r+r}(X_{-r}(X_r - X_{-r}))\Big] + \frac{t+r}{r+r}E[X_{-r}(X_r - X_{-r})] \\
&\qquad + \frac{s+r}{r+r}\frac{t+r}{r+r}E[(X_r - X_{-r})(X_r - X_{-r})] + \frac{(s+r)(r-t)}{r+r} \\
&\quad = \frac{(-r+r')(r'+r)}{r'+r'} \\
&\qquad + \frac{s+r}{r+r} \cdot \frac{(-r+r')(r'-r)}{r'+r'} - \frac{s+r}{r+r} \cdot \frac{(-r+r')(r'+r)}{r'+r'} \\
&\qquad + \frac{t+r}{r+r} \cdot \frac{(-r+r')(r'-r)}{r'+r'} - \frac{t+r}{r+r} \cdot \frac{(-r+r')(r'+r)}{r'+r'} \\
&\qquad + \frac{s+r}{r+r} \cdot \frac{t+r}{r+r}\Big(\frac{(r+r')(r'-r)}{r'+r'} - 2\frac{(-r+r')(r'-r)}{r'+r'} + \frac{(-r+r')(r'+r)}{r'+r'}\Big) \\
&\qquad + \frac{(s+r)(r-t)}{r+r} \\
&\quad = \frac{(-r+r')(r'+r)}{r'+r'} \\
&\qquad - \frac{s+r}{r+r} \cdot \frac{(-r+r')(r+r)}{r'+r'} - \frac{t+r}{r+r} \cdot \frac{(-r+r')(r+r)}{r'+r'} \\
&\qquad + \frac{s+r}{r+r} \cdot \frac{t+r}{r+r}(\frac{(r+r)(r'-r)}{r'+r'} + \frac{(-r+r')(r+r)}{r'+r'}) + \frac{(s+r)(r-t)}{r+r} \\
&\quad = \frac{1}{(r'+r')(r+r)}\big((-r+r')(r+r)(r'-s-t+(-r)) \\
&\qquad\quad + ((s+r)(t+r)((r'-r)+(-r+r')) \\
&\qquad\quad + (s+r)(r-t)(r'+r')\big) \\
&\quad = \frac{1}{(r'+r')(r+r)}\big((-r+r')(r+r)(r'-s-t+(-r)) \\
&\qquad\quad + ((s+r)(r+r)(r'+r')) \\
&\qquad\quad - ((s+r)(t+r)(r+r))\big) \\
&\quad = \frac{1}{(r'+r')}\big((-r+r')(r'-s-t-r) + ((s+r)(r'+r'-t-r))\big) \\
&\quad = \frac{1}{(r'+r')}\big((s+r)(r'-t) + (-r+r')(r'-t)\big) \\
&\quad = \frac{(s+r')(r'-t)}{r'+r'}
\end{aligned}
$$

$\boldsymbol{-r' \leq s \leq -r < t < r}$ In this case $Y_s = 0$ m_r-a.s., so $E[Y_sY_t] = 0$. For the other summand we get

$$\begin{aligned}
&E[H_r(X_s)H_r(X_t)] \\
&\quad = E\Big[X_s(X_{-r} + \frac{t-(-r)}{r-(-r)}(X_r - X_{-r}))\Big] \\
&\quad = E[X_sX_{-r}] + \frac{t+r}{r+r}E[X_sX_r] - \frac{t+r}{r+r}E[X_sX_{-r}] \\
&\quad = \frac{(s+r')(r'+r)}{r'+r'} + \frac{t+r}{r+r}\cdot\frac{(s+r')(r'-r)}{r'+r'} - \frac{t+r}{r+r}\cdot\frac{(s+r')(r'+r)}{r'+r'} \\
&\quad = \frac{(s+r')(r'+r)(r+r) + (t+r)(s+r')(r'-r) - (t+r)(s+r')(r'+r)}{(r+r)(r'+r')} \\
&\quad = \frac{(s+r')(r'+r)(r+r) - (t+r)(s+r')(r+r)}{(r+r)(r'+r')} \\
&\quad = \frac{(s+r')(r'-t)}{(r'+r')}
\end{aligned}$$

$\boldsymbol{-r < s < r \leq t < r'}$ In this case $Y_t = 0$ m_r-a.s., so $E[Y_sY_t] = 0$. For the other summand we get

$$\begin{aligned}
&E[H_r(X_s)H_r(X_t)] \\
&\quad = E\Big[(X_{-r} + \frac{s-(-r)}{r-(-r)}(X_r - X_{-r}))X_t\Big] \\
&\quad = E[X_{-r}X_t] + \frac{s+r}{r+r}E[X_rX_t] - \frac{s+r}{r+r}E[X_{-r}X_t] \\
&\quad = \frac{((-r)+r')(r'-t)}{r'+r'} + \frac{s+r}{r+r}\cdot\frac{(r+r')(r'-t)}{r'+r'} - \frac{s+r}{r+r}\cdot\frac{((-r)+r')(r'-t)}{r'+r'} \\
&\quad = \frac{((-r)+r')(r'-t)(r+r) + (s+r)(r+r')(r'-t)}{(r'+r')(r+r)} \\
&\qquad - \frac{(s+r)((-r)+r')(r'-t)}{(r'+r')(r+r)} \\
&\quad = \frac{((-r)+r')(r'-t)(r+r) + (s+r)(r'-t)(r+r)}{(r'+r')(r+r)} \\
&\quad = \frac{(s+r')(r'-t)}{r'+r'}
\end{aligned}$$

In all cases we get

$$\operatorname{cov}(Z_sZ_t) = \frac{(s+r')(r'-t)}{2r'},$$

hence Z_s has the desired distribution and the proof is complete. $\square$

Using the lemma above we can show property 3 of 2.3.1 for the family $(\pi_r)_{r>0}$:

Lemma 2.3.6. Let $g \in \mathcal{B}_b$ and $f \in \sigma(X_t||t| \geq s)_b$, then we have

$$\pi_r(f\pi_s g) = \pi_r(fg) \tag{2.3.2}$$

Proof. Let $f, g \in \mathcal{F}C_b^\infty$ with $f = F(X_{t_1}, \dots, X_{t_n})$ and $g = G(X_{s_1}, \dots, X_{s_m})$. Then we have:

$$\begin{aligned}
&\pi_r(f\pi_s g) \\
&\quad = \int F(X_{t_1}, \dots, X_{t_n})(\eta) \int G(X_{s_1}, \dots, X_{s_m})(\omega)\pi_s(\eta, \mathrm{d}\omega)\pi_r(\xi, \mathrm{d}\eta) \\
&\quad = \int F(X_{t_1}, \dots, X_{t_n})(\eta + H_r(\xi)) \int G(X_{s_1}, \dots, X_{s_m})(\omega)\pi_s(\eta, \mathrm{d}\omega)m_r(\mathrm{d}\eta) \\
&\quad = \int\int F(X_{t_1}, \dots, X_{t_n})(\eta + H_r(\xi)) \\
&\qquad \times G(X_{s_1}, \dots, X_{s_m})(\omega + H_s(\eta) + H_r(\xi))m_s(\mathrm{d}\omega)m_r(\mathrm{d}\eta)
\end{aligned}$$

since $H_s(\eta)(t_i) = \eta(t_i)$ m_s-a.s

$$\begin{aligned}
&\quad = \int\int F(X_{t_1}, \dots, X_{t_n})(\omega + H_s(\eta) + H_r(\xi)) \\
&\qquad \times G(X_{s_1}, \dots, X_{s_m})(\omega + H_s(\eta) + H_r(\xi))m_s(\mathrm{d}\omega)m_r(\mathrm{d}\eta)
\end{aligned}$$

by Lemma 2.3.5

$$\begin{aligned}
&\quad = \int F(X_{t_1}, \dots, X_{t_n})(\tilde{\omega} + H_r(\xi))G(X_{s_1}, \dots, X_{s_m})(\tilde{\omega} + H_r(\xi))m_r(\mathrm{d}\tilde{\omega}) \\
&\quad = \pi_r(fg).
\end{aligned}$$

□

To show the consistency condition for $\pi_r^{\mathcal{H}}$ we need this more technical result:

Lemma 2.3.7. If $\psi(x, y) = \psi(-x, -y)$ then the family $(\mathcal{H}_r)_{r\in\mathbb{R}}$ is an additive functional, i.e. for $r \geq s$ $\mathcal{H}_r - \mathcal{H}_s$ is $\sigma(\{X_t|t \in (-s, s)^c\})$-measurable.

Proof. Let us calculate the difference:

$$\begin{aligned}
&\mathcal{H}_r(\omega) - \mathcal{H}_s(\omega) \\
&\qquad = \int_{[-r,r]} \varphi(\omega(x))\mathrm{d}x + \frac{1}{2}\iint_{|x|,|y|\leq r} \psi(x - y, \omega(x) - \omega(y))\mathrm{d}x\mathrm{d}y \\
&\qquad + \iint_{|x|\leq r<|y|} \psi(x - y, \omega(x) - \omega(y))\mathrm{d}x\mathrm{d}y \\
&\qquad - \int_{[-s,s]} \varphi(\omega(x))\mathrm{d}x - \frac{1}{2}\iint_{|x|,|y|\leq s} \psi(x - y, \omega(x) - \omega(y))\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&- \iint_{|x|\le s<|y|} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&= \int_{s<|x|\le r} \varphi(\omega(x))\mathrm{d}x + \frac{1}{2}\iint_{\substack{s<|x|\le r\\ s<|y|\le r}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&+ \frac{1}{2}\iint_{\substack{|x|\le s\\ s<|y|\le r}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&+ \frac{1}{2}\iint_{\substack{|y|\le s\\ s<|x|\le r}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&+ \iint_{\substack{s<|x|\le r\\ r<|y|}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&- \iint_{\substack{|x|\le s\\ s<|y|\le r}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&= \int_{s<|x|\le r} \varphi(\omega(x))\mathrm{d}x + \frac{1}{2}\iint_{\substack{s<|x|\le r\\ s<|y|\le r}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&+ \iint_{\substack{s<|x|\le r\\ r<|y|}} \psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

hence $\mathcal{H}_r - \mathcal{H}_s$ is $\sigma(\{X_t | t \in (-s,s)^c\})$-measurable. □

Now we can show the consistency condition for $(\pi_r^{\mathcal{H}})_{r\ge 0}$. Let $s < r$, and define for simplicity $R_{r,s} := \mathcal{H}_r - \mathcal{H}_s$; then

$$
\begin{aligned}
&\pi_r^{\mathcal{H}}(Z\pi_s^{\mathcal{H}}X)(\xi) \\
&\quad= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int Z(\eta)\exp(-\mathcal{H}_r(\eta)) \\
&\quad\times \frac{1}{\pi_s(\exp(-\mathcal{H}_s))(\eta)} \int X(\omega)\exp(-\mathcal{H}_s)(\omega)\pi_s(\eta,\mathrm{d}\omega)\pi_r(\xi,\mathrm{d}\eta) \\
&\quad= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int \exp(-\mathcal{H}_s(\eta)) \\
&\quad\times \frac{Z(\eta)\exp(-R_{r,s})}{\pi_s(\exp(-\mathcal{H}_s))(\eta)} \int X(\omega)\exp(-\mathcal{H}_s)(\omega)\pi_s(\eta,\mathrm{d}\omega)\pi_r(\xi,\mathrm{d}\eta) \\
&\quad= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int Z(\eta)\exp(-R_{r,s})\left(\int X(\omega)\exp(-\mathcal{H}_s(\omega))\pi_s(\eta,\mathrm{d}\omega)\right)\times \\
&\quad\left(\int \exp(-\mathcal{H}_s(\omega))\pi_s(\eta,\mathrm{d}\omega)\right)\frac{1}{\pi_s(\exp(-\mathcal{H}_s))(\eta)}\pi_r(\xi,\mathrm{d}\eta) \\
&\quad= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int Z(\eta)\exp(-R_{r,s})\left(\int X(\omega)\exp(-\mathcal{H}_s(\omega))\pi_s(\eta,\mathrm{d}\omega)\right)\pi_r(\xi,\mathrm{d}\eta)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int Z(\eta)\exp(-R_{r,s})X(\eta)\exp(-\mathcal{H}_s(\eta))\pi_r(\xi,\mathrm{d}\eta) \\
&= \frac{1}{\pi_r(\exp(-\mathcal{H}_r))(\xi)} \int Z(\eta)X(\eta)\exp(-\mathcal{H}_r(\eta))\pi_r(\xi,\mathrm{d}\eta) \\
&= \pi_r^{\mathcal{H}}(ZX)
\end{aligned}
$$

Thus we have finished the proof of 2.3.1.

2.4 Overview of already known existence and uniqueness results

There are already several results on existence and uniqueness of Gibbs-measures for specifications like the one we use. In this section we will give an overview of already known results on existence and uniqueness of Gibbs measures for the specification above. For the convenience of the reader we have adapted the results to our case (in the case of higher dimensions) and we have also changed the notation to fit the notation of the other parts of this text.

2.4.1 Results from [OS99]

In [OS99] Osada and Spohn consider Gibbs measures on $C(\mathbb{R},\mathbb{R})$. They need the following definition:

Definition 2.4.1 (cf [OS99, Definition 2.1])**.** Let $f = f(x)$ and $g = g(x)$ be functions with value on $\mathbb{R} \cup \{\infty\}$ defined on $\mathbb{R}$. We say $g = g(x)$ is a right-dominator (resp. left-dominator) of f if:

1. g is convex and finite on at least two distinct points.
2. $f - g$ is nondecreasing (resp. nonincreasing) in x where we use the convention that $\infty - \infty = 0$ in case of $f(x) = g(x) = \infty$.
3. There exists a constant $a > 0$ such that $g''(x) \geq 2a$ a.e. $x \in \{g < \infty\}$.

Proposition 2.4.2 ([OS99, Theorem 2.2])**.** Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the external potential and let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be the interaction potential. Let $I = (a_-, a_+)$ be an open interval, $-\infty \leq a_- < a_+ \leq \infty$. We assume that $\varphi(x) = \infty$ for $x \in \mathbb{R} \setminus I$ Assume the following conditions hold:

1. Assumptions on the external potential φ.
   a) $\varphi : I \to \mathbb{R}$ is locally integrable and bounded from below.
   b) φ has a right-dominator symmetric around m and a left-dominator symmetric around $-m$ for some $m \geq 0$.
2. Assumption on the interaction potential ψ.

a) $\psi(t,x) = \rho(t)v(x)$ with $\rho \geq 0$, $\rho(t) = \rho(|t|)$ and $v(x) = v(|x|)$.

b) $v(\cdot)$ is convex and piecewise smooth. In addition, for some $p_0 > 1$,

$$u(x) = \operatorname*{ess\,sup}_{y\in\mathbb{R}} \frac{|v'(x) - v'(x-y)|}{1+|y|^{p_0}}$$

is finite and for each $\varepsilon > 0$ there exists a $b = b(\varepsilon) \geq 0$ such that

$$u(x) \leq \varepsilon(v'(x) - v'(x-b)) \text{ for all } x \in \mathbb{R}$$

c) $\rho(|t|) \leq \rho_0(t)$, where $\rho_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is an integrable, convex and nonincreasing function such that $\rho_0(t) > 0$ for all t and $\rho_0(0) < \infty$.

Then there exists a Gibbs measure for (φ, w).

They prove this theorem under the following weaker condition: 2.b) is replaced by: $v(\cdot)$ is convex such that $v = \infty$ for $|x| \geq 0$ ($0 < a \leq \infty$) and

$$|v(x)| \leq C\exp(p_1|x|) \text{ for all } |x| < a \text{ for some } C, p_1 > 0$$

They also have the following result on uniqueness (see [OS99, Theorem 2.3])

Proposition 2.4.3. Assume the same conditions as in 2.4.2. If $\int_0^\infty t\rho(t)\,\mathrm{d}t < \infty$ then there exists exactly one translation invariant Gibbs measure for (φ,ψ) satisfying for some $p_2 > p_0 + 1$: $\int |X_t|^{p_2}\,\mathrm{d}\mu < \infty$. Moreover, any limit points of $\{\mu^{\varphi,\psi}_{T,\xi,\circ}\}_T$ or $\{\mu^{\varphi,\psi}_{T,\xi}\}_T$ for ξ with $\|\xi\|_\infty < 0$ as $T \to \infty$ are unique and, henceforth, translation invariant.

2.4.2 Results from [Har06]

Proposition 2.4.4. Assume that the following conditions hold:

1. φ is bounded from below. There exits φ_0 and φ_1 such that $\varphi = \varphi_0 + \varphi_1$, satisfying the following conditions:

   a) φ_0 is a continuous function such that the associated Schrödinger operator $H_0 = -(1/2)\Delta + \varphi_0$ acting on $L^2(\mathbb{R}; \mathrm{d}x)$ has a strictly positive ground state f_0 of class $C^2(\mathbb{R})$ satisfying the following conditions:

      i. (strict log-concavity) there exists an $\alpha > 0$ such that

      $$(\zeta, \mathrm{Hess}_{u_0}(x)\zeta) \geq \alpha|\zeta|^2 \text{ for all } \zeta, x \in \mathbb{R},$$

      where $u_0 := -\log f_0$.

      ii. there exists a $p_0 \geq 1$ such that

      $$0 < \liminf_{r\to\infty} \frac{1}{r^{2p_0}} \inf_{|x|=r} U(x), \qquad \limsup_{r\to\infty} \frac{1}{r^{p_0}} \inf_{|x|=r} \mathbf{V}(x) < \infty,$$

      where $U = (f_0)^{-2}\operatorname{div}(f_0\nabla f_0)$ and $\mathbf{V} = (f_0)^{-1}\nabla f_0$. Here div denotes the divergence.

b) $\varphi_1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and there exist $b \geq 0$ and $0 \leq p_1 \leq p_0$ such that

$$|\nabla \varphi_1(x)| \leq b\|x\|^{p_1} \text{ for a.e. } x \in \mathbb{R}.$$

2. For each fixed $s \in \mathbb{R}$ and $y \in \mathbb{R}$, $\psi(s, \cdot, y) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. There exists a non-negative, integrable function ψ_0 on $\mathbb{R}$ satisfying:

   a) there exists a $q_o \geq 0$ such that, for a.e. $s \in \mathbb{R}$ and $x, y \in \mathbb{R}$,

   $$|\nabla_x \psi(s,x,y)| \leq \psi_0(s)(\|x\|^{q_0} + \|y\|^{q_0}) \qquad (\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d));$$

   b) for a.e. $s \in \mathbb{R}$ and $x, y \in \mathbb{R}$, $\psi(s,x,y) \geq -\psi_0(s)$.

3. p_0 is strictly larger than q_0.

Then there exists a translation invariant (φ, ψ)-Gibbs measure μ satisfying

$$\int_{C(\mathbb{R},\mathbb{R})} |w(0)|^{2p_0}\, \mu(\mathrm{d}w) < \infty.$$

There is also a part concerning hard-wall Gibbs measures, but this case is not of concern for this paper because in the hard-wall case there is no quasi-invariance (cf. 3.3.1), and so we cannot handle this case.

2.4.3 Results from [Bet03]

First we need the definition for a Kato class function (see [Bet03, p.88, case d=1])

Definition 2.4.5. A measurable function $f : \mathbb{R} \to \mathbb{R}$ is said to be in the *Kato class*, $f \in \mathcal{K}(\mathbb{R})$ if

$$\sup_{x \in \mathbb{R}} \int_{\{|x-y| \leq 1\}} |f(y)|\, \mathrm{d}y < \infty.$$

f is *locally in the Kato class*, $f \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R})$ if $f 1_K \in \mathcal{K}(\mathbb{R})$ for each compact set $K \subset \mathbb{R}$.

f is *Kato-decomposable* if $f = f^+ - f^-$ with $f^- \in \mathcal{K}(\mathbb{R})$, $f^+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R})$, where f^+ is the positive part and f^- is the negative part of f.

Proposition 2.4.6. Assume the following conditions:

1. $\varphi : \mathbb{R} \to \mathbb{R}$ is Kato-decomposable.

2. The Schrödinger operator $H_0 = -(1/2)\Delta + \varphi$ (where Δ denotes the Laplace operator) acting in $L^2(\mathbb{R})$ fulfills $\inf \mathrm{spec}(H_0) = 0$. Moreover, H_0 has a unique strictly positive ground state $\psi_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, i.e. 0 is an eigenvalue of multiplicity one with corresponding eigenfunction ψ_0.

The conditions on ψ are:

1. There exists C_∞ such that

$$\int_{-\infty}^{\infty} |\psi(x_0, x_s, |s|)| \, \mathrm{d}s < C_\infty \text{ and } \int_{-\infty}^{\infty} |\psi(x_s, x_0, |s|)| \, \mathrm{d}s < C_\infty$$

   uniformly in $x \in C(\mathbb{R}, \mathbb{R})$.

2. There exist $D \geq 0$ and $0 \leq C < \alpha$ such that

$$\mathcal{H}_T(x) \leq \mathcal{H}_T(\theta_\tau^{(0)} x) + C_\tau + D$$

   for all $T, \tau > 0$ and all $x \in C(\mathbb{R}, \mathbb{R})$.

Then there exists a Gibbs-measure for (φ, ψ).

Proof. This follows from [Bet03, Theorem 3.2.] and [Bet03, Proposition 3.1.] □

Other results on existence can be found in [BLS05].

3 Quasi-invariance and Closability

In this chapter we will define the bilinear form which we consider in this paper. Furthermore we will show that the form is closable.

3.1 Preliminaries

Let us first define the notions of bilinear forms on a Hilbert space, and of closability.

Definition 3.1.1. Let H be a Hilbertspace and D be a linear subspace of H.

1. A bilinear form an a Hilbert space H is a mapping $\mathcal{E} : D \times D \to \mathbb{R}$, with the following properties: For all $u \in D$ the mappings $\mathcal{E}(u, \cdot) : D \to \mathbb{R}, v \mapsto \mathcal{E}(u, v)$ and $\mathcal{E}(\cdot, u) : D \to \mathbb{R}, v \mapsto \mathcal{E}(v, u)$ are linear.

2. A bilinear form $\mathcal{E}$ on D is called symmetric if for all $u, v \in D$ we have that $\mathcal{E}(u, v) = \mathcal{E}(v, u)$.

3. A bilinear form is called positive definite if $\mathcal{E}(u, u) \geq 0$ for all $u \in D$.

4. A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a symmetric closed form on H if $D(\mathcal{E})$ is a dense linear subspace of H and $\mathcal{E}$ is symmetric and if $D(\mathcal{E})$ is complete with respect to the norm $\sqrt{\mathcal{E}(u, u) + \|u\|_H}$.

5. A bilinear form $\mathcal{E}$ on H with domain D is called closable if for all sequences $(u_n)_{n\in\mathbb{N}} \subset D$ such that $\lim_{n,m\to\infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0$ and $\lim_{n\to\infty} \|u_n\|_H = 0$ it follows that $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$, i.e. a sequence in H which converges to 0 and which is also a Cauchy-sequence with respect to the seminorm $\sqrt{\mathcal{E}(\cdot, \cdot)}$ converges to 0 with respect to this seminorm.

Define first:

$$\mathcal{F}C_b^\infty(C(\mathbb{R}, \mathbb{R})) := \{f(l_1(w), \ldots, l_n(w)) | f \in C_b(\mathbb{R}^n, \mathbb{R}), l_i \in C(\mathbb{R}, \mathbb{R})^*, n \in \mathbb{N}\} \quad (3.1.1)$$

and denote by $\widetilde{\mathcal{F}C_b^\infty}(C(\mathbb{R}, \mathbb{R}))$ the set of $L^2(\mu)$-equivalence classes of $\mathcal{F}C_b^\infty(C(\mathbb{R}, \mathbb{R}))$ and $H = H^{1,2}(\mathbb{R})$. Since we consider only the quasi-invariant case we have that $\operatorname{supp} \mu = C(\mathbb{R}, \mathbb{R})$ and then we have that for $u, v \in \mathcal{F}C_b^\infty$ with $u = v$ μ-a.e. that $u = v$ and hence $\frac{\partial u}{\partial k} = \frac{\partial v}{\partial k}$ for all $k \in H$. Now we can define for $\tilde{u} \in \widetilde{\mathcal{F}C_b^\infty}(C(\mathbb{R}, \mathbb{R}))$ with $u \in \mathcal{F}C_b^\infty(C(\mathbb{R}, \mathbb{R}))$ such that $\tilde{u}$ is the equivalence class of u in $\widetilde{\mathcal{F}C_b^\infty}(C(\mathbb{R}, \mathbb{R}))$ $\frac{\partial \tilde{u}}{\partial k} = \frac{\partial u}{\partial k}$

Let us define the following bilinear forms:

$$D := \widetilde{\mathcal{F}C_b^\infty}(C(\mathbb{R}, \mathbb{R}))$$

$$\mathcal{E}_k(u,v) := \frac{1}{2}\int_{C(\mathbb{R},\mathbb{R})} \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu \qquad\qquad u,v \in D$$
$$\mathcal{E}(u,v) := \frac{1}{2}\int_{C(\mathbb{R},\mathbb{R})} \langle \nabla u, \nabla v\rangle_H\,\mathrm{d}\mu \qquad\qquad u,v \in D.$$

If $\{k_n\}$ is an orthonormal basis of H we have $\mathcal{E}(u,v) = \sum_{n\in\mathbb{N}} \mathcal{E}_{k_n}(u,v)$.

We want to show that $(\mathcal{E}_k, D)$ is closable for all k in an orthonormal basis of H and then conclude by Proposition 3.1.7 that $(\mathcal{E}, D)$ is also closable.

Definition 3.1.2 (see [AR89])**.** Let $k \in E \setminus \{0\}$. μ is called k-quasi-invariant if $\tau_{sk}(\mu)$ is absolutely continuous with respect to μ for all $s \in \mathbb{R}$. In this case we set

$$a_{sk}(z) := \frac{\mathrm{d}\tau_{sk}(\mu)}{\mathrm{d}\mu}(z), z \in E \tag{3.1.2}$$

We will use the following condition. It was first introduced in [Ham75] and then used in [AR89] and [AR90a].

Definition 3.1.3. A $\mathcal{B}(\mathbb{R})$-measurable function $\rho : \mathbb{R} \to \mathbb{R}_+$ fulfills the Hamza-condition if $\rho = 0$ on $\mathbb{R} \setminus R(\rho)$ where

$$R(\rho) := \left\{ t \in \mathbb{R} \,\middle|\, \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho}\,\mathrm{d}s < \infty \text{ for some } \varepsilon > 0 \right\} \tag{3.1.3}$$

Definition 3.1.4 (see [AR89, 1.6 Definition])**.** Let $k \in E \setminus \{0\}$. k is called admissible if for ν_k-a.e. $x \in E_0$, $\rho_k(x, \mathrm{d}s) = \rho_k(x,s)\mathrm{d}s$ for some $\mathcal{B}(\mathbb{R})$-measurable function $\rho(x,\cdot) : \mathbb{R} \to \mathbb{R}_+$ satisfying (H) or equivalently if

$$\frac{\partial}{\partial k}u = \frac{\partial}{\partial k}v \ \mu\text{-a.e. if } u,v \in \mathcal{F}C_b^\infty \text{ with } u = v \ \mu\text{-a.e.} \tag{3.1.4}$$

is satisfied and $(\tilde{\mathcal{E}}_k^{(n)}, \widetilde{\mathcal{F}C_b^n})$ is closable for some (all) $n \in \mathbb{N} \cup \{+\infty\}$, where $\tilde{\mathcal{E}}_k^{(n)}(u,v) = \int_{C(\mathbb{R},\mathbb{R})} \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\,\mathrm{d}\mu$, $u,v \in \mathcal{F}C_b^n$

Remark 3.1.5. It is obvious that the closability of $(\mathcal{E}_k, D)$ is equivalent to the closability of $(\tilde{\mathcal{E}}_k^{(\infty)}, \widetilde{\mathcal{F}C_b^\infty})$, since $\mathcal{E}_k(u,v) = \frac{1}{2}\tilde{\mathcal{E}}_k(u,v)$ and $D = \widetilde{\mathcal{F}C_b^\infty}$.

We want to show closability by using the following two propositions:

Proposition 3.1.6 (see [AR89, 2.4 Corollary])**.** If $(s,z) \mapsto a_{sk}(z)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(E)$-measurable and for μ-a.e. $z \in E$, $s \mapsto a_{sk}(z)$ satisfies (H), then k is admissible.

Proposition 3.1.7 (see [MR92, Proposition I.3.7 (i)])**.** Let $(\mathcal{E}^{(k)}, D^{(k)})$, $k \in \mathbb{N}$, be closable symmetric bilinear forms on $\mathcal{H}$. Define

$$D := \left\{ u \in \bigcap_{k\in\mathbb{N}} D^{(k)} \,\middle|\, \sum_{k=1}^{\infty} \mathcal{E}^{(k)}(u,u) < \infty \right\}$$

and

$$\mathcal{E}(u,v) := \sum_{k=1}^{\infty} \mathcal{E}^{(k)}(u,v); \qquad u,v \in D$$

Then $(\mathcal{E}, D)$ is closable on $\mathcal{H}$.

We will use the Cameron-Martin formula to show k-quasi-invariance for the Gaussian measures $\pi_r(\xi,\cdot)$.

For any topological vector space X we denote by X' its algebraic dual and by X^* its topological dual.

Definition 3.1.8. [see [Bog98, 2.2.7. Definition]] Let X be a locally convex space and let μ be a measure on $\mathcal{E}(X)$ such that $X^* \subset L^2(\mu)$. The element a_μ in the algebraic dual $(X^*)'$ to X^* defined by the formula

$$a_\mu(f) = \int_X f(x)\mu(\mathrm{d}x), \tag{3.1.5}$$

is called the mean of μ. The operator $R_\mu : X^* \to (X^*)'$, defined by the formula

$$R_\mu(f)(g) := \int_X \Big(f(x) - a_\mu(f)\Big)\Big(g(x) - a_\mu(g)\Big)\mu(\mathrm{d}x), \tag{3.1.6}$$

is called the covariance operator of μ; the corresponding quadratic form on X^* is called the covariance of μ.

Proposition 3.1.9. [see [Bog98, 2.4.3. Corollary]] Let γ be a Gaussian measure on a locally convex space X. Then, for any $h \in X$ such that $h = R_\gamma(g)$ and $g \in X^*_\gamma$, where X^*_γ is the L^2-closure of $\{f - a_\gamma(f) | f \in X^*\}$, the measures γ and $\gamma_h = \gamma(\cdot - h)$ are equivalent and the corresponding Radon-Nikodym density is given by the expression

$$\rho_h(x) = \exp\left(g(x) - \frac{1}{2}|h|^2_{H(\gamma)}\right). \tag{3.1.7}$$

3.2 Density for the specification for the free case

In this section we will show that $\pi_r(\xi,\cdot)$ is k-quasi-invariant for certain k and we will calculate the densities $\frac{\mathrm{d}\tau_k(\pi_r(\xi,\cdot))}{\mathrm{d}\pi_r(\xi,\cdot)}$.

To do this we will need to know what is $C(\mathbb{R},\mathbb{R})^*$, $C(\mathbb{R},\mathbb{R})^*_{\pi_r(\xi,\cdot)}$ and $R_{\pi_r(\xi,\cdot)}$.

It is well-known (see e.g. [KA64, p.411]) that

$$C(\mathbb{R},\mathbb{R})^* = \left\{ f = \int_{-\infty}^{\infty} w(t)dg(t) \,\middle|\, \begin{matrix} g \text{ has bdd. var. and is constant on} \\ (-\infty,-r] \text{ and } [r,\infty) \text{ for some } r \in \mathbb{R} \end{matrix} \right\} \tag{3.2.1}$$

Let us first calculate $R_{\pi_r(\xi,\cdot)}(\int \cdot \mathrm{d}f)(\int \cdot \mathrm{d}g)$:

Lemma 3.2.1. Let $f, g \in C_0^1(\mathbb{R},\mathbb{R})$, then $\int \cdot \mathrm{d}f, \int \cdot \mathrm{d}g \in C(\mathbb{R},\mathbb{R})^*$ and

1.
$$a_{\pi_r(\xi,\cdot)}\Big(\int \cdot \,\mathrm{d}f\Big) = \int H_r(\xi)\,\mathrm{d}f. \tag{3.2.2}$$

2.
$$R_{\pi_r(\xi,\cdot)}\Big(\int \cdot \mathrm{d}f\Big)\Big(\int \cdot \mathrm{d}g\Big) = \int\int K_r(s,t)\mathrm{d}f\mathrm{d}g = \int_{-r}^{r} f(t)g(t)\,\mathrm{d}t - \frac{1}{2r}\int_{-r}^{r} f(t)\,\mathrm{d}t\int_{-r}^{r} g(t)\,\mathrm{d}t \tag{3.2.3}$$

Proof. The idea of the proof in both parts is to interchange the order of integration and then some simple calculations. First we will justify, that we may indeed change the order of integration.

For every $r > 0$ we have that $w \mapsto \|w\|_r := \sup\{|w(x)||x \in [-r,r]\}$ is a measurable seminorm on $C(\mathbb{R},\mathbb{R})$. Now we can apply a corollary to Ferniques Theorem [Bog98, Theorem 2.8.5 and Corollary 2.8.6] and know that there exists an $\alpha > 0$ such that $\exp(\alpha\|\cdot\|_r) \in L^1(\pi_r(0,\cdot))$.

Let $-r = t_0 < t_1 < \cdots < t_n = r$ be a partition of $[-r,r]$. Then we have:

$$\begin{aligned}
&\Big|\sum_{i=0}^{n-1} w(t_i)(f(t_{i+1}) - f(t_i))\Big| \\
&\quad\le \sum_{i=0}^{n-1} |w(t_i)|\big|(f(t_{i+1}) - f(t_i))\big| \\
&\quad\le \sum_{i=0}^{n-1} \|w\|_r\big|(f(t_{i+1}) - f(t_i))\big| \\
&\quad= \|w\|_r\sum_{i=0}^{n-1} \big|(f(t_{i+1}) - f(t_i))\big| \\
&\quad\le \|w\|_r\|f\|_{BV}
\end{aligned}$$

and

$$\begin{aligned}
\|w\|_r = \frac{\sqrt{\alpha}}{\sqrt{\alpha}}\|w\|_r &\le \frac{1}{\sqrt{\alpha}}(1 \vee \sqrt{\alpha}\|w\|_r) \le \frac{1}{\sqrt{\alpha}}(1 \vee \sqrt{\alpha}\|w\|_r)^2 \le \frac{1}{\sqrt{\alpha}}(1 \vee \alpha\|w\|_r^2) \\
&\le \frac{1}{\sqrt{\alpha}}(1 + \alpha\|w\|_r^2) \le \frac{1}{\sqrt{\alpha}}\sum_{n=0}^{\infty}\frac{\alpha\|w\|_r^2}{n!} = \frac{1}{\sqrt{\alpha}}\exp(\alpha\|w\|_r^2)
\end{aligned}$$

Thus we have that for every partition $\big|\sum_{i=0}^{n-1} w(t_i)(f(t_{i+1}) - f(t_i))\big|$ is dominated by $\|f\|_{BV}\frac{1}{\sqrt{\alpha}}\exp(\alpha\|w\|_r^2)$, so we may apply Lebesgues Theorem in the next calculation.

Now let $-r = t_0^{(n)} < \cdots < t_{m_n}^{(n)} = r$ be a sequence of partitions of $[-r,r]$, then we have:

$$\begin{aligned}
&\int_{C(\mathbb{R},\mathbb{R})} \int w \,\mathrm{d}f \pi_r(\xi, \mathrm{d}w) \\
&\qquad = \int_{C(\mathbb{R},\mathbb{R})} \lim_{n\to\infty} \sum_{i=1}^{m_n} w(t_i^{(n)})(f(t_i^{(n)}) - f(t_{i-1}^{(n)}))\pi_r(\xi, \mathrm{d}w) \\
&\qquad = \lim_{n\to\infty} \int_{C(\mathbb{R},\mathbb{R})} \sum_{i=1}^{m_n} w(t_i^{(n)})(f(t_i^{(n)}) - f(t_{i-1}^{(n)}))\pi_r(\xi, \mathrm{d}w) \\
&\qquad = \lim_{n\to\infty} \sum_{i=1}^{m_n} \int_{C(\mathbb{R},\mathbb{R})} w(t_i^{(n)})\pi_r(\xi, \mathrm{d}w)(f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \\
&\qquad = \lim_{n\to\infty} \sum_{i=1}^{m_n} H_r(\xi)(t_i^{(n)})(f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \\
&\qquad = \int H_r(\xi) \,\mathrm{d}f.
\end{aligned}$$

Concerning the second part of our lemma, let $-r = t_0 < t_1 < \cdots < t_n = r$ and $-r = s_0 < s_1 < \cdots < s_n = r$ be partitions of $[-r, r]$, then we have

$$\begin{aligned}
&\left| \int w \,\mathrm{d}g \sum_{i=0}^{n-1} w(t_i)(f(t_i) - f(t_{i-1})) \right| \\
&\qquad \le \|w\| \|g\|_{BV} \sum_{i=0}^{n-1} |w(t_i)||f(t_i) - f(t_{i-1})| \\
&\qquad \le \|w\| \|g\|_{BV} \sum_{i=0}^{n-1} \|w\||f(t_i) - f(t_{i-1})| \\
&\qquad \le \|w\| \|g\|_{BV} \|w\| \sum_{i=0}^{n-1} |f(t_i) - f(t_{i-1})| \\
&\qquad \le \|w\|^2 \|g\|_{BV} \|f\|_{BV},
\end{aligned}$$

$$\begin{aligned}
&\left| w(t_i) \sum_{j=0}^{m-1} w(s_j)(g(s_j) - g(s_{j-1})) \right| \\
&\qquad \le \|w\| \sum_{j=0}^{m-1} |w(s_j)||g(s_j) - g(s_{j-1})| \\
&\qquad \le \|w\|^2 \|g\|_{BV}
\end{aligned}$$

and

$$\|w\|_r^2 = \frac{1}{\alpha} \alpha \|w\|_r^2 \le \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\alpha \|w\|_r^2}{n!} = \frac{1}{\alpha} \exp(\alpha \|w\|_r^2).$$

So we may apply Lebesgues' Theorem twice in the following calculation, once for $\lim_{n\to\infty}$ and once for $\lim_{m\to\infty}$. Let $-r = t_0^{(n)} < \cdots < t_{k_n}^{(n)} = r$ and $-r = s_0^{(m)} < \cdots < s_{l_m}^{(m)} = r$

$$
\begin{aligned}
&\int_{C(\mathbb{R},\mathbb{R})} \int w \,\mathrm{d}f \int w \,\mathrm{d}g \pi_r(0,\mathrm{d}w) \\
&\quad = \int_{C(\mathbb{R},\mathbb{R})} \lim_{n\to\infty} \left(\left(\sum_{i=1}^{k_n} w(t_i^{(n)})(f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \right) \int w \,\mathrm{d}g \right) \pi_r(0,\mathrm{d}w) \\
&\quad = \lim_{n\to\infty} \int_{C(\mathbb{R},\mathbb{R})} \left(\left(\sum_{i=1}^{k_n} w(t_i^{(n)})(f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \right) \int w \,\mathrm{d}g \right) \pi_r(0,\mathrm{d}w) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \left((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \int_{C(\mathbb{R},\mathbb{R})} w(t_i^{(n)}) \int w \,\mathrm{d}g \ \pi_r(0,\mathrm{d}w) \right) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \Bigg((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \\
&\qquad\quad \times \int_{C(\mathbb{R},\mathbb{R})} w(t_i^{(n)}) \lim_{m\to\infty} \sum_{j=1}^{l_m} \big(w(s_j^{(m)})(g(s_j^{(m)}) - g(s_{j-1}^{(m)})) \big) \pi_r(0,\mathrm{d}w) \Bigg) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \Bigg((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \\
&\qquad\quad \times \lim_{m\to\infty} \int_{C(\mathbb{R},\mathbb{R})} w(t_i^{(n)}) \sum_{j=1}^{l_m} \big(w(s_j^{(m)})(g(s_j^{(m)}) - g(s_{j-1}^{(m)})) \big) \pi_r(0,\mathrm{d}w) \Bigg) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \Bigg((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \\
&\qquad\quad \times \lim_{m\to\infty} \sum_{j=1}^{l_m} (g(s_j^{(m)}) - g(s_{j-1}^{(m)})) \int_{C(\mathbb{R},\mathbb{R})} w(t_i^{(n)}) w(s_j^{(m)}) \pi_r(0,\mathrm{d}w) \Bigg) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \left((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \lim_{m\to\infty} \sum_{j=1}^{l_m} (g(s_j^{(m)}) - g(s_{j-1}^{(m)})) K_r(t_i^{(n)}, s_j^{(m)}) \right) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{k_n} \left((f(t_i^{(n)}) - f(t_{i-1}^{(n)})) \int K_r(t_i^{(n)}, s) \,\mathrm{d}g(s) \right) \\
&\quad = \int \int K_r(t,s) \mathrm{d}g(s) \,\mathrm{d}f(t)
\end{aligned}
$$

For the last line we also need that $t \mapsto \int K_r(t,s)\,\mathrm{d}g(s)$ is continuous, but this follows from our calculations in the proof of Lemma 3.2.2 and by the fact that $K_r(s,t) = K_r(t,s)$ for all $r, s \in \mathbb{R}$.

□

We will also need $R_{\pi_r(\xi,\cdot)}(F)$ for some elements in $C(\mathbb{R},\mathbb{R})^*_{\pi_r(\xi,\cdot)}$:

Lemma 3.2.2. Let $F \in C(\mathbb{R},\mathbb{R})^*_{\pi_r(\xi,\cdot)}$ such that there exists f of bounded variation such that $F(w) = \int w(x)\,\mathrm{d}f(x) - \int H_r(\xi)\,\mathrm{d}f(x)$. Then we have $R_{\pi_r(\xi,\cdot)}(F) \in C(\mathbb{R},\mathbb{R})$ and

$$R_{\pi_r(\xi,\cdot)}(F) = \int K_r(s,t)\,\mathrm{d}f(s). \tag{3.2.4}$$

Proof. For any $G \in C(\mathbb{R},\mathbb{R})^*_{\pi_r(\xi,\cdot)}$ such that there exists $g \in B$ such that $G(w) = \int w(x)\,\mathrm{d}g(x) - \int H_r(\xi)\,\mathrm{d}g(x)$, we have

$$\begin{aligned}
R_{\pi_r(\xi,\cdot)}(F)(G) &= \int F(w)G(w)\pi_r(\xi,\mathrm{d}w)\\
&= \int\Big(\int w(x)\,\mathrm{d}f(x) - \int H_r(\xi)\,\mathrm{d}f(x)\Big)\\
&\qquad \times \Big(\int w(x)\,\mathrm{d}g(x) - \int H_r(\xi)\,\mathrm{d}g(x)\Big)\pi_r(\xi,\mathrm{d}w)\\
&= \int\Big(\int w(x)\,\mathrm{d}f(x)\Big)\Big(\int w(x)\,\mathrm{d}g(x)\Big)\pi_r(0,\mathrm{d}w)\\
&= \int\Big(\int K_r(s,t)\,\mathrm{d}f(s)\Big)\mathrm{d}g(t)
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\int K_r(s,t)\,\mathrm{d}f(s)\\
&= 1_{[-r,r]}(t)\Big(\int_{-r}^{t}\frac{(s+r)(r-t)}{2r}\mathrm{d}f(s) + \int_{t}^{r}\frac{(t+r)(r-s)}{2r}\mathrm{d}f(s)\Big)\\
&= 1_{[-r,r]}(t)\Big(\frac{r-t}{2r}\int_{-r}^{t}(s+r)\,\mathrm{d}f(s) + \frac{t+r}{2r}\int_{t}^{r}(r-s)\,\mathrm{d}f(s)\Big)\\
&= 1_{[-r,r]}(t)\Big(\frac{r-t}{2r}\int_{-r}^{t}s\,\mathrm{d}f(s) + \frac{r-t}{2}\int_{-r}^{t}\mathrm{d}f(s)\Big)\\
&\quad + 1_{[-r,r]}(t)\Big(\frac{t+r}{2}\int_{t}^{r}\mathrm{d}f(s) - \frac{t+r}{2r}\int_{t}^{r}s\,\mathrm{d}f(s)\Big)\\
&= 1_{[-r,r]}(t)\Big(\frac{r-t}{2r}\Big(tf(t) + rf(-r) - \int_{-r}^{t}f(s)\,\mathrm{d}s\Big) + \frac{r-t}{2}\big(f(t)-f(-r)\big)\Big)\\
&\quad + 1_{[-r,r]}(t)\Big(\frac{t+r}{2}\big(f(r)-f(t) - \frac{t+r}{2r}\Big(rf(r) - tf(t) - \int_{t}^{r}f(s)\,\mathrm{d}s\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 1_{[-r,r]}(t)\Big(\frac{t}{2r}\int_{-r}^{r} f(s)\,\mathrm{d}s + \frac{1}{2}\Big(\int_{-r}^{t} f(s)\,\mathrm{d}s - \int_{t}^{r} f(s)\,\mathrm{d}s\Big)\Big)\\
&= 1_{[-r,r]}(t)\Big(\frac{t-(-r)}{2r}\int_{-r}^{r} f(s)\,\mathrm{d}s - \int_{-r}^{t} f(s)\,\mathrm{d}s\Big)
\end{aligned}
$$

Since the mapping $t \mapsto \int_{-r}^{t} f(s)\,\mathrm{d}s$ is continuous for f of bounded variation, and since $\frac{t-(-r)}{2r}\int_{-r}^{r} f(s)\,\mathrm{d}s - \int_{-r}^{t} f(s)\,\mathrm{d}s = 0$ for $t \in \{-r, r\}$ we have that $R_{\pi_r(\xi,\cdot)}(F) \in C(\mathbb{R},\mathbb{R})$. □

Lemma 3.2.3. Let $h = R_{\pi_r(\xi,\cdot)}(G)$ where $G \in C(\mathbb{R},\mathbb{R})_{\pi_r(\xi,\cdot)}$ with $G(w) = \int w(t)\,\mathrm{d}g(t) - \int H_r(\xi)\,\mathrm{d}g(t)$. Then we have

$$
|h|_{H(\pi_r(\xi,\cdot)} = \|G\|_{L^2(\pi_r(\xi,\cdot))} = \Big\|\int \cdot\,\mathrm{d}g\Big\|_{L^2(\pi_r(0,\cdot))} = \int\int K_r(s,t)\,\mathrm{d}g(s)\,\mathrm{d}g(t) \quad (3.2.5)
$$

Proof. The first equality follows by [Bog98, 2.4.1. Lemma]. To show the remaining parts we make this calculation:

$$
\begin{aligned}
\|G\|_{L^2(\pi_r(\xi,\cdot)} &= \int G(w)^2 \pi_r(\xi,\mathrm{d}w)\\
&= \int\Big(\int w(t)\,\mathrm{d}g(t) - \int H_r(\xi)\,\mathrm{d}g(t)\Big)^2 \pi_r(\xi,\mathrm{d}w)\\
&= \int\Big(\int w(t)\,\mathrm{d}g(t)\Big)^2 \pi_r(0,\mathrm{d}w)\\
&= \int\int K_r(s,t)\,\mathrm{d}g(s)\,\mathrm{d}g(t).
\end{aligned}
$$

□

Now we can put our results together and we get the following:

Proposition 3.2.4. Let $h \in R_{\pi_r(\xi,\cdot)}(C(\mathbb{R},\mathbb{R})^*)$ be defined as

$$
h := R_{\pi_r(\xi,\cdot)}\Big(\int \cdot\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f\Big) = 1_{[-r,r]}(t)\Big(\frac{t-(-r)}{2r}\int_{-r}^{r} f(s)\,\mathrm{d}s - \int_{-r}^{t} f(s)\,\mathrm{d}s\Big).
$$

Then we have

$$
\begin{aligned}
&\frac{\mathrm{d}\pi_r(\xi,\cdot)\circ\tau_h^{-1}}{\mathrm{d}\pi_r(\xi,\cdot)}(w)\\
&= \exp\Big(\int w\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f - \frac{1}{2}\int\int K_r(s,t)\,\mathrm{d}f(s)\,\mathrm{d}f(t)\Big)\\
&= \exp\Big(\int w\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f - \frac{1}{2}\Big(\int_{-r}^{r} f(t)^2\,\mathrm{d}t - \frac{1}{2r}\Big(\int_{-r} f(t)\,\mathrm{d}t\Big)^2\Big)\Big).
\end{aligned}
$$

3.3 Density for the kernels of the interaction case

We start with the following observation:

Proposition 3.3.1. Let $\xi \in C(\mathbb{R},\mathbb{R})$ and suppose that the measure $\pi_r^{\mathcal{H}}(\xi,\cdot)$ is h-quasi-invariant for all $h \in R_{\pi_r(\xi,\cdot)}(C(\mathbb{R},\mathbb{R})^*)$. Then $\pi_r(\{\mathcal{H}_r = \infty\}) \in \{0,1\}$.

Proof. Suppose that $\pi_r(\xi,\{\mathcal{H}_r = \infty\}) > 0$. For simplicity define $A := \{\mathcal{H}_r = \infty\}$. We know that $\pi_r^{\mathcal{H}}(\xi,A) = 0$. Since by assumption $\pi_r^{\mathcal{H}}(\xi,\cdot)$ is h-quasi-invariant we have that $\tau_{-sh}(\pi_r^{\mathcal{H}}(\xi,A)) = 0$, so we have that

$$
\begin{aligned}
0 &= \pi_r^{\mathcal{H}}(\xi,\tau_{-sh}^{-1}(A)) \\
&= \int 1_{\tau_{-sh}^{-1}(A)}(w)e^{-\mathcal{H}_r(w)}\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A(w-sh)e^{-\mathcal{H}_r(w)}\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A(w-sh)1_A(w)e^{-\mathcal{H}_r(w)}\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A(w-sh)1_{A-sh}(w-sh)e^{-\mathcal{H}_r(\cdot+sh)}(w-sh)\pi_r(\xi,\mathrm{d}w) \\
&= \int (1_A 1_{A-sh}e^{-\mathcal{H}_r(\cdot+sh)})(w-sh)\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A 1_{A-sh}e^{-\mathcal{H}_r(\cdot+sh)}\tau_{-sh}\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A 1_{A-sh}e^{-\mathcal{H}_r(\cdot+sh)}\rho_{-sh}\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A(w)1_A(w+sh)e^{-\mathcal{H}_r(w+sh)}\rho_{-sh}(w)\pi_r(\xi,\mathrm{d}w) \\
&= \int 1_A(w)e^{-\mathcal{H}_r(w+sh)}\rho_{-sh}(w)\pi_r(\xi,\mathrm{d}w)
\end{aligned}
$$

Since $\pi_r(\xi,A) > 0$ by assumption and $\rho_{-sh}(w) > 0$ since the family $\{\tau_{sh}\pi_r(\xi,\cdot)|s \in \mathbb{R}|\}$ is a family of equivalent measures we have that $0 = \pi_r(\xi,\{\mathcal{H}_r(\cdot+sh) < \infty\} \cap A) = \pi_r(\xi, A \setminus \{\mathcal{H}_r(\cdot+sh) = \infty\}) = \pi_r(\xi, A \setminus A - sh)$. Now we can apply [Bog98, Corollary 2.5.3] and conclude the proof. □

Since the cases with $\pi_r(\{\mathcal{H}_r = \infty\}) > 0$ are either not quasi-invariant or $\pi_r^{\mathcal{H}}$ can not be defined, because $\int e^{-\mathcal{H}_r}\,\mathrm{d}\pi_r(\xi,\cdot) = 0$, we assume from now on:

For all $\xi \in C(\mathbb{R},\mathbb{R})$ we have that $\pi_r(\xi,\{\mathcal{H}_r = \infty\}) = 0$.

Under this assumption we have:

Lemma 3.3.2. For all $h \in R_{\pi_r(\xi,\cdot)}(C(\mathbb{R},\mathbb{R})^*)$ and $s \in \mathbb{R}$ we have $\tau_s h(\pi_r(\xi,\cdot))(\{\mathcal{H}_r = \infty\}) = 0$.

Proof. The measures $\tau_{sh}(\pi_r(\xi,\cdot))$, $s \in \mathbb{R}$ and $h \in$ are equivalent by 3.1.9 □

We can calculate the densities for the kernels of the specification of the interaction case as follows:

Lemma 3.3.3. Let $g \in B$, $G(w) := \int w \,\mathrm{d}g - \int H_r(\xi)\,\mathrm{d}g$ and $h = R_{\pi_r(\xi,\cdot)}G$. Define $\rho_h := \frac{\mathrm{d}\pi_r(\xi,\cdot)\circ\tau_h^{-1}}{\mathrm{d}\pi_r(\xi,\cdot)} = \exp\big(\int w\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f - \frac{1}{2}\int\int K_r(s,t)\,\mathrm{d}f(s)\,\mathrm{d}f(t)\big)$.

Let $F \in L^1(\pi_r^{\mathcal{H}}(\xi,\cdot))$. Then we have

$$\int F(w)\,\mathrm{d}(\pi_r^{\mathcal{H}}(\xi,\cdot)\circ\tau_k^{-1})(w) = \int F(w)\rho_h \exp(-\mathcal{H}_r(w-k)+\mathcal{H}_r(w))\pi_r^{\mathcal{H}}(\xi,\mathrm{d}w) \quad (3.3.1)$$

Proof. Let $F \in L^1(\pi_r(\xi,\cdot))$, then we have

$$\begin{aligned}
&\int F(w)\,\mathrm{d}(\pi_r^{\mathcal{H}}(\xi,\cdot)\circ\tau_k^{-1})(w)\\
&\quad= \int F\circ\tau_h(w)\,\pi_r^{\mathcal{H}}(\xi,\mathrm{d}w)\\
&\quad= \frac{1}{\int\exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)}\int F\circ\tau_h(w)\exp(-\mathcal{H}_r(w))\pi_r(\xi,\mathrm{d}w)\\
&\quad= \frac{1}{\int\exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)}\int F\circ\tau_h(w)\exp(-\mathcal{H}_r(\cdot-h)\circ\tau_h\pi_r(\xi,\mathrm{d}w)\\
&\quad= \frac{1}{\int\exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)}\int F(w)\exp(-\mathcal{H}_r(w-h))(\pi_r(\xi,\cdot)\circ\tau_k^{-1})(\mathrm{d}w)\\
&\quad= \frac{1}{\int\exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)}\int F(w)\exp(-\mathcal{H}_r(w-h))\rho_h(w)\pi_r(\xi,\mathrm{d}w)\\
&\quad= \frac{1}{\int\exp(-\mathcal{H}_r(\omega))\pi_r(\xi,\mathrm{d}\omega)}\\
&\qquad\times\int F(w)\exp(-\mathcal{H}_r(w))\rho_h(w)\exp(-\mathcal{H}_r(w-k)+\mathcal{H}_r(w))\pi_r(\xi,\mathrm{d}w)\\
&\quad= \int F(w)\rho_h\exp(-\mathcal{H}_r(w-k)+\mathcal{H}_r(w))\pi_r^{\mathcal{H}}(\xi,\mathrm{d}w)
\end{aligned}$$

□

3.4 Density for a Gibbs-measure

We want to do the following: Let $F \in \sigma(\{X_t| -r_0 \le t \le r_0\})$ and h defined as $h(t) = \int K_r(s,t)\,\mathrm{d}f(s)$, i.e. $h = R_{\pi_r(\xi,\cdot)}(\int\cdot\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f)$ for every $\xi \in C(\mathbb{R},\mathbb{R})$ by Lemma 3.2.2, then we have for every $r > r_0$ that

$$\begin{aligned}
\int F\,\mathrm{d}\mu\circ\tau_h^{-1} &= \int F\circ\tau_h\,\mathrm{d}\mu\\
&= \int\int F\circ\tau_h(w)\pi_r^{\mathcal{H}}(\xi,\mathrm{d}w)\,\mathrm{d}\mu(\xi)\\
&= \int\int F(w)\exp\Big(\int w\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f - \frac{1}{2}\int\int K_r(s,t)\mathrm{d}f(s)\mathrm{d}f(t)\Big)
\end{aligned}$$

$$\times \exp(-\mathcal{H}_r(w-k)+\mathcal{H}_r(w))\pi_r^{\mathcal{H}}(\xi, \mathrm{d}w)\,\mathrm{d}\mu(\xi)$$

If we could show that $\tilde{\rho}_h(w) := \exp(\int w\,\mathrm{d}f - \int H_r(\xi)\,\mathrm{d}f - \frac{1}{2}\int\int K_r(s,t)\mathrm{d}f(s)\mathrm{d}f(t)) \times \exp(-\mathcal{H}_r(w-k)+\mathcal{H}_r(w))$ does not depend on ξ then we would get

$$\begin{aligned}
&= \int\int F(w)\tilde{\rho}_h(w)\pi_r^{\mathcal{H}}(\xi, \mathrm{d}w)\,\mathrm{d}\mu(\xi) \\
&= \int F(w)\tilde{\rho}_h(w)\,\mathrm{d}\mu(w).
\end{aligned}$$

If we could also show that $\tilde{\rho}_h$ does not depend on the choice of $r > r_0$ it would be a density for $\mu \circ \tau_h$ with respect to μ for any Gibbs-measure μ for our specification.

3.4.1 Concrete Calculations

Since we need quasi-invariance (and then closability) not for all directions f, but only for an orthonormal basis of H, we can restrict our further calculations to a subset of all possible directions f; this also simplifies the calculations.

Let $f \in C_0^1(\mathbb{R},\mathbb{R})$ and $r_0 > 0$ such that $\operatorname{supp}(f) \subset [-r_0, r_0]$. Then we also have $\operatorname{supp} f' \subset [-r_0, r_0]$.

For $\xi \in C(\mathbb{R},\mathbb{R})$ and $r > r_0$ define $G_\xi \in C(\mathbb{R},\mathbb{R})^*$ via
$G_\xi(w) := \int w\,\mathrm{d}(-f') - \int H_r(\xi)\,\mathrm{d}(-f')$.

Then we have that $R_{\pi_r(\xi,\cdot)}(G_\xi) \in C(\mathbb{R},\mathbb{R})$,

$$\begin{aligned}
R_{\pi_r(\xi,\cdot)}(G_\xi)(t) &= \int K_r(s,t)\,\mathrm{d}(-f')(s) \\
&= 1_{[-r,r]}(t)\left(\frac{t-(-r)}{2r}\int_{-r}^{r} -f'(s)\,\mathrm{d}s - \int_{-r}^{t} -f'(s)\,\mathrm{d}s\right) \\
&= 1_{[-r,r]}(t)\left(\frac{t-(-r)}{2r}(-f(r)+f(-r)) + f(t) - f(-r)\right) \\
&= 1_{[-r,r]}(t)f(t) \\
&= f(t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-r}^{r} H_r(\xi)(t)\,\mathrm{d}(-f')(t) \\
&\quad= \int_{-r}^{r} \xi(-r) + \frac{t-(-r)}{2r}\,\mathrm{d}(-f')(t) \\
&\quad= \xi(-r)\int_{-r}^{r}\mathrm{d}(-f') + \frac{1}{2r}\xi(r)\int_{-r}^{r} t\,\mathrm{d}(-f')(t) + \frac{\xi(r)}{2r}r\int_{-r}^{r}\mathrm{d}(-f')\,\mathrm{d}t \\
&\quad= \xi(-r)(f(r)-f(-r)) + \frac{\xi(r)}{2r}\left(-f'(r)r - (-f'(-r))(-r) - \int_{-r}^{r} -f'(t)\,\mathrm{d}t\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi(r)}{2r}r(-f'(r)+f'(-r)) \\
&=\int_{-r}^{r}-f'(t)\,\mathrm{d}t \\
&=f(r)-f(-r)=0.
\end{aligned}
$$

We also get

$$
\begin{aligned}
\int\int K_r(s,t)\,\mathrm{d}(-f')(s)\,\mathrm{d}(-f')(t) &= \int_{-r}^{r} f(t)\,\mathrm{d}(-f')(t) \\
&= -f(r)f'(r)+f(-r)f'(-r)-\int_{-r}^{r}-f'(t)\,\mathrm{d}f(t) \\
&= \int_{-r}^{r} f'(t)f'(t)\,\mathrm{d}t
\end{aligned}
$$

Using these results we have:

$$
\int F\,\mathrm{d}\mu\circ\tau_f^{-1} = \int\int F(w)e^{-\int w\,\mathrm{d}f'-\frac{1}{2}\int f'(t)^2\,\mathrm{d}t}e^{-\mathcal{H}_r(w-f)+\mathcal{H}_r(w)}\pi_r^{\mathcal{H}}(\xi,\mathrm{d}w)\,\mathrm{d}\mu(\xi)
$$

We already have independence from ξ, now we have to show the independence from r:

$$
\begin{aligned}
&\mathcal{H}_r(\omega-f)-\mathcal{H}_r(\omega)=\int_{-r}^{r}\varphi(\omega(x)-f(x))-\varphi(\omega(x))\mathrm{d}x \\
&+\frac{1}{2}\iint_{|x|,|y|\le r}\psi(x-y,\omega(x)-f(x)-(\omega(y)-f(y)))-\psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y \\
&+\iint_{|x|\le r<|y|}\psi(x-y,\omega(x)-f(x)-(\omega(y)-f(y)))-\psi(x-y,\omega(x)-\omega(y))\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Since the expressions grow too large to handle them together, we will consider the first summand alone and then the two last ones together.

$$
\begin{aligned}
&\int_{-r}^{r}\varphi(\omega(x)-f(x))-\varphi(\omega(x))\mathrm{d}x \\
&\quad=\int\limits_{\operatorname{supp} f}\varphi(\omega(x)-f(x))-\varphi(\omega(x))\mathrm{d}x+\int\limits_{[-r,r]\setminus\operatorname{supp} f}\varphi(\omega(x)-\underbrace{f(x)}_{=0})-\varphi(\omega(x))\mathrm{d}x \\
&\quad=\int_{\operatorname{supp} f}\varphi(\omega(x)-f(x))-\varphi(\omega(x))\mathrm{d}x
\end{aligned}
$$

and

$$
\frac{1}{2}\iint\limits_{|x||y|\le r}\psi(x-y,\omega(x)-f(x)-(\omega(y)-f(y)))-\psi(x-y,\omega(x)-\omega(y)\mathrm{d}x\mathrm{d}y)
$$

$$+ \iint\limits_{|x| \le r < |y|} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y))) - \psi(x-y, \omega(x) - \omega(y) \mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{0 \le |x| < |y| \le r} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y))) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{|x| \le r < |y|} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y))) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

since $\operatorname{supp} f \subset [-r_0, r_0]$ and $r > r_0$ we have

$$= \iint\limits_{0 \le |x| < |y| \le r_0} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y))) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{0 \le |x| \le r_0 < |y| \le r} \psi(x-y, \omega(x) - f(x) - (\omega(y) - \underbrace{f(y)}_{=0})) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{r_0 < |x| < |y| \le r} \psi(x-y, \omega(x) - \underbrace{f(x)}_{=0} - (\omega(y) - \underbrace{f(y)}_{=0})) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{|x| \le r_0 \le r < y} \psi(x-y, \omega(x) - f(x) - (\omega(y) - \underbrace{f(y)}_{=0})) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{r_0 \le |x| \le r < |y|} \psi(x-y, \omega(x) - \underbrace{f(x)}_{=0} - (\omega(y) - \underbrace{f(y)}_{=0})) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{|x| < |y| \le r_0} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y)) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

$$+ \iint\limits_{|x| \le r_0 < |y|} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y)) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y$$

Thus we get:

$$\begin{aligned}
&\mathcal{H}_r(\omega - f) - \mathcal{H}_r(\omega) = \\
&\int_{\operatorname{supp} f} \varphi(\omega(x) - f(x)) - \varphi(\omega(x)) \mathrm{d}x \\
&+ \iint_{|x| < |y| \le r_0} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y)) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y \\
&+ \iint_{|x| \le r_0 < |y|} \psi(x-y, \omega(x) - f(x) - (\omega(y) - f(y)) - \psi(x-y, \omega(x) - \omega(y)) \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.4.1}$$

The expression on the right side is independent of the choice of $r > r_0$.
This gives altogether:

$$
\begin{aligned}
&a_f(w) := \frac{\mathrm{d}\mu \circ \tau_f^{-1}}{\mathrm{d}\mu}(w) = \\
&\exp\Big(-\int w \,\mathrm{d}f' - \frac{1}{2}\int (f'(t))^2 \,\mathrm{d}t\Big) \exp\Big(-\int_{\operatorname{supp} f} \varphi(w(t) - f(t)) - \varphi(w(t)) \,\mathrm{d}t\Big) \\
&\times \exp\Big(-\iint\limits_{|s|<|t|\leq r_0} \psi(s-t, w(s)-w(t)-(f(s)-f(t))) - \psi(s-t, w(s)-w(t)) \,\mathrm{d}t \,\mathrm{d}s\Big) \\
&\quad \times \exp\Big(-\iint\limits_{|s|\leq r_0<|t|} \psi(s-t, w(s)-w(t)-f(s)) - \psi(s-t, w(s)-w(t)) \,\mathrm{d}t \,\mathrm{d}s\Big)
\end{aligned}
\tag{3.4.2}
$$

3.5 Closability

We have the following result:

Theorem 3.5.1. The measure μ is k-quasi-invariant for all $k \in C_0^1(\mathbb{R}, \mathbb{R})$. We have:

$$
\begin{aligned}
&a_{sk}(w) := \frac{\mathrm{d}\mu \circ \tau_{sk}^{-1}}{\mathrm{d}\mu}(w) = \\
&\exp\Big(-\int sw \,\mathrm{d}(k') - \frac{1}{2}\int s^2 (k'(t))^2 \,\mathrm{d}t\Big) \exp\Big(-\int\limits_{\operatorname{supp} k} \varphi(w(t) - sk(t)) - \varphi(w(t)) \,\mathrm{d}t\Big) \\
&\times \exp\Big(-\iint\limits_{0\leq|r|<|t|\leq r_0} \psi(r-t, w(r)-w(t)-s(k(r)-k(t))) - \psi(r-t, w(r)-w(t))) \,\mathrm{d}r \,\mathrm{d}t\Big) \\
&\quad \times \exp\Big(-\iint\limits_{|r|\leq r_0<|t|} \psi(r-t, w(r)-w(t)-sk(r)) - \psi(r-t, w(r)-w(t))) \,\mathrm{d}r \,\mathrm{d}t\Big)
\end{aligned}
\tag{3.5.1}
$$

Suppose that there exists an ONB $(h_n)_{n\in\mathbb{N}} \subset C_0^1(\mathbb{R}, \mathbb{R})$ of H such that $t \mapsto a_{th_n}(w)$ fulfills the Hamza-condition (see 3.1.3) for every $n \in \mathbb{N}$, then $(\mathcal{E}, \mathcal{F}C_b^\infty(C(\mathbb{R}, \mathbb{R})))$ is closable.

Proof. For the calculation of a_{sk} see the section before. The rest follows from 3.1.7 and 3.1.6 $\square$

As an application we will show closability under two different sets of conditions. The first set of conditions is taken from [HO01], but we need a stronger condition on φ, on the other hand we can then show the closability of $\mathcal{E}$ directly without the need of an approximation.

The second set of conditions is taken form [Bet03], where they are conditions that imply the existence of a Gibbsmeasure μ. Again we have to take the same, stronger, condition on φ. The conditions on ψ on the other hand are quite different from those of [HO01].

3.5.1 Conditions similar to [HO01]

Let us assume the following conditions:

1. φ is locally bounded on $\mathbb{R}$.
2. $\psi(x, y) = \psi_1(x)\psi_2(y)$ and ψ_1 and ψ_2 fulfill the following conditions:
   a) $\exists C_1 > 0, \gamma > 1 \forall x \in \mathbb{R} : 0 \leq \psi_1 \leq C_1(1 + |x|)^{-\gamma}$
   b) ψ_2 is convex and $\exists C_2, p > 0 \forall x \in \mathbb{R} : |\psi_2| \leq C_2 e^{p|x|}$.
3. μ is a (φ, ψ)-Gibbs-measure such that $\mu(\{w \in W | \int_{\mathbb{R}} (1+|x|)^{-\gamma} e^{p|w(x)|} \,\mathrm{d}x < \infty\}) = 1$

Then we get for $w \in W$, $s \in \mathbb{R}$ and $f \in C_0^1(\mathbb{R})$ that

$$\begin{aligned}
&\int_{s-\varepsilon}^{s+\varepsilon} a_{tf}^{-1}(w) \,\mathrm{d}t \\
&= \int_{s-\varepsilon}^{s+\varepsilon} \exp\left(t \int_{\operatorname{supp} f} w(x) \,\mathrm{d}(-f')(x) + \frac{1}{2} t^2 \int_{\operatorname{supp} f} (f'(x))^2 \,\mathrm{d}x\right) \\
&\times \exp\left(\int_{\operatorname{supp} f} \varphi(w(x) - tf(x)) - \varphi(w(x)) \,\mathrm{d}x\right) \\
&\times \exp\left(\iint_{|x|<|y|<C_f} \psi(x-y, w(x) - w(y) - t(f(x) - f(y))) - \psi(x-y, w(x) - w(y)) \,\mathrm{d}x\mathrm{d}y\right) \\
&\times \exp\left(\iint_{|x|\leq C_f<|y|} \psi(x-y, w(x) - w(y) - tf(x)) - \psi(x-y, w(x) - w(y)) \,\mathrm{d}x\mathrm{d}y\right) \mathrm{d}t
\end{aligned}$$

We will show that $a_{tf}^{-1}(w)$ is locally bounded for μ-a.e. $w \in C(\mathbb{R}, \mathbb{R})$. Let $w \in C(\mathbb{R}, \mathbb{R})$ be such that $\int_{\mathbb{R}} (1 + |x|)^{-\gamma} e^{pw(x)} \,\mathrm{d}x < \infty$.

The factor in the first line is locally bounded in t so we can estimate it by a constant C. The other factors will be treated separately.

Since f has compact support and w and f are is continuous, so is $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, t) \mapsto w(x) + tf(x)$. For any compact subset $K \subset \mathbb{R}$ we have that $\operatorname{supp} f \times K$ is again compact, hence $F(\operatorname{supp} f \times K)$ is again compact. Then

$$\begin{aligned}
&\exp\left(\int_{\operatorname{supp} f} \varphi(w(x) - tf(x)) - \varphi(w(x)) \,\mathrm{d}x\right) \\
&\quad \leq \exp\left(\int_{\operatorname{supp} f} \sup \varphi(F(\operatorname{supp} f \times K)) + \sup \varphi(w(\operatorname{supp} f)) \mathrm{d}x\right) \\
&\quad < C(K)
\end{aligned}$$

Since ψ_2 is convex by assumption we have

$$\begin{aligned}
&\psi_2(w(x)-w(y)-t(f(x)-f(y)))-\psi_2(w(x)-w(y))\\
&\quad=\psi_2((1-t)(w(x)-w(y))+t(w(x)-w(y)-(f(x)-f(y)))-\psi_2(w(x)-w(y))\\
&\quad=\psi_2((1-t)(w(x)-w(y))+t(w(x)-w(y)-(f(x)-f(y)))-\psi_2(w(x)-w(y))\\
&\quad\le(1-t)\psi_2(w(x)-w(y))+t\psi_2(w(x)-w(y)-(f(x)-f(y))-\psi_2(w(x)-w(y)\\
&\quad=t(\psi_2(w(x)-w(y)-(f(x)-f(y))-\psi_2(w(x)-w(y))).
\end{aligned}$$

For ψ_1 we get:

$$\psi_1(x-y)\le C_1(1+|x-y|)^{-\gamma}=C_1(1+|y|)^{-\gamma}\left(\frac{1+|y|}{1+|x-y|}\right)^{\gamma}$$

and since $|y|\ge|x|$.

$$\frac{1+|y|}{1+|x-y|}\le\frac{1+|y|}{1+\big||y|-|x|\big|}=\frac{1+|y|}{1+|y|-|x|}=\frac{1}{1-\frac{|x|}{1+|y|}}$$

In the case $0<|x|<|y|\le C_f$ we have

$$\frac{|x|}{1+|y|}\le\frac{|x|}{1+|x|}=\frac{1}{\frac{1}{|x|}+1}\le\frac{1}{\frac{1}{+C_f}\frac{1}{}}=\frac{C_f}{1+C_f}(\le 1).$$

In the case $0<|x|<C_f<|y|$ we have

$$\frac{|x|}{1+|y|}\le\frac{C_f}{1+|y|}\le\frac{C_f}{1+C_f}(\le 1).$$

Using this in the above estimate we get:

$$\frac{1+|y|}{1+|x-y|}\le\frac{1}{1-\frac{|x|}{1+|y|}}\le\frac{1}{1-\frac{C_f}{1+C_f}}=\frac{1}{\frac{1+C_f-C_f}{1+C_f}}=1+C_f$$

and finally we have that

$$\psi_1(x-y)\le C_1(1+|y|)^{-\gamma}(1+C_f)^{\gamma}$$

Then we get for the integrand:

$$\begin{aligned}
&\psi(x-y,w(x)-w(y)-t(f(x)-f(y)))-\psi(x-y,w(x)-w(y))\\
&\quad=\psi_1(x-y)\psi_2(w(x)-w(y)-t(f(x)-f(y)))-\psi_1(x-y)\psi_2(w(x)-w(y))\\
&\quad\le t\psi_1(x-y)\big(\psi_2(w(x)-w(y)-(f(x)-f(y))-\psi_2(w(x)-w(y))\big)\\
&\quad\le t\psi_1(x-y)(C_2e^{p|w(x)-w(y)-(f(x)-f(y))|}+C_2e^{p|w(x)-w(y)|}\big)\\
&\quad\le t\psi_1(x-y)C_2\exp(p|w(y)|)(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))\\
&\quad\le t(1+C_f)^{\gamma}C_1(1+|y|)^{-\gamma}C_2\exp(p|w(y)|)(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))
\end{aligned}$$

$$= t\Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big)(1+|y|)^{-\gamma}\exp(p|w(y)|)$$

Then we get for the first integral:

$$\begin{aligned}
&\exp\Bigg(\iint\limits_{|x|<|y|<C_f} \psi(x-y, w(x)-w(y)-t(f(x)-f(y))) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg) \\
&\quad\le \exp\Bigg(\iint\limits_{|x|<|y|<C_f} t\Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big) \\
&\qquad\quad \times (1+|y|)^{-\gamma}\exp(p|w(y)|)\,\mathrm{d}x\mathrm{d}y\Bigg) \\
&\quad= \exp\Bigg(t\iint\limits_{|x|<|y|<C_f} \Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big) \\
&\qquad\quad \times (1+|y|)^{-\gamma}\exp(p|w(y)|)\,\mathrm{d}x\mathrm{d}y\Bigg)
\end{aligned}$$

which is locally bounded in t.

And for the second integral we get:

$$\begin{aligned}
&\exp\Bigg(\iint\limits_{|x|<C_f<|y|} \psi(x-y, w(x)-w(y)-t(f(x)-f(y))) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg) \\
&\quad\le \exp\Bigg(\iint\limits_{|x|<C_f<|y|} t\Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big) \\
&\qquad\quad \times (1+|y|)^{-\gamma}\exp(p|w(y)|)\,\mathrm{d}x\mathrm{d}y\Bigg) \\
&\quad= \exp\Bigg(t\iint\limits_{|x|<C_f<|y|} \Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big) \\
&\qquad\quad \times (1+|y|)^{-\gamma}\exp(p|w(y)|)\,\mathrm{d}x\mathrm{d}y\Bigg) \\
&\quad\le \exp\Bigg(t\int_{-C_f}^{C^f}\int_{\mathbb{R}} \Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big) \\
&\qquad\quad \times (1+|y|)^{-\gamma}\exp(p|w(y)|)\,\mathrm{d}y\mathrm{d}x\Bigg) \\
&\quad= \exp\Bigg(t\int_{-C_f}^{C^f} \Big((1+C_f)^\gamma C_2(\exp(p|w(x)|)(1+2\exp(p\|f\|_\infty))C_1\Big)
\end{aligned}$$

$$\underbrace{\times \int_{\mathbb{R}} (1+|y|)^{-\gamma} \exp(p|w(y)|)\,\mathrm{d}y\,\mathrm{d}x}_{<\infty \text{ by condition (3) for } \mu\text{-a.e. } w}\Bigg)$$

which is again locally bounded in t.

In this case every $s \in \mathbb{R}$ is regular, hence the Hamza condition is fulfilled for all $f \in C_0^1(\mathbb{R},\mathbb{R})$. Since $C_0^1(\mathbb{R},\mathbb{R})$ is dense in H there exists an ONB of H consisting of functions in $C_0^1(\mathbb{R},\mathbb{R})$ and we can apply 3.5.1.

3.5.2 Conditions similar to [Bet03]

Let us assume the following conditions:

1. φ is locally bounded on $\mathbb{R}$.
2. There exists C_∞ such that

$$\int_{-\infty}^{\infty} |\psi(x_0, x_s, |s|)|\,\mathrm{d}s < C_\infty \text{ and } \int_{-\infty}^{\infty} |\psi(x_s, x_0, |s|)|\,\mathrm{d}s < C_\infty$$

   uniformly in $x \in C(\mathbb{R},\mathbb{R}^d)$.

Again we get for $w \in C(\mathbb{R},\mathbb{R})$, $s \in \mathbb{R}$ and $f \in C_0^1(\mathbb{R})$ that

$$\begin{aligned}
&\int_{s-\varepsilon}^{s+\varepsilon} a_{tf}^{-1}(w)\,\mathrm{d}t \\
&\quad= \int_{s-\varepsilon}^{s+\varepsilon} \exp\Big(t \int_{\operatorname{supp} f} w(x)\,\mathrm{d}(-f')(x) + \frac{1}{2}t^2 \int_{\operatorname{supp} f} (f'(x))^2\,\mathrm{d}x\Big) \\
&\quad\times \exp\Big(\int_{\operatorname{supp} f} \varphi(w(x) - tf(x)) - \varphi(w(x))\,\mathrm{d}x\Big) \\
&\quad\times \exp\Big(\iint\limits_{|x|<|y|<C_f} \psi(x-y, w(x)-w(y) - t(f(x)-f(y))) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Big) \\
&\quad\times \exp\Big(\iint\limits_{|x|\le C_f<|y|} \psi(x-y, w(x)-w(y) - tf(x)) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Big)\,\mathrm{d}t \\
&\quad\le \int_{s-\varepsilon}^{s+\varepsilon} \exp\Big(|t|\Big|\int_{\operatorname{supp} f} w(x)\,\mathrm{d}(-f')(x)\Big| + \frac{1}{2}t^2\Big|\int_{\operatorname{supp} f} (f'(x))^2\,\mathrm{d}x\Big|\Big) \\
&\quad\times \exp\Big(\int_{\operatorname{supp} f} |\varphi(w(x) - tf(x))|\,\mathrm{d}x + \int_{\operatorname{supp} f} |\varphi(w(x))|\,\mathrm{d}x\Big) \\
&\quad\times \exp\Big(\iint\limits_{|x|<|y|<C_f} \psi(x-y, w(x)-w(y) - t(f(x)-f(y))) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Big) \\
&\quad\times \exp\Big(\iint\limits_{|x|\le C_f<|y|} \psi(x-y, w(x)-w(y) - tf(x)) - \psi(x-y, w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Big)\,\mathrm{d}t
\end{aligned}$$

the first factor is again locally bounded in t and can be estimated by

$$\exp\Bigg(\max(s-\varepsilon, s+\varepsilon)\int_{\operatorname{supp} f} w(x)\,\mathrm{d}(-f')(x)+\frac{1}{2}\max((s-\varepsilon)^2,(s+\varepsilon)^2)\int_{\operatorname{supp} f}(f'(x))^2\,\mathrm{d}x\Bigg)$$

The second part (φ) poses no problem because φ is again locally bounded as in the section before (3.5.1).

Concerning the ψ-part we have:

$$\begin{aligned}
&\exp\Bigg(\iint\limits_{|x|<|y|<C_f}\psi(x-y,w(x)-w(y)-t(f(x)-f(y)))-\psi(x-y,w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg)\\
&\quad\times\exp\Bigg(\iint\limits_{|x|\le C_f<|y|}\psi(x-y,w(x)-w(y)-tf(x))-\psi(x-y,w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg)\\
&\quad=\exp\Bigg(\iint\limits_{|x|<|y|<C_f}\psi(x-y,w(x)-w(y)-t(f(x)-f(y)))-\psi(x-y,w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg)\\
&\quad\times\exp\Bigg(\iint\limits_{|x|\le C_f<|y|}\psi(x-y,w(x)-w(y)-t(f(x)-\underbrace{f(y)}_{=0}))-\psi(x-y,w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg)\\
&\quad\le\exp\Bigg(\int_0^{C_f}\int_x^{\infty}\psi(x-y,w(x)-w(y)-t(f(x)-f(y)))-\psi(x-y,w(x)-w(y))\,\mathrm{d}x\mathrm{d}y\Bigg)\\
&\quad\le\exp\Bigg(\int_0^{C_f}\int_x^{\infty}\Big|\psi(x-y,w(x)-w(y)-t(f(x)-f(y)))\Big|\,\mathrm{d}x\mathrm{d}y\\
&\qquad+\int_0^{C_f}\int_x^{\infty}\Big|\psi(x-y,w(x)-w(y))\Big|\,\mathrm{d}x\mathrm{d}y\Bigg)
\end{aligned}$$

Then we define for any $w\in C(\mathbb{R},\mathbb{R})$ $w_x:\mathbb{R}\to\mathbb{R}, y\mapsto w(x+y)$. And we get for any $w\in C(\mathbb{R},\mathbb{R})$, hence also for $w+tf$:

$$\begin{aligned}
&\int_0^{C_f}\int_x^{\infty}\big|\psi(x-y,w(x)-w(y))\big|\,\mathrm{d}y\mathrm{d}x\\
&\quad=\int_0^{C_f}\int_0^{\infty}\big|\psi(x-y,w(x)-w(z+x))\big|\,\mathrm{d}z\mathrm{d}x\\
&\quad=\int_0^{C_f}\int_0^{\infty}\big|\psi(x-y,w_x(0)-w_x(z))\big|\,\mathrm{d}z\mathrm{d}x\\
&\quad\le\int_0^{C_f}C_\infty=C_fC_\infty
\end{aligned}$$

Again we have shown that for all $w\in C(\mathbb{R},\mathbb{R})$ every $s\in\mathbb{R}$ is regular, hence the Hamza condition is fulfilled for all $f\in C_0^1(\mathbb{R},\mathbb{R})$. Since $C_0^1(\mathbb{R},\mathbb{R})$ is dense in H there exists an ONB of H consisting of functions in $C_0^1(\mathbb{R},\mathbb{R})$ and we can apply 3.5.1.

4 Quasi-regularity

4.1 Review of Definitions

We consider the following bilinear form: Define $\mathcal{E}$ on $\mathcal{F}C_b^\infty$ as follows:

$$\mathcal{E}(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \,\mathrm{d}\mu$$

and then consider the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E}, \mathcal{F}C_b^\infty)$. The closability has been shown in Chapter 3. Here we have

$$H = H^{1,2}(\mathbb{R}, \mathrm{d}x) \qquad\qquad \|h\|_H = \sqrt{\int h(x)^2\,\mathrm{d}x + \int h'(x)^2 \mathrm{d}x}.$$

We want to show that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular symmetric Dirichlet-form. To prove that it is a symmetric Dirichlet-form we have to check the following conditions (see [MR92, Definition I.4.5]):

For all $u, v \in D(\mathcal{E})$ we have that $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ and $u^+ \wedge 1 \in D(\mathcal{E})$ and

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u,u)$$

Definition 4.1.1. A Dirichlet-form is called quasi-regular if the following conditions are fulfilled:

1. There exists an $\mathcal{E}$-nest of compact subsets of $C(\mathbb{R}, \mathbb{R})$.
2. There exists an $\mathcal{E}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions.
3. There exists $u_n \in D(\mathcal{E}), n \in \mathbb{N}$ with $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n, n \in \mathbb{N}$ and an $\mathcal{E}$-exceptional subset $N \subset C(\mathbb{R}, \mathbb{R})$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $C(\mathbb{R}, \mathbb{R}) \setminus N$.

We recall the definitions used above:

Definition 4.1.2.

1. Define for $F \subset E$, F closed

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) | u = 0 \text{ } m\text{-a.e. on } E \setminus F\}.$$

2. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an *$\mathcal{E}$-nest* if $\bigcup_{k\geq 1} D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$.

3. $\mathcal{E}_1^{1/2}(u) := \sqrt{\mathcal{E}(u,u) + \langle u,u\rangle_H}$ defines a norm on $D(\mathcal{E})$. A subset $A \subset D(\mathcal{E})$ is called $\mathcal{E}_1^{1/2}$-dense, if its closure $\bar{A}$ fulfils $\bar{A} = D(\mathcal{E})$.

4. A property is said to hold $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if there exists an $\mathcal{E}$-exceptional set N such that the property holds on $E \setminus N$.

5. A subset $N \subset E$ is called *$\mathcal{E}$-exceptional* if $N \subset \bigcap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$.

6. An $\mathcal{E}$-q.e. defined function f on E is called *$\mathcal{E}$-quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f_{|F_k}$ is continuous for every $k \in \mathbb{N}$

Definition 4.1.3. Let $(\Omega, \mathcal{M})$ a measurable space. By $\mathcal{P}(\Omega)$ we denote the set of all probability measures on $(\Omega, \mathcal{M})$. The completion of $\mathcal{M}$ with respect to a probability measure P is denoted by $\mathcal{M}^P$. The σ-algebra $\mathcal{M}^*$ is defined by $\mathcal{M}^* := \bigcap_{P\in\mathcal{P}(\Omega)} \mathcal{M}^P$.

Definition 4.1.4. A family $(\mathcal{M}_t)_{t\in[0,\infty]}$ of σ-algebras is called a filtration if $\mathcal{M}_s \subset \mathcal{M}_t$ for $s \leq t$ and $\mathcal{M}_\infty = \sigma(\bigcup_{t\in[0,\infty)} \mathcal{M}_t)$. Let $(\mathcal{M}_t)$ be a filtration. We define then $\mathcal{M}_{t+} := \bigcap_{s>t} \mathcal{M}_s$. A filtration is called *right-continuous* if $\mathcal{M}_{t+} = \mathcal{M}_t$ for all $t \geq 0$.

Definition 4.1.5. Let $(\Omega, \mathcal{M})$ be a measurable space. An $\mathcal{M}$-measurable function $\tau : \Omega \to [0,\infty]$ is called an *$(\mathcal{M}_t)$-stopping time* if $\{\tau \leq t\} \in \mathcal{M}_t$ for all $t \geq 0$. The sub-σ-algebra $\mathcal{M}_\tau$ of $\mathcal{M}$ is then defined as:

$$\mathcal{M}_\tau := \{\Gamma \in \mathcal{M} | \Gamma \cap \{\tau \leq t\} \in \mathcal{M}_t \text{ for all } t \geq 0\}$$

Definition 4.1.6 (cf. [MR92, Definition IV.1.4]). $(X_t)_{t\geq 0}$ is called a *stochastic process with state space* E if $X_t : \Omega \to E$ is an $\mathcal{M}/\mathcal{B}(E)$-measurable map for all $t \in [0,\infty)$. It is called *measurable* if $(t,\omega) \mapsto X_t(\omega)$ is $\mathcal{B}([0,\infty)) \times \mathcal{M}/\mathcal{B}(E)$-measurable. It is called $(\mathcal{M}_t)$-adapted for a filtration $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$ if each X_t is $\mathcal{M}_t/\mathcal{B}(E)$-measurable.

Definition 4.1.7. A stochastic process $\mathbf{M} := (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ is called a Markov process if the following conditions hold:

1. There exists a filtration $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$ such that $(X_t)_{t\geq 0}$ is an $(\mathcal{M}_t)$-adapted stochastic process with state space E.

2. For each $t \geq 0$ there exists a shift-operator $\theta_t : \Omega \to \Omega$ such that $X_s \circ \theta_t = X_{s+t}$ for all $s, t \geq 0$.

3. P_z, $z \in E$, are probability measures on $(\Omega, \mathcal{M})$ such that $z \mapsto P_z(\Gamma)$ is $\mathcal{B}(E)^*$-measurable for each $\Gamma \in \mathcal{M}$ respectively $\mathcal{B}(E)$-measurable if $\Gamma \in \sigma(\{X_s | s \in [0,\infty)\})$.

4. (Markov property) For all $A \in \mathcal{B}(E)$ and $t, s \geq 0$ we have:

$$P_z[X_{s+t} \in A | \mathcal{M}_s] = P_{X_s}[X_t \in A] \qquad P_z - \text{a.s.}, z \in E.$$

Definition 4.1.8. Let $\mathbf{M} := (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ be a Markov process. It is called a *right process* if the following conditions hold:

1. $P_z[X_0 = z] = 1$ for all $z \in E$.
2. For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$.
3. $(\mathcal{M}_t)$ is right-continuous, i.e. and for every $(\mathcal{M}_t)$-stopping time σ and every probability measure μ on E

   $$P_\mu[X_{\sigma+t} \in A | \mathcal{M}_\sigma] = P_{X_\sigma}[X_t \in A]$$

   for all $A \in \mathcal{B}(E)$, $t \geq 0$.

Definition 4.1.9. A right process is called a diffusion if

$$P_z[t \mapsto X_t \text{ is continuous on } [0, \infty)] = 1 \quad \text{for all } z \in E.$$

We also need the following definition (compare with [MR92, Definition IV.2.5])

Definition 4.1.10. A right process $\mathbf{M}$ is called properly associated with $(\mathcal{E}, D(\mathcal{E}))$ if $p_t f$ is an m-version of $T_t f$, where $(T_t)_{t\geq 0}$ is the semigroup associated to $(\mathcal{E}, D(\mathcal{E}))$ (cf. [MR92, Chapter I.2]) and $\mathcal{E}$-quasi-continuous for all $t > 0$, $f \in \mathcal{B}_b(E) \cap L^2(E, m)$. It is called properly coassociated if $p_t f$ is an m-version of $\hat{T}_t f$ and $\mathcal{E}$-quasi-continuous for all $t > 0$, $f \in \mathcal{B}_b(E) \cap L^2(E, m)$.
A pair $(\mathbf{M}, \hat{\mathbf{M}})$ is called properly associated with $(\mathcal{E}, D(\mathcal{E}))$ if $\mathbf{M}$ is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ and $\hat{\mathbf{M}}$ is properly coassociated with $(\mathcal{E}, D(\mathcal{E}))$.

We want to apply the following theorem:

Theorem 4.1.11 ([MR92, V.1.11])**.** A quasi-regular Dirichlet form possesses the local property if and only if it is associated with a pair of diffusions $(\mathbf{M}, \hat{\mathbf{M}})$.

Remark 4.1.12. Since in our case the Dirichlet form is symmetric $\mathbf{M}$ and $\hat{\mathbf{M}}$ coincide.

To prove the first condition of quasi-regularity we closely follow the proofs in [MR92], [RS92] and [RS95].

4.2 $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichletform and some basic estimates

We need the following lemma which is similar to [RS92, Lemma 1.1, Lemma 1.2] and [MR92, Lemma IV.4.1]

Lemma 4.2.1. Let $\varphi \in C_b^1(\mathbb{R}, \mathbb{R})$, then for all $u \in D(\mathcal{E})$ we have $\varphi \circ u \in D(\mathcal{E})$ and for all $u, v \in D(\mathcal{E})$

$$\begin{aligned} \nabla(\varphi \circ u) &= \varphi' \nabla u \\ \mathcal{E}(\varphi \circ u, \varphi \circ v) &= \frac{1}{2} \int (\varphi')^2 \langle \nabla u, \nabla v \rangle_H \, \mathrm{d}\mu \end{aligned}$$

Proof. Let $u \in D(\mathcal{E})$ and $(u_n) \subset \mathcal{F}C_b^\infty$ a sequence such that $u_n \to u$ in $D(\mathcal{E})$. Then we have that $\varphi(u_n) \to \varphi(u)$ in $L^2(E,\mu)$. And $\nabla(\varphi \circ u_n) = \varphi'(u_n)\nabla u_n \xrightarrow[n\to\infty]{} \varphi'(u)\nabla(u)$ in $L^2(E \to H, \mu)$. Since $\mathcal{E}, D(\mathcal{E})$ is closed we have that $\varphi \circ u \in D(\mathcal{E})$ and $\nabla(\varphi \circ u) = \varphi'(u)\nabla u$, hence we have $\mathcal{E}(\varphi \circ u, \varphi \circ v) = \frac{1}{2}\int (\varphi')^2 \langle \nabla u, \nabla v\rangle_H \,\mathrm{d}\mu$. □

Lemma 4.2.2. For $u, v \in D(\mathcal{E}) \cap \mathcal{B}_b(C(\mathbb{R},\mathbb{R}))$ we have

(i) $u \vee v \in D(\mathcal{E})$ and

$$\langle \nabla(u \vee v), \nabla(u \vee v)\rangle_H \le \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H \tag{4.2.1}$$

(ii) $u \wedge v \in D(\mathcal{E})$ and

$$\langle \nabla(u \wedge v), \nabla(u \wedge v)\rangle_H \le \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H \tag{4.2.2}$$

Proof. Since $a \vee b = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$ and $a \wedge b = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$, it is enough to show that $|u| \in D(\mathcal{E})$ for $u \in \mathcal{F}C_b^\infty$ in order to show that $u \vee v$ and $u \wedge v \in D(\mathcal{E})$.

Let δ_n be a Dirac-sequence, (i.e. $\delta_n \in C_0^\infty(\mathbb{R})$, $\delta_n \ge 0$, $\int \delta_n \,\mathrm{d}s = 1$, $\delta_n(s) = \delta_n(-s)$, $s \in \mathbb{R}$ and $\operatorname{supp}\delta_n \subset (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$) and define $f_n(t) := \delta_n * |\cdot|(t) = \int \delta_n(s)|t-s|\,\mathrm{d}s$. Then we have by 4.2.1 $f_n(u-v) \in D(\mathcal{E})$ for $u, v \in D(\mathcal{E})$, $f_n \to |\cdot|$ locally uniformly and

$$\begin{aligned}
f_n'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int |t-s|\delta_n(s)\,\mathrm{d}s \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int |t+h-s|\delta_n(s)\,\mathrm{d}s - \int |t+s|\delta_n(s)\,\mathrm{d}s\right) \\
&= \lim_{h\to 0}\frac{1}{h}\int (|t+h-s| - |t-s|)\delta_n(s)\,\mathrm{d}s \\
&= \lim_{h\to 0}\int \frac{|t+h-s| - |t-s|}{h}\delta_n(s)\,\mathrm{d}s \\
&= \int 1_{(-\infty,t)} - 1_{(t,\infty)}\delta_n(s)\,\mathrm{d}s \\
&\xrightarrow[n\to\infty]{} \begin{cases} 1 & ,t>0 \\ -1 & ,t<0 \\ 0 & ,t=0 \end{cases} \\
&= \operatorname{sign}(t)
\end{aligned}$$

For any bounded $u \in D(\mathcal{E})$ we then have $\nabla f_n \circ u \to \operatorname{sign} \circ u$ and since $u \in D(\mathcal{E})$ and $(\mathcal{E}, D(\mathcal{E}))$ is closed we have that $\nabla|u| = \operatorname{sign}(u)\nabla u$. Applying this result we get:

$$\begin{aligned}
&\langle \nabla(u \vee v), \nabla(u \vee v)\rangle_H \\
&= \langle \nabla(\frac{1}{2}(u+v) + \frac{1}{2}|u-v|), \nabla(\frac{1}{2}(u+v) + \frac{1}{2}|u-v|)\rangle_H \\
&= \langle \frac{1}{2}(\nabla u + \nabla v + \operatorname{sign}(u-v)(\nabla u - \nabla v)), \frac{1}{2}(\nabla u + \nabla v + \operatorname{sign}(u-v)(\nabla u - \nabla v))\rangle_H
\end{aligned}$$

$$
\begin{aligned}
&= \langle 1_{\{u>v\}}\nabla u + 1_{\{u<v\}}\nabla v + \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v), 1_{\{u>v\}}\nabla u + 1_{\{u<v\}}\nabla v\rangle_H \\
&+ \langle 1_{\{u>v\}}\nabla u + 1_{\{u<v\}}\nabla v + \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v), \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v)\rangle_H \\
&= 1_{\{u>v\}}\langle \nabla u, \nabla u\rangle_H + 1_{\{u<v\}}\langle \nabla v, \nabla v\rangle_H \\
&+ \frac{1}{4}(\langle \nabla u, \nabla u\rangle_H + 2\langle \nabla u, \nabla v\rangle_H + \langle \nabla v, \nabla v\rangle_H) \\
&\leq \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H
\end{aligned}
$$

and for (iii) we get:

$$
\begin{aligned}
&\langle \nabla(u \wedge v), \nabla(u \wedge v)\rangle_H \\
&= \langle \nabla(\frac{1}{2}(u+v) - \frac{1}{2}|u-v|), \nabla(\frac{1}{2}(u+v) - \frac{1}{2}|u-v|)\rangle_H \\
&= \langle \frac{1}{2}(\nabla u + \nabla v - \operatorname{sign}(u-v)(\nabla u - \nabla v)), \frac{1}{2}(\nabla u + \nabla v - \operatorname{sign}(u-v)(\nabla u - \nabla v))\rangle_H \\
&= \langle 1_{\{u>v\}}\nabla v + 1_{\{u<v\}}\nabla u + \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v), 1_{\{u>v\}}\nabla v + 1_{\{u<v\}}\nabla u\rangle_H \\
&+ \langle 1_{\{u>v\}}\nabla v + 1_{\{u<v\}}\nabla u + \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v), \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v)\rangle_H \\
&= 1_{\{u>v\}}\langle \nabla v, \nabla v\rangle_H + 1_{\{u<v\}}\langle \nabla u, \nabla u\rangle_H \\
&+ \frac{1}{4}(\langle \nabla u, \nabla u\rangle_H + 2\langle \nabla u, \nabla v\rangle_H + \langle \nabla v, \nabla v\rangle_H) \\
&\leq \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H
\end{aligned}
$$

□

Now we can show that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet-form:

Proposition 4.2.3. $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet-form

Proof. Since $\langle \nabla u, \nabla v\rangle_H = \langle \nabla v, \nabla u\rangle_H$ we have that

$$\mathcal{E}(u,v) = \int \langle \nabla u, \nabla v\rangle_H \,\mathrm{d}\mu = \int \langle \nabla v, \nabla u\rangle_H \,\mathrm{d}\mu = \mathcal{E}(v,u),$$

hence $\mathcal{E}$ is a symmetric bilinear form. The constant functions 1 and 0 are in $\mathcal{F}C_b^\infty \subset D(\mathcal{E})$. Now we apply 4.2.4 and have that for any $u \in D(\mathcal{E})$ u^+ and $u^+ \wedge 1 \in D(\mathcal{E})$. Furthermore we have μ-a.s.:

$$
\begin{aligned}
\langle \nabla(u^+ \wedge 1), \nabla(u^+ \wedge 1)\rangle_H &\leq \langle \nabla u^+, \nabla u^+\rangle_H \vee \underbrace{\langle \nabla 1, \nabla 1\rangle_H}_{=0} \\
&\leq \langle \nabla u, \nabla u\rangle_H \vee \underbrace{\langle \nabla 0, \nabla 0\rangle_H}_{=0} \\
&= \langle \nabla u, \nabla u\rangle_H
\end{aligned}
$$

Using this estimate we have

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) = \frac{1}{2}\int \langle \nabla(u^+ \wedge 1), \nabla(u^+ \wedge 1)\rangle_H \,\mathrm{d}\mu \leq \frac{1}{2}\int \langle \nabla u, \nabla u\rangle_H \,\mathrm{d}\mu = \mathcal{E}(u,u).$$

□

Lemma 4.2.4. For $u, v \in D(\mathcal{E})$ we have

(i) $u \vee v \in D(\mathcal{E})$ and

$$\langle \nabla(u \vee v), \nabla(u \vee v)\rangle_H \leq \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H \tag{4.2.3}$$

(ii) $u \wedge v \in D(\mathcal{E})$ and

$$\langle \nabla(u \wedge v), \nabla(u \wedge v)\rangle_H \leq \langle \nabla u, \nabla u\rangle_H \vee \langle \nabla v, \nabla v\rangle_H \tag{4.2.4}$$

Proof. We will show that $\nabla|u| = \operatorname{sign}(u)\nabla u$ for any $u \in D(\mathcal{E})$. Then we get the result by the same calculations as in 4.2.2. Let $u \in D(\mathcal{E})$ then $|u| \in D(\mathcal{E})$ by [MR92,] since $x \mapsto |x|$ is a normal contraction. Then we have by [MR92, I.4.17] that $(|u| \wedge n) \vee (-n)) \to |u|$ in $D(\mathcal{E})$. So we have that $\nabla(|u| \wedge n) \vee (-n)) \to \nabla|u|$ in $L^2(C(\mathbb{R}, \mathbb{R}), \mu)$. On the other hand we have

$$\begin{aligned}
\nabla(|u| \wedge n) \vee (-n)) &= \nabla|u \wedge n \vee (-n)| \\
&= \operatorname{sign}(u \wedge n \vee (-n))\nabla(u \wedge n \vee (-n)) \\
&= \operatorname{sign}(u)\nabla(u \wedge n \vee (-n)) \\
&\to \operatorname{sign} u \nabla u \text{ in } L^2(C(\mathbb{R}, \mathbb{R}), \mu).
\end{aligned}$$

Putting this together we have $\nabla|u| = \operatorname{sign}(u)\nabla u$. Now we apply the same calculations as in 4.2.2 and finish our proof. □

4.3 Quasi-regularity

For $n \in \mathbb{N}$ we define $C_n := C([-n, n], \mathbb{R})$ with its sup-norm $\|\cdot\|_n$. Denote by C_n^* its topological dual space with norm $\|\cdot\|_n^*$.

Furthermore consider the following mappings:

$p_H : H \to C(\mathbb{R}, \mathbb{R})$	the Sobolev embedding
$p_n : C(\mathbb{R}, \mathbb{R}) \to C([-n, n], \mathbb{R})$	$f \mapsto f_{\vert_{[-n,n]}}$
$i_n : H \to C([-n, n], \mathbb{R})$	$i_n = p_n \circ p_H$
$j_n : C([-n, n], \mathbb{R})^* \to H^*$	$j_n(l)(h) = l(i_n(h))$ the dual mapping to i_n

For a better overview consider the following diagram:

$$\begin{tikzcd}
C(\mathbb{R},\mathbb{R})^* \arrow[dr] & & C(\mathbb{R},\mathbb{R}) \arrow[dd, "p_n"] \\
& H^* \cong H \arrow[ur, "p_H"] \arrow[dr, "i_n"] & \\
C_n^* := C([-n,n],\mathbb{R})^* \arrow[uu] \arrow[ur, "j_n"] & & C([-n,n],\mathbb{R}) =: C_n
\end{tikzcd}$$

Lemma 4.3.1. For all $n \in \mathbb{N}$ we have that $i_n : H \to (C_n, \|\cdot\|_n), h \mapsto h_{[-n,n]}$ is continuous.

Proof. First observe the following:

$$\begin{aligned}
h(x)^2 &= \int_{-\infty}^{x} 2h(t)h'(t)\,\mathrm{d}t \\
&\leq 2\sqrt{\int_{-\infty}^{x} f(t)^2\,\mathrm{d}t}\sqrt{\int_{-\infty}^{x} h'(t)^2\,\mathrm{d}t} \\
&\leq \int_{-\infty}^{x} h(t)^2\,\mathrm{d}t + \int_{-\infty}^{x} h'(t)^2\,\mathrm{d}t \\
&\leq \int_{-\infty}^{\infty} h(t)^2\,\mathrm{d}t + \int_{-\infty}^{\infty} h'(t)^2\,\mathrm{d}t \\
&= \|h\|_H^2
\end{aligned}$$

So we have $\sup_{-n\leq x\leq n}|h(x)| \leq \|h\|_H$. □

Remark 4.3.2. The result from above will be used in the following estimates. It would not work, if we had instead of H and its norm the following (where $\|\cdot\|_{\tilde{H}}$ is just a semi-norm):

$$\begin{aligned}
\tilde{H} &:= \{h \in C(\mathbb{R}, R) | h \text{ is absolutely continuous and } \int h'(t)^2\,\mathrm{d}t < \infty\} \\
\|h\|_{\tilde{H}} &:= \int h'(t)^2\,\mathrm{d}t
\end{aligned}$$

Lemma 4.3.3. For all $n \in \mathbb{N}$ we have that $j : C_n^* \to H^* \equiv H$ is continuous and $\|j_n\|_{L(C_n^* \to H^*)} \leq 1$.

Proof. Denote by $i : H \to C_n$ the embedding from above. Then we have:

$$\begin{aligned}
\|j_n\|_{L(C_n^* \to H^*)} &= \sup\{\|j_n(f)\|_{H^*} | f \in C_n^*, \|f\|_{C_n^*} \leq 1\} \\
&= \sup\{\|f \circ i_n\|_{H^*} | f \in C_n^*, \|f\|_{C_n^*} \leq 1\} \\
&= \sup\{|f(i_n h)| | f \in C_n^*, \|f\|_{C_n^*} \leq 1, h \in H, \|h\|_H \leq 1\} \\
&\leq \sup\{|f(ih)| | f \in C_n^*, \sup_{\substack{h \in H \\ \|h\|_H \leq 1}} |f(ih)| \leq 1, h \in H \|h\|_H \leq 1\} \\
&\leq \sup\{|f(ih)| | f \in C_n^*, \sup_{\substack{h \in H \\ \|h\|_H \leq 1}} |f(ih)| \leq 1, h \in H \|h\|_H \leq 1\} \leq 1
\end{aligned}$$

□

Furthermore fix a function $\varphi \in C_b^\infty$ with $\varphi(x) = x$ for $x \in [-1, 2]$.

Let $(f_k)_k$ be a fixed dense subset of $C(\mathbb{R}, \mathbb{R})$. By the Hahn-Banach theorem for $k, n \in \mathbb{N}$ there exists $l_k^{(n)} \in C_n^*$ such that $l_n^{(n)}(\pi_n(f_k)) = \|\pi_n(f_k)\|_n$ and $\|l_k^{(n)}\|_n^* = 1$. Let us fix this set of functions.

Lemma 4.3.4. For each $n \in \mathbb{N}$ and $g \in C(\mathbb{R}, \mathbb{R})$ we have that $\sup_{k \in \mathbb{N}} l_k^{(n)}(\pi_n(g)) = \|g\|_n$.

Proof. We have $l_k^{(n)}(\pi_n(g)) \leq |l_k^{(n)}(\pi_n(g))| \leq \|\pi_n(g)\|_n$.

Now let $(k_i))_i \in \mathbb{N}$ be a sequence in $\mathbb{N}$ such that $\lim_{i \to \infty} f_{k_i} = g$. Then we have

$$\|\pi_n(g)\|_n = \lim_{i \to \infty} \|\pi_n(f_{k_i})\|_n = \lim_{i \to \infty} l_{k_i}^{(n)}(\pi_n(f_{k_i})) = \lim_{i \to \infty} l_{k_i}^{(n)}(\pi_n(g)) \leq \sup_{k \in \mathbb{N}} l_k^{(n)}(\pi_n(g))$$

□

Lemma 4.3.5. For each $k, n \in \mathbb{N}$ we have that

$$f \mapsto g_{n,k}(f) := \sum_{i=1}^{n} 2^{-i} \sup_{1 \leq j \leq n} (l_j^{(i)}(\pi_i(f - f_k))^+ \wedge 1) \in D(\mathcal{E}). \tag{4.3.1}$$

Proof. We have $\varphi \circ l_j^{(i)} \circ p_i \in \mathcal{F}C_b^\infty$ and

$$(\varphi \circ l_j^{(i)} \circ p_i)^+ \wedge 1 = (l_j^{(i)} \circ p_i)^+ \wedge 1$$

Since the left-hand side is in $D(\mathcal{E})$ by 4.2.2 the right-hand side is also in $D(\mathcal{E})$. Applying 4.2.4 and using induction we have that

$$\sup_{1 \leq j \leq n} (l_j^{(i)}(\pi_i(f - f_k))^+ \wedge 1) \in D(\mathcal{E}).$$

And because $D(\mathcal{E})$ is a linear space we have shown that $g_{n,k} \in D(\mathcal{E})$. □

Lemma 4.3.6. Let $g_{n,k}$ be the sequence of functions on $C(\mathbb{R}, \mathbb{R})$ defined in 4.3.5. Then we have for all $f \in C(\mathbb{R}, \mathbb{R})$ that

$$\sup_{n \in \mathbb{N}} g_{n,k}(f) = d(f, f_k). \tag{4.3.2}$$

Proof. We know that $d(f,f_k)=\sum_{i=1}^{\infty}2^{-i}(\|\pi_i(f-f_k)\|_i\wedge 1)$. Furthermore we know by 4.3.4 that

$$\begin{aligned}\lim_{n\to\infty}\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k)))^+\wedge 1 &= \lim_{n\to\infty}(\sup_{1\le j\le n}l_j^{(i)}(p_i(f-f_k)))^+\wedge 1\\ &=(\lim_{n\to\infty}\sup_{1\le j\le n}l_j^{(i)}(p_i(f-f_k)))^+\wedge 1\\ &=(\sup_{j\in\mathbb{N}}l_j^{(i)}(p_i(f-f_k)))^+\wedge 1\\ &=\|f-f_k\|_i^+\wedge 1\end{aligned}$$

for every i.

Hence the sequence of functions on $\mathbb{N}$ defined as $1_{\{1,\dots n\}}(i)\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k)))^+\wedge 1$ converges pointwise (in $\mathbb{N}$) to $\|p_i(f-f_k)\|_i$. Since the measure $\sum_{i=1}^{\infty}2^{-i}\delta_i$ is a probability measure on $\mathbb{N}$ and since the sequence is bounded by 1, we have by Lebegues Theorem that

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}1_{\{1,\dots,n\}}(i)\sup_{1\le j\le n}\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k)))^+\wedge 1=d(f,f_k)$$

□

Lemma 4.3.7. Let $(g_{n,k})_{n,k\in\mathbb{N}}$ be the set of functions from above, then

$$\sup_{n,k}\langle\nabla g_{n,k},\nabla g_{n,k}\rangle_H\in L^1(E,\mu).$$

Proof. At first we need some estimates:

$$\nabla_H l_j^{(n)}\circ p_n=l_j^{(n)}\circ p_n\circ p_H=l_j^{(n)}\circ i_n$$

$$\|\nabla l_j^{(i)}(p_i(f-f_k))\|_{H^*}=\|l_j^{(i)}\circ i_i\|_{H^*}\le\|l_j^{(i)}\|_{C_n^*}=1$$

Now we have for all $n,k\in\mathbb{N}$:

$$\begin{aligned}&\langle\nabla g_{n,k},\nabla g_{n,k}\rangle_H\\ &=\langle\nabla\sum_{i=1}^{n}2^{-i}\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k))^+\wedge 1),\nabla\sum_{i=1}^{n}2^{-i}\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k))^+\wedge 1)\rangle_H\\ &=\sum_{i=1}^{n}\sum_{\tilde{i}=1}^{n}2^{-i}2^{-\tilde{i}}\langle\nabla\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k))^+\wedge 1),\nabla\sup_{1\le j\le n}(l_j^{(\tilde{i})}(p_{\tilde{i}}(f-f_k))^+\wedge 1)\rangle_H\\ &\le\sum_{i=1}^{n}\sum_{\tilde{i}=1}^{n}2^{-i}2^{-\tilde{i}}\|\nabla\sup_{1\le j\le n}(l_j^{(i)}(p_i(f-f_k))^+\wedge 1)\|\|\nabla\sup_{1\le j\le n}(l_j^{(\tilde{i})}(p_{\tilde{i}}(f-f_k))^+\wedge 1)\|\\ &\le\sum_{i=1}^{n}\sum_{\tilde{i}=1}^{n}2^{-i}2^{-\tilde{i}}\sup_{1\le j\le n}\|\nabla(l_j^{(i)}(p_i(f-f_k))^+\wedge 1)\|\sup_{1\le j\le n}\|\nabla(l_j^{(\tilde{i})}(p_{\tilde{i}}(f-f_k))^+\wedge 1)\|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} \Big(2^{-i} 2^{-\tilde{i}} \sup_{1 \le j \le n} \|\nabla(\varphi(l_j^{(i)}(p_i(f - f_k)))^+ \wedge 1)\| \\
&\qquad \times \sup_{1 \le j \le n} \|\nabla(\varphi(l_j^{(\tilde{i})}(p_{\tilde{i}}(f - f_k)))^+ \wedge 1)\| \Big) \\
&\le \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} 2^{-i} 2^{-\tilde{i}} \sup_{1 \le j \le n} \|\nabla \varphi(l_j^{(i)}(p_i(f - f_k)))\| \sup_{1 \le j \le n} \|\nabla \varphi(l_j^{(\tilde{i})}(p_{\tilde{i}}(f - f_k))\| \\
&= \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} 2^{-i} 2^{-\tilde{i}} \Big(\sup_{1 \le j \le n} |\varphi'(l_j^{(i)}(p_i(f - f_k)))| \|\nabla(l_j^{(i)}(p_i(f - f_k)))\| \\
&\qquad \times \sup_{1 \le j \le n} |\varphi'(l_j^{(\tilde{i})}(p_i(f - f_k))| \|\nabla(l_j^{(\tilde{i})}(p_{\tilde{i}}(f - f_k)))\| \Big) \\
&\le \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} 2^{-i} 2^{-\tilde{i}} C^2 \sup_{1 \le j \le n} \|\nabla(l_j^{(i)}(p_i(f - f_k)))\| \sup_{1 \le j \le n} \|\nabla(l_j^{(\tilde{i})}(p_{\tilde{i}}(f - f_k)))\| \\
&\le \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} 2^{-i} 2^{-\tilde{i}} C^2 \sup_{1 \le j \le n} \|(l_j^{(i)}(p_i)\| \sup_{1 \le j \le n} \|(l_j^{(\tilde{i})}(p_{\tilde{i}}))\| \\
&\le \sum_{i=1}^{n} \sum_{\tilde{i}=1}^{n} 2^{-i} 2^{-\tilde{i}} C^2 1 \\
&= C^2.
\end{aligned}
$$

Since μ is a probability measure we have proven the result. □

Now define the following sequence of functions:

$$f_n(w) := \inf_{m \le n} d(w, w_m) = \inf_{m \le n} \sup_{l \in \mathbb{N}} g_{l,m}(w)$$

Lemma 4.3.8. For the sequence of functions $(f_n)_{n \in \mathbb{N}}$ defined above we have

(i) $f_n \to 0$ in $L^2(C(\mathbb{R}, \mathbb{R}), \mu)$

(ii) $\sum_{k=1}^{n} \frac{1}{n} f_{n_k} \to 0$ in $(D(\mathcal{E}), \mathcal{E}_1^{1/2})$ for a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$

(iii) There exists an $\mathcal{E}$-nest such that $f_n \to 0$ uniformly on each F_k.

(iv) The sets in the $\mathcal{E}$-nest from (iii) are compact.

Proof. (i) The sequence f_n is bounded by f_1. Since $(w_m)_{m \in \mathbb{N}}$ is dense in $C(\mathbb{R}, \mathbb{R})$ we have $f_n \to 0$ pointwise. Altogether we have $f_n \to 0$ in $L^2(C(\mathbb{R}, \mathbb{R}), \mu)$.

(ii)

$$
\begin{aligned}
\mathcal{E}(f_n, f_n) &= \mathcal{E}(\inf_{m \le n} d(w, w_m), \inf_{m \le n} d(w, w_m)) \\
&\le \sup_{m \le n} \mathcal{E}(d(w, w_m), d(w, w_m))
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{m\le n} \int \langle \nabla \sup_{l\in\mathbb{N}} g_{l,m}(w), \nabla \sup_{l\in\mathbb{N}} g_{l,m}(w)\rangle_H \, \mu(\mathrm{d}w) \\
&\le \sup_{m\le n} \int \sup_{l\in\mathbb{N}} \langle \nabla g_{l,m}(w), \nabla g_{l,m}(w)\rangle_H \, \mu(\mathrm{d}w) \\
&\le \int \sup_{m\le n} \sup_{l\in\mathbb{N}} \langle \nabla g_{l,m}(w), \nabla g_{l,m}(w)\rangle_H \, \mu(\mathrm{d}w) \\
&\le \int \underbrace{\sup_{m\in\mathbb{N}, l\in\mathbb{N}} \langle \nabla g_{l,m}(w), \nabla g_{l,m}(w)\rangle_H}_{\in L^1(C(\mathbb{R},\mathbb{R}),\mu) \text{ by } 4.3.7} \, \mu(\mathrm{d}w) \\
&< \infty
\end{aligned}
$$

Now we can apply [MR92, Lemma 2.12.] and we get a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_{n_k} = 0$ in $D(\mathcal{E})$.

(iii) Since all the functions $w \mapsto d(w, w_m)$ are continuous we have that f_n has the continuous μ-version $w \mapsto \inf d(w, w_m)$, so we have that $\frac{1}{n}\sum_{k=1}^{n} f_{n_k}$ is also continuous and hence $\mathcal{E}$-quasi-continuous.

Now we apply [MR92, Proposition III.3.5] and get that there exists $(n_l)_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \frac{1}{n_l}\sum_{k=1}^{n_l} f_{n_k} = 0$ $\mathcal{E}$-quasi-uniformly, i.e. there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $\lim_{l\to\infty} \frac{1}{n_l}\sum_{k=1}^{n_l} f_{n_k} = 0$ uniformly on each F_k.

Fix this $\mathcal{E}$-nest. We will now show that $\lim_{n\to\infty} \inf_{m\le n} d(w, w_m) = 0$ uniformly on F_k for all $k \in \mathbb{N}$: Fix $k \in \mathbb{N}$. Let $\varepsilon > 0$, then there exists $N(\varepsilon)$ such that $\frac{1}{n_l}\sum_{k=1}^{n_l} f_{n_k} < \varepsilon$ for all $l \ge N(\varepsilon)$. Then we have for any $n \ge n_{N(\varepsilon)}$ that there exists an $l \ge N(\varepsilon)$ such that $n \ge n_l$. Then we have:

$$
\begin{aligned}
\inf_{m\le n} d(w, w_m) &= \frac{1}{n_l}\sum_{k=1}^{n_l} \inf_{m\le n} d(w, w_m) \\
&\le \frac{1}{n_l}\sum_{k=1}^{n_l} \inf_{m\le n_k} d(w, w_m) \\
&< \varepsilon
\end{aligned}
$$

(iv) Since $C(\mathbb{R},\mathbb{R})$ is complete and each F_k is bounded it is enough to show that F_k is totally bounded, i.e. for every $\varepsilon > 0$ there exist $w_1, \ldots, w_n \in C(\mathbb{R},\mathbb{R})$ such that $F_k \subset \bigcup_{i=1}^{n} B_\varepsilon(w_i)$.

Take $\varepsilon > 0$ then there exists $N(\varepsilon)$ such that $\inf_{m\le N(\varepsilon)} d(w, w_m) < \varepsilon \forall w \in F_k$ then for any $w \in F_k$ we have: $\inf_{m\le N(\varepsilon)} d(w, w_m) < \varepsilon$, since the infimum is taken over only a finite number of elements, there exists an $i_0 \le N(\varepsilon)$, such that $d(w, w_{i_0}) < \varepsilon$, hence we have $w \in B_\varepsilon(w_{i_0}) \subset \bigcup_{i=1}^{N(\varepsilon)} B_\varepsilon(w_i)$. So we have $F_k \subset \bigcup_{i=1}^{N(\varepsilon)} B_\varepsilon(w_i)$. □

4.4 The Process: Existence and Properties

In this part we show the existence of a diffusion which is associated to $(\mathcal{E}, D(\mathcal{E}))$.

Proposition 4.4.1. The Dirichletform $(\mathcal{E}, D(\mathcal{E}))$ is local.

Proof. It is enough to show that every form $(\mathcal{E}_k, D(\mathcal{E}_k))$ is local. To show this, it is enough to show that $\frac{\partial u}{\partial k} = 0$ on $\operatorname{supp}[u]^c$.

First we will show that

$$\frac{\partial(vw)}{\partial k} = v\frac{\partial w}{\partial k} + w\frac{\partial v}{\partial k}$$

Step 1: Let $v, w \in D(\mathcal{E}_k)$ and v, w are bounded. There exist sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty$ such that $v_n \to v$ and $w_n \to w$ in $D(\mathcal{E}_k)$, so also in $L^2(E, \mu)$. For all $n \in \mathbb{N}$ we have

$$\frac{\partial(v_n w_n)}{\partial k} = v_n\frac{\partial w_n}{\partial k} + w_n\frac{\partial v_n}{\partial k}$$

We also have:

$$\lim_{n\to\infty} v_n\frac{\partial w_n}{\partial k} + w_n\frac{\partial v_n}{\partial k} = v\frac{\partial w}{\partial k} + w\frac{\partial v}{\partial k}$$

Since $(\mathcal{E}_k, D(\mathcal{E}_k))$ is closed we have that $vw \in D(\mathcal{E})$ and $\frac{\partial(vw)}{\partial k} = v\frac{\partial w}{\partial k} + w\frac{\partial v}{\partial k}$

Step 2: Let $v, w \in D(\mathcal{E}_k)$. Define $v_n := (v \wedge n) \vee (-n)$ and $w_n := (w \wedge n) \vee (-n)$. By Step 1 we have that

$$\frac{\partial(v_n w_n)}{\partial k} = v_n\frac{\partial w_n}{\partial k} + w_n\frac{\partial v_n}{\partial k}$$

and by [MR92, I.4.17] we have for the right hand side:

$$\lim_{n\to\infty}(v\wedge n)\vee(-n)\frac{\partial(w\wedge n)\vee(-n)}{\partial k} + (w\wedge n)\vee(-n)\frac{\partial(v\wedge n)\vee(-n)}{\partial k} = v\frac{\partial w}{\partial k} + w\frac{\partial v}{\partial k}$$

Since $(\mathcal{E}_k, D(\mathcal{E}_k))$ is closed we have that $vw \in D(\mathcal{E})$ and $\frac{\partial(vw)}{\partial k} = v\frac{\partial w}{\partial k} + w\frac{\partial v}{\partial k}$

Now let $u \in D(\mathcal{E}_k)$. By [MR92, V.1.7] there exists $v \in D(\mathcal{E}_k) \cap L^\infty(E, \mu)$ such that $0 \le v \le 1_{\operatorname{supp}[u]^c}$ and $v > 0$ μ-a.s. on $\operatorname{supp}[u]^c$. Then we have $uv = 0$ μ-a.s, hence $\frac{\partial(uv)}{\partial k} = 0$. Putting this together with the first part of the proof we get: $0 = \frac{\partial(uv)}{\partial k} = u\frac{\partial v}{\partial k} + v\frac{\partial u}{\partial k}$.

Since the first part is equal to zero on $\operatorname{supp}[u]^c$ and the second on $\operatorname{supp}[u]$, we have that both parts are equal to zero on E. Finally we get that on $\operatorname{supp}[u]^c$ we have $\frac{\partial u}{\partial k} = 0$. □

Theorem 4.4.2. Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet-form defined in Chapter 3. Then there exists a diffusion which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

Proof. We want to apply [MR92, Theorem V.1.11] (cited at 4.1.11).

In order to do this we have to check the conditions of quasi-regularity:

Condition 1 follows immediately by Lemma 4.3.8

Condition 2 is very simple: All Elements of $\mathcal{F}C_b^\infty$ are continuous, hence $\mathcal{E}$-q.c. and $\mathcal{F}C_b^\infty$ is dense in $D(\mathcal{E})$ by construction of the closure.

Condition 3 can also be easily checked. Consider again the ONB $(h_n)_{n\in\mathbb{N}}$ of H as in Chapter 3. Let $(l_n) \subset C(\mathbb{R},\mathbb{R})^*$ such that $l_i(h_j) = \delta_{ij}$ and let $\varphi \in C_b^\infty$ be increasing. Then the family $(\varphi \circ l_n)_{n\in\mathbb{N}}$ is in $\mathcal{F}C_b^\infty$ and it separates the points of $C(\mathbb{R},\mathbb{R})$. We have already shown that $(\mathcal{E}, D(\mathcal{E}))$ is local, hence we can apply [MR92, Theorem V.1.11] and have finished the proof. □

Now we show that $(\mathcal{E}, D(\mathcal{E}))$ is conservative.

Proposition 4.4.3. $(\mathcal{E}, D(\mathcal{E}))$ is conservative.

Proof. We have to show that $T_t 1 = 1$. For this it is enough to show that $L1 = 0$ since we have $T_t f - T_t s = \int_s^t T_r L f \,\mathrm{d}r$ for all $f \in D(\mathcal{E})$ and $t \geq s \geq 0$. To show that $L1 = 0$ we use that $\mathcal{E}(u,v) = (-Lu, v)$. Since $1 \in \mathcal{F}C_b^\infty$ we have for all $v \in \mathcal{F}C_b^\infty$ that $\mathcal{E}(1,v) = \frac{1}{2}\int_{C(\mathbb{R},\mathbb{R})} \langle 0, \nabla_H v\rangle_H \,\mathrm{d}\mu = 0$. So we have $0 = (-L1, v)$ for all $v \in \mathcal{F}C_b^\infty$. Since $\mathcal{F}C_b^\infty$ is dense in $D(\mathcal{E})$ we have that $L1 = 0$. Finally we get: $T_t 1 - 1 = T_t 1 - T_0 1 = \int_0^t T_r L1 \,\mathrm{d}r = \int_0^t T_r 0 \,\mathrm{d}r = 0$ hence we have $T_t 1 = 1$ for all $t > 0$. Since $(\mathcal{E}, D(\mathcal{E}))$ is properly associated to our process we have that $p_t 1 = 1$ and we have finished our proof. □

Finally we state that under some conditions the constructed process solves a stochastic differential equation weakly.

First we need this definition:

Definition 4.4.4. An element $k \in E$ is called well-μ-admissible, if the following holds: There exists β_k in $L^2(E,\mu)$ such that for all $u, v \in \mathcal{F}C_b^\infty$

$$\int \frac{\partial u}{\partial k} v \,\mathrm{d}\mu = -\int u \frac{\partial v}{\partial k} \,\mathrm{d}\mu - \int uv\beta_k \,\mathrm{d}\mu$$

Remark 4.4.5. If k is well-μ-admissible, it is μ-admissible. For a proof see for example [MR92, Proposition II.3.4].

We want to use the following result:

Theorem 4.4.6 (cf. [AR91, Theorem 5.7])**.** Let K_0 be an orthonormal basis in H, separating the points in $C(\mathbb{R},\mathbb{R})$ such that every $k \in K_0$ is well-μ-admissible. Then for q.e. $z \in C(\mathbb{R},\mathbb{R})$ $(\{ {}_{C(\mathbb{R},\mathbb{R})^*}\langle k, X_t\rangle_{C(\mathbb{R},\mathbb{R})} \,|k \in K_0\}, \mathcal{F}_t, P_z)_{t\geq 0}$ solves the following system of stochastic differential equations

$$\begin{aligned} \mathrm{d}Y_t^k &= \mathrm{d}W_t^k + \frac{1}{2}\beta_k((Y_t^k)_{k\in K_0})\,\mathrm{d}t \qquad\qquad k \in K_0 \\ Y_0^k &= {}_{C(\mathbb{R},\mathbb{R})^*}\langle k, z\rangle_{C(\mathbb{R},\mathbb{R})} \end{aligned}$$

where $\{(W_t^k)_{t\geq 0}|k \in K_0\}$ is a collection of independent one dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions starting at zero and where we identify $z \in C(\mathbb{R},\mathbb{R})$ with $({}_{C(\mathbb{R},\mathbb{R})^*}\langle k, z\rangle_{C(\mathbb{R},\mathbb{R})})_{k\in K_0}$.

Proof. This is just [AR91, Theorem 5.7] applied to our situation. □

Remark 4.4.7. We want show the relation between β_k and a_{sk}.

For $k \in C(\mathbb{R},\mathbb{R}) \setminus \{0\}$ such that μ is k-quasi-invariant, choose $l_k \in C(\mathbb{R},\mathbb{R})'$ such that $l_k(k) = 1$ and define $\pi_k(z) := z - l_k(z)k$. Define $E_0^k := \pi_k(C(\mathbb{R},\mathbb{R}))$. Define the measure σ_k by $\sigma_k(A) := \int_{\mathbb{R}} \tau_{sk}(\mu)(A)\,\mathrm{d}s$ for $A \in \mathcal{B}(C(\mathbb{R},\mathbb{R}))$, and $\nu_k := \pi_k(\mu)$. By [AR90b, 4.2 Proposition] we have that $\mu \ll \sigma_k$. Define $\rho_k(z) := \frac{\mathrm{d}\mu}{\mathrm{d}\sigma_k}(z)$.

Suppose furthermore that $\lim_{s\to 0} \frac{1}{s}(\sqrt{a_{sk}} - 1)$ exists in $L^2(E,\mu)$ and that $R(\rho_k(x + \cdot k)) = \mathbb{R}$ for ν_k-a.e. $x \in E_0$. Then we have by [AR90b, 4.4 Remark] and [AR90b, 4.8. Corollary] for every $s \in \mathbb{R}$ that

$$a_{sk}(z) = \exp\left(\int_0^s \beta(k)(z - tk)\,\mathrm{d}t\right) \text{ for } \mu\text{-a.e. } z \in E$$

If $s \mapsto a_{sk}$ is differentiable in 0, we then get that $\frac{\mathrm{d}}{\mathrm{d}_s} a_{sk}|_{s=0} = \exp(\int_0^0)\beta(k)(z - tk)\,\mathrm{d}t)\beta(k)(z) = \beta(k)(z)$.

If $\varphi \in C^1(\mathbb{R},\mathbb{R})$ and ψ is partial differentiable in the second coordinate, we get that:
$\beta_k = \frac{\mathrm{d}}{\mathrm{d}s} a_{sk}|_{s=0} = -\int w dk' + \int \varphi'(w(t)k(t)\,\mathrm{d}t$
$+ \iint \partial_2\psi(r - t, w(r) - w(t))(k(r) - k(t))\,\mathrm{d}r\mathrm{d}t$
$+ \iint \partial_2\psi(r - t, w(r) - w(t))k(r)\,\mathrm{d}r\mathrm{d}t.$

A Some Facts about Stieltjes Integrals

In this chapter we will put together some results about Stieltjes-Integrals.

Lemma A.0.8. Let f, g be of bounded variation, then f and fg are Riemann-integrable and we have

$$\int_a^b g(t)\,\mathrm{d}(\int_a^t f(s)\,\mathrm{d}s) = \int_a^b g(t)f(t)\,\mathrm{d}t.$$

Proof. We have to show that for every $\varepsilon > 0$ there exists a partition P_ε of $[a, b]$ such that for every finer partition $P \supset P_\varepsilon$ we have

$$|\sum_{k=1}^n g(\xi_k)(\int_a^{t_k} f(s)\,\mathrm{d}s - \int_a^{t_{k-1}} f(s)\,\mathrm{d}s) - \int_a^b g(t)f(t)\,\mathrm{d}t| < \varepsilon.$$

Let $\varepsilon > 0$. Then, since fg is Riemann-integrable there exists $\delta > 0$ such that for every partition with $\|P\| < \delta$ we have

$$|\sum_{k=1}^n g(\xi_k)f(\xi_k)(t_k - t_{k-1}) - \int_a^b f(s)g(s)\,\mathrm{d}s| < \frac{\varepsilon}{3}. \tag{A.0.1}$$

Furthermore there exists stepfunctions φ and ψ with $\varphi \le f \le \psi$ and $\int_a^b \psi - \varphi < \varepsilon/(2\sup_{x\in[a,b]}|g(x)|)$. Take now as partition $P_\varepsilon = \{a = t_0, \dots, t_n = b\}$ a partition such that φ and ψ are constant on each interval (t_k, t_{k-1}) and such that $\|P\| < \delta$. Let $P = \{a = t_0 = s_0^{(1)} < \dots < s_{k_1}^{(1)} = t_1 = s_0^{(2)} < \dots < s_{k_n}^{(n)} = t_n = b\}$ be a partition that is finer than P_ε

Let $\tilde\varphi$ be a step function defined as follows:

$$\tilde\varphi(x) = \begin{cases} 0, & x = s_j^{(i)}, 1 \le k \le n \\ f(\xi_j^{(i)}), & x \in (s_{j-1}^{(i)}, s_j^{(i)}) \end{cases}$$

Then we have for $s \in (s_{j-1}^{(i)}, s_j^{(i)})$

$$\begin{aligned} \tilde\varphi(s) &= f(\xi_j^{(i)}) \le \psi(\xi_j^{(i)}) = \psi(s) \\ \tilde\varphi(s) &= f(\xi_j^{(i)}) \ge \varphi(\xi_j^{(i)}) = \varphi(s), \end{aligned}$$

since is φ and ψ are constant on each interval $(t_i, t_{i-1}) \supset (s_{j-1}^{(i)}, s_j^{(i)})$. We then get that $|f - \tilde\varphi| \le \psi - \varphi$. Finally we have:

$$|\sum_{i=1}^n \sum_{j=1}^{k_i} g(\xi_j^{(i)})(\int_a^{s_j^{(i)}} f(s)\,\mathrm{d}s - \int_a^{s_j^{(i)}} f(s)\,\mathrm{d}s) - \int_a^b g(t)f(t)\,\mathrm{d}t|$$

$$
\begin{aligned}
&\leq |\sum_{i=1}^{n}\sum_{j=1}^{k_i} g(\xi_j^{(i)}) \int_{s_{j-1}^{(i)}}^{s_j^{(i)}} f(s)\,\mathrm{d}s - \sum_{i=1}^{n}\sum_{j=1}^{k_i} g(\xi_j^{(i)}) f(\xi_j^{(i)})(s_j^{(i)} - s_{j-1}^{(i)})| \\
&\quad + \underbrace{|\sum_{i=1}^{n}\sum_{j=1}^{k_i} g(\xi_j^{(i)}) f(\xi_j^{(i)})(s_j^{(i)} - s_{j-1}^{(i)}) - \int_a^b g(t)f(t)\,\mathrm{d}t|}_{<\varepsilon/3} \\
&\leq \frac{\varepsilon}{3} + \sum_{i=1}^{n}\sum_{j=1}^{k_i} |g(\xi_j^{(i)})| \cdot |\int_{s_{j-1}^{(i)}}^{s_j^{(i)}} f(s)\,\mathrm{d}s - f(\xi_j^{(i)})(s_j^{(i)} - s_{j-1}^{(i)})| \\
&\leq \frac{\varepsilon}{3} + \sup_{x\in[a,b]} |g(x)| \sum_{i=1}^{n}\sum_{j=1}^{k_i} \int_{s_{j-1}^{(i)}}^{s_j^{(i)}} |f(s) - \tilde{\varphi}(s)|\,\mathrm{d}s \\
&\leq \frac{\varepsilon}{3} + \sup_{x\in[a,b]} |g(x)| \sum_{i=1}^{n}\sum_{j=1}^{k_i} \int_{s_{j-1}^{(i)}}^{s_j^{(i)}} \psi(s) - \varphi(s)\,\mathrm{d}s \\
&= \frac{\varepsilon}{3} + \sup_{x\in[a,b]} |g(x)| \int_a^b \psi(s) - \varphi(s)\,\mathrm{d}s \\
&< \varepsilon
\end{aligned}
$$

□

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