Valuation of financial contingent claims in the presence of model uncertainty

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Diplom-Wirtschaftsmathematiker Jörg Vorbrink

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Contents

Chapter 1

General introduction

1.1 Ambiguity/Knightian uncertainty

Risk is a currently widely used financial factor, and is closely related to probabilities. When dealing with probabilities, it is important to understand how they vary in terms of emergence and verification. We focus on two types, distinguishing between empirical and epistemic probability. The former is characterized by the possibility of verifying its value experimentally, and is also known as objective probability, Savage (1954). Epistemic probability contains roughly similar subjective and logical conclusions about probabilities from knowledge.

When using epistemic probabilities the goal is to predict prospective events from current results. However, when dealing with singular occurrence events such as the weather, where there is no repetition, the approach becomes rather questionable. We will revisit this point when considering financial markets.

The difference between risk and ambiguity in economics was first marked by Knight (1921). Due to Knight (1921), ambiguity is often referred to as Knightian uncertainty.[1]

> "The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculations a priori or from statistics of past experience), while in the case of uncertainty this is not true", Knight (1921).

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[1]In this sense, the terms will be used synonymously.

Thus risk is restricted to situations where an objective probability exists and can be assigned to the uncertain outcomes. For example, if we were to bet on a black or red number in a roulette game, we could argue that both were equally likely due to our experience. Yet, when betting on a horse race event, we face a one-shot event. The assignment of a probability to the events is neither objective nor unique, thus placing us in an uncertainty setting.

Based on this theoretical work, Ellsberg (1961) criticized the subjective expected utility theory (Savage (1954)), and gave empirical evidence for the theory of Knight (1921) summarized in the Ellsberg paradox. The term paradox refers to the violation or contradiction of the, then, well accepted theory of Savage (1954) or Von Neumann and Morgenstern (1944). A distinction between risk and uncertainty was not permitted within the framework of subjective expected utility. Particularly, the independence axiom on preferences could not be maintained in Ellsberg (1961).[2]

One possible way out of this conundrum is the multiple priors[3] model proposed by Gilboa and Schmeidler (1989) which suggests a whole set of probability measures and considers the minimal expected utility due to ambiguity aversion.[4]

1.2 Ambiguity and financial markets

The point of origin of this work is the distinction between risk and uncertainty as indicated and declared in Knight (1921) and Ellsberg (1961). The Ellsberg paradox illustrates the different behaviors of people in risky situations, as opposed to ambiguous situations, when they are not faced with objective probabilities. The latter is typical for financial markets. The market participants do not know the odds with certainty. They can only evaluate historical data to obtain a reasonable understanding of the market and its returns. However, predicting prospective outcomes by means of historical data means implicitly that nothing changes when comparing history and the future. Thus this method implies stationarity of the probabilities, that is, nothing will change when passing from the past or present to the future.

In classical financial markets, the dynamics of the market or a stock price, for example, are assumed to be known. Having assessed the dynamics, one

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[2]Ellsberg's paradox highlights that randomization between indifferent acts can be valuable, Epstein and Schneider (2010).

[3]A prior is a standard term in economics denoting a probability measure.

[4]Another approach is the smooth ambiguity model developed by Klibanoff, Marinacci, and Mukerji (2005).

can calculate future payoffs using the subjective expected utility approach. With this approach, one restricts oneself's view to only one particular model which is not in accordance with Knight's theory or the features exhibited by Ellsberg (1961).

Using only one model implies perfect understanding of the market. Further, it generates model risk when market parameters are stipulated incorrectly.

As can be surmised from the recent financial crisis, market parameters can change abruptly and drastically. Since we are not given objective probabilities in finance, there is no single model appropriate to all uncertainty sources. In this sense, it is reasonable to consider several models with the goal of capturing various risk scenarios so as to reduce the risk of incorrect model parameter specification. Analyzing the situation from different perspectives can help to become more familiar with possible outcomes and to make more robust decisions concerning model risk.

1.3 Scientific work on ambiguity in financial markets

There is an extensive literature on ambiguity as it relates to financial markets. References on the implications of multiple priors models for portfolio choice and asset pricing are gathered in Epstein and Schneider (2010). To illustrate, we describe a few representative models. Epstein and Wang (1994) and Chen and Epstein (2002) studied asset pricing with multiple priors in discrete and continuous time, respectively. Epstein and Schneider (2008) treated the effect of learning in asset pricing, or Trojani and Vanini (2002) reviewed the equity premium puzzle, which is often an issue when dealing with ambiguity.

There is another large field closely related to ambiguity in asset markets, known as risk measures, see Artzner, Delbaen, Eber, and Heath (1999), Föllmer and Schied (2004). The representation of coherent risk measures[5] gives rise to similarities with our approaches in the thesis.

1.4 Content of the thesis

The thesis mainly consists of three chapters, Chapters 2–4. Each chapter is based on an article and is self-contained. All three chapters contain an

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[5]See Artzner, Delbaen, Eber, and Heath (1999) or Peng (2007) among others.

introductory section providing detailed economic motivation and scientific placement. Chapter 3 presents the corresponding continuous time analysis of Chapter 2 which is developed in a discrete time setting. The economic motivation for both is the same and therefore only briefly touched upon in Chapter 3.

In Chapter 2 and 3 we analyze American options from the perspective of an ambiguity averse agent holding the options. Chapter 2 is based on joint work with Tatjana Chudjakow. In Chapter 4, we change gears and address pricing and hedging of claims under Knightian uncertainty using an appropriate concept of no-arbitrage.

The main purpose of the thesis is to point out the effects of ambiguity and extant ambiguity aversion in financial markets. Thus, in all chapters we face multiple priors models.[6] Multiple prior settings relax the assumption of a known distribution of the stock price process, and capture the idea of incomplete information of market data leading to model uncertainty.

In Chapters 2 and 3 we tackle subjective evaluation of American OTC options under ambiguity aversion in the spirit of conservative accounting issues. Chapter 2 considers an ambiguous financial market in discrete time where stock price is modeled by an ambiguous binomial tree. The agent faces uncertainty about the true probability law mirrored in various possible one-step-ahead probabilities. In this model, we analyze American options under ambiguity aversion. Due to the early exercise feature, the agent faces an optimal stopping problem with multiple priors. We use the method of generalized backward induction developed in Riedel (2009) to solve the problem. The method's validity is based on a property of the set of priors. The set is asked to satisfy time-consistency (Riedel (2009)), m-stability (Delbaen (2002)), or rectangularity (Epstein and Schneider (2003b)).[7]

Because of ambiguity aversion, we identify the agent's particular worst-case scenario in terms of the worst-case prior which in turn depends on the respective American option under focus. By detecting each worst-case scenario, we highlight major differences to classical single prior models. These are characterized by their endogenous dynamic structure generated by the agent's model adjustments. These adjustments make it possible to take into account changing beliefs or fears based on particular events.

This point motivates favoring our approach when evaluating (long) positions related to accounting issues. Also from a decision theoretical point of view, our examples clarify that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under subjective

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[6]A comprehensive introduction of each model is given at the beginning of each chapter.

[7]For a survey on these concepts and their equivalence see Delbaen (2002), Riedel (2009).

expected utility.[8]

Chapter 3 constitutes the continuous time version of Chapter 2. It deals with the same questions, albeit with a more sophisticated mathematical instantiation. We consider an ambiguous Black-Scholes-like market. The multiple priors model we use is taken from Chen and Epstein (2002). Similarly to Chapter 2, it relies heavily on a reference measure and imposes the assumption that all priors are equivalent with respect to it.[9] Because of Girsanov's theorem, the multiple priors model (only) leads to uncertainty in the mean of the considered stochastic process, see Chen and Epstein (2002) or Cheng and Riedel (2010). The model leads to drift uncertainty in the stock price when associated with financial markets.[10] We solve the optimal stopping problem in this setting by using reflected backward stochastic differential equations developed in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997).

Chapter 4 differs structurally from the preceding chapters in that it not only treats expectation maximization under ambiguity aversion but also the pricing and hedging issue. Along the same lines, we address a new concept of arbitrage to reflect the interests of both the buyer and seller of European contingent claims in the presence of Knightian uncertainty. We consider an ambiguous Black-Scholes-like market as in Chapter 3, although now featuring volatility uncertainty. This is motivated by the fact that traders can only estimate stock's volatility instead of directly observing it since it is not traded.[11]

To model volatility uncertainty, we need to discard the assumption that the priors of the set are equivalent. Consequently, the multiple priors model has strong structural differences from the first two settings. The set will essentially consist of mutually orthogonal priors. We overcome this problem by utilizing the concept of G-normal distribution and G-Brownian motion as established by Peng (2007).

The additional source of uncertainty embodies model risk, namely, the

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[8]We consider for example, an American up-and-in put option to highlight these behavioral differences. In this case, the agent behaves as two readily distinguishable expected utility maximizers.

[9]From an economic point of view, this assumption implies that the decision maker has perfect knowledge about sure events, an assumption which is not always reasonable. For a detailed justification of this assumption see Epstein and Marinacci (2007).

[10]It is a well known fact in financial markets that the expected return in stock exceeds the riskless interest rate. The estimation of the true drift rate underlies subjectivity as it reflects anticipation.

[11]Besides, this setting can be regarded as an approach to understand the question of volatility arbitrage (Wilmott (2009)).

risk associated with forecasting volatility. The result is an incomplete market which does not enable perfect hedging of European claims generally. We derive an interval of "fair" prices for a claim in the sense that these prices do not admit arbitrage opportunities within the meaning of our new concept. Using a Markovian framework we characterize the upper and lower bounds of the interval of "fair" prices as the solution to the Black-Scholes-Barrenblatt equation (see Avellaneda, Levy, and Paras (1995)), which can be interpreted as a generalized – nonlinear – Black-Scholes PDE.

Thus far we have just introduced the main content of the thesis and placed it in a scientific framework consisting of an economic and decision theoretical level. Although the chapters are closely related, the subject matter of the underlying articles differ. As such, further mention of the related literature and scientific placement of this work within the literature is postponed to the respective chapters. In addition, mathematical notion varies across the chapters as required by the respective mathematical tools.

Chapter 2

American options with multiple priors in discrete time[1]

2.1 Introduction

The increasing trade volume of exotic options both in the plain form and as component of more sophisticated products motivates the more precise study of these structures. The OTC nature of contracts allows for almost endless variety which comes at the price of tractability and evaluation complexity. The payoff of the option is often conditioned on events during the lifetime leading to a path dependent structure which is challenging to evaluate.

Most of the literature on this field concentrates on hedging or replication of such structures analyzing the hedging strategy of the seller or deriving the no-arbitrage price. This analysis is sufficient in the case of European options as it also captures the problem of the buyer. However, in the case of American options the task of the buyer holding the option in her portfolio differs structurally from the hedging problem of the seller. Unlike the bank/the market, the holder of the option is not interested in the risk neutral value of the option but aims to exercise the claim optimally realizing highest possible utility. This valuation in general needs not to be related to the market value of the option as it reflects the personal utility of the holder which depends on investment horizon and objectives and also on the risk attitude of the holder.

Given a stochastic model in discrete time, such as the Cox-Ross-Rubinstein (CRR) model one can easily solve the problem of the buyer using dynamic programming. However, classical binomial tree models impose the

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assumption of a unique given probability measure driving the stock price process. This assumption might be too strong in several cases as it requires perfect understanding of the market structure and complete agreement on one particular model.

As an example we consider a bank holding an American claim in its trading book. The trading strategy of the bank depends on the underlying model used by the bank. If the model specification is error-prone the bank faces model uncertainty. Being unable to completely specify the model, traders rather use multiple priors model instead of choosing one particular model. If the uncertainty cannot be resolved and the accurate model specification is impossible, traders prefer more robust strategies as they perform well even if the model is specified slightly incorrect.

Also, a risk controlling unit assigning the portfolio value and riskiness uses rather a multiple priors model in order to test for model robustness and to measure model risk. Taking several models into account, while performing portfolio distress tests, allows to check the sensitivity of the portfolio to model misspecification. Again in a situation of model uncertainty more robust riskiness assignment is desirable as it minimizes model risks.

Similar reasoning can be applied to accounting issues. An investment funds manager making his annual valuation is interested in the value of options in the book that are not settled yet. In case the company applies coherent risk measures as standard risk evaluation tool for future cash flows on the short side, it is plausible to use a multiple priors model evaluating long positions. Finally, a private investor holding American claims in his depot might exhibit ambiguity aversion in the sense of Ellsberg paradox or Knightian uncertainty. Such behavior may arise from lack of expertise or bad quality of information that is available to the decision maker.

Although for different reasons, all the market participants described above face problems that should not be analyzed in a single prior model and need to be formulated as multiple priors problems. In this chapter we analyze the problem of the holder of an American claim facing model uncertainty that results in a multiple priors model. We characterize optimal stopping strategies for the buyer that assesses utility to future payoffs in terms of minimal expectation, and study how the multiple priors structure affects the stopping behavior.

Multiple priors models have recently attracted much attention. Hansen and Sargent (2001) considered multiple priors models in the context of robust control, Karatzas and Zamfirescu (2003) approached the problem from game theoretical point of view. Delbaen (2002) introduced the notion of coherent risk measures which mathematically corresponds to the approach used in this chapter.

The decision theoretical model of multiple priors was introduced by Gilboa and Schmeidler (1989) and further developed to dynamical settings by Epstein and Schneider (2003b). This is the natural extension of the expected utility model when the information is too imprecise. The methods we use here rely heavily on this work.

Epstein and Schneider (2003a) applied the multiple priors model to financial markets and Epstein and Schneider (2008) addressed the question of learning under uncertainty.

Riedel (2009) considered the general task to optimally stop an adapted payoff process in a multiple priors model and showed that backward induction fails in general. He imposed more structure on the set of priors that ensured the existence of the solution. The cornerstone of the method is the time-consistency of the set of priors which allows the decision maker to change her beliefs about the underlying model as the time evolves. If the set of priors is time-consistent one can proceed as in the classical case,[2] computing the value process of the stopping problem – the multiple priors Snell envelope. It is then optimal to stop as soon as the payoff process reaches the value process. Additionally, the ambiguous optimal stopping problem corresponds to a classical optimal stopping problem for a measure $\hat{P}$ – the so-called worst-case measure (see Riedel (2009), Föllmer and Schied (2004), Karatzas and Kou (1998)).

As an application of the technique, Riedel (2009) solved the exercise problem for the buyer of an American put and call in discrete time. A similar problem was analyzed by Nishimura and Ozaki (2007), they considered the optimal investment decision for a firm in continuous time with infinite time horizon under multiple priors which has been related to the perpetual American call.

In this chapter we follow the lines of Riedel (2009) and analyze several exotic options that have a second source of uncertainty from the perspective of the buyer in a multiple priors setting. We focus on the discrete time version of the problem and develop an ambiguous version of the CRR model. Instead of assuming that the distribution of up- and down- movements of the underlying is known to the buyer we allow the probability of going up on a node to lie in a appropriately modeled set. This leads to a set of models that agree on the size of up- and down-movement but disagree on the mean return.

In this ambiguous binomial tree setting which was first analyzed in Epstein and Schneider (2003a) we aim to apply standard Snell reasoning to evaluate the options. Due to the above mentioned duality result it is enough

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[2]See Snell (1952),Chow, Robbins, and Siegmund (1971) for more detailed analysis.

to calculate the worst-case measure $\hat{P}$ and then to analyze the classical problem under $\hat{P}$. However, the worst-case measure depends highly on the payoff structure of the claim and needs to be calculated for each option separately. If the payoff satisfies certain monotonicity conditions the worst-case measure is easy to derive. The direction/effect of uncertainty is the same for all states of the world and the worst-case measure is then independent on the realization of the stock price process leading to a statical structure that resembles classical one prior models. In the case of more sophisticated payoffs, this stationarity of the worst-case measure breaks down and the worst-case measure changes over time depending on the realization of the stock price. This is due to the fact that uncertainty may affect the model in different ways changing the beliefs of the decision maker, and so the worst-case measure according to the effect that is dominating. This ability to react on information by adjusting the model and to choose the model depending on the payoff is the main structural difference between the classical single measure model and the multiple priors model considered here.

We identify additional sources of uncertainty that lead to the dynamical and path-dependent structure of the worst-case measure. We also analyze the impact of different effects of uncertainty on the overall behavior and the resulting model highlighting differences between the single prior models and the multiple priors model.

In our analysis, we decompose the claims in monotone parts as the worst-case measure for monotone problems is well known. We then analyze each claim separately deriving the worst-case measure conditioned on monotonicity. To complete the analysis we paste the measures obtained on subspaces together using time-consistency. This idea is closely linked to the method of pricing derivatives using digital contracts introduced by Ingersoll (2007) and also used by Buchen (2004). However, this literature focuses on European style options and does not cover the dynamical structure analyzed here.

In the case of barrier options the value of the option is conditioned on the event of reaching a trigger. Unlike the plain vanilla option case, the lifetime of an barrier option becomes uncertain as it depends on the occurrence of the trigger event. This leads to an additional source of uncertainty causing a change in the monotonicity of the value function when the stock price hits the barrier. For example, in the case of an up-and-in put the ambiguity averse decision maker assumes the stock returns to be low and chooses therefore the measure that leads to the lowest drift for the stock price before it reaches the barrier. After hitting the barrier she obtains a plain vanilla put option, monotone in the underlying, and uses therefore the measure leading to the highest drift for the underlying stock price.

The second group of options we focus on are the dual expiry options. Here,

the strike of the option is not known at time zero as it is being determined as a function of the underlying's value on a date different from the issue date of the option – the first expiry. Therefore, additional to the uncertainty about the final payoff, the decision maker faces uncertainty about the value of the strike before first expiry date.

In the case of shout options the first expiration date, the so-called shout/freeze date, is determined by the buyer. Here, the investor has to call the bank if she aims to fix the strike. Therefore, the buyer of an shout option faces two stopping problems: First, she needs to determine the optimal shouting time to set the strike optimally. Second, she needs to stop the payoff process optimally.

The holder of an shout put gets an put after shouting and thus, anticipates high returns after shouting. Before shouting however, she owns a claim whose value is increasing in the price of the underlying which results in low returns, anticipated before shouting.

Finally, we analyze options with payoff function consisting of two monotone pieces. Typical examples are straddles and strangles. The buyer of such options presumes a large change in the underlying's price, but is not sure about the direction of it. Depending on the value of the underlying, the option pays off a call or a put. Consequently, the actual payoff function becomes uncertain.

It is often an option to decompose the value of the claim in an increasing and a decreasing leg. The buyer of the claim changes her beliefs about the returns every time the value switches from decreasing to increasing part of the value function. So, an ambiguity averse buyer of a straddle, for example, presumes the stock price to go down in boom phases and up in bear market phases.

An outline of the chapter is as follows. Section 2.2 introduces the discrete model which is in this form due to Riedel (2009) and recalls the solution for claims with payoffs that are monotone in the underlying stock price.[3] This part builds the base for the subsequent analysis. Section 2.3 provides the solution for barrier options, and Section 2.4 develops the solution for multiple expiry options. Finally, Section 2.5 discusses U-shaped payoffs and Section 2.6 concludes.

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[3]See also Riedel (2009).

2.2 Financial markets and optimal stopping

We first introduce the basic theoretical setup to evaluate options in a multiple priors model. This model has the CRR model as the starting point. It has already been developed in Riedel (2009) and can be seen as a version of the IID model, introduced in Epstein and Schneider (2003a) with a different objective. At the same time, the model is the discrete time version of the κ-ignorance model in Chen and Epstein (2002).

Having established the model, we discuss the market structure and recall the decision problem of the buyer and the solution method – the multiple priors backward induction introduced by Riedel (2009).

2.2.1 The stochastic structure

To set up the model we start with a classical binomial tree. For a fixed maturity date $T \in \mathbb{N}$ we consider a probability space $(\Omega, \mathcal{F}, P_0)$, where $\Omega = \otimes_{t=1}^{T} \mathcal{S}$, $\mathcal{S} = \{0, 1\}$, is the the set of all sequences with values in $\{0, 1\}$, $\mathcal{F}$ is the σ-field generated by all projections $\varepsilon_t : \Omega \to \mathcal{S}$ and P_0 denotes the uniform on $(\Omega, \mathcal{F})$. By construction, the projections $(\varepsilon_t)_{t=1,\dots,T}$ are independent and identically distributed under P_0 with $P_0(\varepsilon_t = 1) = \frac{1}{2}$ for all $t \leq T$. Furthermore, we consider the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ generated by the projections $(\varepsilon_t)_{t=1,\dots,T}$ where $\mathcal{F}_0$ is the trivial σ-field – $\{\emptyset, \Omega\}$. The event $\{\varepsilon_t = 1\}$ represents an up-movement on a tree while the complementary event denotes the down-movement.

Additionally, we define on $(\Omega, \mathcal{F}, P_0)$ a convex set of measures $\mathcal{Q}$ in the following way: We fix an interval $[\underline{p}, \overline{p}] \subset (0, 1)$ for $\underline{p} \leq \overline{p}$ and consider all measures whose conditional one step ahead probabilities of going up on a node of the tree remains within the interval $[\underline{p}, \overline{p}]$ for every $t \leq T$, i.e.,

$$\mathcal{Q} = \left\{ P \in \mathcal{M}_1(\Omega) \mid P(\varepsilon_t = 1 | \mathcal{F}_{t-1}) \in [\underline{p}, \overline{p}] \ \forall t \leq T \right\}.^4 \qquad (2.1)$$

The set $\mathcal{Q}$ is generated by the conditional one-step-ahead correspondence assigning at every node $t \leq T$ the probability of going up.[5] In particular, $\mathcal{Q}$ contains all product measures defined via $P_p(\varepsilon_{t+1} = 1 | \mathcal{F}_t) = p$ for a fixed $p \in [\underline{p}, \overline{p}]$ and all $t < T$. In the following we will denote by $\overline{P}$ the measure $P_{\overline{p}}$, and by $\underline{P}$ the measure $P_{\underline{p}}$.

The state variables $(\varepsilon_t)_{t=1,\dots,T}$ are independent under the product measures above. In general, however, $(\varepsilon_t)_{t=1,\dots,T}$ are correlated. To see this,

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[4] $\mathcal{M}_1(\Omega)$ denotes the space of probability measures on $(\Omega, \mathcal{F})$.
[5] See Epstein and Schneider (2003b).

consider the measure P^τ defined via

$$P^\tau(\varepsilon_{t+1} = 1 | \mathcal{F}_t) = \begin{cases} \overline{p} & \text{if } t \leq \tau \\ \underline{p} & \text{else} \end{cases}$$

for any stopping time $\tau < T$. As the one-step-ahead probabilities remain within the interval $[\underline{p}, \overline{p}]$, the so defined measure P^τ belongs to $\mathcal{Q}$. At the same time, the probability of going up on a node depends on the realized path through the value of τ, and $(\varepsilon_t)_{t=1,\ldots,T}$ are correlated.

The above example reveals an important structural feature of $\mathcal{Q}$, dynamic consistency: The set of measures is stable under the operation of decomposition in marginal and conditional part. Loosely speaking, it allows the decision maker to change the measure she uses as the time evolves in an appropriate manner. In the example above, the decision maker first uses the measure $\overline{P}$ until an event indicated by the stopping time τ and then changes to $\underline{P}$. Mathematically, this property is equivalent to an appropriate version of the law of iterated expectation and is closely linked to the idea of backward induction.[6] The concept has recently attracted much attention and was also discussed under different notions by Delbaen (2002), Epstein and Schneider (2003a), and Föllmer and Schied (2004).

The following lemma summarizes crucial properties of the set $\mathcal{Q}$.

Lemma 2.2.1 *The set of measures defined as in (2.1) satisfies the following properties*

1. *$\mathcal{Q}$ is compact and convex,*

2. *all $P \in \mathcal{Q}$ are equivalent to P_0,*

3. *$\mathcal{Q}$ is time-consistent in the following sense: Let $P, Q \in \mathcal{Q}$, $(p_t), (q_t)$ densities of P, Q with respect to P_0, i.e.,*

$$p_t = \frac{dP}{dP_0}\big|_{\mathcal{F}_t}, \quad q_t = \frac{dQ}{dP_0}\big|_{\mathcal{F}_t}.$$

For a fixed stopping time τ define the measure R via the density (r_t) with respect to P_0,

$$r_t = \frac{dR}{dP_0}\big|_{\mathcal{F}_t} = \begin{cases} p_t, & \text{if } t \leq \tau \\ \frac{p_\tau q_t}{q_\tau}, & \text{else} \end{cases},$$

then $R \in \mathcal{Q}$.

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[6]See Riedel (2009) for a survey on dynamic consistency, the various concepts and their equivalence.

Let us define the set of density processes with respect to the reference measure P_0 as

$$\mathcal{D} = \left\{ \left(\frac{dP}{dP_0}\big|_{\mathcal{F}_t} \right)_{t=0,\dots,T} \big| P \in \mathcal{Q} \right\}.$$

Due to Lemma 2.2.1 we can identify $\mathcal{Q}$ with the set $\mathcal{D}$. A detailed analysis of the structure of these density processes can be found in Riedel (2009), and another formulation in Epstein and Schneider (2003b).

2.2.2 The market model

Within the above introduced probabilistic framework we establish the financial market in the spirit of the CRR model. We consider a market consisting of two assets: a riskless bond with a fixed interest rate $r > -1$ and a risky stock with multiplicative increments. For given model parameters $0 < d < 1 + r < u$ and $S_0 > 0$ the stock $S = (S_t)_{t=0,\dots,T}$ evolves according to

$$S_{t+1} = S_t \cdot \left\{ \begin{array}{ll} u, & \text{if } \varepsilon_{t+1} = 1 \\ d, & \text{if } \varepsilon_{t+1} = 0 \end{array} \right. .$$

Without loss of generality, we assume $u \cdot d = 1$. This is a common and appropriate assumption when dealing with exotic options in binomial models, see Cox and Rubinstein (1985) for example.

For every $t \leq T$ the range of possible stock prices is finite and bounded, we will denote by

$$E_t = \{S_0 \cdot u^{t-2k} | k \in \mathbb{N}, 0 \leq k \leq t\}$$

the set of possible stock prices at time $t \leq T$. Moreover, the filtration generated by the sequence $(S_t)_{t=0,\dots,T}$ coincides with $(\mathcal{F}_t)_{t=0,1,\dots,T}$ and every realized path of S, $(s_1, \dots, s_t)$, can be associated with a realization of $(\varepsilon_s)_{s \leq t}$.

As the state variables are not independent under each probability measure $P \in \mathcal{Q}$, the increments of S are correlated in general. The probability of an up-movement depends on the realized path but stays within the boundaries $[\underline{p}, \overline{p}]$ for every $P \in \mathcal{Q}$. As mentioned above, the returns are independent and identically distributed under the product measures in $\mathcal{Q}$.

Economically, our model describes a market where the market participants are not perfectly certain about the asset price dynamics. To express this uncertainty investors use a class of measures constructed above. The inability to completely determine the underlying probabilistic law may arise from insufficient or imprecise information, or can also be part of the stress

test routine as discussed in the introduction. The set $\mathcal{Q}$ presents the set of possible models the decision maker takes into account. Various choices of $P \in \mathcal{Q}$ correspond to different models. With our specification, mean return on stock is uncertain, and it is easily seen that $\overline{P}$ associates the highest mean return, while $\underline{P}$ corresponds to the lowest mean return on the stock at every node.

The specification of $\mathcal{Q}$ is a part of the model. In practice, it may arise from regulation policies or be imposed by the bank accounting standards, result from statistical considerations or just reflect the degree of ambiguity aversion. The length of the interval $[\underline{p}, \overline{p}]$ determines the range of possible models. As the interval's length decreases, the model converges to the classical binomial tree model. We obtain the classical CRR model as a special case of our model by choosing $\underline{p} = \overline{p}$, or setting both equal to $\frac{1+r-d}{u-d}$ in order to obtain the risk-neutral valuation.

Another difference to the classical binomial tree is the introduction of correlated returns of the stock. This allows to incorporate the decision maker's reaction on new arriving information. In our model the investor is allowed to change the model she uses according to the available information. At this point, the economic implication of time-consistency of $\mathcal{Q}$ becomes clear and appropriate. Based on this feature, our decision maker is allowed to use a measure $P_1 \in \mathcal{Q}$ until an event indicated by a stopping time τ and then to change her beliefs about the right model using another measure $P_2 \in \mathcal{Q}$ after time τ. Thus, the multiple priors decision maker is allowed to adjust the model she uses responding to the state of the market.

We note, however, this notion is not the same as classical Bayesian learning since the decision maker has to little information or market knowledge to learn the correct distribution. While in the learning process, the decision maker updates the model by adjusting the set of possible models, here, the investor keeps the set of possible models fixed, not excluding any of the possible models as time evolves, but chooses a particular model at each point in time, reconsidering her choice when new information arrives.

2.2.3 The decision problem

Let us consider an investor holding an exotic option in her portfolio. As most of the exotic options are OTC contracts there is usually no functioning market for this derivative or the trading of claims involves high transaction costs. Therefore, in absence of a trading partner the buyer is forced to hold the claim until maturity, so we exclude the possibility of selling the acquired contracts concentrating purely on the exercise decision of the investor. In our

analysis we mainly concentrate on institutional investors already holding the derivatives in the portfolio. Therefore, it is plausible to assume risk-neutral agents who discount future payoff by the riskless rate.

Remark 2.2.2 *When having in mind a private investor exposing ambiguity aversion, it seems natural also to introduce risk aversion and to discount by an individual discount rate δ. As these considerations do not change the structure of the worst-case measure obtained here, we do not pursue this issue and maintain risk neutrality.*

We consider an American claim $A : \Omega \to \mathbb{R}_+$ written on S and maturing at T. Since A is written on S, we write the claim's payoff as $A(t, (S_s)_{s \leq t})$ when exercised at time t. Note, we explicitly allow path-dependent structures. The investor holding A in their portfolio aims to maximize their expected payoff by choosing an appropriate exercise strategy. As the expectation in our multiple priors setting is not uniquely defined, the ambiguity averse investor maximizes their minimal expected payoff, i.e.,

$$\text{maximize } \inf_{P \in \mathcal{Q}} \mathbb{E}^P A(\tau, (S_s)_{s \leq \tau}) \text{ over all stopping times } \tau \leq T. \quad (2.2)$$

The choice of the exercise strategy according to the worst possible model corresponds to conservative value assignment. It treats long book positions in the same way as the coherent risk measures treats short positions.[7] The value of the multiple priors problem stated in (2.2), $U_0^{\mathcal{Q}}$, is lower than the value of the single prior problem, U_0^P, for each possible model $P \in \mathcal{Q}$. Therefore, this notion minimizes the model risk as the model misspecification within $\mathcal{Q}$ increases the value of the claim.

Remark 2.2.3 *1. The problem of the long investor stated in (2.2) differs structurally from the task of the seller of the option. The seller of the American claim needs to hedge the claim against every strategy of the buyer. To obtain the hedge she solves the optimal stopping problem under the equivalent martingale measure P^*. In the binomial tree the unique equivalent martingale measure P^* is completely determined by the parameters r, u and d, and does not depend on the mean return, Hull*

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[7]Mathematically, our model is closely related to a representation of coherent risk measures. See Delbaen (2002) or Riedel (2009) for more detailed analysis.

(2006). The situation is different for the buyer as she solves the optimal stopping problem under the physical measure taking the mean return into account and being interested in personal utility maximization rather than in risk-neutral valuation. Although the buyer and the seller use different techniques assigning value to the options and obtaining different values for the claim, there is no contradiction to no-arbitrage condition because of the American structure of the claims considered here.

2. *It is usual to evaluate claims in the book that are not settled yet using mark-to-market approach. The value of the option is then set to be equal to the market price. This makes sense if markets are well functioning or if the investor intends to sell the option on the secondary market rather than hold it until maturity. However, this approach may value the claims incorrectly if the market is malfunctioning or there is no market at all, as it was seen and still is seen at financial markets these days. Multiple prior value assignment through $U^{\mathcal{Q}}$ is an alternative to the fair value accounting as it provides conservative value assignment by using the worst possible scenario. But it also protects the book value from too pessimistic or overoptimistic views of the market that are due to expectations and do not reflect fundamentals. However, $U^{\mathcal{Q}}$ is not the price of the option, it is rather the investor's private value that may differ from the market view.*

2.2.4 The solution method

If $\mathcal{Q}$ is a singleton the problem stated in (2.2) can be solved by classical dynamic programming methods. One backwardly defines the value process of the problem – the Snell envelope – and stops as soon as the value process equals the payoff process. This technique, however, fails to hold in the multiple priors setting.[8] Riedel (2009) extended backward induction to the case of time-consistent multiple priors stating sufficient conditions for the Snell arguments to hold.

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[8]An example is presented in Riedel (2009).

Theorem 2.2.4 *(Riedel (2009)[9]) Given a set of measures satisfying the conditions stated in Lemma 2.2.1, and a bounded payoff process $X = (X_t)_{t=0,...,T}, X_t = A(t, (S_s)_{s \leq t})$, define the multiple priors Snell envelope $U^{\mathcal{Q}}$ recursively by $U_T^{\mathcal{Q}} = X_T$ and*

$$U_t^{\mathcal{Q}} = \max\{X_t, \min_{P \in \mathcal{Q}} \mathbb{E}^P(U_{t+1}^{\mathcal{Q}} | \mathcal{F}_t)\} \quad \text{for } t < T. \tag{2.3}$$

Then,

1. *$U^{\mathcal{Q}}$ is the smallest $\mathcal{Q}$-multiple priors supermartingale[10] that dominates the payoff process X.*

2. *$U^{\mathcal{Q}}$ is the value process of the multiple priors stopping problem for the payoff process X, i.e.,*

$$U_t^{\mathcal{Q}} = \max_{\tau \geq t} \min_{P \in \mathcal{Q}} \mathbb{E}^P(X_\tau | \mathcal{F}_t).$$

3. *An optimal stopping rule is given by*

$$\tau^{\mathcal{Q}} = \min\{t \geq 0 | U_t^{\mathcal{Q}} = X_t\}.$$

The above result ensures the existence of a solution to the problem in (2.2). Moreover, as shown by several authors (for example Föllmer and Schied (2004), Karatzas and Kou (1998), Riedel (2009)) the problem in (2.2) is equivalent to a single prior problem for a particular measure $\hat{P} \in \mathcal{Q}$, i.e., the value process of the multiple priors problem satisfies

$$U^{\mathcal{Q}} = U^{\hat{P}}. \tag{2.4}$$

The measure $\hat{P}$ is called worst-case prior (measure) and can be constructed via backward induction by choosing the worst conditional one-step-ahead probability on every node of the tree and pasting the so obtained densities together at time zero. The worst-case measure is stochastic in general and depends on the payoff process. Thus, it is part of the solution.

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[9]We state the theorem modified to our setting. A more general formulation can be found in the original source. As we face a finite state space, see also Lemma 2.2.1, maxima and minima are well-defined and used here and in the following.

[10]Given a set of measures $\mathcal{Q}$, a $\mathcal{Q}$-multiple priors supermartingale is an adapted process, say $S = (S_t)_{t=0,...,T}$, satisfying $S_t \geq \operatorname{ess\,inf}_{P \in \mathcal{Q}} \mathbb{E}^P(S_{t+1} | \mathcal{F}_t)$ for all $t \leq T - 1$.

Due to the equality in (2.4), the optimal stopping strategies $\tau^{\mathcal{Q}}$ of the multiple priors problem and $\tau^{\hat{P}}$ of the problem for the prior $\hat{P}$ coincide. Therefore, the problem can be solved in two steps. In the first step, one identifies the worst-case measure $\hat{P}$ and solves the problem classically under $\hat{P}$, in the second step. This technique allows to make use of solutions already obtained in the classical case. For problems not having a closed-form solution, this technique reduces numerical complexity by reducing the task to a single prior problem where methods are well developed.

2.2.5 Options with monotone payoffs

We focus on claims whose payoffs only depend on current time and current price of the underlying. We state the solution for claims with payoffs obeying the same monotonicity in the underlying's price at all points in time, Riedel (2009). The results build the foundation for the analysis of more complicated payoffs in the next sections.

We consider an American claim maturing at T and paying off $X_t = A(t, S_t)$ when exercised at t.[11]

Theorem 2.2.5 (Claims with monotone payoffs) *1. If the claim's payoff function $A(t, S_t)$ is increasing in S_t for all t, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\underline{P}}$, and the holder of the claim uses the optimal stopping rule given by $\underline{\tau} = \min\{t \geq 0 : U_t^{\underline{P}} = A(t, S_t)\}$.*

2. If $A(t, S_t)$ is decreasing in S_t for all t, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\overline{P}}$, and an optimal stopping rule under ambiguity is given by $\overline{\tau} = \min\{t \geq 0 : U_t^{\overline{P}} = A(t, S_t)\}$.

The key to this result is the fact that $\underline{P}$, (or $\overline{P}$ respectively), is the worst probability measure in the sense of first-order stochastic dominance, and that the payoff is a monotone function in the underlying stock price. Using Theorem 2.2.5 we can already solve the optimal exercise problem for the call and put option in the multiple priors setting, Riedel (2009).

Corollary 2.2.6 (Call) *A risk-neutral buyer of an American call uses an optimal stopping rule for the prior $\underline{P}$. The value of American call at time*

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[11]Due to the formulation of the decision problem in (2.2), and Theorem 2.2.4, X_t always represents the discounted payoff from exercising. Similarly, $U_t^{\mathcal{Q}}$ denotes the value of the problem at time zero after time t.

zero is given via

$$U_0^{\mathcal{Q}} = U_0^{\underline{P}} = \mathbb{E}^{\underline{P}}\left((S_{\underline{\tau}} - K)^+(1+r)^{-\underline{\tau}}\right),$$

where $\underline{\tau}$ is an optimal stopping time with respect to $\underline{P}$. In particular, if $\underline{p}u + (1 - \underline{p})d > 1 + r$, the American call is never being exercised.

The last statement is proven in Riedel (2009).

Corollary 2.2.7 (Put) *A risk-neutral buyer of an American put uses an optimal stopping rule for the prior $\overline{P}$. The value of American put at time zero is given via*

$$U_0^{\mathcal{Q}} = U_0^{\overline{P}} = \mathbb{E}^{\overline{P}}\left((K - S_{\overline{\tau}})^+(1+r)^{-\overline{\tau}}\right),$$

where $\overline{\tau}$ is an optimal stopping time with respect to $\overline{P}$.

The results of Theorem 2.2.5 help to find the worst-case measure for more complicated options. Using appropriate decompositions we represent the options as claims with monotone payoffs in the sense of the theorem. For those monotone claims we identify the worst-case measure via Theorem 2.2.5. Pasting the so obtained measures together we construct the desired worst-case measure.

2.3 Barrier options

Barrier options are among most traded exotic options and often used as components of more sophisticated derivatives. The knock-in/knock-out feature of the options leads to a lower premium which has to be paid by the buyer. In return, the buyer is exposed to the risk, for instance in the knock-out case, that the underlying hits the barrier and the option becomes worthless. For knock-in options the buyer faces the risk that the underlying firstly has to hit the barrier level before the option becomes valuable.

Before stating the results we prove a technical theorem which enables us to identify the worst-case measure for various path-dependent payoffs such as barrier options.

Throughout this section we suppose that all given barrier levels $H \in \mathbb{R}_+$ lie on the grid of possible asset prices, namely E_T.[12] Given an initial stock

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[12]This assumption simplifies notations. In particular, when letting the grid size tending to zero, this assumption essentially avoids computational errors. See Hull (2006) for an analysis of this issue. He used interpolation method between upper and lower barriers to keep the error low.

price S_0, let $H > S_0$. We define a first-passage time τ by

$$\tau : \Omega \longrightarrow [0, T+1], \quad \tau(\omega) := \inf \{t \geq 0 : S_t(\omega) \geq H\} \wedge T + 1.$$

For $H > S_0$ we call these stopping times depending on the stock price *up-crossing times*. We set $\mathcal{F}_{T+1} := \mathcal{F}_T$ and $\inf \emptyset := \infty$. Similarly, we use the notion *down-crossing times* when $H < S_0$.

In practice, barrier options are said to be weakly path-dependent which emphasizes that their payoffs indeed depend on the whole path of the underlying's price, but considering the two-dimensional process consisting of the underlying's price and its maximum price, (or minimum price respectively), reduces the problem to the Markovian case, i.e., their payoffs at any time t only depend on this two-dimensional process at that time.

In contrast to plain vanilla options, the payoff process of barrier options does not exhibit the same monotonicity in its underlying at each time. The monotonicity rather depends on whether the underlying has already hit the barrier or not. To express this fact mathematically, we use the notion of stochastic intervals. For two first-passage times τ_1 and τ_2 of the same type, that is, both either up-crossing or down-crossing times, we consider the stochastic interval $[\tau_1, \tau_2[$ defined as

$$[\tau_1, \tau_2[:= \left\{ (s, \omega) \in [0, T] \times \Omega \,\middle|\, \tau_1(\omega) \leq s < \tau_2(\omega) \right\}.$$

We only consider first-passage times with $\tau_1 \leq \tau_2$.[13] To guarantee the inequality, we require the thresholds H_1 and H_2 specifying τ_1 and τ_2 to satisfy $H_1 \leq H_2$ in the case of up-crossing times, and $H_1 \geq H_2$ else. With slight abuse of notation we will often write $\mathbb{1}_{[\tau_1, \tau_2[}(t)$ instead of $\mathbb{1}_{[\tau_1, \tau_2[}(t, \omega)$.

We aim to extend Theorem 2.2.5 to more general situations. While not requiring the same monotonicity at all points in time, we only claim the same monotonicity at points in time belonging to the same stochastic interval.

The worst-case measure can be determined recursively using backward induction for computing the Snell envelope $U^{\mathcal{Q}}$, Riedel (2009). One observes, the worst-case conditional one-step-ahead probability at time t, say $\hat{P}_t$, is characterized by the equation $U_{t-1}^{\mathcal{Q}} = \min_{P \in \mathcal{Q}} \mathbb{E}^P \left(U_t^{\mathcal{Q}} | \mathcal{F}_{t-1} \right)$. So, $\hat{P}_t$ is detected by computing $U_{t-1}^{\mathcal{Q}}$. Additionally, in the case of a monotone payoff process X, the monotonicity of $U_{t-1}^{\mathcal{Q}}$ is inherited by the monotonicity of $U_t^{\mathcal{Q}}$ and X_{t-1}, as long as both feature the same.

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[13]By writing $\tau_1 \leq \tau_2$ we require the inequality to hold for all $\omega \in \Omega$. Apart from that, later we write $\tau_1 \leq t$, for example, and do not necessarily mean that all elements of Ω are involved. It is just used as a simplification in place of writing $\tau_1(\omega) \leq t$. We are sure that the reader is able to comprehend the respective coherence.

In the following we will make use of this observation to extend Theorem 2.2.5 to payoff processes which do not exhibit the same monotonicity in the underlying at all times but on various events specified by stochastic intervals as described above.

Let H_2 be a barrier specifying τ_2. If τ_2 is up-crossing time, we define

$$\sigma_i := \inf\{t \in [\sigma_{i-1} + 1, \tau_2[\mid S_t = H_2 \cdot d\} \wedge T + 1$$

for $1 \leq i < T$ with the notation $\sigma_0 := -1$. If τ_2 is down-crossing time, we define for $1 \leq i < T$

$$\sigma_i := \inf\{t \in [\sigma_{i-1} + 1, \tau_2[\mid S_t = H_2 \cdot u\} \wedge T + 1.$$

The introduction of these stopping times has technical reason. They are needed to identify the times/nodes at which there is possibility of reaching the (second) barrier in the subsequent time step.

From now on, we will briefly write $X = (X_t)$ for a process in place of $X = (X_t)_{t=0,\ldots,T}$. We will consider payoff functions of the form $X_t = A(t, S_t) \mathbb{1}_{[\tau_1, \tau_2[}(t)$. Throughout Section 2.3 we will assume that the function $A(t, S_t)$ just consists of a function only depending on S_t and the discounting factor, i.e.,

$$A(t, S_t) = A(S_t)/(1 + r)^t \quad \forall t \leq T. \tag{2.5}$$

Economically it means that the option's payoff from exercising does not depend on time apart from the event specified by $\mathbb{1}_{[\tau_1, \tau_2[}$. It ensures that the option's payoff is stationary on $[\tau_1, \tau_2[$ which is important for the proof of the following theorems, see Appendix A.1. Barrier options comply with this form.

Theorem 2.3.1 *Let H_1, H_2 be the barrier levels specifying τ_1, and τ_2, respectively. Let the payoff process $X = (X_t)$ be given by*

$$X_t = A(t, S_t) \mathbb{1}_{[\tau_1, \tau_2[}(t)$$

where $A(t, \cdot)$ is monotone in S_t for all $t \leq T$, τ_1 and τ_2 are up-crossing times, or constant, satisfying $\tau_1 \leq \tau_2$, (assuming $S_0 < H_1 < H_2$). Let $(U_t^{\hat{P}})$ be the Snell envelope of (X_t) under the measure $\hat{P}$.

1. *If $A(t, \cdot)$ is decreasing for all $t \leq T$, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\hat{P}}$ and the worst-case measure $\hat{P} \in \mathcal{Q}$ is generated by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_1} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u) \right)(1 - \underline{p}) \prod_{u \in]\tau_1, t \wedge T]} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$

   for all $t \leq T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf \left\{ t \in [\tau_1, T] \,\middle|\, U_t^{\hat{P}} = X_t \right\} \wedge T$.

2. *If $A(t, \cdot)$ is increasing for all $t \leq T$, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\hat{P}}$ and the worst-case measure $\hat{P} \in \mathcal{Q}$ is generated by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_2: \ u \neq \sigma_i + 1} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right)$$
$$\prod_{u \leq t: \ u = \sigma_i + 1} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right) \prod_{u \in]\tau_2, t \wedge T]} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$

   for all $t \leq T$ and all occurring $1 \leq i < T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf \left\{ t \in [\tau_1, \sigma_1] \,\middle|\, U_t^{\hat{P}} = X_t \right\} \wedge T$.

Remark 2.3.2 *In the case of $\tau_2 = T + 1$ the stopping times $\sigma_i, i < T$ defined above are not needed and set equal to $T + 1$.*[14]

The proof is shifted to the appendix. It relies heavily on the theory about the multiple priors Snell envelope constructed by backward induction which besides requires time-consistency of $\mathcal{Q}$.

Remark 2.3.3 *Note that the worst-case measure is not unique. If $\tau_2(\omega) < T$, for $\omega \in \Omega$, there exists an index $i < T$ such that $\tau_2(\omega) = \sigma_i(\omega) + 1$. The conditional one-step-ahead probabilities $\hat{P}(\varepsilon_t = 1 \mid \mathcal{F}_{t-1})(\omega)$ for $t > \tau_2(\omega)$ must only attain values within $[\underline{p}, \overline{p}]$ as the claim's payoff is always zero after $\tau_2(\omega)$. Also, optimal exercising only occurs either before or at time $\sigma_1(\omega)$ as the payoff from exercising is immediately 0 afterwards. Thus, the density of the worst-case measure is only relevant for the decision maker up to time $\sigma_1(\omega)$. Afterwards she will not hold the option any longer.*

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[14]We could set $H_2 > S_0 u^T$ in the case of up-crossing times and $H_2 < S_0 d^T$ in the case of down-crossing times.

A similar result as above also holds when dealing with down-crossing times. The only difference is the monotonicity of X and $U^{\mathcal{Q}}$ which changes for down-crossing times. Consequently, the densities of the worst-case measures change. We only state the result without giving the proof as it would be similar to the proof of Theorem 2.3.1.

Theorem 2.3.4 *Take the same assumptions as in Theorem 2.3.1 except the one for τ_1 and τ_2 being now down-crossing times or constant, (assuming $S_0 > H_1 > H_2$ to ensure $\tau_1 \leq \tau_2$).*

1. *If $A(t, \cdot)$ is decreasing for all $t \leq T$, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\hat{P}}$ and the worst-case measure $\hat{P} \in \mathcal{Q}$ is given by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_2: \ u \neq \sigma_i + 1} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$
$$\prod_{u \leq t: \ u = \sigma_i + 1} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right) \prod_{u \in \ ]\tau_2, \, t \wedge T]} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right)$$

   for all $t \leq T$ and all occurring $1 \leq i < T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf \left\{ t \in [\tau_1, \sigma_1] \, \middle| \, U_t^{\hat{P}} = X_t \right\} \wedge T$.

2. *If $A(t, \cdot)$ is increasing for all $t \leq T$, the multiple priors Snell envelope is $U^{\mathcal{Q}} = U^{\hat{P}}$ and the worst-case measure $\hat{P} \in \mathcal{Q}$ is given by the density $\hat{D} = \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_1} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right) \prod_{u \in \ ]\tau_1, \, t \wedge T]} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right)$$

   for all $t \leq T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf \left\{ t \in [\tau_1, T] \, \middle| \, U_t^{\hat{P}} = X_t \right\} \wedge T$.

Remark 2.3.5 *An extension of both theorems to cases where τ_1 is up-crossing time and τ_2 down-crossing time, or vice versa, is also possible. One may also skip the condition $\tau_1 \leq \tau_2$. This is just an assumption made to avoid too many cases that must be distinguished when stating the density and proving the theorem. In particular cases it is also possible to extend the theorem to payoffs which are finite sums of payoff functions such as in the theorems. We will illustrate this for an up-and-out ladder option in Section 2.3.2.*

The theorems above allow to analyze options not having the same monotonicity at all points in time but conditioned on certain events. This is exemplified in the following subsections. The results for the examples are just applications of the theorems stated above.

2.3.1 Simple barrier options

We apply the preceding theory to single barrier options. The payoff of a single barrier option depends on the underlying stock price and a particular trigger event – the underlying's price hits a prescribed barrier during the term of the contract. "In-options" become valuable when the underlying asset price hits a prescribed barrier level H. If this does not happen within the lifetime of the contract, the option remains worthless. In contrast, "out-options" become worthless when the stock price reaches the barrier.

While exercising American put and call can be easily reduced to the single prior case by using monotonicity and first-order stochastic dominance, see Corollaries 2.2.6 and 2.2.7, the picture is quite more involved in the case of American barrier options.

We begin to consider an American up-and-in put with strike price K and barrier H. We assume $H > K$, and to avoid the trivial case, $H > S_0$. Let $T > 0$ be the contract's maturity. Denote by

$$\tau_H := \inf \{t \geq 0 | S_t \geq H\} \wedge T + 1$$

the knock-in time when the up-and-in put becomes valuable. From this time on, the barrier option coincides with an American plain vanilla put initiated at τ_H, expiring at T and strike K.

First, the holder of such option faces uncertainty about whether the option will knock in. After knock-in, she faces the same uncertainty as holding a plain vanilla put. Both uncertainties work in reverse directions. At the very beginning, her ambiguity aversion leads to presuming the lowest possible mean return in the option's underlying's price, and after knock-in, presuming highest mean return.

The precise result is stated in the next corollary.

Corollary 2.3.6 (Up-and-in put) *Consider an American up-and-in put option with data as specified above, and (discounted) payoff $X_t = (K - S_t)^+/(1+r)^t \mathbb{1}_{\{t \geq \tau_H\}}, t \leq T$. The ambiguity averse agent uses the prior $\hat{P} \in \mathcal{Q}$*

generated by the density $\hat{D} \in \mathcal{D}$,

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_H} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right) \prod_{u \in \,]\tau_H, t \wedge T]} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$

for $t \leq T$. *Hence, the value of the option after time* t *from the perspective of the ambiguity averse buyer is given by*

$$U_t^{\mathcal{Q}} = U_t^{\hat{P}} = \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} \mid \mathcal{F}_t] \tag{2.6}$$

with $\hat{\tau} = \inf \left\{ t \in [\tau_H, T] \,\big|\, U_t^{\hat{P}} = X_t \right\} \wedge T$ *an optimal stopping time.*

PROOF: We apply Theorem 2.3.1 part 1. Setting $\tau_1 := \tau_H$ and $\tau_2 := T + 1$, we can rewrite the payoff as $X_t = (K - S_t)^+/(1+r)^t \mathbb{1}_{[\tau_H, T+1[}$ for all $t \leq T$. Since $A(t, S_t) := (K - S_t)^+/(1+r)^t$ is monotone decreasing in S_t for each t, Theorem 2.3.1 part 1 applies. $\qquad \square$

From the density of the worst-case measure we see that the pessimistic buyer presumes a change of mean return at knock-in. Before the option becomes valuable she uses the lowest mean return in her computations, and afterwards, she uses the measure that induces the maximal mean return for the underlying stock price. This corresponds to the lowest conditional one-step-ahead probabilities for up-movements of the stock before knock-in, and the highest afterwards. Besides we see, the worst-case measure $\hat{P}$ is the pasting of $\overline{P}$ after $\underline{P}$ at τ_H. Therefore, it exhibits a non-stationary, stochastic structure.

Using the structure of the worst-case measure, Equation (2.6) can be rewritten as follows. For $t < \tau_H$ we obtain by the law of iterated expectation

$$\begin{aligned} U_t^{\mathcal{Q}} = \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} | \mathcal{F}_t] &= \mathbb{E}^{\hat{P}} \left[\mathbb{E}^{\hat{P}} \left[X_{\hat{\tau}} \mid \mathcal{F}_{\tau_H} \right] \big| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\underline{P}} \left[\mathbb{E}^{\overline{P}} \left[X_{\hat{\tau}} \mid \mathcal{F}_{\tau_H} \right] \big| \mathcal{F}_t \right]. \end{aligned} \tag{2.7}$$

If $t \geq \tau_H$,

$$U_t^{\hat{P}} = \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} | \mathcal{F}_t] = \mathbb{E}^{\overline{P}} \left[(K - S_{\hat{\tau}})^+/(1+r)^{\hat{\tau}} \,\big|\, \mathcal{F}_t \right]$$

which equals the value of a plain vanilla American put in the ambiguity-averse setting discounted to time zero.

Remark 2.3.7 *From a decision theoretical point of view, Equation (2.7) illustrates that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under expected utility. The buyer of an American up-and-in put for example behaves as two readily distinguishable expected utility maximizers. This is so because the worst-case measure $\hat{P}$ depends on the payoff process.*

Using Theorem 2.3.4 part 2 we obtain the analogous result for an American down-and-in call option with barrier $H < S_0$. In this case, the discounted payoff is given by $X_t = (S_t - K)^+/(1 + r)^t \mathbb{1}_{\{t \geq \tau_H\}}$ for all $t \leq T$, where $\tau_H := \inf\{t \geq 0 | S_t \leq H\} \wedge T + 1$. Setting $\tau_1 := \tau_H$ and $\tau_2 := T + 1$ we are in the notion of Theorem 2.3.4 and derive

Corollary 2.3.8 (Down-and-in call) *The ambiguity averse agent uses the prior $\hat{P} \in \mathcal{Q}$ given by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_H} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p})\right) \prod_{u \in]\tau_H, t \wedge T]} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p})\right)$$

for $t \leq T$.

Similar to an up-and-in put option a down-and-in call equals a plain vanilla call option when the underlying hits the barrier level H. On the analogue of (2.7), the value of the down-and-in call at $t < \tau_H$ is

$$U_t^{\mathcal{Q}} = \mathbb{E}^{\overline{P}}\left[\mathbb{E}^{\underline{P}}[X_{\hat{\tau}} | \mathcal{F}_{\tau_H}] \,\big|\, \mathcal{F}_t\right] \tag{2.8}$$

where $\hat{\tau}$ is an optimal stopping time for this considered problem. Under the assumption $\underline{p}u + (1 - \underline{p})d > 1 + r$ we obtain $\hat{\tau} = T$, see Corollary 2.2.6. In that case it is possible to derive a binomial closed-form solution.

In the case of put options, there does not exist a constant early exercise boundary for finite maturity T. Therefore, it is not possible to derive a closed-form binomial expression for the American put or the American up-and-in put, see also Reimer and Sandmann (1995).

Another example is an up-and-out call option. Such an option is knocked out when a prespecified barrier level is reached, consequently, the claim becomes worthless. Let $\tau_H := \inf\{t \geq 0 | S_t \geq H\} \wedge T + 1$ be the knock-out time, and assume $H > S_0$ and $H > K$. We have

Corollary 2.3.9 (Up-and-out call) *The ambiguity averse holder of an up-and-out call with (discounted) payoff $X_t := (S_t - K)^+/(1 + r)^t \mathbb{1}_{\{t \leq \tau_H\}}$ for all $t \leq T$ uses the prior $\hat{P} \in \mathcal{Q}$ given by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t := 2^t \prod_{u \leq \tau_H \wedge t: \ u \neq \sigma_i + 1} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right)$$
$$\prod_{u \leq t: \ u = \sigma_i + 1} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right) \prod_{u \in \ ]\tau_H, t \wedge T]} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$

for all $t \leq T$ and all occurring $1 \leq i < T$. In particular, a sufficient condition for early exercise of the American up-and-out call at time $t < \tau_H$ is given when the underlying's price at that time is larger or equal to $\frac{H}{(1+r)^{T-t}} + K \left(1 - \frac{1}{(1+r)^{T-t}} \right)$.

PROOF: Setting $\tau_1 := 0$, $\tau_2 := \tau_H$ we can rewrite X_t in the form of Theorem 2.3.1 and apply its second part to deduce the worst-case measure $\hat{P}$. Due to the feature of this claim, the payoff from exercising at each time is bounded from above by $H - K$. Therefore, a sufficient condition on early exercise at time t is given by

$$(S_t - K)(1 + r)^{T-t} \geq H - K$$
$$\iff S_t \geq \frac{H}{(1+r)^{T-t}} + K \left(1 - \frac{1}{(1+r)^{T-t}} \right),$$

see also Reimer and Sandmann (1995). $\qquad \square$

Note that the early exercise condition is always satisfied for $t = \sigma_1$. Thus, early exercise occurs at time σ_1 at the latest. Hence, the decision maker always exercises the option when there is knock-out risk at the successive node: the option's underlying S might hit the barrier in the next time period and become worthless. As a consequence, the decision maker does not care about changes of the conditional one-step-ahead probabilities at time $\sigma_1 + 1$ or afterwards because she will have exercised the option at time σ_1, or even earlier.

Remark 2.3.10 *Assuming additionally in Corollary 2.3.9 that the inequality $\underline{p}u + (1 - \underline{p})d > 1 + r$ is satisfied, the ambiguity averse buyer will exercise the American up-and-out call exactly at time σ_1. By the additional assumption, $S_t(1 + r)^{-t}$ is a strict multiple priors submartingale. Thus, early exercise is*

not optimal before σ_1. By the knock-out feature, $S_t = H \cdot d$ generates the maximal option's payoff.

Without early exercise at σ_1, the ambiguity averse buyer of the up-and-out call would switch the conditional one-step-ahead probabilities at all nodes $\sigma_i, i < T$ during the option's lifetime. We will illustrate this in the next subsection when considering a so-called ladder option.

Down-and-out put options behave analogously to up-and-out call options. There are four further types of barrier options exhibited with a single barrier.[15] Due to their structure, the payoffs possess the same monotonicity in their underlying stock price at all times. Thus, the worst-case measures can be identified by using Theorem 2.2.5. Consequently, the worst-case measures do not feature path-depending conditional one-step-ahead probabilities induced by ambiguity as we have seen above.

2.3.2 Multiple barrier options

The above reasoning can also be applied to options endowed with more than one barrier. The theorems above can be used to attain the worst-case measure for options with both a knock-in and a knock-out barrier level; or for out-options additionally exhibited with a further barrier level which replaces the preceding after some prespecified time progress. This will be demonstrated in the following.

We examine ladder options and focus on the special case of an up-and-out ladder call option with two barrier levels H_1 and H_2, and maturity T. We assume $S_0 < H_1 < H_2$.[16] The claim resembles a single up-and-out barrier call option with additional feature that, after some prescribed date $t_1 \in (0, T)$, the knock-out barrier changes from H_1 to the higher level H_2. Thus, the first barrier H_1 is only valid by time t_1. Afterwards, the second barrier H_2 determines the knock-out event.

The change of the barriers during the contract's running time has impact on the buyer's early exercise strategy. This again affects the relevance of the conditional one-step-ahead probabilities at the nodes exhibiting knock-out risk due to the first barrier H_1. At these nodes of the tree, the underlying's price is equal to $H_1 \cdot d$. In difference to a single up-and-out call, the buyer might find it now not optimal to exercise the claim and accepts the knock-out risk due to expected higher asset returns after t_1 when the second barrier

---

[15]For example an up-and-out put, just to name one of them.

[16]We also remind the reader of the assumption that all barriers are nodes of the tree.

is relevant. This might be rational, for instance if the second barrier H_2 is much higher than the first, or H_1 is close to the strike K which would lead to a low early exercise profit before t_1.

The inequality (2.9), (see below), expresses a condition for the illustrated situation. The (discounted) payoff at time t of such a ladder call option with strike price $K < H_1$ and maturity T is defined as

$$
X_t = \begin{cases} (S_t - K)^+/(1 + r)^t, & \text{if } t \leq t_1 \text{ and } t < \tau_{H_1} \\ (S_t - K)^+/(1 + r)^t, & \text{if } t > t_1, t < \tau_{H_2} \text{ and } t < \tau_{H_1} \\ 0, & \text{else} \end{cases}
$$

$$
= (S_t - K)^+/(1 + r)^t \, \mathbb{1}_{[0,\tau_{H_1} \wedge t_1[} + (S_t - K)^+/(1 + r)^t \, \mathbb{1}_{[t_1,\tau_{H_2} \wedge \tau_{H_1}[}
$$

where $\tau_{H_1} := \inf\{t \in [0, t_1] \,|\, S_t = H_1\} \wedge T + 1$, and $\tau_{H_2} := \inf\{t \in \,]t_1, T] \,|\, S_t = H_2\} \wedge T + 1$.[17] To represent the density of the worst-case measure we need the following stopping times similar to above:

$$
\sigma_i := \inf\{t \in [\sigma_{i-1} + 1, \tau_{H_1} \wedge t_1 - 1[\,\big|\, S_t = H_1 \cdot d\} \wedge T + 1
$$

for $1 \leq i < t_1$ with the notation $\sigma_0 := -1$ and

$$
\gamma := \inf\{t \in [t_1, \tau_{H_2} \wedge \tau_{H_1}[\,\big|\, S_t = H_2 \cdot d\} \wedge T + 1.
$$

Furthermore, for $t \leq T$, let $\Omega_{(t)} := \bigotimes_{i=1}^{t}\{0, 1\}$ denote the set of all paths in Ω up to time t.

Corollary 2.3.11 (Ladder call option) *Given all data as above, in particular, let us suppose the strict inequality of Remark 2.3.10. Additionally, suppose that the value function satisfies for all $\omega_{(t)} \in \Omega_{(t)}$ with $S_t(\omega_{(t)}) = H_1 \cdot d$ the inequality*

$$
X_t(\omega_{(t)}) < (1 - \overline{p})U_{t+1}^{\mathcal{Q}}(\omega_{(t)}, 0) \tag{2.9}
$$

for all $t < \tau_{H_1}(\omega_{(t)}) \wedge t_1$. The ambiguity averse buyer of this ladder option uses the prior $\hat{P} \in \mathcal{Q}$ specified by the density process $\hat{D} \in \mathcal{D}$, with

$$
\hat{D}_T := 2^T \prod_{u \leq \tau_{H_2} \wedge T: \; u \neq \sigma_i+1 \; and \; u \neq \gamma+1} \big(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p})\big)
$$

$$
\prod_{u \leq T: \; u = \sigma_i+1 \; or \; u = \gamma+1} \big(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p})\big)
$$

$$
\prod_{u \in \,]\tau_{H_2}, T]} \big(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p})\big)
$$

---

[17]For $t \leq t_1$ the interval $[t_1, t[$ is defined as the empty set.

for all occurring $1 \leq i < t_1$. In particular, the agent uses the canonical optimal stopping rule $\hat{\tau}$ which is equal to $\gamma \wedge T$ in this case.

PROOF: Examining the proof of Theorem 2.3.1 reveals that we can also apply the second part of the theorem to this special situation as the time interval $[0, T]$ is divided into two disjoint intervals and $A(t, S_t) := (S_t - K)^+/(1 + r)^t$, which is increasing in S_t for all $t \leq T$, is the same function on both intervals. Applying the theorem on both subintervals yields the density for the ambiguity averse agent. The optimal stopping rule is also specified by the theorem. The inequality in (2.9) ensures that the decision maker does not exercise the option when the stock price equals $H_1 \cdot d$ before t_1.

From the beginning of time t_1, the same arguments as in the case of an up-and-out call with a single barrier, (see Remark 2.3.10), lead to the optimal stopping time. Thus, this ladder option is held by the agent up to time $\gamma \wedge T$. $\qquad \square$

The ladder call option exemplifies the path-dependent and stochastic structure of the worst-case measure. Depending on the stock price's path the agent adapts the conditional one-step-ahead probabilities frequently. Whenever she faces knock-out risk, she switches the conditional one-step-ahead probability for an up-movement of the stock from the lowest weight $\underline{p}$ to the highest, $\overline{p}$. Thus, she puts highest weight on a stock's up-movement at these nodes. On the other hand, whenever the option does not knock out in that specific period she adapts the model again and presumes the lowest conditional one-step-ahead probability for a stock's up-movement as there is no knock-out risk in the successive time period.

Unlike an American up-and-out call, the decision maker experiences all these varying marginal probabilities as early exercise is not optimal before t_1 due to the assumption in (2.9).

2.4 Multiple expiry options

In this section we analyze exotic options that are characterized by several expiry dates. At every expiry the owner of the option has the right to modify the contract conditions resetting the strike or the maturity in a predefined way. New conditions of the contract depend on the underlying's value at expiry dates and are not known to the buyer at time zero. Therefore, additional to the uncertainty about future underlying's value the decision maker faces uncertainty about future contract conditions while evaluating the option.

The expiry dates can be predefined points in time (forward start options) or random dates chosen by the buyer or seller of the contract (shout options).

Such options can be seen as a sequence of claims where every claim expires at a predefined date and pays off a new born claim expiring at the next expiry date. In the case of European claims, the expiry dates are deterministic corresponding to forward start options. In the case of shout options we face American claims leading to stochastic expiry dates. In general, multiple expiry options can be entitled with any number of expiry dates, here, we consider dual expiry options where contract conditions change exactly once. Kwok, Dai, and Wu (2004) analyze shout options with infinite number of shout possibilities and establish a relation to lookback options.

2.4.1 Shout options

Shout options are contracts that give the buyer the right to reset the strike at a date chosen by her. The event of resetting the contract features is called shouting and gives the structure its name. The reset right allows the investor to benefit from market movements by choosing a favorable strike. At the same time she can lock in already realized profits ensuring against an unfavorable stock movement.

Shout options are often used by professional investors as a cheaper alternative to lookback options. Whereas the buyer of the lookback option has the right to sell the stock at the maximal price, the owner of the shout option has to call her bank and to freeze the price at which she can sell/buy at any time σ before maturity. The structure becomes active. The buyer should have enough understanding of the market in order to set the strike as close as possible to the peak. Mathematically, the buyer faces an optimal stopping problem, aiming to set the strike optimally.

In the following we analyze shout puts focusing on a more special case later on. The same analysis can be performed for call options.

At time zero the buyer of a shout put receives a plain vanilla put option with strike K_0 and maturity T with additional right to modify the strike of the contract once at any time prior to maturity by calling her bank and fixing the strike in a predefined way. At the time of shouting, say σ, the buyer locks in the realized profits by receiving a cash payment $(K_0 - S_\sigma)$, additionally, she receives a new option of European style with strike $K_1 = f(S, \sigma)$, where f is a $\mathcal{F}_\sigma$-measurable function of the whole path $S = (S_1, \ldots, S_\sigma)$ up to time σ. At maturity the buyer receives the positive part of the difference between the strike K_1 and the final stock price, i.e., $(K_1 - S_T)^+$. The contract is then specified by the initial strike K_0, the function f determining the new strike

K_1 and the maturity date T. This structure allows the investor to lock in realized profits protecting himself against downside risk by receiving the cash payment and at the same time to participate on future upside with the new born option.

To simplify the analysis we consider a particular shout option – the so-called single shout floor that allows for closed form solutions even in finite time. The initial strike of the single shout floor K_0 is equal to zero and the strike K_1 is given by $K_1 = f(S_\sigma) = S_\sigma$. The buyer shouts once at $\sigma \leq T$ fixing the strike at S_σ. At the expiry date she receives a payoff that corresponds to the payoff profile of an European put, i.e., $(S_\sigma - S_T)^+$. Thus, the buyer of this shout option has to solve the following problem

Maximize $\min_{P \in \mathcal{Q}} \mathbb{E}^P((S_\sigma - S_T)^+/(1+r)^T)$ over all stopping times $\sigma \leq T$.

$$(2.10)$$

Note that unlike the American put, the exercise date is fixed but the birth date has to be determined optimally by the buyer. Determining the optimal beginning time/shouting time constitutes the optimal stopping problem for the single shout option. The task is to optimally begin the payoff process rather then stop it which can be seen as purchasing a new issued European option with a fixed maturity. We will maintain this parallel during our analysis.

However, we cannot apply our standard theory of backward induction to the problem stated in (2.10) because the payoff $(S_\sigma - S_T)^+/(1+r)^T$ obtained from stopping at any stopping time $\sigma \leq T$ depends on the value of the stock at maturity and is for this reason not adapted to the filtration $(\mathcal{F}_t)$ generated by the path. To overcome this difficulty we condition the payoff on the available information and consider the following payoff process

$$X_t = \min_{P \in \mathcal{Q}} \mathbb{E}^P((S_t - S_T)^+/(1+r)^T | \mathcal{F}_t). \qquad (2.11)$$

For every $t \leq T$ we can interpret X_t as the discounted multiple priors value of the shout floor if shouted at t. At the same time it corresponds to the value of an at-the-money European put issued at t and maturing at T evaluated under multiple priors.[18]

Using the appropriate version of the law of iterated expectations one can easily see that for all stopping times $\sigma \leq T$ we have

$$\min_{P \in \mathcal{Q}} \mathbb{E}^P((S_\sigma - S_T)^+/(1+r)^T) = \min_{P \in \mathcal{Q}} \mathbb{E}^P(X_\sigma).$$

---

[18]Strictly speaking, the value of the European put issued at t and maturing at T differs from the expression (2.11) by a discount term.

Therefore, we can reformulate the problem stated in (2.10) equivalently in the following way

$$\text{Maximize } \min_{P \in \mathcal{Q}} \mathbb{E}^P(X_\sigma) \text{ over all stopping times } \sigma < T \qquad (2.12)$$

where the payoff process X is defined via (2.11). Thus, the optimal stopping time found for (2.12) is also optimal for the problem (2.10) and the values of the two problems coincide. Again, we can interpret the problem as optimal investment in a put with a fixed investment horizon.

We solve the problem in two steps: first we compute X_t – the explicit value of the shout option freezed at t for all $t \leq T$ and derive the worst-case measure after shouting. In the second step, we identify the worst-case measure before shouting reducing the problem to the single prior case.

To compute X_t for a fixed $t \leq T$ we note that the uncertainty about the strike is resolved at the time of shouting. The strike becomes a constant and as a consequence the claim becomes a plain vanilla European put. As the payoff of the put is decreasing in S_t for all $t \leq T$ by Theorem 2.2.5 we conclude that the worst-case measure is given by $\overline{P}$ and we have

$$X_t = \min_{P \in \mathcal{Q}} \mathbb{E}^P \left((S_t - S_T)^+ / (1+r)^T | \mathcal{F}_t \right)$$
$$= \mathbb{E}^{\overline{P}} \left((S_t - S_T)^+ / (1+r)^T | \mathcal{F}_t \right).$$

Additionally, under $\overline{P}$ the increments of the underlying between t and T – $\Delta(S_t, S_T)$ – are independent for all $t \leq T$ which leads to

$$X_t = S_t \cdot \mathbb{E}^{\overline{P}} \left((1 - \Delta(S_t, S_T))^+ / (1+r)^T | \mathcal{F}_t \right) =: S_t \cdot g(\tau)$$

where $\tau = T - t$ and

$$g(\tau) = (1+r)^{-T} \cdot (1 - \overline{p})^\tau \sum_{k=0}^{k^*(\tau)} \binom{\tau}{k} \left(\frac{\overline{p}}{1 - \overline{p}} \right)^k \left(1 - d^{\tau - 2k} \right)$$

with $k^*(t) := \max \left\{ k \, | k < \frac{t}{2} \right\}$ where we have used that $d = \frac{1}{u} < 1$.

The above equation provides the value of the embedded option contained in the shout contract maturing at T at the time of shouting. At the same time it corresponds to the value of the at-the-money European put issued at $t \leq T$ and maturing at T.

The buyer of a shout option uses $\overline{P}$ to evaluate the option after shouting. Moreover, the value of a freezed shout floor is homogeneous of degree one in the current stock price S_t.

As $g(\tau) > 0$ for all $\tau > 0$, X_t as a function of t and S_t is increasing in S_t for all $t \leq T$. Again using Theorem 2.2.5 we conclude that the worst-case measure of problem (2.12) is given by $\underline{P}$.

Remark 2.4.1 *It might be surprising at the first sight that the value of the put contained in the shout contract at the time of shouting increases if the strike increases. The reason for this observation contradicting the usual intuition is the fact that the strike is not a constant at the moment of issuance of the claim. The value of the claim at the time of shouting is increasing with respect to the difference between strike and the current stock price. Economically, a higher S_t at the time of shouting increases the strike of the new born option and enlarges the in-the-money region of the option.*

As a result of the above discussion on the monotonicity of the claim we obtain the following

Corollary 2.4.2 (Shout put) *A risk-neutral buyer of a single shout floor option uses an optimal stopping rule for the prior $\hat{P} \in \mathcal{Q}$ given by the density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t = 2^t \prod_{v=1}^{\sigma \wedge t} \left(\underline{p} \cdot \varepsilon_v + (1 - \underline{p}) \cdot (1 - \varepsilon_v) \right) \prod_{v=\sigma+1}^{t} \left(\overline{p} \cdot \varepsilon_v + (1 - \overline{p}) \cdot (1 - \varepsilon_v) \right)$$

$\forall t \leq T$.

Summing up, we conclude that the value of the shout floor is given by

$$U_t^{\mathcal{Q}} = \begin{cases} \mathbb{E}^{\underline{P}} \left(\mathbb{E}^{\overline{P}} \left((S_\sigma - S_T)^+ / (1+r)^T | \mathcal{F}_\sigma \right) | \mathcal{F}_t \right), & \text{if } t < \sigma \\ \mathbb{E}^{\overline{P}} \left((S_\sigma - S_T)^+ / (1+r)^T | \mathcal{F}_t \right), & \text{else} \end{cases}.$$

The decision maker changes her beliefs about mean returns at the first expiry date. Before shouting and freezing the strike she presumes low returns of the stock that keeps the in-the-money region of the option small and decreases the value of the embedded put; after shouting she receives a put option and therefore changes her belief – being pessimistic, she now presumes that the risky asset will have high returns. This change of beliefs causes the difference in the values of the classical result and the multiple priors result.

To complete the analysis it remains to solve the optimal stopping problem for X under the worst-case measure. The classical solution for the continuous time setting was provided by Kwok, Dai, and Wu (2004). To our knowledge binomial tree analysis has not been conducted for shout options yet.

Lemma 2.4.3 *Denote by $\underline{\mu}$ the mean return under $\underline{P}$, i.e., $\underline{\mu} = \underline{p} \cdot u + (1-\underline{p}) \cdot d$ and by x^* the maximum of the function $g(\tau) \cdot \mu^{T-\tau}$ where $g(\tau)$ is defined as above, $\tau = T - t$. Then an optimal stopping time is given by*

$$\sigma^* = \inf\{t \geq 0 | g(\tau) \cdot \underline{\mu}^{T-\tau} = x^*\} \wedge T.$$

If the maximum x^* is unique, all $\sigma^* \leq t \leq T - 1$ are optimal.

PROOF: To prove the lemma we use the generalized parking technique introduced by Lerche and Beibel (1997). For all $t \leq T$, $\tau = T - t$ we have

$$\mathbb{E}^{\underline{P}}(S_t \cdot g(\tau)) = \mathbb{E}^{\underline{P}}\left(\frac{S_t}{\underline{\mu}^t} \cdot g(\tau)\underline{\mu}^t\right) \leq \mathbb{E}^{\underline{P}}\left(\frac{S_t}{\underline{\mu}^t} \cdot x^*\right)$$

and equality holds for the maximizer t^*. Now since $\frac{S_t}{\underline{\mu}^t}$ is a $\underline{P}$-martingale we get for all stopping times $\sigma < T$

$$\mathbb{E}^{\underline{P}}\left(\frac{S_\sigma}{\underline{\mu}^\sigma} \cdot x^*\right) = S_0 x^*$$

and therefore

$$U_0^{\mathcal{Q}} = \max_{\sigma < T} \mathbb{E}^{\underline{P}}(S_\sigma g(\tau)) = \mathbb{E}^{\underline{P}}(S_{t^*} \cdot g(T - t^*))$$

where t^* satisfies $g(T - t^*)\underline{\mu}^{t^*} = x^*$. $\qquad\square$

The optimal stopping rule is deterministic and does not depend on the level of the stock price S. This follows from the homogeneity of the payoff in S. However, the time of stopping depends highly on the model parameters $u, \overline{p}, \underline{p}$. We suspect that the function $g(\tau) \cdot \mu^{T-\tau}$ is quasi-concave and thus we have a unique maximum but we are not able to prove it. However, we can state a sufficient condition for immediate stopping.

Corollary 2.4.4 *In the above situation we have $\sigma^* = 0$ if $1 - \overline{p} \geq \underline{\mu}$.*

While in the classical CRR market the stopping time depends only on the one step mean return, in the multiple priors model the relation of $\underline{p}$ and $\overline{p}$ plays a crucial role.

2.5 Quasi-convex payoffs

In this section we consider options that consist of two monotone parts. Typical examples are options having U-shaped payoff including straddles, strangles or short option strategies. Investors buying such options are speculating on a change in the underlying's value without specifying the direction of it. Depending on the current price of the underlying, falling or rising stock increases the profit of the investor. This fact leading to different monotonicity types with respect to the underlying causes differing beliefs at different stock prices. One may think of getting various payoff functions conditioned on the current stock price. Thus, stock price uncertainty induces uncertainty about the payoff function.

To illustrate this idea let us consider a straddle: by exercising the straddle above the strike, the buyer gets a payment of $(S_t - K)$ which corresponds to a call. Else she obtains $(K - S_t)$ corresponding to a put option. Thus, depending on the current stock price the payoff function changes. As we will see, this uncertainty about the payoff cannot be resolved over time in general.

Remark 2.5.1 *Mathematically, payoffs described above correspond to quasi-convex/quasi-concave payoff functions. Due to our discrete time setting, we still deal with functions defined on the lattice $E_t, t \leq T$. So we have to be careful when using the term quasi-convex. Strictly speaking, the notion we use corresponds to discrete convexity studied intensively in the context of indivisible goods (see for example Murota (1998) for a general introduction). In the one dimensional setting, discrete convexity reduces to the following: A set $E \subset \mathbb{N}$ is convex if all points in E are contained in the convex hull of E. The definition of quasi-convex is then straight forward.*

We only consider quasi-convex payoffs. In our analysis we concentrate on options with piecewise linear, U-shaped payoffs paying off $f(t, S_t)$ when exercised at $t \leq T$ where f has the following form:

$$f(t, S_t) = c_1 \cdot (K_1 - S_t)^+ + c_2 \cdot (S_t - K_2)^+$$

for c_1, $c_2 \in \mathbb{R}$, $K_1 \leq K_2$. However, our results also apply to more general patterns such as quadratic or ladder functions. We show that the Snell envelope $U_t^{\mathcal{Q}}$ at time $t \leq T$ is a quasi-convex function in S_t if f only depends on current time and current stock price.

Lemma 2.5.2 *If the discounted payoff function $A(t, S_t)$ is quasi-convex in its second variable for every $t \leq T$, then the Snell envelope $U_t^{\mathcal{Q}}$ is given by a quasi-convex function $v(t, x)$, i.e., given $S_t = x_t$,*

$$U_t^{\mathcal{Q}} = v(t, x_t) = \max_{\tau \geq t} \min_{P \in \mathcal{Q}} \mathbb{E}^P(A(\tau, S_\tau)|S_t = x_t)$$

PROOF: We have to show that for every $t \leq T$ the value function $v(t, \cdot)$ only depends on the value of the stock at time t and the quasi-convexity of the payoff function carries over to the value function. We do it via backward induction.

Before applying backward induction we note that in the one-dimensional case a function $g : E_t \to \mathbb{R}, t \leq T$ is quasi-convex if and only if there exists $\hat{x} \in E_t$ such that $g(x) \geq g(\hat{x})$ holds for all $x \in E_t$ with $x \geq \hat{x}$.[19] If $\hat{x}$ belongs to the boundary of E_t the function g is monotone. If $\hat{x}$ belongs to the interior of E_t, g consists of two monotone parts and reaches its minimum at $\hat{x}$. In any case, in one dimension quasi-convexity reduces to the existence of a unique minimum.

For $t = T$ we clearly have for all possible values of $S_T = x_T$

$$U_T^{\mathcal{Q}} = A(T, x_T)$$

where $A(T, \cdot)$ is a quasi-convex function.

For $t + 1 < T$ we assume that for any value of $S_{t+1} = x_{t+1} \in E_{t+1}$ the value function $v(t + 1, \cdot)$ is a quasi-convex function depending only on the current value of the stock. Because of quasi-convexity there exists a unique minimum m_{t+1} and a unique

$$\hat{x}_{t+1} = \min\{x_{t+1} \in E_{t+1}|v(t + 1, x_{t+1}) = m_{t+1}\}.$$

The function $v(t + 1, \cdot)$ is decreasing on the set $\{x_{t+1} \leq \hat{x}_{t+1}\}$ and increasing on the set $\{x_{t+1} \geq \hat{x}_{t+1}\}$.

In $t < T$ we then have for any value $S_t = x_t$

$$
\begin{aligned}
U_t^{\mathcal{Q}} &= \max\{A(t, S_t), \min_{P \in \mathcal{Q}} \mathbb{E}^P\left(U_{t+1}^{\mathcal{Q}}|\mathcal{F}_t\right)\} \\
&= \max\{A(t, x_t), \min_{P \in \mathcal{Q}} \mathbb{E}^P\left(U_{t+1}^{\mathcal{Q}}|S_t = x_t\right)\} \\
&= \max\left\{A(t, x_t), \hat{p}_{t+1}v(t + 1, x_t \cdot u) + (1 - \hat{p}_{t+1})v(t + 1, x_t \cdot d)\right\} \\
&= v(t, x_t)
\end{aligned}
\tag{2.13}
$$

---

[19]By identifying E_t with a subset of $\mathbb{N}$ we are in the setting from above.

where $\hat{p}_{t+1} \in [\underline{p}, \overline{p}]$ is the marginal of the worst-case measure $\hat{P}$ at time t. Since $v(t+1, \cdot)$ is independent of the realized past, the minimizer $\hat{p}_{t+1}$ only depends on the value of x_t. This proves that the value function at time t, $v(t, \cdot)$ only depends on current value of the underlying.

To prove quasi-convexity we analyze the structure of the value in the continuation region in Equation (2.13). Consider the function

$$u(t, x_t) := \hat{p}_{t+1} v(t+1, x_t \cdot u) + (1 - \hat{p}_{t+1}) v(t+1, x_t \cdot d)$$

for different values of $S_t = x_t$.

On the set

$$E_t^d = \{x_t \in E_t | x_t \leq \hat{x}_{t+1} \cdot d\} \tag{2.14}$$

$x_t \cdot d < x_t \cdot u \leq \hat{x}_{t+1}$ and therefore using the induction hypothesis we conclude that the function $u(t+1, \cdot)$ is decreasing as a convex combination of two decreasing functions. Similarly, for all

$$E_t^i = \{x_t \in E_t | x_t \geq \hat{x}_{t+1} \cdot u\} \tag{2.15}$$

we have $\hat{x}_{t+1} \leq x_t \cdot d < x_t \cdot u$ and the function increases on the above set with the same argument.

Because of the binomial tree structure of the state space and the fact that

$$E_{t+1} = \{E_t \cdot u^k | k \in \{-1; 1\}\}$$

equations (2.14) and (2.15) partition the set of possible values of S_t. Thus E_t can be written as

$$\{x_t \in E_t | x_t \leq \hat{x}_{t+1} \cdot d\} \cup \{x_t \in E_t | x_t \geq \hat{x}_{t+1} \cdot u\}$$

which is a disjoint union. Because of monotonicity of $u(t, \cdot)$ on E_t^d and E_t^i, the minimum of $u(t, \cdot)$ is unique. This shows that the function $u(t, \cdot)$ is quasi-convex.

To complete the proof we recall that $A(t, x_t)$ is quasi-convex by assumption. Thus, the function defined by equation (2.13) is a quasi-convex function as maximum of two quasi-convex functions. The value function at time t depends only on the current stock price and given $S_t = x_t$ we can write $U_t^{\mathcal{Q}}$ as a function $v(t, x_t)$. $\qquad \square$

The quasi-convexity of the value function implies that for every $t \leq T$ we can separate the space E_t on which the value of the claim is monotone allowing to determine the worst-case measure. The decomposition point is

the minimizer of the value function $\hat{x}_t$ which has been constructed in the proof of Lemma 2.5.2.

Having analyzed the shape of the value function we can now compute the worst-case measure by the following argument. If asset prices are low, the value function is decreasing. Therefore, with the same argument as for American plain vanilla options, one can show that $\overline{P}$ is the worst-case measure. In the other region on the contrary, $\underline{P}$ is the worst-case measure. At a predefined level $\hat{x}_t$, the investor changes her beliefs and consequently, the stock's mean return. We have the following result.

Lemma 2.5.3 (Straddle) *The buyer of a straddle uses an optimal stopping rule for the measure $\hat{P} \in \mathcal{Q}$ with density $\hat{D} \in \mathcal{D}$,*

$$\hat{D}_t = 2^t \prod_{v \leq t, S_v \in E_v^i} \left(\underline{p} \cdot \varepsilon_v + (1 - \underline{p}) \cdot (1 - \varepsilon_v) \right) \prod_{v \leq t, S_v \in E_v^d} \left(\overline{p} \cdot \varepsilon_v + (1 - \overline{p}) \cdot (1 - \varepsilon_v) \right)$$

$\forall t \leq T$.

PROOF: We consider the value function on the continuation region where for a given $S_t = x_t$ we have $U_t^{\mathcal{Q}} = v(t, x_t)$ and

$$v(t, x_t) = \min_{p_{t+1} \in [\underline{p}, \overline{p}]} \left(p_{t+1} v(t + 1, x_t \cdot u) + (1 - p_{t+1}) v(t + 1, S_t \cdot d) \right)$$

As $v(t, \cdot)$ is decreasing on E_t^d, the worst-case measure on this set is given by $\overline{P}$. With the same argument the worst-case measure $\hat{P}$ is $\underline{P}$ on E_t^i, i.e.,

$$\hat{P}[\varepsilon_{t+1} = 1 | \mathcal{F}_t] = \begin{cases} \underline{p} & \text{on } \{x_t \geq \hat{x}_{t+1} \cdot u\} \\ \overline{p} & \text{on } \{x_t \leq \hat{x}_{t+1} \cdot d\} \end{cases} . \tag{2.16}$$

where $\hat{x}_{t+1}$ is the minimizer of $v(t + 1, \cdot)$. By pasting we obtain the result. $\square$

Under $\hat{P}$ the process (S_t) becomes mean-reverting in an appropriate sense pushing S_t down if it is high and up if it is low. This corresponds to the intuition: the ambiguity averse decision maker anticipates low mean returns in bull market phases and high mean returns when the stock value is low. Unlike previous cases, the uncertainty about the payoff function here cannot be resolved before T in general. The change of the measure occurs every time the stock price crosses the critical value $\hat{x}_t$ forcing the decision maker to change her beliefs about the stock's mean returns. But the threshold at time t depends on the value function of time $t + 1$.

2.6 Conclusion

The chapter studies the optimal exercise strategies of the buyer of various American options in a framework that allows for model uncertainty in discrete time. The imprecise information about the correct probability measure driving the stock price process in the market generates different models with varying conditional one-step-ahead probabilities used by the buyer. The buyer then is allowed to change the measure, and so the model she uses and to assign the value to the claim according to the worst possible model. While the solution for plain vanilla options is straightforward in the model the situation differs if the payoff of the option becomes more sophisticated. The effect of uncertainty differs over time leading to a dynamical structure of the worst-case measure.

This chapter analyzes different effects of uncertainty highlighting the structural difference between the standard models used in finance and the multiple priors models: the buyer of the option adapts her beliefs to the state of the world and the overall effect of model uncertainty.

A natural next step is to extend the theory to continuous market models and to analyze exotic options in that framework.

Chapter 3

American options with multiple priors in continuous time

3.1 Introduction

This chapter builds on Chapter 2, a previous analysis of optimal stopping problems for American exotic options under ambiguity. The motivations and the economic relevance of this study are the same as before, although we move from discrete to continuous time.

In finance it is more appropriate to use continuous time models. Closed-form solutions have the advantage of being easier to interpret, and as such, tend to predominate. They allow for comparative statics that would be otherwise difficult to interpret. In our analysis continuous time also provides a direct relationship to the famous Black-Scholes model, Black and Scholes (1973).

We analyze American options from the perspective of an ambiguity averse buyer in the sense of Ellsberg's paradox. The task of the buyer holding the option is to exercise it optimally realizing the highest possible utility. The valuation reflects the agent's personal utility as it depends on investment horizon, objective, and on risk, as well as ambiguity attitude. Generally this valuation is not related to the market value directly.

Given a classical stochastic model in continuous time such as the Black-Scholes model, one can solve the optimal stopping problem of the buyer using classical theory on optimal stopping, or the relation to free-boundary problems. Despite the abundance of literature on the issue, e.g. Peskir and Shiryaev (2006) or El Karoui (1979), these settings impose the assumption of

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a unique probability measure that drives stock price processes. This assumption might be too strong in many cases since it requires perfect understanding of the market and complete agreement on one particular model. To incorporate uncertainty we drop this assumption. We consider a Black-Scholes-like market whose stock price $X = (X_t)$ evolves according to

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{3.1}$$

where $W = (W_t)$ represents standard Brownian motion under some reference measure P.[1] The various beliefs of the agent are reflected by a set of multiple priors (probability measures) $\mathcal{P}$. Thus she considers the dynamics in (3.1) under each prior Q of the set $\mathcal{P}$ which provides a family of models that come into question to evaluate the claims.

As to the ambiguity model, we use κ-ignorance, see Chen and Epstein (2002). It models uncertainty in the drift rate of the stock price. Under each prior, the stock price in (3.1) obtains an additional drift rate term varying within the interval $[-\kappa, \kappa]$, where κ measures the degree of ambiguity/uncertainty. As noted in Cheng and Riedel (2010), it is essential that the additional terms be allowed to be stochastic and time-varying as this guarantees dynamic consistency.[2]

Dynamic consistency allows the agent to adapt the model according to changing beliefs induced by occurring events. In this setting, the agent holding an American option who is uncertain about the correct drift of the underlying stock price faces the optimization problem

$$V_t := \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right). \tag{3.2}$$

To clarify, at the current time t, the agent aims to optimize her expected discounted payoff $H_\tau \gamma_{\tau-t}^{-1}$ in a worst-case scenario by exercising the claim prior to maturity.

In our analysis the optimization problem is solved by using the relationship to reflected backward stochastic differential equations (RBSDEs).[3] To obtain this relation, the generator of the (reflected) BSDE should be chosen as $f(t, y, z) = -ry - \kappa|z|$ where $-\kappa|z|$ describes the ambiguity aversion and $-ry$ the discounting. This was first established by Chen and Epstein (2002)

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[1]Later we change this point of view slightly, cf. page 50.

[2]See Cheng and Riedel (2010) and Delbaen (2002) for a discussion of the concept of dynamic consistency in dynamic models.

[3]Another approach is the characterization of the value function (V_t) by Cheng and Riedel (2010) as the smallest right-continuous g-supermartingale that dominates the payoff from exercising the claim.

who used the generator $f(z) = -\kappa|z|$ for a BSDE to derive a generalized stochastic differential utility. A similar BSDE framework is used in El Karoui and Quenez (1996) in the context of pricing and hedging under constraints.

BSDEs provide a powerful method for analyzing problems in mathematical finance, (El Karoui, Peng, and Quenez (1997) and Duffie and Epstein (1992)), or in stochastic control and differential games (Hamadene and Lepeltier (1995) & Pham (2009)). BSDEs, in conjunction with g-expectations, play an important role in the theory of dynamic risk measures, (Peng (1997)) and dynamic convex risk measures, respectively, (Delbaen, Peng, and Gianin (2010)). By means of "reflection", the solution is maintained above a given stochastic process, in our case, the payoff process of the respective American claim.

We analyze the problem in (3.2) for several American options exemplifying the effect of ambiguity. As described in Chapter 2 the effect of ambiguity depends highly on the payoff structure of the claim. If the payoff satisfies certain monotonicity behavior as is the case for the American call and put option, the situation resembles the classical one without the emergence of ambiguity. The agent's worst-case scenario is specified by the least favorable drift rate of the stock price process that affects the performance of the agent's option. This scenario is identified by the worst-case prior. In the above described monotone case, the worst-case prior leads to the lowest possible drift rate for the stock price process in case of a call, and the highest possible drift rate in the case of a put option.

For options with more complex payoffs, the worst-case prior generates a stochastic drift rate in (3.1) which is path-dependent and produces endogenous dynamics in the model. These are induced by the ambiguity averse agent and her reaction to the latest information by adjusting the model from time to time as necessary depending on her changing beliefs, or fears, respectively. As such, in the multiple priors setting, changing fears due to transpired events are taken into account when American claims are evaluated and early exercise strategies are determined.

This central difference to classical models is exemplified with the help of barrier options and shout options. In the latter case, the agent will change her beliefs directly after taking action, when she fixes the strike price. In the case of barrier options, here exemplified by means of an up-and-in put option, she adapts the model as a consequence of the trigger event when the underlying stock price reaches the barrier specified in the claim's contractual terms.

From decision theoretical point of view, our examples expose that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under subjective expected utility. For example, the holder of an

American up-and-in put will behave as two readily distinguishable expected utility maximizers.

The chapter is structured as follows. The following section introduces the ambiguity setup in continuous time and relates the resulting multiple priors framework to the financial market. Section 3.3 presents the decision problem of an ambiguity averse agent who holds an American option. It contains a short detour to reflected BSDEs and explains their relationship to the decision problem of the ambiguity averse agent. This section also provides the solution to the optimal stopping problem for American options featuring some monotone payoff structure (see Section 3.3.2). This section builds the base for the subsequent analysis in Section 3.4 concerning American claims with more complex payoffs such as up-and-in put options or shout options. Extensive proofs are given in the appendix, Section 3.5 concludes.

3.2 The setting

We introduce the ambiguity framework in continuous time. We focus on κ-*ignorance*, a particular ambiguity setting, as described by Chen and Epstein (2002) who introduced various ambiguity models. Throughout this chapter we consider an ambiguity framework for a fixed finite time horizon $T > 0$.

First, we depict the ambiguity model κ-ignorance as in Chen and Epstein (2002). Second we introduce the financial market within this ambiguity framework.

Remark 3.2.1 *Given an infinite time horizon, one faces additional technical difficulties according to the underlying filtration arising from Girsanov's theorem and a Brownian motion environment.*[4] *This leads to weaker assumptions on filtration. In particular, the usual conditions on filtration should be relaxed.*[5] *This sometimes causes technical problems since the theory of stochastic calculus and backward stochastic differential equations is usually developed under these conditions.*[6]

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[4]See Remark 3.2.4 as an illustration.

[5]Usually the filtration is assumed to satisfy the usual conditions. This means that the filtration is right-continuous and augmented, cf. Karatzas and Shreve (1991).

[6]The interested reader is referred to von Weizsäcker and Winkler (1990) who develop stochastic calculus in particular Itô calculus without assuming the usual conditions.

3.2.1 The ambiguity model κ-ignorance

Let $W = (W_t)$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the completed Borel σ-algebra on Ω. We denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by the process W and augmented with respect to P. We have $\mathcal{F}_T = \mathcal{F}$ and the filtration satisfies the usual conditions. P serves as a reference measure in the ambiguity model. As we shall see, under κ-ignorance all occurring probability measures $Q \in \mathcal{P}$ are equivalent. So, P has the role of fixing the events of measure zero. Hence, there will be no uncertainty about the events of measure zero.

Remark 3.2.2 *Throughout the analysis, unless stated otherwise, all equalities and inequalities will hold almost surely. The "almost-sure-statements" are to be understood with respect to the reference measure P. Due to the equivalence of all priors $Q \in \mathcal{P}$ the statements will also hold almost surely with respect to any prior $Q \in \mathcal{P}$. If we write $\mathbb{E}$ without any measure we will mean the expectation with respect to the reference measure P.*

Let us depict the construction of the ambiguity model κ-ignorance, Chen and Epstein (2002), Delbaen (2002). It relies heavily on Girsanov's theorem. We only focus on the one-dimensional case. The d-dimensional case works in a straightforward way.

First consider $\mathbb{R}$-valued measurable, $(\mathcal{F}_t)$-adapted, and square-integrable processes $\theta = (\theta_t)$ such that the process $z^\theta = (z_t^\theta)$ defined by

$$dz_t^\theta = -\theta_t z_t^\theta dW_t, \quad z_0^\theta = 1,$$

that is,

$$z_t^\theta = \exp\left\{ -\frac{1}{2} \int_0^t \theta_s^2 ds - \int_0^t \theta_s dW_s \right\} \quad \forall t \in [0, T] \tag{3.3}$$

is a P-martingale. Given $\kappa > 0$ we define the set of density generators Θ by

$$\Theta = \{\theta \mid \theta \text{ progressively measurable and } |\theta_t| \leq \kappa, \ t \in [0, T]\}.^7 \tag{3.4}$$

κ is called the degree of ambiguity (uncertainty). Obviously, for each $\theta \in \Theta$ the Novikov condition $\mathbb{E}\left(\exp\{\frac{1}{2} \int_0^T \theta_s^2 ds\} \right) < \infty$ is satisfied. Therefore,

---

[7] Since we work in a Brownian motion environment we do not need to require predictability in (3.4) as in Delbaen (2002), cf. Theorem 6.3.1 in von Weizsäcker and Winkler (1990).

$\mathbb{E}(z_T^\theta) = z_0^\theta = 1$ and z_T^θ is a P-density on $\mathcal{F}$, Karatzas and Shreve (1991). Consequently, each $\theta \in \Theta$ induces a probability measure Q^θ on $(\Omega, \mathcal{F})$ that is equivalent to P where Q^θ is defined by

$$Q^\theta(A) := \mathbb{E}(\mathbb{1}_A z_T^\theta) \quad \forall A \in \mathcal{F}. \tag{3.5}$$

In other words,

$$\left. \frac{dQ^\theta}{dP} \right|_{\mathcal{F}_t} = z_t^\theta \quad \forall t \in [0, T].$$

According to Girsanov's theorem (cf. Karatzas and Shreve (1991)) we define the set of probability measures $\mathcal{P} := \mathcal{P}^\Theta$ on $(\Omega, \mathcal{F})$ generated by Θ by

$$\mathcal{P}^\Theta := \{Q^\theta \mid \theta \in \Theta \text{ and } Q^\theta \text{ is defined by (3.5)}\}. \tag{3.6}$$

Note that we allow for stochastic and time-varying Girsanov kernels θ. This is important to ensure the dynamic consistency. We otherwise lose this important property.[8]

Additionally, by Girsanov's theorem, the process $W^\theta = (W_t^\theta)$ defined by

$$W_t^\theta := W_t + \int_0^t \theta_s ds \quad \forall t \in [0, T] \tag{3.7}$$

is a standard Brownian motion on $(\Omega, \mathcal{F})$ with respect to the measure Q^θ.

Remark 3.2.3 *κ-ignorance as an ambiguity model has important properties. It allows for explicit results when evaluating financial claims since the range of values of the density processes θ does not change over time as is the case for other models like IID-ambiguity in Chen and Epstein (2002). Consequently we shall see that the worst-case densities become very simple in some examples, meaning without any formal difficulties. Furthermore, under κ-ignorance, the set of priors $\mathcal{P}$ possesses important properties like m-stability or time-consistency, Delbaen (2002), and the existence of worst-case priors, Chen and Epstein (2002).[9]*

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[8]See Chen and Epstein (2002) for details. Also the examples in Section 3.4 illustrate this fact.

[9]See also Chapter 2.

Regarding Remark 3.2.1 the following remark illustrates the importance of relaxing the usual conditions for filtration when κ-ignorance is constructed on an infinite time horizon.

Remark 3.2.4 *(cf. Karatzas and Shreve (1991)) Let P be Wiener measure on $(\Omega, \mathcal{F}) := (\mathcal{C}([0, \infty), \mathbb{R}), \mathcal{B}(\mathcal{C}([0, \infty), \mathbb{R})))$ such that the canonical process $W = (W_t), W_t(\omega) := \omega(t), 0 \le t < \infty, \omega \in \Omega$ is a standard Brownian motion. Denote by $(\mathcal{F}_t^W)$ the (not augmented) filtration generated by W such that $\mathcal{F}_\infty^W = \mathcal{F}$. Let $\theta = (\theta_t)$ be a progressively measurable process with corresponding filtration $(\mathcal{F}_t^W)$, and square-integrable for each $T \in [0, \infty)$. Assume that the process $z^\theta = (z_t^\theta)$ defined as in (3.3) is a P-martingale. Then Girsanov's theorem for an infinite time horizon[10] states that there exists a probability measure Q^θ satisfying*

$$Q^\theta(A) = \mathbb{E}(z_T^\theta 1_A), \ A \in \mathcal{F}_T^W, \ T \in [0, \infty) \tag{3.8}$$

and the process $W^\theta = (W_t^\theta)$ defined as in Equation (3.7) with corresponding filtration $(\mathcal{F}_t^W)$ is a Brownian motion on $(\Omega, \mathcal{F}, Q^\theta)$.

It is essential that $(\mathcal{F}_t^W)$ be raw, unaugmented filtration. Therefore, κ-ignorance can only be constructed with respect to a filtration that does not fulfill the usual conditions.

The difference to the finite time horizon is that now P and Q^θ are only mutually locally absolutely continuous, i.e., equivalent on each $\mathcal{F}_T^W, T \in [0, \infty)$. Viewed as probability measures on $\mathcal{F}$, P and Q^θ are equivalent if and only if z^θ is uniformly integrable. To understand why (3.8) is only required to hold for $A \in \mathcal{F}_T^W, T \in [0, \infty)$, consider the following example.

Example 3.2.5 *Let $\mu > 0$ and fix a process θ with $\theta_t := -\mu \ \forall t \in [0, \infty)$. For this θ consider the P-martingale z^θ defined by*

$$z_t^\theta = \exp\{-\frac{1}{2}\mu^2 t + \mu W_t\} \quad \forall t \in [0, \infty).$$

z^θ is not uniformly integrable. By Girsanov's theorem and the law of large numbers for Brownian motion, Karatzas and Shreve (1991) we obtain for

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[10]See Corollary 5.2 in Karatzas and Shreve (1991).

$A := \{\lim_{t \to \infty} \frac{W_t}{t} = \mu\} \in \mathcal{F}$

$$Q^\theta(A) = 1 \quad and \quad P(A) = 0.$$

Clearly, the P-null event A is in the augmented σ-field $\mathcal{F}_T$ for every $T \in [0, \infty)$. This is the reason why (3.8) is only required to hold for all $A \in \mathcal{F}_T^W, T < \infty$. Otherwise P and Q^θ were mutually singular on $\mathcal{F}_T$ for every $T \geq 0$.

Therefore, κ-ignorance in a Brownian motion environment with infinite time horizon must be set up on a filtration that is not augmented by the P-null sets of $\mathcal{F}$.

3.2.2 The financial market under κ-ignorance

Throughout this chapter we consider a Black-Scholes-like market consisting of two assets, a riskless bond γ and a risky stock X. Their prices evolve according to

$$
\begin{aligned}
d\gamma_t &= r\gamma_t dt, \quad \gamma_0 = 1, \\
dX_t &= \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x > 0
\end{aligned}
\tag{3.9}
$$

where r is a constant interest rate, μ a constant drift rate, and $\sigma > 0$ a constant volatility rate for the stock price.[11] The dynamics in (3.9) are obviously free of ambiguity. To incorporate ambiguity, the decision maker considers Equation (3.9) under multiple priors. She uses the set of priors $\mathcal{P}$ as defined in (3.6). As we shall see, by utilizing the set $\mathcal{P}$ she tries to capture her uncertainty about the true drift rate of the stock.

Let $Q \in \mathcal{P}$, if Q is equal to Q^θ for $\theta \in \Theta$ then the stock price dynamics under Q become

$$dX_t = \mu X_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta.$$

This illustrates that κ-ignorance just models uncertainty about the true drift rate of the stock price.

At this point it is worthwhile mentioning that by changing the prior under consideration, the stock price's volatility rate remains completely unchanged.

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[11]As it is often possible we may also consider a price process with non-constant and stochastic coefficients. To avoid later distinctions of cases and missing the point we assume constant coefficients.

Based on the equivalence of all priors and Girsanov's theorem, κ-ignorance cannot be used to model volatility uncertainty. This requires a set of mutually singular priors. For a detailed study of this issue see Peng (2007) or our Chapter 4.

In the next section, we consider American contingent claims from the perspective of an ambiguity averse decision maker who holds a long position in the claims. The decision maker, a private investor or financial institution, for example, may seek to evaluate or liquidate their position. Both may happen with respect to their subjective probability distribution. They may use their subjective probability distribution to evaluate the claim and to figure out an optimal exercise strategy due to the claim's American feature. In addition, in real option investment decisions, the subjective probability law appears naturally when coming to a decision.[12]

All decision problems are considered under Knightian uncertainty. We focus on a decision maker who is uncertain about market data. As a consequence she does not believe completely in the dynamics proposed in (3.9). For instance she is uncertain about the stock's drift rate which in turn affects the market price of risk.

Contingent claims in finance are typically evaluated with respect to risk-neutral probability measures. Therefore, we assume that the agent will consider the stock's dynamics in (3.9) under the risk-neutral probability measure. Since she does not completely trust in the market, nor all the data, she allows for various market prices of risk.[13] She takes into account prices surfacing around $\frac{\mu-r}{\sigma}$ currently observed at the market. Expanding on this idea, if $Q = Q^\theta$ for some θ defined by $\theta_t = \frac{\mu-r}{\sigma} + \psi_t, \forall t \in [0, T]$, with $\psi = (\psi_t) \in \Theta$ then the dynamics in (3.9) become

$$dX_t = \mu X_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta = r X_t dt - \sigma X_t \psi_t dt + \sigma X_t dW_t^\theta.$$

To stay in the framework of κ-ignorance, as introduced above, we need to change the reference measure. To avoid this step, we prefer to model the stock price dynamics directly under the risk-neutral probability measure, i.e., the agent starts with the reference dynamics

$$dX_t = r X_t dt + \sigma X_t dW_t. \tag{3.10}$$

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[12]See McDonald and Siegel (1986), for example.

[13]As mentioned above the subjective evaluation appears natural. By the variety of considered models subjective beliefs are nevertheless contained. If one prefers the subjective in place of the risk-neutral probability measure as a reference one may also use the model in (3.9) with drift rate μ as a reference.

Now, if she considers (3.10) under $Q = Q^\theta$ for some $\theta \in \Theta$ the dynamics become

$$dX_t = rX_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^\theta. \tag{3.11}$$

Throughout the chapter, Equation (3.11) for varying $\theta \in \Theta$ represents the dynamics our decision maker will take into account when studying optimal stopping problems under the ambiguity aversion modeled by κ-ignorance.

3.3 American options under ambiguity aversion

We focus on American contingent claims under ambiguity aversion.[14] For this issue, we analyze optimal stopping problems under multiple priors. Formally, the optimal stopping problem under ambiguity aversion is defined as

$$V_t := \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right), \quad t \in [0, T] \tag{3.12}$$

where $\gamma_{\tau-t}^{-1}$ is the discounting from current time t up to stopping time τ when the claim is exercised. $H = (H_t)$ represents the payoff process.

We only consider claims with maturity T. The "ess inf" accords with ambiguity aversion which leads to worst-case pricing. The "ess sup" imposes the goal of the agent to optimize the claim's payoff by finding an optimal exercise strategy in the worst-case scenario. All stopping times τ that will come into question in (3.12) are naturally bounded by the time horizon and claim's maturity T. Without ambiguity, V_t represents the unique price for the claim at time t, see Peskir and Shiryaev (2006) for example.

We analyze American options written on X. In general, the claim's payoff from exercising depends on the whole history of the price process. To ensure that the value $V_t, t \in [0, T]$ is well-defined, we impose the following assumption on the claim's payoff process.

Assumption 3.3.1 *Given an American contingent claim H, the payoff from exercising $H = (H_t)$ is an adapted, measurable, nonnegative process with continuous sample paths[15] satisfying $\mathbb{E}\left(\sup_{0 \leq t \leq T} H_t^2\right) < \infty$.*

To solve the optimal stopping problem under multiple priors in (3.12) we utilize the methodology of reflected backward stochastic differential equations (RBSDEs).

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[14]A detailed economic motivation was given in Chapter 2.

[15]It is possible to relax the assumption, see Cheng and Riedel (2010).

3.3.1 A detour: reflected backward stochastic differential equations

At this point we briefly introduce the notion of RBSDEs and point out its relationship to the optimal stopping problem under ambiguity aversion. The proof can be found in Appendix A.2. The Markovian framework contains a very useful connection to partial differential equations (PDEs), a generalization of the Feynman-Kac formula. As a reference for the particular case of backward stochastic differential equations (BSDEs) see El Karoui, Peng, and Quenez (1997). In Section 3.3.2 we employ the results of Chen, Kulperger, and Wei (2005) which strongly exploit the relationship to PDEs.

In this detour we use the same stochastic foundation introduced above. The introduction is taken from El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997).[16] We also introduce the following notation, cf. Pham (2009):

$$\mathbb{L}^2 := \{\xi \mid \xi \text{ is an } \mathcal{F}\text{-measurable random variable with } \mathbb{E}(|\xi|^2) < \infty\},$$

$$\mathbb{H}^2 := \left\{ (\varphi_t) | (\varphi_t) \text{ is a progressively mb. process s.t. } \mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty \right\},$$

$$\mathcal{S}^2 := \left\{ (\varphi_t) | (\varphi_t) \text{ is a progressively mb. process s.t. } \mathbb{E} \left(\sup_{0 \leq t \leq T} |\varphi_t|^2 \right) < \infty \right\}.$$

Given a progressively measurable process $S = (S_t)$, interpreted as an obstacle, the aim is to control a process $Y = (Y_t)$ such that it remains above the obstacle and satisfies equality at terminal time, i.e., $Y_T = S_T$. This is achieved by a RBSDE. We briefly state the definition.

Let $S = (S_t)$ be a real-valued process in $\mathcal{S}^2$, and a generator $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, y, z) \in \mathbb{H}^2 \ \forall (y,z) \in \mathbb{R} \times \mathbb{R}$, and

$$|f(t,y,z) - f(t,y',z')| \leq C(|y - y'| + |z - z'|) \quad \forall t \in [0,T]$$

for some constant $C > 0$ and all $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}$.

Definition 3.3.2 *The solution of the RBSDE with parameters (f, S) is a triple $(Y, Z, K) = (Y_t, Z_t, K_t)$ of $(\mathcal{F}_t)$-progressively measurable processes taking values in $\mathbb{R}, \mathbb{R},$ and $\mathbb{R}_+$, respectively, and satisfying:*

(i) $Y_t = S_T + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0,T]$

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[16]The framework is based on predictable processes. But the arguments rely only on progressive measurability, cf. Pham (2009). Therefore we require the measurability conditions as in Pham (2009).

(ii) $Y_t \geq S_t, \quad t \in [0,T]$

(iii) $K = (K_t)$ *is continuous, increasing,* $K_0 = 0$, *and* $\int_0^T (Y_t - S_t) dK_t = 0$

(iv) $Z = (Z_t) \in \mathbb{H}^2, Y = (Y_t) \in \mathcal{S}^2$, *and* $K_T \in \mathbb{L}^2$

The dynamics in (i) are often expressed in differential form. That is

$$-dY_t = f(t, Y_t, Z_t)dt + dK_t - Z_t dW_t, \quad Y_T = S_T. \qquad (3.13)$$

Intuitively, the process K "pushes Y upwards" such that the constraint (ii) is satisfied, but minimally in the sense of condition (iii). From (i) and (iii) it follows that (Y_t) is continuous. El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997) proved the existence and uniqueness of a solution to the RBSDE as defined here.

Let us consider equation (3.12) for a fixed probability measure Q omitting the operator "ess inf". If $Q = Q^\theta \in \mathcal{P}$ then the process Y^θ defined as the unique solution of the reflected BSDE with obstacle $S = H$[17]

$$Y_t^\theta = H_T + \int_t^T (-rY_s^\theta - \theta_s Z_s^\theta)dt + K_T^\theta - K_t^\theta - \int_t^T Z_s^\theta dW_s, \quad t \in [0,T]$$

also solves Equation (3.12) without ambiguity under the single prior $Q = Q^\theta$. Hence $Y_t^\theta = V_t^Q$ with

$$V_t^Q := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right), \quad t \in [0,T].$$

This follows by Proposition 7.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997) together with Girsanov's theorem. It illustrates that for each $\theta \in \Theta$ the decision maker faces a RBSDE induced by the parameters (f^θ, H) with $f^\theta(t, y, z) = -ry - \theta_t z \ \forall t \in [0,T]$.

The following theorem establishes the link to the optimal stopping problem defined in (3.12). It presents the key to solving the optimal stopping problem under ambiguity aversion.

Theorem 3.3.3 (Duality) *Given a payoff process H, define $f^\theta(t, y, z) :=$ $-ry - \theta_t z$ for each $t \in [0,T]$ and consider the unique solution $(Y_t^\theta, Z_t^\theta, K_t^\theta)$ to the RBSDE associated with (f^θ, H) for each $\theta \in \Theta$.*

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[17]Since we assumed $H = (H_t)$ to be adapted, measurable, and continuous it is progressively measurable, cf. Proposition 1.13 in Karatzas and Shreve (1991).

Let (Y_t, Z_t, K_t) denote the solution of the RBSDE with parameters (f, H) where $f(t, y, z) := \operatorname{ess\,inf}_{\theta \in \Theta} f^\theta(t, y, z) \; \forall t \in [0, T], \forall \; y, z \in \mathbb{R}$. Then there exists $\theta^\star \in \Theta$ such that

$$f(t, Y_t, Z_t) := \operatorname*{ess\,inf}_{\theta \in \Theta} f^\theta(t, Y_t, Z_t) = f^{\theta^\star}(t, Y_t, Z_t)$$

$$= -rY_t - \max_{\theta \in \Theta} \theta_t Z_t = -rY_t - \kappa|Z_t| \quad dt \otimes P \; a.e.$$

Hence,

$$(Y_t, Z_t, K_t) = (Y_t^{\theta^\star}, Z_t^{\theta^\star}, K_t^{\theta^\star}) \quad \forall t \in [0, T] \; a.s. \; and$$

$$Y_t = \operatorname*{ess\,inf}_{\theta \in \Theta} Y_t^\theta = \operatorname*{ess\,inf}_{Q \in \mathcal{P}} V_t^Q \quad \forall t \in [0, T] \; a.s.$$

Furthermore,

$$Y_t = \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t) = \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t) = V_t \quad a.s.$$

Hence, Y also solves the optimal stopping problem of the ambiguity averse decision maker in (3.12). In particular we have

$$\max_{\tau \geq 0} \min_{Q \in \mathcal{P}} \mathbb{E}^Q(H_\tau \gamma_\tau^{-1}) = \min_{Q \in \mathcal{P}} \max_{\tau \geq 0} \mathbb{E}^Q(H_\tau \gamma_\tau^{-1}).$$

An optimal stopping rule is given by

$$\tau_t^\star := \inf\{s \geq t | V_s = H_s\} \quad \forall t \in [0, T].$$

The subscript t indicates that $\tau_t^\star$ is an optimal stopping time when we begin at time t.

PROOF: The proof is mostly given in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997), Theorem 7.2. Since it is not directly related to multiple priors under κ-ignorance, we present the main ideas in Appendix A.2. □

Remark 3.3.4 *The infimum above is an infimum of random variables. Therefore it must be seen as an essential infimum. For time zero there is no ambiguity in the definitions since the σ-algebra $\mathcal{F}_0$ is trivial.*

By interpreting the theorem, the ambiguity averse agent solves the optimal stopping problem under a worst-case prior $Q^\star := Q^{\theta^\star} \in \mathcal{P}$. That is, she first determines the worst-case scenario and then solves a classical optimal stopping problem with respect to this scenario.

The theorem states the relevance of RBSDEs for solving the optimal stopping problem under ambiguity aversion. As indicated in Theorem 3.3.3, from this point on, the payoff process of the claim H will represent the obstacle for the associated RBSDEs. We are interested in the solution of the RBSDE associated with the parameters (f, H). In particular, we target understanding the process $\theta^\star$ that induces the worst-case measure.

3.3.2 Options with monotone payoffs

We focus on American claims whose current payoff can be expressed by a function only depending on the current stock price of the claim's underlying. We assume $H_t = \Phi_t(X_t)$ for each $t \in [0, T]$.[18] In this case the RBSDE with parameters (f, H) becomes a reflected forward backward stochastic differential equation (RFBSDE), cf. El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997). The solution for (3.12) is given by the process Y determined as the solution for

$$\begin{cases} dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x \\ -dY_t = \underbrace{\min_{\theta \in \Theta}(-rY_t - \theta_t Z_t)}_{=-rY_t - \kappa |Z_t| = f(t, Y_t, Z_t)} dt + dK_t - Z_t dW_t, \quad Y_T = \Phi_T(X_T) \end{cases} \tag{3.14}$$

with obstacle $H_t = \Phi_t(X_t) \ \forall t \in [0, T]$.

From this point forward, the mapping $(t, x) \mapsto \Phi_t(x)$ is assumed to be jointly continuous for all $(t, x) \in [0, T] \times \mathbb{R}_+$, and $\Phi_t(X_t) \in L^2(\Omega, \mathcal{F}_t, P) \ \forall t \in [0, T]$. The latter is for instance true if each Φ_t is of polynomial growth (see for example Malliavin (1997), p. 6).

Remark 3.3.5 *If the payoff is zero for each $t \in [0, T)$, i.e., the obstacle only consists of the terminal condition $Y_T = \Phi(X_T)$ the process K is set equal to zero and (3.14) just becomes a forward BSDE without reflection. In this case, the solution Y of (3.14) solves the "optimal stopping problem" under ambiguity aversion for a European contingent claim.*

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[18]Since it is assumed that $H = (H_t)$ has continuous sample paths the mapping $(t, x) \mapsto \Phi_t(x)$ has to be jointly continuous for all $(t, x) \in [0, T] \times \mathbb{R}_+$.

In order to solve the optimal stopping problem in (3.12) we focus on the RFBSDE in (3.14). The characteristic of this setting is that the generator and the obstacle are deterministic. The only randomness of the parameters (f, H) comes from the state of the forward SDE X, a Markov process. We will make use of this observation in the next results. First we derive a result which characterizes the process Z of the solution to (3.14).

Lemma 3.3.6 *Consider the RFBSDE in (3.14) with obstacle* $H_t = \Phi_t(X_t)$ $\forall t \in [0, T]$. *Let* (Y_t, Z_t, K_t) *be the unique solution.*

(i) *If* Φ_t *is increasing for all* $t \in [0, T]$, *we have*

$$Z \geq 0 \quad dt \otimes P \ a.e.$$

(ii) *If* Φ_t *is decreasing for all* $t \in [0, T]$, *we have*

$$Z \leq 0 \quad dt \otimes P \ a.e.$$

PROOF: We only prove (i); (ii) follows analogously.

Without the obstacle requirement in (3.14), and just the terminal condition $Y_T = \Phi_T(X_T)$, it follows from a result in Chen, Kulperger, and Wei (2005)[19] that $Z \geq 0$ $dt \otimes P$ *a.e.* To achieve the passage to reflected BSDEs we employ a penalization method.[20]

Let $n \in \mathbb{N}$, and $(Y_t^{(n)}, Z_t^{(n)})$ be the unique solution of the penalized BSDE with dynamics

$$Y_t^{(n)} = \Phi_T(X_T) + \int_t^T \underbrace{[f(s, Y_s^{(n)}, Z_s^{(n)}) + n(Y_s^{(n)} - \Phi_s(X_s))^-]}_{=: \tilde{f}(s, X_s, Y_s, Z_s)} ds - \int_t^T Z_s^{(n)} dW_s,$$

$t \in [0, T], (x)^- := \max\{-x, 0\}$, and $f(t, y, z) = -ry - \kappa|z|$ as above.

$\tilde{f}$ satisfies the assumptions of a generator for a BSDE as stated in the detour for (reflected) BSDEs.[21] In Chen, Kulperger, and Wei (2005) the

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[19]See Theorem 2 in Chen, Kulperger, and Wei (2005). It is proved by a generalization of the Feynman-Kac formula for BSDEs in connection with the comparison theorem for BSDEs, cf. Peng (1997).

[20]Approximation via penalization is a standard method to transfer results on BSDEs to RBSDEs, see El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997).

[21]The additional dependence on $X.$ in terms of the function $\Phi.$ does not exhibit any further difficulty here, cf. El Karoui, Peng, and Quenez (1997).

generator of the BSDE considered does not depend on $X..$ Fortunately, the map $x \mapsto \tilde{f}(t, x, y, z)$ is increasing for all $t \in [0, T]$, $y, z \in \mathbb{R}$ if and only if $x \mapsto \Phi_t(x)$ is increasing for all $t \in [0, T]$. Thus, a larger x leads to larger generator $\tilde{f}$ and larger terminal payoff. This monotonicity behavior is compatible with the application of the comparison theorem for BSDEs which is necessary to derive the result in Chen, Kulperger, and Wei (2005). Thus, the result in Chen, Kulperger, and Wei (2005) can also be derived for this penalized BSDE. Hence,

$$Z^{(n)} \geq 0 \quad dt \otimes P \ a.e.$$

Now we let n go to infinity. Then $Z^{(n)}$ converges to Z in $L^2(dt \otimes P)$, cf. Section 6 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997). By standard subsequence argument we also obtain $Z \geq 0 \ dt \otimes P \ a.e.$ $\qquad \square$

Using the lemma we can prove the following theorem.

Theorem 3.3.7 (Claims with monotone payoffs) *Consider an American claim H with payoff at current time t given by $H_t = \Phi_t(X_t) \ \forall t \in [0, T]$. The value of the optimal stopping problem under ambiguity aversion in (3.12) is given by*

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau) \, \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right), \quad t \in [0, T].$$

The worst-case prior $Q^\star$ can be specified by its Girsanov density $z_T^{\theta^\star}$.

(i) If Φ_t is increasing for all $t \in [0, T]$, we have $Q^\star = Q^\kappa$, $z_T^{\theta^\star} = z_T^\kappa$ with

$$z_T^\kappa = \exp\{-\frac{1}{2}\kappa^2 T - \kappa W_T\}.$$

(ii) If Φ_t is decreasing for all $t \in [0, T]$, we have $Q^\star = Q^{-\kappa}$, $z_T^{\theta^\star} = z_T^{-\kappa}$ with

$$z_T^{-\kappa} = \exp\{-\frac{1}{2}\kappa^2 T + \kappa W_T\}.$$

In both cases, an optimal stopping time is given by

$$\tau_t^\star := \inf\{s \in [t, T] | V_s = \Phi_s(X_s)\}.$$

PROOF: Let (Y_t, Z_t, K_t) be the unique solution of (3.14). For $t \in [0, T]$ we have $V_t = Y_t = Y_t^{\theta^\star} = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau) \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right)$ by duality, see Theorem 3.3.3. This also verifies the statement about an optimal stopping time.

In case (i), by Lemma 3.3.6 we know that $Z \geq 0$ $dt \otimes P$ a.e. Hence,

$$f(t, Y_t, Z_t) = -rY_t - \kappa Z_t \quad dt \otimes P \text{ a.e.}$$

which implies

$$f(t, Y_t, Z_t) = f^{\theta^\star}(t, Y_t, Z_t) \quad dt \otimes P \text{ a.e.}$$

for $\theta^\star = (\kappa) \in \Theta$. So, the worst-case prior is given by $Q^\star = Q^\kappa$ where Q^κ is identified by its Girsanov density

$$z_T^\kappa = \exp\{-\frac{1}{2}\kappa^2 T - \kappa W_T\}.$$

In case (ii), $f(t, Y_t, Z_t) = -rY_t + \kappa Z_t$ $dt \otimes P$ a.e. Therefore we identify $Q^\star = Q^{-\kappa}$ as the worst-case prior. □

The preceding theorem's proof relies heavily on the close relationship between optimal stopping problems and RBSDEs, the comparison theorem for (reflected) BSDEs, and the Markovian framework which is essential for Lemma 3.3.6. In discrete time, the corresponding theorem has been proven by a generalized backward induction and first-order stochastic dominance, Riedel (2009). As a direct application, we quickly collect the conclusions for the American call and put option.

Corollary 3.3.8 (American call) *Given $L > 0$, let the payoff from exercising the claim be $H_t := (X_t - L)^+$ for all $t \in [0, T]$. Then Q^κ is the worst-case measure. Thus, a risk-neutral buyer of an American call option determines an optimal stopping rule under the prior Q^κ.*

Corollary 3.3.9 (American put) *Given $L > 0$, let $H_t := (L - X_t)^+$ for all $t \in [0, T]$. Then $Q^{-\kappa}$ is the worst-case measure and a risk-neutral buyer of an American put option utilizes an optimal stopping rule for the prior $Q^{-\kappa}$.*

The interpretation of these results is as follows. Exactly as in the corresponding discrete time setting, the ambiguity averse buyer uses for her valuation of a call option for example the prior under which the underlying stock price

possesses the lowest possible drift rate among all priors of the set. That is, under the worst-case prior Q^κ, the stock evolves according to the dynamics of

$$dX_t = (r - \sigma\kappa)X_t dt + \sigma X_t dW_t^\kappa.$$

In the case of an American put option she assumes the highest possible drift rate corresponding to the following stochastic evolution of the stock with respect to $Q^{-\kappa}$

$$dX_t = (r + \sigma\kappa)X_t dt + \sigma X_t dW_t^{-\kappa}.$$

Since X is a Markov process, we write $X_s^{t,x}, s \geq t$ to indicate the price of the stock at time s under the presumption that it is equal to x at time t, i.e., $X_t^{t,x} = x$. As discussed above, by the Markovian structure of (3.14) and X as the only randomness, we also write $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s \in [t,T]}$ for the solution of (3.14) to indicate the Markovian framework. That is, the solution Y can be written as a function of time and state X., (see Section 4 in El Karoui, Peng, and Quenez (1997) or Section 8 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997)).

Using the Markovian structure, the value function $V_t, t \in [0, T]$ in Theorem 3.3.7 simplifies to a function depending solely on the present time and present stock price. That is, under the assumption of $X_t = x$ at time t the value of the optimal stopping problem under ambiguity aversion in (3.12) reduces to

$$\begin{aligned} V_t = Y_t^{t,x} &= \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left(\Phi_\tau(X_\tau) \gamma_{\tau-t}^{-1} | X_t = x \right) \\ &= \sup_{\tau \geq t} \mathbb{E}^{Q^\star} \left(\Phi_\tau(X_\tau^{t,x}) \gamma_{\tau-t}^{-1} \right) =: u(t, x). \end{aligned}$$

Remark 3.3.10 *The value in (3.12) is strictly a function in the above setting, i.e. u of the present time t and the present stock price X_t. Note that we did not assume this to determine the worst-case prior. In particular we did not assume that the value function $u(t, x)$ is differential with respect to x and increasing in x, decreasing, respectively, an assumption often made. The proofs of Lemma 3.3.6 and Theorem 3.3.7 do not require these assumptions, see also Chen, Kulperger, and Wei (2005).*

Besides, the monotonicity of $x \mapsto u(t, x)$ follows directly by comparison theorem. In case (i) of Theorem 3.3.7 for instance, the mapping $x \mapsto$

$\Phi_s(X_s^{t,x})$ *increases because* $x \mapsto X_s^{t,x}$ *increases*[22] *for each* $s \in [t,T]$. *Then, by comparison theorem for RBSDEs, we obtain that* $u(t,x)$ *is monotone increasing in* x.

The usual characterization of Markovian processes yields the following result concerning the remaining maturity of an American put option. The option's American style as well as the fact that the payoff from exercising is just a function depending on the current stock price is essential for this result.

Lemma 3.3.11 *Consider an American put option with strike price* L. *Given* $t \in [0,T]$, *the solution of the optimal stopping problem under ambiguity aversion at time* t V_t *decreases in* t.

PROOF: Let $(t,x) \in [0,T] \times \mathbb{R}_+$ and $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ be the unique solution of the RFBSDE in (3.14) with obstacle $H_s = (L - X_s^{t,x})^+$ $\forall s \in [t,T]$. The Markov property of X and Y, Corollary 3.3.9, and Theorem 3.3.3 yield

$$Y_t^{t,x} = \sup_{0 \leq \tau \leq T-t} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{0,x})^+ \gamma_\tau^{-1} \right).$$

Now let $\varepsilon > 0$ with $t + \varepsilon \leq T$. Again,

$$Y_{t+\varepsilon}^{t+\varepsilon,x} = \sup_{0 \leq \tau \leq T-t-\varepsilon} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{0,x})^+ \gamma_\tau^{-1} \right).$$

Hence, $Y_{t+\varepsilon}^{t+\varepsilon,x} \leq Y_t^{t,x}$ and the claim follows by duality, cf. Theorem 3.3.3. $\square$

For later use let us denote for $t \in [0,T]$ the value in (3.12) for an American put option with strike price L under the assumption of $X_t = x$ by

$$\underline{Y}_t^{t,x} = \sup_{\tau \geq t} \mathbb{E}^{Q^{-\kappa}} \left((L - X_\tau^{t,x})^+ \gamma_{\tau-t}^{-1} \right). \tag{3.15}$$

3.4 Exotic options

In this section we leave the world of Markovian claims with monotone payoffs in the current stock price. We move on to consider the problem in (3.12) for exotic American claims. With the help of two particular examples, we analyze the effect of ambiguity aversion on the optimal stopping behavior in

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[22]See the comparison result for forward SDEs in for example Karatzas and Shreve (1991).

this more involved situation. Examples are a shout option and an American barrier option in terms of an up-and-in put.

Similar to the discrete time setting in Chapter 2, the analysis of these examples demonstrates one of the main differences to the classical situation without ambiguity. Even though multiple priors lead to a more complex evaluation, the approach is more appropriate in the sense of investment evaluation for accounting and risk measurement.

We will see that dynamical model adjustments occur. With these adjustments the agent takes into account changing beliefs based on realized events within the evaluation period. As such, the multiple priors setting induces particular endogenous dynamics. The agent evaluates her stopping behavior under the worst-case scenario, the worst-case prior. This prior will depend crucially on the payoff process as well as on events occurring during the lifetime of the claim under consideration.

3.4.1 American up-and-in put option

An American up-and-in put presents its owner the right to sell a specified underlying stock at a predetermined strike price under the condition that the underlying stock first has to rise above a given barrier level.

Formally, the payoff from exercising the option at time $t \in [0, T]$ is defined as

$$H_t := (L - X_t)^+ \mathbb{1}_{\{\tau_H \leq t\}} \tag{3.16}$$

where $\tau_H := \inf\{0 \leq s \leq T | X_s \geq H\} \wedge T$ denotes the knock-in time at which the option becomes valuable. This is the first time that the underlying reaches the barrier. L defines the strike price and H the barrier. We assume $H > L$ to focus on the most interesting case. We hope not to confuse the reader by the ambiguous use of the letter H denoting the barrier and the claim's payoff process at the same time.

Using previous results and first-order stochastic dominance, we obtain the following evaluation scheme for the American up-and-in put option.

Theorem 3.4.1 (Up-and-in put) *Consider an American up-and-in put with payoff as defined in (3.16). The function*

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\star} \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right)$$

solves the optimal stopping problem under ambiguity aversion in (3.12) whereas the worst-case prior $Q^\star = Q^{\theta^\star}$ is specified by the Girsanov density

$$z_T^{\theta^\star} := \exp\left\{-\frac{1}{2}\int_0^T (\theta_s^\star)^2 ds - \int_0^T \theta_s^\star dW_s\right\}$$

with $\theta^\star$ defined as

$$\theta_t^\star := \begin{cases} \kappa, & \text{if } t < \tau_H \\ -\kappa, & \text{if } \tau_H \le t \le T \end{cases}.$$

An optimal stopping time is given by

$$\tau_t^\star := \inf\left\{t \vee \tau_H \le s \le T | V_s = (L - X_s)^+\right\}.$$

The theorem states that the agent considers the stopping problem under the measure $Q^{\theta^\star}$. It is the pasting of the measures Q^κ and $Q^{-\kappa}$ at the time of knock-in. Thus, she assumes the stock to evolve according to the least favorable drift rate $r - \sigma\kappa$ at the beginning of the contract. During the contract's lifetime, she changes her beliefs and assumes the highest possible drift rate $r + \sigma\kappa$ for the underlying. That is, she adapts her beliefs based on transpired events corresponding to her pessimistic point of view. So at τ_H, the point in time when the option knocks in the agent's beliefs or fears change abruptly. From a decision theoretical point of view, this result illustrates that optimal stopping under ambiguity aversion is behaviorally distinguishable from optimal stopping under expected utility. The buyer of an American up-and-in put for example behaves as two readily distinguishable expected utility maximizers. This is so because the worst-case measure $\hat{P}$ depends on the payoff process.

PROOF: In this section we provide an overview of the main ideas. More details can be found in Appendix A.3.

Given the event $\{\tau_H \le t\}$ the claim equals the usual American put option. Hence, $V_t = \operatorname{ess\,sup}_{\tau \ge t} \mathbb{E}^{Q^{-\kappa}}\left((L - X_\tau)^+ \gamma_{\tau-t}^{-1}|\mathcal{F}_t\right)$.

On $\{\tau_H > t\}$ we have $V_t = \operatorname{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(V_{\tau_H} \gamma_{\tau_H-t}^{-1}|\mathcal{F}_t\right)$, (see the appendix for more details). V_{τ_H} represents the value of the optimal stopping problem under ambiguity aversion at the specific time of knock-in.

Let us write $g(s) := \underline{Y}_s^{s,H}$ where $\underline{Y}_s^{s,H}$ is the value of the American put option under ambiguity aversion, see (3.15). By Lemma 3.3.11 the function $s \mapsto g(s)$ decreases, as is $s \mapsto \gamma_{s-t}^{-1}$. In the appendix we show that τ_H is

stochastically largest under Q^κ in the set of all priors $\mathcal{P}$. That is, for all t, s with $t < s \leq T$, we have on $\{\tau_H > t\}$ and for all $\theta \in \Theta$

$$Q^\kappa \left(\tau_H \leq s | \mathcal{F}_t\right) \leq Q^\theta \left(\tau_H \leq s | \mathcal{F}_t\right).$$

Then the usual characterization of first-order stochastic dominance, Mas-Colell, Whinston, and Green (1995), yields on $\{\tau_H > t\}$

$$\mathbb{E}^{Q^\kappa} \left(g(\tau_H)\gamma_{\tau_H-t}^{-1}|\mathcal{F}_t\right) \leq \mathbb{E}^{Q^\theta} \left(g(\tau_H)\gamma_{\tau_H-t}^{-1}|\mathcal{F}_t\right).$$

Thus the worst-case prior $Q^\star$ is equal to Q^κ on $\{\tau_H > t\}$. Setting both together, $Q^\star$ is given by $Q^{\theta^\star}$ with $\theta^\star$ as defined in the theorem. Since $\theta^\star$ is right-continuous, it is progressively measurable, per Proposition 1.13 in Karatzas and Shreve (1991). Hence $\theta^\star \in \Theta$, which finishes the proof. $\quad\square$

An analogous result holds for the American down-and-in call option. In that case, the agent solves the stopping problem under the worst-case scenario $Q^\star = Q^{\theta^\star}$ where $\theta^\star$ is now defined as

$$\theta_t^\star := \begin{cases} -\kappa, & \text{if } t < \tau_H \\ \kappa, & \text{if } \tau_H \leq t \leq T \end{cases}.$$

Here, τ_H denotes the initial time when the underlying stock price breaks from above through the barrier H.

3.4.2 Shout option

A shout option gives its owner the right to determine the strike price of a corresponding call or put option. It has been studied comprehensively in Chapter 2 of this thesis. We focus on the European put option version. That is, we consider a shout option that gives its buyer the right to freeze the asset price at any time τ^S before maturity to insure herself against later losses. At maturity the buyer obtains the payoff

$$H_T = \begin{cases} X_{\tau^S} - X_T, & \text{if } X_T < X_{\tau^S} \\ 0, & \text{else} \end{cases}. \tag{3.17}$$

The value of the optimal stopping problem under ambiguity aversion for a shout option at time $t \leq \tau^S \leq T$ is defined as

$$V_t = \operatorname*{ess\,sup}_{\tau^S \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left((X_{\tau^S} - X_T)^+ \gamma_{T-t}^{-1}|\mathcal{F}_t\right). \tag{3.18}$$

We only consider the problem for times $t \leq \tau^S$. This is the most interesting case since the owner has not fixed the strike price yet. She still faces the optimal stopping decision which is the decision of shouting.

To evaluate this contract under ambiguity aversion, we first mention the following observation already made in the discrete time setting, Chapter 2. This option is equivalent to the following: upon shouting the owner receives a European put option (at the money) with strike X_{τ^S} and remaining time to maturity $T - \tau^S$. We obtain the following evaluation scheme.

Theorem 3.4.2 (Shout option) *Consider a shout option at its starting time zero with a payoff as defined in (3.17). The solution of (3.18) at time zero simplifies to*

$$V_0 = \sup_{\tau^S \geq 0} \mathbb{E}^{Q^\star} \left((X_{\tau^S} - X_T)^+ \gamma_T^{-1} \right)$$

where the worst-case prior $Q^\star = Q^{\theta^\star}$ is specified by the Girsanov density $z^{\theta^\star}$ with $\theta^\star$ defined by

$$\theta_t^\star := \begin{cases} \kappa, & \text{if } t < \tau^S \\ -\kappa, & \text{if } \tau^S \leq t \leq T \end{cases}.$$

An optimal shouting time is given by

$$\tau^S := \inf \left\{ 0 \leq t \leq T \,|\, V_t = \mathbb{E}^{Q^{-\kappa}} \left((X_t - X_T)^+ \gamma_{T-t}^{-1} | \mathcal{F}_t \right) \right\}.$$

So in this case the ambiguity averse agent changes her beliefs after taking action. Before shouting she assumes the lowest drift rate $(r - \sigma\kappa)$, and the highest rate $(r + \sigma\kappa)$ afterwards. Both rates correspond to the respective least favorable rate, see also Chapter 2. Similarly to the up-and-in put, her pessimistic perspective leads to fearing the lowest possible returns of the risky asset before shouting and the highest possible returns hence.

PROOF: As noted above, at the time of shouting, the value of the contract in (3.18) is

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left((X_{\tau^S} - X_T)^+ \gamma_{T-\tau^S}^{-1} | \mathcal{F}_{\tau^S} \right).$$

This is a European type of monotone problem. The payoff at maturity T is $\Phi_T(x) := (X_{\tau^S} - x)^+$ which is monotone decreasing in x. As a special case of Theorem 3.3.7 we derive the value at the time of action as

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q \left((X_{\tau^S} - X_T)^+ \gamma_{T-\tau^S}^{-1} | \mathcal{F}_{\tau^S} \right) = g(\tau^S, X_{\tau^S})$$

where $g(t, x) := \mathbb{E}^{Q^{-\kappa}}\left((x - X_T^{t,x})^+ \gamma_{T-t}^{-1} | \mathcal{F}_t\right)$.

To determine the value before shouting, consider the following reflected FBSDE with obstacle $(g(t, X_t))_{t \in [0,T]}$

$$
\begin{cases}
dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x \\
-dY_t = -rY_t - \kappa |Z_t| dt + dK_t - Z_t dW_t, \ Y_T = g(T, X_T)
\end{cases}
. \tag{3.19}
$$

At this point it is important to note that the function $g(t, X_t)$ satisfies the assumptions for presenting an obstacle for a reflected BSDE. The joint continuity in (t, x) follows by the properties of solutions to (reflected) BSDEs.[23]

Since g can be rewritten in the following form

$$
g(t, x) = x\mathbb{E}^{Q^{-\kappa}}\left(\left(\left(1 - \exp\{(r - \frac{\sigma^2}{2})(T - t) + \sigma(W_T - W_t)\}\right)^+ \gamma_{T-t}^{-1} | \mathcal{F}_t\right)\right)
$$

we deduce that the function $x \mapsto g(t, x)$ is increasing for all $t \in [0, T]$. Using Theorem 3.3.7 we conclude

$$
V_0 = Y_0 = \sup_{\tau \geq 0} \mathbb{E}^{Q^\kappa}\left(g\left(\tau, X_\tau\right) \gamma_\tau^{-1}\right) = \sup_{\tau \geq 0} \mathbb{E}^{Q^\star}\left((X_\tau - X_T)^+ \gamma_T^{-1}\right).
$$

The last equality follows from the law of iterated expectation. Additionally we obtain an optimal shouting time τ^S. It is determined as the first time that value V. is equal to $g(\cdot, X.)$, the value of the European put under ambiguity aversion. This proves the theorem. $\theta^\star \in \Theta$ since it is right-continuous, again implying progressive measurability. $\qquad\square$

3.5 Conclusion

The chapter studies the optimal stopping problem of the buyer of various American options in a framework of model uncertainty in continuous time. Model uncertainty induced by imprecise information is mirrored in a set of multiple probability measures (priors).

Each measure corresponds to a specific drift rate for the stock price process in the respective market model. The agent then is allowed to adapt the model she uses to assign a value to the claim according to the worst possible model due to her ambiguity averse attitude. We characterize the

---

[23]The value for the European put option is obtained as the solution of a BSDE. Due to the European version of the put option g even belongs to $C^{1,2}([0,T] \times \mathbb{R}_+)$, cf. El Karoui, Peng, and Quenez (1997).

worst possible model by determining a worst-case measure that drives the processes within this model. We established a link to the calculus of reflected BSDEs to solve the optimal stopping problem from arising given the options' American style under multiple priors.

While the solution for plain vanilla options is straightforward, the situation differs if the payoff of the option is more complex. The buyer of such option adapts her beliefs to the state of the world, and to the overall effect of Knightian uncertainty. This leads to dynamical structure of the worst-case measure highlighting the structural differences between standard models in finance and the multiple priors models.

The characteristics are exemplified by solving the problem explicitly for an American barrier option and a shout option. Particularly with regard to risk management objectives, these models are more appropriate since the valuation becomes less sensitive in terms of varying model data and provides more robust exercise strategies.

Chapter 4

Financial markets with volatility uncertainty

4.1 Introduction

Many choice situations exhibit ambiguity. The occurrence of ambiguity aversion, and its effects on economic decisions are well established, at least since the Ellsberg Paradox. One way to model decisions under ambiguity is to use multiple priors. Instead of analyzing a problem in a single prior model as in the classical subjective expected utility approach, the focus is on a multiple priors model used to describe the agent's uncertainty about the correct probability distribution.

These models have recently attracted much attention. The decision theoretical setting of multiple priors was introduced by Gilboa and Schmeidler (1989) and extended to a dynamic model by Epstein and Schneider (2003b). Maccheroni, Marinacci, and Rustichini (2006) generalized the model to variational preferences. Here, multiple priors appear naturally in monetary risk measures as introduced by Artzner, Delbaen, Eber, and Heath (1999) and its dynamic extensions, see Delbaen (2002) and Delbaen, Peng, and Gianin (2010) for example.

Most of the literature tends to concentrate on the modeling of multiple priors with respect to some reference measure.[1] The assumption is that all priors are, at least locally, absolutely continuous with regard to a given reference measure. Although this technical assumption is often made to simplify calculations, it significantly affects the informative value of the multiple

---

[1]See Epstein and Marinacci (2007) or Riedel (2009), for example.

priors model.[2] In diffusion models, by Girsanov's theorem, these multiple priors models only lead to uncertainty in the mean of the considered stochastic process, see Chen and Epstein (2002) or Cheng and Riedel (2010) for examples. When used in finance, they lead to drift uncertainty for stock prices.

It is possible to consider other sources of uncertainty involving the risk described by the standard deviation of a random variable, a question of great relevance to finance. For instance, the price of an option written on a risky stock depends heavily on its underlying volatility. In addition, the value of a portfolio consisting of risky positions is strongly connected with the volatility levels of the corresponding assets. One major problem in practice is to forecast the prospective volatility process in the market. In this sense, it appears quite natural to permit volatility uncertainty.[3] In the sense of risk measuring, it is desirable to seize the risk.

This chapter addresses foundational issues in working with volatility uncertainty in finance. These include the hedging of contingent claims under constraints, Karatzas and Kou (1996), or portfolio optimization, Merton (1990), that should also be formulated and treated in the presence of model uncertainty. Addressing the issues requires an economically reasonable notion of arbitrage, such as that proposed here.

Fernholz and Karatzas (2010) considered the question of outperforming the market when the assumption of no-arbitrage is not imposed and additionally incorporated volatility uncertainty. Our purpose, in contrast, is to model volatility uncertainty on financial markets under the assumption of no-arbitrage. We set a framework for modeling this particular uncertainty and treat the pricing and hedging of European contingent claims. The setting is closely related to model risk, an issue relevant to risk management. As we shall see, allowing for uncertain volatilities will lead to incomplete markets,

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[2]The reference measure plays the role of fixing the sets of measure zero. This means that the decision maker has perfect knowledge about sure events which is obviously not always a reasonable property from an economic point of view. In particular, with a filtration satisfying the "usual conditions" which is mostly arranged one excludes economic interesting models since the decision maker consequently knows already at time zero what can happen and what not. This of course does not reflect reality well as for instance the recent incidents about the Greek government bonds illustrate.

[3]Volatility is very sensitive with respect to changing market data which makes its predictability difficult. It also reflects the market's sentiment. Currently high implicit volatility levels suggest nervous markets whereas low levels rather feature bullish mood. By taking many models into account one may protect oneself against surprising events due to misspecification.

thus effecting the pricing and hedging of claims, while involving model risk. Our solution also provides a method to measure this risk. We consider European claims written on a risky stock S, which additionally features volatility uncertainty. Roughly, S is modeled by the family of processes

$$dS_t^\sigma = rS_t^\sigma dt + \sigma_t S_t^\sigma dB_t$$

where $B = (B_t)$ is a classical Brownian motion and σ_t attains various values in $[\underline{\sigma}, \overline{\sigma}]$ for all t. In this setting we aim to solve

$$\sup_{P \in \mathcal{P}} \mathbb{E}^P(H_T \gamma_T^{-1}) \quad \text{and} \quad \sup_{P \in \mathcal{P}} \mathbb{E}^P(-H_T \gamma_T^{-1}) \tag{4.1}$$

where H_T denotes the payoff of a contingent claim at maturity T, γ_T^{-1} a discounting, and $\mathcal{P}$ presents a set of various probability measures describing the model uncertainty.

It is by no means clear whether the expressions above are well-posed or how to choose $\mathcal{P}$. As seen in Denis and Martini (2006) modeling uncertain volatilities leads to a set of priors which are mutually singular.[4] When dealing with model uncertainty, we need a consistent mathematical framework enabling us to work with processes under various measures at the same time. We utilize the framework of sublinear expectation and G-normal distribution introduced by Peng (2007) to model and control model risk.

We consider a Black-Scholes-like market with uncertain volatilities, i.e., the stock price S is modeled as a geometric G-Brownian motion

$$dS_t = rS_t dt + S_t dB_t, \quad S_0 = x_0, \tag{4.2}$$

where the canonical process $B = (B_t)$ is a G-Brownian motion with respect to a sublinear expectation E_G. E_G is called G-expectation. It also represents a particular coherent risk measure that enables us to quantify the model risk induced by volatility uncertainty. For the construction see Peng (2007) or Peng (2010).

G-Brownian motion forms a very rich and interesting new structure that generalizes the classical diffusion model. It replaces classical Brownian motion as a way to account for model risk in the volatility component. Each B_t is G-normal distributed which resembles the classical normal distribution. The function G characterizes the degree of uncertainty.

Throughout this chapter, we consider the case where G denotes the function $y \mapsto G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-$ with volatility bounds $0 < \underline{\sigma} < \overline{\sigma}$. Each B_t has a mean of zero but an uncertain variance between the bounds $\underline{\sigma}^2 t$

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[4]To understand how things are involved see Example 4.2.1 in Section 4.2.

and $\overline{\sigma}^2 t$. When B_t is evaluated by E_G, we have $E_G(B_t) = E_G(-B_t) = 0$ and $E_G(B_t^2) = \overline{\sigma}^2 t \neq -E_G(-B_t^2) = \underline{\sigma}^2 t$. Consequently, the stock's volatility is uncertain and incorporated in the process (B_t). The quadratic variation process is no longer deterministic. All uncertainty of B is concentrated in its quadratic variation $\langle B \rangle$. It is an absolutely continuous process w.r.t. Lebesgue measure, and its density satisfies $\underline{\sigma}^2 \leq \frac{d\langle B \rangle_t}{dt} \leq \overline{\sigma}^2$.

The related stochastic calculus, especially Itô integral, can also be established with respect to G-Brownian motion, Peng (2010). Notions like martingales are replaced by G-martingales with the same meaning as one would expect from classical probability theory, Definition A.4.16.

Even though at first glance it appears hidden in Equation (4.2), we remain in a multiple priors setting. Denis, Hu, and Peng (2010) showed that the G-framework developed in Peng (2007) corresponds to the framework of quasi-sure analysis.[5] They established that the sublinear expectation E_G can be represented as an upper expectation of classical expectations, i.e., there exists a set of probability measures $\mathcal{P}$ such that $E_G[X] = \sup_{P \in \mathcal{P}} \mathbb{E}^P[X]$.

It should be mentioned that the stipulated dynamics for the stock price in (4.2) imply that the discounted stock price is a symmetric G-martingale. The word "symmetric" implies that the corresponding negative process is also a G-martingale which is not necessarily the case in the G-framework.[6] For this stock price model, we prove that the induced financial market does not admit any arbitrage opportunity. In addition, this accords with classical finance in which problems such as the pricing and hedging of claims are solved with respect to a risk neutral martingale measure where the discounted price process becomes a (local) martingale.

The notion of G-martingale plays an important role in our analysis (see page 112 in the appendix for a deeper discussion).

By Soner, Touzi, and Zhang (2010a) a G-martingale (M_t) solves the following dynamic programming principle, see also Appendix A.4.3:

$$M_t = \operatorname*{ess\,sup}_{Q' \in \mathcal{P}(t,Q)} \mathbb{E}^{Q'}(M_s | \mathcal{F}_t) \quad Q - a.s., \ t \leq s,$$

where $\mathcal{P}(t,Q) := \{Q' \in \mathcal{P} | Q' = Q \text{ on } \mathcal{F}_t\}$. Thus, a G-martingale is a multiple priors martingale as considered in Riedel (2009). The dynamic programming principle states that a G-martingale is a supermartingale for all single priors and a martingale for an optimal prior. By the G-martingale representation theorem of Song (2010c), Theorem A.4.19, and Remark A.4.18 a

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[5]See Denis and Martini (2006), for example.

[6]This characteristic forms a fundamental difference to classical probability theory. Its effect is also reflected in the results of Section 4.3.

symmetric G-martingale will be a martingale with respect to all single priors involved. Also from this point of view, the imposed dynamics on S in (4.2) are economically reasonable.

In such an ambiguous financial market we analyze European contingent claims concerning pricing and hedging. We extend the asset pricing to markets with volatility uncertainty. The concept of no-arbitrage will play a major role in our analysis. Due to the additional source of risk induced by volatility uncertainty, the classical definition of arbitrage will no longer be adequate. We introduce a new arbitrage definition that fits our multiple priors model with mutually singular priors. We verify that our financial market does not admit any arbitrage opportunity in this modified sense.

Using the concept of no-arbitrage, we establish detailed results providing a better economic understanding of financial markets under volatility uncertainty. We determine an interval of no-arbitrage prices for general contingent claims. The bounds of this interval – the upper and lower arbitrage prices h_{up} and h_{low} – are obtained as the expected value of the claim's discounted payoff with respect to G-expectation, see (4.1). They specify the lowest initial capital required to hedge a short position in the claim, or long position, respectively.[7]

Since E_G is a sublinear expectation, we know that $h_{low} \neq h_{up}$ in general. This verifies the market's incompleteness. All in all, any price within the interval (h_{low}, h_{up}) is reasonable initially for a European contingent claim in the sense that it does not admit arbitrage.

In a Markovian setting, when the claim's payoff only depends on the current stock price of its underlying, we deduce more structure about the upper and lower arbitrage prices via the Black-Scholes-Barenblatt PDE. We derive an explicit representation for the corresponding super-hedging strategies and consumption plans.[8] Given the special situation when the payoff function exhibits convexity (concavity), the upper arbitrage price solves the classical Black-Scholes PDE with a volatility equal to $\overline{\sigma}$ ($\underline{\sigma}$), and, vice versa concerning the lower arbitrage price. This corresponds to a worst-case volatility analysis as in El Karoui and Quenez (1998) and Avellaneda, Levy, and Paras (1995).

The same results were also established in Avellaneda, Levy, and Paras (1995) for a Markovian setting. However, the mathematical framework in Avellaneda, Levy, and Paras (1995) is largely intuitive and presumably not adequate for a general study. Our analysis, on the other hand, provides a

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[7]The expression on the left hand side in (4.1) may be interpreted as the ask price the seller is willing to accept for selling the claim, whereas the other represents the bid price the buyer is willing to pay.

[8]It is also called side-payments, cf. Föllmer and Schied (2004).

rigorous foundation for the results in Avellaneda, Levy, and Paras (1995). A recent preprint of Soner, Touzi, and Zhang (2010b) also contains relevant results concerning uncertain volatility.[9] Denis and Martini (2006) also obtained the identity $h_{up} = E_G(H_T \gamma_T^{-1})$ for the case where $H_T \in \mathcal{C}_b(\Omega)$ by using quasi-sure analysis.[10] We formulate the financial market in the presence of volatility uncertainty using the G-framework of Peng (2007). Utilizing the set of multiple priors $\mathcal{P}$ induced by E_G, we are able to investigate the market by a more powerful approach – a no-arbitrage concept which also accounts for volatility uncertainty. The properties of G-Brownian motion, also allow us to obtain explicit results like the PDE derivation in the Markovian setting. In particular, we realize the PDE of Black and Scholes (1973) as a special case when the volatility uncertainty is set equal to zero. Furthermore, we consider the lower arbitrage price h_{low} and obtain the identities for claims with $H_T \in L_G^p(\Omega_T), p \geq 2$.[11] Both bounds together form the basis for an economically reasonable price for the claim.

The chapter is organized as follows. Section 4.2 introduces the financial market we focus on and extends the terminology from mathematical finance. Section 4.3 stresses the concept of no-arbitrage, and the pricing of contingent claims in the general case. In Section 4.4, we restrict ourselves to the Markovian setting and obtain results similar to Avellaneda, Levy, and Paras (1995). Section 4.5 concludes. Appendix A.4 introduces notions of sublinear expectation. It summarizes the necessary definitions, constructions and associated results from the original sources as used here. The appendix serves as a reference work for the reader to obtain a deeper understanding of the mathematics. Whenever we use a new item from the G-framework in the previous sections, we will provide a brief and rough explanation to facilitate comprehension without prior knowledge of the framework presented in the Appendix A.4.

4.2 The market model

Our aim is to analyze financial markets that feature volatility uncertainty. The following example (see also Soner, Touzi, and Zhang (2010b)) illustrates some issues that arise when we deal with uncertain volatilities.

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[9]The authors use an approach different from ours.

[10]$\mathcal{C}_b(\Omega)$ denotes the space of bounded continuous functions on the path space.

[11]$L_G^p(\Omega_T)$ represents a specific space of random variables on which the G-expectation can be defined. $\mathcal{C}_b(\Omega)$ is contained in $L_G^p(\Omega_T)$, see Peng (2010) or Equation (A.7) in Appendix A.4.1.

Example 4.2.1 *Let (B_t) be a Brownian motion with regard to some measure P_0 and consider the price process modeled as $dS_t^\sigma = \sigma_t dB_t$, $S_0 = x$, for various processes $\sigma = (\sigma_t)$. Consider two constant processes $\hat\sigma$ and $\tilde\sigma$. If $\hat\sigma \neq \tilde\sigma$ we have $\langle S^{\hat\sigma}\rangle. \neq \langle S^{\tilde\sigma}\rangle.$ P_0-almost surely, which implies that the distributions $P_0 \circ (S^{\hat\sigma})^{-1}$ and $P_0 \circ (S^{\tilde\sigma})^{-1}$ are mutually singular.*

So, given a family of stochastic processes $X^P, P \in \mathcal{P}$, we need to construct a universal *process X which is uniquely defined with respect to all measures at the same time such that $X = X^P$ P-a.s. for all $P \in \mathcal{P}$. Also, when defining a stochastic integral $I_t^P := \int_0^t \eta_s dB_s$ for all $P \in \mathcal{P}$ simultaneously the same situation arises. Clearly, we can define I_t^P under each P in the classical sense. Since I_t^P may depend on the respective underlying measure P, we are free to redefine the integral outside the support of P. Thus in order to make things work, we need to find a universal integral I_t satisfying $I_t = I_t^P$ P-a.s. for all measures $P \in \mathcal{P}$.*

Let us now come to the introduction of the financial market. All along the chapter we consider a financial market $\mathcal{M}$ consisting of two assets evolving according to

$$
\begin{aligned}
d\gamma_t &= r\gamma_t dt, \ \gamma_0 = 1, \\
dS_t &= rS_t dt + S_t dB_t, \ S_0 = x_0 > 0
\end{aligned}
\tag{4.3}
$$

with a constant interest rate $r \geq 0$. $B = (B_t)$ denotes the canonical process which is a G-Brownian motion under E_G with parameters $\overline{\sigma} > \underline{\sigma} > 0$. See Appendix A.4.1 for the exact definition and construction of the pair B and E_G. The assumption of strict positive volatility is well accepted in finance, as it is of economic relevance, (see also Remark A.4.5). The asset $\gamma = (\gamma_t)$ represents a riskless bond as usual. Since $B = (B_t)$ is a G-Brownian motion, S is modeled as a geometric G-Brownian motion similarly to the original Back-Scholes model, cf. Black and Scholes (1973), where the stock price is modeled by a classical geometric Brownian motion.

As a consequence, in this market the stock price evolution not only involves risk as modeled by the noise part but also ambiguity about the risk due to the unknown deviation of the process B from its mean. In terms of finance, this ambiguity is called volatility uncertainty. If we choose $\sigma = \underline{\sigma} = \overline{\sigma}$ we are in the classical Black-Scholes model, (see Black and Scholes (1973) or any good textbook in finance).

Remark 4.2.2 *Note that the discounted stock price process $(\gamma_t^{-1} S_t)$ is directly modeled as a symmetric G-martingale with regard to the corresponding G-expectation E_G. It is a well known fact that problems like pricing or hedging contingent claims are handled under a risk-neutral probability measure that leads to the favored situation in which the discounted stock price process is a (local) martingale, cf. Duffie (1992).*[12]

The use of G-Brownian motion to model the financial market initially leads to a formulation of $\mathcal{M}$ which is not based on a classical probability space. The representation theorem for G-expectation, (see Theorem A.4.12), establishes a link also to a probabilistic framework. It provides us with a family of probability measures $\mathcal{P}$ on a measurable space $(\Omega_T, \mathcal{F})$ such that the following identity holds

$$E_G(X) = \sup_{P \in \mathcal{P}} \mathbb{E}^P(X)$$

where X is any random variable for which the G-expectation can be defined, for instance when $X : \Omega_T \to \mathbb{R}$ is bounded and continuous. $\mathcal{F} = \mathcal{B}(\Omega_T)$ denotes the Borel σ-algebra on the path space $\Omega_T = C_0([0,T], \mathbb{R})$. The set of probability measures $\mathcal{P}$ can be constructed as follows. Let $W = (W_t)$ be a classical Brownian motion w.r.t. a measure P on $(\Omega_T, \mathcal{F})$. We can consider the filtration $(\mathcal{F}_t)$ generated by W, i.e., $\mathcal{F}_t := \sigma\{W_s | 0 \leq s \leq t\} \vee \mathcal{N}$ where $\mathcal{N}$ denotes the collection of P-null subsets.

Let $\Theta := [\underline{\sigma}, \overline{\sigma}]$, and $\mathcal{A}_{0,T}^{\Theta}$ be the collection of all Θ-valued $(\mathcal{F}_t)$-adapted processes on $[0,T]$. For any $\theta \in \mathcal{A}_{0,T}^{\Theta}$ we define $B_t^{0,\theta} := \int_0^t \theta_s dW_s$ and P^θ as the law of $B^{0,\theta} = \int_0 \theta_s dW_s$, i.e., $P^\theta = P \circ (B^{0,\theta})^{-1}$. Then $\mathcal{P}$ is the closure of $\{P^\theta | \theta \in \mathcal{A}_{0,T}^{\Theta}\}$ under the topology of weak convergence.

Throughout this chapter we consider the tuple $(\Omega_T, \mathcal{F}, (\mathcal{F}_t), W, P)$ together with the set of priors $\mathcal{P}$ as given. Since $\mathcal{P}$ represents E_G, it also represents the volatility uncertainty of the stock price and therefore of $\mathcal{M}$. The G-framework utilized here enables the analysis of stochastic processes for all priors of $\mathcal{P}$ simultaneously. The terminology of "quasi-sure" turns out to be very useful:

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[12]This should also be the case in our ambiguous setting. By modeling the discounted stock price directly as a symmetric G-martingale we do not have to change the sublinear expectation from a subjective to a risk-neutral sublinear expectation and avoid the technical difficulties involved. We require a symmetric G-martingale to ensure that both selling and purchasing the stock is equally favorable to all market participants.

A set $A \in \mathcal{F}$ is called *polar* if $P(A) = 0$ for all $P \in \mathcal{P}$. We say a property holds *"quasi-surely"* (q.s.) if it holds outside a polar set.

Unless stated otherwise, all equations and inequalities should also be understood as "quasi-sure-statements". This means that a property holds almost surely for all conceivable scenarios.

Next, we restate some useful definitions. Although standard in finance, here they have been adapted to this more complex situation. We will need to use the following spaces $L_G^p(\Omega_T), H_G^p(0,T)$, and also $M_G^p(0,T), p \geq 1$, which denote specific spaces in the G-setting. The first concerns random variables for which the G-expectation is defined, see Equation (A.7) in Appendix A.4.1. The other two are particular spaces of processes for which stochastic integrals with respect to B can be defined. Because of the sophisticated setting induced by mutually singular measures it is clear that the stochastic integral cannot be defined for all processes one might think of. These spaces are the closure of collections of simple processes similar to the case when the classical Itô integral is constructed, (see also Appendix A.4.2 page 111).

Throughout this chapter we will presume a finite time horizon denoted by $T > 0$.

Definition 4.2.3 *A trading strategy in the market $\mathcal{M}$ is an $(\mathcal{F}_t)$-adapted vector process $(\eta, \phi) = (\eta_t, \phi_t)$, ϕ a member of $H_G^1(0,T)$ such that $(\phi_t S_t) \in H_G^1(0,T)$, and $\eta_t \in \mathbb{R}$ for all $t \leq T$.*

A cumulative consumption process $C = (C_t)$ is a nonnegative $(\mathcal{F}_t)$-adapted process with values in $L_G^1(\Omega_T)$, and with increasing, right-continuous paths on $(0,T]$, and $C_0 = 0$, $C_T < \infty$ q.s.

Note that the stock's price process S defined by (4.3) is an element of $M_G^2(0,T)$ which coincides with $H_G^2(0,T)$, see Peng (2010). We impose the so-called self-financing condition. That is, consumption and trading in $\mathcal{M}$ satisfy

$$V_t := \eta_t \gamma_t + \phi_t S_t = \eta_0 \gamma_0 + \phi_0 S_0 + \int_0^t \eta_u d\gamma_u + \int_0^t \phi_u dS_u - C_t \quad \forall t \leq T \text{ q.s.}$$
(4.4)

where V_t denotes the value of the trading strategy at time t.

Sometimes, it is more appropriate to consider instead of a trading strategy, a portfolio process which presents the proportions of wealth invested in the risky stock.

Remark 4.2.4 *A portfolio process π represents proportions of a wealth X which are invested in the stock within the considered time interval, whereas a trading strategy (η, ϕ) represents the total numbers of the respective assets the agent holds. There is a one-to-one correspondence between a portfolio process and a trading strategy as defined above. If we define*

$$\phi_t := \frac{X_t \pi_t}{S_t}, \quad \eta_t := \frac{X_t(1 - \pi_t)}{\gamma_t}, \quad \forall t \leq T,$$

(η, ϕ) constitutes a trading strategy in the sense of equation (4.4) as long as π constitutes a portfolio process with corresponding wealth process X as required in Definition 4.2.6 below.

Definition 4.2.5 *A portfolio process is an $(\mathcal{F}_t)$-adapted real valued process $\pi = (\pi_t)$ with values in $L^1_G(\Omega_T)$.*

Definition 4.2.6 *For a given initial capital y, a portfolio process π, and a cumulative consumption process C, consider the wealth equation*

$$dX_t = X_t(1 - \pi_t)\frac{d\gamma_t}{\gamma_t} + X_t\pi_t\frac{dS_t}{S_t} - dC_t$$
$$= rX_t dt + X_t\pi_t dB_t - dC_t$$

with initial wealth $X_0 = y$. Or equivalently,

$$\gamma_t^{-1}X_t = y - \int_0^t \gamma_u^{-1}dC_u + \int_0^t \gamma_u^{-1}X_u\pi_u dB_u, \quad \forall t \leq T.$$

If this equation has a unique solution $X = (X_t) := X^{y,\pi,C}$, it is called the wealth process corresponding to the triple (y, π, C).

In order to have the stochastic integral well defined, $\int_0^T \pi_t^2 X_t^2 dt < \infty$ must hold quasi-surely and we need to impose the requirement that $(\pi_t X_t) \in H^p_G(0, T), p \geq 1$, or $\in M^p_G(0, T), p \geq 2$. We incorporate this into the next definition which describes admissible portfolio processes.

Definition 4.2.7 *A portfolio/consumption process pair (π, C) is called admissible for an initial capital $y \in \mathbb{R}$ if*

(i) the pair obeys the conditions of Definitions 4.2.3, 4.2.5, and 4.2.6

(ii) $(\pi_t X_t^{y,\pi,C}) \in H_G^1(0,T)$

(iii) the solution $X_t^{y,\pi,C}$ satisfies

$$X_t^{y,\pi,C} \geq -L, \quad \forall t \leq T, \ q.s.$$

where L is a nonnegative random variable in $L_G^2(\Omega_T)$

We then write $(\pi, C) \in \mathcal{A}(y)$.

In the above Definitions 4.2.3 – 4.2.7 we need to ensure that the associated stochastic integrals are well-defined. In particular condition (ii) of Definition 4.2.7 ensures that the mathematical framework does not collapse by allowing for too many portfolio processes.

The agent is uncertain about the true volatility, therefore, she uses a portfolio strategy which can be performed independently of the realized scenario at the market. Hence, she is able to analyze the corresponding wealth processes with respect to all conceivable market scenarios $P \in \mathcal{P}$ simultaneously.

These restrictions on the portfolio and consumption processes replace the classical condition of predictable processes. Decisions at some time t must not utilize subsequently revealed information. In our financial setting, the processes have to be members of particular spaces within the G-framework. Based on the construction of these spaces, (by means of (viscosity) solutions of PDEs, Appendix A.4) the portfolio and consumption processes require some kind of regularity, in particular see identity (A.7) in Appendix A.4.1. The economic interpretation is that decisions should not react too abruptly and sensitively to revealed information.

4.3   Arbitrage and contingent claims

As usual in financial markets we impose the concept of arbitrage. Because of this more complex framework, both economically and mathematically, we need a slightly more involved definition of arbitrage.

Definition 4.3.1 (Arbitrage in $\mathcal{M}$) *We say there is an arbitrage opportunity in $\mathcal{M}$ if there exist an initial wealth $y \leq 0$, an admissible pair $(\pi, C) \in \mathcal{A}(y)$ with $C \equiv 0$ such that at some time $T > 0$*

$$X_T^{y,\pi,0} \geq 0 \ \ q.s., \quad and$$
$$P\left(X_T^{y,\pi,0} > 0\right) > 0 \ \ for \ at \ least \ one \ P \in \mathcal{P}.$$

If such a strategy in the sense above existed, one should pursue it since it would be riskless and fortunate in that should the particular P drive the market dynamics, one would make a profit with positive probability. It should be noted that in the given definition of arbitrage we need to require quasi-sure dominance for the wealth at time T in order to exclude the risk in all possible scenarios. So there should not exist a scenario under which there is a positive probability that the terminal wealth is less than zero.

Remark 4.3.2 *The second condition in the definition of arbitrage is just the negation of $X_T^{y,\pi,0} \leq 0$ q.s. Hence, combined with the first condition it excludes that $X_T^{y,\pi,0}$ equals zero quasi-surely.*

We identify $(y, \pi, 0)$ as an arbitrage if there exists profit with positive probability in at least one scenario even though there does not exist profit with positive probability in many others.

Of course, one could also define arbitrage by the requirement that the second condition has to hold for all scenarios, i.e., there existed profit with positive probability in all scenarios. We believe, however, that this kind of arbitrage definition is not wholly reasonable from an economic standpoint, (see Remark 4.3.14).

Lemma 4.3.3 (No-arbitrage) *In the financial market $\mathcal{M}$ there does not exist any arbitrage opportunity.*

PROOF: Assume there exists an arbitrage opportunity, i.e., there exists some $y \leq 0$ and a pair $(\pi, C) \in \mathcal{A}(y)$ with $C \equiv 0$ such that $X_T^{y,\pi,0} \geq 0$ quasi-surely for some $T > 0$. Then by monotonicity we have $E_G(X_T^{y,\pi,0}) \geq 0$. By definition of the wealth process

$$0 \leq E_G\left(X_T^{y,\pi,0}\gamma_T^{-1}\right) \leq y + E_G\left(\int_0^T \gamma_t^{-1} X_t^{y,\pi,0}\pi_t dB_t\right) = y$$

since the G-expectation of an integral with respect to G-Brownian motion is zero.[13] Hence, $E_G\left(X_T^{y,\pi,0}\gamma_T^{-1}\right) = 0$ which again implies $X_T^{y,\pi,0}\gamma_T^{-1} = 0$ q.s. Thus, $(y, \pi, 0)$ is not an arbitrage. □

In the financial market $\mathcal{M}$, we want to consider European contingent claims H with payoff H_T at maturity T. Here, H_T represents a nonnegative,

---

[13]Here we used that $(X_t^{y,\pi,0}\pi_t) \in H_G^1(0,T)$, cf. condition (ii) of Definition 4.2.7.

$\mathcal{F}_T$-adapted random variable. We impose the assumption $H_T \in L_G^2(\Omega_T)$ at all times. The price of the claim at time 0 will be denoted by H_0. In order to find reasonable prices for H we use the concept of arbitrage. Similarly to the above, we define an arbitrage opportunity in the financial market $(\mathcal{M}, H)$ consisting of the original market $\mathcal{M}$ and the contingent claim H.

Definition 4.3.4 (Arbitrage in $(\mathcal{M}, H)$) *There is an arbitrage opportunity in $(\mathcal{M}, H)$ if there exist an initial wealth $y \geq 0$ (respectively, $y \leq 0$), an admissible pair $(\pi, C) \in \mathcal{A}(y)$ and a constant $a = -1$ (respectively, a=1), such that*

$$y + a \cdot H_0 \leq 0$$

at time 0, and

$$X_T^{y,\pi,C} + a \cdot H_T \geq 0 \quad q.s., \quad and$$
$$P\left(X_T^{y,\pi,C} + a \cdot H_T > 0\right) > 0 \quad for \ at \ least \ one \ P \in \mathcal{P}$$

at time T.

The values $a = \pm 1$ in Definition 4.3.4 indicate short ($a = -1$) or long positions in the claim H, respectively. This definition of arbitrage is standard in the literature, Karatzas and Shreve (1998). For the same reasons as before, we again require quasi-sure dominance for the wealth at time T and gain with positive probability for only one possible scenario.

In the following, we show the existence of no-arbitrage prices for a claim H which exclude arbitrage opportunities. Compared to the classical Black-Scholes model, there are many no-arbitrage prices for H in general. We shall see that mostly hedging, or replicating arguments, fail due to the additional source of uncertainty induced by the G-normal distribution causing the incompleteness of the financial market, (see Remark 4.3.11). Thus, in our ambiguous market $\mathcal{M}$ there generally is neither a self-financing portfolio strategy which replicates the European claim, nor a risk-free hedge for the claim since the uncertainty represented by the occurring quadratic variation term cannot be eliminated. Only for special claims H when $\left(E_G[H_T \gamma_T^{-1}|\mathcal{F}_t]\right)$ is a symmetric G-martingale, Remark 4.3.11, we have $h_{up} = h_{low}$.

Clearly, there is only one single G-Brownian motion that occurs in the financial market model. However, due to the representation theorem for G-expectation there are many probability measures involved in $\mathcal{M}$, (see Theorem A.4.12). Each measure reflects a specific volatility rate for the stock

price. Roughly speaking, these measures induce incompleteness since only one scenario is being realized and only in this scenario is stock being traded.

The functional E_G is just a useful method to control the dynamics by giving upper and lower bounds for European contingent claim prices written on the stock, (see Theorem 4.3.6).

The following classes will be relevant to our subsequent analysis.

Definition 4.3.5 *Given a European contingent claim H we define the lower hedging class*

$$\mathcal{L} := \{y \geq 0 | \exists\, (\pi, C) \in \mathcal{A}(-y) :\ X_T^{-y,\pi,C} \geq -H_T \quad q.s.\}$$

and the upper hedging class

$$\mathcal{U} := \{y \geq 0 | \exists\, (\pi, C) \in \mathcal{A}(y) :\ X_T^{y,\pi,C} \geq H_T \quad q.s.\}.$$

In addition, the lower arbitrage price is defined as

$$h_{low} := \sup\{y | y \in \mathcal{L}\}$$

and the upper arbitrage price as

$$h_{up} := \inf\{y | y \in \mathcal{U}\}.$$

To include the case $\mathcal{U} = \emptyset$ we define $\inf \emptyset = \infty$. The main result of this section concerns the lower and upper arbitrage price. It is possible to determine the prices explicitly. We have

Theorem 4.3.6 *Given the financial market $(\mathcal{M}, H)$. The following identities hold:*

$$h_{up} = E_G(H_T \gamma_T^{-1})$$
$$h_{low} = -\, E_G(-H_T \gamma_T^{-1}).$$

Before proving the theorem we establish some results about the hedging classes. As proven in Karatzas and Kou (1996), one can easily show that $\mathcal{L}$ and $\mathcal{U}$ are connected intervals. Precisely we have

Lemma 4.3.7 *Given $y \in \mathcal{L}$, $0 \leq z \leq y$ implies $z \in \mathcal{L}$. Analogously, given $y \in \mathcal{U}$, $z \geq y$ implies $z \in \mathcal{U}$.*

The proof uses the idea that one "just immediately consumes the difference between the two initial wealths".

For $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ let us define the Black-Scholes price of a European contingent claim H

$$u_0^\sigma := \mathbb{E}^{P^\sigma}(H_T \gamma_T^{-1})$$

where $P^\sigma \in \mathcal{P}$ denotes the measure under which S has constant volatility level σ. As mentioned in Appendix A.4.1 it is defined by $P^\sigma := P \circ (X^\sigma)^{-1}$ where $X_t^\sigma := \int_0^t \sigma dW_u = \sigma W_t$. Due to the dynamics of S, per Equation (4.3), P^σ is the usual risk neutral probability measure in the Black-Scholes model with fixed volatility rate σ, see also the proof of Corollary 4.4.3.

Similarly to the case with constraints, (see Karatzas and Kou (1996)), we can prove the following three lemmata. For this let H be a given European contingent claim.

Lemma 4.3.8 *For any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, the following inequality chain holds:*

$$h_{low} \leq u_0^\sigma \leq h_{up}.$$

PROOF: Let $y \in \mathcal{U}$. By definition of $\mathcal{U}$ there exists a pair $(\pi, C) \in \mathcal{A}(y)$ such that $X_T^{y,\pi,C} \geq H_T$ q.s. Using the properties of G-expectation as stated in Appendix A.4.1, in particular Proposition A.4.11 for the first equality, we obtain for any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$

$$\begin{aligned}
y &= E_G \left(y + \int_0^T \gamma_t^{-1} X_t^{y,\pi,C} \pi_t dB_t \right) \\
&\geq E_G \left(y + \int_0^T \gamma_t^{-1} X_t^{y,\pi,C} \pi_t dB_t - \int_0^T \gamma_t^{-1} dC_t \right) \\
&= E_G \left(X_T^{y,\pi,C} \gamma_T^{-1} \right) \geq E_G \left(H_T \gamma_T^{-1} \right) = \sup_{P \in \mathcal{P}} \mathbb{E}^P (H_T \gamma_T^{-1}) \geq u_0^\sigma.
\end{aligned}$$

The first and second inequalities hold due to the monotonicity of E_G, the second equality holds by the definition of the wealth process and due to $y \in \mathcal{U}$, the third equality by the representation theorem for E_G, per Theorem A.4.12, and the last estimate holds because of $P^\sigma \in \mathcal{P}$. Hence, $h_{up} \geq u_0^\sigma$.

Similarly, let $y \in \mathcal{L}$ and $(\pi, C) \in \mathcal{A}(-y)$ be the corresponding pair such that $X_T^{-y,\pi,C} \geq -H_T$ q.s. By the same reasoning, we obtain for any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$

$$
\begin{aligned}
-y &= E_G \left(-y + \int_0^T \gamma_t^{-1} X_t^{-y,\pi,C} \pi_t dB_t \right) \\
&\geq E_G \left(-y + \int_0^T \gamma_t^{-1} X_t^{-y,\pi,C} \pi_t dB_t - \int_0^T \gamma_t^{-1} dC_t \right) \\
&= E_G \left(X_T^{-y,\pi,C} \gamma_T^{-1} \right) \geq E_G \left(-H_T \gamma_T^{-1} \right) \geq -\mathbb{E}^{P^\sigma} \left(H_T \gamma_T^{-1} \right) = -u_0^\sigma
\end{aligned}
$$

which implies $y \leq u_0^\sigma$ and the Lemma follows. $\qquad \square$

Lemma 4.3.9 *For any price $H_0 > h_{up}$ there exists an arbitrage opportunity. Also for any price $H_0 < h_{low}$ there exists an arbitrage opportunity.*

PROOF: We only consider the first case since the argument is similar. Assume $H_0 > h_{up}$ and let $y \in (h_{up}, H_0)$. By definition of h_{up} we deduce that $y \in \mathcal{U}$. Hence there exists a pair $(\pi, C) \in \mathcal{A}(y)$ with

$$
X_T^{y,\pi,C} \geq H_T \quad q.s.
$$

and

$$
y - H_0 < 0.
$$

But this implies the existence of arbitrage in the sense of Definition 4.3.4: $\exists a > 1$ with $ay = H_0$. Then $(\pi, aC) \in \mathcal{A}(ay)$ and $X_T^{ay,\pi,aC} = aX_T^{y,\pi,C}$. Let $P \in \mathcal{P}$, w.l.o.g. we may assume $P(H_T > 0) > 0$. Due to

$$
1 = P(X_T^{y,\pi,C} \geq H_T) \leq P(aX_T^{y,\pi,C} > H_T) + P(X_T^{y,\pi,C} = 0 = H_T)
$$

we deduce $P(X_T^{ay,\pi,aC} > H_T) > 0$. Hence, (ay, π, aC) constitutes an arbitrage. $\qquad \square$

Lemma 4.3.10 *For any $H_0 \notin \mathcal{L} \cup \mathcal{U}$ the financial market $(\mathcal{M}, H)$ is arbitrage free.*

PROOF: Assume $H_0 \notin \mathcal{U}$, $H_0 \notin \mathcal{L}$ and that there exists an arbitrage opportunity in $(\mathcal{M}, H)$. We suppose that it satisfies Definition 4.3.4 for $a = -1$. The case $a = 1$ works similarly.

By definition of arbitrage there exist $y \geq 0$, $(\pi, C) \in \mathcal{A}(y)$ with

$$
y = X_0^{y,\pi,C} \leq H_0
$$

and
$$X_T^{y,\pi,C} \geq H_T \quad q.s.$$

Hence, $y \in \mathcal{U}$, whence $H_0 \in \mathcal{U}$ by Lemma 4.3.7. This contradicts our assumption. $\square$

Now we pass to the proof of Theorem 4.3.6.

PROOF: Let us begin with the first identity $h_{up} = E_G(H_T \gamma_T^{-1})$. As seen in the proof of Lemma 4.3.8, for any $y \in \mathcal{U}$ we have $y \geq E_G\left(H_T \gamma_T^{-1}\right)$. Hence, $h_{up} = \inf\{y | y \in \mathcal{U}\} \geq E_G\left(H_T \gamma_T^{-1}\right)$.

To show the opposite inequality define the G-martingale M by

$$M_t := E_G\left(H_T \gamma_T^{-1} \big| \mathcal{F}_t\right) \quad \forall t \leq T.$$

By the martingale representation theorem (Song (2010c)), see also Theorem A.4.19, there exist $z \in H_G^1(0,T)$ and a continuous, increasing process $K = (K_t)$ with $K_T \in L_G^1(\Omega_T)$ such that for any $t \leq T$

$$M_t = E_G(H_T \gamma_T^{-1}) + \int_0^t z_s dB_s - K_t \quad q.s.$$

For any $t \leq T$ we set $y = E_G(H_T \gamma_T^{-1}) \geq 0$, $X_t \pi_t = z_t \gamma_t \in H_G^1(0,T)$, and $C_t = \int_0^t \gamma_s dK_s \in L_G^1(\Omega_T)$. Then the induced wealth process $X^{y,\pi,C}$ satisfies for any $t \leq T$

$$\gamma_t^{-1} X_t^{y,\pi,C} = y + \int_0^t X_s^{y,\pi,C} \pi_s \gamma_s^{-1} dB_s - \int_0^t \gamma_s^{-1} dC_s = M_t.$$

C obeys the conditions of a cumulative consumption process in the sense of Definition 4.2.3 due to the properties of K. Because of $\gamma_t^{-1} X_t^{y,\pi,C} = M_t \geq 0 \; \forall t \leq T$ the wealth process is bounded from below, whence (π, C) is admissible for y.

As $X_T^{y,\pi,C} = \gamma_T M_T = H_T$ quasi-surely we have $y = E_G(H_T \gamma_T^{-1}) \in \mathcal{U}$. Due to the definition of h_{up} we conclude $h_{up} \leq E_G(H_T \gamma_T^{-1})$.

The proof for the second identity is similar. Again, using the proof of Lemma 4.3.8 we obtain $y \leq -E_G\left(-H_T \gamma_T^{-1}\right)$ for any $y \in \mathcal{L}$ and therefore $h_{low} \leq -E_G\left(-H_T \gamma_T^{-1}\right)$.

To obtain $h_{low} \geq -E_G(-H_T \gamma_T^{-1})$ we again define a G-martingale M by

$$M_t = E_G(-H_T \gamma_T^{-1} | \mathcal{F}_t) \quad \forall t \leq T.$$

The remaining part is almost a copy of above. Again by the martingale representation theorem (Song (2010c)) there exist $z \in H_G^1(0, T)$ and a continuous, increasing process $K = (K_t)$ with $K_T \in L_G^1(\Omega_T)$ such that for any $t \leq T$

$$M_t = E_G(-H_T \gamma_T^{-1}) + \int_0^t z_s dB_s - K_t \quad q.s.$$

As above, for any $t \leq T$ we set $-y = E_G(-H_T \gamma_T^{-1}) \geq 0$, $X_t \pi_t = z_t \gamma_t \in H_G^1(0, T)$, and $C_t = \int_0^t \gamma_s dK_s \in L_G^1(\Omega_T)$. Then the induced wealth process $X^{-y, \pi, C}$ satisfies for all $t \leq T$

$$\gamma_t^{-1} X_t^{-y, \pi, C} = -y + \int_0^t X_s^{-y, \pi, C} \pi_s \gamma_s^{-1} dB_s - \int_0^t \gamma_s^{-1} dC_s = M_t.$$

Again C obeys the conditions of a cumulative consumption process due to the properties of K. Furthermore, for any $t \leq T$

$$\gamma_t^{-1} X_t^{-y, \pi, C} = E_G\left(-H_T \gamma_T^{-1} | \mathcal{F}_t\right) \geq E_G\left(-H_T | \mathcal{F}_t\right)$$

which is bounded from below in the sense of item (iii) in Definition 4.2.7 since $H_T \in L_G^2(\Omega_T)$, thus $E_G\left(-H_T | \mathcal{F}_t\right) \in L_G^2(\Omega_t)$. Hence the wealth process is bounded from below. Consequently, (π, C) is admissible for $-y$.
As $X_T^{-y, \pi, C} = \gamma_T M_T = -H_T$ quasi-surely we have $y = -E_G(-H_T \gamma_T^{-1}) \in \mathcal{L}$. Due to the definition of h_{low} we conclude $h_{low} \geq -E_G(-H_T \gamma_T^{-1})$ which finishes the proof. $\qquad \square$

Remark 4.3.11 *By the last theorem we have $h_{low} \neq h_{up}$ in general since E_G is a sublinear expectation. This implies that the market is incomplete, meaning that not all claims can be hedged perfectly. Thus in general, there are many no-arbitrage prices for H. We always have $h_{low} \neq h_{up}$ as long as $(E_G[H_T \gamma_T^{-1} | \mathcal{F}_t])$ is not a symmetric G-martingale. In the other case, the process K is identically equal to zero, see Remark A.4.18, implying that $(E_G[H_T \gamma_T^{-1} | \mathcal{F}_t])$ is symmetric meaning that there is no mean uncertainty, and H_T can be hedged perfectly due to Theorem A.4.19 and Remark A.4.18.*

As it is showed in Section 4.4, if H for instance is the usual European call or put option this is only the case if $\underline{\sigma} = \overline{\sigma}$ which again implies that E_G becomes a classical expectation.

Remark 4.3.12 *Again under the presumption of $h_{low} \neq h_{up}$ it is not clear a priori whether a claim's price H_0 equal to h_{up} or h_{low} induces an arbitrage*

opportunity or not. In the setting of Karatzas and Kou (1996) there may be situations where there is no arbitrage, while in others there may be arbitrage: for instance, if $H_0 = h_{up} \in \mathcal{U}$ and $C_T > 0$ a.s., then this consumption can be viewed as kind of arbitrage opportunity. The agent consumes along the way, and ends up with terminal wealth H_T almost surely (see Karatzas and Kou (1996)).

As seen in the proof of Theorem 4.3.6, in our setting we always have $h_{up} \in \mathcal{U}$ and $h_{low} \in \mathcal{L}$. We shall see that due to our definition of arbitrage – $P\left(X_T^{y,\pi,C} - aH_T > 0\right) > 0$ only has to hold for one $P \in \mathcal{P}$ – we have that a price $H_0 = h_{up}$ or $H_0 = h_{low}$ induces arbitrage in $(\mathcal{M}, H)$ in the sense of our Definition 4.3.4.

Corollary 4.3.13 *For any price $H_0 \in (h_{low}, h_{up}) \neq \emptyset$ of a European contingent claim at time zero there does not exist any arbitrage opportunity in $(\mathcal{M}, H)$.*

For any price $H_0 \notin (h_{low}, h_{up}) \neq \emptyset$ there does exist arbitrage in the market.

PROOF: The first part directly follows from Lemma 4.3.10. From Lemma 4.3.9 we know that $H_0 \notin [h_{low}, h_{up}]$ implies the existence of an arbitrage opportunity. Therefore we only have to show that an initial price $H_0 = h_{up}$ or $H_0 = h_{low}$ admits an arbitrage opportunity.

We only treat the case $H_0 = h_{up}$, as the second case is analogue. Comparing the proof of Theorem 4.3.6, for $y = E_G(H_T \gamma_T^{-1})$ there exists a pair $(\pi, C) \in \mathcal{A}(y)$ such that

$$\gamma_T^{-1} X_T^{y,\pi,C} = y + \int_0^T X_s^{y,\pi,C} \pi_s \gamma_s^{-1} dB_s - \int_0^T \gamma_s^{-1} dC_s = H_T \gamma_T^{-1} \quad q.s.$$

We had $K_T = \int_0^T \gamma_s^{-1} dC_s$ where K was an increasing, continuous process with $E_G(-K_T) = 0$. Hence we can select $P \in \mathcal{P}$ such that $\mathbb{E}^P(-K_T) < 0$, (see also Remark 4.3.11). Then the pair $(\pi, 0) \in \mathcal{A}(y)$ satisfies

$$\mathbb{E}^P\left(\gamma_T^{-1} X_T^{y,\pi,0}\right) > \mathbb{E}^P\left(\gamma_T^{-1} X_T^{y,\pi,C}\right) = \mathbb{E}^P\left(H_T \gamma_T^{-1}\right).$$

Thus, $P\left(X_T^{y,\pi,0} > H_T\right) > 0$ and we conclude that $(\pi, 0) \in \mathcal{A}(y)$ constitutes an arbitrage. So, possibly the agent may consume along the way, and end up with wealth H_T quasi-surely. $\qquad\square$

Remark 4.3.14 *Note that the second statement of the corollary depends heavily on the definition of arbitrage. Under the assumption of $h_{low} \neq h_{up}$, it states that if H_0 is equal to one of the bounds h_{up} or h_{low} there exists arbitrage in the sense of Definition 4.3.4.*

Coming back to the discussion about the definition of arbitrage begun in Remark 4.3.2, the proofs of the corollary and Theorem 4.3.6 also imply that if we required the last condition in Definition 4.3.4 to be true for all scenarios $P \in \mathcal{P}$, then H_0 equal to one of the bounds would not induce arbitrage in this new sense. Hence, h_{up} and h_{low} would be reasonable prices for the claim.

However, there would exist profit with positive probability in many scenarios. Only the scenarios $P \in \mathcal{P}$ that satisfy $\mathbb{E}^P(-K_T) = 0$ would not provide profit with positive probability. Thus, all $P \in \mathcal{P}$ not being maximizers of $\sup_{P \in \mathcal{P}} \mathbb{E}^P(-K_T)$ would induce arbitrage in the classical sense when only one probability measure was involved.

From our point of view such a situation should be identified as arbitrage, therefore supporting our definition of arbitrage in 4.3.1 and 4.3.4.

Additionally, even though our arbitrage definition requires profit with positive probability for only one scenario, it is simultaneously satisfied for all $P \in \mathcal{P}$ which are not maximizers of $\sup_{P \in \mathcal{P}} \mathbb{E}^P(-K_T)$ – which are quite a few.

Based on the corollary, we call $(h_{low}, h_{up}) \neq \emptyset$ the arbitrage free interval. In the case where a more explicit martingale representation theorem for $(E_G[\gamma_T^{-1} H_T | \mathcal{F}_t])$ holds, Hu and Peng (2010), we obtain a more explicit form for the cumulative consumption process C, see also Theorem A.4.17. In particular, in the Markovian setting where $H_T = \Phi(S_T)$ for some Lipschitz function $\Phi : \mathbb{R} \to \mathbb{R}$, we can give more structural details about the bounds h_{up} and h_{low}. We investigate this issue in the following section.

4.4 The Markovian setting

We consider the same financial market $\mathcal{M}$ as before and restrict ourselves to European contingent claims H which have the form $H_T = \Phi(S_T)$ for some Lipschitz function $\Phi : \mathbb{R} \to \mathbb{R}$.

We will use a nonlinear Feynman-Kac formula established in Peng (2010).

For this issue let us also write the dynamics of S in (4.3) as

$$dS_u^{t,x} = rS_u^{t,x}du + S_u^{t,x}dB_u, \quad u \in [t,T], \quad S_t^{t,x} = x > 0$$

to indicate that the stock price begins in x at time t. Similarly to the lower and upper arbitrage prices at time 0, we define the lower and upper arbitrage prices at time $t \in [0,T]$, $h_{low}^t(x)$ and $h_{up}^t(x)$. We use the dependence on x to indicate that the stock price is at level x at a considered time t, i.e., $S_t = x$.

The following theorem is an extension of Theorem 4.3.6. It establishes the connection of the lower and upper arbitrage prices with solutions of partial differential equations.

Theorem 4.4.1 *Given a European contingent claim $H = \Phi(S_T)$ the upper arbitrage price $h_{up}^t(x)$ is given by $u(t,x)$ where $u : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is the unique solution of the PDE*

$$\partial_t u + rx\partial_x u + G\left(x^2\partial_{xx}u\right) = ru, \quad u(T,\cdot) = \Phi(\cdot). \tag{4.5}$$

An explicit representation for the corresponding trading strategy in the stock and the cumulative consumption process is given by

$$\phi_t = \partial_x u(t, S_t) \quad \forall t \in [0,T],$$
$$C_t = -\frac{1}{2}\int_0^t \partial_{xx}u(s, S_s)S_s^2 d\langle B\rangle_s + \int_0^t G\left(\partial_{xx}u(s, S_s)\right)S_s^2 ds \quad \forall t \in [0,T].$$

Similarly, the lower arbitrage price $h_{low}^t(x)$ is given by $-\underline{u}(t,x)$ where $\underline{u} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ also solves (4.5) but with terminal condition $\underline{u}(T,x) = -\Phi(x) \forall x \in \mathbb{R}_+$. Also, the analog expressions hold true for the corresponding trading strategy and the cumulative consumption process.

The PDE in (4.5) is called the Black-Scholes-Barenblatt equation. It is also established in Avellaneda, Levy, and Paras (1995).

Before passing to the proof, as a preparation let us consider the BSDE

$$Y_s^{t,x} = E_G\left(\Phi(S_T^{t,x}) + \int_s^T f(S_r^{t,x}, Y_r^{t,x})dr|\mathcal{F}_s\right), \quad s \in [t,T]$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given Lipschitz function. Since the BSDE has a unique solution, as indicated by Peng (2010), we can define a function $u : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ by $u(t,x) := Y_t^{t,x}$, $(t,x) \in [0,T] \times \mathbb{R}_+$. Based on a

nonlinear version of the Feynman-Kac formula, per Peng (2010), the function u is a viscosity solution of the following PDE[14]

$$\partial_t u + rx\partial_x u + G(x^2 \partial_{xx} u) + f(x,u) = 0, \quad u(T,\cdot) = \Phi(\cdot). \qquad (4.6)$$

Now we come to the proof of Theorem 4.4.1.

PROOF: It is enough just to treat the upper arbitrage price. For this purpose, define the function

$$\hat{u}(t,x) := E_G\left(\Phi(S_T^{t,x})\gamma_T^{-1}\right).$$

As mentioned above we know $\hat{u}$ solves the PDE in (4.6) for $f \equiv 0$ and with terminal condition $\Phi(\cdot)\gamma_T^{-1}$. Since the function G is non-degenerate, all solutions to PDE (4.6) are classical $C^{1,2}$-solutions, (see Remark A.4.5 or Appendix C, Section 4 in Peng (2010)). Therefore, together with Itô's formula (Theorem 5.4 in Li and Peng (2009)) we obtain with initial stock price $S_0 = x$

$$\hat{u}(t,S_t) - \hat{u}(0,x)$$
$$= \int_0^t \partial_t \hat{u}(s,S_s) + rS_s \partial_x \hat{u}(s,S_s) ds$$
$$+ \int_0^t S_s \partial_x \hat{u}(s,S_s) dB_s + \int_0^t \frac{1}{2} S_s^2 \partial_{xx} \hat{u}(s,S_s) d\langle B \rangle_s$$
$$\overset{(4.6)}{=} \int_0^t S_s \partial_x \hat{u}(s,S_s) dB_s$$
$$+ \underbrace{\frac{1}{2}\int_0^t S_s^2 \partial_{xx} \hat{u}(s,S_s) d\langle B \rangle_s - \int_0^t S_s^2 G\left(\partial_{xx}\hat{u}(s,S_s)\right) ds}_{= -K_t}.$$

Next, consider the function

$$\tilde{u}(t,x) := \gamma_t \hat{u}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}_+.$$

As in Theorem 4.3.6 for $t = 0$, we can deduce that $\tilde{u}(t,x) = h_{up}^t(x) \ \forall (t,x) \in [0,T] \times \mathbb{R}_+$. In addition, one easily checks by plugging $\hat{u}(t,x) = \tilde{u}(t,x)\gamma_t^{-1}$

---

[14]If f satisfies a particular growth condition in x, u is also the unique viscosity solution, see Ishii and Lions (1990). For the definition of viscosity solutions we also refer the reader to Ishii and Lions (1990).

into its PDE that $\tilde{u}$ is a solution of the PDE in (4.5). Also the function u defined by

$$u(t,x) := Y_t^{t,x} = E_G\left(\Phi(S_T^{t,x}) - \int_t^T rY_s^{t,x}ds|\mathcal{F}_t\right) \ \forall(t,x) \in [0,T] \times \mathbb{R}_+$$

solves the PDE in (4.5) due to the nonlinear Feynman-Kac formula with $f(x,y) = -ry$. By uniqueness of the solution in (4.5), see Ishii and Lions (1990) (f is obviously bounded in x), we conclude that $\tilde{u} = u$. Hence, $u(t,x) = E_G\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right) = h_t^{up}(x) \ \forall(t,x) \in [0,T] \times \mathbb{R}_+$ and it uniquely solves the PDE (4.5).

The explicit expressions for the trading strategy ϕ and the cumulative consumption process C follow from the same calculations for u as done above for $\hat{u}$. Then comparing the resulting equation with the wealth equation in Definition 4.2.6 and using the identity $\phi_t = \frac{X_t\pi_t}{S_t}$, Remark 4.2.4 we obtain $\phi_t = \partial_x u(t, S_t) \ \forall t \in [0,T]$ and

$$C_t = -\frac{1}{2}\int_0^t S_s^2 \partial_{xx}u(s, S_s)d\langle B\rangle_s + \int_0^t S_s^2 G\left(\partial_{xx}u(s, S_s)\right)ds.$$

$\square$

Due to Theorem 4.4.1 the functions $u(t,x) = h_{up}^t(x)$ and $\underline{u}(t,x) = -h_{low}^t(x)$ can be characterized as the unique solutions of the Black-Scholes-Barrenblatt equation. In the case of Φ being a convex or concave function, respectively, the PDE in (4.5) simplifies significantly. Due to the following result it becomes the classical Black-Scholes PDE in (4.7) for a certain constant volatility level.

Lemma 4.4.2 *1. If Φ is convex, $u(t,\cdot)$ is convex for any $t \leq T$.*

2. If Φ is concave, $u(t,\cdot)$ is concave for any $t \leq T$.

Analogously, if Φ is convex, $\underline{u}(t,\cdot)$ is concave for any $t \leq T$. If Φ is concave, $\underline{u}(t,\cdot)$ is convex for any $t \leq T$.

PROOF: Again we only need to consider the upper arbitrage price. It is determined by the function $u(t,x) = E_G\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right) \ \forall(t,x) \in [0,T] \times \mathbb{R}_+$.

Firstly, let Φ be convex, $t \in [0,T)$, and $x, y \in \mathbb{R}_+$. Then we have for any

$\alpha \in [0, 1]$

$$u(t, \alpha x + (1 - \alpha)y) = E_G \left[\Phi \left(S_T^{t, \alpha x + (1-\alpha)y} \right) \gamma_{T-t}^{-1} \right]$$

$$= E_G \left[\Phi \left((\alpha x + (1 - \alpha)y) \, e^{r(T-t) - \frac{1}{2} <B>_{T-t} + B_{T-t}} \right) \gamma_{T-t}^{-1} \right]$$

$$\leq E_G \left[\alpha \Phi \left(x e^{r(T-t) - \frac{1}{2} <B>_{T-t} + B_{T-t}} \right) \right.$$

$$\left. + (1 - \alpha) \Phi \left(y e^{r(T-t) - \frac{1}{2} <B>_{T-t} + B_{T-t}} \right) \right] \gamma_{T-t}^{-1}$$

$$\leq E_G \left[\alpha \Phi \left(x e^{r(T-t) - \frac{1}{2} <B>_{T-t} + B_{T-t}} \right) \gamma_{T-t}^{-1} \right]$$

$$+ E_G \left[(1 - \alpha) \Phi \left(y e^{r(T-t) - \frac{1}{2} <B>_{T-t} + B_{T-t}} \right) \gamma_{T-t}^{-1} \right]$$

$$= \alpha E_G \left[\Phi \left(S_T^{t,x} \right) \gamma_{T-t}^{-1} \right] + (1 - \alpha) E_G \left[\Phi \left(S_T^{t,y} \right) \gamma_{T-t}^{-1} \right]$$

$$= \alpha u(t, x) + (1 - \alpha) u(t, y)$$

where we used the convexity of Φ, the monotonicity of E_G and in the second inequality, the sublinearity of E_G. Thus, $u(t, \cdot)$ is convex for all $t \in [0, T]$.

Secondly, let Φ be concave. Define for any $(t, x) \in [0, T] \times \mathbb{R}_+$

$$v(t, x) := \mathbb{E}^P \left[\Phi \left(\tilde{S}_T^{t,x} \right) \gamma_{T-t}^{-1} \right]$$

where

$$d\tilde{S}_s^{t,x} = r \tilde{S}_s^{t,x} ds + \underline{\sigma} \tilde{S}_s^{t,x} dW_s, \quad s \in [t, T], \quad \tilde{S}_t^{t,x} = x.$$

Remember that $W = (W_t)$ is a classical Brownian motion under P. Then by the classical Feynman-Kac formula v solves the Black-Scholes PDE in (4.7) with $\overline{\sigma}$ replaced by $\underline{\sigma}$.

Since $\mathbb{E}^P$ is linear it is straightforward to show that $v(t, \cdot)$ is concave for any $t \in [0, T]$. As a consequence, v also solves (4.5) since $G(\partial_{xx} v) = \frac{1}{2} \underline{\sigma}^2 \partial_{xx} v$. By uniqueness of the solutions to (4.5) we conclude $v = u$. Hence, $u(t, \cdot)$ is concave for any $t \in [0, T]$. $\qquad \square$

Consequently, we have the following corollary.

Corollary 4.4.3 *If Φ is convex, $h_{up}^0(x) = \mathbb{E}^{P^{\overline{\sigma}}} \left(\Phi(S_T^{0,x}) \gamma_T^{-1} \right)$ and*

$$u(t, x) := E_G \left(\Phi(S_T^{t,x}) \gamma_{T-t}^{-1} \right) = \mathbb{E}^{P^{\overline{\sigma}}} \left(\Phi(S_T^{t,x}) \gamma_{T-t}^{-1} \right)$$

solves the Black-Scholes PDE

$$\partial_t u + rx\partial_x u + \frac{1}{2}\overline{\sigma}^2 x^2 \partial_{xx} u = ru, \quad u(T,\cdot) = \Phi(\cdot). \tag{4.7}$$

If Φ is concave, $h_{up}^0(x) = \mathbb{E}^{P^{\underline{\sigma}}}\left(\Phi(S_T^{0,x})\gamma_T^{-1}\right)$ and

$$u(t,x) := E_G\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right) = \mathbb{E}^{P^{\underline{\sigma}}}\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right)$$

solves the PDE in (4.7) with $\underline{\sigma}$ replacing $\overline{\sigma}$.
An analogue result holds for the lower arbitrage price h_{low}, or terminal condition $\underline{u}(T,\cdot) = -\Phi(\cdot)$, respectively.

PROOF: The result directly follows from Theorem 4.4.1 and Lemma 4.4.2. The equality of for instance $E_G\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right)$ and $\mathbb{E}^{P^{\underline{\sigma}}}\left(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}\right)$ follows by the same reasoning as above and the fact that $\tilde{S}$ under P and S under $P^{\underline{\sigma}}$ are identically distributed since B is the canonical process. $\square$

Example 4.4.4 (European call option) *Consider for $K > 0$ the function $\Phi(x) = (x-K)^+$ which represents the payoff of a European call option. Since Φ is convex, and $-\Phi$ concave, we can deduce by means of the last corollary*

$$h_{up}^0(x) = \mathbb{E}^{P^{\overline{\sigma}}}\left((S_T^{0,x} - K)^+ \gamma_T^{-1}\right),$$
$$h_{low}^0(x) = -\mathbb{E}^{P^{\underline{\sigma}}}\left(-(S_T^{0,x} - K)^+ \gamma_T^{-1}\right).$$

Furthermore, the function

$$u(t,x) := \mathbb{E}^{P^{\overline{\sigma}}}\left((S_T^{t,x} - K)^+ \gamma_{T-t}^{-1}\right), \quad (t,x) \in [0,T] \times \mathbb{R}_+,$$

solves the PDE in (4.7). The function

$$\underline{u}(t,x) := \mathbb{E}^{P^{\underline{\sigma}}}\left(-(S_T^{t,x} - K)^+ \gamma_{T-t}^{-1}\right), \quad (t,x) \in [0,T] \times \mathbb{R}_+,$$

solves Equation (4.7) with $\overline{\sigma}$ replaced by $\underline{\sigma}$ and boundary condition $\underline{u}(T,x) = -(x-K)^+ \; \forall x \in \mathbb{R}_+$.

If Φ exhibits mixed convexity/concavity behavior, meaning that for instance, there exists an $x^\star \in \mathbb{R}_+$ such that $\Phi \upharpoonright_{[0,x^\star]}$ is convex whereas $\Phi \upharpoonright_{[x^\star,\infty)}$ is concave, the situation becomes much more involved.

For example, in the case when Φ represents a bullish call spread as considered in Avellaneda, Levy, and Paras (1995), the worst-case volatility will

switch between the volatility bounds $\underline{\sigma}$ and $\overline{\sigma}$ at some threshold $\bar{x}(t)$. The t indicates the time dependence of the threshold. This fact can be verified by solving the PDE in (4.5) numerically, Avellaneda, Levy, and Paras (1995).

The evaluation of Φ becomes economically relevant when Φ represents complex derivatives or a whole portfolio which combines long and short positions. Pricing the whole portfolio is more efficient than pricing the single positions separately. This leads to more reasonable results for the no-arbitrage bounds since the bounds are more closely based on the subadditivity of E_G. Numerical methods for solving the Black-Scholes-Barenblatt PDE in (4.5) can be found in Meyer (2004).

4.5 Conclusion

We present a general framework in mathematical finance in order to deal with model risk caused by volatility uncertainty. This encompasses the extension of terminology widely used in finance such as portfolio strategy, consumption process, arbitrage prices and the concept of no-arbitrage. It is being modified to a quasi-sure analysis framework resulting from the presence of volatility uncertainty.

Our setting does not involve any reference measure, and hence, does not exclude any economically interesting model a priori. We consider a stock price modeled by a geometric G-Brownian motion which features volatility uncertainty based on the structure of a G-Brownian motion. In this ambiguous financial setting, we examine the pricing and hedging of European contingent claims. The "G-framework" summarized in Peng (2010) gives us a meaningful and appropriate mathematical setting. By means of a slightly new concept of no-arbitrage, we establish detailed results which provide a better economic understanding of financial markets under volatility uncertainty.

The current work may form the basis for examining economically relevant questions in the presence of volatility uncertainty in the sense that it extends important notions in finance and shows how to control the different factors. Concrete examples are problems such as hedging under constraints (Karatzas and Kou (1996)) and portfolio optimization (cf. Merton (1990)). A natural step is to extend the above results to American contingent claims and then, for instance, consider the entry decisions of a firm in the sense of irreversible investments as in Nishimura and Ozaki (2007) who solved the problem in the presence of drift uncertainty.

By the natural properties of sublinear expectation any sublinear expectation induces a coherent risk measure, Peng (2010). G-expectation may

appear as a natural candidate to measure model risk. In this context, one might also imagine many concrete applications for finance.

Appendix A

Proofs and supplementary material

A.1 Proof of Theorem 2.3.1

PROOF: We give proof for the case when $A(t, S_t)$ is decreasing in S_t for all $t \leq T$.[1] The second case works analogously.[2] As in Section 2.3.2, we write $\omega_{(t)}$ for an element in $\Omega_{(t)} = \bigotimes_{i=1}^{t} \{0, 1\}, t \leq T$. For a stopping time τ we introduce for each $t \leq T$ the restriction τ^t of τ to paths in Ω running up to time t, that is,

$$\tau^t : \bigotimes_{i=1}^{t} \{0, 1\} \longrightarrow [0, t] \cup \{T + 1\}$$

$$\omega_{(t)} \longmapsto \tau^t(\omega_{(t)}) = \begin{cases} \tau(\omega_{(t)}), & \text{if } \tau(\omega_{(t)}) \leq t \\ T + 1, & \text{else} \end{cases}.$$

One easily checks the stopping time property for these restricted mappings. It is just inherited from τ. We will prove in the following that the value function $\mathcal{U}_t^{\mathcal{Q}}, t \leq T$ can be written as a function u depending on current time, current stock price, and current values of the two stopping times, i.e.,

$$\mathcal{U}_t^{\mathcal{Q}} = u(t, S_t, \tau_1^t, \tau_2^t) = \begin{cases} \tilde{u}_1(t, S_t) \mathbb{1}_{[0, \tau_1^t[}(t) \\ \tilde{u}_2(t, S_t) \mathbb{1}_{[\tau_1^t, \tau_2^t[}(t) \end{cases} \tag{A.1}$$

---

[1]Due to the assumption in (2.5), $A(S_t)$ is asked to be decreasing in its argument.

[2]It is even easier since τ_1 is up-crossing time and $A(t, S_t)$ is increasing in S_t. Thus the agent fears low stock returns by time σ_1, see also Remark 2.3.3.

where $\tilde{u}_1(t, S_t)$ is increasing and $\tilde{u}_2(t, S_t)$ decreasing in S_t.

We start the proof with

Lemma A.1.1 *Let $(U_t^{\mathcal{Q}})$ be the multiple priors Snell envelope of X as defined in Theorem 2.3.1. Assume that $U_t^{\mathcal{Q}}$ is a function of the form as in (A.1) for all $t \leq T$. Then for all $t \in [0, T-1], S \in E_t$ and all $k \in [1, T-t]$*

$$u(t, S, t, T+1) \geq u(t+k, S, t+k, T+1).$$

PROOF: The inequality follows directly by the inequality

$$u(t, S, t, T+1) \geq u(t+k, S, t, T+1) = u(t+k, S, t+k, T+1).$$

The inequality always holds for claims of American style whose non-discounted payoff from exercising only depends on the underlying's current price S at all times.[3] Since $\tau_1^t = t$ and $\tau_2^t = T+1$ it also holds for the considered claims of the theorem. The equality holds since the claim is already knocked in, see (A.1). □

Using theory about the multiple priors Snell envelope, see Riedel (2009), we show by backward induction in t that $U_t^{\mathcal{Q}}$ possesses the representation in (A.1) for all $t \leq T$ such that u additionally exhibits the properties

(i) for $t < \tau_1^t$: $S_t \mapsto u(t, S_t, \tau_1^t(S_t), \tau_2^t(S_t)) \nearrow \ \forall S_t \in E_t$ with $S_t \leq H_1$[4]

(ii) for $t \in [\tau_1^t, \tau_2^t[$: $S_t \mapsto u(t, S_t, \tau_1^t(S_t), \tau_2^t(S_t)) \searrow \ \forall S_t \in E_t$

(iii) for $t \geq \tau_2^t$: $S_t \mapsto u(t, S_t, \tau_1^t(S_t), \tau_2^t(S_t)) = 0 \ \forall S_t \in E_t$

u is well-defined since the payoff process X is uniquely determined by t, S_t, τ_1^t and τ_2^t. Note, this representation for $U^{\mathcal{Q}}$ implies the representation of the worst-case measure $\hat{P}$ using Theorem 2.2.5. We handle this issue simultaneously when proving the representation and properties of u.

---

[3] Of course, it is essential that the set of possible stock price distributions is the same at each node and the option's payoff is independent of time on the event $[\tau_1, \tau_2[$, cf. Equation (2.5).

[4] With slight abuse of notation we may write $\tau_i^t(S_t), i = 1, 2$ whenever the situation is unambiguous.

For $t = T$ we have

$$
U_T^{\mathcal{Q}}(\cdot) = X_T(\cdot) = \mathbb{1}_{[\tau_1^T, \tau_2^T[}(T, \cdot)\, A(T, S_T(\cdot))
$$
$$
= \begin{cases}
0 = u(T, S_T, \tau_1^T, \tau_2^T)\ \forall S_T, & \text{if } \tau_1^T = T + 1 \text{ or } \tau_2^T \le T \\
A(T, S_T) = u(T, S_T, \tau_1^T, T+1)\ \forall S_T, & \text{if } \tau_1^T \le T < \tau_2^T
\end{cases},
$$

and $U_T^{\mathcal{Q}}$ satisfies the representation and the properties because of the assumptions on $A(T, \cdot)$.[5]

In the induction step for $t < T$ we treat the various cases separately.

First, assume $t \in [\tau_1^t, \tau_2^t[$, say $\tau_1^t(\omega_{(t)}) =: k \le t$, $(\tau_2^t(\omega_{(t)}) = T + 1)$: then

$$
U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{ X_t(\omega_{(t)}), \min_{P \in \mathcal{Q}}[U_{t+1}^{\mathcal{Q}} | \mathcal{F}_t](\omega_{(t)}) \right\}
$$
$$
\stackrel{\text{(IH)}}{=} \max\left\{ X_t(\omega_{(t)}), \min_{p_{t+1} \in [\underline{p}, \bar{p}]} \left\{ p_{t+1} u(t+1, S_t u, k, \tau_2^{t+1}(\omega_{(t)}, 1)) \right. \right.
$$
$$
\left. \left. + (1 - p_{t+1}) u(t+1, S_t d, k, \tau_2^{t+1}(\omega_{(t)}, 0)) \right\} \right\}.
$$

By induction hypothesis and due to $\tau_2^{t+1}(\omega_{(t)}, 0) = T + 1 \ge \tau_2^{t+1}(\omega_{(t)}, 1)$, properties (ii) and (iii) for $t + 1$ imply
$u(t+1, S_t d, k, \tau_2^{t+1}(\omega_{(t)}, 0)) \ge u(t+1, S_t u, k, \tau_2^{t+1}(\omega_{(t)}, 1))$. Therefore,

$$
U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{ X_t(\omega_{(t)}), \bar{p}\, u(t+1, S_t u, k, \tau_2^{t+1}(S_t u)) \right.
$$
$$
\left. + (1 - \bar{p}) u(t+1, S_t d, k, T+1) \right\}
$$

and $\hat{P}(\varepsilon_{t+1} = 1 | \omega_{(t)}) = \bar{p}$. In this case we deduce that $U_t^{\mathcal{Q}}$ is a function $u(t, S_t, \tau_1^t, \tau_2^t)$ decreasing in S_t since X_t and $u(t+1, \cdot, k, \tau_2^t(\cdot))$ are decreasing in current stock price S by induction hypothesis, (property (ii) and (iii), respectively).

Second, if $t \ge \tau_2^t(\omega_{(t)}) =: l < T$, $\tau_1^t(\omega_{(t)}) =: k < l$:

$$
U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{ X_t(\omega_{(t)}), \min_{p_{t+1} \in [\underline{p}, \bar{p}]} \right.
$$
$$
\left. \left\{ p_{t+1} u(t+1, S_t u, k, l) + (1 - p_{t+1}) u(t+1, S_t d, k, l) \right\} \right\}
$$
$$
= 0
$$

---

[5]If a function is constant zero, we interpret it as both increasing and decreasing in its argument.

since $X_t(\omega_{(t)}) = 0$ by assumption and $u(t+1, \cdot, k, l) = 0$ by induction hypothesis, (property (iii)). So we are free to choose $p_{t+1} = \overline{p}$.

Third, assume $t < \tau_1^t(\omega_{(t)}) = T + 1$:

In that case, $X_t(\omega_{(t)}) = 0$. We obtain in the <u>first case</u> for $\tau_1^{t+1}(\omega_{(t)}, 1) = T + 1$

$$
\begin{aligned}
U_t^{\mathcal{Q}}(\omega_{(t)}) = \min_{p_{t+1} \in [\underline{p}, \overline{p}]} &\left\{ p_{t+1} u(t+1, S_t u, \tau_1^{t+1}(\omega_{(t)}, 1), T+1) \right. \\
&\left. + (1 - p_{t+1}) u(t+1, S_t d, \tau_1^{t+1}(\omega_{(t)}, 0), T+1) \right\} \\
= &\underline{p} u(t+1, S_t u, T+1, T+1) + (1 - \underline{p}) u(t+1, S_t d, T+1, T+1)
\end{aligned}
$$

by induction hypothesis, (property (i)). Hence, $\hat{P}(\varepsilon_{t+1} = 1 | \omega_{(t)}) = \underline{p}$ and $u(t, \cdot, T+1, T+1)$ is increasing in $S_t \ \forall S_t < H_1$.

In the <u>second case</u> for $\tau_1^{t+1}(\omega_{(t)}, 1) = t + 1$, we obtain

$$
\begin{aligned}
U_t^{\mathcal{Q}}(\omega_{(t)}) = \min_{p_{t+1} \in [\underline{p}, \overline{p}]} &\left\{ p_{t+1} u(t+1, S_t u, t+1, T+1) \right. \\
&\left. + (1 - p_{t+1}) u(t+1, S_t d, T+1, T+1) \right\} \\
= &\underline{p} u(t+1, S_t u, t+1, T+1) + (1 - \underline{p}) u(t+1, S_t d, T+1, T+1)
\end{aligned}
$$

by induction hypothesis, (property (i) since $S_t u = H_1$), and we obtain $\hat{P}(\varepsilon_{t+1} = 1 | \omega_{(t)}) = \underline{p}$.

Again, $U_t^{\mathcal{Q}}(\omega_{(t)})$ is a function of the claimed form $u(t, S_t, T+1, T+1)$. To obtain the validity of property (i) for $u(t, S_t, \tau_1^t(S_t), \tau_2^t(S_t)) \ \forall S_t \leq H_1$ we still need to show $u(t, H_1, t, T+1) \geq u(t, H_1 d, T+1, T+1)$.

Using property (i) of induction hypothesis we derive the first inequality in the estimation below

$$
\begin{aligned}
u(t, H_1 d, T+1, T+1) = &\underline{p} u(t+1, H_1, t+1, T+1) \\
&+ (1 - \underline{p}) u(t+1, H_1 \cdot d^2, T+1, T+1) \\
\leq &\underline{p} u(t+1, H_1, t+1, T+1) \\
&+ (1 - \underline{p}) u(t+1, H_1, t+1, T+1) \\
= &u(t+1, H_1, t+1, T+1) \\
\leq &u(t, H_1, t, T+1).
\end{aligned}
$$

The last inequality is due to Lemma A.1.1. This completes the proof and $(U_t^{\mathcal{Q}})$ satisfies the same recursion as $(U_t^{\hat{P}})$. Thus, $(U_t^{\mathcal{Q}}) = (U_t^{\hat{P}})$ follows and the worst-case measure $\hat{P}$ can be identified by the density $\hat{D}_T$ as it is claimed.

An optimal stopping time is given by $\hat{\tau}$. This follows by general theory, see Riedel (2009). The time boundary τ_1 for the optimal stopping rule is due to the claim's knock-in feature. $\qquad\square$

A.2 Proof of Theorem 3.3.3

This proof is a slight modification of the proof of Theorem 7.2 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997). Since their formulation of the theorem is not directly related to multiple priors, we present the main ideas here.

Let (H_t) define the obstacle and H_T the terminal payoff of all regarded RBSDEs.

Consider the unique solution (Y_t^0, Z_t^0, K_t^0) of the RBSDE with dynamics

$$-dY_t^0 = \underbrace{-rY_t^0}_{=f^0(t,Y_t^0,Z_t^0)} dt + dK_t^0 - Z_t^0 dW_t.$$

Then for each $t \in [0, T]$

$$Y_t^0 = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(\int_t^\tau -rY_s^0 ds + H_\tau | \mathcal{F}_t \right),$$

see Proposition 2.3 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997). Analogously for any $\theta \in \Theta$, the solution $(Y_t^\theta, Z_t^\theta, K_t^\theta)$ of the RBSDE with dynamics

$$-dY_t^\theta = \underbrace{(-rY_t^\theta - \theta_t Z_t^\theta)}_{=f^\theta(t,Y_t^\theta,Z_t^\theta)} dt + dK_t^\theta - Z_t^\theta dW_t$$

satisfies for $t \in [0, T]$

$$Y_t^\theta = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(\int_t^\tau (-rY_s^\theta - \theta_s Z_s^\theta) ds + H_\tau | \mathcal{F}_t \right). \tag{A.2}$$

Now consider for $t \in [0, T]$ and any probability measure Q the equation

$$Y_t^Q = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q \left(\int_t^\tau -rY_s^Q ds + H_\tau | \mathcal{F}_t \right). \tag{A.3}$$

If $Q = Q^\theta$ for some $\theta \in \Theta$ then the solution (Y_t^Q, Z_t^Q, K_t^Q) of the RBSDE with dynamics

$$-dY_t^Q = -rY_t^Q dt + dK_t^Q - Z_t^Q dW_t^\theta$$

satisfies Equation (A.3). Using Girsanov's theorem, $W^\theta = W + \int_0^\cdot \theta_s ds$, we can rewrite the dynamics as

$$-dY_t^Q = (-rY_t^Q - \theta_t Z_t^Q) dt + dK_t^Q - Z_t^Q dW_t.$$

Thus, by uniqueness, we obtain $Y^Q = Y^\theta$, and as a consequence

$$\operatorname*{ess\,inf}_{Q \in \mathcal{P}} Y_t^Q = \operatorname*{ess\,inf}_{\theta \in \Theta} Y_t^\theta.$$

Since $f(t, y, z) \leq f^\theta(t, y, z) \; \forall y, z \in \mathbb{R}, \forall \theta \in \Theta$, we obtain by comparison for RBSDEs, Theorem 4.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997) that

$$Y_t \leq Y_t^\theta \quad \forall \theta \in \Theta.$$

Since Θ is weakly compact in $L^1([0, T] \times \Omega)$,[6] for any real-valued measurable process Z there exists $\theta^\star \in \Theta$ such that $\theta_t^\star Z_t = \max_{\theta \in \Theta} \theta_t Z_t = \kappa |Z_t| \; \forall t \in [0, T]$, by Lemma B.1 in Chen and Epstein (2002). Hence $f(t, y, z) = f^{\theta^\star}(t, y, z) \; dt \otimes P \; a.e. \; \forall y, z \in \mathbb{R}$ and

$$Y_t = Y_t^{\theta^\star} \geq \operatorname*{ess\,inf}_{\theta \in \Theta} Y_t^\theta, \; t \in [0, T].$$

In brief,

$$
\begin{aligned}
Y_t &= \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q \left(\int_t^\tau -r Y_s ds + H_\tau | \mathcal{F}_t \right) \\
&= \operatorname*{ess\,inf}_{\theta \in \Theta} \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(\int_t^\tau -r Y_s - \theta_s Z_s ds + H_\tau | \mathcal{F}_t \right) \\
&= \operatorname*{ess\,inf}_{\theta \in \Theta} Y_t^\theta.
\end{aligned}
$$

Using Proposition 7.1 in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997) and Bayes' rule (Lemma 5.3 in Karatzas and Shreve (1991)) we obtain for each $\theta \in \Theta$

$$
\begin{aligned}
Y_t^\theta &= \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \exp\{ -\int_t^\tau \theta_s dW_s - \frac{1}{2} \int_t^\tau \theta_s^2 ds \} | \mathcal{F}_t \right) \\
&= \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} | \mathcal{F}_t \right) = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^{Q^\theta} \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right).
\end{aligned}
$$

Hence,

$$Y_t = \operatorname*{ess\,inf}_{\theta \in \Theta} \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E} \left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} | \mathcal{F}_t \right) = \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}^Q \left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t \right).$$

---

[6]See Chen and Epstein (2002). This again induces the weak compactness of $\mathcal{P}$ which is that induced by the set of bounded measurable functions.

We clearly have

$$Y_t \geq \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{\theta \in \Theta} \mathbb{E}\left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} \,|\mathcal{F}_t\right).$$

To obtain the other inequality, we use the stopping time $D_t^\theta := \inf\{s \in [t,T] | Y_s^\theta = H_s\}$ which is optimal in Equation (A.2) for each fixed $\theta \in \Theta$, see El Karoui, Kapoudjian, Pardoux, Peng, and Quenez (1997), Theorem 7.2. Then

$$
\begin{aligned}
Y_t &= \mathbb{E}^{Q^{\theta^\star}}\left(H_{D_t^{\theta^\star}} \gamma_{D_t^{\theta^\star}-t}^{-1} | \mathcal{F}_t\right) \\
&= \operatorname*{ess\,inf}_{\theta \in \Theta} \mathbb{E}\left(H_{D_t^\theta} \gamma_{D_t^\theta-t}^{-1} \exp\{-\int_t^{D_t^\theta} \theta_s dB_s - \frac{1}{2}\int_t^{D_t^\theta} \theta_s^2 ds\} | \mathcal{F}_t\right) \\
&\leq \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{\theta \in \Theta} \mathbb{E}\left(H_\tau \gamma_{\tau-t}^{-1} \frac{z_\tau^\theta}{z_t^\theta} \,|\mathcal{F}_t\right) \\
&= \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t\right).
\end{aligned}
$$

This proves for $t \in [0,T]$

$$Y_t = \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q\left(H_\tau \gamma_{\tau-t}^{-1} | \mathcal{F}_t\right) = V_t.$$

By a continuity argument $Y_t = V_t \;\forall t \in [0,T]$ a.s.,[7] and $\tau_t^\star$ is optimal for V_t. Since the minimum for f is attained we conclude the claim for $t = 0$.

A.3 Proof of Theorem 3.4.1

We start with a lemma yielding that τ_H is stochastically largest under Q^κ in the set of priors $\mathcal{P}$ in the following sense.

Lemma A.3.1 *On $\{\tau_H > t\}$ we have for all t,s with $t < s \leq T$ and all $\theta \in \Theta$*

$$Q^\kappa\left(\tau_H \leq s | \mathcal{F}_t\right) \leq Q^\theta\left(\tau_H \leq s | \mathcal{F}_t\right).$$

---

[7]Cheng and Riedel (2010) showed that there exists a version of (V_t) that is right-continuous. Using this version we can deduce the claim.

PROOF: Throughout this proof, all results are conditioned on the event $\{\tau_H > t\}$. Consider for any $u \in (t, s]$ the set $\{X_u \geq H\}$ and define $M_u := \frac{1}{\sigma}[\ln \frac{H}{X_t} - (r - \frac{\sigma^2}{2})(u - t)]$. Let $\theta \in \Theta$. By definition and construction of Q^θ and W^θ by means of Girsanov's theorem we have

$$X_u = X_t \exp\{(r - \frac{\sigma^2}{2})(u - t) + \sigma(W_u - W_t)\}$$
$$= X_t \exp\{(r - \frac{\sigma^2}{2})(u - t) + \sigma(W_u^\theta - W_t^\theta) - \sigma \int_t^u \theta_s ds\}$$

for any $\theta \in \Theta$. Furthermore,

$$Q^\theta\big(\{X_u \geq H\}|\mathcal{F}_t\big) = Q^\theta\big(\{W_u - W_t \geq M_u\}|\mathcal{F}_t\big)$$
$$= Q^\theta\big(\{W_u^\theta - W_t^\theta - \int_t^u \theta_s ds \geq M_u\}|\mathcal{F}_t\big)$$
$$\geq Q^\theta\big(\{W_u^\theta - W_t^\theta - \kappa(u - t) \geq M_u\}|\mathcal{F}_t\big)$$
$$= Q^\kappa\big(\{W_u^\kappa - W_t^\kappa - \kappa(u - t) \geq M_u\}|\mathcal{F}_t\big)$$
$$= Q^\kappa\big(\{W_u - W_t \geq M_u\}|\mathcal{F}_t\big) = Q^\kappa\big(\{X_u \geq H\}|\mathcal{F}_t\big).$$

The inequality holds since for any $\theta \in \Theta$

$$\{W_u^\theta - W_t^\theta - \int_t^u \theta_s ds \geq M_u\} \supseteq \{W_u^\theta - W_t^\theta - \kappa(u - t) \geq M_u\}, \qquad \text{(A.4)}$$

the subsequent equality holds since both W^θ under Q^θ and W^κ under Q^κ are standard Brownian motions and M_u is deterministic on $\mathcal{F}_t$. Due to

$$\bigcup_{u \in (t, s]} \{X_u \geq H\} = \{\tau_H \leq s\} \in \mathcal{F}_s$$

and since the inclusion in (A.4) also holds for the union we conclude the result. □

Cheng and Riedel (2010) verified that the optimal stopped value process is a $\mathcal{P}$-multiple priors martingale in the sense that it, say (M_t) satisfies $M_t = \text{ess inf}_{Q \in \mathcal{P}} \mathbb{E}^Q (M_s|\mathcal{F}_t) \ \forall s, t \in [0, T]$ with $s \geq t$.[8]

To avoid any confusion, let us denote their value process by $(\bar{V}_{t \wedge \tau^\star})_{t \in [0, T]}$, where $\tau^\star$ is an optimal stopping time. In their setting, $\bar{V}_t$ denotes the value of the optimal stopping problem after time t at time zero.[9] In our setting, V_t denotes the value of the optimal stopping problem after time t at time t.

---

[8]Cheng and Riedel (2010) called this a g-martingale. See also Peng (1997).

[9]To fit into our setting the payoff for $(\bar{V}_t)$ has to be the discounted payoff which is $H_t \gamma_t^{-1}$ for each $t \in [0, T]$.

That is, $(V_{t\wedge\tau^\star}\gamma^{-1}_{t\wedge\tau^\star}) = (\bar{V}_{t\wedge\tau^\star})$ is a $\mathcal{P}$-multiple priors martingale. By optional sampling for $\mathcal{P}$-multiple priors martingales, Cheng and Riedel (2010) or Peng (1997) for any stopping time σ with $\sigma \geq t$ a.s.

$$\bar{V}_{t\wedge\tau^\star} = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(\bar{V}_{\sigma\wedge\tau^\star} | \mathcal{F}_{t\wedge\tau^\star} \right)$$

which yields

$$V_{t\wedge\tau^\star} = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\sigma\wedge\tau^\star} \gamma^{-1}_{\sigma\wedge\tau^\star - t\wedge\tau^\star} | \mathcal{F}_{t\wedge\tau^\star} \right). \tag{A.5}$$

Using (A.5) we can rewrite the optimal stopped value process as follows.

Lemma A.3.2 *Given $t \in [0,T]$. We have*

$$V_{t\wedge\tau^\star} = V_{\tau^\star}\mathbb{1}_{\{\tau^\star\leq t\}} + V_t\mathbb{1}_{\{\tau_H\leq t\}}\mathbb{1}_{\{\tau^\star>t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H}\gamma^{-1}_{\tau_H-t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau_H>t\}}.$$

PROOF: First note that exercising the option before knock-in yields payoff zero and therefore cannot be optimal. Hence $\tau^\star \geq \tau_H$ a.s. While keeping this in mind, consider the equality in (A.5) for the stopping time $\sigma := \tau_H \vee t$ yielding

$$\begin{aligned} V_{t\wedge\tau^\star} &= \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau^\star}\mathbb{1}_{\{\tau^\star\leq t\}} + V_{\tau_H\vee t}\,\gamma^{-1}_{\tau_H\vee t-t}\mathbb{1}_{\{\tau^\star>t\}} | \mathcal{F}_{t\wedge\tau^\star} \right) \\ &= V_{\tau^\star}\mathbb{1}_{\{\tau^\star\leq t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H\vee t}\,\gamma^{-1}_{\tau_H\vee t-t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau^\star>t\}} \\ &= V_{\tau^\star}\mathbb{1}_{\{\tau^\star\leq t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_t\mathbb{1}_{\{\tau_H\leq t\}} + V_{\tau_H}\gamma^{-1}_{\tau_H-t}\mathbb{1}_{\{\tau_H>t\}} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau^\star>t\}} \\ &= V_{\tau^\star}\mathbb{1}_{\{\tau^\star\leq t\}} + V_t\mathbb{1}_{\{\tau_H\leq t\}}\mathbb{1}_{\{\tau^\star>t\}} + \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q \left(V_{\tau_H}\gamma^{-1}_{\tau_H-t} | \mathcal{F}_t \right) \mathbb{1}_{\{\tau_H>t\}} \end{aligned}$$

which proves the claim. Besides optional sampling, which heavily requires time-consistency of $\mathcal{P}$, we used that τ_H and $\tau^\star$ are stopping times, and $\operatorname{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q (S + \eta | \mathcal{F}_t) = \eta + \operatorname{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}^Q (S|\mathcal{F}_t)$ for any $\mathcal{F}_t$-measurable random variable η and square-integrable $\mathcal{F}$-measurable S. $\square$

The expectation occurring in Lemma A.3.2 remains to be evaluated. V_{τ_H} corresponds to the value of the American put option under ambiguity aversion. At knock-in when $s = \tau_H$, we know the value is given by

$$g(s) := V_s = \underline{Y}^{s,H}_s = \operatorname*{ess\,sup}_{\tau\geq s} \mathbb{E}^{Q^{-\kappa}} \left((L - X^{s,H}_\tau)^+\gamma^{-1}_{\tau-s} \right).$$

γ_{s-t}^{-1} and $g(s)$ are decreasing in s, per Lemma 3.3.11. Therefore, by Lemma A.3.1 and the usual characterization of first-order stochastic dominance, Mas-Colell, Whinston, and Green (1995) we deduce on $\{\tau_H > t\}$ for any $\theta \in \Theta$

$$V_t = \mathbb{E}^{Q^\kappa} \left(g(\tau_H)\gamma_{\tau_H-t}^{-1}|\mathcal{F}_t \right) \leq \mathbb{E}^{Q^\theta} \left(g(\tau_H)\gamma_{\tau_H-t}^{-1}|\mathcal{F}_t \right).$$

On the complementary event $\{\tau_H \leq t\}$ the claim equals the usual American put option. Hence, it is evaluated with respect to $Q^{-\kappa}$. Setting both together, $\theta^\star$ is as claimed in the theorem. By right-continuity it is progressively measurable. Therefore, $\theta^\star \in \Theta$ and $Q^{\theta^\star}$ is the worst-case prior for the American up-and-in put problem. This finishes the proof.

A.4 Sublinear expectations

We depict notions and preliminaries in the theory of sublinear expectation and related G-Brownian motion. This includes the definition of G-expectation, introduction to Itô calculus with G-Brownian motion and important results concerning the representation of G-expectation and G-martingales. We do not express definitions and results in their greatest generality. Our task, rather, is to present it as it was used in the previous sections. More details can be found in Peng (2010) and Li and Peng (2009).

We also restrict ourselves to the one-dimensional case. However, everything also holds in the *d*-dimensional case. Further, the financial market model can be extended to d risky assets using a d-dimensional G-Brownian motion as is done in classical financial markets with Brownian motion.

A.4.1 Sublinear expectation, G-Brownian motion and G-expectation

Definition A.4.1 *Let $\Omega \neq \emptyset$ be a given set. Let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. ($\mathcal{H}$ can be considered as the space of random variables.) A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: For any $X, Y \in \mathcal{H}$ we have*

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$ for $c \in \mathbb{R}$.

(c) Sub-additivity: $\hat{E}(X + Y) \leq \hat{E}(X) + \hat{E}(Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X) \quad \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Property (c) is also called self-domination. It is equivalent to $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$. Property (c) together with (d) is called sublinearity. It implies convexity:

$$\hat{E}\left(\lambda X + (1 - \lambda)Y\right) \leq \hat{E}(X) + (1 - \lambda)E(Y) \text{ for any } \lambda \in [0, 1].$$

The properties (b) and (c) imply cash translatability:

$$\hat{E}(X + c) = \hat{E}(X) + c \text{ for any } c \in \mathbb{R}.$$

The space $C_{l,Lip}(\mathbb{R}^n)$, where $n \geq 1$ is an integer, plays an important role. It is the space of all real-valued continuous functions φ defined on $\mathbb{R}^n$ such that $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \forall x, y \in \mathbb{R}^n$. Here k is an integer depending on φ.

Definition A.4.2 *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random variable $Y \in \mathcal{H}$ is said to be independent from another random variable $X \in \mathcal{H}$ under $\hat{E}$ if for any test function $\varphi \in C_{l,Lip}(\mathbb{R}^2)$ we have*

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}].$$

Definition A.4.3 *Let X_1 and X_2 be two random variables defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$, respectively. They are called identically distributed, denoted by $X_1 \sim X_2$, if*

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)] \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}).$$

We call $\bar{X}$ an independent copy of X if $\bar{X} \sim X$ and $\bar{X}$ is independent from X.

Definition A.4.4 (G-normal distribution) *A random variable X on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called (centralized) G-normal distributed if for any $a, b \geq 0$*

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2} X$$

where $\bar{X}$ is an independent copy of X. The letter G denotes the function

$$G(y) := \frac{1}{2}\hat{E}[yX^2] : \mathbb{R} \to \mathbb{R}.$$

Note that X has no mean-uncertainty, i.e., one can show that $\hat{E}(X) = \hat{E}(-X) = 0$. Furthermore, the following important identity holds

$$G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-$$

with $\underline{\sigma}^2 := -\hat{E}(-X^2)$ and $\overline{\sigma}^2 := \hat{E}(X^2)$. We write X is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed. Therefore we sometimes say that G-normal distribution is characterized by the parameters $0 < \underline{\sigma} \leq \overline{\sigma}$.

Remark A.4.5 *Throughout this paper we assume $\underline{\sigma} > 0$. From an economic point of view, this assumption is quite reasonable. In finance, volatility is always assumed to be greater than zero. A volatility equal to zero would induce arbitrage.*

The G-framework also works without this condition. However, with this assumption we can do without the notion of a viscosity solution. Our assumption ensures that the function G is non-degenerate and therefore the PDEs induced by the G-normal distribution, Equation (A.6), have classical $C^{1,2}$-solutions, (see page 19 in Peng (2010)).

Remark A.4.6 *The random variable X defined in A.4.4 is also characterized by the following parabolic partial differential equation (PDE for short) defined on $[0, T] \times \mathbb{R}$:*
For any $\varphi \in C_{l,Lip}(\mathbb{R})$, define $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, then u is the unique (viscosity) solution of

$$\partial_t u - G(\partial_{xx} u) = 0, \quad u(0, \cdot) = \varphi(\cdot). \tag{A.6}$$

The PDE is called a G-equation.

Definition A.4.7 *Let $(\Omega, \mathcal{H}, \hat{E})$ be a sublinear expectation space. $(X_t)_{t \geq 0}$ is called a stochastic process if X_t is a random variable in $\mathcal{H}$ for each $t \geq 0$.*

Definition A.4.8 (G-Brownian motion) *A process $(B_t)_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a G-Brownian motion if the following properties are satisfied:*

(i) $B_0 = 0$.

(ii) For each $t, s \geq 0$ the increment $B_{t+s} - B_t$ is $N(\{0\} \times [\underline{\sigma}^2 s, \overline{\sigma}^2 s])$ distributed and independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ for each $n \in \mathbb{N}$, $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

Condition (ii) can be replaced by the following three conditions giving a characterization of G-Brownian motion:

(i) For each $t, s \geq 0$: $B_{t+s} - B_t \sim B_t$ and $\hat{E}(|B_t|^3) \to 0$ as $t \to 0$.

(ii) The increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \le t_1 \le \cdots \le t_n \le t$.

(iii) $\hat{E}(B_t) = -\hat{E}(-B_t) = 0 \quad \forall t \ge 0$.

For each $t_0 > 0$ we have that $(B_{t+t_0} - B_{t_0})_{t\ge0}$ again is a G-Brownian motion.

Let us briefly depict the construction of G-expectation and its corresponding G-Brownian motion. As in the previous sections, we fix a time horizon $T > 0$ and set $\Omega_T = C_0([0, T], \mathbb{R})$ – the space of all real-valued continuous paths starting at zero. We will consider the canonical process $B_t(\omega) := \omega(t), t \le T, \omega \in \Omega$. We define

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \cdots, t_n \in [0, T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}.$$

A G-Brownian motion is firstly constructed on $L_{ip}(\Omega_T)$. For this purpose let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of random variables on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_i is G-normal distributed and ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for each integer $i \ge 1$. Then a sublinear expectation on $L_{ip}(\Omega_T)$ is constructed by the following procedure:
for each $X \in L_{ip}(\Omega_T)$ with $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ for some $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $0 \le t_0 < t_1 < \cdots < t_n \le T$, set

$$E_G[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})]$$
$$:= \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \sqrt{t_2 - t_1}\xi_2), \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

The related conditional expectation of $X \in L_{ip}(\Omega_T)$ as above under $\Omega_{t_i}, i \in \mathbb{N}$, is defined by

$$E_G[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})|\Omega_{t_i}]$$
$$:= \psi(B_{t_1} - B_{t_0}, \cdots, B_{t_i} - B_{t_{i-1}})$$

where $\psi(x_1, \cdots, x_i) := \tilde{E}[\varphi(x_1, \cdots, x_i, \sqrt{t_{i+1} - t_i}\xi_{i+1}, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]$. One checks that E_G consistently defines a sublinear expectation on $L_{ip}(\Omega_T)$ and the canonical process B represents a G-Brownian motion.

Definition A.4.9 *The sublinear expectation $E_G : L_{ip}(\Omega_T) \to \mathbb{R}$ defined through the above procedure is called a G-expectation. The corresponding canonical process $B = (B_t)$ on the sublinear expectation space $(\Omega_T, L_{ip}(\Omega_T), E_G)$ is a G-Brownian motion.*

Let $||\xi||_p := [E_G(|\xi|^p)]^{\frac{1}{p}}$ for $\xi \in L_{ip}(\Omega_T), p \geq 1$. For any $t \in [0, T], E_G(\cdot|\Omega_t)$ can be continuously extended to $L_G^p(\Omega_T)$ – the completion of $L_{ip}(\Omega_T)$ under the norm $||\xi||_p$.

Proposition A.4.10 *The conditional G-expectation* $E_G(\cdot|\Omega_t) : L_G^1(\Omega_T) \to L_G^1(\Omega_t)$ *as defined above has the following properties:*
For any $t \in [0, T], X, Y \in L_G^1(\Omega_T)$ *we have*

(i) $E_G(X|\Omega_t) \geq E_G(Y|\Omega_t)$ *if* $X \geq Y$.

(ii) $E_G(\eta|\Omega_t) = \eta$ *if* $\eta \in L_G^1(\Omega_t)$.

(iii) $E_G(X|\Omega_t) - E_G(Y|\Omega_t) \leq E_G(X - Y|\Omega_t)$.

(iv) $E_G(\eta X|\Omega_t) = \eta^+ E_G(X|\Omega_t) + \eta^- E_G(-X|\Omega_t)$ *for all bounded* $\eta \in L_G^1(\Omega_t)$.

(v) $E_G(E_G(X|\Omega_t)|\Omega_s) = E_G(X|\Omega_{t \wedge s})$.

(vi) $E_G(X|\Omega_t) = E_G(X)$ *for all* $X \in L_G^1(\Omega_T^t)$, *where* $L_G^1(\Omega_T^t)$ *is constructed similarly to* $L_G^1(\Omega_T)$ *but on the time interval* $[t, T]$ *instead of* $[0, T]$.

The following property is often very useful. It also holds for any sublinear expectation if the related conditional expectation is defined reasonably.

Proposition A.4.11 *Let* $X, Y \in L_G^1(\Omega_T)$ *with* $E_G(Y|\Omega_t) = -E_G(-Y|\Omega_t)$ *for some* $t \in [0, T]$. *Then we have*

$$E_G(X + Y|\Omega_t) = E_G(X|\Omega_t) + E_G(Y|\Omega_t).$$

In particular, if $E_G(Y|\Omega_t) = E_G(-Y|\Omega_t) = 0$ *we have* $E_G(X + Y|\Omega_t) = E_G(X|\Omega_t)$.

So far, G-expectation and its corresponding G-Brownian motion has not been based on a given probability space. The next theorem establishes the ramification with probability theory. We obtain a set of probability measures which represents the functional E_G in a subsequently announced sense. Although the measures belonging to the set are mutually singular, the result is similar to the ambiguity setting when the probability measures inducing the ambiguity are absolutely continuous, see Chen and Epstein (2002), Delbaen (2002). References on this representation theorem for G-expectation are Denis, Hu, and Peng (2010) as well as Hu and Peng (2010).

Consider the probability space $(\Omega_T, \mathcal{F}, P)$ with $\mathcal{F} = \mathcal{B}(\Omega_T)$ the Borel σ-algebra. Let $W = (W_t)$ be a classical Brownian motion in this space with corresponding filtration $(\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma\{W_s | 0 \leq s \leq t\} \vee \mathcal{N}$ with $\mathcal{N}$ denoting the collection of P-null subsets. For fixed $t \geq 0$ we define $\mathcal{F}_s^t := \sigma\{W_{t+u} - W_t | 0 \leq u \leq s\} \vee \mathcal{N}$.

Let $\Theta := [\underline{\sigma}, \overline{\sigma}]$ such that $G(y) = \frac{1}{2} \sup_{\theta \in \Theta} y\theta^2$ and denote by $\mathcal{A}_{t,T}^{\Theta}$ the collection of all Θ-valued $(\mathcal{F}_s^t)$-adapted processes on $[t, T]$. For any $\theta \in \mathcal{A}_{t,T}^{\Theta}$ define

$$B_T^{t,\theta} := \int_t^T \theta_s dW_s.$$

Let P^θ be the law of the process $B_t^{0,\theta} = \int_0^t \theta_s dW_s, t \in [0, T]$, i.e., $P^\theta = P \circ (B^{0,\theta})^{-1}$. Define $\mathcal{P}_1 := \{P^\theta | \theta \in \mathcal{A}_{0,T}^{\Theta}\}$ and (the weakly compact set) $\mathcal{P} := \overline{\mathcal{P}}_1$ as the closure of $\mathcal{P}_1$ under the topology of weak convergence.

Using these notations we can formulate the following result.

Theorem A.4.12 *For any $\varphi \in C_{l,Lip}(\mathbb{R}^n), n \in \mathbb{N}, 0 \leq t_1 \leq \cdots \leq t_n \leq T$, we have*

$$E_G[\varphi(B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] = \sup_{\theta \in \mathcal{A}_{0,T}^{\Theta}} \mathbb{E}^P[\varphi(B_{t_1}^{0,\theta}, \cdots, B_{t_n}^{t_{n-1},\theta})]$$

$$= \sup_{\theta \in \mathcal{A}_{0,T}^{\Theta}} \mathbb{E}^{P^\theta}[\varphi(B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})]$$

$$= \sup_{P^\theta \in \mathcal{P}} \mathbb{E}^{P^\theta}[\varphi(B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})].$$

Furthermore,

$$E_G(X) = \sup_{P \in \mathcal{P}} \mathbb{E}^P(X) \quad \forall X \in L_G^1(\Omega_T).$$

The last theorem can also be extended to the conditional G-expectation, see also Soner, Touzi, and Zhang (2010a):[10] we have for $X \in L_G^1(\Omega_T), t \in [0, T]$, and $Q \in \mathcal{P}$,

$$E_G(X|\mathcal{F}_t) = \operatorname*{ess\,sup}_{Q' \in \mathcal{P}(t,Q)} \mathbb{E}^{Q'}(X|\mathcal{F}_t) \quad Q - a.s.$$

where $\mathcal{P}(t, Q) := \{Q' \in \mathcal{P} | Q' = Q \text{ on } \mathcal{F}_t\}$.

As seen in the previous sections of Chapter 4 the following terminology is very useful within the G-framework.

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[10]From now on we write, as we did in Chapter 4, $E_G(\cdot|\mathcal{F}_t)$ instead of $E_G(\cdot|\Omega_t)$.

Definition A.4.13 *A set $A \in \mathcal{F}$ is* polar *if $P(A) = 0$ for all $P \in \mathcal{P}$. We say a property holds "quasi-surely" (q.s.) if it holds outside a polar set.*

Peng (2010) also gave a pathwise description of $L_G^p(\Omega_T)$, helpful to obtain a better understanding of the space. Before passing to this description we need the following.

Definition A.4.14 *A mapping $X : \Omega_T \to \mathbb{R}$ is said to be* quasi-continuous *(q.c.) if $\forall \varepsilon > 0$ there exists an open set O with $\sup_{P \in \mathcal{P}} P(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.*

We say that $X : \Omega_T \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \to \mathbb{R}$ with $X = Y$ q.s.

Peng (2010) showed equality of $L_G^p(\Omega_T), p > 0$ and the closure of the space of continuous and bounded functions on Ω_T, $C_b(\Omega_T)$, when the closure is taken with respect to the norm $||X||_p := (\sup_{P \in \mathcal{P}} \mathbb{E}^P[|X|^p])^{\frac{1}{p}}$. Furthermore, $L_G^p(\Omega_T), p > 0$ is characterized by

$$L_G^p(\Omega_T) = \{X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{n \to \infty} \sup_{P \in \mathcal{P}} \mathbb{E}^P[|X|^p 1_{\{|X| > n\}}] = 0\} \tag{A.7}$$

where $L^0(\Omega_T)$ denotes the space of $\mathcal{F}$-measurable real-valued functions on Ω_T.

This pathwise description of $L_G^p(\Omega_T)$ indicates that one has to be cautious when dealing with G-expectation. A bounded random variable is not necessary an element of $L_G^p(\Omega_T)$ contradicting the classical L^p spaces in probability theory. Thus, the G-expectation of a bounded measurable random variable is not necessarily well-defined.

An example for a bounded random variable that is not in $L_G^1(\Omega_T)$ is constructed in Soner, Touzi, and Zhang (2010a). Worth mentioning, the density of $\langle B \rangle_t$ for a fixed $t \leq T$ is not a member of $L_G^1(\Omega_T)$, see Song (2010a).[11]

The mathematical framework provided enables the simultaneous analysis of stochastic processes for several mutually singular probability measures. In the following, when not stated otherwise all equations and statements are also to be understood as to hold "quasi-surely" meaning that a "property" holds almost surely for all conceivable scenarios.

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[11]Consequently, as we shall see in the next section, the stochastic integral with respect to G-Brownian motion cannot be defined for the process $\left(\frac{d\langle B \rangle_t}{dt} \right)$ as integrand.

A.4.2 Stochastic calculus of Itô type with G-Brownian motion

We briefly present the basic notions on stochastic calculus such as the construction of Itô integral with respect to G-Brownian motion.

For $p \geq 1$, let $M_G^{p,0}(0,T)$ be the collection of simple processes η of the following form: Given a partition of $[0,T]$, $\{t_0, t_1, \cdots, t_N\}, N \in \mathbb{N}, \xi_i \in L_G^p(\Omega_{t_i}) \; \forall i = 0, 1, \cdots, N-1$, for any $t \in [0,T]$ the process η is defined as

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t). \tag{A.8}$$

For each $\eta \in M_G^{p,0}(0,T)$, let $||\eta||_{M_G^p} := \left(E_G \int_0^T |\eta_s|^p ds \right)^{\frac{1}{p}}$. We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $|| \cdot ||_{M_G^p}$.

Definition A.4.15 *For $\eta \in M_G^{2,0}(0,T)$ with the presentation in (A.8) we define the integral mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ by*

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Since I is continuous it can be continuously extended to $M_G^2(0,T)$. The integral has similar properties as in the classical case. For more details see Peng (2010).

The quadratic variation process $(\langle B \rangle_t)$ of B is defined analogously to the classical case as the limit of the quadratic increments of B. We have

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s \quad \forall t \leq T.$$

It is a continuous, increasing process, absolutely continuous with respect to dt. It contains all the statistical uncertainty of the G-Brownian motion. For $s, t \geq 0$ we have $\langle B \rangle_{s+t} - \langle B \rangle_s \sim \langle B \rangle_t$ and it is independent of Ω_s. Furthermore, for any $t \geq s \geq 0$

$$E_G[\langle B \rangle_t - \langle B \rangle_s | \Omega_s] = \overline{\sigma}^2 (t - s),$$
$$E_G[-(\langle B \rangle_t - \langle B \rangle_s) | \Omega_s] = - \underline{\sigma}^2 (t - s).$$

We say that $\langle B \rangle_t$ is $N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$-distributed, i.e., for all $\varphi \in C_{l,Lip}(\mathbb{R})$,

$$E_G[\varphi(\langle B \rangle_t)] = \sup_{\underline{\sigma}^2 \leq v \leq \overline{\sigma}^2} \varphi(vt).$$

Thus, it is a typical process with mean uncertainty. The integral with respect to the quadratic variation of G-Brownian motion $\int_0^t \eta_s d\langle B \rangle_s$ is defined in an obvious way. Firstly, for all $\eta \in M_G^{1,0}(0,T)$ and again in a second step, by a continuity argument for all $\eta \in M_G^1(0,T)$.

The following observation is essential for the characterization of G-martingales. The Itô integral can also be defined for processes of the following form, see Song (2010c): For a partition $\{t_0, t_1, \cdots, t_N\}$ of $[0,T]$, $N \in \mathbb{N}$, and $\xi_i \in L_{ip}(\Omega_{t_i}) \; \forall i = 0, 1, \cdots, N-1$, let η be given by

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t) \quad \forall t \leq T$$

and denote by $H_G^0(0,T)$ the collection of such processes η. For $p \geq 1$ and $\eta \in H_G^0(0,T)$, $||\eta||_{H_G^p} := \left(E_G \left(\int_0^T |\eta_s|^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$. Denote by $H_G^p(0,T)$ the completion of $H_G^0(0,T)$ under this norm. In the case of $p = 2$, $H_G^2(0,T)$ and $M_G^2(0,T)$ coincide. As above we can also construct the Itô integral I on $H_G^0(0,T)$ and extend it continuously to $H_G^p(0,T)$ for any $p \geq 1$. Hence, $I : H_G^p(0,T) \to L_G^p(\Omega_T)$.

A.4.3 Characterization of G-martingales

Definition A.4.16 *A process $M = (M_t)$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $E_G(M_t | \mathcal{F}_s) = M_s$ for all s, t with $s \leq t \leq T$. If both M and $-M$ are G-martingales, M is called a* symmetric *G-martingale.*

Soner, Touzi, and Zhang (2010a) showed that M is a G-martingale if and only if for all $0 \leq s \leq t \leq T, P \in \mathcal{P}$,

$$M_s = \underset{Q' \in \mathcal{P}(s,P)}{\text{ess sup}} \; \mathbb{E}^{Q'}(M_t | \mathcal{F}_s) \quad P - a.s.$$

The identity declares that a G-martingale can be interpreted as a multiple priors martingale which is a supermartingale for any $P \in \mathcal{P}$ and a martingale for an optimal measure.

The next results give a characterization for G-martingales.

Theorem A.4.17 *(Peng (2010)) Let $x \in \mathbb{R}$, $z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$. Then the process*

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \leq T,$$

is a G-martingale.

In particular, the nonsymmetric part $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds$, $t \in [0, T]$ is a G-martingale which is quite surprising compared to classical martingale theory since $(-K_t)$ is a continuous, non-increasing process with quadratic variation equal to zero.

Remark A.4.18 *M is a symmetric G-martingale if and only if $K \equiv 0$, see also Song (2010c).*

The converse statement of Theorem A.4.17 was firstly proven by Soner, Touzi, and Zhang (2010a) and then rewritten in the exact G-framework by Song (2010c) used in Chapter 4. It reads as follows.

Theorem A.4.19 (Martingale representation) *(Song (2010c)) Let $\beta \geq 1$ and $\xi \in L_G^\beta(\Omega_T)$. Then the G-martingale X with $X_t := E_G(\xi|\mathcal{F}_t), t \in [0, T]$, has the following unique representation*

$$X_t = X_0 + \int_0^t z_s dB_s - K_t$$

where K is a continuous, increasing process with $K_0 = 0, K_T \in L_G^\alpha(\Omega_T), z \in H_G^\alpha(0, T), \forall \alpha \in [1, \beta)$, and $-K$ is a G-martingale.

If $\beta = 2$ and ξ are bounded from above, $z \in M_G^2(0, T)$ and $K_T \in L_G^2(\Omega_T)$, see Song (2010b).

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Kurzer Lebenslauf

Jörg Vorbrink

Geboren am 30.04.1979 in Ibbenbüren

Abitur 1998 am Fürstenberg-Gymnasium in Recke

Zivildienst in Ibbenbüren von 07/1998 bis 07/1999

Ausbildung zum Bankkaufmann in Ibbenbüren von 08/1999 bis 01/2002

Bankkaufmann in Ibbenbüren von 02/2002 bis 09/2002

Studium der Wirtschaftsmathematik an der Universität Bielefeld ab
10/2002 mit einem Auslandssemester an der Queen's University Belfast

Diplom-Wirtschaftsmathematiker, Universität Bielefeld 04/2008

Internationales Graduierten Kolleg IGK – Stochastics and Real World
Models – an der Universität Bielefeld von 04/2008 bis 03/2011 mit
Auslandsaufenthalt an der Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing

Promotion zum Dr. rer. pol. an der Fakultät für
Wirtschaftswissenschaften, Universität Bielefeld 07/2011