Stable Firm Structures and the Core of an Economy with Production

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Grad. (Free University of Berlin) 1967

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Economics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

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Committee in Charge

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ABSTRACT

The concept of the core of a private ownership economy without
production is well known. We propose an extension of this equilibrium
concept to the case with production where each group of consumers has
control over some arbitrary production possibility set. This yields
a natural extension for the definition of blocking and thus of the core.
Sufficient conditions for the non-emptiness of the core are given. The
actual proof centers around the possibilities to represent the economy
as a balanced game, which in turn requires the introduction of a bal-
ancedness assumption on the distribution of technical knowledge over
coalitions.

For a comparison of the core and any market equilibrium concept
with production a definition of the producing agents is needed. We
present a model for the determination of firms for a private ownership
economy. The possibility of all or no firms participating in the
market allows an interpretation in terms of entry and exit of firms.
A collection of firms, their production decisions, and their profit
payments to the consumer is called a firm structure. It is called
stable if at the prevailing market price no coalition of consumers.
could guarantee itself higher profit payments. An associated market
equilibrium with a stable firm structure is defined and a main exis-
tence proof is given. Such an equilibrium yields an allocation in
the core. The relationship to earlier results in economic theory as
well as to recent results in game theory is indicated.

Acknowledgments

I am deeply indebted to Professor G. Debreu for his

continued encouragement and stimulating guidance. I also

benefited from many discussions with Professors S. Goldman,

T. A. Marschak, D. Schmeidler, and K. Vind, and with B. Hool.

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CHAPTER I. Introduction

Two conceptually distinct models of an economy have been of

fundamental importance in mathematical economics during the past

years. The classical model of a market economy and its associated

competitive equilibrium have found a complete description in De-

breu's Theory of Value [13] for any finite economy. Market conduct

of each participating agent, i.e., consumers and firms, is determined

by independent maximizing behavior subject to given prices which no

agent can influence through his actions. The resulting competitive

equilibrium, i.e., a price system for which the plans of all agents

are feasible simultaneously is under certain conditions Pareto opti-

mal which supplies the basic argument for a decentralized market

mechanism to achieve efficient and optimal states for the economy

as a whole.

The other basic concept is the core of an economy. To illustrate

the core, consider a list of consumption plans, one assigned to each con-

sumer. This assignment is said to be blocked by a group of consumers

if it can find another assignment for themselves using their own re-

sources and technical possibilities such that each member of the group

can be made better off than with the original assignment. Then the

core consists of all feasible assignments which no group can block.

Such a concept requires cooperation among consumers. In fact it

allows any coalition in the economy to form and to decide on some

joint action, which is the typical feature of a bargaining process.

Hence, the core represents a particular solution of such a bargaining model.

Although the two equilibrium concepts are distinctly different there exists a close relationship between them. It is a well-established result that in pure exchange economies a competitive equilibrium is in the core, which shows that apart from being Pareto optimal a competitive allocation also cannot be blocked by any group of consumers. Hence no group would actually benefit from disregarding market prices and trading or bargaining among themselves in any other way. In general, it is shown that the core as a set is larger than the set of Walras allocations. However, for the case of infinitely many participants in an economy where each agent has only a negligible influence on the outcome, and in the case of replica economies, the two sets coincide, see, e.g., Debreu and Scarf [14], Aumann [4], Vind [37], and Hildenbrand [19].

None of the above results have been extended to economies where production is possible except for very special cases, Debreu and Scarf [14], Champsaur [8], and Hildenbrand [19]. The problem which arises is two-fold.

(a) Since the core describes certain outcomes of a bargaining process over commodity bundles completely independent of any price mechanism, the bargaining power of each coalition with respect to their technical knowledge has to be defined. The two main questions which have to be answered are:

(1) Given the technology of an economy, who has ultimate

control over which parts so that a meaningful definition of blocking, and hence, of the core, can be given?
(2) What is the technology which the union of any two disjoint coalitions will ultimately control?

(b) If one assumes that the ultimate control over technical knowledge is exercised by one or several consumers, one immediately faces the necessity of how firms who carry out most of the production in a market economy are endowed with a certain set of production possibilities. In particular, since a firm produces almost always for an anonymous market and not for the immediate consumption of its owners, a precise definition of a firm and of its "creation" and "destruction" will be needed. This issue has been avoided more or less in the literature of the theory of the firm. The existence of firms as well as their endowment with a certain production possibility set is usually taken as given. Among the few attempts to define a firm are Penrose [29] and Papandreou [28]. However, both authors do not attempt to relate the ownership structure to the dispersion of the technical knowledge among consumers. Even in the discussion of the problems of free entry and exit of a perfectly competitive market, as, e.g., in Baumol [5] and Hicks [18], the problems of ownership and/ or distribution of profits to consumers are neglected.

The examination of these questions has led to the formulation of the two models presented in Chapters II and III. Chapter II

describes the core of a productive economy in a very general way, which can be considered as a model for almost any economy in which consumers exercise control over resources and over production possibilities. Sufficient conditions for the non-emptiness of the core are given which are straightforward extensions of concepts and definitions already common to mathematical economics. The concept of a stable firm structure and the associated equilibrium represent a generalization of existing equilibrium concepts for market economies. An appropriate definition of a firm and hence of a firm structure has to be consistent with the private ownership structure of the economy if comparisons between market equilibria and core allocations are desired. A definition is proposed which is flexible enough to incorporate entry and exit of firms in the market economy. The definition of a stable firm structure then is an immediate consequence of the definition of the firm and of the private ownership structure of the economy. It has a direct interpretation in terms of entry and exit of firms and is independent of any behavioral assumption for firms in the market. Hence it is hoped that it could also be applied to cases where behavioral rules for firms are introduced which is an important aspect of oligopolistic models. Furthermore an equilibrium with a stable firm structure will guarantee an allocation in the core, i.e., there exists no bargaining procedure among consumers which could profitably upset the market equilibrium by disregarding the existing market structure.

Although the concept of an equilibrium with a stable firm structure was defined solely in the spirit of a private ownership economy,

there exists an interesting parallel to it within the theory of
labor-managed market economies as proposed by J. Vanek [35][1/].
He suggests a behavioral procedure for each group of workers of
achieving maximal profits per capita within the firm, which they
manage, and distributing these among themselves. If this is done
with a particular distribution of technological possibilities over
the groups of workers, then the resulting firm structure is indeed
stable in the sense of our definition. However, a comparison with
the core is not possible in this case since resources are not assumed
to be owned by consumers.

CHAPTER II.   The Core of a Productive Economy

1.   Introduction

    The core, originally developed as a solution concept in the
theory of games, has, in recent years, become one of the most power-
ful solution concepts in economics. It represents the final outcome
of a bargaining process in which all possible coalitions may be
formed. Its non-emptiness under minimal assumptions and also its rela-
tionship to the class of balanced games has been demonstrated by var-
ious authors, e.g., Scarf [30] [31], and Aumann [3]. Outside of game ·
theory proper, Edgeworth [16] suggested a recontracting mechanism
for a pure exchange economy the outcome of which is precisely the
core of the associated bargaining model.

[1]This reference is due to Professor T. A. Marschak.

One of the major reasons for the interest of mathematical economists in the core stems from the fact that it contains the competitive market equilibrium, Debreu and Scarf [14], which demonstrates the efficiency of a pricing mechanism to achieve optimal allocations. Most of the existing results, however, deal only with the case of a pure exchange economy, except for some special cases, e.g., Debreu and Scarf [14], Champsaur [8], and Hildenbrand [19][20]. Debreu and Scarf assume that the total production possibility set is a convex cone and that it is available to each coalition. Both assumptions together imply additivity which is also the basic assumption made by Hildenbrand. Champsaur, on the other hand, assumes some *a priori* firm structure in the economy with arbitrary given shares of the firms distributed among the consumers. These shares, combined with a majority voting rule, are then used to define the core. This last approach seems to be an unsatisfactory one since it uses concepts of a market structure which are not necessarily related to a bargaining model.

Since the core describes outcomes of a bargaining process over commodity bundles completely independent of any price mechanism, the definition of the bargaining power of each coalition with respect to their technical knowledge will be a crucial point in a theory of the core with production. The important features of such a definition, however, are not related to the specification of production possibilities per se but rather to the distribution and/or the institutional framework of the total available productive knowledge. A precise definition of these concepts will be given below and it will

be shown that it contains the characterization by Debreu and Scarf and by Hildenbrand as special cases. The main part will be an existence proof of an element in the core using an extension of Scarf's theorem on balanced games [27] [28]. Furthermore, it will be shown that the core always contains Pareto optimal allocations.

2. Definitions and Preliminary Results

The basic framework is an economy as described in Debreu's Theory of Value [13]. The commodity space of the economy is the finite dimensional Euclidean space R^ℓ. There is a finite set of consumers, $I = \{1,\ldots,i,\ldots,n\}$, who are characterized by their consumption sets $X_i \subset R^\ell$, their preference relation $\succsim_i$, and their endowment $e_i \in R^\ell$. Such an economy is usually referred to as a private ownership economy since aggregate resources are defined as $e = \sum_{i \in I} e_i$. Production possibilities will be described in such a way that the notion of private ownership extends naturally to an economy with production.

It is clear that the detailed specification of production technologies lies outside the economist's realm of competence. Hence, in describing production possibility sets of an economy or some subeconomy, one can only allude to some fundamental limitations imposed by the state of technological knowledge and physical laws, leaving it mostly to economic intuition which other qualitative features production possibility sets may have.

Using this definition the analysis will be confined to the framework which has been applied successfully in general equilibrium models, leaving aside the whole array of conceptual problems connected with a

precise definition of technical knowledge. Beyond the specification

of commodities one knows little about the effects of information,

organization, etc., on output. To describe production possibilities,

economists simply list a certain number of determinants which usu-

ally remain unexplained. The application of technical knowledge

always requires some organizational forms which in our context may

be incorporated in the description of the production possibility set

as long as one does not intend to explain changes of such a structure.

However, even for the discussion here the problem does not disappear.

Different organizational structures for the same technology may be

the outcome of different managerial skill of the decision-maker.

However, if one accepts the quite extensive definition of a commodity,

most features of managerial skill can be considered as features of a

person's labor input. Hence they are specifiable as commodities and

not as determinants of technological possibilities. Other features

related to managerial skill cannot be incorporated in the description

of the model. It is mainly the choice of a particular input-output

combination which constitutes the task of a manager in the model.

Similar conceptual problems arise if one wants to define informa-

tion as a determinant of available productive knowledge. Since one

can easily visualize productive knowledge as a tradable item in a

market context its specific features do not seem to fit into a descrip-

tion of it as just another coordinate in the commodity space. If it

were true, then the conventional theory would have to explain also the

production of technical knowledge which would create a circular

argument.

A far more crucial problem for an economic analysis is the dispersion of the technology within the economy among its participating agents. It is clearly of great economic importance who in the economy is able to use specific parts of the technology. Statements about the distribution are usually made in terms of accessibility, technical know-how, and information. This represents again a rather weak verbalization of certain phenomena which traditional economic theory cannot explain. To avoid these complications it is assumed that accessibility of technology and information about it simply mean that the respective agent has full knowledge of and full control over the basic technology which is assigned to him. Hence, in what follows, it is assumed that the total set of available production sets is given as well as its distribution among the consumers and coalitions of consumers.

Let 2^I denote the set of all coalitions of consumers. With each $S \in 2^I$ is associated a non-empty production possibility set Y^S with the convention that $Y^\emptyset = \{0\}$. Then the economy $\mathcal{E}$ is described by the following list:

$$\mathcal{E} = \{I, (X_i), (e_i), (\succsim_i), (Y^S)\} .$$

To complete the description of the production possibilities of the economy one has to define the total production possibility set Y. In general, for any two coalitions S_1 and S_2, the outcome of two production decisions $y^1 \in Y^{S_1}$ and $y^2 \in Y^{S_2}$ will not be related in any fixed fashion to a production decision by the coalition $S = S_1 \cup S_2$. In particular, the following independency assumption is made. Let $\emptyset$ be any collection of coalitions. Then y is a feasible production plan for $\emptyset$ if $y \in \sum_{S \in \emptyset} Y^S$. This leads directly to the following definition of the aggregate production possibility set Y.

Definition 1: $Y = \bigcup\limits_{\emptyset \subset 2^I} \sum\limits_{S \in \emptyset} Y^S.$

One can now define the core of an economy with production.

A list of commodity vectors $x = (x_i)$, $i = 1,\ldots,n$, is

an allocation if $x_i \in X_i$ for every $i \in I$. An allocation is feasible

if there exists a $y \in Y$ such that $\sum\limits_{i \in I} x_i = \sum\limits_{i \in I} e_i + y.$

Definition 2: A non-empty coalition S is said to block an

allocation x if there exist $x_i' \in X_i$, $i \in S$, and $y^S \in Y^S$ such that

(1) $x_i' \underset{i}{\succ} x_i$ $\forall\, i \in S$

(2) $\sum\limits_{i \in S} x_i' = \sum\limits_{i \in S} e_i + y^S$.

Then the core is defined as the set of feasible allocations which

are blocked by no coalition.

This definition is clearly equivalent to the one used in pure

exchange economies if the production sets of all coalitions contain

only the zero production point. In the non-degenerate case with pro-

duction, the definition is independent of the kind and of the distri-

bution of technical knowledge. Hence, with the previous assumptions

on the production sets, it will describe a very general class of

economies. Since blocking is defined only for one coalition without

reference to any subcoalitions, the core may be relatively large. On

the other hand, the definition is general enough to describe more

restricted technologies as well, since interdependence in the forma-

tion of coalitions for production may, in most cases, be incorporated

into the description of the respective production possibility sets.

3. Main Results and Proofs

The purpose of the remaining part of this chapter is to find

sufficient conditions for the non-emptiness of the core. This will be

done by using an extension of Scarf's theorem on balanced games

which appeared first in [30] and [31]. A description of such

games as well as a statement of the results may also be found,

e.g., in Aumann [3] and Billera [6].

Consider a game without side payments given by the triple

(I,v,H), where $I = \{1,\ldots,n\}$ is the set of players, v is the char-

acteristic function which assigns to each coalition $S \subset I$ a non-empty sub-

set $v(S) \subset E^S$ of the utility space E^n. Let $H \subset E^n$ be the set of possible

utility outcomes. In most treatments of games in characteristic func-

tion form it is always assumed that $v(I) = H$. From a conceptual point

of view, however, a distinction between what is enforceable by the

grand coalition and what is possible seems necessary, since games

where the two sets do not coincide are easily conceivable. The

necessary definitions to state Scarf's theorem are now introduced.

Definition 3: A family of non-empty coalitions $\mathscr{J}$ is called

balanced if and only if for all $S \in \mathscr{J}$ there exist weights $d_S > 0$

such that for all $i \in I$,

$$\sum_{\substack{S \in \mathscr{J} \\ S \ni i}} d_S = 1 \quad .$$

Definition 4: A game (I,v) is called balanced if and only if

for all balanced families of coalitions $\mathscr{J}$, $u \in E^n$ and $u^S \in v(S)$ for

all $S \in \mathscr{J}$ implies $u \in v(I)$. u^S denotes the projection of u into the

utility subspace associated with coalition S. Equivalently, a game is

balanced if and only if for every balanced family $\mathscr{J}$

$$\bigcap_{S \in \mathscr{J}} (v(S) \times E^{I \setminus S}) \subset v(I) \quad .$$

Scarf's theorem can now be stated as follows:

Theorem (Scarf): Let $v(I) = H$ *be bounded from above*

and for every $S \subset I$,

(1) $v(S)$ *is non-empty and closed,*

(2) $x \in v(S)$, $y \in E^S$, $x \geq y$ *implies* $y \in v(S)$.

Then every balanced game has a non-empty core.[1]/

For the extension of Scarf's result a natural generalization of

the notion of a balanced game will be used.

Definition 4': A game (I,v,H) is called balanced if and only

if for every balanced collection $\mathscr{C}$

$$\bigcap_{S \in \mathscr{C}} (v(S) \times E^{I \setminus S}) \subset H \ .$$

Lemma 1: *Let* (I,v,H) *be a balanced game. Assume that* H

is non-empty, closed, and bounded from above and that for

every $S \subset I$,

(1) $v(S)$ *is non-empty and closed,*

(2) $x \in v(S)$, $y \in E^S$, $y \leq x$ *implies* $y \in v(S)$,

(3) $x \in H$, $z \in E^I$, $z \leq x$ *implies* $z \in H$.

Then the game has a non-empty core.

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[1]It should be noted that the assumption of individual rationality

given in Scarf's original formulation ([30], p. 53) is omitted here.

A proof of this stronger version of the theorem is given in the

Appendix.

Proof: First one observes that the balancedness implies $v(I) \subset H$

since $\{I\}$ is a balanced family. Consider the enlarged game (I,w)

defined by $w(S) = v(S)$ for $S \neq I$ and $w(I) = H$. Clearly (I,w) is a

game satisfying assumptions (1) and (2) of Scarf's theorem. Since

$H = w(I)$ one has for every balanced family $\mathscr{B}$ not including the all

player coalition

$$\bigcap_{S \in \mathscr{B}} (w(S) \times E^{I \setminus S}) = \bigcap_{S \in \mathscr{B}} (v(S) \times E^{I \setminus S}) \subset w(I).$$

On the other hand, if $\mathscr{B}$ contains I, the inclusion is obvious.

Hence (I,w) is a balanced game which has a non-empty core. Let x

be in the core of (I,w). Clearly x cannot be blocked by any $S \neq I$

in the game (I,v,H). Furthermore, since $v(I) \subset w(I)$ x is also

unblocked by I for the game $(I,v,H.)$. Hence (I,v,H) has a non-empty

core.   Q.E.D.

The preceding results clearly lay out the procedure which will

be followed in the final existence proof. The usefulness of Scarf's

theorem stems from the fact that an exchange economy with convex

preferences can be represented as a balanced game. In order to

guarantee that this is also possible for certain economies with

production one needs some specification of the distribution of the

technology. So far no assumption has been made about the relation-

ship between production sets of different coalitions, except for the

general independency. The following definition describes a relation-

ship which will in general yield a non-empty core.

Definition 5: Let (Y^S) be the technology distribution of the private ownership economy. (Y^S) is called balanced if and only if for every balanced family ϕ and associated weights (d_S)

$$\sum_{S \in \phi} d_S Y^S \subset Y \quad .$$

The following two examples describe balanced technologies. Consider an economy with three consumers, i.e., $I = \{1,2,3\}$ and let $Y^S = \{0\}$ for all S not equal to the grand coalition. Then if $Y^{\{1,2,3\}}$ is "star-shaped" as depicted in Figure 1, (Y^S) is balanced. The significance of this example is that for a balanced technology any individual set as well as the aggregate set may be non-convex.

Consider an economy of the same size but with two types of production sets, type A and type B. Let Y^A be given by the two line segments $\{(OA),(AB)\}$ and Y^B by (OB) (see Figure 2). Now let $Y^{\{1,2,3\}}$ be the zero production point, $Y^{\{1\}} = Y^{\{3\}} = Y^{\{1,3\}} = Y^B$ and $Y^{\{2\}} = Y^{\{1,2\}} = Y^{\{2,3\}} = Y^A$. Clearly the sets $Y^A + Y^B$, $Y^A + Y^A + Y^B$, $2Y^B + Y^A$ are convex. Using this fact it is easy to check that the technology is balanced. In fact one can prove the following more general result.

Lemma 2: Let $0 \in Y^S$ for all $S \in 2^I$ and let Y be convex. Then (Y^S) is balanced.

Proof: First observe that $0 \in Y^S$ for all S implies $Y = \sum_{S \in 2^I} Y^S$, since for any collection ϕ, $\sum_{S \in 2^I} Y^S \supset \sum_{S \in \phi} Y^S$. Hence,

$Y^{\{1,2,3\}}$

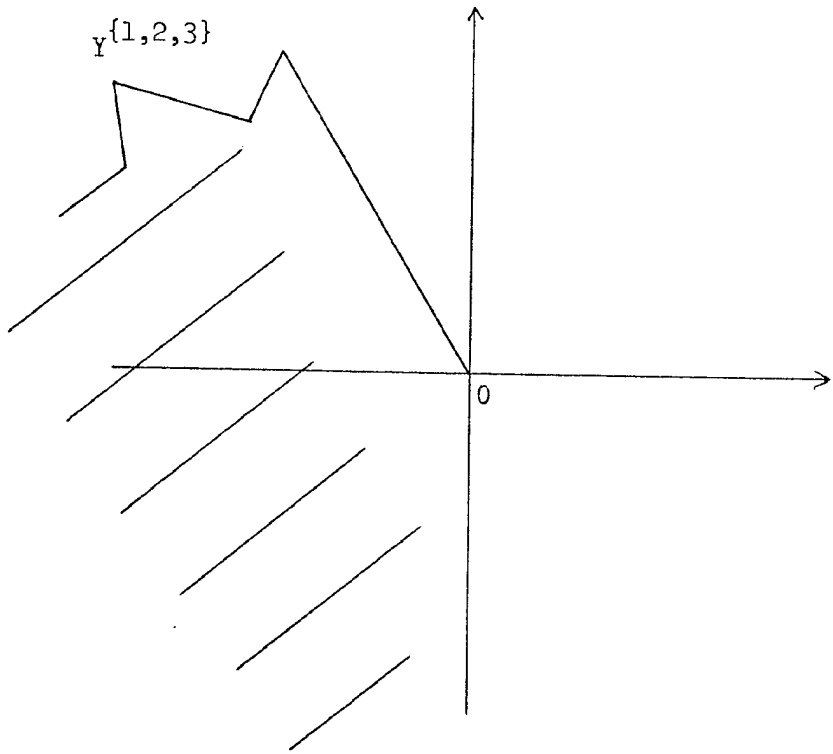Figure 1

B

A   0

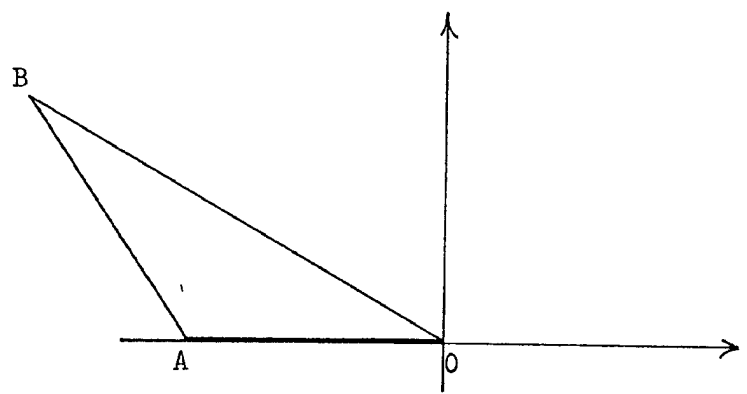Figure 2

$$\sum_{S \in \phi} d_S Y^S = \sum_{S \in \phi} (d_S Y^S + (1 - d_S)\{0\})$$

$$\subset \sum_{S \in \phi} \text{conv } Y^S = \text{conv} \sum_{S \in \phi} Y^S \subset \text{conv} \sum_{S \in 2^I} Y^S$$

where conv Y^S denotes the convex hull of Y^S.   Q.E.D.

In general one would not expect a balanced technology to exhibit convexity in the aggregate which indicates that it describes a wide range of collections of production possibility sets where the individual members as well as the aggregate may embody elements of increasing returns and/or indivisibilities.

Theorem: Let the economy $\mathcal{E} = \{I, (X_i), (e_i), (\succsim_i), (Y^S)\}$ be such that for every $i \in I$,

(1) $X_i = R_+^\ell$, $e_i \geq 0$,

(2) $\succsim_i$ is a complete, transitive, continuous preordering on

X_i such that for any x_i and x_i' with $x_i \succsim_i x_i'$, and for all

$\lambda, 0 \leq \lambda \leq 1$ $\lambda x_i + (1-\lambda) x_i' \succsim_i x_i'$

for every $S \subset I$,

(3) $0 \in Y^S$, Y^S *closed,*

(4) Y *closed and* $AY \cap R_+^\ell = \{0\}$ *where* AY *denotes the asymptotic cone of* Y[1]

(5) (Y^S) *balanced.*

Then $\mathcal{E}$ *has a non-empty core.*

---

[1]For the definition of the asymptotic cone and for the basic results employed in equilibrium theory, see Debreu [13, p. 22].

Proof: First it will be shown that $\mathcal{E}$ is representable as

a game of the form (I,v,H) satisfying conditions $(1)-(3)$ of

Lemma 1, page 12.

Let $x^S = (x_i{}^S)$, $x_i{}^S \in X_i$, $i \in S$ and define $X^S = \{(x_i{}^S) \mid x_i{}^S \in X_i$,

$\sum_{i \in S} x_i{}^S \in Y^S + \{\sum_{i \in S} e_i\}\}$. X^S is the set of feasible allocations for

coalition S. Clearly X^S is non-empty by assumption (1) and (3).

$AY \cap R^\ell_+ = \{0\}$ and $0 \in Y^S$ implies $AY^S \cap R^\ell_+ = \{0\}$ which together with

the closedness of Y^S implies that X^S is compact.

Since for every $i \in I$ the preference relation $\succsim_i$ is continuous

there exist continuous representations $u_i(x_i)$, $i \in I$. Furthermore,

X^S non-empty and compact guarantees that there exists a character-

istic function v from 2^I into the utility space E^I representing the

attainable utility vectors for each coalition S. Moreover, for

every $S \subset I$ $v(S)$ is non-empty and closed. Without loss of generality

one can normalize $v(.)$ such that $v(\{i\}) = \text{Max } \{u_i(x_i) \mid x_i \in Y^{\{i\}} + \{e_i\}\}$

$= 0$ and one can extend v to $\tilde{v}$ by defining $\tilde{v}(S) = v(S) + E^S_-$, where

E^S_- denotes the negative orthant of the utility subspace associated

with coalition S. Assumption (4) guarantees that the set H of

possible utility allocations is closed and bounded from above and H

can be extended to $\tilde{H} = H + R^n_-$ without any loss of generality. Hence

$\mathcal{E}$ is representable as a game $(I,\tilde{v},\tilde{H})$ in characteristic function

form satisfying all assumptions of Lemma 1. For Scarf's theorem and

the extension lemma to be applicable it remains to be shown that it

is a balanced game.

Let $\mathcal{J}$ be a balanced family of non-empty coalitions and let $x^S \in X^S$ be Pareto optimal for S. Define for each $i \in I$

$$x_i = \sum_{\substack{S \in \mathcal{J} \\ S \ni i}} d_S x_i^S \ .$$

It will be shown that (x_i) is a feasible allocation for the economy $\mathcal{E}$.

Since for all $S \in \mathcal{J}$ $\sum_{i \in S} x_i^S = y^S + \sum_{i \in S} e_i$, $y^S \in Y^S$

$$\sum_{i \in I} x_i = \sum_{i \in I} \sum_{\substack{S \in \mathcal{J} \\ S \ni i}} d_S x_i^S = \sum_{S \in \mathcal{J}} \sum_{i \in S} d_S x_i^S = \sum_{S \in \mathcal{J}} d_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \in \mathcal{J}} d_S (y^S + \sum_{i \in S} e_i) = \sum_{S \in \mathcal{J}} d_S y^S + \sum_{i \in I} e_i \sum_{\substack{S \in \mathcal{J} \\ S \ni i}} d_S$$

$$= \sum_{S \in \mathcal{J}} d_S y^S + \sum_{i \in I} e_i \ .$$

Since (Y^S) is balanced, $\sum_{S \in \mathcal{J}} d_S y^S$ is a feasible production bundle.

Hence (x_i) is a feasible allocation. Q.E.D.

4. Remarks

One of the outcomes of the general formulation given above is that in general one cannot expect every point in the core to be a Pareto optimal allocation, which is the typical result in an exchange economy and also in a production economy where the production set of each coalition is the same convex cone with vertex at the origin, see, e.g., Debreu and Scarf [14]. It will also be true in the case where the aggregate production possibility set is equal to the production possibility set of the grand coalition. In general the two sets will not coincide, which is a direct consequence of the definition of the aggregate production possibility set. Then, there may exist

feasible allocations which are not Pareto optimal but Pareto superior to any allocation which can be enforced by the grand coalition. One can say, however, that the core will always contain a subset of the Pareto optimal allocations. To see this, consider an allocation x in the core which is Pareto optimal relative to the core. Now suppose there exists an allocation z outside which is Pareto preferable. Since x is unblocked, z cannot be blocked by any coalition, contradicting that z is not in the core.

Finally, it should be noted that one of the implicit assumptions in the existence proof can be relaxed without changing the result. Instead of using a constructive definition of the aggregate production possibility set, one may take the more abstract approach of defining the private ownership technology by a pair $((Y^S),Y)$. Then the concept of a balanced technology is still well-defined, and the proof will be correct with minor changes. A general characterization of balanced technologies seems to be difficult. The assumption is clearly not necessary for the existence of the core since counter examples can be constructed easily. Balancedness does imply, however, that the distribution of the technology is superadditive for any partition, since any partition with weights equal to one is a balanced family.

CHAPTER III. FIRMS AND MARKET EQUILIBRIA

1. Introduction

Consider a market economy with production. The participating

agents, consumers and producers, carry out their activities accord-

ing to some rules which are determined by the availability of certain

resources or production possibilities and by prices, which enter the

decision-making through an evaluation process. While the consumer

is usually taken as an undisputed primitive concept in economic theory,

the description and existence of certain producers or firms need some

additional justification. The typical feature of decentralized con-

sumption and production decisions manifests itself in the existence

of firms which in most cases produce for an anonymous market rather

than for the immediate need of a particular group of consumers. In

some sense this decentralization is the more meaningful the larger

the economy. If one considers an economy with a given aggregate

technology the set of firms in the market will reflect certain insti-

tutional structures of the economy, usually taken to be defined outside

of economics proper. For an economic analysis, however, there is

still ambiguity as to how the set of firms is determined within the

aggregate technology unless one defines rigidly a fixed set of firms

which makes up the aggregate technology. For many interesting ques-

tions, however, this approach eliminates the problem itself for which

one wants an answer.

If one considers a private ownership economy in the sense of

the preceding bargaining economy one could imagine a variety of ways

to define a market economy with firms from it where the ownership

of resources and the control over technical knowledge are the binding

constraints. For any fixed firm structure, then, one could carry out

the usual analysis of markets and possibly compare the competitive

allocations with the core. This approach would eliminate the flexi-

bility of defining different firm structures or different sets of

firms for the same private ownership economy and of having a market

process select the appropriate one to enable the comparison with the

core. Such a process is actually the desideratum of our model, where

the selection of the firm structure is an outcome at the equilibrium

point and not an *a priori* given datum. The actual determination of

the number of firms in the market at any one time through such a

process could then be viewed as a constructive or quasi-dynamic

mechanism of the tâtonnement type which defines a certain market

equilibrium at the same time. Since the basic model above is com-

pletely static it would be too much to expect a dynamic theory of the

determination of firms. However, the typical features of entry and

exit can be studied quite satisfactorily within a comparative statics

framework. In what follows, a model will be presented for the determ-

ination of firms for a private ownership economy with an equilibrium

concept which has a direct interpretation in terms of a process of

entry and exit of firms taken out of a fixed finite set. Roughly

speaking, an equilibrium of this type where free entry and exit are

allowed will be shown to yield an allocation in the core.

2. A Model of Firms in a Private Ownership Economy

In our framework a firm will be viewed as an independent agent besides the consumers designed to carry out production in a market economy. The firm is an economic construct which derives its existence from actions of some consumers. The original idea of a firm as an independent agent is somewhat related to the size of an economy and also to the existence of markets. Clearly, Robinson and Friday could handle their own production and did not need a separate firm for this purpose. On the other hand, it does not make sense to talk about a firm in an economy if there are no markets where commodities can be exchanged.

The concept of decentralized production and consumption decisions embodies the necessity for a definition of a separate agent which is called a firm. This need did not arise in the previous chapter where no specification of the production decision was needed. For the purposes here a firm j will be identified by an associated coalition S_j and its production possibility set Y^{S_j}. A certain firm j exists in the market or is "created" once a group of consumers $S_j \subset I$ decides to supply their technical knowledge to some "manager" who will carry out a production plan $y_j \in Y_j \equiv Y^{S_j}$. Hence the set of firms J is a subset of 2^I. This set may be empty under certain conditions, e.g., if at prevailing prices no firm could make a nonnegative profit and if no group of consumers is willing to pay for a loss. This reflects the structure of the definitions, namely, that any coalition of consumers has complete control over its technical knowledge as well as

over its being used in a firm. The production possibility set Y^{S_j}

will be used by a firm if and only if S_j has "created" firm j.

Once the firm is created it will decide on some production plan

and also on how its profits will be divided among consumers. It is

assumed throughout that the actual decision-making within the firm,

once it has been created, is not costly and is independent of the

endowment of the particular coalition controlling the production

possibility set. Let θ_{ij} be the profit share of consumer i in the

firm j, where $0 \leq \theta_{ij} \leq 1$. All profits are paid to consumers, so

that $\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. Let $P = \{p \in R_+^\ell \mid \sum_{i=1}^{\ell} p_i = 1\}$ denote the set

of possible prices in the economy. Then, by the convention of signs

for the bundles $y_j \in Y_j$, the scalar product $p \cdot y_j$ will be firm j's

profit, and consumer i will receive an amount of $t_i = \sum_{j \in J} \theta_{ij} p \cdot y_j$. In

what follows, the variables t_i will be used describing the profit

payments in the economy avoiding the lengthy notation for the sum,

keeping in mind that behind a list (t_i) there exists an equivalent

triple $(p, (y_j), (\theta_{ij}))$.

Definition: A triple $(J, (y_j), (t_i))$ is called a firm structure

relative to prices p if

(1) $J \subset 2^I$

(2) $y_j \in Y_j$, $j \in J$

(3) $\sum_{j \in J} p \cdot y_j = \sum_{i \in I} t_i$

with the convention $\sum_{j \in \emptyset} = 0$.

The ultimate goal is to describe and characterize different firm

structures, especially those which are related to allocations in the core. Furthermore, since the definition of a firm structure is very general, one will look for structures which have been reached after free entry and exit of firms into the market has been allowed.

There are several advantages of defining firms in the way described above. Since the specification of the production sets of coalitions implies that each set Y^S is completely controlled and can be used by S, the identification of a firm with such a set in turn guarantees that no other institutional assumptions than the ownership control are needed. The conceptual difficulties of defining a partial production possibility set, which one would have to do in almost any other definition, is avoided. Furthermore, the firms which will actually participate in the economy will be selected from a relatively large set, so that there is some hope of defining entry and exit in a meaningful way.

In this context there would be also some conceptual problems related to organization and information if one tries to describe the decision-making of a producer. If two producers controlling the same technologies decide on production plans with different levels of efficiency, one is tempted to use arguments of different information or organization to explain the difference. However, this can be no more than a descriptive argument unless one has a complete theory of information of technical knowledge. Since no particular behavioral rule for a firm will be used, most such arguments would not contradict any actual outcome or decision in the model. No claim is made, however, that the theory presented here explains such phenomena.

3. Replication of Technology, Free Entry and Exit, and Competitive

    Equilibria

    Firms in the traditional theory of competitive equilibria are,

in most cases, given the behavioral rule of maximizing profits

subject to given prices. If this assumption is made for a fixed

set of firms in an economy the problems of replication and/or

entry of new firms are eliminated. However, the economic notion

of competition always embodies some assumptions about other firms

being able to enter into any market. This is particularly true in

the traditional partial equilibrium analysis of markets. There a

competitive equilibrium is thought of as being established after

any number of firms have been allowed to enter or to leave the

market. If this is really the case in a general equilibrium model,

one has to specify which production sets may be used by entering

firms.

    Closely related to the question of free entry is the problem

of replication of technology. Free entry into the same industry

implicitly assumes that the technology can be replicated any number

of times. Then the effect in a market of a new entering firm and

the replication of the technology within a firm are indistinguish-

able. Moreover, if, in this case, firms maximize profits at

given prices, the profit and the maximizing net output bundle of

any existing firm will be unbounded or undetermined as long as

profits are non-negative. In fact the conventional partial

equilibrium analysis of markets with free entry seems to suffer

from that same inconsistency which allows a newly entering firm

to use the technology without allowing the existing firms to

replicate. From this it seems clear that it is almost impossible

to derive an equilibrium concept where the number of firms in

the market is determined within the model and where free entry

as well as replication is allowed. In fact if one assumes that

a fixed set of firms J comprises all technologically distinct

firms and that each firm is allowed to replicate its technolo-

gy, then a competitive equilibrium in the usual definition will

be equivalent to one where free entry and exit of firms is pos-

sible. However, in such a set-up the distinction among firms

is lost and one can actually only speak of industries or types

of producers.

Using the previous description of an economy one may con-

sider the following special case. Let

$$\mathcal{E} = \{I, (X_i), (e_i), (\succsim_i), (Y^S)\}$$

and let

$$\mathcal{E}' = \{I, (X_i), (e_i), (\succsim_i), J, (Y_j), (\theta_{ij})\},$$

where $J = 2^I$.

Then, $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a competitive equilibrium for $\mathcal{E}'$ if

(1) $\forall i \in I$, $\quad \bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot e_i + \sum_{j \in J} \theta_{ij} \bar{p} \cdot \bar{y}_j$ and $x_i \succ_i \bar{x}_i$ implies

$$\bar{p} \cdot x_i > \bar{p} \cdot e_i + \sum_{j \in J} \theta_{ij} \bar{p} \cdot \bar{y}_j$$

(2) $\forall j \in J$, $\quad \bar{p} \cdot \bar{y}_j \geq \bar{p} \cdot y_j \qquad y_j \in Y_j$

(3) $$\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i + \sum_{j \in J} \bar{y}_j \quad .$$

An immediate consequence of the above definition, of the assumptions on the set of firms, and of the possibilities of replication is the following theorem.

> *Theorem:* Let $((\bar{x}_i), (\bar{y}_j), \bar{p})$ be a competitive equilibrium for the economy $\mathcal{E}'$, and assume that
>
> *(1)* $0 \in Y_j$, $j \in J$;
>
> *(2)* there are no limitations to replication for any firm $j \in J$.
>
> Then $(\bar{x}_i)$ is in the core.

The assertion is equivalent to the result by Debreu and Scarf [14]. The assumptions (1) and (2) guarantee that each production set will be "almost" a cone, so that at a competitive equilibrium, profits have to be zero everywhere. This in turn provides the basic argument in the proof, since any blocking coalition has to achieve at least some small positive profit, which would contradict the profit maximization of each firm if it existed. It should be noted that the proof is carried out without any further specifications of the particular technology and/or its distribution among the coalitions. Since profits are equal to zero everywhere the result is also independent of any particular list of profit shares (θ_{ij}).

The theorem demonstrates the full force of free entry and of replication of technology very clearly. Yet, the conceptualization of it does not seem very satisfactory. Starting with a fixed finite number of firms but allowing each firm to replicate as often as it wants seems to introduce an arbitrary element prejudging the number of firms which will be in the market whether in equilibrium or not. This is clearly undesirable for a definition of an equilibrium which is thought of as the end point of a process determining, among other things, the number of firms in the market.

On the other hand, the procedure of defining the economy $\mathcal{E}'$ cannot be applied to achieve results for economies where replication is somewhat restricted but entry and exit of firms are allowed. In this case, an existence proof along the traditional lines breaks down since production sets will not have the properties implied by assumptions (1) and (2). In particular, if one eliminates assumption (1), then an equilibrium of the economy $\mathcal{E}'$ may have firms producing at negative profits which is not a desirable property. Most important, however, one would want to have a theory of entry and exit of the determination of firm structures which also explains the selection of different firms out of a larger set and not only the relatively trivial case of replication. In the next section an equilibrium definition is proposed which embodies the basic features of entry and exit of firms, and which will also explain the existence of firms at the equilibrium point.

4. A Model of Entry and Exit of Firms

In Part 3 a firm structure was defined as a triple $(J, (y_j),$ $(t_i))$ such that $\sum_{i \in I} t_i = \sum_{j \in J} p \cdot y_j$. Since the ultimate control over the usage of the technology lies in the hands of the consumers, an equilibrium concept should take into account that any coalition, which is dissatisfied with its profit payments from a given firm structure and which can actually achieve higher payments for all of its members by disregarding the existing structure, will actually carry out their own production and thus disturb the structure. Hence some new structure will be the result. This argument, for which a more intuitive justification will be given below, provides the basis for the following definition.

Definition: A list $(J, (y_j), (t_i))$ is called a stable firm structure relative to prices p if

(1) $(J, (y_j), (t_i))$ is a firm structure relative to prices p;

(2) for all $i \in I$ $t_i \geq 0$;

(3) for all $S \subset I$

$$\sum_{i \in S} t_i \geq \sup \{ p \cdot y^S | y^S \in Y^S \}$$

The term stability is used here only to describe a certain triple and is not meant to imply the existence of a particular dynamic process which converges to this structure and thus could also be called stable. Both conditions, however, have a straightforward interpretation in terms of actions which any coalition S_j may take with respect to the existing firm structure. The outcome of those actions will be a new firm structure. Since the structure is defined by three elements, a change in any one of them constitutes a change in the structure.

Suppose condition (2) is not satisfied for a given structure.
Then, some firm in the market may be producing at a loss which
implies that some consumers have to pay for that loss. But then
each of these consumers could refuse to pay for the loss thus
forcing the firm either to change its production plan to a pro-
fitable level or to leave the market. Alternatively, assume that
condition (3) is not satisfied. Then, for some coalition S' of
consumers, $\sum\limits_{i \in S'} t_i < \max \{p \cdot y^{S'} | y^{S'} \in Y^{S'}\}$. Since S' has ultimate
control over $Y^{S'}$, they will be able to guarantee themselves at least
the maximal profit over $Y^{S'}$ without cooperation with any other group.
Hence, in this situation, some of their actions may be the following.

If their technical knowledge is not used already in some exist-
ing firm, they can create a new firm j' and then produce the maxi-
mum profit level and distribute the total profit among themselves.
Hence each member in S' could be made better off in terms of income
regardless of whether the existing firms will continue to pay any
profit shares to them or not. In particular, this will be possible
without any subcoalition of S' entering the market as a firm and
producing simultaneously. On the other hand, if the knowledge $Y^{S'}$
was used already by some firm in the market, the coalition S' may
refuse to make this knowledge available any more unless they re-
ceive at least the amount of profit that can be achieved with their
own knowledge. Again, this desire can be enforced by S'. In either
case, S' can guarantee itself at least that much profit without the
cooperation of any other coalition. The outcome of any of those ac-
tions by S' will be a new firm structure $(J'(y_j'), (t_i'))$.

If one considers the opposite case in which condition (3) is satisfied but some coalition or existing firm attempts to make a change in the structure, then there is no guarantee at all that this coalition will maintain its profit level unless there is a collusive agreement on this point. Hence no coalition really has the power to reach any higher income position by itself.

The definition of a stable firm structure and its interpretation describe the production sector of a free market economy. All production decisions are completely decentralized and made by the individual firms independent of the consumption decisions of the owners of the firm. Although the ultimate control over the available production possibilities is exercised by consumers their influence is only traceable with regard to their desire to achieve a high income level. In this respect the definition guarantees a certain "maximal" income to each consumer relative to his technical knowledge. On the other hand, the definition allows for free entry and exit of all possible firms using only a minimal assumption on cooperation among consumers to guarantee actual formation of any firm. Combining the feature that each consumer maximizes his preference relation subject to his income with the above concept yields the following definition.

Definition: A list $((\bar{x}_i), J, (\bar{y}_j), (\bar{t}_i), \bar{p})$ is a market equilibrium with a stable firm structure if

(1) $\forall i \in I$, $\bar{x}_i$ maximizes $\underset{i}{\succsim}$ in the budget set $\{x_i \in X_i \mid \bar{p} \cdot x_i \leq \bar{p} \cdot e_i + \bar{t}_i\}$;

(2) $(J, (\bar{y}_j), (\bar{t}_i))$ is stable at $\bar{p}$;

(3) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i + \sum_{j \in J} \bar{y}_j$.

Thus an equilibrium has the two main properties that no group of consumers through independent action can increase its total income and no consumer can achieve a higher level of satisfaction using his own income. The concept represents a generalization of the usual competitive equilibrium. In fact, one can show under certain conditions that for an economy where all firms have been formed the competitive equilibrium is also one with a stable firm structure if each firm distributes profits only to its owners. In general, however, the concept is independent of any behavioral assumption for firms; in particular, profit maximization of firms will in general not be present at the equilibrium point.

There exists a second relationship between stable firm structures and competitive behavior of firms which is stated in the following lemma.

Lemma 1: Let $(J,(y_j)(t_i))$ be a stable firm structure relative to p such that the set J defines a partition of I, i.e., for any j' and j" contained in J, $S_{j'} \cap S_{j''} = \emptyset$, and $\bigcup_{j \in J} S_j = I$. Then for all $j \in J$,

(1) $\quad p \cdot y_j = \text{Max}\{p \cdot y \mid y \in Y^{S_j}\}$

(2) $\quad \sum_{i \in S_j} t_i = p \cdot y_j$.

Proof: Consider the partition $\{S_j\}$. The stability implies that for all $j \in J$,

$$\sum_{i \in S_j} t_i \geq \text{Max}\{p \cdot y \mid y \in Y^{S_j}\} \geq p \cdot y_j \ .$$

Hence

$$\sum_{j \in J} p \cdot y_j = \sum_{i \in I} t_i = \sum_{j \in J} \sum_{i \in S_j} t_i \geq \sum_{j \in J} p \cdot y_j$$

which yields (1) and (2). $\hspace{2cm}$ Q.E.D.

One of the main characteristics of a stable firm structure is that it lends itself to an immediate interpretation of the profit payments as returns to technical knowledge. The profit shares which each consumer receives are directly related to his contribution of profitable technical knowledge. The definition of stability guarantees him a certain minimum return. For the specific case of Lemma 1 where each consumer contributes his knowledge only to one firm his return will actually be the maximum share which he could bargain for within the specific firm structure. The specific list of profit payments may in fact be realized by each firm distributing profits only to its owners.

The main result of this section is that an equilibrium with a stable firm structure generates an allocation in the core which extends this result for pure exchange economies to economies with production.

Lemma 2: Let $((\bar{x}_i), J, (\bar{y}_j), (\bar{t}_i), \bar{p})$ *be a market equilibrium with a stable firm structure. Then* $(\bar{x}_i)$ *is an allocation in the core.*

Proof: Suppose the statement were false. Then there would exist a non-empty coalition S which could block $(\bar{x}_i)$, i.e., there exist (x_i), $i \in S$, $y^S \in Y^S$ such that

(1) $x_i \underset{i}{\succ} \bar{x}_i$ $i \in S$

(2) $\sum_{i \in S} x_i = \sum_{i \in S} e_i + y^S$.

Yet (1) implies $\bar{p} \cdot x_i > \bar{p} \cdot e_i + \bar{t}_i$, $i \in S$. Hence

$$\bar{p} \cdot \sum_{i \in S} e_i + \bar{p} \cdot y^S > \bar{p} \cdot \sum_{i \in S} e_i + \sum_{i \in S} \bar{t}_i \geq \bar{p} \cdot \sum_{i \in S} e_i + \text{Max}\{\bar{p} \cdot y | y \in Y^S\}$$

implying $\bar{p} \cdot y^S > \text{Max}\{\bar{p} \cdot y | y \in Y^S\}$

which is a contradiction. Q.E.D.

Hence, the method of construction of firms leads to a meaningful

equilibrium concept which may be viewed as the outcome of a decent-

ralized market process which determines the set of firms producing

the market and the equilibrium profit payments each consumer receives.

Entry and exit of firms have an explicit meaning in this context as a

selection process of the ultimate members of the stable firm struc-

ture out of the large set of possible firms. Lemma 2 established the

relationship of the equilibrium state with the core of the underlying

bargaining economy which was one of the desired properties of the

equilibrium concept.

5. Existence of Equilibria with Stable Firm Structures

The ultimate test whether a new concept is in fact a useful one

is that it is nonvacuous under nonpathological assumptions and, also,

that it explains a wider array of problems than existing concepts.

The second point has been demonstrated in the preceding section. This

section contains a main existence theorem the proof of which is a

straightforward extension of the existence proof for competitive

equilibria given by Debreu in [13]. His notation and definitions will

be followed as closely as possible. Frequent reference will be made .

to proofs of details in [13] which are also used here.

The major differences between Debreu's proof and the one presented

here are a result of the different equilibrium concepts. Since his

method of proof is only directly applicable to an economy with a fixed

firm structure it was necessary to find a procedure which determines a

set of firms and supply bundles at each price. More precisely, for

every price a stable firm structure had to be found. The crucial

argument for the construction is taken directly from the definition

of a stable firm structure which has an immediate interpretation as

a solution of a sidepayment game for each price. Its core, defined

in an appropriate way, yields the necessary continuity property of

the payoffs to show existence of an equilibrium. Lemma 1 and its

proof represent the crucial step. It also supplies the basic

argument for the construction of the set of firms, defined for each

price by the dual variables of a linear program, which is an appli-

cation of the result on cores of balanced games.[1]/

Main Existence Theorem

Let the economy $\mathcal{E}$ be described by

$$\mathcal{E} = \{I, (X_i), (e_i), (\succsim_i), (Y^S)\} .$$

Then $\mathcal{E}$ has a market equilibrium with a stable firm structure if

for all $i \in I$

(C1) $X_i \subset R^\ell$ is closed, convex, and bounded from below,

(C2) i is locally not satiated,

(C3) $\succsim_i$ is a complete, transitive, and continuous preordering on X_i ,

  such that the set $\{x_i \in X_i \mid x_i \succsim_i x_i'\}$ is convex for every $x_i' \in X_i$,

(C4) there exists $x_i^o \in X_i$ such that $x_i^o \ll e_i$,

(C5) $0 \in Y^{\{i\}}$;

and if

---

[1]See, e.g., Shapley [33].

(P1) for all $S \subset I$ Y^S is closed,

(P2) Y is closed,

(P3) $Y \cap (-Y) \subset \{0\}$ irreversibility of production,

(P4) $Y \supset \{-\Omega\}$ free disposal;

and if for every balanced family $\mathcal{J}$ of non-empty coalitions and associated weights (d_S),

(P5) $\sum_{S \in \mathcal{J}} d_S Y^S \subset \sum_{S \in \mathcal{J}} Y^S$,

(P6) $\sum_{S \in \mathcal{J}} Y^S$ is convex.

Assumptions (C1)-(C4) are standard for any existence proof in general equilibrium theory. On the production side (P1)-(P4) are the appropriate generalizations of the assumptions usually made in a competitive model. (C5), (P5), and (P6) reflect the relationship between the sufficient conditions for a non-empty core and for the existence of an equilibrium. (P5) is a slightly stronger assumption for a balanced technology than the one used in Chapter II. Clearly, the assumption $0 \in Y^S$ for every S in conjunction with (P6) also would have been sufficient to guarantee that (Y^S) were balanced. This was proved in Lemma 2 of Section 3, Chapter II. Assumption (P5), however, includes cases where for some S $0 \notin Y^S$.

Consider the economy with three consumers and assign to the coalition {1,2} the set $Y^{\{1,2\}}$ and to all other coalitions the cone Y^S as depicted in Figure 3.
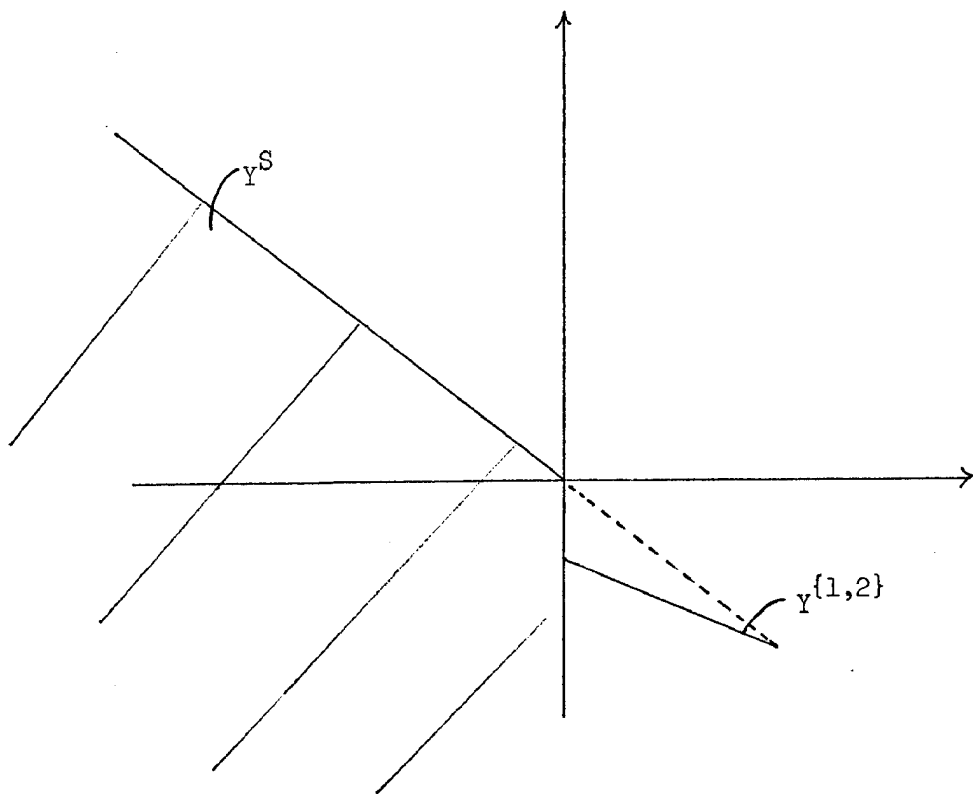
Figure 3

Clearly, for all $0 < d_{\{1,2\}} \leq 1$, $d_{\{1,2\}} Y^{\{1,2\}} \subset Y^{\{1,2\}} + Y^S$,

and for all $\beta > 0$, $\beta Y^S = Y^S$. Hence for any balanced family $\mathscr{A}$

including $\{1,2\}$

$$d_{\{1,2\}} Y^{\{1,2\}} + \sum_{S \neq \{1,2\}} d_S Y^S = d_{\{1,2\}} Y^{\{1,2\}} + (\sum_{S \neq \{1,2\}} d_S) Y^S$$

$$\subset Y^{\{1,2\}} + Y^S = Y^{\{1,2\}} + \sum_{S \neq \{1,2\}} Y^S .$$

Hence this technology satisfies (P5) as well as (C5).

Definition: The set of attainable states of the economy $\mathscr{E}$ is

an $(n+1)$-list of vectors $(x_1, \ldots, x_n, y) \in R^{\ell(n+1)}$ such that for all

$i \in I$ $x_i \in X_i$, $y \in Y$, and $\sum_{i \in I} x_i = \sum_{i \in I} e_i + y$.

Proof: First, one observes that (C1), (P2)-(P4) imply that the set of attainable states of the economy is closed and bounded (Debreu [13], Theorems 1 and 2, p. 77). Hence, most arguments can be carried out in a well-chosen compact cube in the commodity space (Debreu [13], proof of Theorem 1, p. 83). Let K_1 be a closed cube of R^ℓ with center at the origin containing in its interior the set of all attainable consumption and production plans. For $i \in I$, define $X_i^1 = X_i \cap K_1$ and for $S \subset I$, define $Y_1^S = Y^S \cap K_1$. Following Debreu, one can show the existence of an equilibrium for the economy

$$\mathcal{E}_1 = \{I, (X_i \cap K_1), (e_i), (\underset{i}{\Rightarrow}), (Y^S \cap K_1)\}.$$ Although any equilibrium will be contained in this truncated economy, one cannot conclude that any equilibrium with a stable firm structure for $\mathcal{E}_1$ is also an equilibrium with a stable firm structure for $\mathcal{E}$. Therefore, an increasing sequence of cubes K_q with the associated truncated economies $\mathcal{E}_q$ will be constructed, where K_q becomes infinitely large. Arguments similar to the ones used by Debreu ([11], Section 3) and by Hildenbrand ([20], proof of Theorem 2) will then establish that there exists an equilibrium for the unrestricted economy $\mathcal{E}$.

The proof will now be carried out in several steps. Let

$$v_S(p) = \text{Max}\{p \cdot y^S \mid y^S \in Y^S \cap K_1\} \quad S \subset I.$$

Lemma 1: If for all $S \subset I$, Y_1^S is compact, non-empty and if for all $i \in I$ $0 \in Y^{\{i\}}$, then for each $p \in P$ there exists a payoff vector $h \in R^n$, $h \geq 0$, and a generalized characteristic vector $d \in R^{2n}$, $0 \leq d \leq 1$, such that

(1) $\quad \sum_{i \in S} h_i \geqq v_S(p) \qquad$ *for all* $S \subset I$

(2) $\quad \sum_{i \in I} h_i = \sum_{S \subset I} d_S v_S(p)$

(3) $\quad \sum_{S \ni i} d_S = 1 \qquad$ *for all* $i \in I$.

Proof: Since Y_1^S is compact, $v_S(p)$ exists for all S at any p. In particular, $v_{\{i\}}(p) \geqq 0$. Denote by $e_S \in R^n$ the characteristic vector of coalition S, i.e., $(e_S)_i = 1$, if $i \in S$, and zero otherwise; and $e_\emptyset = (0,\ldots,0)$. Let $E = (e_S)$ be the matrix of all 2^n vectors. Arranging the elements in I and E in the appropriate order, one can rewrite (1) as

$\qquad Eh \geqq v(p)$.

Consider the following linear program and its dual.

$\quad$ *Primal:* $\quad$ Min $\sum_{i \in I} h_i$

$\qquad\qquad$ Subject to $\quad Eh \geqq v(p)$

$\quad$ *Dual:* $\qquad$ Max $d \cdot v(p)$

$\qquad\qquad$ Subject to $\quad dE = \mathbf{1} \qquad d \geqq 0$.

$\mathbf{1}$ denotes a vector of appropriate dimension, all of whose elements are equal to one.

$\qquad$ Since $v(p)$ is finite both problems are feasible. Then, by standard duality arguments, both have optimal solutions (d^*, h^*) such that

$$d^* \cdot v(p) = \sum_{i \in I} h_i^* \ .$$

Hence (d^*, h^*) satisfy (1), (2), and (3). $\qquad$ Q.E.D.

Clearly, for a given p, d^* and h^* will not be unique. Define

$$\delta(p) = \{d \mid d \text{ a solution of the dual at } p\}$$

$$\tau(p) = \{h \mid h \text{ a solution of the primal at } p\} \ ,$$

Then Lemma 1 states that $\delta(p) \neq \emptyset$ and $\tau(p) \neq \emptyset$.

Lemma 2:   _If Y_1^S is compact for all $S \subset I$, then δ and τ are upper_

hemi-continuous and convex valued correspondences, and τ admits a

continuous selection.

Proof: Y_1^S compact implies that $v(p)$ is a continuous function.

The dual as the following maximization problem

$$\Pi(p) = \text{Max}\{d.v(p) \mid dE = 1, \ d \geq 0\}$$

yields δ upper hemi-continuous by standard maximization results

since $d.v(p)$ is a continuous function from $\{d \mid dE = 1, \ d \geq 0\} \times P$ into

R and $\{d \mid dE = 1, \ d \geq 0\}$ is trivially continuous in p. Let $d^1 \in \delta(p)$ and

$d^2 \in \delta(p)$ and $0 \leq \lambda \leq 1$. Then $\lambda d^1 + (1-\lambda)d^2 \in \{d \mid dE = 1, \ d \geq 0\}$, a convex

set. Furthermore, $d^1.v(p) = d^2.v(p)$ implies $(\lambda d^1 + (1-\lambda)d^2).v(p) =$

$\lambda d^1.v(p) + (1-\lambda)d^2.v(p) = d^1.v(p)$. Hence, $\delta(p)$ is convex valued.

Similarly, for the primal, one knows that $\beta(p) = \{h \mid Eh \geq v(p)\}$ is

convex valued. Take $h^1 \in \beta(p)$, $h^2 \in \beta(p)$. Then $E(\lambda h^1 + (1-\lambda)h^2) =$

$\lambda Eh^1 + (1-\lambda)Eh^2 \geq \lambda v(p) + (1-\lambda)v(p) = v(p)$.

Furthermore, from the duality property, the objective function

of the primal is continuous in p since $\Pi(p) = \{h.1 \mid h \in \tau(p)\}$ where

$\Pi(p)$ was shown to be continuous. Hence τ maps P into some compact

subset of R^n . For τ to be upper hemi-continuous, it is sufficient

to show that its graph is closed. Consider $p^n \to p$, $h^n \to h$,

$h^n \in \tau(p^n)$. Then the continuity of $\Pi(p)$ and $h^n \in \tau(p^n)$ implies

$h.\mathbf{1} = \Pi(p)$. Hence $h \in \tau(p)$. To show convexity, let $h^1 \in \tau(p)$ and $h^2 \in \tau(p)$. Then $(\lambda h^1 + (1-\lambda)h^2).\mathbf{1} = \lambda h^1.\mathbf{1} + (1-\lambda)h^2.\mathbf{1} = h^1.\mathbf{1}$.

It remains to be shown that τ admits a continuous selection. Consider the following linear program

$$\text{Min} \quad h_1$$

$$\text{Subject to} \quad Eh \geq v(p)$$

$$\mathbf{1}.h \leq \Pi(p)$$

Clearly, the feasible set for this program is $\tau(p)$, a non-empty, compact, and convex subset of R^n of dimension at most equal to n-1, which implies that the program has an optimal solution. Let $f_1(b) = \text{Min}\{h_1 \mid Eh \geq v(p), \mathbf{1}.h \leq \Pi(p)\}$ and $\tau_1(p) = \{h \mid h \in \tau(p), h_1 = f_1(p)\}$. Using the same arguments as before for the correspondence τ , it follows immediately that f_1 is a continuous function, τ_1 is upper hemi-continuous, and $\tau_1(p)$ is non-empty, compact, and of dimension at most n-2. Proceeding in the same fashion, define for $i = 2,\ldots,n$

$$f_i(p) = \text{Min}\{h_i \mid Eh \geq v(p), \mathbf{1}.h \leq \Pi(p), e_{\{i-k\}}.h \leq f_{i-k}(p), k = 1,\ldots,i-1\}$$

and

$$\tau_i(p) = \{h \mid Eh \geq v(p), \mathbf{1}.h \leq \Pi(p), e_{\{i-k\}}.h \leq f_{i-k}(p), k = 1,\ldots,i-1,$$

$$e_{\{i\}}.h = f_i(p)\}$$

$$= \{h \mid h \in \tau_{i-1}(p), e_{\{i\}}.h = f_i(p)\} .$$

Clearly, for all $i = 2, \ldots, n$ f_i is continuous, $\tau_i(p)$ is non-empty,

compact, of dimension at most equal to $\text{Max}\{0, n-i\}$, and τ_i is upper

hemi-continuous. In particular, $\tau_n(p)$ will be the unique point

$(f_1(p), \ldots, f_n(p))$. Since τ_n is upper hemi-continuous the function

$g : P \to R^n$ defined by $g(p) = (f_1(p), \ldots, f_n(p))$ is continuous and

for all $p \in P$ $g(p) \in \tau(p)$.   Q.E.D.

Let $\bar{\eta}_S(p) = \{\bar{y}^S \in Y_1^S \mid p \cdot \bar{y}^S = v_S(p)\}$. Under assumption (P1)

$\bar{\eta}_S(p)$ is non-empty and $\bar{\eta}_S$ is upper hemi-continuous. For each $p \in P$

and $d \in \delta(p)$ define a supply correspondence

$$\eta(d,p) = \sum_{S \subset I} d_S \, \bar{\eta}_S(p)$$

Since the strictly positive components of d define a balanced family

$\mathscr{S}(d)$ it follows that $\eta(d,p) = \sum_{S \in \mathscr{S}(d)} d_S \, \bar{\eta}_S(p)$, $d \in \delta(p)$. Now

define as the aggregate supply correspondence

$$\eta(p) = \text{conv} \bigcup_{d \in \delta(p)} \eta(d,p)$$

where conv denotes convex hull.

Lemma 3: *If* Y_1^S *is compact and non-empty, and if (P5) and (P6)*

hold, then $\eta(p)$ *is non-empty,* η *is an upper hemi-continuous*

correspondence, and $y \in \eta(p)$ *implies*

(1) $p \cdot y = \Pi(p)$,

(2) *there exists a family* $\mathscr{A} \subset 2^I$ *such that* $y \in \sum_{S \in \mathscr{A}} Y^S$.

Proof: The non-emptiness follows from Lemma 1 and from the definition of $\eta(p)$.

Let $y \in \eta(d,p)$, i.e. $y = \sum\limits_{S \subset I} d_S \bar{y}_S$ where $(d_S) \in \delta(p)$ and $\bar{y}_S \in \bar{\eta}_S(p)$. Then

$$p \cdot y = p \cdot \sum_{S \subset I} d_S \bar{y}_S = \sum_{S \subset I} d_S\, p \cdot \bar{y}_S = \sum_{S \subset I} d_S\, v_S(p) = \Pi(p)$$

which proves (1), since the same argument can be used for any finite convex combination of points in $\bigcup\limits_{d \in \delta(p)} \eta(d,p)$.

To prove (2) one uses the fact that with each element in $\eta(d,p)$ is associated a balanced family $\mathscr{A}(d)$. Let $y \in \eta(p)$. Then y can be written as a convex combination of at most $\ell + 1$ vectors $y^k \in \eta(d^k,p)$, $k = 1,\ldots,\ell+1$, i.e., $y = \sum\limits_{k=1}^{\ell+1} \lambda^k y^k$ with $0 \leq \lambda^k \leq 1$ and $\sum\limits_{k=1}^{\ell+1} \lambda^k = 1$. Let $\mathscr{A}^k$ be the balanced family associated with d^k and let $y_S^k \in \bar{\eta}_S(p)$ be such that $y^k = \sum\limits_{S \in \mathscr{A}^k} d_S^k\, y_S^k$.

Then

$$y = \sum_{k=1}^{\ell+1} \lambda^k y^k = \sum_{k=1}^{\ell+1} \lambda^k \sum_{S \in \mathscr{A}^k} d_S^k\, y_S^k = \sum_{k=1}^{\ell+1} \sum_{S \in \mathscr{A}^k} \lambda^k\, d_S^k\, y_S^k \ .$$

First one observes that $\mathscr{A} = \bigcup\limits_{k=1}^{\ell+1} \mathscr{A}^k$ is a balanced family which is defined by the positive components of the associated vector of weights $\gamma = \sum\limits_{k=1}^{\ell+1} \lambda^k d^k$. Clearly, $\gamma \in \delta(p)$ according to Lemma 2. Furthermore,

$$\sum_{\substack{k=1}}^{\ell+1} \sum_{S \in \mathscr{A}^k} \lambda^k \, d_S^k \, y_S^k = \sum_{S \in \mathscr{A}} \sum_{\substack{k \\ \mathscr{A}^k \ni S}} \lambda^k \, d_S^k \, y_S^k$$

$$= \sum_{S \in \mathscr{A}} \sum_{\substack{k \\ \mathscr{A}^k \ni S}} \left(\sum_{\substack{k \\ \mathscr{A}^k \ni S}} \lambda^k \, d_S^k \right) \frac{\lambda^k \, d_S^k}{\sum_{\substack{k \\ \mathscr{A}^k \ni S}} \lambda^k \, d_S^k} \, y_S^k$$

$$= \sum_{S \in \mathscr{A}} \gamma_S \sum_{\substack{k \\ \mathscr{A}^k \ni S}} \frac{\lambda^k \, d_S^k}{\gamma_S} \, y_S^k \in \sum_{S \in \mathscr{A}} \gamma_S \, \text{conv} \, Y^S$$

$$= \sum_{S \in \mathscr{A}} \text{conv} \, \gamma_S \, Y^S = \text{conv} \sum_{S \in \mathscr{A}} \gamma_S \, Y^S \subset \sum_{S \in \mathscr{A}} Y^S$$

where the last inclusion follows from (P5) and (P6). Hence

$y \in \sum_{S \in \mathscr{A}} Y^S$ which proves (2).

The upper hemi-continuity of η will be shown in two steps.
First, it will be demonstrated that $\tilde{\eta}(p) = \bigcup_{d \in \delta(p)} \eta(d,p)$ is upper

hemi-continuous.

Since for all p and all $d \in \delta(p)$ $\eta(d,p)$ is bounded it suffices
to show that $\bigcup_{d \in \delta(p)} \eta(d,p)$ has a closed graph. Consider sequences

$y^n \to y$, $p^n \to p$ such that $y^n \in \bigcup_{d \in \delta(p^n)} \eta(d,p^n)$. Then there exist

sequences $d^n \to d$ and $y^n_S \to y_S$ for every $S \subset I$ such that $d^n \in \delta(p^n)$ and

$y^n_S \in \bar{\eta}_S(p^n)$, $S \subset I$. Since for every $S \subset I$ $\bar{\eta}_S$ and δ have a closed

graph, it follows that $y_S \in \bar{\eta}_S(p)$, $S \subset I$ and $d \in \delta(p)$. Hence

$$y = \sum_{S \subset I} d_S y_S \in \eta(d,p) \subset \bigcup_{d \in \delta(p)} \eta(d,p).$$

It remains to be shown that $\eta(p)$ has a closed graph. Let μ be

any correspondence $\mu : P \to Y$, $Y \subset R^\ell$ and Y compact, and assume that

μ has a closed graph. Consider sequences $z^n \to z$, $p^n \to p$,

$z^n \in \text{conv } \mu(p^n)$. Then there exist sequences $z^n_k \to z_k$, $\lambda^n_k \to \lambda_k$ for

$k = 1,\ldots,\ell+1$ with $0 \leq \lambda^n_k \leq 1$ and $\sum_{k=1}^{\ell+1} \lambda^n_k = 1$ such that $z^n =$

$\sum_{k=1}^{\ell+1} \lambda^n_k z^n_k$ and $z^n_k \in \mu(p^n)$. Since μ has a closed graph it follows

that $z_k \in \mu(p)$ for every $k = 1,\ldots,\ell+1$ and clearly $\sum_{k=1}^{\ell+1} \lambda_k = 1$ with

$0 \leq \lambda_k \leq 1$ for $k = 1,\ldots,\ell+1$. Hence $\sum_{k=1}^{\ell+1} \lambda_k z_k \in \text{conv } \mu(p)$, which

completes the proof of Lemma 3.

Let $g_i(p)$, $i = 1,\ldots,n$ be the i-th component of the continuous

selection $g(p) \in \tau(p)$, i.e., consumer i's profit payment. Then his

budget correspondence β_i can be defined as

$$\beta_i(p) = \{x_i \in X_i \cap K_1 \mid p \cdot x_i \leq p \cdot e_i + g_i(p)\} .$$

Lemma 4: If (C1), (C4), and (C5) hold and if Y_1^S is compact,
then β_i is lower hemi-continuous at every p and has a closed
graph.

Proof: (C4), (C5), and Lemma 2 imply that for all $p \in P$ $\beta_i(p)$

is non-empty. Let $x_i^n \to x_i$, $p^n \to p$ and for all n $x_i^n \in \beta_i(p^n)$.

Hence, $p^n \cdot x_i^n \leq p^n \cdot e_i + g_i(p^n)$ and the continuity on both sides imply

$p \cdot x_i \leq p \cdot e_i + g_i(p)$, i.e. $x_i \in \beta_i(p)$.

Let $x_i \in \beta_i(p)$ and $p^n \to p$. According to (C4) and since $g_i(p) \geq 0$,

$p \cdot x_i^0 < p \cdot e_i + g_i(p)$. Consider the straight line L passing through

x_i^0 and x_i and let $a^n \in L$ be such that $p^n \cdot a^n = p^n \cdot e_i + g_i(p^n)$. Define

$$
x_i^n = \begin{cases} a^n & \text{if } p^n \cdot a^n < p^n \cdot x_i \\ x_i \, , & \text{otherwise} \end{cases}
$$

Clearly, $x_i^n \to x_i$ and, also, $x_i^n \in \beta_i(p^n)$ for all n. Hence, β_i is

lower hemi-continuous. Q.E.D.

Let the demand correspondence ξ_i of each consumer i be defined by

$\xi_i(p) = \{x_i \in \beta_i(p) \mid x_i \underset{i}{\succeq} z_i$ for all $z_i \in \beta_i(p)\}$.

Lemma 5: *Let (C1)-(C5) be satisfied. Then $\xi_i(p)$ is non-empty*

and convex, and the correspondence ξ_i is upper hemi-continuous.

Proof: Since $X_i \cap K_1$ is compact, $\beta_i(p)$ is compact. According to

Lemma 4, β_i is a continuous correspondence. Since $\underset{i}{\succeq}$ is a complete

preorder there exists a maximal element in $\beta_i(p)$, hence $\xi_i(p)$ is

non-empty.

Let $x_i' \in \xi_i(p)$ and $x_i'' \in \xi_i(p)$. For any $0 \leq \lambda \leq 1$, $\lambda x_i' +$

$(1-\lambda)x_i'' \in \beta_i(p)$. Furthermore, by the convexity of $\underset{i}{\succeq}$, $\lambda x_i' +$

$(1-\lambda)x_i'' \underset{i}{\succeq} x_i'$ which implies $\lambda x_i' + (1-\lambda)x_i'' \in \xi_i(p)$.

For ξ_i to be upper hemi-continuous, it suffices to show that ξ_i has a closed graph. Let $x_i^n \to x_i$, $p^n \to p$, and $x_i^n \in \xi_i(p^n)$. Clearly, $x_i \in \beta_i(p)$. Since β_i is lower hemi-continuous, for every $z \in \beta_i(p)$ there exists a sequence $z^n \to z$ and $z^n \in \beta_i(p^n)$. Hence, $x_i^n \succsim_i z^n$ for all n and by the continuity of $\succsim_i$, $x_i \succsim_i z$ for all $z \in \beta_i(p)$, implying that ξ_i is upper hemi-continuous.   Q.E.D.

Let $\xi(p) = \sum_{i \in I} \xi_i(p)$ and define the excess demand correspondence ζ as

$$\zeta(p) = \xi(p) - \sum_{i \in I} e_i - \eta(p)$$

which is non-empty, convex, and upper hemi-continuous. ζ maps P into some compact subset Z of R^ℓ.

Following standard arguments of equilibrium analysis, define a correspondence μ by $\mu(z) = \{p \in P \mid p.z = \text{Max } P.z\}$. Clearly, $\mu(z)$ is non-empty and convex, and μ is upper hemi-continuous. Now let ψ be the correspondence defined by $\psi(z,p) = \zeta(p) \times \mu(z)$. ψ is a map from $Z \times P$ into itself. Furthermore, ψ is upper hemi-continuous and $\psi(p)$ is non-empty and convex. Applying Kakutani's Fixed Point Theorem, there exists a (z^1,p^1) such that $(z^1,p^1) \in \psi(z^1,p^1)$, i.e. $z^1 \in \zeta(p^1)$ and $p^1 \in \mu(z^1)$. It remains to be shown that $z^1 \leqq 0$. For any $p \in P$ and $z \in \zeta(p)$, i.e. $x \in \xi(p)$ and $y \in \eta(p)$, $z = x - \sum_{i \in I} e_i - y$, $p.z = p.x - p.\sum_{i \in I} e_i - p.y \leqq g(p).1 - \Pi(p) = 0$. Hence in particular, $p^1.z^1 \leqq 0$. Since $p^1 \in \mu(z^1)$, $p.z^1 \leqq p^1.z^1 \leqq 0$ for all $p \in P$ implies $z^1 \leqq 0$. Since each consumer is locally not satiated, $p^1.x_i^1 = p^1.e_i + g_i(p^1)$ which implies $p^1.z^1 = 0$. Hence it has been shown that there exists

a list $((x_i^1), J^1, (y_j^1), (t_i^1), p^1)$ where J^1 is determined according to Lemma 3, and $t_i^1 = g_i(p^1)$ $i \in I$. By construction $(J^1, (y_j^1), (t_i^1))$ is stable relative to p^1. Furthermore, for each $i \in I$, x_i^1 is a best element in the restricted budget set, and market excess demand is non-positive.

Now consider an increasing sequence $(K_q)_{q=1,\ldots}$ of closed cubes in R^ℓ with center at the origin and whose diameters tend to infinity. With each K_q associate the truncated economy $\mathscr{E}_q$. Thus, for every $q = 1,\ldots$, there exists a list $((x_i^q), (y_j^q), p^q, (t_i^q), d^q)$ such that

(1) d^q determines the set of firms J^q,

(2) for every $i \in I$

$$x_i \in X_i \cap K_q \quad \text{and} \quad x_i \succ_i x_i^q \quad \text{implies}$$

$$p^q \cdot x_i > p^q \cdot e_i + t_i^q,$$

(3) $(J^q, (y_j^q), (t_i^q))$ is a firm structure relative to p^q, i.e.

$$\sum_{i \in I} t_i^q = \sum_{j \in J^q} p^q \cdot y_j^q,$$

(4) for every $S \subset I$

$$\sum_{i \in S} t_i^q \geq \text{Max}\{p^q \cdot y^S \mid y^S \in Y^S \cap K_q\},$$

(5) $\sum_{i \in I} (x_i^q - e_i) - \sum_{j \in J^q} y_j^q \leq 0$.

By the choice of K_1 and since $K_1 \subset K_q$ for all $q = 1,\ldots$, one knows that for all $i \in I$ $x_i^q \in \text{int } K_1$, $t_i^q \geq 0$, $\sum_{j \in J^q} y_j^q \in \text{int } K_1$,

$0 \leq d^q \leq 1$. Hence the sequences $(x_i^q)_{q=1,...}$ and $(y_j^q)_{q=1,...}$ are bounded as well as (t_i^q) since $(p^q)_{q=1,...}$ is bounded and $t_i^q = g_i(p^q)$. Thus, there exists a converging subsequence with limit point $((\bar{x}_i), (\bar{y}_j), \bar{p}, (\bar{t}_i), \bar{d})$. Clearly, $\bar{x}_i \in X_i$, $\bar{y}_j \in Y_j$, $\bar{t}_i \geq 0$, $\bar{p} \in P$, and $0 \leq \bar{d} \leq 1$. Furthermore, $\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j \leq 0$

and $\sum_{i \in I} \bar{t}_i = \sum_{j \in \bar{J}} \bar{p} \cdot \bar{y}_j$ where $\bar{J}$ is the set of firms determined by $\bar{d}$.

Suppose the firm structure $(\bar{J}, (\bar{y}_j), \bar{t}_i)$ were not stable relative to $\bar{p}$. Then there exists a coalition S and a bundle $y \in Y^S$ such that $\bar{p} \cdot y > \sum_{i \in S} \bar{t}_i$. Clearly, for q large enough $y \in Y^S \cap K_q$ and

$p^q \cdot y > \sum_{i \in S} t_i^q \geq \text{Max}\{p^q \cdot y^S \mid y^S \in Y^S \cap K_q\}$ which contradicts (4).

Let $z_i \in X_i$, $z_i \neq \bar{x}_i$ such that $\bar{p} \cdot z_i \leq \bar{p} \cdot e_i + \bar{t}_i$. There exists a sequence $(z_i^q)_{q=1,...}$ converging to z_i such that $p^q \cdot z_i^q \leq p^q \cdot e_i + t_i^q$ and $z_i^q \in X_i \cap K_q$. Since for all $q = 1,...$ $x_i^q \succsim_i z_i^q$ the continuity of $\succsim_i$ implies $\bar{x}_i \succsim_i z_i$. Hence, $\bar{x}_i$ is a best element in the unrestricted budget set.

It remains to be shown that $\bar{p}$ supports an equilibrium with zero excess demand. Since each consumer is locally not satiated, we have for all $i \in I$, $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot e_i + \bar{t}_i$, which implies $\bar{p} \cdot (\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j) = 0$. If $\bar{p} \gg 0$, then $p \cdot (\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j) \leq$

$\bar{p} \cdot (\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j) = 0$ and $\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j \leq 0$

implies that excess demand is equal to zero. If $\bar{p}$ contains some zero

component, then the assumption of free disposal guarantees that there exist bundles $\bar{y}_j{}'$ such that $\bar{p} \cdot \sum_{j \in \bar{J}} \bar{y}_j{}' = p \cdot \sum_{j \in \bar{J}} \bar{y}_j$ and $\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j{}' = 0$. This completes the proof of the theorem.   Q.E.D.

Appendix

The purpose of this appendix is to show that the assumption of individual rationality which appears in Scarf's original theorem on the non-emptiness of the core of a balanced game ([30], Theorem 1, p. 54) can be omitted without changing the assertion of the theorem.

Theorem (Scarf): Let (I,v) be a non-sidepayment game in characteristic function form and assume for every $S \subseteq I$

(1) $v(S)$ *is non-empty and closed;*

(2) $x \in v(S)$, $y \in E^S$, $y \leq x$ *implies* $y \in v(S)$;

(3) $\{u^S \in v(S) \mid$ for all $i \in S$ $u_i^S \geq v(\{i\})\}$ *is non-empty and bounded.*

If for every balanced family $\mathscr{A}$ of coalitions
$$\bigcap_{S \in \mathscr{A}} (v(S) \times E^{I \setminus S}) \subseteq v(I) ,$$

then (I,v) has a non-empty core.

Lemma: Let (I,v) be a game such that for every $S \subseteq I$

(1) $v(S)$ *is non-empty and closed;*

(2) $x \in v(S)$, $y \in E^S$, $y \leq x$ *implies* $y \in v(S)$;

(3) $v(I)$ *is bounded from above, and for every balanced family $\mathscr{A}$ of coalitions*
$$\bigcap_{S \in \mathscr{A}} (v(S) \times E^{I \setminus S}) \subseteq v(I) .$$

Then (I,v) *has a non-empty core.*

Proof: First one observes, since every partition $\mathcal{P}$ of I is

a balanced family, that

$$\bigcap_{S \in \mathcal{P}} (v(S) \times E^{I \setminus S}) = \prod_{S \in \mathcal{P}} v(S) \subset v(I)$$

where $\prod$ denotes the cartesian product. Hence every $v(S)$ is bounded

from above.

Let $U(I) = \prod_{i \in I} v(\{i\})$ and let $U(S)$ denote the projection of

$U(I)$ into the subspace associated with coalition S.

Consider the game (I,w) where $w(S)$ is defined by $w(S) = v(S) \cup U(S)$. Clearly $w(I) = v(I)$. Furthermore, $w(S)$ satisfies

assumptions (1) and (2) of Scarf's theorem since $v(S)$ and $U(S)$

satisfy (1) and (2). Also, for all $S \subset I$ $w(S)$ is individually

rational; i.e., the set $\{u^S \in w(S) \mid u_i^S \geq w(\{i\}), i \in S\}$ is non-empty.

Since all $v(S)$ are bounded above, the set is also bounded. Hence

(I,w) satisfies assumptions (1)-(3) of Scarf's theorem.

Consider any balanced family $\mathcal{B}$. Then

$$\bigcap_{S \in \mathcal{B}} (w(S) \times E^{I \setminus S}) = \bigcap_{S \in \mathcal{B}} ((v(S) \cup U(S)) \times E^{I \setminus S})$$

$$= \bigcap_{S \in \mathcal{B}} (v(S) \times E^{I \setminus S} \cup U(S) \times E^{I \setminus S})$$

$$= \bigcup_{\mathcal{B}', \mathcal{B}'' \subset \mathcal{B}} \{(\bigcap_{S \in \mathcal{B}'} (v(S) \times E^{I \setminus S})) \cap (\bigcap_{T \in \mathcal{B}''} (U(T) \times E^{I \setminus T}))\}$$

$$\mathcal{B}' \cap \mathcal{B}'' = \emptyset$$

$$\mathcal{B}' \cup \mathcal{B}'' = \mathcal{B}$$

Clearly the members of the union for which $\mathcal{B}' = \mathcal{B}$ and for which

$\mathcal{B}'' = \mathcal{B}$ are subsets of $v(I)$ and hence of $w(I)$. For any mixed

element, i.e., $\mathscr{A}' \neq \emptyset$ and $\mathscr{A}'' \neq \emptyset$ define a new family of coalitions in the following way. Let d_T be the weight associated with $T \in \mathscr{A}''$ and consider the family of singletons $\mathscr{A}(T) = \{ \{i\} \mid i \in T\}$ with associated weights $d_{\{i\}} = d_T$ for $i \in T$. It is easy to check that the collection of coalitions $\mathscr{E} = \{ \{\mathscr{A}(T) \mid T \in \mathscr{A}''\} , \mathscr{A}'\}$ is a balanced family of coalitions. Moreover, for every $T \in \mathscr{A}''$

$$U(T) \times E^{I \setminus T} = \bigcap_{i \in T} (v(\{i\}) \times E^{I \setminus \{i\}}) \ .$$ Hence for any $\mathscr{A}''$

$$\bigcap_{T \in \mathscr{A}''} (U(T) \times E^{I \setminus S}) = \bigcap_{T \in \mathscr{A}''} (\bigcap_{i \in T} v(\{i\}) \times E^{I \setminus \{i\}})$$

$$= \bigcap_{\substack{i \in T \\ T \in \mathscr{A}''}} (v(\{i\}) \times E^{I \setminus \{i\}})$$

which implies that for any $\mathscr{A}'$ and $\mathscr{A}''$

$$(\bigcap_{S \in \mathscr{A}'} (v(S) \times E^{I \setminus S})) \cap (\bigcap_{T \in \mathscr{A}''} (U(T) \times E^{I \setminus T}))$$

$$= \bigcap_{S \in \mathscr{E}} (v(S) \times E^{I \setminus S}) \subset v(I) = w(I) \ .$$

Hence (I,w) is a balanced game. According to Scarf's theorem it has a non-empty core.

Let x be in the core of (I,w). x is feasible for the game (I,v) since $v(I) = w(I)$. Moreover, $v(S) \subset w(S)$ implies that x cannot be blocked by any S in the game (I,v). Hence x belongs to the core of (I,v).   Q.E.D.

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