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INVENTORIES AND MONEY BALANCES IN A DYNAMIC MODEL WITH RATIONING*

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1. Introduction

Recent contributions to the theory of temporary equilibrium models have concentrated on the description of alternative states of equilibrium or disequilibrium depending on the given initial conditions and prices and wages in any particular period of an evolving economic system. While this allows to analyse economic activity i.e. levels of employment, sales, savings etc. in the short run in situations with or without rationing, the characterization of the dynamic evolution of such an economy would have to be the ultimate goal. Given the description of such an economy for any time period, initial conditions of all stock variables and prices and wages define the new initial conditions of the following period. Thus, any real trajectory and the set of stationary states will depend on the stock adjustment process as well as on the particular price and wage adjustment mechanism. In the special case of perfect foresight of agents with respect to prices and wages, stationary sequences of prices and wages will not necessarily imply long run states with market clearing. In such a case the stock adjustment process will get trapped in a situation with rationing, i.e. with long run unemployment or excess demand (see e.g. Honkapohja and Ito (1980)). However, for an arbitrary stationary sequence of prices and wages it is not clear whether the undesirable long run states are due to the wrong price and wage sequence or to the stock adjustment process. In order to separate the two issues a stationary price and wage sequence could be chosen which corresponds to some long run stationary state with market clearing. Then, some statement could be made to the effect that the stock adjustment performs well or not if the long run stationary states are market clearing or with rationing.

In a recent paper of this journal Honkapohja and Ito (1980) present a model with inventory dynamics in a disequilibrium macroeconomic model. As one basic result they establish that long run stationary states in the Keynesian region may be oscillatory whereas those in the repressed inflation region converge monotonically. This asymmetry, which also appeared in a different way in a paper without inventories but with money balances (see Böhm (1978)), may arise from the particular piecewise linear model chosen by Honkapohja and Ito, which does not have an immediate microeconomic foundation. Moreover, money balances as another store of value do not play a role in the production or the consumption sector in their paper.

This paper addresses the question of long run stationary states and the dynamic adjustment of inventories and money balances in a model where producer and consumer behavior is described using a microeconomic intertemporal decision model for both sectors, thus avoiding some of the ad hoc nature of the inventory adjustment process used by Honkapohja and Ito. Moreover, the results represent the appropriate dynamic version of the by now well known model with money and inventories presented with their comparative statics properties by Muellbauer and Portes (1978) and by Böhm (1980, 1981).

The paper discusses the dynamic evolution of an economy with a given aggregate quantity of money at fixed prices and wages. It is clear that long run stationary states cannot be market clearing if prices and wages are not at levels which correspond to some stationary initial conditions of inventories and of cash balances which are market clearing. If prices and wages are wrong in this sense long run stationary states, if they exist, will be of one or more of the four disequilibrium kind. If on the other hand, prices and wages are chosen to correspond to a long run stationary state with market clearing, the question arises whether the long run stationary Walrasian equilibrium will be reached over time. Furthermore, it would be interesting to investigate which disequilibrium states will be observed along any trajectory. Both of these questions will be answered in this paper. The particular indirect utility functions chosen for the producer and the consumer are such that they reflect stationary expectations with respect to prices and wages. Therefore, along any trajectory with fixed prices and wages expectations are self-fulfilling. It will be shown that the economy converges monotonically to the long run stationary equilibrium without rationing, except for initial conditions with low total wealth of the producer which result in long run stationary states with production and employment equal to zero. Moreover, most trajectories will converge either through Keynesian or inflationary states with only finitely many periods in the classical or the underconsumption region.

Sections 2-4 of the paper describe the producer and consumer behavior. Section 5 characterises the long run stationary Walrasian equilibrium. In section 6 the partitioning of the state space is derived and section 7 analyses the dynamic evolution of the economy.

2. Production without Rationing

Consider a single producer who produces from current inputs $q \geqq 0$ of commodities and $z \geqq 0$ of labor the new stock of commodities $\omega \geqq 0$ available at the beginning of the next period with a Cobb-Douglas production function

$$\omega = q^a z^b \quad \text{with} \quad a,b < \tfrac{1}{2},\ a + b > \tfrac{1}{2}.$$

At the beginning of the current period he has at his disposal his initial money balances $m_o^P \geqq 0$ and his initial inventories of commodities $\omega_o \geqq 0$ which are the result of his actions of the preceding period. Since the producer does not buy his own product on the market his input constraint for commodity inputs q is

$$q + y = \omega_o \qquad\qquad y \geqq 0$$

where y are his sales. Given prices and wages (p,w) his current budget constraint is

$$py + m_o^P = wz + m^P$$

where $m^P \geqq 0$ are final cash balances.[1)] Substitution for $y = \omega_o - q$ yields

$$m_o^P + p\omega_o = m^P + pq + wz.$$

Let his expected utility index be given by

$$v(\omega, m^P) = \omega m^P.$$

1) Rather than writing time subscripts as t - 1, t, t + 1, initial conditions for any time period t will be indexed by the subscript zero. The same variables for t + 1 will have no time subscript. No confusion should arise.

Then, the producer determines his optimal production, sales, labor demand and final money holdings as a solution of the following maximization problem:

$$\text{Max } \omega m^P$$

subject to

$$\omega = q^a z^b$$
$$p\omega_o + m_o^P = m^P + pq + wz$$
$$q \leqq \omega_o .$$

Consider the Lagrangean

$$L = q^a z^b m^P + \alpha(p\omega_o + m_o^P - m^P - pq - wz) + \beta(\omega_o - q).$$

For $\omega_o = 0$ one obtains $q = z = 0$ as a solution. One observes that $\omega_o > 0$ implies $q > 0$, $z > 0$, $m^P > 0$, and $\alpha > 0$. If $\beta > 0$, and $m_o^P > 0$, then $\omega_o = q$, $y = 0$ and $m_o^P \geqq \frac{1+b}{a} p\omega_o$. As a solution of the first order conditions one obtains

$$m^P = \frac{1}{1+b} m_o^P$$
$$z = \frac{b}{1+b} \frac{m_o^P}{w}$$

and

$$\omega = \left(\frac{b}{1+b}\right)^b \omega_o^a \left(\frac{m_o^P}{w}\right)^b .$$

The equation

$$m_o^P = \frac{1 + b}{a} p\omega_o$$

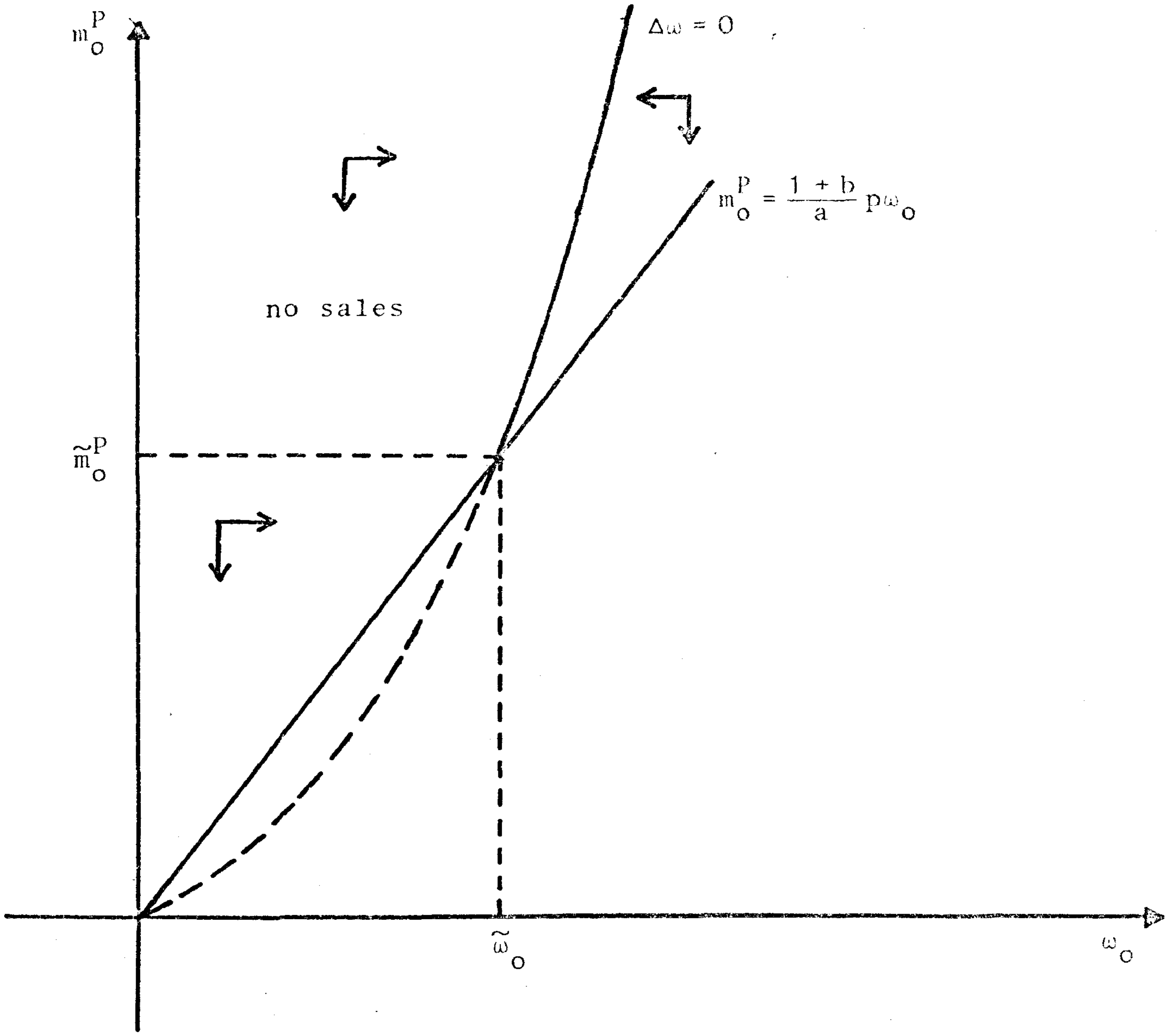
divides the state space for the producer into a no-sales region and a region with positive sales. No sales will occur if the value of stocks is small relative to initial money balances (see Figure 1). If no sales are made money balances decrease at a constant rate which is apparent from the money equation. On the other hand, stocks will increase in the no-sales region, i.e. $\omega > \omega_o$, if and only if

$$m_o > w \frac{1 + b}{b} \omega_o \frac{1 - a}{b} .$$

Since $a < ½$ and $b < ½$, the condition $\Delta\omega = \omega - \omega_o = 0$ defines a power relationship between m_o^P and ω_o (see Figure 1). At $(\tilde{\omega}_o, \tilde{m}_o^P)$, $\Delta\omega = 0$ and $\tilde{m}_o^P = \frac{1 + b}{a} p\tilde{\omega}_o$, which defines the relevant range for $\Delta\omega = 0$ as $\omega_o \geqq \tilde{\omega}_o$. The dynamic development in the no-sales region now follows in a straightforward manner. Since $\omega > 0$ for all $m_o^P > 0$ and $\omega_o > 0$ and since m^P decreases at a constant rate, the process will leave the no-sales region in finite time.

Positive sales occur if

$$m_o < \frac{1 + b}{a} p\omega_o .$$



m_o^P
$\Delta\omega = 0$
$m_o^P = \frac{1+b}{a} p\omega_o$
no sales
$\tilde{m}_o^P$
$\tilde{\omega}_o$
ω_o

FIGURE 1

From the first order conditions one obtains

$$m = \frac{p\omega_o + m_o}{1 + a + b}$$

$$q = \frac{a}{p}\,\frac{p\omega_o + m_o}{1 + a + b}$$

$$z = \frac{b}{w}\,\frac{p\omega_o + m_o}{1 + a + b}$$

$$y = \frac{1}{p}\,\frac{p(1 + b)\,\omega_o - am_o}{1 + a + b}$$

$$\omega = \left(\frac{a}{p}\right)^a \left(\frac{b}{w}\right)^b \left[\frac{p\omega_o + m_o}{1 + a + b}\right]^{a+b} .$$

One observes immediately that the producer's input and money decisions depend on his total wealth $p\omega_o + m_o$ only and not on the particular initial distribution. For any interior solution of the producers optimization problem, one knows that the marginal rate of substitution between inventories and money balances is equal to marginal costs (see Böhm (1981)). For the particular objective function chosen here, this implies that

$$\frac{m^P}{\omega} = C'(\omega, p, w).$$

For the Cobb-Douglas technology this yields

$$m^P = \omega^{\frac{1}{a + b}} \left(\frac{a}{p}\right)^{-\frac{a}{a+b}} \left(\frac{b}{w}\right)^{-\frac{b}{a+b}} ,$$

which defines a unique expansion path E of all optimal pairs (ω, m^P) the producer will ever choose. Therefore, in the sales region the producer will adjust to the expansion path in one step if his initial condition is not already on E and stay on E afterwards.

Apart from the trivial stationary point (0,0) along E, one obtains by straightforward calculations as the unique non-trivial stationary point $(\bar{\omega}, \bar{m}^P)$

$$\bar{\omega} = \left[\left(\frac{p}{a+b}\right)^{a+b} \left(\frac{a}{p}\right)^{a} \left(\frac{b}{w}\right)^{b}\right]^{\frac{1}{1-a-b}}$$

and

$$\bar{m} = \left[\left(\frac{p}{a+b}\right)\left(\frac{a}{p}\right)^{a} \left(\frac{b}{w}\right)^{b}\right]^{\frac{1}{1-a-b}}.$$

It is easily verified that

$$a\bar{m} < (1 + b)\bar{\omega}p,$$

so that $(\bar{\omega}, \bar{m})$ does not lie in the no-sales region. Furthermore

$$\bar{\omega} < \left(\frac{bp}{aw}\right)^{\frac{b}{1-a-b}} = \tilde{\omega} \qquad \text{if} \quad a + b > \tfrac{1}{2}.$$

To complete the description of the dynamic process of inventories and money balances the convergence property along the expansion path E will be established. Since for all t on the expansion path

$$m_{t+1}^{P} = \frac{1}{1 + a + b} \left[p\left(\frac{a}{p}\right)^{a} \left(\frac{b}{w}\right)^{b} \left(m_{t}^{P}\right)^{a+b} + m_{t}^{P} \right],$$

one verifies that m_{t+1} is a strictly increasing and concave function of m_t with

$$\frac{dm_{t+1}}{dm_t}(0) = +\infty \quad \text{and} \quad \frac{dm_{t+1}}{dm_t}(\bar{m}^P) < 1.$$

Therefore, along the expansion path the system converges monotonically to the stationary point $(\bar{\omega}, \bar{m}^P)$.

It is worth noticing that the stationary point $(\bar{\omega}, \bar{m}^P)$ defines the long run total wealth $\bar{V} = p\bar{\omega} + \bar{m}^P$. Therefore, along the expansion path total wealth either decreases or increases monotonically over time depending on whether initial wealth is larger or smaller than $\bar{V}$. From the budget restriction

$$m_{t+1}^{P} + C(\omega_{t+1}) = m_{t}^{P} + p\omega_{t} = V_{t}$$

and the definition of V_{t+1}, one obtains

$$V_{t+1} = p\omega_{t+1} - C(\omega_{t+1}) + V_{t}.$$

This indicates that $V_{t+1} \gtreqless V_t$ if and only if $p\omega_{t+1} \gtreqless C(\omega_{t+1})$. Since the cost function is strictly convex this is clearly equivalent to $\omega_{t+1} \gtreqless \bar{\omega}$. This indicates that the stationary level of inventories $\bar{\omega}$ depends on prices and wages and on the technology, but not on the objective function of the producer.

In order to gain some additional insight into the dynamic process, consider the set of initial states (ω_o, m_o^P) for which $\omega = \omega_o$, i.e. $\Delta\omega = 0$. These are defined by

$$p\omega_o + m_o^P = m^P + C(\omega_o)$$

$$m^P = \omega_o C'(\omega_o)$$

where the second equation defines the expansion path. Substitution yields

$$m_o^P = (C'(\omega_o) - p)\omega_o + C(\omega_o).$$

Since $C'(0) = 0 < p$, there exists $\omega_o' > 0$ such that $(\omega_o', 0)$ solves the equations. For all $\omega_o > \omega_o', m_o^P > 0$ if $\Delta\omega = 0$. Furthermore,

$$\left.\frac{dm_o}{d\omega_o}\right|_{\Delta\omega=0} = 2C'(\omega_o) + \omega_o C''(\omega_o) - p > 0$$

for all $\omega_o \geq \omega_o'$. Alternatively consider all initial conditions (ω_o, m_o^P) for which $\Delta m^P = m^P - m_o^P = 0$. These are defined by

$$p\omega_o = C(\omega)$$

$$m_o^P = \omega C'(\omega).$$

Therefore

$$m_o = C^{-1}(p\omega_o)\ C'(C^{-1}(p\omega_o)).$$

Clearly, $m_o^P = 0$ and $\omega_o = 0$ solve this equation. Furthermore, the function on the right hand side is continuous and increasing in ω_o. Moreover,

$$\left.\frac{dm_o^P}{d\omega_o}\right|_{\Delta m^P=0} = p\left(1 + \omega\frac{C''(\omega)}{C'(\omega)}\right)$$

where $p\omega_o = C(\omega)$. Since for the particular technology

$$\omega\frac{C''(\omega)}{C'(\omega)} = \frac{1-a-b}{a+b}$$

one obtains

$$\left.\frac{dm_o^P}{d\omega_o}\right|_{\Delta m^P=0} = p\frac{1}{a+b} < p\frac{1+b}{a}.$$

Therefore, the set of initial positions with $\Delta m^P = 0$ is a straight line from the origin outside the no-sales region for $\omega_o > 0$.

The complete dynamic characterization is given in Figure 2. Starting from any point in the no-sales region the system will move in finite time to the expansion path and then converge monotonically to $(\bar{\omega},\bar{m}^P)$. From a point (ω_o,m_o^P) outside the no-sales region the trajectory will follow the expansion path after the first iteration and then converge, or it will move into the no-sales region in the first iteration and returning to the expansion path in finite time and then converge. Summarizing the results of this section, one obtains that for each pair of strictly positive prices and wages (p,w) there exists a unique stationary point of strictly positive inventories and money balances with positive sales and labor demand. The stationary state is globally stable and will be attained in the limit if the producer has positive stocks at the initial position. Otherwise, each state with $\omega_o = 0$ is a stationary point.

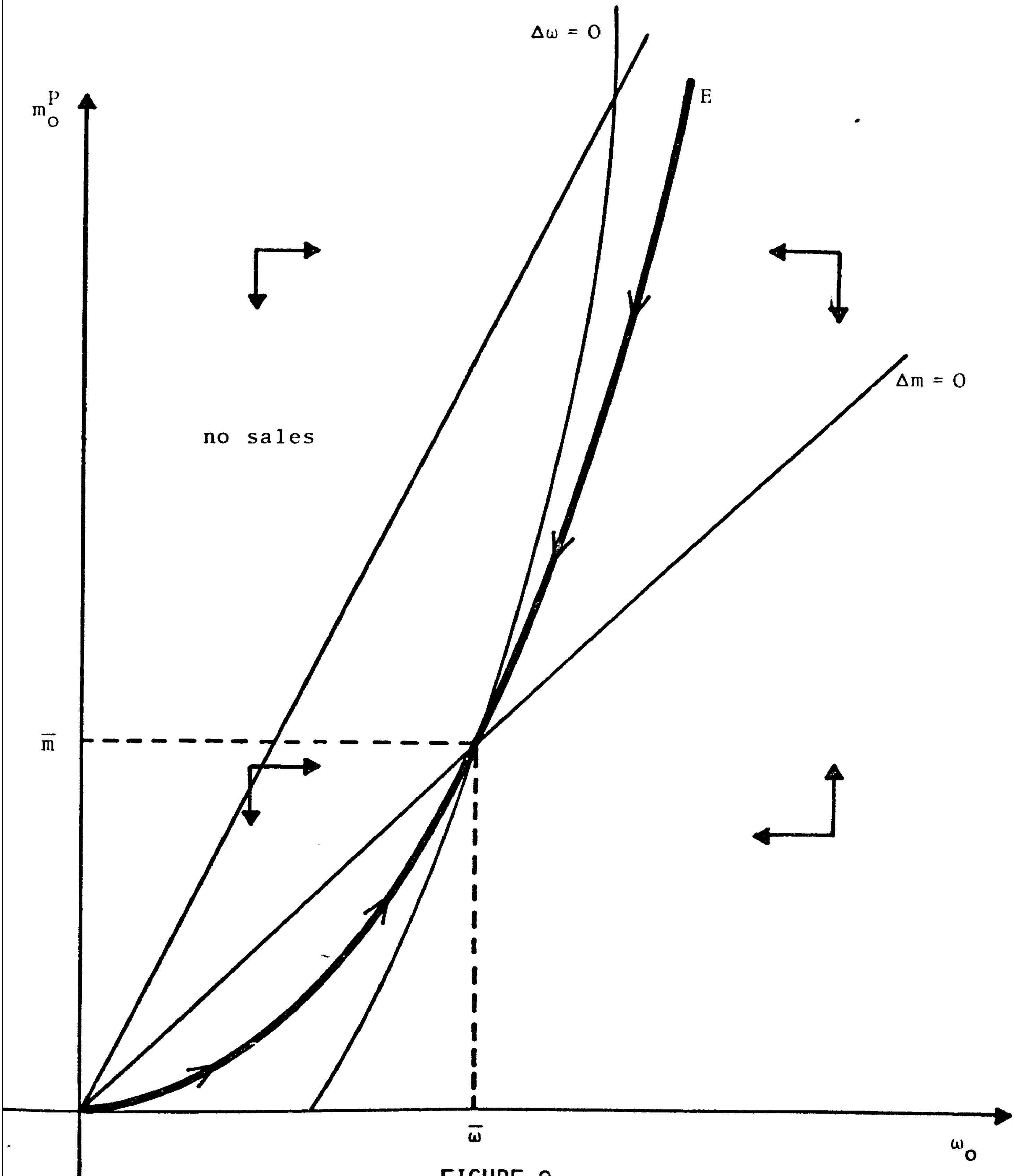


FIGURE 2

3. The Producer under Rationing

Consider a supply constraint $x \geqq 0$ for the producer. x will be binding if and only if $p(1+b)\omega_o - am_o > 0$, i.e. if the initial state of the producer does not lie in the no-sales region. Furthermore,

$$x \leqq \frac{p(1+b)\omega_o - am_o^P}{p(1+a+b)}$$

must hold. Then, the producer's maximization reduces to

$$\underset{(m^P, z)}{\text{Max}} \quad (\omega_o - x)^a z^b m^P$$

subject to

$$m_o^P + px = wz + m^P.$$

As the solution one obtains

$$m^P = \frac{m_o^P + px}{1+b}$$

$$z = \frac{b}{1+b} \frac{m_o^P + px}{w} .$$

One observes that the effective labor demand function $h_y(\omega_o, m_o, p, w, x) = z$ is linear and independent of the level of inventories ω_o, i.e. the labor demand will stay constant if stocks change but the rationing constraint is unchanged.

If the producer is facing a binding demand constraint $\ell \geqq 0$ on the labor market, then

$$\ell \leq \frac{b}{1+b}\,\frac{m_o^P}{w} \qquad \text{or} \qquad \ell \leq \frac{b}{w}\,\frac{p\omega_o + m_o^P}{1+a+b}$$

must hold. Solving the Lagrangean

$$L = q^a \ell^b m^P + \alpha(m_o^P + p\omega_o - m^P - w\ell - pq) + \beta(\omega_o - q)$$

yields

$$\alpha(a\frac{m^P}{q} - p) = \beta.$$

If $\beta > 0$, then demand rationing occurs in the no-sales region and

$$m^P = m_o^P - w\ell$$

with $am^P > p\omega_o$.

Therefore, money balances will decrease as long as the employment level is positive. If $\beta = 0$, one obtains

$$m^P = \frac{m_o^P + p\omega_o - w\ell}{1 + a}$$

and as effective commodity supply function g_z

$$y = \mathrm{Max}\left\{0, \frac{p\omega_o - am_o + aw\ell}{p(1+a)}\right\}.$$

Summarizing the results of the current period decisions of the producer one obtains the characterization given in Figure 3, where the effective supply and demand functions have been drawn for two different values of initial stocks

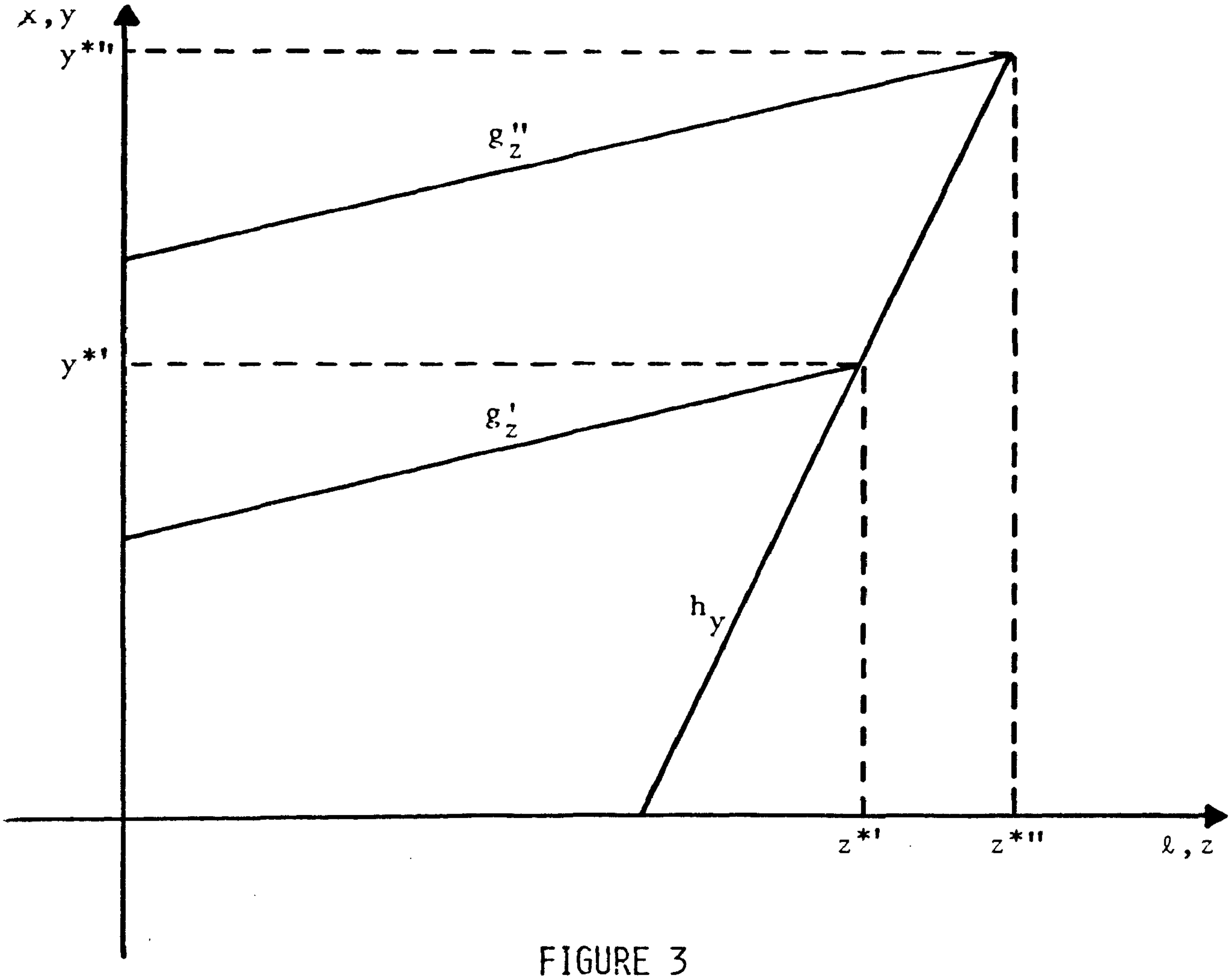


FIGURE 3

ω_o' and ω_o'' with $\omega_o'' > \omega_o'$. $(z^{*\prime}, y^{*\prime})$ and $(z^{*\prime\prime}, y^{*\prime\prime})$ are the corresponding notional decisions of the producer.

4. The Consumer

For the consumption sector the particular example given by Malinvaud (1977) will be chosen. The two functions characterizing the consumer are given by

$$c_u(m_o^C, p, w, z) = \frac{2}{3}\,\frac{m_o^C + wz}{p}$$

$$a_x(m_o^C, p, w, y) = \text{Max}\left\{0, \frac{py + w\hat{\ell} - m_o^C}{2w}\right\}$$

where c_u is effective consumption demand with a rationing level z on the labor market and where a_x is effective labor supply under demand rationing at level y. $\hat{\ell}$ is the maximal amount of labor the consumer is able to supply. The two corresponding functions describing consumption demand and labor supply without rationing are

$$c(m_o^C, p, w) = \frac{w\hat{\ell} + m_o^C}{2p}$$

$$a(m_o^C, p, w) = \text{Max}\left\{0, \frac{3w\hat{\ell} - m_o^C}{4w}\right\}.$$

Therefore, if the consumer is not rationed, the dynamic adjustment is of the form

$$m^C = \text{Max}\left\{\frac{m_o^C + w\hat{\ell}}{4}, \frac{m_o^C - w\hat{\ell}}{2}\right\}$$

which is monotonically converging to the stationary value

$$\bar{m} = \frac{w\hat{\ell}}{3}.$$

One observes that the savings decision and therefore the dynamic adjustment of the consumer's money balances does not depend on prices p. Figure 4 gives the geometric characterization of the process. The stationary levels of consumption and labor supply are easily obtained as

$$\bar{x} = \frac{2}{3}\frac{w}{p}\hat{\ell}$$

and

$$\bar{\ell} = \frac{2}{3}\hat{\ell}.$$

5. Stationary Market Clearing Prices and Wages

The appropriate long run equilibrium concept for a model with inventories and money balances is one in which markets clear with constant i.e. stationary levels of all stock variables through time. Normalizing the maximal amount of labor $\hat{\ell}$ at the level

$$\hat{\ell} = \frac{3}{2}\left(\frac{a+b}{a}\right)^{a/b}$$

and choosing a total quantity of money $\bar{M} = \frac{2+b}{b}\frac{\hat{\ell}}{3}$, one verifies easiliy that $\bar{p} = \frac{2}{3}\frac{a+b}{b}\hat{\ell}$, $\bar{w} = 1$ are the unique market clearing prices at an inventory level $\bar{\omega} = 1$ and cash balances $\bar{m}^P = \frac{2}{3}\frac{\hat{\ell}}{b}$ and $\bar{m}^C = \frac{\hat{\ell}}{3}$ with trades

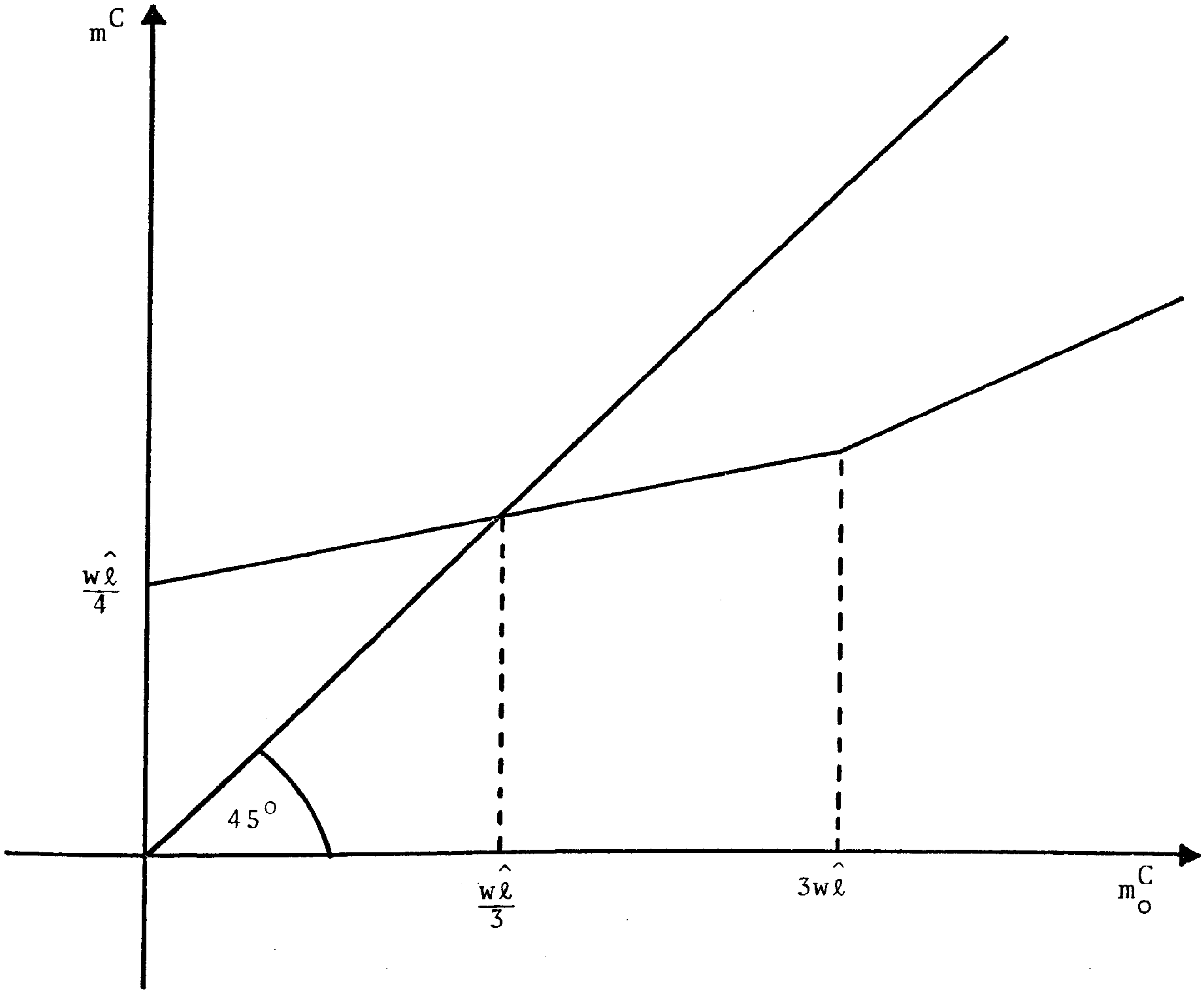


FIGURE 4

$$\bar{x} = \bar{y} = \frac{b}{a+b} \qquad \bar{\ell} = \bar{z} = \frac{2}{3}\hat{\ell} .$$

Moreover $(\bar{\omega}, \bar{m}^P, \bar{m}^C)$ is stationary and $\bar{m}^P + \bar{m}^C = \bar{M}$.

To describe the dynamic evolution of inventories and cash balances at $(\bar{p}, \bar{w})$ it is first necessary to analyse completely the pattern in which the disequilibrium situations are distributed over the state space. Since for all possible states $m_o^P + m_o^C = \bar{M}$, it suffices to consider the set

$$\left\{ (\omega_o, m_o^P) \;\middle|\; \omega_o \geqq 0,\ \bar{M} - m_o^P \geqq 0,\ m_o^P \geqq 0 \right\}$$

as the state space. One easily verifies that the behavioral equations of the production sector as well as of the consumption sector satisfy the sufficient conditions of the uniqueness theorem in Böhm (1981) for all states (ω_o, m_o^P) which generate positive levels of sales and employment. Since the slopes of all effective supply and demand functions with respect to the associated rationing level are constant and independent of the stock variables, it is not difficult to demonstrate that for each (ω_o, m_o^P) one and only one disequilibrium situation is generated.

Due to the uniqueness theorem, for any (ω_o, m_o^P) in the state space the resulting disequilibrium will be either Keynesian K, inflationary I, classical C or of the underconsumption type U. Therefore the state space can be partitioned into four non-overlapping regions which will also be denoted by K, I, C and U respectively.

Consider first states where $m_o^P = \bar{m}^P$ and $\omega_o \geqq \bar{\omega}$. Since the effective labor demand function of the producer is independent of ω_o, higher inventories have no effect on employment, sales or cash balances. Therefore, the producer is supply rationed without rationing for the consumer, which defines the boundary of the Keynesian and the underconsumption region, i.e.

$$K \cap U = \left\{(\omega_o, m_o^P) \;\middle|\; m_o^P = \bar{m}^P,\ \omega_o \geqq \bar{\omega}\right\}.$$

Consider states $(\omega_o, \bar{m}^P + \varepsilon)$, $\bar{M} - \bar{m}^P \geqq \varepsilon > 0$, $\omega_o \geqq \bar{\omega}$. Using the two effective demand functions, the system of equations to be solved is

$$\ell = \frac{b}{1+b}\,\frac{px + m^P + \varepsilon}{w}$$

$$x = \frac{2}{3}\,\frac{w\ell + \bar{m}^C - \varepsilon}{p}\,.$$

One finds as the unique Keynesian solution

$$x = \frac{b}{a+b} - \frac{3b\varepsilon}{(3+b)(a+b)\hat{\ell}}$$

$$\ell = \frac{2}{3}\hat{\ell} + \frac{b\varepsilon}{3+b}\,,$$

which shows that the level of employment ℓ increases with higher money balances of the producer. For all $\varepsilon > 0$ and $(\bar{\omega}, \bar{m}^P + \varepsilon) \in K$, $\omega_o > \bar{\omega}$ has no effect on employment. On the other hand, decreasing ω_o below $\bar{\omega}$ the system will change from K to C and eventually to I, thus passing the boundaries $K \cap C$ and $I \cap C$. Using the unconstrained behavioral equations of the producer and the effective demand function of the consumer, one obtains for the boundary between K and C

$$m_o^P \Big|_{K \cap C} = \frac{2(1 + a + b)}{2 - a} \bar{M} - \frac{3 + b}{2 - a} p\omega_o$$

which holds for

$$\frac{\bar{M}}{p} \frac{(3a + 2b)}{(3 + b)} \leqq \omega_o \leqq \bar{\omega},$$

since $\omega_o = \frac{\bar{M}}{p} \frac{(3a + 2b)}{(3 + b)}$ implies $m_o^P = \bar{M}$.

The border line between C and I is defined by a solution of the unconstrained behavioral equation of the producer and the effective supply function of the consumer. If sales are positive one obtains as a solution

$$m_o^P \Big|_{C \cap I} = \frac{1 + a + b}{1 - b} (\bar{M} - w\hat{\ell}) - p\omega_o = \frac{2(1 + a + b)}{3b} w\hat{\ell} - p\omega_o,$$

which is valid for

$$\frac{1}{p} \frac{a}{b} \frac{2}{3} w\hat{\ell} \leqq \omega_o \leqq \bar{\omega}.$$

At $\omega_o = \frac{2a}{3pb} w\hat{\ell}$ sales are zero and further decreases of ω_o will result in classical situations if

$$m_o^P \geqq \frac{1+b}{b} \frac{2}{3} w\hat{\ell}.$$

Therefore,

$$m_o^P \Big|_{C \cap I} = \text{Min} \left\{ \frac{1+b}{b} \frac{2}{3} w\hat{\ell}, \frac{2(1+a+b)}{3b} w\hat{\ell} - p\omega_o \right\}.$$

Sales and employment levels in the inflationary region are determined by

$$y = \text{Max} \left\{ 0, \frac{p\omega_o - am_o^P + aw\ell}{p(1+a)} \right\}$$

$$\ell = \text{Max} \left\{ 0, \frac{py + w\hat{\ell} - (\overline{M} - m_o^P)}{2w} \right\}.$$

As the solution one obtains

$$y = \text{Max} \left\{ 0, \text{Min} \left\{ \frac{p\omega_o - am_o^P}{p(1+a)}, \frac{2p\omega_o - am_o^P - a(\overline{M} - w\hat{\ell})}{p(2+a)} \right\} \right\}$$

$$\ell = \text{Max} \left\{ 0, \text{Min} \left\{ \frac{m_o^P - (\overline{M} - w\hat{\ell})}{2w}, \frac{p\omega_o + m_o^P - (1+a)(\overline{M} - w\hat{\ell})}{w(2+a)} \right\} \right\}.$$

Therefore, sales and employment are positive if and only if $p\omega_o + m_o^P > (1+a)(\overline{M} - w\hat{\ell})$ and $2p\omega_o - am_o^P > a(\overline{M} - w\hat{\ell})$. Then, however, the level of employment depends on total wealth $p\omega_o + m_o^P$ only and not on the particular distribution of inventories and money balances. On the other hand, if sales are zero, then

$$\ell = \text{Max}\left\{0, \frac{m_o^P - (\overline{M} - w\hat{\ell})}{2w}\right\},$$

which is independent of ω_o. A geometric charact
of the solution is given in Figure 5.

Finally, the boundary $I \cap U$ is obtained fro
strained behavioral equations of the consumer a
effective supply equation of the producer as

$$m_o^P \Big|_{I \cap U} = \frac{2 + 3a}{2 - a}\overline{M} + w\hat{\ell} - \frac{4p}{2 - a}\omega_o$$

which holds for

$$\overline{\omega} \leqq \omega_o \leqq \left[\overline{M}(2 + 3a) + w\hat{\ell}(2 - a)\right]\frac{1}{4p}$$

where the inequality on the right hand side is
$m_o^P = 0$. Figure 6 gives the complete partioning
space into the four regions. For completeness t
ment curves are added. These indicate that the
of employment is attained in the Keynesian regi
total amount of money is held by the producer.
of producer cash balances and low inventories t
level is zero. This feature stems from the fact
consumers's effective and unconstrained labor s
cash balances is equal to zero in the particula
chosen.

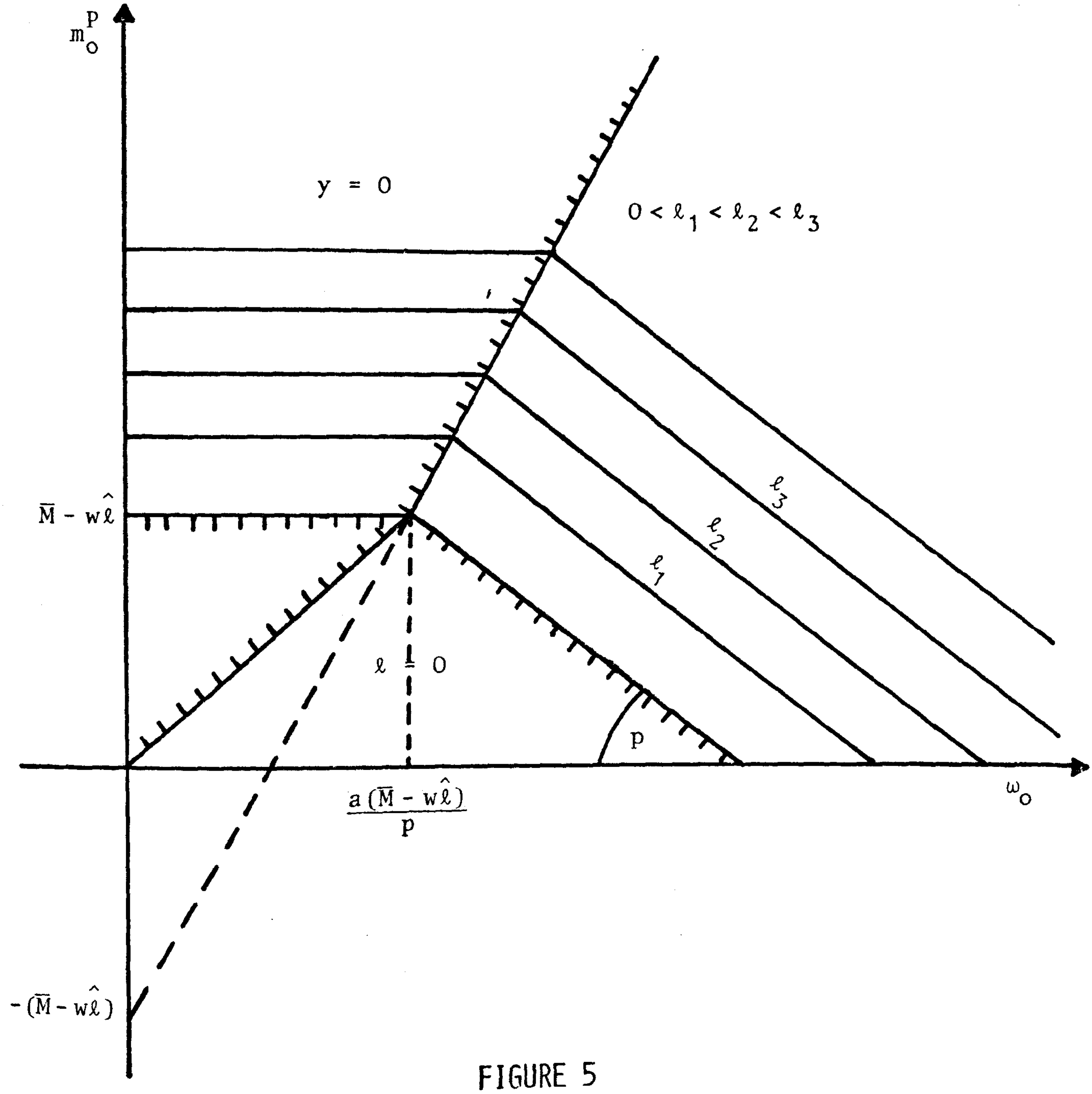
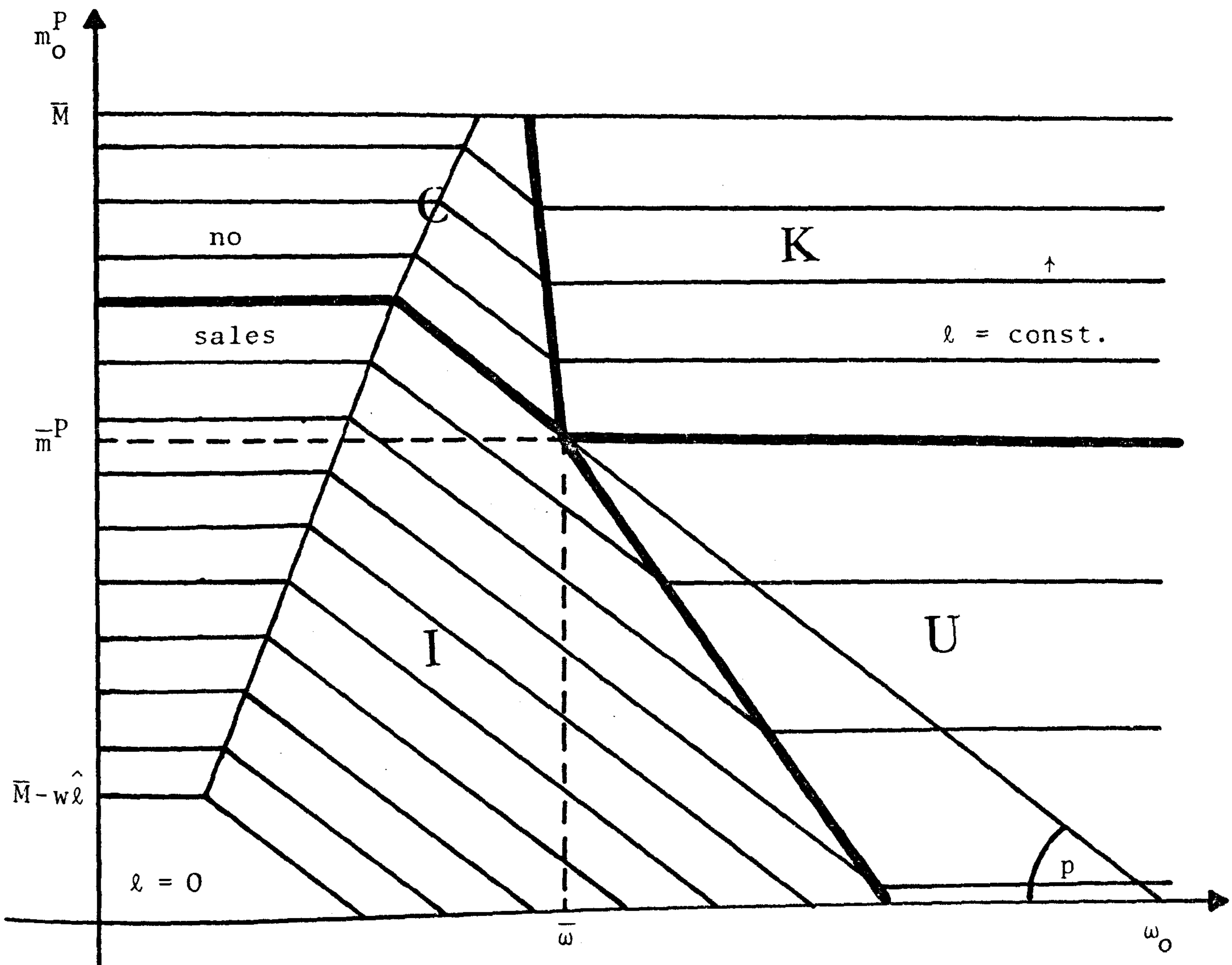

m_o^p
y = 0
$0 < \ell_1 < \ell_2 < \ell_3$
$\bar{M} - w\hat{\ell}$
ℓ_3
ℓ_2
ℓ_1
$\ell = 0$
p
$\frac{a(\bar{M} - w\hat{\ell})}{p}$
ω_o
$-(\bar{M} - w\hat{\ell})$

FIGURE 5



m_o^P
$\overline{M}$
no
sales
C
K
ℓ = const.
$\overline{m}^P$
I
U
$\overline{M}-w\hat{\ell}$
ℓ = 0
p
$\overline{\omega}$
ω_o

FIGURE 6

6. Dynamics of Inventories and Cash Balances

Given the characterization of the state space derived in the previous section, the dynamic evolution of the economy for any arbitrary initial condition (ω_o, m_o^P) can now be described. Since to each initial condition there exists a unique allocation of sales and employment in that period, final money balances and final inventories are uniquely defined, too. Therefore, with each initial condition is associated a unique trajectory in the state space characterizing the evolution of the economy. It is clear that for any classical initial state the trajectory will be identical to the one derived in section 2 with the producer unconstrained as long as the trajectory remains in the classical region. Similarly, the evolution of money balances in the underconsumption region will be determined exclusively by the unconstrained decision of the consumption sector. On the other hand, in the Keynesian and in the inflationary regions full account has to be taken of the spill-over effects between markets.

Utilizing the results of section 2 for the dynamic development in the classical region, one observes immediately that any trajectory starting in C will leave that region in finite time. If sales are zero money balances decrease at a constant rate while inventories increase. Therefore, the trajectory will switch over into the inflationary region or into the classical region with positive sales. Once sales are positive the production sector will move to the unconstrained expansion path in one step with a level of total wealth larger than the stationary level $p\bar{\omega} + \bar{m}^P$, thus moving from the classical into the Keynesian region.

For any starting point in the Keynesian region one observes that sales and employment levels are constant at constant m_o^P and arbitrary ω_o. This stems from the fact that excess inventories are present which result in undesired accumulation for the producer. More precisely, for any $\Delta > 0$ and any point $(\omega_o, m_o^P) \in K \cap C$, employment and sales at $(\omega_o + \Delta, m_o^P)$ are the same as at (ω_o, m_o^P), the latter ones being those of the unconstrained producer at (ω_o, m_o^P). Therefore, new money balances m_K^P are

$$m_K^P = \frac{m_o^P + p\omega_o}{1 + a + b} > \bar{m}$$

for all $\Delta > 0$ and $(\omega_o, m_o^P) \in K \cap C$. Let (y, ℓ) denote unconstrained sales and labor demand of the producer at $(\omega_o, m_o^P) \in K \cap C$. Then, new inventories ω_K are

$$\omega_K = f(\omega_o - y + \Delta, \ell) > f(\omega_o - y, \ell) = \omega$$

where ω solves $C(\omega) = (a + b)m^P$, i.e. (ω, m^P) is on the expansion path E. Therefore,

$$\omega_K > \omega > \bar{\omega} \quad \text{and} \quad m_K^P > \bar{m},$$

so that $(\omega_K, m_K^P) \in K$ and to the right of E. Furthermore, total wealth of the producer

$$V_K = p\omega_K + m_K^P = p\omega_K - C(\omega_K, \ell) + p(\omega_o + \Delta) + m_o^P < p(\omega_o + \Delta) + m_o^P,$$

since for all $\omega_K > \omega > \bar{\omega}$

$$C(\omega_K, \ell) > C(\omega_K) > p\omega_K.$$

Therefore, total wealth of the producer is a monotonically decreasing function in the Keynesian region. Thus, any trajectory starting in that region will remain there and converge monotonically to the stationary state $(\bar{\omega}, \bar{m}^P)$ which is in contrast with the findings of Honkapohja and Ito (1980). Along the adustment path undesired accumulated inventories will be reduced but at a slow enough rate which will not reduce excess inventories and therefore maintain supply rationing for all t.

Consider an initial starting position in the inflationary region. Clearly, if (ω_o, m_o^P) implies $\ell = 0$, then inventories at the next period ω_I will be zero since labor is an essential input for the Cobb-Douglas technology. Moreover, as remarked in section 2, any state with $\omega_o = 0$ is a stationary state. In this case $\ell = 0$, the economic evolution will stop with zero production and zero employment in the inflationary region. Suppose $\ell > 0$ at $(\omega_o, m_o^P) \in I$ and sales are zero. Then, money balances of the producer decrease at a constant rate, i.e.

$$m_I^P \Big|_{y=0} = \frac{m_o^P + \bar{M} - w\hat{\ell}}{2} .$$

Therefore, money balances will not fall below $\bar{M} - w\hat{\ell}$ in finite time which is the critical level at which employment in the no-sales region falls to zero. Thus, employment will be positive. On the other hand, inventories will increase as long as

$$m_o^P > 2w\omega_o^{\frac{1-a}{b}} + (\bar{M} - w\hat{\ell}).$$

Since the right hand side of the inequality is an increasing and convex function of ω_o, the economy will move in finite time outside the area of no-sales and increasing inventories. If inventories decrease and no sales persist, then both inventories and money balances decrease and a state of zero employment and of zero inventories will be reached eventually.

If sales and employment are positive, final money balances and inputs of commodities and labor depend on total wealth only, i.e.

$$m_I^P = \frac{p\omega_o + m_o^P + (\bar{M} - w\hat{\ell})}{(2 + a)}$$

$$\ell_I = \frac{1}{w} \frac{p\omega_o + m_o^P - (1 + a)(\bar{M} - w\hat{\ell})}{(2 + a)}$$

$$q_I = \frac{1}{p} \frac{a\left[p\omega_o + m_o^P + (\bar{M} - w\hat{\ell})\right]}{(2+a)}$$

defining the cost to produce new inventories ω_I as

$$C(\omega_I, \ell_I) = \frac{(1+a)(p\omega_o + m_o^P) - (\bar{M} - w\hat{\ell})}{(2+a)} = (1+a)m_I^P - (\bar{M} - w\hat{\ell}).$$

Therefore, (ω_I, m_I^P) is uniquely defined by total initial wealth and independent of the particular distribution of (ω_o, m_o^P). Hence, as in the unconstrained case for the producer, the dynamic adjustment will follow a unique trajectory along an expansion path E_I defined by the following two equations:

$$m_I^P = \frac{V_o + (\bar{M} - w\hat{\ell})}{(2+a)}$$

$$\omega_I = \left[q_I(V_o)\right]^a \left[\ell_I(V_o)\right]^b .$$

Since final money balances and inputs are linear functions of total wealth and since the production function is strictly concave, the expansion path E_I will be a strictly convex and increasing function of ω_I. Let (ω, m^P) denote the unconstrained decisions of the producer. Since in the inflationary region $V_o = m_o^P + p\omega_o < \bar{m}^P + p\bar{\omega} = \bar{V}$, it follows that $m^P < \bar{m}^P$ and $\omega < \bar{\omega}$. Moreover, one verifies easily that

$$m^P < m_I^P < \bar{m}^P.$$

On the other hand, from the budget equation

$$m_o^P + p\omega_o = m^P + C(\omega) = m_I^P + C(\omega_I, \ell_I)$$

one obtains

$$C(\omega) > C(\omega_I, \ell_I).$$

Since $C(\omega_I, \ell_I)$ is a restricted cost function with ℓ_I as a binding input constraint the inequalities

$$C(\omega, \ell_I) > C(\omega) > C(\omega_I, \ell_I)$$

yield $C(\omega, \ell_I) > C(\omega_I, \ell_I)$ which implies $\omega > \omega_I$. For $p\omega_o + m_o^P = (1 + a)(\bar{M} - w\hat{\ell})$, one has $\ell_I = \omega_I = 0$ and $m_I^P = \bar{M} - w\hat{\ell}$. If $p\omega_o + m_o^P > (1 + a)(\bar{M} - w\hat{\ell})$, $\omega_I > 0$ and $m_I > (\bar{M} - w\hat{\ell})$. Hence, the expansion path E_I is a strictly convex, increasing function from $(0, \bar{M} - w\hat{\ell})$ to $(\bar{\omega}, \bar{m}^P)$, which is to the left of the unconstrained expansion path E. Furthermore, $(\omega_I, m_I^P) \ll (\bar{\omega}, \bar{m}^P)$.

From the equation for final money balances, one obtains for the set of initial states for which $\Delta m = m_I^P - m_o^P = 0$ the function

$$m_o^P \Big|_{\Delta m = 0} = \frac{1}{1 + a}\left[p\omega_o + (\bar{M} - w\hat{\ell})\right]$$

This defines a straight line connecting the point $\left(a\frac{M - w\hat{\ell}}{p}, M - w\hat{\ell}\right)$ with the Walrasian stationary state $(\bar{\omega}, \bar{m}^P)$. In order to describe the dynamic development along E_I it suffices to establish whether $V_I - V_o$ is positive or negative, which is equivalent to showing whether along the line $\Delta m = 0$ one has $\omega_I > \omega_o$ or $\omega_I < \omega_o$, since

$$p\omega_I + m_I^p = V_I \gtreqless V_o = p\omega_o + m_o^p$$

if and only if

$$\omega_I \gtreqless \omega_o.$$

Substituting the condition of $\Delta m = 0$ into the factor input equations and using the production function $\omega_I = q_I^a \ell_I^b$ one obtains

$$\omega_I = \left(\frac{1}{w}\,\frac{p\omega_o - a(\bar{M} - w\hat{\ell})}{(1+a)}\right)^b \left(\frac{1}{p}\,\frac{a(p\omega_o + \bar{M} - w\hat{\ell})}{1+a}\right)^a.$$

Clearly, ω_I is a strictly concave and increasing function of ω_o and $\omega_I = 0$ for $\omega_o = \frac{a(\bar{M} - w\hat{\ell})}{p}$. Moreover,

$$\frac{d\omega_I}{d\omega_o} \to +\infty \quad \text{if} \quad p\omega_o \to a(\bar{M} - w\hat{\ell})$$

and

$$\frac{d\omega_I}{d\omega_o}(\bar{\omega}) = (a+b) < 1.$$

Therefore, continuity implies that there exists a unique $\tilde{\omega}_I$ contained in the open interval $\left(\frac{a(\bar{M} - w\hat{\ell})}{p}, \bar{\omega}\right)$ such that $\omega_I = \tilde{\omega}$ if and only if $\omega_o = \tilde{\omega}_I$, and $\omega_I < \omega_o$ for $\omega_o < \tilde{\omega}_I$ and $\omega_I > \omega_o$ for $\omega_o < \tilde{\omega}_I$. For values $\omega_o > \tilde{\omega}_I$, $V_I - V_o$ is positive and for $\omega_o < \tilde{\omega}_I$, $V_I - V_o$ is negative. Thus, the dynamic development along the expansion path E_I will be monotonically converging to the Walrasian state $(\bar{\omega}, \bar{m}^P)$ if and only if total wealth V_o is larger than $\tilde{V} = p\tilde{\omega}_I + \tilde{m}^P$ where

$$\tilde{m}^P = \frac{1}{1+a}\left[p\tilde{\omega} + (\bar{M} - w\hat{\ell})\right].$$

For $V_o < \tilde{V}$ the economy will converge to a state of zero employment and zero inventories. $(\tilde{\omega}_I, \tilde{m}^P)$ is itself a stationary state with positive sales and positive level of employment. This result is in sharp contrast to the findings about the adjustment in the Keynesian region. There, excess inventories always lead to further downward adjustments preventing the system to get trapped in a stationary state without market clearing. Here, in the inflationary region, small enough total wealth of the producer, i.e. large purchasing power of consumers, will lead to demand rationing on both markets which prevent an adjustment process to a state with market clearing.

Finally, consider initial positions $(\omega_o, m_o^P) \in U$. Since the producer is rationed on both markets, the new level of money balances m_u^P is determined by the unconstrained decision of the consumption sector, i.e.

$$m_u^P = \frac{3\bar{M} - w\hat{\ell} + m_o^P}{4}.$$

If $m_o^P = \frac{3\bar{M} - w\hat{\ell}}{3} = \bar{m}^P$, then $m_u^P = \bar{m}^P$ and the level of employment and sales will stay constant through time. Since the production function is strictly concave ω_u will converge monotonically to $\bar{\omega}$ along the boundary $K \cap U$. If $(\omega_o, m_o^P) \in U$

with $m_o^P < \bar{m}^P$, money balances will increase over time at a constant rate converging to $\bar{m}^P$ independent of initial inventories if the economy remains in the underconsumption region. Consider total wealth $V_u = p\omega_u + m_u^P$. $(\omega_o, m_o^P) \in U$ and $m_o^P < \bar{m}^P$ implies $\omega_o > \bar{\omega}$. Using the budget equation $p\omega_o + m_o^P = m_u^P + C(\omega_u, \ell_u, q_u)$ where (ℓ_u, q_u) are the input levels resulting from the rationing of the producer in both markets, one obtains

$$V_u = p\omega_u - C(\omega_u, \ell_u, q_u) + p\omega_o + m_o^P.$$

If $\omega_u > \bar{\omega}$, then

$$C(\omega_u, \ell_u, q_u) > C(\omega_u) > p\omega_u$$

where $C(\omega_u)$ are the cost of producing ω_u without rationing. Hence, $V_u < p\omega_o + m_o^P$, i.e. total wealth is decreasing. If, on the other hand, $\omega_u \leqq \bar{\omega}$, then $(\omega_u, m_u^P) \in I$. In particular, this argument implies that $V_u < p\omega_o + m_o^P$ if $p\omega_o + m_o^P = p\bar{\omega} + \bar{m}^P$. Since V_u is a continuous function there exists $\varepsilon > 0$ such that $V_u < p\bar{\omega} + \bar{m}^P$ for $p\omega_o + m_o^P = p\bar{\omega} + \bar{m}^P + \varepsilon$ and $m_u^P < \bar{m}^P$. Therefore, for all $(\omega_o, m_o^P) \in U$ with $m_o^P < \bar{m}^P$ and $p\omega_o + m_o^P \leqq \bar{V}$, the economy will switch from the underconsumption region in finite time.

Not all starting points in the underconsumption region will switch over to the inflationary region. This is intuitively clear for points with very large inventories and

$m_o^P = \bar{m}^P - \varepsilon$ for positive but small ε. This pro
on the fact that the adjustment of money bala
with a constant rate independent of inventori
adjustment of inventories proceeds at a rate
by the marginal product of excess inventories
can show that there exist points in U which c
tonically to $(\bar{\omega},\bar{m}^P)$. In order to derive the p
the underconsumption region into those points
monotonic convergence to $(\bar{\omega},\bar{m}^P)$ staying in U
and those for which switching to the inflatic
will occur in finite time, the time map has t
more explicitly.

Consider the time map $F : U \to \mathbb{R}_+^2$ which is

$$m_u^P = F_m(m_o^P) = \frac{3\bar{M} - w\hat{\ell} + m_o^P}{4}$$

$$\omega_u = F_\omega(\omega_o, m_o^P) = \left[\omega_o - \frac{\bar{M} - m_o^P + w\hat{\ell}}{2p}\right]^a \left[\frac{3w\hat{\ell} - \bar{M}}{4w}\right.$$

F is a continuous and monotonic function. Mo
invertible with

$$m_o^P = F_m^{-1}(m_u^P) = 4m_u^P + w\hat{\ell} - 3\bar{M}$$

$$\text{for} \quad \frac{3\bar{M} - w\hat{\ell}}{4} \leq m_u^P \leq \bar{m}^P$$

and

$$\omega_o = F_\omega^{-1}(\omega_u, m_u^P) = \omega_u^{1/a}\left[\frac{3w\hat{\ell} - \bar{M} + F_m^{-1}(m_u^P)}{4w}\right]^{-\frac{b}{a}}.$$

Let F^{-n} denote the n-th iterate of F^{-1}, i.e. $F^{-n} = F^{-1}\left(F^{-(n-1)}\right)$. Then, any starting point in U which will enter into the inflationary region must lie in $F^{-n}(I)$ for some $n \geqq 1$. In fact, it suffices to consider the set $\overset{\circ}{I}$ = int I which is open. Then, all points in

$$\tilde{\tilde{U}} = \left\{\bigcup_{n=1}^{\infty} F^{-n}(\overset{\circ}{I})\right\} \cap \text{int}\, U$$

will leave the underconsumption region in finite time. $\tilde{\tilde{U}}$ is open since F^{-n} is continuous for all n and since $\overset{\circ}{I}$ is open. Therefore, there exists a closed set $U \setminus \tilde{\tilde{U}} \supsetneqq K \cap U$ of all trajectories which remain in U. $\tilde{\tilde{U}}$ must be bounded, since for any $(\omega_o, m_o^P) \in U \setminus \tilde{\tilde{U}}$ there exists n such that $F_m^{-n}(m_o^P) = 0$. Therefore the associated $F_\omega^{-n}(\omega_o, m_o^P) = \tilde{\omega}_o$ must be bounded and starting points $(\omega_o, 0)$ with $\omega_o > \tilde{\omega}_o$ will converge monotonically to $(\bar{\omega}, \bar{m}^P)$ as well. Hence, the monotonicity of F implies that there exists a smallest $\tilde{\tilde{\omega}}_o > \bar{\omega} + \frac{\bar{m}^P}{p}$ such that $(\tilde{\tilde{\omega}}_o, 0)$ converges monotonically to $(\bar{\omega}, \bar{m}^P)$ the trajectory of which remaining in $U \setminus \tilde{\tilde{U}}$.

Let U_I denote the boundary of $U \setminus \tilde{\tilde{U}}$ in U excluding $K \cap U$. U_I must be a line connecting $(\bar{\omega}, \bar{m}^P)$ with $(\tilde{\tilde{\omega}}, 0)$. Monotonicity of F and the linearity of F_m^{-1} imply that U_I has a negative slope. Hence, all points to the right of U_I will converge monotonically to $(\bar{\omega}, \bar{m}^P)$ with each trajectory remaining in U. All points in $\tilde{\tilde{U}}$ will leave the U region in finite time and converge in the inflationary region to $(\bar{\omega},\bar{m}^P)$.

Summarizing the results of this section, (see Figure 7) all classical and an open subset of the underconsumption states are transient which are left in finite time. Once the economy has entered Keynesian or inflationary states it will not cycle back or between these two regions but converge monotonically to the stationary Walrasian state for high enough initial total wealth of the producer. If producer wealth is below the critical level $\tilde{V}$, the economy will converge to a stationary state in the inflationary region with zero sales, zero employment and zero inventories.

7. Conclusions

The overall result of the paper is, that the stock adjustment process is basically a stable one if prices are stationary and correct along the time path. Moreover, the asymmetry between the time paths in Keynesian states and inflationary states, as exhibited by Honkapohja and Ito disappears, which, in their paper of course may be due to the different choice of the price and wage sequence. On the other hand, inflationary states and Keynesian states are in a sense the two more frequently observed disequilibrium situations, supporting the conjecture that classical unemployment and underconsumption states are not likely to be long run phenomena.

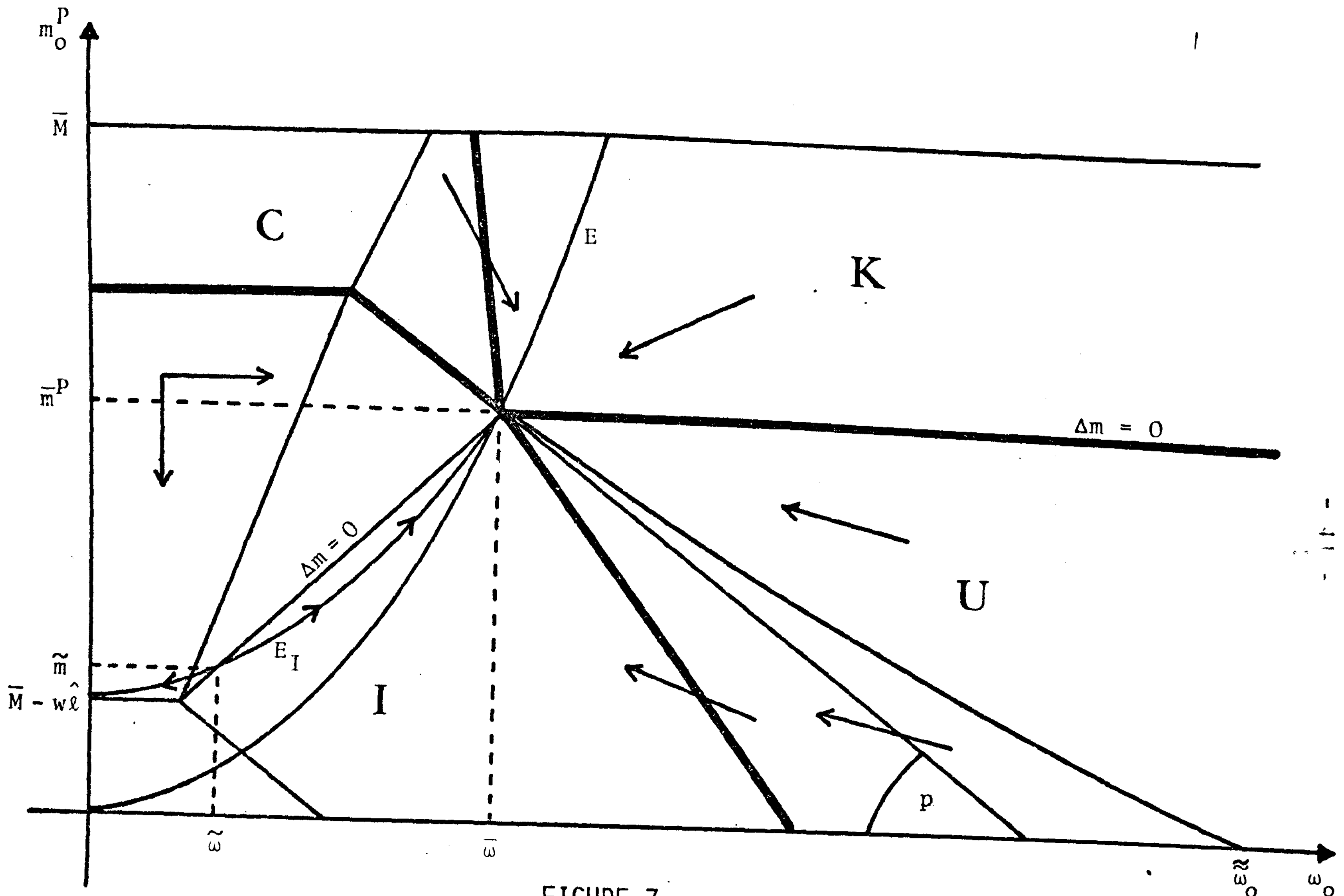

m_o^P
$\bar{M}$
$\bar{m}^P$
$\tilde{m}$
$\bar{M} - w\hat{\ell}$
C
K
U
I
E
E_I
$\Delta m = 0$
$\Delta m = 0$
p
$\tilde{\omega}$
$\bar{\omega}$
$\tilde{\tilde{\omega}}_o$
ω_o

FIGURE 7

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