GENERAL EQUILIBRIUM WITH PROFIT MAXIMIZING OLIGOPOLISTS

by

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Abstract: Monopolistic and oligopolistic equilibria in static convex economies under objective demand are analyzed. In all nondegenerate cases almost any interior real allocation or no allocation may be an oligopolistic equilibrium depending on the price normalization rule.

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1. Introduction

Much of the debate within the theory of monopolistic competition in general equilibrium centers around the issue of the appropriate concept of a demand function against which the monopolist or oligopolist maximizes. The main point of dispute is to be found in the way in which the demand faced by the monopolist takes into account the true income effect generated by the monopolist's own profit. The current generally accepted partition of the theory into the so called subjective and objective demand approaches (see Hart (1985)) reveals a striking inability of much of the literature to deal with the general objective demand situation in a satisfactory way. Hart, apparently convinced of an underlying impossibility result, states in his survey (p. 108): "..., as soon as one moves from subjective to objective demand models, it is necessary, both for reasons of tractability and for reasons of economic sense, to become more specific. It is impossible to construct a completely general 'objective' model of imperfect competition".

It is clear that a theory which is based on the concept of subjective demand in a static model without asymmetric information cannot capture the true general interdependencies of a general equilibrium model. The description of such a subjective demand model must always be very special or artificial regarding the particular assumptions, making it an unsatisfactory conceptual substitute for a general equilibrium model which considers only the true data of the economy. Any model with subjective demand functions or neglected income effects contains arbitrary ad hoc elements which contradict the true demand behavior in many cases. As a consequence the Nash equilibria of such models in general cannot display the best response features one usually associates with static one shot full information economies.

One of the reasons given for the apparent inability to treat general objective models is associated with the existence issue. Once income effects in addition to price effects play an essential role in demand theory, the constraint set as well as the best response of an oligopolist may become highly non-convex or unbounded. In either case none of the standard fix point theorems are applicable and existence can only be shown in very special circumstances (e.g. Gabscewicz and Vial (1972)) or with unacceptably strong assumptions (e.g. Hart (1985), Proposition 3.1), which are essentially tantamount to assuming existence.

The work by Roberts and Sonnenschein (1976, 1977) elaborates on the essential non-convexities which may arise under monopolistic competition. The failure of existence of equilibrium due to boundary problems is hardly ever discussed. More recent publications (e.g. Dierker and Grodal (1986)) provide hints as to a different fundamental cause for non-existence of equilibria in objective models. In their article Dierker and Grodal show that the price normalization rule plays a crucial role. The two important consequences of their findings are that for a given economy (i) the particular allocation which is obtained in a monopolistic equilibrium depends on the normalization rule chosen for prices and (ii) that existence itself fails for some normalization rules. Such a result for the same "real" economy is highly disturbing since it implies that an arbitrary exogenous scheme, which is required to define nominal prices, determines the existence and the real outcome in a monopolistic equilibrium. From their paper, however, it is not clear whether these results are truly general, since the equilibrium price correspondence in their example possesses some special features.

This paper argues that these two findings are in fact "generic" for static economies with profit maximizing oligopolists. Section 2 presents an analysis of objective demand in a simple two commodity economy indicating that the demand set may exhibit quite arbitrary features even under the usual convexity assumptions. In section 3 the class of possible normalization rules for the two commodity economy and the influence of the normalization on monopolistic equilibria is analyzed. If the price elasticity of demand is infinite, then monopolistic equilibria are independent of the normalization rule and they coincide with the Walrasian equilibrium. In almost all other cases it is shown that, depending on the normalization rule, any interior real allocation or no allocation may be a monopolistic equilibrium for any given convex real economy. Section 4 discusses the consequences of these results for the general oligopoly. Section 5 contains some concluding remarks.

2. Objective Demand : Some Examples

Consider an economy with two commodities, labor $l \geq 0$ and consumption $x \geq 0$, and two agents, a consumer with utility function $u(x,l)$ and a producer with an input requirement function $l = c(x)$. Let $p > 0$ denote the price of the consumption good and let $w > 0$ denote the wage rate. Define the Marshallian consumption demand of the consumer as

$$F(p,w,\pi) = \arg\max \{ u[x,(px-\pi)/w] \mid 0 \leq (px-\pi)/w \leq \bar{l} \}.$$

$\bar{l}$ is the maximal amount of labor the consumer is physically able to supply and π is the profit he receives. It will be assumed that $u : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ is at least twice continuously differentiable, quasi-concave, and strongly monotonic, i.e. $u_x = \partial u/\partial x > 0$ and $u_l = \partial u/\partial l < 0$. In this case F is upper hemicontinuous and convex valued.

Assume that the real cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is twice continuously differentiable with $c(0) = 0$, $c'(x) > 0$, and $c''(x) > 0$ for all $x > 0$. Furthermore, suppose that

$$-u_x(0,0)/u_l(0,0) > c'(0).$$

Then there exists a unique Walrasian equilibrium allocation $(x^*,l^*) >> 0$ with an equilibrium real wage $(w/p)^*$ s.t. $c'(x^*) = (p/w)^*$.

Given a wage rate $w_o > 0$, the objective demand set against which the producer would maximize his profit is defined implicitly by

$$OD = \{(x,p) \mid x = F\big(p,w_o,px-w_oc(x)\big),\ 0 \leq x \leq \bar{x}\}$$

where $\bar{x} = c^{-1}(\bar{l})$. Hence, the monopolistic equilibrium is given by

$$(\tilde{x},\tilde{p}) \in \arg\max \{px - w_0 c(x) \mid (x,p) \in OD\}.$$

Apart from the fact that OD is only defined implicitly, the maximization problem of the monopolist is completely analogous in its geometric representation to the standard partial equilibrium model. Since this representation reveals already the fundamental difficulties of any monopolistic general equilibrium model, it is useful and informative to analyze its features by way of some examples.

With one consumer only, the demand set OD has an equivalent explicit representation given by

$$OD = \left\{ (x,p) \;\middle|\; p = -w_0 \frac{u_x\big(x, c(x)\big)}{u_l\big(x, c(x)\big)},\ 0 \le x \le \bar{x} \right\}.$$

Then the function

$$D(x) = -\frac{u_x\big(x,c(x)\big)}{u_l\big(x,c(x)\big)}$$

defines the objective inverse demand function which is simply the supporting real price (p/w) for any producible feasible allocation $\big(x,l\big) = \big(x,c(x)\big)$. Notice that by incorporating all income effects, the demand function depends on preferences and on the technology. The following four examples are chosen to indicate that even in a two commodity model with the standard convexity assumptions the objective demand set may exhibit some unusual features. This implies in turn that equilibria possess some unexpected properties as well.

Example 1

Consider the CES utility function

$$u(x,l) = (1/\rho)[\ x^{\rho} + \delta(\bar{l}-l)^{\rho}]$$

with $\delta > 0$ and $\rho < 1$, $\rho \neq 0$, and an elasticity of substitution $\sigma = 1/(\rho-1)$. Then the objective inverse demand function is

$$D(x) = \frac{1}{\delta}\left[\frac{x}{\bar{l}-c(x)}\right]^{1/\sigma},$$

which is downward sloping for all σ with $-\infty < \sigma < 0$. Moreover, the graph of D becomes a vertical line for $\sigma \rightarrow 0$ and D(x) is constant for $\sigma \rightarrow -\infty$. The profit of the monopolist as a function of x is

$$Q(x) = \frac{w_o}{\delta}\left[\frac{x^{1+1/\sigma}}{(\bar{l}-c(x))^{1/\sigma}}\right] - w_o\ c(x)\ .$$

It is easy to verify that, for $0 > \sigma \geq -1$, Q(x) becomes unbounded for $x \rightarrow 0$. Therefore, no monopolistic equilibrium exists. On the other hand, Q(x) attains an interior maximum for all $\sigma < -1$. Figure 1 contains the objective demand OD for $\sigma = -2$, a linear marginal cost curve MC, and two isoprofit contours of the monopolist. $(\tilde{x},\tilde{p})$ is the monopolistic equilibrium whereas (x^*,p^*) is the Walrasian one. One observes that the usual condition of marginal cost equals marginal revenue holds at $(\tilde{x},\tilde{p})$.

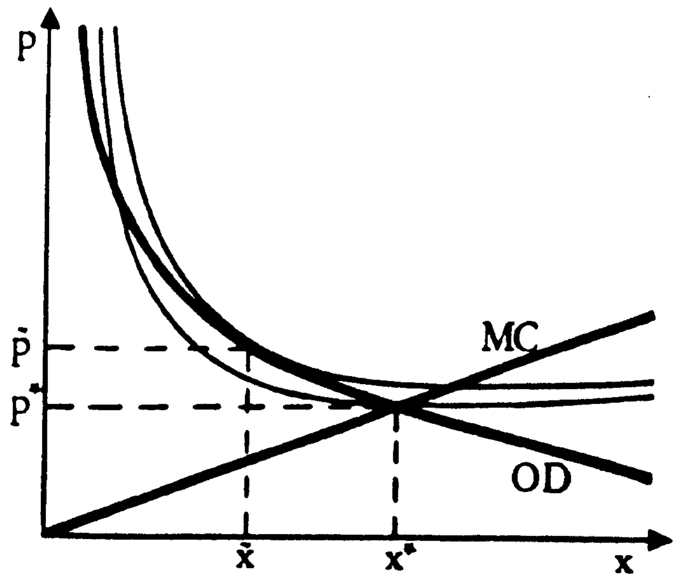

Figure 1

<u>Example 2</u>

Consider an additively separable utility function

$$u(x,l) = v(x) - h(l)$$

where v and h are both strictly increasing with v concave and h convex. This yields the objective inverse demand function

$$D(x) = \frac{v'(x)}{h'\big(c(x)\big)}$$

and the profit function

$$Q(x) = w_o \left[\frac{x\,v'(x)}{h'\big(c(x)\big)} - c(x) \right] .$$

The demand function D is downward sloping if either v or h is nonlinear. Since $h'\big(c(x)\big)$ is non-decreasing, an interior monopolistic equilibrium requires that the function $x\ v'(x)$ be increasing at least in some range. This implies that the relative risk aversion $(x\ v'')/v'$ must be less than one.

Example 3

Consider the utility function

$$u(x,l) = \frac{x - x_o}{l - l_o}$$

where x_o and l_o are constants such that either

(i) $l_o < 0$ and $x_o \leq 0$

or

(ii) $l_o > \bar{l}$ and $x_o > c(\bar{l})$.

u is quasi-concave and monotonic for all $(0,0) \leq (x,l) \leq (\bar{x},\bar{l})$. As inverse objective demand function D one obtains

$$D(x) = \frac{c(x) - l_o}{x - x_o}$$

and as profit function

$$Q(x) = w_o \left[\frac{c(x) - l_o}{x - x_o}\, x - c(x) \right] .$$

One verifies easily that in case (i) D is strictly convex with a global minimum at x^*. For case (ii), D is strictly concave with a global maximum at x^* (see Figure 2). Moreover, the unique Walrasian equilibrium $(x^*, p^*) = \left(x^*, w_0\ c'(x^*)\right)$ is a monopolistic equilibrium in both cases. Finally,

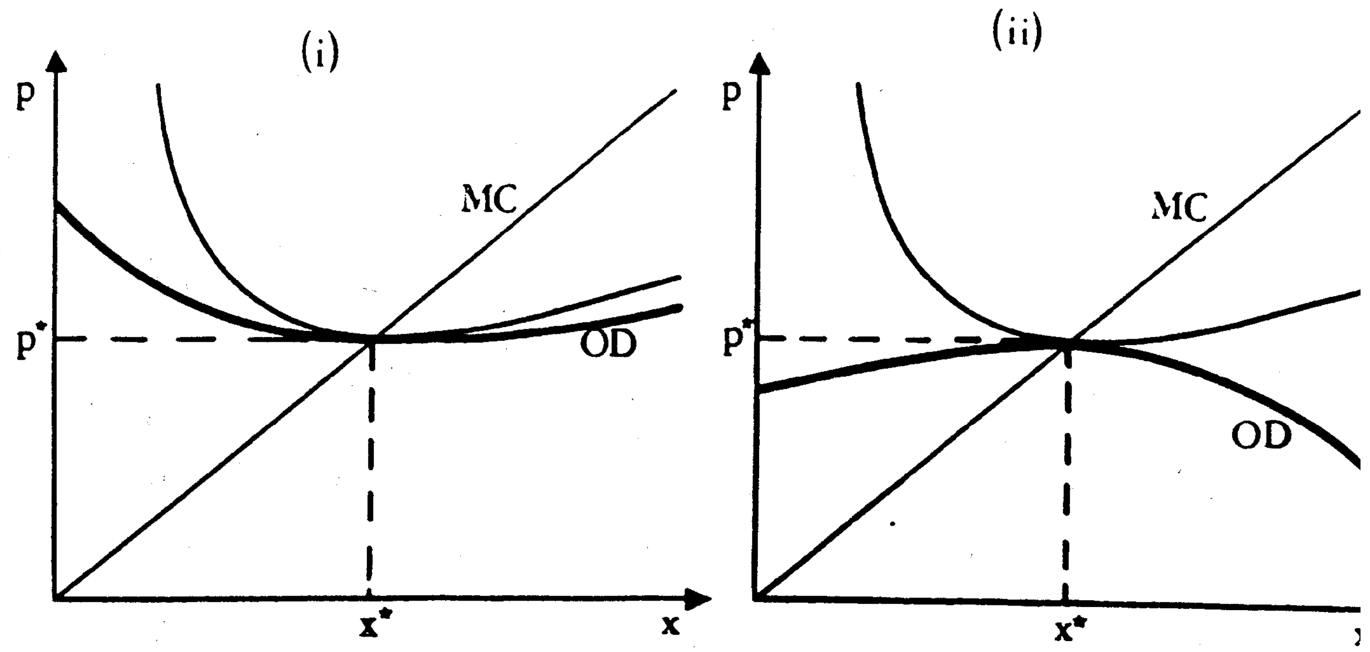

Figure 2

one observes that in case (i) with $x_0 = 0$ the graph of D is an isoprofit contour, so that $Q(x) = Q(\bar{x}) = Q(x^*)$ for all $0 < x < \bar{x}$. Hence, any x is a monopolistic equilibrium. This implies also that there exist optimal plans for the monopolist where he sets prices below marginal costs. The final example investigates the possibility for this property at a unique monopolistic equilibrium.

Example 4

Some experimenting with different utility functions reveals that the objective inverse demand function in a one consumer economy can never generate a unique monopolistic equilibrium $(\tilde{x},\tilde{p})$ such that $\tilde{p} < w_0 c'(\tilde{x})$. However, with more than one consumer the aggregate objective demand set OD can have almost any shape. In particular, an objective inverse demand function may be upward sloping and less steep than the marginal cost curve. Consider an economy with two consumers and with utility functions of the type as in the previous example. The individual objective inverse demand functions are given by

$$D_i(x_i,l_i) = \frac{l_i - l_{oi}}{x_i - x_{oi}} \qquad i = 1,2.$$

Since both pay the same price p and receive the same wage w_0 one must have for an appropriate profit distribution

$$p = w_0 \frac{l_1 - l_{o1}}{x_1 - x_{o1}} = w_0 \frac{l_2 - l_{o2}}{x_2 - x_{o2}} .$$

Since $x_1 + x_2 = x$ and $l_1 + l_2 = l$ this yields

$$p = w_0 \frac{l - (l_{o1} + l_{o2})}{x - (x_{o1} + x_{o2})} .$$

Choose $0 < l^* = l_{o1} + l_{o2} < \bar{l}_1 + \bar{l}_2$ and $x_{o1} + x_{o2} = x^* = c^{-1}(l^*)$, so that

$$p(x) = w_0 \frac{c(x) - c(x^*)}{x - x^*}.$$

It is straightforward to verify that $(x^*, w_0c'(x^*))$ is the unique Walrasian equilibrium. Moreover, $p(0) > 0$, $p'(x) > 0$, $p(\bar{x})$ finite, and

$$p'(x^*) < w_0c''(x^*).$$

Therefore, there exists a monopolistic equilibrium $(\tilde{x},\tilde{p})$ with $\tilde{p} < w_0c'(\tilde{x})$. Figure 3(i) gives a geometric representation of such a situation.

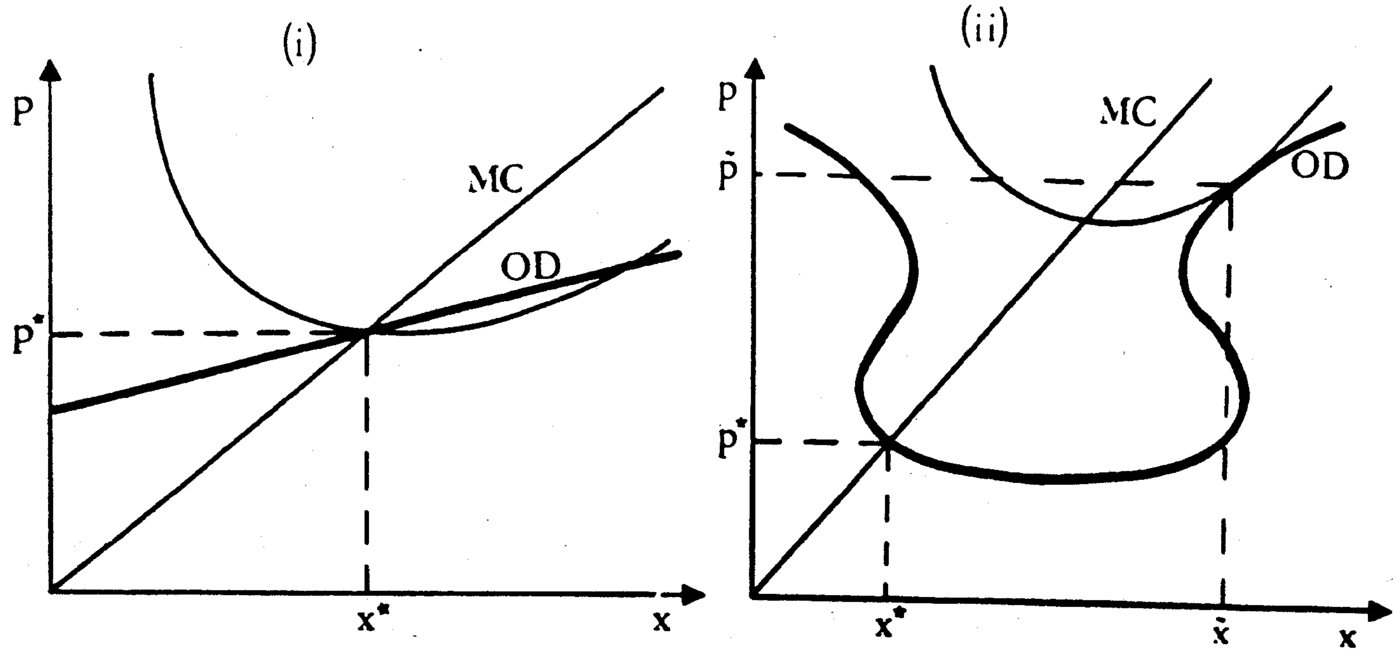

Figure 3

It is clear that the profit distribution implicit in Example 4 plays an important role. In the general case with many consumers the definition of the objective demand set OD for any given profit distribution is straightforward. For $i = 1,...,m$ consumers, let $D_i(x_i,l_i)$ denote consumer i's inverse demand function and let $0 \leq \theta_i \leq 1$ denote i's profit share with $\Sigma\, \theta_i = 1$. Then, the demand set is given by

$$OD = \{ (x,p) \mid x = \sum_{i=1}^{m} x_i , \; c(x) = \sum_{i=1}^{m} l_i , \; p = w_o D_1 (x_1,l_1) \}$$

such that for all $i = 1,\dots,m-1$ the two conditions

$$D_i(x_i,l_i) = D_{i+1}(x_{i+1}, l_{i+1})$$

$$w_o x_i D_i(x_i,l_i) = w_o l_i + \theta_i(px - w_o c(x))$$

hold. This may, even with convex preferences generate a demand set OD as in Figure 3(ii) and therefore the associated monopolistic equilibrium. To summarize the findings of this section one observes that objective demand functions in standard convex economies may in fact look quite different than those usually assumed in the standard partial equilibrium model. The examples show that this may imply unusual properties for monopolistic equilibria as well.

3. Normalization Rules

The examples of the preceding sections as well as any typical partial equilibrium model uses the particular normalization which sets the nominal wage rate w_o to be constant. By choosing nominal prices the monopolist in fact maximizes profits in terms of labor units. It is clear that he would choose a different production plan and a different price if he were to maximize profits in terms of output units. This would be equivalent to the case of a constant output price p_o and the nominal wage rate w being chosen by the monopolist. In many cases this would yield the Walrasian allocation as the monopolistic equilibrium. This occurs precisely in Example 1 for all values of the elasticity of substitution, so that the non-existence for some values of σ disappears.

Choosing one commodity as a numeraire by taking its nominal price to be constant is a particular form of price normalization. In general, a normalization assigns to any list of relative prices their absolute levels which correspond to a particular function defining a constant price level. For the case with two commodities (labor and output) and only one relative price, normalization rules take on a particularly simple form. Let $\alpha = p/w > 0$ denote any possible real price and define a normalization rule $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ as a continuous, strictly positive function which defines nominal prices p and nominal wages w by

$$p = \alpha/ f(\alpha) \text{ and } w = 1/f(\alpha) .$$

Then, $f(\alpha) = 1/w_0$ corresponds to the rule chosen in all examples above, $f(\alpha) = \alpha/p_o$ is the other extreme. $f(\alpha) = (1+\alpha)$ normalizes prices and wages on the unit simplex. But, in principle, any other function increasing or decreasing can be chosen.

Consider now the general problem of the monopolist of choosing prices and quantities on some objective demand set OD. The situation with a general normalization rule implies that the wage rate is no longer a constant, so that the objective demand set OD representing the competitive part of the economy is defined for relative prices and quantities. Hence, the profit of the monopolist is given by

$$Q = \{\, px - wc(x) \mid (x, p/w) \in OD \,\}$$

Assume that OD is given by an objective inverse demand function $D: [0, \bar{x}] \to \mathbb{R}_+$. Given a normalization rule $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$, the profit of the monopolist

$$Q = \{\, px - wc(x) \mid p/w = D(x) \,,\ 0 \le x \le \bar{x} \,\}$$

can be written as

$$Q_f(x) = \left\{ \frac{x\, D(x) - c(x)}{f(D(x))} \right\}, \quad 0 \le x \le \bar{x}\,,$$

a function of x alone. From this expression it is immediately apparent that, for the same real economy represented by the two functions D and c, the normalization rule plays a crucial role. Since $f(D(x))$ enters in the denominator this also implies that the set of maximizers of Q_f will depend on f as well. Thus, the true reason for different equilibria is best understood from the fact that the monopolist's decision consists de facto of a choice of a quantity and of the real wage, and that his objective function in terms of output varies with the normalization rule.

Before stating the two results of this section, let us consider the class of differentiable normalization rules which have a constant elasticity

$$\gamma = \frac{f'(\alpha)\,\alpha}{f(\alpha)},$$

since they mark certain reference cases. For constant γ, this class consists of the functions

$$f(\alpha) = \alpha^{\gamma}/w_o .$$

$\gamma = 0$ and $\gamma = 1$ are the two cases already discussed. For $\gamma \neq 0$ one finds

$$p = \left[w_o \, w^{\gamma-1} \right]^{1/\gamma} .$$

Hence, $\gamma < 0$ or $\gamma > 1$ implies that prices and wages move together, which corresponds to some form of wage indexation, mark up pricing or an escalator clause. However, $0 < \gamma < 1$ implies that p and w are inversely related generating high wages and low prices at high real wages and conversely.

The next two lemmata indicate to what extent the qualitative properties of a monopolistic equilibrium for any real economy depend on the properties of the normalization rule. Let

$$WE = \left\{ x^* \mid D(x^*) = c'(x^*) \, , \, x^* \in [0,\bar{x}] \right\}$$

denote the set of Walrasian equilibria and let

$$ME_f = \arg\max \left\{ Q_f(x) \mid x \in [0,\bar{x}] \right\}$$

denote the set of monopolistic equilibria for a given rule f. Assume that D is continuous on $(0,\bar{x})$ and that

$$c'(0) < \lim_{x \to 0} D(x)$$

$$c'(\bar{x}) \neq \lim_{x \to \bar{x}} D(x) .$$

Define

$$\underline{\alpha} = \inf_{x \in [0,\bar{x}]} D(x) \qquad \bar{\alpha} = \sup_{x \in [0,\bar{x}]} D(x) \, ,$$

and let

$$A = \left\{ \alpha \in [\underline{\alpha},\bar{\alpha}] \mid \exists \; x \in [0,\bar{x}] \, , \, \alpha = D(x) \, , \, x\,D(x) - c(x) > 0 \right\} .$$

LEMMA 1:

(i) *If* $\underline{\alpha} = \bar{\alpha}$ *, i.e.* $D(x)$ *is constant, then* $WE = ME_f$ *for all* f.

(ii) *If* $\underline{\alpha} < \bar{\alpha}$ *, then , for all* $\alpha_o \in A$ *, there exists a normalization rule* f_o *such that* $ME_{f_o} \neq \emptyset$ and $ME_{f_o} \subset D^{-1}(\alpha_o)$.

Property (i) states that the normalization rule does not matter if the inverse demand function is constant, implying that the monopolistic equilibrium is unique and that it coincides with the Walrasian one. On the other hand, (ii) implies that almost any $x \in (0, \bar{x})$, for which profits are positive, may be a monopolistic equilibrium. If D is strictly monotonic, then this holds for all x with $xD(x) - c(x) > 0$.[1]

Proof:
(i) is obvious.

To prove (ii), define

$$\pi(p,w) = \sup \left\{ px - w\,c(x) \mid p/w = D(x) \, , \, x \in [0,\bar{x}] \right\}$$

π is homogeneous of degree one and upper semi-continuous.

Suppose D is invertible. Then

$$\pi(p,w) = pD^{-1}(p/w) - w\, c\big(D^{-1}(p/w)\big)$$

is continuous and positive for $p/w \in A$. Choose $\alpha_o \in A$ and define the normalization rule

$$f_o(\alpha) = \alpha\, D^{-1}(\alpha) - c\big(D^{-1}(\alpha)\big) + |\alpha - \alpha_o| \,.$$

Then, $x_o \neq x = D^{-1}(\alpha)$ implies

$$Qf_o(x) = \frac{x\,D(x) - c(x)}{x\,D(x) - c(x) + |D(x) - D(x_o)|} < 1 = Qf_o(x_o) \,.$$

Therefore, $\{x_0\} = ME_{f_0} = D^{-1}(\alpha_o)$.

If D is not invertible, choose some positive number k and define the upper contour set

$$\pi_k = \Big\{(p,w) \in \mathbb{R}^2_+ \,|\, \pi(p,w) \geq k\Big\} \,.$$

π_k is a closed set and $(p,w) \in \pi_k$ implies $\lambda(p,w) \in \pi_k$ for all $\lambda \geq 1$. Define the function $\lambda : A \rightarrow \mathbb{R}_+$ by

$$\lambda(\alpha) = \text{Min}\Big\{\lambda \geq 0 \mid \pi(\lambda\alpha,\lambda) \geq k\Big\}$$

$$= \text{Min}\Big\{\lambda \geq 0 \mid (\lambda\alpha,\lambda) \in \pi_k\Big\} \,.$$

By construction one has $\lambda(\alpha)\ \pi(\alpha,1) = k$ for all $\alpha \in A$. Moreover, π is lower semi-continuous since π_k is closed. Therefore, for any $\alpha_o \in A$, there exists a strictly positive continuous function $g_o : A \rightarrow \mathbb{R}_{++}$ such that $g_o(\alpha_o) = \lambda(\alpha_o)$ and $\alpha \neq \alpha_o$ implies $g_o(\alpha) < \lambda(\alpha)$. Define the normalization rule $f_o(\alpha) = 1/g(\alpha)$. Then, $\alpha \in A/\{\alpha_o\}$ implies

$$\begin{aligned}\left\{ Q_{f_o}(x) \mid x \in D^{-1}(\alpha) \right\} &\leq \pi\left(\alpha/f_o(\alpha)\ ,\ 1/f_o(\alpha)\right) \\ &= g_o(\alpha)\ \pi(\alpha,1) \\ &< \lambda(\alpha)\ \pi(\alpha,1) = k \\ &= \operatorname{Max}\left\{ Q_{f_o}(x) \mid x \in D^{-1}(\alpha)\ ,\ \alpha \in A \right\} .\end{aligned}$$

Therefore, $\emptyset \neq ME_{f_o} \subset D^{-1}(\alpha_o)$.

QED.

For the situation of a differentiable and downward sloping demand function, the result may be inferred directly from the necessary condition $Q_f^{\prime}(x) = 0$ for an interior maximum, which is

$$f\left(D(x)\right)\left[xD'(x) + D(x) - c'(x)\right] = f'\left(D(x)\right) D'(x) \left[x\, D(x) - c(x)\right] .$$

If $D'(x) = 0$ for all x, (i) follows immediately. If $D'(x)$ is different from zero at $x \in (0,\bar{x})$, then $Q_f^{\prime}(x) = 0$ if and only if

$$\frac{D(x)\,[xD'(x) + D(x) - c'(x)]}{D'(x)\ [xD(x) - c(x)]} = \gamma(\alpha) ,$$

with $\gamma(\alpha) = \alpha f'(\alpha)/f(\alpha)$ as the elasticity of the normalization rule f. Therefore, choosing an f with a constant elasticity equal to the value of the left hand side, makes x a candidate for a monopolistic equilibrium.

Lemma 1 states essentially an indeterminacy result which says, loosely speaking, that any allocation can be a monopolistic equilibrium. At the same time, Lemma 1 is also an existence theorem, stating that for any objective demand function and cost function, there exist many continuous normalization rules f (one for every $\alpha \in A$) such that $ME_f \neq \emptyset$. It is straightforward to see that ME_f is nonempty for any continuous f if D(x) is bounded, i.e. if $\bar{\alpha}$ is finite. Thus, nonexistence occurs only if D(x) becomes unbounded for $x \to 0$ or $x \to \bar{x}$. Lemma 2 provides the associated nonexistence result.

<u>Lemma 2</u>

If $\bar{\alpha} = +\infty$, then there exists a continuous normalization rule f such that $ME_f = \emptyset$.

Proof:

Let $\bar{\alpha} = \lim_{x \to 0} D(x) = +\infty$. There exists a positive number k such that $\lim_{x \to 0} x\big(D(x)\big)^k = +\infty$. Define $f(\alpha) = 1/\alpha^k$. Then

$$\lim_{x \to 0} Q_f(x) = \lim_{x \to 0} \frac{x\,D(x) - c(x)}{f\big(D(x)\big)}$$

$$= \lim_{x \to 0} x\big(D(x)\big)^k \, [D(x) - c(x)/x] = +\infty .$$

The same argument can be used if D(0) is finite but $\lim_{x \to \bar{x}} D(x) = +\infty$.

QED.

Summarizing the results of this section a monopolist's profit maximizing decision under objective demand depends not only on the objective demand but also crucially on how nominal prices are determind from relative prices, unless the price elasticity of demand is infinite anywhere. In this special case the set of monopolistic equilibria for all normalization rules coincides with the unique Walrasian equilibrium. In general, however, almost any allocation with positive profit may be an equilibrium.

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1 Recently, Birgit Grodal brought to my attention that an essay of hers in Danish contains a related result to part (ii) of Lemma 1.

4. Oligopoly

This section is primarily expository and designed to indicate that the type of analysis and the findings of the previous section extend to the general oligopoly situation within a completely objective model of imperfect competition for a private ownership economy. This will be done in two steps. The first extension of the model to the general oligopoly case assumes an aggregate objective inverse demand function for the competitive sector. In this case the effects of different rules of normalization are as apparent as in the one consumer one producer model. The second extension describes the oligopolistic set up for a general Arrow-Debreu private ownership economy. This shows that contrary to Hart's conjecture it is possible to construct a completely objective model with a well defined Cournot-Nash equilibrium.

Consider an economy with n oligopolists who produce n goods from some single non-produced factor of input (labor). The technology of each oligopolist $j = 1,\dots,n$ is given by his input requirement function $c_j(x_j)$ which possesses the same properties as the single real cost function of the previous sections. The competitive sector of the economy is given by an objective inverse demand function $D: \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Let $x = (x_1,\dots,x_n)$. Then $D_j(x)$, $j = 1,\dots,n$ is the relative price for good j with respect to the non-produced good. This formulation clearly includes the special case of the Cournot oligopoly where some or all firms produce a homogeneous product. In this case one simply has

$$D_i(y,x_i,x_j) = D_j(y,x_i,x_j)$$

for all (y,x_i,x_j), $y = (x_k)_{k \neq i,j}$ and any i and j producing the same good.

A price system is an $(n+1)$-tupel $(w,p_1,\dots,p_n) \in \mathbb{R}^{n+1}_+$ of nominal prices. Given a vector of output decisions $x = (x_1,\dots,x_n)$, the profit of the oligopolist $j = 1,\dots,n$ is

$$Q^j = p_j\, x_j - w\, c_j(x_j)$$
$$= w\; [\; x_j\, D_j(x) - c_j(x_j)\;].$$

For the multi commodity case, price normalization now takes the following form. Consider a function $F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_{++}$ which defines a price index

$$P = F\,(w, p_1, ..., p_n).$$

F is assumed to be homogeneous of degree $r > 0$ and differentiable. Normalization therefore is equivalent to the assumption that $F(w, p_1, ...\; p_n)$ is constant for all $(w, p_1, ..., p_n)$. Without loss of generality one may set the price level p equal to one which, in conjunction with homogeneity, yields the normalization rule

$$f(\alpha_1, ..., \alpha_n) = 1/w = [\; F(\alpha_1, ..., \alpha_n, 1)]^{1/r}\,,$$

where $\alpha_j = p_j\; /\; w\;,\; j = 1,...,n$. Absolute prices are then defined by

$$w = 1/f(\alpha_1, ..., \alpha_n) \qquad p_j = \alpha_j / f(\alpha_1, ..., \alpha_n) \qquad j = 1, ..., n\;.$$

Given the normalization f each oligopolist's profit function now becomes

$$Q_f^j(x) = \frac{x_j\, D_j(x) - c_j(x_j)}{f\,\big(D(x)\big)} \qquad\qquad j = 1, ..., n$$

where $f\big(D(x)\big) = f\big(\, D_1(x)\; ,...,\; D_n(x)\,\big)$. This formulation is quite general and it encompasses many of the existing models. The fact that each oligopolist produces only one good is no real restriction, since the profit functions for many product oligopolists are simply sums of the above functions with some multi commodity cost function. Extending the model to situations with $m \geq 2$ non-produced goods is straightforward. This requires the description of at least m-1 additional markets and the associated feasibility conditions, which adds complexity to the model but does not change the structure of the objective function of the oligopolists. The same reasoning applies to cases where a firm j considers itself as a competitor when buying from or selling to another oligopolist. However, any form of bilateral monopoly without a competitive participant in that market seems to require a different model.

The consequences of the results of the previous section for the general oligopoly now seem quite straightforward, since the additional complexity stemming from more producers and/or more consumers clearly does not reduce the potential for the qualitative effects which are responsible for the results of Lemma 1 and 2. Loosely formulated one finds that, "generically" within the space of cost and inverse demand functions, normalization does not matter if and only if all own partial derivatives of the inverse demand functions are constant. This in turn implies that interior monopolistic equilibria which are independent of normalization coincide with Walrasian equilibria.

Finally, to describe the general oligopoly model, consider an Arrow-Debreu private ownership economy $\mathcal{E}$ with $l+1$ commodities, indexed $h = 0,1,...,l$, a set of consumers $I = 1,...,i,...,m$ and a set of producers $J=1,...,j,...,n$. Thus $\mathcal{E}$ is given by

$$\mathcal{E} = \left\{ (X_i, u_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J} \right\}$$

with the usual interpretation (see e.g. Debreu (1982)). Assume that the market for commodity zero is competitive and its price is chosen to be fixed at $w > 0$. The set of producers is divided into the set J_M of oligopolists, $J_M = \{1,...,k\}$, and the set $J_C = \{k+1,...,n\}$ of competitors. Each oligopolist $j \in J_M$ controls prices p_j in l_j

markets, $l \geq 1$ with $\sum_{j=1}^{k} l_j = l$. A list of prices p set by oligopolists is denoted $p = (p_1,...,p_j,...,p_k) \in \mathbb{R}^l$ with $p_j \in \mathbb{R}^{l_j}$, $j = 1,...k$. $\tilde{p} = (w,p) \in \mathbb{R}^{l+1}$ denotes the vector of all prices. For convenience the usual notational conventions will be used to write $p_{-j} = (p_1,...,p_{j-1},p_{j+1},...,p_k)$, $p = (p_j,p_{-j})$, $\tilde{p}_{-j} = (w,p_{-j})$.

Similarly a production plan $y \in Y_j \subset \mathbb{R}^{l+1}$ will be written as $y = (y_j,y_{-j})$ where y_j is the list of production decisions of producer j in the market he controls.

Assuming differentiability of the utility functions u_i, $i = 1,...,m$, define for each $i = 1,...,m$ the individual inverse demand function

$$D^i(x_i) = \frac{1}{\partial u_i/\partial x_{io}} \left[\frac{\partial u_i}{\partial x_{i1}}, ..., \frac{\partial u_i}{\partial x_{il}} \right]$$

In order to minimize notational complexity assume (as for example in Roberts (1980)) that the production sets Y_j, $j = 1,...,n$ are sufficiently well behaved so that profit maximizing behaviour at given prices leads to net demand functions. Then define for each $j = k+1,...,n$

$$h_j(\tilde{p}) = \arg\max \left\{ \tilde{p}y_j \mid y_j \in Y_j \right\}$$

as the profit maximizing net demand of each competitive producer, and for each $j = 1,...,k$

$$h_j(y_j,\tilde{p}_{-j}) = \arg\max \left\{ \tilde{p}_{-j}\, y_{-j} \mid (y_j,y_{-j}) \in Y_j \right\}$$

as the conditional profit maximizing net demand of each oligopolist.

As in the examples in section 1, it is now possible to define the set of feasible choices of prices and production plans by the oligopolist. Let proj $Y_j \subset \mathbb{R}^{l_j}$ denote the projection of producer j's production set into the subspace of markets which he controls. Define

$$Y = \prod_{j=1}^{k} \text{proj } Y_j \subset \mathbb{R}^{l} ,$$

$$P = \left\{ (p_1,\dots,p_k) \mid p_j \geq 0 \right\} ,$$

and $X = \prod_{i=1}^{m} X_i$. Given $w_o > 0$, let $\tilde{M} \subset Y \times P \times X$ denote the set of triples (y,p,x) such that the following conditions hold:

(1) $$w_o \, D^1(x_1) = (p_1,\dots,p_k)$$

for i = 1,...,m-1

(2) $$D^i(x_i) = D^{i+1}(x_{i+1})$$

(3) $$\tilde{p}\,(x_i-e_i) = \sum_{j=1}^{k} \theta_{ij}\big(p_j y_j + \tilde{p}_{-j} h_j(y_j,\tilde{p}_{-j})\big) + \sum_{j=k+1}^{n} \theta_{ij}\, \tilde{p}_j\, h_j(\tilde{p}_j)$$

(4) $$\sum_{i=1}^{m} (x_i-e_i) = \sum_{j=1}^{k} \big(y_j, h_j(y_j,\tilde{p}_{-j})\big) + \sum_{j=k+1}^{n} h_j(\tilde{p}_j).$$

Conditions (1) and (2) stipulate the same supporting prices for all consumers, condition (3) makes actual profit payments consistent with the ownership structure given by (θ_{ij}), and condition (4) imposes feasibility. Thus $\tilde{M}$ describes the set of feasible and consistent choices by all agents.

Let $M \subset Y \times P$ denote the set of production plans $(y_1,\dots,y_k) \in Y$ and prices $(p_1,\dots,p_k) \in P$ such that (1) - (4) hold for some $x \in X$. Then the objective demand set OD^j available as the choice set for oligopolist j is given by

$$OD^j\,(y_{-j},p_{-j}) = \{(y_j,p_j) \mid \big((y_j,y_{-j}),(p_j,p_{-j})\big) \in M\}.$$

Therefore a Nash equilibrium consists of a list (y_j^*, p_j^*) , $j=1,\dots,k$ such that for all $j=1,\dots,k$

$$p_j^* y_j^* + \tilde{p}_{-j}^* h_j (y_j^*, \tilde{p}_{-j}^*) \geq p_j y_j + \tilde{p}_{-j}^* h_j (y_j, \tilde{p}_{-j}^*)$$

$$\text{for all } (y_j,p_j) \in OD^j(y_{-j}^*, p_{-j}^*).$$

The definition of the sets $\tilde{M}$, M and of the mappings OD^j captures in full generality all objective features of the oligopoly model. The formulation makes apparent that in general an oligopolist's choice has to be considered as a simultaneous decision over prices and quantities from a feasible choice set determined by other oligopolists' prices and quantities. Thus the general oligopoly cannot be formalized as a game in normal form but rather as an abstract economy or as a general social system (see e.g. Debreu (1952, 1982)). With one consumer only choice sets may be defined for quantities only, but the dependence on other agents' choices remains.

It is clear that the examples of section 2 may imply some fundamental non-convexities for the set M and for the choice correspondences D_j, $j=1,\dots,k$ in particular, providing further support to the view on existence of equilibria presented by Roberts and Sonnenschein (1977). Moreover, the qualitative nature of the results of section 3 extend fully to the general oligopoly model. As can be seen directly from equations (1) - (4) the normalization rule has an impact on $\tilde{M}$, M and on OD^j, $j=1,...,k$. This implies that the qualitative features of most general equilibrium oligopoly models depend on the chosen normalization. This explains for example why on the one hand the existence proof given by Hart (1985) imposes strong assumptions on consumer preferences in order to bound prices, whereas on the other hand, Gabszewicz and Vial (1972) do not need such a boundary assumption. Another indication of that dependence is the fact that there exists a normalization which provides existence of equilibrium for the model by Roberts and Sonnenschein. Together with the analysis of this paper one finds that boundary problems and non-convexities are not two independent issues.

5. Conclusion

This paper has demonstrated that oligopolistic equilibria in general equilibrium under objective demand with profit maximizing firms crucially depend on the chosen rule of price normalization. In particular it follows from these results that a set of assumptions which guarantees existence for some normalization rule does not necessarily guarantee existence for some other rule. Conversely, the failure of existence may be due to the chosen normalization rule and not exclusively due to inherent non-convexities or boundary properties. On the other hand if existence is to be guaranteed independently of normalization, in many cases either trivial or competitive equilibria are the only outcome. At this stage it seems that the assumption of profit maximization is the villain of these negative results. This assumption has been criticized widely and one should investigate the question with more rigor whether profit maximization or some other objective function for an oligopolist is the appropriate concept. A different question which this paper raises but which leaves profit maximization untouched is whether there exists a generally acceptable and economically appealing normalization rule for any real static economy described by agents' characteristics only. With the results from this paper alone it seems that independence of real allocations from normalization is a unique Walrasian feature.

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