Approximation of generalized connecting orbits with asymptotic rate*

Thorsten Pampel
Department of Economics
University of Bielefeld
P. O. Box 10 01 31
D–33501 Bielefeld, Germany
email: tpampel@wiwi.uni-bielefeld.de

Discussion Paper No. 472

Abstract

We set up the concept of connecting orbits of a generalized form which allows for discontinuities in the system or the solution at time $t = 0$. Moreover, it is possible to select solutions which converge in a strong stable manifold by specifying the asymptotic rates. We embed connecting orbits as defined in the literature, and provide further applications which have the structure of such **generalized connecting orbits**, e. g. the computation of so called "Skiba points" in optimization problems. We develop a numerical method for computing generalized connecting orbits and derive error estimates. In particular, we show that the error decays exponentially with the length of the approximation interval, even in the strongly stable case and for periodic solutions. This is in agreement with known results for orbits connecting hyperbolic equilibria. For our method, we select appropriate asymptotic boundary conditions, which depend typically on parameters. In order to solve these type of boundary value problems we set up an efficient iterative procedure, called **boundary corrector method**. As an example we detect point to periodic connecting orbits in the Lorenz system.

Keywords: Numerical approximation – generalized connecting orbits – asymptotic rate – asymptotic boundary condition
Mathematical Subject Classification (1991): 34C37, 58F22, 65L10, 65L20, 65L70

*This paper is a condensed version of [8, Chap. 4-5], which is available via http://archiv.ub.uni-bielefeld.de/disshabi/mathe.htm.

1 Introduction

Usually, in the literature a connecting orbit is a pair (x, λ) where λ is a parameter and x is a solution of $\dot{x} = f(x, \lambda)$ which converges for $t \to \pm\infty$ to given sets. In particular, these sets are either equilibria or periodic orbits. Therefore, the connecting orbit converges to solutions $v_+(\lambda)$ or $v_-(\lambda)$ in these sets, respectively. More precisely, a connecting orbit (x, λ) solves

$$\dot{x} = f(x, \lambda), \qquad \lim_{t \to \pm\infty} |x(t) - v_\pm(\lambda)(t)| = 0. \tag{1}$$

We reformulate this and solve for $t \in \mathbb{R}_+$

$$\begin{aligned} \dot{x}_+ &= f_+(x_+, \lambda), \qquad x_+(t) \in \mathbb{R}^m, \\ \dot{x}_- &= f_-(x_-, \lambda), \qquad x_-(t) \in \mathbb{R}^m, \\ & g(x_+(0), x_-(0), \lambda) = 0, \\ & |x_\pm(t) - y_\pm(\lambda)(t)| \le C e^{\gamma_\pm t}, \end{aligned} \tag{2}$$

where $f_+ := f$, $f_- := -f$, $g(x_+(0), x_-(0), \lambda) := x_+(0) - x_-(0)$, $y_\pm(\lambda)(t) = v_\pm(\lambda)(\pm t)$, and $C > 0$, $\gamma_\pm < 0$ are admissible constants.[1] Then $x(t) := x_+(t)$ for $t \ge 0$ and $x(t) := x_-(-t)$ for $t < 0$ solves (1) and is smooth in $t = 0$. Here $\gamma_\pm$ are upper bounds for **asymptotic rates**, e. g. if v_+ is a hyperbolic equilibrium, then γ_+ is an upper bound for the real parts of the stable eigenvalues. In particular, $x_\pm(\cdot)$ and $y_\pm(\lambda)(\cdot)$ are called $\gamma_\pm$–asymptotic (see [9]).

The system (2) generalizes the concept of connecting orbits with asymptotic rates as provided in [9] and its solutions will be called **generalized connecting orbits**. This system is rather flexible in the functions $f_\pm$, in the coupling condition g and in the asymptotic rates $\gamma_\pm$. In Sect. 2 we illustrate this by embedding the usual connecting orbits in the concept of generalized connecting orbits and by providing additional applications.

Though (2) has the structure of the problems treated in [9], it is not appropriate to apply the results there directly, because the solutions $y_+(\lambda)$ and $y_-(\lambda)$ have, in general, different asymptotic rates. Hence a gap in the eigenvalue structure might shrink or even vanish. Therefore, we consider here a block partitioning of the system where we take the different asymptotic rates into account. In Sect. 3 we define a **generalized connecting orbit** and its **non-degeneracy** and relate the non-degeneracy to the non-singularity of a linear system.

[1] An assertion where variables are indexed by "$\pm$" means that this assertion holds with both, an index "+" and an index "−".

As numerical method for approximating generalized connecting orbits on finite intervals $J_\pm = [0, T_\pm]$ we solve the boundary value problem

$$\begin{pmatrix} \dot{z}_+ - f_+(z_+, \nu) \\ \dot{z}_- - f_-(z_-, \nu) \\ g\big(z_+(0), z_-(0), \nu\big) \\ M_+(T_+, \nu)(z_+(T_+) - y_+(\nu)(T_+)) \\ M_-(T_-, \nu)(z_-(T_-) - y_-(\nu)(T_-)) \end{pmatrix} = 0,$$

where the matrices $M_\pm(T_\pm, \nu)$ define appropriate asymptotic boundary conditions. In Sect. 4 we analyze the error of this method, which is shown to decay exponentially with the length of the intervals. In particular, choosing asymptotic rates $\gamma_\pm$ and appropriate asymptotic boundary conditions allows to select solutions converging in the strongly stable directions. To avoid the parameter-dependent computation of the asymptotic boundary matrices $M_\pm(T_\pm, \nu)$ we develop the efficient iterative (at most three steps) so called **boundary corrector method** for generalized connecting orbits[2].

The computation of a point to periodic solution in the Lorenz system is shown in Sect. 5 and in Sect. 6 we give some concluding remarks on the theory of generalized connecting orbits.

2 Connecting orbits and other applications

In this section we provide some frameworks to which the notion of a generalized connecting orbit applies.

2.1 Connecting orbits

In the literature (e. g. [3]) a connecting orbit from one compact invariant sets $\mathcal{V}_-(\lambda)$ to another $\mathcal{V}_+(\lambda)$ is a solution $(\bar{x}, \bar{\lambda})$ on $\mathbb{R}$ of a parameterized dynamical system

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p$$

which converges to $\mathcal{V}_\pm(\lambda)$ as $t \to \pm\infty$. In particular, a connecting orbit is called nondegenerate if a transversality condition holds and the number of parameters is

$$p = m_{+u} - m_{-u} - m_{-c} + 1 = m_{+u} + m_{-s} - m + 1$$

[2]This is the same concept as in [9].

where $m_{\pm s}+m_{\pm c}$, $m_{\pm u}+m_{\pm c}$ are the dimensions of the center stable and center unstable manifolds of $\mathcal{V}_{\pm}(\lambda)$ and $m = m_{\pm s}+m_{\pm c}+m_{\pm u}$ holds (see [3]). As in the introduction we reformulate this as generalized connecting orbit (see system (2)). Further details, in particular the choice of additional conditions or parameters to get non-degeneracy (as defined in the Sect. 3), are provided in [8, Sect. 4.2].

Connecting orbits where the invariant sets are either equilibria or periodic orbits are analyzed in [3] and [7]. In [7] "bifurcation functions" are defined and it is proved that connecting orbits exist if and only if the "bifurcation functions" are zero. In [3] the non-degeneracy is related to the regularity of an operator and the non-singularity of a linear map. A similar concept is used here.

An approximation method for periodic to periodic connections, which is similar to ours, is set up in [5]. There, numerical solutions for a specific Hamiltonian system arising from a reduced water-wave problem are computed, but without error estimates for the method.

In our approach we deal with single orbits (depending on λ) on $\mathcal{V}_-(\lambda)$ and $\mathcal{V}_+(\lambda)$, respectively. In case of periodic solutions we fix the phase of the periodic orbit to get a single solution, whereas equilibria are single orbits. As in [8] we may use the phase parameter as additional "free" parameter to find the appropriate phase.

An approximation method for connecting orbits of hyperbolic stationary points is analyzed in [1], [2]. In our generalization we also approximate connections starting or ending in the strong unstable or strong stable manifold, respectively. For example we can apply this to approximate orbit flip solutions (see [10] for details of this codimension-2-bifurcation).

2.2 Solutions with discontinuity

The splitting at 0 allows discontinuities, e. g. f_+ and f_- do not necessarily have the form $f_\pm = a_\pm f$ as in the connecting orbit cases before. We also might approximate solutions with "jumps" at 0, in this case the condition at 0 is $x_+(0)-x_-(0) = v$ with $v \in \mathbb{R}^m$. This can be used to find initial approximations for connecting orbits as follows: First compute solutions which satisfy all conditions of a connecting orbit except $x_+(0) = x_-(0)$ and define the difference vector v. Then use the components of v as continuation parameters and try to continue v to 0. Of course it is not clear whether such (global) continuation works or not, nevertheless it is a heuristic approach for getting initial approximations. A similar method for locating connecting orbits is developed in [6]. It is called "successive continuation" and a local convergence analysis for this method is presented in [6].

2.3 Skiba points

The control problems discussed in [4], lead to $m = 2n$–dimensional dynamical systems (state and costate system), where the first n state-variables are fixed. We want to approximate two solutions $x_\pm = (x^s_\pm, x^c_\pm) \in \mathbb{R}^{2n}$ of $\dot{x} = f(x)$ which converge to different solutions $y_\pm$, but which satisfy $x^s_+(0) = x^s_-(0) = v$ for given $v \in \mathbb{R}^n$. To get unique solutions converging to either equilibria or to periodic orbits the stable manifold has to be n–dimensional (i. e. an equilibrium $y_\pm$ has $m^\pm_b := n$ unstable eigenvalues and a periodic orbit $y_\pm$ has $m^\pm_b := n + 1$ center unstable Floquet multipliers). We set $f_\pm = f$,

$$g\big(x_+(0), x_-(0)\big) = \begin{pmatrix} x_+(0) - v \\ x_-(0) - v \end{pmatrix} \in \mathbb{R}^{2n}$$

and in the periodic case we add the phase of each periodic orbit as "free" parameter, i. e. we have $p = 0$ parameter if the $y_\pm$ are both equilibria, $p = 1$ parameter if one of the $y_\pm$ is a periodic orbit and $p = 2$ parameters if both $y_\pm$ are periodic orbits. Thus we see that

$$\mathcal{W} = \{(x_+, x_-, \lambda) \in \mathbb{R}^{2m+p} | g(x_+, x_-) = 0\} \tag{3}$$

is a manifold in $\mathbb{R}^{2m+p}$ of dimension $2m + p - m = m^+_b + m^-_b$. This is the key condition for non-degeneracy (see Sect. 3). The aim is to compare the values of an objective function $\hat{U}$ for both trajectories. In particular, we want to "free" one component v_i and approximate solutions which satisfy $\hat{U}(x_+(0)) = \hat{U}(x_-(0))$. Thus we substitute for a given index the side conditions $x_{+_i}(0) - v_i = 0$ and $x_{-_i}(0) - v_i = 0$ by $x_{+_i}(0) - x_{-_i}(0) = 0$ and $\hat{U}(x_+(0)) = \hat{U}(x_-(0))$. For details see [4].

3 Nondegenerate generalized connecting orbits

In this section we generalize the concept of connecting orbits and we use a transversality condition to define the **non-degeneracy** of a generalized connecting orbit. Moreover, we relate the non-degeneracy to the non-singularity of a linear operator.

Given two parameterized dynamical systems on $\mathbb{R}_+$

$$\dot{x}_+ = f_+(x_+, \lambda), \quad x_+(t) \in \mathbb{R}^m, \tag{4}$$

$$\dot{x}_- = f_-(x_-, \lambda), \quad x_-(t) \in \mathbb{R}^m, \tag{5}$$

two families of solutions $\{y_+(\lambda)\}_\Lambda$ of (4) and $\{y_-(\lambda)\}_\Lambda$ of (5) and a manifold $\mathcal{W} \subset \mathbb{R}^{2m+p}$. Then a **generalized connecting orbit** consists of (x_+, x_-, λ) where $(x_+(0), x_-(0), \lambda) \in \mathcal{W}$ and x_+ and x_- are solutions of (4) and (5), respectively, such that $x_\pm$ converge with an exponential rate $\gamma_\pm < 0$ to a solution $y_\pm(\lambda)$. This means that each pair $(x_+(0), \lambda)$ and $(x_-(0), \lambda)$ is in the γ_+–stable manifold of $y_+(\lambda)$ evaluated at 0 and in the γ_-–stable manifold of $y_-(\lambda)$ evaluated at 0, respectively.[3] In other words, a generalized connecting orbit $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ has the properties

$$(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \in \mathcal{W}, \quad (\bar{x}_+(0), \bar{\lambda}) \in \mathcal{M}^0_{\gamma_+}, \quad (\bar{x}_-(0), \bar{\lambda}) \in \mathcal{M}^0_{\gamma_-},$$

where $\mathcal{M}^0_{\gamma_\pm}$ are the $\gamma_\pm$–stable manifolds of $y_\pm$ evaluated at 0.

REMARK 1 *The notion of a generalized connecting orbit is not only applied to connecting orbits where we have a positive and a negative part. Nevertheless we index the first two parts of (x_+, x_-, λ) by "+" and "−". Definitions and assertions which hold for both parts are often indexed by "±", e. g. "$(x_\pm, \lambda)$ solves (4), (5)" means that (x_+, λ) solves $\dot{x}_+ = f_+(x_+, \lambda)$ and (x_-, λ) solves $\dot{x}_- = f_-(x_-, \lambda)$. We abbreviate pairs (x_+, x_-) by x and use the notions in [9] by indexing with "+" or "−", e. g. $\mathcal{M}^0_{\gamma_+}$ is the γ_+–stable manifold of y_+ evaluated at 0 (we set an index "±" only at γ and not in addition at $\mathcal{M}$).*

To define a generalized connecting orbit precisely we assume

- **A1** $f_\pm \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $k \geq 2$ and $f_\pm^{(k)}$ is locally Lipschitz with respect to $x_\pm$.
- **A2** $y_\pm \in C^k\big(\mathbf{\Lambda}, BC^1(\mathbb{R}_+, \mathbb{R}^m)\big)$, where $\mathbf{\Lambda} \subset \mathbb{R}^p$ is an open, bounded set. Moreover $y_\pm(\lambda)$ is a solution of $\dot{x} = f_\pm(x, \lambda)$, for each $\lambda \in \mathbf{\Lambda}$ and $y_\pm$ is bounded in $\lambda \in \mathbf{\Lambda}$ and $t \in \mathbb{R}_+$.
- **A3** $\bar{L}_\pm(\lambda) := \frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial}{\partial x} f_\pm\big(y_\pm(\lambda), \lambda\big)$, $\lambda \in \mathbf{\Lambda}$ has a shifted exponential dichotomy with data $\big(\bar{K}_\pm, \bar{\alpha}_\pm, \bar{\beta}_\pm, \bar{P}_a^\pm(\lambda), \bar{P}_b^\pm(\lambda)\big)$, $\bar{\alpha}_\pm < 0$ which are of type C^{k-1} with respect to λ. The ranks of the projectors are independent of λ and given by $m_a^\pm := \dim \mathcal{R}\big(\bar{P}_a^\pm(t)(\lambda)\big)$ and $m_b^\pm := \dim \mathcal{R}\big(\bar{P}_b^\pm(t)(\lambda)\big)$.

We describe the set of points $(x, \lambda) := (x_+, x_-, \lambda)$ where the $x_\pm$ are $\gamma_\pm$–asymptotic with $y_\pm(\lambda)(0)$ by

$$\mathcal{M}^0_\gamma := \big\{(x_+, x_-, \lambda) \big| \lambda \in \mathbf{\Lambda}, (x_+, \lambda) \in \mathcal{M}^0_{\gamma_+}, (x_-, \lambda) \in \mathcal{M}^0_{\gamma_-}\big\} \tag{6}$$

with the interpretation of $\gamma = (\gamma_+, \gamma_-)$ as index. As shown in [8], $\mathcal{M}^0_\gamma$ is an $(m_a^+ + m_a^- + p)$–dimensional C^{k-1}–manifold. To define non-degeneracy we assume in addition to **A1**–**A3**

[3] If $x(0)$ is in the γ–stable manifold of $y(\lambda)$ evaluated at τ, then x converges with an asymptotic rate γ to the solution starting at $y(\lambda)(\tau)$. For an exact definition of a γ–stable manifold evaluated at τ and its parameterization we refer to [9, Prop. 3, Corr. 1].

A4 (dimension condition) $\mathcal{W}$ is an $(m_b^+ + m_b^-)$–dimensional manifold in $\mathbb{R}^{2m+p}$ with $\mathcal{W} = \{(x_+, x_-, \mu) \in \mathbb{R}^{2m+p} | g(x_+, x_-, \mu) = 0\}$, where g is a function with $g \in C^{k-1}(\mathbb{R}^{2m+p}, \mathbb{R}^{m_a^+ + m_a^- + p})$ and $g'(x_+, x_-, \mu)$ has full rank for all $(x_+, x_-, \mu) \in \mathcal{W}$.

A5 (transversality condition) $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} + T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \mathbb{R}^{2m+p}$.

Assumption **A4** implies by $2m + p = (m_a^+ + m_a^- + p) + (m_b^+ + m_b^-)$ that **A5** is equivalent to

$$T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \{0\}.$$

REMARK 2 *Arbitrary choices for the number of parameters p and the manifold $\mathcal{W}$ in different applications are defined in [8, Sect. 4.2].*

DEFINITION 1 (Generalized connecting orbit) *Let* **A1**–**A4** *hold. We call* $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ *a* **generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_+, γ_-)** *if $\bar{x}_\pm$ and $y_\pm(\bar{\lambda})$ are $\gamma_\pm$–asymptotic at $\bar{\lambda}$ and $(\bar{x}_+(0), \bar{x}_-(0), \lambda) \in \mathcal{W}$. If in addition the transversality condition* **A5** *holds, then it is called a* **nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_+, γ_-)**.

For each $(x, \lambda) = (x_+, x_-, \lambda) \in \mathcal{W}$ we define a matrix

$$D(x_+, x_-, \lambda) := \left(\frac{\partial}{\partial x_+} g(x, \lambda), \frac{\partial}{\partial x_-} g(x, \lambda), \frac{\partial}{\partial \lambda} g(x, \lambda) \right) \tag{7}$$

which is in $\mathbb{R}^{m_a^+ + m_a^- + p, 2m+p}$. Thus, $T_{(x,\lambda)}\mathcal{W} = \mathcal{N}\big(D(x_+, x_-, \lambda)\big)$ follows from the definitions of $\mathcal{W}$ and D.

As seen in [9, Prop. 3, Corr. 1] there exists an open neighborhood $V \times \Omega$ of $(0, 0, \bar{\lambda}) \subset \mathcal{R}\big(P_a^+(0)\big) \times \mathcal{R}\big(P_a^-(0)\big) \times \mathbb{R}^p$, such that the $\gamma_\pm$–stable manifold $\mathcal{M}_{\gamma_\pm}^0$ is locally parameterized by $b(\xi_+, \xi_-, \lambda) := \big(x_+(\xi_+, \lambda)(0), x_-(\xi_-, \lambda)(0), \lambda\big)$, where $x_\pm(\xi_\pm, \lambda)$ and $y_\pm(\lambda)$ are $\gamma_\pm$–asymptotic (i. e. $|x_+(\xi_\pm, \lambda)(t) - y_\pm(\lambda)(t)| < C_\pm e^{\gamma_\pm t}$ for some $C_\pm > 0$). Moreover, b maps to

$$W := \big\{ \big(x_+(\xi_+, \lambda)(0), x_-(\xi_-, \lambda)(0), \lambda\big) \bigm| (\xi_+, \xi_-, \lambda) \in V \times \Omega \big\} \subset \mathcal{W}$$

and the tangent map $B(\xi^+, \xi^-, \lambda)(\eta_+, \eta_-, \mu)$ of b is defined by

$$\begin{pmatrix} \frac{\partial}{\partial \xi^+} x_+(\xi^+, \lambda)(0)\eta_+ + \frac{\partial}{\partial \lambda} x_+(\xi^+, \lambda)(0)\mu \\ \frac{\partial}{\partial \xi^-} x_-(\xi^-, \lambda)(0)\eta_- + \frac{\partial}{\partial \lambda} x_-(\xi^-, \lambda)(0)\mu \\ \mu \end{pmatrix}. \tag{8}$$

Hence, for $(x^0, \lambda) := \big(x_+(\xi^+, \lambda)(0), x_-(\xi^-, \lambda)(0), \lambda\big) \in W$ the tangent space of $\mathcal{M}^0_\gamma$ at (x^0, λ) is

$$T_{(x^0,\lambda)}\mathcal{M}^0_\gamma = \mathcal{R}\big(B(\xi^+, \xi^-, \lambda)\big).$$

Let $S^\pm(\cdot, \cdot)$ be the solution operator of $L_\pm x = 0$ on $J_\pm = [0, T_\pm]$ (or $J_\pm = \mathbb{R}_+$, i. e. "$T_\pm = \infty$"), i. e. $L_\pm S^\pm(\cdot, s) = 0$ and $S^\pm(s, s) = \mathrm{Id}_{\mathbb{R}^m}$ for all $s \in J_\pm$. Then,

$$s^\pm_J(w)(t) := \int_0^t S^\pm(t, s)P^\pm_a(s)w(s)\mathrm{d}s - \int_t^{T_\pm} S^\pm(t, s)P^\pm_b(s)w(s)\mathrm{d}s \tag{9}$$

solves $L_\pm x = w$ on $J_\pm$. With this notion the partial derivatives of $x_\pm$ at $(0, \bar\lambda)$ are (see [9, Prop. 3])

$$\frac{\partial}{\partial\xi}x_\pm(0, \bar\lambda) = S^\pm(\cdot, 0)P^\pm_a(0) \tag{10}$$

$$\frac{\partial}{\partial\lambda}x_\pm(0, \bar\lambda) = s^\pm_{[0,\infty)}\big(\Psi_\pm(\cdot)\big) - S^\pm(\cdot, 0)P^\pm_a(0)\frac{\partial}{\partial\lambda}y_\pm(\bar\lambda)(0) + \frac{\partial}{\partial\lambda}y\pm(\bar\lambda)(\cdot), \tag{11}$$

where

$$\Psi_\pm(s) := \left(\frac{\partial}{\partial x}f\big(\bar x_\pm(s), \bar\lambda\big) - \frac{\partial}{\partial x}f\big(\bar y_\pm(s), \bar\lambda\big)\right)\frac{\partial}{\partial\lambda}y_\pm(\bar\lambda)(s) + \left(\frac{\partial}{\partial\lambda}f\big(\bar x_\pm(s), \bar\lambda\big) - \frac{\partial}{\partial\lambda}f\big(\bar y_\pm(s), \bar\lambda\big)\right). \tag{12}$$

Using this and the definitions (7) of D and (8) of B we obtain:

Lemma 1 *Suppose that* **A1**–**A4** *hold, let $\bar x_\pm$ be $\gamma_\pm$–asymptotic with $y_\pm(\bar\lambda)$ at $\bar\lambda$ and let $(x, \lambda) := (x_+(\xi^+, \lambda), x_-(\xi^-, \lambda), \lambda) \in W$ be a generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_+, γ_-).*

Then (x, λ) is nondegenerate if and only if the linear operator $D\big(x_+(\xi^+, \lambda)(0), x_-(\xi^+, \lambda)(0), \lambda\big) \circ B(\xi^+, \xi^-, \lambda)$ is nonsingular.

Proof Analogous to the proof of [9, Lemma 3]. ■

We define for a generalized connecting orbit $(\bar x_+, \bar x_-, \bar\lambda)$ from $\mathcal{W}$ to $y_\pm$ a linear operator $D := D(\bar x_+(0), \bar x_-(0), \bar\lambda)$ and for $(\xi^+, \xi^-, \lambda) = (0, 0, \bar\lambda)$ we define $B := B(0, 0, \bar\lambda)$ and hence

$$B(\eta_+, \eta_-, \mu) = \begin{pmatrix} \eta_+ + \frac{\partial}{\partial\lambda}x_+(0, \bar\lambda)(0)\mu \\ \eta_- + \frac{\partial}{\partial\lambda}x_-(0, \bar\lambda)(0)\mu \\ \mu \end{pmatrix}. \tag{13}$$

In particular, non-degeneracy of $(\bar x_+, \bar x_-, \bar\lambda)$ implies non-singularity of $D \circ B$ (see Lemma 1).

4 The approximation of generalized connecting orbits

A numerical method for approximating generalized connecting orbits on finite intervals is presented in 4.1. Moreover, we analyze the error which is shown to decay exponentially with the length of the intervals. In 4.2 we present the boundary corrector method for generalized connecting orbits.

4.1 The approximation theorem

In this section we set up an approximation theory for generalized connecting orbits. The technique is the same as in [9]. We truncate $\mathbb{R}_+$ to finite intervals $J_+ = [0, T_+]$ and $J_- = [0, T_-]$ and approximate both parts $\bar{x}_+$ and $\bar{x}_-$ of a generalized connecting orbit on J_+ and J_-, respectively. At T_+ and T_- we use asymptotic boundary conditions. The proof and a lot of estimates are similar to those in [9].

Assume **A1**–**A4** and let $(\bar{x}, \bar{\lambda}) = (\bar{x}_+, \bar{x}_-, \bar{\lambda})$ be a nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_-, γ_+), such that $\bar{\alpha}_\pm < \gamma_\pm < \min(0, 2\bar{\beta}_\pm)$. Moreover, suppose that the linearizations $L_\pm := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm, \bar{\lambda})$ have shifted exponential dichotomies with exponents $\alpha_\pm$, $\beta_\pm$ ($\alpha_\pm < \gamma_\pm < 2\beta_\pm$) near $\bar{\alpha}_\pm$, $\bar{\beta}_\pm$, projectors $P_a^\pm$, $P_b^\pm$ and constants $K_\pm$. Furthermore we assume that the boundary conditions are regular in the following sense:

A6 Let $M_\pm(T_\pm, \cdot) \in C^1\big(\mathbf{\Lambda}, \mathbb{R}^{m_b^+ + m_b^-, m}\big)$ be matrix valued functions with $\mathcal{N}\big(M_\pm(T_\pm, \bar{\lambda})\big) \cap \bar{\mathcal{R}}_b^\pm = \{0\}$ for all $T_\pm \in \mathcal{D}_\pm$, where $\mathcal{D}_\pm \subset \mathbb{R}_+$ are sets, such that there exist sequences $\{T_\pm^i\}_{i \in \mathbb{N}} \subset \mathcal{D}_\pm$ with $\lim_{i \to \infty} T_\pm^i = \infty$. Assume that $M_\pm(T_\pm, \lambda)$ and $\frac{\partial}{\partial \lambda} M_\pm(T_\pm, \lambda)$ are uniformly bounded by $M_{M_\pm}$, $M_{M_\pm}^1$ and Lipschitz continuous with constants $L_{M_\pm}$, $L_{M_\pm}^1$ and let the inverses $\bar{N}_\pm := (M_\pm(T_\pm, \bar{\lambda})|_{\bar{\mathcal{R}}_b^\pm})^{-1}$ be bounded by $M_{\bar{N}_\pm} > 0$ uniformly in $T_\pm \in \mathcal{D}_\pm$.

Remark 3 *In the case where the $y_\pm(\lambda)$ are 1–periodic (scaled system) we choose $\mathcal{D}_\pm = \cup_{N \in \mathbb{N}} [N - \tau_-, N + \tau_+]$, $\tau_-, \tau_+ \in [0, 1]$ and in the case where the $y_\pm(\lambda)$ are equilibria we choose $\mathcal{D}_\pm = \mathbb{R}_+$. We abbreviate $\bar{\mathcal{R}}_b^\pm := \mathcal{R}\big(\bar{P}_b^\pm(0)(\bar{\lambda})\big)$ and $\mathcal{R}_b^\pm := \mathcal{R}\big(P_b^\pm(0)(\bar{\lambda})\big)$ and as in [8], we see that the $M_\pm(T_\pm, \bar{\lambda})|_{\mathcal{R}_b^\pm}$ are nonsingular and that their inverses $N_\pm := (M_\pm(T_\pm, \bar{\lambda})|_{\mathcal{R}_b^\pm})^{-1}$ are uniformly bounded by $M_{N_\pm} := 4KM_{\bar{N}_\pm} > 0$ for sufficiently large $T_\pm \in \mathcal{D}_\pm$. To get suitable $M_\pm(T_\pm, \cdot)$ we compute asymptotic boundary conditions by solving an eigenvalue problem (equilibrium) or the adjoint variational equation* *(periodic solution), for details we refer to [8, Sect. 3.2] or [9, Sect. 4.1].*

For some $T_\pm \in \mathcal{D}_\pm$, $J_\pm := [0, T_\pm]$ (notice that $T_- > 0$) we define Banach spaces $\mathcal{Y} = C^1(J_+, \mathbb{R}^m) \times C^1(J_-, \mathbb{R}^m) \times \mathbb{R}^p$ with norm

$$\begin{aligned} ||(x_+, x_-, \lambda)||_{\mathcal{Y}} &= ||x_+||^+_{\mathcal{Y}} + ||x_-||^-_{\mathcal{Y}} + |\lambda|, \\ ||x_\pm||^\pm_{\mathcal{Y}} &= \sup_{t \in J_\pm} \big(|x_\pm(t)| q_\pm(t)\big), \quad q_\pm(t) = \frac{2}{1 + e^{\beta_\pm t}} \end{aligned}$$

and $\mathcal{Z} = C^0(J_+, \mathbb{R}^m) \times C^0(J_-, \mathbb{R}^m) \times \mathbb{R}^{m_a^+ + m_a^- + p} \times \mathbb{R}^{m_b^+} \times \mathbb{R}^{m_b^-}$ with norm

$$\begin{aligned} &||(v_+, v_-, r_0, r_+, r_-)||_{\mathcal{Z}} \\ &= C^{J_+}_{\mathcal{Z}} ||v_+||^+_{\mathcal{Z}} + C^{J_-}_{\mathcal{Z}} ||v_-||^-_{\mathcal{Z}} + |r_0| + C^+_{\mathcal{Z}} |r_+| e^{-\beta_+ T_+} + C^-_{\mathcal{Z}} |r_-| e^{-\beta_- T_-}, \end{aligned}$$

where

$$\begin{aligned} C^{J_\pm}_{\mathcal{Z}} &= (K_\pm + M_{M_\pm}\, M_{N_\pm}\, K^2_\pm) ||R^\pm_0||, \\ C^\pm_{\mathcal{Z}} &= M_{N_\pm}\, K_\pm\, ||R^\pm_0|| \end{aligned}$$

and

$$\begin{aligned} ||v_\pm||^\pm_{\mathcal{Z}} &= ||v_\pm||_{\beta_\pm} + \int_0^{T_\pm} |v_\pm(s)| e^{-\beta_\pm s} \mathrm{ds}, \\ ||v_\pm||^\pm_{\beta_\pm} &= \sup_{t \in J_\pm} (|v_\pm(t)| e^{-\beta_\pm t}). \end{aligned}$$

Here we define $(R_0^+\ R_0^-\ Q_0) := D(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda})$ as in (7) and B as in (13), and we obtain that

$$\begin{aligned} D \circ B(\xi_a^+, \xi_a^-, \mu) &= R_0^+ \big(\xi_a^+ + \frac{\partial}{\partial \lambda} x_+(0, \bar{\lambda})(0)\mu\big) \\ &\quad + R_0^- \big(\xi_a^- + \frac{\partial}{\partial \lambda} x_-(0, \bar{\lambda})(0)\mu\big) + Q_0 \mu \end{aligned} \tag{14}$$

is nonsingular (Lemma 1). Hence $D \circ B(\xi^+, \xi^-, 0) = R_0^+ \xi^+ + R_0^- \xi^-$ implies that $\mathrm{rank}(R_0^\pm) > 0$ and $||R_0^\pm|| > 0$.

For $(\beta, T) := (\beta_+, \beta_-, T_+, T_-)$, we define

$$r(\beta, T) := r^+(\beta_+, T_+) + r^-(\beta_-, T_-), \tag{15}$$

$$r^\pm(\beta_\pm, T_\pm) := \begin{cases} e^{|\beta_\pm| T_\pm} + \frac{1}{|\beta_\pm|}\big(e^{|\beta_\pm| T_\pm} - 1\big) & : \beta_\pm \neq 0, \\ 1 + T_\pm & : \beta_\pm = 0. \end{cases}$$

Before setting up the theorem we define a set of pairs (T_-, T_+) for which the estimate

$$e^{\Delta_+ T_+} + e^{\Delta_- T_-} \leq \frac{\epsilon}{r^+(\beta_+, T_+) + r^-(\beta_-, T_-)} \tag{16}$$

holds (see Fig. 1). The set depends on $\epsilon > 0$, $\beta_\pm \in \mathbb{R}$ and $\Delta_\pm < 0$. Moreover we assume $\Delta_\pm < -|\beta_\pm|$ and define $v(0, T_\pm) = 1 + T_\pm$ and $v(\beta_\pm, T_\pm) = \min(1 + \frac{1}{|\beta_\pm|}, 1 + T_\pm)$, $\beta_\pm \neq 0$.

Then we see that $r^\pm(\beta_\pm, T_\pm) \leq e^{|\beta_\pm| T_\pm} \cdot v(\beta_\pm, T_\pm)$ and that (16) holds for each pair (T_+, T_-) which satisfies

$$\left(e^{\Delta_+ T_+} + e^{\Delta_- T_-}\right)\left(e^{|\beta_+| T_+} v(\beta_+, T_+) + e^{|\beta_-| T_-} v(\beta_-, T_-)\right) \leq \epsilon. \tag{17}$$

The estimate (17) is satisfied, if

$$\begin{aligned}
\Delta_+ T_+ + |\beta_-| T_- + \ln\left(v(\beta_-, T_-)\right) &\leq \ln\left(\frac{\epsilon}{4}\right), && (18)\\
\Delta_- T_- + |\beta_+| T_+ + \ln\left(v(\beta_+, T_+)\right) &\leq \ln\left(\frac{\epsilon}{4}\right), && (19)\\
(\Delta_\pm + |\beta_\pm|) T_\pm + \ln\left(v(\beta_\pm, T_\pm)\right) &\leq \ln\left(\frac{\epsilon}{4}\right) && (20)
\end{aligned}$$

holds. First we choose the minimal $\hat{T}_\pm \in \mathbb{R}_+$, such that (20) holds for all $T_\pm \geq \hat{T}_\pm$. Then we define

$$\begin{aligned}
Q_+(T_-) &= \frac{|\beta_-| T_- + \ln\left(v(\beta_-, T_-)\right) - \ln(\frac{\epsilon}{4})}{-\Delta_+},\\
Q_-(T_+) &= \frac{|\beta_+| T_+ + \ln(v(\beta_+, T_+)) - \ln(\frac{\epsilon}{4})}{-\Delta_-}
\end{aligned}$$

and obtain that (18), (19) holds in the domain

$$\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) = \big\{(T_-, T_+) \;\big|\; T_+ \geq \max\left(\hat{T}_+, Q_+(T_-)\right), \text{ and } T_- \geq \max\left(\hat{T}_-, Q_-(T_+)\right)\big\}.$$

If $\beta_\pm \neq 0$ and $T_\pm \geq \frac{1}{|\beta_\pm|}$, then the functions Q_+ and Q_- are linear with slopes $\frac{|\beta_-|}{-\Delta_+}$ and $\frac{|\beta_+|}{-\Delta_-}$, respectively. Therefore the assumption $\Delta_\pm < -|\beta_\pm|$ implies that $|\beta_+| \cdot |\beta_-| < \Delta_+ \cdot \Delta_-$ and hence $\frac{|\beta_+|}{-\Delta_-}$, the slope of Q_- is less than $\frac{-\Delta_+}{|\beta_-|}$, the slope of Q_+^{-1}. Thus, Q_- and Q_+^{-1} intersect and $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) \neq \emptyset$. In particular, for each $a \in (\frac{|\beta_+|}{-\Delta_-}, \frac{-\Delta_+}{|\beta_-|})$ there exists some $\bar{T}$, such that $(T, aT) \in \mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ for all $T \geq \bar{T}$. Similar results hold for $\beta_+ = 0$ or $\beta_- = 0$ by using definition of $Q_\pm$. Thus we obtain the following Lemma:

LEMMA 2 *Let $\epsilon > 0$, $\beta_\pm \in \mathbb{R}$, and $\Delta_\pm < -|\beta_\pm|$. Then (16) is satisfied for all $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ and $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm) \neq \emptyset$.*

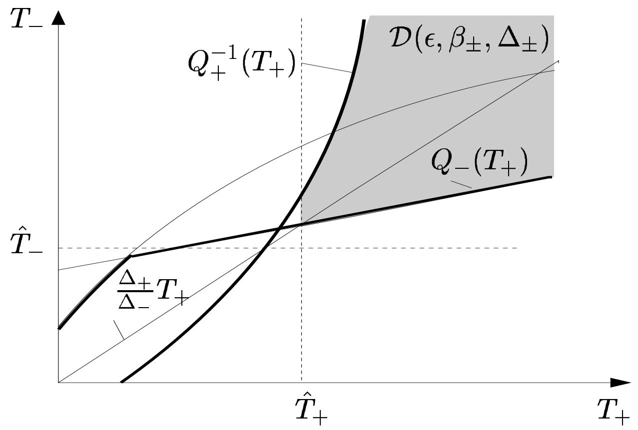

Figure 1: Typical diagram for $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$, where $\beta_\pm \neq 0$

REMARK 4 *Typically, ϵ is small. If $\epsilon < 4$, then (20) is satisfied at $\hat{T}_\pm$ with equality and $Q_+(\hat{T}_-) = \frac{\Delta_-}{\Delta_+}\hat{T}_-$, $Q_-(\hat{T}_+) = \frac{\Delta_+}{\Delta_-}\hat{T}_+$. If $\mathcal{D}_+ \times \mathcal{D}_-$ is defined as in Remark 3 then $\mathcal{D}(\epsilon, \beta_\pm, \Delta_\pm)$ has a sufficiently large intersection with $\mathcal{D}_+ \times \mathcal{D}_-$.*

THEOREM 1 *Suppose that* **A1**–**A6** *hold, let $d_\pm = 1$ and let $(\bar{x}, \bar{\lambda})$ be a nondegenerate generalized connecting orbit from $\mathcal{W}$ to $y_\pm$ of type (γ_-, γ_+). Let the linear operators $L_\pm := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm, \bar{\lambda})$ have shifted exponential dichotomies with dichotomy data $(K_\pm, \alpha_\pm, \beta_\pm, P_a^\pm, P_b^\pm)$ and $\alpha_\pm < \gamma_\pm < \min(0, 2\beta_\pm)$.*

Then there exist $\delta > 0$, $C^\pm > 0$, $\epsilon > 0$ and $\bar{T}_\pm$ sufficiently large, such that for all $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, d_\pm\gamma_\pm - \beta_\pm) \cap \mathcal{D}_+ \times \mathcal{D}_-$, $T_\pm \geq \bar{T}_\pm$ the operator equation

$$H_J(z_+, z_-, \nu) = \begin{pmatrix} \dot{z}_+ - f_+(z_+, \nu) \\ \dot{z}_- - f_-(z_-, \nu) \\ g\big(z_+(0), z_-(0), \nu\big) \\ M_+(T_+, \nu)(z_+(T_+) - y_+(\nu)(T_+)) \\ M_-(T_-, \nu)(z_-(T_-) - y_-(\nu)(T_-)) \end{pmatrix} = 0 \quad (21)$$

has a unique solution (z_J^+, z_J^-, ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}_+|_{J_+}, \bar{x}_-|_{J_-}, \bar{\lambda})$ in $\mathcal{Y}$. Furthermore, the following pointwise estimates with $d_\pm = 1$ hold

$$|\nu_J - \bar{\lambda}| \leq C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}, \quad (22)$$

$$||z_J^\pm(t) - \bar{x}_\pm|_{J_\pm}(t)|| \leq \left(C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}\right) \begin{cases} e^{\beta_\pm t} & : \beta_\pm > 0 \\ 1 & : \beta_\pm \leq 0 \end{cases} \quad (23)$$

where $\Delta_\pm := d_\pm\gamma_\pm - \beta_\pm < -|\beta_\pm|$.

If $M_+(T_+,\lambda)$ satisfies

$$|\bar{M}_+(T_+,\bar\lambda)(\bar x(T_+) - y(\bar\lambda)(T_+))| \le C^+_{\bar M}|\bar x_+(T_+) - y_+(\bar\lambda)(T_+)|^2 \quad (24)$$

for a constant $C^+_{\bar M} > 0$ and sufficiently large $T_+ \in \mathcal{D}_+$, then the estimates hold for $d_+ = 2$ and we can replace $\alpha_+ < \gamma_+ < \min(0, 2\beta_+)$ by $\alpha_+ < \gamma_+ < \min(0,\beta_+)$. The analogous result holds in the "$-$" case.

Proof The proof is similar to that of [9, Theorem 4] and a lot of ideas and estimates can be carried over.

Sketch of the proof of Theorem 1

We apply the following perturbation lemma (see also [2], [13]) with $F = H_J$ and $y_0 = (\bar x_+|_{J_+}, \bar x_-|_{J_-}, \bar\lambda) =: (\bar x|_J, \bar\lambda)$.

Lemma 3 (Perturbation lemma) *Let $F : K_\delta(y_0) \to \mathcal{Z}$ be a C^1-function from a ball in $\mathcal{Y}$ into $\mathcal{Z}$ (Banach spaces). Assume that $F'(y_0)$ is a homeomorphism and there exist constants κ and σ, such that*

$$\begin{aligned} ||F'(y) - F'(y_0)|| \le \kappa &< \sigma \le ||F'(y_0)^{-1}||^{-1}, \\ ||F(y_0)|| &\le (\sigma-\kappa)\delta \end{aligned}$$

holds for all $y \in K_\delta(y_0)$. Then F has a unique zero $\bar y$ in $K_\delta(y_0)$ and

$$\begin{aligned} ||\bar y - y_0|| &\le (\sigma-\kappa)^{-1}||F(y_0)||, \\ ||y_1 - y_2|| &\le (\sigma-\kappa)^{-1}||F(y_1) - F(y_2)|| \end{aligned}$$

for all $y_1, y_2 \in K_\delta(y_0)$.

Here and in the remainder part we abbreviate sometimes for example $(w_+, w_-, r_0, r_+, r_-)$ by (w, r_0, r). We prove the assumptions of Lemma 3 by the following steps:

- **S1** There exists $C_{lin} > 0$ such that $||(v,\mu)||_{\mathcal{Y}} \le C_{lin}||(w,r_0,r)||_{\mathcal{Z}}$ holds for each solution (v,μ) of $H_J'(\bar x|_J,\bar\lambda)(v,\mu) = (w,r_0,r)$.
- **S2** $H_J'(\bar x|_J,\bar\lambda)$ is a homeomorphism, $\sigma := \frac{1}{C_{lin}} \le ||H_J'(\bar x|_J,\bar\lambda)^{-1}||^{-1}$.
- **S3** There exists a constant $C_{Lip} > 0$ such that for $\delta := \frac{\sigma}{2C_{Lip}} > 0$ and $\tilde\delta := \frac{\delta}{r(\beta,T_+)}$ the following holds: $||H_J'(z,\nu) - H_J'(\bar x|_J,\bar\lambda)|| \le \kappa := \frac{\sigma}{2}\ \forall (z,\nu) \in \mathcal{B}_{\tilde\delta}(\bar x|_J,\bar\lambda)$.
- **S4** There exist constants $\epsilon > 0$, $\tilde C^\pm > 0$ and $\bar T_\pm$ sufficiently large, such that $||H_J(\bar x|_J,\bar\lambda)|| \le \tilde C^+ e^{\Delta_+ T_+} + \tilde C^- e^{\Delta_- T_-} \le (\sigma-\kappa)\frac{\delta}{r(\beta,T_+)}$ for all $(T_+,T_-) \in \mathcal{D}(\epsilon,\beta_\pm,d_\pm\gamma_\pm - \beta_\pm) \cap \mathcal{D}_+ \times \mathcal{D}_-$, $T_\pm \ge \bar T_\pm$.

S5 Lemma 3 implies that (21) has a unique solution (z_J, ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}|_J, \bar{\lambda})$ with $||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}$, where $C^{\pm} := 2C_{lin}\tilde{C}^{\pm}$.

Now we derive the details.

S1 Let $(w_+, w_-, r_0, r_+, r_-) \in \mathcal{Z}$ be arbitrary and let $(v_+, v_-, \mu) \in \mathcal{Y}$ be a solution of the inhomogeneous equation

$$H'_J(\bar{x}_+|_{J_+}, \bar{x}_-|_{J_-}, \bar{\lambda})(v_+, v_-, \mu) = (w_+, w_-, r_0, r_+, r_-).$$

This is equivalent to the variational equation

$$\begin{aligned} \dot{v}_+ - A_+(\cdot)v_+ - V_+(\cdot)\mu &= w_+, & (25)\\ \dot{v}_- - A_-(\cdot)v_- - V_-(\cdot)\mu &= w_-, & (26)\\ R_0^+ v_+(0) + R_0^- v_-(0) + Q_0\mu &= r_0, & (27)\\ R_+ v_+(T_+) + Q_+\mu &= r_+, & (28)\\ R_- v_-(T_-) + Q_-\mu &= r_-, & (29) \end{aligned}$$

where

$$\begin{aligned} A_\pm(\cdot) &= \tfrac{\partial}{\partial x} f_\pm(\bar{x}_\pm(\cdot), \bar{\lambda}),\\ V_\pm(\cdot) &= \tfrac{\partial}{\partial \lambda} f_\pm(\bar{x}_\pm(\cdot), \bar{\lambda}),\\ R_0^\pm &= \tfrac{\partial}{\partial x_\pm} g(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}),\\ Q_0 &= \tfrac{\partial}{\partial \lambda} g(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}),\\ R_\pm &= M_\pm(T_+, \bar{\lambda}),\\ Q_\pm &= \tfrac{\partial}{\partial \lambda} M_\pm(T_\pm, \bar{\lambda})\big(\bar{x}_\pm(T_\pm) - y_\pm(\bar{\lambda})(T_\pm)\big)\\ &\quad -M_\pm(T_\pm, \bar{\lambda})\tfrac{\partial}{\partial \lambda} y_\pm(\bar{\lambda})(T_\pm). \end{aligned}$$

Defining $\xi_b^\pm = P_b^\pm(T_+)\big(v_\pm(T_\pm) - \frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu\big) \in \mathcal{R}\big(P_b^\pm(T_+)\big)$ and $\xi_a^\pm = P_a^\pm(0)v_\pm(0) \in \mathcal{R}(P_a^\pm(0))$ the unique solution $v_\pm$ of

$$\begin{gathered} L_\pm v_\pm = \big(V_\pm(\cdot)\mu + w_\pm\big), \quad t \in J_\pm,\\ P_a^\pm(0)v_\pm(0) = \xi_a^\pm, \quad P_b^\pm(T_\pm)v_\pm(T_\pm) = \xi_b^\pm + P_b^\pm(T_\pm)\tfrac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu \end{gathered}$$

is (see also [9, Eqs. (51),(54),(55)])

$$\begin{aligned} v_\pm(t) &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) + s_J^\pm(V_\pm(\cdot)\mu)(t)\\ &\quad + S_\pm(t,T_\pm)P_b^\pm(T_\pm)\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu\\ &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) + \frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(t)\mu\\ &\quad - S_\pm(t,0)P_a^\pm(0)\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(0)\mu + s_J^\pm\big(\Psi_\pm(\cdot)\big)(t)\mu \qquad (30)\\ &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(w_\pm)(t) + \frac{\partial}{\partial\lambda} x_\pm(0,\bar{\lambda})(t)\mu\\ &\quad + S_\pm(t,T_\pm)\int_{T_\pm}^{\infty} S_\pm(T_\pm,s)P_b^\pm(s)\Psi_\pm(s)\mu \mathrm{d}s. \qquad (31) \end{aligned}$$

Here $S_\pm$ and $s_J^\pm$ are defined as in (9) and the derivatives of $x_\pm$ are defined by (10)-(12). Therefore $v_\pm$ solves (25), (26). Using the definitions of D and B, and (14), (27) and (31) we obtain

$$\begin{aligned}
&(D \circ B \quad R_0^+ S_+(0,T_+)|_{\mathcal{R}_b^+} \quad R_0^- S_-(0,T_-)|_{\mathcal{R}_b^-}) \begin{pmatrix} (\xi_a^+, \xi_a^-, \mu) \\ \xi_b^+ \\ \xi_b^- \end{pmatrix} \\
&= \; D \circ B(\xi_a^+, \xi_a^-, \mu) + R_0^+ S_+(0,T_+)\xi_b^+ + R_0^- S_-(0,T_-)\xi_b^- \\
&= \; r_0 + R_a^+(T_+) + R_a^-(T_-)
\end{aligned}$$

with

$$\begin{aligned}
R_a^\pm(T_\pm) \;&:=\; -R_0^\pm S_\pm(0,T_\pm) \int_{T_\pm}^\infty S(T_\pm, s) P_b^\pm(s) \Psi_\pm(s) \mu \mathrm{d}s \\
&\quad + R_0^\pm \int_0^{T_\pm} S_\pm(0,s) P_b^\pm(s) w_\pm(s) \mathrm{d}s.
\end{aligned}$$

From $s_J^\pm\big(\Psi_\pm(\cdot)\big)(T_\pm) = P_a^\pm(T_\pm) s_J^\pm\big(\Psi_\pm(\cdot)\big)(T_\pm)$, (11), (28), (29) and (30) we get

$$\begin{aligned}
M_\pm(T_\pm, \bar\lambda)\xi_b^\pm &= r_\pm - \frac{\partial}{\partial\lambda} M_\pm(T_\pm, \bar\lambda)\mu\big(\bar x(T_\pm) - y_\pm(\bar\lambda)(T_\pm)\big) \\
&\quad + M_\pm(T_\pm, \bar\lambda) P_a^\pm(T_\pm)\big(\frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm) - \frac{\partial}{\partial\lambda} x_\pm(0,\bar\lambda)(T_\pm)\big)\mu \\
&\quad - M_\pm(T_\pm, \bar\lambda) S_\pm(T_\pm, 0)\xi_a^\pm - M_\pm(T_\pm, \bar\lambda) s_J^\pm(w_\pm)(T_\pm) \\
&=: \; R_b^\pm(T_\pm).
\end{aligned}$$

Defining $\hat\xi_b^\pm := \xi_b^\pm e^{-\beta_\pm T_\pm}$ and $R_a(T_+, T_-) := R_a^+(T_+) + R_a^-(T_-)$ we get the linear system

$$\begin{pmatrix} D \circ B & R_0^+ S_+(0,T_+)|_{\mathcal{R}_b^+} e^{\beta_+ T_+} & R_0^- S_-(0,T_-)|_{\mathcal{R}_b^-} e^{\beta_- T_-} \\ 0 & M_+(T_+, \bar\lambda)|_{\mathcal{R}_b^+} & 0 \\ 0 & 0 & M_-(T_-, \bar\lambda)|_{\mathcal{R}_b^-} \end{pmatrix} \begin{pmatrix} \xi_a^+, \xi_a^-, \mu \\ \hat\xi_b^+ \\ \hat\xi_b^- \end{pmatrix}
= \begin{pmatrix} r_0 + R_a(T_+, T_-) \\ R_b^+(T_+) e^{-\beta_+ T_+} \\ R_b^-(T_-) e^{-\beta_- T_-} \end{pmatrix}.$$

We estimate $|R_a^\pm(T_\pm)|$ and $|R_b^\pm(T_\pm)|e^{-\beta T_\pm}$ by

$$\begin{aligned}
|R_a^\pm(T_\pm)| \;&\leq\; ||(\tilde w_\pm, 0, 0)||_{\mathcal{Z}} + ||R_0^\pm|| K_\pm c_{\Delta,\beta}^\pm e^{(\gamma_\pm - \beta_\pm)T_+} |\mu|, \\
|R_b^\pm(T_\pm)| e^{-\beta_\pm T_\pm} \;&\leq\; \frac{||(\tilde w_\pm, 0, \tilde r_\pm)||_{\mathcal{Z}} + \frac{\tilde c_\pm}{C_{\xi,\mu}} e^{(\gamma_\pm - \beta_\pm)T_\pm}(|\xi_a^\pm| + |\mu|)}{K_\pm \, M_{N_\pm} \, ||R_0^\pm||}.
\end{aligned}$$

with $\tilde w_+ := (w_+, 0)$, $\tilde r_+ := (r_+, 0)$, $\tilde w_- := (0, w_-)$, $\tilde r_+ := (0, r_-)$, $C_{\xi,\mu} := ||(D \circ B)^{-1}||$ and some constants $c_{\Delta,\beta}^\pm > 0$ and $\tilde c^\pm > 0$.

To estimate $|(\xi_a^+, \xi_a^-, \mu)|$ we notice that μ is involved in both parts of the problem, thus we have to choose $\bar{T}_\pm$ as follows: Let $a > 1$ and $T_\pm \geq \bar{T}_\pm := \frac{\ln(\tilde{c}_\pm) - \ln(1-\frac{1}{a}) + \ln(2)}{\beta_\pm - \gamma_\pm}$, then $\tilde{c}_\pm e^{(\gamma_\pm - \beta_\pm)T_\pm} \leq \frac{1}{2}(1-\frac{1}{a})$ holds. Now we can estimate

$$\begin{aligned}
|(\xi_a^+, \xi_a^-, \mu)| &\leq ||(D \circ B)^{-1}|| \, ||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}} \\
&\quad + \tilde{c}_+(|\xi_a^+| + |\mu|)e^{(\gamma_+ - \beta_+)T_+} + \tilde{c}_-(|\xi_a^-| + |\mu|)e^{(\gamma_- - \beta_-)T_-} \\
&\leq a \cdot C_{\xi,\mu}||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}}
\end{aligned}$$

and

$$\begin{aligned}
|\hat{\xi}_b^\pm| &\leq \frac{1}{K_\pm \, ||R_0^\pm||}||(\tilde{w}_\pm, 0, \tilde{r}_\pm)||_{\mathcal{Z}} \\
&\quad + \frac{1}{C_{\xi,\mu} K_\pm \, ||R_0^\pm||} \cdot \tilde{c}_\pm \, (|\xi_a^\pm| + |\mu|)e^{(\gamma_\pm - \beta_\pm)T_\pm} \\
&\leq \frac{1+a}{2} \frac{1}{K_\pm \, ||R_0^\pm||} \, ||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}} \\
&\leq \frac{a}{K_\pm \, ||R_0^\pm||}||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}}.
\end{aligned}$$

The estimate of $||v_\pm(t)||$ is

$$|v_\pm(t)| \leq \frac{1 + e^{\beta_\pm t}}{2} C_v^\pm \, a \, ||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}}$$

where $C_v^\pm > 0$ is some constant (for details see [8]). Therefore we get $||v_\pm||_{\mathcal{Y}}^\pm = \sup_{t \in J_\pm} \left(||v_\pm(t)|| q_\pm(t)\right) \leq a C_v^\pm ||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}}$ and $||(v_+, v_-, \mu)||_{\mathcal{Y}} \leq C_{lin}||(w_+, w_-, r_0, r_+, r_-)||_{\mathcal{Z}}$ with the constant $C_{lin} = a \, (C_{\xi,\mu} + C_v^+ + C_v^-)$.

S2 The equation $H_J'(\bar{x}|_J, \bar{\lambda})(v_+, v_-, \mu) = (w_+, w_-, r_0, r_+, r_-)$ is by assumption **A4** a linear boundary value problem of dimension $m_a + p + m_b = m + p$ for which the Fredholm alternative holds. Thus $\sigma := \frac{1}{C_{lin}} \leq ||H_J'(\bar{x}|_J, \bar{\lambda})^{-1}||^{-1}$.

S3 For any $(z, \nu), (v, \mu) \in \mathcal{Y}$ we can estimate as in [9]

$$\begin{aligned}
&\left\| H_J'(z, \nu) - H_J'(\bar{x}|_J, \bar{\lambda}) \begin{pmatrix} v \\ \mu \end{pmatrix} \right\|_{\mathcal{Z}} \\
&\quad \leq C_{Lip}||(z - \bar{x}|_J, \nu - \bar{\lambda})||_{\mathcal{Y}} \cdot ||(v, \mu)||_{\mathcal{Y}} \cdot r(\beta, T)
\end{aligned}$$

with some constant $C_{Lip} > 0$ (for details see [8]). Therefore with $\kappa := \frac{\sigma}{2}$ and $\delta := \frac{\sigma}{2C_{Lip}}$ we see for any $(z, \nu) \in \mathcal{B}_{\frac{\delta}{r(\beta,T)}}(\bar{x}|_J, \bar{\lambda})$

$$||H_J'(z, \nu) - H_J'(\bar{x}|_J, \bar{\lambda})||_{\mathcal{YZ}} \leq \frac{\sigma}{2} = \kappa < \sigma \leq ||H_J'(x|_J, \bar{\lambda})^{-1}||_{\mathcal{YZ}}^{-1}.$$

S4 On the other hand the truncation error is

$$||H_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Z}} = \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ M_+(T_+)\big(\bar{\lambda}\big)(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)\big) \\ M_-(T_-)\big(\bar{\lambda}\big)(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)\big) \end{pmatrix} \right\|_{\mathcal{Z}}$$

$$\begin{aligned} &= C^+_{\mathcal{Z}}\big|M_+(T_+,\bar{\lambda})\big(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)\big)\big|e^{-\beta_+T_+} \\ &\quad +C^-_{\mathcal{Z}}\big|M_-(T_-,\bar{\lambda})\big(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)\big)\big|e^{-\beta_-T_-} \\ &\leq C^+_{\mathcal{Z}}\, \tilde{C}^+_{tr}||\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)||^{d_+}e^{-\beta_+T_+} \\ &\quad +C^-_{\mathcal{Z}}\, \tilde{C}^-_{tr}||\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)||^{d_-}\, e^{-\beta_-T_-} \\ &\leq \tilde{C}^+e^{(d_+\gamma_+-\beta_+)T_+} + \tilde{C}^-e^{(d_-\gamma_--\beta_-)T_-} \\ &\leq \bar{C}(e^{(d_+\gamma_+-\beta_+)T_+} + e^{(d_-\gamma_--\beta_-)T_-}), \end{aligned}$$

where $\tilde{C}^{\pm}_{tr} := \begin{cases} M_{M_\pm} & : d_\pm = 1, \\ C^{\pm}_{\bar{M}} & : d_\pm = 2 \qquad (C^{\pm}_{\bar{M}} \text{ s. t. (24) holds}), \end{cases}$
$\tilde{C}^{\pm} := C^{\pm}_{\mathcal{Z}}\tilde{C}^{\pm}_{tr}C^{d_\pm}_{\bar{x}}$ and $\bar{C} = \max(\tilde{C}^+,\tilde{C}^-)$. Lemma 2 implies that

$$\bar{C}\big(e^{(d_+\gamma_+-\beta_+)T_+} + e^{(d_-\gamma_--\beta_-)T_-}\big) \leq \frac{\delta}{r(\beta,T)}\frac{\sigma}{2}$$

is satisfied for all $(T_+,T_-) \in \mathcal{D}(\epsilon,\beta_\pm,d_\pm\gamma_\pm-\beta_\pm)\cap\mathcal{D}_+\times\mathcal{D}_-$, $T_\pm \geq \bar{T}_\pm$, where $\epsilon = \frac{\sigma\delta}{2\bar{C}}$.

S5 Lemma 3 implies that there exists a unique solution (z_J,ν_J) in $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J,\bar{\lambda})$ with

$$||(z_J,\nu_J) - (\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}} \leq C^+e^{(d_+\gamma_+-\beta_+)T_+} + C^-e^{(d_-\gamma_--\beta_-)T_-}$$

where $C^{\pm} = 2C_{lin}\tilde{C}^{\pm}$. Moreover, we see

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\leq C^+\, e^{\Delta_+T_+} + C^-\, e^{\Delta_-T_-}, \\ ||z^{\pm}_J(t) - \bar{x}_\pm|_{J_\pm}(t)|| &\leq \big(C^+e^{\Delta_+T_+} + C^-e^{\Delta_-T_-}\big)\cdot\frac{1+e^{\beta_\pm t}}{2}, \end{aligned}$$

and hence the estimates (22) and (23) hold. ■

For $\beta_\pm \leq 0$ the following estimate holds

$$||z^{\pm}_J - \bar{x}_\pm|_{J_\pm}||_\infty \leq C^+e^{(d_+\gamma_+-\beta_+)T_+} + C^-e^{(d_-\gamma_--\beta_-)T_-}.$$

We can apply Theorem 1 with $\hat{\beta}_\pm \in (\frac{\gamma_\pm}{2}, \beta\pm]$. Hence, if $\beta_+ > 0$ and $\beta_- < 0$, then we can apply Theorem 1 with $\beta_+ = 0$. The result is an approximation which satisfies the following estimates:

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\le C^+ \, e^{d_+\gamma_+T_+} + C^- \, e^{(d_-\gamma_- - \beta_-)T_-}, \\ ||z_J^\pm - \bar{x}_\pm|_{J_\pm}||_\infty &\le C^+ e^{d_+\gamma_+T_+} + C^- e^{(d_-\gamma_- - \beta_-)T_-}. \end{aligned}$$

We get analogous results if $\beta_+ < 0$ and $\beta_- > 0$ or if both are positive. To get a super-convergence in the parameter as in [1], [2], [11] for $\beta_\pm > 0$ we restrict to those pairs (T_+, T_-) satisfying $\beta_+T_+ = \beta_-T_-$. For these pairs we see

$$\begin{aligned} |\nu_J - \bar{\lambda}| &\le C^+ \, e^{(d_+\gamma_+ - \beta_+)T_+} + C^- \, e^{(d_-\gamma_- - \beta_-)T_-}, \\ ||z_J^\pm - \bar{x}_\pm|_{J_\pm}||_\infty &\le C^+ e^{d_+\gamma_+T_+} + C^- e^{d_-\gamma_-T_-}. \end{aligned}$$

4.2 The boundary corrector method

As in [9] we set up a boundary corrector method for generalized connecting orbits and assume

A7 There exists a neighborhood $\mathbf{\Lambda}_0$ of $\bar{\lambda}$ and functions $\bar{M}_\pm(T_\pm, \cdot) \in C^1(\mathbf{\Lambda}_0, \mathbb{R}^{m_b^\pm, m})$, such that **A6** and (24) hold for both, "+" and "−" and each $\lambda \in \mathbf{\Lambda}_0$.

The **Boundary Corrector Method** for generalized connecting orbits is defined as follows:

1. Start with some $\mu_0 \in \mathbf{\Lambda}_0$, $i = 0$
2. Calculate $(z_{i+1}, \mu_{i+1}) = (z_{i+1}^+, z_{i+1}^-, \mu_{i+1})$ as the solution of

$$H_J^{\mu_i}(z_+, z_-, \nu) = \begin{pmatrix} \dot{z}_+ - f_+(z_+, \nu) \\ \dot{z}_- - f_-(z_-, \nu) \\ g\big(z_+(0), z_-(0), \nu\big) \\ M^+(T_+, \mu_i)(z_+(T_+) - y_+(\nu)(T_+)) \\ M^-(T_-, \mu_i)(z_-(T_-) - y_-(\nu)(T_-)) \end{pmatrix} = 0$$

3. Repeat with **2.** ("$i = i + 1$") once if $\beta_\pm \ge 0$ or twice if $\beta_+ < 0$ or $\beta_- < 0$.

The Proposition 1 states that the error of the solution of the boundary corrector method has the same exponential rate $(2\gamma_\pm - \beta_\pm)$ as the solution of Theorem 1 with $d_\pm = 2$.

PROPOSITON 1 *Suppose that the assumptions of Theorem 1 and* **A7** *hold. Moreover, let $\bar{T}_\pm$ be sufficiently large.*

Then there exists some $C_{bcm} > 0$, such that for all $T_\pm \ge \bar{T}_\pm$ with $(T_+, T_-) \in \mathcal{D}(\epsilon, \beta_\pm, d_\pm\gamma_\pm - \beta_\pm)$ for both, $d_\pm = 1$ and $d_\pm = 2$, the *result* (z_J, ν_J) *of the* **boundary corrector method** *can be estimated by*

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq C_{bcm}\big(e^{(2\gamma_+ - \beta_+)T_+} + e^{(2\gamma_- - \beta_-)T_-}\big).$$

PROOF The proof is analogous to that of [9, Prop. 4], noticing that $e^{(\gamma_+ - \beta_+)T_+} \cdot e^{(\gamma_- - \beta_-)T_-} \leq \max(e^{2(\gamma_+ - \beta_+)T_+}, e^{2(\gamma_- - \beta_-)T_-})$.

5 The Lorenz system

In this section we apply our theoretical results to the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - bx_3. \end{aligned}$$

We detect the point to periodic connecting orbit plotted in Fig. 2 and continue this connecting orbit by varying the parameter σ. To

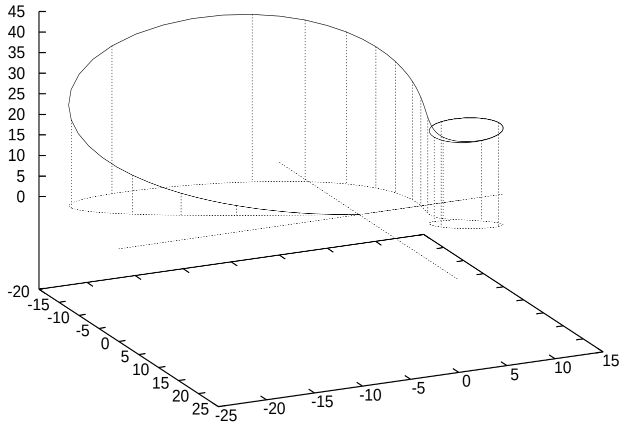

Figure 2: Approximation of a point to periodic connecting orbit with parameters $\sigma = 10$ and $b = \frac{8}{3}$

get an initial approximation of a point to periodic connecting orbit we apply the strategy in [9]. We approximate the "first part" (a solution in the unstable manifold of 0 which intersects the hyperplane

$\{(0, x_2, x_3) | x_2, x_3 \in \mathbb{R}\}$ at $t = 0$) for different parameter values r and define an approximation of the intersection points which is linear in r. For the "second part" (a solution in the stable manifold of the periodic orbit) we compute a solution which has its initial value on the linearization (mentioned above) and which is asymptotic to the periodic orbit. As result we get an approximation for the unknown parameter $r \approx 24.05803$, the unknown period $\bar{T} \approx 0.677167$ and initial solutions for the "first part" of the connecting orbit $z(t) \in \mathbb{R}^3$, the "second part" $x(t) \in \mathbb{R}^3$ and the periodic orbit $y(t) \in \mathbb{R}^3$. To

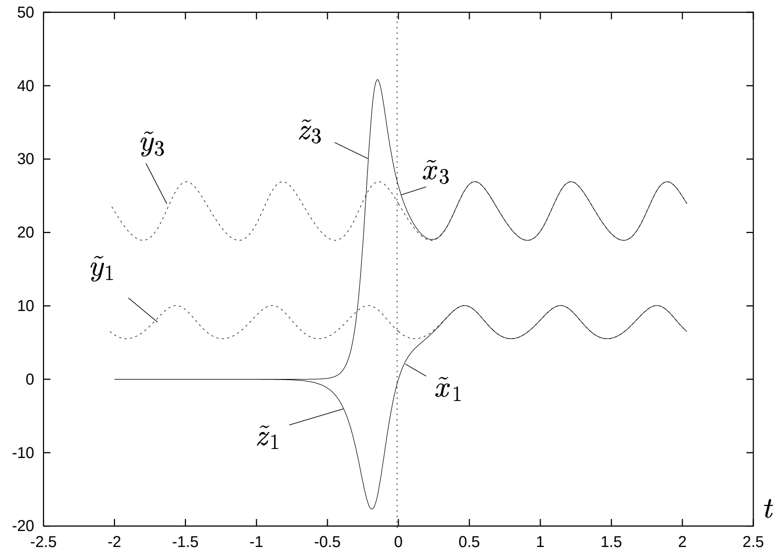

Figure 3: The first and third component of the connecting orbit (—) and the periodic orbit (- -) in the rescaled version

apply the results of this paper we solve the (forthcoming) boundary value problem (32). We truncate the scaled solution on the interval $[0, 3]$, such that we compute three periods of the periodic orbit. The periodic orbit is fixed at $t = 2$ and $t = 3$ by $y(2) = y(3)$ and the phase fixing function $\chi(y(2), \phi) = y_1(2) - \phi$, where the value $\phi = 6.5043$ as well as the matrices $V_+, V_- \in \mathbb{R}^{2,3}$ for the asymptotic boundary conditions are also defined by the initial approximation.

The boundary value solver restricts us to compute on the interval $[0, 3]$ even for the first part z of the connecting orbit. Thus we also scale the first part by a constant $\tau = \frac{2}{3}$, which is chosen to get roughly similar exponents $\alpha_+ \approx -7.845$ and $\alpha_- \approx -6.76$ for the scaled system. At $r = 24.0$ the "unstable eigenvalue" of 0 is 10.1365 and the "stable Floquet multiplier" of the periodic orbit is $0.93 \cdot 10^{-5}$. This yields for the systems scaled by $\bar{T} \approx 0.677167$ and $\tau = \frac{2}{3}$ the exponents $\alpha_\pm$ mentioned above. Therefore we solve

$$\begin{pmatrix} \dot{x} - \bar{T} f(x, r; \sigma, b) \\ \dot{y} - \bar{T} f(y, r; \sigma, b) \\ \dot{z} + \tau f(z, r; \sigma, b) \\ x(0) - z(0) \\ \chi\big(y(2), \phi\big) \\ y(2) - y(3) \\ V_+\big(x(3) - y(3)\big) \\ V_- z(3) \end{pmatrix} = 0. \tag{32}$$

The phase portrait is plotted in Fig. 2 and in Fig. 3 we plot the first and third component of the connecting orbit (—) and the periodic orbit (- -) in the rescaled version, i. e. $\tilde{x} = x(\cdot/\bar{T})$, $\tilde{z} = z(-\cdot/\tau)$ and $\tilde{y} = y(\cdot/\bar{T})$ for $t \geq 0$ and $\tilde{y} = y\big(\cdot/\bar{T} + 3\big)$ for $t < 0$.

Parameter continuation with respect to σ yields a branch of point to periodic connecting orbits. In Fig. 4 we plot the pairs of parameters (r, σ) corresponding to these point to periodic connecting orbits.

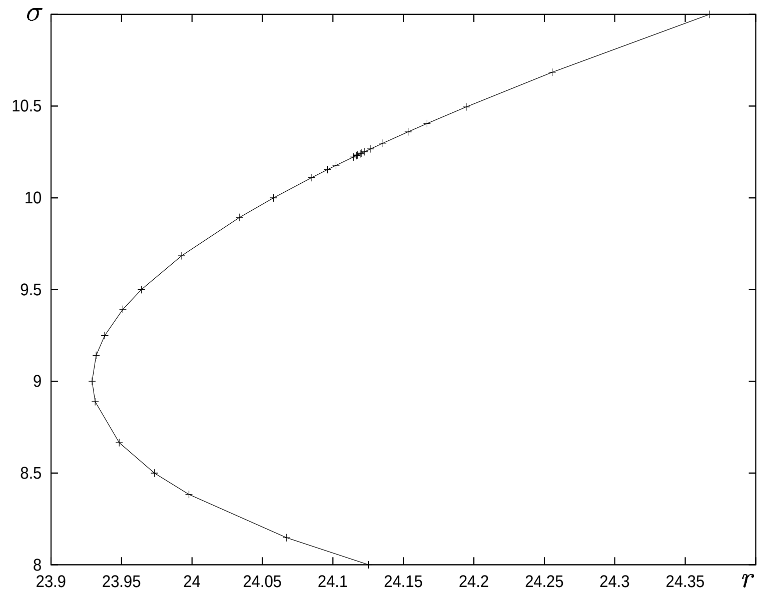

Figure 4: Parameterspace r–σ. Branch of point to periodic connecting orbits

6 Concluding remarks on the theory

We have developed an approximation theory for a general type of connecting orbits, which includes most of the common connecting orbits. However, there is an additional kind of connecting orbit not covered by our theory. These are solutions which converge, but not with an exponential rate (in our notion this means $\alpha = 0$). The case of a homoclinic connecting orbit of a semi-hyperbolic equilibrium is analyzed in [11], [12]. Continuation of the stable manifold of an equilibrium to a Hopf-bifurcation yields that α has to tend to 0 and hence the approximation interval must be enlarged and the error estimates become worse. Therefore we cannot apply the theory to approximate a Hopf-Shilnikov bifurcation.

It seems straightforward to transfer the results of this paper from the case with two subproblems to the case with a finite number of subproblems.

Acknowledgements This research was supported by DFG-Schwerpunktprogramm “Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme” and by DFG under contract No. BO 635/9-1,3. I am very grateful to Prof. W.-J. Beyn for helpful comments and exciting discussions.

References

[1] W.-J. Beyn. Global bifurcations and their numerical computation. In: D. Roose, A. Spence, B. DeDier (eds.), Continuation and Bifurcations: Numerical Techniques and Applications, pages 169–181. Dordrecht, Kluwer, 1990

[2] W.-J. Beyn. The numerical computation of connecting orbits in dynamical systems. IMA J. Numer. Anal., **10**, 379–405, 10 (1990)

[3] W.-J. Beyn. On well–posed problems for connecting orbits in dynamical systems. In: P. Kloeden, K. Palmer (eds.), Proceedings of ‘Chaotic Numerics’, volume 172, pages 131–168. Contemporary Mathematics, 1994

[4] W.-J. Beyn, T. Pampel, W. Semmler. Dynamic optimization and Skiba sets in economic examples. Discussion paper 469, University of Bielefeld, Department of Economics, 2001. Accepted by Optimal Control Applications and Methods

[5] A. R. Champneys, G. J. Lord. Computation of homoclinic solutions to periodic orbits in a reduced water–wave problem. Physica D, **102**, 101–124, 102 (1997)

[6] E. J. Doedel, M. J. Friedman, B. I. Kunin. Successive continuation for locating connecting orbits. Numer. Algorithms, **14**, 103–124, 14 (1997)

[7] J. K. Hale, X.-B. Lin. Heteroclinic orbits for retarded functional-differential equations. J. Differential Equations, **65**, 175–202, 65 (1986)

[8] T. Pampel. Numerical approximation of generalized connecting orbits. PhD-Thesis, Department of Mathematics, University of Bielefeld, 1999. Available via http://archiv.ub.uni-bielefeld.de/disshabi/mathe.htm

[9] T. Pampel. Numerical approximation of connecting orbits with asymptotic rate. Numerische Mathematik, (2001), forthcoming, the original publication is available on LINK at http://dx.doi.org/10.1007/s002110100302

[10] B. Sandstede. Verzweigungstheorie homokliner Verdopplungen. PhD-Thesis, University of Stuttgart, 1993

[11] B. Sandstede. Convergence estimates for the numerical approximation of homoclinic solutions. IMA J. Numer. Anal., **17**, 437–462, 17 (1997)

[12] S. Schecter. Numerical computation of saddle-node homoclinic bifurcation points. SIAM J. Numer. Anal., **30**, 1155–1178, 30 (1993)

[13] G. Vainikko. Funktionalanalysis der Diskretisierungsmethoden. Teubner Texte zur Mathematik, 1976