Cephoids: Minkowski Sums of Prisms

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Abstract

We discuss the structure those polytopes in $\mathbb{R}^n_+$ that are Minkowski sums of prisms. A prism is the convex hull of the origin and n positive multiples of the unit vectors. We characterize the defining outer surface of such polytopes by describing the shape of all maximal faces. As this shape resembles the view of a cephalopod, the polytope obtained is called a "cephoid". The general geometrical and combinatorial aspects of cephoids are exhibited.

1 Sums of Prisms

We consider the class of polytopes in $\mathbb{R}^n_+$ which are obtained as Minkowski sums of prisms. It is our aim to exhibit the structure of the surface of these polyhedra and to present a combinatorial classification. We also suggest a recursive enumeration and description of the maximal faces.

A well known class of polyhedra with a simple prime decomposition are given by the zonoids (see [8]). The polytopes generating a zonoid are line segments. The fact that each line segment generates a beltshaped area on the surface refers to the terminology.

In the present paper the generating polyhedra (the "prisms") generate a more complicated structure. There appears a shape which resembles an octopus or a squid or, more generally, a cephalopod. Therefore we call the polytopes of our family "cephoids". Recall that a cepheid is type of variable star, the first specimen observed in the Cepheus configuration.

We describe cephoids formally as follows.

Let us denote by $\boldsymbol{I} := \{1, \ldots, n\}$ the set of coordinates of $\mathbb{R}^n$ and by $\boldsymbol{e}^i$ the i^{th} unit vector of $\mathbb{R}^n$ $(i \in \boldsymbol{I})$. Also write $\boldsymbol{e} = (1, \ldots, 1)$. Let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0 \in \mathbb{R}^n_+$. Put $\boldsymbol{a}^i := a_i \boldsymbol{e}^i$ $(i \in \boldsymbol{I})$ and associate with $\boldsymbol{a}$ the ***prism*** $\Pi^{\boldsymbol{a}}$ which is given by

$$\Pi^{\boldsymbol{a}} := \operatorname{conv}\left\{\mathbf{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}. \tag{1.1}$$

The (outward) face of this prism is the ***simplex*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} := \operatorname{conv}\left\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}. \tag{1.2}$$

For any $\boldsymbol{J} \subseteq \boldsymbol{I}$ we obtain the ***subprism*** of $\Pi^{\boldsymbol{a}}$ given by

$$\Pi^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Pi^{\boldsymbol{a}} \mid x_i = 0 \ (i \notin \boldsymbol{J})\}, \tag{1.3}$$

a similar notation is used for the simplex, $\Delta^{\boldsymbol{a}}$ we write for the subface generated by the coordinates $i \in \boldsymbol{J}$

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \mid x_i = 0 \ (i \notin \boldsymbol{J})\}. \tag{1.4}$$

Now we consider the Minkowski sum of prisms.

Definition 1.1. *Let* $\boldsymbol{a}^{\bullet} := (\boldsymbol{a}^{(k)})_{k=1}^K$ *denote a family of positive vectors and let*

$$\Pi = \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} \tag{1.5}$$

be the (algebraic) sum. Then Π *is called a* ***cephoid****.*

We start out with a few examples in order to become familiar with cephoidal polyhedra. First of all, we note that the representation of a cephoid by a family of prisms is, in general, by no means unique.

Example 1.2. A prism may be represented as a cephoid in various ways. E.g., let $\Pi = \Pi^{\boldsymbol{e}}$ be the unit prism and let Π be represented as the sum $\Pi = \Pi^{\alpha\boldsymbol{e}} + \Pi^{\beta\boldsymbol{e}}$ with $\alpha, \beta \geq 0, \quad \alpha + \beta = 1$. The outer surface, i.e., the unit simplex $\Delta = \Delta^{\boldsymbol{e}} = \Delta^{\alpha\boldsymbol{e}} + \Delta^{\beta\boldsymbol{e}}$ is the union of the two translates $\alpha\boldsymbol{e}^1 + \Delta^{\beta\boldsymbol{e}}$, $\beta\boldsymbol{e}^2 + \Delta^{\alpha\boldsymbol{e}}$ and a "diamond" $\Delta_{13}^{\alpha\boldsymbol{e}} + \Delta_{23}^{\beta\boldsymbol{e}}$. (cf. Figure 1.1)

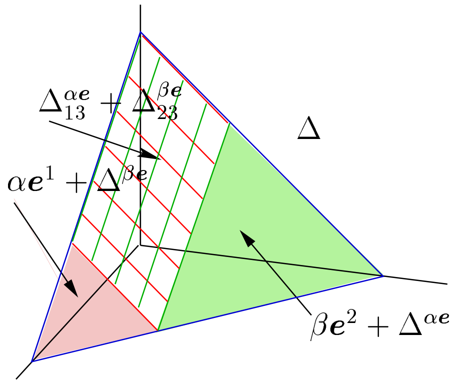

Figure 1.1: The unit simplex is a cephoid

However, the representation is not unique as indicated by Figure 1.2.

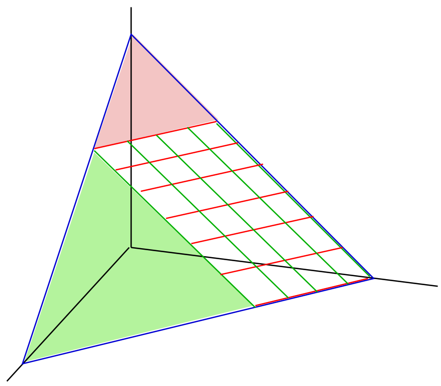

Figure 1.2: Another representation of $\Delta^{\boldsymbol{e}}$

As all prisms involved are homothetic, it turns out that the vector used to translate a prism is rather arbitrary.

A slight generalization of the above consideration shows that the unit prism (or for that matter, any prism) may be represented as an arbitrary sum of homothetic prisms of smaller size. E.g., Figure 1.3 shows $\Delta^{\boldsymbol{e}}$ decomposed into four homothetic simplices plus diamonds, this is the result of summing up four homothetic prisms.

In (Figure 1.4) the unit prism is the sum of seven homothetic prisms each one $\frac{1}{7}$ of the size of the original one. The "diamonds" are sums of one–

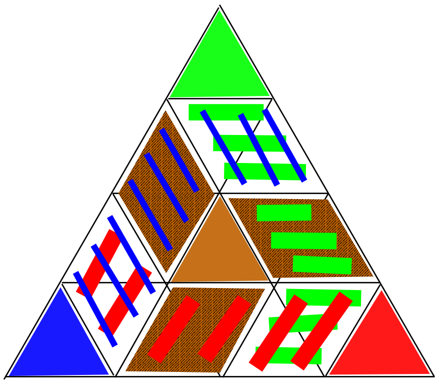

Figure 1.3: $\Delta^{\boldsymbol{e}}$ as the sum of four prisms

dimensional subsimplices and the central prism generates a diamond with each of the other ones. Here the "cephoidal" structure is clearly recognized.

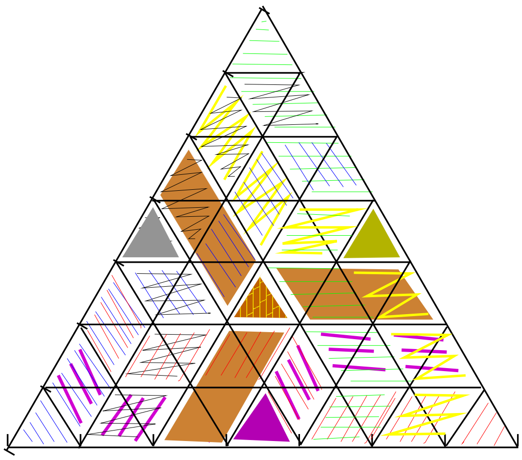

Figure 1.4: The unit prism as a sum of 7

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The next example uses two nonhomothetic prisms.

Example 1.3. Again there are the translates of the two prisms involved, i.e. $\Delta^{\boldsymbol{a}} + \boldsymbol{b}^1$ and $\Delta^{\boldsymbol{b}} + \boldsymbol{a}^1$. The "diamond" is the sum $\Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$. The situation is depicted in Figure 1.3.

There is a similarity in the surface structure exhibited between Figure 1.3 and Figure 1.3 (the planar version resulting from the decomposition of the unit simplex). Again the representation of the surface is not unique. E.g., each of the two prisms might be decomposed into a sum of two homothetic smaller prisms. But the representation indicated in Figure 1.3 *is unique* if one requires in addition a minimal sum of summands.

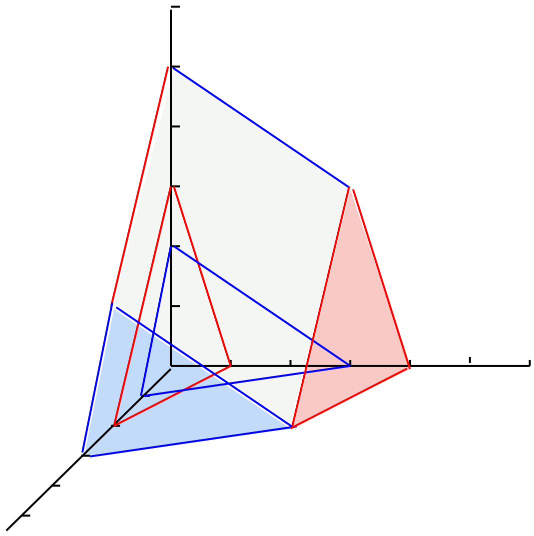

Figure 1.5: Adding two prisms

This latter property vanishes again – even for nonhomothetic simplices – if two of the subsimplices of the surface are homothetic (Figure 1.6).

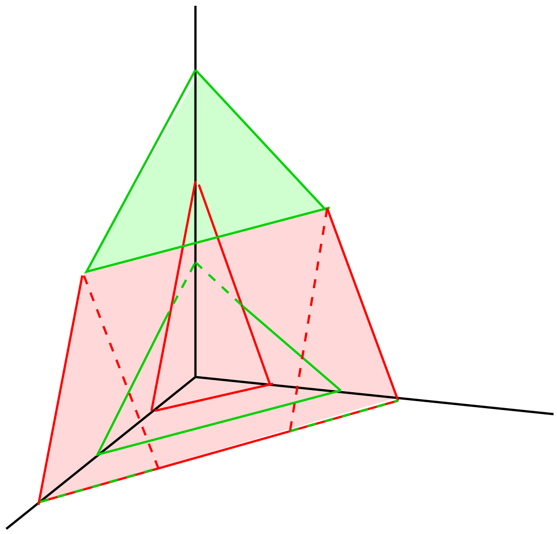

Figure 1.6: The sum of two prisms – parallel subfaces

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Now we consider the sum of three prisms.

Example 1.4. In the example represented by Figure 1.7, the slopes in each $x_i x_j$-plane are ordered in a cyclic way.

The sum shows a translate of each simplex located in the appropriate corner. Each translate generates exactly one nontrivial "arm" consisting of two diamonds. Every diamond is the sum of two two subfaces of the simplices involved. There appears a new central extremal point which is the sum of three vertices of the simplices involved.

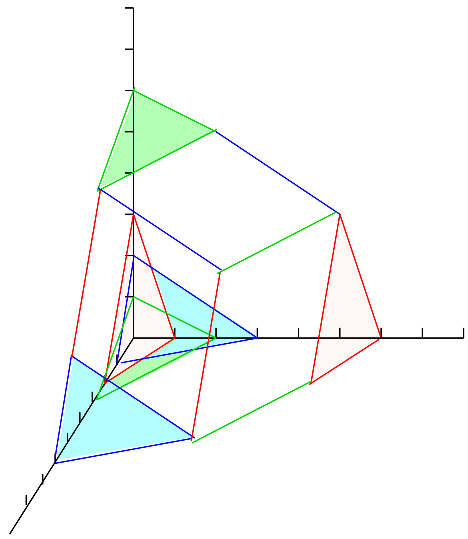

Figure 1.7: The sum of three prisms

Now we shave off the central vertex and replace it by a further simplex. This we achieve by adding a further prism having a joint normal with the central vertex. The result is a polyhedron as indicated in Figure 1.8. It shows certain symmetries, the new triangle having replaced the central vertex.

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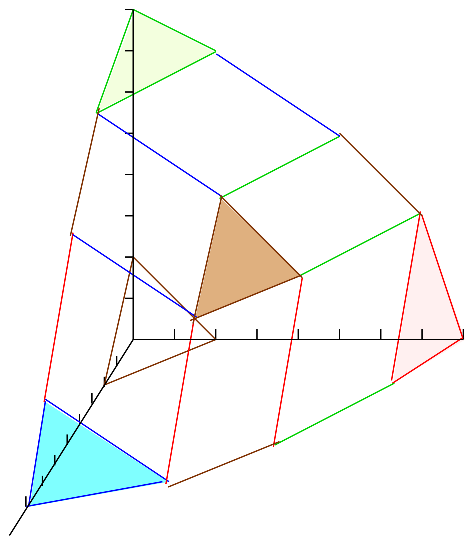

Figure 1.8: Adding a further prism

Observe the cephoidal structure of this example. The central simplex generates a "tentacle" in each direction while those located at the boundary generate just one. If one compares this with Figure 1.3, then clearly the tentacle structure is the same.

The planar case is in some way "degenerate". However it serves to represent the surface structure of a cephoid.

The appropriate version of a nondegenerate family is captured by the following definition.

Definition 1.5. *Let* $\boldsymbol{a}^{\bullet} = (\boldsymbol{a}^{(k)})_{k=1}^{K}$ *denote a family of positive vectors and let*

$$\left(\Pi^{\boldsymbol{a}^{(k)}}\right)_{k=1}^{K} \quad =: \quad \left(\Pi^{(k)}\right)_{k=1}^{K} \tag{1.6}$$

be the corresponding family of prisms. We shall say that the family (of vectors or prisms) is ***nondegenerate*** *or* ***in general position*** *if the following conditions hold true:*

1. *For any system of nonempty index sets* $\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)} \subseteq \boldsymbol{I}$ *with*

$$\bigcup_{k \in \boldsymbol{K}} \boldsymbol{J}^{(k)} = \boldsymbol{I}$$

   the system of linear homogeneous equations in the variables $x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_K$ *given by*

$$a_i^{(k)} x_i - \lambda_k = 0 \quad (i \in \boldsymbol{J}^{(k)},\ k \in \boldsymbol{K}) \tag{1.7}$$

   has a space of solutions $\mathbb{U}$ *of dimension*

$$\dim \mathbb{U} = n + K - \sum_{k \in \boldsymbol{K}} j_k \tag{1.8}$$

   with $j_k = |\boldsymbol{J}^{(k)}|$.

2. *For any* $\boldsymbol{I}^{(0)} \subseteq \boldsymbol{I}$ *the restricted system*

$$\boldsymbol{a}^{\bullet}\Big|_{\boldsymbol{I}^{(0)}} := \left(\boldsymbol{a}^{(k)}\Big|_{\boldsymbol{I}^{0}}\right)_{k \in \boldsymbol{K}} \tag{1.9}$$

   obtained by restricting the vectors to $\boldsymbol{I}^{(0)}$ *satisfies the condition of* item *1 in the subspace* $\mathbb{R}^{\boldsymbol{I}^{(0)}}$.

3. *The term nondegenerate is also applied to the cephoid generated by a nondenerate family* $\boldsymbol{a}^{\bullet}$.

Theorem 1.6. *A nondegenerate cephoid is uniquely represented as a sum of nonhomothetic prisms.*

The proof follows from [7] Theorem 3.2.8.

The aim of this paper is to analyze the structure of cephoids generated by nondegenerate families of prisms. As we shall see, the general structure of a cephoidal surface is at best represented on (a multiple of) the unit simplex: there is a "canonical" mapping between the two surfaces preserving the partially ordered set of faces. (E.g. Figure 1.3 is the "canonical representation" of Figure 1.8) This geometric structure is accompanied by a combinatorial structure we shall exhibit. Eventually, it is seen that in higher dimensions there is an abundance of types of cephoids. The surfaces exhibit an ever increasing family of polyedra. Yet, the geometric structure is characterized by its combinatorial counterpart: a set of orderings or permutations which is genuine to a cephoid.

2 The Canonical Representation

Recall the similar structure exhibited in Figures 1.3 and 1.7. There is a mapping of the surface structure of a cephoid on a suitable multiple of the unit simplex such that both structures are "combinatorically equivalent", i.e., the posets (partially ordered sets) of subfaces are isomorphic (see [1]).

In order to simplify the notation, we use $\boldsymbol{K} := \{1, \ldots, K\}$ for the index set of a family of prisms. We consider a family $(\boldsymbol{a}^{(k)})_{k \in \boldsymbol{K}}$ of vectors in general position; the prism $\Pi := \sum_{k \in \boldsymbol{K}} \Pi^{(k)}$ and its surface $\Delta := \sum_{k \in \boldsymbol{K}} \Delta^{(k)}$ are defined as previously.

We take K copies of $\boldsymbol{e}$ which we denote by $\boldsymbol{a}^{0(1)}, \ldots, \boldsymbol{a}^{0(K)}$. As in SECTION 1 we write $\boldsymbol{a}^{0(k)i} := a_i^{0(k)} \boldsymbol{e}^i$, where $a_i^{0(k)}$ denotes the $i'th$ coordinate of $\boldsymbol{a}^{0(k)}$.

For every $k \in \boldsymbol{K}$ let $\Pi^{0(k)} := \Pi^e$ and $\Delta^{0(k)} := \Delta^e$ be a copy of the unit prism and simplex respectively. The (homothetic) sums generated are denoted by

$$\Pi^0 := \sum_{k \in \boldsymbol{K}} \Pi^{0(k)} = \Pi^{K\boldsymbol{e}} = K\Pi^{\boldsymbol{e}}$$

and

$$\Delta^0 := \sum_{k \in \boldsymbol{K}} \Delta^{0(k)} = \Delta^{K\boldsymbol{e}} = K\Delta^{\boldsymbol{e}}$$

respectively. As all prisms involved are homothetic, the simplex Δ^0 has the (trivial) face poset of the unit simplex. However, we will indicate a simple way to generate a copy of the face poset of Δ on Δ^0.

A direct way to generate a "grid" on the surface Δ is provided by the integer vectors

$$\boldsymbol{k} = (k_1, \ldots, k_n), \quad k_i \in \mathbb{N}_0 \ (i \in \boldsymbol{I}), \sum_{i \in \boldsymbol{I}} k_i = K \ . \tag{2.1}$$

All these vectors are a sum of vertices in various ways, i.e., we have

$$\boldsymbol{k} = \sum_{k \in \boldsymbol{K}_1} \boldsymbol{a}^{0(k)1} + \ldots + \sum_{k \in \boldsymbol{K}_n} \boldsymbol{a}^{0(k)n} \tag{2.2}$$

with arbitrary pairwise disjoint sets $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n$ the union of which is $\boldsymbol{K}$. With the vertices of Δ this is different: by nondegeneracy every vertex is a unique sum of certain vertices of the $\Delta^{\boldsymbol{a}^{(k)}}$ involved. More precisely, for every vertex $\boldsymbol{u}$ of Δ, there is a unique mapping $\mathbf{i}^\bullet$ such that $\boldsymbol{u}$ can be written via

$$\begin{aligned} \mathbf{i}_\bullet : \quad & \boldsymbol{K} \to \boldsymbol{I} \\ \boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_\bullet} := \quad & \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \end{aligned} \tag{2.3}$$

Now we have

Definition 2.1. *1. Let* $\boldsymbol{u}$ *be a vertex on* Δ *and let* $\mathbf{i}_\bullet$ *be the corresponding mapping as described by* (2.3)*. Then*

$$\boldsymbol{u}^0 \ := \ \kappa(\boldsymbol{u}) \ := \ \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{0(k)\mathbf{i}_k} \tag{2.4}$$

is the ***canonical representation*** *of* $\boldsymbol{u}$ *on* Δ^0.

2. Let $\boldsymbol{F}$ *be a face of* Δ *and let* $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^L$ *be its extremal points. Then the convex hull of the images, i.e.,*

$$\kappa(\boldsymbol{F}) \ := \ \boldsymbol{F}^0 \ := \ \operatorname{conv}\{\kappa(\boldsymbol{u}^1), \ldots, \kappa(\boldsymbol{u}^L)\}, \tag{2.5}$$

is the ***canonical representation*** *of* $\boldsymbol{F}$ *on* Δ.

3. Let $\mathcal{V}$ *be the poset of faces of* Δ *and let*

$$\mathcal{V}^0 \ := \ \kappa(\mathcal{V}) \ := \ \{\kappa(\boldsymbol{F}) \,|\, \boldsymbol{F} \in \mathcal{V}\} \tag{2.6}$$

be the collection of images of faces under the mapping κ*. Then* $\mathcal{V}^0$ *is the* ***canonical representation*** *of* $\mathcal{V}$ *on* Δ.

Theorem 2.2. $\mathcal{V}^0$ *is a poset which is isomorphic to* $\mathcal{V}$*. Hence* $(\Delta, \mathcal{V})$ *and* $(\Delta^0, \mathcal{V}^0)$ *are combinatorically equivalent.*

Proof:

This is a standard procedure in convex geometry (see [4]). The mapping κ is bijective between the vertices of Δ and the appropriate subset of grid vectors as described in equations (2.1) and (2.2). The minimum of two faces (whenever it exists) is obtained by taking the intersection of the corresponding two sets of extremal points. Similarly, if the maximum of two faces exists, then it is obtained via the union of the sets of extremal points. Each representation of a vertex is one one hand a vector $\boldsymbol{k}$ as described in (2.1). On the other hand, given the natural ordering on $\boldsymbol{K} = \{1, \ldots, K\}$, it is described or "labelled" via some function $\mathbf{i}_\bullet$ by $(\mathbf{i}_1, \ldots, \mathbf{i}_K)$.

q.e.d.

The canonical representation is the suitable projection of the outer surface Δ of a cephoid Π on an $(n-1)$–dimensional subset. E.g., the poset of faces of Figures 1.3 and 1.7 are combinatorically equivalent. Also, we can visualize the surface of 4–dimensional cephoids on a suitable multiple of the unit simplex of $\mathbb{R}^3$ (a tetrahedron), which will serve to discuss several important examples in Section 6.

3 Faces and Normals

Let $(\boldsymbol{a}^{(k)})_{k\in\boldsymbol{K}}$ denote a family of positive vectors in general position and let

$$\Pi \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{(k)}, \quad \Delta \;=\; \sum_{k\in\boldsymbol{K}} \Delta^{\boldsymbol{a}^{(k)}} \;=\; \sum_{k\in\boldsymbol{K}} \Delta^{(k)}.$$

denote the sum (the cephoid generated) and its outer surface. If $\boldsymbol{J} \subseteq \boldsymbol{I}$ is a subset of $\boldsymbol{J}$, then we indicate the intersection of a set $\boldsymbol{F}$ with $\mathbb{R}_{\boldsymbol{J}} \;=\; \mathbb{R}^{\boldsymbol{J}}$ by a subscript $_{\boldsymbol{J}}$. In case of a prism, this is also the projection.

The following theorem describes the typical maximal face of Π simultaneously indicating the nature of its normal.

Theorem 3.1 (The Coincidence Theorem). *Let $2 \leq K \leq (n-1)$ and let $\boldsymbol{F}$ be a maximal face of Π and let $\boldsymbol{n}^{\star}$ be its normal. Then the following holds true.*

1. *For each $k \in \boldsymbol{K}$ there is an index set $\boldsymbol{J}^{(k)}$ and a corresponding subface $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ satisfying*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \;. \tag{3.1}$$

2. *There exist a unique index set $\boldsymbol{L} = \{i_1, \ldots, i_L\} \subseteq \boldsymbol{I}$ satisfying $L = |\boldsymbol{L}| \leq K-1$ and such that*

   (a) $1 \leq \left|\; \boldsymbol{J}^{(k)} \cap \boldsymbol{L} \;\right| \leq 2$

   (b) With a suitable reordering $\{k_1, \ldots, k_K\}$ of the indices of $\boldsymbol{K}$ the index sets

$$\boldsymbol{J}^{(k_1)}, \ldots, \boldsymbol{J}^{(k_K)} \tag{3.2}$$

   are arranged in a way such that each one has exactly one common index with his neighbors and this index is an element of $\boldsymbol{L}$.

   (c) The intersection $\boldsymbol{F}_{\{i_1,\ldots,i_L\}} \;=\; \boldsymbol{F}\cap\Delta_{\{i_1,\ldots,i_L\}}$ of $\boldsymbol{F}$ with the $i_1, \ldots, i_L$ boundary of Δ has dimension $L-1$.

3. *There are positive constants c_k $(k = 1, \ldots, K)$ (unique up to a multiple) such that the following holds true:*

   (a) $\boldsymbol{n}^{\star}$ is (up to a multiple) exactly the normal of the prism

$$\Pi^{\star} \;=\; \operatorname{conv}\left(\bigcup_{k\in\boldsymbol{K}} c_k \Pi^{(k)}\right). \tag{3.3}$$

   (b) $\boldsymbol{n}^{\star}$ is a normal to each of the prisms $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$.

(c) For each neighboring pair k_l, k_{l+1} in the ordering suggested in (3.2) *the prisms $c_{k_l}\Delta^{(k_l)}_{\boldsymbol{J}^{(k_l)}}$ and $c_{k_{l+1}}\Delta^{(k_{l+1})}_{\boldsymbol{J}^{(k_{l+1})}}$ have a joint vertex, this is exactly the one corresponding to the joint index which is a member of $\boldsymbol{L}$.*

Proof:

1^st^STEP :

As $\boldsymbol{F}$ is maximal, we can apply Theorem 1.15 in EWALD [1], see also Theorem 3.1.1 in PALLASCHKE–URBANSKI [4]. Accordingly, there is, for each $k \in \boldsymbol{K}$, a subface $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ such that

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}. \tag{3.4}$$

holds true.

Let $|\boldsymbol{J}^{(k)}| \;:=\; j_k \;\; (k \in \boldsymbol{K})$. Then each summand yields a dimension of $j_k - 1$. In order to produce the proper dimension $(n-1)$ of a maximal face, we must have

$$j_1 - 1 + j_2 - 1 + \cdots + j_K - 1 \geq n - 1$$

and by our nondegeneracy assumption we obtain

$$j_1 - 1 + j_2 - 1 + \cdots + j_K - 1 = n - 1$$

or

$$j_1 + j_2 + \cdots + j_K = K + n - 1 \geq n + 1 \ . \tag{3.5}$$

As there are n indices available in $\boldsymbol{I}$, some of them must appear at least twice within the index sets $\boldsymbol{J}^{(k)}$ so that a total of $K-1$ additional indices occurs. We list the indices that appear at least twice in a set $\boldsymbol{L} \;=\; \{i_1, \ldots, i_L\}$.

Assume, for suitable integers $K_1, \ldots K_L \geq 1$, that

$$\begin{array}{cccc} i_1 & \text{appears} & K_1 + 1 & \text{times} \\ i_2 & \text{appears} & K_2 + 1 & \text{times} \\ \cdots & \cdots & \cdots & \cdots \\ i_L & \text{appears} & K_L + 1 & \text{times} \end{array} \tag{3.6}$$

such that

$$K_1 + \cdots + K_L = K - 1 \tag{3.7}$$

holds true.

2^nd^STEP :

Note that no index can appear twice in one of the index sets $\boldsymbol{J}^{(k)}$ as the latter ones refer to the vertices of the subprisms $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. Hence, each $\boldsymbol{J}^{(k)}$ has to contain at least one of the indices $i_1, \ldots, i_L$. On the other hand, there are $2K-2$ appearances of the indices in $\boldsymbol{L}$ so there are two of these indices in all of the K sets $\boldsymbol{J}^{(k)}$ except 2. Therefore, the sets $\boldsymbol{J}^{(k)}$ can be rearranged as indicated by (3.2).

The two extremal cases are that just one index i_1 appears K times (hence in all index sets) or that $K-1$ indices appear twice (so $K-1$ pairs of index sets have a common index).

3$^{\text{rd}}$STEP :

Consequently, the face $\boldsymbol{F}$ contains sums of vectors $\boldsymbol{a}^{(k)i}$ where i is chosen from $\boldsymbol{L} = \{i_1, \ldots, i_L\}$ *only*. For those $\boldsymbol{J}^{(k)}$ that contain two of these indices, we can take two sums, each one using one of the indices, and construct convex combinations of these two sums. This generates an $L-1$–dimensional subsimplex on the outer surface Δ of Π such that only coordinates $i_1, \ldots, i_L$ appear. Hence, the intersection of $\boldsymbol{F} \cap \Delta_{\boldsymbol{L}}$ has indeed dimension $L-1$.

4$^{\text{th}}$STEP :

Next consider the linear system of equations in variables $c_1, \ldots, c_K$ and $\lambda_1 \ldots \lambda_L$ given by

$$
\begin{array}{rcl}
c_k a_{i_1}^{(k)} & = & \lambda_1 \quad (\text{ for all k with } i_1 \in \boldsymbol{J}^{(k)}), \\
c_k a_{i_2}^{(k)} & = & \lambda_2 \quad (\text{ for all k with } i_2 \in \boldsymbol{J}^{(k)}), \\
\ldots & = & \ldots \\
c_k a_{i_L}^{(k)} & = & \lambda_L \quad (\text{ for all k with } i_L \in \boldsymbol{J}^{(k)}).
\end{array}
\tag{3.8}
$$

The number of variables is $K+L$. The number of equations is computed as i_1 occurs K_1+1 times, i_2 occurs K_2+1 time, …etc.

$$K_1 + 1 + \cdots + K_L + 1 \; = \; K - 1 + L.$$

Some inspections shows that indeed all coefficients can be computed successively, the solution space has dimension 1. As all vectors $\boldsymbol{a}^{(k)}$ $(k \in \boldsymbol{K})$ are positive, we can choose a positive solution.

5$^{\text{th}}$STEP :

Given the normal $\boldsymbol{n}^\star$ of $\boldsymbol{F}$ denote the tangential hyperplane of $\boldsymbol{F}$ by $\mathcal{H}_{t^\star}^{\boldsymbol{n}^\star}$. As the faces $\Delta_{\boldsymbol{J}^{(k)}}^{(k)}$ $(k \in \boldsymbol{K})$ all admit of the normal $\boldsymbol{n}^\star$, so do the faces $c_\kappa \Delta_{\boldsymbol{J}^{(k)}}^{(k)}$ $(k \in \boldsymbol{K})$. However, the first K_1 faces as well as the second K_2 two faces, …etc. have a common extremal point, as we have

$$c_k a_{i_1}^{(k)} \; = \; \lambda_1 \quad (i_1 \in \boldsymbol{J}^{(k)}),$$

etc. in view of 3.8.

Hence, they are all contained in the *same* hyperplane, say $\mathcal{H}_{t^\circ}^{\boldsymbol{n}^\star}$ parallel to $\mathcal{H}_{t^\star}^{\boldsymbol{n}^\star}$. Moreover, in view the $2^{nd}STEP$ it follows that any two of these systems which have a common index, the tangential hyperplane is the same and so (because of connectedness) all the faces $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ $(k \in \boldsymbol{K})$ admit of the common tangential hyperplane $\mathcal{H}^{\boldsymbol{n}^\star}_{t^\circ}$.

From this it follows that the prism $\Pi^\star = \bigvee_{k\in\boldsymbol{K}} c_k \Pi^{(k)}$ admits $\mathcal{H}^{\boldsymbol{n}^\star}_{t^\circ}$ as a tangential hyperplane. But the normal of a prism is uniquely defined up to some multiple, hence $\mathcal{H}^{\boldsymbol{n}^\star}_{t^\circ}$ is *the* tangential hyperplane of $\Delta^\star$ and $\boldsymbol{n}^\star$ is the normal.

q.e.d.

Definition 3.2. *Given a maximal face* $\boldsymbol{F}$ *of* Π, *we shall say that the index sets* $\boldsymbol{J}^{(k)}$ $(k \in \boldsymbol{K})$ ***determine*** *the face. The elements of the set* $\boldsymbol{L}$ *as given by Theorem 3.1 are called the* ***boundary indices*** *associated to* $\boldsymbol{F}$.

The boundary indices determine the smallest boundary of Δ that the face will intersect properly. On the other hand they refer to the adjustment process described by the Concidence Theorem.

If we turn to the canonical representation, then the indices $i_1, \ldots, i_L$ determine a subface of the representing multiple $K\Delta^{\boldsymbol{e}}$ of the unit simplex. Again this is the smallest subface that $\boldsymbol{F}$ will intersect properly.

Remark 3.3. For any vertex $\sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)i}$ there is a unique maximal face containing it. To see this, it is sufficient to consider the case $n = 3, K = 2$. If there are two maximal faces containing $\boldsymbol{a}^i + \boldsymbol{b}^i$, then there must be two maximal faces $\boldsymbol{F}$ and $\boldsymbol{F}'$ containing $\boldsymbol{a}^i + \boldsymbol{b}^i$ that are *adjacent*. These faces have a joint subface of dimension $n - 2 = 1$ containing $\boldsymbol{a}^i + \boldsymbol{b}^i$, say $\boldsymbol{F}_{\{ij\}} = \Delta^{\boldsymbol{a}}_{\{ij\}} + \Delta^{\boldsymbol{b}}_{\{i\}} = \boldsymbol{F}'_{\{ij\}} = \Delta^{\boldsymbol{a}}_{\{i\}} + \Delta^{\boldsymbol{b}}_{\{ij\}}$. Hence, there exists $0 \leq t \leq 1$ such that

$$t\boldsymbol{a}^i + (1-t)\boldsymbol{a}^j + \boldsymbol{b}^i = \boldsymbol{a}^i + \boldsymbol{b}^j$$

holds true. This implies

$$(1-t)a_j = b_j,\ (1-t)a_i = b_i,$$

which contradicts our nondegeneracy assumption (Definition 1.6). Similarly, it follows easily that two faces $\boldsymbol{F}, \boldsymbol{F}'$ having the same set of boundary indices $\boldsymbol{L}$ necessarily do *not* fully cut into $\Delta_{\boldsymbol{L}}$, more precisely, $(\boldsymbol{F}\cap\Delta_{\boldsymbol{L}})\cap(\boldsymbol{F}\cap\Delta_{\boldsymbol{L}})$ has dimension strictly less than L. On the other face, restricting the generating family to $\mathbb{R}_{\boldsymbol{L}}$ generates maximal subfaces, non of which can be in two different maximal subfaces of Δ.

Hence the assignment of a maximal face to its L–dimensional subface generated by its boundary indices is unique.

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It will be useful to choose a representation of a face by just mentioning the index sets $\boldsymbol{J}^{(k)}$. Thus, we write conveniently

$$\text{(3.9)} \qquad \begin{matrix} \boldsymbol{a}^{(1)} & \boldsymbol{a}^{(2)} & \ldots & \boldsymbol{a}^{(K)} \\ \boldsymbol{J}^{(1)} & \boldsymbol{J}^{(2)} & \ldots & \boldsymbol{J}^{(K)}, \end{matrix}$$

whenever we wish to indicate the subface given by 3.4. We may even omit mentioning $\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(K)}$ when the family of positive vectors *and the ordering* cannot be confused. However, if the ordering is given by (3.2), then the common boundary indices of $\boldsymbol{L}$ appear successively (possibly in multiple way), indicating the joint vertices of the subprisms involved.

The same considerations as presented for maximal faces can be repeated for lower dimensional faces. Thus,the concept of boundary indices (as resulting from the Coincidence Theorem) is well defined for lower dimensional faces.

Theorem 3.4. *Let $\boldsymbol{F}$ be a maximal face of Π with a set of boundary indices $\boldsymbol{L}$ and let $\boldsymbol{F}^{(0)}$ be an (n-2)-dimensional subface of $\boldsymbol{F}$. Then, for $\boldsymbol{F}^{(0)}$, there is a set of boundary indices $\boldsymbol{L}' = \{i_1, \ldots, i_{L'}\} \subseteq \boldsymbol{L}$ such that $L-1 \leq L' \leq L$ holds true.*

Proof: The same considerations presented for a face of dimension $(n-1)$ hold true for a face of dimension $(n-2)$. If the latter one is subset of the former one, then there is just one index missing from one of the index sets $\boldsymbol{J}^{(k)}$ $(k \in \boldsymbol{K})$ representing $\boldsymbol{F}$. The missing index may or may not be one of the specified set of boundary indices determined by $\boldsymbol{F}$.

q.e.d.

Theorem 3.5 (The Neighborhood Theorem). *Let $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ be maximal faces of Π that are adjacent. Suppose both are given by their index sets as in 3.4. Then*

1. *K-2 of the index sets are equal.*
2. *Suppose without loss of generality that these are the last $K-2$ index sets. Then the remaining two index sets (i.e., with the indices 1 and 2) satisfy*

   $$\boldsymbol{J}'^{(1)} \subseteq \boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)} \subseteq \boldsymbol{J}'^{(2)} \quad , \quad |\boldsymbol{J}^{(1)} - \boldsymbol{J}'^{(1)}| = |\boldsymbol{J}'^{(2)} - \boldsymbol{J}^{(2)}| \tag{3.10}$$

   (or vice versa)
3. *The difference in each case is obtained by switching just one index not in $\boldsymbol{L}$ or exchanging two indices of $\boldsymbol{L}$ i.e.,*
   (a) *Either there is $q \notin \boldsymbol{L},\ q \in \boldsymbol{J}^{(1)},\ q \notin \boldsymbol{J}^{(2)}$ satisfying*

   $$\boldsymbol{J}'^{(1)} = \boldsymbol{J}^{(1)} - \{q\}, \boldsymbol{J}'^{(2)} = \boldsymbol{J}^{(2)} \cup \{q\}, \tag{3.11}$$

   (b) *or else, there are indices $i \in \boldsymbol{L},\ j \in \boldsymbol{L}'$ with $i \in \boldsymbol{J}^{(1)} \cap \boldsymbol{J}^{(2)}$ and $j \in \boldsymbol{J}'^{(1)} \cap \boldsymbol{J}'^{(2)}$, satisfying*

   $$\boldsymbol{J}'^{(1)} = \boldsymbol{J}^{(1)} - \{i\}, \boldsymbol{J}'^{(2)} = \boldsymbol{J}^{(2)} \cup \{j\}, \tag{3.12}$$

   holds true.

That is, given some face $\boldsymbol{F}$, an adjacent maximal face $\boldsymbol{F}'$ has either the same boundary indices, in which case there is just a non–boundary index switched. Or else one of the boundary indices is diminished in the number of appearances (possibly becoming a non–boundary index thereby) and another boundary index is created or increased in his number of appearances.

Proof: The proof follows from Theorems 3.1 and 3.4. Note that the number of L of multiple appearances is the same for all maximal faces, hence, if $\boldsymbol{L}$ is diminished by the passage to a $(n-2)$–dimensional subface, it must be increased again in the next adjacent maximal subface. **q.e.d.**

Corollary 3.6. *If $K \leq n-1$ holds true, then each maximal face intersects at least one boundary face of dimension $K-1 \leq n-2$. Within the canonical representation, the image of every maximal subface intersects a boundary simplex of dimension $K-1$.*

Remark 3.7. If $K = n-1$ is the case, then the number of boundary indices is at most $n-2$. This occurs if the indexsets that determine the face contain 2 indices each.

If $K < n-1$ prevails, then, with $m := K+1$, the projection on any m–dimensional subspace of $\mathbb{R}^n$ reflects the situation $K = m-1$. Hence all maximal faces and the corresponding boundary indices (at most $K-1 = m-2$) can be found by projecting the family $(\boldsymbol{a}^{(k)})_{k \in \boldsymbol{K}}$ of positive vectors into $\mathbb{R}^m_+$. Thus, any face of Δ has its counterpart in any m–dimensional projection of Δ and the geometric (and combinatorial) structure is revealed already in lower dimensions.

On the other hand, if $K \geq n$ holds true, a simple count of dimensions reveals that at least $K-n$ subsimplices involved in the representation (3.4) must be vertices. Thus, the results of the Coincidence Theorem can be applied to any group of $n-1$ prisms adding suitable vertices from the remaining prisms. (See also Lemma 5.8 and Corollary 5.9).

Hence, one can say that the case $K = n-1$ is the decisive one. The details will be explained in Section 5.

In particular, regarding Section 1, the decisive case $n = 3$ and $K = 2$ is represented in Figure 1.3 with canonical representation given in Figure 1.1. Each face has one boundary index which corresponds to a vertex of the canonical representation. For more then two prisms in $\mathbb{R}^2_+$, any two of them form a "diamond" and there may be two or three boundary indices. That is, a face may properly cut a subsimplex and not contain a vertex or it may even be "interior". Yet, diamonds (i.e., sums of two subsimplices (line segments)) are the most complicated type of maximal face that appears in 3 dimensions.

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Before we proceed to discuss this in more detail, let us discuss the combinatorial structure that goes together with a maximal face. We will first exhibit this for arbitrary n and $K = 2$ in the following section.

4 Summing Two Prisms: The Tentacle System

We shall treat the sum $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ of two prisms explicitly within this section as it provides an essential inside into the nature of maximal faces and, in addition, supplies the basis for further developments. First of all, we specify our previous result for $K = 2$.

Theorem 4.1. *Let $\boldsymbol{F}$ be a maximal face of Π. Then there exists uniquely $i \in \boldsymbol{I}$ such that $\boldsymbol{a}^i + \boldsymbol{b}^i \in \boldsymbol{F}$ holds true. On the other hands, each vertex $\boldsymbol{a}^i + \boldsymbol{b}^i$ of Δ is contained in a unique maximal face $\boldsymbol{F}$. Moreover, there are positive constants $c_{\boldsymbol{a}}$ and $c_{\boldsymbol{b}}$ (unique up to a multiple) such that the following holds true*

1. *$c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}}$ and $c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}}$ have exactly one common vertex, this is the one in i–direction, i.e., the vertex*

   (4.1) $$c_{\boldsymbol{a}}\boldsymbol{a}^i = c_{\boldsymbol{b}}\boldsymbol{b}^i \ .$$

2. *The normal $\boldsymbol{n}^\star$ of $\boldsymbol{F}$ is (up to a multiple) exactly the normal of the prism*

   (4.2) $$\Delta^\star \ = \ c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}} \vee c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}} \ .$$

3. *There are two sets $\boldsymbol{J}^1, \boldsymbol{J}^2 \subseteq \boldsymbol{I}$ and some $i \in \boldsymbol{I}$ such that*

   (4.3) $$\boldsymbol{F} \ = \ \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^1} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^2}$$

   with

   $$|\boldsymbol{J}^1| + |\boldsymbol{J}^2| = n + 1 \ .$$

   and

   (4.4) $$\boldsymbol{I}^1 \cap \boldsymbol{I}^2 = \{i\}.$$

Proof: This is clearly Theorem 3.1 and Remark 3.3 specified for $K = 2$.

q.e.d.

We can now exactly describe the structure of the sum of two polyhedra. To this end we introduce the following notation. Let $\prec$ be a total ordering of $\boldsymbol{I}$. We denote by

(4.5) $$T_k^{\prec} \ := \ \{i \in \boldsymbol{I} \,|\, i \prec k\} \cup \{k\}$$

the set of predecessors of $k \in \boldsymbol{I}$ including k. Similarly, let

(4.6) $$S_k^{\prec} \ := \ \{i \in \boldsymbol{I} \,|\, k \prec i\} \cup \{k\}$$

denote the set of successors of k including k. Clearly

$$S_i^{\prec} \cap T_i^{\prec} \ = \ \{i\} \quad (i \in \boldsymbol{I})$$

holds true. Now we have the following

Theorem 4.2. *1. The sum of two prisms in $\mathbb{R}^n$ has exactly n maximal faces.*

2. For any sum of two prisms $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ there exists uniquely an ordering $\prec$ of $\boldsymbol{I}$ such that the maximal faces are exactly described by

$$\boldsymbol{F}^{\prec i} := \Delta^{\boldsymbol{a}}_{S_i^{\prec}} + \Delta^{\boldsymbol{b}}_{T_i^{\prec}} \quad (i \in \boldsymbol{I}). \tag{4.7}$$

3. There are exactly $n!$ types of sums of two prisms (in general position) Each type corresponds to an ordering of $\boldsymbol{I}$ such that the faces are given by equation (4.7).

Proof: Each maximal face $\boldsymbol{F}$ of the surface contains exactly one vertex $\boldsymbol{a}^i + \boldsymbol{b}^i$ of Δ for some $i \in \boldsymbol{I}$ and, the other way around, for every $i \in \boldsymbol{I}$, there is exactly one face containing $\boldsymbol{a}^i + \boldsymbol{b}^i$. Thus , there is a one–to–one correspondence between vertices and faces of the shape indicated by (4.3) and (4.4).

Now the Neighborhood Theorem (Theorem 3.5) requires the index sets for $\boldsymbol{a}$ as well as for $\boldsymbol{b}$ to be ordered, there has to be a "tight" sequence of sets $S_1, S_2, \ldots, S_n \subseteq \boldsymbol{I}$ such that $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n$ and $|S_k| = k (k \in \boldsymbol{I})$ holds true. This defines uniquely an ordering as claimed in our theorem.

q.e.d.

Faces of the type mentioned in formula (4.7) will be subsequently used in this paper. Therefore, whenever the two vectors involved are $\boldsymbol{a}^{(k)}, \boldsymbol{a}^{(k')}$ instead of $\boldsymbol{a}$ and $\boldsymbol{b}$, then formula (4.7) is rewritten as

$$\boldsymbol{F}^{kk';\prec i} := \Delta^{(k)}_{S_i^{\prec}} + \Delta^{(k')}_{T_i^{\prec}} \quad (i \in \boldsymbol{I}). \tag{4.8}$$

Remark 4.3. If the ordering involved is the natural one, we can obviously generate all maximal faces by listing the system of index sets $\boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}$ according to formula (4.7) as follows

$$\begin{array}{ll} 1 & 1234\ldots n \\ 12 & 234\ldots n \\ 123 & 34\ldots n \\ \ldots & \ldots \\ 123\ldots n & n \end{array} \tag{4.9}$$

Obviously each face is obtained from its neighbor by moving the doubly appearing index by one step to the right. We refer to this procedure as to the ***moving index principle*** for one index. There is a rather obvious generalization for more than one index.

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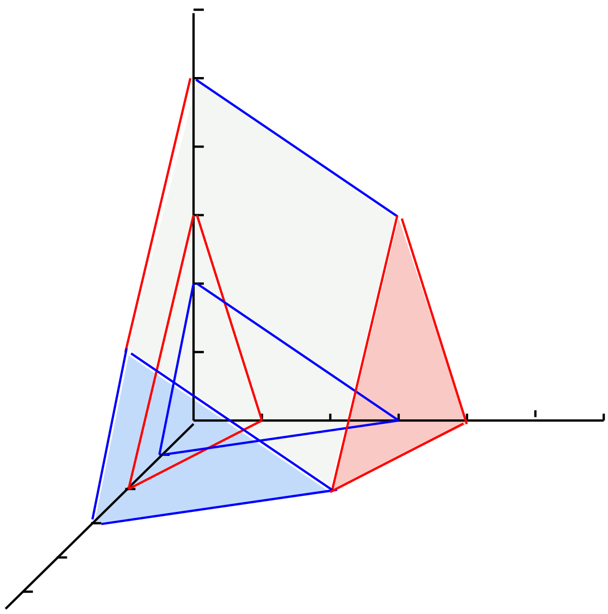

Figure 4.1: The sum of two prisms for $n = 2$

Example 4.4. For $n = 3$ the sum of two prisms is seen in Figure 4.1. Note that the faces $\boldsymbol{F}^1 = \Delta^{\boldsymbol{a}} + \boldsymbol{b}^1$ and $\boldsymbol{F}^2 = \Delta^{\boldsymbol{b}} + \boldsymbol{a}^2$ determine the third face uniquely to be $\boldsymbol{F}^3 = \Delta^{\boldsymbol{a}}_{\{23\}} + \Delta^{\boldsymbol{b}}_{\{13\}}$.

$$\begin{aligned} \boldsymbol{F}^{\prec 2} &= \Delta^{\boldsymbol{a}}_2 + \Delta^{\beta}_{231} \\ \boldsymbol{F}^{\prec 3} &= \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{31} \\ \boldsymbol{F}^{\prec 1} &= \Delta^{\boldsymbol{a}}_{231} + \Delta^{\boldsymbol{b}}_{1} \, . \end{aligned} \tag{4.10}$$

That is, index 2 appears first, index 3 appears second, ... according to the ordering which, in the case depicted, is $\prec\; = (2, 3, 1)$. At each face, the decisive index appears twofold, representing the fact that it indicates the basis vector at which a coincidence is enforced according to Theorem 4.1.

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Example 4.5. For $n = 4$ we can offer a sketch of the canonical representation, see Figure 4.2.

Assuming that the translate of $\Delta^{\boldsymbol{a}}$ occupies the first vertex of the sum (i.e., $2\boldsymbol{e}^1$), and the translate of $\Delta^{\boldsymbol{b}}$ the second one, the left hand version of Figure 4.2 corresponds to the ordering 2341. For, the maximal faces are given by

$$\begin{aligned} \boldsymbol{F}^{\prec 2} &= \Delta^{\boldsymbol{a}}_2 + \Delta^{\boldsymbol{b}}_{2341} \\ \boldsymbol{F}^{\prec 3} &= \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{341} \\ \boldsymbol{F}^{\prec 4} &= \Delta^{\boldsymbol{a}}_{234} + \Delta^{\boldsymbol{b}}_{41} \\ \boldsymbol{F}^{\prec 1} &= \Delta^{\boldsymbol{a}}_{2341} + \Delta^{\boldsymbol{b}}_{1} \, . \end{aligned} \tag{4.11}$$

The reader may check that the right hand side version of Figure 4.2 corresponds to the ordering 2431.

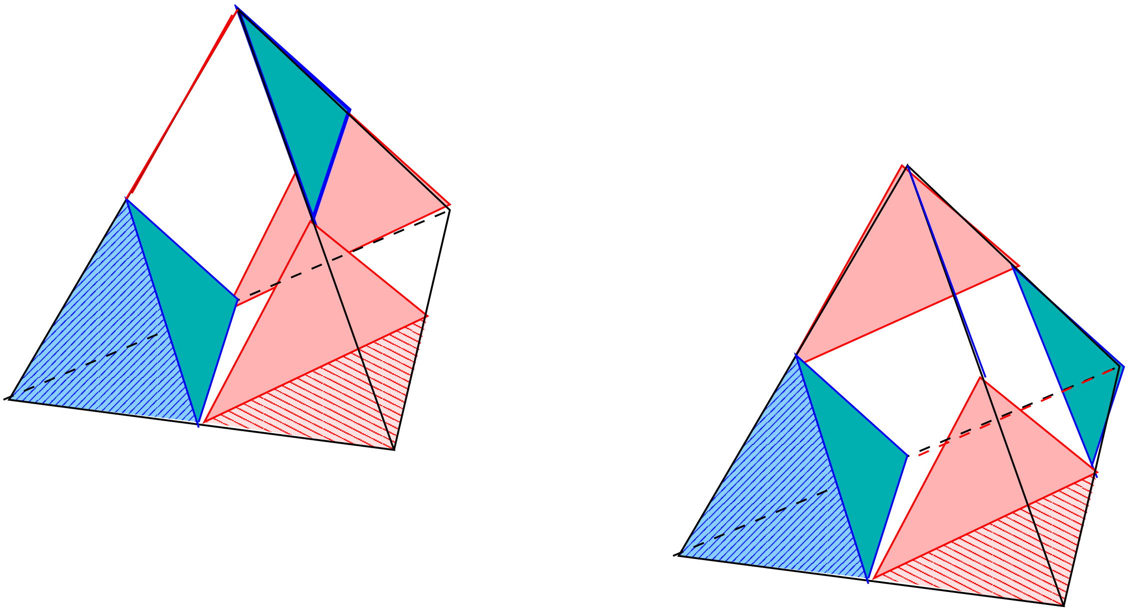

Figure 4.2: The sum of two prisms for $n = 4$

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We conclude by deriving a few facts regarding an *arbitrary* sum of K prisms. This will be a useful preparation for the general discussion of cephoids to be presented later on. First of all, it is easy to see that the translates of all prisms will appear on the surface.

Theorem 4.6. *Let* $(\boldsymbol{a}^{(k)})_{k\in\boldsymbol{K}}$ *be a family of positive vectors in general position. Then, for any* $k, l \in \boldsymbol{K}$*, the simplex* $\Delta^{(k)}$ *has a joint normal with exactly one vertex of* $\Delta^{(l)}$ $(l \neq k)$.

Proof: The normals that belong to all the vertices of $\Delta^{(l)}$ span the octant $\mathbb{R}^n_+$. If two of these normals are joint to the one of $\Delta^{(k)}$, then the normal cone of a two dimensional subface of $\Delta^{(l)}$ equals the corresponding one of $\Delta^{(k)}$, which we have ruled out by nondegeneracy.

q.e.d.

Corollary 4.7 (The translates of prisms, cylinders, . . . etc. on Δ). *Let* $\left(\boldsymbol{a}^{(k)}\right)_{k\in\boldsymbol{K}}$ *be a family of positive vectors in general position and let* $\Pi := \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$, $\Delta := \sum_{k\in\boldsymbol{K}} \Delta^{(k)}$.

1. *Each* $\Pi^{(k)}$ *yields a translate of* $\Delta^{(k)}$ *on* Δ*. This translate is the sum of each* $\Delta^{(k)}$ *with* $K-1$ *vertices, each one from a different simplex* $\Delta^{(l)}$, $l \neq k$*. The vertices to each translate of a simplex are uniquely defined.*

2. *For any two prisms* $\Pi^{(k)}, \Pi^{(k')}$ *and any maximal face* $F^{kk';\prec i}$ *of the sum of these two prisms which is constructed according to Theorem 4.2, formula* (4.7) *(see the version in* (4.8)*), there is a translate on* Δ*. This translate is the sum of* $F^{kk';\prec i}$ *with* $K-2$ *vertices, each one from a different simplex* $\Delta^{(l)}$*,* $l \neq k, k'$*. The vertices are uniquely defined.*

The proof employs obvious generalizations of the one for Theorem 4.6.

A particular case of the above corollary is obtained by considering, for each pair of prisms, the sum of an edge and an $(n-1)$–simplex. We call such a sum a ***cylinder***. As we will see, each translate of a simplex stretches a sequence of cylinders from of its $(n-1)$–subsimplices not located at the boundary of Δ to the corresponding $(n-1)$–subsimplex of Δ. This generates the appearance of an "cephalopod" and motivates the name "cephoid" we have chosen. Technically, the following is a consequence of Theorem 4.2 but can also be seen in the context of Corollary 4.7.

Corollary 4.8. *Let* $\left(\boldsymbol{a}^{(k)}\right)_{k\in\boldsymbol{K}}$ *be a family of positive vectors in general position and let* $\Pi := \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ *,* $\Delta := \sum_{k\in\boldsymbol{K}} \Delta^{(k)}$.

1. *For every pair* $k, k' \in \boldsymbol{K}, k \neq k'$*, there exists uniquely* $i \in \boldsymbol{I}$ *such that for some* $j \neq i$ *the simplices*

$$\Delta^{(k)}_{\boldsymbol{I}-i} \quad \textit{and} \quad \Delta^{(k')}_{ij}$$

admit of a joint normal, this is the normal $\boldsymbol{n}^\star = \boldsymbol{n}^{\star kk'}$ *of the prism*

$$\Pi^{\star kk'} = c_k \Pi^{(k)} + c_{k'} \Pi^{k'}$$

with the constants determined by Theorem 4.1.

2. *For every pair* $k, k' \in \boldsymbol{K}, k \neq k'$*,* $i \in \boldsymbol{I}$ *given as above, and* $l \in \boldsymbol{K} - \{k, k'\}$ *there exists a unique* $i_l \in \boldsymbol{I} - \{i\}$ *such that* $\Pi^{(l)}$ *admits of a joint normal with* $\Pi^{\star kk'} = c_k\Pi^{(k)} + c_{k'}\Pi^{k'}$ *in* $\boldsymbol{a}^{(l)i_l}$*; this is exactly the one* i_l *for which* $\boldsymbol{n}^\star$ *is admitted as a normal in* $\boldsymbol{a}^{(l)i_l}$*.*

3. *Hence, for every pair* $k, k' \in \boldsymbol{K}, k \neq k'$*, there is (uniquely)* $i \in \boldsymbol{I}$ *and a sequence* $(i_l)_{i\neq k,k'}$ *such that*

$$\boldsymbol{F}^{kk';\prec i;i_\bullet} := \Delta^{(k)}_{\boldsymbol{I}-i} + \Delta^{(k')}_{ij} + \sum_{l\neq k,k'} \boldsymbol{a}^{(l)i_l} = \boldsymbol{F}^{kk';\prec i} + \sum_{l\neq k,k'} \boldsymbol{a}^{(l)i_l} \tag{4.12}$$

is a maximal face of $\boldsymbol{P}$.

Definition 4.9.

1. *The face*

$$\boldsymbol{F}^{kk'} = \boldsymbol{F}^{kk';\prec i;i_\bullet} \tag{4.13}$$

described by item 3 *of Corollary 4.8 is the* ***cylinder*** *generated by* k *and* k'*. In particular, we refer to an* ***i–cylinder*** *whenever* i *is given by item* 1 *of Corollary 4.8.*

2. *For every $k \in \boldsymbol{K}$ the set of i–cylinders*

$$\left\{ \boldsymbol{F}^{kk'} \,\middle|\, k' \in \boldsymbol{K} - k,\ i^{kk'} = i \right\} \tag{4.14}$$

   *is the **tentacle** in direction i (the **i–tentacle**) generated by $\Delta^{(k)}$.*

3. *The system*

$$\left\{ \boldsymbol{F}^{kk'} \,\middle|\, k' \in \boldsymbol{K} - k \right\} \tag{4.15}$$

   *is the **system of tentacles** generated by $\Delta^{(k)}$.*

Corollary 4.10. *The surface of any sum of K prisms contains for any prism involved a system of tentacles consisting of exactly $K-1$ cylinders.*

1. *For $n > 3$, there are K translates of prisms and $K(K-1)$ cylinders on this surface.*

2. *For $n = 3$ any pair of prisms generates just one cylinder, hence the number of cylinders is $\frac{K(K-1)}{2} = \binom{K}{2}$.*

The cylinders for $n = 3$ have been called "diamonds" in our introductory remarks because this is what they look like.

5 Enumerating the Faces

Let $(\boldsymbol{a}^{(k)})_{k=1}^{K}$ denote a family of positive vectors in general position. Let Π denote the cephoid generated and let Δ be its surface. We develop recursive procedures describing the number and nature of maximal faces of Π.

Definition 5.1. *We say that a maximal face* $\boldsymbol{F}$ *has a* ***proper cut*** *with a subface* $\Delta_{\boldsymbol{J}}$ *of* Δ *if*

$$dim(\boldsymbol{F} \cap \Delta_{\boldsymbol{J}}) = |\boldsymbol{J}| - 1 = dim\Delta_{\boldsymbol{J}} \tag{5.1}$$

holds true. That is, the intersection has the same dimension as the subsimplex of the boundary. $\boldsymbol{F}$ *is called an* l–***face*** *if the minimal dimension of a proper cut is* $l-1$.

Thus, a 1–face contains a vertex, a 2–face cuts properly into a 2 dimensional subface of Δ but does not contain a vertex, etc. E.g., we know that for $K = 2$ every face is 1–face (Theorem 4.2).

Corollary 5.2. *Any maximal face* $\boldsymbol{F}$ *is an* l*–face for some* $l \leq \min\{K-1, n\}$. *Given the representation of* $\boldsymbol{F}$ *by means of the index sets* $\boldsymbol{J}^{(k)}$ $(k \in \boldsymbol{J})$*, the boundary* $\Delta_{\boldsymbol{L}}$ *of* Δ *that yields the minimal maximal cut is uniquely defined by the set* $\boldsymbol{L}$ *of boundary indices (Definition 3.2).*

This follows immediately from the Coincidence Theorem 3.1.

Remark 5.3. The projection of a prism $\Delta^{(k)}$ onto some subspace yields the corresponding subsimplex. Restricting the summation to a subspace amounts to adding prisms within this subspace and generating a cephoid of lower dimension. In general, (maximal) faces could disappear be the restriction to lower dimensions. However, if a face intersects an octant of lower dimension, then the intersection is a face.

Consider a maximal face with boundary index set $\boldsymbol{L}$. If the restriction to some lower dimensional $\mathbb{R}_{\boldsymbol{J}}$ respects $\boldsymbol{L}$ (i.e., $\boldsymbol{L} \subseteq \boldsymbol{J}$), then $\boldsymbol{F} \cap \Delta_{\boldsymbol{J}}$ is indeed a maximal face. This is so as the number of indices necessary in order to generate the correct dimension is preserved. Thus, $\boldsymbol{F} \cap \Delta_{\boldsymbol{J}}$ is indeed an l–face with the same set of boundary indices.

The recursive procedure is essentially based on this property of cephoids: l–faces appear already in lower dimensions, hence can be enumerated and characterized recursively.

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Definition 5.4. *The number of maximal faces of* Π *is denoted by* $f(K, n)$. *The number of* l*–faces is denoted by* $h(K, l)$.

We note that $f(K, 1) = h(K, 1) = 1$ and $f(K, 2) = K$, $h(K, 2) = K - 2$ holds true immediately.

Lemma 5.5. *For $n \leq K$*

$$f(K,n) = h(K,n) + \binom{n}{n-1} h(K,n-1) + \ldots + nh(K,1). \tag{5.2}$$

For $n \geq K-1$

$$f(K,n) = \binom{n}{K-1} h(K,K-1) + \ldots + nh(K,1). \tag{5.3}$$

Proof: We collect the faces according to the minimal subsimplex they are sharing a proper cut with. In view of the Coincidence Theorem, each maximal face is represented uniquely by its minimal proper cut (cf. Remarks 3.3, 5.3). The $(n-2)$–faces of Δ can be obtained by counting the $(n-2)$–faces in each of the n restrictions of Δ with dimension $(n-2)$ etc. This can be seen more clearly by inspection of the canonical representation: The representing simplex $K\Delta^{\mathbf{e}}$ has n subsimplices of dimension $(n-2)$, each maximal $(n-2)$–face represented in $K\Delta^{\mathbf{e}}$ appears in exactly one of these subsimplices etc.

The second formula follows in view of Corollary 5.2,

q.e.d.

On the other hand, if we know the total number of faces for some dimension n, then we can compute the number of "interior" faces by subtracting all faces that properly cut into some boundary face, formally:

Corollary 5.6. *For $K \geq n$*

$$h(K,n) = f(K,n) - \left(\binom{n}{n-1} h(K,n-1) + \ldots nh(K,1)\right) \tag{5.4}$$

Definition 5.7. *A maximal face*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{5.5}$$

*is said to be r–**full** if*

$$\left|\left\{k \;\middle|\; \left|\boldsymbol{J}^{(k)}\right| \geq 2\right\}\right| = r$$

holds true.

Any maximal face $\boldsymbol{F}$ is r–full with some r, $1 \leq r \leq K$, and there are $K - r$ summands in the representation (5.5) which are just vertices.

Lemma 5.8. *Let $n \leq K$. Then any maximal face is at most $(n-1)$–full.*

Proof: Suppose, a maximal face is given as in (5.5) and let, as previously $j_k \;:=\; \left|\boldsymbol{J}^{(k)}\right|$ denote the number of indices in $\boldsymbol{J}^{(k)}$ $(k \in \boldsymbol{K})$. A simple dimension consideration as in the proof of the Coincidence Theorem shows that

$$j_1 + \ldots + j_K = K + n - 1 \tag{5.6}$$

holds true. Let $\boldsymbol{F}$ be r–full and suppose w.l.o.g that $j_1, \ldots, j_r \geq 2$ is the case. The we find

$$j_1 + \ldots + j_k \geq 2r + (K - r) = r + K. \tag{5.7}$$

Combining both equations we obtain the inequality claimed.

q.e.d.

E.g., for $n = 3$ dimensions, we know that each maximal face has dimension 2, we have triangles and diamonds. Accordingly, for $K = 2$, any such face is either the sum of two line segments (subsimplices of the simplices involved) or the sum of a simplex and a vertex. If K exceeds 2, then any maximal face looks essentially equal to the case $K = 2$ up to the fact that a number of vertices is added. Thus, any maximal face is either a sum of two line segments plus vertices or of a simplex plus vertices.

Corollary 5.9. *Let $n \leq K$. Then each maximal face is composed by means of at least $K - n + 1$ vertices. Hence, there are at most $(n-1)$–subsimplices of dimension ≥ 2 involved in the representation* (5.5).

Corollary 5.10. *For $K \geq n$ we have*

$$f(K, n) \;=\; \binom{K}{n-1} f(n-1, n) \tag{5.8}$$

Proof: According to Lemma 5.8 we know that each face is at most $(n-1)$–full. Each group of $(n-1)$ prisms chosen among the K prisms involved generates $f(n-1, n)$ maximal faces which, together with a vertex from each of the remaining prisms, appear as maximal faces of Δ. Now we can choose $\binom{K}{n-1}$ such groups.

q.e.d.

Theorem 5.11. *The number of maximal faces $f(K, n)$ can be recursively computed using the numbers $f(K', n)$ $(K' < K)$.*

Proof: According to Corollary 5.10 it suffices to compute the number $f(K, n)$ for $K \leq n - 1$.

In view of Formula (5.3), we can compute the number $f(K, n)$ by means of the numbers

$$h(K, l) \quad 1 \leq l \leq K - 1 \leq n - 2. \tag{5.9}$$

The number $h(K, l)$ can be computed successively in terms of the number $f(K, l)$ via Formula (5.4) of Corollary 5.6.

Now, as $l \leq K-1$, the numbers $f(K,l)$ in turn can be computed by means of the numbers $f(l-1,l)$ via Corollary 5.10, we have

$$f(K,l) = \binom{K}{l-1} f(l-1,l). \tag{5.10}$$

Now, $l-1 \leq K-2$ shows that we obtain all necessary data by employing quantities computed during the recursive procedure,

q.e.d.

Example 5.12. Consider the case $K = 4$.

First consider the case $n = 3$; we know already that the total number of faces is $K + \binom{K}{2} = 10$. The number of 1–faces is 3 and the number of 2–faces is $\binom{3}{2}(K-1) = 3 \cdot 2 = 6$, hence we have exactly 1 interior or 3–face in each triangle. Compare Figure 1.3 for an example.

Next let $n = 4$. Each maximal face is uniquely a 1–face or a 2–face. There are 4 vertices each one located in a 1–face. Moreover, the $\binom{n}{2}$ edges each show two 2–faces (the vertices being covered by the 1–faces). Hence, the total number of maximal faces is $4 + 6 \cdot 2 = 16$.

Essentially, it suffices to compute $f(4,n)$ for $n \leq 4$. Indeed, for $n \geq 5$ each maximal face is at most a 3–face. Exactly one 3–face appears on each triangle (as discussed above for $n = 3$). Consequently, we have to count the interior faces for at most the triangles. Thus the number of total faces is computed as

$$\begin{aligned} &\text{vertices} \times \text{number of 1–faces} \\ &+ \text{edges} \times \text{number of 2 faces} \\ &+ \text{triangles} \times \text{number of 3–faces} \end{aligned}$$

which is

$$\begin{aligned} & n \times 1 + \binom{n}{2} \times (K-2) + \binom{n}{3} \times (K-3) \\ & = n + 2\binom{n}{2} + \binom{n}{3} \,. \end{aligned} \tag{5.11}$$

This settles the case $K = 4$ completely.

Actually, the case $K = 4$ and $n = 4$ can be computed from the case $K = 3$, $n = 4$ (Corollary 5.10). We will treat $K = 3$ in all detail within SECTION 6.

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For small numbers K and n we can come up with the following table.

Number of prisms K	$f(K,n)$ for arbitrary n	Comments
$K = 2$	n	The Ordering Theorem 4.2
$K = 3$	$n+\binom{n}{2}$	The results of SECTION 6
$K = 4$	$n+2\binom{n}{2}+\binom{n}{3}$	See Example 5.12
$K = 5$	$n+3\binom{n}{2}+3\binom{n}{3}\binom{n}{4}$	Similarly to 5.12
Dimension n	**$f(K,n)$ for arbitrary K**	**Comments**
$n = 3$	$K+\binom{K}{2}$	See Corollary 4.10
$n = 4$	$K+2\binom{K}{2}+\binom{K}{3}$	See Corollary 4.10

We continue by pointing out a similar recursive procedure that provides the representation of all maximal faces of a cephoid recursively.

To this end we write

$$\boldsymbol{a}^{\bullet} \ := \ \left(\boldsymbol{a}^{(k)}\right)_{\kappa\in\boldsymbol{K}} \tag{5.12}$$

for the family involved. The procedure we have in mind is then described by the mapping

(5.13)
$$\mathcal{F}(K,n;\star) \ : \ \left\{\boldsymbol{a}^{\bullet} \ := \ \left(\boldsymbol{a}^{(k)}\right)_{\kappa\in\boldsymbol{K}} \,\middle|\, \boldsymbol{a}^{\bullet} \text{ is nondegenerate } \right\} \to \mathcal{P}\left((\mathcal{P}(\boldsymbol{I}))^{\boldsymbol{K}}\right)$$

which associates with a set of positive vectors in $\mathbb{R}^n_+$ a finite set $\mathcal{F}(K,n;\boldsymbol{a}^{\bullet})$ the elements of which are k–tuples $(\boldsymbol{J}^{(1)},\ldots,\boldsymbol{J}^{(K)})$ which correspond to the faces of Π via $\boldsymbol{F} = \sum_{k=1}^{K}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. According to Theorem 5.11 we know that $|\mathcal{F}(K,n;\boldsymbol{a}^{\bullet})| = f(K,n)$ can be recursively computed (independently on $\boldsymbol{a}^{\bullet}$). We now show that the same is true for the function itself.

Theorem 5.13. *The function $\mathcal{F}(K,n;\star)$ can be recursively computed using the functions $\mathcal{F}(K',n;\star)$ $(K' < K)$.*

Proof:

1$^{\text{st}}$STEP : We start out with $n = 2$. In this case, any (comprehensive) polyhedron is a sum of line segments (prisms), a fact that has been observed by MASCHLER–PERLES [2]. For arbitrary K assume that the slopes of line segments $\frac{a_1^{(k)}}{a_2^{(k)}}$ are strictly decreasing in k. Then the faces of Π are given by

$$\boldsymbol{F}^{(k)} \ := \ \sum_{l<k} a^{(l)1} + \Delta^{\boldsymbol{a}^{(k)}} + \sum_{l>k} a^{(l)2} \ . \tag{5.14}$$

Equivalently, the corresponding index sets are given by

$$
\begin{array}{rcccl}
\boldsymbol{J}^{(1)} & = \ldots = & \boldsymbol{J}^{(k-1)} & = & \{1\}, \\
 & & \boldsymbol{J}^{(k)} & = & \{1,2\}, \\
\boldsymbol{J}^{(k+1)} & = \ldots = & \boldsymbol{J}^{(K)} & = & \{2\}.
\end{array} \tag{5.15}
$$

This way, all faces are completely described. In this simple situation, the maximal faces are just the translates of the prisms, the proper vertices follow from Corollary 4.7.

2^nd STEP : Let $\boldsymbol{C} \subset \mathbb{R}^n_+$ be a convex comprehensive polyhedron with dimension $(n-1)$ and positive (outer) normal $\boldsymbol{n}$ and let $\boldsymbol{b}$ be a positive vector. Assume that none of the 2–dimensional subfaces of $\boldsymbol{C}$ has a normal parallel to $\boldsymbol{b}$. Then there is a unique $i \in \boldsymbol{I}$ such that the vertex $\boldsymbol{b}^i$ of $\Delta^{\boldsymbol{b}}$ admits of the normal $\boldsymbol{n}$. We assume that there is a procedure that computes the index i given $\boldsymbol{C}$ and $\boldsymbol{b}$.

3^rd STEP : Next we consider the case $K = 2$ that has already been treated in SECTION 4.

We know that for any two prisms $\Pi^{\boldsymbol{a}}$ and $\Pi^{\boldsymbol{b}}$ there is an ordering of $\boldsymbol{I}$ that completely describes all faces via formula (4.7).

Now, for $n = 3$, determine $i \in \boldsymbol{I} = \{1,2,3\}$ such that $\boldsymbol{b}^i$ admits the normal of $\Delta^{\boldsymbol{a}}$. Next, determine $j \in \boldsymbol{I} = \{1,2,3\}$ such that $\boldsymbol{a}^j$ admits the normal of $\Delta^{\boldsymbol{b}}$. Let k be the third index. Then the ordering is $i \prec k \prec j$.

For $n > 3$, suppose the procedure is known for $(n-1)$. Compute the ordering on $\boldsymbol{I}-\{n\}$ by projecting all prisms and the sum in $\mathbb{R}\big|_{\boldsymbol{I}-\{n\}}$. Likewise proceed in order to obtain an ordering on $\boldsymbol{I} - \{1\}$. The orderings are necessarily consistent and define an ordering on $\boldsymbol{I}$. This generates all faces and index sets via (4.7).

4^th STEP :

Now we proceed by induction in K. As previously, it is sufficient to assume that $n \geq K+1$ holds true. See Corollary 5.10 which shows that, whenever $K \geq n$ holds true, it suffices to compute the faces for any group of $(n-1)$ prisms and adding the appropriate vertices for the remaining ones (the latter procedure involves the one mentioned in the $3^{rd} STEP$).

5^th STEP : So we assume $K \leq n-1$. The following observation explains the basic idea.

Let $\boldsymbol{F}$ be a maximal face, we know that $\boldsymbol{F}$ is an l–face with $l \leq K-1 \leq n-2$. Consider the representation by means of the corresponding index sets, say

$$
\boldsymbol{F} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{5.16}
$$

and let $\boldsymbol{L} \in \boldsymbol{I}$ with $|\boldsymbol{L}| = l$ be the set of boundary indices (Definition 3.2). As $l \leq n-2$ is true, there are *at least two* indices in $\boldsymbol{I}$ that are *not* elements of $\boldsymbol{L}$. Assume for the sake of simplicity that these indices are $1, n$.

Now, first of all, consider the projection on $\mathbb{R}_{\boldsymbol{I}-\{n\}}$. We write $\boldsymbol{I}-n$ and $\mathbb{R}_{\boldsymbol{I}-n}$ for simplicity.

Then we have $\Delta_{\boldsymbol{J}^{(k)}} \,\big|\, _{\boldsymbol{I}-n} = \Delta_{\boldsymbol{J}^{(k)}-n}$. As $\boldsymbol{F} \,\big|\, _{\boldsymbol{I}-n}$ is maximal with respect to $\Pi_{\boldsymbol{I}-n} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{I}-n}$ we note that

$$\boldsymbol{F} \Bigg|_{\boldsymbol{I}-\{n\}} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}-n}. \tag{5.17}$$

Hence, $\boldsymbol{F} \,\big|\, _{\boldsymbol{I}-\{n\}}$ is an l face with the same set of boundary indices $\boldsymbol{L}$. Now $\boldsymbol{F} \,|\, _{\boldsymbol{I}-\{n\}}$ or rather the corresponding index sets can indeed be computed using the procedures defining the functions $\mathcal{F}(K', n; \star)$ for $K' < K$. This is so as the dimension has been reduced by one in view of the $4^{th}STEP$. Indeed, the argument is quite the same as for the reduction of the function f or, for that matter, the one emphasized in Remark (5.3).

The same procedure can now be followed using the projection on $\mathbb{R}_{\boldsymbol{I}-1}$.

For any k we have then computed recursively the index sets

$$\boldsymbol{J}^{(k)} - n \ , \ \boldsymbol{J}^{(k)} - 1 \quad ,$$

and as

$$\boldsymbol{J}^{(k)} = (\boldsymbol{J}^{(k)} - n) \cup (\boldsymbol{J}^{(k)} - 1) \quad ,$$

In fact it should be clear that all l faces are uniquely determined by their projection $\boldsymbol{F} \,|\, _{\boldsymbol{I}-\boldsymbol{L}}$ as we can successively eliminate one index $i \notin \boldsymbol{L}$ after the other and determine by the above procedure the $\boldsymbol{J}^{(k)}$ containing i.

6$^{\text{th}}$STEP :

To perform the induction, we may assume that we have computed all systems of index sets provided by $\mathcal{F}(K, n, \boldsymbol{a}^{\bullet} \,|\, _{\boldsymbol{I}-i})$ which involves the values of $\mathcal{F}$ for $K' < K$ in view of the $4^{th}STEP$.

Suppose $\boldsymbol{F}_{-i}$ is an l face of $\Pi_{\boldsymbol{I}-i}$ with boundary indices collected in $\boldsymbol{L}$. denote the index sets by $\boldsymbol{J}^{(k)}_{-i}$, these are obtained by induction. Then, for $j \neq i$, $j \notin \boldsymbol{L}$, there is an l–face $\boldsymbol{F}_{-j}$ of $\Pi_{\boldsymbol{I}-j}$ with the same set $\boldsymbol{L}$ of boundary indices such that

$$\boldsymbol{F}_{-i} \,|\, _{\boldsymbol{L}} = \boldsymbol{F}_{-j} \,|\, _{\boldsymbol{L}} \tag{5.18}$$

holds true and there are at least 2 such indices i and j. For any k, the index set $\boldsymbol{J}^{(k)}$ is then determined via

$$\boldsymbol{J}^{(k)} = \bigcup_{i \notin \boldsymbol{L}} \boldsymbol{J}^{(k)}_{-i} \tag{5.19}$$

which defines a face $\boldsymbol{F}$ of Π or an element of $\mathcal{F}(K, n; \boldsymbol{a}^{\bullet})$,

q.e.d.

Remark 5.14. The procedures developed within this section clearly induce algorithms for computing the number of maximal faces and their nature. In particular, Theorem 5.11 and Theorem 5.13 call for a corollary to actually produce this algorithm in a closed form. Within this presentation which is considered to be of a structural nature, is not our aim to actually produce this algorithm.

Besides, we will point out a non recursive algorithm which directly produces the data of the maximal faces from those of the family $\boldsymbol{a}^{(\bullet)}$ involved, see [3].

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6 Sums of 3 Prisms: Blocks

Within this section we shall discuss the case $K = 3$, i.e., the sum of three prisms in detail as it is the last case that can be visualized and simultaneously explains the general version for arbitrary K and n. Presently we write $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{d}b + \Pi^{\boldsymbol{c}}$ and Δ is the surface of Π.

Let us restate the Coincidence Theorem for this particular case.

Theorem 6.1. *Let $\boldsymbol{F}$ be a maximal face of Π. Then*

1. *Either there exists uniquely $i \in \boldsymbol{I}$ such that $\boldsymbol{a}^i + \boldsymbol{b}^i + \boldsymbol{c}^i \in \boldsymbol{F}$ holds true. That is, $\boldsymbol{F}$ contains a unique vertex of the Π and hence is a 1–face with $\boldsymbol{L} = \{i\}$. Thus, with suitable $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ $(k = 1, 2, 3)$*

$$\boldsymbol{F} \;=\; \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^{(1)}} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^{(2)}} + \Delta^{\boldsymbol{c}}_{\boldsymbol{J}^{(3)}} \tag{6.1}$$

   such that $i \in \cap_{k=1}^{3} \boldsymbol{J}^{(k)}$ holds true.

2. *Or else there is a unique pair $i, j \in \boldsymbol{I}$ such that $\boldsymbol{F}$ is a 2–face with $\boldsymbol{L} = \{i, j\}$. That is, $\boldsymbol{F} \cap \Delta_{\{ij\}}$ is a nondegenerate interval located within the relative interior of Δ_{ij}. Thus, with suitable $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ $(k = 1, 2, 3)$*

$$\boldsymbol{F} \;=\; \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^{(1)}} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^{(2)}} + \Delta^{\boldsymbol{c}}_{\boldsymbol{J}^{(3)}} \tag{6.2}$$

   such that $\{ij\} \subseteq \boldsymbol{J}^{(k)}$ holds true for one k while the other two index sets contain either i or j and not both.

Moreover, there are positive constants $c_{\boldsymbol{a}}, c_{\boldsymbol{b}}$ and $c_{\boldsymbol{c}}$ (unique up to a multiple) such that the normal $\boldsymbol{n}^{\star}$ of $\boldsymbol{F}$ is (up to a multiple) exactly the normal of the prism

$$\Delta^{\star} \;=\; c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}} \vee c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}} \vee c_{\boldsymbol{c}}\Pi^{\boldsymbol{c}} \;. \tag{6.3}$$

Example 6.2. If $n = K = 3$, then we obtain 3 translates of simplices and 3 diamonds on the surface Δ of Π. Each simplex $\Delta^{\boldsymbol{a}}$ has an image on the surface Δ. The tentacle system generated by each simplex consists of two diamonds. Any two simplices share exactly one diamond. As there are no further maximal faces, the number of faces is always $3 + 3 = 6$.

Consider the sum of 3 prisms in $\mathbb{R}^3$ represented by Figure 6.1. This version is dubbed the ***circle***. As we know, each pair of the prisms yields a sum the surface of which is represented by an ordering. The orderings are indeed "cyclic" as they are induced by the cyclic subgroup of permutations of three elements, we find

the orderings

$$123 \quad 231 \quad 312 \;. \tag{6.4}$$

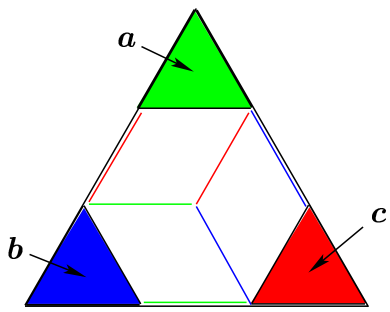

Figure 6.1: The circle

E.g., the sum of $\boldsymbol{a}$ and $\boldsymbol{b}$ has three maximal faces, these are described by

(6.5)

$\boldsymbol{a}$	$\boldsymbol{b}$
123	3
12	23
1	123

A similar diagram holds for $\boldsymbol{b}$ vs. $\boldsymbol{c}$ and $\boldsymbol{c}$ vs. $\boldsymbol{a}$, employing two further permutations. As there a three prisms involved, we obtain the complete description of the maximal faces generated by $\boldsymbol{a}$ and $\boldsymbol{b}$ by adding a suitable vertex of $\boldsymbol{c}$ which yields the following list.

(6.6)

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
123	3	3
12	23	1
1	123	1

Listing all three diagrams as induced by the 3 permutations we obtain the complete structure of Δ. Noted that the three diagrams list 9 faces, but each simplex appears twice, so we have indeed 6 maximal faces.

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Example 6.3. The next example is called the *windmill* and represented by Figure 6.2. This cephoid involves

the three orderings

$$132, \quad 321, \text{ and } 213 \tag{6.7}$$

between the three pairs which refer to the acyclic subgroup of permutations of three elements. Again a complete description has to involve a suitable vertex of the third prism.

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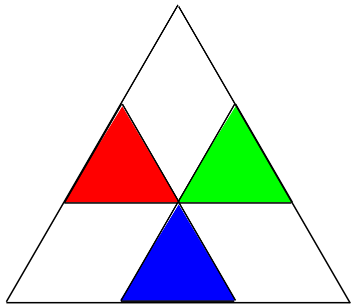

Figure 6.2: The windmill

For $n = 4, K = 3$ we can still represent the sum of three prism canononically, i.e., by depicting the projection into the subsimplex spanned by $3\boldsymbol{e}^1, 3\boldsymbol{e}^2, 3\boldsymbol{e}^3$, and $3\boldsymbol{e}^4$ of $\mathbb{R}^4$. this simplex is 3–dimensional and has the shape of a tetrahedron in $\mathbb{R}^3$.

It is clear from Corollary 4.7, that there are three translates of simplices on Δ. Each of these generates tentacles consisting of 2 cylinders (Corollary 4.8). Thus, we find immediately nine maximal faces that are 2–full, hence involve a vertex.

Now, in addition to these nine faces, there is exactly 1 *block*, i.e., a maximal face that is 3–full. Indeed, we have

Lemma 6.4. *For $n = 4$ and $K = 3$. Then Δ has exactly 10 faces.*

Proof: We imitate the general procedure explained in Theorem 5.13, which is quite simple for $K = 3$. Indeed, any maximal face containes either a vertex or cuts the interior of an edge. On the other hand, each edge intersects exactly 3 maximal faces. The 4–dimensional unit simplex (and its multiples) has 4 vertices and 6 edges, hence there must be 10 maximal faces. **q.e.d.**

The situation can be observed using the canonical representation as the lattice of faces is completely preserved. Indeed, we find exactly 3 prisms, 6 cylinders and 1 block in the following examples.

Example 6.5. We start out with a *circle of* 3 in $\mathbb{R}^4$. The canonical representation of this polyhedron is presented in Figure 6.3.

The translates of the simplices are located in the corners of Δ, hence each one is an $\boldsymbol{F}^i$ type. The fourth $\boldsymbol{F}^i$ type is a cylinder. All the other cylinders and the block are $\boldsymbol{F}^{ij}$ type. In particular, the block is an $\boldsymbol{F}^{23}$ type, more precisely

$$\Delta_{12} + \Delta_{23} + \Delta_{34}. \tag{6.8}$$

We list the three orderings referring to each sum of two prisms, these are

$$\begin{array}{cc|cc|cc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{c} & \boldsymbol{a} \\ \multicolumn{2}{c|}{1234} & \multicolumn{2}{c|}{2341} & \multicolumn{2}{c}{4312} \end{array} \tag{6.9}$$

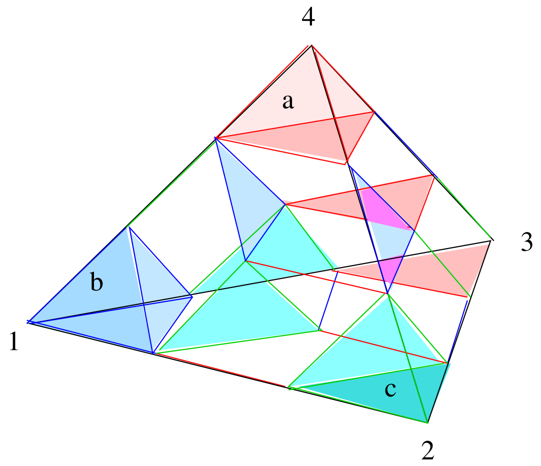

Figure 6.3: A circle of 3 prisms in 4 dimensions.

Of course this is not sufficient, but the orderings appear again in the full description of all maximal faces, where a suitable vertex is added to each of the faces generated. As it turns out, the single block corresponds to a further ordering, this one is already suggested by (6.8) to be 1234. Thus, a full description of all maximal faces is given as follows:

(6.10)

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{a}$	$\boldsymbol{c}$	$\boldsymbol{a}$	$\boldsymbol{b}$
1234	4	4	2341	1	1	4312	2	2
123	34	4	234	41	1	431	12	2
12	234	4	23	341	1	43	312	3
1	1234	1	2	2341	2	4	4312	4

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
12	23	34

Generally we have:

Theorem 6.6. *Let $n = 4$ and $K = 3$. Let $(\boldsymbol{a}^{(k)})_{k=1}^{K}$ denote a family of positive vectors in general position and let*

$$\Pi = \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} = \sum_{k=1}^{K} \Pi^{(k)}.$$

Then the maximal faces of Π are given as follows:

1. *There are three orderings $\overset{ij}{\prec}$, each one referring to a pair of prisms $\Pi^{(i)}, \Pi^{(j)}$ $(i, j \in \boldsymbol{I})$, which yield the maximal faces in the corresponding sum of these* two *prisms .*
2. *To each of these faces there correponds a unique vertex of the third polyhedron such that the result is a maximal face of Π.*
3. *There is a further ordering representing exactly one block. This block is uniquely defined by either one of the following requirements:*

(a) *The block covers exactly the missing vertex or interval in an edge that is* not *covered by the above faces constructed from the sums of two.*

(b) *The block is adjacent to at least one face generated by each of the sums of two.*

This is an obvious result.

Going back to the table (6.10) in Example 6.5, we can deduce from the upper set of three matrices (representing the sums of two plus a vertex) that the interior interval of edge 23 is not covered by a maximal face and that indeed the edge 23 does not intersect any translate of $\Delta^{\boldsymbol{b}}$. As the edge 23 intersects a translate of $\Delta^{\boldsymbol{a}}_{12}$ and of $\Delta^{\boldsymbol{c}}_{24}$ it is clear that $\Delta^{\boldsymbol{b}}_{23}$ is the missing edge.

As for the second argument, observe that

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 12 & 234 & 4 \end{array} \tag{6.11}$$

is adjacent to the block. This cylinder stems from the sum of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$. It is, by the way, also adjacent to its predecessor

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 123 & 34 & 4 \end{array} \tag{6.12}$$

which precedes within the same ordering, as the third vertex (i.e. 4) does not change. Similar, if we look to the second ordering (referring to $\boldsymbol{b}$ and $\boldsymbol{c}$), then we observe that

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 1 & 23 & 341 \end{array} \tag{6.13}$$

is adjacent to the block as well as to its predecessor in the ordering.

Finally, let us look to the third ordering, the one defined by $\boldsymbol{a}$ and $\boldsymbol{c}$. Here indeed the block has *two* neighbors which are

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 21 & 2 & 134 \\ 213 & 3 & 34 \end{array} \tag{6.14}$$

These two have been adjacent as far as the sum of $\Pi^{\boldsymbol{a}}$ and $\Pi^{\boldsymbol{c}}$ was concerned. But the unique vertex of $\boldsymbol{b}$ that renders these faces to become faces of Π changes from 2 to 3, so they are no longer adjacent but both adjacent to the block.

In the second sketch two translates of simplices are $\boldsymbol{F}^i$ types (actually $\boldsymbol{F}^1$ and $\boldsymbol{F}^3$). The faces $\boldsymbol{F}^2$ and $\boldsymbol{F}^4$ are cylinders and again the diamond is $\boldsymbol{F}^{24}$.

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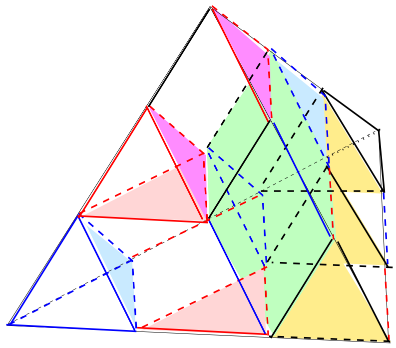

Figure 6.4: A further sum of 3 prisms

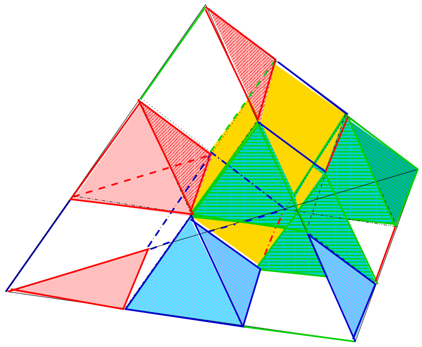

Figure 6.5: The marriage of a windmill and a circle

We can construct a 4–dimensional cephoid such the two 3-dimensional subfaces resemebles the two examples above. Thus, Figure 6.5 is "the marriage of a windmill and a circle".

We are now in the position to describe the case $K = 3$.

Theorem 6.7. *Let $K = 3$ and $K \leq n-1$. Then Π has $n + \binom{n}{2}$ maximal faces.*

Proof: Consider the canonical representation.

There are $\binom{n}{2}$ edges of the prism Δ^{Ke} used for the representation and each of them has a proper cut with exactly 3 maximal faces. On the other hand, each maximal face is either a 1–face (hence containes a vertex) or a 2–face (hence intersects exactly one edge properly and contains no vertex. Thus, the total number of faces is indeed $n + \binom{n}{2}$, i.e., the number of vertices plus the number of edges in the canonical representation.

q.e.d.

Theorem 6.8. *Let Π be a sum of 3 prisms in $\mathbb{R}^n_+$ and assume that no block contains a vertex. Then Π is characterized by 4 orderings. Three orderings correspond to each pair of prisms. These generate all together $(n-3)$ maximal faces according to the moving index principle for 1 index (see Remark 4.3). A further order which is connecting all three prisms generates $\binom{n-3}{2}$ faces according to the moving index principle.*

According to Theorem 4.2 any two prisms generate an ordering and hence n maximal faces of their sum according to the moving index principle 4.3. Each of these generates a maximal face within the sum of three prisms when combined with a proper vertex of the third prism (Theorem 4.6, Corollary 4.7). Clearly, the three translates of the prisms (each one with a suitable vertex of the other two) appear twice within this scheme, hence the total number of faces that correspond to the pairs of two prisms equals $3n - 3$.

The number of the remaining faces is now (Theorem 6.7)

$$n + \binom{n}{2} - 3(n-1) = \binom{n-2}{2}$$

These faces have to be sums involving at least an edge from each simplex, hence the size of each index set $\boldsymbol{J}^{(k)}$ $k = 1, 2, 3$ is at least two. As they have to be neighbors each of them has to be obtained from another one by the neighborhood theorem (Theorem 3.5). Thus, the two common indices have to be moved according to the moving index principle.

Note that the number of sets $\boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}, \boldsymbol{J}^{(3)}$ to be obtained by the moving index principle is indeed $\binom{n-3}{2}$. To see this, take the natural ordering $1, 2, \ldots, n$. Then, there is one system of index sets of the type

$$1, 2, \ldots, n-2 \quad * \quad n-2, n-1 \quad * \quad n-1, n,$$

there are two systems involving the first $(n-3)$ for the first index set:

$$1, 2, \ldots, n-3 \quad * \quad n-3, n-2, n-1 \quad * \quad n-1, n,$$

and

$$1, 2, \ldots, n-3 \quad * \quad n-3, n-2, \quad * \quad n-2, n-1, n,$$

(two versions obtained by moving the second index), three systems obtained by fixing the first $(n-4)$ indices etc.

Thus we have

$$1 + 2 + 3 + \ldots + n - 3 \;=\; \binom{n-2}{2}$$

systems which exactly generate the missing number of maximal faces.

Example 6.9. E.g., for $n = 7$ the blocks are suggested by the moving index principle for two indices as follows, assuming that the ordering is the natural one:

12345	56	67
1234	456	67
1234	45	567
123	3456	67
123	345	567
123	34	4567
12	23456	67
12	2345	567
12	234	4567
12	23	34567

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