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Regrouping of endowments in exchange markets with indivisible goods*

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Abstract

In this paper we are interested in efficient and individually rational exchange rules for markets with heterogeneous indivisible goods that exclude the possibility that an agent benefits by regrouping goods in her initial endowment. We present a suitable environment in which the existence of such rules can be analysed, and show the incompatibility of efficiency, individual rationality and regrouping-proofness even if agents' preferences are additive separable.

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Keywords: exchange markets; indivisible goods; regrouping-proofness.

1 Introduction

We consider exchange markets with heterogeneous indivisible objects in which each agent is initially endowed with a set of objects, and there is no divisible object one can use as a medium of exchange (cf. Papái (2004)). An exchange market can, for example, be seen as a generalization of a housing market (cf. Shapley and Scarf (1974)) in which each agent owns exactly one object. Moreover, it is also a generalization of the case in which the agents are

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allowed to trade different types of indivisible objects and each agent's initial endowment consists of one object of each type (cf. Moulin (1995) and Konishi et al. (2001)). An exchange rule for this market assigns to each trader a set of objects, and the main interest of study is in the existence of rules that satisfy such compelling properties like efficiency, individual rationality (no agent is worse off after trading with other agents) and strategy-proofness (no agent ever benefits from misrepresenting her preferences). We refer the reader to Roth (1982), Roth and Postlewaite (1977) and Ma (1994) for a study of the above question in the context of the classical housing market model, to Sönmez (1999) for a related study in general matching problems, and to Konishi et al. (2001) and Papái (2004) for an examination in the general model of an exchange market with heterogeneous indivisible goods.

However, the fact that each agent initially owns a set of objects opens more possibilities one may use in order to manipulate the outcome of an exchange rule. For example, an agent may have an incentive to manipulate the outcome of a rule via hiding or destroying a part of her initial endowment, or via transferring some objects to another agent. The study of hiding-proofness and destruction-proofness in the context of classical exchange economies goes back to Postlewaite (1979), while transfer-proof rules were studied by Sertel and Özkal-Sanver (2002) for the case of a two-sided matching model with endowments. In the context of exchange markets with indivisible goods, these three manipulation possibilities were examined in a recent work of Atlamaz and Klaus (2005). In particular, these authors show that efficient and individually rational rules are generally not immune to manipulations via endowments with some exceptions to two-agents exchange markets.

In the present paper we are also interested in exchange rules that prevent the possibility of manipulation via endowments but, in contrast to the cited papers, we would like to exclude the possibility that an agent benefits by regrouping goods in her initial endowment (regrouping-proofness). One can imagine for example an agent who owns a house in which there is an integrated kitchen by default, and bedroom furniture. Then, depending on the exchange rule, it may be worthy for the agent to separate the kitchen from the house and offer it as a single object on the market. Another possibility could be the bundling of the house with the integrated kitchen together with the bedroom furniture. In order to take such regrouping activities into account in our analysis we will make in what follows an explicit distinction between agents' *pre-endowments* and their *initial endowments*. More specifically, a *pre-endowment* is a set of indivisible objects an agent may offer for a possible trade, while an *initial endowment* is simply a *partition* of the corresponding agent's pre-endowment. In other words, we assume that each agent enters the market with an initial endowment (in which some indivisible objects from her pre-endowment may be already bundled), and an exchange rule assigns to each agent a set of indivisible objects. However, an allocation for such a market is a list of sets of objects (one for each agent) that respects not only the indivisibility of the objects but also the fact that some objects may be already bundled in the corresponding initial endowments.

Given such an environment, we define an exchange rule to be *regrouping-proof* if no agent benefits by announcing a partition of her pre-endowment that *differs* from her original initial endowment. As it turns out, combining regrouping-proofness with efficiency and individual rationality generates impossibility results even if agents' preferences are additive separable.

The paper is organized as follows. In Section 2 we introduce the basic components of our exchange market model. Section 3 presents our impossibility results. We conclude in Section 4 with a final discussion.

2 An exchange market setting

In an exchange market with indivisible objects there is a set of $n \geq 2$ *agents* $N = \{1, \ldots, n\}$ and a finite set K of *heterogeneous indivisible objects*. Each agent $i \in N$ is equipped with a *preference relation* R_i (i.e., a reflexive, transitive and complete binary relation) defined over 2^K. The associated strict preference and indifference are denoted by P_i and I_i, respectively. We let $\mathcal{R}_i$ denote the set of all preferences for agent i, $R = (R_i)_{i \in N}$ a *preference profile*, and $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_n$ the set of all preference profiles.

Each agent $i \in N$ has a *pre-endowment* $I_i \in 2^K$, where $I_i \cap I_j = \emptyset$ for $i \neq j$ and $\cup_{i \in N} I_i = K$. An *initial endowment* E_i of agent $i \in N$ is a *partition* of her pre-endowment, i.e., $E_i \in \mathcal{I}_i$ where $\mathcal{I}_i$ is the set of all partitions of I_i. An *initial endowment distribution* is a vector $E = (E_i)_{i \in N} \in \mathcal{I}$, where

$\mathcal{I} = \mathcal{I}_1 \times \ldots \times \mathcal{I}_n$.

Given a preference profile $R \in \mathcal{R}$ and an initial endowment distribution $E \in \mathcal{I}$, we denote by (R, E) an *exchange market* (with heterogeneous indivisible objects). Since the focus in this paper is on misrepresentation of initial endowments (in a way to be specified) and not on misrepresentation of preferences, we assume in what follows that the preference profile remains fixed while initial endowment distributions may vary. Hence, we denote an exchange market by its initial endowment distribution $E \in \mathcal{I}$.

An *allocation* for an exchange market $E \in \mathcal{I}$ is a list $S = (S_i)_{i \in N} \in \left(2^K\right)^n$ for which the following three conditions hold: (1) $\cup_{i \in N} S_i = K$, (2) for all $i, j \in N$ with $i \neq j$, we have $S_i \cap S_j = \emptyset$, and (3) for all $i \in N$ and for each $X \in E_i$ there is $j \in N$ such that $X \subseteq S_j$. The first two conditions are standard - the first requirement says that all indivisible objects are distributed among the agents, and the second guarantees that no two different agents own the same indivisible object. The third condition is specific to our setting and simply says that an allocation should respect the fact that some indivisible goods in agents' initial endowments may are bundled. More specifically, this condition requires goods belonging to the same bundle in the agents' initial endowments to be assigned to one agent only. For all $E \in \mathcal{I}$, we denote by $\mathcal{A}(E)$ the set of all allocations for E.

An (exchange) *rule* φ is a function that associates with each exchange market $E \in \mathcal{I}$ an allocation $\varphi(E) = (S_i)_{i \in N} \in \mathcal{A}(E)$. For each $i \in N$, we call $\varphi_i(E)$ the *allotment* of agent i at $\varphi(E)$.

We will be interested in rules that select (Pareto) efficient and individually rational allocations. A rule φ is called *efficient* if for all $E \in \mathcal{I}$ there is no allocation $(S_i)_{i \in N}$ such that $S_i R_i \varphi_i(E)$ for all $i \in N$, and $S_j P_j \varphi_j(E)$ for some $j \in N$. A rule φ is *individually rational* if $\varphi_i(E) R_i \left(\cup_{X \in E_i} X\right)$ for all $i \in N$.

Given that individual initial endowments are private information, an agent may manipulate the outcome to her advantage by announcing a partition of her pre-endowment that *differs* from her initial endowment. Let, for example, agent i's initial endowment be $E_i = \{\{ab\}, \{c\}\}$, where $\{ab\}$ denotes the bundle consisting of a and b. Agent i may then either decompose the existing bundle and announce $E_i' = \{\{a\}, \{b\}, \{c\}\}$, or bundle all goods and announce $E_i'' = \{\{abc\}\}$, or first decompose the existing bundle and then create a new bundle. In the latter case she may announce either $E_i''' = \{\{a\},\{bc\}\}$ or $E_i'''' = \{\{ac\},\{b\}\}$. If a rule is regrouping-proof, no agent should benefit from either of these possibilities.

In order to formally introduce regrouping-proofness, let us fix an exchange market $E \in \mathcal{I}$, a rule φ and an agent $i \in N$. Suppose $\varphi(E) = S$, and let $S' \in \left(2^K\right)^n$. We say that S' *is reachable for* i *from* S *via regrouping* if there is $E_i' \neq E_i$ such that $\varphi(E_i', E_{-i}) = S'$. We say that the rule φ is *manipulable at* S *by* i *via regrouping* if there is $S' \in \left(2^K\right)^n$ such that S' is reachable for i from S via regrouping, and $\varphi_i(E_i', E_{-i})\, P_i \varphi_i(E)$. The rule φ *is manipulable at* S *via regrouping* if there is $i \in N$ such that φ is manipulable at S by i via regrouping. Finally, φ is *regrouping-proof* if there is no $E \in \mathcal{I}$ such that φ is manipulable at $\varphi(E)$ via regrouping.

3 Two impossibility results

We start with the case in which agents are allowed to be indifferent between two different sets of indivisible objects, and consider the additive separable preference domain. Recall that agent i's preferences are *additive separable* if there is a function $u_i : K \to \mathbb{R}$ such that for all $T, T' \in 2^K$ we have that $T R_i T'$ if and only if $\sum_{k\in T} u_i(k) \geq \sum_{k\in T'} u_i(k)$. We show that on this domain no rule is efficient, individually rational and regrouping-proof. Clearly, the result holds on any preference domain that contains the domain of additive separable preferences.

Proposition 1 *For exchange markets with additive separable preferences, no rule is efficient, individually rational and regrouping-proof.*

Proof. Suppose that φ is efficient, individually rational and regrouping-proof. Let $N = \{1,2\}$, $E = (\{\{ab\}\},\{\{cd\}\})$, and $R = (R_1, R_2)$ with the following utility representation: $u_1(a) = u_1(b) = u_1(c) = 1$, $u_1(d) = -1$, and $u_2(b) = u_2(d) = 1$, $u_2(a) = u_2(c) = -1$.

The set of allocations for E consists of $A^1 = (\{a,b\},\{c,d\})$, $A^2 = (\{a,b,c,d\},\emptyset)$, and $A^3 = (\emptyset,\{a,b,c,d\})$. Notice that only A^1 and A^2 are individually rational and efficient. Hence, $\varphi(E) \in \{A^1, A^2\}$. Note also that, no matter which efficient and individually rational allocation the rule φ selects, agent 2 has an incentive to announce $E'_2 = \{\{c\}, \{d\}\}$ instead of $E_2 = \{\{cd\}\}$.

To see why, let us have a look at the set of allocations for (E_1, E'_2). The latter consists of the following elements: $B^1 = (\{a,b\}, \{c,d\})$, $B^2 = (\{a,b,c,d\}, \emptyset)$, $B^3 = (\emptyset, \{a,b,c,d\})$, $B^4 = (\{b\}, \{a,c,d\})$, $B^5 = (\{a,c,d\}, \{b\})$, $B^6 = (\{d\}, \{a,b,c\})$, and $B^7 = (\{a,b,c\}, \{d\})$. The individually rational allocations are B^1, B^2, and B^7, and only B^7 is efficient. Hence, $\varphi(E_1, E'_2) = B^7$. Notice that agent 2 prefers $\{d\}$ (her allotment at B^7) over $\{a,b\}$ (her allotment at A^1) and over $\emptyset$ (her allotment at A^2) in violation of regrouping-proofness.

Hence, we have incompatibility of efficiency, individual rationality and regrouping-proofness for $n = 2$. For $n > 2$, one adds agents who prefer their initial endowments to any possible trade. Since only agents 1 and 2 trade in this case, the incompatibility of these properties persists for $n > 2$. ■

Notice that the preferences in the example we used in the proof of Proposition 1 are perfectly dichotomous (cf. Dimitrov et al. (2004) and Ju (2003)). Perfectly dichotomous preferences constitute a very small subdomain of the domain of additive separable preferences. Furthermore, in this example, agents' allotments according to the efficient and individually rational allocations for the original exchange market (A^1 and A^2) lie in the same indifference classes. Hence, one can get the impression that the above impossibility result comes off because of the very large indifference classes the domain of perfectly dichotomous preferences allows for. However, as we show next, an analogous impossibility result holds on the additive separable *strict* preference domain (and on each domain that contains it).

Proposition 2 *For exchange markets with additive separable strict preferences, no rule is efficient, individually rational and regrouping-proof.*

Proof. Suppose that φ is efficient, individually rational and regrouping-proof. Let $N = \{1,2\}$, $E = (\{\{ab\}\}, \{\{cd\}\})$, and $R = (R_1, R_2)$ with the following utility representation (cf. Atlamaz and Klaus (2005)): $u_1(a) = 5$, $u_1(b) = 2.1$, $u_1(c) = 3$, $u_1(d) = 4$, and $u_2(a) = 6$, $u_2(b) = 3$, $u_2(c) = 1.1$, $u_2(d) = 4$.

The set of allocations for E consists of $A^1 = (\{a,b\}, \{c,d\})$, $A^2 =$

$(\emptyset, \{a, b, c, d\})$, $A^3 = (\{a, b, c, d\}, \emptyset)$. Notice that the allocations A^2 and A^3 are not individually rational and, hence, the only efficient and individually rational allocation is A^1, i.e., $\varphi(E) = A^1$.

Suppose now that agent 1 announces $E_1' = \{\{a\}, \{b\}\}$ instead of $E_1 = \{\{ab\}\}$. Then, the set of allocations for the exchange market (E_1', E_2) consists of $B^1 = (\{a, b\}, \{c, d\})$, $B^2 = (\emptyset, \{a, b, c, d\})$, $B^3 = (\{a, b, c, d\}, \emptyset)$, $B^4 = (\{a\}, \{b, c, d\})$, $B^5 = (\{b, c, d\}, \{a\})$, $B^6 = (\{b\}, \{a, c, d\})$, and $B^7 = (\{a, c, d\}, \{b\})$. The allocations that are individually rational are B^1 and B^5, and only B^5 is efficient. Notice that agent 1 prefers $\{b, c, d\}$ (her allotment at B^5) over $\{a, b\}$ (her allotment at A^1) in violation of regrouping-proofness.

Hence, we have incompatibility of efficiency, individual rationality and regrouping-proofness for $n = 2$. For $n > 2$, one adds agents who prefer their initial endowments to any possible trade. Since only agents 1 and 2 trade in this case, the incompatibility of these properties persists for $n > 2$. ■

4 Discussion

The setup presented in this paper differs from the model of an exchange market with heterogeneous indivisible goods in the sense that it allows for more structure in agents' initial endowments that are now partitions of the corresponding pre-endowments. Our impossibility results in this framework are in accordance with the non-existence of efficient, individually rational and hiding-proof rules for exchange markets with indivisible goods (cf. Atlamaz and Klaus (2005)) and, as a general observation, with the non-existence of efficient, individually rational and strategy-proof rules for the case in which the agents are allowed to trade different types of indivisible objects and each agent's initial endowment consists of one object of each type (cf. Konishi et al. (2001)).

Let us now explain in more details the intuition that is behind the incompatibility of efficiency, individual rationality and regrouping-proofness by taking a closer look at the examples used in the proofs of our results. Notice that, in these examples, the agent who manipulates in the corresponding market via her regrouping activities creates more (individually rational) allocations a rule may select from. The corresponding agent's hope is that her new allotment will be one she strictly prefers over her original allotment. Since regrouping-proofness dashes this hope, one would expect to easily find an exchange rule with the desired properties. However, creating more individually rational allocations is connected with the creation of more allocations that may Pareto dominate a given allocation. Indeed, in these examples, there is no efficient and individually rational allocation in the manipulated version of the markets that is efficient and individually rational for the corresponding original markets as well. Moreover, the efficient and individually rational allocations after the manipulation are not even allocations for the corresponding original market versions. Hence, there is a general tension based on the difficulty to prevent a rule from selecting an (efficient and individually rational) allocation in the manipulated version of the exchange market that is not an allocation in the original exchange market. The examples we use in the proofs of our impossibility results rely exactly on this point.

Finally, we would like to mention a possible relaxation of the notion of regrouping-proofness. As defined, an agent manipulates via regrouping just by announcing a partition that *differs* from her original one. One could also define the notions of *decomposition-proofness* (*bundling-proofness*) by requiring an agent not to benefit by announcing a partition of her pre-endowment that is *finer* (*coarser*) than the original one. Notice that the regrouping activities of the agents in the previous section were in fact decomposition activities and, hence, no rule is efficient, individually rational and decomposition-proof. The existence of efficient, individually rational and bundling-proof rules seems to us to be an interesting topic for further research.

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