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On top coalitions, common rankings, and semistrict core stability

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Abstract

The top coalition property of Banerjee et al. (2001) and the common ranking property of Farrell and Scotchmer (1988) are suffi cient conditions for core stability in hedonic games. We introduce the semistrict core as a stronger stability concept than the core, and show that the top coalition property guarantees the existence of semistrictly core stable coalition structures. Moreover, for each game satisfying the common ranking property, the core and the semistrict core coincide.

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1 Introduction

The dependence of a player's utility on the composition of members of her coalition can be examined in the context of hedonic coalition formation games (cf. Dréze and Greenberg (1980)). The formal model of a hedonic game was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002). In their work, the focus on the identity of the members of a coalition determines the structure of the game: the latter consists of a preference ranking, for each player, over the coalitions that player may belong to. Given a hedonic game, one is usually interested in the existence of

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stable outcomes, i.e., partitions of the set of players into coalitions. For instance, Banerjee et al. (2001) introduce a top coalition property and show that it guarantees the existence of core stable partitions, that is, partitions for which there is no group of individuals who can all be strictly better off by forming a new deviating coalition. This condition is a weaker version of the common ranking property of Farrell and Scotchmer (1988), and it is satisfied in many interesting economic applications, e.g., in the context of cost sharing problems.

However, neither the top coalition property nor the common ranking property guarantees that the strict core of the corresponding game is nonempty, i.e., it may exist a group of players in which everyone is weakly better off and at least one player is strictly better off in comparison to the corresponding coalitions in the partition under study. In this note we introduce the semistrict core as a stability notion for hedonic games that is stronger than the core but weaker than the strict core, and show that the top coalition property guarantees the existence of semistrictly core stable partitions in hedonic games. Moreover, for each game satisfying the common ranking property, the core and the semistrict core coincide.

Basic definitions are provided in Section 2, and Section 3 contains an example illustrating the discriminative power of the semistrict core. Our results are presented in Section 4.

2 Definitions

Consider a finite set of players $N = \{1, 2, \ldots, n\}$. A *coalition* is a non-empty subset of N. For each player $i \in N$, we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A collection Π of coalitions is called a *coalition structure* if Π is a partition of N. For each collection of coalitions Π and each $i \in N$, by $\Pi(i)$ we denote the coalition in Π containing i. Each player $i \in N$ has a preference $\succeq_i$ over $\mathcal{N}_i$, i.e., a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. We denote by $\succeq = (\succeq_1, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. Moreover, we assume that the preference of each player $i \in N$ over coalition structures is *purely hedonic*, i.e., it is completely characterized by $\succeq_i$ in such a way that, for each Π and

Π', each player i weakly prefers Π to Π' if and only if $\Pi(i) \succeq_i \Pi'(i)$. The pair $(N, \succeq)$ is called a *hedonic game*.

A coalition structure Π is *strictly core stable* for $(N, \succeq)$ if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ holds for all $i \in X$ and $X \succ_j \Pi(j)$ is true for some player $j \in X$. We say that Π is *core stable* for $(N, \succeq)$ if there does not exist a nonempty coalition X such that $X \succ_i \Pi(i)$ holds for each $i \in X$.

The following two properties have been shown to suffi ce for nonemptiness of the core (but not of the strict core) of a hedonic game. Let $(N, \succeq)$ be a hedonic game. Given a player set $V \subseteq N$, a coalition $S \subseteq V$ is a *top coalition* of V if for any $i \in S$ and any $T \subseteq V$ with $i \in T$, we have $S \succeq_i T$. We say that $(N, \succeq)$ satisfies the *top coalition property* if every player set has a top coalition. A game $(N, \succeq)$ satisfies the *common ranking property* if there exists an ordering $\trianglerighteq$ over $2^N \setminus \{\emptyset\}$ such that for any $i \in N$ and any $S, T \in \mathcal{N}_i$ we have $S \succeq_i T$ if and only if $S \trianglerighteq T$. Clearly, the common ranking property implies the top coalition property; the fact that the converse relation does not hold is illustrated by means of Game 4 in the work of Banerjee et al. (2001).

3 Example

Let us consider a hedonic game with player set $N = \{1, \ldots, 5\}$ and players'preferences as displayed in the following table[1]:

Player 1	**Player 2**
$12, 123, 124, 125, 1345$	$12, 123, 124, 125, 2345$
$134, 135, 145, 1234, 1235, 1245$	$234, 235, 245, 1234, 1235, 1245$
12345	12345
$1, \ldots$	$2, \ldots$

[1] Each player is indifferent between any two coalitions on the same row and strictly prefers a coalition on a higher row over a coalition on a lower row; in particular, each player is indifferent between being single and any coalition (she is a member of) not displayed in the corresponding column. We simplify notation for coalitions by using, e.g., "134" instead of $\{1, 3, 4\}$.

Player 3	**Player 4**	**Player 5**
$134, 135, 234, 235$	$134, 145, 234, 245$	$135, 145, 235, 245$
$1234, 1235, 1345, 2345$	$1234, 1245, 1345, 2345$	$1235, 1245, 1345, 2345$
12345	12345	12345
$3, \ldots$	$4, \ldots$	$5, \ldots$

One can easily check that the strict core of this game is empty (cf. Dimitrov and Haake (2005)). Let us examine in more detail the core stable partitions $\Pi' = \{12, 345\}$ and $\Pi'' = \{123, 45\}$. Consider the coalition $X = 1345$ and the following partitions of it - $\mathcal{X}^{\Pi'} = \{1, 345\}$ and $\mathcal{X}^{\Pi''} = \{13, 45\}$. Clearly, each player in X weakly prefers to be in X instead to be in her corresponding coalition either according to Π' or according to Π''. Notice however the following difference between Π' and Π'' in terms of $\mathcal{X}^{\Pi'}$ and $\mathcal{X}^{\Pi''}$: in each element of $\mathcal{X}^{\Pi''}$ there is at least one player who strictly benefits from being in X, while for $\mathcal{X}^{\Pi'}$ this is not the case (we have $X \sim_1 \Pi'(1)$ and $\mathcal{X}^{\Pi'}(1) = \{1\}$). One can easily check that there is no coalition that is a deviation from Π' in the described sense. In what follows we qualify Π' as being semistrictly core stable.

4 Semistrict core stability

Let $(N, \succeq)$ be a hedonic game. For any coalition $X \subseteq N$ and for any coalition structure Π of N, let $\mathcal{X}^{\Pi} := \{X \cap P \mid P \in \Pi\}$. We say that Π is *semistrictly core stable* if there does not exist a nonempty coalition $X \subseteq N$ such that

$$\text{for all } i \in X : X \succeq_i \Pi(i), \tag{1}$$

and

$$\text{for all } X' \in \mathcal{X}^{\Pi} : X \succ_j \Pi(j) \text{ for some } j \in X'. \tag{2}$$

Put in other words, in the definition of the semistrict core a more precise structure of the set of players who are strictly better off in a deviation is added; this addition is made by requiring that at least one player from each original coalition (according to

Π) strictly prefers to be in the corresponding deviating coalition.[2] Observe that strict core stability implies semistrict core stability that, in turn, implies core stability. In what follows, we denote by $C(N, \succeq)$ and $SSC(N, \succeq)$ the core and the semistrict core, respectively, of a hedonic game $(N, \succeq)$.

Proposition 1 *If $(N, \succeq)$ satisfies the top coalition property, then $SSC(N, \succeq) \neq \emptyset$.*

Proof. Let $V_0 = N$ and $S_1 \subseteq V_0$ be a top coalition of V_0. Next, define $V_1 = V_0 \setminus S_1$ and let S_2 be a top coalition of V_1. Continue in this way till the set N is exhausted, i.e., till $V_K = \emptyset$ and $V_{K-1} \neq \emptyset$ for some positive integer K. Let $\Pi = \{S_1, \ldots, S_K\}$. We show that Π is semistrictly core stable.

Suppose to the contrary that there is a deviation from Π, i.e., there exists a nonempty coalition $X \subseteq N$ satisfying (1) and (2). If $S_1 \cap X \neq \emptyset$, then, by the top coalition property, $S_1 \succeq_i X$ for all $i \in S_1 \cap X$. Thus, by noticing that $S_1 \cap X \in \mathcal{X}^{\Pi}$, we have a contradiction to (2), i.e., it is not possible X to contain members from S_1. If $S_2 \cap X \neq \emptyset$, then, again by the top coalition property, $S_2 \succeq_i X \setminus S_1 = X$ for all $i \in S_2 \cap X$. Since $S_2 \cap X \in \mathcal{X}^{\Pi}$, we have again a contradiction to (2). Thus, X does not include any members from S_2 either. By the same argument repeatedly applied, we conclude that no deviation (satisfying (1) and (2)) from Π is possible. Hence, $\Pi \in SSC(N, \succeq)$. ∎

Since the common ranking property implies the top coalition property, the following result follows immediately.

Corollary 1 *If $(N, \succeq)$ satisfies the common ranking property, then $SSC(N, \succeq) \neq \emptyset$.*

Finally, we show that the common ranking property is strong enough to guarantee that all core stable partitions in a hedonic game are semistrictly core stable as well.

Proposition 2 *If $(N, \succeq)$ satisfies the common ranking property, then $SSC(N, \succeq) = C(N, \succeq)$.*

Proof. Suppose to the contrary that $C(N, \succeq) \setminus SSC(N, \succeq) \neq \emptyset$ and let $\Pi \in C(N, \succeq) \setminus SSC(N, \succeq)$. Then, there is a deviation from Π, i.e., there exists a nonempty

[2] The idea of semistrict core stability can already be found in the work of Kirchsteiger and Puppe (1997).

coalition $X \subseteq N$ satisfying (1) and (2). Since $\Pi \in C(N, \succeq)$, there is a player $i^* \in X$ such that $\Pi(i^*) \succeq_{i^*} X$ which, in combination with (1), implies $\Pi(i^*) \sim_{i^*} X$. Thus, by the common ranking property, $\Pi(i^*)$ and X are commonly indifferent. Hence, again by the common ranking property, we have $\Pi(i^*) \sim_j X$ for all $j \in \Pi(i^*) \cap X$ in contradiction to (2). ■

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