Working Papers

Institute of Mathematical Economics

368

June 2005

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http://www.wiwi.uni-bielefeld.de/˜imw/Papers/showpaper.php?368
ISSN: 0931-6558

Convex games, clan games, and their marginal games*

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June 2005

Abstract

We provide characterizations of convex games and total clan games by using properties of their corresponding marginal games. As it turns out, a cooperative game is convex if and only if all its marginal games are superadditive, and a monotonic game satisfying the veto player property with respect to the members of a coalition C is a total clan

*D. Dimitrov gratefully acknowledges financial support from the Alexander von Humboldt Foundation.

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game (with clan C) if and only if all its C-based marginal games are subadditive.

AMS Classification: 91A12.

JEL Classification: C71.

Keywords: characterization, convex games, marginal games, subadditive games, superadditive games, total clan games.

1 Introduction

In this paper we consider two important classes of cooperative games - the class of convex games and the class of total clan games - and study the relationships between each game in the corresponding class and its appropriately defined marginal games.

The class of *convex games* was introduced by Shapley (1971) and has attracted a lot of attention because the games in this class have very nice properties. We mention here that the core of a convex game is the unique stable set and its extreme points can be easily described, the Shapley value of a convex game is in the barycenter of the core in the sense that it is the average of the marginal vectors, and that convex games are *totally convex* since each subgame of a convex game is also convex. Many equivalent characterizations of this class of games can be found in the cooperative game literature. For example, the supermodularity of the characteristic function, the increasing marginal return properties for individual players and for groups of players, and characterizations that deal with the relation between the core and the Weber set (cf. Shapley (1971), Ichiishi (1981), Curiel and Tijs (1991), Curiel (1997)); a characterization of a convex game using the exactness of its subgames can be found in the work of Biswas et al. (1999) and Azrieli and Lehrer (2005).

The class of *clan games* has been introduced by Potters et al. (1989)

to model social conflicts between a "powerful" group of players (the clan) and "powerless" players (non-clan members). Economic applications of such games include bankruptcy problems, production economies, information acquisition and holding situations (cf. Muto et al. (1988), Potters et al. (1989), Branzei et al. (2001), Tijs et al. (2005)). In the work of Voorneveld et al. (2002) *total clan games* were introduced as monotonic clan games whose subgames inherit the structure of the original (clan) game.

It is worth mentioning that both convex games and total clan games have monotonic allocation schemes; specifically, convex games have population monotonic allocation schemes (cf. Sprumont (1990)) and total clan games have bi-monotonic allocation schemes (cf. Branzei et al. (2001) and Voorneveld et al. (2002)).

In this paper we present another common feature of these classes of games that deals with their characterization by means of certain properties of appropriately defined *marginal games*. In Section 3 we show that a game is convex if and only if all its marginal games are superadditive. Further, in Section 4, we consider monotonic games with veto group C, introduce the notion of a C-based marginal game and prove that a veto monotonic game with veto group C is a total clan game (with clan C) if and only if all its C-based marginal games are subadditive.

2 Preliminaries

A cooperative game with transferable utility (a TU-game) is a pair $\langle N, v \rangle$, where $N = \{1, \dots, n\}$ is a *set of players* and $v \colon 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the *worth* of coalition S, i.e., the members of S can obtain a total payoff of $v(S)$ by agreeing to cooperate. The *subgame* $\langle S, v_{|S} \rangle$ is obtained from $\langle N, v \rangle$ by restricting attention to $S \subseteq N$. For each coalition $S \in 2^N \setminus \{\emptyset\}$ and each player $i \in S$, define

$$M_i(S, v) := v(S) - v(S \setminus \{i\})$$

to be the *marginal contribution of player i to coalition S*. A player $i \in N$ is a *veto player* if $v(S) = 0$ whenever $i \notin S$. A game $\langle N, v \rangle$ is *monotonic* if for each $S_1, S_2 \in 2^N$ with $S_1 \subseteq S_2$ we have $v(S_1) \leq v(S_2)$. A game $\langle N, v \rangle$ is called

- *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$;
- *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

A game $\langle N, v \rangle$ is *subadditive* (*concave*) if $\langle N, -v \rangle$ is superadditive (convex). Clearly, each convex (concave) game is also superadditive (subadditive). In what follows we will use also the following alternative characterization of concavity (cf. Curiel (1997)). A game $\langle N, v \rangle$ is concave if for every pair of coalitions $S_1, S_2 \in 2^N$ and every $i \in N$ we have that

$$i \in S_1 \subseteq S_2 \text{ implies } M_i(S_1, v) \geq M_i(S_2, v).$$

3 Convex games

In order to provide a characterization of convex games we will make use of the following notion of a marginal game. Given a game $\langle N, v \rangle$ and a coalition $T \subseteq N$, the *T-marginal game* $v_T : 2^{N \setminus T} \to \mathbb{R}$ is defined by

$$v_T(S) := v(S \cup T) - v(T)$$

for each $S \subseteq N \setminus T$.

Marginal games turned out to be useful for proving the fact that the core of a game is a subset of the Weber set (cf. Weber (1988)); they also play a key role for generating the constrained egalitarian solution for convex games (cf. Dutta and Ray (1989)) and the equal split-off set for arbitrary TU-games (cf. Branzei et al. (2004a)).

It is known that if a game is convex then all its marginal games are also convex (cf. Dutta and Ray (1989)). The next example shows that the superadditivity of a game is not necessarily inherited by its marginal games.

Example 1 *Let* $N = \{1,2,3\}$ *and* $v(\{1\}) = 10$, $v(\{1,2\}) = 12$, $v(\{1,3\}) = 11$, $v(\{1,2,3\}) = 12\frac{1}{2}$, *and* $v(S) = 0$ *for all other* $S \subset N$. *Clearly, the game* $\langle N, v\rangle$ *is superadditive. Its* $\{1\}$*-marginal game is given by* $v_{\{1\}}(\{2\}) = v(\{1,2\}) - v(\{1\}) = 2$, $v_{\{1\}}(\{3\}) = 11 - 10 = 1$, *and* $v_{\{1\}}(\{2,3\}) = 2\frac{1}{2}$. *Since* $v_{\{1\}}(\{2,3\}) = 2\frac{1}{2} < 3 = v_{\{1\}}(\{2\}) + v_{\{1\}}(\{3\})$, *the marginal game* $\left\langle \{2,3\}, v_{\{1\}} \right\rangle$ *is not superadditive.*

We show next that the superadditivity of all marginal games of a TU-game v assures a stronger property than the superadditivity of v, namely the convexity of v. This result has been independently obtained by Branzei et al. (2004b) and Martinez-Legaz (2004).

Theorem 1 *A game* $\langle N, v\rangle$ *is convex if and only if for each* $T \in 2^N$ *the* T*-marginal game* $\langle N \setminus T, v_T\rangle$ *is superadditive.*

Proof. (i) Suppose $\langle N, v\rangle$ is convex and let $T \subseteq N$. Take $S_1, S_2 \subseteq N \setminus T$.

Then

$$
\begin{aligned}
& v_T(S_1 \cup S_2) + v_T(S_1 \cap S_2) \\
= \;& v(T \cup S_1 \cup S_2) + v(T \cup (S_1 \cap S_2)) - 2v(T) \\
= \;& v((T \cup S_1) \cup (T \cup S_2)) + v((T \cup S_1) \cap (T \cup S_2)) - 2v(T) \\
\geq \;& v(T \cup S_1) + v(T \cup S_2) - 2v(T) \\
= \;& (v(T \cup S_1) - v(T)) + (v(T \cup S_2) - v(T)) \\
= \;& v_T(S_1) + v_T(S_2),
\end{aligned}
$$

where the inequality follows from the convexity of v. Hence, v_T is convex (and superadditive as well).

(ii) Suppose that for each $T \in 2^N$ the game $\langle N \setminus T, v_T \rangle$ is superadditive. Take $S_1, S_2 \subseteq N$. We have to prove that

$$v(S_1 \cup S_2) + v(S_1 \cap S_2) \geq v(S_1) + v(S_2).$$

If $S_1 \cap S_2 = \emptyset$, then the assertion easily follows from the superadditivity of the game $\langle N \setminus \emptyset, v_\emptyset \rangle = \langle N, v \rangle$ and $v(\emptyset) = 0$.

Suppose now $S_1 \cap S_2 \neq \emptyset$ and let $T := S_1 \cap S_2$. Because $\langle N \setminus T, v_T \rangle$ is superadditive, we have that

$$v_T(S_1 \setminus T) + v_T(S_2 \setminus T) \leq v_T((S_1 \cup S_2) \setminus T)$$

from which follows

$$v(S_1) - v(T) + v(S_2) - v(T) \leq v(S_1 \cup S_2) - v(T)$$

implying

$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(T).$$

■

Using the above characterization, we can answer immediately the following question: "Under which conditions are all marginal games of a superadditive original game superadditive?"

Corollary 1 *Let $\langle N, v\rangle$ be a superadditive game. Then $\langle N \setminus T, v_T\rangle$ is superadditive for each $T \in 2^N$ if and only if $\langle N \setminus T, v_T\rangle$ is convex for each $T \in 2^N$.*

Remark 1 *In view of Theorem 1, a game $\langle N, v\rangle$ is concave if and only if for each $T \in 2^N$ the marginal game $\langle N \setminus T, v_T\rangle$ is subadditive.*

4 Total clan games

Clan games have been introduced in Potters et al. (1989) to model social conflicts between "powerful" players (clan members) and "powerless" players (non-clan members). In a clan game there is a group of "powerful" players which has veto power and the "powerless" players operate more profitably in unions than on their own. More precisely, a game v is a *clan game* with clan $C \in 2^N \setminus \{\emptyset, N\}$ if it satisfies the following four conditions:

(a) $v(S) \geq 0$ for all $S \subseteq N$;

(b) $M_i(N, v) \geq 0$ for each $i \in N$;

(c) Clan property: every player $i \in C$ is a veto player, i.e., $v(S) = 0$ for each coalition S with $C \nsubseteq S$;

(d) Union property: $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(N, v)$ if $C \subseteq S$.

For notational convenience, define $\mathcal{P}^C := \{S \subseteq N \mid C \subseteq S\}$ as the collection of all coalitions containing the clan C.

A game v is a *total clan game* with clan $C \in 2^N \setminus \{\emptyset, N\}$ if v is monotonic and $v_{|S}$ is a clan game (with clan C) for every $S \in \mathcal{P}^C$. Notice that in the definition of a total clan game attention is restricted to coalitions that contain the clan C, since the clan property of v implies that in the other subgames the characteristic function is simply the zero function. Note further that monotonicity implies (a) and (b). As shown by Voorneveld et al. (2002) a game v is a total clan game with clan C if and only if v is monotonic, every player $i \in C$ is a veto player, and for all coalitions $S_1, S_2 \in \mathcal{P}^C$ the following *C-concavity property* holds:

$$S_1 \subseteq S_2 \text{ and } i \in S_1 \setminus C \text{ imply } M_i(S_1, v) \geq M_i(S_2, v).$$

Let $N = \{1, \ldots, n\}$ and $C \in 2^N \setminus \{\emptyset, N\}$. In what follows we denote by MV_C^N the set of all monotonic games on N satisfying the veto player property w.r.t. each player $i \in C$.

Given a game $v \in MV_C^N$ and a coalition $T \in 2^{N \setminus C}$, the *C-based T-marginal game* $v_T^C : 2^{N \setminus T} \to \mathbb{R}$ is defined by

$$v_T^C(S) := v(S \cup T \cup C) - v(T \cup C)$$

for each $S \subseteq N \setminus T$.

We consider now the effect of requiring subadditivity for each C-based marginal game of an original game $v \in MV_C^N$.

Theorem 2 *Let $v \in MV_C^N$. Then v is a total clan game if and only if for each $T \in 2^{N \setminus C}$ the C-based T-marginal game $\langle N \setminus T, v_T^C \rangle$ is subadditive.*

Proof. (i) Suppose $\langle N, v \rangle$ is a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$ and let $T \in 2^{N \setminus C}$. We show that the game v_T^C is concave and hence, subadditive. Thus, we have to show that for every pair of coalitions $S_1, S_2 \in 2^{N \setminus T}$ and every $i \in N$ we have that

$$i \in S_1 \subseteq S_2 \text{ implies } M_i(S_1, v_T^C) \geq M_i(S_2, v_T^C).$$

Take S_1, S_2 and i as above. We distinguish the following two cases:

(i.1) $i \in S_1 \setminus C$. We have

$$\begin{aligned}
& M_i\left(S_1, v_T^C\right) \\
= \;& v_T^C(S_1) - v_T^C(S_1 \setminus \{i\}) \\
= \;& v\left(S_1 \cup T \cup C\right) - v(T \cup C) - v\left((S_1 \setminus \{i\}) \cup T \cup C\right) + v(T \cup C) \\
\geq \;& v\left(S_2 \cup T \cup C\right) - v\left((S_2 \setminus \{i\}) \cup T \cup C\right) \\
= \;& \left(v\left(S_2 \cup T \cup C\right) - v(T \cup C)\right) - \left(v\left((S_2 \setminus \{i\}) \cup T \cup C\right) - v(T \cup C)\right) \\
= \;& v_T^C(S_2) - v_T^C(S_2 \setminus \{i\}) \\
= \;& M_i\left(S_2, v_T^C\right),
\end{aligned}$$

where the inequality follows from the C-concavity of v.

(i.2) $i \in S_1 \cap C$. We have

$$\begin{aligned}
& M_i\left(S_1, v_T^C\right) \\
= \;& v_T^C(S_1) - v_T^C(S_1 \setminus \{i\}) \\
= \;& v\left(S_1 \cup T \cup C\right) - v(T \cup C) - v\left((S_1 \setminus \{i\}) \cup T \cup C\right) + v(T \cup C) \\
= \;& 0 \\
= \;& v\left(S_2 \cup T \cup C\right) - v(T \cup C) - \left(v\left((S_2 \setminus \{i\}) \cup T \cup C\right) - v(T \cup C)\right) \\
= \;& v_T^C(S_2) - v_T^C(S_2 \setminus \{i\}) \\
= \;& M_i\left(S_2, v_T^C\right),
\end{aligned}$$

where the third and the fourth equalities follow from $S_1 \cup C = (S_1 \setminus \{i\}) \cup C$ and $S_2 \cup C = (S_2 \setminus \{i\}) \cup C$, respectively.

(ii) Suppose that for each $T \in 2^{N \setminus C}$ the game $\left\langle N \setminus T, v_T^C \right\rangle$ is subadditive. Take $S_1, S_2 \in \mathcal{P}^C$ with $S_1 \subseteq S_2$. We show that the inequality

$$M_i\left(S_1, v\right) = v\left(S_1\right) - v\left(S_1 \setminus \{i\}\right) \geq v(S_2) - v\left(S_2 \setminus \{i\}\right) = M_i\left(S_2, v\right)$$

holds for each $i \in S_1 \setminus C$.

Let $i \in S_1 \backslash C$ and $T := S_1 \backslash (C \cup \{i\})$. Take the coalitions $C \cup \{i\} \subseteq N \backslash T$ and $S_2 \setminus S_1 \subseteq N \setminus T$, and notice that $(C \cup \{i\}) \cap (S_2 \setminus S_1) = \emptyset$. Because $\langle N \setminus T, v_T^C \rangle$ is subadditive, we have

$$v_T^C (C \cup \{i\}) + v_T^C (S_2 \setminus S_1) \geq v_T^C ((S_2 \setminus S_1) \cup C \cup \{i\})$$

iff

$$v_T^C (C \cup \{i\}) \geq v_T^C ((S_2 \setminus S_1) \cup C \cup \{i\}) - v_T^C (S_2 \setminus S_1)$$

iff

$$\begin{aligned} & v (C \cup \{i\} \cup T) - v (T \cup C) \\ \geq \;& v ((S_2 \setminus S_1) \cup C \cup \{i\} \cup T) - v (T \cup C) - v ((S_2 \setminus S_1) \cup T \cup C) + v (T \cup C) \end{aligned}$$

iff

$$v (C \cup \{i\} \cup T) - v (T \cup C) \geq v ((S_2 \setminus S_1) \cup C \cup \{i\} \cup T) - v ((S_2 \setminus S_1) \cup T \cup C)$$

iff

$$v (S_1) - v (S_1 \setminus \{i\}) \geq v (S_2) - v (S_2 \setminus \{i\})$$

implying that v is C-concave and hence, a total clan game (with clan C). ■

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