Auction Prices and Asset Allocations of the Electronic Equity Trading System *Xetra*

Li Xihao

Bielefeld Graduate School of Economics and Management

Jan Wenzelburger

Department of Economics

Bielefeld University, P.O. Box 100 131
D-33501 Bielefeld, Germany

xli@wiwi.uni-bielefeld.de
jwenzelb@wiwi.uni-bielefeld.de

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Abstract

This paper formalizes the price and asset allocation mechanism of multi-unit double auctions in *Xetra*, the electronic equity trading system operated by the German stock exchange. The trading principles are imbedded into the classical theory of quantity rationing. The properties of Xetra auctions are investigated and possibilities to improve the auction mechanism are discussed.

Keywords: Multi-unit double auctions, market mechanism, electronic trading systems, Xetra, financial markets, disequilibrium, rationing.

JEL Classification: D44, D40, G12.

1 Introduction

In the past decades, the amount of worldwide equity transactions that were processed by electronic trading platforms increased significantly. In Germany, over 90% of equity transactions are executed by the *Xetra* System operated by German stock exchange, cf. Gruppe Deutsche Börse (2003). Other well-established European equity trading platforms are the Pan European stock exchange, Euronext, which connects the stock exchanges of Amsterdam, Brussels, and Paris, the Portuguese stock exchange BVLP (Bolsa de Valores de Lisboa e Porto), and the London International Financial Futures and Options Exchange (LIFFE). Many countries including China are currently in the process of establishing their own domestic electronic equity trading platforms.

There are at least three advantages of using electronic equity trading platforms instead of trading on conventional floor markets. First, electronic platforms provide more real time information during the trading process. Second, electronic trading platforms are more transparent than conventional markets. Equity prices are stipulated according to well-specified market models while market makers in conventional markets have considerable influence on the price determination. This 'black-box' argument applies in particular for prices which are negotiated among a small number of dealers. Third, transaction costs of electronic equity trading platforms are on average lower than those of conventional floor markets.

Despite the popularity of electronic trading systems, little is known about a microeconomic foundation of the market models and the investment strategies that are adapted to these markets, e.g., see Harris (1990) and Huang & Stoll (1991). Electronic equity markets have attracted only relatively little attention in the theory of financial markets. The classical closed-form analysis in the literature derives asset market prices from intertemporal equilibrium conditions assuming that asset markets clear and expectations are always rational, e.g., see Ingersoll (1987), Pliska (1997), or LeRoy & Werner (2001). Böhm, Deutscher & Wenzelburger (2000) pointed out that this classical approach involves two implicit conditions: One for the assumption of market clearing in each trading period and the other for the rational-expectations hypothesis. The latter condition may be replaced by introducing the notion of a forecasting rule along with the concept of a *perfect* forecasting rule as an operational concept for rational expectations. The market-clearing condition, however, still remains an unresolved conceptional problem as it is easy to construct an asset market for which market-clearing prices do not exist generically, e.g., see Böhm (2002). This theoretical insight provides the motivation to study the market mechanisms of 'real' financial markets which handle a great diversity of traders every day. A prominent example of such markets is an electronic equity trading platform in which buyers and sellers trade with each other through a computer system.

One of the well-established electronic equity trading systems is organized by the German stock exchange (Deutsche Börse) in Frankfurt, Germany. Deutsche Börse operates an electronic equity trading platform called *Xetra* which is an order-driven system in which agents can trade equities by submitting certain types of order specifications through a computer interface. A description of this interface along with the market models is found in a brochure distributed by Gruppe Deutsche Börse (2003). Despite the clarity of the Xetra market models, literature in financial markets so far has provided only little understanding of the nature of the formation of auction prices and final transactions in electronic equity markets and its implication for possible investment strategies. The price mechanism of electronic equity markets has intuitively been described in Sharpe, Alexander & Bailey (1999), however, without formal rigor.

Xetra market models consist essentially of two trading forms, continuous trading and (Xetra) auctions. This paper focuses on the Xetra auction which is composed of three phases: a call phase, a price determination phase, and an order book balancing phase. During the call phase, traders may submit order specifications. Orders will be tagged with a time-priority index and collected in an order book for each equity. The call phase has a random end after a fixed minimum time span and is followed by the price determination phase in which the auction price is determined in light of the Xetra market model. As soon as the auction price has been stipulated, orders are matched and transactions are carried out. For equities without market imbalance information, the surplus is offered again to traders in the order book balancing phase when not all orders in the order book can be fully executed. At the end of the auction process, all orders which were not or only partially executed are transferred to the next possible trading form or deleted according to their trading restrictions.

This paper aims at a formal description of the price and allocation mechanism of Xetra auctions. Section 2 depicts a formal representation of the demand and supply schedules. Section 3 formalizes the Xetra auction price mechanism as well as an investigation of the properties of this price mechanism. Section 4 describes the Xetra auction allocation mechanism along with a discussion of its properties. Section 5 concludes this paper with an outline of future research.

2 Demand and Supply Schedule

We describe the Xetra auction of a single equity. Xetra auctions handle several types of orders: limit orders, market orders, market-to-limit orders, iceberg orders, stop orders and quotes. Xetra auctions consider Market-to-limit orders as market orders, iceberg orders as limit orders, and stop orders as either market orders or limit orders depending on the imposed trading restrictions. Quotes are handled as two orders (a limit bid and another limit ask simultaneously) in the order book. Hence, in essence, Xetra auctions handle two types of orders: limit orders and market orders. An order specification with a claim to sell is called an ask (limit/market) order and an order with a claim to buy is called a bid (limit/market) order.

During the call phase, Xetra collects all asks and bids quoted by traders in an order book, labeled with a time-priority index. Assume that there are $I+1>2$ bids, indexed by $i \in \{0,1,\ldots,I\}$ and $J+1>2$ asks indexed by $j \in \{0,1,\ldots,J\}$. In particular, assume that bid 0 and ask 0 are market orders while the rest are limit orders. Therefore, $\{1,\ldots,I\}$ is the index set of limit bids and $\{1,\ldots,J\}$ the index set of limit asks.

All these orders constitute the demand and the supply side of the market. To formulate the price and the allocation mechanism of Xetra auctions, we first focus on a convenient presentation of individual bids and asks or, in other words, of individual demand and supply schedules.

2.1 Demand-to-buy schedule

For bid 0, which is a market order with a non-negative quantity d_0, the individual demand function is:

$$\mathcal{L}_0^D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto d_0 \end{cases}.$$

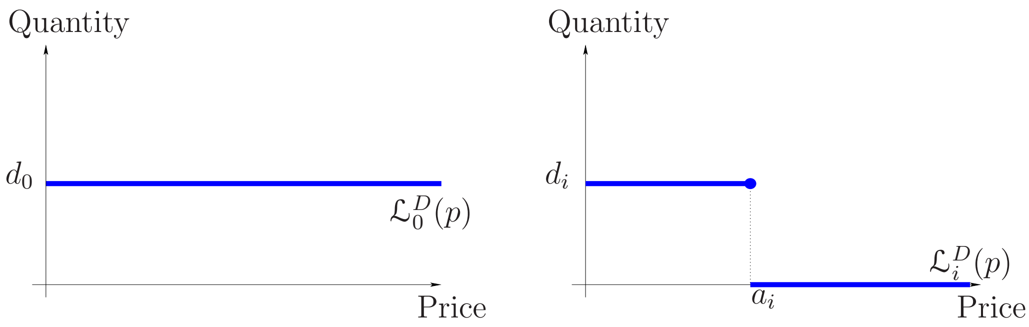
Each bid $i \in \{1,\ldots,I\}$ consists of a price-quantity pair (a_i, d_i), which states the intention to buy d_i shares when the auction price is no higher than a_i. Bid i can be represented as an individual demand function as follows.

If $1_{A_i^D}(p)$ denotes a characteristic function of the compact interval $A_i^D = [0, a_i]$ such that

$$1_{A_i^D}(p) := \begin{cases} 1 & \text{when } \ p \in A_i^D, \\ 0 & \text{when } \ p \in \mathbb{R}_+ \setminus A_i^D, \end{cases}$$

we define the individual demand function that represents bid i by the step function

$$\mathcal{L}_i^D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto d_i 1_{A_i^D}(p) \end{cases}. \tag{1}$$



(a) The market order: bid 0

(b) The limit order: bid i

Figure 1: The Individual demand function.

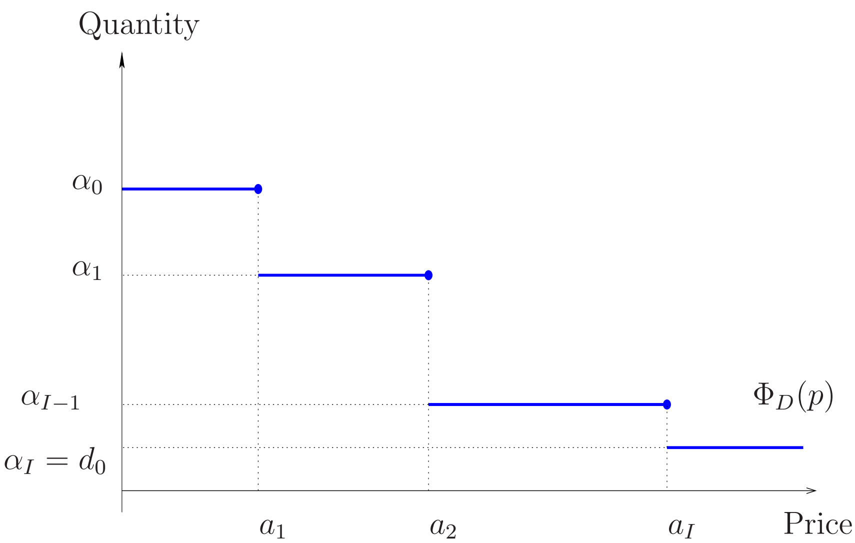


Figure 2: The Aggregate demand function.

The aggregate demand function is defined as the sum of the individual demand functions:

$$\Phi_D : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \sum_{i=0}^{I} \mathcal{L}_i^D(p). \end{cases} \tag{2}$$

Without loss of generality, assume $a_I > \ldots > a_2 > a_1 > 0$. Then we obtain the following lemma:

Lemma 1. *Let $a_I > \ldots > a_2 > a_1 > 0$. The aggregate demand function $\Phi_D(p)$ is non-increasing and takes the form:*

$$\Phi_D(p) = \sum_{i=0}^{I} \alpha_i 1_{A_i}(p), \quad p \in \mathbb{R}_+, \tag{3}$$

where $\alpha_i := d_0 + \sum_{k=i+1}^{I} d_k$, $i = 0, 1, \ldots, I-1$, $\alpha_I := d_0$ *and* $A_0 := [0, a_1]$, $A_i := (a_i, a_{i+1}]$, $i = 1, \ldots, I-1$, $A_I := (a_I, +\infty)$.

Proof. Notice that bid 0, which is the market order, is executable for any $p \in \mathbb{R}_+$. $\{A_0, \ldots, A_I\}$ is by construction a partition of $\mathbb{R}_+$. Let $i_* \in \{0, \ldots, I-1\}$ be arbitrary but fixed. Then $p \in A_{i_*}$ implies that all bids $i = 0, i_* + 1, \ldots I$ are executable. The corresponding aggregate volume is $\alpha_{i_*} = d_0 + \sum_{k=i_*+1}^{I} d_k$. $p \in A_I$ implies that only bid 0 is executable because $p > a_I$. The corresponding aggregate volume is $\alpha_I = d_0$. This establishes the specific presentation of the aggregate demand function. And Φ_D is non-increasing since $\alpha_0 > \alpha_1 > \ldots > \alpha_I$. $\square$

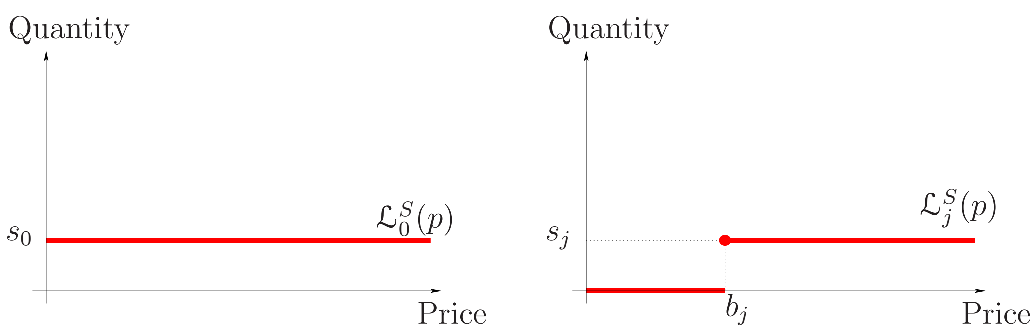
2.2 Supply-to-sell schedule (asks)

For ask 0, which is a market order with a non-negative quantity s_0, the individual supply function is:

$$\mathfrak{L}_0^S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto s_0 \end{cases}.$$

Each ask $j \in \{1, \ldots, J\}$ consists of a price-quantity pair (b_j, s_j), which states the intention to sell s_j shares when the auction price is no lower than b_j. Analogous to the bids, ask j can be represented as an individual supply function as follows. If $1_{B_j^S}(p)$ denotes a characteristic function of the interval $B_j^S = [b_j, +\infty)$, the individual supply function that represents ask j is given by the step function:

$$\mathfrak{L}_j^S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto s_j 1_{B_j^S}(p) \end{cases}. \tag{4}$$



(a) the market order: ask 0

(b) the limit order: ask j

Figure 3: The Individual supply function.

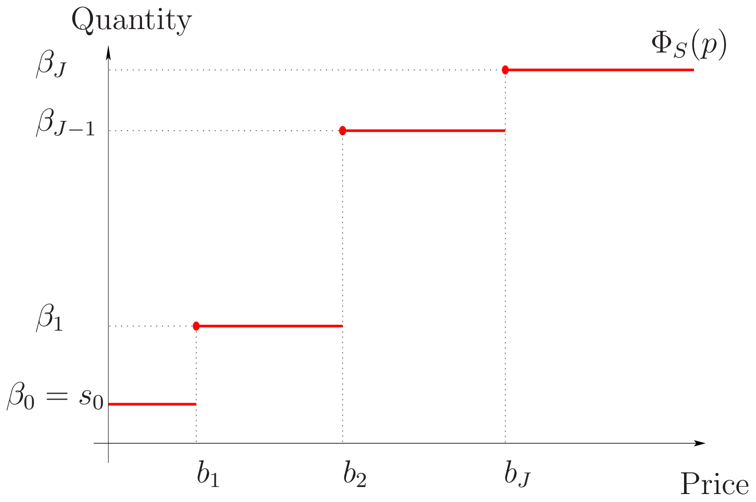


Figure 4: The Aggregate supply function.

The aggregate supply function is defined as the sum of the individual supply functions

$$\Phi_S : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \sum_{j=0}^{J} \mathcal{L}_j^S(p) \end{cases}. \tag{5}$$

Without loss of generality, let $b_J > \ldots > b_2 > b_1 > 0$. Then we obtain the following lemma:

Lemma 2. *Let $b_J > \ldots > b_2 > b_1 > 0$. The aggregate supply function $\Phi_S(p)$ is non-decreasing and takes the form:*

$$\Phi_S(p) = \sum_{j=0}^{J} \beta_j 1_{B_j}(p), \quad p \in \mathbb{R}_+, \tag{6}$$

where $\beta_0 := s_0$, $\beta_j := s_0 + \sum_{k=1}^{j} s_k$, for $j = 1, \ldots, J$;
and $B_0 := [0, b_1)$, $B_j := [b_j, b_{j+1})$, for $j = 1, \ldots, J-1$, $B_J := [b_J, +\infty)$.

Proof. Notice that ask 0, which is the market order, is executable for any $p \in \mathbb{R}_+$. $\{B_0, \ldots, B_J\}$ is by construction a partition of $\mathbb{R}_+$. $p \in B_0$ implies that only ask 0 is executable. The corresponding aggregate volume is $\beta_0 = s_0$. Let $j_* \in \{1, \ldots, J\}$ be arbitrary but fixed. Then $p \in B_{j_*}$ implies that all asks $j = 0, 1, \ldots, j_*$ are executable. The corresponding aggregate volume is $\beta_{j_*} = s_0 + \sum_{k=1}^{j_*} s_k$. This establishes the specific presentation of the aggregate supply function. And Φ_S is non-decreasing since $\beta_J > \ldots > \beta_1 > \beta_0$. $\square$

3 *Xetra* Auction Price Mechanism

The call phase stops with a random end after a fixed minimum time span and is followed by the price determination phase during which the auction price is determined and all feasible transactions are executed. In price determination phase the order book is closed and no new orders will be accepted. The Xetra auction price is determined by Xetra according to a set of well-specified pricing rules described in the brochure published by Gruppe Deutsche Börse (2003):

> A limit price with the highest executable order volume and the lowest surplus is called a candidate price.
>
> *Rule 1.* The auction price is the candidate price if there is only one candidate price.
>
> *Rule 2.* If there is more than one candidate price, then there are four cases:
>
> > *Rule 2.1.* If the surplus for all the candidate prices is on the demand side, then the auction price is stipulated as the highest candidate price.
> >
> > *Rule 2.2.* If the surplus for all the candidate prices is on the supply side, then the auction price is stipulated as the lowest candidate price.
> >
> > *Rule 2.3.* If there is no surplus for all the candidate prices, a reference price P_{ref} designated by Xetra is included as an additional criterion. The auction price is determined as follows:
> >
> > > *Rule 2.3.1.* The auction price is the highest candidate price if the reference price is higher than the highest candidate price.
> > >
> > > *Rule 2.3.2.* The auction price is the lowest candidate price if the reference price is lower than the lowest candidate price.
> > >
> > > *Rule 2.3.3.* The auction price is the reference price if the reference price lies between the highest candidate price and the lowest candidate price.
> >
> > *Rule 2.4.* If there are some candidate prices with a surplus on the supply side and others with a surplus on the demand side, then the *upper bound price* is chosen as the lowest candidate price with a surplus on the supply side and the *lower bound price* is chosen as the highest candidate price with a surplus on the demand side, which is always less than the upper bound price as we will show later. Xetra determines the auction price with these two prices and the reference price P_{ref}:
> >
> > > *Rule 2.4.1.* The auction price is the upper bound price if the reference price is higher than the upper bound price.
> > >
> > > *Rule 2.4.2.* The auction price is the lower bound price if the reference price is lower than the lower bound price.

Rule 2.4.3. The auction price is equal to the reference price if the reference price lies between the upper bound price and the lower bound price.

Rule 3. If there are only market orders on both sides of the order book, i.e. there is no limit price, then the auction price is the reference price P_{ref}.

Rule 4. If Rule 1 to Rule 3 fail, there exists no auction price.

Rule 4 implies that there could be no executable order volume in Xetra such that no Xetra auction price exists. In order to describe how the Xetra auction price is determined formally, it is useful to represent the concepts of *executable order volume* and *surplus* in Xetra.

3.1 Executable order volume and surplus

Let $p \in \mathbb{R}_+$ be some arbitrary price such that the aggregate demand $\Phi_D(p)$ may be unequal to the aggregate supply $\Phi_S(p)$, then only the minimum of $\Phi_D(p)$ and $\Phi_S(p)$ could possibly be traded. The quantity which is feasible to trade will henceforth be called *executable order volume* and is defined by

$$\Phi_V : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R}_+ \\ p & \longmapsto \min\{\Phi_D(p),\ \Phi_S(p)\} \end{cases}. \tag{7}$$

The function (7) is also referred as the *trading volume function*. The *excess demand function* is as usual given by

$$\Phi_Z : \begin{cases} \mathbb{R}_+ & \longrightarrow \mathbb{R} \\ p & \longmapsto \Phi_D(p) - \Phi_S(p) \end{cases}. \tag{8}$$

Xetra refers to the absolute value of the excess demand $|\Phi_Z(p)|$ as the *surplus*.

3.2 The *Xetra* Auction Price Model

Only limit prices are taken into account by the pricing rules of Xetra auctions. Let $\Omega_0 := \{a_1, \dots, a_I, b_1, \dots, b_J\}$ denote the set of all limit prices associated to a given order book. The highest executable order volume $V_{\max}$ is the maximum value of the trading volume function in the domain Ω_0 and given by

$$V_{\max} := \max\ \{\Phi_V(p) \mid p \in \Omega_0\}.$$

Notice that $V_{\max}$ exists and is finite: the image of the trading-volume function Φ_V is bounded and Ω_0 is finite. The set of *volume maximizing prices* is defined by

$$\Omega_V := \{p \in \Omega_0 \mid \Phi_V(p) = V_{\max}\}$$

which is non-empty and finite. When $V_{\max} = 0$, the executable order volume is zero and no transaction will be carried out in Xetra. Hence no Xetra auction price exists.[1]

Suppose that a non-zero executable order volume exists so that $V_{\max} > 0$ for a non-empty set Ω_V. In the next step Xetra looks for the lowest surplus $Z_{\min}$ which is defined by

$$Z_{\min} := \min\big\{|\Phi_Z(p)| \bigm| p \in \Omega_V\big\}.$$

The prices corresponding to the highest executable order volume and the lowest surplus, referred to as *candidate prices* by the Xetra pricing rules, are given by

$$\Omega_Z := \big\{p \in \Omega_V \bigm| |\Phi_Z(p)| = Z_{\min}\big\}.$$

Since Ω_V is well defined and finite, $Z_{\min}$ exists and Ω_Z is also well defined and non-empty. Denote by

$$\overline{P_Z} := \max \Omega_Z \quad \text{and} \quad \underline{P_Z} := \min \Omega_Z$$

the highest candidate price and the lowest candidate price in Ω_Z, respectively. When there is only one candidate price, the unique candidate price $\overline{P_Z} = \underline{P_Z}$ is chosen as the uniquely determined *Xetra auction price.*

Obviously, there could be more than one candidate price, i.e., $\#\Omega_Z > 1$. According to Rule 2.1, when all the candidate prices are with a surplus on the demand side, $\Phi_Z(p) > 0$ for all $p \in \Omega_Z$, the highest candidate price $\overline{P_Z}$ is chosen as the Xetra auction price.

According to Rule 2.2, when all the candidate prices are with a surplus on the supply side, $\Phi_Z(p) < 0$ for all $p \in \Omega_Z$, the lowest candidate price $\underline{P_Z}$ is chosen as the Xetra auction price.

Otherwise, there could be no surplus for all candidate prices or there could be a surplus on the demand side for some candidate prices while others with a surplus on the supply side. In these two cases, a reference price P_{ref} designated by Xetra is included to determine the Xetra auction price.

In the first case, which corresponds to Rule 2.3, Xetra compares the reference price P_{ref} with the highest candidate price $\overline{P_Z}$ and the lowest candidate price $\underline{P_Z}$ and chooses one of these three prices as the Xetra auction price using the pricing rules described in *Rule 2.3.1*, *Rule 2.3.2*, and *Rule 2.3.3*.

[1] Note that $\Omega_V = \Omega_0$ when $V_{\max} = 0$.

In the second case, which corresponds to Rule 2.4, Xetra chooses the upper bound price $P_{\max}$ which is the lowest candidate price with a surplus on the supply side and the lower bound price $P_{\min}$ which is the highest candidate price with a surplus on the demand side. Formally, the upper bound price and the lower bound price are defined as:

$$\begin{aligned} P_{\max} &:= \min\{p \in \Omega_Z \mid \Phi_Z(p) = -Z_{\min}\}, \\ P_{\min} &:= \max\{p \in \Omega_Z \mid \Phi_Z(p) = Z_{\min}\}. \end{aligned}$$

The upper bound price $P_{\max}$ and the lower bound price $P_{\min}$ are well defined in this case, and $P_{\max} > P_{\min}$ since $\Phi_Z(p)$ is non-increasing so that any price with a surplus on the supply side is always greater than any price with a surplus on the demand side. Xetra compares the reference price P_{ref} with the upper bound price $P_{\max}$ and the lower bound price $P_{\min}$ and chooses one of these three prices as the Xetra auction price according the pricing rules stated in *Rule 2.4.1*, *Rule 2.4.2*, and *Rule 2.4.3*

We are now in a position to formalize the pricing rules determining the Xetra auction price P_{Xetra} in the following theorem.

Theorem 1. *If $\Omega_0 \neq \emptyset$ and $V_{\max} > 0$, then a unique auction price P_{Xetra} exists and is determined by the following equations:*

(i) If $\#\Omega_Z = 1$, then $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$.

(ii) If $\#\Omega_Z > 1$, then

$$P_{\text{Xetra}} = \begin{cases} \overline{P_Z}, & \textit{if } \; \Phi_Z(\overline{P_Z}) > 0, \\ \underline{P_Z}, & \textit{if } \; \Phi_Z(\underline{P_Z}) < 0, \\ \max\{\underline{P_Z}, \min\{P_{\text{ref}}, \overline{P_Z}\}\}, & \textit{if } \; \Phi_Z(\underline{P_Z}) = \Phi_Z(\overline{P_Z}) = 0, \\ \max\{P_{\min}, \min\{P_{\text{ref}}, P_{\max}\}\}, & \textit{if } \; \Phi_Z(\underline{P_Z}) > 0 \\ & \quad \textit{and } \; \Phi_Z(\overline{P_Z}) < 0. \end{cases} \tag{9}$$

If only market orders exists so that $\Omega_0 = \emptyset$ with $d_0 > 0$ and $s_0 > 0$, then the auction price is $P_{\text{Xetra}} = P_{\text{ref}}$.

Otherwise, the auction price P_{Xetra} remains unspecified.

Proof. $\Omega_0 \neq \emptyset$ implies that there exists at least one limit order, and $V_{\max} > 0$ implies that $\Omega_Z \neq \emptyset$. Thus, Rule 1 and Rule 2 are considered when $V_{\max} > 0$ and $\Omega_0 \neq \emptyset$.

Rule 1 states that $P_{\text{Xetra}} = \overline{P_Z} = \underline{P_Z}$ when $\#\Omega_Z = 1$, and Rule 2 describes four cases when $\#\Omega_Z > 1$, which corresponds to equation (9).

Rule 2.1 corresponds to the case of all the prices in Ω_Z having the same surplus on the demand side $\Phi_Z(p) > 0$. This implies $P_{\text{Xetra}} = \overline{P_Z}$ when $\Phi_Z(\overline{P_Z}) > 0$. Analogously, Rule 2.2 corresponds to all the prices in Ω_Z having the same surplus on the supply side $\Phi_Z(p) < 0$. This implies $P_{\text{Xetra}} = \underline{P_Z}$ when $\Phi_Z(\underline{P_Z}) < 0$.

According to Rule 2.3, the Xetra auction price $P_{\text{Xetra}} = \max\left\{\underline{P_Z}, \min\{P_{\text{ref}}, \overline{P_Z}\}\right\}$ when there is no surplus for all $p \in \Omega_Z$, which is $\Phi_Z(\underline{P_Z}) = \Phi_Z(\overline{P_Z}) = 0$.

Rule 2.4 states the case that $\Phi_Z(\overline{P_Z}) < 0$ and $\Phi_Z(\underline{P_Z}) > 0$. Derived from Rule 2.4.1, Rule 2.4.2, and Rule 2.4.3, the Xetra auction price is transformed as: $P_{\text{Xetra}} = \max\{P_{\min}, \min\{P_{\text{ref}}, P_{\max}\}\}$.

Rule 3 is applied when there exists market orders on both market sides, $d_0 > 0$ and $s_0 > 0$, and no limit orders, $\Omega_0 = \emptyset$. In this case $P_{\text{Xetra}} = P_{\text{ref}}$.

Rule 4 is applied when all the rules above fail. In this case there is no Xetra auction price P_{Xetra}. $\square$

Theorem 1 is a comprehensive characterization of the pricing rules of Xetra auctions. Given any order book situation, a unique Xetra auction price P_{Xetra}, if it exists, is determined by Theorem 1. Figure 5 illustrates the formulation of P_{Xetra} in the case of surplus on the supply side.

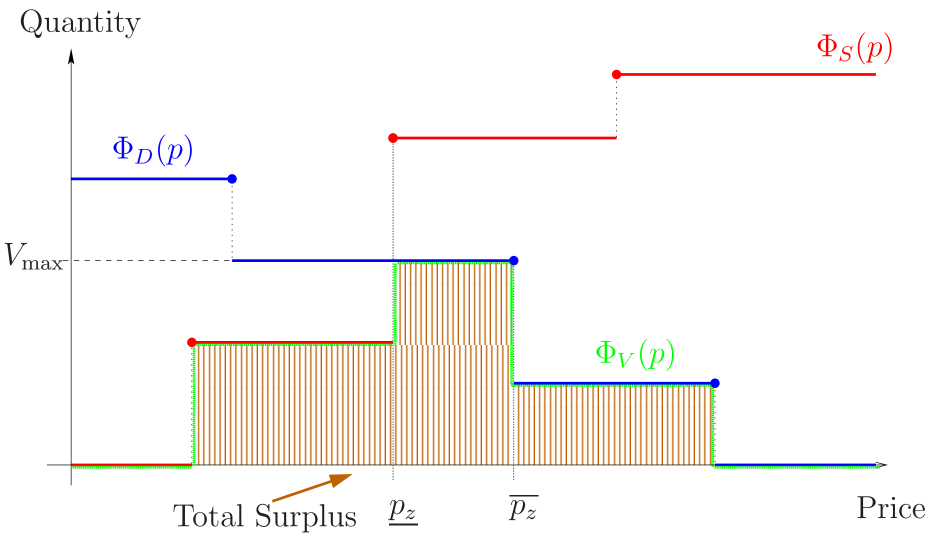


Figure 5: Surplus on the supply side: $\Phi_Z(\underline{p_z}) < 0$, $P_{\text{Xetra}} = \underline{p_z}$.

3.3 Properties of *Xetra* Auction Price Mechanism

In this section, we compare the Xetra auction pricing rules with the classical benchmark of market clearing prices of the Walrasian type and investigate its properties.

Non-market-clearing auction price. In the traditional microeconomic literature, e.g., see Pindyck & Rubinfeld (2001), the aggregate demand and the aggregate supply schedule are represented by continuous curves in price-quantity space such that the market price and allocation are determined by the intersection point of the demand curve and the supply curve. The continuity of these two curves ensures that the market price, if it exists, is with the property of the highest trading volume and with zero surplus simultaneously. The market price of this type is market clearing in the sense that the quantity supplied is equal to the quantity demanded, i.e., zero surplus.

The fact that the aggregate demand function and the aggregate supply function in Xetra auctions are step functions which are not continuous makes the Xetra auction price mechanism different from the price mechanism in conventional market mechanism. Xetra auctions follow two principles:

1. The auction price is with the highest executable order volume;
2. The auction price is with the lowest surplus.

They are constructed in such a way that the Xetra auction price can be determined even when market clearing is impossible.

Market welfare. To design a market mechanism, one has to predefine a set of design objectives which serve as measurement of evaluating a market mechanism, e.g., see Phelps, McBurney, Parsons & Sklar (2002). Considering market welfare, which is the social welfare of a market, is one of the traditional measurements of evaluating market mechanisms. It is therefore interesting to see how well Xetra auction price mechanism fits for this measurement.

Derived from the traditional concept of market welfare, the market welfare of a multi-unit double auction market is measured by the total surplus which is composed of the sum over each bid of the demand surplus which is the bid quantity times the positive difference between the bid price and the actual auction price and the sum over each ask of the supply surplus which is the ask quantity times the positive difference between the actual auction price and the ask price, see Domowitz (1990) or O'Hara (1995).[2] In the case of Xetra auction, the total surplus is the shadowed area in Figure 5.

For a competitive market mechanism, the maximum market welfare is automatically ensured when the market price is in equilibrium, i.e., with zero surplus. This is the case simply because the property of the highest executable trading quantity, which guarantees the maximum market welfare, coincides with the property of zero surplus in competitive market where demand and supply functions are continuous.

Comparing with competitive markets, Xetra auction price achieves the maximum market welfare when the auction price is with the property of the highest executable order volume, which is any price $p \in \Omega_V$. Thus, the Xetra auction price P_{Xetra} is also with the property of the maximum market welfare since $P_{\text{Xetra}} \in \Omega_V$. Notice that the auction price with the maximum market welfare in Xetra auction market mechanism is not always unique which is ascribed to the step functional forms of demand and supply in Xetra. This is responsible for the non-uniqueness of the price associated with the highest executable order volume.

The result above can be generalized to the case of any multi-unit double auction market in which the demand function and the supply function are step functions, stating that the property of the maximum market welfare is ensured when the price is with the property of the highest executable order volume, even though many auction prices guaranteeing maximum market welfare might exist due to the step functional forms of demand and supply.

Limitation of limit prices. Apart from a preset reference price, Xetra auctions consider only limit prices for possible auction prices. This seems to be attributed to the operationality of auction price computation. However, the following example will show that this restriction discards prices which would lead to a market clearing situation. It corresponds to Example 4 on page 36 in Gruppe Deutsche Börse (2003).

Example: Consider one equity with one market order for buying 100 shares, another market order for selling 100 shares, one bid with price-quantity $(199, 100)$, and one ask with price-quantity $(202, 100)$. The order book situation is shown in Figure 6.

In this example, $\Omega_0 = \{199, 202\}$, $V_{\max} = 100$ and the set of the corresponding volume-maximizing prices is $\Omega_V = \{199, 202\}$. $Z_{\min} = 100$ and the set of the candidate prices is $\Omega_Z = \{199, 202\}$. Since there is surplus on the demand side as

[2] As stated in Pindyck & Rubinfeld (2001), the traditional concept of market welfare is defined as the sum of the consumer surplus and the producer surplus. In a multi-unit double auction market, the notion of the consumer surplus is translated into the notion of the demand surplus and the notion of the producer surplus is replaced by the notion of the supply surplus.

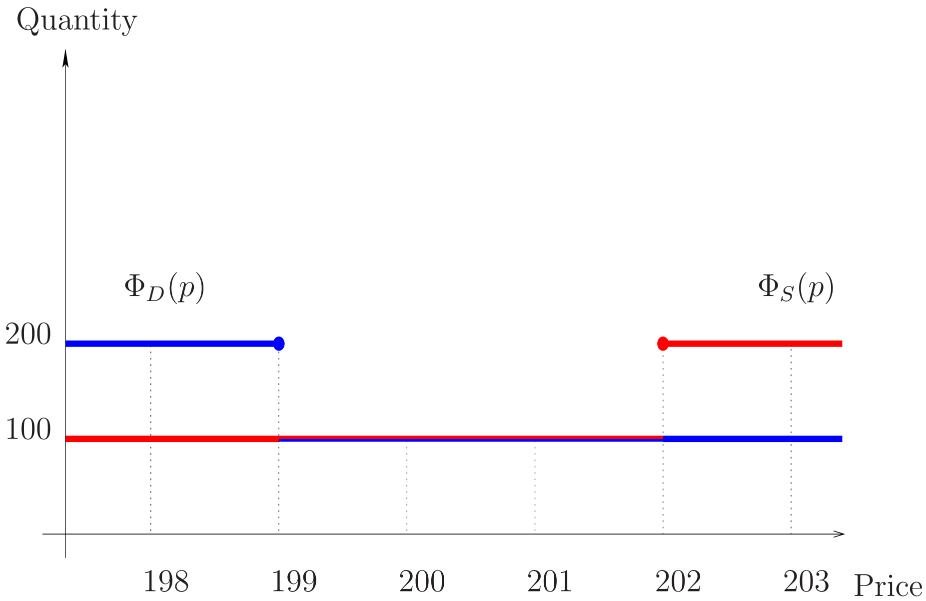


Figure 6: Possibility of Market Clearing Price

well as on the supply side, Xetra applies Rule 2.4 leading to $P_{\text{Xetra}} = 200$ when $P_{\text{ref}} = 200$, $P_{\text{Xetra}} = 202$ when $P_{\text{ref}} = 203$, and $P_{\text{Xetra}} = 199$ when $P_{\text{ref}} = 199$.

As readily seen from Figure 6, any price $p \in (199, 202)$ is market clearing but not a limit price. Only when the reference price satisfies $P_{\text{ref}} \in (199, 202)$ could the Xetra auction price be market clearing, since then $P_{\text{Xetra}} = P_{\text{ref}}$. In this case, market clearing could be obtained, for example, by taking the midpoint of the interval $(199, 202)$ and setting $P_{\text{Xetra}} = 201.5$. The Xetra auction pricing rules thus exclude the possibility of market clearing by not taking into account prices other than limit prices.

Li (2005) investigates a modified auction price mechanism which is also implementable on a computer system. It is shown that this modified auction price is with the highest executable order volume and the lowest surplus and hence obeys the principles of Xetra.

4 *Xetra* Auction Allocation Mechanism

Xetra computes the auction market allocation given the Xetra auction price P_{Xetra}. Final transaction for each order specification is determined by Xetra auction allocation mechanism for a given P_{Xetra}.

4.1 The *Xetra* Auction Allocation Model

When an order is submitted to the order book, it is labeled with a time tag which determines the time priority with which it is executed. The time tag attached to each order determines the ranking of the execution in the order book. Given a Xetra auction Price P_{Xetra}, executable orders are executed by time priority. There are two execution sequences corresponding to the demand side (bids) and the supply side (asks).

Denote the execution priority of bid i by $\iota_d(i)$ and the execution priority of ask j by $\iota_s(j)$ respectively, where $\iota_d(i) \in \{0, 1, \ldots, I\}$ and $\iota_s(j) \in \{0, 1, \ldots, J\}$. The position in the execution sequence of bid i then is $\iota_d(i)$, which implies that there are $\iota_d(i)$ bids which will be executed before bid i. Analogously, there are $\iota_s(j)$ asks which will be executed before ask j.

Market orders always have higher priority than limit orders, thus bid 0 and ask 0 are with the ranking of $\iota_d(0) = 0$ and $\iota_s(0) = 0$.

The final transaction for each order is highly affected by its position in the execution sequence since Xetra applies the rule of ***First Come First Serve (FCFS)*** for the order execution.[3] Given the fixed ranking of the execution sequence, bid i will not be executed until all higher ranked bids are executed. The maximum feasible quantity that bid i can get is therefore the quantity which higher ranked bids have left over, that is, the positive difference between the highest executable order volume $\Phi_V(P_{\text{Xetra}}) = V_{\max}$ and the aggregate executed order volume before bid i is handled. Thus, the maximum feasible quantity for bid $i \in \{0, 1, \ldots, I\}$ is given by

$$\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}}) := \max\Big\{0, \Phi_V(P_{\text{Xetra}}) - \sum_{m=0}^{\iota_d(i)-1} \mathfrak{L}^D_{\iota_d^{-1}(m)}(P_{\text{Xetra}})\Big\}, \qquad (10)$$

where $\iota_d^{-1}(m)$ denotes the index of the bid which is in position m of the execution sequence of bids. If the individual demand $\mathfrak{L}_i^D(P_{\text{Xetra}})$ of bid i is less than $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$, then bid i is fully served and it receives

$$\mathfrak{L}_i^D(P_{\text{Xetra}}) = \begin{cases} d_i & \text{if } \; P_{\text{Xetra}} \in [0, a_i], \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathfrak{L}_i^D(P_{\text{Xetra}})$ is greater than $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$, bid i can only be partially executed. The final transaction is $\bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})$ and bid i is rationed. Denoting the final transaction of bid i by X_i^D, we have

$$X_i^D(P_{\text{Xetra}}) := \min\Big\{\mathfrak{L}_i^D(P_{\text{Xetra}}), \bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I. \qquad (11)$$

[3]FCFS is equivalent to the rule *First In First Out (FIFO)*.

As regards the supply side, the maximum feasible quantity for ask $j \in \{0, 1, \ldots, J\}$ is the positive difference between the highest executable order volume $\Phi_V(P_{\text{Xetra}}) = V_{\max}$ and the aggregate executed order volume before ask j is handled. The maximum feasible quantity for ask j is given by

$$\bar{\mathfrak{L}}_j^S(P_{\text{Xetra}}) := \max\Big\{0, \Phi_V(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathfrak{L}^S_{\iota_s^{-1}(n)}(P_{\text{Xetra}})\Big\}, \tag{12}$$

where $\iota_s^{-1}(n)$ denotes the index of the ask which is in position n of the execution sequence of asks. Denoting the final transaction for ask j by X_j^S, similarly we have

$$X_j^S(P_{\text{Xetra}}) := \min\Big\{\mathfrak{L}_j^S(P_{\text{Xetra}}), \bar{\mathfrak{L}}_j^S(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J, \tag{13}$$

where

$$\mathfrak{L}_j^S(P_{\text{Xetra}}) = \begin{cases} s_j & \text{if } \; P_{\text{Xetra}} \in [b_j, +\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing, the Xetra auction allocation mechanism for any given Xetra auction price P_{Xetra} is given by

$$\begin{cases} X_i^D(P_{\text{Xetra}}) := & \min\Big\{\mathfrak{L}_i^D(P_{\text{Xetra}}), \bar{\mathfrak{L}}_i^D(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I; \\ X_j^S(P_{\text{Xetra}}) := & \min\Big\{\mathfrak{L}_j^S(P_{\text{Xetra}}), \bar{\mathfrak{L}}_j^S(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J. \end{cases} \tag{14}$$

Notice that the aggregate final transaction of bids is equal to aggregate final transaction of asks, that is,

$$\sum_{i=0}^{I} X_i^D(P_{\text{Xetra}}) = \sum_{j=0}^{J} X_j^S(P_{\text{Xetra}}) = \Phi_V(P_{\text{Xetra}}) = V_{\max}.$$

Also notice that the market-clearing situation is included as a special case in which all orders are fully executed:

$$\begin{cases} \mathfrak{L}_i^D(P_{\text{Xetra}}) = & X_i^D(P_{\text{Xetra}}), \quad i = 0, 1, \ldots, I; \\ \mathfrak{L}_j^S(P_{\text{Xetra}}) = & X_j^S(P_{\text{Xetra}}), \quad j = 0, 1, \ldots, J. \end{cases} \tag{15}$$

4.2 Properties of *Xetra* Auction Allocation Mechanism

The Xetra auction allocation mechanism satisfies some well-known properties of rationing mechanisms which are a allocation mechanisms under the assumption of fixed market prices, see Benassy (1982) or Böhm (1989) for more details.

Voluntary exchange. The property of voluntary exchange states that no trader is forced to trade more than she claims. Intuitively, this property holds by the very definition of Xetra orders. More formally, the Xetra auction allocation mechanism (14) satisfies this property because for all i, j,

$$\begin{aligned} X_i^D(P_{\text{Xetra}}) &\leq \mathfrak{L}_i^D(P_{\text{Xetra}}), \\ X_j^S(P_{\text{Xetra}}) &\leq \mathfrak{L}_j^S(P_{\text{Xetra}}). \end{aligned}$$

The short-side rule. According to Benassy (1982), the 'short' side of a market is the market side where the aggregate transaction volume is smallest. It is thus the demand side if there is excess supply, the supply side if excess demand exists. The other side is called the 'long' side.

An allocation mechanism is called *'efficient'*, or frictionless, if no mutually advantageous trade can be carried out from the transaction attained. This implies that traders on the short side of a market will realize their desired transactions.

Combining the property of voluntary exchange and market efficiency, we obtain the so-called *'short-side rule'* stating that traders on the short side will realize all of their individual demand (supply). Formally, the Xetra auction allocation mechanism (14) satisfies the short-side rule if

$$\Phi_D(P_{\text{Xetra}}) \geq \Phi_S(P_{\text{Xetra}}) \;\Rightarrow\; X_j^S(P_{\text{Xetra}}) = \mathfrak{L}_j^S(P_{\text{Xetra}}), \quad \forall j; \tag{16}$$

$$\Phi_D(P_{\text{Xetra}}) \leq \Phi_S(P_{\text{Xetra}}) \;\Rightarrow\; X_i^D(P_{\text{Xetra}}) = \mathfrak{L}_i^D(P_{\text{Xetra}}), \quad \forall i. \tag{17}$$

By analogy, we only verify condition (16). Clearly, $\Phi_D(P_{\text{Xetra}}) \geq \Phi_S(P_{\text{Xetra}})$ implies $\Phi_V(P_{\text{Xetra}}) = \Phi_S(P_{\text{Xetra}})$ and hence

$$\Phi_S(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathfrak{L}^S_{\iota_s^{-1}(n)}(P_{\text{Xetra}}) \geq \mathfrak{L}_j^S(P_{\text{Xetra}}), \quad j = 0, 1, \ldots, J,$$

which implies that (16) holds.

Anonymity. Trading in Xetra is anonymous in the sense that traders cannot identify which trader enters an order specification before the final transaction is determined, see Gruppe Deutsche Börse (2003). But traditional rationing theory cares also about the property of anonymity of the allocation mechanisms.

Loosely speaking, an allocation mechanism is called anonymous, if any two traders with the same characteristics attain the same final transaction. In the case of Xetra auctions, any two bids i and i' with the same time priority and the same order specification attain the same final transaction $X_i^D(P_{\text{Xetra}}) = X_{i'}^D(P_{\text{Xetra}})$. The same holds true for asks. Hence, the Xetra allocation mechanism satisfies the property of anonymity in that sense.

Notice, however, that the concept of time priority in Xetra might be subject to various influences which are beyond the control of the system in the sense of queuing theory. In view of stochastic rationing mechanisms (Weinrich 1984), the property of anonymity would hold only if the same orders attain the same final transactions on average.

Manipulability. An allocation mechanism is called *non-manipulable* in quantity if the final transaction of a trader, when she is rationed, faces a bound which depends solely on the quoted quantities of the other traders that she can not manipulate. It is called manipulable in quantity if the trader can, when she is rationed, increase her final transaction by increasing her quoted quantity. Intuitively, non-manipulability implies that the individual quantity quoted by a trader has no impact on her maximum feasible quantity and vice versa.

In Xetra, orders face upper bounds $\bar{\mathcal{L}}_i^D(P_{\text{Xetra}})$ and $\bar{\mathcal{L}}_j^S(P_{\text{Xetra}})$ for their final transactions, should they be rationed. In the case of excess demand $\Phi_D(P_{\text{Xetra}}) > \Phi_S(P_{\text{Xetra}})$, only bids will be rationed. The maximum feasible quantity of bid i is

$$\bar{\mathcal{L}}_i^D(P_{\text{Xetra}}) = \max\Big\{0, \Phi_S(P_{\text{Xetra}}) - \sum_{m=0}^{\iota_d(i)-1} \mathcal{L}^D_{\iota_d^{-1}(m)}(P_{\text{Xetra}})\Big\}, \quad i = 0, 1, \ldots, I$$

which is independent of its individual quantity $\mathcal{L}_i^D(P_{\text{Xetra}})$.

Analogously, in the case of excess supply $\Phi_S(P_{\text{Xetra}}) > \Phi_D(P_{\text{Xetra}})$, only asks are rationed. The maximum feasible quantity of ask j is

$$\bar{\mathcal{L}}_j^S(P_{\text{Xetra}}) = \max\Big\{0, \Phi_D(P_{\text{Xetra}}) - \sum_{n=0}^{\iota_s(j)-1} \mathcal{L}^S_{\iota_s^{-1}(n)}(P_{\text{Xetra}})\Big\}, \quad j = 0, 1, \ldots, J$$

which is independent of its individual quantity $\mathcal{L}_j^S(P_{\text{Xetra}})$.

At first sight, this observation seems to imply that the Xetra mechanism is non-manipulable in the sense of classical rationing theory. However, traders do influence the Xetra auction price by revising their order specifications. Hence, the situation in Xetra auctions are more complicated than traditionally assumed in market with conventional rationing mechanisms in which prices are always presumed to be fixed. To attain a profound understanding of the property of manipulability in Xetra, one would have to further investigate the relationship between individual order specifications and the Xetra auction price. This is left for future research.

5 Concluding Remarks

This paper presented a formalization of the market mechanism of Xetra auctions which belong to the class of multi-unit double auctions. It should be seen as a first step towards a better understanding of electronic equity trading systems such as Xetra. Much work has to be done in order to obtain a more comprehensive theory for such markets. One direction is to develop a microeconomic foundation of portfolio selection. The traditional approach is to assume that investors are price takers. Investors using electronic equity trading systems, however, have not only full knowledge on how market prices and final transactions are determined but also on the current order book situation. This raises the problem to what extent this knowledge could be exploited for their trading strategies. Among other issues, this problem will be treated in Li (2005).

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