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A taxonomy of myopic stability concepts for hedonic games*

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Abstract

We present a taxonomy of myopic stability concepts for hedonic games in terms of deviations, and discuss the status of the existence problems of stable coalition structures. In particular, we show that contractual strictly core stable coalition structures always exist, and provide suffi cient conditions for the existence of contractually Nash stable and weak individually stable coalition structures on the class of separable games.

JEL classification: C71, A14, D20.

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1 Introduction

One possibility to study the process of coalition formation is to model it as a hedonic coalition formation game. In such a model each player's preferences over coalitions depend only on the composition of members of her coalition. The formation of societies, social clubs and groups are examples in which the hedonic aspect of coalition formation (cf. [11]) plays an important role. Given a hedonic game, the main interest is then in the existence of outcomes (partitions of the set of players) that are stable in some sense. For example,

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the focus in [1], [3], [5], [6], [7], [8], and [10] is on the existence of core stable partitions, while [4] and [5] contain suffi cient conditions for the existence of Nash and individually stable partitions as well.

Most of the stability concepts studied in the literature presuppose that the players in the game are *myopic* in the sense that they do not take into account how their decisions to form a coalition will affect in the future the decisions of other players. Furthermore, these stability concepts are based either on *coalitional deviations* (core and strict core stability) or on *individual deviations* (Nash, individual, and contractual individual stability).

In this paper we suggest a unified look at the nature of the possible deviations from a given coalition structure, and offer a taxonomy of myopic stability concepts for hedonic games. In doing so, we require, no matter how the additional properties of the deviation look like, that there should always exist at least one player who has a strong incentive to move. In this way we describe, including the five stability notions mentioned above, ten different stability concepts for hedonic games. However, not all of these ten notions deserve a special attention because, as it can be shown, there are always coalition structures that are stable in the sense of four of these stability concepts and, moreover, without any preference restrictions. This is the reason why we focus mainly on the existence of two of the new stability concepts - contractual Nash stability and weak individual stability.

The paper is organized as follows. We introduce in Section 2 some preliminaries on hedonic games and different stability concepts that can be found in the literature. A taxonomy of myopic stability concepts is then presented in Section 3, where different implications between the stability notions are discussed as well. Sections 4, 5, and 6 are devoted to *contractual strict core stability*, *contractual Nash stability* and *weak individual stability*, respectively. We show that contractual strictly core stable coalition structures always exist. Moreover, on the class of separable games, a *weak mutuality* condition suffi ces for the existence of contractual Nash stable partitions, while a*solidarity* property guarantees the existence of weak individually stable coalition structures. We conclude in Section 7 with some final remarks.

2 Preliminaries

Consider a finite set of players $N = \{1, 2, \ldots, n\}$. A *coalition* is a non-empty subset of N. For each player $i \in N$, we denote by $\mathcal{A}^i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A collection Π of coalitions is called a *coalition structure* if Π is a partition of N, i.e., the coalitions in Π are pairwise disjoint and $\bigcup_{X\in\Pi} X = N$. For each coalition structure Π and each player $i \in N$, by $\Pi(i)$ we denote the coalition in Π containing i, i.e., $\Pi(i) \in \Pi$ and $i \in \Pi(i)$.

We assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{A}^i$, i.e., a binary relation over $\mathcal{A}^i$ which is reflexive, complete, and transitive. We denote by $\succeq = (\succeq_1, \succeq_2, \dots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. Moreover, we assume that the preference of each player $i \in N$ over coalition structures is *purely hedonic*, i.e., it is completely characterized by $\succeq_i$ in such a way that, for each coalition structure Π and Π', each player i weakly prefers Π to Π' if and only if $\Pi(i) \succeq_i \Pi'(i)$. A *hedonic game* $\langle N, \succeq \rangle$ is a pair of a finite set N of players and a preference profile $\succeq$.

Now we define stability concepts based on coalitional deviations and on individual deviations, which can be found in the literature. Let $\langle N, \succeq \rangle$ be a hedonic game and let Π be a coalition structure. We say that

- Π is *core stable* if there does not exist a coalition X such that
  - $X \succ_i \Pi(i)$ for all $i \in X$;
- Π is *strictly core stable* if there does not exist a coalition X such that
  - $X \succeq_i \Pi(i)$ for all $i \in X$, and
  - $X \succ_j \Pi(j)$ for some $j \in X$;
- Π is *Nash stable* if there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that
  - $X \cup \{i\} \succ_i \Pi(i)$;
- Π is *individually stable* if there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that
  - $X \cup \{i\} \succ_i \Pi(i)$, and
  - $X \cup \{i\} \succeq_j X$ for all $j \in X$;

- Π is *contractual individually stable* if there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that
  - $X \cup \{i\} \succ_i \Pi(i)$,
  - $X \cup \{i\} \succeq_j X$ for all $j \in X$, and
  - $\Pi(i) \setminus \{i\} \succeq_j \Pi(i)$ for all $j \in \Pi(i) \setminus \{i\}$.

Observe that strict core stability implies core stability, Nash stability implies individual stability, and individual stability implies contractual individual stability. Moreover, strict core stability implies individual stability as well.

3 Taxonomy and interpretations

In this section, we first define several sets of coalitions capturing the nature of deviating coalitions for those stability concepts introduced in the previous section. Then, in terms of these sets, some new stability concepts are introduced.

Let $\langle N, \succeq \rangle$ be a hedonic game, and let Π be a coalition structure. By $\textsc{all}(\Pi)$ we denote the set of all possible deviations from Π, i.e.,

$$\textsc{all}(\Pi) = 2^N \setminus (\Pi \cup \{\emptyset\}).$$

We next define the sets $\textsc{weak}(\Pi)$, $\textsc{strong}(\Pi)$, $\textsc{nash}(\Pi)$, and $\textsc{cont}(\Pi)$ as follows:

$$\begin{aligned}
\textsc{strong}(\Pi) &= \{X \in \textsc{all}(\Pi) \mid \forall i \in X\, [X \succ_i \Pi(i)]\},\\
\textsc{weak}(\Pi) &= \{X \in \textsc{all}(\Pi) \mid \forall i \in X\, [X \succeq_i \Pi(i)] \text{ and } \exists i \in X\, [X \succ_i \Pi(i)]\},\\
\textsc{nash}(\Pi) &= \{X \in \textsc{all}(\Pi) \mid \exists i \in X\, [X \setminus \{i\} \in \Pi \cup \{\emptyset\} \text{ and } X \succ_i \Pi(i)]\},\\
\textsc{cont}(\Pi) &= \{X \in \textsc{all}(\Pi) \mid \forall i \in N \setminus X\, [\Pi(i) \setminus X \succeq_i \Pi(i)]\}.
\end{aligned}$$

Then, we have the following observation.

Observation 1 *The stability concepts introduced in the previous section can be described in terms of* $\textsc{weak}(\Pi)$*,* $\textsc{strong}(\Pi)$*,* $\textsc{nash}(\Pi)$*, and* $\textsc{cont}(\Pi)$ *as follows:*

Π *is core stable*	$\Leftrightarrow$	$\textsc{strong}(\Pi) = \emptyset$,
Π *is strictly core stable*	$\Leftrightarrow$	$\textsc{weak}(\Pi) = \emptyset$,
Π *is Nash stable*	$\Leftrightarrow$	$\textsc{nash}(\Pi) = \emptyset$,
Π *is individually stable*	$\Leftrightarrow$	$\textsc{nash}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$,
Π *is contractual individually stable*	$\Leftrightarrow$	$\textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$.

Our first example is meant to illustrate the usefulness of defining stability concepts in terms of the above sets of deviations.

Example 1 *Consider a hedonic game* $\langle N, \succsim \rangle$ *with* $N = \{1, 2, 3\}$ *and* $\succsim = (\succsim_1, \succsim_2, \succsim_3)$ *defined as follows:*

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\},$$
$$\{1,2,3\} \succ_2 \{1,2\} \sim_2 \{2,3\} \succ_2 \{2\},$$
$$\{1,2,3\} \succ_3 \{1,3\} \sim_3 \{2,3\} \succ_3 \{3\}.$$

From $|N| = 3$*, there are five possible coalition structures, and according to the preference profile* $\succsim$*, the sets* WEAK(Π)*,* STRONG(Π)*,* NASH(Π)*, and* CONT(Π) *for each coalition structure* Π *are as follows.*

Π	WEAK(Π)	STRONG(Π)
$\{\{1\},\{2\},\{3\}\}$	$\{\{1,2\},\{2,3\}\}$	$\{\{1,2\},\{2,3\}\}$
$\{\{1,2\},\{3\}\}$	$\{\{2,3\}\}$	$\emptyset$
$\{\{1,3\},\{2\}\}$	$\{\{1\},\{1,2\},\{2,3\},\{1,2,3\}\}$	$\{\{1\},\{1,2\},\{1,2,3\}\}$
$\{\{1\},\{2,3\}\}$	$\{\{1,2\}\}$	$\emptyset$
$\{\{1,2,3\}\}$	$\{\{1\}\}$	$\{\{1\}\}$

Π	NASH(Π)	CONT(Π)
$\{\{1\},\{2\},\{3\}\}$	$\{\{1,2\},\{1,3\},\{2,3\}\}$	$\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$
$\{\{1,2\},\{3\}\}$	$\{\{1,2,3\}\}$	$\{\{1,2,3\}\}$
$\{\{1,3\},\{2\}\}$	$\{\{1\},\{1,2\},\{1,2,3\}\}$	$\{\{3\},\{2,3\},\{1,2,3\}\}$
$\{\{1\},\{2,3\}\}$	$\emptyset$	$\{\{1,2,3\}\}$
$\{\{1,2,3\}\}$	$\{\{1\}\}$	$\{\{2,3\}\}$

For this hedonic game, we have

- *two core stable coalition structures* $\{\{1,2\},\{3\}\}$ *and* $\{\{1\},\{2,3\}\}$*,*
- *no strictly core stable coalition structure,*
- *one Nash stable coalition structure* $\{\{1\},\{2,3\}\}$*,*
- *two individually stable coalition structures* $\{\{1,2\},\{3\}\}$ *and* $\{\{1\},\{2,3\}\}$*,*
- *three contractual individually stable coalition structures* $\{\{1,2\},\{3\}\}$*,* $\{\{1\},\{2,3\}\}$*, and* $\{\{1,2,3\}\}$*.*

	$\textsc{all}(\Pi)$	$\textsc{weak}(\Pi)$	$\textsc{strong}(\Pi)$
$\textsc{all}(\Pi)$		**strict core stability**	**core stability**
$\textsc{nash}(\Pi)$	**Nash stability**	**individual stability**	weak individual stability
$\textsc{cont}(\Pi)$		contractual strict core stability	contractual core stability
$\textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$	contractual Nash stability	**contractual individual stability**	weak contractual individual stability

Table 1: Correspondences between stability concepts and sets of coalitions

From the other combinations of $\textsc{weak}(\Pi)$, $\textsc{strong}(\Pi)$, $\textsc{nash}(\Pi)$, and $\textsc{cont}(\Pi)$, we obtain several other stability concepts, namely

$$
\begin{aligned}
&\Pi \text{ is weak individually stable} && \Leftrightarrow \textsc{nash}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset,\\
&\Pi \text{ is contractual strictly core stable} && \Leftrightarrow \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset,\\
&\Pi \text{ is contractually core stable} && \Leftrightarrow \textsc{cont}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset,\\
&\Pi \text{ is contractually Nash stable} && \Leftrightarrow \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) = \emptyset,\\
&\Pi \text{ is weak contractual individually stable} && \Leftrightarrow \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset.
\end{aligned}
$$

The correspondences between the stability concepts and the sets of coalitions $\textsc{all}(\Pi)$, $\textsc{weak}(\Pi)$, $\textsc{strong}(\Pi)$, $\textsc{nash}(\Pi)$, and $\textsc{cont}(\Pi)$ are shown in Table 1.

For example, Π is *weak individually stable* if $\textsc{nash}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset$, which means that there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that every player $j \in X \cup \{i\}$ is strictly better off when player i joins X (i.e., $X \cup \{i\} \succ_j \Pi(j)$ for all $j \in X \cup \{i\}$). Clearly, weak individual stability is implied by core stability, because $\textsc{nash}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset$ when $\textsc{strong}(\Pi) = \emptyset$. On the other hand, weak individual stability is also implied by individual stability, because $\textsc{strong}(\Pi) \subseteq \textsc{weak}(\Pi)$ and hence $\textsc{nash}(\Pi) \cap \textsc{strong}(\Pi) = \emptyset$ when $\textsc{nash}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$.

Example 2 *Consider the hedonic game $\langle N, \succeq \rangle$ defined in Example 1. For this game, we have*

- *two weak individually stable coalition structures* $\{\{1,2\},\{3\}\}$ *and* $\{\{1\},\{2,3\}\}$*;*
- *three contractual strictly core stable coalition structures* $\{\{1,2\},\{3\}\}$*,* $\{\{1\},\{2,3\}\}$*, and* $\{\{1,2,3\}\}$*;*
- *three contractually core stable coalition structures* $\{\{1,2\},\{3\}\}$*,* $\{\{1\},\{2,3\}\}$*, and* $\{\{1,2,3\}\}$*;*
- *two contractually Nash stable coalition structures* $\{\{1\},\{2,3\}\}$*, and* $\{\{1,2,3\}\}$*;*
- *three weak contractual individually stable coalition structures* $\{\{1,2\},\{3\}\}$*,* $\{\{1\},\{2,3\}\}$*, and* $\{\{1,2,3\}\}$*.*

Observe that, for every coalition structure Π, we have the following inclusions:

$$\textsc{strong}(\Pi) \subseteq \textsc{weak}(\Pi) \subseteq \textsc{all}(\Pi).$$

Hence, in each row of Table 1, the stability notion in the first column implies the others, and the stability notion in the second column implies the one in the third column. Moreover, the following inclusions also hold:

$$\begin{array}{ccc} \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) & \subseteq & \textsc{nash}(\Pi) \\ \supseteq & & \supseteq \\ \textsc{cont}(\Pi) & \subseteq & \textsc{all}(\Pi). \end{array}$$

Hence, in each column of Table 1, the stability notion in the first row implies the others, the stability notion in the second row implies the one in the fourth row, and the stability notion in the third row implies the one in the fourth row as well.

Now let us explain why there are two empty cells in Table 1. The first empty cell corresponds to $\textsc{all}(\Pi) \cap \textsc{all}(\Pi) = \textsc{all}(\Pi) = \emptyset$. Observe that we have $\textsc{all}(\Pi) = \emptyset$ if and only if $|N| = 1$, which is the trivial case with the unique coalition partition $\Pi = \{N\}$. The second empty cell corresponds to a coalition structure Π such that $\textsc{cont}(\Pi) \cap \textsc{all}(\Pi) = \textsc{cont}(\Pi) = \emptyset$. Notice that in this case it is not even indicated why a coalition $X \in \textsc{cont}(\Pi)$ deviates from a coalition structure Π, and hence, the notion of a deviation does not make sense. Indeed, $\textsc{cont}(\Pi) = \emptyset$ only if $\Pi = \{N\}$; otherwise, $N \in \textsc{cont}(\Pi)$.

4 Contractual strict core stability

We start our study of the new stability notions presented in the previous section by showing that a contractual strictly core stable coalition structure always exists. As a related result, it was shown in [2] that, on any preference domain, a contractual individually stable coalition structure always exists, where a coalition structure Π is contractual individually stable if and only if $\textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$. Here, we slightly extend this result. Namely, we show that, on any preference domain, a contractual strictly core stable coalition structure always exists. Recall that a coalition structure Π is *contractual strictly core stable* if

$$\textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset,$$

i.e., there does not exist a coalition X such that $X \succeq_i \Pi(i)$ for all $i \in X$, $X \succ_j \Pi(j)$ for some $j \in X$, and $\Pi(i) \setminus X \succeq_i \Pi(i)$ for all $i \in N \setminus X$.

According to the arguments in the previous section, for every coalition structure Π, we have the following inclusions:

$$\begin{array}{ccc} \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{strong}(\Pi) & \subseteq & \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) \\ \supseteq & & \supseteq \\ \textsc{cont}(\Pi) \cap \textsc{strong}(\Pi) & \subseteq & \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi). \end{array}$$

Hence, contractual strict core stability implies contractual core stability, contractual individual stability, and weak contractual individual stability. Our result implies that, on any preference domain, there always exists a coalition structure which is contractual strictly core stable, contractually core stable, contractual individually stable, and weak contractual individually stable.

Proposition 1 *A contractual strictly core stable coalition structure always exists.*

Proof. A contractual strictly core stable coalition structure can be constructed by the following algorithm:

Step 1. Set $\Pi := \{N\}$.

Step 2. Repeats the following until $\textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$:

- Find an $X \in \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi)$.
- Set $\Pi := \{Y \setminus X \mid Y \in \Pi \text{ and } Y \not\subseteq X\} \cup \{X\}$.

Step 3. Return Π.

Let Π be an arbitrary coalition structure such that $\text{CONT}(\Pi) \cap \text{WEAK}(\Pi) \neq \emptyset$, and let $X \in \text{CONT}(\Pi) \cap \text{WEAK}(\Pi)$. Then consider $\Pi' = \{Y \setminus X \mid Y \in \Pi \text{ and } Y \not\subseteq X\} \cup \{X\}$.

Observe that Π' is a coalition structure as well. Thus, by starting with the coalition structure $\{N\}$, a coalition structure will be obtained when the algorithm halts. Since the algorithm halts when $\text{CONT}(\Pi) \cap \text{WEAK}(\Pi) = \emptyset$, the outcome Π of the algorithm is a contractual strictly core stable coalition structure.

Moreover, for each $i \in N$,

- if $\Pi(i) \cap X = \emptyset$, we have $\Pi'(i) = \Pi(i)$,
- if $\Pi(i) \cap X \neq \emptyset$ and $i \in X$, we have $\Pi'(i) = X \succeq_i \Pi(i)$ from $X \in \text{WEAK}(\Pi)$, and
- if $\Pi(i) \cap X \neq \emptyset$ and $i \notin X$, we have $\Pi'(i) = \Pi(i) \setminus X \succeq_i \Pi(i)$ from $X \in \text{CONT}(\Pi)$.

In other words, no player i is worse off being in $\Pi'(i)$ than being in $\Pi(i)$. From $X \in \text{WEAK}(\Pi)$, there is at least one $i \in X$ such that $\Pi'(i) = X \succ_i \Pi(i)$. Observe that, without being worse off, each player i can be better off at most $|\mathcal{A}^i| - 1 = 2^{n-1} - 1$ times. It follows then that Step 2 in the algorithm repeats at most $n2^{n-1} - n$ times, and therefore, the algorithm halts. ■

Remark 2 *Indeed, in [2] a similar algorithm was proposed for showing the existence of a contractual individually stable coalition structure. Here we essentially show that a similar argument works for a stronger stability concept as well.*

5 Contractual Nash stability

The notion of contractual Nash stability applies to situations in which, in order to move to another coalition, the corresponding player needs only the permission of her current coalition to leave. Imagine for example a criminal society that is already partitioned into groups. In such an environment it seems very plausible that it is easier for someone to join a criminal group than to get a permission to leave an already existing group she is a member of.

More formally, let $\langle N, \succeq \rangle$ be a hedonic game, and let Π be a coalition structure. As defined previously, a partition Π is *contractually Nash stable* if

$$\text{NASH}(\Pi) \cap \text{CONT}(\Pi) = \emptyset,$$

i.e., there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that $X \cup \{i\} \succ_i \Pi(i)$, and $\Pi(i) \setminus \{i\} \succeq_j \Pi(i)$ for all $j \in \Pi(i) \setminus \{i\}$.

Before we precede to our result on contractual Nash stability, let us introduce the domain of separable preferences and additive preferences.

Definition 1 *A preference profile $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ is* ***separable*** *if, for every $i, j \in N$ with $i \neq j$ and for each $X \in \mathcal{A}^i$ with $j \notin X$,*

- *$\{i, j\} \succeq_i \{i\}$ if and only if $X \cup \{j\} \succeq_i X$, and*
- *$\{i, j\} \preceq_i \{i\}$ if and only if $X \cup \{j\} \preceq_i X$.*

Definition 2 *A preference profile $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ is* ***additive separable*** *if, for every $i \in N$, there exists a real-valued function $v_i : N \to \mathbb{R}$ such that for every $X, Y \in \mathcal{A}^i$*

- *$X \succeq_i Y$ if and only if $\sum_{j \in X} v_i(j) \geq \sum_{j \in Y} v_i(j)$.*

For further purposes in this paper, we will redefine separability in the following manner. For each $i \in N$, let G_i, U_i, and B_i be the sets of desirable, neutral, and undesirable coalitional partners, respectively, of player i, i.e.,

- $G_i = \{j \in N \setminus \{i\} \mid \{i, j\} \succ_i \{i\}\}$,
- $U_i = \{j \in N \setminus \{i\} \mid \{i, j\} \sim_i \{i\}\}$, and
- $B_i = \{j \in N \setminus \{i\} \mid \{i, j\} \prec_i \{i\}\}$.

Obviously, (G_i, U_i, B_i) is a partition of $N \setminus \{i\}$. Then, separability can be defined in terms of (G_i, U_i, B_i) as follows. A preference profile $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ is *separable* if, for every $i, j \in N$ with $i \neq j$ and for each $X \in \mathcal{A}^i \setminus \mathcal{A}^j$,

- $j \in G_i$ if and only if $X \cup \{j\} \succ_i X$,
- $j \in U_i$ if and only if $X \cup \{j\} \sim_i X$, and
- $j \in B_i$ if and only if $X \cup \{j\} \prec_i X$.

Let us now redirect our attention to the existence of contractually Nash stable coalition structures. First, as a related result, it was shown in [4] that if the additive separable preference domain is under consideration, then imposing *symmetry* (i.e., $v_i(j) = v_j(i)$ for every $i, j \in N$) on players'preferences guarantees the existence of a Nash stable coalition structure. Moreover, symmetry is a critical condition for this result in the sense that a Nash stable coalition structure may fail to exist by weakening symmetry to mutuality. Recall that a preference profile satisfies *mutuality* if, for every $i, j \in N$, $i \neq j$,

- $j \in G_i$ if and only if $i \in G_j$,
- $j \in U_i$ if and only if $i \in U_j$, and
- $j \in B_i$ if and only if $i \in B_j$.

Clearly, under additive separability, mutuality means that, for every $i, j \in N$, $i \neq j$,

- $v_i(j) \geq 0$ if and only if $v_j(i) \geq 0$.

Notice further that, even with mutuality and on the additive separable preference domain, contractual Nash stability is *strictly weaker* than Nash stability. This is illustrated by our next example containing a game for which a contractual Nash stable coalition structure exists and no Nash stable coalition structure exists. This example is given in [3] and used in [4] to show the nonexistence of individually stable coalition structures, which implies the nonexistence of Nash stable coalition structures.

Example 3 *Consider the hedonic game $\langle N, \succeq \rangle$ with $N = \{1, 2, 3, 4, 5\}$ and an additive separable preference profile $\succeq$ defined by the following v_is.*

$$
\begin{array}{lllll}
v_1(1) = 0, & v_1(2) = 1, & v_1(3) = -4, & v_1(4) = -4, & v_1(5) = 2, \\
v_2(1) = 2, & v_2(2) = 0, & v_2(3) = 1, & v_2(4) = -4, & v_2(5) = -4, \\
v_3(1) = -4, & v_3(2) = 2, & v_3(3) = 0, & v_3(4) = 1, & v_3(5) = -4, \\
v_4(1) = -4, & v_4(2) = -4, & v_4(3) = 2, & v_4(4) = 0, & v_4(5) = 1, \\
v_5(1) = 1, & v_5(2) = -4, & v_5(3) = -4, & v_5(4) = 2, & v_5(5) = 0.
\end{array}
$$

It can easily be verified that $v_i(j) \geq 0$ if and only if $v_j(i) \geq 0$ for each $i, j \in N$, i.e., mutuality is satisfied. Observe that, for each coalition structure Π, $\text{NASH}(\Pi) \neq \emptyset$ if $|X| \geq 3$ or $|X| = 1$ for some $X \in \Pi$. In fact, each coalition structure Π contains at least one such a coalition X. Therefore, a Nash stable coalition structure does not exist.

Now consider the coalition structure $\Pi = \{N\}$, and we show that Π is contractually Nash stable. Observe that $\textsc{nash}(\Pi) = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$. *Further, for each $i \in N$, there exists $j \in N$ such that $j \neq i$ and $v_j(i) > 0$. It follows that $\Pi(j) \setminus \{i\} = N \setminus \{i\} \prec_j N = \Pi(i)$ for some $j \in N \setminus \{i\}$. Hence, $\{i\} \notin \textsc{cont}(\Pi)$ for each $i \in N$. Therefore,* $\textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) = \emptyset$*, i.e., Π is contractually Nash stable.*

The next example shows that, again on the additive separable preference domain, contractual Nash stability is *strictly stronger* than contractual individual stability. Namely, for the hedonic game shown in the example, no contractual Nash stable coalition structure exists and a contractual individual stability exists.

Example 4 *Consider the hedonic game $\langle N, \succeq \rangle$ with $N = \{1,2,3,4\}$ and an additive separable preference profile $\succeq$ defined by the following v_is.*

$$\begin{array}{llll} v_1(1) = 0, & v_1(2) = 0, & v_1(3) = -2, & v_1(4) = 1, \\ v_2(1) = -2, & v_2(2) = 0, & v_2(3) = 0, & v_2(4) = 1, \\ v_3(1) = 0, & v_3(2) = -2, & v_3(3) = 0, & v_3(4) = 1, \\ v_4(1) = 0, & v_4(2) = 0, & v_4(3) = 0, & v_4(4) = 0. \end{array}$$

As mentioned in the previous section, on any preference domain, a contractual individually stable coalition structure always exists. Let $\Pi = \{\{1\},\{2\},\{3,4\}\}$. Then, we have $\textsc{weak}(\Pi) = \{\{1,4\},\{2,4\}\}$*, but $\{1,4\},\{2,4\} \notin \textsc{cont}(\Pi)$ from $\{3\} \prec_3 \{3,4\}$. Thus,* $\textsc{nash}(\Pi) \cap \textsc{cont}(\Pi) \cap \textsc{weak}(\Pi) = \emptyset$*, i.e., Π is contractual individually stable.*

Now we show that a contractually Nash stable coalition structure does not exist. Observe that we have $v_1(3) = v_2(1) = v_3(2) = -2$ and $\sum_{j \in X} v_i(j) \leq 1$ for each $i \in N$ and for each $X \in \mathcal{A}^i$. Thus, for each coalition structure Π,

- $\{2\} \in \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$ *if* $\Pi(1) = \Pi(2)$,
- $\{1\} \in \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$ *if* $\Pi(1) = \Pi(3)$,
- $\{3\} \in \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$ *if* $\Pi(2) = \Pi(3)$,

and hence, Π is contractually Nash stable only if $\Pi(1)$, $\Pi(2)$, $\Pi(3)$ are three different coalitions. Let Π be a coalition structure for which this is indeed the case. Then, we have

- $\{1,4\} \in \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$ *if* $\Pi(4) = \{4\}$,
- $\{1,3,4\} \in \textsc{nash}(\Pi) \cap \textsc{cont}(\Pi)$ *if* $\Pi(4) = \{1,4\}$,

- $\{1,2,4\} \in \text{NASH}(\Pi) \cap \text{CONT}(\Pi)$ *if* $\Pi(4) = \{2,4\}$,

- $\{2,3,4\} \in \text{NASH}(\Pi) \cap \text{CONT}(\Pi)$ *if* $\Pi(4) = \{3,4\}$.

Therefore, $\text{NASH}(\Pi) \cap \text{CONT}(\Pi) \neq \emptyset$ *for each coalition structure* Π, *i.e., a contractually Nash stable coalition structure does not exist for this game.*

We are ready now to provide a suffi cient condition for the existence of a contractually Nash stable coalition structure. In order to state our result, we allow for a larger domain, namely the domain of separable preference profiles, and impose a weaker version of mutuality.

Let $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ be a preference profile. We say that $\succeq$ is *weakly mutual* if, for each $i \in N$,

- $i \in G_j$ for some $j \in N$ if $G_i \neq \emptyset$.

Clearly, weak mutuality is implied by mutuality.

Proposition 2 *Let* $\langle N, \succeq \rangle$ *be a separable hedonic game satisfying weak mutuality. Then, a contractually Nash stable coalition structure exists.*

Proof. Let $\langle N, \succeq \rangle$ be as above and let $S = \bigcup_{i \in N} G_i$. By definition, for each $i \in S$, there exists $j \in N$ such that $i \in G_j$, and conversely, for each $i \in N \setminus S$, we have $i \notin G_j$ for each $j \in N$. Then, let Π be a coalition structure defined as follows:

$$\Pi = \begin{cases} \{S\} \cup \{\{i\} \mid i \in N \setminus S\} & \text{if } S \neq \emptyset, \\ \{\{i\} \mid i \in N\} & \text{otherwise.} \end{cases}$$

Observe that we have $\Pi(i) = S$ if $i \in S$, and otherwise $\Pi(i) = \{i\}$. In the following, we show that Π is contractually Nash stable if $\succeq$ satisfies weak mutuality, i.e., $i \in G_j$ for some $j \in N$ if $G_i \neq \emptyset$. By definition of $\text{NASH}(\Pi)$, it suffi ces to show that$\{i\} \cup \Pi(j) \notin \text{NASH}(\Pi) \cap \text{CONT}(\Pi)$ for each $i, j \in N$.

Let $i \in N \setminus S$. By definition, we have $\Pi(i) = \{i\}$ (no matter S is empty or not). Moreover, we have $i \notin G_j$ for each $j \in N$, and by weak mutuality, we have $G_i = \emptyset$ (i.e., $j \in U_i \cup B_i$ for each $j \in N$). It follows that $\{i\} \cup \Pi(j) \preceq_i \{i\} = \Pi(i)$ for each $j \in N$, and therefore, $\{i\} \cup \Pi(j) \notin \text{NASH}(\Pi)$ for each $j \in N$.

Let $i \in S$. By definition, we have $\Pi(i) = S$, and there exists $k \in N$ such that $i \in G_k$, and thus, $G_k \neq \emptyset$. By weak mutuality, we have $k \in G_\ell$ for some $\ell \in N$, and thus, $k \in S$

as well. From $i \in G_k$ and $i \in S = \Pi(k)$, we have $\Pi(k) \setminus (\{i\} \cup \Pi(j)) \prec_k \Pi(k)$ for each $j \in N \setminus S$, and thus, $\{i\} \cup \Pi(j) \notin \text{CONT}(\Pi)$ for each $j \in N \setminus S$. Moreover, we have $\{i\} \cup \Pi(j) = \Pi(i) \notin \text{CONT}(\Pi)$ for each $j \in S$. Therefore, $\{i\} \cup \Pi(j) \notin \text{CONT}(\Pi)$ for each $j \in N$. ■

Remark 3 *Observe that players'preferences in the hedonic game of Example 4 are separable (more precisely, additive separable), but they do not satisfy weak mutuality. Hence, weak mutuality is a critical condition for the existence of contractually Nash stable coalition structures.*

6 Weak individual stability

We turn now to the study of weak individual stability. Recall that a coalition structure Π is *weak individually stable* if

$$\text{NASH}(\Pi) \cap \text{STRONG}(\Pi) = \emptyset,$$

i.e., there does not exist a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that $X \cup \{i\} \succ_j \Pi(j)$ for all $j \in X \cup \{i\}$.

Notice that, as it can be illustrated by means of Example 3 (cf. [4]), there are additive separable hedonic games satisfying mutuality with no weak individually stable coalition structures. It follows that even if preferences are additive separable, requiring mutuality does not suffi ce for the existence of weak individually stable coalition structure.

In order to present an existence result for weak individually stable coalition structures, we introduce a solidarity condition. It has a very intuitive interpretation and says that if a player j likes another player i, then all "undesirable" players for i are also "undesirable" for j.

Let $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ be a preference profile. We say that $\succeq$ satisfies *solidarity* if, for all $i, j \in N$,

- $i \in G_j$ implies $B_i \subseteq B_j \cup \{j\}$.

In our next lemma, we show an important implication of this condition.

Lemma 4 *Let $\langle N, \succeq \rangle$ be a hedonic game satisfying solidarity. For every $i, j \in N$ with $i \neq j$, $G_i \cup U_i \cup \{i\} = G_j \cup U_j \cup \{j\}$ if $i \in G_j$ and $j \in G_i$.*

Proof. Let $\langle N, \succeq \rangle$ be as above, and let $i, j \in N$ be such that $i \neq j$, $i \in G_j$, and $j \in G_i$. By solidarity, we have $B_i \subseteq B_j \cup \{j\}$ and $B_j \subseteq B_i \cup \{i\}$. It follows that $G_j \cup U_j \subseteq G_i \cup U_i \cup \{i\}$ and $G_i \cup U_i \subseteq G_j \cup U_j \cup \{j\}$. Therefore, $G_i \cup U_i \cup \{i\} = G_j \cup U_j \cup \{j\}$. ■

As it turns out, this implication of the solidarity condition guarantees the existence of weak individually stable coalition structures on the class of separable games.

Proposition 3 *Let $\langle N, \succeq \rangle$ be a separable hedonic game satisfying solidarity. Then, a weak individually stable coalition structure exists.*

Proof. Let $\langle N, \succeq \rangle$ be as above and let Π be the coalition structure constructed by the following algorithm.

Step 1. Set $\Pi := \emptyset$ and $R := N$.

Step 2. Repeats the following until $R = \emptyset$:

- Find one of the largest coalitions $X \subseteq R$ such that for each nonempty proper subset Y of X, there exists pair (i, j) of $i \in Y$ and $j \in X \setminus Y$ satisfying $i \in G_j$ and $j \in G_i$.
- Set $\Pi := \Pi \cup \{X\}$ and $R := R \setminus X$.

Step 3. Return Π.

In graph theoretical terms, each $X \in \Pi$ is a connected component of the undirected graph $G = (N, E)$ with node set N and edge set $E = \{\{i, j\} \subseteq N \mid i \neq j, i \in G_j, j \in G_i\}$. Hence, Π is the unique partition of N into connected components of $G = (N, E)$. Obviously, Π constructed by this algorithm is a coalition structure. By applying Lemma 4 it can be easily shown that $G_i \cup \{i\} = G_j \cup \{j\}$ for each $X \in \Pi$ and for every $i, j \in X$, which implies that $X \subseteq G_i \cup \{i\}$ for each $i \in X$.

Now we show that Π is weak individually stable by contradiction. Suppose there is a pair (i, X) of $i \in N$ and $X \in \Pi \cup \{\emptyset\}$ such that $X \cup \{i\} \succ_j \Pi(j)$ for all $j \in X \cup \{i\}$, i.e., $X \cup \{i\} \in \text{NASH}(\Pi) \cap \text{STRONG}(\Pi)$. For each $i \in N$, we have $\Pi(i) \subseteq G_i \cup \{i\}$ from the construction of Π, which implies $\Pi(i) \succeq_i \{i\}$. Therefore, $\{i\} \notin \text{NASH}(\Pi)$, i.e., $X \neq \emptyset$.

Let $X \in \Pi$. From $X \cup \{i\} \succ_i \Pi(i)$, it is obvious that $X \neq \Pi(i)$, and by separability, there exists $k \in X$ such that $k \in G_i$. From $X \cup \{i\} \succ_j X$ for each $j \in X$ and separability, we have $i \in G_k$. Thus, $X \cup \Pi(i)$ is such that, for each nonempty proper subset Y of

$X \cup \Pi(i)$, there exists pair (i, j) of $i \in Y$ and $j \in (X \cup \Pi(i)) \setminus Y$ satisfying $i \in G_j$ and $j \in G_i$, which contradicts to the largestness of each coalition in Π. ■

Notice finally that if we narrow the domain of separable preferences by requiring that each player views every other player either as a desirable or as a undesirable coalitional partner (i.e., $U_i = \emptyset$ for all $i \in N$), then the solidarity condition guarantees the existence of individually stable coalition structures as well. This is due simply to the fact that in such an environment weak individual stability and individual stability coincide.

7 Conclusion

The taxonomy of stability concepts for hedonic games offered in this paper relies on the simple observation that each deviation from a coalition structure reflects different degrees of social intervention in one's strong wish to migrate to another group of players. The differences in the social intervention were taken into account when constructing the different sets of coalitional deviations that, in turn, led to several new stability notions. It was shown that contractual strictly core stable coalition structures always exist, while on the class of separable games one needs additional conditions in order to assure the existence of contractually Nash stable and weak individually stable coalition structures.

As mentioned in the Introduction, our taxonomy considers only myopic stability concepts. However, it would be worthy to place the newly introduced stability notions in a framework in which players are farsighted in the sense that they take into account how their decisions to form a coalition will affect in the future the decisions of other players. If players' preferences are strict, it was shown in [9] that all core stable structures are coalitional farsightedly stable as well, and that a corresponding result holds true for Nash stability but neither for individual stability nor for contractual individual stability. Since both contractual strict core stability and weak individual stability are weaker concepts than individual stability, one would expect that the corresponding farsighted notions would refine their myopic counterparts. On the other hand, contractual Nash stability is weaker than Nash stability and stronger than contractual individual stability. Hence, one needs further investigations on how the relationship between the corresponding farsighted and myopic counterparts of this stability notion would look like.

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