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Stable coalition structures in simple games with veto control*

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Abstract

In this paper we study hedonic coalition formation games in which players' preferences over coalitions are induced by a semi-value of a monotonic simple game with veto control. We consider partitions of the player set in which the winning coalition contains the union of all minimal winning coalitions, and show that each of these partitions belongs to the strict core of the hedonic game. Exactly such coalition structures constitute the strict core when the simple game is symmetric. Provided that the veto player set is not a winning coalition in a symmetric simple game, then the partition containing the grand coalition is the unique strictly core stable coalition structure.

JEL Classification: D72, C71.

Keywords: Banzhaf value, hedonic game, semi-value, Shapley value, simple game, strict core.

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1 Introduction

In this paper we address the question which players in a monotonic simple game with veto control "should" form a winning coalition. Inspired by the work of Shenoy (1979), we fix a *semi-value* (i.e., a symmetric probabilistic value (cf. Weber (1988), Monderer and Samet (2002)) for the simple game and use it to extract players' preferences over coalitions in a *hedonic coalition formation game* (cf. Bogomolnaia and Jackson (2002) and Banerjee et al. (2001)). A solution of this (and each) hedonic game is a partition of the set of players into coalitions. In this way, we have a suitable environment in which the question of stability can be approached. As it can be easily seen, it is not possible a coalition structure to be stable (to be defined later) if it does not contain a winning coalition. Hence, the answer to the question which partitions are stable is at the same time an answer to the question which winning coalitions should form with respect to stability concerns.

We have chosen the *strict core* as our stability concept for hedonic games, being the strongest stability notion based on coalitional deviations. As it turns out, if the winning coalition in a coalition structure contains the *union of all minimal winning coalitions*, then the coalition structure belongs to the strict core of the hedonic game. In order to fully characterize the strict core, we consider symmetric simple games with veto control and show that the winning coalition in each strictly core stable partition contains the union of all minimal winning coalitions. Further, provided that the veto player set is not a winning coalition in a symmetric simple game, the partition containing the grand coalition turns out to be the unique strictly core stable coalition structure.

The way of modelling we follow in this paper is a stylized one since, by using a semi-value to induce preferences over coalitions, we assume players to be purely office seeking. This line of study has a long tradition since Riker's (1962) classical monograph (see Laver and Schofield (1990) for an extensive survey). Peleg (1981) and Einy (1985) develop a theory of coalition formation in simple games with dominant players, whereas Carreras (1996) studies, among others, the formation of partnerships (cf. Kalai and Samet (1987)) in simple games. In contrast to these papers, we do not presuppose any (additional) internal structure on the winning coalition that forms; its internal structure is rather determined by the notion of strict core stability applied to the induced hedonic game. More precisely, we bring together a power index (applied to a monotonic simple game with veto control) with the notion of (strict) core stability, arriving at the "most stable" winning coalition containing the union of all minimal winning coalitions. The methodology can of course be applied to a broader class of problems, which leads us to a property of the simple game that appears to be crucial. Shenoy (1979, Theorem 7.4) provides a sufficient condition for nonemptiness of the core of an abstract game appropriately induced by the Shapley value (which is the unique efficient semi-value) of a proper monotonic simple game. The condition says that the simple game should not exhibit the paradox of smaller coalitions (to be defined later) with respect to the Shapley value. However, as we show in Section 3, a monotonic simple game with veto control (being proper) satisfies Shenoy's condition with respect to the corresponding semi-value if and only if the veto player set is a winning coalition. Thus, the classes of simple games considered by Shenoy (1979) and in the present paper are rather complementary.

Surprisingly, up to our knowledge, situations of coalition formation involving veto players have never been considered in the literature. One particular reason is that in a simple game every coalition of parties which can constitute the necessary majority to form a government is deemed as a winning coalition. However, this way of modeling neglects many aspects a political party may consider as important when forming its preferences over the possible governments it may be a member of – for instance, the political views of the other members. And if these considerations are taken into account while incorporating a government formation situation into the framework of simple games, the resulting simple game will have a high probability to be veto-controlled. That's why our study provides an alternative point of view for the analysis and explanation of government formation situations. Moreover, our results are valid for both of two classical power indices, the Shapley-Shubik index (cf. Shapley (1953), Shapley and Shubik (1954)) and the Banzhaf index (cf. Bhanzaf (1965)), and also for the whole class of semi-values which are considered by Carreras et al. (2003) as very consistent alternatives to the mentioned classical power indices.

The paper is organized as follows. Section 2 includes basic notions and solution concepts from the theory of simple games and hedonic games. The main results are presented in Section 3. We conclude in Section 4 with some final remarks.

2 Preliminaries

2.1 Simple games

Let N be a finite *set of players*, which we will keep fixed throughout the paper. A transferable utility game (TU-game) with player set N is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Each subset of N is a called a *coalition*. We denote the set of TU-games with player set N by $\mathcal{G}^N$.

A game $v \in \mathcal{G}^N$ is *monotonic* if $v(S) \geq v(T)$ for every $S, T \in 2^N$ with $T \subseteq S$. A player $i \in N$ is a *null player* in v if $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are *symmetric* in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Given $v \in \mathcal{G}^N$ and $S \in 2^N$, the restriction of v to S (a *subgame* of v) is denoted by v_S and is defined by $v_S(T) = v(T)$ for every $T \subseteq S$.

A game $v \in \mathcal{G}^N$ is called *simple* if v is monotonic, $v(S) \in \{0, 1\}$ for every $S \in 2^N$ and $v(N) = 1$. We refer to a coalition $S \subseteq N$ with $v(S) = 1$ as a *winning coalition.* A winning coalition S is called *minimal winning* if there does not exist a coalition $T \subset S$ which is winning. We denote by $\mathcal{W}^v$ the *set of winning coalitions* and by $\mathcal{MW}^v$ the *set of minimal winning coalitions* in the simple game v (cf. Shapley (1962)). Notice that every simple game v is characterized by the set $\mathcal{MW}^v$ of its minimal winning coalitions.

A simple game v is *proper* if $v(S) = 1$ implies $v(N \setminus S) = 0$. A player $i \in N$ is a *veto player* in a simple game v if for all $S \subseteq N$, $S \in \mathcal{W}^v$ implies $i \in S$; the set of all veto players in v is denoted by $veto\,(v)$. Notice that $(veto\,(v) \cap S) \subseteq veto\,(v_S)$ is valid for each $S \subseteq N$. The *set of all monotonic simple games with veto control* on the player set N will be denoted by $\mathcal{S}^N$. Observe that $v \in \mathcal{S}^N$ implies the properness of v.

A *solution* (of a TU-game) is a mapping φ: $\mathcal{G}^N \rightarrow \mathbb{R}^N$ taking each $v \in \mathcal{G}^N$ to a single vector in $\mathbb{R}^N$, i.e., it assigns a real number $\varphi_i\,(v)$ to each player $i \in N$. A solution φ is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$ and it is *symmetric* if $\varphi_i\,(v) = \varphi_j\,(v)$ for all $i, j \in N$ who are symmetric in v. A solution φ satisfies the *null player property* if $\varphi_i(v) = 0$ for all $i \in N$ who are null players in v.

An efficient, symmetric solution satisfying the null player property is the *Shapley value* (cf. Shapley (1953), Shapley and Shubik (1954)) $Sh : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ defined by

$$Sh_i(v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right) \quad (i \in N).$$

A probabilistic value also assigns to each player an average of his marginal contributions and hence, it keeps an essential feature of the Shapley value. However, it might fail to satisfy either efficiency or symmetry. To be more precise, let P_N^i denote the set of probability distributions on $2^{N \setminus \{i\}}$, the family of coalitions not containing i. A solution $F : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ is called a *probabilistic value* (cf. Weber (1988)) if for every $v \in \mathcal{G}^N$ and $i \in N$,

$$F_i(v) = \sum_{T \subseteq N \setminus \{i\}} p^i(T) \left(v(T \cup \{i\}) - v(T)\right),$$

where $p^i \in P_N^i$ can be interpreted as player i's subjective evaluation of the probability of joining different coalitions. For instance, the probabilistic value which is defined by $p^i(T) = \frac{1}{|N|}\binom{|N|-1}{|T|}^{-1}$, $i \in N$, is the Shapley value and the one which is defined by $p^i(T) = \frac{1}{2^{|N|-1}}$, $i \in N$, is the Banzhaf value (cf. Banzhaf (1965)). A symmetric probabilistic value is called a *semi-value* (cf. Monderer and Samet (2002)). The set of all semi-values on $\mathcal{G}^N$ is denoted by $\mathcal{F}$.

2.2 Hedonic games

For each player $i \in N$ we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A partition Π of N is called a *coalition structure*. For each coalition structure Π and each player $i \in N$, we denote by $\Pi(i)$ the coalition in Π containing player i, i.e., $\Pi(i) \in \Pi$ and $i \in \Pi(i)$. The set of all coalition structures of N will be denoted by $\mathbf{C}^N$.

Further, we assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, i.e., a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. Denote by $\succ_i$ and $\sim_i$ the strict and indifference relation associated with $\succeq_i$ and by $\succeq := (\succeq_1, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. A player's preference relation over coalitions canonically induces a preference relation over coalition structures in the following way: For any two coalition structures Π and Π', player i weakly prefers Π to Π' if and only if he weakly prefers "his" coalition in Π to the one in Π', i.e., $\Pi \succeq_i \Pi'$ if and only if $\Pi(i) \succeq_i \Pi'(i)$. Hence, we assume that players' preferences over coalition structures are *purely hedonic*, i.e., they are completely characterized by their preferences over coalitions. Finally, a *hedonic game* $(N, \succeq)$ is a pair consisting of the set of players and a preference profile.

Unlike solution concepts for (simple) cooperative games do, there is no worth to distribute in hedonic games. The relevant question is rather, which coalition structure should form, taking players' preferences into account. The basic property that we require is strict core stability.

Given a hedonic game $(N, \succeq)$, a partition Π of N is *strictly core stable* for $(N, \succeq)$, if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ holds for all $i \in X$ and $X \succ_j \Pi(j)$ is true for some player $j \in X$. Π is *core stable* if there does not exist a nonempty coalition X such that $X \succ_i \Pi(i)$ holds for each $i \in X$. Put in other words, a coalition structure Π is strictly core stable if no group of players are willing to form a coalition, so that each player is at least as well off with this new coalition and some player is better off compared to the corresponding coalitions in Π. Clearly, a weaker notion of coalitional deviation is incorporated in the definition of core stability

- everyone in the deviating coalition should be better off. Observe that strict core stability implies core stability. In what follows, we denote by $SC(N, \succeq)$ the set of all strictly core stable coalition structures of a hedonic game $(N, \succeq)$. Alternatively, we call $SC(N, \succeq)$ the *strict core* of $(N, \succeq)$.

3 Coalition formation

Given a game $v \in \mathcal{S}^N$ and a semi-value $F \in \mathcal{F}$, we define a hedonic game $(N, \succeq)$ by inducing players' preferences over coalitions in the following way (cf. Shenoy (1979), Dimitrov and Haake (2005)). For each $i \in N$ and for all $S, T \in \mathcal{N}_i$,

$$S \succeq_i T \text{ if and only if } F_i(v_S) \geq F_i(v_T). \tag{1}$$

According to (1), player i's preferences over any two coalitions S and T he may be a member of are induced by i's semi-value in the simple game restricted to S and T, respectively. Notice that paying attention to the corresponding coalitions is compatible with the very definition of a hedonic game - each player in such a game evaluates any two coalition structures based only on his preferences over the coalitions in the two partitions he belongs to (cf. Aumann and Dréze (1974), Shenoy (1979)).

3.1 Strict core existence

We now turn to the question whether there exist strictly core stable coalition structures for hedonic games induced as in (1). For $v \in \mathcal{S}^N$, let

$$\mathbf{P}^v := \{S \subseteq N \mid S \supseteq (\cup_{S' \in \mathcal{MW}^v} S')\}$$

and

$$\mathbf{CP}_N^v := \left\{\Pi \in \mathbf{C}^N \mid \Pi \cap \mathbf{P}^v \neq \emptyset\right\}.$$

In other words, the set $\mathbf{CP}_N^v$ consists of all coalition structures containing a winning coalition which includes all minimal winning coalitions. Our main result in this paper states that all coalition structures from $\mathbf{CP}_N^v$ are strictly core stable. In order to present this result, we need the characterization of semi-values defined on games with finite support provided by Dubey et al. (1981).

Theorem (Dubey et al. (1981), Theorem 1(a)) *Let U be an infinite set of players and Γ denote the space of all TU-games on U with a finite carrier. Then F is a semi-value on Γ if and only if there exists a Borel probability measure $\mathcal{P}$ on $[0,1]$ such that for every finite coalition N and for every $v \in \Gamma$ with carrier N, $F(v) = F_{\mathcal{P}}(v)$, where*

$$F_{\mathcal{P},i}(v) = \sum_{S \subseteq N \setminus \{i\}} \beta^{|N|}(|S|) \left(v(T \cup \{i\}) - v(T)\right), \quad i \in N \tag{2}$$

with

$$\beta^{|N|}(|S|) = \int_0^1 t^{|S|}(1-t)^{(|N|-|S|-1)} d\mathcal{P}(t). \tag{3}$$

Moreover, the correspondence $\mathcal{P} \to F_{\mathcal{P}}$ is one-to-one.

Taking into account the above characterization, the following two lemmas will be helpful.

Lemma 1 *Let $v \in \mathcal{S}^N$, $F \in \mathcal{F}$, $T \in \mathcal{W}^v \setminus \{N\}$ and $j \in N \setminus T$. Then, $F_i\left(v_{T\cup\{j\}}\right) \geq F_i\left(v_T\right)$ for each $i \in veto(v)$.*

Proof. For $Q \subseteq N$, let MW_Q^v denote the set of all minimal winning coalitions in v that are contained in Q. Observe that $MW_{T\cup\{j\}}^v \supseteq MW_T^v$ and let $i \in veto(v)$. We have by (2),

$$F_i(v_{T\cup\{j\}}) = \textstyle\sum_{\{Q \subseteq (T\cup\{j\})\setminus\{i\} | v(Q)=0, v(Q\cup\{i\})=1\}} \beta^{|T|+1}(|Q|)$$

and

$$F_i(v_T) = \textstyle\sum_{\{Q \subseteq T\setminus\{i\} | v(Q)=0, v(Q\cup\{i\})=1\}} \beta^{|T|}(|Q|).$$

We establish the inequality $F_i(v_{T\cup\{j\}}) \geq F_i(v_T)$ by first showing that $\mathcal{A} \supseteq \mathcal{B} \cup \mathcal{C}$, where the sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined as follows:

$$\begin{aligned} &\mathcal{A} := \{Q \subseteq (T\cup\{j\}) \setminus \{i\} \mid v(Q) = 0, v(Q\cup\{i\}) = 1\}, \\ &\mathcal{B} := \{Q \subseteq T \setminus \{i\} \mid v(Q) = 0, v(Q\cup\{i\}) = 1\}, \\ &\mathcal{C} := \{(Q\cup\{j\}) \subseteq (T\cup\{j\}) \setminus \{i\} \mid Q \in \mathcal{B}\}. \end{aligned}$$

Let $R \in \mathcal{B} \cup \mathcal{C}$. If $R \in \mathcal{B}$, then R is obviously a member of $\mathcal{A}$. Suppose now that $R \in \mathcal{C}$. Then, there exists $R' \in \mathcal{B}$ such that $R = R' \cup \{j\}$. Moreover, R is a losing coalition since $i \notin R$ is a veto player. We have also $R \cup \{i\} \in \mathcal{W}^v$ by $R' \cup \{i\} \in \mathcal{W}^v$ and the monotonicity of v. Hence, $R \in \mathcal{A}$ which implies $\mathcal{A} \supseteq \mathcal{B} \cup \mathcal{C}$. Furthermore, it can easily be observed that the inclusion is strict, i.e., $\mathcal{A} \supset \mathcal{B} \cup \mathcal{C}$, when $MW^v_{T \cup \{j\}} \supset MW^v_T$, and $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ when $MW^v_{T \cup \{j\}} = MW^v_T$.

Then,

$$\begin{aligned}
F_i(v_{T\cup\{j\}}) &= \textstyle\sum_{Q \in \mathcal{A}} \beta^{|T|+1}(|Q|) \\
&\geq \textstyle\sum_{Q \in \mathcal{B} \cup \mathcal{C}} \beta^{|T|+1}(|Q|) \\
&= \textstyle\sum_{Q \in \mathcal{B}} \left[\beta^{|T|+1}(|Q|) + \beta^{|T|+1}(|Q|+1)\right] \\
&= \textstyle\sum_{Q \in \mathcal{B}} [\int_0^1 t^{|Q|}(1-t)^{(|T|+1-|Q|-1)} d\mathcal{P}(t) + \\
&\quad \textstyle\int_0^1 t^{|Q|+1}(1-t)^{(|T|+1-(|Q|+1)-1)} d\mathcal{P}(t)] \\
&= \textstyle\sum_{Q \in \mathcal{B}} [\int_0^1 t^{|Q|}(1-t)^{(|T|-|Q|-1)} d\mathcal{P}(t)] \\
&= \textstyle\sum_{Q \in \mathcal{B}} \beta^{|T|}(|Q|) \\
&= F_i(v_T)
\end{aligned}$$

where the third equality follows from (3) and hence, the assertion follows. Notice that the inequality is strict when $MW^v_{T \cup \{j\}} \supset MW^v_T$ and $F_i(v_{T\cup\{j\}}) = F_i(v_T)$ when $MW^v_{T \cup \{j\}} = MW^v_T$. ■

Lemma 2 *Let $v \in \mathcal{S}^N$, $F \in \mathcal{F}$ and $S = \cup_{S' \in \mathcal{MW}^v} S'$. Then, for each $T \subseteq N$, $F_i(v_S) \geq F_i(v_T)$ for each $i \in veto(v)$.*

Proof. Let $T \subseteq N$. If T is a losing coalition, by the monotonicity of v, we are done. Obviously, $MW^v_T = MW^v_S$ for every $T \supseteq S$. Then, by Lemma 1,

$$F_i(v_T) = F_i(v_S) \text{ for each } i \in S \text{ and each } T \supseteq S. \tag{4}$$

So, assume that $T \in \mathcal{W}^v$ and $T \not\supseteq S$, and let $j \in N \setminus T$. In view of the proof of Lemma 1, $F_i\left(v_{T\cup\{j\}}\right) = F_i(v_T)$ for each $i \in veto(v)$ if $MW^v_{T\cup\{j\}} = MW^v_T$, and $F_i\left(v_{T\cup\{j\}}\right) > F_i(v_T)$ for each $i \in veto(v)$ if $MW^v_{T\cup\{j\}} \supset MW^v_T$. Consider a sequence of players $j_1, \ldots, j_\ell$ such that $\{j_1, \ldots, j_\ell\} = S \setminus T$; thus, $T \cup \{j_1, \ldots, j_\ell\} \supset S$. Notice that, by the definition of S, there is $k \in \{0, \ldots, \ell-1\}$ such that $MW^v_{T\cup\{j_1,\ldots,j_{k+1}\}} \supset$

$MW^v_{T\cup\{j_1,\ldots,j_k\}}$ (if $k=0$, we set $T\cup\{j_1,\ldots,j_0\}=T$). Then, by (4) and the repeated use of Lemma 1,

$$\begin{aligned} F_i(v_S) &= F_i(v_{T\cup\{j_1,\ldots,j_\ell\}}) \geq F_i(v_{T\cup\{j_1,\ldots,j_{\ell-1}\}}) \geq \ldots \geq F_i\left(v_{T\cup\{j_1,\ldots,j_{k+1}\}}\right) \\ &> F_i\left(v_{T\cup\{j_1,\ldots,j_k\}}\right) \geq \ldots \geq F_i(v_T) \end{aligned}$$

for each $i \in veto(v)$. ■

We are ready now to present our strict core existence result.

Proposition 1 *Let* $v \in \mathcal{S}^N$, $F \in \mathcal{F}$ *and* $(N, \succeq)$ *be induced as in* (1). *Then,* $\mathbf{CP}^v_N \subseteq SC(N,\succeq)$.

Proof. Let Π be a partition of N containing $S = \cup_{S'\in\mathcal{MW}^v} S'$. Since $\mathbf{CP}^v_N \subseteq SC(N,\succeq)$ follows easily by Lemma 2 if $\Pi \in SC(N,\succeq)$, we proceed by showing the strict core stability of Π.

If $\mathcal{MW}^v = \{veto(v)\}$, then each player in $N \setminus veto(v)$ is a null player in v (and thus, in each of its corresponding subgames). Hence, there is no coalition $T \in 2^N \setminus \Pi$ that makes any of its members strictly better off in comparison to the corresponding coalitions in Π.

Suppose now that $\mathcal{MW}^v \neq \{veto(v)\}$ and to the contrary, let there be a (winning) coalition $T \subseteq N$ such that

$$F_i(v_T) \geq F_i\left(v_{\Pi(i)}\right) \text{ for each } i \in T \tag{5}$$

and

$$F_j(v_T) > F_j\left(v_{\Pi(j)}\right) \text{ for some } j \in T. \tag{6}$$

Consider the following two possible cases:

Case 1: $S \subseteq T$. By (4), $F_i(v_T) = F_i(v_S)$ for each $i \in S$ and each $T \supseteq S$. Hence, $F_i(v_T) = F_i\left(v_{\Pi(i)}\right) = F_i(v_S)$ for each $i \in S\cap T$, i.e., (6) should hold for some $j \in T\setminus S$. Notice however that, by the monotonicity of v, (6) implies $F_j(v_T) > 0$ which is, since $j \in N \setminus S$ is a null player in v (and thus, in v_T), a contradiction to $F_j(v_T) = 0$.

Case 2: $S \nsubseteq T$. In view of the proof of Lemma 2, we have $F_i\left(v_{\Pi(i)}\right) = F_i(v_S) > F_i(v_T)$ for each $i \in veto(v) \subseteq T$, a contradiction to (5). ■

We would like finally to mention that, given a simple game with veto control, inducing a hedonic game by a semi-value (as in (1)) is crucial for the nonemptiness of the strict core. Our first example illustrates this point.

Example 1 *Let $N = \{1, 2, 3\}$ and the game $v \in \mathcal{S}^N$ be given by its minimal winning coalitions $\mathcal{MW}^v = \{12, 13\}$. Let φ be a solution such that*

$$\varphi_1(v_S) = \begin{cases} \frac{1}{2} & \text{if } S \in \{12, 13, 123\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_2(v_S) = \begin{cases} \frac{1}{2} & \text{if } S = 12, \\ \frac{3}{8} & \text{if } S = 123, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_3(v_S) = \begin{cases} \frac{1}{2} & \text{if } S = 13, \\ \frac{3}{8} & \text{if } S = 123, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that φ is inefficient since $\varphi_1(v) + \varphi_2(v) + \varphi_3(v) \neq 1$. Let us first show that φ is not a semi-value. If the opposite were the case, then, by (2),

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \beta^{|N|}(|S|)\left(v(T \cup \{i\}) - v(T)\right), \quad i \in N.$$

Since $\varphi_2(v) = \beta^{|N|}(|\{1\}|) = \frac{3}{8}$, $\beta^{|N|}(1)$ must be equal to $\frac{3}{8}$. But then $\varphi_1(v) = \beta^{|N|}(|\{2\}|) + \beta^{|N|}(|\{3\}|) + \beta^{|N|}(|\{2,3\}|) \geq \frac{6}{8}$ contradicting with $\varphi_1(v) = \frac{1}{2}$. Hence, φ is not a semi-value.

Taking the payoffs according to φ to extract preferences over coalitions, the players evaluate coalitions as follows:

$$12 \sim_1 13 \sim_1 123 \succ_1 1.$$

$$12 \succ_2 123 \succ_2 2 \sim_2 23.$$

$$13 \succ_3 123 \succ_3 3 \sim_3 23.$$

Collecting all preferences, we obtain a hedonic game $(N, \succeq)$ with preferences induced by φ. Inspecting $(N, \succeq)$, one finds that $SC(N, \succeq) = \emptyset$.

3.2 Symmetric games

Notice that the inverse inclusion to the one in Proposition 1 can be proved only in very special cases. For instance, it is easy to show that if either $veto(v) \in \mathcal{W}^v$ or $|N| \leq 3$, then $\mathbf{CP}_N^v = SC(N, \succeq)$. However, in general and as exemplified next, the strict core might be strictly larger than $\mathbf{CP}_N^v$.

Example 2 *Let $N = \{1, 2, 3, 4\}$ and the game $v \in \mathcal{S}^N$ be given by its minimal winning coalitions $\mathcal{MW}^v = \{12, 134\}$. Players' payoffs according to the Shapley value are*

$$Sh_1(v_S) = \begin{cases} \frac{7}{12} & \text{if } S = N, \\ \frac{1}{2} & \text{if } S \in \{12, 123, 124\}, \\ \frac{1}{3} & \text{if } S = 134, \\ 0 & \text{otherwise.} \end{cases} \qquad Sh_2(v_S) = \begin{cases} \frac{1}{2} & \text{if } S \in \{12, 123, 124\}, \\ \frac{3}{12} & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

$$Sh_3(v_S) = \begin{cases} \frac{1}{3} & \text{if } S = 134, \\ \frac{1}{12} & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases} \qquad Sh_4(v_S) = \begin{cases} \frac{1}{3} & \text{if } S = 134, \\ \frac{1}{12} & \text{if } S = N, \\ 0 & \text{otherwise,} \end{cases}$$

Taking this to extract preferences over coalitions, the players evaluate coalitions as follows:

$$1234 \succ_1 12 \sim_1 123 \sim_1 124 \succ_1 134 \succ_1 1 \sim_1 13 \sim_1 14.$$
$$12 \sim_2 123 \sim_2 124 \succ_2 1234 \succ_2 2 \sim_2 23 \sim_2 24 \sim_2 234.$$
$$134 \succ_3 1234 \succ_3 3 \sim_3 13 \sim_3 23 \sim_3 34 \sim_3 123 \sim_3 234.$$
$$134 \succ_4 1234 \succ_4 4 \sim_4 14 \sim_4 24 \sim_4 34 \sim_4 124 \sim_4 234.$$

Collecting all preferences, we obtain a hedonic game $(N, \succeq)$. Inspecting $(N, \succeq)$, one finds that $\mathbf{CP}_N^v = \{\{1234\}\} \subset \{\{1234\}, \{12, 34\}, \{12, 3, 4\}\} = SC(N, \succeq)$.

In order to provide a full characterization of the strict core of the induced hedonic game, we will require the underlying simple game to be symmetric. Recall that $v \in \mathcal{S}^N$

is *symmetric*, if $S \in \mathcal{W}^v$ implies $T \in \mathcal{W}^v$ for each coalition T with $veto(v) \subseteq T$ and $|T| = |S|$.

Proposition 2 *Let $v \in \mathcal{S}^N$ be symmetric, $F \in \mathcal{F}$ and $(N, \succeq)$ be induced as in* (1)*. Then, $SC(N, \succeq) = \mathbf{CP}_N^v$.*

Proof. In view of Proposition 1 it is enough to show that if a partition Π contains a winning coalition $T \notin \mathbf{P}^v$, then $\Pi \notin SC(N, \succeq)$.

Notice first that by $T \notin \mathbf{P}^v$, we have $T \neq N$. Let $j \in N \setminus T$ and $i \in T \setminus veto(v)$ (such an i exists since, otherwise, $T \in \mathcal{W}^v$ and $T \setminus veto(v) = \emptyset$ would imply $T \in \mathbf{P}^v$). Consider the coalition $T' = (T \setminus \{i\}) \cup \{j\}$. By the symmetry of v, all non-veto players in T are symmetric in v_T and all non-veto players in T' are symmetric in $v_{T'}$. Hence, by $|T| = |T'|$, $F_i(v_{T'}) = F_i(v_T) = F_i\left(v_{\Pi(i)}\right)$ for each $i \in T' \setminus \{j\}$, and $F_j(v_{T'}) > 0 = F_j\left(v_{\Pi(j)}\right)$. It follows then that T' is a deviation (in the sense of the strict core) from Π and thus, $\Pi \notin SC(N, \succeq)$. ■

The case in which the monotonic simple game v is proper and symmetric was also analyzed by Shenoy (1979)[1]. In his Theorem 7.6, he shows that the core of the hedonic game (induced as in (1) with $F = Sh$) consists in this case only of partitions containing a minimal winning coalition with minimal cardinality. Consider for instance the game (N, v) with $N = \{1, 2, 3\}$ and $\mathcal{MW}^v = \{12, 13, 23\}$. If the hedonic game is induced as in (1) with $F = Sh$, then the reader can easily check that the core of the game consists of the following three partitions: $\{12, 3\}$, $\{13, 2\}$, $\{23, 1\}$. Notice however that, in contrast to Proposition 1, neither of these partitions is strictly core stable.

Finally, we show that the partition containing the grand coalition is the "most" stable coalition structure if $veto(v) \notin \mathcal{W}^v$.

Corollary 1 *Let $v \in \mathcal{S}^N$ be symmetric, $F \in \mathcal{F}$ and $(N, \succeq)$ be induced as in* (1)*. If $veto(v) \notin \mathcal{W}^v$, then $SC(N, \succeq) = \{\{N\}\}$.*

[1] In this work, a monotonic simple game v is defined to be symmetric if $S \in \mathcal{W}^v$ implies $T \in \mathcal{W}^v$ for each coalition T with $|T| = |S|$. Notice that, with this definition of symmetry, a player has veto power in v if and only if $\mathcal{W}^v = \{N\}$.

Proof. It follows from $veto(v) \notin \mathcal{W}^v$ that there is a player $i \in N \setminus veto(v)$ who belongs to a minimal winning coalition. Hence, by the symmetry of v, each player from $N \setminus veto(v)$ is member of a minimal winning coalition; thus, $\cup_{S' \in \mathcal{MW}^v} S' = N$. In view of the proof of Proposition 1, $SC\,(N, \succeq) = \{\{N\}\}$. ■

3.3 Veto games and the paradox of smaller coalitions

As already mentioned, the strict core of a hedonic game is the strongest stability notion based on coalitional deviations. Another possibility, pursued by Shenoy (1979), is to consider the weaker notion of the core. In his Theorem 7.4, Shenoy (1979) shows that if players' preferences over coalitions are induced as in (1) with $F = Sh$, and the simple game does not exhibit the paradox of smaller coalitions with respect to the Shapley value, then the core is nonempty. More precisely, a simple game v *does not exhibit the paradox of smaller coalitions w.r.t. a cooperative solution concept* φ, if for all $S, T \in \mathcal{W}^v$, $S \subseteq T$ implies $\varphi_i\,(v_S) \geq \varphi_i\,(v_T)$ for all $i \in S$. The absence of this paradox in simple games respects the fact that if players form a smaller winning coalition, then their power (as measured by φ) should not decrease since there are fewer players to share the same amount of power. It is worth mentioning that Table A.1 in Shenoy (1979) lists all monotonic and proper simple games with up to four players and verifies presence or absence of the paradox with respect to the Shapley value. In what follows, we present the conditions under which the paradox of smaller coalitions w.r.t. a semi-value $F \in \mathcal{F}$ is not present in a monotonic simple game with veto control.

Proposition 3 *Let $v \in \mathcal{S}^N$, $F \in \mathcal{F}$ and $(N, \succeq)$ be induced as in* (1)*. The game v does not exhibit the paradox of smaller coalitions w.r.t. F if and only if $\mathcal{MW}^v = \{veto(v)\}$.*

Proof. Let $\mathcal{MW}^v = \{veto(v)\}$ and take $R, T \in \mathcal{W}^v$ with $T \supseteq R$. Then, $F_i\,(v_R) \geq F_i\,(v_T)$ for each $i \in R$ follows easily from Lemma 2 by noticing that all players in $T \setminus veto(v)$ are null players in v.

Suppose next that v does not exhibit the paradox of smaller coalitions with respect to F. We show that $|\mathcal{MW}^v| \geq 2$ leads to a contradiction.

Let $S_1, S_2 \in \mathcal{MW}^v$, $S_1 \neq S_2$ and Π be a partition containing $S = \cup_{S' \in \mathcal{MW}^v} S'$. Since $S_1 \subset S$ and v does not exhibit the paradox w.r.t. F,

$$F_i(v_{S_1}) \geq F_i(v_S) \text{ for each } i \in S_1. \tag{7}$$

Since $S_2 \in \mathcal{MW}^v$ with $S_2 \neq S_1$, it follows from Lemma 2 that $F_i(v_S) > F_i(v_{S_1})$ in contradiction to (7). ■

4 Concluding Remarks

In this paper we focussed on the stability of coalition structures containing the union of all minimal winning coalitions (or one of its supersets) in simple games with veto control. An important question about any stable coalition structure is whether there exists a natural coalition formation dynamics which ensures the formation of that coalition structure. For instance, Çiftçi et al. (2006) considers bilateral agreements as an important coalition formation procedure in voting/government formation situations. This work focuses on the Shapley value as an appropriate measure of voting power and analyzes, inspired by Sprumont (1990), the existence of sequences of bilateral agreements that are *population monotonic* in the sense that each player's voting power does not decrease as the coalition to which he belongs grows through the agreements in the sequence. As a result, these authors show that starting from any coalition structure which does not contain any winning coalition, there exists a sequence of population monotonic bilateral agreements among the elements of the starting coalition structure which results in the formation of the union of all minimal winning coalitions (or one of its supersets) if and only if the set of veto players in the simple game is nonempty. Moreover, if the set of veto players is a winning coalition ($SC(N, \succeq) = \mathbf{CP}_N^v$), then every such sequence results in the formation of the set of veto players (or one of its supersets).

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